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A FOUNDATIONAL STUDY IN THE PEDAGOGY OF ARITHMETIC

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A FOUNDATIONAL STUDY

IN THE

PEDAGOGY OF ARITHMETIC

BY

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PREFACE.

The problem of arithmetical teaching has been attacked in the past from many avenues of approach. Several interesting treatises, written from almost every point of view except from the insights gained by careful first-hand study of cases and experimental trial of adults and children, have appeared. While these have been suggestive and valuable according to their lights, manifestly they are makeshifts awaiting more thoroughgoing and systematic attempts to render available to teachers and students of education the scholarly investigations that have been going on for the past decade or more and that have been entombed for the most part in the libraries of our universities. Many of these need popularizing and to an extent rewriting, since, unfortunately, our best investigators are not always our best expositors.

If the author has succeeded in summarizing and organizing to some extent these widely scattered studies into the genesis, psychology, statistics, and didactics of number in a manner to make them readable and interesting as well as more concisely contributory to the kind of material which he feels must be the subject-matter of the prolegomena to any future arithmetical didactic, he will feel duly rewarded.

In all of this work as well as in the original work of part two, heartfelt acknowledgment of the helpfulness and guidance of Assistant Professor Paul R. Radosavljevich, of the faculty of the School of Pedagogy of New York University, is hereby made; also to Professor Robert McDougall and especially to Dean Balliet of this School for encouragement and counsel. Thanks must also be given to the teachers of P. S. No. 27, Jersey City, for cheerful assistance in many of the little attractive but extremely valuable phases of the work of preparation; and to Professor S. A. Courtis, of the Detroit Home and Day School, for the use made of his tests and of some of his published as well as unpublished discussions.

INTRODUCTORY NOTE

The following treatise is based on a thesis on the psychology and pedagogy of arithmetic prepared by the author in the department of education of the graduate school of New York University and accepted by the faculty of that school in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

The first part of the book constitutes a very complete résumé of the extensive experimental literature on this subject and makes it readily accessible to the general student. It will serve equally well as a convenient guide to the special student who desires to make a first-hand study of it.

The second part forms the author's own research, and is an important contribution both because of its positive results and as a scholarly critique of the technique employed by other investigators.

The experimental method will do for pedagogy in the future what it has done so effectively for psychology during the last half century, by placing it on a more scientific basis and eliminating the element of speculation from the study of problems which lend themselves readily to experimental methods of investigation.

The definite results already attained, although relatively meager, are yet of such general interest and importance that we cannot afford to disregard them in making school programs and determining methods of teaching. The author of this book has therefore rendered an important service, besides making an original contribution of much interest, in making these results accessible, in convenient form, to the classroom teacher as well as to the busy superintendent and principal of schools.

Thomas M. Balliet, Ph.D.
Dean of School of Pedagogy.

New York University,
Washington Square, New York,
August 17, 1914.

INTRODUCTION.

A study in the pedagogy of arithmetic will include at least the following lines of inquiry: (a) the origin and development of number concepts in primitive men and in children (anthropogenetic studies): (b) the mental functioning of adults capable of introspection and description with reference to the processes involved (psychological studies): (c) the objective study of abilities and efficiency and their economical development with a view to the possible discovery of relationships, causes and hindrances (statistical studies); (d) the objective study of the school child's apprehension of number (experimental didactics). In addition to these four main lines of inquiry and, indeed, involved in them to a greater or less extent, it will be desirable to consider somewhat such topics as the hygiene of arithmetic (normal and healthful learning); arithmetical prodigies (geniuses, but primitive in their arithmetical processes); transfer (how far does an ability developed to deal with one set of materials pass over to the dealing with another set); ideational types (arithmetical working-types of imagery). Upon the basis of such studies only, can be founded a scientific curriculum and a scientific method.

"No pedagogue has anywhere even attempted to sum up the copious but scattered anthropological, experimental, pedagogical, psychogenetic and phylogenetic resources now available" [in arithmetic] (G. Stanley Hall).[1] In the following pages an attempt, though far from exhaustive, will be made to give this summary.

Our purpose then is (1) to review at some length and in a

[1] 61, p. 393.

measure criticize, supplement and organize the recent literature bearing on the lines of inquiry mentioned above, and (2) to give an account of a series of experiments tried by the author in order to study at first hand (a) the arithmetical abilities of certain school children, (b) the problem of the school child's concepts of number.

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PART I.

A REVIEW OF REPRESENTATIVE STUDIES OF NUMBER.

A REVIEW OF REPRESENTATIVE STUDY OF NUMBER.

1. Genetic Studies.[1]

(a) *Primitive Men.*

The origin and development of the number sense in primitive man has been studied in considerable detail by L. L. Conant (1) and W J McGee (2). Of course, primitive men have long since turned to dust; so that the method of procedure must be to investigate the conditions in existing peoples who are still in varying stages of savage culture. The anthropological assumption is that the course of mental progress is approximately uniform and is by *natural* means, so that the growth stage of a people may be determined by studying the mentality of other peoples developed to a corresponding stage.

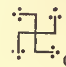
It might be supposed that the development of the number sense could only proceed *pari passu* with the development of number names, and that the limits of the latter would set the limitations of the former; or again that the convenience for counters of the fingers and ten toes would give rise, very early in human history, to decimal or vigesimal number systems. But a study of the beginnings of number systems shows "that (1) the origin of number names is at the bottom of the scale of human development, that is to say, comes late, (2) primeval man does not cognize quinary and decimal systems, (3) does not use his toes, fingers, etc., as mechanical adjuncts to nascent notation" (2, p. 654). Many primitive peoples count by fingers and hands, sometimes with the addition of toes and feet, and thereby fix quinary, decimal and vigesimal systems; but the burden of evidence is that these are far from primeval.

The paucity of number names among savage tribes is illustrated by the following examples cited by McGee (2). Some Australian

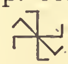
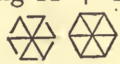
[1] This review of Genetic Studies appeared originally in Educational Foundations, 1914, XXV, pp. 5–10.

tribes count laboriously up to 2, 3, 4, or 6, sometimes doubling 2 to make 4 or 3 to make 6 and in other ways revealing a quasi-binary system; certain Brazilian tribes are described as counting only to 2, 3, 4, usually with an additional name for many; the Tasmanians counted commonly to 2, sometimes to 4, and were able to reach 5 by the addition of one to the limital number (four and one more). But it should be noted that this paucity of number names did not set the limit to numerical capacity. Most tribesmen reveal the germ of notation in the use of sticks, notches, etc. By means of these and of gesture language they symbolized numerical values beyond the limits of the number names. In the savage mind, and (as we shall see) in the child's mind, the series-idea as an abstraction precedes the naming. Modern names are the product of gradual evolution motivated by a felt necessity to express the series-idea already present. Our savage ancestors found it a tremendous task to reach a very meager number expression; that their conception of number far outran their ability to express is shown by much anthropological evidence, Conant (1), Phillips (5); e.g., the original inhabitants of Victoria had no numerals above two, yet they counted and even recorded the phases of the moon.

The foundational study of McGee (2) bids us consider as fundamental to an understanding of the number systems of the earliest men that these men were beyond everything egoists and mystics with strange and (to us) mysterious ways of orientation. Binary, quaternary and senary systems become to them subconsciously ternary, quinary and septenary respectively. Quaternary does not remain purely quaternary, but by the addition of a vague unity (the ego, from which all things procecd) becomes quinary, the system of fours becomes in thought as well as in graphic representation a system of fives, sixes becomes sevens, and these by a process of *augmentation* form the bases of the larger numbers which belong to the respective systems.

An examination of some of the mechanical symbolisms will serve to make the method clear. The four-five system is represented by + or ∴̇∴. (Note that the point of intersection of the lines of the cross, or the middle dot represents the unity that is always counted in by the primeval man, himself, the It.) This symbol is raised from 4 + 1 to 8 + 1 by adding a line or dot to each of the four cardinal extremities, thus  or  "The mystical middle is persistent and can be counted but once howsoever the value be augmented" (2, p. 662). Again the 8 + 1 is raised to 12 + 1 by the addition of dots, thus ; or still higher,  representing 20 + 1, etc.; or by the development of the "meander,"  representing 16 + 1, etc. The augmentation may proceed thus indefinitely:

Four-five system— $\frac{4+1}{5}$ $\frac{8+1}{9}$ $\frac{12+1}{13}$ $\frac{16+1}{17}$ $\frac{20+1}{21}$ $\frac{24+1}{25}$ $\frac{28+1}{29}$ $\frac{32+1}{33}$, etc.

The six-seven system is produced by the superposition of a binary system on the quaternary system. "It is more complicated and modified through the difficulty of depicting tridimensional relations on a bidimensional surface. Among the Pueblo peoples this is overcome by bisecting two of the quadrants, thus  but mechanical tendency operates to produce the regular figure  (2, p. 662). From the first figure augmentation produces  representing 12 + 1, which later is modified into the hexagram 

Six-seven system— $\frac{6+1}{7}$ $\frac{12+1}{13}$ $\frac{18+1}{19}$ $\frac{24+1}{25}$ $\frac{30+1}{31}$ $\frac{36+1}{37}$ $\frac{42+1}{43}$, etc.

The binary-ternary system, from which the quatern-quinary system is derived by augmentation, is lacking in graphic representation. It was produced in a manner similar to the others.

Two-three system— $\frac{2+1}{3}$ $\frac{4+1}{5}$ $\frac{6+1}{7}$ $\frac{8+1}{9}$ $\frac{10+1}{11}$ $\frac{12+1}{13}$ $\frac{14+1}{15}$.

It will be observed that some of the numbers occur in more than one system; these were deemed of unusual mystical significance. "These number systems are distinct from Aryan arithmetic, both in motive and mechanism. They are devices for divination, for binding the real world to the supernal, and it is only later or in an ancillary way that they are prostituted to practical uses; yet by reason of the extraordinary potency imputed to them they dominate thought and action in the culture stages to which they belong and profoundly affect the course of intellectual development. The base of the system is a measure of the intellectual capacity normal to the culture stage to which it belongs" (2, p. 660).

The investigation shows that the binary-ternary system is the earliest; higher in the scale of human development are the quatern-quinary and senary-septenary systems. It may seem strange that having hit upon a concept of five in the quatern-quinary thenceforth the development of the convenient decimal system should not be rapid. But the *definite* quinary concept from which the step to ten would have been easy was missed by many primitive tribes, although it was possessed by the ancestors of the Arabs with their decimal system, by the Mexicans who had a vigesimal system, and a few others; and we find in many lands a distinct development of the senary-septenary system instead. Finger counting, then, is a comparatively late development, for if it had existed early, it would have led to the early rise of the decimal system. Direct observation of savage tribes in the earliest stages of culture confirm this absence of finger counting, for many of them are found to be unable to count their fingers without the use of other symbols.

It would be of more anthropological than mathematical interest to trace the influence of the almacabala described above on the social and fiducial systems of primeval man and their relation to his ways of orientating himself. The dualistic (binary) conception of the cosmos existing among the earliest men (and further back among animals), the danger side in front, the safety side in the rear, merges into the antithesis of sea and land; the sea full of horrors, the land the haven of safety. Still later the strand of the shore stretching on either hand, as the sea is faced, characterizes the quaternary stage of culture. More important from our point of view is the *law of augmentation* which lies at the basis of the source of certain vestigial features still persisting in Aryan culture.

The three number systems described have all left vestiges, some of which persist even to modern times among civilized peoples. They may be traced to peoples still living in the lower culture stages, thus serving to establish the course of the development of number concepts and to throw light upon the numeration of our savage ancestors.

A few examples cited by McGee follow: The 6–7 system survives as the bridge connecting almacabala and mathematics. In the graphic form it became the Pythagorean hexagon of two superposed triangles, the hexagram of Brianchon (Paracelsus), the subrational hexagram of Pascal, etc. The astrologic seven retarded acceptance of the discovery of the eighth planet, Neptune. In the numerical form six, and more especially seven, play large rôles in the classical and sacred literature revived during the Elizabethan period. Nine (from the 2–3 and 4–5 systems) survives in the Muses, nine lives of the cat, effeminacy of the tailor; it even survived in the school books of the early

part of the century in the more curious than useful arithmetic process of casting out nines; thirteen in all three systems is still the messenger of evil in the minds of many; seventh son of a seventh son needs no training for medical craft, nor seventh daughter of seventh daughter as seeress.

It seems, then, that we have here the historical presentation of the beginning of mathematics. The archaic method was "to use integral numbers as tokens of extra-natural potencies rather than as symbols for natural values; to combine them by a simple rule tending to develop into algorismic processes; and to represent the numerical combinations by mechanical devices tending to develop into geometrical forms; the system being characterized by the method of reckoning from an ill-defined unity counted but once in each combination" (2, p. 664), "so that in all cases the exoterically mystical number carried an esoteric complement in the form of a simple unity reflecting the egoistic personality or subjectivity of the thinker" (2, p. 656).

The quinary, decimal and vigesimal systems set forth by Conant (1) with their accompaniment of counting by fingers and hands, sometimes with toes and feet, mark later stages of culture and form transitional steps to our modern system. As mankind progressed, counting was divorced from its supernal connections and became motivated more and more by practical ends of more precise adjustment to the environment. But for a long, long time man remained in the mere counting stage objectifying his series by objects and marks for the most part (number names being but a meager possession); it is only when the savage becomes a thinking human being that number in the mathematical sense can be grasped by him. At this point "mere reckoning ceases and arithmetic begins" (1, p. 73).

To go back into our animal ancestry to search for earlier beginnings is interesting but futile. It is certain that the highest animals and the lowest savages have in common mathematical ability of a certain (or uncertain) sort which enables them to distinguish differences between small groups, and in some cases apparently to determine absolute number. The classical instance of the crow which could tell that one was absent from a group of not more than four men (simultaneous presentation); the nightingale which could tell when it had finished its meal of three worms given one at a time (successive presentation); the parent wasp which supplies its young with a food supply of five to twenty-four victims, no more, no less, according to its species; these are cases in point. Did these animals count or was their apprehension merely quantitative? Whatever the answer it applies equally well to the animal or the savage, since the objective product is the same in both. It would seem that we have here a capacity ultimate in its nature, with its roots lying far back in prehuman evolution, which because of its ultimate nature must, as Conant says, be left in the region of pure speculation.

(b) *Children.*

All will agree, perhaps, that progress beyond mere rudimentary quantitative discrimination depends upon *the ability to count*. There seems to be little difference of opinion concerning this among observers, experimenters, psychologists and philosophers (Lay, Meumann, Phillips, Messenger, Dewey). However intuitive and instantaneous the apprehension of number may seem to the mature mind, psychogenetic analysis shows it to be generated by slow degrees through counting. The child, then, must and does count. Does the child count spontaneously, instinctively, or must he await the development of a power, however low, of abstraction and generalization?

This problem has been quite thoroughly investigated by Phillips (5). He experimented on kindergarten children, consulted primary teachers, and collected data from 616 persons by means of a questionnaire, 235 men, 319 women, 62 neutral; 72 per cent of these teachers, 40 per cent having taught over five years.

Phillips found the following facts which called for interpretation.

Children, before they have learned the number names, are observed to follow a succession of stimuli or to create a succession long before there is any conscious idea of number. They will repeat a series of sounds, as the strokes of the clock, throw down a given number of blocks time and again. A boy under two rolled one after the other ten mud balls down an incline, marking one each time until each contained a little cross. Tallying with the fingers or toes, nodding the head, rhythmical articulations of various sorts, etc., are the characteristic responses in this stage.

Having learned the number names as a series of auditory symbols mechanically associated with one another (this he does very soon through imitation before he learns to read or to write), he uses them indiscriminately and without reference to objecta of any kind. Thirty-three out of thirty-nine children in a kindergarten counted without reference to objects. Primary teachers consulted say that for some time children count in this manner, and, if an attempt is made to apply the counting, the series of names constantly tends to run far ahead of the objects. Most children are found to learn the names independently of the objects. "I placed before little Willie little sticks and told him to touch each stick as he counted; he was just as apt to say six or any other number when he touched three as to say it correctly" (5, p. 262). Children show a passion for counting. They eagerly seek the names, catch them readily and use them delightedly. They will use the few names they know at first, repeating them over and over again, for a series of indefinite length. They recognize three or four objects at first as individuals, calling the fourth one four even when set aside by itself. They change the order of the number names, the particular articulatory response being often a matter of indifference in such cases as elsewhere in their counting. To sum up negatively these results in a word: embryonic counting is not a response to outer stimuli; it is not the result of the observation of sensible things. The positive correlate to this is, counting at first is a motor response to an *inner* series. But what is the nature of this inner series and how is it established?

Phillips (5) says that (a) it is a consciousness of succession that has resulted from a long experience of successions in consciousness. Moreover, the changes (b) are naïvely rhythmical. This rhythmical subjectivity is an ultimate fact, a possession common to all, though not the same in all. Our minds are attuned to rhythmical responses but they do not all play the same tune. Experiments performed upon the rhythmical sense show an indifference point, a length of interval between the beats of a metronome, say, more easily and accurately judged than any other above or below, but it varies with individuals and with the different senses. (c) Series of metronome beats tend to group themselves into rhythmic multiples. This is a matter of common experience in listening to the tick of a clock. Children inquire why some ticks are longer than others. This synchronization, as I understand it, is a category of the subjective ordering of experiences which we cannot help if we would. (d) The experiences themselves begin at birth. Some of these are rhythmical *per se* and may serve at times as symbols for an internal rhythm, as the pulse, respiration, walking, etc. (e) Changes in consciousness are continually taking place produced by the varying impressions from all the senses. The earliest and most rudimentary form of knowledge is a knowledge of a series of changes. "All the experimental work substantiates James's thought that number is primarily strokes of attention." (f) "The tactile sense very early produces an endless series of changes in consciousness which soon become vaguely recognized as distinct both in time and space.

Sound continually plays its part after the first few hours of life; random noises, voices, ticking and striking of the clock all contribute to the formation of the *series-idea*" (5, p. 228). All the rhythms and the recurrences of life tend to establish (1) the fact of succession, (2) the idea of succession. Professor Dewey says that changes in consciousness and consciousness of change (from the latter of which, according to our hypothesis, the idea of number originates) are not the same. Granted; but from what is the latter derived if not from the former?

Changes of consciousness, then, pass over into consciousness of change and the series-idea is thus established. As thus established it is abstract, a potentiality of numbering rather than a numbering process, a category according to which concrete things tend to express themselves. In other words touches, sounds, visions, clock strokes, counters of various sorts are the concrete *symbols* of an inner abstract series seeking expression in these ways in the absence of a more sophisticated symbolism. These symbols at first may be highly particularized, the concreting being limited in the case of any individual, but later give rise to a general idea expressible by any series of successions. The thing to be noted here is that in this stage objects are symbols, the group of objects is used in the place of a name. This symbolization is in advance of the power to name, and enables an early vague concept in children and, as we have seen, in savages to grow without the aid of a mathematical vocabulary.

Here ontogeny and phylogeny are in accord. But the child in his individual development is at an immense advantage over his savage ancestor. "The fact that the child has the number names ready to hand gives an immense momentum to the development of the number series-idea" (5, p. 238). Enough has probably been set down to show that the child's use of names at first is not an application of them to groups of objects, but in this (second) stage the names are merely a motor symbolism for the inner series which he has gained in ways described. Articulatory response to a prompting subjective succession-motif now takes the place of other motor and objective responses, which were found either in his own bodily economy or in sensible presentations with which he created or to which he imputed appropriate responses. Having attained the articulatory response, he does not at once and altogether abandon his more primitive responses; the tendency to use rhythmical movement, e.g., remains. A child forbidden to use his fingers will wiggle toes inside his shoes, etc. ?preschool

This stage of consciousness is essentially a counting period, and any words that can be arranged into a series would do and often does as well as the one which society has furnished him, e.g., eeny, meeny, minee, mo, etc. The counting is spontaneous and entirely free from sensible observation. The abstraction and generalization of number from a group of objects (Dewey) and the application of numbers to objects come later and furnish the essential problem with the beginning school child. school

The development of the number concept in the mathematical sense is mainly a problem that concerns the child of school age. Heretofore his number sense has been an idea of succession with objects and motor activities (including naming) as symbols of a subjective series. Now he must be led to *apply* his number series, not only to successive but to simultaneous presentations, in order that he may describe, if need be, his experiences in accurate numerical terms. There is substantial agreement that he must derive this abstraction, number in the strict sense, through perception of things. The apparatus, materials and devices of the school room and the guidance of the teacher are the aids that secure the systematic and economical furtherance of the growing concept. But it should not be overlooked that the child comes to school, as a rule, with a dawning knowledge, at least, of the meaning and the art of applied number, and this signifies that concepts are present up to a certain point. The observations and experiments on children of pre-school age afford some degree of information on this point.

Theophil Fries[1] (1906) studied the question how the command of the numbers represented by the first digits arises in the first years of life. He began with a child of two, using nine-pins. When the child's back was turned he took away as many as three pins, but the child seemed to notice nothing unusual. (Preyer had said that a ten-months-old child noticed the taking away of *one* pin.) After eight days' experimenting Fries reached only negative results. The difference noticed by Preyer's child must have been quantitative (change in specific magnitude), not numerical. Not until the number was reduced to two or three would Fries's child cry: "others." Further experiment showed that the child really commanded no number higher than two. The child could count, but at first *his counting had no reference to his making out two.* Nine months later the relation between the *numbers* and the *number-words* for one and two had considerably cleared up. But *one* and *two* were also associated with *small* and *large*, showing that *form* and *magnitude* were in a sense their background. This is important, as will appear later (page 50 ff.). Later the child commanded the number three, but did not definitely connect his memorized number-words with it. At the age of three and a half, three was found to be mastered, but the child did not know more than three things when presented. Threeness was found to be mastered not only in special adjacency, but also in temporal sequence or succession. Just after the child had finished its fourth year, experiments on number *four* were made, which was found to be not established—a concept still in the making.

John I. Jegi's daughter (1901) at two years named the numerals in correct order to 12, occasionally to 17, but had little sense of their real value except in the case of 1 and 2. There seemed a

[1] This study and the following one from Jegi are taken substantially from Hall's Educational Problems, Vol. 2, Chap. 18.

vague feeling that the higher numbers applied to greater magnitudes; three was a favorite expression for a small number less than 6 or 7, while 8 and 11 were used in designating more. She would count the 17 stairs and then say there were 11 or 13, rarely giving the right number. At first the numerals were not given in the right order. Little progress was made during the four months following the second birthday; although she showed marked improvement in the ability to compare magnitudes, she made no attempt to state her conclusions numerically.

(c) *Prodigies.*

Prodigies, or lightning calculators, as they are popularly called, have seemed to most people who have given them only casual consideration to be in a class by themselves, inexplicable freaks who perform in a manner inconsistent and unconformable with ordinary mental laws. However, while those who have studied them find them precocious indeed, yet with a precocity that is not freakish, but, on the contrary, entirely natural. I have elected to discuss them under the head of genetic studies, because, while their performances could well be treated from the standpoint of pure psychology or of objective efficiency, considerations from which standpoints must enter in at any rate, yet they are chiefly important for pedagogy, so far as they are important at all in the present state of our knowledge of them, on account of the light they incidentally throw on the development of the number sense. The prodigies are essentially children or primitive men in their numerical operations. It is related that some of them spent their early days as prodigies in arranging and piling up pebbles, thus trying to symbolize the steps in the series-idea which for them was highly abstracted in a manner similar to that of savages and children. This same struggle for expression eventuates in other instances, as we shall see, in number forms, thus furnishing another example of analogy in this particular to childhood and, may we say, also to the childhood of the race. The biographies of most of them reveal that so far as they are mental calculators of the unartificial type, they know nothing of books or the learning of the schools. They develop short cuts that are marvelous in their effects, but these short cuts are unsophisticated, naïve, uninfluenced by education in the ordinary sense. In short, their power seems to be the result of an unusual native capacity to discover concepts of the relations of numbers[1] unknown to most. This capacity together with a tenacious memory for numbers gives them their precocity. Now if we could know in a measure how they came about these concepts or even what they are, new light would undoubtedly be thrown on one aspect of the development of the number concept. Perhaps, as Dr. Hall (6) intimates, much number power inherent in the race has been strangled and become atrophied through ill-advised artificial methods and disuse of natural ones which have passed out like the "lost chord." It is a matter of common experience that the knowledge of a few short cuts, even with ordinary mortals, increases calculating power amazingly. Prodigies provide a largely unexplored field. If there are any that are in captivity or that can be caught, let the pedagogues get at them.

The chief studies on this subject are by Scripture (13) and Mitchell (12). Mitchell's is the more careful and thoroughgoing. He himself is a minor prodigy and through a study of his own case has been able to penetrate the secret of much of the wonderful work of others. "Natural" calculators develop *spontaneously*, at least in the first instance; artificial calculators use external aids from the start. A calculator may begin in the natural way and later use artificial methods as Gauss (see biographical sketch page 28) did in his use of logarithms. George P. Bidder, Jr., whose natural gift is doubtful, was greatly dependent upon artificial aids.

