FIRST-YEAR MATHEMATICS

BRESLICH

John Freeman
Freashman Class
1923

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RENÉ DESCARTES

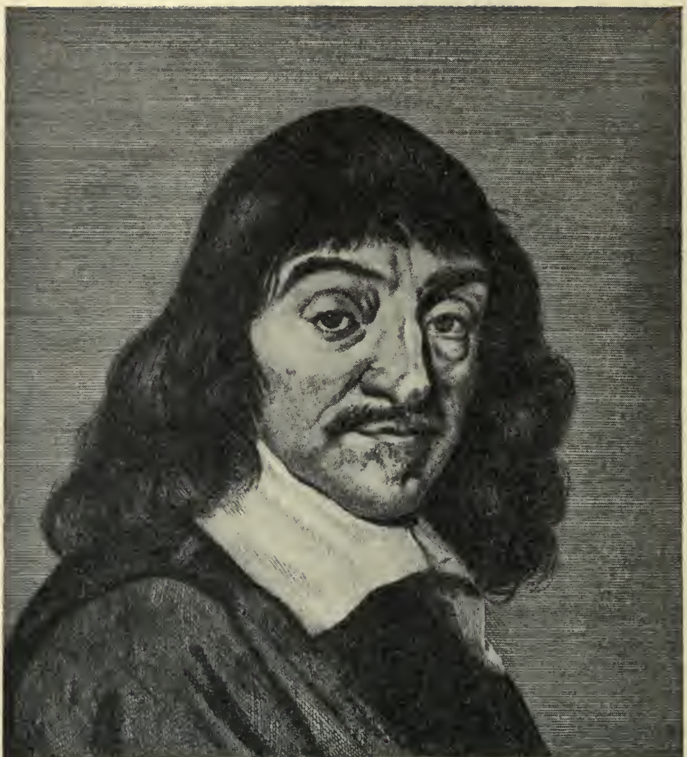
RENÉ DESCARTES was born at La Haye, near Tours, March 31, 1596, and died at Stockholm, February 11, 1650. As a boy his health was delicate, and at the age of eight his father sent him to a Jesuit school at La Flêche. Finishing his education at this school in 1612, he went to Paris. Here he devoted two years to mathematical studies with Mydorge and Mersenne. In his day the only callings open to the sons of the nobility were the army and the church. He chose the former and joined the army of Prince Maurice of Orange, then at Breda, a town in Holland. He then devoted himself to the study of philosophy, science, and mathematics. Read the story of his asking Isaac Beeckman to decipher a Dutch placard for him in Ball's *History of Mathematics*, pp. 269–70 (5th ed.).

Descartes was a small man with a large head, projecting brow, prominent nose, and black hair that grew down nearly to his eyebrows. His voice was feeble and he was cold and selfish in disposition. He is said to have despised all learning and art unless something tangible could be gotten from them. He went to Stockholm at the invitation of the Queen of Sweden in 1649 and died there of lung trouble after a few months.

In the year 1637 he wrote a book, *Discourse on Methods*, which contained an appendix on geometry that constitutes his title to enduring fame. The appendix showed how to study geometrical figures by means of algebraic equations and contained the following contributions to algebra:

1. Established the custom of denoting known numbers by letters at the beginning of the alphabet and unknowns by letters at the end of the alphabet.
2. Introduced the system of indices now used in mathematics.
3. Contributed the earliest recognition of the advantage of taking all terms of an equation to its first member.
4. Realized the true meaning of negative numbers and used them freely.
5. Gave the rule for finding the number of positive and the number of negative roots of an equation, and this is still called Descartes' Rule.
6. Introduced indeterminate coefficients in solving equations.
7. Gave the first statement of the so-called Euler Theorem connecting the faces, edges, and angles of a polyhedron.

These contributions and other minor ones give him a better right than Vieta to the cognomen "Father of modern algebra." Furthermore, his geometry is regarded as containing the outlines of analytical geometry. He is regarded as the originator of this branch of mathematics.



Ren. Cartesius.

First-Year Mathematics *for* Secondary Schools

BY

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EDITORIAL PREFACE

The course of study in American high schools is in process of extensive change. The change commenced with the introduction of new subjects. At first science began to compete with the older subjects; then came manual training, commercial and agricultural subjects, the fine arts, and a whole series of new literary courses. In the beginning the traditional subjects saw no reason for mixing in this forward movement, and such phrases as "regular studies," "substantial subjects," and "serious courses" were frequently heard as evidences of the complacent satisfaction with which the well-established departments viewed the struggles for place of the newer subjects. Today, however, the teachers of mathematics and classics are less anxious than formerly to be classified apart. Even the more conservative now write books on why they do as they do and they speak with a certain vehemence which betokens anxiety. They also prepare many editions of their familiar type of textbook, saying of each that it is something which is both old and new. All these indications make it clear that the change in the high-school curriculum which began with the introduction of new subjects will not come to an end until many changes have been made in the traditional subjects also.

Over against the obstinate conservatism of some teachers is to be set the vigorous movement within all subjects to fit them effectively to the needs of students. The interest of today is in supervised study, in better modes of helping students to think, in economy of human energy and enthusiasm. This means inevitably a reworking of the subjects taught in the schools. It is the opportunity of this generation of teachers to work out the changes that are needed to make courses more productive for mental life and growth.

During this process of reform, mathematics has changed perhaps less than any other subject. The textbooks in algebra have modified but little their list of topics or their mode of exposition. Most of the later books introduce graphs and have graded their problems better and have omitted some of the intricacies which were included a generation ago. These improvements are welcome but insufficient, and if algebra has been conservative, what words shall one find to describe Euclidian geometry! Most teachers of mathematics continue to indict themselves by failing abnormally large percentages of their students; and, what is more, the extreme conservatives among these teachers regard it as a virtue that they do not bring their students to the passing level. It is useless to argue with a teacher who puts on the student body the blame when 25 per cent of them are unable to profit by contact with himself. Such a teacher has no insight into the social relations of which he is a part; he is absorbed in subject-matter or in some other considerations remote from real school life. He fails to realize the significant historical fact that the time has passed when the chief duty of the teacher is to eliminate students.

Evidences are not wanting, however, that a thorough reorganization of the body of secondary-school mathematics is at hand. Since the presidential address before the American Mathematical Society of one of the undersigned (cf. *Science*, March 13, 1903), many papers have agreed in recommending radical changes, both in organization and content of the courses and in method of instruction. The School of Education of the University of Chicago as a center for educational experimentation under test conditions has since 1903 been developing a solution of this difficult problem of mathematical reorganization. The solution is sought along the lines of fusion. The reasons for this type of solution as detailed in former editions of this book are generally understood, and during recent years the trend of enlightened opinion has set strongly toward the educational desirability of this type of material for beginners in mathematics.

The salient historical facts of the experiment of which this book is a product are as follows:

In the school year of 1903–4 a tentative program for mathematics along fusion lines was worked out in conferences of the teachers of mathematics of the college and the high school of the School of Education. This program was revised, and used in mimeograph form as a text for the first-year classes in 1904–5. It was again revised, remimeographed, and the improved form was the text of these classes in 1905–6. After another revision the material was published by the University High School office in the summer of 1906 under the title of *First-Year Mathematics for Secondary Schools.* The University Press soon began supplying other high schools that desired to try out the material, and a little later took over the publishing end of the enterprise. The teachers co-operating with Mr. George W. Myers in the authorship of the first edition were Messrs. William R. Wickes, Ernst R. Breslich, Harris F. MacNeish, and Ernest A. Wreidt.

To continue the fusion type of work with the second-year classes of 1906–7 the little manual, *Geometric Exercises for Algebraic Solution,* was compiled that year, and published by the University Press in the summer of 1907. This manual was to supplement a standard text in geometry which was then in use with second-year students.

In the summer of 1909 the University Press published the revised and completer edition of *First-Year Mathematics* under the same title as before, Messrs. Arnold Dresden and Ernest L. Caldwell having been added to the list of participating authors.

In the spring of 1910 the University Press published *Second-Year Mathematics for Secondary Schools* from manuscripts that had been in use in mimeograph form in second-year classes for over a year. This was also avowedly a tentative form of the second-year material for provisional use, though most of the critics have failed or refused to see the avowal.

First- and *Second-Year Mathematics* have been the only texts used in the respective years in the University High School since their publication.

The present edition of *First-Year Mathematics,* which is soon to be followed by a revised form of its companion, *Second-Year Mathematics,* bears the sole name of Mr. Breslich, head of the High School department of mathematics, as author. While both books are the natural outgrowth of the experiment begun twelve years ago, Mr. Breslich has entirely recast and rewritten the texts. His earnest and untiring work on the experiment from the outset has peculiarly fitted him for the task. Authorial credit for the present form of the material is entirely due to him.

In certain important respects the plan of the present edition is a distinct departure from that of former editions. The work here having been done by a single author, who has himself been teaching the material to his own classes, some very specific gains have been brought to the book, a few of which may well be enumerated:

1. Greater homogeneity of the material and closer and more persistent correlation of matter drawn from the several mathematical branches. This means a much smoother gradation of difficulties and a better sustained conformity to an original plan.

2. The frank and full criticism of the author's work by his colleagues, both privately and in conference, has resulted in many both major and minor alterations that make the work more easily teachable by those who are inexperienced in the use of such material. The organization of the material has thus acquired a high degree of objectivity and practicality.

3. A much more searching scrutiny of the psychology of the final organization was thus realized. This subject-matter may accordingly claim to be, not only empirically and experimentally suitable, but also to be psychologically justifiable.

4. Much greater emphasis is here placed upon experimental and inductive geometry than was done in former editions. Moreover, the inductive work shades over into deductive procedures more gradually here than in the former books.

These are not the only gains of this edition over former editions, but they are the gains of greatest scope and significance. The language of the book has been carefully studied with reference to its easy comprehensibility by beginners. The exercises have been very carefully chosen, graded, and related. In short, no pains have been spared and no labor stinted to make the present edition both an easily workable text for those who try the material for the first time, and a highly profitable body of ideas for beginners.

To those who may examine this book from the point of view of the critical mathematician, it ought to be said that it is designedly a pedagogical rather than a logical organization of general and fundamental mathematical notions. Rigor in the pure mathematical sense is not attempted in definitions, axioms, or principles. Insight has everywhere been the controlling consideration. Experimentation, intuition, and induction are freely employed. It takes time to learn deduction and the approach to this goal is gradual but sure. Clearness and comprehensibility under normal classroom conditions and in public-school environments have been guiding motives. Austere ideals of abstract rigor have been sacrificed. Still it is believed that as much rigor is demanded from stage to stage of the development as can profitably be attempted by the present-day thirteen- or fourteen-year-old boy or girl.

As to incommensurables, little more is attempted than to make clear to the student that they exist and the nature of the difficulty they present. His mind will be opened toward them and he will be ready for fuller ideas in good time.

The School of Education and the Department of Mathematics of the University of Chicago share the settled conviction of Mr. Breslich and his colleagues that this new book will contribute to the solution of the problems which confront the mathematical sciences in their efforts to be a vital part of the new course of study in American high schools.

Eliakim H. Moore
George W. Myers
Charles H. Judd

AUTHOR'S PREFACE

In planning the work of the first year, the following facts have been kept in mind:

1. Each of the various divisions of secondary mathematics—algebra, geometry, and trigonometry—includes simple principles relatively easy to master, and also difficult, complex principles. The simpler principles are best suited for beginners, and may therefore be brought together in an introductory course which leads up to more complex aspects of these various branches of mathematical science.

2. Because they make the acquaintance of only one of the three subjects during the first year, many students fail to get an insight into secondary mathematics and are discouraged from continuing the study. Thus it is commonly the case that the student is brought into contact only with algebra in the first year. When he finds algebra very difficult he frequently misses the opportunity to discover that he can be successful in geometry. If an introductory course can be formulated in which algebra and geometry are taught together, success in one field will arouse an interest and enthusiasm which will encourage the student to attack the other with increased vigor. The result will be a gain of mathematical power and no loss in general training.

3. The relationship between algebra and geometry has long been recognized by technical students of mathematics, quite apart from any consideration of the desirability of teaching them together. Algebra and geometry supplement each other. Both are used to express facts about quantity; e.g., the graph and the formula both express the law of a group of numerical facts. Both give these facts in a form easily taken in by the trained eye; both state the facts in generalized form and thus make the deduction of any number of particular instances possible. By correlating the two related forms of thought in a

single course of instruction the student's comprehension of quantity is at the same time simplified and deepened; simplified, because the double method of attack makes it easier to overcome difficulties; deepened, because of the more enduring impression made upon the mind. In the course presented in this book, geometry is used throughout the book to illustrate algebraic processes, while algebra carries on the reasoning in the compact and abstract symbols which generalize quantitative facts in a degree which is impossible in graphic expression.

4. The number of mathematics courses required for graduation from the high school is constantly being reduced. A student taking only one year of mathematics will, under ordinary circumstances, for this reason fail to come into contact with that very important body of geometrical ideas necessary to increase his understanding of his space environment.

5. A student will be most interested in subjects in which practical values are most clearly exhibited. If instruction in various branches of mathematics is given in the introductory course, the student will see the usefulness of various modes of treatment of the facts of quantity. He is made to realize the value of algebra by explicit references whenever the superiority of algebraic over geometric methods appears. Up to this point only the relation of algebra to geometry has been commented on. It is also true that the fundamental notions of trigonometry, which are commonly kept from the student until the third or fourth year of the high school, appeal to him because of their usefulness as tools in problem-solving. Hence, these notions introduced at an early stage are presented in a way not difficult for the beginner. Practical applications, especially to surveying, have therefore in the following pages received considerable attention.

When the various branches of mathematics are treated as separate subjects, there is a tendency for each to take on the rigid form of the final science. This tends inevitably to a certain formalism in mode of presentation. Such formalism is not the best method for the beginner. Correlation helps to avoid excessive formalism. Rigor is not carried beyond the understanding of the pupil. Indeed, it must be said that when rigor is attempted beyond the comprehension of the student it is only apparent. For correlated mathematics it is relatively easy to adopt a method of approach which is largely inductive. In geometry the peculiar properties of the appropriate figures are studied and the results are then combined into a theorem. This brings about an easier and a much better understanding than a beginner can obtain from a logical proof. Not until toward the end of the geometry of the first year does the demonstration take the form of a logical proof. Axioms usually assumed to be self-evident are in the following pages illustrated in order to make their meaning apparent and vital. Algebra is introduced as a natural means of expressing facts about number and gradually becomes a symbolic language especially well adapted to stating the conditions of a problem in a natural and helpful way. The growing difficulty and complexity of problems then lead to the necessity of learning how to manipulate algebraic symbols. The symbolism of algebra thus becomes a highly clarifying instrument of problem-analysis and problem-solving. The laws of algebra are carefully illustrated, thus avoiding the danger of symbol-juggling without insight into the real meaning.

There are certain processes which belong together logically but which should be separated in treatment because they make difficulties for the beginner. Hence, wherever the processes are not needed as instruments of instruction they are taught separately; e.g., the meaning of positive and negative numbers, the laws of signs, and the operations with positive and negative numbers are not studied until the pupil has become thoroughly familiar with unsigned literal numbers and with the operations and laws of such literal numbers. The fusion plan makes possible a wide choice of process for the particular difficulty in hand and thus very materially facilitates conformity to the pedagogical dictum, "One major difficulty at a time."

Until recently the character of secondary texts has been nearly uniform. An attempt to reorganize traditional material will not bring the best results unless this material is presented in a form in which even the inexperienced teacher can use it successfully. In making the experiment from which the text resulted, the author has intentionally gone about the work deliberately. He has watched for several years the difficulties encountered by new teachers coming into the mathematics department of the University High School and he has also had the good fortune of getting the frank and most helpful criticism of the material from teachers who have been in the department for some time. In addition to this, he has had the advice and criticism of several professors of the College of Education, who, from their interest in the work as an educational experiment, have made a detailed study and criticism of the material. The result of it all is that the book has become easily teachable.

One of the aims of every high-school teacher is to teach his pupils to work independently and to be able to use their books. The author shares this aim with his colleagues and has accordingly made a careful study of the difficulties met by high-school pupils in preparing their work. The organization of the material is such that a student will be aided in distinguishing the essential from the less important. Summaries given at the end of each chapter will be helpful in reviewing the work of the chapter. The list of typical problems in chapter XIX will make it possible for the student to review the whole course with increasing interest, because the ground is here covered differently from the way in which it was gone over the first time.

The chief gains of this text over the traditional treatment may be summarized as follows:

The student receives a broader mathematical preparation. At the end of a year he will know a number of important geometrical facts; he will know enough algebra to manipulate formulas and to solve equations in one or more unknowns. He has learned to use both algebraic and geometric methods of solving problems.

The program is richer in content. There is much dissatisfaction on the part of parents and students with a course in mathematics offering only one subject during the whole year. The teaching of several subjects makes it possible to bring in practical problems from each, with the result that the course will lose in formalism and be better suited for beginners. The student sees the superiority of algebraic methods over geometric methods and of trigonometric methods over both, and comes to appreciate the value of the course and is more likely to decide to go on with more advanced work.

This results in an increased measure of economy of time, because progress is continuous; whatever is learned is kept available, and topics commonly treated in two or more mathematical branches are here treated once for all.

The teacher will receive new suggestions as to methods of teaching. For in the attempt to open up new fields and to treat traditional material partly according to new methods it became necessary to organize this material with unusual care and to supply a large number of illustrative examples and suggestions as to the aim and methods involved. The new material also makes it much easier for the teacher to elicit greater spontaneity of effort from pupils.

The author desires to render full acknowledgment to Professor Charles H. Judd for his interest and very substantial aid in the way of suggestions and criticisms, and to Principal Franklin W. Johnson, whose encouragement, assistance, and continued interest have made this educational experiment possible. He is also indebted to his colleagues in the department of mathematics, Messrs. William D. Reeve, Raleigh Schorling, and Horace C. Wright, who have read all of the chapters in detail and whose constructive criticisms have been most valuable.

The portraits appearing as inserts, with the exception of that of John Wallis, have been taken from *Philosophical Portrait Series*, published by The Open Court Publishing Co., Chicago.

ERNST R. BRESLICH

CONTENTS

STUDY HELPS FOR STUDENTS[1]

The habits of study formed in school are of greater importance than the subjects mastered. The following suggestions, if carefully followed, will help you make your mind an efficient tool. Your daily aim should be to learn your lesson in *less* time, or to learn it *better* in the *same* time.

1. Make out a definite daily program, arranging for a definite time for the study of mathematics. You will thus form the habit of concentrating your thoughts on the subject at that time.

2. Provide yourself with the material the lesson requires; have on hand textbook, notebook, ruler, compass, special paper needed, etc. When writing, be sure to have the light from the left side.

3. Understand the lesson assignment. Learn to take notes on the suggestions given by the teacher when the lesson is assigned. Take down accurately the assignment and any references given. Pick out the important topics of the lesson before beginning your study.

4. Learn to use your textbook, as it will help you to use other books. Therefore understand the purpose of such devices as index, footnotes, etc., and use them freely.

5. Do not lose time getting ready for study. Sit down and begin to work at once. Concentrate on your work, i.e., put your mind on it and let nothing disturb you. Have the will to learn.

[1] These study helps are taken from *Study Helps for Students in the University High School.* They have been found to be very valuable to students in *learning* how to study and to teachers in *training* students how to study effectively.

6. As a rule it is best to go over the lesson quickly, then to go over it again carefully; e.g., before beginning to solve a problem read it through and be sure you understand what is given and what is to be proved. Keep these two things clearly in mind while you are working on the problem.

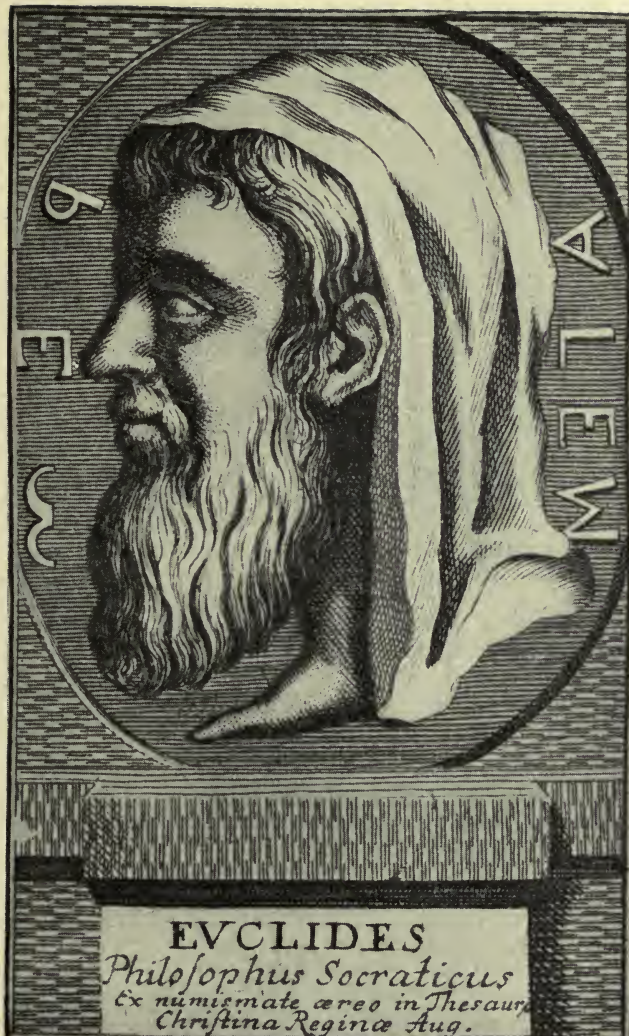
7. Do individual study. Learn to form your own judgments, to work your own problems. Individual study is honest study.

8. Try to put the facts you are learning into practical use if possible. Apply them to present-day conditions. Illustrate them in terms familiar to you.

9. Take an interest in the subject. Read the corresponding literature in your school library. Talk to your parents about your school work. Discuss with them points that interest you.

10. Review your lessons frequently. If there were points you did not understand, the review will help you to master them.

11. Prepare each lesson every day. The habit of meeting each requirement punctually is of extreme importance.



EVCLIDES
Philosophus Socraticus
Ex numismate æreo in Thesauro
Christinæ Reginæ Aug.

EUCLID OF ALEXANDRIA

THE dates of Euclid's birth and death are unknown. He lived and taught mathematics at Alexandria in Egypt from 306 to 283 B.C. He had probably studied in Plato's school at Athens and in Aristotle's school at Stagira, both in Greece. Being well versed in both mathematics and Aristotelian logic, when he became head of the mathematical school at Alexandria he undertook to cast all Greek mathematics into the form of syllogistic reasoning. The result was his *Elements of Geometry*, which has become the most celebrated mathematical text ever written. The Greeks had always regarded the proofs of theorems as a real part of geometrical study. Most other mathematical peoples included only the results and conclusions. The form Euclid gave to these proofs was so excellent that his *Elements* soon replaced all other texts of his time, gave him the nickname of "The Author of the Elements," and for nearly 2,500 years has made his name a synonym for his science. In England boys are even today said to be studying Euclid when it is meant they are studying geometry.

The form of our modern texts in geometry is nearly the same as Euclid gave his text. Euclid's text is at least the basis of our American texts. You will be interested to read the story of Euclid's ordering his slave to give a student a coin for studying geometry, in order to show the contempt he had for those who studied geometry for gain; also the story of his telling the young Ptolemy "There is no royal road to geometry." Reason's highway is the only road we know of which leads to a knowledge of geometry, even today. Remember that if the road seems rough and steep at places, there are charming views at the top.

Read Euclid's biography in some encyclopedia or history of mathematics. While reading think how long men and boys have been trying to learn how to prove theorems as Euclid proved them three centuries before the birth of Christ. A study that has stood about the same for so long is not likely to be much different during your lifetime.

CHAPTER I

THE STRAIGHT LINE

Measurement of Line-Segments

1. Straight lines. Lay a ruler on a sheet of notebook paper and pass the point of a sharp pencil along the edge of the ruler. The drawing obtained is what in ordinary speech is called a **straight line.**

We use the same term when we refer to a tight telegraph wire, a stretched string, a ray of sunlight passing into a dark room through a small opening, etc.

Give other examples of straight lines that can be seen in the classroom.

2. Geometric lines. The different examples in § 1 all have characteristics which differ, but they have one characteristic in common, namely, their length. When we speak in exact scientific terms we use the word "line" to refer to the length of each of the above, not to their width and thickness. Thus, **geometric lines** have length only, but not width nor thickness.

3. Points. A particle of dust in the air, a small chalk dot on the blackboard, the sharp end of a needle, are examples of points as we ordinarily use that word. Again, for purposes of scientific exactness we must neglect the whiteness of the crayon point, the grayness of the dust point, and the spread-outness, however small, of each of the objects. In science a **point** is merely the position, but not length, breadth, nor thickness.

4. Notation for points. The position of a point on a straight line is indicated by a small cross-line or dot.

Thus in Fig. 1 three points are marked. They are named A, B, and C. It is customary in science to use *capital letters* in naming points.

FIG. 1

5. Line-segments. The portion of a line bounded by two points on the line is a **line-segment**, or, briefly, a **segment**, as AB or BC, Fig. 1.

6. Measurement of length. One edge of a ruler is graduated in inches. Place the edge of your ruler along the line-segment AB, Fig. 2, and count the number of inches contained in AB.

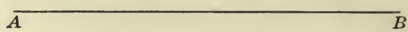


FIG. 2

When we compare the length of a line-segment, as AB, Fig. 2, with such a well-known and well-established segment as an inch, or a yard, we are **measuring** the line-segment; or, put in general terms, to **measure** a line-segment is to find out how many times it contains another line-segment, called the **unit-segment**.

7. Units of length. Most of the civilized nations have derived a unit of length from the length of the human foot. The result was that they differed in the standard length. A commission appointed by the National Assembly of France devised a standard length which was adopted by France in 1793 and is now used very generally in scientific work in all countries. This standard unit is called the *meter*. It is a bar of platinum, equal approximately to 1.1 yards, which is approximately one ten-millionth of the distance from the North Pole to the equator, measured along a meridian. The meter is divided into 1,000 equal parts, called *millimeters*. Ten millimeters make a *centimeter*, ten centimeters make a *decimeter*, and ten decimeters make a meter.

EXERCISES

1. Using a ruler whose edge is graduated in centimeters, measure AB, Fig. 3, in centimeters.

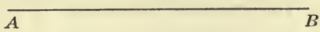


FIG. 3

2. With a ruler measure in centimeters the edge of a page of this textbook.

8. Compass. A compass, Fig. 4, is an instrument used for measuring. Because a good compass is accurate, it is useful in many forms of exact work. One who expects to do exact work should learn to use it freely.

FIG. 4

9. Use of the compass. The compass may be used to measure line-segments. Instead of laying the ruler directly on AB, Fig. 2, as was done in § 6, the sharp points of the compass are placed at A and B. Then the points are placed on the marks of the ruler and the number of inches or centimeters between them is counted.

EXERCISES

1. With the compass measure AB, Fig. 2, in inches.

2. With the compass measure AB, Fig. 3, in centimeters.

3. Measure the length of the page of this book with the compass.

If an inch is not contained exactly, leaving a remainder less than an inch, count the number of eighths or sixteenths of an inch left over.

4. Mark two points on paper and estimate the distance between them. Test the correctness of the estimate by measuring the same distance with the compass.

10. Use of squared paper for measuring. The following exercises explain the use of squared paper in measuring line-segments.

EXERCISES

1. Measure the length of AB, Fig. 5, in centimeters, using squared paper.

Place the points of the compass at A and B. Then place the points on one of the heavy lines of the squared paper, as at C and D. Each side of a large square being 1 cm. long, count the number of centimeters in CD and estimate the remainder to tenths of a centimeter. Express the length of CD in the form of a decimal.

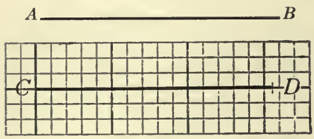


Fig. 5

2. Draw a line-segment and measure its length in centimeters. Express the result in the form of a decimal fraction.

3. Draw a line-segment and measure its length approximately to *two decimal places.*

To find the length of a segment to two decimal places use as a unit a segment equal to ten times the length of the side of a small square on centimeter paper. Then AB, Fig. 6, is equal to 1 and CD is equal to .1. Why?

Imagine CD divided into 10 equal parts. Then .1 of CD equals .01 of AB.

Line-segment AB, Fig. 7, equals 2.35. Why?

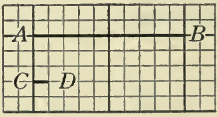


Fig. 6

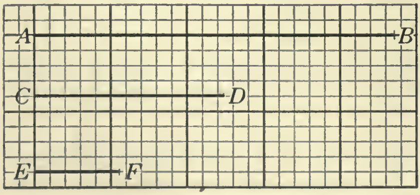


Fig. 7

4. Measure to two decimal places the length of segments CD and EF, Fig. 7.

5. Draw several line-segments and using compass and squared paper measure them to two decimal places.

6. Measure to two decimal places the segments AB, BC, and CA, Fig. 8.

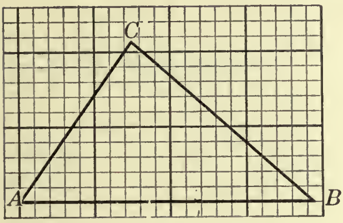


Fig. 8

7. Draw a triangle and measure the three sides to two decimal places.

11. Equal segments. If a segment can be placed upon another so that the end-points of one coincide with (exactly fit upon) the end-points of the other, the segments are said to be **equal.** In Fig. 9 segments a and b are equal. This is read a equals b and is written $a = b$.

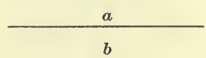


Fig. 9

Draw two equal line-segments.

12. Unequal segments. If a segment b when placed upon a segment a covers only a part of a, b and a are said to be **unequal.** This is written $a \neq b$ (read a is not equal to b).

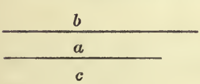


Fig. 10

In Fig. 10, a is less than b and greater than c. The first is written $a < b$ and the second $a > c$.

13. Notation for line-segments. In the preceding paragraphs line-segments have been denoted in two ways: (1) by marking two of the points of the segment by capital letters, (2) by marking the segment with a small letter placed near the middle of the segment. In the second case the small letter generally stands for the length of the segment and is therefore a *number* that can be found by measuring the segment.

14. Literal numbers. A number denoted by a letter is a **literal number.**

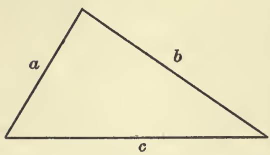


Fig. 11

In Fig. 11 find, by measuring, the numbers denoted by a, b, and c.

15. Representation of numbers. Numbers may be represented either by line-segments or by letters. The numbers that the line-segments represent are found by measuring the segments.

EXERCISES

1. Represent the following numbers by segments: 3, 14, 4.5, 7.8, 2.47, 1.64, .32.

2. Draw several line-segments and find, by measuring, the numbers represented by them.

16. Graphical representation. A line-segment used to denote a number is called a **graph** of the number.

The diagram below represents graphically the amounts in millions spent in 1909 by the people of the United States for certain necessities and luxuries.

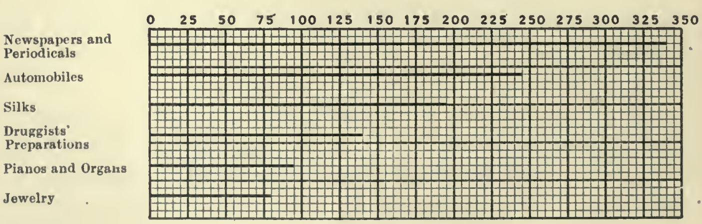


Fig. 12

17. Ways of expressing facts about quantity. Graphs are an effective help when used to illustrate numerical statistics and scientific data. There are various ways in which the same facts about quantity can be expressed.

First, we may take the exact figures from the census reports about the population of the United States from 1790 to 1910. This gives the following table of numbers:

Year	Population	Year	Population
1790	3,929,214	1860	31,443,321
1800	5,308,483	1870	38,558,371
1810	7,239,881	1880	50,155,783
1820	9,638,453	1890	62,947,714
1830	12,860,702	1900	75,994,575
1840	17,063,353	1910	91,972,266
1850	23,191,876		

Second, we may represent these facts graphically by line-segments. Thus, Fig. 13 shows at a glance the whole

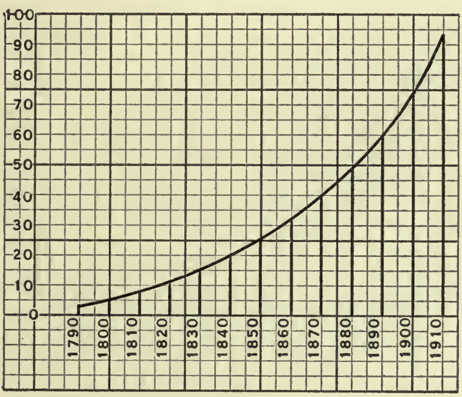


Fig. 13

group of facts given in the preceding table. One can also see the change in the population without knowing the exact values of the numbers represented by the line-segments. This is brought out even better by joining the end-points of the vertical segments. We may even predict what the population will be in 1920. This mode of expressing quantity is especially good when we wish to compare magnitudes with each other; e.g., national debts, navies of various countries, consumption of coal, etc.

Compare the two methods above as to their advantages and disadvantages.

Third, we may represent facts by literal numbers, as may be seen from the following illustration:

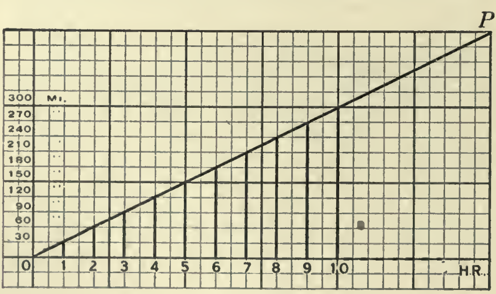
A train traveling in equal time-intervals over equal distances is said to have *uniform motion.* The distance passed over in the unit of time (e.g., an hour) is called the *velocity* of the train. Thus, the velocity of a train is 30 mi. per hour, if the train travels 30 mi. every hour. It follows that the train travels in 2 hours a distance of 2×30 mi., in 3 hours 3×30 mi., in 4 hours 4×30 mi., etc., and in t hours a distance of $t\times30$ miles.
In general, *distance = time × velocity,* or in letters

$$d=t\times v \qquad \text{.......................}(A)$$

Thus, $t\times v$ represents the following facts, if $v=30$

Time in hours..	1	2	3	4	5	6	7	8	9	10	etc.
Distance in miles	30	60	90	120	150	180	210	240	270	300	etc.

(B)



.....(C)

FIG. 14

The same facts may be represented graphically, as in Fig. 14. Notice that the graph in Fig. 14 is regular, in fact it is a straight line. This enables us to read from the figure the distances passed over during a given number of hours.

EXERCISES

1. Tell from the graph the number of miles the train travels in 15 hours; in 12 hours.

2. A train travels at the rate of 20 mi. per hour. Represent in a table, as in (*B*) p. 8, the distance passed over in the first ten hours. Represent the same facts graphically, as in (*C*).

18. Graphing data. Make the graphs for the following and tell what the graphs show.

EXERCISES

1. Graph these average heights of boys and girls:

Age	2	4	6	8	10	12	14	16	18	20
Boys......	1.6 ft.	2.6	3.0	3.5	4.0	4.8	5.2	5.5	5.6	5.7
Girls......	1.6 ft.	2.6	3.0	3.5	3.9	4.5	4.8	5.2	5.3	5.4

Mark off on squared paper the heights, as in Fig. 15, calling the vertical side of a large square one foot. Connect the points as shown. The broken dotted line so obtained is the graph of the heights of girls, the full line is the graph of the heights of boys.

At what age do boys grow most rapidly?

The answer may be seen from the table or from the graph.

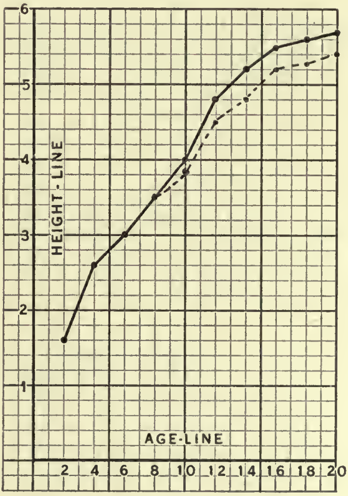


FIG. 15

2. The populations, in millions, of the United States for each ten years, beginning 1790, are: 3.9, 5.3, 7.2, 9.6, 12.9, 17.1, 23.2, 31.4, 38.6, 50.2, 62.9, 76.0, 92.0. Make the graph.

3. The monthly average rainfall or snowfall, in inches, at a certain place for 30 years is given below.

Jan.	2.8	May	3.59	Sept.	2.91
Feb.	2.30	June	3.79	Oct.	2.63
Mar.	2.56	July	3.61	Nov.	2.66
Apr.	2.70	Aug.	2.83	Dec.	2.71

Graph these data, letting twice the side of a large square represent one inch, and tell what the connecting line shows.

4. In a certain school the eyes of the pupils of all classes were tested two or three times each year for sixteen years. The following table gives the percentage of short-sighted pupils for each grade and the increase from one grade to the next. Graph the data. When is the percentage of increase lowest? When is it highest? Point out striking peculiarities of the graph.

Grade	1	2	3	4	5	6	7	8	9	10	11	12
Percentage...	2.8	4.6	7.8	11.7	12.1	15.3	17.0	22.5	29.7	36.0	41.7	47.7
Increase	...	1.8	3.2	3.9	.4	3.2	1.7	4.5	7.2	6.3	5.7	6.0

5. The following table gives the population, in millions, of the United States in 1910, by sex, race, nativity, and parentage. Draw the graphs.

	Native White Native Parentage	Native White Foreign Parentage	Foreign White	Negro	Indian	Japanese	Chinese
Male...	25.2	9.4	7.5	4.9	.13	.063	.067
Female.	24.3	9.5	5.8	4.9	.13	.009	.005

6. Graph these average lengths of day from sunrise to sunset in latitude 42°.

	Hours		Hours		Hours
Jan. 16	9.5	May 16	14.5	Sept. 15	12.5
Feb. 15	10.5	June 15	15.0	Oct. 16	11.2
Mar. 16	11.9	July 16	14.9	Nov. 15	9.6
Apr. 15	13.3	Aug. 16	13.9	Dec. 16	9.1

7. Using the same sheet, scale, and dates as in problem 6, graph the average days' lengths

in latitude 38°: 9.7 10.8 12.0 13.3 14.4 14.9 14.6 13.7 12.5 11.2 10.5 9.5

in latitude 45°: 9.1 10.4 11.9 13.5 14.9 15.6 15.3 14.1 12.6 11.1 9.6 8.8

What differences in the change of the day's length in different latitudes do the three graphs show?

8. A boy saved two dimes a week, placing them in a savings box. Tabulate, as in (*B*), p. 8, the contents (number of dimes) for each of the first ten weeks. Represent these facts graphically. Represent the same facts in letters, as in (*A*), p. 8, knowing that the contents c is always 2 times the corresponding number of weeks w.

19. To test a ruler. To make sure that lines drawn with his ruler are straight, a carpenter makes the following test: He sights along the edge, making the end-points appear to fall together. If all other points in the edge fall together with the end-points, the edge of the ruler is straight.

A ruler may also be tested as follows: Mark two points on a sheet of paper. Lay the ruler on the paper so that the edge passes through the two points. Draw a line-segment through these points. Then lay the ruler on the opposite side of this line, making the edge pass through the two points. If the edge falls exactly along the line, the ruler is straight.

Test your ruler in both ways.

20. Coinciding lines. From § 19 it is seen that two straight lines fall together if two points of one fall on two points of the other. The lines are said to **coincide**. Fig. 16 shows that any number of lines can be drawn from one point to another, but only one of them is a straight line; i.e., *Through two points one and only one straight line can be drawn.* The two points are said to **determine** the straight line.

FIG. 16

21. To produce a segment. Since only one straight line can be drawn between two points, a segment may be extended by moving the straight-edge along the segment.

FIG. 17

The segment, Fig. 17, has been extended in the directions indicated by the arrows; it is said to be **produced**.

Mark three points A, B, and C not on the same straight line. Draw line-segments AB, BC, and CA. Show by measuring AB, BC, and CA that CA is shorter than the path CBA.

22. *The shortest path between two points is the straight line-segment joining the points.* Even when the exact scientific statement of this principle is not known to us, our ordinary behavior conforms to it; e.g., when hurrying from one place to another we choose the straight line as the shortest path between them. Apparently the fact is known even to animals; e.g., the dog when called by his master runs to him in a straight line.

23. Axiom. Statements like those printed in italics in §§ 20 and 22 when assumed to be true are called **axioms.** Usually axioms are statements so simple that they seem evident. Axioms that belong especially to geometry are sometimes called **postulates.** For example, some writers would call the statements in §§ 20 and 22 postulates. In this book they will be referred to as axioms 1 and 2 respectively.

24. Point of intersection. Through a given point any number of straight lines can be drawn. When two straight lines have only one point in common, they are said to **intersect** and the point is called the **point of intersection.**

EXERCISES

1. Through a given point A draw several straight lines.

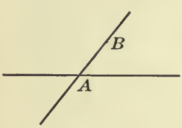


FIG. 18

2. If A is the point of intersection of two segments, Fig. 18, show that any point B on one of the segments cannot lie on the other.

25. Axiom 3. *Two straight lines can intersect in only one point.*

For, if they had two points in common, they would coincide (§ 20).

Summary

26. This chapter has taught the meaning of the following terms and phrases: straight line, line-segment, point, measurement of length, unit of length, equal and unequal segments, literal number, to produce a segment, coinciding lines, axiom, point of intersection.

27. The use of the following instruments has been learned: the ruler, for drawing and measuring straight line-segments; the compass, for laying off line-segments; and squared paper, for measuring segments and graphing data.

28. The following symbols are used: $=$, meaning *is equal to;* $>$ for *is greater than;* $<$ for *is less than;* $\neq$ for *is not equal to;* $\times$, meaning *multiplied by.*

29. Three ways of representing statistics and scientific data are studied:

(1) tabulating them.

(2) representing them graphically by means of line-segments.

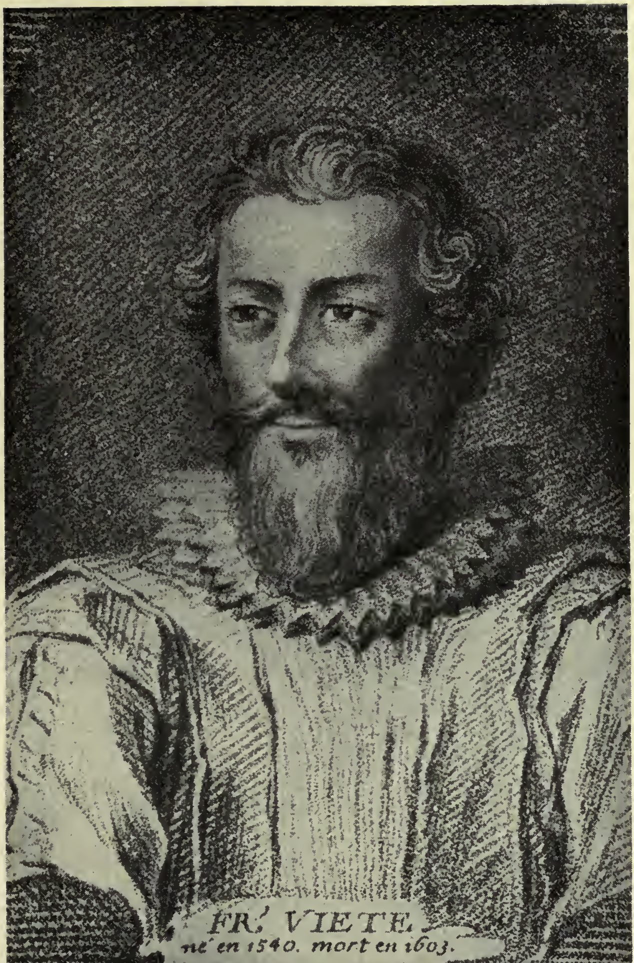
(3) representing facts by means of literal numbers.

30. Axioms.

1. *Through two points one and only one straight line can be drawn.*

2. *The straight line-segment joining two points is the shortest path between them.*

3. *Two straight lines can intersect in only one point.*


FR.ˢ VIETE.
né en 1540. mort en 1603.

FRANÇOIS VIÈTA

FRANÇOIS VIÈTA, better known by his Latin name, Franciscus Vieta, a French mathematician of great originality, was born at Fontenay in Poitou in 1540 and died at Paris in 1603. In his book entitled *Isagoge in artem analyticam*, published in 1591, he introduced so many new and important algebraic ideas that he has been called the "Father of modern algebra." He used vowels for unknown numbers and consonants for knowns. He discussed cubic and quartic equations, devised an excellent method of finding the approximate roots of numerical equations, and solved a 45th-degree equation that was famous for its difficulty in his age. He used trigonometrical ideas in solving it. He displayed an ingenuity that was almost uncanny in deciphering the Spanish secret signal code, by which he was able to read all the Spanish military dispatches that fell into the hands of the French, to the great profit of his king. His mathematical knowledge made him of great service to his country in his public life. He studied mathematics as a pastime and for the love of it.

Besides the *Isagoge*, he wrote several other important mathematical works. Read his biography in some encyclopedia or history of mathematics.

CHAPTER II

ADDITION AND SUBTRACTION

Graphical Addition and Subtraction

31. Number scale. Beginning from a point O, Fig. 19, a unit-segment is laid off repeatedly, forming a geometric picture of the series of numbers 1, 2, 3, etc. According to this arrangement, to every number 1, 2, 3 . . . . . . corresponds a point on the line OA, the number indicating how many units the corresponding point is from O. A series of numbers, as 1, 2, 3 . . . . . . thus arranged, is a **number scale.** Such scales are found on the meterstick, the thermometer, the engineer's steel tape, etc.

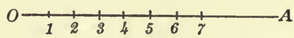


FIG. 19

32. Sum. If two line-segments, 2 cm. and 1 cm. long, respectively, are placed upon an indefinite straight line, as AB, Fig. 20, adjoining each other and having only one end-point in common, the segment AD thus formed is the **sum** of 2 cm. and 1 cm. This is written $2+1=3$.

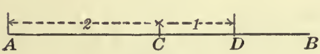


FIG. 20

EXERCISES

1. By use of line-segments find the sum of 4 and 3; 2.2 and 3.1; 6 and 1.4.

2. Let AB and CD, Fig. 21, be *any two* line-segments. Denoting their lengths by a and b respectively, find the sum $a+b$, as EH.

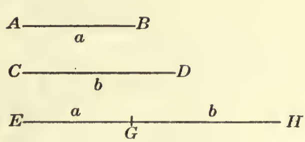


FIG. 21

The problem shows how a letter may be used to denote different segments. One advantage of expressing lengths by letters is that we can find the sum before we know the numbers for which the letters stand.

3. Denoting the length of EH, Fig. 21, by c, show by measuring AB, CD, and EH that $a+b=c$.

Fig. 22

4. Show by measuring that the whole segment AB, Fig. 22, equals the sum of all its parts AC, CE, and EB.

Problem 4 illustrates the following axioms:

33. Axiom 4. *The whole is equal to the sum of all its parts.*

34. Axiom 5. *The whole is greater than any of its parts.*

EXERCISES

1. Draw the segments $a=3.4$ cm., $b=2.3$ cm., $c=1.5$ cm. Draw the sum $a+b+c$.

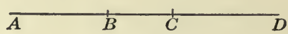


Fig. 23

2. A, B, C, and D are four consecutive points on a straight line, Fig. 23, such that $AB=CD$. Show by measuring that $AC=BD$.

3. Show without measuring that $AC=BD$, Fig. 23.

Since it is given that $AB=CD$, Fig. 23
and since $\underline{BC=BC,}$
it follows that $AB+BC=CD+BC$
or $AC=BD$ (Axiom 4)

In the third step of the solution of problem 3 the following axiom is used:

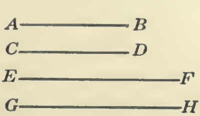


Fig. 24

35. Axiom 6. *If the same number be added to equals the sums are equal.*

1. Let AB and CD, Fig. 24, be two equal segments. Also let EF and GH

be equal. Find the sums $AB+EF$ and $CD+GH$. Show by measuring that these sums are equal.

2. Letting a, b, c, d be the lengths of four segments such that

$$a=b$$

and

$$c=d$$

show by measuring that $a+c=b+d$

Problem 2 illustrates the following axiom:

36. Axiom 7. *Equals added to equals give equal sums* (addition axiom).

37. Axioms 4–7, although so far related only to line-segments, hold equally for all other things. Thus, the number of pupils in a class is greater than the number of pupils in one row. What axiom is used? If the same number of apples is placed in each of two boxes already containing an equal number, the boxes will again contain an equal number of apples. Why? If the number of boys a in one classroom equals the number of boys b in another, and if the number of girls c in the first room equals the number of girls d in the second, then by axiom 7, $a+c=b+d$; i.e., the number of pupils in both rooms is the same.

38. Difference. Place a line-segment, as AB, Fig. 25, upon another, as CD, so that the two have one end-point in common. The amount ED by which CD is greater then AB is the **difference** between CD and AB. This may be stated as follows: CD minus AB equals ED, or in written form: $CD-AB=ED$.

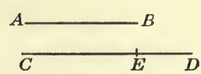


FIG. 25

39. Symbols. The symbols $+$ and $-$ used in the preceding pages have the same meaning as in arithmetic.

They are read *plus* and *minus* and indicate the operations of addition and subtraction.

EXERCISES

1. Subtract by means of line-segments (graphically) 3.5 from 6.

As in Fig. 25, draw $CD=6$ cm., $AB=3.5$ cm., and measure ED.

2. Subtract, graphically, a number, as s, from another larger number m. Denoting the difference between m and s by d, show by measuring that $m-s=d$.

40. Equation. The statements $a+b=c$, $m-s=d$, $CD-AB=ED$, etc., express equalities and are called **equations.**

EXERCISES

1. A man had 22 acres and sold 8 acres. He had l acres left. Express this in equation form.

2. A man had 22 acres and sold s acres. How many had he left? Give answer in the form of an equation.

3. A man possessed p acres and sold s acres. How many had he left?

Express by equations the relations of the numbers in the following problems:

4. A boy has m marbles and buys b more. He has then M marbles.

5. There are b boys and g girls in a class of p pupils.

6. A boy earns c cents a day for d days. He then has C cents.

7. A profit of p dollars added to a capital of c dollars amounts to a dollars.

8. An article bought for b dollars is sold for s dollars at a loss of l dollars.

9. Express a number 5 greater than n; c greater than n; 7 less than b; x less than n.

10. The difference of two numbers is 110. If the smaller is s, what is the larger?

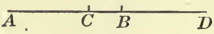


FIG. 26

11. The difference of two numbers is d. If the larger is l, what is the smaller?

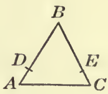


FIG. 27

12. In Fig. 26, $AB=CD$. Show by measuring AB, CD, and CB that $AB-CB=CD-CB$.

13. In Fig. 27 $AB=BC$ and $\underline{AD=CE}$. Show by measuring that $BD=BE$; i.e., $AB-AD=BC-CE$

Problems 12 and 13 illustrate the following axiom:

41. Axiom 8. *If the same number or equal numbers be subtracted from equals the differences are equal* (subtraction axiom).

Give the reasons for the following conclusions:

1. $a=9$, $b=3$, therefore $a+b=12$
2. $m=10$, $s=4$, therefore $m-s=6$
3. $a=x$, $b=y$, therefore $a+b=x+y$
4. $m=r$, $s=t$, therefore $m-s=r-t$

42. Axiom 9. *The sums obtained by adding unequals to equals are unequal in the same order as the unequal addends.*

E.g., if $AB>CD$	Or, with numbers, $18>12$
and if $\underline{EF=GH}$	and $\underline{10=10}$
show that $AB+EF>CD+GH$	Hence, $18+10>12+10$

43. Axiom 10. *The sum obtained by adding unequals to unequals in the same order are unequal in the same order.*

E.g., if $AB>CD$	Or, with numbers, $7>2$
and if $\underline{EF>GH}$	and $\underline{8>3}$
show that $AB+EF>CD+GH$	Hence, $7+8>2+3$

44. Axiom 11. *The differences obtained by subtracting unequals from equals are unequal in the order opposite to that of the subtrahends.*

E.g., if	$AB = CD$	Or, with numbers,	$15 = 15$
and if	$EF < GH$	and	$3 < 10$
show that	$AB - EF > CD - GH$	But	$15 - 3 > 15 - 10$

45. Historical note. About 300 B.C. the mathematician Euclid of Alexandria wrote a mathematical text called *The Elements*, in which he added, subtracted, and multiplied by use of line-segments. The first to use algebraic addition, subtraction, and multiplication seems to have been Diophantus, about 250 A.D.* The Italian Pacioli in 1494 was the first to give rules for all processes of addition, subtraction, multiplication, and division.

It is the purpose of the remainder of this chapter to study some of the laws of algebraic addition and subtraction.

46. Multiples. The line-segment AB, Fig. 28, is laid off repeatedly on the segment CD, making $CE = EF = FG$, etc. Denoting the length of AB by l, then $CE = l$, $CF = 2l$, $CG = 3l$, etc. The segments CF, CG, CH, etc., are called **multiples** of AB.

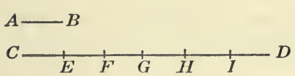


Fig. 28

47. Graphical addition and subtraction. Literal numbers may be added and subtracted by means of line-segments.

* See Cajori, *History of Mathematics*, pp. 35 and 74, or Ball's *History of Mathematics*, pp. 52 and 103.

EXERCISES

1. Let a and b denote two numbers. Find the sum $3a+2b$.

Represent a and b by line-segments. Since $3a=a+a+a$, and $2b=b+b$ lay off a on a straight line three times and then b twice. Then the distance AB, Fig. 29, from the starting-point to the end-point is $3a+2b$.

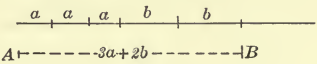


FIG. 29

2. If $a=1$ and $b=2$, find graphically the sum $3a+2b$.

Let a be represented by 1 cm., b by 2 cm.

3. Find graphically $3x+y-2z$, x, y, and z denoting three numbers.

4. Find graphically $3x+y-2z$, if $x=4.4$, $y=2.3$, $z=1.2$.

5. If $m=2.1$ find graphically $3m+5m$.

Perimeters

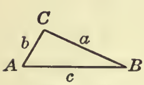


FIG. 30

48. Triangle. Three points A, B, C, Fig. 30, and the three segments a, b, and c joining them, form a **triangle**. A triangle has three **vertices** (corners) A, B, and C and three **sides** a, b, and c.

49. Perimeter. The sum of the three sides of a triangle, as $a+b+c$, Fig. 30, is the **perimeter** of the triangle.

EXERCISES

1. A lot has the form of an equal-sided (equilateral) triangle, Fig. 31, each side being x rods long. How many rods of fence will be needed to inclose it?

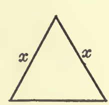


FIG. 31

2. What is the sum of the sides of a triangle, Fig. 32, whose sides are $2x$ ft., $2x$ ft., and $3x$ ft. long?

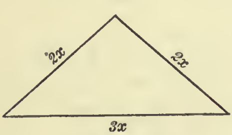


FIG. 32

3. What is the perimeter of a triangle whose sides are $2a$, $5a$, and $6a$?

50. Polygon. The figure $ABCDEF$, Fig. 33, formed by joining the points A, B, C, D, E, and F by line-segments, is a **polygon.** The word "polygon" comes from the Greek and means *many-cornered.* Polygons having 4, 5, 6 ...... n sides are called **quadrilateral, pentagon, hexagon** ...... **n-gon**, respectively. The sum of the sides of a polygon is its *perimeter.*

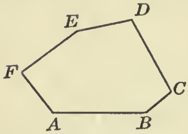


FIG. 33

51. Equilateral. A polygon having all sides equal is an **equilateral** polygon.

EXERCISES

Of what kind of polygons may the following equations express the perimeters, p?

1. $p=3x$	5. $p=7x$	9. $p=12x$
2. $p=4x$	6. $p=8x$	10. $p=20x$
3. $p=5x$	7. $p=9x$	11. $p=nx$
4. $p=6x$	8. $p=10x$	12. $p=ax$

52. Value. A **value** of a letter is a number for which the letter stands. A letter may have a value even before the value is numerically determined.

Thus, in exercises 1–4, § 51, what values has x in order that the perimeter in each case be 60?

EXERCISES

1. Determine the value of x in equations 8–10, § 51, which makes the perimeter in each case 60.

2. Find the values of the perimeters in exercises 1–12, § 51, for $x=2$ in.; $x=3$ ft.; $x=6$ yards.

$p=3x=x+x+x=2+2+2=6$, etc.

3. Show, by a sketch, polygons whose perimeters, p, are given by $p=6x+4$; $p=4x+16$; $p=3x+3y$; $p=4x+2y$; $p=5x+3y$.

4. Find the values of the perimeters in exercise 3 for $x=5$, $y=2$; $x=12$, $y=4$; $x=1$, $y=7$.

5. Show that $4s$ has the same value as $4\times s$.

E.g., let $s=3$, then $4s=3+3+3+3=4\times 3$.

6. Show that $8p=8\times p$.

53. Monomials. Numbers like $4s$, $8p$, $5c$, etc., are **monomials**, or **terms**.

54. Coefficient.‡ The arithmetical factors 2, 3, 4, etc., in $2x$, $3x$, $4x$, etc., are the **coefficients** of the literal factor x. When no coefficient is written, as in x, a, n, the coefficient is understood to be 1. Thus, a means $1a$, n means $1n$.

55. Similar and dissimilar numbers. Monomials having the same (common) literal factor are called **similar numbers**, as $5b$ and $8b$, but $4b$ and $4a$ are **dissimilar numbers.**

56. Polynomial. An algebraic number consisting of two or more monomials, as $4a+10$, $5x+6a-b$, is a **polynomial.** "Polynomial" means *many-termed.*

57. Binomial. Trinomial. Polynomials of two terms are called **binomials**, as $2a+3b$. Polynomials of three terms are **trinomials.**

Algebraic Addition and Subtraction

58. Addition of similar monomials. The sum of similar numbers, as $4a$ and $3a$, can be simplified (reduced) according to a law which may be illustrated as follows:

Reduce to the simplest form the sum $4a+3a$.

Since $4a+3a=(a+a+a+a)+(a+a+a)^{*}=7a$, it follows that $4a+3a=7a$.

‡ The term "coefficient" was first used by the French mathematician Vieta (1540–1603) in a pamphlet on calculation published in 1591.

* The symbols ( ) are used to show the parts that come from $4a$ and $3a$, respectively.

Thus, *the sum of similar monomials is a monomial having the coefficient equal to the sum of the coefficients of the given monomials and having the same literal factor as the given monomials.*

Using this law, reduce to the simplest form each of the following sums: $5x+6x$; $18n+5n+4n$; $a+3a+10a$; $12s+6s+s+16s$.

59. The advantage of adding numbers according to the law of § 58 may be seen from the solution of the following problem:

The tickets for a football game are sold at 25 cents by John, Henry, Kenneth, William, and James. They report sales as follows: John sold 56 tickets, Henry 75, Kenneth 27, William 83, James 69. At the gate, 123 tickets are sold. Find the total receipts.

Solution I:	John,	56×25c. =	$ 14.00
	Henry,	75×25c. =	18.75
	Kenneth,	27×25c. =	6.75
	William,	83×25c. =	20.75
	James,	69×25c. =	17.25
	Gate,	123×25c. =	30.75
	Total receipts		= $108.25

Solution II:	John,	56 tickets,	56×25c.
	Henry,	75 "	75×25c.
	Kenneth,	27 "	27×25c.
	William,	83 "	83×25c.
	James,	69 "	69×25c.
	Gate,	123 "	123×25c.
		433 tickets,	433×25c. = $108.25

Solution II is the simpler. Because the different terms to be added have the common factor 25, they are added by prefixing the sum of the coefficients to the common factor.

EXERCISES

1. The tickets being sold at x cents, John sells 60 tickets, Henry 78, Kenneth 45, William 36, and James 84. At the gate 137 tickets were sold. Find the total receipts.

Total receipts: $60x+78x+45x+36x+84x+137x=440x$.

Again the terms have a common factor x and the sum was simplified by prefixing the sum of the coefficients to the common factor.

Is solution I possible in this case? Give a reason for your answer.

2. Is it possible to reduce to a simpler form the sum of 10 apples, 5 pears, and 4 plums? Give a reason for your answer.

3. Which of these sums can be reduced?

$$4x+2,\ 3y+2y,\ 5a+2a+b$$

4. The length of the school hall is l feet. I go through the hall 6 times on Monday, 8 times on Tuesday, 4 times on Wednesday, 6 times on Thursday, and 10 times on Friday. How many feet do I travel along the hall during the week?

5. The running track in the playground is y yards. While in training, I run around it 6 times on Monday, 8 times on Tuesday, 10 times on Wednesday, 12 times on Thursday, and 14 times on Friday. How many yards do I run during the week?

6. Add as indicated:

(1) $4x+20x+7x+11x$	(4) $\frac{1}{2}b+\frac{3}{8}b+\frac{1}{4}b$
(2) $4y+1y+5y+2y+3y$	(5) $(5z+z)+(4z+2z)$
(3) $10m+4m+2m+2n+3n$	(6) $12.5c+1.2c+4c$

7. Two trains leave a station at the same time traveling in opposite directions at the rate of m miles per hour. How far apart are two towns, if the trains reach them in 8 and 12 hours respectively?

60. Subtraction of similar monomials. The law for simplifying the difference of similar monomials is similar to the law for addition and may be illustrated as follows:

Subtract $4a$ from $7a$.

$$7a = a+a+a+a+a+a+a$$

and $$4a = a+a+a+a$$

Subtracting equals from equals, $7a-4a=a+a+a=3a$

Thus, *the difference of similar monomials is a monomial having the coefficient equal to the difference of the coefficients of the given monomials and having the same literal factor.*

EXERCISES

1. Reduce to the simplest form each of the following differences:

$10e-7e$; $13a-5a$; $14n-2n$; $18w-6w$; $3.48g-.25g$; $1.04x-.08x$; $\frac{2}{3}m-\frac{1}{4}m$; $\frac{5}{8}p-\frac{4}{7}p$.

2. Reduce the following sums and differences to their simplest forms:

(1) $5a+7a-4a$
$5a+7a-4a\ (5+7-4)a=8a$

(2) $14x-x+3x$

(3) $13.5c+2.4c-c$

(4) $\frac{1}{3}b+\frac{7}{8}b-\frac{1}{4}b$

(5) $(8z-z)+(5z-2z)$

(6) $15x+(6x-4x)$

61. Commutative law. The following problems illustrate the commutative law:

1. John has two kinds of marbles, 8 of one and 3 of the other. How many has he in all?

To find the number of marbles he has he may add either the 3 marbles to the 8 or the 8 to the 3, thus: $8+3=3+8$.

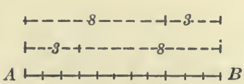


FIG. 34

2. Show by adding line-segments that $8+3=3+8$ (Fig. 34).

3. Show as in problems 1 and 2 that $4+3=3+4$.

4. Show that $a+b=b+a$.

5. Show that $a+b+c=c+a+b=b+c+a$.

Problems 1 to 4 show that *the value of a sum remains unchanged by changing the order of the addends.* This is called the **Commutative Law of Addition.**

Add in the most advantageous way, making use of the commutative law:

$$875+316+25;\ 9,993+4,287+7$$

62. Parentheses. The symbols ( ), [ ], { }, called **parenthesis, bracket,** and **brace,** respectively, are used to inclose numbers. Sometimes one of the symbols is inclosed within another, thus:

$$8+4+7+2=[(8+4)+7]+2$$

and $$8+4+7+2+3=\{[(8+4)+7]+2\}+3$$

EXERCISES

1. A boy has three kinds of marbles, 4 of one kind, 7 of another, and 5 of a third. How many marbles has he?

To find the sum, the first kind may be added to the second and the result to the third; i.e., $4+7+5=(4+7)+5=16$. Or he may add the marbles of the second and third kinds and these to the first kind, giving the equation $4+(7+5)=16$. It follows that $(4+7)+5=4+(7+5)$. Why?

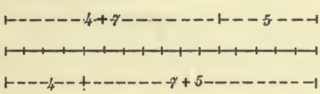


Fig. 35

2. Show by means of line-segments (Fig. 35) that

$$(4+7)+5=4+(7+5).$$

3. Show by means of segments that $(a+b)+c=a+(b+c)$.

4. Show by means of segments that

$$a+b+c+d=a+(b+c)+d=(a+b)+(c+d).$$

Problems 1 to 4 illustrate the following law:

63. Associative Law of Addition. *The sum of several numbers is the same in whatever way two or more of the numbers are combined into a sum before adding in the rest.*

EXERCISES

Using the commutative and associative laws the following sums are to be found in the most advantageous way:

1. $(4356+1483)+(4356-1483)$
2. $(34+128)+66$
3. $381+(436+19)$

Simplify the following:

4. $x+\{7x+(4x-2x)+(3x-x)\}$
5. $[15a-(3a+2a)]+[(8a-a)-4a]$
6. $3m+[m+(7m-4m)+3m]-(5m-4m)$
7. $3y-2y+[\{12y-8y-(2y+y)+7y\}-y]$

Summary

64. This chapter has taught the meaning of the following terms: number scale, sum, difference, equation, triangle, perimeter, polygon, quadrilateral, pentagon, hexagon, n-gon, equilateral polygon, value of a letter, monomial, coefficient, similar and dissimilar numbers, polynomial, binomial, trinomial.

65. The following symbols have been introduced: $+$ for addition, $-$ for subtraction, the parenthesis ( ), the bracket [ ], and the brace { }.

66. Axioms.

4. *The whole is equal to the sum of all its parts.*

5. *The whole is greater than any of its parts.*

6. *If the same number be added to equals the sums are equal.*

7. *Equals added to equals give equal sums.*

8. *If the same number or equal numbers be subtracted from equals the differences are equal.*

9. *The sums obtained by adding unequals to equals are unequal in the same order as the unequal addends.*

10. *The sums obtained by adding unequals to unequals in the same order are unequal in the same order.*

11. *The differences obtained by subtracting unequals from equals are unequal in the order opposite to that of the subtrahends.*

67. Laws.

1. *The sum (difference) of similar monomials is a monomial having the coefficient equal to the sum (difference) of the coefficients of the given monomials and having the same literal factor.*

2. *The value of a sum remains unchanged by changing the order of the addends* (Commutative Law of Addition).

3. *The sum of several numbers is the same in whatever way two or more of the numbers are combined into a sum before adding in the rest* (Associative Law of Addition).

68. Algebraic numbers may be added or subtracted graphically.

CHAPTER III

THE EQUATION

Use of Axioms in Solving Equations

69. Uses of the equation. The equation was used in the first two chapters to state the equality of two numbers or of two geometric magnitudes and to express verbal statements in brief form. Thus, the statement that the distance passed over by a body moving with uniform velocity is obtained by multiplying the velocity by the time, takes the simple form $d=v\times t$. Or, denoting the minuend by m, the subtrahend by s, and the difference by d, the statement that the difference equals the minuend less the subtrahend is expressed in equation form by $d=m-s$. In mathematics the equation is of great importance, especially as a tool for solving problems, for the statement of a problem in most cases takes the form of an equation.

In making a study of the equation we must begin with some very simple problems in order that we may clearly understand the new laws to be developed. If these laws are mastered in connection with simple cases, it will be easy to apply them later to more complicated and difficult cases.

A bag of grain of unknown weight, w ounces, together with an 8-oz. weight just balances an 18-oz. weight. How much does the bag of grain weigh?

The problem may be stated in an equation, thus:

$$w+8=18. \quad \text{Find } w.$$

Suppose 8 oz. to be taken from each pan, giving

$$w=10$$

The bag of grain weighs 10 oz.

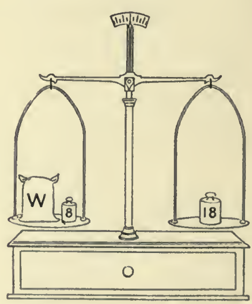


FIG. 36
Showing $w+8=18$

70. Member of an equation. An equation, as $w+8=18$, may be regarded as an expression of balance between the numbers on the two sides of the equality sign. The number to the left of the equality sign is the left side, or **left member** of the equation, the number to the right is the right side, or **right member.**

Thus, in the equation $a+5=7$, $a+5$ is the left member and 7 is the right member.

EXERCISES

1. State in words problems in weighing expressed by the following equations:

$$w+5=7;\ w+2=4;\ 3+w=8;\ 10=5+w$$

2. Show how by the aid of the balance the value of w in each of the equations of problem 1 can be found.

71. Determining the value of the unknown number. An equation, as $w+8=18$, may be regarded as stating the question: *What number added to 8 gives 18?* It has been shown that the answer may be found by interpreting the equation as a problem in weighing and then taking 8 oz. from both pans of the balance. *Just as the scales will balance if the same number of weights are taken from each pan, we may subtract the same number from both sides of an equation and get another equation.* The work of finding the unknown number this way may be arranged in written form thus:

$$\begin{array}{lr} \text{Let} & w+8=18 \\ & \underline{8=\ 8} \\ \text{Then } w & =10 \end{array}$$

For, if the same number be subtracted from equal numbers, the remainders are equal (subtraction axiom, § 41).

To test the correctness of the result, replace the unknown number in the original equation by 10, obtaining

$10+8=18$. Since both members of the equation reduce to the same number, the result $w=10$ is correct.

72. Substitution. When a number is put in place of a literal number, it is said to be **substituted** for the literal number.

73. Satisfying an equation. When both sides of an equation reduce to the same number for certain values of the unknown number, the equation is said to be **satisfied.** Thus, 2 satisfies the equation $x+4=6$. Why?

74. Root. A number that satisfies an equation is a **root** of the equation.

75. Solving an equation. The process of finding the values of the unknown number or numbers which satisfy an equation is called **solving** the equation.

76. Check. A test, or a **check**, of the correctness of the result obtained by solving the equation can be made by substituting in the original equation the result in place of the unknown number. If the equation is satisfied, the result is a *root* of the equation.

EXERCISES

Solve the following equations and problems and check the results:

1. $2+n=17$

$$\begin{array}{ll} \text{Let} & 2+n=17 \\ & \underline{2\quad=\;2} \\ \text{Then} & n=15, \text{ by subtracting 2 from both members} \\ \textit{Check:} & 2+15=17, \text{ by substitution} \\ \text{Hence,} & 17=17 \end{array}$$

2. $w+2=10$
3. $x+4=36$
4. $l+\ 7=18$
5. $d+10=14$
6. $8+v=15$
7. $13=s+3$

8. Two equal, but unknown, weights, together with a 1-lb. weight just balance a 16-lb. and a 1-lb. weight together. How heavy is each unknown weight?

In the form of an equation, the problem is stated thus:

$$2w+1=17. \quad \text{Find } w.$$

Suppose 1 lb. to be taken from each pan, giving

$$2w=16$$

Then $w=8$

FIG. 37.—An equation is an expression of balance of values.

9. A man finds that three bags of shot, of equal, but unknown, weights, together with two 2-lb. weights on the left scale-pan and a 12-lb. and a 4-lb. weight on the right scale-pan just balance. Find the weight of each bag of shot.

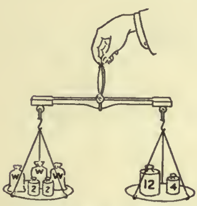


FIG. 38.—If the weight in one pan is changed, the weight in the other must be changed correspondingly.