[1] Numbers in the sense of mere counters for calculating.

Our interest is in the born calculators, cases of mathematical precocity, not of deliberately practiced operations.

Let us get into touch with some of the prodigies through the biographical interest. The following sketches are taken in the main from Mitchell. (Where they are not the fact is noted.)

Tom Fuller was born in Africa in 1710, and was brought to Virginia as a slave at the age of 14. He never received any schooling and was wholly illiterate. His principal feats in mental calculation were the reduction of a number of years, months, days, etc. to seconds, finding the sum of a geometrical progression, and multiplying as many as nine figures by nine figures. He was extremely slow, his longer calculations requiring weeks; his method seems to have been exclusively a counting method, more or less abbreviated by practice. There is no evidence that he ever learned what school boys know as the multiplication table; this ignorance, strange as it may seem, would not necessarily militate at all against his expertness as a calculator, nor does it furnish any explanation of his lack of speed, since, as we shall see, one may become even a "lightning" calculator without knowing this table.

Jedidiah Buxton was born in Derbyshire, England, in 1702. His father and grandfather were men of some education, but he himself had none. Moreover, he was very stupid from childhood, not even showing any mathematical ability. Not until the age of 12 did he show interest in calculation. Once his well of genius was tapped, however, he developed into a major prodigy, and became able to handle immense numbers, once (at least) squaring a 39-figure number, and making other surprising calculations. He was very slow (it took him over two months to perform the feat named above) and never got much beyond the counting stage. He could calculate while working and carry on two different calculations at once. The fact that he had practically no interests outside of calculation is an illustration of the extreme isolation in which the calculating ability may develop. The possessors often make no practical application of their facility, but exploit it merely for exhibition and money-making purposes. To become proficient, however, the prodigies must have found an inherent interest in calculation for its own sake, and to find audiences must have found many persons similarly interested. This inherent interest in figures and processes, as such, which has been observed also in children, has its pedagogical application in this: that teachers probably agonize too much over the concretion of number work for children, much of their work in this direction being "love's labor lost." (Compare Dr. G. S. Hall, Ed. Problems, page 365.)

To stimulate interest in this review, it may be well before going further with our brief biographies to give an example of Buxton's method of procedure, to show how differently from ordinary persons (including teachers) the calculator operates. To multiply 456 by 378, he multiplied successively by 5, 20 and 3 to get 300 times 456; then multiplied 456 by 5 and that product by 15, and added the result to 300 x 456; and finally completed the operation by adding 3 x 456. It will be noted that, although some writers who have discussed prodigies (Scripture, Proctor) have thought that an enlarged multiplication table (beyond 12 x 12, say, up to 100 x 100) was needed for the larger calculations, there is no item, with a single exception, in the foregoing calculation requiring a table beyond 12 x 12, or, indeed, any table at all. If he used a table at all, he would need a slightly extended one to multiply by 15. But in all probability he simply *counted* in the series of multiples of the multiplicand, in each of the several operations involved. Mitchell, a minor prodigy himself, who through his knowledge of his own methods and his training in psychology has gained an insight into the methods of others greater than that of any one else who has discussed the subject, has shown in his own case how this may be done. For example, to multiply 48 by 64 he would count by 48's to 384 (= 8 x 48), then by 384's to 1536, then to 3072 (= 8 x 384 = 64 x 48), the required answer.

Zerah Colburn, son of a farmer of little education, was born in Vermont in 1804. He began to calculate when 5 years old. His power developed gradually and gradually deteriorated through lack of practice. His education was fairly good but was subject to interruptions. He had only a moderate liking for mathematics, preferring the languages. His feats were the ordinary arithmetical operations, multiplication by five figures, extracting square and cube roots (of perfect powers) and factoring large numbers. In extracting roots and factoring he used the method of two-figure endings, which is a method more or less common with the prodigies. I quote a description of it from Mitchell. "Given the last two figures of a number, the last two figures of the square are known; but given only the last two figures of a perfect square, the last two figures of the square root are not definitely known, although the possible values are usually only four in number. Similarly, an odd ending has only one possible cube root, but an even ending has either none, or two which differ from each other by 50. Now, suppose a given number is known or suspected to be a perfect square or cube and its root contains only three figures. The first figure can readily be determined by inspection and the last two figures must be one of a limited number of possible roots of the ending of a given number. It is usually easy after a little practice to tell almost at a glance which of the possible roots to choose in a given case. In doubtful cases (multiples of 5, e.g., where the number of possible roots is greater) such expedients as casting out 9's, squaring or cubing one of the suspected answers or some number near it, or using the three-figure instead of the two-figure ending, will help to decide which is the correct root" (12, p. 93).

It is in this class of problems that Mitchell thinks he might compete with the real prodigies, but "skill in solving such problems does not imply special skill or quickness in other branches of mental arithmetic. . . . This is a method of guessing by inspection of the ending of the given number, not a real calculation" (12, p. 94). In short, it is one of the tricks of the trade and is cited here, for one reason, to show the necessity, in striving to gain some understanding of the feats of prodigies, for distinguishing between such a method and genuine calculations. The application of this method to factoring is still simpler and seems to give some promise of usefulness in the school room; it is therefore given in Appendix V.

Henri Mondeux, born in Tours, France, 1826, was the son of a wood cutter. Tended sheep at the age of seven and learned calculation by the use of pebbles. Later, like several of the other prodigies, he became a professional calculator. Unlike many of them, he was able to profit by instruction in mathematics and showed considerable aptitude up to a certain point. His knowledge of mathematics enabled him to formulate many ingenious processes and to deal with algebraic problems as well as arithmetical operations. But he belonged to the natural family of great calculators, his methods in the first instance taking shape independently of books.

Jacques Inaudi, born 1867 in Italy, came of a not talented family and early, like Mondeux, became a tender of sheep. While thus occupied at the age of six he began calculating; at seven he could multiply five figures by five figures mentally. His education was very limited and he did not learn to read and write until he was 20. His customary feats were subtractions with 21-figure numbers; addition of five 6-figure numbers; multiplication of 4-figure numbers by four figures; division of two 4-figure numbers; simple algebraic problems by trial. In Binet's tests on Inaudi, 2-figure multiplications were performed in 2 seconds, 3-figure in 6.4 seconds, 4-figure, the limit of his ordinary stage exhibitions, in

21 seconds, 5-figure multiplications in 40 seconds; showing that in passing from five to six figures he exceeded the limits of his ordinary practice and became confused.

Ugo Zaneboni, a compatriot of Inaudi and born in the same year, had only a fair education. He began calculating at the age of 12, his power being well developed at 14. His feats were based on railway and similar statistics, evolution with the aid of 2-figure endings, also roots of imperfect powers probably by trial and memory. It is possible that he made use of a simple number form. The use of a number form by prodigies has been later referred to (page 47) and the reason for it briefly stated. How the number form can aid the calculator will appear to some extent in our notice of Mlle. Uranie Diamandi, whose case has been set forth at some length by Dr. Mlle. I. Ioteyko in *La Revue Psychologique*, III, 1910. It will be possible also to picture the number form of George P. Bidder, Jr., since it has been preserved in the collection of Galton.

Johann Martin Zacharias Dase, born in 1824, Hamburg, was in a class by himself as a calculator; there seems to have been scarcely any limit to his powers. He attended school at the age of two and one-half years, took to the stage as a professional at fifteen, but while he was probably precocious (i. e., began at an early age) in calculation, he was slightly stupid in everything else, even including mathematics. He had a wonderful figure memory and more than usual power of apprehending the number of objects in a group. He upon occasion multiplied mentally 100 figures by 100 figures in less than nine hours, extracted the square root of a 100-figure number in less than an hour; he required 54 seconds for 8-figure multiplications, 40 minutes (41 figures per minute) for 40-figure multiplications, and in his 100-figure multiplications maintained a speed of 20 figures per minute. Dase never displayed any confusion in his work, and it seems that only physical fatigue and not the size of the numbers could limit the extent of his calculations. What tricks (devices for saving calculation) he introduced into his work does not appear. Doubtless he made good use in this connection of his remarkable figure memory. He may have, as Mitchell says, memorized the squares and cubes of all numbers up to 100. This would, with the aid of the method of endings, enable him to obtain "instantly" three- or four-figure roots (of perfect powers) and to approximate the roots of imperfect powers, and so relieve some of the tedium of computation, but, of course, is inadequate by itself to explain such monumental feats as his. They must have been, in the main, genuinely labored calculations.

We have proceeded far enough, perhaps, to appreciate Mitchell's statement of the three grades of mathematical ability in the calculators.

The lowest grade never get beyond the stage of pure counting, though practice abbreviates this. At this stage the point of view is not even arithmetical; the calculator thinks not of operations, but of properties of numbers and of series, and the short cuts he uses are of a relatively simple sort, showing no mathematical insight. (Fuller, Buxton, Inaudi, Zaneboni.)

The second grade have a well developed knowledge of arithmetic and an arithmetical point of view. It is operations of calculation, not mere properties, in which they are interested, and the various short cuts used are, we may suppose, suggested by practice rather than by mathematical keenness. In this class may be placed Dase, also Colburn.

The highest grade have real mathematical ability, power to take the algebraic point of view, to generalize and hence to discover all sorts of ingenious short cuts and symmetries. (Mondeux, Bidder, Sr., Diamandi, Safford, Gauss.)

Mitchell does not expect this classification to be taken too seriously; there are no hard and fast lines, but it serves to bring out the point that mental calculation and mathematical ability are essentially independent.

George Parker Bidder was born in Devonshire, England, 1806, son of a mason. One brother had a remarkable memory for Bible texts, another was a good mathematician; a nephew had great mechanical talent; his son was an excellent mathematician and calculator; two granddaughters were above average ability in mental arithmetic. The hereditary strain of talent in the family is notable, and, while significant possibly in this particular case, is not significant in general for prodigies, since the explanation by heredity lacks a basis in too many cases. Bidder learned to count at the age of six, using pebbles, shot, etc. at first, and soon became very proficient in calculation, retaining the power through life and using it in his profession. He became a man of wide interests, well educated, able both mathematically and generally. Besides the ordinary arithmetical operations, he did multiplications up to 12 figures by 12 figures; compound interest examples; roots and factors by the aid of 2-figure endings, etc. His methods were often original, highly ingenious, and rapid. A fact bearing on method appears in connection with Bidder. He fancied at first that he possessed a table up to 100 x 100, but found that he was in error. If he had 89 to multiply by 73, he would say instantly 6497, but, in what appeared to be merely an instant of time, according to his own account, he multiplied 80 by 70, 80 by 3, 9 by 70 and 9 by 3 and then added up the products. Note the order of multiplication. It begins at the left instead of the right. Many of the prodigies use this order. Indeed, as Mitchell says, there is no reason except custom why we should not all do this; it is more *natural* and just as convenient. In *mental* calculation it favors combining at each separate stage and so the partial products may be forgotten and anxiety relieved.

George P. Bidder, Jr., son of George Parker Bidder, is classed by Mitchell as an "artificial" calculator who was stimulated to his comparatively few feats, performed slowly and with occasional errors, by the example of his father; artificial in distinction from a born or natural calculator who usually begins early before he is "spoiled" by contact with schools and inclines to use a counting rather than a conventional method. The fact, however, that he had a number form affords a presumption, at least, that some of his methods were or had been primitive. It is clear that his main interests lay outside of calculating. He was educated for the bar and practiced law. That he was possessed of considerable ability is shown by his having been seventh wrangler at Cambridge in 1858, at the age of 21.

Among other things he was able to multiply up to 15-figure numbers by 15 figures, by the use of cross multiplication, so called. As at least one other calculator (P. Diamandi) used this method, which has a curious interest of its own, a description of it more detailed than that of Mitchell follows:

Rule—To multiply a 5-figure number by a 3-figure number by the method of cross multiplication:

Multiply each figure of the multiplicand in turn beginning at the right by each figure of the multiplier in the following order: By the units figure twice in succession; by the tens figure once; by the units figure; by the tens figure; by the hundreds figure; by the units figure; by the tens figure; by the hundreds figure; by the units figure; by the tens figure; by the hundreds figure; by the tens figure (next in order since the multiplication by the units figure has been completed); by the hundreds figure; by the hundreds figure again (since the multiplication by both units figure and tens figure has been completed). As each simple product is obtained write it or visualize it in its place; add each vertical column of single digits as soon as it is completed and write or visualize the sum; lastly perform the reductions in the total product, or do this as soon as the sums of the vertical columns are severally obtained; write or visualize the figures of the total product in their places.

THE CONVENTIONAL FORM.

$$
\begin{array}{r}
46273 \\
\underline{729} \\
416457 \\
92546 \\
\underline{323911} \\
33733017
\end{array}
$$

The form by cross multiplication partially worked out (to 7 × 7)

4	6		2		7		3
			7		2		9
					2		7
						5	
			6		3		
	(1)			(1)4			
	1	6	8		6		
5	4						
	1						
	4	5	4				
	2		1				
4	9	(1)1					
	12		9		(1)1		7
	(1)		(1)				
	(1)3		(1)0				

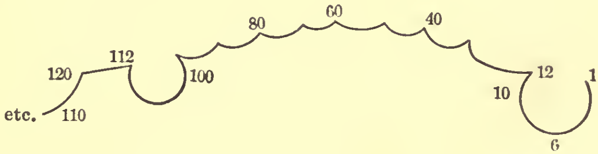
The order of multiplications according to the rule for cross multiplication.

9 × 3	9 × 6	9 × 4	2 × 4
9 × 7	2 × 2	2 × 6	7 × 6
2 × 3	7 × 7	7 × 2	7 × 4
9 × 2			
2 × 7			
7 × 3			

It will be observed that the figures of the partial products are found in vertical lines; first 7; then 5, 6; then 4, 4, 1; then 6, 5, 1, etc. Each column is added before the numbers that compose the next column are found. The advantage is that the various figures of the partial products can be forgotten almost as fast as obtained, since that figure of the total product which depends on a given column of the partial product is found and recorded as soon as the column is known and the numbers in that column therefore play no further part in the calculation.

While this method has its advantages, it is obviously a modification of written multiplication, a derivative from the learning of the schools and not a naïve procedure such as that of his father (see page 21, and Appendix VI). It could only be used by strong visualizers such as Bidder, Jr., and P. Diamandi were; the whole process was doubtless pictured by them and seen as plainly as if written. It is known that Diamandi learned mental arithmetic after written arithmetic, and probably Bidder did the same.

As has been stated, Bidder had a number form which I reproduce here from Galton (Plate 1, fig. 20, opp. page 380, Inquiries into Human Faculty).



That he did not use it, however, at least in his more elaborate calculations, appears from his statement to Galton that when he was multiplying together two large numbers his mind was engrossed in the operation, and the idea of locality in the series sinks out of prominence.

Pericles Diamandi, born 1868 in Greece, belonged to a family of calculators, having a brother and sister who shared his gift. He excelled in mathematics at school and had a more than usual talent for languages. His talent for calculating did not appear until he entered business at the age of 16. He had good memory of the visual type, was possessed of a number form and colored audition for some names. This color imagery also appears in the case of his sister Uranie, but was of the visual type instead of auditory. Diamandi could multiply mentally up to five figures by five figures, but not rapidly, using the method of cross multiplication.

Uranie Diamandi [1] possesses (at the age of 22) the same aptitudes for mental calculation and the same visual type of memory

[1] This account is taken from Les Calculateurs prodigies (I. Ioteyko), La Revue Psychologique, III, 1910, pp. 320–328, and Visualization colorée et sens chromatique chez Mlle. Uranie Diamandi (V. Kipiani), same volume, pp. 329–335.

as her brother, with besides a very remarkable power of color visualization. It is from their mother that the two calculators think they get their excellent memory. She has received a classical education and is facile in the languages and history. In mathematics her knowledge does not go beyond elementary arithmetic, but she uses many unconventional processes. At school, at the age of seven, she first noticed that she performed the little calculations proposed more quickly than her fellow pupils. But it was at the age of thirteen that the success of her brother stimulated her to cultivate her peculiar talents. At fifteen, she gave her first exhibition. The operations performed by her in public were additions, subtractions, multiplications, divisions; squaring a number of 4 to 6 figures; extracting the square or cube root of a number of 8 to 10 figures; extracting fourth to eighth roots of numbers of 8 to 12 figures; writing on the board without pause the square of all the numbers from 2 to 100; calculating the number of minutes and seconds elapsed since a given date, e. g., in the year 1453, keeping account of the leap years; indicating the day of the week corresponding to any given date, the day being known to the questioner. Her general memory is excellent, but her special memory for figures is phenomenal. A feat which illustrates this is her grasping and reproducing in two or three minutes as many as 25 figures arranged in a square. The figures seem literally photographed upon her brain and are projected upon an imaginary table lying before her eyes. The image, instead of being vague and fugitive as it would be in the case of ordinary persons, remains with a remarkable clearness for a certain time in a conscious condition in all of its original complexity. Once having disappeared, it is susceptible to recall after an astonishingly long time.

In the matter of retentivity Mlle. Diamandi differs from her brother. She is prodigal of her power, as, perhaps, at her age she can afford to be, and violates the law of economical mental functioning. Her brother says that the figures remain graven in his memory for only two days, but that it would be easy to prolong the time by making a certain effort. His habit, however, in order to avoid fatigue in his frequent séances is not to do this. He has acquired the faculty of forgetting, to the end that he may not uselessly encumber his memory. This is good psychology. The old forms of memory ought to disappear entirely as do the intellectual elements which have served as a basis for them. Forgetfulness, relative amnesia, is to be considered as a condition of psychic renewal. As she grows older Mlle. Diamandi should certainly take account of this law, especially in view of the fact that her retentivity will diminish.

Mlle. Diamandi's number form is of interest because she has been able to give a good description of it and makes constant use of it in her work. The naïve demands which she makes upon it mark her as having some of the characteristics, at least, of the born calculator, in distinction from her brother, who is classed by Mitchell as wholly of the artificial type.

She began to see it at the age of fifteen during her first public séance. The form by reason of its constancy and relative fixity appears with the characters of a stereotyped image. It is a square inclosed by figures with a free space in the interior which is the field of operations. Every number proposed appears at once at its place in the chain of numbers. Each operation carries in the central part the mental inscription of the several principal numbers. If it is necessary to make a clear place in view of a new operation, the image of the figures is relegated to the higher portion of the field of operations, where it appears to hide itself, but always within the form. Later, after two or three days, the numbers appear to her elsewhere; they are seen in deposit, as it were, in the upper part of her subjective table on the *outside* of the form. The figures to be retained, on the contrary, do not cross the chain, but they are not continuously seen, appearing only at the moment of need. They are for the time being vague, but become definite by an effort of memory. The part outside the chain appears to be a "magazine" of numbers to be retained longer than the others. The part inside next to the chain will be, on the contrary, affected by the numbers which are not in play at the moment, but which are known as just beyond to be recalled without delay. Finally the numbers actually in play appear in the central part of the field of operations.

The part played by color imagery in Mlle. Diamandi's case is probably unique in the history of prodigies. If she thinks of a number, a letter of the alphabet or the name of a day, the images which she has in her mind appear colored with perfect clearness. In the number form the several figures of each composite number have their own peculiar colors, always the same, and the number as a whole has its color. The chain of numbers is as a flowery garland projected upon a gray background. A peculiarity of this colored number form is that closing the eyes is fatal to it, nothing remaining except a vague form without figures or colors. With the eyes open, the mental coloration of the figures enters as a cause into their recall and consequently into the facility of mental calculation. When a figure is forgotten its recall is facilitated if its color forms a contrast with the neighboring figure (already in mind). For example, 104 (black, white, maroon) is easy to apprehend and retain because 0, which is white, is found placed between two dark colors; likewise 129 (black, light yellow, brown) is more easily grasped and retained by reason of contrast. It is of psychological interest to know that, while the numbers thought of bring with them the colors, the contrary does not happen. The colored visualization serves to enrich the associations and thus to furnish additional clues in the effort of number recall.

Truman Henry Safford was born in Vermont in 1836. His father was interested in mathematics and his mother had been a school teacher. He exhibited an all-round precocity; began to calculate between the ages of three and five, studied higher mathematics at eight, computed and published almanacs at nine and ten. His education was extended and he displayed an interest in all studies, but especially in mathematics and astronomy. He could perform easily and rapidly all of the feats of calculation already described, including the extraction of roots and finding of factors (by the aid of two-figure endings) and the multiplication of one large number by another (in one case, each factor was composed of 18 figures). His memory, of the auditory type, was encyclopedic in scope, not confined to figures. An idiosyncrasy which must have lent interest to his performances was a marked habit of nervous contortion and restless movement during calculations, sometimes amounting to almost a riot of convulsive manifestations, an exhibition doubtless also of psychological interest if one could understand its full significance.

André Marie Ampère, born in 1775 in Lyon, France, was also precocious, beginning calculation with the aid of pebbles, etc. at the age of three or four. Became an all-round scholar and especially distinguished as a physicist. Specific information concerning his work as a calculator is largely lacking.

Carl Friedrich Gauss, born in Germany in 1777, began calculating in his third year and probably retained his power through life. He became a mathematician of the highest rank and used his knowledge of mathematics (e.g., logarithms) as an aid in calculating. He was, however, as we gather from his early beginning, a born calculator and one of the few cases in which calculating power and mathematical ability are found together. Here again the data with respect to work as a prodigy are meager.

Frank D. Mitchell, upon whom we have drawn most largely in our review of prodigies, himself had considerable aptitude for mental calculation and may be regarded as a minor prodigy of the natural type. His interest in calculation began at the age of three or four years. He learned to count to 10, then to 100, then beyond; also by 2's, 3's, etc. In the series 2 x 2, 2 x 2 x 2, 3 x 3, 3 x 3 x 3, etc., in short, the powers of the number by which he was counting were natural resting places and awakened his interest, so that before long he began to count in the power series of different numbers (2, 4, 8, 16, 32, etc.; 3, 9, 27, 81, etc.) for considerable distances. At first he simply emphasized the powers as they occurred in the *complete* series of multiples, but gradually he learned to omit the intermediate multiples and simply count in the power series proper: 2, 4, 8, 16, etc.; 3, 9, 27, 81, etc. But almost always when the number exceeded 100, he emphasized the last two figures and gradually got into the habit of ignoring all the others. Thus instead of saying 3, 9, 27, 81, 243, 729, 2187, 6561, etc., he usually counted 3, 9, 27, 81, 43, 29, 87, 61; and in this simplified form counted along for considerable distances. It may be said here that Mitchell's mental calculations take the form almost exclusively of tracing the last two figures through the different operations, ignoring all the other figures, the problems which he solves by preference being of the general form of finding the last two figures of any power (or integer root) of any number. Thus the counting cited above is a typical computation of the last two figures of the 8th power of 3 (6561). At first the process was 3, (6), 9, (18), 27, (54), 81, (62), 43, (86), 29, (58), 87, (74), 61. (To make this procedure clear, I have worked out more details than Mitchell gives; for which, see Appendix III.) He would pass over the intervening multiples (in parentheses above) lightly, in time learning to omit them altogether, and before long the process came to be simply 3, 9, 81, 61; that is, simply squaring each number to get the next, the intermediate counting taking place so rapidly and automatically as hardly to appear in consciousness at all except in brief flashes. And even these flashes were sometimes almost absent, so that only the 3 and the 61 stood out, the rest remaining a mere blur.

Multiplication naturally grew out of this counting process, but it was really counting rather than multiplication proper, since he did not learn the multiplication table until some time later when he went to school. Thus to find 9 x 7 at this time he would count 9, 18, etc. to 63, and even at the time of writing his study of prodigies, except within the limits of the multiplication table as he learned it to 12 x 12, his mental multiplications were abbreviated countings of this sort (skipping most of the intermediate links) rather than true multiplication. There is reason to believe, says Mitchell, that at least two of the major prodigies never got beyond this counting process; none of them, with the possible exception of Gauss and Dase, used an enlarged table (memorized so as to be automatic in its action like the school table), and even in these two cases there is no direct evidence, only the bare possibility. The feats of such men as Buxton, Colburn, Mondeux, Dase, and Bidder (who belong to the *natural* family of great calculators), whose methods in the first instance took shape independently of books, can be explained without presupposing a table reaching beyond 10 x 10 or 12 x 12. The child who becomes a calculator begins to multiply soon after he learns to count; his habits and methods are definitely formed before a table beyond 10 x 10 is needed.

Mitchell has a strong preference for working with even numbers. By a special method he practically always changes odd numbers into even numbers for calculating purposes where only the last two figures are required. The even number thus obtained is readily converted into the desired odd number. (For method of conversion, see Appendix IV.)

In the course of his calculations or countings a number of properties gradually attracted his attention, such as that every power of a number ending with 0 or 5 ends with 0 or 5, that the 4th power of any other number ends with 1 or 6 according as it is odd or even, that the 5th power ends with the same figure as the first, the 6th power with the same figure as the 2nd, etc., and that if 76 or any number ending with 76 is multiplied by a multiple of 4, the last two figures of the product are the same as those of the multiplier (e. g., 12 x 76 = 912).