In the language of an equation we may write

$$3w+4=16$$

Taking 4 lb. from each pan, we have

$$3w=12$$

Then $w=4$

Each bag of shot weighs 4 lb.

10. Show how the values of w in the following equations may be found with the balance:

$$2w+6=16;\ 3w+7=19;\ 3w+10=28;\ 5w+7=27$$

77. Use of axioms in solving equations. The equations in problems 9 and 10 may be solved by use of axioms as follows:

$$\begin{aligned} \text{Let } 3w+4&=16 \\ 4&=\ 4 \\ \hline \text{Then } 3w &= 12, \text{ by subtracting 4 from both sides} \\ \frac{3w}{3} &= \frac{12}{3}, \text{ by dividing both sides by 3.} \\ w &= 4, \text{ by reducing the fractions} \end{aligned}$$

State the axiom used in obtaining the equation $3w=12$.

In obtaining the equation $w=4$ from $3w=12$ the following axiom was used:

78. Axiom 12. *If equals be divided by equal numbers (excluding division by 0) the quotients are equal* (division axiom). E.g., if a segment $AB=CD$, then $\frac{1}{2}AB=\frac{1}{2}CD$. Or, since $8+4=2+10$, it follows that $\frac{1}{2}$ of $(8+4)$ equals $\frac{1}{2}$ of $(2+10)$.

Solve by use of axioms the equations in problem 10, § 76, stating the axiom used in each step of the solutions.

79. The equation $n-2=5$ may be regarded as stating the question: *What number diminished by 2 gives 5?* The number is 7 and may be found by adding 2 to both sides of the equation.

In written form the work of solving the equation is arranged as follows:

Let $\quad n-2=5$

$\qquad\quad \underline{\phantom{n-{}}2=2}$

Then $n+2-2=2+5$, for equals added to equals give equal sums.

Or, $\quad n=7$

EXERCISES

State each of the following equations in the form of a question. Then solve the equations, giving in each step the axiom used, and check.

1. $x-8=3$	8. $2x+6=16$	15. $9x-7=74$
2. $x-5=7$	9. $3x+7=19$	16. $4m+3.2=15.2$
3. $n-2=4$	10. $3t+6=27$	17. $5n-1.4=8.6$
4. $w-7=14$	11. $2t-11=21$	18. $3x+2=2x+5$
5. $x-10=1$	12. $4x-5=23$	19. $5x-4=4x+7$
6. $10-x=3$	13. $9k-15=93$	20. $2y+3=3y-5$
7. $2x-1=9$	14. $9x+8=116$	21. $7s-2=6s+8$

80. Axiom 13. *If equals be multiplied by the same number or equal numbers, the products are equal* (multiplication axiom).

$$\text{E.g., since } 3\tfrac{1}{2} = \tfrac{7}{2}$$
$$\text{and } 2 = 2$$
$$\text{it follows that } 2\times 3\tfrac{1}{2} = 2\times\tfrac{7}{2}$$

Axiom 13 may be used to remove the fractions in an equation. To illustrate this,

Let $\frac{1}{2}x = 9$

Multiplying both members by 2, $2\times\frac{1}{2}x = 2\times 9$ (by axiom 13)

Reducing to simplest form, $x = 18$

EXERCISES

Solve the following equations:

1. $\frac{s}{7}=6$
2. $\frac{t}{8}=9$
3. $\frac{x}{6}=3$
4. $\frac{n}{4}=2$
5. $\frac{w}{10}=10$
6. $\frac{l}{5}=15$

81. Problems to be solved by arithmetic or algebra. Many problems may be solved either by arithmetic or by the use of the equation. When the solution of a problem is made by the use of the equation, it is commonly called an **algebraic solution.**

1. Divide a pole 20 ft. long into two parts so that one part shall be 4 times as long as the other.

Arithmetic Solution

The shorter part is a certain length.
The longer part is 4 times this length.
The whole pole is then 5 times as long as the shorter part.
The pole is 20 ft. long.
The shorter part is $\frac{1}{5}$ of 20 ft. or 4 ft.
The longer part is 4×4 ft., or 16 ft.
Hence, the parts are 4 ft. and 16 ft. long.

Algebraic Solution

Let n be the number of feet in the shorter part.
Then $4n$ is the number of feet in the longer part,

$$\begin{aligned} \text{and } n+4n, \text{ or } 5n &= 20 \\ n &= 4 \\ 4n &= 16 \end{aligned}$$

Hence, the parts are 4 ft. and 16 ft. long.

2. A farmer wishes to inclose a rectangular pen with 80 ft. of wire fencing. He wishes the pen to be 3 times as long as it is wide. How long shall he make each side?

Algebraic Solution

Let x be the number of feet in the smaller side.
Then $3x$ is the number of feet in the longer side,
and $x+3x$, or $4x$, is the number of feet half-way round the pen.

$$\begin{aligned} 4x &= 40 \\ x &= 10 \\ 3x &= 30 \end{aligned}$$

Hence, the sides are 10 ft. and 30 ft. long.

Problems to Be Solved by the Aid of the Equation

82. Important steps in the algebraic solution of problems. The solutions of the foregoing problems 1 and 2 illustrate the following important steps:

a) In every problem certain facts are given, or known, and others are to be determined.

b) In solving the problem one of the unknown numbers is denoted by a symbol, as x.

c) Then all the given facts are expressed in algebraic language, the number x being used as if it were known.

d) Two different expressions for (i.e., denoting) the same number are equated (joined by the sign $=$ of equality).

e) The solution of the equation thus obtained gives the required values of the unknown number.

f) The correctness of these values is tested by substituting them in the conditions of the problem. The result is correct if these conditions are satisfied.

EXERCISES

83. Geometric problems. The following problems contain geometric relations and are to be solved by the aid of the equation:

1. The perimeter of an equilateral quadrilateral is 48 ft. Find a side.

Denoting the side by s the perimeter may be represented by $4s$ or by 48. Hence, $4s = 48$.

2. The perimeter of an equilateral hexagon (6-side) is 186. Find a side.

3. The perimeter of an equilateral decagon (10-side) is 285. Find a side.

4. The perimeter of an equilateral dodecagon (12-side) is 264. Find a side.

5. The opposite sides of the quadrilateral, Fig. 39, are equal. The perimeter is 432 yd. and the length is 26 yd. greater than the width. Find the dimensions.

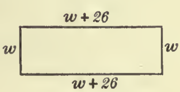


FIG. 39

6. A Maypole 22 ft. high breaks into two pieces so that the top piece, hanging beside the lower piece, lacks 6 ft. of reaching the ground. How long is each piece? Draw a figure.

7. Draw a figure representing a regulation football field, the length of which is $56\frac{2}{3}$ yd. greater than the width, the sum of the length and width being $163\frac{1}{3}$ yards.

8. Make a sketch of a rectangle, one side of which is 12 ft. longer than 3 times as long as the other, the perimeter being 80 feet.

9. A garden is 5 times as long is it is wide. It takes 120 yd. of fence to inclose it. Find the length and width.

10. A side of a lot 126 ft. long is to be divided into two parts so that one is 6 times as long as the other. Find the length of the short part.

84. Problems expressing number relations. The following problems give training in expressing given number relations in algebraic form:

1. If 4 times a number is decreased by 5 the result is 35. Find the number.

2. If $\frac{8}{10}$ of a number is increased by 12 the result is 20. Find the number.

3. If 12 is subtracted from 2 times a number the result is the same as the number increased by $\frac{3}{4}$ of itself. Find the number.

4. Six times a number increased by $\frac{1}{2}$ of itself equals 11. Find the number.

5. Four times a number increased by $\frac{1}{5}$ of itself is the same as twice the number increased by 11. Find the number.

6. John was asked to think of a number, to treble it, and to add 4. He obtained 25 as a result. What was the original number?

85. Consecutive number problems.

1. Find two consecutive numbers whose sum is 203.

   Let x be one number and $x+1$ the other.

2. Find three consecutive numbers whose sum is 474.

3. Find two consecutive odd numbers whose sum is 204.

   Let x be one number and $x+2$ the other.

4. Find two consecutive even numbers whose sum is 378.

5. Find three consecutive even numbers whose sum is 372.

86. Miscellaneous problems.

1. James has 3 times as much money as Charles and 4 times as much as William. All together they have 57 cents. How many cents has each?

2. A farmer has 3 times as many sheep as his neighbor. Having sold 22 sheep, he has the same number as his neighbor. How many did each have before the sale?

3. Two men divide $5,247 between them. One receives $324 more than twice as much as the other.. How much does each receive?

4. A purse of $75 was divided between two persons. One received $27 more than the other. How much did each receive?

5. Three men, A, B, and C, wish to divide 1,584 shares of stock among themselves so that A shall have 25 more than B and C shall have 50 more than B. How many shares must each receive?

6. At an election in the Freshman class of a certain high school 84 votes were cast. Candidates A and B each received a certain number of votes; candidates C and D each received 3 times as many as candidate A; and candidate E received 4 times as many as D plus 4. How many votes did each candidate receive?

7. A man divides his 160-acre farm as follows: He takes a certain number of acres for lots, 4 times as much for pasture, 4 times as much for corn as for pasture, $\frac{1}{2}$ as much for wheat as for corn, and 15 acres for meadow. How many acres does he assign to each purpose?

8. The manager of a high-school football team received from the manager of a visiting team a check for $63 for expenses. Cashing the check he obtained $2 and $5 bills, the same number of each. How many bills of each kind were there?

9. A box of oranges was sold for $3.60, giving a profit equal to $\frac{1}{5}$ of the cost. What was the cost?

10. Two partners in business make a profit of \$5,248. Of this amount they decide to give \$325 to charity. The remainder is to be divided between them so that one receives twice as much as the other. How much does each receive?

11. The annual income of a family is divided as follows: One-tenth is used for clothing, one-third for groceries and meat, milk, and help, and one-fifth for rent. This leaves \$1,320 for other expenses, and for the savings account. How much is the income?

12. The manager of a business receives a salary of \$2,000 a year and 1 per cent of the year's profit. In one year the profit of the business was 10 per cent of an income of \$700,000. What was the manager's income that year?

13. A boy has \$4 in his bank and saves 40 cents each week. His brother has \$16 and draws 10 cents each week. After how many weeks will they have equal amounts in the bank?

14. A bicyclist plans to take a trip of 148 mi. in 6 days. He intends to ride a certain number of miles on Monday, $\frac{7}{8}$ as many on Tuesday, as many on Wednesday, $\frac{5}{8}$ as many on Thursday, and $\frac{1}{2}$ as many on Friday. Having followed this schedule he finds it necessary to ride 20 mi. on Saturday. How many miles did he ride each day?

15. After 8 years a father will be 4 times as old as his son is now. How old is the father, the present age of the son being 12 years?

16. Show how to divide a sum of \$1,278 among three persons in such a way that the share of the first shall be 3 times that of the second, and the share of the second twice that of the third.

17. Divide \$260 into two parts such that one is \$24 more than the other.

EXERCISES

87. Solve the following equations:

1. $\frac{y}{2}+\frac{y}{4}=\frac{1}{3}$

Solution: Multiplying both sides of the equation by the least common multiple of the denominators, i.e., by 12:

$$\frac{12y}{2}+\frac{12y}{4}=\frac{12\times 1}{3}$$

Then $6y+3y=4$, by reducing the fractions. This clears the equation of fractions

$9y=4$, by combining like terms

$y=\frac{4}{9}$, by dividing both sides of the equation by 9

Check the result.

2. $25z-17=113$
3. $28x+14=158$
4. $28x-9=251$
5. $20x+2x-18x=22$
6. $17y-3y+16y=105$
7. $17s+7s-13s=88$
8. $16t+2t-13t=22\frac{1}{2}$
9. $321x-109x+8x=22$
10. $404y-304y+12y=560$
11. $3.4x-1.2x+4.8x=70$
12. $3.5x+7.6x-8.6x=15$
13. $5.8y-3.9y+12.6y=58$
14. $6s-3.5s+5.5s=68$
15. $6.82s+1.18s-3.54s=42$
16. $8x-4.5x+5.2x=87$
17. $16.5x+15.8-2.3x=186.2$
18. $6.15y-1.65y+7.8=57.3$
19. $8y+6.875+2y=46.875$
20. $z+5.37z-8.73=61.34$
21. $13t-8.75t+6.87=57.87$
22. $15x+3.73x-9.23=65.69$
23. $3x+7x+15x-2x+5=74$
24. $2x+7x-3x-6=24$
25. $\frac{1}{4}x=6$
26. $\frac{1}{2}x=3$
27. $\frac{2}{3}x=8$
28. $\frac{5}{6}x=20$
29. $\frac{15}{a}=5$
30. $\frac{4}{x}=12$
31. $\frac{5}{z}=20$
32. $x+\frac{1}{2}x=6$
33. $x-\frac{1}{2}x=7$
34. $\frac{x}{2}+\frac{x}{4}=3$
35. $\frac{t}{3}-\frac{t}{6}=10$
36. $\frac{t}{5}+\frac{t}{3}=8$
37. $\frac{y}{4}-\frac{y}{7}=6$
38. $\frac{3}{4}x+\frac{1}{2}x-\frac{1}{8}x=18$
39. $6x-\frac{2}{3}x-\frac{1}{4}x+7=129$

Summary

88. This chapter has taught the meaning of each of the following terms: members of the equation, substitution, satisfying the equation, root of the equation, solving and checking equations.

89. In solving equations the following axioms are used:

6, 7. *If the same number or equal numbers be added to equals the sums are equal* (addition axiom).

8. *If the same number or equal numbers be subtracted from equals the remainders are equal* (subtraction axiom).

12. *If equals be divided by equal numbers (excluding division by 0) the quotients are equal* (division axiom).

13. *If equals be multiplied by the same number or equal numbers the products are equal* (multiplication axiom).

90. In many problems the algebraic solution is simpler than the arithmetical solution.

91. In solving verbal problems algebraically it is helpful to observe the following steps:

Denote by a letter the unknown number called for in the problem.

Express the given data in algebraic form.

Obtain the equation by equating two expressions denoting the same number.

Solve the equation and check the results.

92. An equation containing fractions may be cleared of fractions by multiplying every term in the equation by the least common multiple of the denominators and then reducing all fractions to the simplest form.

93. Some geometric problems may be solved by the aid of the equation.


TALES MILESIVS
VII Sapientum Primus
Apud Achillem Maffium

THALES OF MILETUS

THALES was born at Miletus in Asia Minor about 640 B.C. and died there about 542 B.C. He was probably of Phoenician parentage. In early life he was a merchant and traveled in Egypt in the pursuit of business. While there he studied Egyptian science, and it is said he amazed the king by determining the height of a pyramid by measuring its shadow. On his return home he abandoned business, opened at Miletus the first Greek school of philosophy, about 600 B.C., and devoted his life thenceforth to scientific and philosophical pursuits. He became famous as an astronomer, a mathematician, and a philosopher. He was the first to study mathematics scientifically. In proving mathematical truths he used the superposing of figures and other reasoning methods. He was the originator of the mathematical method of "thinking it out."

His school lasted over a hundred years after his death. The Pythagorean school grew out of it. The geometry of Thales' school consisted of half a dozen of our most elementary theorems, but it is worth while noting that he insisted that they be proved by sound reasoning.

Students will be interested to look up in some biographical dictionary or history of mathematics the story of his cornering the olive presses to illustrate his shrewdness, the story of his trouble with the badly behaved donkey to illustrate his sagacity, the story of his tumbling into a ditch to show his absent-mindedness, and the story of his prediction of the eclipse which later gave him a place among the Seven Sages of Greece.

Gow's *History of Greek Mathematics* gives a good account of Thales. See also Smith's *Dictionary of Greek and Roman Biography.*

CHAPTER IV

ANGLES

Classification of Angles

94. Angle. Clockwise and counter-clockwise direction of turning. What part of a complete turn does the long hand of a clock make (Fig. 40) when it rotates (turns) about the center post from **12** to **3**? **12** to **6**? **12** to **9**? **12** to **4**? **12** to **8**? **12** to **2**? **12** to **10**?

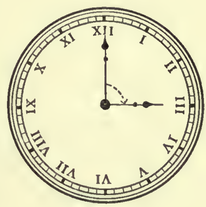


FIG. 40

If a line, as OA, Fig. 41, be imagined to rotate in a plane about a fixed point, O, in the direction indicated by the arrowheads, until it reaches the position OB, it is said to turn through an angle, a. The same angle may be formed by rotating OB about O until it takes the position OA. In the latter case the direction of turning is **clockwise**; in the first case it is **counter-clockwise**, i.e., opposite to the direction in which the hands of a clock move. The **amount of rotation** needed to bring OA to the position OB is the **angle** formed by OA and OB. The word "angle" comes from the Latin *angulus*, meaning *corner*.

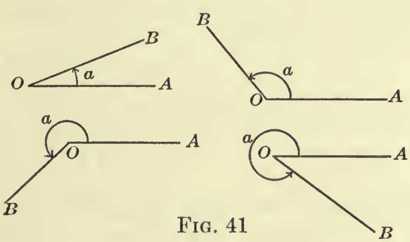


FIG. 41

95. Symbols for angle. The symbol for angle is $\angle$; for angles it is $\measuredangle$.

96. Notation for angles. An angle may be denoted by a small letter written within the angle, as a in Fig. 41. Often it is convenient to denote an angle by the point of intersection of the lines forming the angle prefixed by the angle symbol, as $\angle O$. Sometimes three letters are used, as $\angle AOB$ or $\angle BOA$, the first and last letters denoting points on the lines forming the angle, the middle letter denoting the point of intersection.

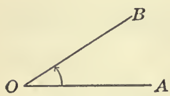


Fig. 42

97. Sides. The lines AO and BO, Fig. 42, forming the angle AOB, are the **sides** of $\angle AOB$.

98. Vertex. The point of intersection of the sides of an angle is the **vertex** of the angle. "Vertex" is a Latin word, meaning *turning-point.*

99. Right angle, straight angle, perigon. If a line rotating in a plane about one of its points makes $\frac{1}{4}$ of a complete turn, the angle formed is a **right angle**, Fig. 43. If the line makes a half turn, the angle is a straight angle,

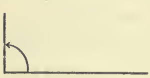


Fig. 43


Fig. 44


Fig. 45

Fig. 44. Thus, a **straight angle** is an angle whose sides lie in the same straight line on opposite sides of the vertex. If the line makes a complete turn, the angle is a **perigon**, Fig. 45.

1. Draw a right angle; a straight angle; a perigon.

2. Draw an angle made by a line which has rotated $\frac{3}{4}$ of a complete turn; $\frac{1}{3}$ of a complete turn.

3. Show that a perigon is equal to two straight angles or four right angles.

100. Acute and obtuse angles. An angle less than a right angle is an **acute** (sharp) angle, Fig. 46. An angle *greater* than a right angle and *less* than a straight angle is an **obtuse** (blunt angle, Fig. 47).

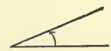


Fig. 46
Acute angle

Fig. 47.—Obtuse angle.

1. Draw an acute angle; an obtuse angle.

2. Point out a number of right angles in the classroom.

3. In the letter *A*, Fig. 48, point out the acute angles; the obtuse angles.

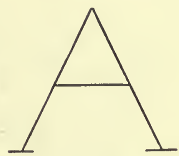


Fig. 48

The Measurement of Angles

101. Comparison of angles. Angles are compared by placing their planes one over the other so that the vertex and a side of one angle coincide (fit) with the vertex and a side of the other. If the other sides of the angles coincide, the angles are **equal.** If the other sides do not coincide, the angles are **unequal**, the smaller angle being the one whose second side falls within the other angle.

1. Compare angles *A* and *B*, Fig. 49; *B* and *C*.

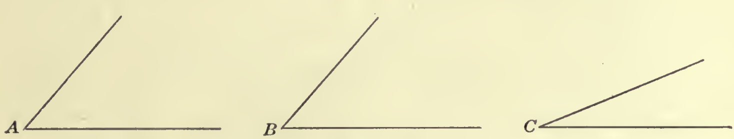


Fig. 49

To place angle *B* on *A* make a trace of $\angle B$ on thin paper and fit this on $\angle A$.

2. Draw two angles that appear to you to be equal and compare them as in problem 1.

102. Size of an angle. Since an angle is formed by rotating a line about a fixed point, the size of the angle depends entirely upon the amount of turning and not upon the length of the sides. Thus, $\angle P$ is greater than $\angle O$, Fig. 50, although the sides of $\angle O$ are longer than the sides of $\angle P$.

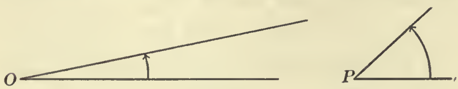


FIG. 50

103. Graphical addition and subtraction of angles. If one angle is placed *adjacent* to another, so that the vertex and one side coincide, the angles are said to be **added**. Thus,

$$\angle AOC = \angle AOB + \angle BOC,$$

Fig. 51. To **subtract** one angle from another, place the first *on* the second so that they have the vertex and one side in common. Thus,

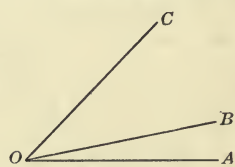


FIG. 51

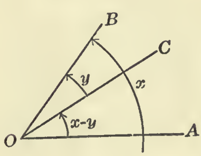


FIG. 52

$$\angle AOC = \angle AOB - \angle BOC, \text{ Fig. 52.}$$

1. Draw two unequal angles and find the sum; the difference.

Use thin paper, as in problem 1, § 101.

2. Let x, y, and z denote three angles so that $x > y, y > z$. Draw $x+y+z$; $x-y+z$; $x+y-z$.

104. Degree, minute, second. The right angle XOY, Fig. 53, is divided into 90 equal

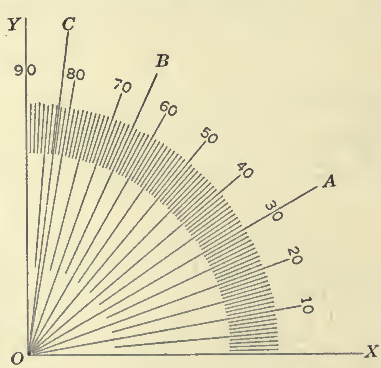


FIG. 53

parts. Each of these is a **degree** (1°). The degree is used as angle unit. A degree can be divided into 60 equal parts, called **minutes** (1′). Each minute contains 60 equal parts, called **seconds** (1″). The Greek mathematician Hypsicles (180 B.C.) was the first to divide an angle into degrees.*

EXERCISES

1. What part of a perigon is a degree? Of a straight angle? Of a right angle?

2. Read the following: 13°24′3.5″.

3. How many seconds are contained in an angle of 20°14′22″?

4. How many degrees are in $\angle XOA$, Fig. 53? In $\angle XOB$? In $\angle AOB$? In $\angle XOC$? In $\angle AOC$?

5. How many degrees are contained in the angle made by the long hand of the clock when it has rotated from 12 to 3? 12 to 6? 12 to 9? 12 to 4? 12 to 8? 12 to 2? 12 to 10?

6. How many degrees, right angles, straight angles, are in the angle formed by the hands of a clock at 9 o'clock? At 6 o'clock? At 4 o'clock? At 2 o'clock?

7. Draw freehand angles containing 45°, 60°, 30°, 90°, 180°, 360°.

8. How many degrees are in two straight angles? In four right angles? In $\frac{2}{3}$ of a right angle? In $\frac{7}{9}$ of a right angle? In $\frac{1}{5}$ of a right angle? In r right angles?

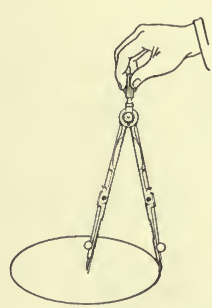


FIG. 54

105. Circle. Center. Opening the compass and placing the sharp point at a point P, Fig. 54, the pencil can be made to draw a curved line, called a *circle*. Thus, the **circle** is a closed curved line, all points of which lie in the

* See Ball, p. 85.

same plane and are equally distant from a fixed point within the curved line. The fixed point is the **center** of the circle.

Some writers use the word "circumference" in the sense in which we use "circle."

The Use of the Protractor in Measuring Angles

106. The protractor. The protractor, Fig. 55, is an instrument for measuring angles. The circular rim is divided into 180 equal parts. If straight lines were

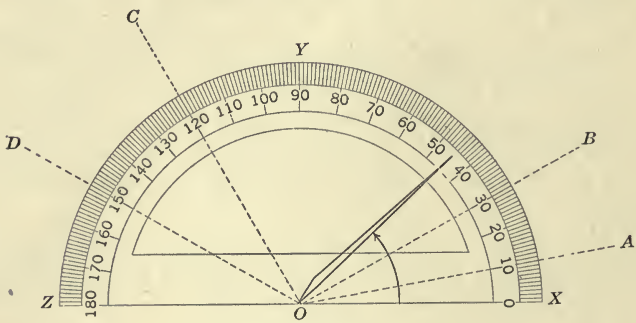


FIG. 55

drawn from the center, O, to each mark on the rim, 180 equal angles would be formed at O, each of which would be one degree (1°).

107. Arc-degree. A part of the circle, Fig. 55, included between two consecutive marks is an **arc-degree.**

1. Suppose a pointer to rotate about the fixed point, O, Fig. 55, in the direction indicated by the arrow. In rotating from the position OX to the position OZ, through how many right angles would it turn? Through how many straight angles? Through how many degrees?

2. Find the number of degrees and the number of right angles in each of the following angles of Fig. 55: *XOZ; AOZ; COZ; XOA; XOB; XOY; XOC; XOD; AOB; AOY; BOC; BOD.*

108. Radius. A line-segment drawn from the center to a point on the circle is a **radius**. Thus, OX is the radius of the circle XYZ, Fig. 55.

109. Arc. A part of a circle is an **arc**. An angle whose vertex is at the center of a circle **intercepts** the arc cut off by its sides.

110. Semicircle. Quadrant. An arc equal to one-half of a circle is a **semicircle**. An arc equal to one-fourth of a circle is a **quadrant**.

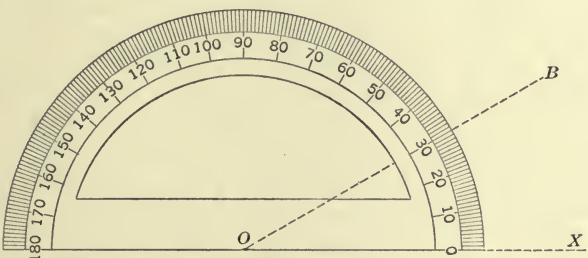


FIG. 56

111. To measure an angle. To measure an angle, as XOB, Fig. 56, by means of a protractor, place the protractor on the angle, making the center of the circle fall on the vertex O of the angle, and making the line from the center to the zero-reading fall along one side, OX, of the angle. Then read along the circle from the zero-reading to the point where the side OB cuts the circle. Thus, $\angle XOB = 30°$.

EXERCISES

1. Draw an angle. By use of the protractor find the number of degrees in the angle.

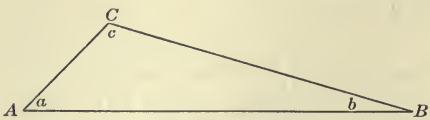


Fig. 57

2. Draw a triangle. How many angles does it have? Read each angle in three letters.

3. Fill out the table, Fig. 58, with reference to the triangle *ABC*, Fig. 57.

4. Draw a triangle. Fill out a table like Fig. 58 with reference to this triangle.

Angle	Classify Angle (Acute, Obtuse, Etc.)	Amount of Rotation (Complete, Half, Less than Half, Etc.)	No. of Degrees (Estimated)	No. of Degrees (Measured)	Error
a					
b					
c					
Sum					

Fig. 58

The Sum of the Angles of a Triangle

112. The sum of the angles of a triangle. From exercises 3 and 4 of § 111, it is seen that *the sum of the angles of a triangle is a straight angle, or 180°.*

113. Theorem. It can be demonstrated (proved) by geometry that the sum of the angles of *any* triangle is a straight angle.* A statement to be proved is called a **theorem.**

* The first mathematician to prove this theorem probably was Thales (640 B.C.–550 B.C.). Ball, pp. 16–17.

EXERCISES

1. Draw a triangle. Tear off the corners and place the angles adjacent to each other as in Fig. 59. What seems to be the sum of the angles of the triangle?

FIG. 59

2. That the sum of the angles of a triangle is 180° can be shown by rotating a stick or a pencil successively through the angles, as follows: Draw a triangle, Fig. 60. Place a pencil or stick in position 1 and note the direction it is pointing. Rotate the pencil through angle x. Then move it along AB to position 2. Turn the pencil through angle y and move it along BC to position 3. Turn it through angle z to position 4. The pencil has now rotated through an amount equal to $x+y+z$. Note the direction the pencil is pointing in the last position. Through what part of a complete turn has it rotated? Through how many right angles? Through how many degrees?

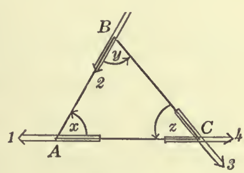


FIG. 60

3. State by an equation the number of degrees in the sum of the angles x, y, and z of a triangle.

By means of this equation it is possible to find an angle of a triangle when the other two are known. This is of great importance to surveyors, as it enables them to find all angles of a triangle although they may be able to measure directly only two angles.

114. Problems. Solve the problems below, observing the following steps: Make a sketch of the triangle, denoting the angles as given in the problem. Use the theorem of § 112 to obtain the equation. Solve the equation and find the values of the angles.

1. The angles of a triangle are $3x$, x, and $6x$. Find the numerical values.

2. Find the value of each angle of a triangle, the first angle of which is twice the second, the third being 3 times the first.

3. Find the angles of a triangle if the first angle is 6 times the second, and the third is $\frac{1}{2}$ of the first.

4. The three angles of a triangle are equal. Find them.

5. One angle of a triangle is 27°. The second angle is 27° larger than the third. How large is each angle?

6. One angle of a triangle is $\frac{2}{7}$ as large as another. The third is 3 times as large as the first. How large is each angle?

7. Find the angles of a triangle if the first is $\frac{1}{3}$ of the second, and the third is $\frac{1}{2}$ of the first.

8. Find the angles of a triangle if the first angle is 18° more than the second, and the third is 12° less than the second.

9. The difference between two angles of a triangle is 20°, and the third angle is 36°. Find the unknown angles.

10. Find the angles of a triangle if the first is 25° more than the second, and the third is 3 times the first.

11. Find the angles of a triangle if the first angle is double the second, and the third is 3 times the first, less 9°.

12. Find the angles of a triangle if the first is $\frac{1}{4}$ of the second, and the third is $\frac{1}{7}$ of the first, plus 18°.

13. Find the angles of a triangle if the first is $3\frac{1}{2}$ times the second, minus 8°, and the third is $\frac{1}{5}$ of the second.

14. Find the angles of a triangle if the first is 6 times the second, plus 18°, and the third is $\frac{1}{2}$ of the first, minus 7°.

15. Show that a triangle cannot contain more than one right angle; not more than one obtuse angle.

16. Two angles, a and b, of one triangle are equal respectively to the angles r and s of another triangle. Show that the third angle, c, of the first triangle equals the third angle, t, of the other.

The Sum of the Exterior Angles of a Triangle

115. Exterior angles. If a side as AB of a triangle ABC, Fig. 61, is extended the angle s, formed by the consecutive side BC and the extension of AB, is an **exterior angle**.

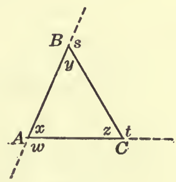


FIG. 61

1. How many exterior angles are there at each corner?

2. How many exterior angles has a triangle?

3. Draw a triangle. Prolong one side at each vertex and measure the three exterior angles formed. What is the sum of the exterior angles, if one angle is taken at each vertex?

4. In Fig. 62, find the sum of the three exterior angles by rotating a pencil as indicated.

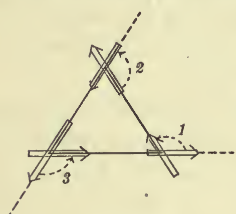


FIG. 62

5. Show that *the sum of the exterior angles of a triangle, taking one at each vertex, is 360°.*

Suggestion: How many degrees are in the sum $x+w$, Fig. 61?

In $z+t$? In $y+s$?

Show that

$$(x+w)+(z+t)+(y+s)=3\times 180°.$$
$$=540°.$$

This equation may be written:

$$(x+y+z)+(w+s+t) = 540°. \quad \text{Why?}$$
$$\text{But } \underline{x+y+z} \quad \underline{=180°.} \quad \text{Why?}$$
$$\text{Therefore, } w+s+t = 360°. \quad \text{Why?}$$

116. Interior angles. The angles of a triangle are called **interior angles** when contrasted with the exterior angles.

117. Proof. Reasoning like that of problem 5, § 115, is very common in geometry. Such reasoning is called **proving**, or **proof**.

EXERCISES

1. The three exterior angles of a triangle are equal. Find the value of each exterior and interior angle.

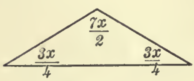


FIG. 63

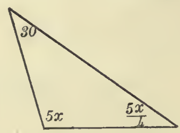


FIG. 64

2. Find the unknown interior and exterior angles of the triangles of Figs. 63 and 64.

3. Draw freehand the following angles: 30°, 45°, 60°, 90°, 135°, and by means of a protractor test the accuracy of your drawings.

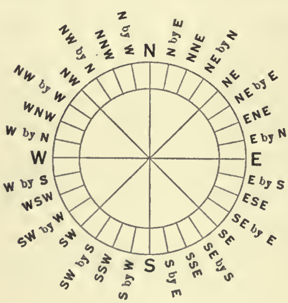


FIG. 65

4. In the mariner's compass, Fig. 65, state what angles are formed by these directions: E and SE; W and NE; N and E; NW and SE; S and NW; WSW and E; ENE and WNW.

5. Divide an angle into two equal parts, first by estimating, then by means of the protractor.

6. Fold a piece of paper, Fig. 66, as in Fig. 67. Fold again, making the edge OA fall along OB. Unfold the paper, Fig. 68. The

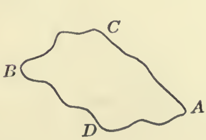


FIG. 66

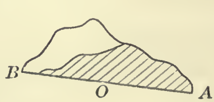


FIG. 67

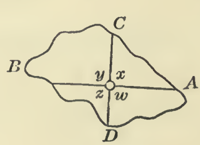


FIG. 68

creases are seen to form four angles x, y, z, w. Why are these angles equal? Why are they right angles?

118. Problem 6 illustrates the theorem that *all right angles are equal.*

1. Show by measuring with the protractor that *an exterior angle of a triangle equals the sum of the two remote interior angles.*

2. Show by measuring that the two exterior angles at the same vertex of a triangle are equal.

119. Classification of triangles as to angles. A triangle all of whose angles are acute is an **acute triangle.** A triangle one of whose angles is obtuse is an **obtuse triangle.** A triangle one of whose angles is a right angle is a **right triangle.** An **equiangular triangle** is a triangle whose angles are equal.

Show that if one angle of a triangle is a right angle the other two are acute angles.

120. Measurement of angles out of doors. To measure angles in the open, e.g., the angle formed by two roads, an angle-measurer can be made as follows:

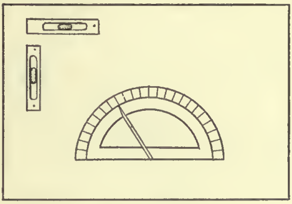


FIG. 69

On a drawing board, Fig. 69, tack a protractor. Place the board upon a table or fasten it to a tripod and place the center of the circle of the protractor as nearly as possible over the vertex of the angle to be measured. Bring the table into horizontal position. Use a ruler, with a pin stuck in it at each end, to sight in the direction of each side of the angle and each time note the reading on the protractor. The difference between these readings is the number of degrees in the angle.

When accuracy is important, as in astronomy and surveying, better instruments are used (see the engineer's transit, Fig. 70). The principal parts of these instruments are the graduated circles for taking the readings for the sides of the angles, and the small telescope for sighting in the direction of the sides.

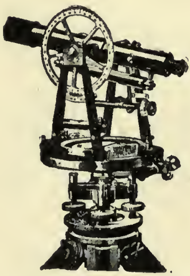


FIG. 70.—Engineer's transit.

To Draw an Angle Equal to a Given Angle

121. By use of protractor to draw an angle of given size. Let it be required to draw an angle of 60°.

Draw an indefinite line as AB, Fig. 71. Place the protractor with the center of the circle on A and with the zero reading on AB. At the 60° mark place a point C. Remove the protractor and draw AC. Angle BAC is the required angle.

1. With a protractor draw the following angles: 80°; $92\frac{1}{2}$°; 170°; 36°; 91°; 160°.

2. Which of the angles in problem 1 are acute? Obtuse?

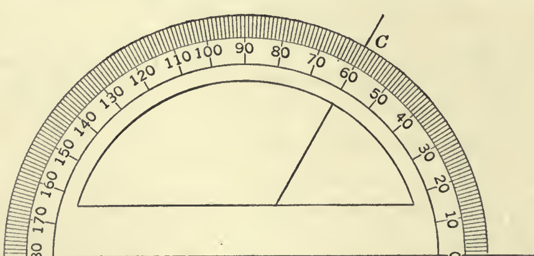


FIG. 71

122. Oblique angles, oblique lines, perpendicular lines. Acute and obtuse angles are **oblique angles**. The sides of oblique angles are said to be **oblique** to each other;

the sides of a right angle are said to be **perpendicular** to each other.

123. By use of the protractor to draw an angle equal to a given angle.

Measure the given angle with the protractor and construct an angle containing the same number of degrees, using the method of § 121.

An angle may be constructed equal to a given angle by use of ruler and compass only. This construction is a consequence of the following relation between angles, whose vertices are at the center of a circle (central angles), and their intercepted arcs.

124. Theorem: *Equal central angles in the same or equal circles intercept equal arcs.*

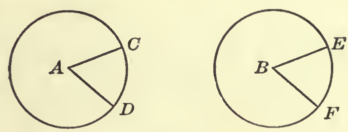


Fig. 72

For angle A, Fig. 72, may be placed upon angle B, making $\angle A$ coincide with $\angle B$. (Why is this possible?) Then circle A will coincide with circle B. (Why?) This will make C fall on E and D on F. Therefore arc CD coincides with arc EF.

125. Theorem: *In the same or equal circles equal arcs are intercepted by equal central angles.*

For, circle A, Fig. 72, may be made to coincide with circle B so that arc CD coincides with arc EF. (Why can this be done?) Then $\angle A$ must coincide with $\angle B$. (Why?)

126. The theorems in §§ 124 and 125 explain why the protractor may be used to measure angles. For every angle-degree at the center intercepts an arc-degree on the rim of the protractor. Every angle therefore contains as many angle-degrees as there are arc-degrees in the intercepted arc. This may be expressed briefly thus: *A central angle is measured by the intercepted arc.*

127. Problem: *At a given point on a given line to construct an angle equal to a given angle.**

Let $\angle ABC$, Fig. 73, be the given angle and let D be the point on the given line EF.

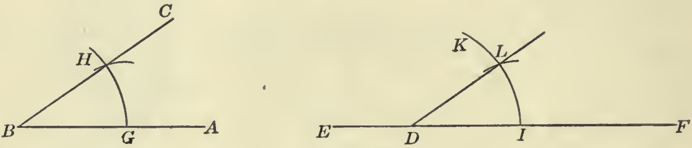


FIG. 73

Using B and D as centers and the same radius draw arcs GH and IK.

With a radius equal to GH and I as center draw an arc meeting IK at L.

Draw line DL.

Angle IDL is the required angle.

Check the correctness of the construction by measuring $\measuredangle GBH$ and IDL with the protractor.

EXERCISES

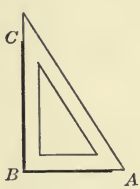


FIG. 74

1. Draw a right angle by using a right triangle, Fig. 74.

2. Draw a right angle by using the protractor (see § 121).

3. Draw a right angle by using ruler and compass.

Draw a line-segment, as AB, Fig. 75.

Using a point C on AB and a convenient radius, draw arcs of a circle meeting AB in the points D and E.

With D and E as centers and a convenient radius draw arcs meeting at F.

Draw CF. Angle BCF is the required angle.

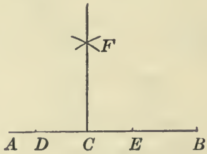


FIG. 75

* Oenopides of Chios (500 B.C.–430 B.C.) has the reputation of being the first to solve this problem (see Cajori, p. 19).

4. Draw an angle equal to the sum of two given angles, using the protractor.

Measure the given angles with a protractor and draw an angle containing as many degrees as are contained in the given angles together.

5. Using ruler and compass draw an angle equal to the sum of two given angles.

Let $\measuredangle ABC$ and DEF, Fig. 76, be the given angles whose sum is to be constructed.

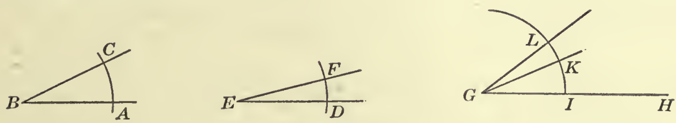


FIG. 76

Construction: Draw a line-segment, as GH.
With B, E, and G as centers and the same radius draw arcs AC, DF, and IL.
Draw arc IK = arc AC and arc KL = arc FD.
Angle IGL is the required sum.

Proof:

$\angle ABC = \angle IGK$. Why?

$\underline{\angle DEF = \angle KGL.}$ Why?

$\angle ABC + \angle DEF = \angle IGK + \angle KGL$. Why?

$\angle ABC + \angle DEF = \angle IGL$. Why?

6. Let x be a given angle. Draw $2x$; $3x$; using the methods of problems 4 and 5.

7. Let x, y, and z be three angles. Construct $x+y+z$.

8. Draw the differences of two given unequal angles, using the protractor.

9. Using ruler and compass draw an angle equal to the difference of two given unequal angles.

Proceed as in problem 5, but lay off KL from K in the direction opposite to KL, i.e., on KI.

10. Of the given angles x, y, z, angle x is greater than y and y is greater than z. Draw an angle equal to $x-y+z$; $x+y-z$.

11. Using a protractor, divide an angle into 2, 3, 4 . . . . . ., etc., equal parts.

Measure the angle and take $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ . . . . . ., etc., of the numerical measure.

12. Using ruler and compass *divide an angle into two equal parts.*

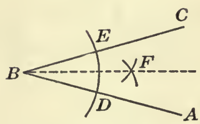


FIG. 77

Let $\angle ABC$, Fig. 77, be the given angle.

With B as center and any radius draw an arc cutting BA and BC at D and E.

With D and E as centers and a convenient radius, draw arcs meeting at F.

Draw BF.

BF divides $\angle ABC$ into two equal parts.

128. Bisecting a magnitude. To divide a magnitude into two equal parts is to **bisect** it.

1. Two ports A and B are 15 miles apart. They contain cannon of 11-mile range. Make a figure showing the space these guns can shell.

2. Bisect a straight angle, using the method of problem 12, § 127. Compare the construction with that of problem 3, § 127.

3. Using the construction of problem 2, *construct a perpendicular to a given line at a given point in it.*

Summary

129. In this chapter the meaning of the following terms was taught: angle; acute, right, obtuse, straight, and oblique angle; clockwise and counter-clockwise direction of turning; perigon; sides and vertex of an angle; equal and unequal angles; size of an angle; angle-degree and arc-degree; minutes, seconds; circle; center and radius of a circle; arc of a circle; quadrant and semicircle; intercepted arcs; theorem; proof; interior and exterior angles of a triangle; acute, obtuse, right, oblique, and equiangular triangle; to bisect a magnitude.

130. The following symbols have been introduced: $\angle$ for angle, $\measuredangle$ for angles, ($^\circ$) for degree, ($'$) for minute, ($''$) for second.

131. Angles may be denoted by a single small letter written within the angle, or by a capital letter written outside of the angle at the vertex, or by three letters, the first and last denoting points on the sides and the middle letter denoting the vertex.

132. The truth of the following theorems has been shown:

1. *The sum of the angles of a triangle is a straight angle.*

2. *The sum of the exterior angles of a triangle is 360°.*

3. *An exterior angle of a triangle equals the sum of the two remote interior angles.*

4. *Equal central angles in the same or equal circles intercept equal arcs and equal arcs are intercepted by equal central angles.*

5. *A central angle is measured by the intercepted arc.*

133. The protractor is an instrument for measuring angles and for drawing angles equal to given angles.

134. Ruler and compasses were used *to add and subtract given angles; to draw an angle equal to a given angle; to bisect an angle; to construct a perpendicular to a given line at a given point in the line.*

CHAPTER V

AREAS AND VOLUMES. MULTIPLICATION

The Area of a Square

135. Original meaning of geometry. When land is to be sold or divided, it becomes necessary to measure it. The Greek writer Herodotus tells that in Egypt originally the land had been divided equally among the Egyptians. Whenever the river Nile washed away part of the land, the Egyptians actually had to rediscover their lands after the flood, and in order to adjust the taxes overseers were sent out by the king to measure out by how much the land had become smaller. Thus, measuring was the beginning of geometry.

The word "geometry" comes from the Greek words *ge*, meaning *earth*, and *metron*, meaning *measure*.

136. Kinds of quadrilaterals. Pieces of land, fields, pastures, lots, etc., often have the form of polygons. The following are the most familiar quadrilaterals.

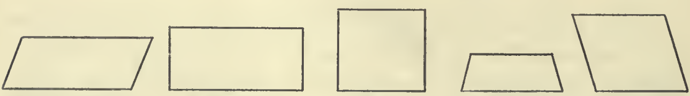


FIG. 78

EXERCISES

1. What quadrilaterals in Fig. 78 are equilateral? Equiangular?

2. What quadrilaterals contain right angles?

3. In what respect does the square differ from the rectangle? From the parallelogram? From the rhombus?

4. In what respect does the rhombus differ from the parallelogram?

137. Area of a square the length of whose side is a whole number. To measure a square place it upon squared paper, as $ABCD$, Fig. 79, and count the number of square centimeters contained in it.

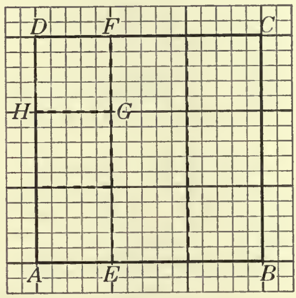


FIG. 79

This may be done as follows: Suppose the side AB to be 3 cm. long. The square can then be divided into 3 vertical rectangular strips, as $AEFD$. Each strip, being 3 cm. high, can be divided into 3 equal sq. cm., as $HGFD$. Therefore $ABCD$ contains 3×3 sq. cm., or 9 sq. cm.

The result, 9, is the **area** of $ABCD$. One square centimeter, as $HGFD$, is the **unit of area.**

1. Find in a similar way the area of a square whose side is 4 cm. long.

2. Find the area of a square whose side is 6 cm. long.

The foregoing problems show that the number of square units contained in a square may be obtained by multiplying the number of units of length in one side by itself. This may be stated briefly thus:

The area of a square is equal to a side multiplied by itself.

Since the length (base) of a square is equal to the width (altitude), the area A of a square whose side is a units long is given by the equation $A = a \times a$. This may be expressed in another form, thus: *A equals a-square,* or in symbols, $A = a^2$, a^2 meaning $a \times a$.

138. Formula. In the two preceding paragraphs two methods of finding the area of a square are given. First, *the geometric method.* A drawing of the square is made and the number of unit squares contained in it is counted.

Second, *the algebraic method.* The length of the side is substituted in the general expression $A=a^2$. The advantage of the last method over the first is apparent. The equation $A=a^2$ is called a **formula** for finding the area of a square.

Find by means of the formula the areas of squares whose sides have the following lengths: 4; 6; 10; 100; x; l; s.

139. Units of area. The standard units of area are the square meter (sq. m.), square decimeter (sq. dm.), square centimeter (sq. cm.), square millimeter (sq. mm.), square foot (sq. ft.), square inch (sq. in.), and square yard (sq. yd.). Their sides are respectively of the following lengths: 1 m., 1 dm., 1 cm., 1 mm., 1 ft., 1 in., and 1 yd.

The area of land is expressed in acres, an acre being a quantity of land containing 4,840 square yards.

Show that 1 sq. m. = 100 sq. dm.; 1 sq. dm. = 100 sq. cm.; 1 sq. cm. = 100 sq. mm.

140. Area of a square the length of whose side is a fraction. In § 137 it was shown that the area of a square the length of whose side is a whole number (i.e., an integer) may be found from the formula $A=a^2$. It will be shown that the same formula may be used to compute the area of a square the length of whose side is a fraction.

EXERCISES

1. Find the area of a square whose side is $2\frac{1}{2}$ cm.

Draw a square on a line-segment $2\frac{1}{2}$ cm. long and divide it as in Fig. 80. Find the sum of the parts of the large square.

2. Verify the result of exercise 1 by letting $a=2\frac{1}{2}$ in the formula $A=a^2$.

3. Find the area of a square whose side is 1.5 m.

By taking a smaller unit of length the side of this square may be expressed as an integral number: since 1 m. = 10 dm., 1.5 m. = 15 dm. Using the formula $A=a^2$, we have $A=(15)^2$ sq. dm. = 225 sq. dm. = 2.25 sq. m., since $\frac{1}{100}$ sq. m. = 1 sq. dm.

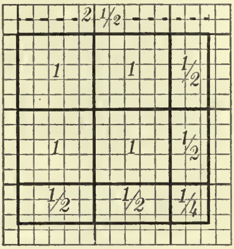


FIG. 80

4. Verify the result of exercise 1 by solving it by the method of exercise 3.

Change $a=2\frac{1}{2}$ cm. to $a=2.5$ cm. = 25 mm.

5. Find the area of a square whose side is $2\frac{3}{4}$ m.

Change $2\frac{3}{4}$ m. to 2.75 m. = 275 cm. Then apply $A=a^2$.

6. In the formula $A=a^2$ let $a=2\frac{3}{4}$ m. and show that the result obtained agrees with that of exercise 5.

Exercises 1–6 show that when the length of the side of a square is fractional, as $a=1\frac{1}{4}$ m., this fraction may be changed to a decimal, $a=1.25$ m., which may be expressed as an integral number in terms of a smaller unit of length, $a=125$ cm. By means of the formula $A=a^2$ the area of the given square may then be expressed in terms of a smaller unit square, $A=15{,}625$ sq. cm. Since 1 sq. cm. = $\frac{1}{10{,}000}$ sq. m., the area may now be expressed in terms of the original square unit. Thus, $A=1.5625$ sq. m. However, this result agrees with the result obtained by letting $a=1\frac{1}{4}$ m. in $A=a^2$. For, $A=(1\frac{1}{4})^2$ sq. m. $=\frac{25}{16}$ sq. m. $=1.5625$ square meters.

It follows that the formula $A=a^2$ may be applied to squares the lengths of whose sides are fractional or integral numbers.

This result may be expressed in the form of the following theorem:

Theorem: *The area of a square is equal to the square of one side.*

$$A=a^2$$

Area of a Rectangle

141. Formula for the area of a rectangle. The area of a rectangle may be expressed by a formula found in a manner similar to finding the formula for the square.

EXERCISES

1. Denoting the base of a rectangle by b and the altitude by h, draw on squared paper a rectangle for which $b=3$ cm. and $h=$ 4 cm. Find the area by counting the number of square centimeters in the rectangle.

2. Draw rectangles for the following pairs of values of b and h: $b=3.4$ cm., $h=2$ cm.; $b=4.2$ cm., $h=2$ cm., and find the areas by counting the number of square centimeters.

3. By the method of § 141 find the area of rectangles having $a=2.8$ m., $h=3.6$ m., $a=4.8$ m., $h=1.8$ meters.

4. Show that the results of exercises 1–3 may be obtained from the following theorem:

Theorem: *The area of a rectangle equals the product of the base by the altitude.*

$$A=b\times h$$

EXERCISES

1. Find the perimeter and area of a square whose side is 4 ft.; x ft.; .06 cm.; 2.48 meters.

2. The perimeter of a square floor is 64 ft.; $4x$ ft; $8a$ ft. Find the area.

3. How many feet of wire are needed to fence in a square piece of ground whose area is 2,500 sq. ft.? a^2 sq. m.? x^2 square yards?

4. The base of a rectangle is 3 times as long as the altitude. The sum of the base and altitude is 16 inches. Find the sides and area of the rectangle.

5. Show by an equation the other dimension of a rectangle, having one side equal to 9 ft. and an area equal to 18 sq. ft.
" " " " " 9 ft. " " " " " 15 sq. ft.
" " " " " 6 yd. " " " " " 12 sq. yd.
" " " " " 4 yd. " " " " " A sq. yd.

6. The frame of a picture of rectangular form of dimensions 22 in. by 17 in. is $2\frac{1}{2}$ in. wide. Find the area of the frame and the outer perimeter.

7. A mantel is 36 in. high and 42 in. wide. The grate is 30 in. high and 30 in. wide. Find the area of the mantel and the number of square tiles contained in it if each tile is 3 in. long.

8. How many square tiles of 8 in. length are needed to make a walk 28 ft. long and 3 ft. wide?

9. In a rectangular garden 25 ft. wide and 95 ft. long a 3-ft. wide walk is laid along the whole edge. The midpoints of the long sides are joined by a 2-ft. path. How many square feet are left for the garden?

Cube and Parallelopiped

142. Cube. A solid like the one represented in Fig. 81 is called a **cube**. A cube has six *faces*, all of which are squares. Two faces come together in an *edge*. Thus, there are 12 edges in the cube. The cube has 8 *corners*.

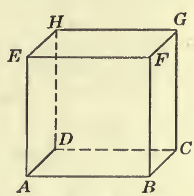


FIG. 81

143. Parallelopiped. The solid represented in Fig. 82 is a **rectangular parallelopiped**. The faces of a rectangular parallelopiped are rectangles.

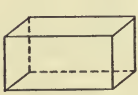


FIG. 82

How many faces has a parallelopiped? How many edges? How many corners?

Volume of Cube and Rectangular Parallelopiped

144. Unit of volume. To measure a solid a cube is used whose edge equals the unit-length. The number of times this cube is contained in a given solid is called the **volume** of the solid, the cube being the **unit volume.**

145. To find the volume of a parallelopiped. Let Fig. 83 represent a rectangular parallelopiped 5 cm. long, 4 cm. deep, and 3 cm. high. Since the face on which the figure stands is a rectangle 5 cm. long and 4 cm. wide, a layer of 5×4 unit cubes may be placed on it. Since the solid is 3 cm. high, it contains three such layers and therefore $3\times5\times4$ unit cubes. Thus, the parallelopiped contains as many units of volume as the number of units obtained by multiplying the length by the width and their product by the height.

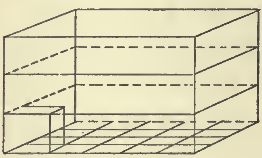


FIG. 83

This is usually expressed in the form of a theorem, thus:

Theorem: *The volume of a parallelopiped equals the product of the length by the height by the width.*

$$V=l\times h\times w$$

146. Volume of a cube. The volume of a cube is computed in the same way as for the parallelopiped. Since the edges of a cube are all equal, the theorem of § 145 takes the following form:

The volume of a cube whose edge is e units long is given by the formula $V=e\times e\times e$.

The formula $V=e\times e\times e$ may be written briefly $V=e^3$, read V equals *e-cube*, e^3 meaning $e\times e\times e$.

In words, this formula is expressed as follows:

Theorem: *The volume of a cube is equal to the cube of one edge.*

$$V=e^3$$

147. Graphing equations. In chapter I, § 17, three ways of representing data were shown. Let us apply them to the following problem:

A number of rectangles are 3 in. wide and their lengths are 4″, 5″, 6″, 7″, 8″, 9″, 10″, 11″, 12″, 13″. Calculate the areas of these rectangles.

The first mode of representing the areas for various lengths is to tabulate for each value of the length, l, the corresponding value of the area, A [see Table (I) in Fig. 84].

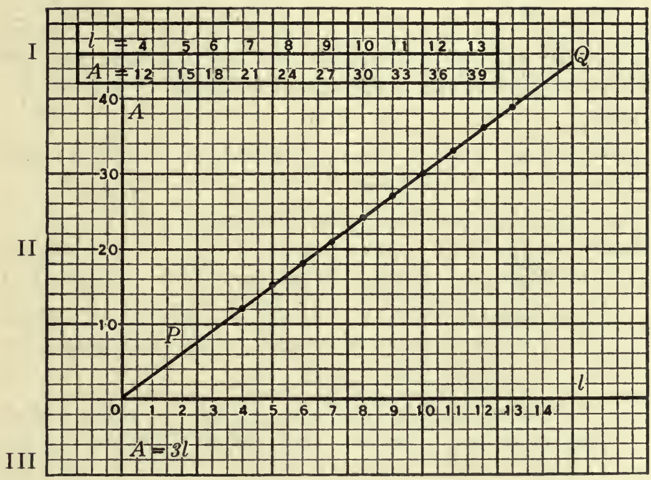


FIG. 84

Second, the same data may be represented graphically.

From the point O, Fig. 84, lay off the length horizontally and from the end-point of the length lay off the area vertically. Then join the end-points of the vertical lines by straight lines.

Third, the equation $A = 3l$ represents the data of the table and graph. The straight line PQ, Fig. 84, is said to be the graph of $A = 3l$.

1. From the graph, Fig. 84, find the area when $l = 1, 2, 3, 14$.

2. Graph the following equations:

1. $A=5l$	3. $L=2c+1$	5. $C=w^2$
2. $D=t+1$	4. $y=2x+3$	6. $y=x^2+3$

148. Linear equations. Equations whose graphs are straight lines are **linear** equations.

Multiplication of Monomials

149. Symbols for multiplication. The formulas $A=b\times h$, $V=l\times h\times w$, $A=a^2$, $V=e^3$ suggest that the products of numbers may be represented geometrically; e.g., the product of any two numbers may be expressed graphically by a rectangle whose dimensions are equal to the given numbers.

The product of two *equal* numbers, or of three *equal* numbers, may be represented by the area of a square or the volume of a cube, respectively. Hence, the notation a-square and e-cube. The product of four or more equal numbers cannot be expressed geometrically. However, in algebra the process is extended. Thus, we write $a\times a=a^2$, $a\times a\times a=a^3$, $a\times a\times a\times a=a^4$, read a-fourth, $a\times a\times a\times a\times a=a^5$, read a-fifth, etc.

The product of two *different* literal numbers, as x and y, is shown by writing the letters side by side, as xy, with no sign between them. We are familiar with the form $x\times y$ from arithmetic. The form xy is most used in algebra. It is often convenient to use the form $x\cdot y$.

150. Exponent. Base. Power. In $x^2, x^3, x^4 \ldots\ldots x^n$ the $2, 3, 4 \ldots\ldots n$ are called **exponents** of x.

What is the meaning of x-fifth? x-sixth? x-seventh? x-tenth? x-nth?

Write these numbers in symbols.

The **exponent** of a number is the small figure or letter written to the right and a little above the number symbol, to denote the number of equal factors in a product. In 6^3, meaning the product $6 \times 6 \times 6$, the 3 is the exponent, the number 6 is the **base**, and the product, 6^3, is the **power**. Thus 216 ($=6^3$) is the third power of 6. (Why?)

When no exponent is written, as in ax, the exponent is understood to be 1 for each letter, as though the number were written a^1x^1.

Notice that $4x$ means $4 \cdot x$, or that x is to be used *as an addend* 4 times, while x^4 means $x \cdot x \cdot x \cdot x$, or that x is to be used *as a factor* 4 times, and similarly for the other forms, as $3x$ and x^3, $5x$ and x^5, etc.

EXERCISES

1. Letting $x=6$, give the meaning and value of each of these numbers.

1. $2x$	4. x^3	7. $5x$	10. $3x^2$
2. x^2	5. $4x$	8. x^5	11. $5x^3$
3. $3x$	6. x^4	9. $2x^2$	12. $2x^4$

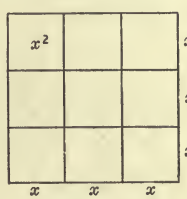


FIG. 85

2. Show from Fig. 85, (1) that the perimeter, p, of a square of $3x$ is given by $p=12x$; (2) that the area A is given by $9x^2$.

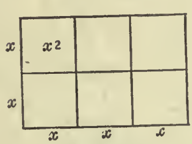


FIG. 86

3. Express by equations the perimeter and area of the rectangle, Fig. 86.

4. Show from a figure the perimeter and area of a rectangle $3a$ by $5a$.

5. Write the following products in briefest form:

$$x \cdot x \cdot x \cdot x; \ \tfrac{1}{6} \cdot \tfrac{1}{6} \cdot \tfrac{1}{6}; \ \tfrac{2}{3} \cdot \tfrac{2}{3} \cdot \tfrac{2}{3}; \ \frac{m}{n} \cdot \frac{m}{n}; \ 1\tfrac{2}{3} \cdot 1\tfrac{2}{3} \cdot 1\tfrac{2}{3}$$

6. Find the values of the monomials: 2^3; 11^2; $(\frac{1}{3})^3$; 3^4; $(1.2)^2$; 7^3.

7. Letting $a=4$, $x=3$, $b=1$, $y=2$, find the values of the polynomials:

$$x^2+2x+1;\ x^2+3xy+y^2;\ a^3-2a^2b+3ab^2-b^3$$

8. Two messengers leave a town at the same time, traveling in opposite directions. They travel a and b mi. an hour respectively. How far apart are they after t hours?

9. Find the values of the following numbers, if $x=4$; $2x^2$; $(2x)^2$; $3x^3$; $(3x)^3$

10. Letting $a=2$, $b=1$, $c=5$, $d=3$, $e=4$, find the value of

$$\frac{ab+bc+cd+de}{a+b+c+d};\ \frac{a^2+b^2+c^2+d^2+e^2}{a+b+c+d};\ \frac{a^3+3a^2b+3ab^2+b^3}{a+b}$$

151. Product of powers having the same base. The product of two powers having the same base can be simplified. Thus, $x^2 \cdot x^3 = xx \cdot xxx = x^5$; $x^3 \cdot x = xxx \cdot x = x^4$.

Give orally the products of the following pairs of factors in briefest form:

1.	$4^2 \cdot 4^2$	9.	$b^2 \cdot b^5$	17.	$b \cdot b^3$
2.	$8 \cdot 8^3$	10.	$c \cdot c^8$	18.	$K^2 \cdot K$
3.	$10^2 \cdot 10^4$	11.	$c^3 \cdot c^6$	19.	$K^2 \cdot K^3$
4.	$a^2 \cdot a$	12.	$x^3 \cdot x^8$	20.	$a \cdot x^2$
5.	$12^3 \cdot 12^4$	13.	$a \cdot x$	21.	$a^2 \cdot x$
6.	$a^3 \cdot a^3$	14.	$b \cdot c$	22.	$g \cdot t^2$
7.	$a \cdot a^5$	15.	$b \cdot b$	23.	$a^2 \cdot b$
8.	$a^4 \cdot a^7$	16.	$a \cdot a^6$	24.	$x^2 \cdot y^2$

152. Comutative law (law of order). To illustrate the equation $5\times 2=2\times 5$ draw a rectangle 5 units long and 2 units wide, Fig. 87. The area is 10 sq. units $=5\times 2$ sq. units. A rectangle 2 units long and 5 units wide differs from the rectangle

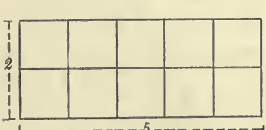


FIG. 87

in Fig. 87 only in position. Its area is 2×5 sq. units. Since 5×2 and 2×5 represent the area of the same rectangle in different positions it follows that $5 \times 2 = 2 \times 5$.

1. Show that $7 \times 8 = 8 \times 7$.

2. Show that $ab = ba$.

These problems illustrate that in algebra, as in arithmetic, *the factors of a product may be changed in order without changing the value of the product.* This is the **Commutative Law of Multiplication.**

Simplify the following products, doing all you can mentally:

1. $4xy \cdot 5x^3y^2$

$$4xy \cdot 5x^3y^2 = 4 \cdot 5 \cdot x \cdot x^3 \cdot y \cdot y^2 \quad \text{(Why?)}$$
$$= 20x^4y^3$$

2.	$4b^2c \cdot 5b^3c^2$	5.	$(2xy)^2$	8.	$2 \cdot 3 \cdot 5 \cdot 2 \cdot 5 \cdot 4$
3.	$abc \cdot a^2bc^2 \cdot 2abc^3$	6.	$(2m^3n^2)^3$	9.	$3 \cdot 5 \cdot 4 \cdot 4 \cdot 4$
4.	$4a^3 \cdot 5a^2 \cdot 2a$	7.	$(3a^2x)^2$	10.	$(3ax)\ (4a^2x)\ (8a^3x^2)$

Addition of Monomials

153. Coefficient. The **coefficient** of any factor in a term is the product of all the other factors of the term. Thus, in $4axy$ the coefficient of axy is 4, of xy is $4a$, of ax is $4y$. In $\frac{bc}{3}$ the coefficient of c is $\frac{b}{3}$, of b, is $\frac{c}{3}$, etc. When the coefficient of the *term* is spoken of, the arithmetical factor is usually meant. It is common to say the coefficient of the term $4axy$ is 4; of $\frac{bc}{3}$ is $\frac{1}{3}$, etc. If no coefficient is written, as in a, x^4, 1 is understood to be the coefficient, as though $1a$, $1x^4$ were written.

154. Similar terms. Terms which have a *common factor* are said to be similar with respect to that factor and are called **similar** or **like** terms, as $2x^2y^2$, $12x^2y^2$, $8x^2y^2$.

155. Dissimilar terms. Terms which have no common factor are called **dissimilar** or **unlike** terms, as $4a^2$ and $3cb^2$.

EXERCISES

In each of the following polynomials point out with respect to what factor the terms are similar, state in each case the coefficient of the common factor, and then reduce to the simplest form. Do all you can without writing down your work.

1. $4a^2b+a^2b+5a^2b+3a^2b$

The common factor is a^2b. The coefficients are 4, 1, 5, 3.
Hence, $4a^2b+a^2b+5a^2b+3a^2b=(4+1+5+3)a^2b=13a^2b$.

2. $4x+7x+20x+35x$
3. $ax+25x+bx+46x$
4. $ax+bx+cx+dx$
5. $8pq^2r+14pq^2r+12pq^2r$
6. $3a^2b+5a^2b+7a^2b+3a^2b$
7. $2ax+3xa+7xa+5ax$
8. $3pq^2+6tq^2+8rq^2+12sq^2$
9. $4axz+7cxz+5dxz+9exz$
10. $abm+pmq+xmy+mdc$
11. $3(a+b)+4(a+b)+12(a+b)$
12. $8(x^2+y^2)+10(x^2+y^2)+12(x^2+y^2)$
13. $3\frac{2}{5}(pr-q^2)+5\frac{1}{2}(pr-q^2)+4\frac{3}{10}(pr-q^2)$

Multiplication of a Polynomial by a Monomial

156. Graphical multiplication of a sum by a monomial. Solve the following problems:

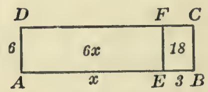


FIG. 88

1. Express by an equation the area of a rectangle of dimension 6 and $x+3$, Fig. 88.

The area of $ABCD=6(x+3)$. Why? ...... (1)

The parenthesis ( ) means that the number within is to be multiplied by 6.

Dividing the rectangle $ABCD$ into two rectangles by drawing line EF, we have

$$ABCD = AEFD + EBCF. \quad \text{Why?} \ldots\ldots (2)$$

But $AEFD = 6x$. Why?......(3)

and $EBCF = 18$. Why?......(4)

Substituting for $ABCD$, $AEFD$, and $EBCF$ their equals obtained from equations (1), (3), and (4), equation (2) takes the form $6(x+3) = 6x+18$.

2. Show from Fig. 89 that $a(m+n) = am + an$.

FIG. 89

3. Show from a figure that

$$d(a+b+c) = da + db + dc$$

The equations in problems 2 and 3 exemplify the axiom: *A whole equals the sum of all its parts.*

157. Partial products. The products am and an, problem 2, are the **partial products** of $a(m+n)$

What are the partial products of $d(a+b+c)$?

158. Product of a polynomial by a monomial. Problems 1, 2, and 3, § 156, illustrate the following principle:

A polynomial is multiplied by a monomial by multiplying each term of the polynomial by the monomial and then adding the partial products.

1. Multiply as indicated $2(3x+4y)$; $(x+2y)4a$; $3a+4b\cdot 2c$; $5b\cdot 4x+2y$.

2. Simplify by carrying out the multiplications before the additions and subtractions $a(a+b)-b$; $(x+y)m-ym$; $3x^2+4xy\cdot 2x+10$.

3. Letting $a=4$, $b=1$, $c=3$, find the values of the following numbers: $(a+b)c$; $2(a+b)-c$; $4a+2(b+c)$; $8a+(3b+4c)$; $3a(2ab+8-5c)$; $25c^3$; a^b; $(2a+3)5+(4b+5)2$.

159. Graphical multiplication of a difference by a monomial. The number $(a-b)c$ means that b is subtracted from a and the difference multiplied by c. The product $(a-b)c$ may be represented geometrically, thus:

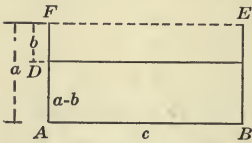


FIG. 90

$$\text{In Fig. 90, } ABCD = ABEF - DCEF$$
$$ABEF = ac$$
$$DCEF = bc$$
$$ABCD = (a-b)c$$

Substituting in the first equation for $ABCD$, $ABEF$, and $DCEF$ their equals $(a-b)c$, ac, and bc, we have $(a-b)c = ac - bc$.

EXERCISES

1. State the principle illustrated by the preceding problem.

2. Express the following products as polynomials:
$(2x-3y)a$; $2(4x-6y)$; $2x(4a+5b-3c)$; $(a^2-2ab+b^2)a$.

3. Multiply as indicated: $a^2bz^3 \cdot b^2$; $8m^2n \cdot 4mn^2$; $(2x+4y)2$; $(4x^2-2x+3)2x$; $x^2y^2 \cdot x^2z$; $4rs^2t \cdot 3st^2$; $3x(x-y)$; $4y(x+y-2z)$.

4. Simplify $5(2x-3y)+2(2x-y)+(x-4y)2x$.

5. Give the factors of the following products: $ab+bc$; $xy+xz$; $2a+6b$; $2xy+10x$; $3x-6y$; $2ax-8ay$; $4xa^2-12xb^3$; $3xa^2b+9xy^2b$.

6. The width of a rectangle is $x-3$, the length x, and the perimeter 66 yards. Find the width and the length in yards.

7. The area of a square equals that of a rectangle whose sides are 4 cm. less and 8 cm. greater respectively than the side of the square. Find the area of the rectangle.

8. After 12 years a man will be twice as old as he was 12 years ago. What is his age?

9. Solve the following equations:

   1. $3(x+4) = 22+x$
   2. $9(x+35) = 5(2x+45)$
   3. $3(x+15)+5 = 2(2x+9)+4(x+3)$

4. $\frac{2(x+2)}{3}=8$

5. $\frac{5(y+\frac{1}{2})}{7}=2\frac{6}{7}$

6. $\frac{x+7}{5}+1=\frac{x+11}{14}+2$

7. $9x=\frac{7x-34}{5}+22$

Multiplying Polynomials by Polynomials

160. Graphical multiplication. The product of $a+b$ by $c+d$ is written $(a+b)(c+d)$. It may be represented geometrically by a rectangle of dimensions $a+b$ and $c+d$, Fig. 91.

The rectangle $ABCD$ is composed of four rectangles ac, bc, ad, and bd.

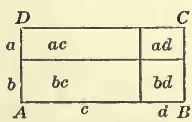


FIG. 91

Therefore,

$$(a+b)(c+d)=ac+bc+ad+bd,$$

both sides of the equation representing the area of rectangle $ABCD$.