When the calculation takes account of all the figures, Mitchell's powers of mental arithmetic are little above the average. The multiplication of two-figure numbers takes him longer mentally than on paper, and with three-figure numbers it is such an effort to remember the partial products that usually each one must be repeated aloud two or three times, and even then he is apt to forget the first partial product by the time he has found the third. With small two-figure numbers he can on paper, using one figure of the multiplicand at a time, multiply in a single operation, especially where the number is even, e. g., 24 or 36. With 19 or 23, also, it would probably be easier to multiply in a single operation than in two operations in the ordinary way; but in such a case after the products exceed 100 the multiplication would often tend to resolve itself into counting, rapid and automatic, but counting nevertheless. Thus, up to 5 x 23 = 115, he would probably count by 23 directly or depend on his memory; but after that, to pass to 6 x 23 = 138, he would first count in the 3, then the 20, thus reaching 138 from 115 via 118 and 128.

To square a number, as 162, which contains no prime factors except 2 or 3, he multiplies successively by 3, 3, 2, 3, 3 (486, 1458, 2916, 8748, 26244); in other cases of squaring comparatively small numbers he often uses the algebraic formula $(a + b)^2$.

Psychological Considerations. In the light of our biographical review, and the data regarding calculating procedure therein included, it is possible to get some clue to the psychology of the prodigy.

1. One fact stands out, I think, undeniably. The prodigy is a genius. Now the phenomenon, called genius, is ultimate in its nature and is its own explanation; one cannot explain a *summum genus* further than to make it appear that it is a

summum genus, which places it outside the sphere of explanation. One can *describe* the conditions of its manifestations; one can answer the question, "how is the genius?" but not, "why is the genius?"

At least four signs appear: (a) The tremendous interest of the prodigies in their work; (b) their unusual insight into the properties and relations of numbers, which leads to (c) their amazing performances; (d) their precocity.

(a) Their work is the result of interest, but the interest is the result of genius. They proceed with calculation because, as the saying is, they have a genius for it. True, in a sense, the interest is a result of the work because of the joy the endowed man takes in his work, illuminated as his pathway is by facets of light unperceived by the general; the two are reciprocal in their causative action, but interest is primary as between the two, and genius is elemental to both.

(b) Their insight into properties and relations comes as flashes, whereas, in the case of the ordinary man it come but slowly or not at all. In a sense, too, it is acquired; all genius needs appropriate stimuli for its development and with practice its field of operations becomes enlarged. But stimuli and practice might be present to you and to me with improving but certainly not astounding results. It is not meant that the knowledge of these properties is the result of mathematical insight. So far as it may be, it lacks the naïveté and spontaneity that characterize the devices of our born prodigies. Nor is it meant that the properties and relations are necessarily such as would be recognized as valuable by the mathematician. Yet the prodigies on occasion made discoveries of facts theretofore undiscovered or believed to be impossible by the mathematicians, and it may be that some of their devices await, through proper study and formulation, assimilation into the theory of number.

(c) Their amazing performances are explained only in part by their genius for numerical properties and relations; memory, as we have seen, plays a large part. Not such a memory as you and I have, but a hypertrophied memory, a memory sublimated to such an extent as to amount to a supernormal activity in itself and therefore another phase of the genius of the operator. It must be remembered, however, that no faculty of figure memory, however prodigious, could, as such, produce a calculator. It is of importance in this connection only as it stands in the service of calculation genius, as such. Besides memory, we must add (to explain rapidity, when the calculation is a genuine one) subconscious functioning. All of the steps do not appear in the focus of consciousness; some are slurred over; as in the case of Mitchell when computing the 8th power of 3. Results often come before the full consciousness of the numbers involved. Introspection of our ordinary process in the addition of a column of figures will serve to enable us to understand this in a measure. If we are at all expert, we simply glance up (or down) the column and name the several sums (or motorize them), scarcely being conscious of the figures added. While this power of subconscious functioning doubtless may be improved by practice in the ordinary individual, it is developed far beyond the ordinary in the arithmetical prodigy.

(d) Their precocity is a sign of genius. Precocity is not always a sign of genius, although genius is often accompanied by precocity. But if we can determine on other grounds that the prodigies are geniuses, their precocity can be taken as confirmatory evidence.

With scarcely an exception, the prodigies were precocious, and the universality of this trait in them, and the constancy of its eventuation in actual achievement in their cases, make us ready to accept their performances as instances of what Carpenter (Mental Physiology, page 231) in discussing prodigies describes as "numerical intuition or congenital aptitude for recognizing the relations of numbers."

Mitchell says, "Precocity in calculation is natural and normal; popular amazement over it is groundless; there is no need even to regard it as remarkable. Owing to the origin of mental calculation in ordinary counting and the complete independence and self-sufficiency of mental arithmetic, mere mathematical precocity falls into a different class from musical precocity" (12, p. 130). To this we say that isolation of mental arithmetic and the untrammeled conditions under which it may go on may account for its free development, but cannot account for its initiation. It may be admitted, too, that the precocity is natural if by that is meant that it is not supernatural. But that it is normal we cannot agree. Children do not do normally what the prodigies did as children; so that it is no wonder that the neighbors should be amazed at them. He says further, "Given a knowledge of how to count and later a few definitions, any child of average ability can go on *once his interest is aroused* (italics mine) and construct unaided practically the whole science of arithmetic." This we grant, "once his interest is aroused." But we do not believe it possible to arouse his interest (in the degree here intimated) unless he has a "congenital aptitude" far above the average. That prodigies do have an aptitude which the average child has not is shown by their very precocity. The precocity is the cause of his interest, not the effect of it.

2. It seems clear that the prodigy is of more than ordinarily sharply defined mental type. An interesting comparison[1] between P. Diamandi and J. Inaudi bearing on this point is given by Dr. Ioteyko in her article (14) previously referred to, based upon certain observations and studies by Professor Manouvrier, Binet, and others. Diamandi is found to be of a pure visual-motor type, Inaudi of pure auditive-motor. Their aptitudes are so one-sided that the two calculators are, as it were, the living embodiments of two types of memorization. To arrive at his results Inaudi must receive the problems to be solved in an auditive way. The assistants speak the figures. It is estimated that 300 is the number of figures which he engraves on his memory in this way at a single séance. The sight of the figures is of no service.

The opposite is the fact in the case of Diamandi.

When problems are given to him through audition he appears embarrassed, hesitates, commits errors and asks that the figures be repeated to him many times. But when the figures are presented to him in writing, he memorizes large numbers rapidly and accurately. In his case figure memory is but a phase of a general power; all the visual sphere appears well developed; colors, letters and written words are easily remembered though his retentivity of these is weaker than of figures. The auditive memory of Inaudi, on the other hand, except with reference to figures, is much inferior to the normal.

Mitchell's memory type is auditive-motor. He learned to count orally and his calculations began at once without further aid. He cannot remember ever counting on his fingers or using pebbles or the like, and even when he learned to make written figures later on they never became associated with his mental calculations which remained strictly auditory (or auditive-motor) throughout. Ordinarily the motor element is almost

[1] Les Calculateurs prodigies, in La Revue Psychologique, III, p. 320.

entirely absent; when the calculations remain in familiar fields they are accompanied by no perceptible innervation of the muscles of speech. When he attempts unpracticed feats, however, the tendency to motorization is marked. Much the same thing was true of Safford. Compare this with Browne's subjects, who in unpracticed stages and in times of confusion or uncertainty tended to reinstate the verbalism of the tables (p. 73). It would seem, at first sight, that the motor-minded person is at a distinct advantage because when his imagination or thought is at a low ebb he can "crank up" by means of his motor apparatus; however, this supposed advantage disappears when it is remembered that visualization itself, when sharply analyzed, is found to be largely made up of motor elements.

Mitchell's type is not so sharply defined as that of Diamandi, Inaudi and others, but Mitchell is not a great calculator. His case approaches more nearly to that of the ordinary person, in that while one type of imagery (the auditory in his case) slightly predominates, and, so far as calculation is concerned, is the prevailing type, he also makes use, with no special difficulty, of visual images in geometry and similar fields, and is, generally speaking, of the mixed type.

Mitchell has made a careful study of the psychological types of the prodigies, gathering up all the available evidence, and finds that "many of the calculators heretofore supposed to be of the visual type turn out on closer examination to belong to the auditory or auditory-motor type, at least in calculation" (12, p. 132). Only two of the major prodigies (P. Diamandi and Bidder, Jr.) are certainly of the visual type, and both of these, as we have seen, belong to the class of "artificial" calculators. It is to be expected that the natural calculators should be of the auditory type. We have seen that our calculators usually begin as soon as they learn to count, in most cases before they can read and write. "Since counting is essentially a verbal process, the calculator, who begins from counting before he learns to read and write, will usually belong to the auditory type, and will make relatively little use of visual images in his actual calculations" (12, p. 132).

3. It would seem to go without saying that a more than usual power of attention is demanded by the feats of the calculators. "The attitude taken by calculators," says Dr. Ioteyko, "during mental work displays all the characteristics and signs of attention; eyes half closed, head bent, the figure expressing withdrawal and detachment from everything in the exterior world. Distractions coming from outside incommode them but little" (14, p. 325). The ability of some of them to carry on a running conversation or to go about other work or to carry on two calculations at once presents an anomaly of attention which must be explained by the fact that much of the work has been seduced to an unusual degree of automatism, unconscious cerebration or subconscious functioning, call it by what name we may. Unusual concentration, however, would still be needed to maintain a grip on the whole situation. Chronometric records of reaction-times show in both Inaudi and Diamandi great rapidity (half the ordinary in the case of Inaudi). While these records may be taken as a measure of attention (the mind wanderer reacts slowly), and are so cited by Dr. Ioteyko, they may be also regarded as measures of the sensitivity and impressibility of the nervous system, and, therefore, of retentivity.

4. The memory factor has already been referred to under the discussion of the genius of prodigies; also its relation to the calculation psychosis. It is mentioned again in order to note that so far as it is a matter of retentivity it is explicable by the experimentally determined impressibility just referred to. Again, the wonder of the feats of memory somewhat dimin-

ishes when it is known in some cases, and surmised in others, that what appears to be memory differs scarcely at all from perception since it is *the reproduction of a still persistent image.* Also, it is necessary to distinguish between genuine memory feats and quick, automatic calculations of which the results simulate memory. Bidder, Sr., it will be remembered, was for a time under the illusion that he possessed a multiplication table up to 100 x 100. A calculator might, for example, be able with practice to multiply 48 x 64 so readily and rapidly by more or less automatic processes that he would get the answer (3072) as quickly as if he had relied on a direct act of memory; and if the process of calculation happened to be mainly or wholly automatic, he might even be ignorant of its existence and suppose he had actually found the result by direct and unaided memory; disillusion would come, however, when under stress of fatigue or lack of practice his calculations become more conscious and brief flashes of the intermediate processes pass through his mind. The true nature of the psychosis is then revealed not as memory but as calculation, for direct memory would not discover such intermediate links.

5. Does the psychology of the prodigy present any pathological aspects? There is room for suspicion that the superiority displayed in certain processes is due to a disequilibration among the faculties, one center predominating to the detriment of others. This suspicion becomes certainty in two of the cases which we have reviewed (Buxton, Dase). The evidence is the lack of other interests, and even of ordinary general intelligence. Buxton's case was the more marked of the two in this respect, and it is hardly too much to say that he was a victim of arrested development, in other words, a mental defective. He also showed the symptoms of arithmomania, since in listening to a sermon his whole occupation was to count the number of words, etc. One should not be surprised

that the mental defective is capable of becoming a calculator of degree. He may be incapable of developing the number concept, abstract notion of number, but it would seem that computation can go on very well without such concept. Dr. H. H. Goddard[1] gives an account of an experiment conducted in the Vineland School, which tends to confirm this view. "Twelve of our most trainable boys were taken at the most trainable age and drilled on number. They were trained until they could cover a blackboard with figures and, drawing a line underneath, add up the total mass. Day after day and week after week they did this and gloried in their accomplishment. I need not tell you that they did not understand what they were doing." Some time later after this drill was stopped they could not tell the sum of 3 and 2. He also cites in the same article a number of cases of children who could perform the fundamental operations very well, but who could grasp and reproduce simple dictation involving number notions with only slight success. But whatever may be true of the mental hygiene of Buxton and Dase, there is no evidence of lack of equilibrium or development in the other cases. These with the possible exception of Fuller all show average or superior intelligence, though in only two or three cases is the mathematical ability great. But we have already seen in our biographical review that mathematical and calculating ability are not correlated. Goddard's experiment throws some light on the reason why. Number, in the mathematical sense, is the abstract notion of the relation between concrete magnitudes. (See our discussion of McLellan and Dewey's Psychology of Number, p. 91 ff.) The rise of the number concept is the child's initiation into mathematics. But, by dint of practice, calculation can go on moderately in children and

[1] See Supplement No. 1, The Training School No. 46, 1907, pp. 20–24. A Side Light on the Development of the Number Concept.

prodigiously in persons specially gifted without blossoming into the apprehension of true number relations. Such apprehension is not needed, but rather of properties and relations of figures, as such. Nor is whatever of mathematical theory may underlie such properties necessary.

Pedagogical Applications. We can enumerate some of the possible pedagogical applications only briefly.

1. The importance of having recourse in teaching children to the processes of memory peculiar to the individual may be inferred from the cases of Inaudi and Diamandi. The superiority of Diamandi would pass unperceived if one should propose to him problems in the auditive way; the opposite would be true for Inaudi.

2. Calculation *per se* is of importance. It can be cultivated. It would probably pay, therefore, to study the methods of prodigies to discover feasible short cuts now not much noted. An example is possibly the method of two-figure endings. Mitchell says, "They (two-figure endings) are, from a mathematical standpoint, trivial and of limited interest. To answer questions in evolution and factoring, the mathematician would turn to his tables of factors and roots or to a logarithm table; he would regard the properties of the mathematical prodigy's two-figure endings as unimportant special cases of more general propositions in the theory of numbers. They are, then, of merely curious interest in mental calculation. It is conceivable, of course, that if a new and comprehensive theory of their properties were worked out, it might find a subordinate place in the theory of numbers" (12, p. 113). It may be added that it might find a subordinate but somewhat valuable place, also, as a basis for arithmetical practice. Hail to the genius who will work out for us the arithmetic of two-figure endings.

3. Out of calculation *may* blossom the concept and the mathematical sense generally. (See our discussion of McLellan and Dewey's Psychology of Number, p. 91 ff.) Whether this will happen will depend, in part, upon the aims of the teacher and her skill in securing them. (Of course, it may happen in spite of her.) It is certain that nothing of the concept will emerge, no matter what the practice and effort, until the time is ripe. As a child will not walk until his neuromuscular apparatus for walking has reached a certain stage of growth and adjustment and the instinct for walking arises, so he will not resolve abstract number notions from concrete things until a proper degree of maturity arrives.

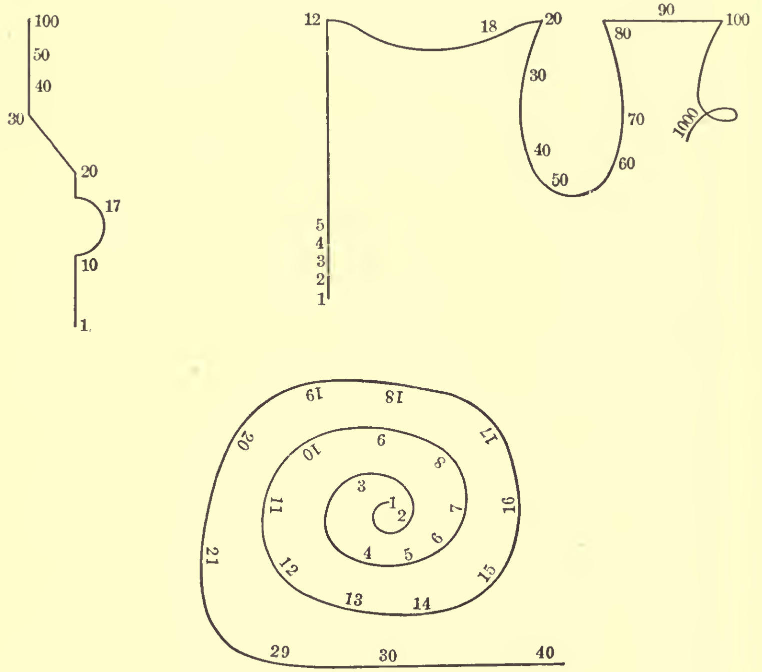
4. A lesson that the modern pedagogue hardly needs to be taught may be gathered from the fact that Fuller, Ampere, Bidder, Mondeux, Buxton, Gauss, Whately, Colburn, and Safford learned numbers and their values before figures. Says Bidder (quoted by Scripture[1]), "The reason for my obtaining the peculiar power of dealing with numbers may be attributed to the fact that I understood the value of numbers before I knew the symbolical figures. In consequence numbers have always had a significance and a meaning to me very different to that which figures convey to children in general."

(d) *Number Forms.*

The treatment of the genesis of the number sense would not approach completeness without some consideration of the curious phenomenon which often accompanies its beginnings, viz., the occurrence in the consciousness of many children and the persistence in that of some adults of a kind of imagery that seems to help out the early strivings of the more or less abstract number series for some sort of symbolic expression. This imagery takes

[1] Arithmetical Prodigies—American Journal of Psychology IV, 1891, p. 58.

the character of so-called number forms, and while the evidence of the existence of these forms is taken largely from the testimony of more or less mature persons, yet these usually agree that they originated early in life, many saying that they have always possessed them and expressing surprise at finding that everybody does not possess them. I present below drawings of three of the simpler forms, after Phillips (7).



Number Forms (after Phillips)

Several pages of these drawings are to be found in Galton's discussion (11) and in Phillips's article (7). They are arranged in zigzags or long curves or lines going in different directions, the more important numbers (to the possessor) being situated at the turns. Some of them are very complicated, requiring tridimensional space to represent them. The numbers may appear either written or printed, and the general mental movement in connection with their appearance is that whenever a number is thought of it appears in the same place on a visual diagram which is invariably called up, viewed by the mental eye and often definitely located. These diagrams are often very much larger than the drawings. Galton says that the most common way is to see only two or three figures of the diagram at once, but Phillips says that this depends upon whether the mind is performing mental calculations or the form is viewed as a whole.

What proportion of people possess these forms depends upon what extension is given to the term "number." For it is not only in the field of number strictly speaking that diagrams occur. Persons who have no number form have schemes for the days of the week, days of the month, months of the year, letters of the alphabet, etc. In some investigations these, probably altogether different in their origin and explanation, have not been sharply separated from true number forms. Galton found 1 in 30 males or 15 females. Patrick (10) found 1 in 6 adults, a larger proportion among children, slightly more common among women than among men. Phillips found that 15% males, 17% females have *some* form; 7% males, 8% females have *number* forms.

Galton (1883), Patrick (1893), Flournoy (1893), and later Calkins and Phillips have studied many number forms and published their results. As Phillips's investigation is the latest and takes into account previous results and interpretations, I follow him here in the main.

Phillips (7) formulated a number of questions which he submitted to 332 normal school students, 974 school children, 7th, 8th, and 9th grades, and 343 miscellaneous adults. The children were asked to draw whatever form or forms they had. In all the rooms, except five, an effort was made to keep them from obtaining an imaginary form for the occasion. Each pupil giving a form was questioned privately; and in the five rooms opportunity was given to fake a form by presenting drawings and giving explanations; but the forms collected in these rooms show no signs of fraud.

The drawings were examined as to their general direction, first turns and endings, and the results tabulated including the drawings collected by Galton, Flournoy and Patrick.

TABLE

Turn			End					
Number of Forms 263	To Left 47	To Right 167	At 10 91	At 12 64	At 20 21	Below 100 109	At 100 94	Above 100 55

The table shows that the large majority turn to the right; out of 263 forms 91 make their first turn at 10, 64 at 12; only 55 extend beyond 100. "There is at least a suggestion that both the first turning and the limits of these forms may be related to the manner of counting and indicate that their formation begins with the earliest counting" (7, p. 516).

Of 280 answering the question as to the age at which their form first appeared 241 did not remember a time when it was not at call; 17 say at 6 years; 11 say they learned to add or multiply on their scheme. Several things in the forms point to the fact that "number forms begin with the naming of numbers, and go on as the child learns to count to 100. Most children can count 100 before they can recognize anything written or printed. It is only natural that they should try to visualize the abstract and try to cast it into some concrete form or space relation" (7, p. 516).

As to the origin of the particular form possessed by individuals few had any explanation to offer. Some of the simplest forms were ascribed to imaging the clock face or other objects more or less frequently before the eye. Such are only more or less modified mental images and were located on blackboards, books, blocks, etc. This class does not exist among adults.

Is it useful or troublesome?

Two hundred and eleven answered this question. Ninety-seven are sure of its utility, 1 finds it troublesome; 113 neither useful nor troublesome. Many, who said at first "Neither," afterward corrected this statement by saying they found themselves continually using it. At first they were unconscious of using it, using their form just as language is used as a medium without giving it a thought.

Where is it located?

Diamandi, an Italian mathematical prodigy of whom we have given an account, located his on the convolutions of the brain; but the rule is to locate it in space near the head, when located at all. Several did not answer this question and many were not conscious of any localization.

The direction of turn lacks explanation. Right- or left-handedness is not found to have any influence.

Are they inherited?

Galton emphasized heredity as an explanation and Miss Calkins (9) is also much in favor of heredity, but this only pushes the question further back. Flournoy says that only predisposition can be attributed to heredity. Phillips's questionnaire obtained 128 answers to this question. Six relatives had a number form; seventeen relatives had other diagrams. Phillips says, "I find no more signs of or reason to look for heredity here, than in any highly developed power of imagination, memory, art, music, etc., all of which are much questioned since Weisman's theory of heredity" (7, p. 520).

Number forms are not found most frequently, as might be supposed and as Galton thought, among the most *imaginative;* have no connection with love of mathematics, nor with general intelligence. Besides the number forms consisting of lines, straight, curved, spiral, which the (comparatively few) owners say are as distinctly seen as if actually before their eyes, there are many cases in which lines are absent and the arrangement more or less vague, the numbers simply being located in a certain direction or at a certain distance or *being felt* to be at differing distances away. Some say the numbers thought of recede, others that they go up, the large ones getting very high. "It seems," says Phillips, "that nearly all persons possess some idea of extension of numbers, more or less indefinite" (7, p. 525).

Out of 480 pupils above the 7th grade 74% visualize all mental operations with numbers in some way and but few had thought it could be otherwise. In the 1st, 2nd, and 3rd grades 785 children were asked to count and then to tell if the numbers take any direction. The most unexpected directions were asked for first. Thirty-four said they went down; 60 to the right; 29 to the left; 74 right forward; 470 some upward movement. The same material is here as in the more definite number forms and many of these are probably as distinct as the complex forms.

All the evidence seems to point to the fact that this manifestation, whether of the sort just described or of the more complex type, takes its rise very early in life, is almost universal among children at a certain primeval stage in their development, drops out in many cases after it has served its initial function, but persists and develops in complexity in a few persons. It seems most probable that it begins with and runs parallel to the establishment of the series idea which, as we have seen, is the background of the number concept. If this be true, it would find in this fact a basis for explanation. It will be remembered that the series idea is developed before number is applied to objects or number names have been learned. Counting in this stage is a motor response to an inner rhythm. But in the absence of number names, which later serve to free the motor mechanism by giving it adequate and easily pursued avenues of discharge, the field of response is much restricted. Now the number forms serve as the means of bursting the bounds of these restrictions, and as a kind of concrete symbolism analogous to the sticks, pebbles, etc. of the prenaming period. The motor impulses find their outlet and their relief in a geometrizing psychosis (doubtless instinctive), which leads to localization in space or imaginary movement along imagined lines, which movement, of course, must have a spatial background. We can better understand perhaps how limited the prenaming period is in available responses and how favorable a time it is for the birth of a more or less fantastic symbolism, if we remember that even number names do not long suffice the child; he soon delights in making numberless columns of figures. Prodigies make long calculations before, in some cases, they can make or read written symbols. It is not surprising to learn that some of them used number forms; these prodigies were in the same stage with reference to available modes of motor response as children before they can make figures.

But number forms, having originated, persist and are sometimes developed after number names and written symbols are freely used. Any collection of number forms shows that most of them are shaped more or less by the relations of numbers to each other. Here they serve the purpose, working through the geometrizing instinct, of preserving in a visual field, numbers in helpful relations to one another; also to continue a series, and to lay it off so that it can be grasped as a whole.

Dr. Hall suggests that it might pay to cultivate in children good number forms as aids to memory and reproduction, and thus to the elementary operations. Elements from quite a number of the recorded or yet to be discovered ones, might be selected for use at the very beginning in the primary text-books. Adelia R. Hornbrook (8) claims an advantage in giving to all children a uniform number form by having it constantly hung before them. This was suggested to her by experiences with children who had number forms, some of whom performed remarkable operations with their aid. For the number forms used by her see her article Educational Review Vol. 5, No. 5, pp. 467–480.