The product $(a-b)(c+d)$ is represented by a rectangle having the dimensions $(a-b)$ and $(c+d)$, Fig. 92.

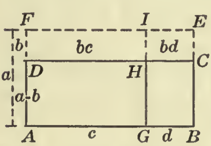


FIG. 92

The rectangle $ABEF=ac+ad$.

Subtracting from this the rectangles bc and bd, we obtain the rectangle $ABCD$.

Therefore, $(a-b)(c+d)=ac+ad-bc-bd$, both sides of the equation representing the area of rectangle $ABCD$.

Show from a rectangle that

$$(a+b+c)(m+n)=am+bm+cm+an+bn+cn$$

161. Multiplication of polynomials. In § 160 the following principle is illustrated:

A polynomial is multiplied by a polynomial by multiplying each term of one polynomial by every term of the other and adding the partial products.

EXERCISES

1. Using the principle of § 161 express the following products as polynomials:

1. $(x+y)(a+b)$
2. $(x+y)(m+n)$
3. $(a+x)(b+y)$
4. $(r+s)(a+x)$
5. $12(7+6-4)$
6. $2a(3a^2-4a+2)$

2. Solve the equation:

$$(s+2)(s+5)=(s+1)(s+3)+22$$

3. Find for $x=3$, $y=2$ the value of

$(3x-2y)(2x+3y)$; $(2x^2-5x+6)(5x-3)$; $(7x^2y-3xy^2)(x^2y-y^2x)$

4. By means of figures express as polynomials the following squares:

1. $(a+b)(a+b)$ or $(a+b)^2$
2. $(c+d)^2$
3. $(x+y)^2$
4. $(m+n)^2$

5. Draw squares whose areas are expressed by the following trinomials:

1. $a^2+2ax+x^2$
2. $b^2+2bc+c^2$
3. $k^2+2kb+b^2$
4. $s^2+2st+t^2$
5. x^2+6x+9
6. $c^2+8c+16$

6. What are the factors of the trinomials in problem 5?

7. The area of a rectangle is 96 sq. yd., the base is $(8+x)$ yd., and the altitude 8 yards. Find the base.

8. One side of a rectangle is 5 yd., the other is 7 yards. By how much must the longer side be increased so that the area is $1\frac{3}{7}$ times as large as before?

162. Check. Problems in multiplication may be **checked** by substituting in the problem and in the product convenient values for the letters. Both should reduce to the same number.

EXERCISES

Multiply as indicated and check. Use pencil only when needed.

1. $3x(x^3+4xy+2y^2)$

   $3x(x^3+4xy+2y^2)=3x^4+12x^2y+6xy^2$

   Check: Let $x=1$, $y=2$

   $3x(x^3+4xy+2y^2)=3(1+8+8)=51$

   $3x^4+12x^2y+6xy^2=3+24+24=51$

2. $5\frac{1}{3}mk(9m^2+6mk+18k^2)$
3. $(a^2+2a+1)(a-3)$
4. $(x^2+xy+y^2)(x^2-xy+y^2)$
5. $(p+q+r)^2$
6. $(.4x-.3y+.7z)(10x+20y+30z)$
7. $(.4a+.3b)^2+(.6a+.2b)^2+(.2a+.3b)(.5a+.1b)$
8. $(2x^3-7x+3)(2x+5)$
9. $(a^3-3y+5)(a^2+10)$
10. $(p^4+2p^2+7p)(2p-1)$

Area of Parallelogram and Triangle

163. Area of parallelogram. To find the area of a parallelogram, as $ABCD$, Fig. 93, draw line BE at right angles to AB, dividing the parallelogram into two parts, the triangle BEC and the quadrilateral $ABED$. The triangle is then cut off and moved to the left until BC coincides with AD, forming a rectangle as $ABEF$, Fig. 94.

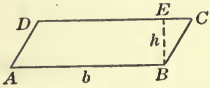


FIG. 93

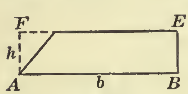


FIG. 94

Then rectangle $ABEF$ = parallelogram $ABCD$. Why? But rectangle $ABEF=b \cdot h$. Why? Therefore, parallelogram $ABCD=b \cdot h$. Why? This gives the following theorem:

Theorem: *The area of a parallelogram is equal to the product of the base by the altitude.*

$$P=bh$$

Find the area of a parallelogram (1) if $b=28$, and $h=19$

(2) if $b=16.3$, and $h=14.6$

164. Area of a triangle. A triangle, as ABC, Fig. 95, is one-half of a parallelogram obtained by drawing the lines DC and AD. Since the triangle and the parallelogram have the same base BA and equal altitudes h it follows that the area of ABC is equal to one-half of the product of AB by h. (Why?) This leads to the following theorem:

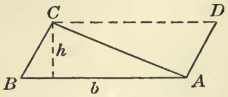


FIG. 95

Theorem: *The area of a triangle is equal to one-half of the product of the base by the altitude.*

$$T = \frac{1}{2} b \cdot h$$

1. Find the area of a triangle
   (1) if $b=12$ ft., and $h=16$ ft.
   (2) if $b=8.2$ ft., and $h=7.78$ ft.

2. Show by an equation the other dimensions of a triangle having

a base $=6$ ft.	and an area $=24$ sq. ft.
" " $=6$ ft.	" " " $=9$ sq. ft.
an altitude $=4$ yd.	" " " $=16$ sq. yd.
" " $=a$ ft.	" " " $=a$ sq. ft.
" " $=c$ in.	" " " $=b$ sq. in.

165. Quotient. The **quotient** of x divided by y is written $\frac{x}{y}$ or $x \div y$ and is read "x over y" and "x divided by y" respectively.

166. The area of a trapezoid. To determine the area of a trapezoid, as $ABCD$, Fig. 96, draw the line-segment BD, dividing the trapezoid into two triangles.

Triangle $ADB = \frac{1}{2}hb$. Why?

Triangle $DBC = \frac{1}{2}ha$. Why?

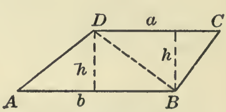


FIG. 96

Therefore, trapezoid $ABCD = \frac{1}{2}hb + \frac{1}{2}ha$. Why?

or trapezoid $ABCD = \frac{1}{2}h(b+a)$. Why?

Calling a and b the **bases** of the trapezoid this result expresses the following theorem:

Theorem: *The area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases.*

The altitudes, bases, and areas of parallelograms are as given below. Find the value of the unknown dimension.

Altitude	Base	Area
$3x+2$	8	56
16	$5x-7$	288
$2x+1$	15	105
10	$3x+2$	110

Summary

167. In this chapter the meaning of the following terms was taught: parallelogram, rectangle, square, trapezoid, rhombus; area of a figure, unit of area; formula; cube, parallelopiped; volume, unit of volume; exponent, power, base; coefficient; similar and dissimilar terms; partial products; quotient.

168. The following symbols were introduced: the exponent, to indicate the number of equal factors. To indicate the product of unequal factors use the symbols $\times$, $\cdot$, or no symbol at all. Division is indicated by the symbol $\div$ or by placing one letter or figure (number) over another with a line between.

169. Equations may be graphed. Equations whose graphs are straight lines are *linear* equations.

170. The product of two numbers may be represented geometrically as the area of a rectangle.

171. Problems in multiplication may be checked by substituting convenient values for the letters in the problem and in the result.

172. The following algebraic principles were taught:

1. **Commutative law.** *The factors of a product may be changed in order without changing the value of the product.*

2. *A polynomial is multiplied by a monomial by multiplying each term of the polynomial by the monomial and then adding the partial products.*

3. *A polynomial is multiplied by a polynomial by multiplying each term of one polynomial by every term of the other and adding the partial products.*

173. The following theorems and formulas have been obtained:

1. *The area of a square is equal to the square of one side.*

2. *The area of a rectangle is equal to the product of the base by the altitude.*

3. *The volume of a rectangular parallelopiped is equal to the product of the length by the height by the width.*

4. *The volume of a cube is equal to the cube of an edge.*

5. *The area of a parallelogram is equal to the product of the base by the altitude.*

6. *The area of a triangle is equal to one-half of the product of the base by the altitude.*

7. *The area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases.*

JOHN WALLIS

J O H N   W A L L I S

JOHN WALLIS was born at Ashford in England, November 22, 1616, and died at Oxford, October 28, 1703. During a school holiday, when he was only fifteen years old, he was struck with curiosity at seeing the odd symbols and signs of an arithmetic in the hands of his brother. He borrowed the book and in a fortnight, with his brother's assistance, he had mastered the book. He studied to become a physician, and he was the first to maintain that the blood circulates in the human body. His main interest however was in mathematics.

Wallis became professor of geometry at Oxford in 1649 and lived there thenceforth until his death. Besides many mathematical works, he wrote on theology, logic, philosophy, and he devised a system for teaching deaf-mutes. His genius thus revealed itself in many ways.

He wrote an arithmetic and an algebra that were long the standard texts. Newton is said to have learned algebra from Wallis' text. In his algebra he introduced into mathematics the symbol || for parallelism. He was one of the pioneers in the field of calculus and he developed the theory of interpolation very fully. He discovered

$$\pi = 2\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\ldots\ldots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\ldots\ldots}$$

He was familiar with all the mathematics of his day, added greatly to the subjects of higher arithmetic and algebra, and did much toward laying the foundations in English science for what mathematicians now call *analysis.*

Read the life of Wallis in Ball's *History of Mathematics,* pp. 288–93 (5th ed.).

CHAPTER VI

ANGLE-PAIRS

Adjacent Angles

174. Adjacent angles. Two angles that have the same vertex and a common side between them are **adjacent angles.** The sides which are not common are the **exterior sides.**

EXERCISES

1. Are b and d, Fig. 97, adjacent angles? Give reason for your answer. Are a and b adjacent angles? b and c?

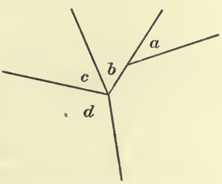


Fig. 97

2. Read the exterior sides of $\measuredangle x$ and y, Fig. 98; the common side.

3. Draw two adjacent acute angles whose sum is a right angle.

4. Draw three acute angles whose sum is a right angle.

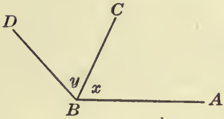


Fig. 98

5. Draw two adjacent obtuse angles whose sum is 3 right angles.

6. Draw two adjacent angles, one obtuse and the other acute, whose sum is 2 right angles.

7. With a protractor draw adjacent angles of 75° and 85°; of 103° and 57°; of $31\frac{1}{2}°$ and $2\frac{1}{2}°$. In each case check the work by finding the sum arithmetically and then measuring the sum with a protractor.

8. Draw two intersecting straight lines making a pair of adjacent angles equal and show by measuring that the angles are right angles.

175. Perpendicular lines. If two straight lines intersect making a pair of adjacent angles equal, each line is **perpendicular** to the other.

1. In Fig. 99 which lines are perpendicular? Point out the equal adjacent angles.

2. Draw two lines perpendicular to each other, using only the ruler.

FIG. 99

176. Theorem. *At a given point in a given line only one perpendicular can be drawn to the line.*

For, if two perpendiculars to AC, Fig. 100, were drawn at B, the two right angles DBC and EBC would be unequal. This contradicts the theorem that all right angles are equal and is therefore impossible.

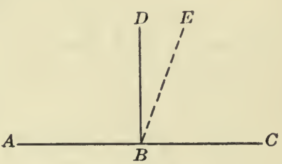


FIG. 100

177. Draw two adjacent angles of 56° and 124°; of $19\frac{1}{2}$° and $160\frac{1}{2}$°; of 92° and 88°. With a ruler, or straightedge, see if the exterior sides of each pair of angles form a straight line. What term is applied to each angle-sum?

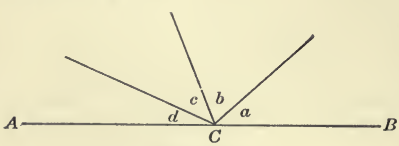


FIG. 101

This problem leads to the following theorem:

Theorem: *If the sum of two adjacent angles is a straight angle, the exterior sides are in the same straight line.*

178. From a point C on a straight line AB, Fig. 101, draw three lines as in the figure. By estimating find the number of degrees in each angle. Then measure each angle with the protractor. Fill out the table in Fig. 102.

What seems to be the sum of all the angles about a point on one side of a straight line? This illustrates the theorem of § 179 below.

Angle........	a	b	c	d	Sum
Estimate.....	40				
Measure	39				

FIG. 102

The Sum of the Adjacent Angles about a Point on One Side of a Straight Line

179. Theorem: *The sum of the adjacent angles about a point, on one side of a straight line, is a straight angle, or 180°.*

EXERCISES

1. Find the number of degrees in each angle of Fig. 103.

We may write

$9x+x+(37-2x)+(5x-26)=180.$ Why?

Changing the order of the terms,

$9x+x+5x-2x+37-26=180.$ What law is used?

$13x+11=180.$ Why?

$13x=169.$ Why?

$x=13.$ Why?

Therefore

$$9x=117$$
$$37-2x=11$$
$$5x-26=39$$

Check: $x+9x+37-2x+5x=180$

2. With a protractor make a drawing of Fig. 103.

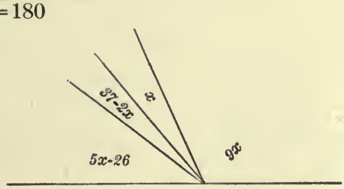


FIG. 103

3. All the angular space about a point in a plane, on one side of a straight line, is divided into angles represented by the

following expressions. Find x and each angle in degrees. Draw figures for exercises 1, 2, and 3.

1. x, $5x$, $7x-2$
2. $8x$, $48-3x$, $5x-22$, $4x-14$
3. $25\frac{2}{3}+5x$, $8x+8\frac{5}{6}$, $3x$, $9\frac{1}{2}+x$
4. $3x$, $2(x+9)$, x, $42-x$
5. $2x$, $2(x-10)$, $x-18$, $3(36-x)$
6. $3(x-3)$, $x+33$, $2(41-x)$
7. $2.8x+39.33$, $1.2x-32.09$, $x+7.16$
8. $6.93x$, $4.82x$, $1.27x+5.09$, $138.91-9.02x$
9. $\frac{3}{5}x$, $88+\frac{1}{5}x$
10. $\frac{2}{3}x+10$, $86-\frac{1}{3}x$
11. $\frac{5}{7}x+14$, $97-\frac{2}{7}x$

The Sum of the Angles at a Point

180. The sum of the angles just covering the angular space about a point may be found from the following problem.

Draw four lines from a point, Fig. 104. Find the number of degrees in each angle, first, by estimating, then by measuring with a protractor. What seems to be the sum of the angles that just fill the angular space about a point? This may be stated in the form of a theorem as follows:

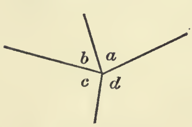


FIG. 104

Theorem: *The sum of all the angles at a point just covering the angular space about the point is a perigon, or 360°.*

EXERCISES

1. Find by solving an equation the number of degrees in each angle of Fig. 105.

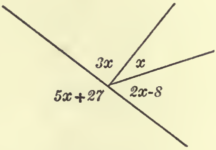


FIG. 105

All the angular space about a point in a plane is divided into angles represented by the following expressions; find x and each angle in degrees. With protractor draw figures for exercises 2, 3, and 4.

2. $2x,\ x,\ 4x+40,\ 180-3x$
3. $x,\ 36+5x,\ 3x-9$
4. $3x,\ 5x,\ 5x+45,\ 27-x$
5. $15x+16\frac{1}{2},\ 37\frac{1}{2}-2x,\ 8x-9$
6. $3x,\ 5x+26\frac{2}{5},\ 2x,\ 9x+143\frac{3}{5}$
7. $7x+24,\ 14x+53\frac{3}{8},\ 120\frac{5}{8}-3x$

Solve the following equations:

8. $15-6t+8t=25$
9. $6t-7+4t=13$
10. $7y-3y+10y=39$
11. $2(x-3)+12=18$
12. $8+5(s+7)=63$
13. $18r+13-10r=75$
14. $\frac{x}{4}+5=9$
15. $\frac{r}{3}+8-\frac{r}{5}=10$
16. $\frac{9x}{2}-\frac{5x}{3}=28$
17. $\frac{26y}{3}-15+\frac{5y}{2}=52$
18. $3.7l-3.6-2.9l=2.4$
19. $2x+4(x+10)+3x=130$

Supplementary Angles

181. Supplementary angles. Two angles whose sum is a straight angle (180°) are **supplementary angles.** Each angle is said to be the **supplement** of the other.

If the supplementary angles are adjacent they are called **supplementary adjacent** angles.

EXERCISES

1. Draw two supplementary adjacent angles.

2. Are 50° and 130° supplementary? 37° and 133°? 60° and 120°? 90° and 90°?

3. How many degrees are there in the supplement of an angle of 45°? Of $120\frac{1}{2}$°? Of 90°? Of a°? Of x°?

4. Write the supplement of $a°$; of $b°$; of $5d°$; of $\frac{3c°}{2}$.

5. If angles of 120° and $a°$ are supplementary, what does a represent?

6. If $x°+80°=180°$, what is the supplement of $x°$? What is the value of x?

7. In the equation, $a°+b°=180°$, what is the supplement of $a°$? Of $b°$? Why?

8. State by an equation that the following pairs of angles are supplementary:

(1) $x°$ and $60°$	(4) $50°$ and $x°+70°$
(2) $70°$ and $y°$	(5) $2x°+3°$ and $27x°-2°$
(3) $b°$ and $c°$	(6) $\frac{3}{4}x°$ and $\frac{17}{5}x°+112\frac{1}{8}°$

9. The supplement of $x+3$ degrees is $2x+27$ degrees. Find x, $x+3$, and $2x+27$.

We may write	$x+3+2x+27=180.$	Why?	(1)
Combining like terms	$3x+30=180$		(2)
Subtracting 30	$3x=150.$	What axiom is used?	(3)
Dividing by 3	$x=50.$	What axiom?	(4)
Whence	$x+3=53.$	Why?	(5)
	$2x+27=127$		(6)
Check:	$x+3+2x+27=180$		

10. $x°$ is the supplement of $x°+84°$. Find the angles.

11. One of two supplementary angles is 98° larger than the other. Find the angles.

12. One of two supplementary angles is 27° smaller than the other. Find the angles.

13. One of two supplementary angles is $3\frac{1}{2}$ times as large as the other. Find the angles.

14. The difference of two supplementary angles is 110°. Find them.

Let $x°$ be one angle and $x° + 110°$ the other.

15. Find two supplementary angles whose difference is 21°; $36\frac{1}{2}°$; $73\frac{1}{4}°$; $d°$.

16. The difference between an angle and its supplement is 37°. Find the angle.

17. How many degrees are there in the angle, $x°$, if it is the supplement of $5x°$? Of $7x°$? Of $3\frac{1}{2}x°$?

18. How many degrees are there in an angle that is the supplement of 4 times itself? Of 8 times itself? Of 10 times itself? Of $2\frac{1}{2}$ times itself? Of $\frac{3}{4}$ of itself? Of $\frac{1}{8}$ of itself?

19. Express in algebraic language:

(1) the double of an angle, x
(2) 15° added to 3 times the angle
(3) 29° subtracted from 6 times the angle
(4) 4 times the sum of the angle and 13°
(5) Two-thirds of the sum of the angle and 17°

20. Express in algebraic symbols:

(1) the supplement of an angle, x
(2) 5 times the supplement
(3) 3 times the supplement
(4) 14° added to 3 times the supplement
(5) 16° subtracted from 3 times the supplement
(6) the supplement increased by 10°
(7) the supplement diminished by 18°
(8) the supplement divided by 4
(9) one-third of the supplement
(10) 17° added to the supplement
(11) 20° added to one-third of the supplement
(12) 19° subtracted from $\frac{3}{5}$ of the supplement

21. If an angle is doubled and its supplement is increased by 20°, the sum of the new angles thus obtained is 280°. Find the two supplementary angles.

Let x be one angle, and $180-x$ the other; then by the conditions of the problem $2x+(180-x+20)=280$.

22. If an angle is trebled, and its supplement is diminished by 112°, the sum of the angles obtained is 168°. Find the supplementary angles.

23. The sum of an angle and $\frac{1}{3}$ of its supplement is 90°. Find the angle.

24. If an angle is increased by 12°, and its supplement is divided by 5, the sum of the angles obtained is 80°. Find the supplementary angles.

25. If 20° is added to 5 times an angle, and 15° is subtracted from 2 times the supplement of the angle, the sum of the angles obtained is 401°. Find the supplementary angles.

Solve the following equations:

26. $2(x-5)+\frac{1}{2}(x+1)=3$

27. $5(x-5)+(x+3)=2$

28. $\frac{15z}{2}-20+(z+12)=9$

29. $\frac{x+5}{3}+\frac{2x-5}{5}-4=4$

30. $\frac{6(x-2)}{3}+\frac{x+3}{4}-3=5$

31. Find x and each of the following supplementary angle-pairs:

$$\frac{x}{5}+\frac{x}{2}+172\tfrac{1}{2} \text{ and } \frac{3x}{10}-\frac{x}{4}; \quad x-\frac{x}{7} \text{ and } \frac{3x}{4}+90$$

32. Draw a figure showing that *the supplements of equal angles are equal.*

Complementary Angles

182. Complementary angles. Two angles whose sum is a right angle are **complementary angles.** Each angle is said to be the **complement** of the other.

EXERCISES

1. What is the complement of a, Fig. 106? Of b?

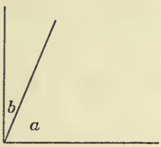


FIG. 106

2. Draw two adjacent complementary angles. May either angle be obtuse? Point out two perpendicular lines.

3. Show whether 22° and 68° are complementary; $43\frac{2}{3}°$ and $46\frac{2}{3}°$; $89\frac{3}{8}°$ and $\frac{5}{8}°$.

4. What is the complement of 60°? Of 30°? Of $10\frac{3}{8}°$? Of $45\frac{5}{7}°$? Of $a°$? Of $n°$? Of $x°$?

5. Write the complement of $d°$; of $3c°$; of $\frac{5d°}{3}$; of $\frac{3s°+2t°}{5}$; of $7(a+b)$ degrees; of $5x^2$ degrees; of $7y^3$ degrees; of $3x^2-5y^4$ degrees.

6. If angles of 40° and $d°$ are complementary, how many degrees does d stand for?

7. If $y°+70°=90°$, what is the complement of $y°$? Why? What is the value of y?

8. In the equation, $c°+d°=90°$, what is the complement of $c°$? Of $d°$? Why?

9. State by equations that the following pairs of angles are complementary:

(1) $y°$ and 50° (2) 30° and $z°$ (3) $w°$ and $x°$
(4) $a°+30°$ and $a°-20°$ (5) $2x°+7°$ and $5x°-2°$
(6) $3(x+7)$ degrees and $5(2x-8)$ degrees
(7) $\frac{2}{3}x-15\frac{7}{9}$ degrees and $26\frac{1}{2}x+43\frac{5}{8}$ degrees.

10. $x°$ is the complement of $x°+48°$. Find the angles.

11. One of two complementary angles is 24° larger than the other. Find the angles.

12. One of two complementary angles is 28° smaller than the other. Find the angles.

13. How many degrees are there in the angle x, which is the complement of $4x$? Of $6x$? Of $5\frac{1}{2}x$?

14. How many degrees are there in an angle that is the complement of 3 times itself? Of 7 times itself? Of 6 times itself? Of $3\frac{1}{3}$ times itself? Of $\frac{4}{5}$ of itself? Of $\frac{1}{9}$ of itself?

15. The difference of two complementary angles is 83°. Find them.

16. Find two complementary angles whose difference is 21°; $36\frac{1}{2}°$; $73\frac{1}{4}°$; $d°$.

17. The difference between an angle and its complement is 27°. Find the angle.

18. If an angle is doubled, and its complement is increased by 40°, the sum of the new angles thus obtained is 160°. Find the complementary angles.

19. If an angle is trebled, and its complement is diminished by 40°, the sum of the angles obtained is 130°. Find the complementary angles.

20. The sum of an angle and $\frac{1}{2}$ of the complement is 75°. Find the angle.

21. If an angle is increased by 15°, and the complement is divided by 3, the sum of the angles obtained is 75°. Find the complementary angles.

22. If 20° is added to 3 times an angle, and 6° is subtracted from $\frac{2}{3}$ of the complement, the sum of the angles obtained is 102°. Find the complementary angles.

Solve the following equations:

23. $\frac{3x}{5}+88+\frac{x}{5}=180$

24. $\frac{4t}{7}+10-\frac{3t}{14}=15$

25. Draw a figure showing that *the complements of equal angles are equal.*

Opposite Angles

183. Opposite angles. Two angles having a *common vertex,* and having sides in the *same straight line,* but in *opposite directions,* are called **opposite** or **vertical** angles (as x and z, Fig. 107).

EXERCISES

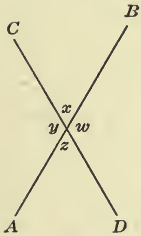


FIG. 107

1. Read both pairs of **opposite** angles in Fig. 107.

2. On tracing paper make a trace of angles y and z (Fig. 107). Put this trace on angles x and w and see whether z coincides with (fits on) x, and y with w. How do the opposite angles compare in size?

3. Test your conclusion in exercise 2 by drawing two intersecting straight lines and measuring both pairs of opposite angles with a protractor.

4. Show that in Fig. 107 $x+y=180$
Show that $y+z=180$
Show that $x+y=y+z$
Then $x=z$. Why?

5. Show, as in exercise 4, that $y=w$.

Exercises 4 and 5 show the truth of the following theorem:

Theorem: *If two lines intersect, the opposite angles are equal.*

EXERCISES

1. Two intersecting straight lines form four angles a, b, c, and d. If $a=40°20'10''$, how large are b, c, and d?

2. Show that the bisectors of two adjacent supplementary angles are perpendicular to each other.

Proof: $a+b+c+d=180°$, Fig. 108. Why?

$a=b$. Why?

$c=d$. Why?

Therefore $b+b+c+c=180$. Why?

$2b+2c=180$. Why?

$b+c=90$. Why?

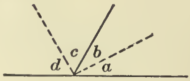


FIG. 108

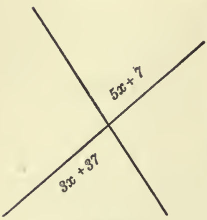


FIG. 109

3. Find x and the four angles made by two intersecting straight lines, if two opposite angles (Fig. 109) are $3x+37$ and $5x+7$.

Since the given angles are opposite angles,

$5x+7=3x+37$. Why? (1)

Subtracting 7 $5x=3x+30$. What axiom? (2)

Subtracting $3x$ $2x=30$. What axiom? (3)

Dividing by 2 $x=15$. What axiom? (4)

$5x+7=5\cdot15+7=82$

$3x+37=3\cdot15+37=82$

Each of the other two angles is $180-82=98$. Why?

Check: $82+98+82+98=360$

4. Find x and each of the following pairs of opposite angles made by two intersecting straight lines. Draw figures for exercises (1) and (2).

(1) $7x+27$ and $4x+87$

(2) $3x-17$ and $x+103$

(3) $\frac{3}{5}x+16\frac{1}{2}$ and $\frac{2}{5}x+24\frac{1}{2}$

(4) $2\frac{4}{11}x-13$ and $1\frac{1}{11}x+57$

(5) $\frac{1}{3}x+\frac{5}{3}x$ and $\frac{3}{2}x+42$

(6) $\frac{2}{3}x+\frac{1}{2}x-28$ and x

(7) $5x+\frac{3x}{4}$ and $\frac{5x}{2}+130$

(8) $\frac{7x}{4}-\frac{x}{6}$ and $\frac{2x}{3}+66$

(9) $\frac{x}{3}+\frac{x}{6}$ and $\frac{x}{4}+18$

(10) $\frac{7x}{2}-\frac{3x}{4}$ and $\frac{3x}{5}+8\frac{3}{5}$

5. Solve the following equations:

(1) $\frac{z}{3}+16+\frac{z}{2}-14=7$

(2) $\frac{2r}{3}-15+\frac{3r}{4}=8$

The Acute Angles of a Right Triangle

184. Theorem: *The acute angles of a right triangle are complementary angles.* Show that this is true.

Find the values of the acute angles of a right triangle if one angle is 3 times the other; 5 times the other; $\frac{2}{3}$ of the other; $1\frac{1}{5}$ of the other; 6 more than 7 times the other; $\frac{1}{2}$ of the other diminished by 33.

185. One of the acute angles of a right triangle is twice as large as the other. Find the acute angles. Draw a right triangle containing these angles.

If the side opposite the 90° angle is twice as long as the side opposite the 30° angle the construction is correct.

This problem illustrates the following theorem:

Theorem: *In a right triangle whose acute angles are 30° and 60° the side opposite the 90° angle (hypotenuse) is twice as long as the side opposite the 30° angle.*

The acute angles of a right triangle are given equal. Find the number of degrees in each angle. Make a drawing of the triangle. How do the lengths of the sides about the right angle compare?

186. Isosceles triangle. A triangle having two sides equal is an **isosceles triangle.**

EXERCISES

1. In Fig. 110 $\triangle ABC$, ADC, and BDC are right triangles. Show that x and a are complements of the same angle and therefore equal.

Show that y and b are equal angles.

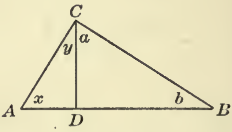


FIG. 110

2. Solve the following equations:

(1) $9+\frac{x}{5}-12+x=9$

(2) $\frac{3x-2}{7}-1=\frac{3}{7}$

(3) $9x-\frac{x}{9}-1=7\frac{8}{9}$

(4) $\frac{15r}{7}+15-\frac{3r}{14}=28\frac{1}{2}$

3. The following angle pairs are acute angles of a right triangle. Find x and the angles.

$$\frac{x}{8}+2x \text{ and } \frac{87}{2}-\frac{5x}{6}\,;\quad \frac{x}{8}+x \text{ and } \frac{x}{12}+\frac{35}{2}$$

Angle-Pairs Formed by Two Lines Intersected by a Third

187. When two lines are intersected by a third line (transversal), eight angles are formed, Fig. 111.

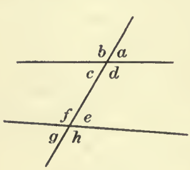


FIG. 111

1. Make a list of the supplementary adjacent angles and for each pair state the equation expressing the relation between the angles.

2. Make a list of the opposite angles and state the equation for each pair.

188. When two lines, Fig. 112, are cut by a transversal,

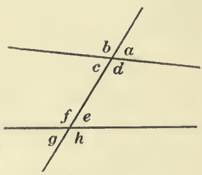


FIG. 112

$$\text{the angles of the angle-pairs}\left\{\begin{array}{l} a \text{ and } e\\ b \text{ and } f\\ d \text{ and } h\\ c \text{ and } g\end{array}\right\}\text{are called } \textbf{corresponding} \text{ angles,}$$

angles c, d, e, f are called **interior** angles,
angles a, b, g, h are called **exterior** angles,

the angles of the angle-pairs	d and e c and f	are called **interior** angles on the *same side* of the transversal,
the angles of the angle-pairs	d and f c and e	on *opposite* sides of the transversal, are called **alternate interior** angles,
the angles of the angle-pairs	b and h a and g	on *opposite* sides of the transversal, are called **alternate exterior** angles.

EXERCISES

1. If $a=e$ prove that $c=e$.
2. If $a=e$ prove that b and e are supplements.
3. If $a=e$ prove that h and a are supplements.
4. If $c=e$ prove that c and f are supplementary angles.
5. If $c=e$ prove that $a=g$.
6. If $c=e$ prove that $f=d$.
7. If $c+f=180°$ prove that $b=f$.
8. If $c+f=180°$ prove that $f=d$.
9. If $c+f=180°$ prove that $a+f=180°$.
10. If $c+f=180°$ prove that $c+h=180°$.

Summary

189. This chapter has taught the following terms: adjacent angles, opposite angles; perpendicular lines; supplementary angles; complementary angles; isosceles triangle; corresponding angles formed by two lines cut by a transversal; alternate interior angles, alternate exterior angles, interior angles on the same side of the transversal; hypotenuse of a right triangle.

190. The truth of the following theorems has been shown:*

1. *At a given point in a given line only one perpendicular can be drawn to that line.*

2. *If the sum of two adjacent angles is a straight angle the exterior sides are in the same straight line.*

3. *The sum of the adjacent angles at a point on one side of a straight line is a straight angle.*

4. *The sum of all the angles at a given point covering the angular space about the point is a perigon.*

5. *Supplements of equal angles are equal.*

6. *Complements of equal angles are equal.*

7. *If two lines intersect, the opposite angles are equal.*

8. *The acute angles of a right triangle are complementary.*

9. *In a right triangle whose acute angles are 30° and 60° the hypotenuse is twice as long as the side opposite the 30° angle.*

* These theorems were probably all first proved by the School of Pythagoras, founded at Croton in Southern Italy about 529 B.C. See Ball, p. 19.

PYTHAGORAS

P Y T H A G O R A S

PYTHAGORAS was born at Samos about 569 B.C. of Phoenician parents and died, probably at Metapontum, in Southern Italy, about 500 B.C. He was primarily a moral reformer and philosopher, but he was celebrated also as a mystic, a geometer, an arithmetician, and as a teacher of astronomy, mechanics, and music. His system of morals and his philosophy were founded on mathematics. He is said to have been the first to employ the word mathematics. The meaning he gave it was what we understand by general science. With him geometry meant about what high-school people today mean by mathematics.

He divided his mathematics into numbers absolute or arithmetic, numbers applied or music, magnitudes at rest or geometry, and magnitudes in motion or astronomy. His successors for many years regarded this as the necessary and sufficient course of study for a liberal education. It is the origin of the famous "quadrivium" that constituted higher education for 2,000 years.

After completing his studies near his home under Pherecydes of Syros and Anaximander, the latter a disciple and successor of Thales of Miletus, Pythagoras traveled extensively, studying mathematics in Egypt, Chaldea, and Asia Minor. Returning from his travels he settled at Samos and gave lectures with indifferent success until some time near 529 B.C., when he migrated to Tarentum. After a brief stay here he removed to Croton in Southern Italy, where he opened his famous school of philosophy and mathematics in 529 B.C. Here his school was attended by enthusiastic audiences.

He divided his hearers into probationers and Pythagoreans. The probationers were much the larger group. He formed the Pythagoreans into a brotherhood, like a modern fraternity. All possessions were to be held in common and all discoveries were to be referred to the founder. The chief mathematical discoveries were revealed only to the Pythagoreans. Read in some history of mathematics the story of the drowning of a Pythagorean for divulging a mathematical discovery and claiming it as his own, and other more significant facts about this secret order and its wonderful founder. The leading teachings of the brotherhood were self-command, temperance, purity, and obedience. Its secrecy and strict discipline soon gave the society such power in the state as to arouse the jealousy and hatred of certain influential classes in that democratic community. Instigated by his political opponents, a mob murdered many of Pythagoras' followers and finally, after his flight, probably to Metapontum, murdered the leader himself. After the death of their leader the Pythagoreans were dispersed over Southern Italy, Sicily, and the Grecian peninsula. They renounced secrecy, opened schools at divers centers, and they and their disciples continued publicly to teach Pythagorean doctrines for a hundred years after the death of Pythagoras.

Pythagoras' geometry consisted of the substance of what is contained in the first book of our school geometries about triangles, parallels, and parallelograms, together with some few isolated theorems about irrational magnitudes. His reasoning was often not rigorous, e.g., he sometimes assumed the converse to be true without a proof. His most original work was in the theory of numbers, called by the Greeks *Arithmetica*. Pythagoras left no books or other writings, so that what we know of him is traditional. He himself believed, not in publicity, but in secrecy.

CHAPTER VII

PARALLEL LINES. LINES AND PLANES IN SPACE

Parallel Lines

191. Parallel lines. Two lines are said to be **parallel** if they lie in the same plane but do not meet however far extended.

EXERCISES

1. Point out the parallel edges of a table top, of a sheet of notebook paper, of a rectangular box, of a cube. Give other examples of parallel lines.

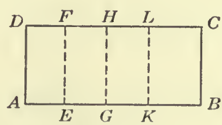


FIG. 113

2. Show by measuring that the opposite sides of a rectangle are everywhere equally distant, i.e., $EF = GH = KL$, Fig. 113.

3. Extend the opposite sides of a carefully drawn rectangle and show that the extensions are everywhere equally distant.

192. The last two problems illustrate the following fact:

Parallel lines are everywhere equally distant.

193. Symbol for parallelism.* The symbol for parallelism is $\parallel$. Thus, the statement AB is parallel to CD is written $AB \parallel CD$.

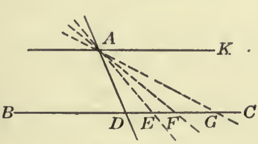


FIG. 114

194. Let A, Fig. 114, be a point not on the straight line BC. Let AD be a straight line through A, intersecting BC at D. If AD

* The symbol $\parallel$ for parallelism was introduced by John Wallis.

is made to rotate about A, point D moves along BC, taking positions as E, F, G, etc. We will assume that there is a position of the rotating line, as AK, such that it does not intersect BC, i.e., when the rotating line is parallel to BC. Further, we will assume that this is the *only position* in which the rotating line does not meet BC. Thus, when it has passed the position AK by the least amount, it will intersect BC to the left of D. These assumptions are stated in the form of the following axiom:

Axiom: *One and only one parallel can be drawn to a line from a point outside the line.*

EXERCISES

1. Draw a line, as AB, Fig. 115, and place along this line one side PQ of a triangle T, cut from paper or wood. Along the second side, PR, of the triangle draw CD. Move triangle T, letting PQ move along BA until it takes the position P_1Q_1. Along P_1R_1 draw line C_1D_1. Notice that the size of $\angle x$ has remained unchanged. Show by measuring, as in § 191, that any two points on CD are equally distant from C_1D_1. Thus, $CD \parallel C_1D_1$. Why?

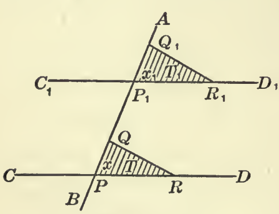


FIG. 115

2. Draw a line, as AB, Fig. 116. At any two points of AB, as P and P_1, draw with the protractor two equal angles, x and x_1. Show as in exercise 1 that $CD \parallel EF$.

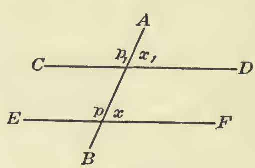


FIG. 116

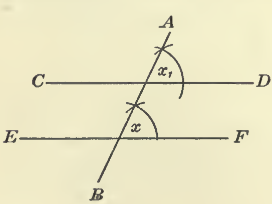


FIG. 117

3. Draw a line, as AB, Fig. 117. With the compass draw $\angle x = \angle x_1$. Show that $CD \parallel EF$.

195. Exercises 1, 2, and 3 show how to draw parallel lines by aid of the triangle, protractor, and compass, respectively. All three ways are based upon the following theorem:

Theorem: *If two lines are cut by a transversal making the corresponding angles equal the lines are parallel.*

EXERCISES

1. To a given straight line, l, draw a parallel line passing through a point, A, not on l.

First method: Move a triangle along the edge of a ruler.

Second method: Draw a line through A intersecting l at B. Construct an angle at A equal to the angle formed at B, making $\measuredangle A$ and B equal corresponding angles.

2. Show that *two lines perpendicular to the same line are parallel.*

Use the theorem of § 195.

3. Prove the following theorem: *Two lines are parallel if two alternate interior angles formed with a transversal are equal.*

First show algebraically that the corresponding angles are equal if the alternate interior angles are equal. Then use the theorem of § 195.

4. Prove as in exercise 3 that *two lines are parallel if the interior angles on the same side formed with a transversal are supplementary.*

5. Prove that *two lines parallel to the same line are parallel to each other;* i.e., if AB and CD, Fig. 118, are parallel to EF, then prove $AB \parallel CD$.

Proof: If AB and CD were not parallel, they would intersect if far enough extended, as is indicated by the dotted lines. Then from the point of intersection K there would be two lines KA and KC parallel to EF. Since this is impossible (why?), AB is parallel to CD.

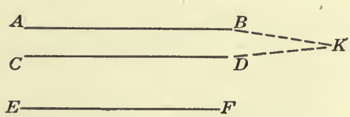


FIG. 118

6. State the conditions which will show that two lines are parallel to each other.

7. Draw two parallel straight lines and a transversal, as in Fig. 119. Measure and compare the corresponding angles a and b; c and d; e and f; g and h.

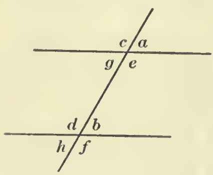


FIG. 119

8. Draw two lines that are not parallel, and a transversal. Measure and compare the corresponding angles.

196. Exercises 7 and 8 illustrate the following theorem:

Theorem: *If two parallel lines are cut by a transversal the corresponding angles are equal.*

EXERCISES

1. Two parallels and a transversal make angles designated as shown in Figs. 120, 121. Find the value of x and of all the 8 angles in each figure.

Use theorem of § 196.

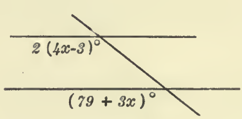


FIG. 120

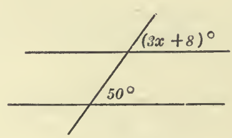


FIG. 121

2. Prove that *if two parallels are cut by a transversal the alternate interior angles are equal.*

First show by § 196 that the corresponding angles are equal, i.e., $a=e$, Fig. 122. Then prove that the alternate interior angles are equal, i.e., that $e=c$.

3. Prove that *if two parallels are cut by a transversal the interior angles on the same side are supplementary.*

Proof: In Fig. 122, $a=e$. Why? $a+d=180$. Why? Therefore $e+d=180$. Why?

4. In Fig. 122 lines AB and CD are given parallel. State which angles are equal and which angles are supplementary.

5. With two parallels and a transversal the alternate interior angles are $7(x+1)°$ and $(181-2x)°$. Find x and all the 8 angles.

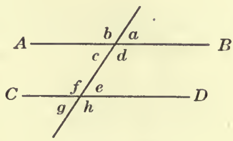


FIG. 122

6. With two parallels and a transversal the alternate interior angles are $(3x-5)°$ and $5(x-7)°$. Find x and all the 8 angles.

7. Let $\angle A$, Fig. 123, be a given angle. Through a point P draw two lines parallel respectively to the sides of $\angle A$.

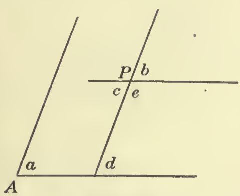


FIG. 123

Prove that $a=b$.

Proof: $a=d$. Why? $d=b$. Why? Therefore $a=b$.

8. Prove that $a=c$, Fig. 123.

9. Prove that a and e, Fig. 123, are supplementary.

197. Exercises 8 and 9 show that *if two angles have their sides parallel respectively they are either equal or supplementary.*

Prove this theorem for the angle-pairs in Fig. 124.

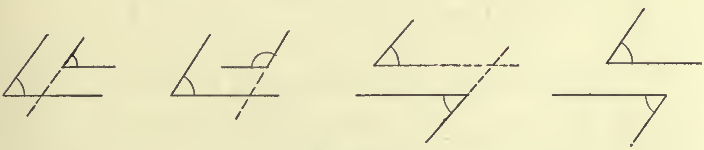


FIG. 124

198. Using Fig. 125, in which DE is parallel to AC, prove that *the sum of the interior angles of a triangle is two right angles.*

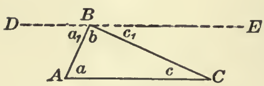


FIG. 125

Proof: $a_1 + b + c_1 = 180$. Why?
$a_1 = a$. Why?
$c_1 = c$. Why?
Therefore $a + b + c = 180$. Why?

199. Using Fig. 126 prove that *an exterior angle of a triangle equals the sum of the two remote interior angles.*

Proof: $a = a_1$. Why?
$b = b_1$. Why?
$a + b + c = 180$. Why?
$a_1 + b_1 + c = 180$. Why?
Therefore $a + b + c = a_1 + b_1 + c$. Why?
or $a + b = a_1 + b_1$. Why?

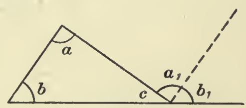


FIG. 126

EXERCISES

1. Prove that the sum of the interior angles of a quadrilateral is 360°. (See Fig. 127.)

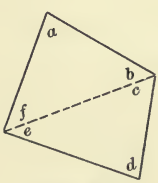


FIG. 127

2. Find the angles of a quadrilateral if each angle is 30° greater than the consecutive angle.

3. The angles of a quadrilateral are to each other as 1:2:3:4. How large is each?

Let the first angle be denoted by a, the second by $2a$, etc.

Angles of the Parallelogram and Trapezoid

200. Parallelogram. A quadrilateral whose opposite sides are parallel is a **parallelogram.**

201. Consecutive and opposite angles of a quadrilateral. The angle-pairs x and y, y and s, s and t, t and x, Fig. 128, are **consecutive** angles. The pairs x and s, y and t are **opposite** angles.

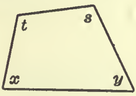


Fig. 128

EXERCISES

1. With a ruler and protractor draw a parallelogram having two consecutive sides 3 in. and 5 in. respectively and the angle included between them equal to 60°.

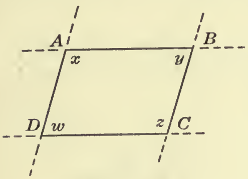


Fig. 129

2. Prove that *the consecutive angles of a parallelogram are supplementary.* (See Fig. 129.)

3. Prove that *the opposite angles of a parallelogram are equal.*

4. Find the angles of a parallelogram if one angle is 3 times as large as the consecutive angle.

5. The difference of two consecutive angles of a parallelogram is 20°. Find the values of all the angles of the parallelogram.

6. Two consecutive angles of a parallelogram are so related that 3 times one angle diminished by the other is equal to 30°. Find the values of both angles.

202. Trapezoid. A quadrilateral having only one pair of parallel sides is a **trapezoid.**

EXERCISES

1. In the trapezoid, Fig. 130, prove that x and y are supplementary angles.

2. Prove that the sum of the interior angles of a trapezoid is four right angles.

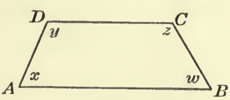
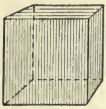


Fig. 130

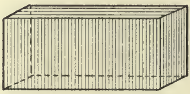
3. In the trapezoid, Fig. 130, C is $\frac{5}{3}$ times as large as B and D is 4 times as large as A. How large is each?

203. Solid. Surface. The **cube**, Fig. 131, and the rectangular **parallelopiped**, Fig. 132, are **solids**. Other



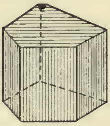
Cube

FIG. 131



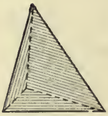
Rectangular Parallelopiped

FIG. 132



Prism

FIG. 133



Pyramid

FIG. 134

solids that are frequently found are the **prism**, Fig. 133, the **pyramid**, Fig. 134, the **cylinder**, Fig. 135, the **cone**, Fig. 136, and the **sphere**, Fig. 137.

Cylinder

FIG. 135

Cone

FIG. 136

Sphere

FIG. 137

A solid consists of matter. It may be hard or soft, heavy or light. It has color. It fills a limited portion of space and is separated from the surrounding space by its **surface**. In geometry when we study a solid we are interested only in the form of the solid, its size and its shape. We disregard the color, weight, density, etc., and think only of the limited portion of space which it occupies. Such a solid is a **geometric solid**.

Which of the solids, Fig. 131–137, have flat surfaces? Which have curved surfaces?

204. Plane. A flat surface is called a **plane surface** or a **plane**.

Point out plane surfaces in the classroom. Point out surfaces that are not plane.

To test whether a surface is plane a straight edge is applied to it. If in every position the straight edge coincides entirely with the surface it is said to be a plane surface.

Since only one straight line can be drawn through two points it follows that *a straight line two of whose points lie in a plane lies entirely in the plane.*

How may a carpenter making a plane surface determine whether it is a plane?

205. In § 191 it was seen that two lines in the *same plane* are parallel if they do not meet, however far extended. When two lines in *space* do not meet they are not necessarily parallel.

Thus in Fig. 138 lines A_1D_1 and AB do not meet, yet they are not parallel lines.

Point out lines in the classroom which do not meet and are *not* parallel; others which *are* parallel.

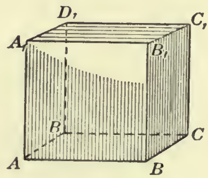


FIG. 138

206. Parallel planes. Two planes which do not meet however far extended are **parallel planes.**

Thus, the planes of faces $ABCD$ and $A_1B_1C_1D_1$ in Fig. 138 are parallel.

Point out parallel planes in the classroom.

207. Parallel lines and planes. No line drawn in plane $A_1B_1C_1D_1$, Fig. 138, can meet plane $ABCD$. (Why?) A line which does not meet a plane however far extended is **parallel to the plane.**

In the classroom point out lines which are parallel to planes.

Line BB_1, Fig. 138, is perpendicular to BC and BA. BB_1 is also perpendicular to any line through B drawn in plane $ABCD$. This may be verified by means of a try-square on a wooden model of a cube. BB_1 is said to be perpendicular to plane $ABCD$.

Point out in the classroom lines that are perpendicular to planes; e.g., one edge of a door is perpendicular to the plane of the floor. (Why?)

208. Lines perpendicular to a plane. If a line intersects a plane and is perpendicular to any line in that plane passing through the point of intersection it is said to be **perpendicular to the plane.** The symbol for expressing that one line is perpendicular to another is $\perp$. Thus, the statement AB is perpendicular to CD is written $AB \perp CD$.

Models of Geometrical Solids

209. The cube. The cube may be constructed from a figure like Fig. 139. On cardboard draw the figure. Cut out the figure along the heavy lines. Then fold along the dotted lines. Join the edges by means of gummed paper. This will form a model of a cube. Measure the edge of the cube and compute the area of the whole surface. Find the volume of this cube.

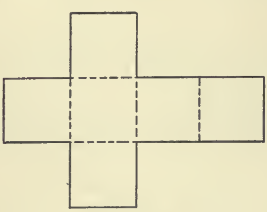


FIG. 139

210. The parallelopiped. A rectangular parallelopiped may be constructed from a figure like Fig. 140. Make the model. Compute the volume of the solid and the area of the surface.

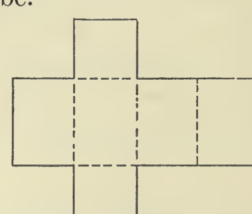


FIG. 140

211. The prism. Construct a prism from a drawing like that in Fig. 141. Find the area of the entire surface.

To find the area of the hexagon divide each into 6 equilateral triangles. Measure the base and altitude of one triangle and compute the area.

To draw the hexagons, draw a circle, step around it, using the radius for a step, Fig. 142, and connect consecutive marks.

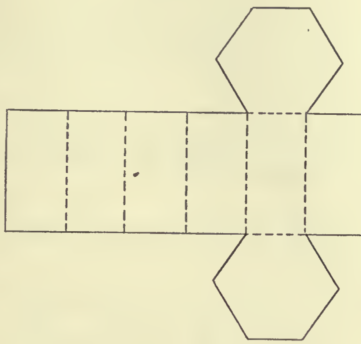


FIG. 141

212. The pyramid. To construct models of pyramids use Figs. 143 and 144.

213. The cone. Cut out two unequal circles, Fig. 145. Place the circles so that they touch each other, making point C fall on D. Then turn the circles, keeping them always touching each other, until C meets circle B at D'. This will make arc DED' equal to the length of circle A. Cut out angle DBD' and use the remainder for the curved surface of the cone. Use circle A as base.

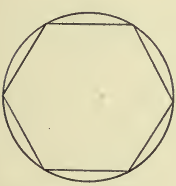


FIG. 142

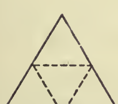


FIG. 143

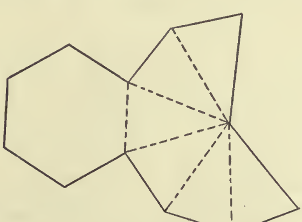


FIG. 144

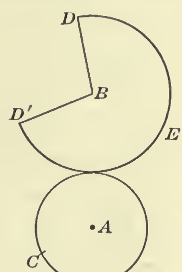


FIG. 145

Summary

214. The chapter has taught the meaning of the following terms: parallel lines; opposite and consecutive angles of a quadrilateral; solid, prism, pyramid, cone, cylinder, sphere; plane, parallel planes; lines parallel or perpendicular to a plane.

215. The symbol for parallelism is $\parallel$. The symbol for perpendicularity is $\perp$.

216. The following theorems and axioms have been taught:

1. *Parallel lines are everywhere equidistant.*

2. *One and only one parallel can be drawn to a line from a point not on the line.*

3. *If two lines are cut by a transversal making the corresponding angles or alternate interior angles equal the lines are parallel.*

4. *If two lines are cut by a transversal making the interior angles on the same side supplementary they are parallel.*

5. *Two lines perpendicular to the same line are parallel.*

6. *Two lines parallel to the same line are parallel.*

7. *If two parallel lines are cut by a transversal the alternate interior angles are equal, the corresponding angles are equal, and the interior angles on the same side are supplementary.*

8. *If two angles have their sides parallel respectively they are either equal or supplementary.*

9. *The exterior angle of a triangle is equal to the sum of the two remote interior angles.*

10. *The opposite angles of a parallelogram are equal.*

11. *The consecutive angles of a parallelogram are supplementary.*

217. The following construction was taught:

Through a given point outside of a given line to draw a line parallel to the given line.

218. It was seen how models can be made of the following geometrical solids: cube, rectangular parallelopiped, prism, pyramid, and cone.

CHAPTER VIII

MEASUREMENT OF LINES IN SPACE. SIMILAR FIGURES

Drawing to Scale

219. Indirect measurement. In the preceding chapters lengths of distances and sizes of angles were determined by direct measurement with ruler, compass, and protractor. However, in many cases problems call for the lengths of distances which cannot be measured directly, e.g., the distance across a river, the height of a tree, etc. The following problems will show that when it is impossible to determine distances or angles by direct measurement other related lines or angles may be measured which will enable us to determine the required parts. This is called **indirect measurement.**

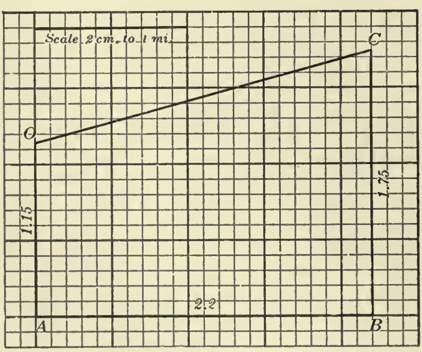


Fig. 146

A man starting from a point O, Fig. 146, walks 1.15 mi. south, then 2.2 mi. east, and then 1.75 mi. north. How far is he from the starting-point?

Letting 2 cm. on squared paper represent 1 mi., make a drawing of the measured distances, as Fig. 146. Then measure CO in the drawing. This is the required length.

220. Drawing to scale. The distances of the last problem as graphed in Fig. 146 are said to be **drawn to**

the scale: 2 cm. to 1 mi. On all scale drawings the scale should be indicated.

EXERCISES

1. Draw to the scale 2 cm. to 1 mi. the following distances: 2.34 mi.; 1.06 mi.; 3.90 mi.; 0.15 mi.; 2.63 miles.

2. A man starting from a point S walks 45 yd. east and then 60 yd. north. Find the direct distance from the stopping-point to the starting-point.

Make a drawing to the scale 1 cm. to 10 yards.

221. Surveying instruments. The instruments commonly used by surveyors for measuring distances are the **surveyor's chain**, Fig. 147, and **tape**, Fig. 148. Angles are measured with the transit (p. 56).

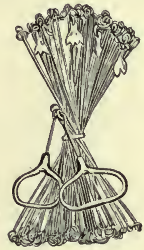


Fig. 147.—Surveyor's chain.

The chain is made up of a succession of links having loops at the ends, which are connected by rings.

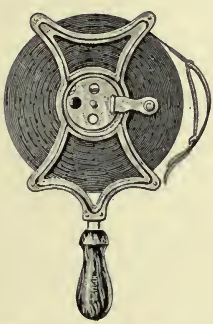


Fig. 148.—Steel Tape

In measuring distances, **chaining pins**, Fig. 149, are used to mark the end-point of the chain or tape.

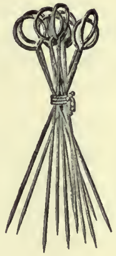


Fig. 149.—Chaining pins.

EXERCISES

1. A man walks 80 yd. south, then 144 yd. east, and then 120 yd. north. Find the direct distance from starting-point.

Use the scale 1 cm. to 12 yards.

2. Two men start from the same point. One walks 5 mi. west, and then 3 mi. north; the other walks 4 mi. south, and then 5 mi. east. How far apart are they?

3. Draw to the scale 1 cm. to 2 ft. a plan of a room 24 ft. by 18 ft. and find the distance diagonally across the floor.

4. Draw a plan of a rectangular field 16 rods long and 12 rods wide, using the scale 1 in. to 4 rods, and find the distance diagonally across the field.

5. Draw to the scale 1 cm. to 10 ft. a plan of the end of a house, Fig. 150, and find the height of the top of the roof from the ground.

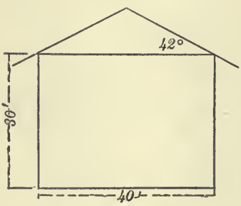


Fig. 150

6. The Union Station in Washington, D.C., is 760 ft. long and 343 ft. wide. The main waiting-room covers an area of 220×130 sq. ft. The station exceeds the United States Capitol in dimensions, the latter being 751 ft. long and 350 ft. wide. Draw rectangles whose dimensions are scale drawings of the dimensions of both buildings and the waiting-room.

7. A railroad surveyor wishes to measure across the swamp AB, Fig. 151. He measures the distance from a tree A to a stone C and finds it to be 165 feet. The distance from a tree at B to the stone C is 150 feet. Find the distance in feet across the swamp, the angle at C being 80°.

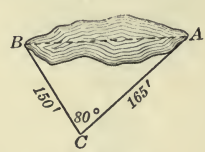


Fig. 151

8. To measure the width AC of a stream, Fig. 152, without crossing it, an engineer lays off a line, BC, on one side of the river, and measures the angles at B and C. Draw a triangle to scale from the data in the figure and determine the width of the river.

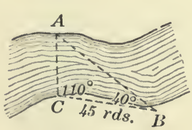


Fig. 152

9. A boy wishes to determine the height, HK, Fig. 153, of a factory chimney. He places the angle-measurer first at B and then at A and measures the angles x and y. The angle-measurer lies on a box, or tripod, $3\frac{1}{2}$ ft. from the ground. A and

B are two points in line with the chimney and 50 ft. apart. What is the height of the chimney if the ground is level and if $x=63°$ and $y=33\frac{1}{2}°$?

10. The line AB, Fig. 154, passes through a building. Explain how to find by means of a scale drawing the distance between A and B.

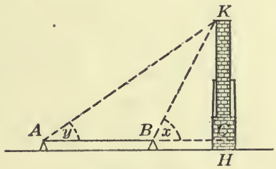


FIG. 153

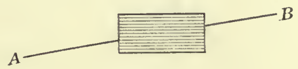


FIG. 154

222. Angle of elevation. A telescope is pointed horizontally in the direction EH, Fig. 155, toward the tower HT, and the farther end is then raised (elevated) until the telescope points to the top, T, of the tower. The angle HET through which the telescope turned is the **angle of elevation** of the top of the tower, from the point of observation.

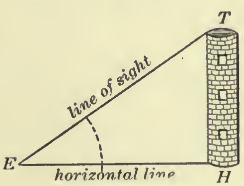


FIG. 155

EXERCISES

1. From point A, Fig. 153, what is the angle of elevation of the top of the chimney? From point B?

2. When the angle of elevation of the sun is 25°, a building casts a shadow 90 ft. long, on level ground. Find the height of the building.

3. Find the angle of elevation of the sun when a tree 40 ft. high casts, on level ground, a shadow 60 ft. long.

4. The angle of elevation of the top of a tree is 38°, the observer standing 20 yd. from the tree. How high is the tree?

5. On the top of a tower stands a flagstaff. At a point A, on level ground, 50 ft. from the base of the tower, the angle of elevation of the top of the flagstaff is 35°. At the same point A

the angle of elevation of the top of the tower is 20°. Find the length of the flagstaff.

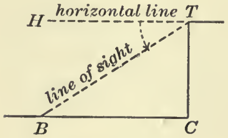


FIG. 156

223. Angle of depression. A telescope at T, on the top of a cliff (Fig. 156), is pointed horizontally, and then the farther end is lowered (depressed) until the telescope points to the buoy at B. The angle HTB through which the telescope turned is the **angle of depression** of the buoy from the point T.

EXERCISES

1. If the height of the cliff, Fig. 156, is 100 ft., and the angle of depression of the buoy, as seen from T, is 40°, what is the distance of the buoy from the bottom of the cliff?

2. A boat passes a tower on which is a searchlight 120 ft. above sea-level. Find the angle through which the beam of light must be depressed from the horizontal so that it may shine directly on the boat when it is 400 ft. from the base of the tower.

3. From the top of a cliff 150 ft. high the angle of depression of a boat is 25°. How far is the boat from the top of the cliff?

4. From a lighthouse, situated on a rock, the angle of depression of a ship is 12°, and from the top of the rock it is 8°. The height of the lighthouse above the rock is 45 feet. Find the distance of the ship from the rock.

224. Bearing of a line. The angle which a line makes with the north-south line is the **bearing of the line.**

The bearings of the arrows in Fig. 157 are read 30° east of north, 20° west of south, 60° west of north, 75° east of south. These may be written briefly: N 30°E, S 20°W, N 60°W, S 75°E.

EXERCISES

1. With a ruler and protractor draw lines having the following bearings: 45° east of north (northeast); 45° west of south

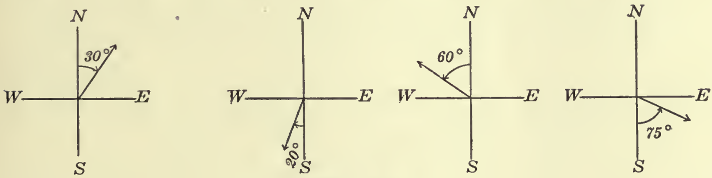


FIG. 157

(southwest); 65° east of south; 65° east of north; $47\frac{1}{2}$° west of south; 40° west of north.

2. Write in abbreviated form the bearings of the lines in exercise 1.

225. Bearing of a point. An observer standing at a point A is looking in the direction of another point B. The bearing of the line from A to B with reference to the north-south line through the point of observation A is called the **bearing of the point B from A.**

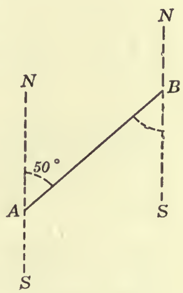


FIG. 158

Thus, in Fig. 158, the bearing of B from A is N 50°E; the bearing of A from B is S 50°W.

Why are angles NAB and SBA, Fig. 158, equal?

EXERCISES

1. The bearing of a fort B from A, both on the seacoast, is N 65°W. An enemy's vessel at anchor off the coast is observed from A to bear northwest, from B northeast. The forts are known to be 7 mi. apart. Find the distance from each fort to the vessel.

Make a drawing to the scale 1 cm. to 2 meters. To get the directions of the lines draw north-south lines first through A, and through B when the position of B is determined.

2. Point Q is 6.4 mi. east and 9.8 mi. north of P. Find the distance from P to Q. What angle does PQ make with the north-south line through P?

3. The view from a battery at B, Fig. 159, to the enemy's fort at F is obstructed by a hill H. A point P is found near the bottom of the hill from which F is observed to bear 4 mi. northeast. P is 6.25 mi. northwest of B. Find the distance and bearing of F from B.

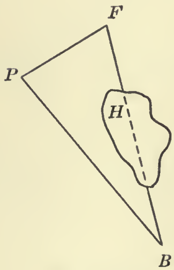


FIG. 159

4. A man wishes to measure the width of a river without crossing it. The river flows due west. Standing at A, on the bank, he observes a tree on the other bank in the direction N 20°E. He walks along the bank 50 rods east to B and there observes the tree in the direction N 60°W. Find the width of the river.

Ratio

226. Ratio of numbers. The quotient found by dividing 6 by 3 is the ratio of 6 to 3. This is sometimes written 6:3, or 6/3. For the present purpose $\frac{6}{3}$ will be considered the most convenient form. Thus the ratio of 3 to 4 is $\frac{3}{4}$; of a to b is $\frac{a}{b}$. The **ratio** of any number to another number is the quotient found by dividing the first number by the second.

Any fraction may be regarded as an expression of the ratio of the numerator to the denominator. A whole number expresses the ratio of itself to 1.

227. Ratio of line-segments. If a line-segment is 2 in. long and another 3 in., the first is $\frac{2}{3}$ of the second. The number $\frac{2}{3}$ is the ratio of the two line-segments.

What is the ratio of two line-segments of length 5 in. and 7 inches?

In practice, to find the ratio of two line-segments each is measured in a convenient unit, as inch or centimeter, and one measure is then divided by the other. Thus, the **ratio of two line-segments** is the ratio of the numerical measures of the segments, both being measured with the same unit.

228. Notation for ratio of segments. According to § 227 the ratio of two line-segments, as AB and CD, Fig. 160, means $\frac{\text{measure of } AB}{\text{measure of } CD}$, with the agreement that a *common unit* is used in measuring. This is usually written briefly $\frac{AB}{CD}$.

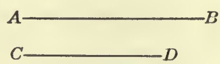


FIG. 160

EXERCISES

1. Using 2 cm. as a unit find the ratio $\frac{AB}{CD}$, Fig. 160, and express the result as a decimal fraction.

2. Two line-segments are measured in yards and are found to be 7 yd. and 3 yd. long respectively. Find the ratio.

3. What are the measures of the segments in exercise 2 in feet? What is the ratio?

4. Draw two line-segments and measure them in centimeters and in inches. In each case find the ratio expressed as a decimal fraction. How do the results compare?

229. Exercises 2, 3, and 4 illustrate the fact that *a change in the unit of measure does not change the ratio.*

EXERCISES

1. Draw two line-segments having the ratio $\frac{2}{3}$. How many pairs of line-segments have the ratio $\frac{2}{3}$?

2. Give several pairs of numbers having the ratio $\frac{5}{6}$.

3. Show that $3x$ and $7x$ represent all number pairs whose ratio is $\frac{3}{7}$.

4. Divide 85 into two parts in the ratio 2:3.

Let the parts be denoted by $2x$ and $3x$.

5. Divide 84 into three parts which are as 3:4:5.

6. Two numbers are in the ratio $\frac{5}{6}$. If 12 is subtracted from each, the differences are in the ratio $\frac{3}{4}$. What are the numbers?

7. What number added to 12 and subtracted from 30 gives results that are in the ratio $\frac{5}{10}$?

8. The ratio of two line-segments is $\frac{2\frac{1}{2}}{3\frac{3}{4}}$. The longer line is 30 centimeters. Find the shorter line.

230. Ratio of angles and polygons. The ratio of two angles or of two polygons is the ratio of their numerical measures.

EXERCISES

1. Draw two angles and measure each with a protractor. Find the ratio of the angles.

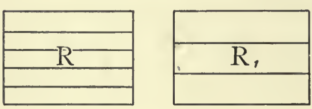


FIG. 161

2. Rectangles R and R_1, Fig. 161, are divided into equal parts. Find the ratio.

3. The angles of a triangle are in the ratio 1:2:3. Find them.

4. The angles covering the plane around a point are in the ratio 2:3:4. Find them.

5. The ratio of 2 times an angle to 8 times the supplement is $\frac{1}{2}$. Find the angles.

6. If 6° be taken from an angle and added to its complement the ratio of the angles thus formed is $\frac{2}{7}$. Find the angles.

Similar Figures

231. Construction of triangles. In §§ 219–225 it was shown how unknown distances and angles could be determined by indirect measurement. In the following sections an algebraic method of finding distances will be developed which is more advantageous than the geometric method of using scale drawings.

EXERCISES

1. Draw a triangle whose angles are respectively 35°, 65°, and 80° and measure each side.

Draw a line-segment of convenient length, as *AB*, Fig. 162. At *A* draw on *AB* an angle of 35°. At *B* draw an angle of 65°. Check the accuracy of the construction by measuring angle *C*. Measure to two decimal places *AB*, *AC*, and *BC*.

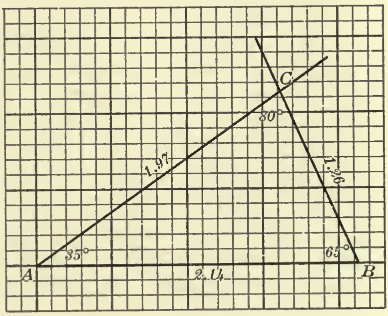


FIG 162

2. Starting from a length of *AB* different from that in Fig. 162, construct a second triangle whose angles are 35°, 65°, and 80° respectively and measure its sides.

3. Find the ratio, to two decimal places, of each side of the triangle in exercise 1 to the corresponding side of the triangle in exercise 2. How do your results compare with the results obtained by other members of your class?

4. How do the triangles constructed in exercises 1 and 2 compare as to shape? Are these triangles of the same shape as the triangles made by the other members of the class? Are the triangles of the same size?

5. Draw two triangles of different sizes in each of which the angles are 30°, 60°, and 90°. Compare the triangles as to shape. Measure all sides of the triangles. Find the ratios of the sides of one triangle to the corresponding sides of the other.

6. Draw two triangles of different sizes each of which has angles equal to 90°, 25°, and 65° respectively. Compare the

triangles as to size and shape. Compare the ratios of the corresponding sides.

7. Draw a triangle. Draw another having the angles equal respectively to the angles of the first triangle. Are the triangles of the same shape? Are they necessarily of the same size? Compare the ratios of the corresponding sides.

232. Similar triangles. Triangles having the same shape are called **similar triangles.** Similar triangles are not necessarily of the same size.

233. Exercises 1–7, § 231, illustrate the following theorem:

Theorem: *If the angles of one triangle are respectively equal to the angles of another the triangles are similar.*

It was also seen that if the corresponding angles of two triangles are equal, it follows that the ratios of the corresponding sides are equal.

This explains the following definition of similar triangles and polygons.

234. Similar polygons. If two polygons have their corresponding angles equal and the ratios of the corresponding sides equal they are called **similar polygons.**

Similar figures are of frequent occurrence, e.g., two squares, two circles, two equilateral triangles, a figure and its scale drawing, a photograph and an enlarged or reduced picture, and so on.

Similar triangles may be regarded as the same triangle *magnified* or *minified* to a definite scale, or both may be regarded as scale drawings of the same triangle to different scales.

Similarity of figures is indicated by the symbol $\backsim$. Thus, $\triangle XYZ \backsim \triangle X_1Y_1Z_1$ means that $\triangle XYZ$ and $X_1Y_1Z_1$ are similar.

235. Problems in similar figures. The fact that the ratios of the corresponding sides of similar polygons are equal suggests an algebraic method of finding distances.