We have attempted above to state clearly the most favorable genetic explanation of number forms. Explanations have been attempted from other points of view, all of which Flournoy says are analyzable into two great classes: (a) psychological, (b) physiological, harmonizable by the principle of psychophysical parallelism. These are only of mild interest to the pedagogue and will be omitted.

2. Psychological Studies.

(a) *Perception.*

We now pass from the consideration of number beginnings in the race and in the individual to a psychological inquiry into its manifestations in the relatively mature person. It is not sufficient to know how the number sense and the number concept begin; we must also know to a degree, also, their eventuation. For the child will become a man; without knowing the man we cannot know the child. The child may be *sui generis* and have his own psychology, but he is also the future man; whatever the man is he began as a child; he does not entirely put away "childish things."

Psychological studies of the number consciousness of older children and adults are bound to throw light on the general psychology of the subject whenever the results obtained are sufficiently in common with one another to give at least hypotheses of likely verifiable validity, or in common with results of the study of younger children whose minds are still largely naïve in their functioning and not overlaid with habit. The tremendous advantage of work with adults as subjects is their power of introspection, which enables them to give a subjective analysis of processes which must remain almost wholly objective with children. So that results, if any, obtained from adults in common with those from children will thus get a psychological account otherwise impossible.[1] Whatever results may be obtained for general psychology, it must be remembered that, after all, large indi-

[1] Wundt writes: "In the investigation of children and savages only objective symptoms are in general available; any psychological interpretation of these symptoms is possible only on the basis of mature adult introspection which has been carried out under experimental conditions." Outlines of Psychology p. 330.

vidual differences will obtain and in any pedagogical procedure such differences must be taken into account.

It must be admitted that such a program as that intimated above has not, as yet, been adequately carried out. Still a number of important contributions have been made.

Messenger (19) presents some important results of an experimental study of the perception of objects in groups (1) when the presentations were tactile, (2) when they were visual.

The experiments on touch were performed with two brass rods four inches long sharpened at one end and rounded on the other, to make, when desired, the contacts qualitatively different. The subject sat with the palm of the right hand on the back of the left and his fingers interlaced. The backs of the fingers were stimulated on the second phalanges with the sharp end of one rod and the blunt end of the other and the subject asked to tell whether the sharp end was to the right or the left. Thirty stimulations were given in a series with only a few moments rest between stimulations.

It was found that the perception of the number of contacts is a matter of distance and direction, or in other words of localization in space; but further it is also a matter of association. So far as *immediate* excitation is concerned, space and difference in direction are the determining factors. Inverse variation of the two is necessary for the perception of two points as separate. But number is not given immediately. A subject may distinguish a qualitative difference between two points of the æsthesiometer and one point, even though he does not distinguish a difference in the number of points. The perception of number only arises with the association of the number with certain perceived qualitative differences.

In the experiments on the perception of visual objects, white cards containing black dots and lines were exposed for .01 second by means of a drop screen electrically controlled. First three cards were exposed in succession, each three times, and the judgments totaled. Subjects were allowed to make a drawing before stating the number, if they chose.

Fig. A

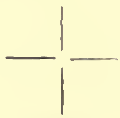


Fig. B

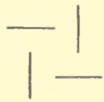


Fig. C

For A the average % of error (six subjects) was 49, for B 16, for C 46. B and C were equally visible (black), but B is more easily judged than C, hence the perception does not depend upon visibility. B is also more easily judged than A, although more complex, while C and A are equally easy, although C is more complex, so that as far as this experiment is concerned, perception does not depend on complexity. There can be no difference between B and C on the basis of difference in the direction of lines, for the directions are the same. (The inference is that the form of the figure as a whole is the important factor.)

To test this matter further cards containing lines arranged as shown in figure D and E were used, five cards for each figure in a series, each card shown three times.

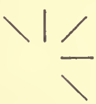


Fig. D

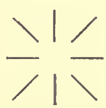


Fig. E

In each series the distances of the lines from the center varied from 2 cm. to 10 cm., but this variation was found to have no effect on perception. For D the average error (nine subjects) was 30, for E 17; only one subject overestimates E; five overestimate D and four underestimate. The author says that those who overestimated D gave their attention to what they saw and inferred the rest, while those who underestimated attended to the missing part and neglected the rest. Presumably he bases this statement on introspection reports, though he does not say so. The reason for the greater accuracy and uniformity in E is, the author infers, that the lines in E make a complete figure, thus confirming the initial experiments. Three subjects saw no appreciable difference between the two series. (This is an illustration of the importance mentioned above for noting exceptional cases.) The unusualness and *apparent* irregularity of D are doubtless also factors. The author made an incidental test showing that there is a tendency not to make so many errors with a regular arrangement, but the errors are greater when made.

In testing the influence of size and distribution, a further series of experiments was made, using nine cards of the general character of Fig. F.

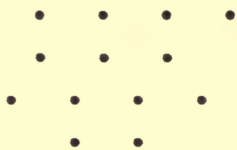


Fig. F

The spots were of three sizes, small, medium, large. (Fig. F represents the small size.) Each size was given on separate cards three different distributions: close together, medium distribution, scattered. Regularity was avoided except in the matter of distribution between spots. Each of the nine cards was shown three times, by the drop screen method as before, in the order of the size of the dots from small to large in each distribution, from close to scattered. The results for *distribution*, size unchanged, (eight subjects) were as follows (correct score for each distribution, a total of 135 dots):

(a) Close 112 (b) Medium 133 (c) Scattered 137

For all observers there is a higher score as the distribution-distance increases, except in two cases in which the medium distance is judged higher than (a) or (c). The average steadily increases.

The interpretation of these results is doubtless, as the author says, to be made on the basis of the spatial conditions. The dots when more scattered fill more space as a whole and are therefore judged to be more, in the absence of other data. Though the spaces between the dots are larger, attention is directed mainly to the dots so that allowances that may be made for the spaces are not apt to be sufficient.

The results for *size*, distribution unchanged (eight subjects), is shown below. The first three cards were shown six times, making the score for each size a total of 180 dots.

(a) Small 154 (b) Medium 160 (c) Large 184

It will be seen that the tendency to increase the estimate is not so great in the case of size. The results show that larger objects in a given space give the idea of a greater number, but this is counterbalanced by the knowledge, gained from experience, that into a given space can be put a greater number of small objects than of large ones.

In both cases, it will be noted that the estimated number is an *inference from the characteristics* (differing sizes) or conditions (differing distances apart or space occupied as a whole), the re-resulting judgment being modified by what in the impression is attended to. Space conditions being the same, other data are sought by the percipient upon which to base his judgment. A further experiment to show this was tried with cards containing

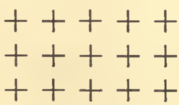


Fig. G

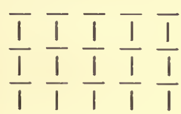


Fig. H

marks arranged as in Fig. G and Fig. H, each group occupying as a whole the same space.

In general the number of lines in G was regarded as about twice the number that are in H. The subjects gave as their procedure that they estimated the number of crosses in G and doubled it; this was not possible in H. The arrangement in H is so unfamiliar that the estimate in .01 second must have been almost a pure guess. In this experiment number is inferred from *prominent characteristics*.

It may be concluded, then, that the apprehension of the number in a simultaneous group is not an intuitive factor in the perception, like sensation, but a judgment factor as in all of our common everyday perceptions. If this be true, we should expect that it would be subject to errors and illusions. And this proves to be the fact, as we have seen. Not only does the occupancy of an actually greater space by a group give the illusion of greater number when the number is really the same, but the illusion of a greater space causes a corresponding illusion of a greater number. Fig. I and Fig. J containing lines of the same size, covering the same

— — — —

Fig. I

| | | |

Fig. J

distance, were each shown to the subjects, first in the horizontal position and then in the vertical. The vertical positions always yielded the higher estimates, corresponding to the well-known tendency to overestimate distance in the vertical direction.

The perception of number is a judgment mediated in part by spatial data, in part by other qualitative characteristics. This judgment is made possible by repeated prior experiences of association of certain qualitative differences in the unit impression with certain numbers of parts derived *by actual count*. (Many subjects made a drawing of their unit impressions and then counted the parts, a procedure reminiscent of earlier experience.)

When the association has become established, the experiences that have led up to the fixation of association have dropped out of consciousness and the apprehension seems to be immediate. Now when a group is presented the impression is at once associated with the number assigned to it before. "Four, for example, is a name for a certain object of sensation, and for sensation it is a unit, but a unit of such a character that it can be separated into parts and each part in turn become an object of sensation. It is a unit of the sort that has heretofore been separated into four parts and has habitually been recognized as four" (19, p. 42). On this account any lack of familiarity with the grouping, whether regular (see Figs. C, D) or irregular (see Fig. F), or, generally speaking, with the qualitative aspect of the unit-impression, leads to more or less uncertainty.

The most important conclusion is that the numerical apprehension of a group takes its rise in the application of the number series to the group. Counting is fundamental. Phillips is right when he intimates that there is no such thing as an eye span (except for very small groups). The apprehension of larger groups is the result of very rapid counting or the association of number with form. (Phillips instances the case of a teacher who was very much surprised at the inability of her class to tell the number in a group when one day she changed the forms of the groups on the blackboard.) Any arrangement of dots or lines for presentation to the child that will facilitate such rapid counting and association and give drill in them would seem to be a valuable contribution to the didactics of arithmetic. The experimental determination of such an arrangement best from the point of view of economical learning has recently engaged the attention of a number of school men, particularly in Germany, to whom reference will be made later (pp. 59 ff. and 149 ff.).

A study of the factors that influence the estimation of number made by C. J. Burnett (18) confirms the results of Messenger in several particulars, and adds a consideration of some factors not studied by him. Burnett found that (1) area is to a large extent a determinant of the judgment of *relative* number, but that different subjects (and the same subject at different times) show tendencies to give opposite judgments. It should be noted that the exposures in Burnett's experiments for the estimate of relative number were simultaneous (two group apparatus) (see appendix for the brief statement of the details of Burnett's procedure); while Messenger's exposures were successive. In the latter's experiments greater area almost invariably gave greater number. The important result gained from both, however, is that other factors besides pure numerical judgment enter into the estimate and that one of these is *area*. (2) As between a homogeneous group and a group irregularly gathered into nuclei, the majority of subjects estimated the homogeneous group as larger than the nucleated group, although some shifting of tendencies was noted. In Messenger's experiments on the influence of internal arrangement the groups compared (see pp. 53, 54, Figs. G, H and Figs. I, J) were homogeneous, but differed in the case of Figs. G, H in suggestiveness of estimation-aids, and in the case of Figs. I, J in their relation to a certain frequently occurring illusion of perception. An internal arrangement that gives a clue to a calculation-help (may not this explain, in part at least, the tendency to favor a homogeneous group?), or one that calls for judgment of a group arranged vertically was estimated as larger. Burnett and Messenger here supplement one another. (3) The influence of complexity in group composition was studied by comparing a group consisting wholly of gray and one consisting of three colors, red, yellow-orange, and blue-green. The subjects favored the noncomplex group. Two further experiments on complexity were tried (with successive instead of simultaneous exposures). A group of circles (plain) were compared with a group with small red circles in the center. The group with the complex members was estimated greater. Again, two sizes of balls were used in one group and one size in another. The complex group was estimated smaller. Complexity is evidently a factor, but the kind of complexity determines the relative judgment with opposite tendencies often present. Messenger found on very meager data (see page 51) that complexity is not a factor. It seems safe to say that complexity is sometimes a factor, sometimes not, depending on the kind of complexity. (Messenger's complexity was an added number of lines.) (4) In the matter of size, group areas being the same, opposite tendencies were found as in the case of area, a small majority favoring the larger size. Messenger found the larger size almost invariably favored, but his difference of procedure must be remembered. The introspective evidence given by Burnett's subjects differs, also, from Messenger's, the former being inclined to explain their judgments as a function of group vacancies and as a result of a distribution error, grounded in a fundamental tendency to base the judgment of relative number upon the character of the vacancies in a group. The principle of fluctuating attention will explain the influence now of objects, now of vacancies, and, of course, will vary in its working in individuals and in groups of individuals, the subjective character of vividness attaching now to one, now to the other factor, determining which shall be operative.

Burnett's experiments on the judgment of *absolute* number are more comparable with those of Messenger and the results are more in accordance with his. He found, for instance, using successive exposures (one group apparatus), (1) that scattering objects raises the apparent number, (2) that smaller size tends to reduce the apparent number. (It should be mentioned that Messenger's time of exposure (.01 second) is less than that of Burnett; but on the other hand Burnett found that apparent number (absolute judgment) is *inversely* proportional to the length of exposure.) He found also in absolute judgments (1) wide variation from objective correctness, (2) wider discrepancy with larger numbers than with small ones, (3) a general tendency to judge in multiples of five, (4) a general tendency to underestimate.

Other factors studied by Burnett (not by Messenger) were: (a) *Color:* Red increases relative number, on the whole, as against gray; brightness and vividness may account for this. (b) *Form of figures in the group:* Circles against squares set irregularly were judged by some as larger in number, by others as smaller. The introspective notes were not illuminating. "Form may influence the judgment merely through its space-characteristics, but possibly also through the vividness of intrinsic interest" (18, p. 380). (c) *Brightness as between gray and white:* All observers favor darker group. (d) *Complexity of environment:* Drift of tendency is toward the group with the more complex environment.

The estimation of relative number in the visual field is modified, then, by group, area, internal distribution, order and complexity in group composition; by the size, form, color, brightness and complexity of individual members; and by the character of the environment. Experiments were also made showing that the factors contributed by the objects through other senses than sight at the time they were being estimated, as the weight of the objects and conditions of eye-muscle strain, modify the judgments, but that irrelevant stimulations during the time of observation as touches, noises (factors outside the objects) have little or no effect.

In the judgment of absolute number, as we have seen, there is a tendency to underestimate; scattering the objects increases, compacting diminishes the apparent number; the smaller the size of objects the fewer do they appear; heterogeneity in group composition usually lessens the number.

On the whole the most influential factors were those lying in the space characters of the groups. In his interpretation of his results Burnett agrees substantially with Messenger that the principle of association adequately accounts in many cases for the more or less accurate estimate of number, as influenced by qualitative differences in the groups to be numbered. Says he, "Our practical experience in the simultaneous variability of number and certain other characteristics of a group of objects has been such as to lead us into illusions when the two no longer vary together. In such a case, when we have no time to count, we are actually led to see a group as smaller or larger in accordance with the variations perceived in the associated factor" (18, p. 386).

The pedagogical bearing of such investigations lies chiefly in their contributions toward the solution of the question: Is there an original grasp of the number of impressions simultaneously (or successively) presented, or is it attained gradually through counting and association of the number thus ascertained with the presented group until counting is no longer necessary? The conclusion was that there is no such original (innate) power of apprehension. Immediately arise then other questions: How is this apparently immediate apprehension of number to be developed? Within what bounds is it possible with adults and especially with children? In what measure is it conditioned by practice and completed through practice? What special outer and inner conditions facilitate or hinder it?

To make clearer at this point the practical bearing of this on the didactics of arithmetic we may anticipate our later discussion (p. 149 ff.) to the extent of saying that it has been found by general experience and exacter experiment that children are more positive in their fundamental operations if their ability to grasp groups of counters visually presented has been developed as sharply and completely as possible.

I take from Meumann's discussion (57) a review of the experiments made in this country and abroad relative especially to the scope of immediate number apprehension.

Cattell (23), in Wundt's laboratory, presented simultaneously for estimate with an exposure of .01 second a number of short vertical lines. Four to five lines were found to be rightly estimated.

Dietze (*über den Bewusstseinsumfang, daselbst,* 2. 1885), in the same laboratory, made a research with metronome strokes in order to determine how many strokes can be estimated without counting (successive presentation). It was shown that the number of rightly comprehended impressions depends on the quickness of the strokes. He found .2 to .3 second to be a favorable rate. Furthermore the estimate depended on the rhythmical grouping of the strokes. The most favorable arrangement was 8 x 5 strokes by means of which forty strokes was rightly judged, the highest number attained. It appears, however, that the judgments in Dietze's experiments were relative judgments, as his procedure was to have the subjects compare two sound series of different lengths separated by a short pause. It is the judgment of *absolute* number that is of importance here, and as Meumann says; whether we are grasping the number of impressions will not be determined by the comparison of two series of sound impressions following one another. A further criticism by Meumann is that with Dietze's method it is possible for retention (either immediate or as memory) to enter as a factor.

Warren (Princeton Contributions to Psychology 2, 1898, vol. 3) used the reaction method. He showed to the observers black circles, arranged in a circle, exposed for .131 second; they reacted by opening the mouth when they had perceived the group (1 to 8 circles). He found that adults rightly estimated three simultaneous sight stimuli and five successive, without counting. The criticism of this is that both numbers are too low; also the time of exposure is much longer than is necessary to finish successive apprehension. [It must have been a sore trial to the subject and was certainly a great exhibition of self-restraint not to count in the second part of this experiment, when there was ample time for it. Indeed it would be difficult to devise a method of visually presenting successively the members of a group in which the results would not be more or less vitiated by counting].

Nanu (Zur Psychologie der Zahlaufassung, Würzburger Diss. 1904) found that in all cases, with four grown-ups, up to eleven sound stimuli would be determined rightly without counting, and, in several cases, up to 49 strokes. All participants rhythmatize unconsciously the impressions whereby, as with Dietze, the number grasp was much strengthened. The observation of Dietze that with even numbers as starting point the determination becomes more positive Nanu found not justifiable. Again, an experiment was made on the visual grasp of bright circles on a dark background in which the exposure was .033 second. The circles were arranged variously: (a) in a straight line running diagonally through the field of vision, (2) in the form of a cross, (3) in the form of a parallelogram, (4) in the form of a hexagon, (5) in a circle. It was shown that arrangement in figures favors the number grasp as against the arrangement in lines. The highest number of impressions which in all cases of all observers was rightly judged was *five*, if presented only in rows; if presented in rows alternated with figures, this number amounts to *six*. By the arrangement in figures, cross and parallelogram were relatively favorable, circle and hexagon unfavorable. The highest number of impressions which by all observers were rightly estimated in all cases with the same figure was: for the parallelogram, *ten;* for the cross and the circle, *eight;* for the hexagon, *eight* (in 75% of right cases only). In these experiments two apprehension-types show themselves, a synthetical and an analytical; the former places together in the conception the group out of the elements, the latter has at first the impression of the whole and first turns its attention to the parts in two lines. The number of impressions was overestimated by the analytical type, underestimated by the synthetical. Two important results from Nanu's experiments are (1) the linear arrangement of the elements for the sense-perception of number works more unfavorably than the symmetrical; the circle formed and polygon arrangement are purposeless; (2) the number of simultaneously comprehended elements, without counting, is greater for the grown than one many times imagines; while by a more favorable arrangement, and without much practice, eight to ten elements visually simultaneous, and up to fifteen audibly successive can be grasped, provided definite number-concepts have already been attained. Meumann adds (1) that the estimation of a definite number of impressions without counting, as an innately accomplished function, stands against all our psychological knowledge, (2) the recorded observations of Preyer and others show that children actually attain number-conceptions by counting.

The experiments described above were all with adult subjects. Lay, Walsemann and others come closer to the school problem with their experiments on school children with special reference to favorable arrangement for the economical development of definite concepts. These we shall postpone for treatment in a later connection (p. 149 ff).

(b) *Counting.*

Counting, which looms so large as the *sine qua non* of arithmetical development, has an interest of its own considered as a psychophysical act. Briefly described from this point of view it is the matching up or approximate synchronization of the terms of two series of events in consciousness; the one being a series of innervations tending to motor discharge, in other words to articulatory response or naming, the other a series of impressions from sense-stimuli. The separation of these two factors in order to determine how far the rate of counting and also of adding is dependent on the articulation factor forms an interesting as well as difficult study in number consciousness. It might be supposed that counting and adding, at least by adults, would become by practice so facile as far as the association factor is concerned, as to be practically instantaneous, so that differences in rate will depend on relative difficulties of articulation. This is found to be true to a certain extent but not wholly so. Judd (29), after determining the maximum rate at which the short series of numerals (1 to 10) or similar groups of letters can be articulated, made experiments to determine the rate at which sounds, tactual impressions or successive visual impressions can be counted; in other words the time required by the process of relating the internal series to the external. The rapidity of articulation was found to set the limit, four times in five, of counting a succession of sounds (the easiest). The limit to counting of sounds seems to be set by some internal rhythm and for most of the persons investigated this was the rhythm of articulation. In the same individuals the other forms of sensory experience offered much greater difficulty and the adjustment between internal and external successions is a more complicated process. Incidentally it was found that for 9 out of 15 counting aloud was either more rapid than silent counting or equally rapid with it. In some cases the effort to inhibit articulation in counting retarded the process of silent counting so that it was 20 or 30 per cent slower than counting aloud. Arnett (25) found that "the range of variations in adding times is on the whole so little greater than those found for simple reading that it is impossible to determine without more prolonged and careful experiments relative ease of adding certain combinations. Subjects state that they gained many sums by associations as simple as reading" (25, p. 335).

Arnett's study of counting (25), using educated adults as subjects, has brought out some interesting aspects. He had the subjects count small black labels (½ inch x ¾ inch) one inch apart on a light smooth background, placed in a horizontal row on a level with the eye of the observer. First they counted simply from left to right, then from right to left, then in groups of 2s, 3s, 4s, 5s in both directions. Observers were directed to note any points concerning the psychophysiology of the process. Number of labels in each case was about 30, the observer never knowing exactly beforehand.

Results.—(1) Numbers were usually spoken, actually or incipiently, as eye came to rest on one label or group of labels. (2) The sudden stop and some slight muscular punctuation (voluntary movement) brought the coördination of the two series to consciousness. (3) Involuntary wink was a source or error. (4) Counting by twos, threes, fours, fives was more difficult but more rapid than counting by ones.

TABLE

Counting by	Average Time (seconds)	Average of errors
1s	20.4	5.0
2s	14.0	2.0
3s	11.5	3.8
4s	10.7	2.3
5s	9.6	3.5

Table shows that time decreases with size of the group but not proportionately. Greatest accuracy is with double and quadruple grouping. Grouping by twos is most accurate. (5) Counting in reverse direction from right to left was observed to be more certain. No explanation given.

Arnett tried a second series of experiments on counting an irregular series of clicks. Two pendulums were used with different rates of vibration. Mercury contacts at central point of the arc of vibration served to make and break the circuit of an electromagnet. The two pendulums vibrating nonsynchronously caused the clicks to come irregularly, the set recurring after a fixed interval, about 30 clicks in each set.

Results.—(1) The lack of rhythm caused the count to be very difficult and inaccurate. (2) Corollary, our counting mechanism is rhythmical in its action. (3) Irregular counting may be made easier and more accurate by practice.

An important factor in the counting process, perhaps not generally recognized, brought out deserves special mention, viz., that of voluntary movement which punctuates the process and seems necessary to start the mechanism and keep it going. Touching the objects, pointing or telling off with fingers or hand or pencil strokes, or, if nothing more, the special fixation of the eyes (in visual counting): such movements are the constant accompaniments and excitants of the inner (speech) series. Pauses in the former mean concomitant pauses in the latter, and when punctuation is difficult as in the unrhythmical series the counting becomes difficult. Counting without a trace of voluntary movement to set the psychophysical mechanism in order is very difficult for all and doubtless would be impossible for many. Try it.

Looking at counting from a pedagogical standpoint, it is, of course, more than a psychophysical act. A new element enters in—that of its motivation—why count at all?

At early stages in the child's life counting is automatic, spontaneous (at first without reference to objects), grows often into a passion which, as Burnham and others have shown, sometimes manifests itself pathologically. It is play, and like other play needs no motive other than the fun of it. But now civilization must have its inning with the little savage; his parents and teachers and playmates from time to time demand numerical descriptions of situations; they keep asking "how many" until he feels the need to know "how many" and the fact that a successful adjustment to his environment often involves exact numbering must perforce gradually dawn upon his consciousness. Here then is the motive without which the whole process would be meaningless from the point of view of serious living, viz., the need for exact measurement. The measurement motif, as we shall see, runs as well through his later calculations and is partly social in its nature, partly mathematical *per se*. The demands of society and the nature of the content ulterior to methodology as they may at first be thought, must be taken into consideration in any pedagogical view; for psychological considerations alone might lead us to produce children who are prodigies of facile functioning, like in a measure to the historical prodigies, but prodigies to no purpose. The play counting and the facile process work are the alphabet of arithmetic and must not be neglected. Children having learned their letters like to say the alphabet forward and backward and even delight in the a-b *abs* and other nonsensical combinations, but, as everyone knows, these sporting proclivities tend to drop out as soon as the serious business of spelling begins.

The importance of counting and its peculiar function in the definition of number will appear more clearly in our later review of the psychology of the number processes (p. 73 ff). A word or two here as to the several phases it assumes in the child's history.

1. Counting is a motor response to an inner rhythmical series without reference to objects. The responses at first are certain muscular movements, as repeating strokes with the clock, rolling mud balls, nodding the head, arranging pebbles, moving toes and fingers, etc., later articulating, using number names. (See our review of Phillips's article, pp. 7–11.)

2. It is a mechanical application of number names to objects.

3. It is rational counting. "This involves the putting of units (parts) in a certain ordered relation to one another, as well as marking them off or discriminating them. If when the child discriminates one thing from another, he loses sight of the identity, the link which connects them, he gains no idea of a group, and hence there is no counting. There is to him simply a lot of unrelated things. When we reach 'two' in counting, we must still keep in mind 'one'; if we do not, we have not '*two*' but merely *another* one. Two things may be before us and the word 'two' may be uttered, but the concept two is absent. The concept two involves the act of putting together and holding together two discriminated ones" (32, p. 31).

It is in this sense of the word that counting may be said truly to require "a considerable power of intellectual abstraction." The difference which makes the individuality of each object must be noted, and yet the different individuals must be grouped as one whole. Moreover, it is the kind of counting that must be done in order to arrive at the recognition of objects as forming one connected whole or group.

The controversy between Dewey and Phillips (Ped. Sem. vol. V, pp. 290–298 and pp. 426–434) seems largely to turn on the definition of the word counting. Dewey regards (3) above as the true counting and recognizes freely (2) as a childish sport; but denies the name absolutely to (1). Phillips fails to credit Dewey with the recognition of (2) and minimizes (3). The whole debate seems at this distance much like a tempest in a tea-pot. The three phenomena, interrelated, given above undoubtedly exist as described, call them by what name we will, and must be reckoned with in our dealings with children.

Sooner or later, then, the child must take up his "white man's burden" of exact numbering and for a purpose. As many school men believe, among them Dr. Hall, he should assume this burden much later than he is now called upon to do. Better to allow him to enjoy to the full his naïve interest in pure numbers and number relations, an interest which, as studies of child life show, displays itself in many directions. (Pure numbers, however they may seem to the teacher, are not abstract to the child in this stage.) As soon as his number sense originates in the discrimination of a this and a that, it is "inevitably and incessantly applied." So concrete is his imagery connected with symbols and names that in many cases he develops a personal feeling toward them. He selects favorites, personifies, socializes, dramatizes numbers in ways which often appear fantastic.

Four was found by Phillips to be a very favorite number. Odd numbers are generally disliked. Most number systems turn upon ten, but some are founded on *six*, *seven*, *eight* and *nine*. "The most devilish thing," says Marjorie Fleming in John Brown's book, "is eight times eight and seven times seven. It is what human nature cannot endure." The investigations of this topic are few. Sanford's (28) study on the guessing of numbers makes a slight contribution inasmuch as it may be assumed that persons in guessing will unconsciously name numbers that have a personal appeal.

The material for this study was derived from a guessing contest for a prize conducted for advertising purposes by a Worcester merchant. The guesses were upon the number of beans in a five pint bottle filled to the cork with small white beans and conspicuously displayed in the show window. Customers were given with their purchases cards with places marked for the inscription of a number and for a name and address. These cards were filled out at the time or later and deposited in a box conveniently placed for the purpose. The cards deposited furnish the data for the study. Two thousand five hundred seventy-three guesses were made by men and boys and 244 by women and girls. The actual number of beans was 8,834 and the winner a man. The main part of the study is on the group of guessers whose names appeared not more than five times; this group consisted of 535 persons (men and boys); the results were worked out on the basis of 1,043 cards. The range of guesses was 285 to 1,000,000. The median guess was 7,257. The medians of the upper and lower halves of the series which give the limits within which falls approximately one-half the total number of guesses are 4,173 and 9,536. The range from 1,200 to 16,000 includes about 9/10 of the guesses.

TABLE I

FREQUENCY PER 1,000 GUESSES OF THE VARIOUS DIGITS WHEN SET IN THE UNITS PLACE

0	1	2	3	4	5	6	7	8	9
231	107	67	132	58	79	75	105	59	85

NOTE—Compare the frequency of the odd digits with those of the even.

TABLE II

NUMBERS GUESSED 3 TIMES OR MORE

1500 guessed 7 times
2000 guessed 5 times
10000 guessed 5 times
3000 guessed 4 times
6000 guessed 4 times
7500 guessed 4 times
8500 guessed 4 times
9000 guessed 3 times
2500 guessed 3 times
2850 guessed 3 times

5550 guessed 3 times
7250 guessed 3 times
7850 guessed 3 times
8000 guessed 3 times
11000 guessed 3 times
15000 guessed 3 times
....
7840 guessed 3 times
7989 guessed 3 times
10101 guessed 3 times

Repetitional

9999 guessed 7 times
8888 guessed 4 times
6666 guessed 3 times
7777 guessed 3 times
9997 guessed 3 times

Serial Numbers

6543 guessed 4 times

TABLE III

FREQUENCY PER 1000 GUESSES OF THE VARIOUS DIGITS WHEN SET IN THE TENS PLACE.

0	1	2	3	4	5	6	7	8	9
185	69	88	77	82	112	95	105	97	88

The order of preference for digits in the units place (derived from Table I) is:

0 — 3 — 1 — 7 — 9 — 5 — 6 — 2 — 8 — 4

In the tens place (from Table III):

0 — 5 — 7 — 8 — 6 — (2 — 9) — 4 — 3 — 1 (2 and 9 have equal frequency).

Leaving out of account the round numbers, repetitional and serial numbers (see Table II), the combinations of high frequency were found to be as follows:

High frequency—75 (20, 60, 63), 76, 43, 87;

Low frequency—66 (48, 88, 94), 14, 46, 95.

Numbers in parentheses have equal frequency.

Note—Read from highest to lowest in first case and from lowest to highest in second.

The guesses of the women and girls were also tabulated as above, but no evidence was discovered that the guessing habits of women and girls differed from those of men and boys.

Conclusions. Odd numbers are preferred in the units' place (Table I). The advantage of odd numbers over even in the tens' place almost completely disappears (Table II). For digits in the hundreds place the uniformity is still greater (not tabulated). It is evident that the final figure is most significant for the guesser.

Two-thirds of the guessers made use of particularized numbers (not round numbers).

About one-third guessed round numbers or those adjacent to them or numbers showing a repetitional or serial character in the digits chosen.

The important point brought out in this study is that there are undoubtedly distinct tendencies to assume differing attitudes toward certain numbers. Further investigation only can show whether anything of value to pedagogy lies in the matter, aside from the point already made that pure numbers have a personal appeal that render their early concreting for children unnecessary.

Material for the study of the personification of numbers is more plentiful.

Examples of *personification* given by Phillips (5): Thinking of 4 as a fat duck, 7 a tall man, 5 a pig-tail, 9 a lazy man, etc. These are probably suggested by the character of the symbols. Functional in their origin: 7 is always trying to break into 14, (says one), but is crowded out by 2; it then makes a try at 28 and 35, and finally does get into 49, etc. Here we have dramatization as well as personification.

Extreme cases of personification and dramatization are those of Miss Whiting (27) and of Mrs. X as given by Leuba (26). In Miss Whiting's case, 1, 2, and 3 were children; 4 was a good generous woman; 5 a reckless scamp; 6 an amiable but easily cheated prince; 7 an arrant rogue; 8 a haughty and very distinguished lady; 9, noble but generous to a fault; 10, a cold but great lord; 11, the herald of the king; and 12, the king. Five coaxed a child to help him run away, which he did till he got to 10, but before he reached there stopped to try to put 6 into 12, where he wanted to go, but could only drop him next door. The designing 7 tempted 5 to help him instead of 6 to get into 12. This angered 8, who thrust 5 into 13, where he stayed until the kind 9 rescued him and put him into 14, etc. Mrs. X's personifications have some curious details. Figure 1 is distinctly negative, it does not interest me, but is a *him*, goes about in trousers, is grown up and slender. Two is a good-looking, fresh-complexioned, blonde, commonplace man, a sort of brother to 3, though not intimate with him. Three is a pretty girl about sixteen with curly hair and rosy complexion. . . . Six is a young man of about 20, a noble counterpart of 3, for whom he is suspected of entertaining tender feelings; his hair is curly and blonde; he is a beauty, broad shouldered, 5 feet 11 inches, an English type, fond of boating. Seven is a decorative old piece of bric-a-brac, in the shape of a retired lawyer, perhaps; tall, thin, very neat, using beautiful English, and busied in trifling matters. . . . One reason why 4 and 8 wound each other's feelings is that 4 used to trifle with 6, a nice boyish fellow; now she does so less. Three and 6 are very open in their regard. Three is in awe of 8 who used to tease her pleasantly, etc. As soon as Mrs. X dwelt an instant on a number, its personality materialized and she thought that perhaps the irksomeness of arithmetic made her dwell on the personal side of figures. She also personified capitals and printed letters. A is a handsome, distinguished matron. B is a middle-aged man, sometimes a woman. The small letters are the children of the large ones. Also objects in nature. A tall thin tree is likely to be a woman; tulips, poplars and beeches are always so, while oaks are masculine. Some species have both sexes.

Miss Whiting deems her own phantasies as the mere diversions of a child who has tried to find entertainment from an irksome study. Dr. Hall cites her case and says he "believes it to be an unusual efflorescence of tendencies akin to those which most children pass through" (6, p. 356). Dr. Hall (6) made a study of a boy of thirteen, who came under his observation, and comes to the conclusion that the small numbers have an individuality due to their *functional* relations to other numbers and that these traits can be clearly distinguished from two other groups which spring (1) from name functions, (2) from the symbols, both of which have their own associations. "These individual traits suggest perhaps one factor in the psychic processes involved in elementary mathematics" (6, p. 359).

(c) *Fundamental Processes.*

The following review is based mainly on the experimental study of C. E. Browne, 1906 (30) supplemented to some extent by that of L. A. Arnett, 1905 (25). For brief statement of Browne's procedure, see Appendix II.

The processes of addition, subtraction, multiplication and division considered as psychophysical acts, afford a field of inquiry of considerable pedagogical significance. They all take their rise in counting and the association series formed in learning to count discover themselves as helps (sometimes as hindrances) in later process work.

Addition.—The slower counting is soon replaced in the school history of the child by memorized tables, through which he is expected to form associative bonds between process and result that function automatically. The learning of these tables is (usually) a motor (linguistic) response to visual stimuli. He "says" his tables. At first the full verbalism of the tables is apt to be carried over into his process work; as, 9 and 7 are 16 and 7 are 23, etc. The ideal of his teacher, however, is to get him to drop the tabular formulas and to announce (in oral adding) his results, using only direct numerical associations; as, 9, 16, 23, etc. When he can do this with a fair degree of rapidity, he has become a good adder; the slow adders are those who either actually or mentally reproduce as they add the complete formulas, or who, worse still, are still in the counting stage. Observation and experiments show a tendency of even practiced subjects to revert to these earlier stages in times of confusion, uncertainty, or fatigue; they are a necessary resort in cases of lack of practice or feeble association.

In adding a column of figures, as in counting, an inner series of innervations and an outer series of motor responses are observed. The successive recognition of results are the stimuli to the inner series initiating the incipient or complete naming of the results. The motorization of results (always present) is an ideo-motor process depending for its promptness, of course, upon the prompt presence of the idea (the addition result); this, in turn, depends upon the smoothness of the association between the two numbers to be added, which association (presumably) has been reduced to automatism previously through the learning of the tables. The factors, stated in the order of functioning, are: (1) The recognition of a result (except when beginning the addition of a column when the first digit takes the place of a result), (2) the motorizing of the result, (3) the recognition of a digit to be added, (4) the associative process leading to the sum of the two. The first is conscious; the second, subconscious; the third conscious, usually not motorized; the fourth, subconscious. The points of attention, then, in the process are (1) and (3). Any imperfection in either phase of attention or in (4) results in slowness or in mistakes or in both. Any retardation or difficulty in (2) in some cases leads to the reinstatement of the full verbalism of the tables (sometimes repeated again and again for the particular combination in view); (the combination 8 + 4, for example, is motorized thus: 8 and 4 are 12, instead of thus: 8, 12; or a reversion may even be had to counting, thus: 8—9, 10, 11, 12); in other cases to a more strongly accented and conscious saying of results.

It has been intimated above that the prerequisite to expertness in adding is the possession by the subject of strong fundamental associations such as have been afforded by drill in the tables. (The same remark applies, of course, to the other processes.) Some of the combinations are more difficult than others and continue for a long time to be a source of friction and in some cases are never reduced to perfect mechanism. These more difficult combinations may be psychologically differentiated at once from the others in a general way by eliminating those which are relatively easy for every one. In the number space from 1 to 10, the longest step is limited to four members, as 6 + 4, 5 + 4, 4 + 4, 3 + 4, 2 + 4; 2 + 8, 3 + 7, 4 + 6, etc., are only apparent exceptions since they may be combined in the reverse order. [From a practical standpoint, it would be uneconomical to reverse these combinations; they should be learned as they stand, because of the time-loss which would be suffered by their reversal each time.] 5 + 5 is a real exception, but it belongs to counting by 5s and is one of the easiest associations. Beyond 10 also some of the combinations are relatively easy. 9 + 9 belongs to counting by 9s, 10 + 10 is a count. 10 + any digit affords no difficulty. This holds for all 20s, 40s, 50s plus any digit. Once more, all results based upon results in the space 1–10, as 12 + 4, 13 + 5, 45 + 4, etc. are nearly as easy as the simple combinations.

This elimination leaves a range of results 11 to 17 inclusive, derived by combining single digits, in which the character of the associative bonds differs from the lower range and where the main difficulties of fixing these bonds may be expected to arise. The maximum range in this space is eight members. It is with the digit relations in this space that the adding psychosis has to do, since all adding is found to be a continuous referring to numbers under 20. Any difficulties remaining here will be certain to crop up in column adding. Difficulty with 7 + 6 means difficulty with 27 + 6, etc.

The introspective data in Browne's experiments furnish evidence that the relative difficulty of the different combinations finds its explanation in certain laws of memory as dependent upon previous associative experiences. These laws were formulated by Ebbinghaus as the result of his study of the memory for a series of nonsense syllables. They seem to hold as well in single digit adding. Ebbinghaus concluded from his study "that, in the process of impressing any series upon the mind by repetition, bonds of association are formed between all the individual members of the series. Every member of such a series acquires a tendency to bring the other members along with it when it reënters consciousness. These bonds or tendencies are of different degrees of strength. For remote members of the series, they are weaker than for neighboring members. The associative bonds for given distances backward are weaker than for the same distance forward. The strength of all bonds increases with the number of repetitions. But the stronger bonds between neighboring members are much more quickly strengthened than are the weaker bonds between more distant members. Therefore the more the number of repetitions increases, so much stronger become these bonds absolutely and relatively to those of more separated members" (Quoted by Browne, 30, p. 9).

Now the analogy between remembering a series of nonsense syllables and remembering combinations of digits is the outcome of a more fundamental analogy between learning to count and learning nonsense syllables. "Counting is purely a verbal formula, the one law being that the members always follow one another in the same order. The laws of association found by Ebbinghaus hold much more for the count than for nonsense syllables, in that the child goes through his series of nonsense syllables, the one, two, three, etc., until it is impossible to say it wrong, thus fastening upon his after experience in adding a bondage to these laws of which the introspective date gave evidence at every turn" (30, p. 12).

That primitive addition is counting there can be no doubt. The crude beginnings of addition are observed to be of this nature in children and in unpracticed adders generally. In times of confusion even practiced adders return to the count. Now the law, as applied to the counting series, means simply that 9, for instance, has a stronger association for 11 (say) than for the more remote 2, so that, if 9 reënters consciousness, 11 is much more likely to be brought along with it than 2. To make the application to addition, let us compare two combinations:

$$\begin{aligned}&(a)\ 9 + 3 = 12\\ &(b)\ 3 + 9 = 12\end{aligned}$$

In (a) 9 reënters consciousness and brings with it numbers closely associated with it in the count, among which is 12. In other words, the associative bond between 9 and 12 is strong and this may be expected, if the law holds throughout, to facilitate the forming of a strong bond between the result (12) and the combination (9 + 3). On the other hand, in (b), the 3 to which 9 is to be added has a weaker affinity for 12 (being further removed from it in the count), and therefore the associative experiences of the count may be expected to give comparatively little aid to the addition. Hence 9 + 3 will be an easy combination, while 3 + 9 will be comparatively difficult. Following this matter up, it will be seen that, according to the law, 9 + 2 will be easier than 9 + 3 and 9 + 4 harder; or the difficulty will increase in the order 9 + 2, 9 + 3, 9 + 4, 9 + 6, 9 + 7, 9 + 8, etc. (except where other factors enter in to ease up what would legally be difficult; e. g., 9 + 9 has been eased, let us say, through the count by 9s). It will be noticed in the order given above that as the difficulty increases, the difference between the digits becomes less. The most concise statement of the law of difficulty in the light of the Ebbinghaus law is, perhaps, this: (a) Addition of the smaller of two digits to the larger is easier than the reverse, (b) such addition (smaller digit to larger) will increase in difficulty in direct proportion to the difference between the digits. Where introspective evidence gives apparent contradictions to this law, it is found that certain devices have been used to ease up what is inherently difficult. For example 9 + 7 is made easier this way:

$9 + 7 = 10 + 6 = 16$, that is to say, enough is added to the larger digit to make it 10 and the same amount subtracted from the small digit, it then being necessary only to pronounce the remainder and annex the syllable ten (teen). In general, this is a workable short step whenever the difference between the digits is small (not more than 1 or 2).

This psychological testimony to the applicability of the Ebbinghaus law to single-digit adding is very interesting but cannot, of course, be deemed conclusive upon the basis of a single study, however carefully conducted, or however well trained in introspection the subjects. Arnett's chronoscopic study (25) of adding-times (equally inconclusive through fewness of subjects, though interesting) shows in contradiction to Browne that adding to the larger number in case of some of the larger numbers is a relatively slow process (and by inference a relatively more difficult one); that even numbers are not easier to add than odd (do not take less time); that the order of the digits, generally speaking, does not affect time (by inference does not affect ease). These points and others bearing on the relative difficulty of the combinations, before any valid pedagogical conclusions can be drawn, demand further psychological inquiry and, more important still, studies on school children with objective criteria such as the amount and percentage of error discovered in the use of the several combinations by large groups of children. It must be freely conceded, however, that the Ebbinghaus law affords the best hypothesis so far at hand to explain in a psychogenetic way certain phenomena discovered in adding, while the adding, so to speak, still remains in or at times reverts to a state of nature. As we have seen and shall see again in considering subtraction, civilized short cuts and well memorized direct associations tend to obscure the working of the law but do not necessarily contravene it. So that any attempts to verify the law, say through comparative studies of the efficiency of school children in the several combinations, would need to be carefully scrutinized before their results could be held for or against the truth of the law. It would seem, for instance, that the more expert adders must be ruled out and attention given to those who are still in the barbaric stage of their mathematical evolution, that is to say, to the comparatively inefficient. (Compare p. 288.)

A few other results in the study by Browne may be mentioned briefly.

Even numbers, as intimated above, are found easier to add than odd, except in cases of doubling as $3 + 3$, $7 + 7$, etc. And in such combinations as $9 + 3$, in which the common factor 3 closely relates it to the 3 count.

Certain *special* tendencies to error are noted, caused by incidental relations, peculiar juxtapositions, etc. "When any preceding digit remains in or near the focus of consciousness (as often happens when there is uncertainty regarding the accuracy of a result or in case the attention has been called in particular manner to a particular digit) such a digit is likely to displace or change the digit of the result which is or should be in the focus of consciousness at that instant" (30, p. 5).

Again, it is found that if the adding goes too fast or too slow, the results deteriorate. In the former case, the motorization of results is slurred over, and the feeling of certainty, which has much to do with effective work, resident in this (normally) subconscious phase of the adding process, is interfered with. In the latter case, while there is ample time for motorization to occur, it is counterbalanced by the fact that attention has between its normal focal points more time to wander, and accidental arrangements of digits and results become more suggestive of false associations.

Combination adding (adding columns by grouping digits) is not found by Arnett to have any special advantages over adding by single digits. Those who add by simple digits are more rapid and more accurate. While occasionally grouping may help, it may safely be left to take care of itself. Above all, as Arnett's study shows, no special tendency to group particular combinations is observable. It is a matter of idiosyncrasy. No attempt should therefore be made to impose one person's combinations on another, especially on children.

Subtraction.—Column adding by single digits (so called in distinction from adding columns by grouping digits) has for its companion process subtraction by single digits. Starting from 100 say, one may subtract one by one a series of digits, as $100 - 6 - 7 - 3 - 9$, etc. This process has the same four stages as the addition process, discussed above, viz.: (1) The recognition of a result (except when beginning), (2) the motorizing of the result, (3) the recognition of a digit to be subtracted, (4) the associative process leading to the difference of the two. Just as in addition (and in counting) the successive recognitions of results (in counting, the successive recognitions of events or objects) are the stimuli to the inner series initiating the incipient or complete naming of results. In fact the whole discussion of addition applies equally well to subtraction. The same laws of association are operative, but *with this difference*—they operate in the reverse order. This proves to be a large *but.* The reverse lever is not easily applied at first. The consequence is that primitive subtraction is addition and more primitively still, counting.

The introspective data of Browne's subjects show that the more unpracticed of them subtracted at first by adding, and while they soon emerged from this stage into that of direct association, during the time they were in it they observed a peculiar kind of prolonged motorization which served as part of the cause of the prolongation of the process as a whole; and also certain phenomena tending to further confirm the Ebbinghaus law. When the subtraction falls in the range 1 to 10, that is, when the digit of the minuend is larger than the digit of the subtrahend, it takes place usually by direct association. But when the digit to be subtracted is the larger, as soon as it is perceived in the light of the digit of the preceding result, it becomes a "cue" to a tentative motorizing of a *partially* apprehended result. To illustrate, let us follow the subtraction, 44 — 6, through the four stages: (1) 44 (a previous remainder) is recognized, (2) it is motorized, (3) 6 is recognized and compared with 4, (4) there is with the unpracticed no smooth working of a direct association, leading to the prompt recognition of the complete result and its motorization, but only the recognition of the partial result (thirty "something") and *its* motorization; this continues into thirty "something" and 8 are 44, *thirty-six* and 8 are 44. This may be termed a "natural" procedure. The associations derived from the counting stage are forward associations, hence associations backward are weaker. The tendency at first, therefore, is to translate the backward associations of subtraction into the forward ones of addition, so that as far as the fundamental laws of association are concerned, subtraction by addition is easier. It is a natural method. It is hardly necessary to add that it is a primitive method, that is to say, barbarous. It is awkward, roundabout, time-destroying. It was found that the subjects by a small amount of practice showed great gain in deriving remainders directly and so dispensing with addition. The interest of teachers, therefore, in addition-subtraction is, or should be, paleontological. Children having gained the direct associations, whatever reversions they may make involuntarily to earlier evolutionary stages, surely should not be *forced* into a revivification of a ghostly, or shall we say, ghastly past.

In subtraction, the Ebbinghaus law seems to be working in several particulars.

1. Subtraction is harder than addition, which fact stated in terms of the law takes the form—associations backward are

weaker than for the same distances forward, decreasing in strength as the distance between members increases.

2. While subtraction is still in the primitive stage certain subtractions are by preference performed directly, instead of additively. We have seen that certain hard additions (by the law and by introspection) become easy when aided by subtraction ($8 + 7 = 10 + 5$). (Practiced performers tend to use such easy cuts and so obscure the law.) Similarly, when the additive performance of a subtraction involves a weak associative bond, the tendency is to subtract directly because the subtraction association (count backward), though reversed, is at least as strong or stronger than the forward association. Thus, $12 - 3 = 9$ gives a relatively weak association forward (3 and ? are 12); hence subtraction is easier; $12 - 9 = 3$ gives a strong association forward (9 and ? are 12); hence addition in this case is apt to be used (primitively speaking, of course).

3. After the subject has emerged into the stage of direct association, as a rule he does not trust his remainders. His remainders now are obtained by a pure *tour de force* of memory, a new set of associations imposed, as it were, upon those that took their roots in a primitive stage of his mathematical evolution; the latter served him well then, but now have no organic connection with his new set. He constantly tends to revert to the old set for verification; he is always wanting to "prove" his subtraction by addition. This is only another illustration of the greater stability of earlier modes of behavior, whether racial or individual.

The practical aspect of subtraction is written subtraction. Column subtraction is practiced only as a school-room device.

The point of importance in connection with it, not covered by the preceding discussion, is the different methods that may be employed. What method is most economical and least liable to error? Such a question is best approached by studies in objective efficiency. Nevertheless, results so obtained would still await explanation, and any light from psychological experimentation may be welcomed.

The so-called Austrian method (by addition) is ruled out by considerations previously presented. This narrows the available methods to two, the prevalent one of borrowing one, when necessary, from the preceding minuend figure; and another in which one under like necessity, is added to the subtrahend. The psychology of the matter hinges upon the number of points of attention required in obtaining a given figure of the remainder; the fewer the points, the less time required and the less liability to error.