EXERCISES

1. To measure the height of a tree.

The tree, the length of the shadow, and the sun's rays passing over the top of the tree form a triangle, Fig. 163. The shadow is measured and is found to be 80 ft. long. At the same time the shadow of a vertical stick 3 ft. high is measured and found to be 8 ft. long, Fig. 164. The stick, the shadow, and the sun's rays form a triangle similar to the first triangle. Why? Letting h denote the height of the tree, we have

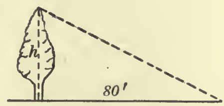


FIG. 163

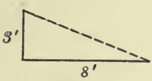


FIG. 164

$$\frac{h}{3}=\frac{80}{8}.\quad \text{Why?} \qquad \text{Whence, } h=30^{*}$$

2. To measure the height of a flagpole a boy holds a pencil AB, Fig. 165, in vertical position and 2 ft. from his eye so that

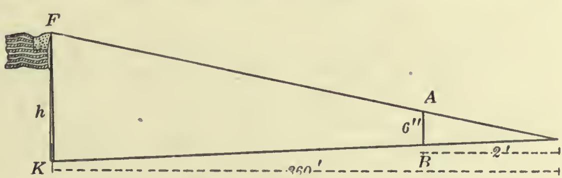


FIG. 165

it covers the pole. If the pole is 360 ft. away and if the pencil is 6 in. long, how high is the pole?

* According to a report by Plutarch this method of finding distances was devised by Thales (640 B.C.–550 B.C.) who used it to find the height of a pyramid. (See Ball, p. 16.)

3. The gables of a house and a porch have the same shape. The sides of the porch-gable are 7 ft., 7 ft., and 10 feet. The longest side of the house-gable is 25 feet. How long are the other two sides of the house-gable?

4. The sides of a triangle are 8, 10, and 13. The shortest side of a similar triangle is 11. Find the other sides.

5. The sides of a triangle are 4.6 cm., 5.4 cm., and 6 centimeters. The corresponding sides of a similar triangle are x cm., y cm., and 15 centimeters. Find x and y.

6. The sides of a triangle are 1, 2, and 3, and the longest side of a similar triangle is 20. Find the other sides of the second triangle.

7. Two rectangular flower beds are to be made of the same shape, but of different size. One is to be 3 ft. wide and 5 ft. long; the other 12 ft. wide. How long should it be?

8. A city block and a lot within the block have the same shape. The lot is 100 ft. by 150 ft. and the block is 300 ft. wide. How long is it?

9. Prove that if a line is drawn parallel to one side of a given triangle, meeting the other two sides, a triangle is formed similar to the given triangle.

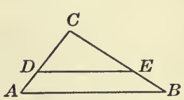


Fig. 166

Show that the angles of triangle ABC, Fig. 166, are equal to the corresponding angles of triangle DEC. Then apply § 233.

10. In Fig. 166 let $AC=21$, $BC=35$, $DC=3$. Find EC.

11. In Fig. 166 let $AC=3+x$, $DC=3$, $BC=4$, and $EC=1$. Find x and the length of AD.

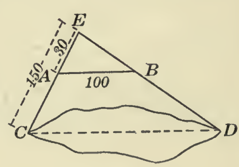


Fig. 167

12. To find the distance across a lake CD, Fig. 167, triangle CEB is drawn and EC, AE, and AB are measured. Find CD.

13. Line AB, Fig. 168, is 20 ft. long. Divide it into parts having the ratio $\frac{2}{3}$. How long is each part?

Draw $AC \perp AB$ and 2 units long. Draw $BD \perp AB$ and equal to 3. Draw CD. Show that triangles AEC and EBD are similar and determine x by means of an equation.

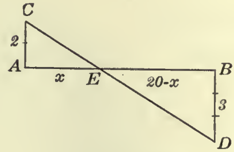


FIG. 168

14. Divide a line 24 ft. long into parts having the ratio $\frac{7}{5}$; $\frac{4}{7}$; $\frac{6}{5}$.

15. Draw a triangle. Draw a second triangle whose sides are respectively twice as long as the sides of the first.

Let ABC, Fig. 169, be the first triangle. On an indefinite line lay off A_1B_1 equal to twice AB. With A_1 as center and radius

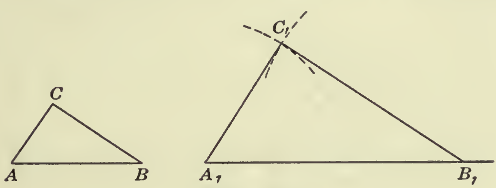


FIG. 169

twice as long as AC strike an arc, as at C_1. With B_1 as center and radius twice as long as BC strike an arc meeting the first arc. Draw A_1C_1 and B_1C_1. Then $A_1B_1C_1$ is the required triangle.

16. Compare the triangles constructed in exercise 15 as to shape. Are they similar? Measure the corresponding angles. What are the ratios of the corresponding sides?

17. Draw a triangle with sides 3 times as long as the corresponding sides of another triangle. How do the ratios of the corresponding sides compare? Are the triangles similar? Measure the corresponding angles.

236. Exercises 15 to 17 illustrate the following theorem:

Theorem: *Two triangles are similar if the ratios of the corresponding sides are equal.*

Summary

237. This chapter has taught the meaning of the following terms: indirect measurement, drawing to scale; angle of elevation, angle of depression; bearing of a line, of a point; ratio of numbers, of line-segments, of angles, of polygons; similar figures. The symbol $\backsim$ is used to indicate similarity.

238. The following instruments have been shown to be used by surveyors and engineers: the transit, surveyor's chain, surveyor's tape, chaining pins.

239. The truth of the following theorems has been shown:

1. *Two triangles are similar if the angles of one are respectively equal to the angles of the other.*

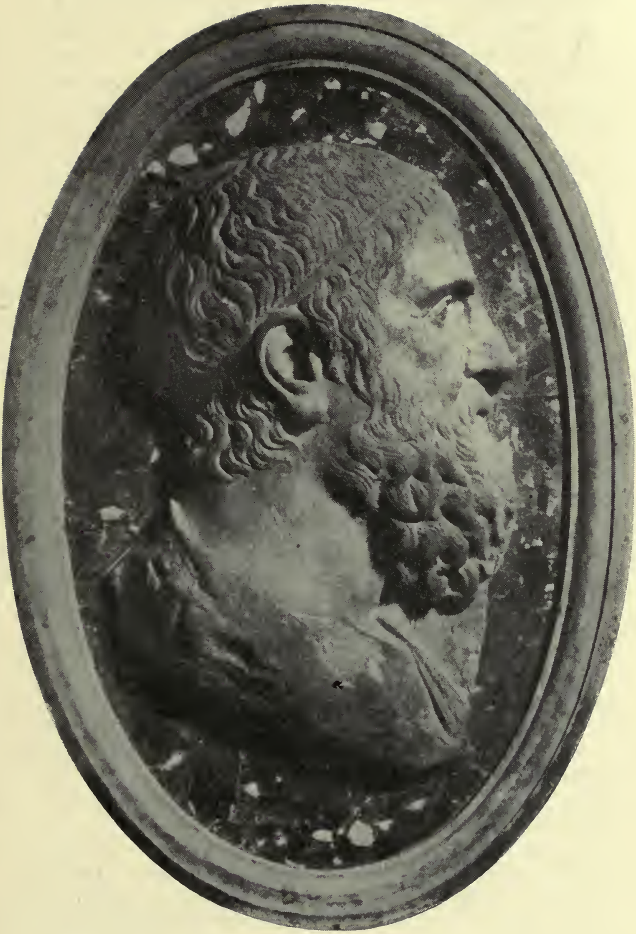
2. *Two triangles are similar if the ratios of the corresponding sides are equal.*

240. Two ways of finding distances by indirect measurement have been used:

1. The required distance is found by means of a scale drawing obtained from measurement of lines and angles related to the required distance.

2. The required distance is found by solving an equation obtained from the similarity of two triangles, one of which contains the required distance as a side.

241. Similar polygons are defined in geometry as polygons whose corresponding angles are equal and whose corresponding sides have equal ratios.



ARCHIMEDES

A R C H I M E D E S

ARCHIMEDES was born at Syracuse on the island of Sicily 287 B.C. and died there 212 B.C. It is said that he was related to the royal family of Syracuse. He studied mathematics under Conon at the University of Alexandria in Egypt. His great mechanical ingenuity was often called into the service of his government. He held it to be beneath the dignity of a scientist to apply his science to practical use; nevertheless he was the inventor of numerous practical devices and mechanical contrivances. Read about his detecting the fraudulent goldsmith, his invention of burning-glasses, his lever device for launching ships, and his device for pumping the water out of ships and even of inundated fields, etc., in Ball's *History of Mathematics.*

It was on the occasion of launching one of the king's large new ships that he remarked that he could move the earth if he but had a fulcrum to place his lever on.

He wrote many mathematical and scientific works, including important contributions to almost every field of science then known. He did especially valuable work in plane and solid geometry. In a book on the *Circle* he showed for the first time that the ratio of the length of a circumference to its diameter is between $3\frac{1}{7}$ and $3\frac{10}{71}$. He also worked out in another place the relations as to volume and surface of the cone, cylinder, and sphere. He regarded his discoveries on the round bodies as so important and so beautiful that he requested that the figure of a sphere inscribed in a cylinder be carved on his tombstone.

His contributions to pure and applied science were so numerous and so important that he is often referred to as "The Newton of antiquity." See whether you think the title appropriate after reading in Ball or elsewhere what both he and Newton did for science.

In his Sand Counter, Archimedes undertook to calculate and to express in numbers the number of grains of sand it would take to fill the universe. It is thought that the reason he did this was to show his scientific countrymen that there could be devised a great deal more powerful way of writing numbers than the way the Alexandrian scholars were teaching people to write them.

CHAPTER IX

RATIO. VARIATION. PROPORTION

Trigonometric Ratios

242. In chapter VIII the importance of ratio* in determining distances has been recognized. What follows here will illustrate further the use of ratios in the solution of problems.

I wish to know the height of a recently constructed chimney, AB, Fig. 170. I have determined that I am about 250 ft. from the chimney and that the angle of elevation of the top is approximately 30°.

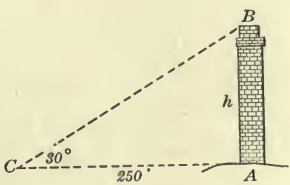


FIG. 170

The problem is easily solved by means of a scale drawing.

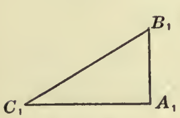


FIG. 171

Let $\triangle A_1B_1C_1$, Fig. 171, be a scale drawing of $\triangle ABC$.

Then $\triangle A_1B_1C_1 \backsim \triangle ABC$. Why?

Hence, $\dfrac{h}{A_1B_1} = \dfrac{250}{A_1C_1}$, or $h = 250 \cdot \dfrac{A_1B_1}{A_1C_1}$. Why?

The last equation shows that if the ratio $\dfrac{A_1B_1}{A_1C_1}$ were known, h could be found by multiplying this ratio by 250.

* Ratio and proportion is one of the oldest topics of mathematics. It was employed by the Egyptians, Babylonians, Chinese, Greeks, Arabs, and mediaeval Europeans. The equation has largely supplanted the proportion in modern mathematics.

However, the ratio $\frac{A_1B_1}{A_1C_1}$ can be obtained from *any* right triangle one of whose acute angles is 30°. For, let $\triangle A_1B_1C_1$, Fig. 171, and $\triangle ABC$, Fig 170, represent any two right triangles having $\angle C = \angle C_1 = 30°$. Then, since these two triangles are similar, $\frac{AB}{A_1B_1} = \frac{AC}{A_1C_1}$.

Multiplying both sides of this equation by A_1B_1 we have

$$AB = \frac{AC \times A_1B_1}{A_1C_1}.$$

Dividing both sides by AC, we have $\frac{AB}{AC} = \frac{A_1B_1}{A_1C_1}$.

Thus, the problem of finding AB may be solved as follows: From any right triangle containing an angle of 30°, determine the ratio $\frac{A_1B_1}{A_1C_1}$. Then find h by substituting the result in the equation

$$h = 250 \cdot \frac{A_1B_1}{A_1C_1}.$$

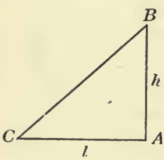


FIG. 172

243. Table of tangents. Denoting the angle of elevation by C, Fig. 172, the height by h, and the distance from the point of observation to the chimney by l, the equation

$$h = l \cdot \frac{AB}{AC}$$

may be used as a formula for computing h. The value of the ratio $\frac{AB}{AC}$ may be computed for various values of $\angle C$ and tabulated. Any height may then be found by measuring the angle of elevation C and the distance l, and then multiplying l by the value of the ratio $\frac{AB}{AC}$ corresponding to C in the table. The ratio $\frac{AB}{AC}$ is called the **tangent** of $\angle C$ and the table is the **table of tangents.**

TABLE OF TANGENTS OF ANGLES FROM 1° TO 89°

Angle	Tangent	Angle	Tangent	Angle	Tangent	Angle	Tangent	Angle	Tangent
1°	.02	19°	.34	37°	.75	55°	1.43	73°	3.27
2	.03	20	.36	38	.78	56	1.48	74	3.49
3	.05	21	.38	39	.81	57	1.54	75	3.73
4	.07	22	.40	40	.84	58	1.60	76	4.01
5	.09	23	.42	41	.87	59	1.66	77	4.33
6	.10	24	.44	42	.90	60	1.73	78	4.70
7	.12	25	.47	43	.93	61	1.80	79	5.14
8	.14	26	.49	44	.96	62	1.88	80	5.67
9	.16	27	.51	45	1.00	63	1.96	81	6.31
10	.18	28	.53	46	1.03	64	2.05	82	7.11
11	.19	29	.55	47	1.07	65	2.14	83	8.14
12	.21	30	.58	48	1.11	66	2.25	84	9.51
13	.23	31	.60	49	1.15	67	2.36	85	11.43
14	.25	32	.62	50	1.19	68	2.47	86	14.30
15	.27	33	.65	51	1.23	69	2.60	87	19.08
16	.29	34	.67	52	1.28	70	2.75	88	28.64
17	.31	35	.70	53	1.33	71	2.90	89	57.29
18	.32	36	.73	54	1.37	72	3.08		

EXERCISES

Solve the following problems:

1. The rope of a flagpole, BC, Fig. 173, is stretched out so that it touches the ground at a point 20 ft. from the foot of the pole, and makes an angle of 73° with the ground. Find the height of the flagpole.

$$h = 20\frac{BC}{AC} \text{ by the formula}$$

$$h = 20(3.27); \text{ from the table}$$

Hence, $h = 65.4$

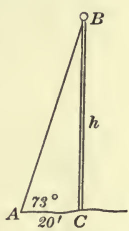


FIG. 173

2. A balloon is anchored to the ground at a point A by a rope, making an angle of 67° with the ground. The point C on the ground directly under the balloon is 139 ft. from A. Assuming the rope-line to be straight, find the height of the balloon.

3. The angle of elevation of an aeroplane at a point A on level ground is 60°. The point C on the ground directly under the aeroplane is 300 yd. from A. Find the height of the aeroplane.

4. The angle of elevation of the top of a tower is 27° and the distance from the foot of the tower is 259 feet. Find the height of the tower.

5. A vertical pole 8 ft. long casts on level ground a shadow 9 ft. long. Find the angle of elevation of the sun.

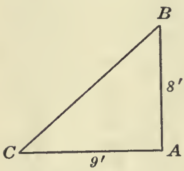


FIG. 174

Since the ratio $\frac{AB}{AC}=\frac{8}{9}=.89$ approximately, Fig. 174, the table of tangents shows the corresponding value of angle C to be approximately 42°.

6. A tree is broken by the wind into two parts forming a right triangle with the ground. The upper part makes an angle of 40° with the ground. The top of the tree is 48 ft. from the foot, the distance being measured along the ground. How high was the tree?

7. From the roof of a building the angle of depression of a point directly opposite on the other side of the street is 74°. The street being 45 ft. wide, find the height of the building.

Ratio

244. On squared paper draw a triangle, as ABC, Fig. 175. Through a point on AC, as D, draw line DE parallel to AB. Find the ratios $\frac{CD}{DA}$ and $\frac{CE}{EB}$ to two decimal places.

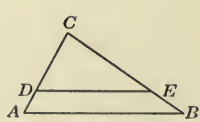


FIG. 175

How do the ratios compare?

This problem illustrates the following theorem:

Theorem: *A line drawn parallel to one side of a triangle divides the other two sides into corresponding parts having equal ratios.**

EXERCISES

1. Find EB, Fig. 175, if $DC=8$, $DA=2.5$, and $CE=10.2$.

2. Find EB, if $DC=4$, $AD=.27$, and $CE=1\frac{1}{3}\,DC$.

3. Find EB, if $DC=a$, $AD=b$, and $CE=c$.

4. If $AD=DC$, Fig. 175, show that $BE=EC$.

5. State exercise 4 in the form of a theroem.

245. Draw a triangle, as ABC, Fig. 176. Bisect angle ACB and find the ratios $\frac{AC}{CB}$ and $\frac{AD}{DB}$. How do they compare? Express the result as a theorem and compare your statement with the following theorem:

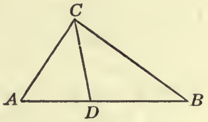


FIG. 176

Theorem: *A line bisecting an angle of a triangle divides the side opposite that angle into parts whose ratio is equal to the ratio of the other two sides.*

EXERCISES

1. If in Fig. 176 $AC=3''$, $BC=6''$, and if $AD=2''$ less than DB, find the lengths of AD and DB, using the foregoing theorem:

2. Find the segments of AB, Fig. 176, if $AB=8$, $AC=4$, and $CB=7.6$.

* Eudoxus (408–355 B.C.) gave the first satisfactory proof of this theorem. Archimedes (287–212 B.C.) extended it to apply even if the parallel cuts the extensions of the sides.

246. Draw a triangle, as ABC, Fig. 177, measure AB, and divide it into 7 equal parts.

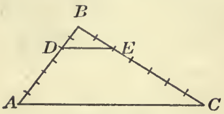


Fig. 177

Lay off $BD = \frac{2}{7} AB$, thus dividing AB at D in the ratio $\frac{2}{7}$. Similarly divide CB in the ratio $\frac{2}{7}$. Draw DE. Measure angles BDE and DAC.

If the construction is well done, $\angle\!\!\!s\ BDE$ and DAC will be found to be equal.

Show that DE is parallel to AC.

This problem illustrates the following theorem:

Theorem: *If two sides of a triangle are divided into parts having the same ratio, the line joining the points of division is parallel to the third side of the triangle.*

EXERCISES

1. In triangle ABC, Fig. 177, find EC, if $AD = 4$, $DB = 6$, and $BE = 10$.

2. The distance AB across a swamp is to be found, Fig. 178.

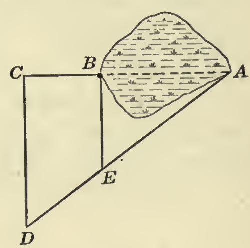


Fig. 178

A point C is found in the same line with A and B.

At C and B lines CD and BE are drawn perpendicular to CB and the line AD is drawn. The measured values of CB, DE, and EA are 160 ft., 175 ft., and 642 ft. respectively. Find the distance AB.

3. Divide a line-segment into two equal parts.

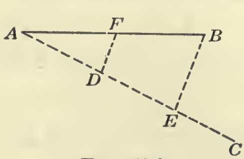


Fig. 179

Let AB, Fig. 179, be the given segment.

Draw AC through A making an acute angle with AB.

On AC lay off $AD = DE$.

Join E to B.

Draw $DF \parallel EB$.

Show that $AF = FB$. (See exercise 4, § 244.)

4. Divide a line-segment into two parts whose ratio is $\frac{2}{3}$.

Let AB, Fig. 180, be the given segment.

Draw AC making an acute angle with AB.

On AC lay off $AD=2$ units and $DE=3$ units.

Join E to B.

Through D draw $DF \parallel EB$.

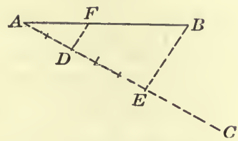


FIG. 180

Show that $\frac{AF}{FB}=\frac{2}{3}$.

5. Divide a line-segment into parts having the ratio $\frac{3}{4}$; $\frac{1}{2}$; $\frac{7}{9}$.

247. Use of compass in finding ratios of line-segments. Let AB and CD, Fig. 181, be two segments whose ratio is to be found.

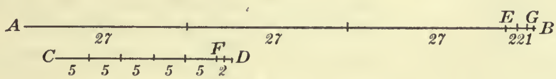


FIG. 181

Let us assume that AB and CD contain a common unit of measure. (It will be shown later that there are line-segments that have no common unit of measure.) To find the common unit, proceed as follows:

Lay off the smaller segment CD on the larger AB as often as possible, leaving a remainder EB, which is less than CD.

Lay off EB on CD, leaving a remainder FD, which is less than EB.

Lay off FD on EB, leaving a remainder GB.

Lay off GB on FD, leaving no remainder.

The last remainder, GB, is a common unit of measure of AB and CD.

Using GB as unit, show that $AB=86$, and $CD=27$.

Therefore $\frac{AB}{CD}=\frac{86}{27}$.

248. Greatest common divisor. The method of finding the ratio of two line-segments by means of the compass as given in § 247 is not as practical as the method of chapter VIII in which squared paper or the marked ruler was used for measuring. However, the method of § 247 is valuable for other purposes; e.g., it serves to illustrate a method of finding the greatest common divisor (g.c.d.) of arithmetic and algebraic numbers, so important in reducing ratios and fractions to lowest terms.

Find the greatest common divisor of 86 and 27.

Compare the following solution step for step with the solution in § 247.

27)86(3 81 — 5	Divide the smaller number, 27, into the larger, 86, leaving a remainder 5.
5)27(5 25 — 2	Divide this remainder into the smaller given number, leaving a remainder 2.
2)5(2 4 — 1	Divide the last remainder into the preceding divisor, leaving a remainder 1.
1)2(2 2 —	Divide the last remainder into the preceding divisor, leaving no remainder.

The last divisor, 1, is the greatest common divisor of 86 and 27.

249. Prime numbers. When two numbers have no common divisor except unity they are said to be **prime** to each other. A number which has no divisor except itself and unity is a **prime number**.

EXERCISES

1. Find the g.c.d. of the following pairs of numbers:

1. 73 and 16	5. 263 and 765
2. 1,155 and 52	6. 350 and 425
3. 174 and 273	7. 3,542 and 5,016
4. 174 and 275	8. 3,795 and 2,865

2. Find geometrically, i.e., by means of line-segments and the use of the compass, the greatest common divisor of 73 and 16; of 70 and 36; of 45 and 42.

250. Reduction of ratios. In algebra, as in arithmetic, fractions are reduced to **lowest terms** either by dividing both numerator and denominator by all factors common to them, or by dividing both terms by their g.c.d.

EXERCISES

Reduce to lowest terms.

1. $\frac{6}{15}$

$\frac{6}{15}=\frac{2\cdot\cancel{3}}{5\cdot\cancel{3}}=\frac{2}{5}$

2. $\frac{7}{14}$

3. $\frac{22}{66}$

4. $\frac{10}{36}$

5. $\frac{12}{18}$

6. $\frac{15}{24}$

7. $\frac{18}{30}$

8. $\frac{20}{36}$

9. $\frac{24}{48}$

10. $\frac{75}{100}$

11. $\frac{102}{141}$

12. $\frac{192}{240}$

13. $\frac{666}{909}$

14. $\frac{960}{1{,}728}$

15. $\frac{805}{966}$

First find g.c.d. then reduce.

16. $\frac{2{,}924}{5{,}117}$

17. $\frac{803}{1{,}752}$

18. $\frac{741}{1{,}254}$

19. $\frac{1{,}707}{2{,}845}$

20. $\frac{550}{660}$

21. $\frac{455}{2{,}310}$

22. $\frac{24x^3y}{18xy^2}$

$\frac{24x^3y}{18xy^2}=\frac{\cancel{6}\cdot 4\cancel{x}\cdot x\cdot x\cdot\cancel{y}}{\cancel{6}\cdot 3\cancel{x}\cdot y\cdot\cancel{y}}=\frac{4x^2}{3y}$

23. $\frac{ab}{ac}$

24. $\frac{xy}{xz^2}$

25. $\frac{abx}{aby}$

26. $\frac{x^2y}{x^2z}$

27. $\frac{xy^2z}{xy^2w}$

28. $\frac{xyz}{m\cdot nx}$

29. $\frac{a(a+b)}{b(a+b)}$

30. $\frac{x^5y}{x^7z}$

31. $\frac{28m^3n^2k}{14m^2nk}$

32. $\frac{12abc^2}{4a^3b}$

33. $\frac{30ab^3cx}{6a^2c^2x^2}$

34. $\frac{m^2n^3pqr}{p^2q^3mnr}$

35. $\frac{15a^4bc^2}{30a^4bc}$

36. $\frac{12a^2b^2c^2d}{60abc^2d^2}$

37. $\frac{18bx^2y}{12axy^2}$

38. $\frac{2dx^2z^2}{3cxy^2z}$

39. $\frac{63a^3v^4x^4}{7a^2b^5x^5}$

40. $\frac{15x^4yz^3}{3x^8yz^2}$

41. $\frac{25m^5n^6r^7}{5m^3n^4r^5}$

42. $\frac{7a^3b^4cx}{56a^5b^2c^3xy}$

Direct Variation

251. Function. If a man walks at the rate of 2 mi. an hour, the distance d he walks in a given time t is found from the equation $d = 2t$.

The following table gives the distance passed over in 1 hr., 2 hr., 3 hr. . . . . . This table shows that a change

Number of hours	1	2	3	4	5	6	7	8
Distance passed over . . .	2	4	6	8	10	12	14	16

in the time causes a change in the distance passed over. The distance is said to **depend upon** the time, or the distance is said to be a **function** of the time. The time and distance are *variables*, the rate is a *constant*.

Dependence of one magnitude upon another is met frequently. Thus, the amount a man earns in a given time depends upon the number of days he works; the price paid for a piece of goods depends upon the number of yards; the greater a man's property is the higher will be his taxes; the weight of a piece of iron depends upon its size, etc. In geometry the area of a square depends upon the length of the side; the area of a circle depends upon the length of the radius; the perimeter of an equilateral triangle depends upon the side, etc.

252. Constant. A number that has a fixed value is a **constant.**

253. Variable. A number that changes, assuming a series of values, is a **variable.**

EXERCISES

1. In the table above find the ratio of each distance to the corresponding time. How do the ratios compare? It is seen

that although the distance and time vary the ratio does not change (remains constant).

2. A man earns \$20 per week. Show that the amount he earns in a given time depends upon the time. Show that as the time changes, the amount changes, but that the ratio of the amount to the time remains constant.

3. Write the equation for the weight w in pounds, and the cost c of flour at $3\frac{1}{2}$ cents a pound. Find the values of c as w takes the integral values from 1 to 10.

What is the ratio of the values of w to the corresponding values of c?

4. Tabulate the areas of rectangles with the altitude 5 and bases 1, 2, 3 . . 6. How does the area vary as the base varies?

Write the equation for the area A and the base b of rectangles with altitude 5. Graph the equation.

Does a change in value of b change the value of A?

Does it change the value of the ratio $\frac{A}{b}$?

254. Direct variation. When two numbers vary so that one depends upon the other for its value, leaving the ratio of any value of one to the corresponding value of the other constant, then one is said to **vary directly** as the other, or to be **directly proportional** to the other.

Thus, the number x is said to vary directly as y, if the ratio $\frac{x}{y}$ remains constant, x and y both changing, or varying. The equation $\frac{x}{y}=c$ expresses algebraically, and is equivalent to, the statement that x **varies directly as** y.

EXERCISES

Express the following statements by means of equations:

1. A man's pay p is directly proportional to the number of days t.

By definition $\frac{p}{t}=k$, a constant.

2. The weight of a steel rail varies directly as the length.

3. The weight of a mass of silver varies directly as the volume.

4. If a body moves at a uniform rate, the distance varies directly as the time.

5. The area of an equilateral triangle varies directly as the square of the side.

Solve the following problems:

6. The area of a rectangle varies directly as the base if the altitude remains constant; and when the area is 27, the base is 3. What is the constant ratio of the area to the base?

Since $\frac{A}{b}=k$, $\frac{27}{3}=k$ or $k=9$

Therefore $\frac{A}{b}=9$

7. When the base of the rectangle in problem 6 is 8, what is the area?

8. The turning-tendency caused by a weight moved along a bar, or lever, varies directly as the lever-arm. The turning-tendency is 20 when the arm is 5. Find the turning-tendency when the arm is 7.

9. The length of a circle varies directly as the diameter. The constant ratio of the circle to the diameter is 3.14, approximately. Write the equation for the circle c and the diameter d. When the circle is 157, what is the diameter?

10. The distance d through which a body falls from rest varies directly as the square of the time t in which it falls; and a body is observed to fall 400 ft. in 5 seconds. What is the constant ratio of d to t^2?

Write the equation for d and t.

How far does a body fall in 1 second? In 2 seconds? In 3 seconds?

11. x varies directly as y, and when $x=20$, $y=4$. Find the value of x when $y=17$.

12. If z varies as x, and $z=48$ when $x=4$, find z when $x=11$.

13. Cut from a cardboard circles with diameters of various lengths. By rolling the circles along a straight line determine the lengths of the circles. Determine the ratio of each circle to the diameter. Compare your result with that obtained by other members of the class.

Does the length of the circle vary as the diameter? Give reasons.

14. The speed of a falling body varies directly as the time. Write the equation for the speed v and the time t.

A body, falling from rest, moves at the rate of 160 ft. a second 5 seconds after it begins to fall. What will be the speed attained in 8 seconds?

15. The distance passed over by a body moving at a constant rate varies directly as the time. Find the rate of a train which travels, at uniform rate, the distance of 225 mi. in 6 hours.

16. A stone fell from a building 560 ft. high. In how many seconds did it reach the ground? (See problem 10.)

17. The time t (number of seconds) of oscillation of a pendulum varies directly as the square root of the length l. A pendulum 39.2 in. long makes one oscillation in 1 second. Find the length of a pendulum which makes an oscillation in 2 seconds.

18. Show that the area A of a square varies directly as the square of the side s.

19. The area of the equilateral triangle varies directly as the square of the side. The area of an equilateral triangle whose side is 2 equals $\sqrt{3}$. What is the area of an equilateral triangle whose side is 4?

20. The altitude of an equilateral triangle varies directly as the side. The altitude of an equilateral triangle whose side is $2\sqrt{3}$ is equal to 3. Find the altitude of an equilateral triangle whose side is $\frac{4}{\sqrt{3}}$.

21. The simple interest on an investment varies directly as the time. If the interest for 6 years on a sum of money is $200, what will be the interest for 8 years and 3 months?

22. The surface of a sphere varies as the square of the radius. The surface of a sphere with radius 7 in. is 616 sq. in. What is the radius of a sphere twice as large in surface?

Inverse Variation

255. Inverse variation defined. In §§ 251–254 certain magnitudes were found to depend upon each other so that an *increase* in one caused an *increase* in the other, or a *decrease* in one caused the other to *decrease*. However, some magnitudes depend each upon each other so that an *increase* in one causes the other to *decrease;* e.g., an increase in the number of men doing a piece of work causes a decrease in the time it takes to do the work. If 8 men do a piece of work in 12 days, 16 men will need only 6 days to do it. The number of men is said to **vary inversely** as the number of days, or to be **inversely proportional** to the number of days.

Suppose a man wishes to fence in a rectangular piece of ground containing 140 square feet. If he makes it 14 ft. long, the width must be 10 feet. If he makes it 20 ft. long, he can allow only 7 ft. for the width, etc. Thus, the length and width must vary in a way which leaves the area, i.e., their product, constant. This leads to the following definition: A number x **varies inversely as** y if the product xy remains constant as both x and y vary, i.e., if $xy=k$, a constant.

EXERCISES

1. Express the following statements by means of equations:

(1). The apparent size of an object varies inversely as its distance.

(2). The time needed to go a certain distance varies inversely as the rate of travel.

(3). The force of gravity varies inversely as the square of the distance.

(4). The heat of a stove varies inversely as the square of the distance from it.

2. In $y=\frac{5}{x}$ or $y=\frac{a}{x}$ show that x varies inversely as y.

3. The variable y varies inversely as x. When $x=1$, $y=2$. Write the equation for x and y and find the value of the constant product. Find the value of y when $x=8$.

4. The number of men doing a piece of work varies inversely as the time. If twelve men can do a piece of work in 28 days, in how many days can 3 men do the same?

5. When gas in a cylinder is put under pressure, the volume is reduced as the pressure is increased. It is found in physics that the volume varies inversely as the pressure. The volume of a gas is 4 cubic cm. when the pressure is 3 pounds. What is the volume under a pressure of 6 pounds?

256. Graphing direct variation. The equation $c=3.14d$ states that the length of the circle varies as the diameter (see problem 13, § 254). Complete the following table for corresponding values of c and d.

d . . .	0	1	2	3	4	5	6	7	8	9	10	11
c . . .	0	3.1	6.3	9.4								

Represent the equation $c=3.14d$ graphically as in Fig. 182.

Graph the laws of variation in problems 8, 10, 11, and 14, § 254.

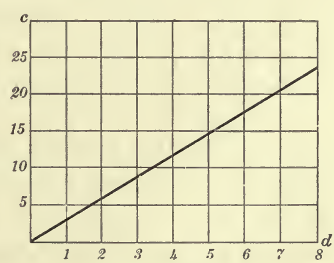


Fig. 182

257. Graphing inverse variation. A distance of 42 mi. is passed over by a train at various rates. Find the time it takes the train to travel this distance if it moves uniformly at the rate of 45 mi. per hour; 40 mi. per hour; 35 mi. per hour, etc.

Letting t denote the time and r the rate, then $t=\frac{42}{r}$ expresses the conditions of the problem. Thus, the time varies inversely as the rate. Why?

The facts expressed by the equation $t=\frac{42}{r}$ as t and r take different values may be tabulated as follows:

r........	45	40	35	30	25	20	15	10	5
t........	0.9	1.0	1.2	1.4	1.7	2.1	2.8	4.2	8.4

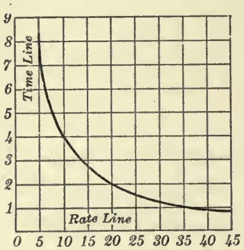


FIG. 183

The same facts are given graphically in Fig. 183. State from the graph how the time varies as the rate varies from a very small to a very large value.

Graph the following equations:

$xy=8$; $xy=24$; $xy=30$.

Proportion

258. Proportion. Problems in similar triangles have been solved by means of equations of equal ratios, as $\frac{x}{3}=\frac{5}{27}$. An equation of two equal ratios is a **proportion**.

Thus, $\frac{4}{6}=\frac{2}{3}$, $\frac{1}{5}=\frac{3}{15}$, $\frac{a}{b}=\frac{c}{d}$ are all proportions. A proportion as $\frac{a}{b}=\frac{c}{d}$ may be read "*a* over *b* equals *c* over *d*," or "*a* is to *b* as *c* is to *d*." The proportion is sometimes written in the form $a:b=c:d$.

Is the statement $\frac{2}{5}=\frac{5}{13}$ a proportion? Is $\frac{2}{7}=\frac{8}{25}$ a proportion? Give reasons for your answers.

259. Means and extremes. The first and last terms in a proportion are called the **extremes**; the second and third, the **means**. Thus, in $\frac{a}{b}=\frac{c}{d}$, a and d are the extremes, b and c the means.

EXERCISES

1. Compare the product of the extremes with the product of the means in the proportions: $\frac{2}{5}=\frac{6}{15}$; $\frac{3}{7}=\frac{6}{14}$; $\frac{20}{2}=\frac{10}{1}$; $\frac{12}{3}=\frac{4}{1}$. What do you find true of the products?

2. Make up several proportions and compare the product of the means with the product of the extremes.

Exercises 1 and 2 illustrate the following theorem:

Theorem: *In a proportion the product of the means is equal to the product of the extremes.*

This theorem is a convenient test of proportionality, as it is usually simpler to find the products than to reduce the ratios to lowest terms.

Proof: The equation $\frac{a}{b}=\frac{c}{d}$ may be cleared of fractions by multiplying both ratios by bd and then reducing them to lowest terms. It follows that $ad=bc$.

EXERCISES AND PROBLEMS

Solve the following problems:

1. Divide $2,400 into two parts having the ratio 2:1.

Find x in the equation $\frac{2{,}400-x}{x}=\frac{2}{1}$.

2. If 5 and 3 are added to a certain number and 3 and 4 subtracted from it the resulting four numbers are in proportion. Find the number.

3. Two numbers are to each other as 3 to 7 and their sum is 50. Find the numbers.

4. If 3 is subtracted from a number, 6 added to 2 times the number, 12 subtracted from 2 times the number, and 6 subtracted from 4 times the number, the resulting numbers are in proportion. What is the number?

5. The dimensions of a rectangle are 7 and 4 and the length of a similar rectangle is 8. What is the width?

6. Find the value of x in the following proportions. Check the results by substituting in the original equation.

1. $\frac{x}{160}=\frac{3}{5}$
2. $\frac{4}{x}=\frac{132}{3}$
3. $\frac{18}{810}=\frac{x}{3}$
4. $\frac{200-x}{x}=\frac{7}{18}$
5. $\frac{5+x}{3-x}=\frac{2+x}{4-x}$
6. $\frac{1-3a}{2a}=\frac{-6a}{4a-1}$
7. $\frac{x-12}{1-3x}=\frac{2}{1}$
8. $\frac{x+1}{10}=\frac{3x+2}{28}$
9. $\frac{x-4}{x-13}=\frac{x-8}{x-14}$
10. $\frac{2-x}{3-x}=\frac{3-x}{1-x}$
11. $\frac{x+5}{x+3}=\frac{x-3}{x-4}$
12. $\frac{4x-3}{2x+3}=\frac{4x+2}{2x+6}$
13. $\frac{x-5}{4}=\frac{x+4}{16}$
14. $\frac{x+2}{x-2}=\frac{2x-3}{2x+1}$

7. The shadows of a tree and a 5-ft. rod are 96 ft. and 8 ft. respectively. Find the height of the tree.

8. If 2″ on a map represent 21 mi., what distance on the map represents 50 miles?

9. Solve the following equations:

$$\frac{y+\frac{1}{4}}{y-\frac{1}{4}}=\frac{y+\frac{5}{3}}{y-1} \qquad \frac{1+\frac{x}{3}}{1-\frac{x}{4}}=\frac{2-\frac{x}{3}}{1+\frac{x}{4}}$$

Alloy Problems

Solve the following problems:

1. In an alloy of silver and copper weighing 90 oz., there are 6 oz. of copper. Find how much silver must be added so that 10 oz. of the new alloy shall contain $\frac{2}{5}$ oz. of copper.

Let x be the number of ounces of silver added.

Then $90+x$ is the number of oz. in the new alloy.

Since $(90+x)$ oz. of the new alloy contain 6 oz. of copper,

$\frac{90+x}{6}$ oz. contain 1 oz. of copper.

Similarly, since 10 oz. of the new alloy contain $\frac{2}{5}$ oz. of copper, $\frac{10}{\frac{2}{5}}$ oz. contain 1 oz. of copper.

$$\text{Therefore } \frac{90+x}{6}=\frac{10}{\frac{2}{5}}$$

$$\tfrac{2}{5}(90+x)=6\cdot 10$$

$$36+\frac{2x}{5}=60$$

$$x=60$$

2. If 80 lb. of sea-water contain 4 lb. of salt, how much fresh water must be added to make a new solution of which 45 lb. contain $\frac{2}{3}$ lb. of salt?

3. In a mass of alloy for watch-cases, which contains 60 oz., there are 20 oz. of gold. How much copper must be added so that in a case weighing 2 oz. there will be $\frac{1}{2}$ oz. of gold?

4. In an alloy weighing 80 grams, there are 34 grams of gold. How much nickel must be added so that a ring made from the new alloy and weighing $1\frac{1}{2}$ grams shall contain $\frac{1}{4}$ gram of gold?

5. In an alloy weighing a oz. there are b oz. of gold. How much of another metal must be added so that a portion weighing c oz. shall contain d oz. of gold?

6. Gun metal is composed of tin and copper. 4,100 lb. of gun metal of a certain grade contain 3,444 lb. of copper. How much tin must be added so that 2,100 lb. shall contain 1,722 lb. of copper?

Lever Problems

Two weights, w_1 and w_2, Fig. 184, will balance on a beam that lies across a stick when the distances, d_1 and d_2, of weights from the stick are in the inverse ratio of their weights; i.e., when $\frac{d_1}{d_2}=\frac{w_2}{w_1}$.

FIG. 184

1. Find d_1, if (1) $d_2=18$ ft., $w_2=60$ lb., $w_1=50$ lb.
   (2) $d_2=27$ in., $w_2=36$ lb., $w_1=24$ lb.
2. Find d_2, if (1) $d_1=40$ in., $w_2=16$ lb., $w_1=18$ lb.
   (2) $d_1=25$ in., $w_2=3.8$ lb., $w_1=2.85$ lb.
3. Find w_1, if (1) $d_1=2.5$ in., $d_2=7.5$ ft., $w_2=10.5$ lb.
   (2) $d_1=6.6$ ft., $d_2=9.9$ ft., $w_2=17$ lb.
4. Find w_2, if (1) $d_1=3.5$ ft., $d_2=8.5$ ft., $w_1=30$ lb.

Mixture Problems

1. What per cent of evaporation must take place from a 6 per cent solution of salt and water (salt-water of which 6 per cent by weight is salt) to make the remaining portion of the mixture an 8 per cent solution?

Let x be the number of per cent evaporated.

Then $100-x$ is the number of per cent remaining.

Since 6 per cent of the first solution is salt, $\frac{6}{100}\cdot 100$ is the amount of salt.

Since 8 per cent of the new solution is salt, $\frac{8}{100}(100-x)$ is also the amount of salt.

Therefore $\frac{6}{100}\cdot 100=\frac{8}{100}(100-x)$

$$600=800-8x$$

$$x=25$$

2. What per cent of evaporation must take place from a 90 per cent solution to produce a 95 per cent solution?

3. A physician having a 6 per cent solution of a certain kind of medicine wishes to dilute it to a $3\frac{1}{2}$ per cent solution. What per cent of water must be added?

4. A druggist has a 95 per cent solution. What must he do to change it to an 80 per cent solution required in a prescription?

Proportionality of Areas

Prove the following theorems:

260. Theorem: *The areas of two rectangles are to each other as the products of their dimensions.*

Proof: Denoting the areas of the rectangles by R and R_1, and their dimensions by b, h, and b_1, h_1, we have

$$R = bh. \quad \text{Why?}$$
$$R_1 = b_1h_1. \quad \text{Why?}$$

Therefore $$\frac{R}{R_1} = \frac{bh}{b_1h_1}. \quad \text{Why?}$$

261. Theorem: *If two rectangles have equal bases, they are to each other as the altitudes.*

For, $\frac{R}{R_1} = \frac{bh}{b_1h_1}$. Reducing the ratio $\frac{bh}{b_1h_1}$ to lowest terms,

$$\frac{R}{R_1} = \frac{\cancel{b}h}{\cancel{b}_1h_1} = \frac{h}{h_1}$$

262. Theorem: *If two rectangles have equal altitudes they are to each other as the bases.* Prove.

EXERCISES

1. The equation $\frac{5}{4} = \frac{x}{8}$ expresses the relation between the ratio of the areas and the ratio of the altitudes of two rectangles. Find the altitude x.

2. The area of a rectangle is 80 sq. ft., and the base is 10 yards. What is the area of a rectangle having the same altitude and a base equal to 24 yards?

Prove the following theorems:

263. Theorem: *The areas of parallelograms are to each other as the products of the bases and altitudes.*

264. Theorem: *The areas of two triangles are to each other as the products of the bases and altitudes.*

265. Theorem: *The areas of two parallelograms having equal bases are to each other as the altitudes.*

266. Theorem: *The areas of two triangles having equal bases are to each other as the altitudes.*

Summary

267. The following new terms have been taught in this chapter: prime numbers, reduction of fractions to lowest terms; greatest common divisor; direct and inverse variation, variables, constant, function; proportion; means and extremes. The symbol $\sqrt{\ }$ expresses square root.

268. The truth of the following theorems* has been shown:

1. *A line parallel to one side of a triangle divides the other two sides into corresponding parts having equal ratios.*

2. *A line bisecting an angle of a triangle divides the side opposite that angle into parts whose ratio is equal to the ratio of the other two sides.*

3. *If two sides of a triangle are divided into parts having the same ratio, the line joining the points of division is parallel to the third side of the triangle.*

4. *In a proportion the product of the means equals the product of the extremes.*

5. *Two rectangles, two parallelograms, or two triangles are to each other as the products of the bases and altitudes.*

6. *Two rectangles, two parallelograms, or two triangles having equal bases are to each other as the altitudes.*

* Most of these theorems were first proved by the Pythagorean School from 500 B.C. to 400 B.C.

7. *Two rectangles, two parallelograms, or two triangles having equal altitudes are to each other as the bases.*

269. A common unit of two line-segments may be found by use of the compass. The process of finding the greatest common divisor of two numbers is similar to the process of finding the greatest common unit of two line-segments.

270. Ratios and fractions may be reduced to lowest terms by dividing numerator and denominator by all factors common to them.

271. The statements x varies directly as y and x varies inversely as y are equivalent to the statements $x=ky$ and $xy=k$ respectively. Both equations have been represented graphically.

272. Many problems, as alloy problems, mixture problems, and lever problems, lead to proportions.

CHAPTER X

CONGRUENCE OF TRIANGLES

Congruence

273. Congruent figures.* In chapter VIII problems in finding distances were solved by indirect measurement. A drawing was made of the same shape as the figure containing the distance to be determined, and the corresponding distance in the drawing was then measured, or found algebraically by means of an equation. It will be shown in this chapter how to determine a distance from a figure having the same *size* and *shape* as the figure containing the unknown distance. Figures having the same size and the same shape are called **congruent figures.**

One of the problems of this chapter is to find out *under what conditions two triangles are necessarily congruent.*

EXERCISES

1. *To construct a triangle having given two sides and the angle between them.*

Construction: Suppose the given sides to be 3 in. and 4 in. long respectively and the given angle to be 60°.

On a line AB, Fig. 185, lay off AC equal to the 4-in. segment.

On AC at A construct $\angle DAC$ equal to the given angle.

On AD lay off AE equal to the 3-in. segment.

Draw EC. Then $\triangle AEC$ is the required triangle.

*The principle of congruence was employed somewhat by the School of Thales and very extensively by the School of Pythagoras. Pythagoras systematized it and gave it the dignity of a mathematical principle.

2. Cut out the triangle in exercise 1 and see if it can be made to coincide with the triangles constructed by other members of the class.

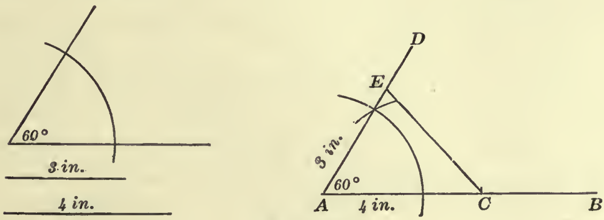


FIG. 185

3. Draw a triangle. Construct a second triangle having two sides and the included angle equal respectively to two sides and the included angle of the first triangle. See if the two triangles can be made to coincide.

Exercises 1, 2, and 3 illustrate the following theorem:

274. Theorem: *Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

The truth of this statement may be established by reasoning as follows:

Let ABC and $A_1B_1C_1$, Fig. 186, represent two triangles having $b = b_1$, $c = c_1$, $\angle A = \angle A_1$.

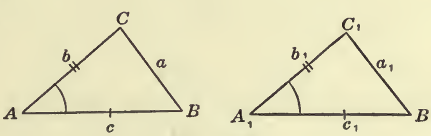


FIG. 186

Imagine $\triangle A_1B_1C_1$ placed on $\triangle ABC$.

Then $\angle A_1$ can be made to fit on $\angle A$, since $\angle A$ is given equal to $\angle A_1$.

Side c_1 can be made to coincide with side c, thus making B_1 fall on B. Why?

Side b_1 will coincide with b, making C_1 fall on C. Why?

Then a_1 and a must coincide. For only one straight line can be drawn between two points.

Therefore triangles $A_1B_1C_1$ and ABC coincide throughout and are congruent.

In Fig. 187 the inaccessible distance AB across a lake is to be determined.

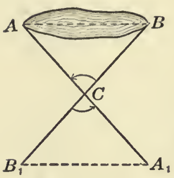


FIG. 187

A point C is selected from which A and B can be seen.

Line AC is drawn and extended to A_1 making $AC = A_1C$.

Similarly B_1C is made equal to BC. Then B_1A_1 is drawn.

Show that triangles A_1B_1C and ABC are congruent.

Hence, AB is determined by measuring A_1B_1. Why?

EXERCISES

1. *To construct a triangle having given two angles and the side included between their vertices.*

Construction: Suppose the angles to be 35° and 50°, and the side to be 4 in. long.

Draw a line, as AB, Fig. 188.

On AB lay off AC equal to the given 4-in. segment.

On AC at A construct angle DAC equal to the 50° angle.

On AC at C construct an angle equal to the 35° angle.

$\triangle ACE$ is the required triangle. Why?

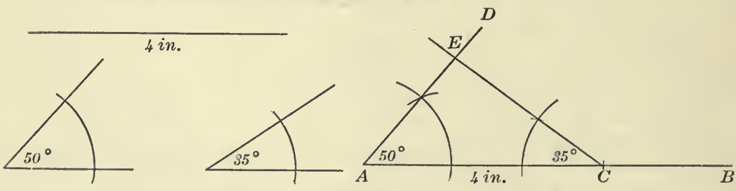


FIG. 188

2. Cut out the triangle constructed in exercise 1 and see if it can be made to coincide with the triangles constructed by other members of the class.

3. Draw a triangle. Construct another triangle having two angles and the side between their vertices equal respectively to the corresponding parts of the first triangle. See if the triangles can be made to coincide.

Exercises 1, 2, and 3 illustrate the following theorem:

275. Theorem: *Two triangles are congruent if two angles and the side included between their vertices in one triangle are equal respectively to the corresponding parts in the other.*

For, let any two triangles as ABC and $A_1B_1C_1$, Fig. 189, have $c = c_1$, $\angle A = \angle A_1$, $\angle B = \angle B_1$.

Imagine $\triangle A_1B_1C_1$ placed upon $\triangle ABC$.

Then A_1B_1 can be made to coincide with AB. Why?

Angle A_1 must then coincide with angle A. Why?

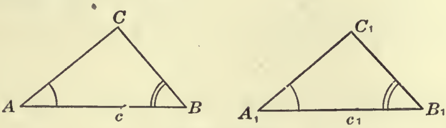


Fig. 189

Angle B_1 must coincide with angle B. Why?

Point C_1 must therefore lie on lines AC and BC.

Thus, C_1 must fall on C, the only point common to AC and BC.

Therefore triangles ABC and $A_1B_1C_1$ coincide and are congruent.

276. Methods of proof: The theorems of §§ 274 and 275* were proved by placing one figure over the other and then showing that one fits exactly upon the other. This method of proof is known as the *method by superposition*. It is one of several kinds of proof used in geometry.

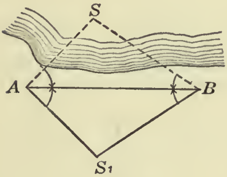


Fig. 190

In Fig. 190 the distances of two points A and B from a lighthouse S are to be determined.

On AB, at A, angle BAS_1 is constructed equal to angle BAS, and angle ABS_1 equal to angle ABS. AS_1 is measured and found to be 2,680 ft. BS_1 is found to be 3,420 ft. What are the lengths of AS and BS? Give reasons for your answer.

* The theorem of § 275 was proved for the first time by Thales, who used it to find the distance of a ship at sea from the shore. (See Ball, p. 15.)

277. Notation for corresponding parts. Corresponding sides and angles of congruent figures may be denoted in various ways. The subscripts in Fig. 191 indicate that point C_1 corresponds to point C, A_1 to A, and B_1 to B. Similarly side a_1 corresponds to side a, b_1 to b, and c_1 to c.

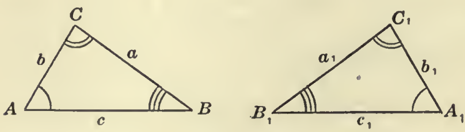


Fig. 191

One arc in an angle means that the angle corresponds to the angle in the congruent figure marked with one arc. Angles marked with two arcs, three arcs, etc., are corresponding angles.

Frequently the symbol (′) is used to indicate correspondence. Thus A' (A-prime) corresponds to A, Fig. 192, B' to B, and C' to C.

Correspondence of sides is also indicated by one, two, or more strokes across the side, as in Fig. 192.

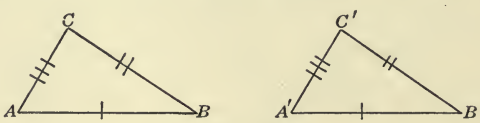


Fig. 192

278. Symbol for congruence. The symbol for **congruence** is $\cong$, the $\backsim$ indicating similarity and $=$ equality of the figures. Thus, the statement $\triangle ABC$ and $A'B'C'$ are congruent is written $\triangle ABC \cong \triangle A'B'C'$, meaning of the same *size* and *shape*.

The Isosceles and the Equilateral Triangle

279. Conditions under which triangles are congruent. The theorems in §§ 274 and 275 suggest a method of proving triangles congruent, which is more advantageous than the method of superposition. According to these theorems to prove two triangles congruent it is not necessary to know that *all* parts of one triangle are equal to the corresponding parts of the other, but it is sufficient to show that one of the following two conditions is satisfied:

1. That the triangles have two sides and the included angle equal respectively, briefly expressed by "side, angle, side" or (s.a.s.).

2. That the triangles have two angles and the side included between their vertices equal respectively (a.s.a.).

The application of this method of proof will be illustrated by proving some of the properties of the isosceles triangle.

280. Theorem: *The base angles of an isosceles triangle are equal.**

Let $\triangle ABC$, Fig. 193, be isosceles, i.e., let $a=b$. It is to be proved that $\angle A=\angle B$.

Proof: Draw the helping line CD bisecting $\angle C$, i.e., making $x=y$.

Prove that $\triangle ADC \cong \triangle BDC$. (s.a.s.)

It follows that $\angle A = \angle B$, because corresponding parts of congruent triangles are equal.

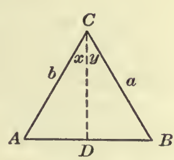


FIG. 193

281. Congruent-triangle method. The method of proving two line-segments, or two angles, equal by showing that they are corresponding parts of congruent triangles is called the **congruent-triangle method.**

* This theorem was first proved by Thales. Ball, p. 15, indicates the method of proof supposed to have been used by him.

EXERCISES

Prove the following:

1. **Theorem:** *An equilateral triangle is equiangular.*

Apply the theorem in § 280.

2. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

Use the congruent-triangle method.

3. **Theorem:** *All points on the perpendicular bisector of a line-segment are equidistant from the end-points of the segment.*

4. If a line bisects an angle of a triangle and is perpendicular to the opposite side, the triangle is isosceles.

5. **Theorem:** *If two angles of a triangle are equal, the triangle is isosceles.*

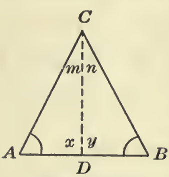


FIG. 194

Let A and B be the equal angles, Fig. 194.

Draw the helping line $CD \perp AB$, i.e., making $x=y$.

Since two angles of $\triangle ADC$ are equal to two angles of $\triangle BDC$, it follows that $m=n$. (See p. 53, problem 16.)

Prove $\triangle ADC \cong \triangle BDC$. (a.s.a.)

Then $AC=BC$. Why?

6. An equiangular triangle is equilateral.

Apply the theorem in exercise 5.

7. If the perpendicular bisector of one side of a triangle passes through the opposite vertex, the triangle is isosceles.

8. **Theorem:** *If two sides of a triangle are unequal the angles opposite them are unequal, the greater angle lying opposite the greater side.*

Let CB, Fig. 195, be greater than CA.

Lay off on CB the segment $CD=CA$.

Draw AD and through B draw $BE \parallel AD$.

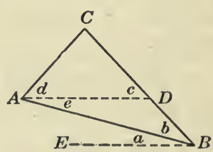


FIG. 195

	$a+b=c.$	Why?
	$c=d.$	Why?
Hence	$a+b=d.$	Why?
But	$d<d+e.$	Why?
Therefore	$a+b<d+e.$	Why?

Since $a=e$, it follows that $b<d$. For, equals subtracted from unequals give unequals in the same order as the minuends.

9. **Theorem:** *If two angles of a triangle are unequal the sides opposite them are unequal, the greater side lying opposite the greater angle.*

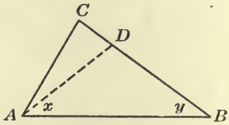


FIG. 196

Through A draw AD making $x=y$, Fig. 196.

Then $AD=DB$. Why?

Since $AD+DC>CA$, it follows that

$$DB+DC>CA. \quad \text{Why?}$$

Thus, $\quad BC>CA$.

282. Construction of a triangle all of whose sides are given.

EXERCISES

1. *Given the three sides of a triangle, to construct the triangle.*

Construction: Suppose the sides to be 2 in., 3 in., and 4 in. long respectively.

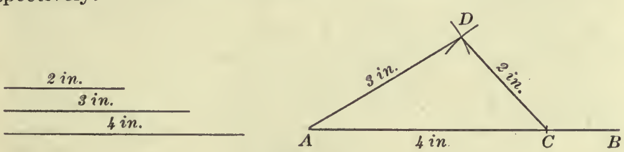


FIG. 197

Draw a line, as AB, Fig. 197.

On AB lay off AC equal to the 4-in. segment.

With A as center and with the 3-in. segment as radius draw an arc at D.

With B as center and with the 2-in. segment as radius draw an arc intersecting the first arc at D.

Join the intersection point D to A and C.

$\triangle ADC$ is the required triangle.

2. Cut out the triangle constructed in exercise 1 and see if it can be made to coincide with the triangles constructed by other members of the class.

Exercises 1 and 2 illustrate the fact that two triangles are congruent if their sides are equal respectively.

3. By taking various lengths of the given segments, e.g., by making the sum of two segments equal to or less than the third,

show that the construction in exercise 1 is possible only when the sum of any two of the given segments is greater than the third.

4. Construct an equilateral triangle on a side 2 in. long.

5. Draw a triangle having sides of $\frac{1}{2}$ in., $\frac{3}{4}$ in. and 1 in.

6. With crayon and string or with a blackboard compass draw on the blackboard a triangle having sides 6 in., 8 in., and 10 in. long.

7. Describe how with a 100-ft. steel tape stakes may be set in the ground to be the corners of a triangle of sides 30′, 50′, and 60′ long.

8. Given the base and one of the equal sides of an isosceles triangle, to construct the triangle.

283. Theorem: *If the three sides of one triangle are equal, respectively, to the three sides of another triangle, the triangles are congruent.* (s.s.s.)

For, let ABC and $A_1B_1C_1$, Fig. 198, represent two triangles having $AB = A_1B_1$, $BC = B_1C_1$, and $CA = C_1A_1$.

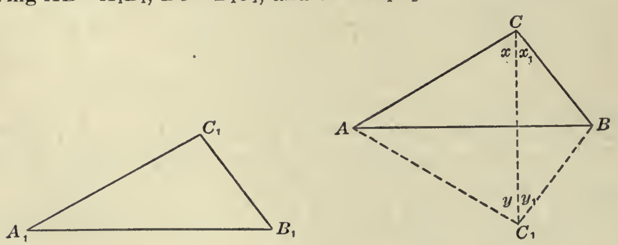


FIG. 198

Imagine $\triangle A_1B_1C_1$ to be placed adjacent to $\triangle ABC$ so that A_1B_1 coincides with AB and that C_1 and C lie on opposite sides of AB.

Draw CC_1

Then $AC_1 = AC$. Why? Hence $x = y$. Why?

Since $BC_1 = BC$, it follows that $x_1 = y_1$. Why?

Therefore $x + x_1 = y + y_1$, or $\angle C = \angle C_1$. Why?

Prove $\triangle ABC \cong \triangle ABC_1$. (s.a.s.)

Hence $\triangle ABC \cong \triangle A_1B_1C_1$. Why?

NICCOLO FONTANA

NICCOLO FONTANA

NICCOLO FONTANA, nicknamed Tartaglia (Stammerer), was born at Brescia in Italy about 1500, and died at Venice, December 14, 1557. In 1512 his native city was captured by the French army, and during the sack of the city his father was killed and he himself so severely injured as to produce the stoppage in his speech from which he was nicknamed and from which he never recovered. His mother was too poor to pay for more than fifteen days of schooling for him, and she could not afford even to buy paper for him to use in private study. In spite of all these obstacles he rose by dint of his own efforts to be the ablest and best known algebraist of his time.

Mathematical contests were the vogue in Tartaglia's day. When a mathematician made a discovery he challenged some well-known mathematician to a contest. Each contestant proposed the same number of problems for the other to solve in a specified time. The one who solved the largest number in the time set was declared the victor. It was thus he won his title as a mathematician. Tartaglia engaged in two such contests, winning both times. See an account of these contests in Ball's *History of Mathematics*, pp. 218 and 223 (5th ed.).

The scientific service for which Tartagalia is best known today is his discovery of the solution of the cubic equation, about 1530. He began public by lecturing on mathematics in Verona, and in 1535 he was appointed to a chair of mathematics in Venice. He wrote three important works on mathematics, besides publishing an edition of Euclid in 1543 and an edition of Archimedes the same year. His *Inventioni*, published in Venice in 1546, contains his solution of the cubic equation, which had been published by Cardan as his own the year before.

EXERCISES

Prove the following:

1. **Theorem:** *If a point is equidistant from the end-points of a line-segment, it is on the perpendicular bisector of the segment.*

Let C, Fig. 199, be equidistant from A and B, i.e., let $AC = BC$.

Draw the helping line CD from C to the mid-point D of AB.

Prove $\triangle ADC \cong \triangle BDC$. (s.s.s.)

Then $x = y$ and $CD \perp AB$.

Therefore CD is the perpendicular bisector of AB.

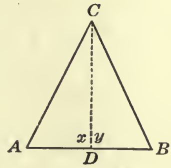


FIG. 199

2. **Theorem:** *If each of two points of one line is equally distant from two points of another line the lines are perpendicular.*

For, by exercise 1 each of the given points A and B, Fig. 200, lies on the perpendicular bisector of the second line CD.

Since only one straight line can be drawn through two points, the line AB and the perpendicular bisector of CD must be the same line.

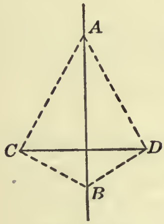


FIG. 200

284. Locus of points. In exercise 1, § 283, it was shown that *every point equidistant from the end-points of a segment lies on the perpendicular bisector of the segment.* In exercise 3, § 281, it was shown that *every point on the perpendicular bisector of a line-segment is equidistant from the end-points of the segment.*

From these two theorems it follows that *the perpendicular bisector of a segment is the place,* or **locus,** *of all points equidistant from its end-points.*

EXERCISES

1. What is the locus of all points in a plane which are at a distance of 10 ft. from a given point P in the plane? At a distance a from the given point?

2. What is the locus of all points in space known to have a given distance from a given point?

3. What is the locus of all points in a plane at a given distance from a given line?

4. What is the locus of all points in a plane at equal distances from two parallel lines?

5. What is the locus of all points in space having a given distance from a given line?

6. What is the locus of all points in space having a distance of 2 ft. from the floor of the room?

7. What is the locus of all points in space known to be at equal distances from two given points?

The Right Triangle

285. Theorem: *Two right triangles are congruent if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.*

Let ABC and $A_1B_1C_1$, Fig. 201, be two right triangles having the hypotenuse (side opposite the right angle) AC equal to A_1C_1 and $BC = B_1C_1$.

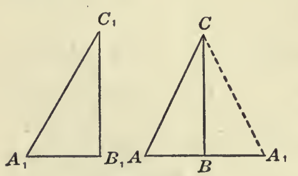


FIG. 201

Place $\triangle A_1B_1C_1$ adjacent to $\triangle ABC$, making B_1C_1 coincide with the equal side BC, so that A and A_1 lie on opposite sides of BC.

Then ABA_1 is a straight line and the figure ACA_1 a triangle. Why?

Show that $\triangle ACA_1$ is an isosceles triangle.

Hence $\angle A = \angle A_1$. Why?

Prove $\triangle ABC \cong \triangle A_1BC$. (a.s.a.)

Then $\triangle A_1B_1C_1 \cong \triangle ABC$. Why?

EXERCISES

Prove the following:

1. Two right triangles are congruent if the two sides about the right angle of one are respectively equal to the two sides about the right angle of the other.

2. Two right triangles are congruent if the hypotenuse and an acute angle of one are respectively equal to the hypotenuse and an acute angle of the other.

3. **Theorem:** *The shortest distance from a point to a line is the perpendicular from the point to the line.*

For, let AB, Fig. 202, be perpendicular to DE and let AC be any other line from A to DE.

In $\triangle ABC$ $\angle C$ is acute. Why? Therefore $\angle C$ is less than $\angle B$. Why? Hence $AC > AB$. Why?

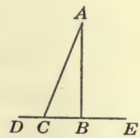


FIG. 202

4. Oblique lines drawn to a line from a point on a perpendicular to the line and making equal angles with the perpendicular are equal.

5. Oblique lines drawn from a point on a perpendicular to a line and making equal angles with the line are equal.

Summary

286. The following new terms have been taught in this chapter: congruent figures, corresponding parts of congruent figures; locus of points; methods of proof.

287. The symbol $\cong$ indicates congruence, i.e., can be made to coincide.

288. Problems in finding unknown distances and angles can be solved by construction of congruent triangles.

289. The following constructions of triangles were taught:

1. When two sides and the included angle are given.
2. When two angles and the side between the vertices are given.
3. When the three sides are given.

290. Two methods of proof have been taught:

1. The superposition method.
2. The congruent-triangle method.

291. The following theorems state the conditions that determine the congruence of triangles:

1. *Two triangles are congruent if they have two sides and the included angle equal respectively.*

2. *Two triangles are congruent if they have two angles and the side between their vertices equal respectively.*

3. *Two triangles are congruent if the corresponding sides are equal.*

4. *Two right triangles are congruent if the hypotenuse and a side of one are respectively equal to the hypotenuse and a side of the other.*

292. The following theorems on the equilateral triangle were proved:

1. *An equilateral triangle is equiangular.*

2. *An equiangular triangle is equilateral.*

293. The following theorems on the isosceles triangle were proved:

1. *The base angles of an isosceles triangle are equal.*

2. *If two angles of a triangle are equal the triangle is isosceles.*

3. *The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.*

4. *If a line bisects an angle of a triangle and is perpendicular to the opposite side the triangle is isosceles.*

5. *If the perpendicular bisector of one side of a triangle passes through the opposite vertex the triangle is isosceles.*

294. The following theorems on inequalities were proved:

1. *If two sides of a triangle are unequal the angles opposite them are unequal, the greater angle lying opposite to the greater side.*

2. *If two angles of a triangle are unequal the sides opposite them are unequal, the greater side lying opposite the greater angle.*

295. The following theorems on perpendicular lines were proved:

1. *All points on the perpendicular bisector of a segment are equally distant from the end-points of the segment.*

2. *If a point is equidistant from the end-points of a line-segment it is on the perpendicular bisector of the segment.*

3. *The perpendicular bisector of a line-segment is the locus of all points equidistant from its end-points.*

4. *If two points of a line are each equidistant from two points of another line the lines are perpendicular.*

5. *The shortest distance from a point to a line is the perpendicular from the point to the line.*

6. *Oblique lines drawn to a line from a point on a perpendicular to the line and making equal angles with the perpendicular are equal.*

7. *Oblique lines drawn from a point on a perpendicular to a line and making equal angles with the line are equal.*

CHAPTER XI

CONSTRUCTIONS. SYMMETRY. CIRCLE

The Fundamental Constructions

296. In the preceding chapters several construction problems were taught without proof of their correctness. These constructions will now be summarized and proofs will be given.

EXERCISES

1. *To bisect an angle.* (See § 127, exercise 12.)

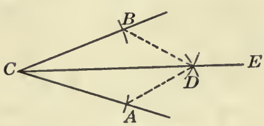


Fig. 203

Make the construction indicated in Fig. 203.

To prove this construction, draw helping lines AD and BD.

Prove $\triangle BDC \cong \triangle ADC$. (s.s.s.)

Then $\angle DCB = \angle DCA$. Why?

2. *At a given point on a given line to construct a perpendicular to the line.* (See § 127, exercise 3; § 128, exercise 3.)

Make the construction indicated in Fig. 204.

According to this construction F and C are each equidistant from D and E.

Hence, FC is the perpendicular bisector of DE. (Why?) It follows that $FC \perp AB$ at C.

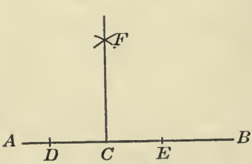


Fig. 204

3. Show that *only one perpendicular can be drawn to a line at a given point on the line.* (See § 176.)

4. *To bisect a given line-segment.*

With A as center and a convenient radius draw arcs as at C and D, Fig. 205.

With B as center and the same radius draw arcs intersecting the first two arcs.

Draw CD intersecting AB at E. Then $AE = EB$.

For, according to the construction C and D are each equidistant from A and B, and hence CD is the perpendicular bisector of AB. Why?

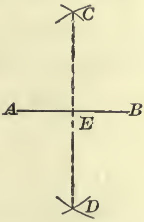


FIG. 205

5. *To construct the perpendicular bisector of a line-segment.*

Construction and proof are the same as in exercise 4.

6. *From a point outside of a line to draw a perpendicular to the line.*

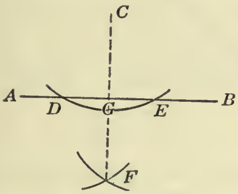


FIG. 206

The directions for this construction are the same as in exercise 2.

Proof: Since C and F, Fig. 206, are equidistant from D and E, by construction, CF is the perpendicular bisector of DE. Why?

Therefore $CG \perp AB$.

7. *Only one perpendicular can be drawn from a point to a line.*

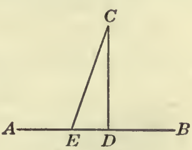


FIG. 207

For, if CD and CE, Fig. 207, were both perpendicular to AB, two angles of $\triangle EDC$ would be right angles.

This is impossible, since the sum of the three angles of triangle EDC would then be greater than 180.

8. *At a given point on a given line to draw a line making an angle with the given line equal to a given angle.** (See § 125.)

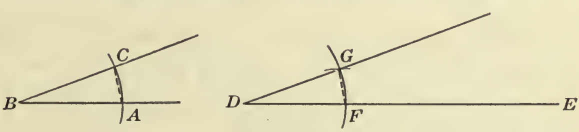


FIG. 208

Draw the helping lines CA and GF, Fig. 208.
Prove $\triangle CBA \cong \triangle GDF$. (s.s.s.)
Hence, $\angle ABC = \angle FDG$.

* This construction was first made by Oenopides of Chios (500–430 B.C.). See Ball, p. 30.

297. The fundamental constructions in § 296 are now to be used in more difficult constructions. Let it be understood that the unmarked ruler and compass* are the only instruments to be used in these problems.

Applications of the Fundamental Constructions

EXERCISES

1. Through a given point outside of a given line to draw a line parallel to the given line. (§ 194, exercise 3.)

2. Construct a triangle having given two sides and the included angle. (§ 273, exercise 1.)

3. Construct a triangle having given two angles and the side between their vertices. (§ 274, exercise 1.)

4. Construct a triangle having given the three sides. (§ 282, exercise 1.)

5. Construct an angle equal to 60°.

Proceed as in the construction of an equilateral triangle; show that one of the angles of the triangle is 60°.

6. Construct the following angles: 30°, 15°, 120°.

7. Construct angles of 90°, 45°, 22°30′.

8. Construct angles of 135°, 75°165°.

9. Construct a triangle having given two sides and the angle opposite one of them.

Let m and n, Fig. 209, be the given sides and let $\angle K$ be the given angle.