In the common method of borrowing from the minuend, when the minuend figure is smaller than the subtrahend, one is taken from the preceding minuend figure and imaged along with the minuend figure to be used. By analysis of the further process into steps this taking of the one is seen to be separated from the next minuend figure by four points: (a) the resulting figure of the remainder, (b) the recognition of the next figure of the subtrahend, (c) the recognition of the relative size of this figure and the corresponding figure of the minuend, (d) the recognition of the minuend figure (to be reduced).

In the method of adding one to the subtrahend, the habit is to image the upper figure at once as a teen, then between this and the increasing of the figure of the subtrahend, which is the final point of attention in the process, only one point intervenes, viz., the recognition of the result.

Hence, it is concluded, and it would seem upon good grounds, that the latter method is the better. Some interest attaches here to the effect on the process of the ideational type of the subject. From a psychological standpoint it would be worth while to trace this in some detail. The foregoing account is based upon the assumption that the subject is of the visual type in his conscious imagery. It is found that four focal points of attention occur in the first method. The same would be true of the motor-type. The mixed type would not differ here, but in the second method an additional point would intervene.

In any case the conclusion as to the relative merit of the two methods is not altered.

Multiplication.—Multiplication is a short cut to the sum of equal numbers. The race could conceivably have gotten along without multiplication and division, addition and subtraction answering all the necessities of computation, but at great expense of time. Multiplication and division must be regarded as more highly developed processes than their predecessors and doubtless should come somewhat later in the history of the child as they came much later in the history of the race. Both are highly artificial, but multiplication is the more primitive, being more closely affiliated to the count. Division, while genetically a derivative from subtraction, is practically the reverse of multiplication and greatly dependent upon it.

In multiplying a series of digits by a given multiplier, the parallelism between outer and inner series (the recognition of results [products] and motor responses more or less incipient), which were noted in addition and subtraction, is found again operative. Any lack of promptness in the functioning of the associative factor (process with result) leads to a more decided motorizing of the delayed result, sometimes to a full reinstatement, more or less conscious, of the complete verbalism of the tables. The ideal is, given a multiplier (8) and a multiplicand (9), to pronounce or write at once the product (72). In a series of multiplications the multiplier is almost an unconscious element in the process, the multiplicand is the focus of attention, the product is expected to be immediately associated therewith, the verbalism, which was established through the auditory-motor learning of the tables, having through practice dropped away. Children, as teachers know, in whom the tabular associations are not fully established revert to the count in times of delayed association. This close affiliation of multiplication to the count renders it more stable than division with greater sense of dependableness on results through the consciousness of the ever-present possibility of the restoration of lost results.

Division, with its direct and isolated associations (in practiced performers) of process with result, yields less dependableness and the constant tendency, therefore, is to "prove" division by multiplication.

In written multiplication the motorization of results suffers modification. The act of writing the unit figure of each product which is automatic tends to take the place of the linguistic motorizing so far as the unit figure is concerned, and the linguistic motorizing goes largely to the tens figure of each product. Thus the tens figure is objectified and held until it has properly functioned in the carrying process.

Carrying in written multiplication is the source of many errors. Though it is a mere matter of addition even good adders make more mistakes than might be expected. One explanation is that the conditions of normal adding are reversed, the digit to be added coming first, the product to which it is to be added second.

Division.—In dividing a series of multiples by a common factor the familiar synchronization of two series of phenomena again appears. The motorization of results in some form is always present and any interference with it results in a feeling of uncertainty. This is true of all the four processes. At first, traces of the multiplication formula appear, but after practice grow less and direct associations are used.

Written division is complicated by the fact that the successive partial dividends to be dealt with in an example are not always exact multiples of the divisor, and by the multiplication and subtraction operations that must be brought in. The source of errors in quotient figures will often be found not in the division operation but in one of the other operations involved. The writing of the quotient figure tends to become automatic and to go along as a parallel but separate series. Attention is thus freed and passes at once to the subtraction. "The difficulty of the process as a whole increases proportionally with the size of the divisor, because of the increased range of possibilities as to the dividend numbers falling above the multiple" (30, p. 37).

Browne gives the following kinds of errors occurring in division proper, as the result of his study.

Any factor commonly used as a divisor of a given dividend is liable to appear as the quotient digit, or even the divisor itself may so appear. Three kinds of such errors follow:

> 1. Errors making the divisor or some other factor of the dividend the quotient digit, as $24 \div 8$, giving a quotient of 8 or 4. Numbers such as 24, 16, 12, containing more than two factors commonly used as divisors, were especially liable to this kind of error. Such errors are insidious because the subject generally passes over them with no sense of inaccuracy.
>
> 2. Dividing a digit by itself, as $5 \div 5 = 5$.
>
> 3. When only a difference of one exists between the divisor and quotient digits, the subject frequently has difficulty in selecting the required quotient.

In $72 \div 8 = 9$, 8 and 9 are contiguous members of the counting series; as one comes into consciousness it tends to bring the other along with it and this creates a doubt as to which is really right.

It will be interesting to see whether tests of school children confirm these results. (See page 290 ff.)

Remarks.—One phenomenon was observed in all four processes. When the recognition series gets too far ahead of motorizing series the subconsciously following train of motor results is apt to be lost sight of. In any case, this was always felt to be a danger, and a uniform time per digit seemed desirable. In addition, for example, where the attention moves on to the digit to be added before the motorization of the previous sum has taken place, the error of substituting the perceived digit for the digit of the sum often occurs.

Another phenomenon noted common to all the processes is the assumption at the beginning of a certain attitude of mind corresponding to the intention, to perform the process. Its office is to direct the association processes and hold them within the proper field. The "set," as it is called, is much strengthened by the first steps in the series in each of the four simple processes, but is harder to initiate in subtraction than in the others. In multiplication it is intimately connected with anticipation and its principal element is the multiplier.

There is no great difference in the imagery employed in the several processes. In all subjects the terms used were largely motor or motor-auditory (linguistic). Visual imagery, though present, apparently is less fundamental.

Psychological studies such as the foregoing and, we may add, studies in objective intelligence which will come later, contribute results of only partial significance for the teaching problems (problems of method). These studies make the person central; his capabilities and his limitations afford the primary question. But there are other questions which are constantly arising in connection with such studies. One of these is that of the orderly content and purpose of studies (choice and organization of stimuli—the what and the why). To be able to know how and when the child's mind functions best along certain lines and most economically for certain results is of great importance to the pedagogue in this connection, for it puts him upon searching out from the complex environment to which the child is to be adjusted just those stimuli and just that order of stimuli which will occasion the appropriate timely economical mental activity. But since in the varied content there is a choice of stimuli, and from the purely psychological standpoint one stimulus may be as good as another, it follows that stimuli should be selected and organized not only with reference to the mental functioning of the child, but also with reference to what may be termed ulterior demands, the demands, say, of social or economic utility or of thoroughgoing scholarship in a subject. Otherwise a psychological monstrosity might be produced, a prodigy of facile functioning, but a creature out of tune with the universe.

The nature of the subject matter itself and its organization with reference to some aim are important considerations, then, in determining teaching procedure, whether it be in arithmetic or in any other subject. Not only how does the child learn number, but what of number should he learn and why should he learn this at all are the questions. As the latter question is differently answered, the procedure of the best pedagogues within the limits set by the results of psychological and objective studies will vary. The one who believes that the aim is to make of the child a calculating machine will organize the subject to that end and will use whatever insight he may have acquired to secure a presentation that will insure most advantageous mental functioning on the part of the child to that end. The one who believes that the aim is to occasion the acquirement of as much as possible of mathematical insight, particularly if this can be done without sacrificing facility in computation, will strive to discover and to apply the nature of number, the psychology of number processes from the point of view of rational procedure, with occasional excursions even into the philosophy of mathematics.

The latter aim I believe to be the true one, because while it affords a broad basis of organization, it does not, as the epilogue will show, prevent the attainment of more proximate and perhaps more immediately appealing aims. If it be the true one, then the one to decide the aim of elementary instruction in arithmetic is the university mathematical scholar; or, if we cannot catch his ear, then the educational thinker who has sufficient knowledge of mathematics to appreciate what it essentially is. If we ask the scholar (or the thinker) what to teach the child, he will, we believe, be apt to answer, thinking of mathematics as a unitary subject and not made up of separate compartments (with his vision pointing beyond merely immediate aims), "Teach him that number is ratio; this is fundamental."

The adoption of this far-reaching aim does not preclude attention to many proximate aims making for practical efficiency which will continually crop up along the pathway of the practical teacher. Here the teacher will need all the light that can be shed upon persistent problems by experimental pedagogy. He may be willing to accept from the mathematician or the philosopher the *ultimate* goal, but once having accepted it, he will be very careful not to allow his zeal to overrun his discretion by accepting uncritically all the philosopher or the mathematician may have to say about his everyday work. He will know that the recognition of the distinction between number and quantity, that number is the measure of quantity, that number is psychical, not physical, will come late in the mental development of the children and that for some children, during the years of the elementary school, it will not come at all. At the same time he will realize that the incompetence of some children to rise to the full (or indeed partial) stature of arithmetical insight affords to him no reason for depriving others of their birthright. He will be careful to give to his children all that the mere drill-master can give, but he will give much more besides to those that have the ability to receive. Much that he does will be caviare to the general. Indeed, there may be periods when even with his best pupils the sledding appears difficult, at which times it will be the part of wisdom to wait for more snow (greater maturity).

It is hoped that it will appear plain before the end of this part of the discussion that nothing will be lost to minor aims by keeping in mind the large aim of rational procedure with the end of conferring arithmetical insight; those who stop short of the higher mathematics will lose nothing in the way of calculating facility, on the contrary something will be gained toward easing the processes themselves, as such; while those who pursue their mathematical studies in high school and college will have been furnished in their elementary arithmetical training a groundwork (implicit, at least, but also to a considerable extent explicit with the better minds) for later growth in power. In elementary instruction the rationality of procedure, let us concede, will be explicit at all stages only in the mind of the teacher; but it will be *implicit* (at least) also at all stages in the mind of the child. The implicitness for the child will consist in habits of performance with respect to the arithmetical processes, the full import of which he may not at first grasp, but which, nevertheless, are intelligible and which will not need to be unlearned when his dawning mathematical consciousness brings him to the point, now and again, of apprehending their rational significance. He will not be a victim of arrested development induced by too long dwelling on the merely mechanical aspects of number processes, a semiidiotic chatterer of numerical nonsense syllables to whom it is vain to attempt to present the more thoughtful aspects; for all along the way every thought-spark that is potential in him will have been given a chance to flash.

First, then, for the child the inculcation of right habits giving correct mental attitude is indicated; the interpretation thereof, (the result of inner growth) as his strength permits, to accompany or to follow; this is the psychological key to the problem. Is it not the psychological key to more than one educational situation? It is so in moral education. First moral habits (to the extent of automatism if possible), habits moral for the child in an implicit sense only; later ethical interpretations. In religious education; first social service (explicitly religious only for the relatively mature person), then theological interpretations.

This discussion naturally brings us to a consideration of the classic production in this country on the subject, McLellan and Dewey's Psychology of Number (New York, 1898).

The word psychology has hitherto been used in the sense of the psychophysics of the number processes. In this book, the emphasis, so far as the book is a psychology, is mainly for the child upon the motivation of these processes and for the teacher upon the consequences that flow into method from keeping steadily in mind the *raison d'être* of the processes.

I have attempted to present its main thesis concretely by setting forth below briefly the material presented in chapter 6 and 7 (Arithmetical Operations or the Development of Number), going beyond these chapters occasionally for supplementary matter. These chapters present the essence of the whole matter in applied form, and, it may be added, that these and other concrete chapters have been largely ignored by those who have discussed the book unfavorably. As a practical school man I shall find it necessary to differ from the book on a few points of school-room procedure, and on one or two points of fact.

Every numerical operation involves three factors: (1) a quantity or magnitude to be measured, (2) a unit of measurement, (3) a ratio of the first to the second; *this last factor is the number.* Example: one coat cost $4, two coats can be bought for $8. The measured quantity is $8, the measuring unit is $4. ($8 ÷ $4 = 2). Again, four coats cost $8, one coat costs $2. Here the $8 are conceived of as 8 one-dollars and arranged in four groups:

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$ $ $ $

Each group contains \$2 (\$8 ÷ 4 = \$2). The measured quantity is \$8, the measuring unit, which in this case is *sought*, is \$2, the number (a given factor in this case) is 4.

The operation can be naturally and completely apprehended only when these three factors are introduced. This does not mean that they are always to be formulated. On the contrary, the formulation at the outset would be confusing; it would be too great a tax on attention. But the three factors must be present and must be *used* to the end that out of such habituation the true nature of number may gradually be evolved.

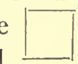
It is plain that the three factors are involved in division, as appears above; also in multiplication: one coat costs \$2, 4 coats will cost \$8. (4 × \$2 = \$8). The measured quantity is \$8; the unit, \$2; the number, 4. But does this appear in addition? Take a simple case. John has \$2, James \$3, Alfred \$4; how much have they altogether? (1) The total magnitude the amount (muchness) altogether, is here the thing *sought for*. (2) The unit of measurement is the one dollar. (3) The number is the measuring of how many of these units there are in all, namely, nine. When discovered it defines or measures the how much of the magnitude which is at first but vaguely conceived. In other words, it must be borne in mind that the thought of some inclusive magnitude must, psychologically, precede the operation, if its real meaning is to be apprehended. The conclusion simply defines or states exactly how much is that magnitude which at the outset is grasped only vaguely as mere magnitude. (The last three statements may be accepted provided they be understood as defining a point of view into which the child is gradually to be inducted. But that there should be much practice in the processes without conscious evaluation goes without saying; that the authors do not mean otherwise is gathered from scattered statements urging the importance of practice with pure numbers.)

Further examples illustrating the definition of number are: the yard of cloth costs *seventeen* cents; the box will hold *thirty-six* cubic inches; the purse contains *eight* ten-dollar gold pieces. The *seventeen*, *thirty-six*, *eight* express the numerical values of the quantities; they are pure numbers, the results of a purely mental process (psychical products, not physical aggregations); they represent the relative value or ratio of the measured quantity to the unit of measurement. The numerical value and the unit of measurement taken together express the *absolute* value (or magnitude) of the measured quantity. Number is thus seen to be the product of the mere repetition of a unit of measurement; it simply indicates *how many there are;* it is purely abstract, denoting the series of acts by which the mind constructs defined parts into a unified and definite whole. Absolute value (quantity numerically defined) is represented by the application of this *how many* to magnitude, to quantity; that is, to limited quality. Number, simply as number, always signifies how many times one "so much" (the unit of measurement) is taken to make up another "so much," the magnitude to be measured. It is due to the fundamental activities of mind, discrimination and relation working upon a qualitative whole. Quantity, the unity measured, whether a "collection of objects" or a physical whole, is *continuous*, and undefined *how much;* number as measuring value is *discrete*, how *many*. The magnitude (muchness) *before* measurement is mere *unity; after* measurement it is an aggregation of parts (units) making up one whole; number (showing how many) refers to the units which put together make the sum (or product). Quantity, measured magnitude, is always concrete; it is a certain kind of magnitude, length, volume, weight, area, amount of cost, etc. Number, as simply defining the how many units of measurement, is always abstract.

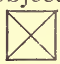
From what has been said above, McLellan and Dewey's doctrine of the nature and definition of number should be clear. It will be further brought out, however, as we proceed through their discussion of the fundamental operations. Their purpose is to show that the operations represent the internal and intrinsic development of number (and to point out the educational bearing of this fact); that they are processes of the measurement of magnitude—transforming a vague idea of quantity into a definite one, not something externally performed upon or done with existing or ready made numbers; that they are intrinsic in the idea of number itself [and are motivated by an instinctive mental movement to ascertain the numerical value of the qualitatively limited objects in the environment—they constitute one mode of exploring the world]. [The world takes on a greater and greater degree of mathematical definition, a more and more refined numerical evaluation, as the processes grow in accuracy and definiteness, the fundamental operations of addition, subtraction, multiplication and division being the successive phases of this growth.]

Addition and Subtraction.—In the addition or subtraction of quantities we are not conscious of their ratio; we do not even *use* the idea of their ratio. Even in this phase, however, the child may be led to evaluate the sum or the difference (e. g., $256 as 256 times $1), and to think of it as quantity to be measured. But addition is mere aggregation and subtraction the finding of how much more or less. This is the discovery of a kind of relationship, but not of ratio. The whole is the sum of all its parts and the difference is a comparison of greater with less. The idea of relative value in this one point of moreness or aggregation is brought out, but not the idea of what multiple or part one of the quantities is of the other. This is a more complex conception and so a later mental product.

Addition, as having a reason for being, is a process of breaking up a whole into parts and then putting together these parts into a whole. Thus, what was originally a vague magnitude is measured or defined and given a precise numerical value. This point of view alone gives any reason for performing the operation and sets a limit to it.

To teach the addition facts in any small number we may begin with an arrangement of sticks or splints that will lend itself to the grasping of the number as a whole, for instance  for *six*. Here we have a unity which the child may grasp as a whole, but indefinitely. The procedure indicated, then, is the analysis of this whole into parts for the purpose of changing the vague unity into a measured unity. So the child proceeds to count off the various sticks. If the square  has already been mastered, the figure will be recognized as one 4 + one 2. Or if one of the diagonals is changed thus  it will be recognized as two triangles — that is, as one 3 + one 3. Of course, it may be taken all to pieces and put together again, and recognized as 6 *ones*; or they may be arranged as 3 *twos* ∧ ∧ ∧. In this way the pupil is always working within a unity, gradually securing its completer definition, while gaining facility in computation. He is being trained in the recognition of six *at a glance* (which all primary teachers aim to secure by whatever method). He is given the maximum opportunity for the exercise of power in answer to his *constructive* interest; he is left the minimum for mere mechanical drill. In a word, the rational teaching direction here indicated both for power and practical efficiency is: *Begin with a whole which may be intuitively presented and proceed by the partition of this original whole into minor wholes, and their recombination.*

[I have stated the plan of teaching beginning addition in some detail because it affords an excellent illustration of the point that there need be no conflict between rational procedure and practical efficiency, indeed the latter is aided by the former. All primary teachers agree, I suppose, that at this stage the desideratum is the grasping of a whole intuitively presented. This numerical apprehension of a group of objects can reach certainty only by means of the counting habit which by practice and certain groupings approximates more and more nearly to instantaneousness, until it is scarcely to be distinguished from intuition. Now the prevailing procedure with reference to subtraction and division has necessarily been analytical, but with reference to addition and multiplication, synthetical from the start. The result has been the cultivation of two mental habits where one would do; the synthetical habit, viz., of merely aggregating quantities into a sum which does not look back to the parts composing it is a bad one, because it secures only the proximate aim of mechanical association of numbers to be added or multiplied with sum or product; and occasions the slowness with which groups are grasped as wholes, a matter not surprising when it is considered that it has been wholly unemphasized by the teacher, having been left to be caught accidentally or providentially by the pupil.]

By this method the subtractions are given along with the additions because the reaction to a group of objects may be either in terms of addition or of subtraction. Thus  may mean 4 + 2 or 6 — 4 or 6 — 2. For purposes of practice in discovering the number facts in a group, a symmetrical arrangement of dots symbolical of any sort of objects may, perhaps, be more advantageously used than objects. (Objects of course should always precede any sort of symbolism.) Thus six may be represented thus : : : . Four plus two, six minus two, six minus four may then be shown thus :: | : , etc.

Subtraction, however, is slightly more complex than addition, in that it is necessary to note at the same time both the whole and one of the parts. In addition, the whole emphasis is upon the result—the parts are merged at once into it. As motivating the process, the approach to it should be made analytical by calling attention from the beginning to the undefined whole, which, through addition, is to become definite.

Multiplication.—The principle of measuring with an exact unit, i. e., a unit which is itself made up of minor units in the same scale, gives rise to multiplication and division. Such measurement discovers in each case a certain relation of the magnitude measured to the measuring unit. Hence arises the ratio idea. No such idea appears in addition or subtraction, although some preparation has been made for its appearance by the addition of equal addends and the successive subtractions of equal subtrahends. But the operation, $3 + 3 + 3 + 3 = 12$, treated simply as addition, does not give rise to the multiplication idea. It is necessary to count the addends and to become conscious that the number, four, is a factor of a product, twelve, expressing the ratio of the unit 3 to the product. This abstract notion, which is involved in all multiplication, renders multiplication psychologically more complex than addition, though it logically depends upon it and genetically is derived from it. Although multiplication is essentially a synthetic process, yet the approach should be analytical or else the operation is "figuring" and nothing more. The order of attention is as follows: (1) the vague sense of some whole which is to be more accurately determined by some product, (2) the multiplicand, a quantity or magnitude made up of a number of "primary units," (3) the multiplier, the number of times this derived unit is to be repeated; in other words, the ratio, (4) the product, the definite measurement of the former vague whole in terms of the primary unit. It is important that the quantity which is to be finally expressed by the product should be first suggested, in order that the habit of regarding number as measuring quantity may be permanently formed; or, if not suggested at the beginning, then interpreted when obtained as the accomplished measurement of a measured quantity in terms of some unit of reference. This plan works toward giving the work an inherent interest and cultivates mathematical insight. Work with pure numbers, at the same time, must by no means be neglected.

Division.—Division has its genesis in subtraction but is not identical with it. The number of 4s in 12, for example, may be found primitively by successive subtractions of 4 from 12, *provided* the number of 4s is *counted* and registered as the measure of the 12 in terms of the unit 4. Here, again, we see the function of counting in the measuring process. Its service to measurement as a means to an end lifts it above the level of mere enumeration for its own sake and gives it rational significance.

The successive subtractions in parting a group of objects compose the presentative phase of division. But the division idea does not arise in its fullness until the parts are counted and the whole again thus reconstructed. Through constructive exercises of this sort, division is gradually developed as the inverse of multiplication, and the multiplication tables are seen to be useful in short circuiting the process of division, as they were previously so found in the addition of equal addends. Whereas the problem in multiplication is: 4 times 5 feet = 20 feet, the problem in division is: given 20 feet and either of the factors to find the other factor. In either case, "we first of all determine what the problem demands, times or parts, then operate with the pure number symbols and interpret the result according to the conditions of the problem." It is unnecessary to force the learner into making refined distinctions in this connection between two kinds of division, one of which may be called "partition." As far as the bare operation is concerned there is only one kind of division, but there is no escape from the alternative interpretation of the result in each case, even with the beginner. Still less necessary is it (here we oppose McLellan and Dewey) to introduce the law of commutation. Having given careful consideration to the very clear presentation of this law on pp. 74–77, 113, 117, 120–122, 137, 163, 164, 168–170 of the book, my conclusion is that it should be very sparingly adverted to in multiplication and religiously abjured in division. So clear is the presentation that it becomes very plain to the reader that there *are* as a matter of fact *two* kinds of division—witness the procedure with objects, the actual process which discovers the nature of the operation. For example, 20 apples ÷ 5 apples requires quite a different procedure with objects from 20 apples divided by 5, that is, into 5 groups. But it appears further that *upon reflection* (the phrase "upon reflection" is added by the writer as it is unaccountably omitted in the book) these two processes are so closely correlated as to be identical. That is to say, in both there is a mental sequence of five acts (a count of five), which is repeated until the 20 is used up, and finally the number of such sequences is counted and found to be four. This is undoubtedly true and is calculated to appeal tremendously to the child with the philosophical mind. Puzzle—find the child with the philosophical mind. The two processes may be admitted to be correlates, or even identities provided that they be abstracted from the concrete problems that gave them birth. To show what happens in the application of this philosophical correlation to concrete work, take the problem: Divide 20 apples among 5 boys; here the 5 shows the number of groups into which 20 apples are to be separated. Shall we, then, for the sake of making division a process of subtraction (in the interests of its genetic origin) call 5, 5 apples? "Yes," say McLellan and Dewey, "since 20 apples ÷ 5 apples and 20 apples ÷ 5 are by the law of commutation psychic identities." So we have for our solution 20 apples ÷ 5 apples = 4. We can imagine the consternation of the ordinary child at this mysterious and sudden transmutation, at the joyous touch of the philosopher's stone, of boys into apples. We can also sympathize with his cheerful struggles to interpret the four of the quotient, should his teachers be so unkind as to insist upon it.

So much for the philosophical correlate as a school-room device. McLellan and Dewey call these processes *psychological* correlates. As a matter of fact, what is the psychological correlate of 20 apples ÷ 5 apples = 4? Without answering this dogmatically I would suggest to any one interested in discovering it to make the following simple experiment on one or more primary children. Count out four groups of 5 objects each, thus:

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Get the numerical reaction of the children upon this grouping by asking them to write as many different kinds of equations as they can think of, suggested by the grouping. The division equations obtained will be the true *psychological* correlates or identities for that process, according to my way of thinking.

McLellan and Dewey do well to intimate in one place that nothing is to be said about the law of commutation *until the time is ripe*. It appears certain that the time is *not* ripe during the elementary period of school life.