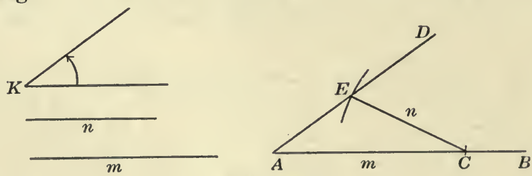


FIG. 209

* The Greek philosopher Plato (429–384 B.C.) taught geometry as a basis for the study of philosophy. As a convenient means of defining the field of elementary geometry he decided that the unmarked straightedge and the compass should be the only instruments to be used in a construction. See Ball, p. 43, or Cajori, p. 31.

Construction: Draw a line of convenient length, as AB. Lay off on AB a segment equal to m.

At A on AB construct an angle equal to $\angle K$.

With radius equal to n and center at C draw an arc meeting AD at E.

Draw CE.

$\triangle ACE$ is the required triangle.

Discussion: In this construction problem a solution is not always possible.

For, if n is sufficiently small, the arc and line AD do not intersect. Therefore no triangle exists containing the given parts. This will always be the case when n is less than the perpendicular CF, Fig. 210, from C to AD.

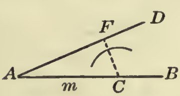


FIG. 210

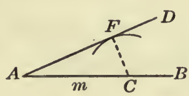


FIG. 211

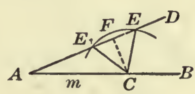


FIG. 212

When $n=CF$, Fig. 211, the arc will just touch AD at F and $\triangle ACF$ is the required triangle.

When $n>CF$ and $n<m$, Fig. 212, show that the arc intersects AD in two points, as E and E_1 and that the two triangles ACE and ACE_1 satisfy the conditions of the problem.

When $n=m$, Fig. 213, and when $n>m$, Fig. 214, show that there will be only one solution of the problem, i.e., $\triangle ACE$.

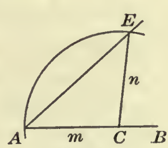


FIG. 213

298. The constructions in exercise 9 in which only one triangle is possible, i.e., those of Figs. 211, 213, and 214, illustrate the following theorems of congruence:

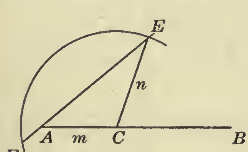


FIG. 214

1. *Two right triangles are congruent if they have the hypotenuse and one of the other sides equal respectively.* (Fig. 211.)

2. *Two isosceles triangles are congruent if the equal sides and the base angle of one are respectively equal to the corresponding parts of the other.* (Fig. 213.)

3. *Two triangles are congruent if two sides and the angle opposite the greater side are respectively equal.* (Fig. 214.)

EXERCISES

1. To divide a right angle into three equal parts.*

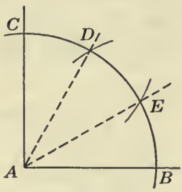


FIG. 215

With A as center, Fig. 215, and a convenient radius draw an indefinite arc intersecting the sides of the right angle at B and C.

With B as center and the same radius draw an arc at D.

With D as center and radius equal to CD draw an arc at E.

Lines AD and AE trisect the angle BAC. For, $\angle BAD = 60°$. Why? Therefore $\angle DAC = 30°$. Why? Show that $\angle DAE = 30°$.

2. To construct a right triangle having given the hypotenuse and one of the other sides.

3. To construct a right triangle having given the hypotenuse and one acute angle.

* The problem of dividing a right angle into three equal parts was solved very early. However, the *general* problem, i.e., to trisect *any* angle by use of unmarked straightedge and compass, presented great difficulties. It became one of the famous problems of geometry and much intellectual energy has been expended on these problems. However, this has not been wasted, as many discoveries were made by mathematicians in the attempt to obtain a solution. It has been proved that the trisection of the general angle with ruler and compass is impossible. (See Cajori's *History of Mathematics*, p. 24.)

4. Construct an isosceles triangle having given one of the base angles and the altitude.

5. Construct a right triangle whose acute angles are 60° and 30°.

Draw an equilateral triangle and divide it into two right triangles by means of the altitude.

How does the length of the hypotenuse in this right triangle compare with that of the side opposite the 30° angle?

6. Construct an equilateral triangle having given the altitude.

Symmetry

299. Symmetry. If one hand is held in front of a plane mirror, the image obtained in the mirror is of the same size and shape as the other hand. The hand and the image are said to be **symmetric** with respect to the plane of the mirror.

Other illustrations of symmetry are: a pair of gloves, the two doors of an automobile opposite to each other, the two posts at an entrance to a park, a printed page and the type-page from which it was made, etc.

300. Symmetry of a single body or figure. The human head is symmetric with respect to a plane passing midway between the eyes. The cube is symmetric with respect to several planes; indicate the position of some of these planes of symmetry. How many planes of symmetry can be passed through a sphere? Give other examples of symmetric solids.

301. Symmetry of plane figures. If a figure is drawn in ink on paper and if the paper is folded before the ink is dry, an image of the figure is obtained. The image and the original figure are symmetric.

EXERCISES

1. Construct a figure symmetric to line AB, Fig. 216, with respect to the line CD.

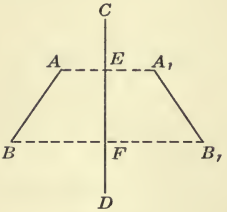


FIG. 216

From A draw $AA_1 \perp CD$, making $AE = EA_1$.

Similarly draw $BB_1 \perp CD$, making $BF = FB_1$.

Draw A_1B_1.

Then A_1B_1 and AB are symmetric with respect to CD.

2. Construct a triangle symmetric to triangle ABC, Fig. 217, with respect to the line DE.

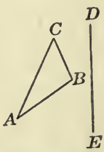


FIG. 217

As in exercise 1 draw lines symmetric to AB, BC, and CA.

3. Construct a figure symmetric to $ABCDEFGH$, Fig. 218.

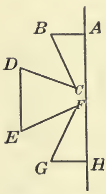


FIG. 218

302. Axis of symmetry. Two figures are symmetric with respect to a line if the line is the perpendicular bisector of all line-segments joining the corresponding parts of the figures. The line is called the **axis of symmetry.**

EXERCISES

1. Show that two plane figures are congruent if they are symmetric with respect to an axis of symmetry.

Use superposition to show that every point of one figure can be made to coincide with the corresponding point of the other.

2. Draw the axis of symmetry of an isosceles triangle.

3. Show that the bisector of the vertex angle of an isosceles triangle is the axis of symmetry. Show how some of the properties of an isosceles triangle stated in § 293 follow from the symmetry of the figure.

4. How many axes of symmetry has an equilateral triangle? A square? A rectangle?

5. Show how the principle of axial symmetry stated in exercise 1 may be used in designing, pattern-making, printing, building, tailoring, etc.

303. Some geometric facts are easily established from the symmetry of the figures. Show that the following are true:

1. Any point on the perpendicular bisector of a line-segment is equidistant from the end-points.

Let CD, Fig. 219, be the perpendicular bisector of AB.

Since CD is the axis of symmetry of AB, $\angle AEC$ can be made to coincide with $\angle BEC$.

Hence AC coincides with BC.

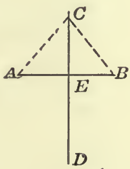


FIG. 219

2. Any point not on the perpendicular bisector of a line-segment is not equally distant from the end-points.

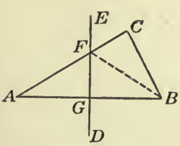


FIG. 220

Let ED, Fig. 220, be the perpendicular bisector of AB.

Rotate AGF about ED as an axis. Then AFC takes the form of the broken line BFC, and $AC = AF + FC = BF + FC > BC$. Why?

3. Any point on the bisector of an angle is equidistant from the sides.

Let CE be the bisector of $\angle GCF$, Fig. 221.

Let $AD \perp CF$ and $AB \perp CG$.

The bisector CE is the axis of symmetry of the angle, since FC can be made to coincide with GC by revolving $\angle GCA$ about CA as an axis.

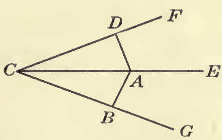


FIG. 221

Then AB must coincide with AD, since only one perpendicular can be drawn from a point to a line.

4. Any point not on the bisector of an angle is not equidistant from the sides of the angle.

Let CE be the bisector of $\angle BCD$, Fig. 222.

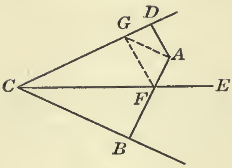


Fig. 222

Let $AD \perp CD$ and $AB \perp CB$.

Rotating $\angle FCB$ about CE as an axis, line AFB takes the form of the broken line AFG.

But $AF+FG>AG$. Why?

and $AG>AD$. Why?

Therefore $AF+FG>AD$. Why?

304. The last two problems combined express the following theorem:

Theorem: *The bisector of an angle is the locus of points within the angle which are equidistant from the sides.*

The rectangle $CDEF$, Fig. 223, represents the top of a billiard table. Find the point P on the cushion FE to which a ball at A, cued at the center, must be directed to strike a ball at B.

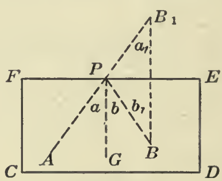


Fig. 223

Locate a point B_1, symmetric to B with respect to FE. The point of intersection P of AB_1 with FE is the required point.

For, it is learned from physics that if a perpendicular is drawn to FE at P and if the angle of incidence, a, equals the angle of reflection, b, a ball directed from A striking at P will rebound and strike B.

To prove that $a=b$, show that $a=a_1$, $b=b_1$, and $a_1=b_1$.

The Circle

305. The circle and circular arc have been used in the construction of many figures. Before taking up the following constructions of this chapter it is necessary to become acquainted with some of the properties of the circle.

306. Tangent. A line that touches a circle in only one point, however far produced, is a **tangent**.

Line AB, Fig. 224, is tangent to circle D at C.

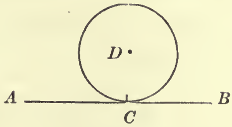


FIG. 224

307. Contact point. The point common to a circle and a tangent is called the **point of tangency**, or **point of contact**.

308. Theorem: *The radius drawn to the point of contact of a tangent is perpendicular to the tangent.*

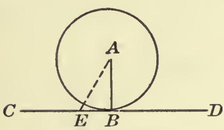


FIG. 225

For, any point E of the tangent, not the contact point, Fig. 225, lies outside of the circle.

Hence, line AE must be longer than a radius, and $AE > AB$.

Thus, AB is the shortest of all lines drawn from the center A to CD. It is therefore perpendicular to CD.

309. Theorem: *A line perpendicular to a radius at the outer end-point is tangent to the circle.*

For, if AB, Fig. 225, is perpendicular to CD, then it is shorter than any other line, as AE, drawn from A to CD.

Hence, AE is longer than a radius and E must be outside of the circle.

Since E is any point on CD, not B, it follows that B is the only point on CD and on the circle, and that CD is tangent to the circle.

310. Regular polygon. A polygon that is equilateral and equiangular is a **regular polygon**.

311. Inscribed polygon. A polygon whose vertices lie on a circle is an **inscribed polygon**. The circle is *circumscribed about* the polygon.

312. Circumscribed polygon. A polygon whose sides are tangent to a circle is a **circumscribed polygon.** The circle is *inscribed in* the polygon.

EXERCISES

1. At a given point on a circle to construct the tangent to the circle.

Draw the radius to the given point and erect a perpendicular to it at that point.

2. To find the center of a given circle.

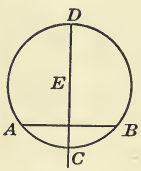


FIG. 226

Draw a chord, as AB, Fig. 226.

Draw the perpendicular bisector CD of the chord AB.

CD must pass through the center. Why?

Bisect CD at E.

E is the center of the circle.

3. Draw a circle passing through two given points.

How many circles can be drawn through those points?

4. Circumscribe a circle about a triangle.

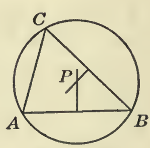


FIG. 227

Let ABC be the given triangle, Fig. 227.

Draw the perpendicular bisectors of two sides, as AB and CD. Both perpendiculars must contain the center of the circle. Why?

Hence, they must intersect at the center of the circumscribed circle.

Let P be the point of intersection. Since P is equally distant from A and B and from C and B, it must be equally distant from A and C.

Hence, a circle with P as center and radius PC must pass through A, B, and C.

5. Show that the perpendicular bisector of AC passes through P, Fig. 227.

6. To inscribe a circle in a triangle.

Let ABC, Fig. 228, be the given triangle.

Draw the bisectors of two angles, as A and B.

With their point of intersection, P as center and a perpendicular from P to AB as radius draw a circle.

This is the required circle.

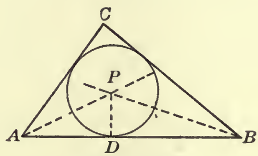


Fig. 228

7. Prove that the circle constructed in exercise 6 is tangent to the sides of triangle ABC.

Show that P is equally distant from BA and BC.

Show that P is equally distant from AB and AC.

Then P is equally distant from CA and CB.

Hence, a circle with P as center and radius PD will be tangent to the sides of $\triangle ABC$. (§ 309.)

8. Show that the bisector of angle C, Fig. 228, passes through P.

9. To inscribe a square in a circle.

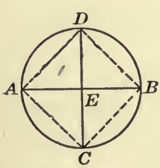


Fig. 229

Draw two diameters perpendicular to each other, as AB and CD, Fig. 229.

Join the successive points A, C, B, and D.

The quadrilateral $ACBD$ is the required square.

10. Prove that $ACBD$ constructed in Fig. 229 is a square.

Prove ▵ AED, DEB, BEC, AEC congruent. (s.a.s.)

Hence, $AD = DB = BC = CA$. Why?

Prove that ∡ EAD, ADE, EDB, DBE, etc., are 45° angles.

Prove that ∡ DAC, ADB, DBC, BCA are right angles.

11. To circumscribe a square about a circle.

Draw tangents at A, C, B, and D, Fig. 229. The quadrilateral formed is the required square.

12. To inscribe a regular hexagon in a circle.

Let circle A, Fig. 230, be the given circle.

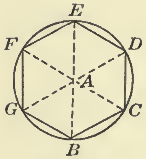


Fig. 230

With any point on the circle, as B, as center and radius equal to AB draw an arc at C. With C as center and the same radius draw an arc at D.

Similarly draw arcs at E, F, and G.

Join the consecutive points B, C, D, etc.

The polygon $BCDEFG$ is the required hexagon.

13. Prove that the hexagon constructed in exercise 12 is regular.

Prove $\triangle BAC$, CAD......GAB equilateral.

Hence, $BC = CD = DE$, etc.

Prove that $\angle BCD$, CDE, etc., are equal to 120° and therefore equal to each other.

Summary

313. This chapter has taught the meaning of the following terms: symmetry, plane of symmetry, axis of symmetry; tangent to a circle, contact point; regular polygon; inscribed and circumscribed polygons.

314. Proofs were given for the fundamental constructions taught in the preceding chapters.

315. The fundamental constructions were applied in a number of more complicated construction problems.

316. Symmetry of figures suggests the solution of problems of construction and proofs of theorems.

317. The following locus theorem has been proved:

The locus of points within an angle which are equally distant from the sides is the bisector of the angle.

318. The following theorems were proved:

1. *A line perpendicular to a radius at the outer end-point is tangent to the circle.*

2. *The radius drawn to the point of contact of a tangent is perpendicular to the tangent.*

319. The following problems of construction were taught:

1. To find the center of a given circle.
2. To inscribe a circle in a triangle.
3. To circumscribe a circle about a triangle.
4. To inscribe a square in a circle.
5. To inscribe a regular hexagon in a circle.

CHAPTER XII

POSITIVE AND NEGATIVE NUMBERS. THE LAWS OF SIGNS

Uses of Positive and Negative Numbers

320. In the preceding chapters problems in geometry have been solved by geometric and by algebraic methods. It is the aim of this and the following chapters to develop skill in adding, subtracting, multiplying, and dividing algebraic numbers, to learn more about the solution of equations, and to make use of algebra, not only in the solution of geometric problems, but also in solving problems arising outside of the field of geometry.

Two trains A and B leave Columbus, Ohio, at 8:00 A.M. Two hours later A is 60 mi. and B is 90 mi. from Columbus. How far is B from A? Show from Figs. 231 and 232 that there are two answers to this question, according as the trains travel in the same or the opposite direction.

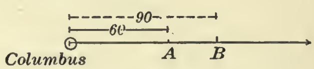


FIG. 231

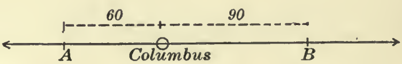


FIG. 232

321. Notation for directed line-segments. It is customary to distinguish line-segments when measured in opposite directions by the *positive* or *plus* (+) sign and the *negative* or *minus* (−) sign, the plus sign indicating direction to the right (or upward), the minus sign indicating direction to the left (or downward). Moreover, the plus or minus sign is prefixed to the number expressing the measure (length) of a directed segment. Thus, in Fig. 231, $OA = +60$, $OB = +90$. But in Fig. 232 $OA = -60$ and $OB = +90$.

322. Number-scale. With the agreement of § 321 it is possible to express both length and direction of line-segments by numbers. In Fig. 233 these numbers are represented geometrically on a straight line with reference to a point O in that line.

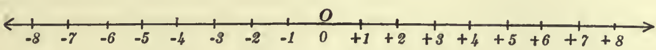


FIG. 233

This arrangement is called the **number-scale**, or the *algebraic scale.*

323. Positive and negative numbers. A number preceded by a plus sign is a **positive number**. A number preceded by a minus sign is a **negative number**. The plus sign need not always be written. Thus, when no sign is prefixed to a number it is understood to be a positive number. The negative sign is never omitted.

324. Absolute value. The value of a number regardless of sign is called the **absolute value**, or the **numerical value**. Thus, the absolute value of $+4$ is 4, of -7 is 7.

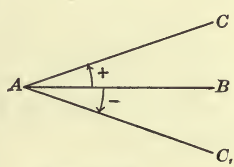


FIG. 234

325. Positive and negative angles. By rotating line AB, Fig. 234, around A until it takes the position AC, the angle BAC is formed. By rotating AB in the opposite direction, angle BAC_1 is formed. To distinguish between these directions one angle may be denoted by the *plus* sign, whereupon the other is denoted by the *minus* sign. It is customary to consider an angle **positive** when it is formed by rotating a line counter-clockwise, and **negative** when it is formed by clockwise rotation.*

EXERCISES

1. Make drawings on the following angles, always starting from a line in horizontal position, as AB, Fig. 234: $+90°$, $+45°$, $+160°$, $+270°$, $-45°$, $-30°$, $-180°$, $-270°$, $-360°$.

2. A line rotates through an angle of $+75°$ and then through an angle of $-40°$. Give the size and direction of the angle formed by the first and the last positions of the line.

326. Positive and negative temperature. Temperature readings are taken with reference to a definite point on the thermometer, called the zero point. A temperature reading above the zero point is plus; below, minus.

Give the meaning of the following temperature readings: $-4, -2, \ 0, +3, +8, +8, +4, +3, +1, \ 0, -1, -2$.

EXERCISES

In arithmetic the $+$ sign always denotes addition, the $-$ sign, subtraction.

The following problems illustrate some of the uses of positive and negative numbers in algebra.

1. If a man's debts be indicated by the minus sign and his possessions by the plus sign, what is the condition of a man's affairs if his debts and possessions are given by $+\$1,200$ and $-\$1,000$? By $+\$73$ and $-\$50$? $-\$75$ and $+\$60$? $-\$300$ and $+\$1,000$?

* Euler (1707–1783) was the first to apply positive and negative signs to angles. This he did in a book called *Introductio*, published in 1748. Gauss (1777–1855) completed the modern science of positive and negative angles.

2. A bicyclist starts from a point and rides 18 mi. due northward (+18 mi.), then 10 mi. due southward (−10 mi.). How far is he from the starting-point?

3. How far and in what direction from the starting-point is a traveler who goes eastward (+) or westward (−) as shown by these pairs of numbers: +16 mi. then −6 mi.? −20 mi. then +28 mi.? −18 mi. then +18 mi.? +100 mi. then +50 miles?

4. Denoting latitude north of the equator by the plus sign and latitude south by the minus sign, give the meaning of the following latitudes: +28°, +2°, −18°, +12°, −10°.

5. A boy starts with no money. If he earns 50 cents (+50 cents) and spend 40 cents (−40 cents), how much money has he then?

6. What does the minus sign denote if the plus sign denotes above? forward? upward? to the right? east? north? gain? possession? income ? addition? increase?

7. The statistics below give the value in millions of dollars of the excess of merchandise imported into or exported from the United States to the Philippine Islands for the years 1892–1912. Denoting an excess of exports by +, of imports by −, read the following: −6.2, −9, −6.8, −4.6, −4.8, −4.3, −3.7, −4, −3.3, −.4, −1.3, −7.3, −7.2, −6.4, −6.9, −2.8, +1.3, +1.7, −.5, +2.3, +.5.

Graphing Data

327. Positive and negative quantities may be represented graphically.

EXERCISES

1. On a winter day the thermometer was read at 6:00 A.M. and every hour afterward until 5:00 P.M. The hourly readings were −10°, −8°, −7°, −5°, 0°, +2°, +8°, +10°, +10°,

+5°, 0°, −5°. Mark off these readings on squared paper (Fig. 235). Connect the points thus obtained as shown in the figure. This forms a broken line, called a temperature-line.

Time	Temperature
6:00 A.M.	−10°
7:00	− 8°
8:00	− 7°
9:00	− 5°
10:00	0°
11:00	+ 2°
12:00	+ 8°
1:00 P.M.	+10°
2:00	+10°
3:00	+ 5°
4:00	0°
5:00	− 5°

From the temperature-line one may obtain some information regarding the changes of the temperature: When was it coldest? Warmest? When was the change most rapid?

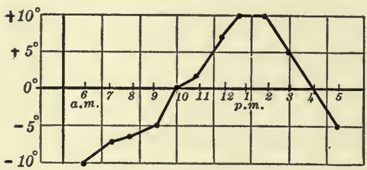


FIG. 235

When was the change a rise? etc. A more accurate picture of the changes in temperature could be obtained by taking the readings oftener than every hour.

2. On squared paper draw a line to show the following hourly readings, beginning at 8:00 A.M.: +2°, −2°, −4°, −2°, +2°, +4°, +4°, +8°, +10°.

3. The average monthly temperatures for a northern town are

Jan.	− 4°	May	+42°	Sept.	+48°
Feb.	− 7°	June	+52°	Oct.	+37°
Mar.	+14°	July	+62°	Nov.	+25°
Apr.	+26°	Aug.	+60°	Dec.	+ 2°

Draw the temperature-line.

4. The daily average temperatures for 14 days at a certain place were +8°, 0°, −10°, +12°, −6°, +14°, +15°, +2°, −5°, +15°, +20°, 0°, 0°, +10°. Graph these readings.

5. A ship's latitude from week to week was +42°, +38°, +30°, +20°, +12°, +2°, −1°, −6°, −3°, +12°. Graph these latitudes and tell when the ship crossed the equator.

6. The following table gives the lowest, highest, and average temperature for Chicago taken for 38 years previous to December 31, 1911.

Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
−20	−21	−12	17	27	40	50	49	32	14	− 2	−23
64	63	81	88	94	98	103	98	98	87	75	68
24	25	34	46	56	66	72	71	65	53	39	29

Graph the three temperature-lines on the same sheet of squared paper.

7. The following numbers indicate in millions of dollars the value of the excess of exports from the United States or imports into the United States from Austria-Hungary for the years 1892–1912. An excess of exports is positive, an excess of imports, negative:

$-6.19, -9.48, -6.37, -4.38, -5.20, -4.13, +.98$
$+.83, -2.03, -2.84, -3.98, -3.41, -2.14, +1.07,$
$+1.02, -.87, +.75, -1.21, -2.45, +2.56, +5.67.$

Make the graph.

Historical Note.—The earliest instances of the regular use of the signs + and − occur in the fifteenth century. John Widmann, of Eger, born about 1460, and probably a physician, wrote a *Mercantile Arithmetic.* In this book these signs are used merely as marks signifying excess or deficiency.* The French mathematician Vièta (1540–1603) seems to have been the first to make regular use of the + and − signs as shorthand symbols for addition and subtraction.† Descartes (1596–1650) showed how to represent positive and negative numbers along a line.

* See Ball, p. 206.

† See Cajori, p. 150.

Addition of Positive and Negative Numbers

328. Graphical addition. Addition of positive and negative numbers may be performed graphically.

EXERCISES

1. Find graphically the sum of $(+5)$ and (-2).

Let $(+5)$ be a distance traveled eastward and (-2) a distance traveled westward. Find the distance and direction of the stopping-point from the starting-point of an automobile going first $(+5)$ mi., then (-2) miles. This distance is considered the sum of the distances passed over.

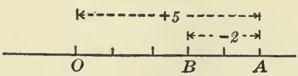


Fig. 236

Starting from O, Fig. 236, lay off five units to the right to point A. From A lay off two units to the left to point B. The position of B is three units to the right of O, i.e., $(+3)$. This may be expressed in symbols by the equation $(+5)+(-2)=(+3)$.

2. Find graphically the following sums: $(+6)+(+2)$; $(-6)+(+2)$; $(+6)+(-2)$; $(-6)+(-2)$; $(+15)+(-10)$; $(+15)+(-20)$; $(-18)+(+24)$; $(-12)+(-9)$.

329. Positive and negative numbers may be added by letting the $+$ and $-$ signs denote two opposite senses, as gain and loss, income and expenditure, etc.

EXERCISES

1. Find the sum of $(+25)$ and (-31).

Let $(+25)$ denote a gain of \$25 and (-31) a loss of \$31; the final standing, a loss of \$6, is the sum. Thus,

$$(+25)+(-31)=(-6)$$

2. In a way similar to that of exercise 1 find the following sums:

$+15$	-15	$+15$	-15	$+38$	-38	$+38$	-38
$+\ 8$	$-\ 8$	$-\ 8$	$+\ 8$	$+19$	-19	-19	-19

330. Algebraic addition. Show from examination of the results of exercise 2 that positive and negative numbers may be added according to the following laws:

1. *To add two numbers having like signs, find the sum of their absolute values and prefix to this sum the common sign.*

2. *To add a positive and a negative number find the difference of their absolute values and prefix to it the sign of the number having the greater absolute value.*

EXERCISES

Solve the following exercises with the aid of positive and negative numbers. In finding the sums use the laws of § 330. Then verify the results by the method of § 329.

1. At noon a thermometer read 3° below 0°. In the evening it was 8° warmer. How many degress did the thermometer read in the evening?

2. A boatman rows, at a rate that would carry him 3 mi. an hour through still water, down a river whose current is 2 mi. an hour. What is his rate per hour? What is his rate per hour if he rows up the river?

3. A motor boat, having a speed that would make it go 12 mi. an hour in still water, is going down a river whose current is 2.4 mi. an hour. How fast does the boat move? How fast can the boat move upstream?

4. If a man's property is worth $3,600 and his debts amount to $1,400, what is his financial standing?

5. A toy balloon pulls upward with a force of 9 oz. If a weight of 6 oz. is attached, will the balloon rise or fall? With what force?

6. A balloon pulls upward on a stone, weighing 6 oz., with a force of 8 oz. What is the sum of the forces?

331. Addition of three or more numbers. The following problems call for the addition of three or more positive and negative numbers.

EXERCISES

1. John received \$4 from his father and \$3.50 from his mother. He paid \$3 to the grocer and \$3.75 to the hardware man. How much money had he left?

$$(+4)+(+3.50)+(-3)+(-3.75)=+.75$$

2. Translate the following into problems like exercise 1 and solve each:

$$(+4)+(3.50)+(-3.75)+(-3)$$
$$(+4)+(-3)+(3.50)+(-3.75)$$
$$(+3.50)+(-3)+(+4)+(-3.75)$$

3. How do the results in exercises 1 and 2 compare?

4. Interchange in every possible way the addends in $(+8)+(-6)+(+4)$ and in each case find the value of the sum.

332. Commutative law. Exercises 2, 3, and 4 show that the commutative law holds for positive and negative numbers; i.e., *The value of a sum of positive and negative numbers is the same whatever the order of the addends.*

EXERCISES

1. An elevator starts at a certain floor, goes up 65 ft., down 91 ft., up 52 ft., down 13 ft., up 65 ft., and stops. How far and in what direction is the stopping-point from the starting-point?

2. A vessel starting in latitude $+20°$ sails $+13°$ in latitude, then $-60°$, then $+40°$, then $-10°$. What is its latitude after the sailings?

3. What is the latitude of a ship starting in latitude $-50°$ after these changes of latitude: $+10°$, $-5°$, $+18°$, $-7°$, $+38°$, $-12°$, $+60°$?

4. What is the most advantageous way of adding several positive and negative numbers?

5. A certain wholesale house owes various factories the following amounts: \$475.50, \$240.00, \$638.50. Several merchants owe the wholesale firm these sums: \$360.20, \$159.45, \$520.70. The firm has on hand \$1,254.00. What is its financial standing?

6. Find the following sums:

$+50$	$+35$	-45	$+75$	-236	$+8x$	$-14a$
$+25$	-38	-20	$+13$	$+780$	$-6x$	$-46a$
-18	$+24$	$+60$	-86	-95	$-4x$	$+77a$
-6	-15	$+55$	$+8$	$+45$	$+7x$	$-5a$

7. Show that the sum of two numbers having unlike signs but the same absolute value is zero.

8. Find the value of $(-3x)+(-2x)+(+6x)+(+10x)+(-5x)$ and test the result by substituting $x=2$ in the given sum and in the result.

9. The heat of a metal was increased by 20.4°. The metal was then cooled 4° and finally heated 2° more. What is the change in the original temperature?

10. Add the following, doing all you can orally:

1. $+5$ -3	5. -7 $+10$	9. $+6$ $+10$
2. $+8$ -5	6. -2 $+5$	10. -8 -4
3. $+7$ -10	7. -16 $+4$	11. $+3$ $+10$
4. $+6$ -8	8. -3 $+1$	12. -7 -2

13. $\begin{array}{r}-8a\\+4a\\\hline\end{array}$

14. $\begin{array}{r}+\ 7x\\-10x\\\hline\end{array}$

15. $\begin{array}{r}-12m\\+16m\\\hline\end{array}$

16. $\begin{array}{r}+24n\\-\ 6n\\\hline\end{array}$

17. $\begin{array}{r}+\frac{3}{4}\\+\frac{5}{8}\\\hline\end{array}$

18. $\begin{array}{r}-\frac{7}{8}\\+\frac{3}{4}\\\hline\end{array}$

19. $\begin{array}{r}+\frac{11}{16}\\-\frac{5}{6}\\\hline\end{array}$

20. $\begin{array}{r}+3\frac{1}{2}\\-2\frac{1}{3}\\\hline\end{array}$

21. $\begin{array}{r}-5\frac{1}{4}\\-2\frac{3}{8}\\\hline\end{array}$

22. $\begin{array}{r}+4\frac{3}{4}\\-6\frac{1}{8}\\\hline\end{array}$

23. $\begin{array}{r}-6\frac{2}{3}\\+8\frac{5}{6}\\\hline\end{array}$

24. $\begin{array}{r}+7\frac{3}{5}\\-7\frac{4}{5}\\\hline\end{array}$

25. $\begin{array}{r}-12\frac{1}{2}\\-\ 2\frac{1}{2}\\\hline\end{array}$

26. $\begin{array}{r}-18\frac{1}{3}\\+26\frac{1}{4}\\\hline\end{array}$

27. $\begin{array}{r}-2.12\\-1.88\\\hline\end{array}$

28. $\begin{array}{r}+3.16\\-4.08\\\hline\end{array}$

29. $\begin{array}{r}-13\frac{1}{2}\\+23\frac{3}{8}\\\hline\end{array}$

30. $\begin{array}{r}-6.69\\+8.04\\\hline\end{array}$

31. $\begin{array}{r}+\ 8.95\\-11.25\\\hline\end{array}$

32. $\begin{array}{r}-16r\\+18r\\\hline\end{array}$

33. $\begin{array}{r}-3.2s\\-6.8s\\\hline\end{array}$

34. $\begin{array}{r}+7\frac{2}{3}x\\-6\frac{5}{6}x\\\hline\end{array}$

35. $\begin{array}{r}+32b^2c\\-28b^2c\\\hline\end{array}$

36. $\begin{array}{r}+18v^2y^3\\-24v^2y^3\\\hline\end{array}$

37. $\begin{array}{r}+\ 8ax\\-\ 6ax\\\hline\end{array}$

38. $\begin{array}{r}-12x^2y\\-\ 7x^2y\\\hline\end{array}$

39. $\begin{array}{r}-68mr^3\\-75mr^3\\\hline\end{array}$

40. $\begin{array}{r}-6\frac{1}{2}cd^2\\+3\frac{2}{3}cd^2\\\hline\end{array}$

41. $\begin{array}{r}+4.5x^2yz\\-8x^2yz\\\hline\end{array}$

42. $\begin{array}{r}+2.4ab^2c\\-6.2ab^2c\\\hline\end{array}$

Subtraction of Positive and Negative Numbers

333. Graphical subtraction. Positive and negative numbers may be subtracted graphically.

EXERCISES

1. Subtract $(+b)$ from $(+a)$.

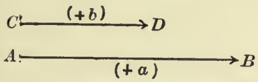


FIG. 237

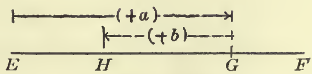


FIG. 238

Let AB, Fig. 237, represent in magnitude and direction the number $(+a)$ and let CD represent $(+b)$. To subtract graphically $(+b)$ from $(+a)$ means to lay off the minuend $(+a)$ on an indefinite line, as EF, Fig. 238, in its *own* direction, i.e., to point G; and to lay off the subtrahend $(+b)$ from G in the direction *opposite* to that of $(+b)$, to point H.

EH is the difference between $(+a)$ and $(+b)$.

Thus, $EH = (+a) - (+b)$.

2. Show that EH, Fig. 238, may be constructed by adding $(+a)$ and $(-b)$, i.e., that $(+a) - (+b) = (+a) + (-b)$.

3. Show graphically that the following expressions are equal: $(+5) - (+3)$ and $(+5) + (-3)$; $(+8) - (+10)$ and $(+8) + (-10)$; $(+6) - (+7)$ and $(+6) + (-7)$

4. Subtract $(-b)$ from $(+a)$.

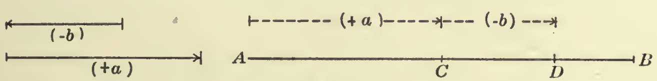


FIG. 239

On AB, Fig. 239, lay off the minuend $(+a)$ in its own direction to C. From C lay off the subtrahend $(-b)$ in the direction *opposite* to that of $(-b)$ to point D.

AD is the difference between $(+a)$ and $(-b)$.

Thus, $AD = (+a) - (-b)$.

5. Show that AC, Fig. 239, may be constructed by adding $(+a)$ and $(+b)$, i.e., that $(+a) - (-b) = (+a) + (+b)$.

6. Show graphically that the following expressions are equal: $(+6) - (-3)$ and $(+6) + (+3)$; $(+4) - (-5)$ and $(+4) + (+5)$; $(+8) - (-8)$ and $(+8) + (+8)$; $(+7) - (-4)$ and $(+7) + (+4)$

7. Show graphically that $(-4)-(+6)=(-4)+(-6)$; $(-8)-(-4)=(-8)+(+4)$; $(-7)-(+2)=(-7)+(-2)$; $(-3)-(-4)=(-3)+(+4)$; $(-a)-(+b)=(-a)+(-b)$; $(-a)-(-b)=(-a)+(+b)$.

334. Algebraic subtraction. The exercises of § 333 show that numbers may be subtracted according to the following law:

To subtract a number change the sign of the subtrahend and add the result to the minuend.

Arithmetic numbers may be subtracted by this law by considering them as positive numbers. Thus, $7-5=(+7)-(+5)=(+7)+(-5)=2$. In arithmetic a number can be subtracted only from a larger number. In algebra subtraction is always possible, e.g.,

$$7-10=(+7)-(+10)=(+7)+(-10)=-3.$$

EXERCISES

Subtract the lower from the upper number of the following:

1. $\begin{array}{r} +19 \\ -10 \\ \hline \end{array}$

2. $\begin{array}{r} -66 \\ -25 \\ \hline \end{array}$

3. $\begin{array}{r} -75 \\ +25 \\ \hline \end{array}$

4. $\begin{array}{r} +\ 8 \\ +10 \\ \hline \end{array}$

5. $\begin{array}{r} +\frac{7}{8} \\ +\frac{3}{4} \\ \hline \end{array}$

6. $\begin{array}{r} -\frac{5}{8} \\ -\frac{1}{4} \\ \hline \end{array}$

7. $\begin{array}{r} +\frac{7}{9} \\ -\frac{2}{3} \\ \hline \end{array}$

8. $\begin{array}{r} -\frac{9}{10} \\ +\frac{2}{3} \\ \hline \end{array}$

9. $\begin{array}{r} -6v^2 \\ +\ v^2 \\ \hline \end{array}$

10. $\begin{array}{r} -13c^3 \\ +\ 8c^3 \\ \hline \end{array}$

11. $\begin{array}{r} -\ 9c \\ -15c \\ \hline \end{array}$

12. $\begin{array}{r} +7b^2 \\ -9b^2 \\ \hline \end{array}$

13. $\begin{array}{r} -8m^3 \\ +3m^3 \\ \hline \end{array}$

14. $\begin{array}{r} +a \\ -a \\ \hline \end{array}$

15. $\begin{array}{r} +6.75a^3 \\ -7.25a^3 \\ \hline \end{array}$

16. $\begin{array}{r}-3.16c^2\\-0.89c^2\\\hline\end{array}$	19. $\begin{array}{r}-4.76t\\+9.67t\\\hline\end{array}$	22. $\begin{array}{r}+\ .80m\\+1.42m\\\hline\end{array}$
17. $\begin{array}{r}+6(x+z)\\-7(x+z)\\\hline\end{array}$	20. $\begin{array}{r}+\ .82a^2\\-3.75a^2\\\hline\end{array}$	23. $\begin{array}{r}-3.26s\\+7.49s\\\hline\end{array}$
18. $\begin{array}{r}-9(c-s)\\+3(c-s)\\\hline\end{array}$	21. $\begin{array}{r}-0.75c\\-0.90c\\\hline\end{array}$	24. $\begin{array}{r}+2.03y\\-5.24y\\\hline\end{array}$

Law of Signs in Multiplication

335. Graphical multiplication. The absolute value and sign of the product of two numbers may be determined geometrically.

EXERCISES

1. Find the product of $(+4)$ by $(+3)$.

Since $(+3)\ (+4)$ is the same as $(3)\ (+4)$ it follows that $(+3)(+4)$ equals $(+4)+(+4)+(+4)=(+12)$. Geometrically this means that to multiply $(+4)$ by $(+3)$ is to lay off $(+4)$ three times in its *own* direction, Fig. 240. Thus,

$(+3)\ (+4)=(+12)$.

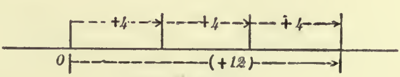


FIG. 240

2. Find the product of (-4) by $(+3)$.

Since $(+3)\ (-4)$ is the same as $(3)\ (-4)$, $(+3)\ (-4) =(-4)+(-4)+(-4)=(-12)$. Make a drawing for $(+3)(-4)$, i.e., lay off (-4) three times in its *own* direction. Thus, $(+3)\ (-4) =(-12)$.

3. Find the product of $(+4)$ by (-3).

Assuming that the commutative law holds for positive and negative numbers, i.e., that the value of a product is not changed by changing the order of the factors, it follows that $(-3)\ (+4) =(+4)\ (-3)=(-12)$, according to exercise 2. The same result is obtained if we agree to mean by $(-3)\ (+4)$ that $(+4)$ is to be laid off three times in the direction *opposite* to its own direction. Thus, $(-3)\ (+4)=-12$. Make a drawing for this product.

4. Find the product of (-4) by (-3).

According to the agreements made in exercises 1, 2, and 3, $(-3)(-4)$ will be understood to mean that (-4) is to be laid off three times in the direction *opposite* to that of (-4). Thus, $(-3)\ (-4) = (+12)$.

336. Law of signs for multiplication. Exercises 1–4, § 335, illustrate the following law for determining the sign of a product:

The product of two factors having like signs is positive.

The product of two factors having unlike signs is negative.

Find the value of the following products, (1) using the law of signs, (2) geometrically:

$(-2)\ (-3)$; $(+3)\ (-2)$; $(-2)\ (+4)$; $(+2)\ (+8)$; $(-3)\ (-5)$; $(+3)\ (-5)$; $(-2)\ (+5)$; $(-2)\ (-5)$; $(-3)\ (+6)$

337. Turning-tendency. The laws of signs may be illustrated with the bar, Fig. 241. A light bar supplied with equally spaced pegs is balanced about its middle point M. With a number of equal weights, w, the following experiments are performed:

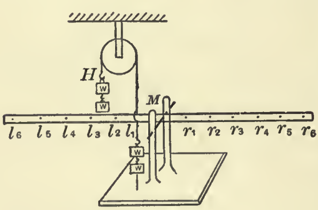


FIG. 241

1. Hang a weight of $2w$ on the peg l_1. This weight tends to turn the bar. Note the number of weights that must be attached to the hook H to balance this turning-tendency. Now hang the same weight $2w$ on peg l_2 and measure its turning-tendency by attaching to the hook, H, weights sufficient to balance the bar.

2. In a similar manner find the turning-tendency caused by the weight $2w$ on the peg l_3; on l_4; on l_5.

A large number of experiments like 1 and 2 have shown the following:

1. *The turning-tendency, or leverage, varies as the distance of the peg from the turning-point, M.*

2. *The turning-tendency is equal to the product of the weight by the distance of the peg, where the weight hangs, from the turning-point.*

The same two facts are true on the right side as on the left, but the bar turns, or tends to turn, in the opposite direction.

338. Direction of turning. The following method has been agreed upon to distinguish between the two directions of turning. When the bar turns, or tends to turn, *with* the hands of the clock it is said to turn *clockwise;* if it turns, or tends to turn, *against* the hands of the clock, it is said to turn *counter-clockwise.* Weights attached to the peg are downward-pulling weights, and are designated by the $-$ sign. Weights attached at H pull upward on the bar and are designated by the $+$ sign.

339. Lever-arm. The distance from the turning-point to the peg where the weight, or force, acts will be called the *lever-arm* or *arm* of the force. Lever-arms measured from the turning-point toward the right will be marked $+$; those toward the left, $-$.

For example, if the distance from M to peg r_1 be represented by $+1$, then the distance from M to r_4 will be $+4$; from M to l_3 by -3, and so on.

340. Multiplication of positive and negative numbers. By means of the apparatus, Fig. 241, the product of positive and negative numbers is now to be found.

EXERCISES

1. Find the product $(+4)\ (+3)$.

Whatever device may be used for finding this product, the result must agree with the result from arithmetic; i.e., it must be $(+12)$.

Let $(+4)$ represent an arm 4 pegs to the right and $(+3)$ a force pulling upward on the bar. The direction of turning is counter-clockwise. If it is agreed to call this the *plus* direction then the turning-tendency,

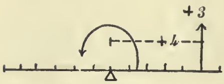


FIG. 242

$$T=(+4)(+3)=(+12),$$

agrees with the arithmetical product 4×3.

2. Find by means of the bar the following products:

$$(+2)(+4);\quad (+10)(+6);\quad (+a)(+b).$$

3. Find the product $(+4)(-3)$.

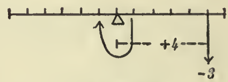


FIG. 243

This means that at the fourth peg to the right a downward force of 3 weights is attached, Fig. 243. The direction of turning is clockwise and will be called minus, since it is opposite to the positive direction.

Hence, $T=(+4)(-3)=(-12)$.

4. Show by means of the bar that $(-4)(+3)=(-12)$.

5. Show by means of the bar that $(-4)(-3)=(+12)$.

6. Compare the result of exercises 1, 3, 4, and 5 with the sign law (§ 336).

7. Using the sign law find the products of the following, doing all you can mentally.

1. $(+5)(+3)$	13. $(-\frac{2}{3})(+\frac{5}{6})$	25. $(-c)(-pr)$
2. $(-5)(+3)$	14. $(+\frac{3}{4})(+\frac{5}{9})$	26. $(-3\frac{4}{5})(+xy)$
3. $(-5)(-3)$	15. $(-\frac{7}{8})(-\frac{3}{14})$	27. $(-5\frac{1}{6})(-xz)$
4. $(+5)(-3)$	16. $(+\frac{7}{11})(-\frac{5}{6})$	28. $(-5\frac{1}{2}a)(+b)$
5. $(+3)(+6)$	17. $(-6\frac{3}{4})(-6\frac{3}{4})$	29. $(-2\frac{1}{2}c)(-2\frac{1}{2}d)$
6. $(+2)(-2)$	18. $(+6\frac{3}{4})(-6\frac{3}{4})$	30. $(+7\frac{1}{3}x)(-7\frac{1}{3}y)$
7. $(-6)(-2)$	19. $(-6\frac{3}{4})(+6\frac{3}{4})$	31. $(+5)(-gl^2)$
8. $(-3)(-4)$	20. $(+6\frac{3}{4})(+6\frac{3}{4})$	32. $(-7)(-g^2l)$
9. $(-4)(+2)$	21. $(+3)(-a)$	33. $(-10)(+v^2x)$
10. $2\cdot -3$	22. $(-3)(-x)$	34. $(+12)(-ay^3)$
11. $-3\cdot -4$	23. $(-7)(+rs)$	35. $(-16)(+abc^2)$
12. $(\frac{2}{3})(+\frac{1}{4})$	24. $(-a)(+pq)$	36. $(+bc)(-a)$

Multiplication by Zero

341. The product 4×0 means $0+0+0+0=0$. In general $a \times 0=0$. Why? Since $a \times 0=0 \times a$, it follows that $0 \times a=0$. Why? This illustrates the following theorem: *The value of a product is zero if one of the factors is zero.*

EXERCISES

1. Show by means of a bar that $a \times 0=0$; $0 \times a=0$.

2. The dimensions of a rectangle are a and b. Keeping b the same, bisect a and form a rectangle of dimensions $\frac{a}{2}$ and b. Bisect $\frac{a}{2}$ and form a rectangle of dimensions $\frac{a}{4}$ and b. What are the areas of the rectangles formed?

3. Find the areas of rectangles having the following dimensions: $\frac{a}{8}$, b; $\frac{a}{16}$, b; $\frac{a}{32}$, b; ...... As one side of the rectangle becomes smaller and smaller approaching nearer to zero, how does the area change? Show from this that $0 \times b=0$.

4. Show by means of the bar or by means of the rectangle that $0 \times 0=0$.

Product of Several Factors

342. The product of several factors is obtained by multiplying the first factor by the second, the result by the third, etc. Often, however, it is more advantageous to rearrange the factors before multiplying.

EXERCISES

1. Find the values of the following products:

$$(+3) \cdot 4 \cdot (-6) \cdot 5 \cdot (-2);$$
$$3 \cdot 5 \cdot (-2.8) \cdot (-1.2);$$
$$(-\tfrac{6}{5}) \cdot (\tfrac{10}{3})\ (-\tfrac{4}{3})\ (\tfrac{9}{2}).$$

2. Find the values of the following powers: $(-1)^2$, $(-1)^3$, $(-1)^4$; $(-2)^2$, $(-2)^3$, $(-2)^4$, $(-2)^5$; $(-3)^2$, $(-3)^3$, $(-3)^4$.

3. Find the value of $2x^3-4x^2-5x+3$ for $x=-3$.

4. Find the value of $x^3-3x^2y+4xy^2-5y^3$ for $x=-2$.

5. Find the value of $5x^3+6x^2+3x+4$ for $x=10$.

6. Compare the values of the following: $(-2)^3$ and -2^3; $(-2)^4$ and -2^4; $(-x)^3$ and $-x^3$; $(-x)^4$ and $-x^4$.

7. What is the sign of the product of 7 factors, 4 of which are positive and 3 of which are negative?

8. What powers of negative numbers are positive? Negative?

Law of Signs for Division

343. Division. Dividend. Divisor. Quotient. To *divide* a number 12 by 3 means to find the number which multiplied by 3 gives 12. Thus, the product 12 and one factor 3 being given, the process of finding the other factor is **division.** Division of 12 by 3 is indicated in the following ways: $\frac{12}{3}$, 12:3, $12 \div 3$, or 12/3.

Division in algebra has the same meaning as in arithmetic: the product of two factors and either factor being given, the operation of finding the other factor is called division. The given product is the **dividend**, the given factor the **divisor**, the result of the division the **quotient**.

EXERCISES

1. Since $-10=(+2)\,(-5)$, what must $(-10)\div(+2)$, or $\frac{-10}{+2}$ be? $\frac{-10}{-5}$?

2. Answer the following questions, giving reasons for your answers:

1. $(+12)\div(+3)=$?	5. $(-12)\div(-3)=$?
2. $(+12)\div(+4)=$?	6. $(-12)\div(+4)=$?
3. $(-12)\div(+3)=$?	7. $(+12)\div(-3)=$?
4. $(-12)\div(-4)=$?	8. $(+12)\div(-4)=$?

344. According to the definition of division:

$$(+ab) \div (+a) = (+b) \text{ for } (+a)\,(+b) = (+ab)$$
$$(+ab) \div (-a) = (-b) \text{ “ } (-a)\,(-b) = (+ab)$$
$$(-ab) \div (+a) = (-b) \text{ “ } (+a)\,(-b) = (-ab)$$
$$(-ab) \div (-a) = (+b) \text{ “ } (-a)\,(+b) = (-ab)$$

Examine these equations and state the sign of the quotient:

1. If the signs of dividend and divisor are *alike* (i.e., both $+$, or both $-$).

2. If the signs of dividend and divisor are *unlike* (i.e., one $-$ and the other $+$).

State the law of signs for division and compare the statement with the following:

345. Law of signs in division: *If dividend and divisor have **like** signs the quotient is **positive**; if dividend and divisor have **unlike** signs the quotient is **negative**.*

EXERCISES

1. Using this law give answers to the following questions:

1. $\dfrac{+18}{+2} = ?$	5. $\dfrac{+18}{+9} = ?$	9. $\dfrac{xy}{y} = ?$
2. $\dfrac{+18}{-2} = ?$	6. $\dfrac{+18}{-9} = ?$	10. $\dfrac{-xy}{y} = ?$
3. $\dfrac{-18}{+2} = ?$	7. $\dfrac{-18}{+9} = ?$	11. $\dfrac{-xy}{-x} = ?$
4. $\dfrac{-18}{-2} = ?$	8. $\dfrac{-18}{-9} = ?$	12. $\dfrac{-xy}{-y} = ?$

2. On a winter day the thermometer read $-4°$ in the morning, $-1°$ at noon, and $-6°$ in the evening. What was the average temperature?

To find the average (mean) temperature divide the sum of the readings by the number of readings.

3. Define *quotient* and show that a number divided by itself gives the quotient 1.

4. Show that a number divided by 1 gives that number as quotient.

5. Show that 0 divided by any other number gives 0.

346. Division by 0. The quotients $\frac{1}{0}$, $\frac{2}{0}$, $\frac{-5}{0}$, $\frac{a}{0}$ ...... etc., have no meaning, for a number multiplied by 0 cannot give 1, 2, -5, a, etc.

The quotient $\frac{0}{0}$ is undetermined, as every number multiplied by 0 gives 0.

Therefore *it is assumed that in all quotients hereafter the divisor is not zero nor equal to zero.*

EXERCISES

1. Solve the following equations:

1. $x+14=10$
2. $-4x=8$
3. $3 \div x=-3$
4. $\frac{2}{3}x=-\frac{10}{6}$
5. $x \div \frac{3}{4}=-\frac{8}{6}$
6. $2x \div \frac{4}{5}=\frac{6}{25}$

2. Find the quotients of the following, doing all you can mentally:

1. $(+\frac{3}{4}) \div (+\frac{1}{3})$
2. $(-\frac{7}{8}) \div (+\frac{3}{4})$
3. $(+\frac{7}{8}) \div (-\frac{3}{4})$
4. $(+\frac{5}{7}) \div (+\frac{2}{7})$
5. $(+\frac{5}{7}) \div (-\frac{3}{7})$
6. $(+\frac{5}{6}) \div (-\frac{2}{3})$
7. $(-\frac{5}{6}) \div (+\frac{3}{5})$
8. $(+\frac{5}{7}) \div (-\frac{3}{4})$
9. $(-\frac{5}{7}) \div (-\frac{3}{4})$
10. $(-\frac{7}{8}) \div (+\frac{5}{6})$
11. $(-\frac{5}{6}) \div (-\frac{3}{7})$
12. $(+\frac{3}{7}) \div (+\frac{5}{6})$
13. $(-\frac{3}{7}) \div (-\frac{5}{6})$
14. $(-\frac{5}{6}) \div (+\frac{2}{7})$
15. $(+\frac{11}{12}) \div (-\frac{3}{5})$
16. $(-2a) \div (+2)$
17. $(-2a) \div (-a)$
18. $(-5b) \div (+5)$
19. $(+12x) \div (-4x)$
20. $(-a) \div (-\frac{1}{2}a)$
21. $(+a) \div (-\frac{1}{4}a)$
22. $(-a^3) \div (+a)$
23. $(+a^2) \div (-a^2)$
24. $(-a^4) \div (+a^3)$
25. $(-6a^3) \div (+3a)$
26. $(+4ab) \div (-b)$
27. $(+4ab) \div (-2a)$
28. $(-3ax) \div (+3x)$
29. $(-2xy) \div (+xy)$
30. $(+6\frac{2}{3}r) \div (-3\frac{1}{4}r)$

31. $(+av^3) \div(-av)$
32. $(-av^3) \div+v^2)$
33. $(-abc) \div(-a)$
34. $(+abc) \div(+ac)$
35. $(-abc^2) \div(-ac)$
36. $(-avr) \div(+ar)$
37. $(+7\frac{2}{3}ax)\div(+23a)$
38. $(-7\frac{2}{3}yz) \div(-11\frac{1}{2}y)$
39. $(-16\frac{1}{2}yz^3) \div(+4\frac{1}{8}yz)$
40. $(+6.82az)\div(-31a)$
41. $-3(a+b^2)\div(a+b)$
42. $(-72a^8b^{10})\div(-6a^8b^{10})$
43. $-96x^3z^4 \div -12xz^4$
44. $-36m^3n^4x^2 \div +6m^4n^3$

3. Find the values of the following expressions:

1. $(+3x^4)\div(-x^2)\div(-x)$; $(-3x)\cdot(-4x^3)\div(-6x^2)$; $(5x^3-3x^2)\div 2x$; for $x=-2$.

2. $\dfrac{a^3-a^2b}{a^2}$ for $a=6$, $b=5$.

3. $\dfrac{-14x^4(x^2-4x+4)}{7(-x)^3}$ for $x=-2$.

Summary

347. The chapter has taught the following:

1. Positive and negative numbers may be used to distinguish between the opposite senses, as upward and downward, to the right and to the left, gain and loss, etc.

2. Statistics containing positive and negative numbers may be represented graphically.

3. Positive and negative numbers may be added and subtracted graphically.

4. *The sum of two numbers having like signs is the sum of the absolute values with the common sign prefixed.*

5. *The sum of two numbers having unlike signs is the difference of the absolute values prefixed by the sign of the number having the greater absolute value.*

6. *The value of a sum is not changed by changing the order of the addends.*

7. *To subtract a number change the sign of the subtrahend and add the result to the minuend.*

8. *The product of two numbers having like signs is positive; of two numbers having unlike signs, negative.*

9. *The turning-tendency caused by a force acting upon a bar is equal to the product of the force by the lever-arm.*

10. *The value of a product is zero if one of the factors is zero.*

11. *Factors may be changed in order before finding the product.*

12. *The quotient of two numbers having like signs is positive; of two numbers having unlike signs, negative.*

13. Numbers may be multiplied graphically or a turning-bar may be used to explain multiplication of positive and negative numbers.

BLAISE PASCAL

BLAISE PASCAL

BLAISE PASCAL, a natural but somewhat erratic genius, was born at Clermont, France, on June 19, 1623, and died at Paris, August 19, 1662. He had displayed exceptional ability by the age of eight, and, despite the discouragements of his father and his teacher, became greatly interested in geometry at twelve years of age. Deprived of books on geometry, he discovered for himself many of the properties of figures. Seeing the boy's determination to study geometry, his father gave him a copy of Euclid's *Elements*, which he mastered in a few weeks.

At the age of fourteen Pascal was admitted to the weekly scientific meetings of the French geometricians; at sixteen he wrote an essay of marked originality on conic sections, and at eighteen he constructed an important calculating machine. Thereafter he studied for a time experimental science, then religion, then returned again to mathematics. He formulated a new theorem of conics, still known as "Pascal's theorem," and invented and employed his arithmetical triangle for figurate numbers from which the coefficients of the expansion of a binomial are obtained. He laid down the foundations of the theory of probability, did much work on the cycloid, and exerted himself on the theory of indivisibles. He is said to have worn himself out completely through excessive hard work, so that he died of old age at the age of thirty-nine. See an account of his life and work in some history of mathematics.

CHAPTER XIII

ADDITION AND SUBTRACTION

Review of the Laws of Addition

348. The laws of algebra which in chapter II were shown to hold for addition of literal numbers hold also for positive and negative numbers.

1. Commutative law. Show graphically that

$$(+8)+(-3)=(-3)+(+8).$$

This illustrates the **commutative law**:

The value of a sum remains unchanged by changing the order of the addends.

2. Associative law. Show graphically that

$$(-5)+(+3)+(-2)=[(-5)+3]+(-2)=(-5)+[(+3)+(-2)]$$

This problem illustrates the **associative law**:

In adding several numbers the sum is the same in whatever way two or more of the numbers are combined into a sum before adding in the rest.

349. Similar terms. Terms which have a *common factor* are said to be **similar with respect to that factor** and are called **similar terms.** Thus,

$\frac{2}{3}x^2y^2$, $12x^2y^2$, $-8x^2y^2$ are similar terms. Why?

Also $\sqrt{5}a^2bc^3$, $10a^2bc^3$, $-3a^2bc^3$. Why?

350. Dissimilar terms. Terms which have *no common factor* are called **dissimilar terms,** as $4a^2$ and $-3cb^2$.

EXERCISES

Point out with respect to what factor the following terms are similar and give the coefficients of the common factor:

1. $4x, -7x, 20x, -35x$
2. $ax, -25x, -bx, 46x$
3. $-8pq^2r, -14pq^2r, -12pq^2r$
4. $3a^2b, -5a^2b, +7a^2b, -3a^2b$
5. $2ax, 3ax, -7ax, -5ax$
6. $-3pq^2, 6tq^2, -8kq^2, 12sq^2$

Addition of Monomials

351. Express the sums of the following numbers in the form of a polynomial and combine like terms:

EXERCISES

1. $4a^2b, -a^2b, -5a^2b, a^2b, -3a^2b$

The common factor is a^2b. The coefficients of a^2b are 4, -1, -5, 1, -3. The sum of the coefficients is -4. Hence, the required sum is $-4a^2b$.

2. $10xy^2, -4xy^2, -2xy^2, +xy^2, -5xy^2$

The common factor is xy^2. The coefficients of xy^2 are 10, -4, -2, $+1$, -5. The sum of the coefficients is 0. Hence, the required sum is 0. Why?

3. $+15a, -7a, +18a$
4. $-18x^2, -12x^2, +15x^2, -3x^2$
5. $+2\frac{1}{2}ab, +3\frac{1}{3}ab, -4\frac{1}{2}ab, -5\frac{2}{3}ab$
6. $+27abc, -35abc, +10abc, -2abc$
7. $3(a+b), -4(a+b), 12(a+b)$
8. $-8(x^2+y^2), -24(x^2+y^2), 17(x^2+y^2)$
9. $-3\frac{2}{5}(pr-q^2), +5\frac{2}{3}(pr-q^2); -4\frac{7}{10}(pr-q^2)$
10. $18(mp-3s)^2, -15(mp-3s)^2, -37(mp-3s)^2, 14(mp-3s)^2$
11. $a(x+y+z), -b(x+y+z), -c(x+y+z), d(x+y+z)$

12. $15ax^2, \; -7bx^2, \; 8dx^2, \; -5cx^2$

13. $5s^2t, \; -12x^2y, \; +7x^2y, \; -3s^2t$

14. $-27\frac{2}{3}ab, \; +18\frac{1}{3}cd, \; 15\frac{1}{5}ab, \; 14\frac{2}{5}cd$

15. $5ax, \; -3x, \; x$

16. $9a^2b^2, \; -3c^3y^3, \; 4a^2b^2, \; -4c^3y^3, \; -3a^2b^2$

17. $3mp^2, \; -8mp^2, \; +5a^2x, \; -3a^2x, \; -4mp^2, \; 2a^2x$

18. $-4st, \; -t, \; 3t, \; +5st$

19. $ar^2p, \; -r^2p, \; -br^2p, \; pr^2, \; -cpr^2$

20. $27(a+b)^2, \; +4c, \; +15a, \; -12c, \; -15(a+b)^2, \; -8a$

21. $a(a-b), \; b(a-b), \; -c(a-b)$

22. $a^2(a+b), \; -2ab(a+b), \; b^2(a+b)$

23. $-6\frac{1}{3}(a+b), \; \frac{1}{2}(a^2+b^2), \; -\frac{1}{4}(a^2+b^2), \; +8\frac{2}{3}(a+b)$

24. $r^3(r+p), \; -3r^2(r+p), \; 3r(r+p), \; r+p$

Addition of Polynomials

352. The law for adding polynomials may be seen from the following two problems:

1. The main stairway in a school consists of four flights of stairs, having a, b, c, and d steps, respectively.

If a pupil goes up and down 8 times in one day, how many steps does he take?

If a pupil goes up and down the stairs 5 times on Monday, 6 times on Tuesday, 4 times on Wednesday, 5 times on Thursday, and 4 times on Friday, how many steps does he take on the stairway every week?

On Monday:	$10a+10b+10c+10d$
On Tuesday:	$12a+12b+12c+12d$
On Wednesday:	$8a+8b+8c+8d$
On Thursday:	$10a+10b+10c+10d$
On Friday:	$8a+8b+8c+8d$
	$48a+48b+48c+48d$

Thus, these polynomials are added by adding similar terms.

2. Add: $\begin{array}{r} +27x^2-15xy+18y^2 \\ -13x^2+30xy-\ \ 5y^2 \\ \hline \end{array}$

The sum, $14x^2+15xy+13y^2$ is obtained by adding similar terms in the polynomials.

Frequently the work is arranged as follows:

$$(27x^2-15xy+18y^2)+(-13x^2+30xy-5y^2)=+14x^2+15xy+13y^2$$

353. Degree of a number. The **degree of a number** is denoted by the exponent of the number. The numbers x^2, y^3, z^4 are of the second, third, and fourth degree respectively.

The monomial $5ab^2x^3y^2$ is of the first degree in a, of the second degree in b and y, and of the third degree in x.

354. Degree of a monomial. The sum of the exponents of the literal factors of a monomial is the **degree of the monomial.**

Thus, x^2, $4xy^2$, $3x^2yz^3$ are respectively of second, third, and sixth degree.

355. Degree of a polynomial. When a polynomial has been reduced to the simplest form, the degree of the term having the highest degree is the **degree of the polynomial.** Thus, $x^2+xy+2x+2y+4$ is of the second degree; $x^4+2x^3+2x^4-3x^2-6x-3x^4+7$ is of the third degree, although in form it is of the fourth degree.

356. Descending power. A polynomial is said to be arranged in **descecding powers of *x*** when the term of the highest degree in x is placed first, the term of next lower degree second, etc., and the term not containing x placed last. Thus, the polynomial $x^3+2x^4-3x^2+4x^3+7+x$, when arranged according to descending powers of x, takes the form $2x^4+5x^3-3x^2+x+7$.

357. Ascending powers. When arranged in the order $7+x-3x^2+5x^3+2x^4$, the polynomial is arranged according to **ascending powers of x.**

Find the sum of the following polynomials:

$$-4x^2+2x^3-3x+6,\ 7-x^2,\ 6x^3-4x+3,\ -2x^3-2x^2-7$$

Arranging according to descending powers of x:

$$\begin{array}{rrrr} +2x^3 & -4x^2 & -3x & +\ 6 \\ & -\ x^2 & & +\ 7 \\ 6x^3 & & -4x & +\ 3 \\ -2x^3 & -2x^2 & & +\ 7 \\ \hline 6x^3 & -7x^2 & -7x & +23 \end{array}$$

Adding

Check: Let $x=1$. Then

$$\begin{array}{rrrrl} 2x^3 & -4x^2 & -3x & +6 & = 1 \\ & -\ x^2 & & +7 & = 6 \\ 6x^3 & & -4x & +3 & = 5 \\ -2x^3 & -2x^2 & & +7 & = 3 \\ \hline 6 & -7 & -7 & +23 & =15 \end{array}$$

358. The problems in §§ 352 and 357 illustrate the following law for adding polynomials:

Addition of polynomials. *Polynomials are added by adding similar terms.*

EXERCISES

Add the following polynomials and reduce the sum to the simplest form. Test by substituting values for the letters.

1. $5a+3b+9c,\ 2a-10b-2c$
2. $3x+4y+2z,\ 5x+6y-5z$
3. $2x+5y+7z,\ x-8y+3z$
4. $6a+15c-17b-8d,\ -7a+21d+15b-12c$
5. $7m-n+3p,\ 5m-4p,\ 2m+6n-5p$
6. $-3a-7b+14c,\ -11a+20b-34c$
7. $3m+16n+4p,\ m-4n+6p,\ 2m-2n-10p$
8. $14k-11l+12m,\ -3k+12l-6m,\ -12k+l-2m$

9. $25x-6y+14z,\ -22x+11y+2z,\ 11x-10y+8z$

10. $7a^2-15c^2+23b^2,\ -6a^2+12c^2-21b^2$

11. $x^2+2xy+y^2,\ x^2-2xy+y^2,\ x^2-4xy,\ 4xy+y^2$

12. $29xy-7y^2+24x^2,\ -14xy-36x^2+36y^2$

13. $6.2x^2-12.5xy-2.5y^2,\ -4.1x^2+6xy+.03y^2$

14. $\frac{5}{8}x^2-\frac{1}{3}xy-\frac{1}{4}y^2,\ -x^2-\frac{2}{3}xy+2y^2,\ \frac{2}{5}x^2-xy-5y^2$

15. $8a^3-7a^2-11a,\ -6a^2+2a+10,\ 4a^3+9a-5$

16. $5x^3-10x^2-2x-12,\ -9x^3+x^2-7x+14,\ 3x^3-x+5$

17. $9x^3-3x^2+4x-7,\ -4x^3+3x^2+2x+8,\ 4x^3-2x^2+8x-4$

18. $p^3+3p^2+4p-6,\ -p^2-2p+1,\ p^2-1,\ 3p^3+2p+2$

19. $-18a^2b^2+12a^4-8a^3b,\ 4\frac{1}{2}a^3b-a^2b^2+3\frac{3}{8}a^4$

20. $-23a^2b+41a^2c+56c^2b-15b^2c,$
$-6a^2b+26a^2c+59c^2b-26b^2c,\ 25a^2c+19b^2c-18c^2b$

21. $5(a+b)-7(a^2+b^2)+8(a^3+b^3),$
$-4(a^3+b^3)+5(a^2+b^2)-5(a+b)$

22. $5a^3(a+b)-6a^2b(a^2+b^2)+3ab^2(a^3+b^3),$
$a^2b(a^2+b^2)+a^3(a+b)-7ab^2(a^3+b^3)$

23. $\frac{3}{2}(a+b+c)-\frac{3}{4}(a-b+c)+\frac{4}{5}(a+b-c)+1\frac{1}{5}(-a+b+c),$
$-\frac{2}{3}(a+b+c)+\frac{4}{3}(a-b+c)-\frac{5}{4}(a+b-c)-\frac{5}{6}(-a+b+c)$

24. $5(a+b)^3-7(a+b)^2+4(a+b),$
$3(a+b)^3-2(a+b)^2-5(a+b)$

25. $2(x+y)^2-6(x+y)+1,\ -5(x+y)^2+3(x+y)-6,$
$(x+y)^2-(x+y)+2$

26. $6(lr+t)+7(l-n)+tz,\ 5tz-8(lr+t)-5(l-n),$
$3(lr+t)-(l-n)-4tz$

Subtraction of Monomials

359. It has been shown (§ 334) that subtraction of algebraic numbers may always be changed into addition by the following law:

To find the difference of two numbers the sign of the subtrahend is changed and the result added to the minuend.

Thus, $\left.\begin{array}{r}+5x\\-7x\end{array}\right\}$ may be replaced by $\left\{\begin{array}{r}+5x\\+7x\end{array}\right.$

$\overline{}$ **S** $\qquad\qquad$ $\overline{}$ **A**

S means subtract, **A** means add.

EXERCISES

Subtract the lower monomials from the upper:

1. $\begin{array}{r}16x\\-5x\\\hline\end{array}\quad\begin{array}{r}-\ 4a\\+17a\\\hline\end{array}\quad\begin{array}{r}-10s\\-\ 2s\\\hline\end{array}\quad\begin{array}{r}-7\frac{1}{2}a\\+3\frac{1}{2}a\\\hline\end{array}\quad\begin{array}{r}+15x^2\\-17x^2\\\hline\end{array}\quad\begin{array}{r}-15\ ab\\-18\frac{2}{3}ab\\\hline\end{array}$

2. $\begin{array}{r}+5m^2px\\-4m^2px\\\hline\end{array}\quad\begin{array}{r}-\ 8\frac{2}{3}(p+q)\\+14\frac{1}{3}(p+q)\\\hline\end{array}\quad\begin{array}{r}-7\frac{2}{3}(x-y)\\-5\frac{1}{3}(x-y)\\\hline\end{array}\quad\begin{array}{r}+11m^2(a-2b^3)\\+29m^2(a-2b^3)\\\hline\end{array}$

360. Instead of writing the subtrahend under the minuend, it is often written on the same line with the minuend, connected by a minus sign, $-$. Thus, $(+5x)-(-7x)$ is equivalent to $(+5x)+(+7x)$ or $+5x+7x$ or $+12x$.

Omitting the second step the work may be written as follows:

$$(+5x)-(-7x)=5x+7x=+12x$$

EXERCISES

Reduce the following to the simplest form, doing all you can orally:

1. $(+5ab)-(+12ab)$; $(5x^2y)-(-3x^2y)$; $(-12ab^4)-(+8ab^4)$
2. $(-2x^3y)-(+2x^3y)$; $(-75a^2rl^2)-(-54a^2rl^2)$
3. $(-18p^3f^3)-(63p^3f^3)$; $(+25s^2gh)-(-75s^2gh)$
4. $(-3\frac{1}{2}hk^2v)-(-2.4hk^2v)$; $(+8.7p^3q^2s^4)-(-4\frac{1}{5}p^3q^2s^4)$
5. $\{-5(a+b)\}-\{+7(a+b)\}+\{2(a+b)\}$

6. $\{+18\frac{2}{3}(a+b+c)\}-\{25\frac{1}{3}(a+b+c)\}$

7. $\{+4(t^2-f^3)\}-\{-2(t^2-f^3)\}+\{10(t^2-f^3)\}$

8. $\{-3.4(v^2-h^2)\}+\{-4.5(v^2-h^2)\}-\{2.1(v^2-h^2)\}$

Subtraction of Polynomials

361. When the subtrahend consists of more than one term the subtraction may be performed by subtracting each term of the subtrahend from the minuend.

For example, when we wish to subtract 7 dollars, 4 quarters, and 10 dimes from 15 dollars, 8 quarters, and 30 dimes, we subtract 7 dollars from 15 dollars, leaving 8 dollars; 4 quarters from 8 quarters, leaving 4 quarters; and 10 dimes from 30 dimes, leaving 20 dimes.