It is in the division process that number as measurement is afforded the most adequate field, so far reached, for exploitation. Starting as it does with a definite magnitude to be operated upon, the necessity of conceiving of a vague magnitude to be measured as in multiplication is not encountered. By working the two processes together the nature of multiplication as analytical in its first intention is cleared up, since division, which is analytical throughout, is seen to be working with the same data, inversely.

The expression of measured quantity, once more, has two elements, the unit of measure and the number of these units constituting the quantity. The cost of a farm of sixty acres at fifty dollars an acre is sixty fifties. Here the unit is $50 and the number is 60. But note that the unit is itself made up of minor units (one dollar), it is itself a measured quantity; and further, that, when the cost of the farm is completely expressed, it is defined in terms of the minor units, viz., 300 one-dollars. We thus have in the *complete* expression of the measured quantity, besides the two elements already named, a third, viz., the minor or primary unit (in this case \$1) from which the actual unit of measurement (\$50) is derived.

It is interesting to see that the fraction, as McLellan and Dewey point out, gives complete expression through its notation to the process of measurement. In $\$\frac{3}{4}$ for example, we have what may be viewed in the aspect of a measured whole, the same as the 300 dollars above. It is defined in terms of the primary unit (one dollar); the actual unit of measure is derived from the primary by dividing it into four equal parts and taking one of the parts (one-fourth dollar); the numerator 3 shows how many of these units make up the given quantity, and expresses the ratio of this quantity ($\$\frac{3}{4}$) to the derived unit ($\$\frac{1}{4}$). (McLellan and Dewey say that 3 expresses the ratio of the quantity to the standard unity; this, of course, is a mistake, this ratio being expressed by the fraction itself.) In the expression $\$\frac{3}{4}$, note that the number is the numerator 3; fourths designate the repeated unit; there are three *fourths.* This is true whether the primary unit is a single dollar or a number of dollars. For example, in the expression $\frac{3}{4}$ x \$8, the primary unity, or, as it may be called, the standard unity or unit of reference, is \$8; the derived unit (measuring unit) is \$2 (one of the four equal parts of \$8); completing the process we get \$6, which is 3 times \$2

Considered as a measured quantity, then, the *numerical* aspect of the fraction centers in the numerator; but the fractional notation has the advantage over integral of giving compactly, by means of the reciprocal of the denominator, along with the number the unit numbered. Hence every fraction may be conceived of as expressing the ratio of a measured quantity to a measuring unit. In the complex fraction, this appears as well as in the simple fraction. For example, take $\frac{9}{3/4}$. Factoring $\frac{9}{3/4}=9\times\frac{4}{3}$, which, being interpreted from our present point of view, means that the ratio of $\frac{9}{3/4}$ to $\frac{4}{3}$ is 9.

It may be remarked at this point that complex fractions (and compound, as well) possess for the school man only an academic interest. Concrete problems that might be allowed to give rise to them can always obtain their numerical solutions in processes that involve simple fractions. From the standpoint of the pedagogue the nomenclature and all that appertains to it may very well be passed over to the limbo of the curiosity shop.

We may agree with McLellan and Dewey that the fraction is to be viewed as a concrete measured quantity and is to be apprehended in terms of (1) the measuring unit determined by the relation of the denominator to the unit of reference, and (2) the ratio of the measured quantity (the fraction) to this measuring unit. This is the insight to be gained sooner or later in the interest of sound mathematical training, toward which the teacher should consciously lead the child. But it still remains true (and with this McLellan and Dewey would doubtless agree) that the full measure of this conception must be a somewhat mature product of earlier work with fractions in which the ratio idea lies fallow, although it is ever implied. As a matter of detail (with which McLellan and Dewey may not agree), for the beginner the unit of reference should be *fixed* at one, a single thing. Let us say, then, that provisionally and for the sake of simplicity a fraction is one or more of the equal parts of a *single thing*. What is three-fourths? It is three of the four equal parts into which a single thing is divided. Nothing is now said about the ratio of three-fourths to one-fourth, but every time the child generates this fraction from his (single) thing (pie, apple, disk, what-not), he multiplies one-fourth by three, and so the ratio idea is implied constantly in such manipulation, and, we may hope, will come to consciousness in due time in an explicit way. We must respectfully submit, also, that the pie is a better object to work with at this stage than the foot-rule (McLellan and Dewey object to much work with the pie on the ground that it is not a measured whole and therefore not a quantitative unity). If we may be allowed to be slightly facetious at this point (when life is so serious) we may say that the pie possesses more intrinsic interest for the child than the foot-rule. (I know of no experimental studies on this point, but observation seems to confirm it.) It is possible, of course, to make the foot-rule of interest, even painfully so, but this interest, all will probably agree, is apt to be negative rather than positive. Seriously speaking, it is a matter of common experience that disk-like (and spherical) objects bring out the fractional idea (relation of part to whole) better than linear objects.

The pupil, once having derived parts by breaking up wholes, can readily be led to regard the parts as individual things and to operate upon them with about as much ease as upon integers. But hold! The philosophers say we must not do this. Says Dr. Harris (Editor's Preface, McLellan and Dewey's The Psychology of Number, p. vii), "The methods in vogue in elementary schools are chiefly based on the idea that it is necessary to eliminate the ratio idea by changing one of the terms of the fraction to a qualitative unit and by this to change the thought to that of simple number." The inference from this is that they would have us present the ratio idea explicitly from the beginning. This procedure would certainly require in pupils a "high consciousness of the nature of quantity." "Hence," says

Dr. Harris, "the difficulty of teaching this subject in the elementary school." We should say so!

It seems clear, moreover, that the procedure according to "methods in vogue" does not necessarily "eliminate the ratio idea," but rather allows it to remain in abeyance for the time being. To the fruitfulness, for simplicity and clearness, of the prevailing procedure every school man will testify; and this applies to the common distinction between the proper and the improper fraction, which McLellan and Dewey characterize as a "mystery," since from an advanced view point all fractions are proper. But our contention is that the view point of the beginner is and should be that the fraction is one or more of the equal parts of a single thing. Hence, it is logically improper to unqualifiedly call $\frac{4}{3}$ a fraction since it does not conform to the definition. In the light of a deeper insight gained from the later expansion of his concept, all fractions will become quite "proper" and all hands will be satisfied. This sort of thing is continually happening in the mental history of all of us. Things that have seemed "improper" to us in the half light of partial truth became "proper" in the greater illumination of profounder understanding. Witness the changing attitude of Methodists toward card playing. Even the so-called fixed-unit "fallacy" is not so much a fallacy as a half-way station on the road to completer truth.

In conclusion, we may agree with the following: "The primary step in the *explicit* (italics mine) teaching of fractions—that is, in making the habit of fractioning already formed [in whole numbers (and I may now perhaps be permitted to add in earlier work in fractions)] an object of analytical attention—is to make perfectly definite the child's acquaintance with certain standard measures, their subdivisions and relations. In all fractions—because in all exact measurement—there must be a *definite* unit of measure. This implies two things: (a) The definition of a standard of reference (the primary unit) in terms of its own unit of measure; (b) the measurement of the given quantity by means of this derived unit. If the foot is a unit of measure, it is unmeaning in itself; it must be mastered, must be given significance by relating it to other units in the scale of length; it is 1 yard $\div$ 3 in one direction; or (taking the usual divisions of the scale) it is ($\frac{12}{12}$ i. e., $\frac{1}{12}$ x 12) in the other direction, i. e., as measured in inches. The [explicit] teaching of fractions, then, should be based on the ordinary standard scales of measurement; on the fundamental process of parting and wholing in measurement, and not upon the qualitative parts of an undefined unity" (32, p. 141).

The doctrine of the book may be summed up briefly as follows:

1. Number is psychical in its nature. Sense facts may be attended to without giving the idea of number. Objects aid the mind in its work of constructing numerical ideas, but the objects are not number. No clearly defined numerical ideas can enter into consciousness till the mind orders the objects, that is, compares them and relates them in a certain way.

2. Number is the ratio of a measured quantity to a unit of measurement.

3. Hence the methods of arithmetic should be largely methods of measurement.

Moreover, there is a certain order of growth in complexity and demand upon conscious attention. This order, upon the whole, is addition, subtraction, multiplication, division. This order may be correlated to the ripening instincts and interests of the growing child. The purpose and means of instruction will be determined on the one hand by the nature of number and on the other by the nature of the child.

(d) *Reasoning.*

The important thing for the teacher to know in connection with problematic work in arithmetic is that the reasoning involved is deductive in its nature, and syllogistic in form when the argument is explicitly stated. The laws of deductive reasoning, as everyone knows, have come down to us from Aristotle; no essential contribution to this aspect of thinking has been made by any subsequent logician. It is not to be expected, therefore, that psychological studies of children's (or of adults') thinking processes will reveal new laws of thinking; they can reveal only how minds function (or fail to function) in well established modes. In short, they will be studies in efficiency providing data for psychology and not psychological *per se*. No systematic objective study of reasoning calculated to give psychological insight has, so far as I know, as yet been made. A careful analysis and classification of children's mistakes in problematic arithmetic with a view to discovering wherein the fault usually lies, whether in failure to recognize the major premise or to distribute the middle term or to infer the concluding judgment, would, it seems to me, be one method of attack. Necessary presuppositions to such experimentation would be: (a) ability of the child to get thought from the printed page, (b) experiences by the child of situations like or similar to those presented in the problem. It might be supposed (and has been supposed) that, given these apperceptive data, success in reasoning is for the most part assured. Teachers, however, find that such is not the fact. They call failure in such cases lack of "gumption." Psychologists give it a higher sounding name. Whatever we call it, it would be interesting and important to know specifically *at what point* in the process lack of "gumption" enters in and whether it is a lack of native ability or of "concept of method" (in which case it can be remedied by training).

Observation of children and a few experiments seem to show that native ability is a larger factor in this phase of arithmetic than in any other and that therefore it is less susceptible to improvement through training. Bonser (34) found (1910) in his experiments on the reasoning ability of fourth, fifth and sixth year children that the mathematical judgment of the younger children in many instances surpassed that of the older ones. "Children with 3 to 5 years of experience and training more than a corresponding number of other children frequently do not do so well as the group of fewer years and less experience but with greater ability by nature" (34, page 33).

Thorndike (48) found instances of the rarity of this sagacity in an unexpected quarter. He says, after testing 40 adults (all school teachers and many of them college graduates) on problems, that the vast majority use no arithmetical analysis or reasoning. An interesting side-light is thrown on the question of why children fail in this as in other subjects of school study by Dr. Radosavljevich's discussion (78) of apperception. He shows by implication that the child may assimilate a situation without passing on to complete apperception of it; lack of sagacity, we may say, prevents him from apperceiving the relation of the particular question to the general principle involved. The studies of Courtis show incidentally that the introduction of unconventional language and of irrelevant numbers upset completely those children who sometimes simulate reasoning through their memory of typical problems and solutions as ordinarily presented in the arithmetics or by their teachers.

We have said that arithmetical reasoning is deductive. That this is so will appear from a statement of the steps involved in the solution of a problem. "First, the analysis of the situation by which the essential features of the problems are conceived and abstracted; second, the recall of an appropriate principle to be applied to the abstracted problem, a search among various principles which may suggest themselves for the right one; and third, √

involved in the second, the inference, the recognition of the identity between the known principle and the new situation. . . . Clearly these are examples of deductive reasoning of the usual scientific type involving data, principles, and inferences" (34, p. 14).

Take for example the problem:

If 3 hats cost \$12, what will 1 hat cost?
Major premise: 1 hat cost $\frac{1}{3}$ the cost of 3 hats;
Minor premise: 3 hats cost \$12;
Conclusion: 1 hat costs $\frac{1}{3}$ of \$12.

The first step described above gives the minor premise, the second the major premise, the third the conclusion. The crux of the child's attack would seem to be the second step or the establishment of the major premise. It is not absolutely certain that once this is determined the rest would follow, but it is certain that the process cannot go on without it.

The following are typical examples of failure to resolve problems not, we will assume, through inability to get thought from print or through lack of familiarity with the situation, but through inability from one cause or another to establish a correct major premise.

If three men fall nine feet, how far does one man fall?
Major premise (false): One man falls $\frac{1}{3}$ of the distance of three men;
Minor premise: Three men fall 9 feet;
Conclusion: One man falls $\frac{1}{3}$ of 9 feet.

A dog standing on 1 leg weighs 15 pounds, how much will he weigh standing on 4 legs?
Major premise (false): A dog standing on 4 legs weighs four times as much as a dog standing on 1 leg;
Minor premise: A dog standing on 1 leg weighs 15 pounds;
Conclusion: A dog standing on 4 legs weighs 4 times 15 pounds.

The cause of failure to grasp the major premise may be either lack of native ability (sagacity, power to *completely* apperceive the situation) or lack of a "concept of method." In so far as it may be the latter, it is a matter that can be remedied through training. To anticipate what has been said in another connection (p. 145): "It is undoubtedly true that much practice in seeking and applying the major premise, without which little success can be hoped for, can be given in the problematical portions of arithmetic and the habit thus formed can be carried over." "It is not meant that any such formal, logical statement as that set forth above shall be used, but rather simple direct means for getting at the heart of the matter from the start, from which point forward the child must be given the chance to work out his own salvation" (p. 146).

The validity of the conclusion in syllogistic reasoning, assuming that no fallacies enter the course of the argument, depends on the truth of the major premise. This, as everyone knows, is established inductively (usually). And so the child must be led to establish his major premise in at least a quasi-inductive way by a study of cases. For example, in the first syllogism cited the relation of one hat to three hats (and therefore of the cost of one hat to the cost of three hats) is originally established by showing concretely that if three hats are separated into three equal groups (which is what we mean by finding one-third of three) there will be one hat in each group. Similarly the falsity of the major premise in the second syllogism cited is easily discovered by attention to a concrete case.

The suggestions of Professor Suzallo (33) are of psychological significance in this connection as they provide a plan of proceeding from simple (one-step) to the more complicated (two-step) problems and of making sure that the child gets training in modes of attack, with reference to the different types that may occur, in pedagogical order.

In one-step reasoning there are four types, one for each of the fundamental processes. It is not necessary, however, to exhaust these before proceeding to two-step reasoning. In the latter the four fundamental steps can be combined in sixteen different ways. Even in the same problem one-step reasoning can sometimes be combined into two-step in more than one way.

Example: A hardware merchant had 14 crowbars in stock. He sold 5 on Wednesday and 4 on Thursday. How many remained?

Solution 1. $14 - 5 = 9$; $9 - 4 = 5$ (Two subtraction steps)
Solution 2. $5 + 4 = 9$; $14 - 9 = 5$ (Addition and subtraction)

It is suggested by Professor Suzallo that in the primary grades typical problems be taught involving operations in the following order, to be followed by miscellaneous examples:

1. $+$	6. $-\ +$	11. $\times\ \div$	16. $\times\ -$
2. $-$	7. $\times$	12. $\div\ \times$	17. $+\ \div$
3. $+\ +$	8. $\div$	13. $+\ \times$	18. $\div\ +$
4. $-\ -$	9. $\times\ \times$	14. $\times\ +$	19. $-\ \div$
5. $\div\ -$	10. $\div\ \div$	15. $-\ \times$	20. $\div\ -$

"The idea of inclusion, togetherness, separateness, left-overness, etc., should be presented in many varying situations so that the process is associated with the essential and not the accidental part of the situation."

In the process work of arithmetic the procedure is sometimes quasi-inductive, sometimes deductive.

For example, $\frac{1}{3}$ of $\frac{4}{5}$.

The first step, $\frac{1}{3}$ of $\frac{1}{5}$, is established by concrete (objective) work. Then the procedure becomes deductive.

Major premise: $\frac{1}{3}$ of $\frac{4}{5} = 4 \times (\frac{1}{3}$ of $\frac{1}{5})$;
Minor premise: $\frac{1}{3}$ of $\frac{1}{5} = \frac{1}{15}$;
Conclusion: $\frac{1}{3}$ of $\frac{4}{5} = 4 \times \frac{1}{15} = \frac{4}{15}$.

If it is desired to establish a rule for the multiplication of a fraction by a fraction, the pupil is led to observe that the product $\frac{4}{15}$ can be obtained by multiplying the numerators together for the numerator of the product and the denominators together for the denominator of the product (a quasi-induction). The application of the rule to further examples is deduction.

It is to be noted that in the strict sense of the word there is no inductive reasoning in arithmetic. Concrete examples there are leading to a rule or principle, but the rule or principle when reached is an absolute certainty, not a more or less probable hypothesis awaiting verification. One example is sufficient to establish the principle; a hundred more cannot make it more certain; the only reason we multiply examples is the pedagogical one of emphasis and repetition for the sake of clearness and has no *logical* significance.

3. Statistical Studies.

(a) *Efficiency.*

Psychological studies of the number consciousness, such as have been reviewed on the preceding pages, are important to pedagogy chiefly as contributions to theory. As contributions to theory, they are also, of course, contributions to practice, since all good practice must have sound theory back of it, whether the theory be consciously formulated or not. Presumably the procedure in our present day schools is an endeavor to put into practice what is known, or thought to be known, of educational psychology. The attempt in the past has been to train teachers in psychology and methods based upon it; and teachers and schools have been rated largely upon the orthodoxy of their procedure from this point of view, *regardless of measured or measurable results.*

It has been only within the last decade that the idea has come to full consciousness among leaders in educational thought that no amount of so-thought orthodoxy of methods on the part of the teachers can atone for lack of efficiency on the part of the pupil; in short, that no teaching is good teaching that fails to produce *measurable* results.

Efficiency tests to discover what is actually being accomplished as the schools are at present conducted, are a desirable antidote and supplement to theoretical conclusions as to what ought to be accomplished, as well as a source of information to pedagogues in many practical directions. Such experimental tests, also, while not psychological *per se*, are calculated to furnish a vast fund of data awaiting a psychological interpretation which shall place them upon a scientific basis; so the pendulum may swing from theory to practice to theory to practice, each movement serving to bring us closer to a realization of the dream of a predictive pedagogy.

Dr. J. M. Rice (47), 1902, made the pioneer investigation on arithmetical abilities, his main thesis being the enforcement of the idea that schools and teachers must be judged by results; and, incidentally, the discovery of the factors determining the widely varying abilities found from school to school. He tested children in seventeen schools in seven cities. He concludes that the controlling factors in the successful schools are the establishment of standards and regular testing for results. He eliminates as factors, that might be supposed to control, the following: (1) Home environment, (2) size of classes, (3) age of pupils, (4) time of day, (5) time devoted to arithmetic, (6) home work, (7) method of teaching, (8) teaching ability, (9) course of study, (10) superintendent's training of teachers.

The elimination of some of these factors is surprising and must be regarded as exceedingly doubtful until later investigation confirms or rejects the conclusion. It may, perhaps, be said that (4),(5), and (6) are now experimentally confirmed.

Dr. Rice's study is open to the criticism of lack of rigor and completeness in his statistical methods and (so far as one can determine) of lack of uniformity of conditions. Such criticism, of course, is easy at this distance and is in no way derogatory to the value and acuteness of his pioneer effort. Its value in general lies in the stimulus given to similar studies and the impulse given to the shifting of the point of view in regard to the manner of judging the efficiency of teacher and school.

C. W. Stone (46), 1908, under conditions as nearly uniform as possible and with approved statistical methods, made an extended investigation of the arithmetical abilities of children of the upper 6th grade. He drew his materials from 26 school systems, ranging from Massachusetts to Illinois, and he personally conducted the tests in each of the 152 class-rooms, securing over 6,000 test papers. In addition, he collected data as to the time spent on arithmetic by these pupils both in and out of school, in each of the grades from one to six. Finally, he obtained the course of study from each of the 26 systems, and had these arranged by competent judges in order of excellence. Two test papers were used with each pupil, one for the fundamental arithmetical operations, the other for arithmetical reasoning. The paper in fundamentals contains 14 test problems, those from 1 to 9 were arranged in order of increasing difficulty, the time allowed was 12 minutes, and the results were scored on the basis of 1 for each step of each problem. The paper in reasoning presented 12 problems of increasing difficulty, the time allowed was 15 minutes, and, on the basis of preliminary experiments, the results were scored from 1 to 2 for each problem. In the comparison of systems the scores of 100 pupils from each system were taken at random and combined. The following are some of the more important results:

1. The scores for the 26 systems vary in fundamentals from 1841 to 4099 and in reasoning from 356 to 914. The same system often occupies a decidedly different rank in fundamentals from that held in reasoning.

2. Comparing the scores of all systems, the correlation of reasoning with the average of the fundamentals is quite low (.40), —it is lowest with addition (.32), and highest with subtraction (.50). The correlation of the different fundamentals with each other is very high (.90 to .95).

3. An examination of the individual scores of 500 pupils chosen at random from 4 systems shows a wide variability, ranging from 3 to 63 in fundamentals, and from 0 to 15.2 in reasoning. With a median of 6 in the latter, 33 pupils score only 2 or less. The boys are no more variable than the girls.

4. In the individual scores reasoning shows a still lower correlation with fundamentals (.32) than in systems, and the correla-

tion is lowest with addition (.28) and highest with division (.36). Addition shows a much lower correlation with the other fundamentals (.50 to .65) than these do with each other .89 to .95.

5. In general it would seem that division is most like reasoning, subtraction comes next, multiplication is a close third, and addition is farthest removed. Moreover, ability in addition is the least guarantee of ability in other fundamentals.

6. It is inadmissible to speak of arithmetical ability as a single function,—rather we have to do with a number of abilities or products, and there is less connection between arithmetical reasoning and ability in the fundamental operations than between English and geography.

7. Supervision by the superintendent or a special supervisor does not seem to be a potent factor in developing arithmetical abilities.

8. The total time devoted to arithmetic in the first six grades varies in the different systems from 7% to 22% of the total school time, yet an elaborate comparison of time expenditure with accomplishment indicates that difference in time plays an almost negligible rôle. More important factors are the distribution of time among the grades and the use of time within a grade. In many systems there is a deplorable waste of time.

9. The correlation of excellence of the course of study, as rated by the judges with arithmetical reasoning (.56) is much higher than with fundamentals (.13). Future improvement in the course of study lies in the direction of indicating the place of drill in the educative process.

(Note.—The foregoing abstract of Stone's investigation is taken with but little change from the Journal of Educational Psychology, December, 1910.)

The investigation of Dr. Stone has inspired several others along the same line, notably that of Mr. S. A. Courtis, head of the department of mathematics in the Detroit Home and Day School, who has published his results in volumes X, XI, XII, of the Elementary School Teacher, Chicago (42), and finally, on the basis of these results, has devised a series of standard tests, with a manual of directions, in such form that they may be readily and fruitfully used by any teacher or principal for the quantitative determination of efficiency and growth.

From the beginning of his experimental work (1909) the purpose of Mr. Courtis was to establish standards from which to measure the success or the failure of work in arithmetic and to trace the development of ability through the several grades. The work of Stone had shown that arithmetical ability, so-called, is not a single ability but a complex of abilities existing in varying conditions of relation or non-relation with one another in the same individual, as well as in classes and school systems. Courtis's experiments present this fact even more strikingly, as he covers the broader field of all the elementary grades (except the very lowest) and of the high school grades, and show furthermore, that classes and individuals as they go up the grades are not, as things exist, harmoniously progressing in the several abilities but present bewildering differences in rate and quality of improvement. Making due allowances for differences in native capacity, which, of course, cannot be overcome, these variations in objective efficiency present a problem the solution of which will appeal to every school man. If intelligent and efficient work is to be done by teachers they must know (a) how their classes and its members stand relatively to the established norms of any ability, (b) what to do in the way of drill or other exercise to remedy deficiencies; as a corollary, what to refrain from doing with children or classes having reached a desirable (not always the maximum) rank. The second of these questions is going to require continued experimental work with drills and exercises different as to kind, duration and times of application, with uniform tests before and after, so that, after a while, data may be secured upon which recommendations as to procedure may be based with some confidence. Meanwhile, by the use of the Courtis tests, the condition of classes and pupils may be diagnosed, the teacher may proceed according to the light that exists to remedy defects, and, by means of a second test, uniform with the first, discover the measure of her success.

It is my purpose to illustrate the character and the application of these tests, so far as diagnosis is concerned, by an account of the tests as given by me in a city school, rather than by a digest of the experiments of Courtis in his own school. Space does not permit going into the details of the gradual elaboration of the tests into their present form with measurably equal units of work and standard scores. Recourse may be had to the original articles (42, 43) by those who are curious in regard to the historical aspect of the matter. As an introduction, however, to the account of work with our own children (see p. 261 ff) we must refer briefly to some of Courtis's results.