The subtraction of algebraic polynomials is then not different from the subtraction of monomials and may again be reduced to addition.

Example:

$$\left.\begin{array}{r}+9x^2-14xy-12y^2\\-7x^2+\ \ 5xy-15y^2\end{array}\right\}\text{ may be replaced by }\left\{\begin{array}{r}+9x^2-14xy-12y^2\\+7x^2-\ \ 5xy+15y^2\end{array}\right.$$

$$\underline{\qquad\qquad\qquad}\ \mathbf{S}\qquad\qquad\qquad\underline{\qquad\qquad\qquad}\ \mathbf{A}$$

The result is: $16x^2-19xy+3y^2$

EXERCISES

1. Subtract the lower from the upper polynomial:

1. $$\begin{array}{l}a^2+2ab+b^2\\ \underline{a^2-2ab+b^2}\end{array}$$

2. $$\begin{array}{lll}4x^2-2xy & & +8y^2\\ \underline{4x^2} & \underline{-4x^2y-} & \underline{y^2}\end{array}$$

3. $$\begin{array}{r}a^2b^2+a^4-b^4-3a^3b+4ab^3\\ \underline{-a^2b^2-a^4-b^4-3a^3b-4ab^3}\end{array}$$

4. $$\begin{array}{r}x^3+3x^2y+3xy^2+\ y^3\\ \underline{-3x^2y+3xy^2-3y^3}\end{array}$$

2. From $c^3-2a^2c-d^3-r^2$ subtract $-a^2c-3d^3-r^2-c^3$.

Without rewriting the following polynomial is obtained:

$$c^3-2a^2c-d^3-r^2+a^2c+3d^3+r^2+c^3$$

Combining similar terms, we have $2c^3-a^2c+2d^3$.

3. From $-17x^4+16y^4+4x^3y-29xy^3-35x^2y^2$ subtract $+15y^4-41x^2y^2+8x^3y-18x^4-25xy^3$, without rewriting the polynomials.

4. From $12ab-3cd+12xy$ subtract $3ab+2cd-11xy$.

5. From $8xy-3x+4y$ subtract $-4xy+3x-4y$.

6. From $16m^3-8mn^2+4n^3$ subtract $7m^3-4m^2n+14n^3$.

The preceding problems have illustrated the following law:

Subtraction of polynomials. *Polynomials may be subtracted by changing mentally the sign of each term of the subtrahend and then adding the resulting polynomial to the minuend.*

EXERCISES

Subtract as indicated, doing as much of the work as you can mentally:

1. $(+17x^2-14xy-15y^2)-(-16x^2+12xy-9y^2)$
2. $(45x^2y^2-27x^4+81y^4)-(73x^4+45y^4+65x^2y^2)$
3. $(27x^3-6x^2y+8y^3)-(-15x^3+8xy^2-4y^3)$
4. $(-56g^4+27g^3h-14gh^3)-(-13g^3h+18g^2h^2-6gh^3+25h^4)$
5. $(-5a^2x+10bxy+24b^2y-18axy)$
$-(-6b^2y+12axy-4a^2x-2bxy)$
6. $(-9m^2pq-5m^3p-14m^2q^2)-(-6m^3p-10m^2pq-13m^2q^2)$
7. $(2r^3+r^2s+rs^2+2s^3)-(r^3+3r^2s+3rs^2+s^3)$
8. $(+4x^3-3x^2y+12y^3-7xy^2)-(2y^3-3xy^2+4x^2y+7x^3)$
9. $(\frac{1}{2}l^3-3\frac{1}{2}lm^2+4\frac{1}{3}m^3-3l^2m)-(\frac{3}{4}lm^2+5\frac{1}{2}l^2m-2l^3-3m^3)$
10. $(5\frac{2}{3}abc-7\frac{1}{2}a^2b-8\frac{2}{5}b^2c-6\frac{1}{3}c^2a)$
$-(4\frac{1}{3}a^2b-5\frac{2}{3}c^2a+3\frac{1}{5}abc+7\frac{1}{3}b^2c)$
11. $(3.4v^3s^4-5.7v^4s^3+9.8v^6s^2)-(-1.7v^4s^3-3.2v^6s^2-4v^3s^4)$
12. $(4t^4-7\frac{1}{2}f^3)-(3\frac{1}{2}t^3f-7.6t^4+5\frac{2}{3}tf^3+6f^3)+(2\frac{1}{2}t^3f-3\frac{1}{3}tf^3)$

13. $(-2\frac{1}{2}k^3ml+7.5k^2m^2l-3.24km^3l)$
$-(3\frac{1}{4}km^3l-3.6k^2m^2l-5.4k^3ml+6kml)$

14. $(3ax^3-4syz^3)-(-2ax^3+4y^3-5z^3+3syz^3)-(2y^3+3z^3)$

15. $(-5mv^2-3mvu+4mu^2)-(+3mv^2-6muv-4nuv+9mu^2)$

16. Compare the signs of the terms of the subtrahend in the separate parts of exercises 1–15 before and after the parentheses are removed.

17. State a rule as to the effect of a minus sign in front of a polynomial in parenthesis.

18. State a similar rule as to the effect of a plus sign.

Removal of Parentheses

362. Parentheses. Numbers are grouped by inclosing them in a **parenthesis**, thus, ( ). Other symbols used for this purpose are the **brackets** [ ], the **brace** { }, and the **vinculum** $\overline{}$. Thus, the expressions $(a+b)\div(a-b)$, $[a+b]\div[a-b]$, $\{a+b\}\div\{a-b\}$, and $\dfrac{a+b}{a-b}$ all indicate that $a+b$ is to be divided by $a-b$. Similarly, to indicate that $c+d$ is to be subtracted from $a+b$, any of the following symbols may be used: $(a+b)-(c+d)$, $[a+b]-[c+d]$, $\{a+b\}-\{c+d\}$, or $a+b-\overline{c+d}$.*

EXERCISES

1. Give the meaning of the following:

   1. $7-\{5+(8-2)\}$

   2. $-3-[5-\{-4+6\}]$

* The Italian Bombelli in an algebra of 1572 used a symbol which amounted to a parenthesis. Vièta (1540–1603) was the first to employ the parenthesis ( ), the brackets [ ], and brace { }.

Girard (died 1632) perfected the parenthesis to the modern usage.

Descartes in 1637 introduced the vinculum $\overline{}$, in the way we use it.

3. $2(a+b)-4\{a-\overline{2a-3b}\}$

2. Perform the following operations and simplify results:

1. $4-\{5-(t^2-4)\}$
2. $8\frac{1}{2}-\{4k^3-(3k^3-5\frac{1}{2})\}$
3. $4t^2-\{t^2-3t^3+\overline{3t^2-t^3}\}$
4. $2f-[6f-\overline{3g-4f}-(2g-4f)]$
5. $9x-\{5y-(6y+7z)-(7y-4z)\}$
6. $-3a^4+[4a^3-(3a^3-5a^2)-(4a^2+3a)]$
7. $16e^2-\{-42e^2-(3e^2+2)\}-(50e^2+3)$
8. $7.5p^3-[3.4p^3-\overline{4.2p^3+1.6p^2}-\overline{3.4p^2-4.5p}]$
9. $3k^3-\{2p^2+k^3-\overline{p^2+r}\}-[p^2-\mathrm{r}-(k^3+r)]$
10. $105s^2-\overline{14st+3t^2}+\{4st-2t^2-(5s^2-2st)\}$
11. $-[3rl-(3r^2-3l^2)]-\{4r^2-\overline{5rl-2l^2}\}+(2r^2+2l^2)$

Summary

363. The meaning of the following terms was taught in this chapter: degree of a number, of a monomial, of a polynomial; ascending and descending powers; brackets, brace, vinculum.

The chief uses of the vinculum in modern algebra are:

1. The dividing-line of a fraction is a vinculum over the denominator.

2. The bar of a radical sign is a vinculum over the number underneath it.

364. The following laws have been extended to apply to positive and negative numbers:

1. Commutative law. *The value of a sum remains unchanged by changing the order of the addends.*

2. Associative law. *In adding several numbers the sum is the same in whatever way two or more of the numbers are combined into a sum before adding in the rest.*

3. *To add similar monomials prefix to the common factor the sum of the coefficients.*

4. *To add polynomials add the similar terms.*

5. *To find the difference of two numbers, change the sign of the subtrahend and add.*

6. *Polynomials may be subtracted by changing the sign of each term of the subtrahend and then adding the resulting polynomial to the minuend.*

7. *A parenthesis preceded by a + sign may be removed without making any other changes.*

8. *A parenthesis preceded by a minus sign may be removed if the sign of every term within the parenthesis is changed.*

LEONARDO OF PISA

LEONARDO OF PISA

LEONARDO FIBONACCI, the greatest mathematician of Europe of the period from the fourth to the sixteenth century, was born at Pisa, Italy, about 1175 A.D., and died about 1250 A.D. While still a boy he went with his father to live at Bugia on the coast of Algiers, at which place he was educated. Here he learned the Arabic system of notation and numeration and became acquainted with the algebra of the great Arabian scholar, Alkarismi. He traveled later in Egypt, Syria, and other Mediterranean countries, and acquired all the mathematical learning that he could. Returning to Pisa about 1200, he began writing a book on arithmetic and algebra, which he published in 1202 under the title *Algebra et almuchabala*, which he took from Alkarismi's great work. This book is commonly referred to as the *Liber abaci* (Book of the Abacus).

In *Liber abaci* Leonardo explains the Arabic system of numeration and emphasizes its advantages over the Roman system then in vogue. He gave an account of Arabic algebra and pointed out the convenience of using geometry to get demonstrations of algebraic formulas, just as geometry is used in chapter XV of this text. He also solved simple equations, showed how to solve a few quadratics, and illustrated his rules by problems on numbers, much as is done here. The *Liber abaci* contained much material of historical value about Arabian mathematics, which was then more highly developed than European mathematics and was also unknown in Europe. This book was the source and authority for arithmetics and algebras in Europe for centuries.

Leonardo wrote a book of *Geometrical Practise*, another on *Squares*, a tract on determinate algebraic problems, and a few other minor works, none of which ever acquired the celebrity of his *Liber abaci*. For fuller details about Leonardo's work, see Ball's *History of Mathematics*, pp. 167–70.

CHAPTER XIV

MULTIPLICATION AND DIVISION

Multiplication of Monomials

365. In finding the product of two or more monomials it is assumed that the commutative law, § 152, holds for positive and negative numbers.

Find the product: $(2a^2b)(-3ab^2)(-4b^3)$

The sign is determined by the sign law, § 336, and is found to be $+$. Why?

The factors are then rearranged as follows: $-2\cdot3\cdot4a^2\cdot\mathrm{a}\cdot\mathrm{b}\cdot\mathrm{b}^2\cdot\mathrm{b}^3$, i.e., the product of the arithmetical factors is formed first and then the product of the literal factors.

Multiplying, we have

$$(2a^2b)(-3ab^2)(-4b^3) = +24a^3b^6$$

This problem illustrates the following method of multiplying monomials:

Multiplication of monomials. *To find the product of two or more monomials:*

1. *Determine the sign of the product according to the law of signs in multiplication.*

2. *Find the product of the arithmetical factors.*

3. *Multiply this product by the product of the literal factors.*

Find the following product and test the result for $a=1$, $b=1$, $x=2$, $y=3$:

$$\begin{aligned}(2ay)(-4a^2bx)(-3ab^2x^3y^2) &= +2\cdot4\cdot3a\cdot a^2\cdot a\cdot b\cdot b^2\cdot x\cdot x^3y\cdot y^2\\ &= +24a^4b^3x^4y^3\end{aligned}$$

Test: $(6)(-8)(-216) = 10{,}368$

$24\cdot1\cdot1\cdot16\cdot27 = 10{,}368$

EXERCISES

1. Simplify the following products, doing as much of the work as you can mentally:

1. $(+17)(+25)(-8)$
2. $(2mt^2)(-3m^2t)(-5m^3)$
3. $(6a^2bc^2)(4a^2b^2c)(3a^2b^2c^2)(5ab^2c^2)$
4. $(+8.2a^2mb)(-3.5amn^2b^2)(4.3m^2n^2b)$
5. $(-3xpq^2r)(-2\frac{1}{3}xp^2qr)(-4zpqr^2)$
6. $(\frac{3}{5}a^2b)(-\frac{10}{9}ab^2x)(-\frac{2}{15}abx^2)$
7. $(-2\frac{1}{2}x^2y)(-5\frac{1}{2}x^2y^3z)(+16x^3yz^4)$
8. $(-5\frac{1}{3}p^2q^3)(+5\frac{1}{4}q^2r^3)(-1\frac{1}{7}r^2p^3)$
9. $(4\frac{1}{2}t^3s^2u^3)(-2\frac{2}{3}t^2s^5u^{12})(-\frac{1}{12}t^{10}s^7u^8)$
10. $(\frac{3}{5}a^2x)(-10aby)(-6ay)(\frac{1}{4}xy)(\frac{1}{6}ab)$
11. $(\frac{2}{3}by^2)(\frac{6}{5}a^2bx^2y)(-\frac{10}{4}ax^2y)$
12. $(-2)^2$; $(-2)^3$; $(-2)^4$; $(-2)^5$
13. $(-3)^1$; $(-3)^2$; $(-3)^3$; $(-3)^4$
14. $(-2)^2(-3)^2(-4)(2)^3$
15. $(-a)^2(-a^2)(-a^3)(-a)^3$
16. $(-a^2)(+a^4)(-a^3)(+a)^2$
17. $(-a)^2(-a)^4(-a)^3(+a^4)$
18. $(a^5)^2(-a^3)^4(-a^2)^5(a^3)$
19. $(3p^2)^2(-p^3)^5(-5p^4)^2$
20. $(2\frac{1}{2}x^3)^2(-\frac{2}{5}x^5)^2(3x^7)^2$
21. $(a+b)^3(a+b)^5(a+b)^2$
22. $3(x+y)^5(x-y)^4 \cdot 4(x+y)^3(x-y)^6$

2. Find the value of the following:

1. $(-3)(-2)+(4)(-5)-(-3)(2)-(-2)^3$
2. x^2-6x+9 for $x=2$; for $x=-3$
3. $x^4-3x^3+2x^2+4x+6$ for $x=-2$

3. Find two factors of each of the following products:

$$36x^2y^3z;\quad -72a^4b^2;\quad 51p^4qr^2;\quad -18a^2bx^3$$

4. Solve the following equations:

$$\frac{x}{6} = -\frac{1}{12};\quad -\frac{x}{a^2} = -a^5;\quad \frac{x}{a} = -b$$

Multiplication of Polynomials by Monomials

366. Distributive Law of Multiplication. It was found in chapter V, § 158, that a polynomial may be multiplied by a monomial as follows:

To multiply a polynomial by a monomial multiply every term of the polynomial by the monomial and add the resulting products.

This is called the **Distributive Law of Multiplication.**

EXERCISES

Multiply as indicated and test by substituting values for the letters:

1. $5x(x^2 - 3xy + 5y^2)$

$$\begin{array}{ll} & x^2 - 3xy + 5y^2 \\ & 5x \\ \hline \textit{Product:} & 5x^2 - 15x^2y + 25xy^2 \end{array}$$

Check: Let $x = 1$, $y = 2$.

Then $5x(x^2 - 3xy + 5y^2) = 5(1 - 6 + 20) = 5 \cdot 15 = 75$

and $5x^3 - 15x^2y + 25xy^2 = 5 - 30 + 100 = 75$

2. $6\frac{2}{3}ab(3a^2 - 6ab + 12b^2)$

3. $3a(4a - 5b) + 5b(6a - 3b)$

4. $-5.7x^3(\frac{12}{19}abx - 1\frac{4}{19}aby - 2\frac{2}{19}xyz + \frac{7}{19}byz)$

5. $(3a^2b^3 + 4ab^2 + 2b^3)2ab^2$

6. $(x^3y^4 - 4x^4y^5 + 6x^2y^7 - 9x^5y^4)3.5x^2y^3$

7. $2(3a - 2b) - 5(a + 3b) + 2(3a - b)$

8. $2a+b(a-2b)+(4a-b)3+b$

9. $(x^2-5x+6)2x-(x+1)3x^2-x(x^2-7x+4)$

10. $5a^2(2\frac{1}{5}a^2-4\frac{1}{2}ab-3.5b^2)-4b^2(3\frac{1}{3}a^2+2ab-2\frac{3}{8}b^2)$

11. $4x^2(3x-2y)-2xy(3x-2y)+(3x-2y)6y^2$

12. $2(3a^2-4\frac{1}{2}ab+7b^2)-(4a^2+5\frac{2}{3}ab-8\frac{1}{2}b^2)$

13. $x^2-[3x(x^2-2)-2x^2(x+1)]$

14. $4a[(1-a)2a^2+(3a+1)3a^2]$

15. $5a\{4a-2(3a-4b)+5(4a-3b)\}$

16. $2a\{5(4a-7b-3c)-6(5a+4b-8c)\}$

17. $-4x[2x^2+3x\{4(x-1)-5(x-2)\}]$

18. $5y^3-[3y^3-2y\{4y(y+3)-5y(2y+6)\}]$

Multiplication of Polynomials by Polynomials

367. When the dimensions of a rectangle are polynomials the area can be found by separating the rectangle into other rectangles whose dimensions are simpler numbers.

EXERCISES

1. Using the heavy dividing line, express the area of the rectangle of Fig. 244 as the sum of two rectangles. Express the area also as the sum of four rectangles.

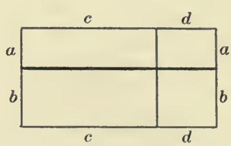


FIG. 244

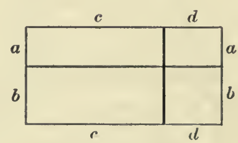


FIG. 245

2. Express the area of the rectangle of Fig. 245, (1) as the sum of two rectangles; (2) as the sum of four rectangles.

3. Write the areas of the rectangles whose dimensions are given below as the sum of two or more rectangles. Simplify the polynomials thus obtained.

	Length	Width
1.	$3+4$	$6+2$
2.	$a+6$	$a-2$
3.	a^2+b^2+2ab	$a+b$
4.	$x^2+2xy+y^2$	$x-y$
5.	$a-b$	$a^2-2ab+b^2$
6.	$5+x$	$25+10x+x^2$

The rectangles of exercises 1, 2, and 3 are first expressed as the sum of two rectangles. Thus, for the rectangle in exercise 1, $(a+b)(c+d)=a(c+d)+b(c+d)$. These separate products are called **partial products** of the given polynomials $a+b$ and $c+d$. To obtain the final product, the partial products are simplified.

This gives $(a+b)(c+d)=ac+ad+bc+bd$.

This illustrates the following method of multiplying two polynomials:

Multiplication of polynomials. *Multiply every term of one polynomial by each term of the other and add the partial products.*

EXERCISES

Multiply as indicated:

1. $(a^2+ab+b^2)(a+b)$

The work may be arranged as follows:

$$\begin{array}{lll}
 & a^2+ab+b^2 & \\
 & \underline{a+b} & \\
\text{First partial product:} & a^3+a^2b+ab^2 & =a(a^2+ab+b^2) \\
\text{Second} \quad \text{“} \quad \text{“} & \underline{a^2b+ab^2+b^3} & =b(a^2+ab+b^2) \\
\text{Hence,} & a^3+2a^2b+2ab^2+b^2 & =(a+b)(a^2+ab+b^2)
\end{array}$$

2. $(ax+by)(ax-by)$

3. $(a^3+ab^2+b^3)(a+b)$

4. $(p^2+2p+1)(p-3)$

5. $(2\frac{1}{2}a^2-3\frac{1}{3}b^3)(12a^3-6b^2)$
6. $(\frac{1}{3}pt-\frac{1}{2}ts)(\frac{3}{4}pt-\frac{2}{3}ts)$
7. $(a+b+c+d)^2$; $(a+b-2c)^2$
8. $(-2a-3b+c)^2$; $(a^2+b^2-3c^3)^2$
9. $(2a^2+b^2+3c^2)(2a^2+b^2-3c^2)$
10. $(.5x-.4y-.3z)(10x-20y+30z)$
11. $(3y^2-45xy+6x^2)(3y^2-5x^2)$
12. $(ax+by-cz)(ax+by+cz)$
13. $(ax+by-cz)^2$; $(ax-by-cz)^2$
14. $(\frac{1}{3}ms-\frac{1}{4}st-\frac{2}{3}tm)(6m-12s+18t)$
15. $(-1.4l^2-2.5ml+.9m^2)(.2l^2-lm-.1m^2)$
16. $(a^2+ab+b^2)(a-b)(a+b)$
17. $(4x^2+2xy+y^2)(2x+y)(2x-y)$
18. $(a+b)^3$; $(a-b)^3$; $(2x-y)^3$; $(x+2y)^3$
19. $(4.6abc+1.2ab^2)(4a^2c-5a^2b)$
20. $(p^2+pr-r^2)(p^2+pr+r^2)$
21. $(9k^2-3kt+t^2)(9k^2+5kt-t^2)$
22. $(9x^2-6xy+4y^2)(9x^2+6xy+y^2)$
23. $(5a+2b)^3-(5a-2b)^3$
24. $4\frac{1}{2}mn(4m^2-6mn+9n^2)-3mn^2(13\frac{1}{2}n-9m)$
25. $(2p^2+q^2)^2-(2p^2-q^2)^2-4p^2(p-2q)^2$
26. $(2f-3h)^2-(2f+3h)^2+(2f-3h)(2f+3h)$
27. $(.5x-.6y)^2-(.5x+.6y)^2-(.5x+.6y)(.5x-.6y)$

Multiplication of Arithmetical Numbers

368. Multiplication of arithmetical numbers is a special case of multiplication of polynomials.

For, in the polynomial $a \cdot 10^3+b \cdot 10^2+c \cdot 10+d$ let $a=2$, $b=5$, $c=1$, $d=3$.

Then the polynomial takes the form $2 \cdot 10^3+5 \cdot 10^2+1 \cdot 10+3 =2{,}000+500+10+3=2{,}513$.

EXERCISES

1. Multiply 482 by 347.

The number 482, written as a polynomial, takes the form $400+80+2$, or $4 \cdot 10^2+8 \cdot 10+2$. The number 347 takes the form $3 \cdot 10^2+4 \cdot 10+7$. Thus,

$$(482)(347)=(4 \cdot 10^2+8 \cdot 10+2)(3 \cdot 10^2+4 \cdot 10+7)$$

Multiplying,
$$\begin{array}{rrrrr}
\multicolumn{5}{l}{(4 \cdot 10^2+8 \cdot 10+2)(3 \cdot 10^2+4 \cdot 10+7)} \\
\hline
12 \cdot 10^4 & +24 \cdot 10^3 & +\ 6 \cdot 10^2 & & \\
 & 16 \cdot 10^3 & +32 \cdot 10^2 & +\ 8 \cdot 10 & \\
 & & +28 \cdot 10^2 & +56 \cdot 10 & +14 \\
\hline
12 \cdot 10^4 & +40 \cdot 10^3 & +66 \cdot 10^2 & +64 \cdot 10 & +14
\end{array}$$

This is equal to

$$120{,}000+40{,}000+6{,}600+640+14, \text{ or } 167{,}254$$

2. Write 32,569 as a polynomial arranged in descending powers of 10.

3. Using the method of exercise 1, find the product 3,462 by 3.

4. Using the method of exercise 1, multiply 287 by 453.

369. Decimal system. Because the arithmetical numbers are expressible as polynomials in 10, they have been called **decimal numbers.**

Thus, a decimal number is obtained by substituting $x=10$ in such a polynomial as $2x^3+5x^2+7x+8$.

370. Other systems of numbers. Substitution of other values for x gives a system of numbers different from the one with which we are familiar; e.g., $x=12$ gives the number $2 \cdot 12^3+5 \cdot 12^2+7 \cdot 12+8$. Such a system of numbers would even have certain advantages over the decimal system.* It is called the duodecimal system. We know that the Hindoos used 5 as a base for their system of numerals.

* See Cajori, *History of Elementary Mathematics*, p. 2.

Division of Monomials

371. In § 367 the area of a rectangle was found when both dimensions are known. It will now be shown how to find one of the dimensions if the area and the other dimension are given.

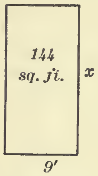


FIG. 246

EXERCISES

1. If the area of a rectangle is 144 sq. ft. and the base 9 ft., Fig. 246, what is the altitude?

2. Find the altitude of a rectangle of area 144 sq. ft., if the base is 16 ft.; 12 ft.; 8 ft.; 72 feet.

372. Division. To divide 8 by 2 means to find a number which multiplied by 2 gives 8. (See § 343.) In general, *to divide one number by another is to find a third number which, multiplied by the second number, gives the first.*

EXERCISES

1. Show that a number divided by itself gives 1.

Since $n \cdot 1 = n$, it follows by § 372 that $\frac{n}{n} = 1$.

2. Show that a number divided by 1 gives the number.

3. Show that 0 divided by a number, not 0, gives 0.

4. Show that *a product of two numbers divided by one of the factors gives the other factor.*

5. The following sometimes purports to be a proof demonstrating that $4 = 7$. Find the fallacy:

Two numbers a and b are given equal, $a = b$.

Then $a - b = 0$. Hence, $4(a - b) = 0$ and $7(a - b) = 0$.

It follows that $4(a - b) = 7(a - b)$. Why?

Dividing both sides of the equation by $a - b$ gives $4 = 7$.

6. Divide $25a^2b$ by $5a$.

First method: Since $5ab \cdot 5a = 25a^2b$, $25a^2b \div 5a = 5ab$ (by § 372).

Second method:

Since $25a^2b = 5a \cdot 5ab$, $\dfrac{25a^2b}{5a} = \dfrac{5ab \cdot \cancel{5a}}{\cancel{5a}} = 5ab$ (by exercise 4)

7. Find the following quotients: $\dfrac{a^6}{a^4}$; $\dfrac{a^5}{a^3}$; $\dfrac{a^{10}}{a^2}$; $\dfrac{a^7}{a^5}$

8. Divide as indicated:

$8x \div 4$	$18a^4b^5c^6 \div 9ab^5c^3$
$-25a^5b^6 \div 5a^3b^4$	$(x+y)^4 \div (x+y)^2$
$30a^4b^3 \div -6ab^2$	$(-a)^4b^3(-c)^3 \div (-a)^2b(-c)$

Reduction of Quotients

373. Reduction of quotients. In algebra, as in arithmetic, the quotient is not altered if dividend and divisor are both divided by the same factor. Dividing dividend and divisor by the highest common factor is to **reduce the quotient to the simplest form.**

EXERCISES

Reduce to the simplest form:

1. $\dfrac{63a^3b^4c^4}{-7a^2b^5c^6}$

The sign of the quotient is $-$. Why?

The numerical factors can be divided by 7.

The literal factors a^3 and a^2 are divisible by a^2; b^4, and b^5 are divisible by b^4; and c^4 and c^6 by c^4.

The work of reducing the quotient may be arranged as follows:

$$\frac{63a^3b^4c^4}{-7a^2b^5c^6} = \frac{\overset{9}{\cancel{63}} \cdot \overset{a}{\cancel{a^3}} \cdot \overset{1}{\cancel{b^4}} \cdot \overset{1}{\cancel{c^4}}}{\underset{-1}{\cancel{-7}} \cdot \underset{1}{\cancel{a^2}} \cdot \underset{b}{\cancel{b^5}} \cdot \underset{c^2}{\cancel{c^6}}} = \frac{9a}{-bc^2} = -\frac{9a}{bc^2}$$

2. $\dfrac{xy}{xz}$

3. $\dfrac{56a^5}{a^8}$

4. $\dfrac{6a^3b^2}{-2a^2b^3}$

5. $\dfrac{-6x^5y^2z^4}{3xy^2z^3}$

6. $\dfrac{15r^4st^3}{3r^8st^2}$

7. $\dfrac{6x^2yz^3}{2xy}$

8. $\dfrac{-25a^5b^6c^7}{5a^3b^4c^5}$

9. $\dfrac{-1.69x^4y^5z^6w^5a^8}{-1.3a^4w^2x^2y^3z^4}$

10. $\dfrac{+12\frac{1}{2}t^2u^5s^4}{-2\frac{1}{2}ts^4u^3}$

11. $\dfrac{12(-a)^4b^5(-c)^6}{3(-a)^2b^3(-c)^4}$

12. $\dfrac{14(a+b)^3}{-7(a+b)^2}$

13. $\dfrac{-94(x^2-y^2)^5}{-2(x^2-y^2)^3}$

14. $\dfrac{-3.43(a^2+2ab+b^2)^8}{49(a^2+2ab+b^2)^4}$

15. $\dfrac{5\frac{2}{12}x^5y^4(a^2+b^2)^7}{2\frac{1}{2}\frac{3}{2}x^2y^3(a^2+b^2)^5}$

16. $\dfrac{-45m^4n^3w^2z}{.9m^2n^4w^3}$

17. $\dfrac{-51x^6y^5z^6}{1.7ax^4y^5z^6}$

Monomial Factors

374. In § 367 we have learned to find the product when the factors are given. We will now see how the factors may be found when the product is given.

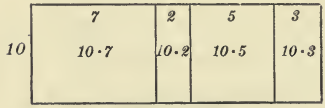


Fig. 247

EXERCISES

1. Show that the total area of four adjacent flower-beds, Fig. 247, may be expressed in either of the following ways:

$10 \cdot 7+10 \cdot 2+10 \cdot 5+10 \cdot 3$ or $10(7+2+5+3)$

Hence, $10 \cdot 7+10 \cdot 2+10 \cdot 5+10 \cdot 3=10(7+2+5+3)$

Which way is the more advantageous? Give reasons for your answer.

2. Represent in two different ways the combined area of four rectangles of length a and of bases a, $3ab$, b, and c, respectively. Express the equality of the two representations by an equation.

3. The total area of three adjacent lots is $x^4+3x^3y+4x^2y^2$ sq. m., each term representing the area of a lot. The lots all have the same length. What may their dimensions be?

4. Sketch a rectangle whose area is

$4x^2-3xy$	$3m^5-12m^3n+6mn^4$
$5x^3-10x^2y+15xy^2$	$15x^4-10x^3+5x^2$
$14a^2b^3c^2-21a^3b^2c^2+35a^2b^2c^3$	$10x^3-4x^2+6x+2$

375. Exercises 1–4, § 374, illustrate the following method of finding the monomial factor of a polynomial:

Determine by inspection the highest monomial factor contained in each term of the polynomial. This is one factor of the polynomial.

Divide the polynomial by this factor. The result is the other factor of the polynomial.

EXERCISES

1. Factor the polynomial $14x^2y^2z-7x^3y^3z^2+28xy^2z^2$.

By inspection the highest common factor is found to be $7xy^2z$.

Dividing each term of the polynomial by $7xy^2z$, the quotient is $2x-x^2yz+4z$.

Hence, $14x^2y^2z-7x^3y^3z^2+28xy^2z^2=7xy^2z(2x+x^2yz+4z)$

2. Factor the polynomials in exercise 4, § 374.

3. Factor the following:

$5a-10b$	$ax+ay-az$
$17x^2-289x^3$	$4a^2x^3-12a^3x^4-20a^4x^3$
$16x^2-2abx$	$5mx+10m^2x^2-40m^3x^3$

Reduction of Quotients

EXERCISES

376. Reduce the following quotients to lowest terms:

1. $\dfrac{27a^4b^5c^6-18a^5b^6c^7-54a^6b^7c^8}{-9a^7b^8c^9}$

The factors of the numerator are $9a^4b^5c^6$ and $(3-2abc-6a^2b^2c^2)$. The fraction may now be written:

$$\frac{9a^4b^5c^6(3-2abc-6a^2b^2c^2)}{-9a^7b^8c^9}$$

Dividing dividend and divisor by $9a^4b^5c^6$, gives

$$-\frac{3-2abc-6a^2b^2c^2}{a^3b^3c^3}$$

2. $\dfrac{39ab+9a^2}{3a^2}$

3. $\dfrac{36a^3x+8ax^3}{-12a^4x^4}$

4. $\dfrac{36x^2y^4-42x^5y^6z}{-12x^3y^5}$

5. $\dfrac{20a^3b-15a^2b^2+30ab^3}{5ab}$

6. $\dfrac{10x^2y-15x^3y^2+5x^3y}{-5xy}$

7. $\dfrac{16a^3x^2y-44a^2x^3z}{4a^2x^2}$

8. $\dfrac{35ab^8c^3-42a^3b^4c^3-49a^8b^4c^3+21a^3b^5c^7}{7ab^3c^3}$

9. $\dfrac{12am^2n^3-16bm^3n^2+40abm^2n^2-28a^2m^2n^2}{-4mn}$

10. $\dfrac{-15p^{15}q^{10}-12p^{27}q^{14}+9a^3p^{14}q^{16}-27abcp^5q^{10}}{3p^2q^3}$

11. $\dfrac{4abc(3abc-5a^2b^2c^2-7ab^2c+6ab^4c)}{-2a^2bc^2}$

12. $\dfrac{2x(4yz-6x^2y^2)-8y^2(5xz^2-1\frac{1}{2}x^3)}{4xyz}$

13. $\dfrac{15a^3b^4c^5-9a^4b^3c^2-45a^2b^3c^5+21a^5b^4c^3-3a^4b^5c^4}{3a^2b^3c^2}$

14. $\dfrac{21abdpqs-35abcpqt-42acdpts}{-7ap}$

15. $\dfrac{15(x+y)^5-25(x+y)^8-35(x+y)^3}{-5(x+y)^3}$

Division of Polynomials

377. In the division of polynomials, all the preceding operations find application. It is therefore a subject by means of which addition, subtraction, multiplication, and division by binomials are reviewed.

Divide 86,932 by 412.

The process in full is as follows:

$$\begin{array}{r|l} 86{,}932 & 412 \\ 824 & \text{Quotient} = 211 \\ \hline 453 & \\ 412 & \\ \hline 412 & \\ 412 & \\ \hline \end{array}$$

Arranging the numbers 86,932 and 412 in the form of polynomials, they may be written $80{,}000+6{,}000+900+30+2$ and $400+10+2$.

The process of dividing 86,932 by 412 may now be arranged as follows:

$$\begin{array}{l|l} 80{,}000+6{,}000+900+30+2 & 400+10+2 \\ 80{,}000+2{,}000+400 & \text{Quotient} = 200+10+1 \\ \hline \qquad 4{,}000+500+30 & \\ \qquad 4{,}000+100+20 & \\ \hline \qquad\qquad 400+10+2 & \\ \qquad\qquad 400+10+2 & \\ \hline \end{array}$$

Writing the polynomials arranged according to descending powers of 10, the division takes the form:

$$\begin{array}{l|l} 8\cdot10^4+6\cdot10^3+9\cdot10^2+3\cdot10+2 & 4\cdot10^2+1\cdot10+2 \\ 8\cdot10^4+2\cdot10^3+4\cdot10^2 & \text{Quotient} = 2\cdot10^2+1\cdot10+1 \\ \hline \qquad 4\cdot10^3+5\cdot10^2+3\cdot10 & \\ \qquad 4\cdot10^3+1\cdot10^2+2\cdot10 & \\ \hline \qquad\qquad 4\cdot10^2+1\cdot10+2 & \\ \qquad\qquad 4\cdot10^2+1\cdot10+2 & \\ \hline \end{array}$$

378. Inspection of the process of dividing one polynomial by another brings out the following facts:

1. *The first term of the dividend divided by the first term of the divisor gives the first term of the quotient.*

2. *The divisor is multiplied by the first term of the quotient and the product subtracted from the dividend.*

3. *The first term of the remainder divided by the first term of the divisor gives the second term of the quotient.*

4. *The divisor is multiplied by the second term of the quotient and the product subtracted from the remainder.*

5. *To get the other terms of the quotient proceed in the same way as for the first and second terms.*

6. *Throughout the process all polynomials are to be arranged according to powers of the same letter.*

EXERCISES

1. Divide $8x^4+6x^3+9x^2+3x+2$ by $4x^2+x+2$.

$$\begin{array}{lll|l}
8x^4+6x^3+9x^2+3x+2 & & & \underline{4x^2+x+2} \\
\underline{8x^4+4x^3+4x^2} & & & \underline{\text{Quotient}=2x^2+x+1} \\
\quad 2x^3+5x^2+3x & & & \\
\quad \underline{2x^3+\ x^2+x} & & & \\
\qquad 4x^2+2x+2 & & & \\
\qquad \underline{4x^2+2x+2} & & &
\end{array}$$

2. Divide $a^3+3a^2b+3ab^2+b^3$ by $a+b$.

$$\begin{array}{l|l}
a^3+3a^2b+3ab^2+b^3 & \underline{a+b} \\
\underline{a^3+\ a^2b} & \underline{\text{Quotient}=a^2+2ab+b^2} \\
\quad 2a^2b+3ab^2 & \\
\quad \underline{2a^2b+2ab^2} & \\
\qquad ab^2+b^3 & \\
\qquad \underline{ab^2+b^3} &
\end{array}$$

3. Divide x^3+y^3 by $x+y$.

$$
\begin{array}{l|l}
x^3+y^3 & x+y \\
\underline{x^3+x^2y} & \text{Quotient}=x^2-xy+y^2 \\
\quad -x^2y+y^3 & \\
\quad \underline{-x^2y-xy^2} & \\
\qquad\quad +xy^2+y^3 & \\
\qquad\quad \underline{+xy^2+y^3} &
\end{array}
$$

379. Checking division. Whenever the division is exact, there is no final remainder. The equation: *quotient times divisor equals dividend,* or, in symbols, $Q\times d=D$, may then be used as a check.

EXERCISES

Divide and test by multiplying:

1. (a^3-a^2-4a+4) by (a^2-3a+2)
2. $(a^3-5a^2+10a-12)$ by $(a-3)$
3. $(1+5x+6x^2)$ by $(1+2x)$
4. $(9t^2+24st+16s^2)$ by $(3t+4s)$
5. $(1.2x^2+.5xy-2.8y^2)$ by $(4x+7y)$
6. $(6k^3-31k^2t+47kt^2-42t^3)$ by $(2k-7t)$
7. $(10x^3-5xy^2-54y^3-4x^2y)$ by $(x-2y)$
8. $(27a^3-54a^2b+36ab^2-8b^3)$ by $(3a-2b)$
9. $(27a^3-54a^2b+36ab^2-8b^3)$ by $(9a^2-12ab+4b^2)$
10. $(9t^6-12t^3s^3+4s^6)$ by $(3t^3-2s^3)$
11. $(4x^6+12x^3y^3+9y^6)$ by $(2x^3+3y^3)$
12. $(3y^4+15x^3y-9xy^3-25x^4+10x^2y^2)$ by $(5x^2-3y^2)$
13. $(7x^5+10x^4-26x^3+17x^2-11x+3)$ by $(7x^3-4x^2+3x-1)$
14. $(15+8a-32a^2+32a^3-15a^4)$ by $(3+4a-5a^2)$

15. $(64u^6+14.4u^4v^2+1.08u^2v^4+.027v^6)$ by $(4u^2+.3v)$

16. (x^3-y^3) by $(x-y)$; $(27t^3+64s^3)$ by $(3t-4s)$

17. (x^5-1) by $(x-1)$; (x^6-1) by $(x-1)$

18. (x^5-y^5) by $(x-y)$; (x^6-y^6) by $(x-y)$

19. $(64a^6-b^6)$ by $(8a^3-b^3)$; $(16x^4-25r^4)$ by $(4x^2-5r^2)$

20. $(8x^3-y^3)$ by $(4x^2+2xy+y^2)$

21. $(.008s^3t^3+v^3)$ by $(.2st+v)$

22. $(64a^6-b^6)$ by $(16a^4+4a^2b^2+b^4)$

23. $(.008s^3t^3+v^3)$ by $(.04s^2t^2-.2stv+v^2)$

24. $(27m^6-8n^6)$ by $(9m^4+6m^2n^2+4n^4)$

25. $(27m^6+8n^6)$ by $(3m^2+2n^2)$

26. $(a^3+a^2b+ab^2+ac^2+bc^2+b^3)$ by $(a+b)$

27. $(a^3+a^2b+ab^2+ac^2+bc^2+b^3)$ by $(a^2+b^2+c^2)$

Summary

380. The chapter has taught the meaning of the following terms: partial products; decimal system of numbers; factoring; division.

381. The chapter has taught the processes for the following operations:

Multiplication of monomials, of polynomials by monomials, of polynomials by polynomials.

Division of monomials; reduction of quotients to simplest form; division of polynomials.

382. Distributive law of multiplication. Polynomials are multiplied by monomials according to the following law:

Multiply every term of the polynomial by the monomial and add the resulting products.

383. It was seen:

That a product is divided by a number if one of the factors is divided by that number.

That a number divided by itself gives 1.

That a number divided by 1 gives the number.

That 0 divided by a number, not 0, gives 0.

384. Arithmetical numbers may be arranged in the form of polynomials according to powers of 10.

385. Monomial factors of a polynomial are found by inspection. The other factor is then found by dividing the polynomial by the monomial factor.

386. The process of dividing one polynomial by another is essentially the same as the process of dividing arithmetical numbers.

CHAPTER XV

SPECIAL PRODUCTS. FACTORING. QUADRATIC EQUATIONS

The Square of a Binomial

387. It is of advantage to be able to carry out rapidly certain multiplications that occur frequently in algebra.

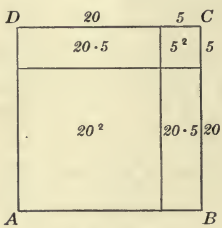


Fig. 248

$(20+5)^2=20^2+2\cdot20\cdot5+5^2$

388. Quadratic trinomial square. Let $ABCD$, Fig. 248, represent a square whose side is 25 ft. long. Show that the area of $ABCD$ is equal to the sum of the areas of the four parts, 20^2, $20\cdot5$, $20\cdot5$, and 5^2. This may be expressed by the equation $(20+5)^2=20^2+2\cdot20\cdot5+5^2$.

Similarly show that the area of the square in Fig. 249 gives the equation

$$(a+b)^2=a^2+2ab+b^2.$$

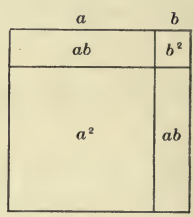


Fig. 249

$(a+b)^2=a^2+2ab+b^2$

The trinomial $a^2+2ab+b^2$ is of the second degree and is called a **quadratic trinomial.** Since this trinomial represents the area of a square, it is called a **quadratic trinomial square.**

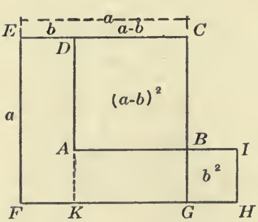


Fig. 250

$(a-b)^2=a^2-2ab+b^2$

389. Let $ABCD$, Fig. 250, represent a square $(a-b)$ ft. long.

Show that the area of $ABCD$ equals

$$EFGC+GHIB-FKDE-KHIA$$

Therefore $(a-b)^2=a^2+b^2-ab-ab$. Why? This may be written

$$(a-b)^2=a^2-2ab+b^2.$$

The trinomial $a^2-2ab+b^2$ is also a *quadratic trinomial square.*

Express by means of a drawing the following squares of binomials as quadratic trinomial squares:

$$(p+q)^2;\ (p-q)^2;\ (x+y)^2;\ (x-y)^2;\ (m+n)^2;\ (m-n)^2$$

390. Squaring binomials. The square of binomials can be obtained without a figure, by multiplication.

Express by multiplication the following squares in the form of trinomials:

1. $(m-x)^2$
   $(m-x)^2=(m-x)(m-x)=m^2-2mx+x^2$
2. $(a+b)^2$
3. $(p-k)^2$
4. $(p+k)^2$
5. $(x-y)^2$
6. $(m+x)^2$

Notice that the *squares of these binomials may be found by inspection from the terms of the binomials by squaring the first term, adding the square of the second term, and then adding or subtracting 2 times the product of the two terms, according as the sign of the second term is + or −.*

EXERCISES

Find by inspection the trinomials equal to the following squares:

1. $(x+p)^2$
2. $(v+a)^2$
3. $(x-p)^2$
4. $(x+5)^2$
5. $(y-7)^2$
6. $(4a-1)^2$
7. $(\frac{1}{3}x+\frac{1}{2}y)^2$
8. $(.4a-.3t)^2$
9. $(k-r)^2$
10. $(r+y)^2$
11. $(k+n)^2$
12. $(r-y)^2$
13. $(h-p)^2$
14. $(g-f)^2$
15. $(3px-4qy)^2$
16. $(.6xyz+1)^2$
17. $(5a+2b)^2$
18. $(7xy+z)^2$
19. $(3a-4b)^2$
20. $[(a+b)+3]^2$
21. $[(a-b)+z]^2$
22. $[(k+b)-s]^2$
23. $[(a-b)-2c]^2$
24. $[(ax+y)-z]^2$

391. Squaring arithmetical numbers. Squares of arithmetical numbers are easily found by writing the numbers in the form of binomials.

EXERCISES

Square the following numbers:

1. 53

$53^2=(50+3)^2=50^2+2\cdot 50\cdot 3+3^2=2{,}500+300+9=2{,}809$

2. 68

$68^2=(70-2)^2=4{,}900-280+4=4{,}624$

3. 13	6. 21	9. 43	12. 91
4. 14	7. 31	10. 72	13. 89
5. 15	8. 38	11. 81	14. 103

Factoring Trinomial Squares

392. It was found in § 390 that the square of a binomial is a trinomial consisting of the square of the first term of the binomial, plus or minus (according as the binomial is a sum or a difference) twice the product of the first and second terms of the binomial, plus the square of the second term. It follows that *a trinomial, in which two terms are squares* (and positive) *and the other term is plus or minus twice the product of the square roots of those two terms, is the square of the sum or difference of those two square roots according as the remaining term is plus or minus.*

This enables us to find by inspection the two equal factors of a trinomial square.

EXERCISES

Factor the following trinomials by inspection:

1. $k^2+6kl+9l^2$

k^2 and $9l^2$ are the squares of k and $3l$ respectively, and $6kl$ is twice the product of their square roots.

Hence, $k^2+6kl+9l^2=(k+3l)\ (k+3l)=(k+3l)^2$.

2. $x^2+2xy+y^2$
3. $a^2-2ab+b^2$
4. $m^2+4mn+4n^2$
5. b^2-6b+9
6. $k^2+8k+16$
7. $a^2+10ab+25b^2$
8. $49-140n^2+100n^4$
9. $a^2b^2c^2+8abc+16$
10. $16x^2+24xy+9y^2$
11. $25k^2+70ks+49s^2$
12. $16p^2+72pq+81q^2$
13. $121a^2-242ab+121b^2$
14. $81x^2-126xy+49y^2$
15. $169r^2-260rs+100s^2$
16. $256m^2+96mn+9n^2$
17. $49x^6-154x^3y+121y^2$
18. $625m^4+50m^2v^2+v^4$
19. $169r^2-78rt^2+9t^4$
20. $25x^2y^2+40xyz+16z^2$
21. $36r^2s^2-84rstw+49t^2w^2$
22. $9a^2b^2+48abmn+64m^2n^2$
23. $49a^2k^2+42akxy+9x^2y^2$
24. $25p^2q^2r^2-60pqrs+36s^2$
25. $81a^2+180abcd+100b^2c^2d^2$
26. $64m^2n^2+320mnp+400p^2$
27. $225k^2-120kl+16l^2$
28. $100x^2+280xy+196y^2$
29. $64x^2y^2+80xyz+25z^2$
30. $16x^2-56xyz+49y^2z^2$
31. $4p^2-36pqr+81q^2r^2$

Product of the Sum of Two Numbers by Their Difference

393. The product of the sum of two numbers by their difference can be found from a rectangle whose dimensions are the sum of two segments and the difference of the same two segments.

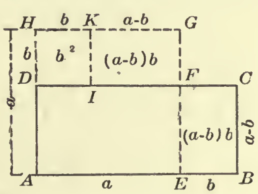


FIG. 251

$(a+b)(a-b)=a^2-b^2$

Let $ABCD$, Fig. 251, represent a rectangle having the dimensions $(a+b)(a-b)$.

$ABCD=AEFD+EBCF$. Why?
$=AEFD+IFGK$. Why?
$=AEGH-DIKH-IFGK+IFGK$. Why?
$=AEGH-DIKH$.
$=a^2-b^2$

Hence, $(a+b)(a-b)=a^2-b^2$

From a drawing express the following products as the difference of two squares:

$$(a+x)(a-x);\quad (m+n)(m-n);\quad (y+4)(y-4)$$

394. The product of the sum of two numbers by their difference can be readily found by multiplying.

EXERCISES

Express as the difference of two squares by multiplying:

1. $(a+x)(a-x)$

$(a+x)(a-x)=a^2+ax-ax-x^2=a^2-x^2$

2. $(r+4)(r-4)$
3. $(t+q)(t-q)$
4. $(p+s)(p-s)$
5. $(y+b)(y-b)$

The results obtained in these problems suggest the following way of obtaining the product of the sum of two numbers by their difference by inspection: *Square each number and subtract the second square from the first.*

EXERCISES

Express by inspection the following products as the difference of two squares:

1. $(2a+3b)(2a-3b)$
2. $(3x-4z)(3x+4z)$
3. $(5p+3s)(5p-3s)$
4. $(\frac{1}{2}r+\frac{1}{3}t)(\frac{1}{2}r-\frac{1}{3}t)$
5. $(\frac{2}{5}xy-\frac{2}{3}xz)(\frac{2}{5}xy+\frac{2}{3}xz)$
6. $(4abc-3)(4abc+3)$
7. $(1+5p^2q^2)(1-5p^2q^2)$
8. $(1-a)(1+a)(1+a^2)$
9. $(u-v)(u+v)(u^2+v^2)$
10. $(10a+7bc)(10a-7bc)$

Factoring the Difference of Two Squares

395. The equation $(a+b)(a-b)=a^2-b^2$ suggests a method of obtaining the factors of a^2-b^2. In words, the factors of the difference of two squares are the sum of their square roots and the difference of their square roots.

EXERCISES

Factor the following binomials and test the results by multiplication:

1. $256x^2-y^2$

The square root of $256x^2$ is $16x$ and the square root of y^2 is y.
Hence, $256x^2-y^2=(16x-y)(16x+y)$

2. x^2-y^2
3. $16k^2-25b^2$
4. $49a^2-9b^2$
5. $m^2n^2-144r^2$
6. $289u^2-81v^2$
7. $16-25y^2$
8. $169d^2h^2s^2-225t^4$
9. $9m^4-121n^2g^4$
10. $49a^2-100b^4g^2$
11. $196-361a^4k^6x^2$
12. $225b^{10}-f^4g^6h^{14}$
13. r^4-s^4
14. $81m^4-16n^4r^4$
15. x^8-y^8
16. $256a^4-625c^4d^8$
17. m^6-n^6
18. $a^{10}-b^{10}$
19. $64x^6-9$
20. $(r+3s)^2-16t^2$
21. $(2a+b^2)^2-9c^4$
22. $(5x^2-3y^3)^2-16z^4$
23. $(x^2-y)^2-x^6$

396. Factoring the difference of two square numbers enables us to find the value of the difference of two squares of arithmetical numbers by inspection.

Find the value of the following:

1. 81^2-19^2
2. 137^2-37^2
3. 137^2-63^2
4. 126^2-26^2
5. $1,017^2-17^2$
6. 511^2-489^2

397. To find the square of a trinomial. Draw a square whose side is $a+b+c$ and show that

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac$$

How may a trinomial be squared by inspection?

Find by inspection: $(a+2b+3c)^2$; $(x-3y+z)^2$

The Product of Two Binomials of the Form $(ax+b)(cx+d)$

398. A quick way of multiplying two binomials of the form $ax+b$ and $cx+d$ can be seen from a study of the result obtained by multiplication.

Multiply $(ax+b)(cx+d)$

$$\begin{array}{l} \quad ax+b \\ \quad \underline{cx+d} \\ acx^2+bcx \\ \qquad\quad \underline{+adx+bd} \\ acx^2+(bc+ad)x+bd \end{array}$$

It is seen that the first term of the product is the product of the first terms of the binomials, that the last term is the product of the last terms of the binomials, and that the middle term is the sum of the two cross-products.

EXERCISES

Multiply by inspection, doing all you can mentally:

1. $(3x+5)(2x+3)$

Writing the second binomial under the first.

$3x+5$
$\underline{2x+3}$, we see that the product of the first terms is $6x^2$, the product of the last terms is 15, the sum of the cross-products is $9x+10x=19x$.

Hence, $(3x+5)(2x+3)=6x^2+19x+15$

2. $(5x+4)(3x+2)$
3. $(3x-5)(2x+3)$
4. $(5x+4)(3x-2)$
5. $(2x-7)(4x+3)$
6. $(3x-5)(2x-3)$
7. $(5x-4)(3x-2)$

Factoring Trinomials of the Form ax^2+bx+c

399. The trinomials found in exercises 1–7, § 398, are all of the same form: ax^2+bx+c. The method of factoring a trinomial of this form can be seen from the following illustration: Factor $3x^2+17x+10$.

The various possibilities are

$$\begin{array}{c} +3x \quad +10 \\ \times \\ +\;x \quad +\;1 \\ \hline \end{array} \qquad \begin{array}{c} +3x \quad +\;1 \\ \times \\ +\;x \quad +10 \\ \hline \end{array} \qquad \begin{array}{c} +3x \quad +5 \\ \times \\ +\;x \quad +2 \\ \hline \end{array} \qquad \begin{array}{c} +3x \quad +2 \\ \times \\ +\;x \quad +5 \\ \hline \end{array}$$

The last case being the one with the sum of the cross-products equal to $17x$ gives the required factors.

Hence, $3x^2+17x+10=(3x+2)(x+5)$

EXERCISES

In the same way, factor the following trinomials:

1. $2x^2+11x+12$
2. $8c^2+46c-12$
3. $3x^2-17x+10$
4. $8z^2-31z+21$
5. $5x^2-38x+21$
6. $11a^2-23ab+2b^2$
7. $7k^2+123k-54$
8. $12t^2+31st-15s^2$
9. $5m^2-29mn+36n^2$
10. $10r^2-23r-5$
11. $6b^2-29b+35$
12. $6f^2-f-77$
13. $102-11a-a^2$
14. $15+37z-8z^2$
15. $1-6xy+5x^2y^2$
16. $2x^2+11x+12$
17. $14x^2+53xy+14y^2$
18. $5x^2+13x-6$
19. $17a^2+6a-11$
20. $8p^2-14p-39$

Miscellaneous Exercises for Practice

400. Factor the following:

1. c^2-c-56
2. $48x^2-3y^2$
3. $a^2b^2+a^2b+ab^2$
4. $72r^2+41r-45$
5. $x^2y^2-5xy+4$
6. a^2-64
7. $x^2yz+xy^2z+xyz^2$
8. $a^2bc^3-2abc^3-8bc^3$
9. $4ax^2-9ay^2$
10. $18x^2+37x+19$
11. $7bx^2+42bxy+63by^2$
12. $6a^2b+12ab^2+18a^2b^2$
13. p^4-16
14. $7b^2+41bc-6c^2$

15. $810a^4c^2-10b^4c^3$
16. p^2q^2r-r
17. $38x^3y^4+57x^4y^3-19x^3$
18. $9x^2-4xy-13y^2$
19. $a^3b^3c^2+a^2b^3c^3+a^3b^2c^3$
20. $450-2a^2$
21. $c^2-16c+64$
22. $b+15b^2-16$
23. $a^4b^2-6a^3b+9a^2$
24. $12a^2xy+12axy+3xy$
25. x^2+4x-5
26. $15+19a^2-34a$
27. $81a^2-1$
28. $5b^2+10b-15$
29. $5a^2-5b^2$
30. $6x^2-56y^2+41xy$
31. $a^2k^3-2abk^3+b^2k^3$
32. $3a^2-9ab-210b^2$
33. ax^3+ax^2+ax+a
34. $30c^2-31c+8$
35. ax^4-100a
36. $9m^2-24mn+16n^2$
37. a^3+a^2+a
38. $36a^2+27ab+2b^2$
39. $a^2b^2+18ab+80$
40. $4x^2+32xy+39y^2$

The Theorem of Pythagoras

401. There exists a simple relation between the sides of a right triangle which is used in solving some geometric problems.

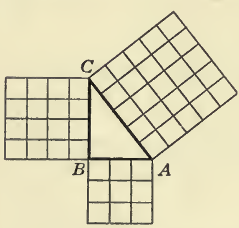


Fig. 252

1. Construct a right triangle, making the sides including the right angle 3 and 4 units long respectively, as $\triangle ABC$, Fig. 252. Using the same unit find the length of AC. On each side draw a square and divide each into unit squares. Counting these squares, find how the square on the hypotenuse compares in size with the sum of the squares on the other two sides. Express the result by means of an equation.

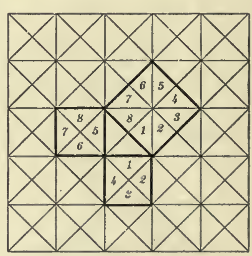


Fig. 253

2. How does the square on the hypotenuse, Fig. 253, compare with the sum of the squares on the other sides?

402. The last two problems illustrate the following theorem:

Theorem of Pythagoras: *In a right triangle the sum of the squares on the sides including the right angle is equal to the square on the hypotenuse.*

This theorem is one of the most famous theorems of geometry. It was named after the Greek mathematician, Pythagoras (569–500 B.C.). He is believed to have been the first to give a general proof of it.

EXERCISES

1. Show by counting the small squares, Fig. 254, that the square on the hypotenuse equals the sum of the squares of the other two sides.

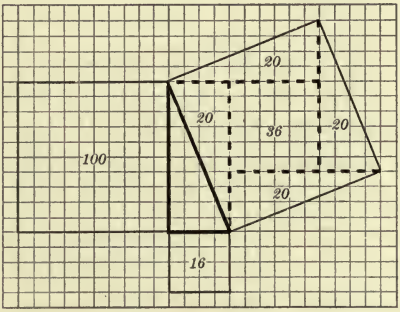


Fig. 254

2. Let c, Figs. 255 and 256, denote the length of the hypotenuse and let a and b denote the lengths of the sides including the right angle.

Draw squares on line-segments equal to $(a+b)$. Divide these squares as indicated.

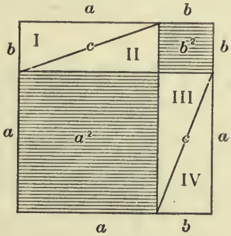


Fig. 255

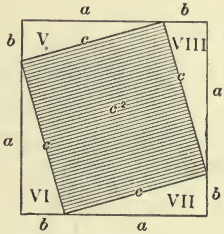


Fig. 256

Show that $\triangle$I, II ...... VIII are congruent. (s.a.s.)

Show that the square on $(a+b)$, Fig. 255, equals

$$a^2+b^2+(\text{I}+\text{II}+\text{III}+\text{IV})$$

Show that the square on $(a+b)$, Fig. 256, equals

$$c^2+(\text{V}+\text{VI}+\text{VII}+\text{VIII})$$

Hence, if from each square on $(a+b)$ four of the congruent triangles are taken, the remainders a^2+b^2 and c^2 are equal.

Therefore $a^2+b^2=c^2$

3. The two short sides of a right triangle are 6 and 8. What is the length of the hypotenuse?

Denote the hypotenuse by x, Fig. 257.

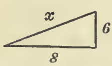


Fig. 257

Then
$$x^2=8^2+6^2=64+36$$
$$x^2=100$$
$$x=10$$

Hence, the hypotenuse is equal to 10.

4. A wall is 16 ft. long and 12 ft. high. How long is a string stretched from a lower corner to the opposite corner?

5. The hypotenuse of a right triangle is 30 ft. and one of the other sides is 18 ft. long. How long is the third side?

6. The sides of a right triangle are as 3 is to 4 and the hypotenuse is 35 ft. long. How long are the sides?

Call the sides $3x$ and $4x$.

7. A ladder 20 ft. long just reaches a window 16 ft. above the ground. How far is the bottom of the ladder from the foot of the wall if the ground is level?

8. The diagonal of a square is 12. What is the side? The perimeter?

The square root of a number, as 80, or a, or $x+y$, that is not a perfect square, is indicated thus: $\sqrt{80}$, or $\sqrt{a}$, or $\sqrt{x+y}$.

9. The diagonal of a square is a. Find the perimeter.

10. Find the hypotenuse, h, of a right triangle whose area is 54 sq. ft. and whose base is 12 feet.

11. Find the hypotenuse, h, of a right triangle whose area is r sq. rd. and whose base is b rods.

12. Find the perimeter of a right triangle whose area is 216 sq. rd. and whose base is 48 rods.

13. Find the perimeter of a right triangle whose area is s sq. rd. and whose base is b rods.

14. A tree, standing on level ground, was broken 36 ft. from the ground and the top struck the ground 27 ft. from the stump, the broken end remaining on the stump. How tall was the tree before breaking?

15. Under conditions of exercise 14 suppose the top piece t ft. long and that the top struck the ground f ft. from the stump. How tall was the tree before it broke?

16. The sides of a rectangle are a and b and the diagonal is d. Show that $d^2=a^2+b^2$ and $d=\sqrt{a^2+b^2}$.

17. Show that the diagonal, d, of a square of side a is given by the formula $d=a\sqrt{2}$.

18. The hypotenuse and one side of a right triangle are c and b. Show that the other side a equals $\sqrt{c^2-b^2}=\sqrt{(c+b)(c-b)}$.

19. Using the result of exercise 17 as a formula, find the diagonals of squares whose sides are 4 in.; 1.2 cm.; 3.42 centimeters.

20. Using the result of exercise 18 as a formula, find the side a when $c=40$ in., $b=24$ in.; $c=.625$ cm.; $b=.375$ centimeters.

In the case $c=4$, $b=3$, show that $a=\sqrt{7}$. The length of the side cannot be expressed exactly without the radical sign, but can be approximated by extracting the square root of 7.

Square Root of Arithmetical Numbers

403. The square of a number is found easily by multiplying the number by itself. Thus, the square of 346 is $346 \cdot 346=119{,}716$. However, the process of finding the number which multiplied by itself gives 119,716 is much more complicated.

404. The square of a binomial is obtained from the formula

$$(a+b)^2=a^2+2ab+b^2 \ldots\ldots\ldots\ldots \text{I}$$

This formula is represented geometrically in Fig. 258.

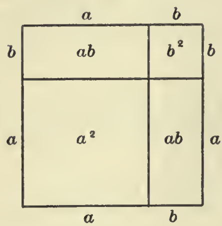


FIG. 258

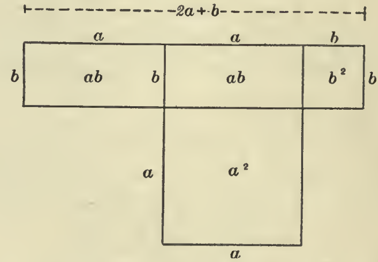


FIG. 259

Show from Fig. 259 that by placing one of the rectangles *ab* in a different position, formula I can be changed to the form II:

$$(a+b)^2=a^2+(2a+b)b \ldots\ldots\ldots\ldots \text{II}$$

Formula II will be used in finding the square root of a given number.

405. The number of digits in the square root. The following exercises show how to determine the number of digits in the square root of a given number.

EXERCISES

1. Show that the square of a number of 1 digit contains 1 or 2 digits.

Give the squares of the integral (whole) numbers from 1 to 9.

2. Show that the square root of a number of 1 or 2 digits contains 1 digit in its integral part.

3. Find the integral part of the square roots of the following numbers: 3, 5, 7, 18, 27, 39, 50, 65, 89.

4. Show that the square of a number containing 2 digits in the integral part has 3 or 4 digits in its integral part.

5. Show that the square root of a number of 3 or 4 digits has 2 digits in the integral part.

6. Find the integral parts of the square roots of the following numbers: 110, 150, 209, 630, 1,625, 8,274.

The following table is a summary and extension of exercises 1 to 6.

Number of digits in a given number........	1	2	3	4	5	6
Number of digits in the square................	1 or 2	3 or 4	5 or 6	7 or 8	9 or 10	11 or 12

Thus, the number of digits in the integral part of the square of a number is twice as large or one less than twice as large as the number of digits in the integral part of the given number.

This suggests the following device for determining the number of digits in the integral part of the square root of a number. Beginning at the decimal point mark off toward the left periods of 2 digits. Then the number of digits in the square roots will be the same as the number of periods. Thus, since 54,783 is divided into three periods: 5′47′83, the period farthest left containing only 1 digit, the square root of 54,783 contains 3 digits in its integral part.

Find the square root of 729.

Beginning at the decimal point mark off to the left periods of 2 digits: 7′29.

Hence, the square root contains 2 digits. Why?

The tens digit of the square root is the largest integer whose square is less than 7, i.e., 2.

Denoting the unit digit by x, $\sqrt{729}=20+x$

and $729=(20+x)^2=400+(2\cdot 20+x)x$, by formula II

Hence, $729-400=(2\cdot 20+x)x$, or $329=(2\cdot 20+x)x$

The value of x is found by trial: Since $2\cdot 20=40$ and since $8\cdot 40=320$, 8 might be tried. But $(2\cdot 20+8)8=384$ and is larger than 329. Trying the next smaller number, 7, we find $(2\cdot 20+7)7=329$. Therefore $x=7$ and $\sqrt{729}=20+7=27$.

406. The process of finding the square root of 729 may be given in the following condensed forms:

(*a*)

$$\sqrt{7'29}=20+7$$
$$20^2=400$$
$$\overline{329}$$
$$(2\cdot 20+7)7=\underline{329}$$

(*b*)

Omitting zeros: $\sqrt{7'29}=27$
$$2^2=4$$
$$\overline{329}$$
$$47\cdot 7=\underline{329}$$

407. The process of finding the square root. The process of finding the square root, omitting zeros, consists of the following steps:

(1) *Point off periods of 2 digits, beginning at the decimal point* .... 7′29

(2) *Find the largest integer whose square is less than 7*, i.e., 2.

This is the tens digit of the square root.

(3) *Subtract the square of the digit just found from 7* .... $\underline{4}$
and bring down the next period of digits 329

(4) *Neglecting the units digit in the remainder, divide the number so formed by 2 times the tens digit of the root*, i.e., by 4.

(5) *The result found in* (4) *is used tentatively as unit digit of the roots. It is also adjoined to 2 times the tens digit of the square root and the result obtained is then multiplied by the unit digit of the root* .... $\underline{329}$

(6) *The product in* (5) *is subtracted from the remainder obtained in* (3). *If the product is larger than the minuend the unit digit of the root is decreased by 1 and step* (5) *repeated with it. If there is a remainder the given number is not a perfect square.*

EXERCISES

Extract the square roots of the following numbers:

4,096, 1,444, 676, 2,116, 784, 4,761

Quadratic Equations

408. Quadratic equations. Geometric problems sometimes lead to equations in which the highest power of the unknown is the second power. Such equations are of the second degree and are called **quadratic equations.**

The diagonal of a rectangle is 8 units longer than one side and 9 units longer than the other. Find the length of the diagonal.

Denoting the length of the diagonal by x, Fig. 260, the sides of the rectangle are $x-8$ and $x-9$.

Therefore $x^2=(x-8)^2+(x-9)^2$. Why?

This equation reduces to the *normal form*

$$x^2-34x+145=0$$

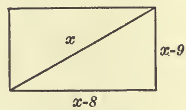


FIG. 260

It is our next aim to learn how to solve the equation.

409. Quadratic function. The expression $x^2-34x+145$ is called a **quadratic function** of x, or a **function of the second degree**, and $x^2-34x+145=0$ is a second degree, or quadratic, **equation.**

410. Graphical solution. The equation $x^2-34x+145=0$ may be solved graphically.

Denoting the function $x^2-34x+145$ by y, we find the following table of corresponding values of x and y (Fig. 261):

The graph shows that the function becomes 0 at two places, for

x	y
0	145
2	81
4	25
6	23
10	−95
15	−140
20	−135
25	−80
30	25
35	175

FIG. 261

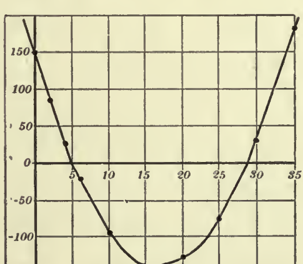


FIG. 262

$x=5$ and for $x=29$. Thus, the equation $x^2-34x+145=0$ is satisfied by $x=5$, and by $x=29$. Test by substituting these values in the equation.

PROBLEMS AND EXERCISES

1. A stone, falling from rest, goes in a given time 16 ft. multiplied by the square of the number of seconds it has fallen; i.e., $s=16t^2$.

Find s, if (1) $t=4$ seconds
(2) $t=11.5$ seconds

Find t, if (1) $s=64$ feet
(2) $s=1{,}600$ feet

Make a graph of the function $16t^2$.

2. A stone, thrown downward, goes in a given time 16 ft. multiplied by the square of the number of seconds it has fallen, plus the product of the velocity with which it is thrown and the number of seconds fallen; i.e., $s=vt+16t^2$.

Find s, when $v=3$ and $t=3, 7, 12$.

Make the graph of $vt+16t^2$ for $v=3$.

3. Solve the following equations graphically:

1. $x^2-8x+12=0$	5. $x^2-10x+24=0$
2. $x^2-6x+5=0$	6. $x^2-10x+25=0$
3. $x^2-5x-6=0$	7. $4x^2-12x+5=0$
4. $x^2-3x-10=0$	8. $4x^2+8x-5=0$

411. Quadratic equations solved by factoring.

Solve the equation $x^2-8x+12=0$

$$x^2-8x+12=0$$

Factoring,

$$(x-2)(x-6)=0 \qquad (1)$$

This equation is satisfied if $x-2=0$ (2)

and if $x-6=0$ (3)

because:

The product of two or more numbers is zero only if at least one of the numbers is zero.

From equation (2), $x=2$

Check: $2^2-8\cdot2+12=4-16+12=0$

Likewise from (3),

Check as above. $x=6$.

Consequently, both 2 and 6 are roots of the equation.

PROBLEMS AND EXERCISES

Solve the following quadratic equations by the method of factoring, and test the results:

1. $x^2-3x+2=0$
2. $y^2-4y+3=0$
3. $m^2=m+2$
4. $n^2+5n=6$
5. $a^2+7a+6=0$
6. $m^2=4m+12$
7. $k^2+k=56$
8. $r^2+51=20r$
9. $b^2=4b+77$
10. $c^2+112=23c$
11. $\frac{x^2}{15}=\frac{x}{5}+\frac{2}{3}$
12. $\frac{x^2}{10}-3=\frac{13x}{10}$
13. $\frac{x^2}{21}-\frac{4x}{7}=\frac{4}{3}$
14. $\frac{x^2}{6}=\frac{x}{2}+9$
15. $\frac{4x}{5}=\frac{x^2}{15}+\frac{7}{3}$
16. $\frac{x}{2}+\frac{15}{7}=\frac{x^2}{14}$
17. $\frac{x^2}{13}+\frac{6x}{13}=7$
18. $\frac{4x}{x-1}+\frac{x-10}{x}=4$
19. $\frac{5y-1}{9}+\frac{3y-1}{5}=\frac{10y}{9}-\frac{4}{9y}$
20. $\frac{a+7}{9-4a^2}=\frac{1-a}{2a+3}+\frac{4}{2a-3}$

Solve the following problems:

21. The base of a triangle exceeds the altitude by 4 in., and the area is 30 square inches. Find the base and altitude.

22. A rectangular field is twice as long as wide. If it were 20 rd. longer and 24 rd. wider, the area would be doubled. What are the dimensions?

23. The perimeter of a rectangular field is 60 rods. The area is 200 square rods. Find the dimensions.

24. A tree standing on level ground was broken over so that the top touched the ground 50 ft. from the stump. The stump was 20 ft. more than two-fifths of the height of the tree. What was the height of the tree?

412. Quadratic equations solved by completing the square.

EXERCISES

Solve the following problems:

1. The area of a rectangle 10 units long is 21 square units greater than the area of a square whose side equals the unknown dimension of the rectangle. Find the unknown dimension.

Show that $x^2-10x+21=0$ is the algebraic statement of the problem.

Write the equation thus:

$$x^2-10x=-21$$

Add 25 to both sides to make the first side a trinomial square:

$$x^2-10x+25=4$$

or,

$$(x-5)^2=4$$

Take the square root of both sides, remembering that 4 has two square roots, $+2$ and -2; thus:

$$x-5=\pm 2^*$$

Using the $+$ sign

$$x-5=\quad 2, \text{ whence } x=+7$$

Using the $-$ sign,

$$x-5=-2, \text{ whence } x=+3$$

Check both values of x by substituting in the equation,

$$x^2-10x+21=0$$

The algebraic method just given is called the method of solving the equation by **completing the square.**

2. One dimension of a rectangle is 5 units and the other is equal to the side of a square. The sum of the areas of the rectangle and of the square is 36 square units. Find the unknown dimension of the rectangle.

Show that $y^2+5y-36=0$ is the algebraic statement of the problem.

* **Square-root axiom:** *Any number has two square roots of the same absolute value, but of contrary sign.*

Solve the equation, $y^2+5y-36=0$ by the method of completing the square:

$$y^2+5y+\frac{25}{4}=36+\frac{25}{4}=\frac{169}{4}$$

$$y+\frac{5}{2}=\pm\frac{13}{2}$$

$$y=\frac{-5\pm 13}{2}=4, \text{ or } -9$$

Hence, the unknown dimension is 4 units.

413. Though a quadratic equation generally has *two* solutions, this does not mean that every *problem* that leads to a quadratic equation has two solutions. The nature of the conditions of the problem may be such as to make one, or even both, of the solutions of the quadratic impossible, or inappropriate, or meaningless. When neither of the two solutions of the quadratic is a solution of the problem it usually means that the conditions of the problem are impossible, or are contradictory, or that the problem is erroneously stated. To decide which solution, if either, meets the conditions stated in a problem, it is necessary to substitute the solutions in the conditions of the problem, and to reject solutions of the equation which do not meet the conditions.

The graphical method of solving quadratic equations may furnish only approximate values of x for the corresponding quadratic equations. The algebraic method furnishes exact solutions for quadratic equations.

PROBLEMS AND EXERCISES

Solve the following quadratic equations by the method of completing the square, and check:

1. $4x^2-12x+5=0$

Dividing both sides of the given equation by the coefficient of x^2,

$$x^2-3x+\frac{5}{4}=0$$

This may be written $x^2-3x=-\frac{5}{4}$

Adding to both sides of the equation the square of one-half of the coefficient of x,

$$x^2-3x+\frac{9}{4}=1$$

or,

$$\left(x-\frac{3}{2}\right)^2=1$$

Extracting the square root of both sides of this equation,

$$x-\frac{3}{2}=\pm 1$$

Whence,

$$x=\frac{3}{2}\pm 1$$

and

$$x=\frac{5}{2}, \text{ or } \frac{1}{2}$$

Check by substituting in the given equation:

2. $6x^2-17x-14=0$
3. $6x^2+7x-20=0$
4. $9x^2+30x-24=0$
5. $4s^2+45s-36=0$
6. $10s^2-21s-10=0$
7. $12s^2-71s+42=0$
8. $4x^2+8x-5=0$
9. $x^2+2x-3=0$
10. $x^2+4x+3=0$
11. $x^2+4x-5=0$
12. $a^2+8a-20=0$
13. $y^2+14y+45=0$
14. $t^2+14t-51=0$
15. $k^2-85=12k$
16. $z^2=10z+24$
17. $w^2-91=6w$
18. $r^2+3r+2=0$
19. $m^2+5m+6=0$
20. $10x^2+21x-10=0$
21. $h^2+40=13h$
22. $x^2+x=42$
23. $x^2+6x+5=0$
24. $\frac{x^2}{24}-\frac{x}{3}-=\frac{1}{2}$
25. $\frac{5}{z}=\frac{z-1}{z}+\frac{z}{z-1}$
26. $\frac{2t+3}{t+8}-\frac{2t+9}{3t+4}=0$
27. $\frac{k+1}{k+2}-\frac{k+3}{k+4}=\frac{8}{3}$

Solve the following problems:

28. The length of a rectangular field is 4 yd. more than the width, and the area is 60 square yards. Find the dimensions.

29. The hypotenuse of a right triangle is 10 ft. and one of the sides is 2 ft. longer than the other. Find the length of the sides.

30. The sum of the area of two square fields is 61 sq. rd., and a side of one is 1 rd. longer than a side of the other. Find the sides of both squares.

31. What must be the dimensions of a coal-bin to hold 6 tons of coal, if the depth is 6 ft. and the length is equal to the sum of the width and depth, allowing 40 cu. ft. of space per ton of coal?

32. Telegraph poles are placed at equal distances along a railway. In order that there be two less per mile it would be necessary to increase by 24 ft. the distance between every two consecutive poles. Find the number of poles to the mile.

Summary

The chapter has taught the following:

414. Certain multiplications occurring frequently in algebra may be performed mentally. They are: the square of a binomial, the square of a trinomial, the product of the sum of two numbers by their difference, and the product of two factors of the form $ax+b$ and $cx+d$.

415. Factors of the following polynomials may be found by inspection: The quadratic trinomial square, the difference of two squares, and the quadratic trinomial of the form ax^2+bx+c.

416. Arithmetical numbers may be squared according to the formula $a^2+2ab+b^2$.

417. Several illustrations of the theorem of Pythagoras.

418. The extraction of the square root of arithmetical numbers.

419. Quadratic equations may be solved by graph, by factoring, and by completing the square.

CHAPTER XVI

PROBLEMS LEADING TO EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN

Solution of Problems and Equations

420. Arithmetic and algebraic solution of problems. The problems of the foregoing chapters have shown the advantage of using letters to represent numbers when stating and solving problems. Many problems in arithmetic are simplified when solved by algebra, as may be seen by comparing the following arithmetic and algebraic solutions.

A man after traveling 9 mi. finds that he has yet $\frac{7}{10}$ of his trip to make. How long is his trip?

Arithmetic Solution

$\frac{10}{10}$ of his trip equals the whole distance
$\frac{7}{10}$ " " " " " distance still to be traveled
$\frac{10}{10}-\frac{7}{10}=\frac{3}{10}$ " " " already traveled
9 mi. " " " " "
Hence, $\frac{3}{10}$ of his trip equal 9 mi.
$\frac{1}{10}$ " " " " 3 mi.
$\frac{10}{10}$ " " " " 30 mi.
Therefore his trip is 30 mi. long.

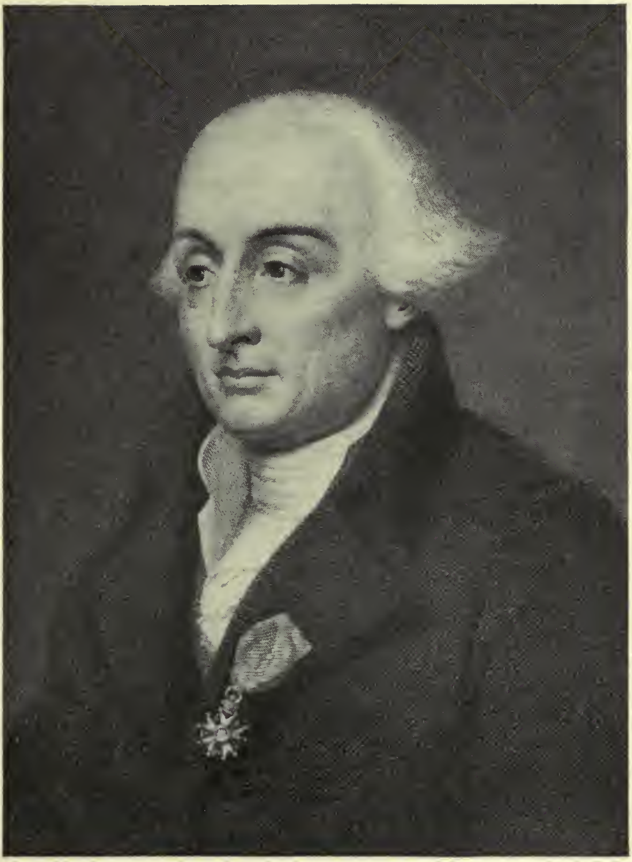
Algebraic Solution

Let x be the number of miles in the whole trip.

Then $9+\frac{7x}{10}=$ the whole distance

Hence
$$x=\frac{7x}{10}+9$$
$$10x=7x+90$$
$$3x=90$$
$$x=30$$

Therefore the trip is 30 mi. long.



JOSEPH LOUIS LAGRANGE

JOSEPH LOUIS LAGRANGE

JOSEPH LOUIS LAGRANGE, the greatest mathematician of the eighteenth century, was born at Turin, Italy, January 25, 1736, and died at Paris, April 10, 1813. It was by his own industry and ability that he rose to the front rank of mathematicians. He showed little taste for mathematical studies before he was seventeen. Happening upon a memoir by Halley that interested him, he threw himself into the study of mathematics, and after a year of hard work became so accomplished a mathematician that he was made a lecturer in the artillery school. He at once took up an isoperimetrical problem of long-standing difficulty among mathematicians, solved it by methods of his own devising, winning the admiration of Euler, and, by nineteen years of age, a place in the very front rank of living mathematicians.

In 1758 he established the Turin Academy, and most of his early writings are found in the five volumes of its transactions. In 1761 he stood without a rival as the foremost living mathematician. In 1766 Euler left Berlin for St. Petersburg, and Frederick the Great wrote to Lagrange saying "the greatest king in Europe" wished "the greatest mathematician in Europe" to reside at his court. Lagrange accepted the invitation, and during the 22 years of his stay here did a prodigious amount of work. In 1787 Frederick the Great died, and in 1788 Lagrange accepted an offer of Louis XVI of France to move to Paris. Here he remained, and wrote and taught mathematics during those troublous times in France, until his death. He was befriended by three rulers—Frederick the Great, Louis XVI, and Napoleon. For a fuller account of the labors of Lagrange at Berlin and at Paris read Ball's *History of Mathematics*, pp. 404–11 (5th ed.).

Ball says of his personal appearance: "he was of medium height, and slightly formed, with pale blue eyes and a colorless complexion." Of his character Ball remarks: "he was nervous and timid, he detested controversy, and to avoid it willingly allowed others to take the credit for what he had himself done."

421. Stating and solving problems. Stating a problem in algebra usually means expressing in algebraic symbols certain number relations that are given in words in the problem.

Skill in stating and solving problems can be acquired best through much practice. The statement of a problem in most cases takes the form of an equation. To obtain this equation the following rules may be useful:

I. *Denote the unknown number by a letter, then translate the verbal statement of the number relations into a symbolic statement in equation form.*

The first letters of words are convenient letters to denote numbers while they are yet unknown, as n for number, t for time, a for age, w for weight, etc.

EXERCISES

Give statements of the following problems, then solve and check.

1. What number increased by 6 gives 13?

Statement:

$$n \quad + \quad 6 \quad = \quad 13$$

Solve the equation and check by substitution.

2. Two-thirds of a number diminished by 12 equals 4. Find the number.

Statement:

Two-thirds of a number diminished by 12 equals 4

$$\tfrac{2}{3} \quad \times \quad n \quad - \quad 12 \quad = \quad 4$$

Solve the equation and check by substitution.

3. Translate the following equations into words and find the value of x:

1. $x-17=19$	4. $2x-3=3x-7$
2. $66=x+48$	5. $18-x=4x-3$
3. $2.8=2x+1.9$	6. $8x-16=7x-9$

Sometimes the equation may be obtained as follows:

II. *By the aid of literal and of arithmetical numbers obtain two different number expressions for some number of the problem, and write the two expressions equal to one another.*

Thus, an equation states in symbols that two different number expressions stand for the same number.

EXERCISES

1. A rectangle is 3 times as long as wide, and the perimeter is 48 inches. Find the width and length.

Equate two different expressions of the perimeter.

2. The angles of a triangle are x, $2x$, and $3x$ degrees. Find the numerical values.

3. The acute angles of a right triangle are $4x$ and $5x$. Find the acute angles.

422. Notation for unknown numbers. While any letter may be used to denote the *primary unknown* it is customary to use x, y, or z or some one of the later letters of the alphabet for this purpose.

The earlier letters of the alphabet, as a, b, l, m, k, etc., are commonly used to denote *known* numbers.

Solve the following equations for x; i.e., find the value of x in terms of the other letters in the equation:

1. $2x-28a=x-11a$

2. $4x+b=5x-19b$

423. Solving equations. The solution of an equation consists in getting another equation in which the unknown is alone on one side. This equation is obtained by means of the axioms stated in § 89.*

One of the important form changes is the removal of parentheses, as in the solution of the following problem:

The difference of double a boy's age and 3 times his age 10 years ago is his present age. How old is the boy?

Statement:	$2x-3(x-10)=x$	(1)
Since $-3(x-10)=-3x+30$, hence	$2x-3x+30=x$	(2)
Since $2x-3x=-x$, hence	$-x+30=x$	(3)
Adding x to both sides,	$30=2x$	(4)
Dividing by 2,	$15=x$	(5)

The boy's age is 15 years.

Check by testing in the conditions of the problem. State the axioms used to get equations (4) and (5).

EXERCISES

Solve the following equations, in each case checking the result:

1. $5x+9=3x+17$

	$5x+9=3x+17$	
Subtracting $3x$ from both sides,	$3x=3x$	
	$2x+9=17$	(State axiom used.)
Subtracting 9,	$9=9$	
	$2x=8$	
Dividing both sides by 2,	$\dfrac{2x}{2}=\dfrac{8}{2}$	
or,	$x=4$	(State axiom used.)

Check: $5x+9=5\cdot4+9=29$
$3x+17=3\cdot4+17=29$

Hence, 4 is a root of the equation $5x+9=3x+17$

*Mohammed ben Musa Al Hovarezmi, who lived during the reign of Caliph Al Mamun (813–33), was the first notable Arabian author of mathematical books. The title of a book in which he explains the solution of equations is *Aldshebr walmukabala*. By "aldshebr" is meant the transposing of terms from one side of an equation to the other. This is the origin of the word "algebra." See Ball, pp. 156–57.

The preceding solution may be shortened by doing mentally some of the work written out in full.

Thus, $5x+9=3x+17$

Subtracting $3x+9$ from both sides, $2x=8$

Dividing by 2, $x=4$

2. $5x+17=8x-2$
3. $3x+5=5x-15$
4. $6x-7=10x+1$
5. $5x-17+3x-5=x-1$
6. $6x-7-8x+115=0$
7. $5(x-1)=3(x+1)$
8. $8(3-2x)=2(5-x)=28$
9. $-3x-24=33(2-x)$
10. $5(1-13x)=35-105x$
11. $11-3(x-2)=x-8$
12. $3x-2(x+5)=6x-20$
13. $3(x-2)+15=5x-3$
14. $\frac{3x}{4}+5=x+3$

    Multiply both sides by 4
15. $\frac{x}{4}+2x-8=3x-5$
16. $\frac{3x}{4}-91=-5-10x$
17. $8+2x+\frac{x}{4}=1\frac{3}{4}+\frac{2x}{3}$
18. $\frac{2x+3}{5}-\frac{x-2}{3}=\frac{7}{5}$
19. $\frac{3x-2}{7}-\frac{1-4x}{3}=8\frac{4}{21}$
20. $\frac{3(5-x)}{4}+\frac{6-x}{5}=2x$
21. $\frac{1}{2}(5y-3)-\frac{1}{3}(5y-2)=5$
22. $\frac{3}{4}(7n+4)+\frac{2}{3}(7n-1)=10n+2$
23. $(y-7)(y-8)=(y-5)(y-9)$
24. $(4m-5)(3m+1)=12m(m+1)$
25. $x(x-1)-x(x-2)=4(x-3)$

Geometric Problems

424. The following problems contain geometric relations. Besides expressing these relations in algebraic symbols, a figure representing them should be drawn in each case before obtaining the equation.

1. In an isosceles triangle the exterior angle at the base is twice as large as the exterior angle at the vertex.

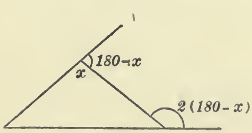


Fig. 263

Find the interior angles of the triangle.

Denoting the vertex angle, Fig. 263, by x, the exterior angle at the vertex is $180-x$. The exterior angle at the base is therefore $2(180-x)$. Hence, the interior angle at the base is $180-2(180-x)$. Why? What is the third interior angle of the triangle?

Show that $x+2[180-2(180-x)]=180$ and solve the equation.

2. One of two supplementary adjacent angles is 4.5 as large as the other. How large is each?

3. The difference of the acute angles of a right triangle is 18°12′. How large is each?

4. A rectangle is 2 units longer than 3 times the width, and the perimeter is 60. Find the length and width.

5. The perimeter of a rectangle is 20 inches. If the base is decreased by 4 in. and the altitude increased by 2 in., the area remains unchanged. What are the dimensions of the rectangle?

6. Find the sides of an isosceles triangle whose perimeter is 360 in. and whose base is 75 inches.

7. A classroom of a certain high school is $\frac{8}{5}$ as long as it is wide. If the length were diminished 3 ft. and the width increased by the same amount the room would be square. Find the dimensions.

8. The area of a square is equal to that of a rectangle having its base 12 ft. greater and its altitude 4 ft. shorter than the side of the square. Find the dimensions of both figures.

9. The area of a triangle, whose altitude is 5 units less than the base, is equal to 10 square units less than half the area of a square on the base of the triangle. Find the base and the altitude of the triangle.

10. The United States Treasury building in Washington, D.C., is 222 ft. longer than it is wide. The entire distance around the building is 1,500 feet. Find its length and width.

11. The length of the Pennsylvania Station in New York is 80 ft. less than twice the width and $\frac{1}{4}$ as long as twice the distance around diminished by 700 feet. How long and how wide is this station?

12. The perimeter of a rectangular steel plate is 240 in. and the length is 85.24 inches. Find its width and area.

What is the weight of this plate, if it is $\frac{5}{8}$ in. thick, and if 1 sq. ft. of $\frac{1}{8}$-in. plate weighs 5.1 pounds?

13. The sides of a triangle are in the ratio 3:4:5. Find the three sides, if the perimeter is 108 inches.

14. One of the acute angles of a right triangle is 8 times as large as the other. Find the two acute angles.

15. Find the angles of an isosceles triangle if one of the base angles is 50°.

16. One angle of a triangle is 5° greater than 5 times another. Find the third angle, if the smaller of the first two angles is 19°. Make a scale drawing of the triangle, if the side opposite the greatest angle is 60 feet.

17. The interior angles on the same side formed by two parallel lines cut by a transversal are $\frac{3}{5}x$ and $88+\frac{1}{5}x$. Find x and the angles.

18. The acute angles of a right triangle are denoted by $\frac{1}{3}x+7$ and $41-\frac{1}{6}x$. Find x and the angles. Draw the triangle.

19. Two supplementary adjacent angles are denoted by $\frac{5}{7}x+29$ and $97-\frac{2}{7}x$. Find x and the angles.

20. The interior angles of a triangle are 69°, $\left(\frac{x}{4}+17\right)^\circ$, and $\left(\frac{x}{3}+80\right)^\circ$. Find x and the two unknown angles of the triangle.

Problems Involving Number Relations

425. The following problems give training in expressing number relations in algebraic form.

1. Express that one-eighth of the double of n, increased by 25, equals one-half of n.

2. Write in symbols an expression for double a number, increased by 3 times the sum of the number and 4.

3. Write an expression for double a number, decreased by 3 times the difference between the number and 4.

In the phrases "the difference between 8 and 4," "the difference of 8 and 4," and the like, it is understood that the first-mentioned number (the 8) is the minuend; thus $8-4$, not $4-8$, is the difference between 8 and 4.

4. Write the double of a number, decreased by 3 times the difference of the number and 2.

5. Write in symbols 4 times the difference of a number and 3, decreased by 3 times the difference of the number and 1.

6. Show, in symbols, that the double of a number, increased by 3 times the difference of the number and 4, equals 13.

7. One-fourth of the difference of 3 times a number and 8 is 10. Find the number.

8. Three times a number is 56 greater than one-third of the number. Find the number.

9. What number multiplied by 2.5 gives 40?

10. Three times a number, increased by 5 times the number, gives 72. Find the number.

11. If 1 is added to 4 times a number and subtracted from the number, the ratio of the results is 5. Find the number.

12. Five times a number is increased by 3 and the sum is divided by the sum of the number and 4. The result is then equal to 4. Find the number.

13. I have in mind a certain number. You can determine it from the following data: If you multiply the number increased by 3 by the number decreased by 5 and divide the product by the number decreased by 7, the result is the same as the number increased by 1.

14. A father is 40 years old and his son is 7. In how many years will the father be twice as old as the son?

15. The combined age of a father, mother, and son is 75 years. The mother is 3 times as old as the son and the father $1\frac{1}{6}$ times as old as the mother. How old is each?

16. Double the number of years in a boy's age is 16 more than his age 2 years ago. How old is the boy?

17. The difference of double a boy's age and 3 times his age 10 years ago is his present age. How old is the boy?

18. Express in symbols that the sum of three consecutive integers, differing by 3, equals 27. Find the three integers.

19. The difference of the squares of two consecutive numbers is 19. Find the numbers.

20. The difference of the squares of two consecutive numbers is 273. Find the numbers.

21. The difference of the squares of two consecutive numbers is a. Find the numbers.

22. The difference of the squares of two consecutive even numbers is 28. Find the numbers.

23. The difference of the squares of two consecutive even numbers is 100. Find the numbers.

24. The difference of the squares of two consecutive even numbers is a. Find the numbers.

25. The difference of the squares of two consecutive odd numbers is 48. Find the numbers.

26. The difference of the squares of two consecutive odd numbers is s. Find the numbers.

Motion Problems

426. Rate. Distance. Time. If a train travels 30 mi. an hour, it is said to move at a **rate** of 30 mi. an hour. If a man walks 35 yd. a minute, he is said to walk at the rate of 35 yd. a minute.

The distance traveled by a body depends upon the time it travels and upon the rate at which it travels. The following problems will show the relation between the distance, rate, and time.

EXERCISES

1. The rate of a train is 40 mi. an hour. If it leaves the station at 1:00 P.M. how far is it at 2:00 P.M., 3:00 P.M., 4:00 P.M., 5:00 P.M., etc.? How far is it at 3:20, 4:30, 6:45?

2. Denoting the distance traveled by d, find d when the rate is 60 mi. an hour and the number of hours is 5.

3. Let d be the distance traveled in t hr. at the rate of r mi. an hour. Show that $d=rt$. Translate this equation into words.

4. Show how to obtain from the equation $d=rt$, the equations $r=\frac{d}{t}$ and $t=\frac{d}{r}$. Translate these equations into words.

427. Relation between distance, rate, and time. The equations of problems 3 and 4 above show that if two of the three numbers d, r, and t are known or expressed in algebraic symbols, the third can always be expressed in terms of them.

EXERCISES

1. A bird flies a distance of 80 mi. in 2 hr. 30 minutes. Find the rate, supposing it to be uniform.

2. Sound travels 1,080 ft. a second. If the sound of a stroke of lightning is heard 3.5 sec. after the flash, how far away is the stroke?

3. If sound travels f ft. a second, how far away is a lightning stroke if the sound is heard s seconds after the flash?

4. A tree 2,376 ft. distant was struck by lightning. It took $2\frac{1}{5}$ sec. for the sound to reach the ear. Find the rate at which the sound traveled.

428. Graphical solution of motion problems. Motion problems may be solved graphically.

1. A man walks along a straight road, AB, Fig. 264, in the direction of the arrows ($\rightarrow$), at the rate of 3 mi. an hour. How far from the house, H, is he:

4 hr. before reaching it?	1 hr. after reaching it?
3 hr. before reaching it?	2 hr. after reaching it?
2 hr. before reaching it?	3 hr. after reaching it?
1 hr. before reaching it?	4 hr. after reaching it?

Denoting the number of hours before reaching the house by $-$ and the number of hours after by $+$, lay off these numbers on the time-line to the right or left of O according as they are $+$ or $-$. At each of the points -4 to $+4$ on the time-line lay off vertically the corresponding distances from the house, letting the vertical side of a square represent 3 mi. In this way the points P, Q, R, S ...... X are obtained. If these points are joined by straight lines, it will be found that they all lie in the same straight line, PX.

Time	Distance
-4	-12
-3	-9
-2	-6
-1	-3
1	3
2	6
3	9
4	12

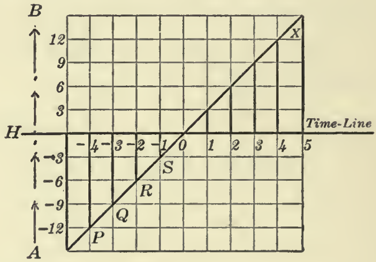


Fig. 264

The straight line PX represents geometrically the uniform motion of the man walking from A to B.

2. Show from the graph how far from the house the man was $3\frac{1}{2}$ hr. after reaching it. Show from the graph when the man was $4\frac{1}{2}$ mi. from the house; $3\frac{1}{3}$ mi. from the house.

3. At 10:00 A.M. a freight train leaves a station O going north at the rate of 30 mi. an hour. At 1:00 P.M. an express passes the station going in the opposite direction at the rate of 60 mi. per hour. When and where did the trains meet?

Graphic Solution

On the time-line (Fig. 265) lay off the time, letting one side of a square represent one hour. At each hour-point lay off vertically the distance from the station, letting a vertical side of a square represent 30 mi. The points A, B, C . . . . . . thus obtained are joined by the straight line PQ. PQ is the graph for the freight train.

Similarly, the graph LM for the express train is obtained. The point of intersection C of PQ and LM gives the distance of each train from the station at 12:00 o'clock. Why? Thus, the trains meet at 12:00 o'clock 60 mi. from the station.

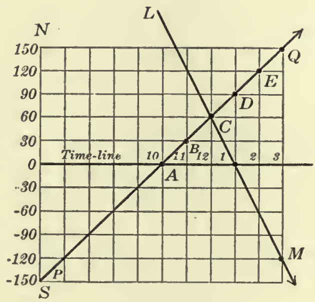


FIG. 265

Algebraic Solution

(1) Let x be the number of hours from 10:00 A.M. until the trains meet.

(2) Then the time t; rate r; and distance, d, for each train are as follows:

For the freight train $\begin{cases} t = x \\ r = 30 \\ d = 30x \end{cases}$

For the express train $\begin{cases} t = 3 - x, \text{ since it passed the station 3 hr.} \\ r = 60 \qquad \text{after the freight train left} \\ d = 60(3 - x) \end{cases}$

(3) The fact that the distance from the station to the meeting-point is the same for both trains gives the equation

$$30x = 60(3 - x)$$

(4) Solve the equation in (3).

4. Solve graphically the following problem:

A, starting from point P, moves downstream at the rate of 8 mi. an hour. B starts 3 minutes later and moves at the rate of 10 mi. an hour. When and how far from P will B overtake A?

EXERCISES

Solve the following problems algebraically:

1. A local train goes at the rate of 30 mi. an hour. An express starts two hours later and goes at the rate of 50 mi. an hour. In how many hours, and how far from the starting-point, will the second train overtake the first?

(1) Let x be the number of hours it takes the second train to overtake the first train.

(2) Then, according to the data of the problem,

for the express train	for the local train
$t=x$	$t=x+2$
$r=30$	$r=50$
Hence $d=30x$	Hence $d=50(x+2)$

(3) Since both trains travel the same distance, the following equation holds:

$$50(x+2)=30x$$

(4) Solve this equation to determine the value of x, and then solve for d.

2. An express train whose rate is 40 mi. per hour starts 1 hr. 4 min. after a freight train, and overtakes it in 1 hr. 36 min. How many miles per hour does the freight train run?

3. Two friends, A and B, live at the distance of 33 mi. from each other. In order to meet A, B leaves home an hour earlier than A. If A travels at the rate of 5 mi. and B at the rate of $4\frac{1}{2}$ mi. an hour, when and where will they meet?

4. Two towns, A and B, are 288 mi. apart. Two men travel toward each other, starting at the same time from A and B respectively, and going at the rates that are to each other as 3:5. They meet in three days. How many miles a day does each travel?

5. A is 160 yd. east and B 112 yd. west of a gate. Both start at the same time to walk toward the gate, A going 3 yd. and

B 2 yd. a second. When will they be at equal distances from the gate?

The drawing, Fig. 266, will be helpful in obtaining the equation.

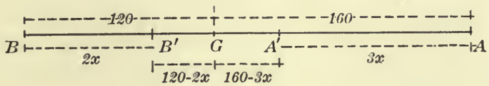


FIG. 266

6. Two trains start at the same time from Chicago and Kansas City respectively, 498 mi. apart. The train from Chicago travels at a rate of 40 mi. an hour, the other at a rate of 50 mi. an hour. When and where will they meet?

Make a drawing representing the distances and the equation.

7. At 6:00 A.M. a train leaves New York for Buffalo on the New York Central. At 9:00 A.M. a second train follows. The first train travels 30 mi. an hour, the second 50 mi. an hour. When and how far from New York will the second overtake the first?

8. A man rows downstream at the rate of 6 mi. an hour and returns at the rate of 3 mi. an hour. If he has 9 hours at his disposal, how far downstream can he go and return?

9. Two trains start at the same time from S, one going east at the rate of 35 mi. an hour, and the other going west at a rate $\frac{1}{7}$ faster. How long after starting will they be 100 mi. apart?

10. A, B, and C are three towns on a straight road. The distance from A to B is 20 miles. A man leaves B for C and travels at the rate of 5 mi. an hour. At the same time an automobile traveling at the rate of 30 mi. an hour leaves A to go to C. When and where will the auto overtake the man?

11. At 10:00 A.M. a freight train leaves the station A, traveling at the rate of 32 mi. an hour. An express train going at the rate of 72 mi. an hour passes the station at 11:00 A.M. and follows the freight train. When and where will the trains pass?

12. A train moves at a uniform rate. If the rate were 6 mi. an hour faster the distance it would go in 8 hr. is 50 mi. greater than the distance it would go in 11 hr. at a rate 7 mi. an hour less than the actual rate. Find the actual rate of the train.

429. Motion along a circular path. In the following problems the path of motion is assumed to be circular and the motion uniform.

EXERCISES

1. Two boys were running in the same direction along a circular path at the rates of 110 yd. and 98 yd. a minute, respectively. The length of the circle is 36 yards. Supposing they start from the same point, when will they be together again?

(1) Show that in x minutes they run $110x$ yd. and $98x$ yd., respectively.

(2) Then $110x=98x+36$ expresses the fact that the faster runner has made a gain of a complete circle, which is necessary to enable him to meet the slow runner.

(3) Solve the equation in (2).

2. Suppose the boys in exercise 1 are running in opposite directions. How long will it take them to meet?

3. Two automobiles travel over a road which runs around a lake practically circular in shape. They make the circuit in 2 hr. 45 min. and 3 hr. 30 min., respectively. How long after passing each other will they meet?

(1) Let x be the required number of hours

(2) $\begin{cases} \dfrac{1}{2\frac{3}{4}} \text{ is the part of the circuit made by one automobile in 1 hr.} \\ \dfrac{1}{3\frac{1}{2}} \text{ is the part of the circuit made by the other automobile in 1 hr.} \end{cases}$

(3) $\begin{cases} \dfrac{x}{2\frac{3}{4}} \text{ is the number of circuits made by the first automobile in } x \text{ hr.} \\ \dfrac{x}{3\frac{1}{2}} \text{ is the number of circuits made by the second automobile in } x \text{ hr.} \end{cases}$

(4) Since the faster automobile makes the circuit once more than the slower, it follows that $\frac{x}{2\frac{3}{4}}=\frac{x}{3\frac{1}{2}}+1$.

4. Venus makes its orbit in 224.7 days, or about $7\frac{1}{2}$ months, the earth starting as shown in Fig. 267. In how many days will Venus next be in line between the earth and the sun?

The rate per month of Venus is $\frac{2}{15}$ of the orbit, that of the earth $\frac{1}{12}$. Let x be the required number of months.

The equation may be obtained as in exercise 3.

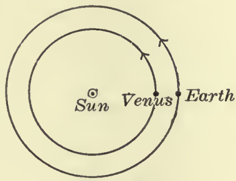


FIG. 267

5. Calling the time of revolution of Venus about the sun $7\frac{1}{2}$ months, and that of Mercury 3 months, how many months after Mercury is in the line between Venus and the sun will it next be in the same relative position?

Clock Problems

430. The movement of the hands of a clock furnishes a mechanical illustration of circular motion.

1. At what time between 3:00 and 4:00 o'clock are the hands of the clock together?

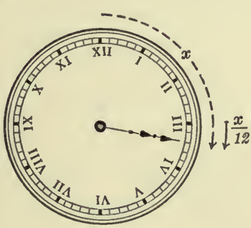


FIG. 268

(1) Let x, Fig. 268, be the number of minutes after 3:00 o'clock when the hands are together; i.e., x is the number of minute-spaces over which the minute hand passes from 3:00 o'clock until it first overtakes the hour hand.

(2) Then $\frac{x}{12}$ is the number of minute-spaces over which the hour hand passes in the same time. Why?

(3) Since the number of minute-spaces from 12:00 to 3:00 is 15, it follows that $x=15+\frac{x}{12}$. (The whole is equal to the sum of the parts.)

(4) The equation in (3) determines the value of x.

2. At what time between 2:00 and 3:00 o'clock are the hands of the clock together?

Represent in a drawing the number of minute-spaces passed over by the hour hand and by the minute hand.

3. At what time between 3:00 and 4:00 o'clock are the hands of the clock at right angles?

4. At what time between 3:00 and 4:00 o'clock are the hands of the clock 16 minute-spaces apart?

5. At what time between 7:00 and 8:00 o'clock are the hands of the clock pointing in opposite directions?

6. At 12:00 o'clock the hands of a clock are together. When will they be together the next time?

7. What angle is formed by the hands of a clock at 2:30?

8. In how many minutes does the minute hand of a clock gain 15 minute-spaces on the hour hand?

9. At what time between 5:00 o'clock and 6:00 o'clock will the hands be together?

10. When between 8:00 o'clock and 9:00 o'clock is the minute hand 10 minutes behind the hour hand? When is it over the hour hand? When is it 10 minutes ahead of the hour hand?

EXERCISES

Solve the following equations:

1. $4(2x+9)+3(x-9)=\frac{13(x-5)}{7}$
2. $2(x-1)+3(x-2)=4(x+3)$
3. $5x-2k=2k+3x$
4. $ax+ad=bd-bx$
5. $a^2x+b=b^2x+a$
6. $(2y-5)\ (4y-7)=8y^2+52$
7. $(a+x)^2-x^2=b^2$
8. $(n+4)\ (n+3)-(n+2)\ (n+1)=42$

9. $\frac{3x}{4}+37+\frac{2x}{3}+109=180$

10. $\frac{2x}{3}+93+\frac{x}{2}-180=18$

11. $2\left(\frac{x}{5}+\frac{x}{6}\right)-180=\frac{1}{2}\left(309-\frac{x}{3}\right)$

12. $2\left(\frac{x}{2}+x\right)-90+\frac{1}{6}(x+210)=90$

13. $100-4\left(x+\frac{x}{16}\right)=\frac{1}{3}(261-5x)-80$

14. $\frac{5}{6}(12y-\frac{6}{5})-80=4+\frac{6}{7}(14y+\frac{7}{6})$

15. $\frac{1}{7}(5n-1)-\frac{1}{3}(4n-2)=8$

16. $\frac{1}{4}(1-s)=\frac{1}{5}(2-s)+\frac{1}{6}(3+s)$

17. $t-15-\frac{1}{6}(9t-2)-\frac{3}{4}t-\frac{1}{3}=0$

18. $5.8x+3.69=3.96+2.8x$

19. $.374x-.53+1.2x+.06=.8+1.32x$

20. $.3(1.5x-.8)=.6(5.1+.2x)$

21. $.05(20x-3.2)=.8(4x+.12)-11.256$

22. $1.4x-1.61-\frac{.21x+.012}{.8}=1.3x$

23. $\frac{1-2x}{.25}-\frac{2x-.5}{12.5}+\frac{2x-\frac{1}{3}}{5}=\frac{6.35-.5x}{3}$

24. $\frac{.4x+.39}{7}-\frac{.2x-.66}{.9}=\frac{.08x+.38}{.2}$

Problems on Percentage and Interest

431. Some problems on percentage and interest lead to equations of the first degree.

1. Find the percentage of \$120 at 1%.

The expression 1 per cent means $\frac{1}{100}$. Thus, 1 per cent of \$120 means $\frac{1}{100}$ of \$120 or \$120 multiplied by $\frac{1}{100}$.

Therefore the percentage of \$120 at 1 per cent is $\frac{1}{100}\cdot \$120$ or $\frac{\$120}{100}$.

2. Find the percentage of \$120 at 4%.

Since the percentage of \$120 at 1% equals $\frac{1}{100}\cdot\$120$, it follows that the percentage at 4% equals $4\cdot\frac{1}{100}\cdot\$120=\frac{4\cdot\$120}{100}$.

3. Find the percentage of \$120 at 6%; at $7\frac{1}{2}$%; at r%.

4. Find the percentage at 8% of \$25; of \$250; of \$$b$.

5. Find the percentage at r% of \$20; of \$80; of \$$b$.

6. Calling p the percentage, b the base, and r the rate, show that

$$p=b\times\frac{r}{100} \tag{1}$$

and show that, by one of the laws of multiplication of fractions, this may be written

$$p=\frac{b\times r}{100} \tag{2}$$

7. Read (2) of exercise 6 and translate it into words.

8. Find the interest on \$175 at 4% for 2 years; for 5 years; for $\frac{3}{4}$ of a year; for $2\frac{7}{8}$ years; for t years.

9. Find the interest on \$600 for 5 years at 3%; at 5%; at 8%; at $6\frac{1}{2}$%; at r%.

10. Find the interest on \$160 for t years at 6%; at $3\frac{1}{2}$%; at r%.

11. Find the interest at 6% for 3 years on \$200; on \$360; on \$756; on \$$p$.

12. Find the interest at 5% for t years on \$$p$.

13. Find the interest i, at r% for t years on \$$p$.

14. State in words the meaning of

$$i=p\times\frac{r}{100}\times t=\frac{p\times r\times t}{100}$$

432. The result of exercise 14 above is a general formula for finding the interest when principal, rate, and time are given.

EXERCISES

1. Find the interest on a principle of \$225 for 5 years at $2\frac{1}{2}\%$.

Let $p=225$, $r=2.5$, $t=5$, and substitute these values in the formula of exercise 14 above.

2. Find the income from an investment of \$2,800 at 5% for 8 years.

3. Find the percentage of \$450 at 5%; of 375 bushels at 20%; of 1,800 men at 15%.

4. On a certain day a grocer sells \$315 worth of goods. His profit is 12% of the purchase price. How much did he gain?

Show that $x+\frac{x \cdot 12}{100}=315$.

5. A merchant lost \$345 in a certain sale. His loss was 9% of the amount invested. How much was the investment?

6. The number of pupils enrolled in the United States in 1910 was 17,813,852. During the following year there was an increase of 1.25%. How large was the enrolment in 1911?

7. Into what two parts can \$1,000 be divided so that the income of one at 6% shall equal the income of the other at 4%?

8. How can a man divide \$2,000 so that the income of part at 4% shall be the same as that of the rest at 5%?

9. How many dollars must be invested at 4% to give the same income as that of \$2,500 at 6%?

10. A certain sum invested at $5\frac{1}{2}\%$ gave the same interest in 4 years as \$3,300 gave in 8 years at 3%. How large was the sum?

11. Show how to divide \$1,400 into two parts so that one part at 4% shall produce twice as much income as the other at 3%.

12. A sum of \$2,200 is divided and invested so that the simple interest on one part at 5% equals the interest on the other at 6%. Find how the money is divided.

Mixture Problems

433. The total amount of the mixture in the following problems may be expressed in two ways. Solve the equations thus obtained.

1. Bell-metal is by weight 5 parts tin and 16 parts copper. How many pounds of tin and copper are there in a bell weighing 4,800 pounds?

First method: Let x be the number of pounds of tin, then show that $x+\frac{16x}{5}=4{,}800$, and solve for x; then find $\frac{16x}{5}$.

Second method (without fractions): Let $5x$ be the number of pounds of tin, and solve: $5x+16x=4{,}800$.

After finding x, calculate $5x$ and $16x$.

2. Gunpowder contains, by weight, 6 parts saltpeter, 1 part sulphur, and 1 part charcoal. How many pounds of saltpeter, of sulphur, and of charcoal are there in 120 lb. of gunpowder?

3. If gunpowder were composed of 4 parts, by weight, of saltpeter, 2 parts sulphur, and 3 parts charcoal, how many pounds of each would there be in 200 lb. of gunpowder?

4. With ingredients as in exercise 3, how much saltpeter is burned in the discharge of a cannon using 50 lb. of powder to the cartridge?

5. Baking powder is composed of 4 parts, by weight, of cream of tartar, 1 part starch, and 1 part soda. How much of each ingredient is there in 18 lb. of baking powder?

6. A certain mixture weighing 240 lb. contains 3 parts, by weight, of copper, 5 parts of iron, and 4 parts of carbon. How may pounds of each ingredient are there in the mixture?

7. In a watchcase weighing 2 oz. the gold is 14 carats fine; i.e., there are 14 parts of gold in every 24 parts of the whole alloy. How many ounces of pure gold are there in the case?

8. A certain compound contains, by weight, 5 parts carbon to every 3 parts of iron, and 7 parts of iron to every 2 parts of copper. In 124 lb. of the compound how many pounds are there of carbon, of iron, and of copper?

Lever Problems

434. Law of leverages. In order to solve by means of the equation certain problems arising out of the common uses of forces it is necessary to know a law of these forces.

A bar or lever (Fig. 269) has loadings as follows: A force of (+3) on an arm of (−6) and a force of (−3) on an arm of (−2). Adding the turning-tendencies, the total turning-tendency is

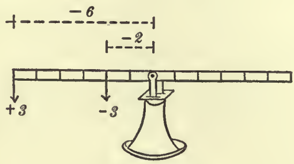


FIG. 269

$$(+3)(-6)+(-3)(-2)=(-18)+(+6)=-12$$

Thiss ays in mathematical language that the bar *turns* in *negative* direction, i.e., *clockwise.*

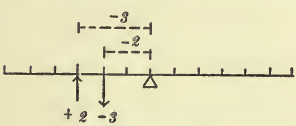


FIG. 270

If a force of (+2) on an arm of (−3) and a force of (−3) on an arm of (−2) are on the bar, Fig. 270, the total turning-tendency is

$$(+2)(-3)+(-3)(-2)=(-6)+(+6)=0$$

which means that the bar does *not* turn.

If a force of (+3) on an arm of (+2), a force of (−4) on an arm of (−9), and a force of (−12) on an arm of (+3) are on the bar, Fig. 271, the total turning-tendency is

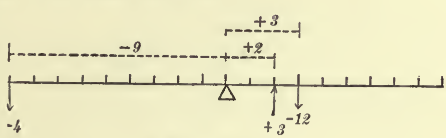


FIG. 271

$$(+3)\ (+2)+(-4)\ (-9)+(-12)\ (+3)=+6+36-36=+6$$

which shows that the bar *turns* in the *positive* direction.

Experiments like the one above show that: If two or more forces are acting on the bar at the same time the *total turning-tendency* is found by *adding* algebraically the separate turning-tendencies. If the *algebraic sum* is zero, the bar balances. If the sum is not zero, the bar turns in the direction indicated by the sign of the sum.

Thus, the following law will furnish an equation in lever problems:

Law of turning-tendencies or leverages: *For balance, the algebraic sum of all the turning-tendencies must equal zero.*

EXERCISES

Using the law of leverages solve the following problems:

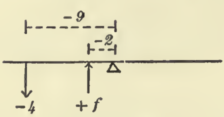


Fig. 272

1. If a force of -4 on an arm of -9, Fig. 272, and a force of $+f$ on an arm of -2, are on the bar, what must f be for balance?

If the bar is balanced, the sum of the turning-tendencies must be zero. We may then write

$$(f)(-2)+(-4)(-9)=0 \quad (1)$$

Multiplying, $-2f+36=0$ (2)

Subtracting 36, $-2f=-36$. What axiom? (3)

Dividing by -2, $f=18$. What axiom? (4)

Check: From equation (1),

$$(+18)(-2)+(-4)(-9)=(-36)+(+36)=0$$

2. A bar, Fig. 273, is balanced by a force of $+10$ on an arm of -6, and a force of $s+3$ on an arm of $+5$. Find the values of s and $s+3$.

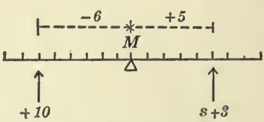


Fig. 273

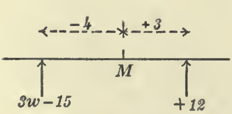


Fig. 274

3. A force of $3w-15$, Fig. 274, on an arm of -4 is balanced by a force of $+12$ on an arm of $+3$. Find the values of w and $3w-15$.

4. AB, Fig. 275, is a crowbar, $6\frac{1}{2}$ ft. long, supported at F, $\frac{1}{2}$ ft. from A. A stone presses down at A with a force of 1,800 pounds. How many pounds of force must be exerted by a man pressing down at B to raise the stone? The weight of the crowbar is to be disregarded.

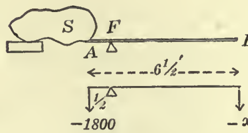


FIG. 275

5. With other conditions as in exercise 4, what would be the pressure at B if the fulcrum F (point of support) were 3 in. from A?

6. With the fulcrum $\frac{1}{2}$ ft. from A (Fig. 275) what weight would be held in balance by a pressure of 200 lb. at B?

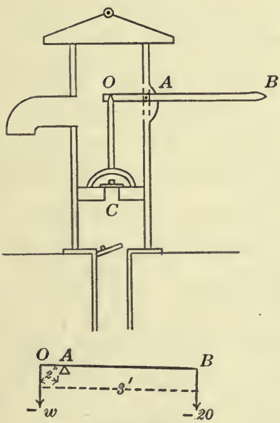


FIG. 276

7. A suction pump, Fig. 276, is a device for raising water from wells. The handle, OB, works against a pin at A, so that the hand pushing downward at B raises a mass of water by the aid of a piston C connected with the handle OB by the rod OC. If $OA = 2$ in. and $OB = 3$ ft., what load at O will be raised by a force of 20 lb. pushing downward at B?

8. With other conditions as in exercise 7 what force will be exerted at O by a downward force of 68 lb. at B?

9. A stone slab S, Fig. 277, weighing 2,400 lb., rests with its edge on a point B, 6 in. from the fulcrum F of a crowbar FA 6 ft. long. How many pounds of force must be exerted at A to raise the slab?

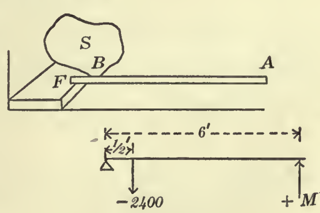


FIG. 277

10. Solve the following equations:

1. $3x+15=x+25$
2. $9x-8=25-2x$
3. $16+3x=6+x$
4. $20-4x=8-10x$
5. $\frac{x-3}{15}+\frac{x+2}{4}=6$
6. $x-\frac{-x}{5}-\frac{2}{7}=7\frac{5}{7}$
7. $\frac{7s+5}{11}-\frac{s+5}{3}=0$
8. $\frac{s-1}{s+1}=\frac{s+7}{s+17}$
9. $x-\frac{6-x}{2}=2-(x-5)$
10. $cx-\frac{a^2(x-3)}{c}=\frac{3a^2}{c}-\frac{a-c}{c}$
11. $s(s-1)-s(s-2)=3$
12. $(s+3)=s+3(s+8)$
13. $-\frac{s-1}{3}-\frac{s-2}{4}=2$
14. $(s+1)\ (s+3)=(s+2)\ (s+5)-13$
15. $4(s+6)-2(s-3)=38$
16. $\frac{1}{(s+3)(s+6)}=\frac{1}{(s+2)(s+8)}$
17. $\frac{s-4}{3}+\frac{s+4}{7}=\frac{s+20}{10}+\frac{s-5}{5}$
18. $\frac{2s-3}{4}-\frac{s+5}{2}-\frac{s-1}{2}=-1\frac{3}{4}$
19. $(5y+6)\ (2y+3)=11y+26+10y^2$
20. $0=(9y+4)\ (8y+9)-72y^2+39y-75$
21. $(120y^2+75y-165)\div 15=2y(4y-3)$
22. $(78y^2+37y-63)\div(6y+7)=64y-5(11y-15)$
23. $ay+by+cy=d+e$
24. $\frac{y}{a}+\frac{c}{b}=\frac{y}{b}+\frac{c}{a}$
25. $\frac{y-2a}{3b}=\frac{y-3b}{2a}$
26. $\frac{c}{y}+\frac{b}{c}=\frac{b}{y}+\frac{c}{b}$

Summary

435. The chapter has taught the following suggestions for obtaining the equation of the problem:

I. Denote the unknown number by a letter and translate the verbal statement of the number relations of the problem into a symbolic statement.

II. Obtain two different number expressions for some number of the problem and write them equal to one another.

436. The letters x, y, z or some of the other later letters of the alphabet are generally used to denote the *unknown numbers*. The earlier letters, as a, b, k, l, m, etc., are commonly used to denote *known numbers*.

437. An equation is solved by obtaining another equation in which the unknown is alone on one side. This is done by means of the addition, subtraction, multiplication, and division axioms.

438. Special devices have been taught, as being helpful in the solution of problems of certain types. Thus, in problems containing geometric relations a figure should be drawn. Often the equation is found by expressing algebraically some geometric theorem. Motion problems are solved by expressing algebraically two of the three elements—distance, rate, and time. The third is then found from these two by means of the equation: *distance equals rate times time*, or, $d=r \cdot t$.

439. Clock problems, mixture problems, percentage and interest problems, and lever problems all give practice in expressing algebraically the verbal statements of problems.

440. Law of leverages. For balance, the algebraic sum of all the turning-tendencies must equal zero.

CHAPTER XVII

LINEAR EQUATIONS CONTAINING TWO OR MORE UNKNOWN NUMBERS

A System of Two Linear Equations

441. Some problems lead to two linear equations in two unknowns.

Find two numbers such that 3 times the first diminished by 2 times the second is equal to 9, and 3 times the second is 4 greater than 2 times the first.

Let x and y denote the required numbers; then, by the first condition of the problem

$$3x-2y=9 \quad \ldots\ldots (1)$$

By the second condition of the problem

$$3y=4+2x \quad \ldots\ldots (2)$$

Solving both equations for y

$$y=\frac{3x-9}{2} \quad \ldots\ldots (1')$$

and $$y=\frac{2x+4}{3} \quad \ldots\ldots (2')$$

Equations $(1')$ and $(2')$ express y in terms of x. To every value assigned to x there is a corresponding value of y; e.g., from $(1')$ it follows that $y=6$, when $x=7$.

442. Solution of a system. *A pair of values of x and y* satisfying either equation is a **solution of that equation.** A pair of values of x and y satisfying both equations is a **common** solution of the two equations. Two equations having a common solution are called **simultaneous equations** and the pair of equations is referred to as a **system** of equations.

To solve the problem in § 441 means to find the solution of the system of equations (1) and (2), or of the system (1′) and (2′).

In the following paragraphs, two methods of obtaining the solution will be presented.

Graphical Method of Solving a System of Equations

443. Graphs of equations in two unknowns.

A quick way of obtaining the graph of the equation $y=\frac{3x-9}{2}$ is

(1) to let $x=0$. This gives $y=-\frac{9}{2}=-4.5$

(2) to let $y=0$. Then $\frac{3x-9}{2}=0$, $3x=9$, $x=3$

Thus, we have two solutions of the equation: $x=0$, $y=-4.5$, and $x=3$, $y=0$.

This is sufficient for drawing the graph. For the purpose of greater accuracy a third value of x, as $x=-5$, may be taken. The corresponding value of y is -12. Tabulate these three solutions of the equation, as in Fig. 278 (table 1).

Similarly obtain three solutions of the equation $y=\frac{2x+4}{3}$. (See table 2 to the left in Fig. 278.) The value $y=1.3+$ in this table is an approximation for $\frac{4}{3}$.

Each solution in the tables locates one point on the graph; e.g., to graph the solution $(-5, -12)$ pass from 0 5 units to the left and from there 12 units downward, locating point A. In the same way points B, C, D, E, and F are determined. Then draw lines ABC and DEF.

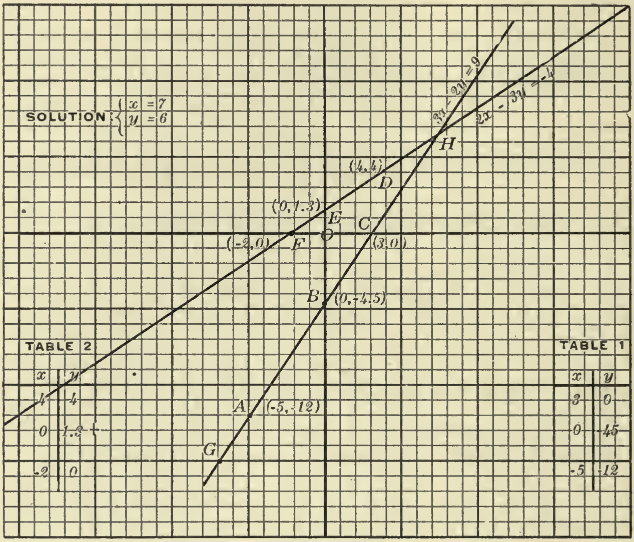


Fig. 278

444. Co-ordinates. The pairs of values of x and y locating points A, B, C . . . . . . are the **co-ordinates** of A, B, C . . . . . . , etc.

EXERCISES

1. If the drawing, Fig. 278, is made accurately, any solution of the equation $y=\frac{3x-9}{2}$ will locate a point on line ABC. To show this, find various solutions of the equation, not found in the tables, and graph them.

2. Any point on ABC, Fig. 278, will furnish a solution of the equation $3x-2y=9$. E.g., the co-ordinates of point G are $x=-7$, $y=-15$. Show that they satisfy the equation. Determine other solutions of the equation from points on ABC.

445. Indeterminate equations. Exercises 1 and 2 above show that there are as many solutions of the equation as there are points on the graph, i.e., an indefinitely large number. For this reason equations in two unknowns (variables) are called **indeterminate equations.*** However, a *system of two equations* may be **determinate.**

For, the co-ordinates of any point on the graph ABC determine a solution of $y=\frac{3x-9}{2}$ and the co-ordinates of any point on DEF determine a solution of $y=\frac{2x+4}{3}$. Hence, the point of intersection, H, of ABC and DEF determines the common solution of these equations, i.e., $x=7$, $y=6$.

446. Summary. The following is the process of solving a system of equations graphically:

1. *Find at least two solutions of each equation, preferably three.*

This is done quickly by substituting for x the value 0 and finding the corresponding value of y and then substituting $y=0$ and finding the corresponding value of x. Any third value of x may be used to obtain the third solution. If any values come out fractional, it is best to use approximations to one decimal place.

2. *Tabulate the solutions.*

3. *Graph the solutions.*

* Diophantus of Alexandria (fourth century A.D.) made a study of indeterminate equations. Indeterminate equations are therefore known as Diophantine equations.

René Descartes (1596–1650) made the discovery that a point in a plane could be completely determined if its distances, say x and y, from two perpendicular lines were known. He saw that though an equation in two unknowns was indeterminate, the values of x and y satisfying the equation determined the points on a line representing the equation. (See Ball, pp. 272, 273.)

4. *Determine the point of intersection of the graphs and the co-ordinates of that point.*

5. *The co-ordinates of the point of intersection are the solution of the given system of equations.*

EXERCISES

Solve the following systems by the graphical method:

1. $x+2y=17$
   $3x-y=2$

2. $2x-\frac{3y}{2}=3$
   $x+2y=7$

3. $7x+3y=-36$
   $5x+2y=7$

4. $9x-6y=36$
   $13x+5y=11$

5. $2x-3y=4$
   $4x+5y=30$

6. $x+2y=11$
   $x-\frac{y}{2}=1$

7. $8x+5y=44$
   $2x-y=2$

8. $11x+7y=40$
   $3x-5y=4$

9. $x-2y=0$
   $x=4$

10. $x-2=0$
    $y=-3$

447. Normal form. When a linear equation in two or more unknowns has all terms containing the unknowns on one side and all other terms on the other side, and when all similar terms have been combined, as in $2x-3y=7$, the equation is said to be in the **normal form.**

EXERCISES

Put the following equations into normal form:

1. $7x=8-9y$

2. $\frac{4}{3}w+5=\frac{t}{3}$

3. $\frac{m+3p}{5}-\frac{3m-p}{2}=4$

4. $\frac{x+y-3}{2}-\frac{x-y+4}{3}=0$

5. $(x+2)(y-4)-(x+3)(y-6)=14$

448. Disadvantages of the graphical solution. One of the disadvantages of the graphical method of solving a system of equations is the difficulty of finding accurate solutions of some systems. In the following pages an algebraic method will be shown which always gives accurate results. The algebraic method also has the advantage that it is easily extended to solve equations in three or more unknowns. For practical purposes, as in the solution of problems in science, the approximate solutions obtained by the graphical method are generally sufficient.

Algebraic Solution of Equations in Two Unknowns

449. Elimination by addition or subtraction.

(*a*) A boy sets out on a walking trip and travels at a uniform rate for 5 hours, when he meets with an accident. He continues, however, at a slower pace, and 3 hours later reaches a point 26 miles from home. (*b*) If he had turned back at the time of the accident he would have reached in 3 hours a point 14 miles from home. What was his rate of speed both before and after the accident?

Letting x and y denote the rate before and after the accident respectively,

$$\begin{array}{lrll} \text{Condition } (a) & 5x+3y & =26 & \\ \text{Condition } (b) & 5x-3y & =14 & \\ \hline & 10x & =40 & \text{(Add. Ax.)} \\ & x & =\ 4 & \text{(Div. Ax.)} \\ & 6y & =12 & \text{(Sub. Ax.)} \\ & y & =\ 2 & \text{(Div. Ax.)} \end{array}$$

Check the solution $x=4$, $y=2$ by substituting in the problem.

The equation $10x=40$ was obtained from equations (1) and (2) by what operation? What terms were eliminated from equations (1) and (2) by this operation?

What is true of the coefficients of the y-terms which makes possible the elimination of these terms *by addition?*

The equation $6y=12$ was obtained from equations (1) and (2) by what operation? What terms were eliminated from equations (1) and (2) by this operation? What is true of the coefficients of the y-terms which makes possible the elimination of these terms *by subtraction?*

This method of elimination is called **elimination by addition or subtraction.**

Solve by this method the following systems:

1. $l+s=24.5$
   $l-s=8.5$

2. $m+4n=4$
   $m-2n=16$

450. In the preceding systems the coefficients of the unknown numbers were numerically equal. We must learn what to do in case the coefficients are not equal, as in the following example:

Solve this system:

$$5x+3y=26 \qquad (1)$$
$$4x-7y=2 \qquad (2)$$

If equation (1) is multiplied by the coefficient of x in equation (2), and equation (2) by the coefficient of x in equation (1), what is the coefficient of x in each of the resulting equations?

What operation will then eliminate the x-terms from the resulting equations?

How may the y-terms be eliminated from equations (1) and (2)?

The following gives the complete solution:

$7\times(1)$	$35x+21y=182$	$4\times(1)$	$20x+12y=104$
$3\times(2)$	$12x-21y=6$	$5\times(2)$	$20x-35y=10$
Hence,	$47x=188$	Hence,	$47y=94$
	$x=4$		$y=2$

Having found the value of x as shown above, the value of y might have been found by substituting the value of x in one of the given equations, thus,

$5x+3y=26$	Check in (1)	$5\cdot4+3\cdot2=26$
$5\cdot4+3y=26$		
$3y=6$	Check in (2)	$4\cdot4-7\cdot2=2$
$y=2$		

451. Summary. To eliminate by addition or subtraction proceed as follows:

1. *Make the coefficients of one of the unknown numbers numerically the same in both equations.*

This is done by multiplying one or both equations by the proper number.

2. *Then eliminate one of the unknowns either by adding or by subtracting the equations according as the coefficients of the unknown have unlike signs, or like signs.*

EXERCISES

Solve the following systems of equations:

1. $\begin{cases} 4x+3y=13 \\ 3x+2y=9 \end{cases}$

2. $\begin{cases} 5x+4y=22 \\ 3x-7y=-15 \end{cases}$

3. $\begin{cases} 3x-5y=51 \\ 2x+7y=3 \end{cases}$

4. $\begin{cases} 2t-7s=58 \\ -9t+4s=69 \end{cases}$

5. $\begin{cases} 13x+10y=59 \\ 11x-\ \ 9y=15 \end{cases}$

6. $\begin{cases} -5s+9p=4 \\ 7s+6p=-80 \end{cases}$

7. $\begin{cases} 25R+14r=385 \\ -15R+\ \ 9r=30 \end{cases}$

8. $\begin{cases} 21u+69v=111 \\ 14u-26v=2 \end{cases}$

9. $\begin{cases} 33A-28B=38 \\ 22A+35B=79 \end{cases}$

10. $\begin{cases} 66m+55n=308 \\ 77m-15n=201 \end{cases}$

11. $\begin{cases} 12x+15y=66 \\ 16x-25y=-2 \end{cases}$

12. $\begin{cases} 8h-21y=33 \\ 6h+35y=177 \end{cases}$

13. $\begin{cases} 7x+5y=17 \\ 11x-3y=5 \end{cases}$

14. $\begin{cases} 6a-4b=2 \\ 5a+7b=43 \end{cases}$

15. $\begin{cases} -11x+9y=16 \\ 4x+8y=28 \end{cases}$

16. $\begin{cases} 3u-\ \ 2v=4 \\ -7u+13v=-1 \end{cases}$

17. $\begin{cases} 9R-2r=44 \\ 6R-\ \ r=31 \end{cases}$

18. $\begin{cases} 13u-6v=22 \\ 4u+9v=61 \end{cases}$

19. $\begin{cases} \dfrac{2h}{7}+\dfrac{5k}{2}=33 \\ \dfrac{3h}{4}-\dfrac{2k}{5}=17 \end{cases}$

20. $\begin{cases} 4x+5y=14 \\ 3x-2y=-1 \end{cases}$

21. $\begin{cases} \frac{1}{2}x+\frac{3y}{5}=9 \\ \frac{x}{3}+\frac{1}{2}y=7 \end{cases}$

22. $\begin{cases} 7x-2y=8 \\ 3x+4y=18 \end{cases}$

23. $\begin{cases} 2m+11k=50 \\ 11m+\ \ 2k=41 \end{cases}$

24. $\begin{cases} s+6p=4s+5p \\ (s+4)-(5p-s)-s=0 \end{cases}$

25. $\begin{cases} \frac{4R}{3}-\frac{1}{4}r=5 \\ \frac{5R}{2}-\frac{2r}{3}=7 \end{cases}$

Geometric Problems

452. The following geometric problems lead to equations in two unknowns:

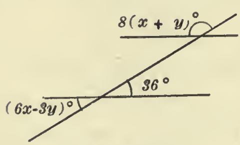


FIG. 279

1. With parallels, transversal, and angles as shown in Fig. 279 find x, y, and all the 8 angles.

2. With parallels, transversal, and angles as shown in Fig. 280 find x, y, and all the 8 angles.

3. With two parallels and a transversal, a pair of corresponding angles are $(x+2y)°$ and $2(x-y)°$, the angle adjacent to the latter being 120°. Find x, y, and the unknown angles.

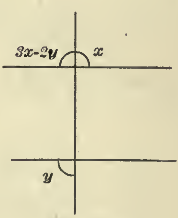


FIG. 280

4. With two parallels and a transversal a pair of alternate exterior angles are $(5y-2x)°$ and $(9x+y)°$. The angle adjacent to the latter is 86°. Find x, y, and the unknown angles.

5. With two parallels, the interior angles on the same side of a transversal are $(14x-y)°$ and $(6x+y)°$ and their difference is 14°. Find x, y, and all the 8 angles.

6. With two parallels, the interior angles on the same side of the transversal are $(4x-y)°$ and $5(2y+x)°$. Two alternate

interior angles are $5(2y+x)°$ and $125°$. Find x, y, and the unknown angles.

7. One angle of a triangle is $64°$ less than the sum of the other two and $16°$ less than the difference. How large is each?

8. The angles made by two pairs of parallels intersecting as in Figs. 281 and 282 are designated as shown. Find x, y, and all the angles.

9. In the trapezoid $ABCD$, Fig. 283, angle C is 3 times as large as angle A and angle D is 5 times as large as angle B. How large is each angle?

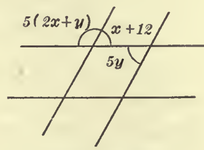


FIG. 281

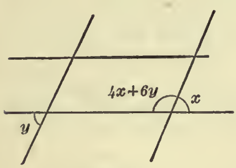


FIG. 282

10. The difference of the acute angles of a right triangle is $36°$. Find the number of degrees in each acute angle.

11. One dimension of a rectangle is 5 and one dimension of another is 3. The sum of the areas is 65 and the difference 35. Find the dimensions of the rectangles.

12. The sum of the areas of two rectangles of dimensions 5 and x, and 3 and y, is 49, and the area of a rectangle of dimensions 3 and x exceeds the area of a rectangle of dimensions 5 and y by 9. Find x and y.

13. The areas of two triangles having equal bases are 72 sq. in. and 60 sq. in. Twice the altitude of the first plus 3 times the altitude of the second is equal to 54 in. Find the altitudes.

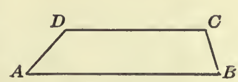


FIG. 283

14. The altitude of a trapezoid is 8 and the area is 56. If the lower base is increased by a length equal to the upper base, the area is 72. Find the bases of the trapezoid.

15. The lower base of a trapezoid is 24 and the area is 150. If a length equal to $\frac{1}{6}$ of the lower base is added to the upper base the area is 170. Find the altitude and upper base of the trapezoid.

16. The triangles in Fig. 284 have equal corresponding parts as indicated. The letter x has the same value throughout. The same is true of y. Find x and y.

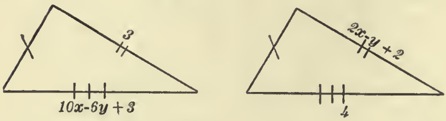


FIG. 284

17. The ratio of the lengths of two circles is 2. Six times the radius of the first minus 4 times the radius of the second is equal to 14. Find the radii and lengths of the circles.

The length of a circle is $2\pi r$, where $\pi = \frac{22}{7}$ approximately.

18. The areas of two circles are to each other as 4:36. One-half the radius of the first plus $\frac{1}{7}$ of the radius of the second is equal to $6\frac{1}{2}$. Find the radii and areas.

The area of a circle is πr^2. Hence $\frac{\pi x^2}{\pi y^2} = \frac{4}{36}$. Extract the square root of both sides of the equation.

Motion Problems

453. Absolute and relative velocity. While a freight train is moving at the rate of x mi. an hour a brakeman walks along the top of the cars of the train from rear to front at the rate of y mi. an hour. Show that the velocity with which the brakeman moves over the ground, his **absolute velocity**, is equal to his own velocity on top of the cars, his **relative velocity**, increased by the velocity of the train. Express his absolute velocity in terms of x and y.

Determine the absolute velocity of a brakeman walking along the top of the cars of a train in the direction opposite to that of the train if his relative velocity is y and the velocity of the train x.

EXERCISES

1. Two trains A and B pass each other going in the same direction at rates of 30 mi. and 40 mi. an hour

respectively. What is the speed of A relative to B? What would it be if the trains were moving in opposite directions?

2. Two trains pass each other going in the same direction with a relative speed of 10 mi. an hour. Going in opposite directions they would pass with a relative speed of 70 mi. an hour. Find the speed of each train.

3. A boat crew rows 4 mi. downstream in 20 minutes and the same distance upstream in 35 minutes. Find the rate of the boat crew in still water and the rate of the current.

4. A steamboat can run $7\frac{1}{2}$ mi. per $\frac{1}{2}$ hour upstream and $4\frac{1}{2}$ mi. per $\frac{1}{2}$ hour downstream. What is its rate in still water and what is the rate of the current?

5. A and B run a race of 450 yards. In the first trial B begins with a start of 60 yd. ahead of A and A wins by 18 seconds. In the second trial B begins with a start of 30 seconds and wins by 10 yards. Find the rates of A and B.

Miscellaneous Problems

454. Problems involving number relations. The following exercises give practice in expressing number relations in the form of equations:

1. The difference of two numbers is 17 and the sum is 167. Find the numbers.

2. If 13 times one number be subtracted from 3 times another the difference obtained is 41. The sum of 11 times the first number and 8 times the second is 18. What are the numbers?

3. Three tons of hard coal and two tons of soft coal cost \$32. The price remaining the same, 2 tons of hard coal and 6 tons of soft coal cost \$43.50. What were the prices per ton of the two kinds of coal?

4. A street railway company receives a certain sum for each cash fare and a different sum for each transfer. On one trip 13 cash fares and 18 transfers were taken, amounting to \$1.10 for

the company. On the return trip there were 7 cash fares and 24 transfers and the amount for the company was \$0.95. What does the company receive for a cash fare? For a transfer?

455. Digit problems.

1. The tens digit of a number is 4 and the units digit is 3. Indicate in symbols the numbers of units in the tens. Indicate the number of units in the number.

2. The tens digit of a number is t and the units digit is u. Indicate the number of units in the number.

3. A number is denoted by $10t+u$; what will denote the number formed by reversing the order of the digits?

4. Show, in symbols, a number whose digits are x and 4; x and y; y and x.

5. Show, in symbols, a three-digit number whose digits are a, b, and c; c, a, and b; x, z, and y; y, z, and x.

6. Express, in symbols, that a number whose units digit is 3 less than the tens digit equals 27 more than the number obtained by writing the digits in the reverse order.

7. A number having 2 digits is 4 times as large as the sum of the digits. If the order of the digits is reversed the resulting number is 27 larger than the original number. What is the number?

8. The difference of a number of 2 digits and a number having the same digits but in reverse order is 18. The sum of the digits is 8. What are the numbers?

9. A number of 2 digits is equal to 9 less than 7 times the sum of the digits. If the order of the digits is reversed the number obtained is 18 less than the original number. What is the original number?

456. Income problems.

1. A man gained 8 per cent on one investment and lost 3 per cent on another. If the money invested amounted to \$22,000 and the net gain was \$440, what was the amount of each investment?

2. Two investments, one at $3\frac{1}{3}$ per cent, and the other at $5\frac{1}{2}$ per cent, yield annually \$150. If the first had been at $8\frac{3}{4}$ per

cent, and the other at $3\frac{1}{2}$ per cent, the annual income would have been \$175. What was the amount of each investment?

3. A man invested two sums at 4 per cent and 5 per cent respectively, receiving annually \$158. He reinvested the money at 5 per cent and 6 per cent respectively, receiving \$36 per year more than before. Find the sums invested.

4. A part of \$2,500 was invested at 3.5 per cent and the other at 4 per cent. The second investment brings \$25 less per year than the first. How much was each investment?

5. A part of \$4,000 is invested at 3.5 per cent and the remainder at 4.5 per cent. The first investment yields in four years \$15 less than the second investment in two years. Find the sums invested.

Fractional Equations

457. Solve the following equations:

1. $\begin{cases} \dfrac{x+y}{2}-\dfrac{x-y}{3}=\dfrac{7}{6} \\ \dfrac{x+y}{3}+\dfrac{x-y}{4}=\dfrac{5}{4} \end{cases}$

2. $\begin{cases} \dfrac{6-4R}{3}=\dfrac{3H-8}{2} \\ 3R-4=\dfrac{8H-2}{5}-1 \end{cases}$

3. $\begin{cases} (2u+3):5=(3v+5):7 \\ 7u:4v=77:40 \end{cases}$

4. $\begin{cases} 3x-\dfrac{2x-y}{4}=5+\dfrac{x+7}{5} \\ 5x-18+\dfrac{2x}{3}=\dfrac{1}{2}+\dfrac{7-5y}{2} \end{cases}$

5. $\begin{cases} c-27=\dfrac{17-2c}{3}-\dfrac{5d-8}{2} \\ 16-2d=\dfrac{2c-d}{4}-\dfrac{55+c}{5} \end{cases}$

6. $\begin{cases} \dfrac{r-2}{r-3}=\dfrac{v+4}{v+5} \\ \dfrac{r+1}{r+2}=\dfrac{v+3}{v-4} \end{cases}$

7. $\begin{cases} \dfrac{2m}{5}-\dfrac{5n}{6}=-\dfrac{1}{2} \\ \dfrac{m}{6}+\dfrac{5n}{9}=\dfrac{5}{2} \end{cases}$

8. $\begin{cases} \dfrac{2.2g+.8}{1.5}=\dfrac{.5h-.4}{.3} \\ \dfrac{.5g-.3}{.2}=\dfrac{1.5h-.5}{2.5} \end{cases}$

9. $\begin{cases} \dfrac{1u}{7}-\dfrac{2}{3}(w+1)=\dfrac{1}{3}(u-1)-9 \\ \frac{1}{6}(w+1)-u=\frac{1}{2}(w-1)-10 \end{cases}$

10. $\begin{cases} 3b-5+\dfrac{2a-b}{4}=\dfrac{a+7}{5} \\ \dfrac{2a}{3}-\dfrac{1}{2}+5a-18=\dfrac{7-5b}{2} \end{cases}$

11. $\begin{cases} a+\frac{1}{2}(3a-b-1)=\frac{1}{4}+\frac{3}{4}(b-1) \\ \frac{1}{10}(7b+24)=\frac{1}{5}(4a+3b) \end{cases}$

12. $\begin{cases} \dfrac{6-H}{2}-\dfrac{R+5}{5}=-12 \\ \frac{1}{3}(12+H)-\frac{1}{9}(R-4)=4 \end{cases}$

13. $\begin{cases} -\dfrac{20-2K}{3}+\dfrac{5K}{6}=\dfrac{H}{6}+\dfrac{4K-19}{3} \\ 5-\dfrac{2K+21}{3}=-\dfrac{H+5K}{6} \end{cases}$

458. Systems of three or more linear equations.

Solve the following problem:

The sum of 3 times the first, 5 times the second, and 3 times the third of three numbers is equal to 22. The sum of 5 times the first and 3 times the second, diminished by 4 times the third, is equal to -1. If from the sum of 4 times the first and twice the second, 5 times the third is subtracted, the remainder is -7. What are the numbers?

Letting x, y, and z denote the first, second, and third numbers respectively, the equations are:

$$3x+5y+3z=22 \qquad (1)$$
$$5x+3y-4z=-1 \qquad (2)$$
$$4x+2y-5z=-7 \qquad (3)$$

Eliminating z from (1) and (2), and again from (1) and (3) or (2) and (3), what two unknowns will the resulting equations contain? Solve these equations for x and y:

Eliminating z from (1) and (2):

$$\begin{array}{lrl} 4\times(1) & 12x+20y+12z= & 88 \quad (4) \\ 3\times(2) & 15x+9y-12z= & -3 \quad (5) \\ \hline \text{(Add. Ax.)} & 27x+29y \quad = & 85 \quad (6) \end{array}$$

Next eliminate z from (1) and (3), thus:

$$\begin{array}{lrl} 5\times(1) & 15x+25y+15z= & 110 \quad (7) \\ 3\times(3) & 12x+6y-15z= & -21 \quad (8) \\ \hline & 27x+31y \quad = & 89 \quad (9) \end{array}$$

Solving (6) and (9), $x=1$ and $y = 2$

Substituting in (2), $5\cdot1+3\cdot2-4z=-1$

$$z=3$$

Thus, the solution of the given system is the set of numbers: $\begin{cases} x=1 \\ y=2 \\ z=3 \end{cases}$

The solution is checked by substituting these values of x, y, and z into equations (1), (2), and (3):

Check in (1):

$$3\cdot 1+5\cdot 2+3\cdot 3=22;\ \text{i.e.,}\quad 22=22$$

Check in (2):

$$5\cdot 1+3\cdot 2-4\cdot 3=-1;\ \text{i.e.,}\quad -1=-1$$

Check in (3):

$$4\cdot 1+2\cdot 2-5\cdot 3=-7;\ \text{i.e.,}\quad -7=-7$$

The method used in solving the system of equations of the foregoing problem consists of three principal steps:

1. *Make two different pairs of equations out of the three equations, and eliminate the same unknown from both of these pairs.*

In the problem the unknown was eliminated by the method of addition and subtraction.

2. *This gives two equations in two unknowns. Solve the two resulting equations as a system of two linear equations.*

3. *Substitute the values of the two unknowns just found in any one of the given equations containing the third unknown, and solve the resulting equation.*

EXERCISES

1. The sum of two angles of a triangle exceeds the third angle by 26°. Five times the difference of the first two is 8° more than the third angle. Find the number of degrees in each angle.

2. The sum of two numbers is 2 greater than 3 times a third number, the difference is equal to the third number, and twice the third number increased by the second is 4 greater than the first. Find the numbers.

3. The angles of a triangle are A, B, and C; $\frac{1}{4}A+\frac{B}{8}=C$, and $\frac{1}{2}A+\frac{1}{10}B=\frac{1}{2}C+30$. Find the values of A, B, and C.

4. Solve the following systems:

1. $\begin{cases} 3x+y-2z=-1 \\ -4x+2y+3z=9 \\ 5x+3y-2z=5 \end{cases}$

2. $\begin{cases} 2u+2v+w=9 \\ u+3v+2w=13 \\ 3u-3v+4w=9 \end{cases}$

3. $\begin{cases} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=\frac{23}{12} \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{2}z=\frac{7}{3} \\ \frac{1}{4}x+\frac{1}{2}y+\frac{1}{3}z=\frac{9}{4} \end{cases}$

4. $\begin{cases} a+3b+5c=21 \\ 3a-2b-4c=22 \\ 4a-3b-6c=28 \end{cases}$

5. $\begin{cases} 2p+2s+3v=4 \\ 3p+4s+6v=7 \\ p+2s+6v=4 \end{cases}$

6. $\begin{cases} 4x-4y+8z=3 \\ x+y+z=0 \\ x-y-4z=2 \end{cases}$

7. $3x+4y+6z=2x+6y+5z=4x+2y+9z=68$

8. $\begin{cases} \dfrac{A}{2}+\dfrac{B}{4}-\dfrac{C}{5}=1 \\ \dfrac{A}{3}-\dfrac{B}{2}+\dfrac{C}{3}=12 \\ \dfrac{A}{4}-\dfrac{B}{4}-\dfrac{C}{2}=-13 \end{cases}$

9. $\begin{cases} 2x+3y+z=\frac{3}{4} \\ 5x-2y+z=\frac{17}{20} \\ 3x+y+2z=\frac{4}{5} \end{cases}$

10. $\begin{cases} 7x+5y-2z=\frac{53}{6} \\ 5x-2y+3z=5 \\ 2x+9y-5z=\frac{29}{6} \end{cases}$

11. $\begin{cases} x+y+z=3 \\ x-y+z=6 \\ -y+3z+x=12 \end{cases}$

12. $\begin{cases} 4u-v=0 \\ 5v-2w=40 \\ 6w-u=15 \end{cases}$

13. $\begin{cases} 5x+3y=45 \\ 7y-2z=27 \\ 3z-4x=-12 \end{cases}$

14. $\begin{cases} 3v-2u=\frac{2}{5} \\ 7u-4w=\frac{1}{2} \\ 10w-2v=\frac{1}{10} \end{cases}$

15. $\begin{cases} H+K=2 \\ K+R=4 \\ R+H=6 \end{cases}$

16. $\begin{cases} x+y+z=1 \\ 2x+3y+4z=-3 \\ 3x-4y-5z=14 \end{cases}$

17. $\begin{cases} x+2y+3z=14 \\ 2x+y+z=7 \\ .3x+.2y+.2z=1.3 \end{cases}$

18. $\begin{cases} x+1=2(y+1) \\ y+2=z+1 \\ \frac{4}{5}(z+3)=x+1 \end{cases}$

19. $\begin{cases} \dfrac{x}{2}+\dfrac{y}{2}=3 \\ 1-\dfrac{y}{2}=4 \\ 3z-2x=10 \\ s-\dfrac{t}{2}=6 \\ t+z=14 \end{cases}$

20. $\begin{cases} x+y+z+u=10 \\ x-y+z+u=6 \\ x+y-z+u=4 \\ x+y+z-u=2 \end{cases}$

Problems

459. Solve the following problems:

1. The sum of the 3 digits of a number is 16. If the order of the digits is reversed, the new number is 396 less than the original number. If the middle digit be placed first, the resulting number is 90 less than the original number. What is the number?

2. The sum of the 3 digits of a number is 15. The second digit is 2 times the first less the third. If the order of the digits is reversed the resulting number is 198 less than the given number. What is the original number?

3. A sum of $18,000 is invested as follows: One part at 3.5 per cent, a second part at 5 per cent, and the rest at 4 per cent. The total interest is $730. If the first part had been invested at 4 per cent, the second at 3 per cent, and the third at 6 per cent, the total annual interest would have been $840. How much was each part?

4. A man invested three sums of money in the following ways: The first sum at 5 per cent and the second at 4 per cent, giving an income of $490. The second at 5 per cent and the third at 3 per cent, giving an income of $540. The three sums together amounted to $19,000. How much was each sum?

Summary

460. The meaning of the following terms was taught in this chapter: system of equations in two or more unknowns; solution of a system of equations; method of elimination by addition and subtraction; co-ordinates of a point; indeterminate equations; normal form of an equation in more than one unknown; absolute and relative velocity.

461. A system of equations in two or more unknowns may be solved by the method of elimination by addition and subtraction. A system of equations in two unknowns may be solved by the graphic method.

462. Some of the following typical problems lead to equations in several unknowns: geometric problems, motion problems, digit problems, and income problems.

463. To solve a system of equations graphically, find two, or three, solutions of each equation, graph these solutions, draw the lines determined by them, determine their point of intersection, and find the co-ordinates of that point.

464. To solve a system of equations by elimination, make the coefficients of one of the unknown numbers numerically the same, then by adding or by subtracting one equation from the other, one of the unknowns may be eliminated.

465. A system of equations in three unknowns is solved by eliminating the same unknown from both of two pairs of equations made from the three given equations. This gives a system of two equations in two unknowns. By solving this system the values of two of the unknowns are found. The value of the third unknown is obtained by substituting the values of the other unknowns (now known) in any of the given equations. Systems in more than three unknowns are treated similarly.

LEONHARD EULER

LEONHARD EULER

LEONHARD EULER was born at Basle on April 15, 1707, and died at St. Petersburg on September 7, 1783. He was educated under the direction of the great mathematical teacher, John Bernouilli, at Basle in Switzerland. In 1725 he went to St. Petersburg as professor of mathematics, taught there until 1741, when he moved to Berlin, Germany, at the request of Frederick the Great. He lived in Berlin until 1766, when he returned to St. Petersburg. Within two or three years of his return to Russia he became totally blind, but in spite of this he worked on until his death.

He wrote an immense number of papers dealing with all sorts of mathematical subjects. He reformed almost all the branches of pure mathematics that were known, added numerous details and proofs, created much of what is now called analysis, and arranged the entire material of mathematics in the consistent form that we have today. He was probably the most prolific mathematical writer the world has known, and most of his work was of a very high order of merit, though his style was diffuse and prolix.

In 1748 he wrote his *Introductio in analysin infinitorum*, the first part of which contains about what our modern texts contain on algebra, theory of equations, and trigonometry. He separated trigonometry from astronomy and made it a branch of mathematics. He here introduced the use of e for the Napierian base and π for the ratio of a circumference to the diameter of a circle. The second part was on analytic geometry.

He wrote a book on differential calculus, one on integral calculus, and much too many other books and memoirs to permit of even mention here. See an account of Euler in any good encyclopedia, or in Ball's or Cajori's histories of mathematics, and note what is said about his ways of working in mathematics; for he is one of the very best mathematical authors to read and study to learn how to become a really independent worker in mathematics.

CHAPTER XVIII

THE FORMULA

The Formula as a General Rule

466. Formula. The ideas of mathematical statements can often be expressed in algebraic symbols with gain in clearness and conciseness. Thus, the statement, "The area of a rectangle equals the length by the height," takes the form $A = lh$, l denoting the length, h the height, and A the area; the statement "distance = rate × time" may be expressed by the equation $d = rt$; etc. The symbolic form of such statements not only is more comprehensible than the verbal form, but may easily be applied to a particular case by assigning definite values to the letters; e.g., the area of a rectangle 2 in. long and 3 in. high is found by letting $l = 2$ and $h = 3$, in the equation $A = lh$.

A general statement, or rule, in which letters and symbols are used in place of numbers or words is a **formula**. A formula is an algebraic equation. If the values of all but one of the letters are known, the value of the unknown is then found by solving the equation. Thus, the volume of a sphere is expressed by the formula $V = \frac{4}{3}\pi r^3$, where V is the volume, $\pi = \frac{22}{7}$ approximately, and r is the radius. To find the radius of a sphere whose volume is $14\frac{1}{7}$ cubic inches, let $V = 14\frac{1}{7}$. Then $14\frac{1}{7} = \frac{4}{3} \times \frac{22}{7} r^3$, or $\frac{99}{7} = \frac{4}{3} \times \frac{22}{7} r^3$. Dividing both sides by 11, $\frac{9}{7} = \frac{4}{3} \times \frac{2}{7} r^3$, from which it follows that $r = 1\frac{1}{2}$ inches.

467. Motion-problem formula. The formula is used in solving motion problems.

EXERCISES

1. An express train whose rate is 45 mi. per hour starts $1\frac{1}{2}$ hr. after a freight train and overtakes it in $2\frac{3}{4}$ hr. Find the rate per hour of the freight train.

The formula for uniform motion is $d=r\cdot t$, d denoting the distance, r the rate, and t the time.

Let r be the rate of the freight train

Then $4\frac{1}{4}$ = time the freight train travels

Hence, $4\frac{1}{4}r$ = distance the freight train travels

45 = rate of the express train

$2\frac{3}{4}$ = time of the express train

Hence, $123\frac{3}{4}$ = distance the express train travels

Therefore, $4\frac{1}{4}r=123\frac{3}{4}$

$$r=29\tfrac{2}{17}$$

2. An express train whose rate is R mi. per hour starts h hr. after a freight train and overtakes it in t hours. Find the rate per hour of the freight train.

Let r = rate of the freight train

Then $h+t$ = time the freight train travels

$(h+t)r$ = distance the freight train travels

R = rate of the express train

t = time the express train travels

Rt = distance the express train travels

Hence, $(h+t)r=Rt$

$$r=\frac{Rt}{h+t}$$

3. Using the equation $r=\dfrac{Rt}{h+t}$ as a formula, solve exercise 1.

Let $R=45$; $t=2\frac{3}{4}$; $h=1\frac{1}{2}$

4. An express train whose rate is 40 mi. per hour starts 1 hr. 4 min. after a freight train and overtakes it in 1 hr. 36 min. How many miles per hour does the freight train run?

Use the formula in exercise 2.

5. Find h in terms of R, t, and r.

Clearing the equation $r=\frac{Rt}{h+t}$ of fractions, we have $hr+tr=Rt$.

Hence, $hr=Rt-tr$ and $h=\frac{Rt-tr}{r}$

468. Work-problem formula. The work-problem is a type of problem for which a formula is easily obtained.

EXERCISES

1. A can do a piece of work in 5 days and B in 7 days. How long will it take them to do it together?

Let n be the number of days it will take them together

Then $\frac{1}{n}$ = the amount of work they can do in 1 day

$\frac{1}{5}$ = the amount A can do in 1 day

$\frac{1}{7}$ = the amount B can do in 1 day

Hence, $$\frac{1}{5}+\frac{1}{7}=\frac{1}{n}$$

Multiplying by $35n$, $7n+5n=35$

$$12n=35$$

$$n=2\tfrac{11}{12}$$

It is clear that numbers other than 5 and 7 would be used just as the 5 and 7 are used here.

2. A can do a piece of work in a days and B in b days. How long will it take them to do it together?

Let n be the number of days it will take them together

Then $\frac{1}{n}$ = the amount of work they can do in 1 day

$\frac{1}{a}$ = the amount A can do in 1 day

$\frac{1}{b}$ = the amount B can do in 1 day

Hence, $$\frac{1}{a}+\frac{1}{b}=\frac{1}{n}$$

Multiplying by abn, $bn+an=ab$

$$(b+a)n=ab$$

$$n=\frac{ab}{a+b}$$

Any problem of the type of exercise 1 may be solved by using the equation $n=\frac{ab}{a+b}$ as a formula. Thus, to solve exercise 1, let $a=5$, $b=7$. Then $n=\frac{5\times 7}{5+7}=\frac{35}{12}=2\frac{11}{12}$

3. A can build a wall in 10 days and B in 14 days. How long will it take them to do it together?

Use the formula of exercise 2.

4. A and B can build a fence in 7 days. B can do it alone in 12 days. How long will it take A to do it alone?

Use the formula.

5. Solve the equation $n=\frac{ab}{a+b}$ for a; for b.

Clearing of fractions,

$$\begin{aligned} na+nb&=ab\\ na-ab&=nb\\ a(n-b)&=-nb\\ a&=-\frac{nb}{n-b}\end{aligned}$$

EXERCISES

1. Express in symbols the following laws of arithmetic:

(1) The product equals the multiplicand times the multiplier.

(2) The dividend equals the divisor times the quotient plus the remainder.

(3) The product of a fraction by a whole number is the product of the whole number by the numerator, divided by the denominator.

(4) The sum of two fractions having the same denominator equals the sum of the numerators divided by the common denominator.

(5) The quotient of a fraction divided by a whole number is equal to the numerator divided by the product of the whole number by the denominator.

(6) The quotient of two fractions equals the dividend multiplied by the inverted divisor.

(7) The square root of a fraction equals the square root of the numerator divided by the square root of the denominator.

(8) The square of a fraction is the square of the numerator divided by the square of the denominator.

2. Translate into words the following formulas expressing laws of arithmetic:

(1) $p=\frac{b\times r}{100}$ (percentage law)

(2) $i=\frac{prt}{100}$ (interest law)

(3) $\frac{n_1}{d_1}+\frac{n_2}{d_2}=\frac{n_1d_2+n_2d_1}{d_1d_2}$ (addition of fractions)

(4) $\frac{n_1}{d_1}-\frac{n_2}{d_2}=\frac{n_1d_2-n_2d_1}{d_1d_2}$ (subtraction of fractions)

(5) $\frac{n_1}{d_1}\times\frac{n_2}{d_2}=\frac{n_1n_2}{d_1d_2}$ (multiplication of fractions)

(6) $\frac{n_1}{d_1}\div\frac{n_2}{d_2}=\frac{n_1}{d_1}\times\frac{d_2}{d_1}$ (division of fractions)

3. Express the following statements in symbols:

(1) The distance passed over by a body is equal to the rate times the time.

Denote the distance by d, the rate by r, and the time by t.

(2) The power (or rate of doing work) is equal to the amount of work divided by the time in which it is done.

Denote the power by P, the amount of work done by W, and the time by t.

(3) The time of vibration, t, of a pendulum is equal to π times the square root of the quotient of the length, l, divided by the acceleration, g, due to gravity.

(4) The velocity is equal to the space passed over divided by the time.

(5) The momentum of a mass is equal to its velocity multiplied by the mass.

(6) The time it takes a body projected upward to reach the greatest height is equal to the velocity of projection divided by the retardation due to gravity. The greatest height is equal to the square of the velocity of projection divided by 2 times the retardation due to gravity.

4. Translate the following formulas into words:

(1) If the velocity obtained by a falling body in t seconds is denoted by v_t and the acceleration due to gravity by g, then $v_t = gt$.

(2) The distance, s, passed over by a falling body in t seconds is $s = \frac{1}{2}gt^2$. The velocity obtained in the same time is $v = \sqrt{2gs}$.

(3) If a fluid of density d moves with a velocity v, the diminution of pressure due to the motion is $p = \frac{1}{2}dv^2$.

(4) A body of volume V immersed in a liquid of density D is buoyed up by a force $F = DgV$ (Archimedes' principle).

(5) For a perfect gas, changing from pressure p and volume v to pressure p' and volume v' without change of temperature, $pv = p'v'$ (Boyle's law).

Evaluation of Formulas

469. Find the value of the letter called for in the following formulas:

1. The distance traversed by a moving body is equal to the rate multiplied by the time; that is, $D = rt$.

Find D, if (1) $r = 30$ ft. per second, and $t = 5$ seconds.
(2) $r = 5$ mi. per hour, and $t = 17$ hours.

2. The area of a rectangle is equal to the product of the base and the altitude; that is, $A = ba$.

Find A, if (1) $b = 13$ ft., and $a = 24$ feet.
(2) $b = 10.2$ in., and $a = 3.5$ inches.

3. The area of a triangle is equal to $\frac{1}{2}$ the product of the base by the altitude; that is, $A = \frac{1}{2}ba$.

Find A, if (1) $b = 12$ ft., and $a = 16$ feet.
(2) $b = 8.2$ rd., and $a = 7.78$ rods.

4. The area of a parallelogram is equal to the product of the base by the altitude; that is, $A=ba$.

Find A, if (1) $b=28$, and $a=19$.
(2) $b=16.3$, and $a=14.6$.

5. Two weights, w_1 and w_2, will balance on a beam that lies across a stick when the distances, d_1 and d_2, of the weights from the stick are in the inverse ratio of their weights; i.e., when $\frac{d_1}{d_2}=\frac{w_2}{w_1}$.

Find d_1, if (1) $d_2=18$ ft., $w_2=60$ lb., $w_1=50$ pounds.
(2) $d_2=27$ in., $w_2=36$ lb., $w_1=24$ pounds.

Find d_2, if (1) $d_1=40$ in., $w_2=16$ lb., $w_1=18$ pounds.
(2) $d_1=25$ in., $w_2=3.8$ lb., $w_1=2.85$ pounds.

6. The weight, w, of any mass is equal to the volume, v, multiplied by the density, d; i.e., $w=vd$.

Find w, if $v=64$ cu. in., and $d=16.2$ pounds.
Find v, if $w=648$ lb., and $d=12.2$ pounds.
Find d, if $w=800$, and $v=160$.

7. The weight, w, that a force, p, pulling up a smooth slope, h ft. high and l ft. long, will just move is given by $w=\frac{l}{h}p$.

Find h if $w=120$ lb., $l=20$ rd., $p=60$ pounds.

8. A stone falling from rest goes in a given time 16 ft. multiplied by the square of the number of seconds it has fallen; i.e., $s=16t^2$.

Find s, if $t=4$ sec.; 11.5 seconds.
Find t, if $s=64$ ft.; 1,600 feet.

9. A stone, thrown downward, goes in a given time 16 ft. multiplied by the square of the number of seconds it has fallen, plus the product of the velocity with which it is thrown and the number of seconds fallen; i.e., $s=16t^2+vt$.

Find s, if $t=12$ sec., and $v=3$ ft. per second.
if $t=8$ sec., and $v=7$ ft. per second.
Find v, if $t=5$ sec., and $s=500$ feet.

10. The time, t, taken for a pendulum to make a single vibration equals $\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum. We take g approximately equal to 32, and π equal to $\frac{22}{7}$.

Find t, if $l=8$ ft.; if $l=\frac{392}{121}$ feet.
Find l, if $t=1$ sec.; if $t=4$ seconds.

11. The length of a circle is approximately equal to $\frac{22}{7}$ of the diameter; i.e., $C=\pi d$.

Find C, if $d=21$ ft.; 7 ft.; $\frac{1}{2}$ foot.
Find d, if $C=88$ ft.; 66 ft.; 16 feet.

12. The volume of a sphere equals $\frac{4}{3}\pi$ times the cube of the radius; i.e., $V=\frac{4}{3}\pi r^3$.

Find V, if $r=\frac{7}{2}$ ft.; 11 ft.; 2 feet.

13. The area of a circle is π times the square of the radius; i.e., $A=\pi r^2$.

Find A, if $r=3$ ft.; 7 ft.; 21 feet.
Find r, if $A=154$ sq. ft.; 220 square feet.

14. Find the length of a belt connecting two pulleys whose diameters D and d are 22 in. and 24 in. respectively, the distance, a, between the centers being 9 ft. The length is given by the formula

$$l=\pi\frac{D+d}{2}+2a$$

15. The diagonal, d, of a rectangular parallelopiped of dimensions a, b, and c is given by the formula $d=\sqrt{a^2+b^2+c^2}$. Find d, if $a=5$, $b=4$, $c=3$.

16. Find the area A, of a circular ring, formed by two concentric circles of radii R and r, if $R=12$, $r=10$, and $A=\pi(R^2-r^2)$.

17. If a, b, and c denote the lengths of the sides of a triangle and s one-half of the perimeter, the area is $\sqrt{s(s-a)(s-b)(s-c)}$.

Find the area of a triangle whose sides are 10, 6, and 8; 10, 17, and 21; 3, 4, and 5; 5, 12, and 13.

18. $h=\frac{v^2}{2g}$ (law of falling bodies). Find h, if $v=11$, $g=32.2$; if $v=2$, $g=32.2$

19. $\frac{Wl}{4}=\frac{sbh^2}{6}$. Find h, if $W=6{,}748$, $l=5.5$, $s=3{,}500$, $b=6$
if $W=4{,}954$, $l=3$, $s=2{,}500$, $b=5.5$

20. $X=\frac{Er}{Ir-E}$. Find X, if $E=0.056$, $r=2$, $I=1.4$

21. $R=\frac{rr'}{r+r'}$. Find R, if $r=11.5$, $r'=6.5$
if $r=13$, $r'=15$

22. $E=\frac{MV^2}{2}$. Find E, if $M=12$, $V=5$
if $M=11$, $V=9$
Find M, if $E=8$, $V=4$
if $E=50$, $V=5$

23. $\frac{F}{P}=\frac{h}{2\pi r}$. Find P, if $F=25$, $r=18$, $h=\frac{3}{4}$

24. $h=\frac{s^2}{4(2r-h)}$. Find r, if $s=425$, $h=8$

25. $\frac{P}{W}=\frac{dr}{2\pi lR}$. Find W, if $R=20$, $l=10$, $r=9$, $P=75$, $d=\frac{1}{2}$

470. Expressing one of the letters of a formula in terms of the others.

Solve the following equations:

1. $A=\frac{b+b'}{2}\cdot h$, for h; for b; or b' (area of a trapezoid)

2. $i=\frac{prt}{100}$ for p; for r; for t; for pr; for rt (interest formula)

3. $V=\frac{1}{3}bh$ for b; for h (volume of a pyramid)

4. $C=\frac{5}{9}(F-32)$ for F (centigrade in terms of Fahrenheit)

5. $pd=PD$, for p; for d; for P; for D (law of levers)

6. $C=\dfrac{E}{R+r}$ for E; for R; for r (current in a circuit including external and internal resistance)

7. $\dfrac{1}{F}=\dfrac{1}{f_1}+\dfrac{1}{f_2}$ for F; for f_1; for f_2. (F = principal focal distance of lens, f_1 and f_2 = conjugate focal distances)

8. $pv=p_0v_0\left(1+\dfrac{t}{273}\right)$ for p_0v_0; for t (expansion of gases)

9. $R=\dfrac{l^2}{6d}+\dfrac{d}{2}$ for l (R = radius of curvature from spherometer readings)

10. $C=\dfrac{1}{C_1}+\dfrac{1}{C_2}$ for C_1; for C_2

11. $\dfrac{1}{8}WL=\dfrac{IS}{c}$ for W; for S; for $\dfrac{I}{c}$

12. $ax-bx=6a-6b$, for x

13. $cx+ax=a^2+c^2+2ac$, for x

14. $ax-\dfrac{x-a}{c}=a^2$, for x

15. $\dfrac{x}{a}+\dfrac{x}{b}=\dfrac{1}{ab}$, for a

16. $cx-\dfrac{a^2(x-3)}{c}=\dfrac{3a^2}{c}-\dfrac{a-c}{c}$, for x

17. $\dfrac{x}{a}-\dfrac{x}{b}=\dfrac{b^2-a^2}{ab}$, for x

18. $\dfrac{x}{4b}-\dfrac{x}{3b}=\dfrac{6a-8b}{12ab}$, for x

Summary

471. The chapter has taught the following:

1. The formula expresses in algebraic symbols the ideas of a mathematical statement with gain in clearness and conciseness.

2. The formula may be used to solve any particular problem of a type by assigning definite values to the letters.

3. The laws of arithmetic may be expressed conveniently by formulae.

4. Any letter of a formula may be expressed in terms of the others, by solving the equation for that letter.

CHAPTER XIX

REVIEW AND SUPPLEMENTARY QUESTIONS AND PROBLEMS

CHAPTER I

472. Measurement of line-segments.

1. Give the meaning of the following terms: straight line, geometric line, geometric point, line-segment, and unit of length.

2. Draw a line-segment and measure the length approximately to two decimal places.

3. Draw a triangle and measure the sides approximately to two decimal places.

4. A train travels at the rate of 24 mi. an hour. Represent graphically the distances passed over in each of the first 12 hours.

5. In 1913 and 1914 a certain magazine contained approximately as many lines of advertisements as given in the table below. Make the graph and tell what it shows.

Months		Jan.	Feb.	Mar.	Apr.	May	June
Number of lines	1913	12,000	11,700	13,600	15,900	18,400	16,600
	1914	13,600	16,000	19,000	20,200	21,400	20,100

Months		July	Aug.	Sept.	Oct.	Nov.	Dec.
Number of lines	1913	14,800	12,700	13,300	16,000	18,700	25,000
	1914	15,800	15,000	15,700	16,900	20,000	26,100

6. The following table gives the temperature readings for a certain day:

2:00 A.M.	4:00	6:00	8:00	10:00	12:00	2:00 P.M.	4:00	6:00	8:00	10:00	12:00
70°	66°	65°	66°	68°	77°	81°	81.5°	83°	82°	80°	77°

Draw the temperature curve and tell what the graph shows.

CHAPTER II

473. Graphical addition and subtraction.

1. Give the reasons for the following conclusions and illustrate them graphically:

(1) If $a=3$, $b=5$, then $a+b=8$.

(2) If $m=8$, $n=2$, then $m-\mathrm{n}=6$.

(3) If $a=x$, $b=y$, then $a+b=x+y$.

(4) If $a=s$, $b=t$, then $a-b=s-t$.

2. Construct $3a+4b-2c$.

3. If $m=2.4$, construct $3m+4m$.

474. Perimeters.

1. Find the perimeter of a triangle whose sides are $2l$, $6l$, and $5l$, respectively. What is the length of the perimeter if $l=1.2$?

2. Give the meaning of the following terms: triangle, perimeter, polygon, quadrilateral, pentagon, hexagon, n-gon.

3. Give the meaning of the following: coefficient, term, monomial, binomial, trinomial, polynomial.

4. Show by a sketch figures whose perimeters, p, are given by the equation: $p=4x$; $p=6x+18y$; $p=3x+4x+5x$.

5. Find the value of p in problem 4 for $x=4$, $y=2$; $x=1.3$, $y=2.7$.

475. Algebraic addition and subtraction.

1. State how to add similar monomials.

2. Add as indicated:

$$4a+7a+10a;\ 2a+5b+6a+3b;\ 12.4x+3.8x$$

3. State how to find the difference of similar monomials.

4. Reduce to the simplest form: $12x-8x$; $5.7y-2.8y$; $(8a-a)-(5a-2a)$.

5. Illustrate by an example the commutative law of addition.

6. Illustrate the associative law of addition.

7. Simplify:

$$a+\{10a+(8a-6a)+(4a-a)\}$$
$$10b-b+[\{16b-2b-(3b+b)+5b\}-b]$$

CHAPTER III

476. Use of axioms in solving equations.

1. Give the meaning of the following terms: equation, solving an equation, checking, root of an equation, satisfying an equation.

2. Show how the value of p in the equation $4p+9=17$ may be found with the balance.

3. State the axioms used in solving equations.

4. Solve the following equations, giving in each step the axiom used, and check:

$$10x-9=65;\ \frac{n}{5}=14;\ a+\frac{1}{3}a=12;\ \frac{5}{t}=20;\ x+5.37x-8.73=61.34$$

Solve the following problems:

5. Three times a certain number exceeds 40 by as much as 40 exceeds the number. What is the number?

6. Six times a number less 5 is the same as 4 times the number increased by 1. Find the number.

7. A line 20 in. long is to be divided into 2 parts, one of which increased by 14 in. shall be equal to the other increased by 10 inches. Find the length of each part.

8. Find two consecutive numbers whose sum is equal to 247.

9. The length of a rectangular field is 3 times the width, and the length of a fence surrounding the field is 744 feet. How wide is the field?

CHAPTER IV

477. Classification of angles.

1. How many degrees are there contained in the angle formed by the hands of a clock at 2:00 o'clock? at 6:00 o'clock? at 8:00 o'clock?

2. Draw a right angle, a straight angle, a perigon.

3. Draw an acute angle, an obtuse angle, an oblique angle.

478. Measurement of angles.

1. Using only ruler and pencil, draw angles containing 40°, 75°, 135°, 270°. Check the accuracy of your angles by measuring them with a protractor.

2. What part of a right angle is 1 degree?

3. How many minutes are there in a degree? How many seconds in a minute?

479. The use of the protractor.

1. Give the meaning of the following: circle, arc, radius, semicircle, quadrant. Illustrate each by means of a figure.

2. Draw an obtuse triangle. Measure each angle and find the sum of the three angles.

480. The sum of the angles of a triangle.

1. If x, y, and z denote the number of degrees in the angles of a triangle, what equation expresses a relation between x, y, and z?

2. Show that the sum of the interior angles of a triangle is 180°.

3. One angle of a triangle is 30° greater than another. The third angle is 10° greater than twice the first. How large is each angle?

4. Show that the sum of the exterior angles of a triangle is 360°.

5. Find the exterior angles of a triangle if the first is 30° greater than the second, and the third is $\frac{1}{2}$ of the first, plus 15°.

6. Give the meaning of the following: obtuse triangle, acute triangle, right triangle, equiangular triangle, equilateral triangle. Illustrate each by means of a figure.

7. Show that in the same, or equal, circles equal central angles intercept equal arcs.

8. Show that in the same, or equal, circles equal arcs are intercepted by equal central angles.

481. Problems of construction.

1. Draw an angle equal to a given angle, using the protractor.

2. Using ruler and compass only, *construct at a given point on a given line an angle equal to a given angle.*

3. Draw a right angle, using the protractor.

4. *Construct a right angle, using ruler and compass.*

5. Construct an angle equal to the sum of two given angles, using ruler and compass.

6. Construct an angle equal to the difference of two given angles, using ruler and compass.

7. *Bisect a given angle.*

8. *At a given point in a line construct a perpendicular to the line.*

SIR ISAAC NEWTON

SIR ISAAC NEWTON

SIR ISAAC NEWTON, the world's greatest mathematician and physicist, was born at Woolsthorpe, Lincolnshire, England, December 25, 1642, and died at London, March 20, 1727. At Trinity College, Cambridge, where he was educated, his genius for mathematics showed itself even before graduation. He was prepared for his life-work by reading Oughtred's *Clavis*, Descartes' *Geometrie*, Kepler's *Optics*, Vieta's works, Van Schooten's *Miscellanies*, and Wallis' *Arithmetica*.

He took his B.A. degree in 1665, and within a year had discovered the binomial theorem and invented the fluxional calculus. He discovered numerous laws of physics, the law of gravitation, and wrote numerous works, which he usually published only at the urgent insistence of his friends, and generally much later than he finished them. He wrote a work which he entitled *Universal Arithmetic*, which was really an algebra. In this work we find the first use of positive and negative fractions and irrational numbers used as exponents. His masterpiece, and one of the greatest works of all times, is the *Principia*, in three books. It was published in 1687.

Newton at one time represented his university in Parliament, in 1705 he was knighted by Queen Anne, and for twenty-five years he was president of the Royal Society. He held the Lucasian chair of geometry at Cambridge from 1669 to 1701.

Ball says of his appearance: "Newton was short, and towards the close of his life rather stout, but well set, with a square lower jaw, brown eyes, a broad forehead, and rather sharp features. His hair turned gray before he was thirty, and remained thick and white as silver till his death."

Bishop Burnet called him "the whitest soul" he ever knew.

Ball says, "Newton was always perfectly straightforward and honest; but in his controversies . . . . though scrupulously just, he was not generous."

Newton said of himself: "I do not know what I may appear to the world: but to myself I seem to have been only like a boy, playing on the sea-shore, and diverting myself in now and then finding a smoother pebble, or a prettier shell, than ordinary, whilst the great ocean of truth lay all undiscovered before me."

Ball's *History of Mathematics* gives an excellent account of Newton's life and works from p. 319 to p. 352 (5th ed.). Read it for the sake of the interest and inspiration it will give you.

CHAPTER V

482. Area of a square.

1. Draw figures representing the following quadrilaterals: parallelogram, rectangle, square, trapezoid, rhombus.

2. State the formula for the area of a square.

3. Find the area of squares of sides 2.4 cm.; $5\frac{2}{3}$ cm.; 13 cm.

4. The perimeter of a square is $16a$. Find the area.

483. Area of a rectangle.

1. State the formula for the area of a rectangle.

2. The frame of a picture of rectangular form of dimensions 18 in. by 12 in. is $2\frac{1}{4}$ in. wide. Find the area of the frame.

484. Cube and parallelopiped.

1. State the formula for the volume of a cube.

2. State the formula for the volume of a rectangular parallelopiped.

3. The edge of a cube is 12 inches. Find the area of the total surface.

4. The length of a parallelopiped is 6 in., the width is 8 in., and the height is 10 inches. Find the area of the total surface.

5. The length of an edge of a cube is l. Find the surface and volume.

6. The three dimensions of a rectangular parallelopiped are a, b, and c. Find the area of the surface and the volume.

485. Graphing equations.

1. Graph the equation $y=10x-2$.

2. Graph the equation $y=2x^2+1$.

486. Multiplication of monomials.

1. Give the meaning of the following: exponent, base, power. Illustrate each by giving an example.

2. Find the values of the following for $x=2$: $4x$, x^4, $4x^3$, $(2x)^3$, $\frac{x^2+4}{x+2}$, x^3+x^2+x-7, $x^2 \cdot x^3$, $x \cdot x^4$

3. Illustrate by an example the commutative law of multiplication.

4. Simplify the following products: $3a^2bc \cdot 2ab^2c \cdot 4abc^2$; $10a^2 \cdot 5a^3 \cdot 2a$; $(3ab)(2a^2b)(10a^2b^2)$; $(6a^2vxy^2)^3$

487. Addition of monomials.

1. What is meant by the coefficient of a factor in a term?

2. In $2a^2bx$, what is the coefficient of a^2bx? Of x? Of a^2x?

3. Reduce the following polynomials to the simplest form:

$$4a+7a+15a$$
$$mx+4x+nx+3x$$
$$4(x+y)+2(x+y)+(x+y)$$
$$2.5(a^2+b^2)+3.4(a^2+b^2)+a^2+b^2$$

488. Multiplication of a polynomial by a monomial.

1. Find by means of a rectangle the product of $x+4$ by a.

2. Multiply as indicated: $x(a+5b)$; $x+y \cdot m$; $a^4+3a^2 \cdot 2b^2+b^4$; $2b(a+b-3c)$; $5+x(2x+3)$; $2m(3m^2+2m+7)$

3. Factor the following: $ab+ac$; x^2-xy; $5xy-15x^2$; $ab^2c+a^2bc^2$; $6x^2y+12xy^2+18x^2y^2$; $38a^3y^4+57a^4y^3-19a^3$

4. Solve the following equations:

$$\frac{x+3}{7}=\frac{x-2}{6};\ \frac{4}{n}+3=\frac{3}{n}+4;\ \frac{1}{3f}+\frac{1}{2f}=1$$

489. Multiplication of polynomials by polynomials.

1. By means of figures express as polynomials the following products: $(a+b)(a+b)$; $(m+n)(2x+3y)$

2. Draw rectangles whose areas are expressed by the following trinomials: $ab+ac$; $2xy+4x^2$

3. Draw squares whose areas are expressed by $m^2+2mn+n^2$; a^2+6a+9.

4. Multiply as indicated:

$$(x^2+2x+1)(x+1);\quad (a^2+2ab+b^2)(a^2+2ab+b^2)$$

490. Area of parallelogram and triangle.

1. Show that *the area of a parallelogram is equal to the product of the base and altitude.*

2. Show that *the area of a triangle is equal to one-half of the product of the base by the altitude.*

3. Show that *the area of a trapezoid is equal to one-half of the product of the altitude by the sum of the bases.*

CHAPTER VI

491. Adjacent angles.

1. What are adjacent angles? Make a drawing of two adjacent angles.

2. What are perpendicular lines?

3. Show that at a given point in a given line only one perpendicular can be drawn to the line.

4. What is the sum of the adjacent angles about a point, on one side of a straight line?

5. The angular space about a point, on one side of a straight line, is divided into angles denoted by $3x$, $2(x+9)$, x, and $42-x$. Find x and each angle in degrees. Draw the figure.

6. What is the sum of the angles at a point just covering the angular space about the point?

7. All the angular space about a point is divided into angles represented by x, $36+5x$, and $3x-9$. Find x and each angle in degrees. Draw the figures.

492. Supplementary angles.

1. What are supplementary angles? Illustrate your answer with a drawing.

2. The difference of two supplementary angles is 110. Find them.

3. If an angle is increased by 12°, and the supplement is divided by 5, the sum of the angles obtained is 80°. Find the supplementary angles.

493. Complementary angles.

1. What are complementary angles? Illustrate your answer with a drawing.

2. The difference between an angle and its complement is 27°. Find the angle.

3. If an angle is increased by 15° and the complement is divided by 3 the sum of the angles obtained is 75°. Find the complementary angles.

494. Opposite angles.

1. What are opposite angles? Make a drawing of opposite angles.

2. Show that if two lines intersect, the opposite angles are equal.

3. Two opposite angles are denoted by $5x+\frac{3x}{4}$ and $\frac{5x}{2}+130$. Find x and the angles. Draw a figure representing these angles.

495. The acute angles of a right triangle.

1. Show that the acute angles of a right triangle are complementary.

2. The acute angles of a right triangle are denoted by $\frac{x}{8}+2x$ and $\frac{87}{2}-\frac{5x}{6}$. Find x and the angles.

3. If in a right triangle the acute angles are 30° and 60° respectively, how does the hypotenuse compare with the shortest side?

496. Equations.

1. Solve $\frac{3a-2}{14}-\frac{1}{2}=\frac{3}{14}$

2. Solve $\frac{30x}{7}+30-\frac{6x}{14}=57$

3. Solve $2(a-2)+\frac{a+3}{4}-4=4$

497. Angle pairs formed by two lines intersected by a third.

1. If the corresponding angles are equal prove that the alternate interior angles are equal.

2. If the corresponding angles are equal prove that the interior angles on the same side are supplementary.

3. If the alternate interior angles are equal show that the corresponding angles are equal.

CHAPTER VII

498. Parallel lines.

1. When are two lines said to be parallel?

2. If two parallel lines are cut by a transversal how do the corresponding angles compare? If two given lines are cut by a transversal making the corresponding angles equal, what is known about the given lines?

3. Show that two lines perpendicular to the same line are parallel.

4. Show that *two lines are parallel if two alternate interior angles formed with a transversal are equal.* (Use problem 3, § 497.)

5. Show that *two lines are parallel if the interior angles formed with a transversal are supplementary.*

6. Show that *two lines parallel to the same line are parallel to each other.*

7. Show that *if two parallels are cut by a transversal the alternate interior angles are equal and the interior angles on the same side are supplementary.* How may this be used to draw a line parallel to a given line?

8. Prove that if two angles have their sides parallel they are either equal or supplementary.

9. The corresponding angles formed by two parallels and a transversal are denoted by $2(4x-3)°$ and $(79+3x)°$. Find x and the angles.

10. Show that *the sum of the interior angles of a triangle is 180°.*

11. Show that an exterior angle of a triangle is equal to the sum of the two remote interior angles.

12. What is meant by a parallelogram?

13. Prove that *the opposite angles of a parallelogram are equal.*

14. Prove that *the consecutive angles of a parallelogram are supplementary.*

15. The consecutive angles of a parallelogram are so related that 3 times one angle diminished by the other is equal to 30°. Find the number of degrees in each angle.

16. Make drawings representing the following: cube, parallelopiped, prism, pyramid, cylinder, cone, sphere.

17. Name on the figures of problem 16 the following: parallel lines, parallel planes, lines parallel to planes, parallel lines cut by a transversal, non-parallel lines cut by a transversal, lines perpendicular to each other, lines perpendicular to a plane.

CHAPTER VIII

499. Drawing to scale.

1. What is meant by indirect measurement?

2. To measure the width, AC, of a stream an engineer lays off a line, BC, on one side of the river, and measures the angles ACB and ABC. If $BC=45$ rd., $\angle ACB=110°$, and $\angle ABC=40°$, find AC by means of a scale drawing.

3. The angle of elevation of the top of a pole is 38°, the observer standing 20 yd. from the pole. How high is the pole?

4. From the top of a cliff 150 ft. high the angle of depression of a boat is 25°. How far is the boat from the top of the cliff?

5. The view from a battery at B to the enemy's fort at F is obstructed. A point P is located from which F is observed to bear 4 mi. northeast. P is 6.24 mi. northwest of B. Find the distance and bearing of F from B.

500. Ratio.

1. What is meant by the ratio of two numbers? Illustrate.

2. What is the ratio of two line-segments?

3. What number added to 12 and subtracted from 30 gives results that are in the ratio $\frac{5}{10}$?

4. Divide 81 into three parts that are in the ratio 2:3:4.

501. Similar figures.

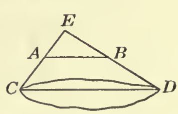
1. What are similar figures?

2. State the relation existing between the sides of similar triangles. How do the angles compare?

3. The sides of a triangle are 8, 10, and 13. The shortest side of a similar triangle is 11. Find the other sides.

4. A tree casts a shadow 90′ long. At the same time a vertical stick 4′ long casts a shadow 50′ long. How high is the tree?

5. In the adjoining figure CD represents the distance across a swamp. AB is parallel to CD. If $AB = 100'$, $AE = 30'$, and $CE = 150'$, find CD.



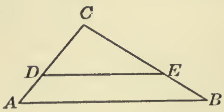
CHAPTER IX

502. Trigonometric ratios.

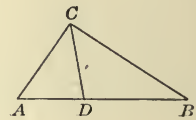
The angle of elevation of the top of a tower is 27° and the distance from the foot of the tower is 259 ft. Find the height of the tower: first, by a scale drawing; second, by trigonometry.

503. Ratio.

1. In the adjoining figure, DE is parallel to AB. Find EB, if $DC = 4$, $AD = .27$, and $CE = 1\frac{1}{3}DC$. State the principle used in obtaining the equation.



2. In the adjoining figure CD bisects angle C. If $AC = 3''$, $BC = 6''$, and AD is $2''$ less than DB, find AD. State principle used in obtaining the equation.



3. Bisect a line-segment.

4. Divide a line-segment into parts having the ratio $\frac{3}{4}$.

5. Using ruler and compass only, find the ratio of two given line-segments.

6. Find the greatest common divisor of 3,542 and 5,016.

7. Reduce to simplest form $\frac{63a^3b^3x^4}{7a^2b^4x^6}$

8. How is a ratio reduced to the simplest form?

504. Variation.

1. Give the meaning of the following terms: variable, constant, function. Illustrate by an example the meaning of each of these terms.

2. What is meant by the statement: x varies directly as y?

3. The simple interest on an investment varies directly as the time. If the interest for 6 years on a sum of money is \$200, what will be the interest for 8 years?

4. The distance, d, through which a body falls from rest varies directly as the square of the time, t, in which it falls. State the equation for d and t for a body observed to fall 400 ft. in 5 seconds. Graph the equation.

5. What is the meaning of the statement: x varies inversely as y?

6. The apparent size, s, of an object varies inversely as the distance, d. Express the statement by means of an equation.

505. Proportion.

1. What is a proportion? Illustrate your statement by giving several examples.

2. Prove that in a proportion the product of the means equals the product of the extremes.

3. Divide \$2,400 into two parts having the ratio $\frac{5}{7}$.

4. Solve the equation $\frac{x+2}{x+3}=\frac{x+3}{x+1}$

5. If 80 lb. of sea-water contain 4 lb. of salt, how much fresh water must be added to make a new solution of which 45 lb. contain $\frac{2}{3}$ lb. of salt?

6. If $\frac{d_1}{d_2}=\frac{w_2}{w_1}$, find d_1 if $d_2=27$ ft., $w_2=36$ lb., and $w_1=24$ pounds.

7. What per cent of evaporation must take place from a 90 per cent solution to produce a 95 per cent solution?

8. Prove that the areas of two rectangles are to each other as the products of the dimensions.

9. Prove that if two rectangles have equal bases they are to each other as the altitudes.

10. Prove that the areas of two triangles are to each other as the product of the bases and altitudes.

11. Prove that if two triangles have equal bases they are to each other as the altitudes.

CHAPTER X

506. Congruence.

1. What are congruent figures?

2. Prove that *two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

3. How may the theorem in problem 2 be used to find inaccessible distances?

4. Prove that *two triangles are congruent if two angles and the side included between their vertices in one triangle are equal respectively to the corresponding parts in the other.*

5. Prove that *the base angles of an isosceles triangle are equal.*

6. Prove that an equilateral triangle is equiangular.

7. Prove that the bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

8. Prove that all points on the perpendicular bisector of a line-segment are equidistant from the end-points of the segment.

9. Prove that if a line bisects an angle of a triangle and is perpendicular to the opposite side the triangle is isosceles.

10. Prove that *if two angles of a triangle are equal the triangle is isosceles.*

11. Prove that an equiangular triangle is equilateral.

12. Prove that if the perpendicular bisector of one side of a triangle passes through the opposite vertex the triangle is isosceles.

13. Prove that *if two sides of a triangle are unequal the angles opposite them are unequal.*

14. Prove that *if two angles of a triangle are unequal the sides opposite them are unequal.*

15. Prove that *if three sides of one triangle are equal respectively to three sides of another the triangles are congruent.*

16. Prove that *if each of two points of one line is equally distant from two points of another line the line joining the first two points is the perpendicular bisector of the segment joining the other two.*

17. What is the locus of all points equidistant from the end-points of a line-segment?

18. Prove that two right triangles are congruent if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.

19. Prove that *the shortest distance from a point to a line is the perpendicular from the point to the line.*

20. Prove that oblique lines drawn from a point on the perpendicular to the line and making equal angles with the perpendicular are equal.

CHAPTER XI

507. Fundamental constructions.

1. *To bisect an angle.* What method of proof is used to prove this construction? Make the construction and give the proof.

2. *At a point on a given line to construct a perpendicular to the line.* Make the construction. State the principal theorem upon which the proof of this construction is based. Give the proof.

3. *To bisect a given line-segment.* Make the construction. What is the principal theorem used to prove the construction? Give proof.

4. *To construct the perpendicular bisector of a line-segment.* Make the construction and give the proof. Compare this construction with that in problem 3. Compare the proofs of problems 3 and 4.

5. *From a point outside of a line to construct a perpendicular to the line.* Make the construction. Upon what principal proposition is the proof based? Give proof.

6. *At a given point on a given line to draw a line making an angle with the given line equal to the given angle.* Make the construction. What method of proof is used? Give the proof.

508. Application of the fundamental constructions.

1. Construct a triangle having given the following parts: two sides and the included angle; two angles and the side between their vertices; three sides; two sides and the angle opposite one of them.

2. Construct a right triangle having given the following parts: the hypotenuse and one of the other sides; the hypotenuse and one of the acute angles.

3. Construct an isosceles triangle having given one of the base angles and the altitude.

4. Construct an equilateral triangle having given the altitude.

5. Construct a right triangle whose acute angles are 30° and 60°.

6. To trisect a right angle.

7. Construct angles of 60°, 30°, 15°, 120°, 90°, 45°, 22°30′, 135°, 75°, 165°.

8. Through a point outside of a given line to draw a line parallel to the given line.

509. Theorems.

1. *Only one perpendicular can be drawn to a line at a given point on the line.* Give reason.

2. *Only one perpendicular can be drawn from a point to a line.* Give reason.

510. Symmetry.

1. Give several illustrations of symmetric bodies.

2. Name several single bodies symmetric with respect to a plane.

3. Draw several figures symmetric with respect to a line.

4. What is an axis of symmetry?

5. Establish the following theorems by means of the symmetry of figures:

a) *A point on the perpendicular bisector of a line-segment is equidistant from the end-points.*

b) *A point not on the perpendicular bisector of a line-segment is not equidistant from the end-points.*

c) *A point on the bisector of an angle is equidistant from the sides of the angle.*

d) *A point not on the bisector of an angle is not equidistant from the sides.*

6. *Show that the locus of points, within an angle, equidistant from the sides is the bisector of the angle.*

511. The circle.

1. Give the meaning of the following and draw a figure for each to illustrate your statement: tangent, point of contact, regular polygon, inscribed polygon, circumscribed polygon.

2. Prove the following theorems:

a) *The radius drawn to the point of contact of a tangent is perpendicular to the tangent.*

b) *A line perpendicular to a radius at the outer end-point is tangent to the circle.*

3. Make the following constructions:

a) At a point on a circle construct the tangent.

b) Find the center of a given circle.

c) Draw a circle passing through two given points.

d) Circumscribe a circle about a triangle. Give proof.

e) Inscribe a circle in a triangle. Give proof.

f) Inscribe a square in a circle. Give proof.

g) Circumscribe a square about a circle. Give proof.

h) Inscribe a regular hexagon in a circle. Give proof.

i) Prove that perpendicular bisectors of the sides of a triangle pass through the center of the circumscribed circle.

k) Prove that the bisectors of the angles of a triangle pass through the center of the inscribed circle.

CHAPTER XII

512. Uses of positive and negative numbers.

1. What are positive numbers? Negative numbers?

2. Give the meaning of absolute, or numerical, value.

3. State some of the uses of positive and negative numbers. Illustrate each case.

4. Graph the following hourly temperature readings:

8:00 A.M.	9:00	10:00	11:00	12:00	1:00 P.M.	2:00	3:00	4:00	5:00	6:00
$-6°$	$-4°$	$-2°$	$+2°$	$+4°$	$+3°$	$+3°$	$+2°$	$+0°$	$-2°$	$-3°$

513. Addition of positive and negative numbers.

1. Find the following sums graphically:

$(+5)+(+3)$; $(+5)+(-3)$; $(-5)+(+3)$; $(-5)+(-3)$

2. State a rule for adding positive and negative numbers.

3. Find the sums in problem 1 algebraically.

4. What is the most advantageous way of adding three or more algebraic numbers?

5. Find the sums of the following: $+49$, $+35$, -45, $+75$, -236; $25x$, $-38x$, $-20x$, $30x$, $-6x$.

6. Add the following:

$+25a$	$+3\frac{1}{3}x$	$-3.14k$	$-24a^2b$	$-2.3x^2yz^2$
$-\ 7a$	$-2\frac{1}{2}x$	$+4.26k$	$+18a^2b$	$-6.5x^2yz^2$

514. Subtraction of positive and negative numbers.

1. State the rule for subtracting algebraic numbers.

2. In the following subtract the lower number from the upper:

$-\frac{7}{6}$	$+\frac{9}{5}$	$+18c$	$-6.3a^2$	$-4.2(a+b)$
$+\frac{2}{3}$	$+\frac{7}{10}$	$-12c$	$-7.2a^2$	$+2.7(a+b)$

515. Multiplication of positive and negative numbers.

1. State the law of signs in multiplication.

2. Find the value of the following products:

$(+2)(+8)$; $(-\frac{2}{3})(+\frac{6}{5})$; $(+3)(-x)$; $(-7\frac{1}{2}a)(-3\frac{1}{3}b)$

3. Express the turning-tendency, or leverage, in terms of the force and arm.

4. In the following the first factor is the force, the second is the arm. Find the leverage.

$$(+2)(-3); \quad (-2)(-3); \quad (-2)(+3); \quad (+2)(+3)$$

5. Give the value of the following products:

$$0\times b; \quad b\times 0; \quad (+3)(0)(+4); \quad (1.4)(-3.2)(0)$$

6. Find the values of $3x^3-2x^2+7x-4$ for $x=0, 3, 1, -2, -4$.

516. Division of positive and negative numbers.

1. State the law of signs for division.

2. Show that $\dfrac{+x}{-y}=\dfrac{-x}{+y}=-\dfrac{+x}{+y}$.

3. Find the quotients of the following:

$$(-\tfrac{7}{4})\div(-\tfrac{3}{8}); \quad (-2x)\div(+6x); \quad (+4ab)\div(+b);$$
$$(-3.6x^2)\div(+4xy)$$

CHAPTER XIII

517. The laws of addition.

1. Illustrate the commutative law of addition.

2. Illustrate the associative law of addition.

518. Addition of monomials.

1. Add the following monomials:

$+12x, \ -3x, \ +14x$

$2(m+n), \ -6(m+n), \ (m+n)$

$14(a^2b-c), \ -8(a^2b-c)$

$(-5ak), \ (+2bk), \ (+ck)$

$a(x+y), \ b(x+y), \ -c(x+y)$

$3a^2b, \ -5xy, \ +7a^2b, \ +2xy$

519. Addition of polynomials.

1. State how to add polynomials.

2. Arrange according to ascending powers of x; according to descending powers of x:

$$x^5+x^4y-2x^2y^3-7y^5-3x^3y^2+8xy^4$$

3. Add the following polynomials:

$$-4\tfrac{1}{2}a^2b+16ab^2-7b^3+a^3;\quad -4a^3+8ab^2-2b^3-2\tfrac{1}{2}a^2b$$

4. Add $7(x-y)-4(x+y)+4\cdot 7$; $9(x+y)+3(x-y)-9\cdot 7$; $6(x-y)+2\cdot 7-3(x+y)$. Verify the results by letting $x=2$, $y=1$. (Chicago)*

520. Subtraction of monomials.

1. State how to find the difference of two monomials.

2. Subtract the lower from the upper:

$-12a$	$-7.5x$	$+16a^2bc$	$+17.2(x-3y^2)$
$+\ 3a$	$-2.3x$	$-20a^2bc$	$+22.6(x-3y^2)$

3. Subtract as indicated: $(-8.4a^2b^2c^3)-(-6.3a^2b^2c^3)$; $\{-4.5(x^2-y^2)\}-\{-7.6(x^2-y^2)\}$

521. Subtraction of polynomials.

1. State how to find the differences of polynomials.

2. Subtract as indicated:

$$(2.4x^2-7.3xy+16y^2)-(-3.7x^2+2.4xy-10y^2)$$

3. From $16-15\cdot 30+14(x-5yz)-13(5y-x)$ subtract $32-16\cdot 30+8(5y-z)$ without performing any of the indicated multiplications. (Chicago)

522. Removal of parentheses.

Add and subtract as indicated and simplify:

1. $3x-[4y-(3y+7x)]$

2. $5p^3-\{3p^2-(2p^2+7p^3)-(3p^2+4p^3)\}-(p^3+10p^2)$

3. Remove the parentheses: $a^2-\{2ab-[b^2-(c^2-\overline{2cd-d^2})]\}$ (Chicago)

* (Chicago) means: taken from an entrance examination given by the University of Chicago.

CHAPTER XIV

523. Multiplication of monomials.

1. State how to multiply monomials.

2. Find the following products and test the result by substituting values for the letters:

$(+22)(-14)(-3)$; $(-2\frac{1}{3}ax^2y)(-4\frac{1}{6}a^2xy)$; $(-3x^2)^2(-2x)^3(x^2)^4$

3. Find the value of $x^2-12x+4$ for $x=2$; for $x=-3$.

524. Multiplication of polynomials by monomials.

1. State how to multiply a polynomial by a monomial.

2. Multiply as indicated and test by substituting values for the letters:

$4x+3x(x-2y)+(2x-3y)x$; $2a\{2b-3(4a-b)+4(3a-5b)\}$

525. Multiplication of polynomials by polynomials.

1. State how to multiply polynomials by polynomials.

2. Multiply $pa^2+qb^2-rc^2$ by $a+b-c$. (Chicago)

3. Multiply $x^6+2x^3y^3+y^6$ by $x^6-2x^3y^3+y^6$.

4. Multiply $13x-12y-xy+3$ by $5x-3y+xy+5$.

526. Reduction of quotients.

1. State how to reduce quotients to the simplest form.

2. Reduce the following to the simplest form:

$$\frac{-25a^4bc^3}{12a^8bc^2} \qquad \frac{60x^2y-45xy^2+90xy}{15xy}$$

$$\frac{(-x)^2y(-z)^3}{5(-x)^4y^2(-z)} \qquad \frac{20(a+b)^5-15(a+b)^6+30(a+b)^8}{-5(a+b)^3}$$

527. Division of polynomials by polynomials.

1. Divide $x^3+x^2y+xy^2+xz^2+yz^2+z^3$ by $x+y$.

2. Divide $10x^3y^2-14x^4y+12x^5-x^2y^2-8xy^4+4y^5$

by $2y^3-3xy^2-4x^2y+6x^3$,

and check the result by substituting $x=y=1$. (Chicago)

CHAPTER XV

528. Special products.

1. Draw a figure to show that $(c+d)^2=c^2+2cd+d^2$.

2. State how to find the square of a binomial without a figure, i.e., by inspection.

3. Find the trinomials equal to the following products:

$$(p-r)^2;\quad (.4xy-.3z)^2;\quad [(m+n)-t]^2$$

4. Give by inspection the following products as the difference of two squares:

$$(3b-2c)(3b+2c);\quad (1+\tfrac{2}{3}x^2)(1-\tfrac{2}{3}x^2);\quad (x+y)(x-y)(x^2+y^2)$$

5. Give by inspection the following squares of trinomials:

$$(2m-4s+t)^2;\quad (a-.3b+2)^2$$

529. Factoring.

1. Factor the following trinomials:

$$49-140n^2+100n^4;\quad 49m^2n^2+42mnxy+9x^2y^2$$

2. Find the factors of the following binomials:

$$1-25y^2;\quad 196-225d^2h^2y^2;\quad x^6-y^6;\quad (a^2-b)^2-c^4$$

3. Find the factors of

$$6x^2-29xy+35y^2;\quad 2a^2+11a+12;\quad 102-11m-m^2$$

530. The theorem of Pythagoras.

1. State the theorem of Pythagoras.

2. Express the diagonal of a square in terms of the side.

531. Square root.

Find the square roots of 6,241; 643,204.

532. Quadratic equations.

1. Solve the following equations graphically:

$$x^2-8x=-27;\ m^2-\frac{3m}{2}=\frac{27}{16}$$

2. Solve by factoring:

$$n^2+28=11n;\ 3c^2+c-2=0$$

3. Solve by completing the square:

$$x^2+10x=24;\ 4y^2+20y=-9$$

4. The hypotenuse of a right triangle is 5 ft. and one of the sides is 1 ft. longer than the other. Find the length of the sides.

CHAPTER XVI

533. Solution of equations.

1. State how to solve an equation of the first degree in one unknown.

2. Solve the following equations:

(1) $4x+16=9x+11$

(2) $5+3x+\frac{2x}{3}=1\frac{2}{3}+\frac{x}{6}$

(3) $\frac{3x+1}{4}-\frac{x-3}{2}=\frac{7}{4}$

(4) $(2x-1)(x-3)=(x-5)(2x+4)$

(5) $\frac{1}{3}(1-x)=\frac{1}{4}(2-x)+\frac{1}{6}(3+x)$

(6) $.8(10x-2.3)=.05(5x+.4)-2.45$

(7) $\frac{3x+5}{x-5}-\frac{6x+13}{3}=2x$

(8) $ay-\frac{3b^2}{a}=\frac{b^2(y-3)}{a}-\frac{b-a}{a}$

3. Solve the following problems:

(1) The difference of the acute angles of a right triangle is $22°14'$. How large is each?

(2) The perimeter of a rectangle is 24 inches. If the altitude is increased by 4 in. and the base decreased by 2 in. the area remains unchanged. What are the dimensions of the rectangle?

(3) The sides of a triangle are in the ratio $5:6:13$. Find the three sides, if the perimeter is 144 inches.

(4) The angles of a triangle are denoted by $(x+8)°$, $2(x+8)°$, and $(2x-4)°$. Find x and the angles.

(5) The interior angles on the same side formed by two parallel lines cut by a transversal are $\frac{2}{7}x$ and $\frac{3}{2}x-20$. Find x and the angles.

(6) Three times a number is increased by 21 and the sum is divided by the sum of the number and 7. The result is then equal to 3. Find the number.

(7) A father is 46 years old and his son is 8. In how many years will the father be 3 times as old as the son?

(8) The difference of the squares of two consecutive even numbers is 108. Find the numbers.

(9) Two men are 25 mi. apart and walk toward each other at the rate of $3\frac{1}{2}$ and 4 mi. an hour respectively. After how long do they meet? (Yale)

(10) A man travels 50 mi. in an automobile in $3\frac{1}{4}$ hours. If he runs at the rate of 20 mi. an hour in the country and at the rate of 8 mi. an hour when within city limits, find how many miles of his journey is in the country. (Yale)

(11) The length of a room is 8 ft. greater than its width. If each be increased by 2 ft., the area of the room will be increased by 60 sq. ft. What is the actual area of the room? (Yale)

(12) Show how to divide \$2,000 into two investments, one at 4% and the other at 3%, so that the former shall produce twice as much income as the latter. (Chicago)

(13) Find two consecutive odd numbers such that the difference of their squares shall be 152. (Chicago)

(14) The tickets of admission to a game are 25 cents each for adults and 10 cents each for children. It is found that $18.75 is taken in and the turnstile shows that 117 persons attend. How many children are there at the game? (Yale)

(15) A sum of $1,050 is divided into two parts and invested; the simple interest on the one part at 4% for 6 years is the same as the simple interest on the other at 5% for 12 years; find how the money is divided. (Princeton)

(16) At what time between 8:00 and 9:00 o'clock are the hands of the clock together? At right angles? In opposite directions?

(17) A beam 18 ft. long and weighing 40 lb. is supported at a point 3 ft. from the center. What force must be exerted at the end farthest from the fulcrum for balance?

(18) A, standing 5 ft. from the fulcrum, balances B, who stands 7 ft. from it. A weighs 126 pounds. Find the weight of B.

CHAPTER XVII

534. System of two linear equations in two unknowns.

1. What is a simultaneous system of equations?

2. State the graphical method of solving simultaneous equations.

3. Solve the following systems graphically:

(1) $\begin{cases} 2x-3y=4 \\ 4x+2y=1 \end{cases}$ (2) $\begin{cases} 4x+7y=-27 \\ x-2y=12 \end{cases}$

4. When is a linear equation in several unknowns said to be in normal form?

5. State how to solve simultaneous equations by elimination.

6. Solve the following systems by elimination by adding or subtracting:

(1) $\begin{cases} \dfrac{x}{5}+\dfrac{y}{3}=\dfrac{37}{15} \\ \dfrac{2x}{3}-\dfrac{y}{3}=1 \end{cases}$ (2) $\begin{cases} \dfrac{x}{3}-y=1 \\ x+\dfrac{y}{2}=4 \end{cases}$

7. With two parallels the interior angles on the same side of a transversal are $(8y-10)°$ and $(5x+y+5)°$. Two alternate interior angles are $(x+4y)°$ and 70°. Find x, y, and the unknown angles.

8. A man has two sons, one six years older than the other; after two years the father's age will be twice the combined ages of his sons; and six years ago his age was four times their combined ages. How old is each? (Princeton)

9. The rates of two trains differ by 5 mi. an hour. The faster requires one hour less time to run 280 miles. Find the rate of each. (Yale)

10. The sum of the two digits of a 2-digit number is 10. If 54 be subtracted from the number the result will be equal to the number obtained by reversing the digits of the original number. Find the number.

11. A man invested two sums, one at 4%, the other at 5%, and received annually an income of \$500. He then reinvests these sums at 5% and 6%, respectively, receiving an income of \$600 annually. Find the two sums invested.

12. Two angles of a triangle are equal and the third angle is equal to their sum. How many degrees are in each angle?

13. Solve for m and n:

$$\frac{7m+8}{5}-\frac{7n-1}{4}=-2$$

$$\frac{2m-4}{2}+\frac{n-1}{3}=-\frac{1}{3}$$

Make a graph for the equations and then illustrate and verify the solution. (Chicago)

14. Solve for a, b, and c:

$$3a-2c+b=-1$$
$$2a-b+3c=9$$
$$b+3a-c=2 \quad \text{(Chicago)}$$

15. Solve the following simultaneous equations:

$$\begin{aligned} 2x + 3y + 5 &= 0 \\ 6y + 5z &= 7 \\ 3x + 10z - 1 &= 0 \end{aligned}$$

Indicate your answers clearly, and verify by substituting in the given equations. (Harvard)

CHAPTER XVIII

535. The formula as a general rule.

1. What is a formula?

2. A man rides a distance of p mi. and walks back at a rate of q mi. an hour. The entire trip took t hours. Find his rate of riding. (Yale)

3. An express train whose rate is 45 mi. an hour leaves a station a hr. after a freight train. The express overtakes the freight train in b hours. Find the rate of the freight train.

536. Evaluation of formulas.

1. If $s = 16t^2 + vt$, find s for $t = 10$, $v = 5$.

2. If $v = \frac{4}{3}\pi r^3$ and $r = \frac{d}{2}$, find v for $d = 7$.

3. If $h = \frac{v^2}{2g}$, find v for $h = 25$ and $g = 32$.

4. If $R = \frac{rr'}{r + r'}$, find r for $R = 50$, $r' = 15$.

537. Express any one of the letters of each of the following formulas in terms of the others:

1. Solve $C = \frac{5}{9}(F - 32)$ for F.

2. Solve $s = \frac{n}{2}(a + l)$ for a.

3. Solve $A=\frac{h}{2}(b_1+b_2)$ for b_2.

4. Solve $s=\frac{rl-a}{r-l}$ for l.

5. Solve $\frac{x}{a}+\frac{a-2x}{3}+4=3a$ for x.

6. Solve $1-\frac{4(3a+4x)}{a}=\frac{5(3a+2x)}{a}$ for x.

7. Solve $\frac{1}{E}-\frac{1}{P}=\frac{1}{S}$ for E; for P; for S.

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SYMBOLS

FORMULAS

Right triangle	$a^2+b^2=c^2$
Circle	$c=2\pi r,\ \pi d$
Area of rectangle	$A=bh$
Area of square	$A=s^2$
Area of parallelogram	$A=bh$
Area of triangle	$A=\frac{1}{2}bh$
Area of trapezoid	$A=\frac{1}{2}h(b+b')$
Area of circle	$A=\pi r^2$
Motion	$d=rt$