The points deduced from Stone's investigation that chiefly attracted the attention of Courtis were: (a) that taking the efficiency of the school making the highest score in fundamentals as 100%, the efficiency represented by the median score was 76%, that of the lowest score 45%; (b) that in reasoning the median score represented an efficiency of 60%, the lowest an efficiency of 39%; (c) that the per cent. of school time given to arithmetic varied from 7 to 23% and high ability appeared more commonly with the small rather than with the great expenditure of time. These facts seemed to demand further investigation to locate more specifically if possible the critical points that give rise to these variations and ultimately to devise ways measurably to remove them. For this purpose Courtis used Stone's tests at first, giving them to every grade from the 3rd to the 13th; this could be done as parts of the test were within the abilities of 3rd grade children, while the length of the tests was sufficient to occupy the 13th grade children during the time given; later Courtis devised tentative tests of his own, viz., speed tests in the fundamental tables (time $1\frac{1}{2}$ min.), a speed test in one-step reasoning, process to be indicated without numerical work (time 2 min.), a test in fundamentals similar to that of Stone (time 10 min.), a test in problem-work (time 8 min.), and a test in copying figures (time 1 min.).[1] The same tests being given to all the grades under uniform conditions, the comparative scores made became a measure of the development of abilities through the several grades. Courtis soon adopted the method of scoring simply by examples as right or wrong, as this was found to be as satisfactory for comparative purposes as any more elaborate scheme.

Some classes were found to have high ability in both fundamentals and reasoning or to have low in both, others high in one and low in the other. This suggested that the truest measure of abilities of the grades is a combination score from both tests. The combination scores were found to decrease the marked variations in ability present in the scores for either test alone. "The meaning of this," says Courtis (42a, p. 73), "is not apparent. It has been a puzzling fact of my teaching experience that ability to reason and ability to be exact in abstract work seldom go together. I am inclined to believe that there is a psychological principle at work which, if known, would solve more riddles than one in educational procedure." Accuracy was found gradually to decrease through the grammar grades and to increase through the high school grades at about the same rate. "If this result is confirmed by future tests there is an important lesson here. If inaccuracy in grammar grades is due to some natural cause outside of arithmetic proper, to insist on accuracy or to spend much time working for it may be not only wasteful but harmful" (42a, p. 74). Accuracy in tabular operations does not insure

[1] These tests were of the same general character as the more recent ones described on pp. 252–260 (q. v.), but the units of work as equalities had not yet been worked out.

equivalent efficiency in working examples involving these operations. As a matter of fact, the result of the test (in the Detroit school) of the 13th grade showed that 93% class accuracy in the tables meant 83% in examples. In individuals the difference is still more striking. In a personal conference with Mr. Courtis, he exhibited some data not yet published, from which it would be a fair but rather startling inference that the correlations between knowledge of the tables and accuracy in fundamental examples is very slight; that, in very truth, some individuals, for instance, work examples in multiplication seemingly without knowing their tables, while, on the other hand, some fail on the examples who are excellent in tables. Such facts as these are calculated to make us humbly acknowledge our almost total ignorance of the working of children's minds in such particulars. Again, as showing the variability within the grades and from grade to grade, it was found that the 5th grade, for example, was strong on multiplication and division and weak in addition, the 9th grade strong in addition and weak in multiplication and division, etc. Because a class does well in addition it will not necessarily do well in any other operation, not even subtraction. Such results have some bearing on the question of transfer. "If skill in addition does not influence skill in subtraction, it would seem that there could be little hope of transfer to any other subject" (42a, p. 179).

"The results [of a study of the relative difficulty of the four operations] make it probable that accuracy of work is a function of the development of the child and in a way independent of his arithmetical training. Each grade seems to have a certain general degree of accuracy, which after the early years of learning does not seem to increase as one would expect. Within the limits of the accuracy of a class, all combinations of accuracies in the different operations are found. Addition is found to be the hardest, multiplication or division least difficult. . . . It seems practically certain that in the present state of our arithmetical teaching each operation and each part or division of a topic is learned by the child as a separate unrelated activity" (42a, p. 180).

The mistakes made in the fundamental examples and in problems were analyzed and the mistakes made in copying and in carrying (attention and memory) were separated from the others to see how far such mistakes influence accuracy. It was found that of those making the mistakes at any given time in a class at least one-third and usually two-thirds will be making mistakes in copying and carrying. The mere introduction of carrying into multiplication examples caused an increase of 30% in the number of examples missed. The effect was least in subtraction; most of the classes actually solve the more difficult examples involving carrying more accurately than the simpler ones without it. The explanation is offered that "borrowing in subtraction becomes so habitual because most of the school work demands it that the habit persists in those cases in which no borrowing is necessary" (42c, p. 363). The effect in addition was apparently midway between that in subtraction and in multiplication and division. This ability is therefore judged to be a factor of sufficient importance to receive direct classroom attention.

A general study of the distribution of ability in the grades discovered a very irregular distribution after the 5th grade; in the early grades the classes were units, but as pupils pass from grade to grade, individuals react to training in different ways so that the unity is broken up. The composition of grade 12, according to combined scores in fundamentals and reasoning, expressed in terms of the average performance of members of the other grades was found to be as follows:

(Note.—At this time no norm had been established for the several grades. Since then Courtis has worked out norms based on tests of about 9,000 children; the work of confirming or revising these standards is still being carried on by him with the coöperation of many superintendents, principals and teachers.)

1 member of 3rd grade ability;
7 members of 4th grade ability;
12 members of 5th grade ability;
3 members of 6th grade ability;
1 member of 12th grade ability;
1 member of 13th grade ability;
7 members of about 13th grade ability.

Thus over one-half the class was found of less ability than the average ability of the 6th grade.

For the account of an investigation with the Courtis tests, see pages 252–299.

(b) *Ideation.*

Can individual children be classed as visual-minded, auditory-minded, etc., and their type be observationally or experimentally determined? If so, what is the relation of the several types to the learning and reproduction of numbers?

The studies of these questions have been few and the results tentative. F. W. Smedley made a pioneer investigation which he published (1900) in the Report of the Department of Child Study, Board of Education, Chicago. This is interesting and suggestive but lacks scientific value on account of undeveloped statistical methods. Of more value is W. A. Lay's Anschauungs und Gedachtnistypen, Weisbaden, 1903. Lay reaches the following conclusions:

1. There are no ideational types in the sense that an individual will think entirely in a visual or auditory or motor way, but we can name types visual, auditory or motor, meaning by it that this particular kind of imagery seems to give best results in apprehension and recall.

2. The mixed type comprises a whole series of subordinate types and variations.

3. It does not follow that a child belonging to one type in language reproduction must necessarily belong to the same type as far as arithmetic is concerned. (Lay was the first to discover this important fact.)

M. W. Meyerhardt (37) found (1906) that a visual presentation of a series of letters or numbers brought better immediate memory results than an auditory presentation, but he refers to the time required for recall rather than the accuracy of it. He says that the visualizer sees in his imagination a table such as has been shown him in reality; he will require for reciting little more time than for first reading; but the auditory type has in his imagination no table which he can read but must hear an inner voice repeat to him the series localized not in space but in time; this requires more time. He concludes that the memory type can, therefore, be determined by the rate of recall.

K. Eckhardt (38) investigated (1907) the relation between the ideational type to which the individual belongs and the value of the types in the actual teaching of arithmetic. He gave three tests. In the first the teacher pronounced a number and the children (aged 8 to 10) wrote what images they had in mind in response; in the second, two two-place numbers were announced to them to be added, and they were asked, "what images do you have in addition?" In the third test pupils were asked to imagine the series, 3, 6, 9 to 30.

The best results were shown by the visual type, and the number possessing that type was found to be relatively large. Eckhardt concludes that visual images are of decided assistance to number memory and that the elementary operations can be made easier by means of them; also that they can be cultivated by exercise.

One might pertinently ask what is to become of the auditory or other types in view of all this catering to the visual. It would seem to be better to determine, if possible, the predominant imagery of the several minds and adopt varying modes of presentation in accordance with the results obtained.

Eckhardt's investigation cannot be regarded seriously since results based upon the introspection of children must on *a priori* grounds appear to be exceedingly doubtful. It has later been shown by Springer (35) that children's statements about the character of their imagery are entirely untrustworthy

The latest and most systematic investigation of this subject is that by I. Springer (35) in his Mental Reproductive Types in Arithmetic, 1911.

The purpose of the experiments was to discover whether there are distinct ideational types in arithmetic, the relation of these types to immediate memory of numbers and the distribution and development of these types in the grades. Experiments were made on about 500 children of the male sex in each grade from the first to the eighth.

Series of numbers of varying lengths were presented to the children by each of four modes of presentation, viz., auditory, auditory-motor, visual, visual-motor. The purpose of giving series of different lengths was to determine (1) the effect of length of the series upon immediate memory, (2) to ascertain what form of presentation gave best results with the longer series, (3) to compare the best results in each of the series given. I give below the several series of numbers used in the visual presentation which will serve as a sample of those used in the other presentations. (The numbers were changed for each kind of presentation but the length of the several series remained respectively the same.)

It will be seen from the tabulation that the 2-number series was presented to the 1st and 2d grades, the 3-number series to the 1st and 2d grades, a 4-number series to the 1st, 2d, 3d and 4th, 5th, 6th, 7th and 8th grades, a 5-number series to the 3d and 4th grades, a 6-number series to the 3d, 4th, 5th, 6th, 7th and 8th grades, and a 10-number series to the 5th, 6th, 7th and

8th grades. It was thus made possible to compare the grades in each kind of presentation on the basis of length of series and this was done with reference to 4, 6 and 10-number series; that is to

Grades 1 and 2:

1st Series	2d Series	3d Series
87	65	34
54	92	47
..	39	64
..	..	81

Grades 3 and 4:

1st Series	2d Series	3d Series
49	83	85
23	57	43
64	34	87
92	68	47
..	29	38
..	..	55

Grades 5, 6, 7 and 8:

1st Series	2d Series	3d Series
18	47	95
22	26	87
84	35	24
59	51	39
..	83	65
..	62	73
..	..	28
..	..	37
..	..	45
..	..	76

say, 4 numbers, grades 1 to 8; 6 numbers, grades 3 to 8; 10 numbers, grades 5 to 8.

Preliminary experiments were used to fix upon numbers in the several series that should not be too easy or too hard. Two seconds was the time given to the presentation of each number.

Auditory Presentation.—The teacher read once the series and at a signal the children were required to write the numbers they remembered.

Auditory-motor Presentation.—The teacher read the three series as before and after each number the class repeated the number aloud. At the completion of a series the children wrote the numbers they remembered.

Visual Presentation.—Each number in the several series was presented to sight and at the end of each series the children wrote.

Visual-motor Presentation.—Each number was exposed on the blackboard and the class repeated the number aloud. When the series was completed it was covered and the children wrote. Two seconds was allowed for repeating a number.

The results were worked out under three rubrics:

1. Immediate memory for numbers.

A. Relation of different modes of presentation to immediate memory of numbers.

Here the three series were combined for each grade in each kind of presentation and the following conclusions reached:

(a) The immediate memory for numbers increases with the grade and age of school children.

(b) Visual motor presentation gives the best results in all the grades except the first and fifth. In the first grade auditory is $\frac{1}{10}$ better than visual-motor; in the fifth grade visual is superior by $\frac{6}{10}$.

(c) In general we can make the hypothesis that the child upon entering school and during the first grade will give best results by an auditory presentation, but from second to eighth grades it will give best results by the visual-motor presentation.

B. Effect of length of series on immediate memory (and continuation of relation of mode of presentation to reproduction). Here the 4-number series was given by each kind of presentation in grades 1 to 8; 6-number in grades 3 to 8; and 10-number in grades 5 to 8.

The conclusions reached were as follows:

(a) Length of series is a great factor in immediate memory.

(b) Immediate memory for long series seems to be best in 5th and 6th years.

(c) Visual-motor presentation is superior to all other forms in the 6th, 7th and 8th year and inferior only to the visual presentation in the 5th grade.

(d) Best results in early ages are obtained from auditory presentation, in later years from visual and visual-motor presentation.

(e) Auditory presentation gives very poor results throughout the grades, showing that loud speech movements are a hindrance instead of a help.

2. Relation of the reproductive results of the several methods of presentation to intelligence.

The teachers of the children classified them as bright, average, and dull with special reference to work in arithmetic.

The scores of the three classes of children in immediate memory of 4-number, 6-number and 10-number series were compared with respect to effect of length of series and of mode of presentation. The following conclusions were reached:

(a) All three classes of children (bright, average, dull) obtain the best results in the visual-motor presentation. Hence visual-motor presentation is a help to immediate memory of numbers regardless of length of series.

(b) Auditory-minded pupils of all classes find the addition of the motor element a hindrance instead of a help.

(c) Bright pupils obtain better results from the visual and visual-motor presentation than from auditory or auditory-motor presentations.

3. The determination of ideational types and their distribution throughout the grades. Are there distinct types of number imagery?

The objective method used by Springer, which undoubtedly has greater validity than the introspective method of Eckhardt, was to tabulate the results in each of the various forms of presentation and assume that the child belongs to the type in which he had secured the best results. The table below shows the results in each grade of each mode of presentation expressed in terms of the per cent. of the number of pupils in the grade:

TABLE

Grade	Auditory	Visual	Aud.-Motor	Vis.-Motor
1	44%	6%	25%	25%
2	12	17	29	42
3	5	27	11	57
4	6	37	7	50
5	13	49	12	26
6	44	16	5	35
7	19	35	9	37
8	16	41	13	30

Nearly one-half (44%) of the children of the first grade are auditory, but in the second grade the number decreases to 12%, and in the third grade to 5%. The auditory-motor type is practically negligible in all the grades though fairly prevalent in the first and second. The visual-motor predominates in four of the eight grades and is a strong factor in the others. Springer concludes further:

1. The vast majority are of the mixed type. There are no types in number imagery in the sense that children think exclusively in one kind of imagery. In this he agrees with W. A. Lay, cited above.

2. Although most of the children are of the mixed type, yet certain methods of presentation will produce better results than certain other methods. We can therefore classify children into the four types according to their results.

An interesting question which arises in connection with ideational types is, are they due to heredity or training?

Meumann in discussing this question says: "Ideational types can be explained as the result of four possibilities. (a) They are determined by heredity alone and, if such be the case, then training and practice have no effect and whatever type the child is endowed with must remain unchanged. (b) They are to some degree predisposed by heredity and are subject to change, but the influence of training and practice is less than that of heredity. (c) They depend upon heredity and training but the influence of training is greater than that of heredity. (d) Heredity is of no account in determining ideational types, training being the sole factor" (Ökonomie und Technik des Gedächtnisses, 1908, p. 166, quoted by Springer, p. 118).

Meumann maintains that though instruction may have some influence yet the factor of heredity is more important. Springer on the basis of his results disagrees with this. He ascribes the drop in audiles from 44% in the first grade to 12% and 5% in the second and third grades respectively to the fact that upon careful examination of the methods of instruction in these grades he found that in the second and third grades teaching of arithmetic was mainly written, in the first grade mainly oral.

Springer closes his thesis (p. 139 ff.) with an interesting commentary, substantially as follows:

The investigation suggests one cause for retardation. A teacher of the auditory type, say, may have a class 75% of another type, say visual or visual-motor. He fails to reach them. A careful investigation of retarded pupils will no doubt show that there are many children in our schools who have failed to comprehend instruction and who have not succeeded in keeping up with their grade because the mental type of imagery was not understood, recognized and taken into account by the teacher in the presentation of the subject matter of the curriculum.

In the group system at present pupils are classified according to their brightness or dullness in certain subjects. It seems that a better and more scientific classification might be secured by taking account of the varying mental types.

Bright children, especially, found the auditory or auditory-motor presentation a decided hindrance to the acquisition of knowledge. It was also discovered that for children who are of the auditory type the addition of the motor element not only is not an assistance but actually hampers the pupil in his intellectual operations. Hence if we carry out our maxim "Appeal to all the senses possible" we shall find that instead of assisting the children we are actually hindering some of them, though with others the addition of the motor element of writing or pronouncing may be a help. Discrimination must be used. I have only to add that while it has been found that ordinary mortals are of the mixed type of mental imagery, extraordinary individuals, if we may judge from our study of prodigies, are of an extremely pure type. Their one-sidedness is thus partially accounted for. Compare especially P. Diamandi and J. Inaudi with F. D. Mitchell in our account of prodigies (pp. 35, 36).

(c) *Transfer.*

It is difficult to escape the conclusion from the investigations of Stone, Courtis and others that arithmetical ability is not a general one but is a complex of abilities made up of specific abilities among which a low degree of correlation exists. In other words, there is no guarantee that a high degree of attainment in one branch of the subject implies a high degree in another branch or in the subject as a whole. Pupils who do good work in fundamentals are found to do poor work in reasoning and vice versa; and a similar variability is found among the fundamental processes themselves. The question arises, then, do the effects of practice in a particular subject matter transfer at all to other subject matter? If so, to what extent and under what conditions? This is the objective statement of the question. Stated more broadly and in its subjective aspect, it may be worded as follows: Does not the repeated functioning of the mind as perception, memory, imagination, thought, etc., on a given content so train these powers that an ability will be gained to function indifferently on any content, however dissimilar? Those who answer this question affirmatively are the adherents of the doctrine of formal discipline, which, in its traditional form, seems to maintain that the several "faculties" are kinds of abstract forms or modes, which are developed from certain subjects peculiarly adapted to the purpose (as mathematics for reasoning, literature for imagination, etc.), and which may then be applied successfully to other and quite unlike situations (for example, the power gained from mathematics to reasoning in other fields, or even to work in such a widely separated field as foreign languages).

It may be said, at once, that few now hold to the traditional form of the dogma (or to the "faculty" psychology which fathers it). The doctrine of *specific* discipline expresses the trend of modern thinking; for example, the ability to reason gained from mathematics is good for mathematics and for nothing else. Accordingly there is no such thing as general ability in the old sense; on the contrary, the mind is a complex of more or less distinct knowledges and abilities, each of which demands its own specific instruction and training and each of which functions efficiently only with reference to a content or to activities of the same character as the original or of similar character. All but the most extreme adherents of this view, however, accept the modification that certain general results from specific training, very important to education, are noticeable, viz., general concepts of method (generalized ideas of attention, attack, procedure, inductively derived from properly conducted special studies, which may be transformed into habits generally applicable); and, not less important for younger pupils, general attitudes toward their school work which shall determine whether they shall be parrot repeaters or seekers after understanding.

Our interest, however, is not the general problem of transfer from one subject to another but the more particular one of transfer from one part of the same subject to another. It might be supposed (and has been supposed) that a pupil is either good in arithmetic or he is not. The experiments on the performances of pupils heretofore described point toward a different conclusion. But these experiments were not directed specifically toward the question of transfer and the procedure, therefore, cannot be regarded as conclusive on that point. The procedure in the investigation of transfer is now well settled to be as follows: (1) to measure the ability in each subject (or branch of subject) at the beginning of the test, (2) to concentrate on increasing the ability in one of the subjects (or one branch), (3) to measure again the ability in the other subject (or branch) to see if there has been any increase following the increase made in the subject concentrated upon.

The following experiments were conducted under the supervision of W. H. Winch (50) in three London municipal schools for girls and one school for boys. The purpose of the experiments was to seek an answer to the question, "Does improvement in accuracy of numerical computation transfer to arithmetical reasoning?"

"In each case a whole class, working according to the same syllabus of work and under one teacher, was divided into two equal groups. The division was effected on the results of several tests in problematical arithmetic. In order that the natural ability of the children rather than their memory of recent teaching should be tested in these exercises, it was arranged that no problems should be given in a form with which the pupils were well acquainted. The tests were marked solely with reference to the accuracy of the arithmetical reasoning and entirely without reference to the accuracy of numerical computation. No attention was paid to the *right answers;* marks were given with reference to the process only" (50, p. 561).

"When the two equal groups had been obtained, one of them was practiced in a series of exercises in rule sums which every child knew how to do; the other group being meanwhile engaged in some other branch of school work. In every other respect the curriculum for both groups was precisely the same during the period of the experiment. No other arithmetical work was done during the time the experiment lasted" (50, p. 561).

"Finally the two groups, namely, the one practiced in accurate numerical computation and the one not so practiced were placed together again, and final tests were given in arithmetical reasoning. There had been in all cases a considerable improvement in the accuracy of numerical computation during the series of practice exercises. How far was this improvement in the accuracy of numerical computation transferred to accuracy in arithmetical reasoning? Did the practiced or the non-practiced groups do better work when the groups worked the same test exercises at the end?" (50, pp. 561, 562).

The method of marking the problems was to credit each necessary rational step taken in the solution with *one* mark. "It is not maintained that this system of marking is theoretically perfect, but it gives steady and reliable results, and is a system which teachers readily appreciate and understand" (50, p. 566).

"The method of marking adapted for accuracy in computation, though extremely laborious, was, in principle, extremely simple.

It was intended that no sum should be set in these exercises which the children did not know how to do; but, of course, though the processes were well known, there was much variation in accuracy of computation. It is obvious to any one who has kept records of arithmetical work from week to week that the usual method of marking sums 'right' or 'wrong' and allowing for each sum so many marks, all or most of which are lost if there is any numerical inaccuracy, will not give us a series of results which will adequately show improvement in accuracy. A child with one or two figures wrong in several sums may really have worked more accurately than another child who has all the sums right but one, provided that that one be wrong in many places. Further, a child may make much improvement even though its total number of sums right does not increase. Having these considerations in mind it was decided to give one mark for each correct unit of addition, subtraction, multiplication and division" (50, p. 567). "Some—a very few—able children telescope processes which average children work out at greater length, but their units are always calculated on the most generous basis" (50, p. 368). (I take it that this means that such pupils were credited with all the units involved up to each correct point in their solutions, though the units of work actually performed by them were fewer.) "A child on this method of marking does not lose all the rest of the marks for that sum because one number early in the sum is found wrong, even though subsequent operations on that number lead to further errors in the sum as usually understood. He may not really have made any more [further] errors; it is necessary to go through the sum, unit by unit, to see if he has or not" (50, p. 368).

The following tables summarize the results for three of the four schools: (50, pp. 571, 578, 585).

School I

Marks in Preliminary Exercises	Nonpracticed Group A			Practiced Group B		
	Number of Children	Average Mark Per Child in Preliminary Exercises	Average Mark in Final Exercise	Number of Children	Average Mark in in Preliminary Exercises	Average Mark in Final Exercise
30 and over....	1	33.0	15.0	1	32.0	16.0
25 to 30........	5	27.4	14.6	4	28.7	13.5
20 to 25........	2	20.5	7.5	3	22.0	7.0
15 to 20........	4	18.0	9.2	4	18.0	9.2
10 to 15........	4	14.2	6.0	4	14.0	4.2
Group Averages.	16	20.6 (5.8)	10.2 (3.6)	16	20.6 (5.9)	9.1 (4.7)

School II

Over 35........	2	37.0	33.5	2	36.0	35.0
35 to 30........	2	32.5	32.0	3	32.7	32.7
30 to 25........	5	29.0	28.8	4	28.7	28.0
25 to 20........	3	24.0	23.3	4	22.5	27.7
20 to 15........	5	18.6	18.8	4	17.5	16.0
15 to 5........	5	9.0	12.2	5	9.8	15.4
Group Averages.	22	22.5 (7.9)	22.7 (7.7)	22	22.5 (7.7)	24.2 (7.4)

School III

35 to 40........	4	37.8	38.3	4	37.8	38.8
30 to 35........	3	34.0	38.7	3	34.0	37.0
25 to 30........	6	28.0	28.2	5	28.0	32.8
20 to 25........	4	22.8	23.8	5	22.6	28.6
15 to 20........	8	17.8	19.5	8	17.5	20.3
10 to 15........	5	13.2	18.8	4	12.0	13.8
5 to 10........	5	8.4	8.6	4	7.8	13.3
Group Averages.	35	21.8 (8.1)	23.6 (9.2)	33[1]	22.0 (8.0)	25.5 (8.5)

[1] Two boys omitted on account of absence.

In the foregoing tabulations the children in each of the two groups are sectioned off by convenient score limits (10 to 15, etc.)

on the basis of the preliminary exercises in reasoning, and the average mark in the final exercises is recorded for each section. If we compare the final marks of the two groups in any of the tables section by section, the differences are seen to be slight and not always in favor of the practiced group.

Winch (51) tried some further work on the transfer of numerical accuracy in a municipal boys' school in London. A class of 72 boys, with an average age of slightly over ten years, was divided into two equal groups on the basis of the pupils' ability in arithmetical reasoning. One group was given ten practice exercises in numerical computation while the other was occupied with drawing. Both groups were then tested with the same problems in arithmetical reasoning. This is the same general procedure as was used in the first series of tests, but the method was modified by not working out the problems numerically. This plan was regarded by the experimenter "as a better means for testing the transfer (if any) of improvement in numerical accuracy to accuracy in reasoning and a safer basis from which to calculate correlations" (51, p. 263).

Sixty-four boys finished the tests. The results are shown in the following table condensed from that of the author:

Marks in Preliminary Tests	Nonpracticed Group		Practiced Group	
	No. of Boys	Av. Mark Final Tests	No. of Boys	Av. Mark Final Tests
Over 60	5	72.8	5	79.2
50 to 60	5	59.8	5	53.0
40 to 50	5	48.4	6	53.7
30 to 40	7	36.7	6	32.3
20 to 30	7	32.9	7	32.0
Under 20	3	23.7	3	16.0
Group Averages	32	45.7 (14.2)	32	45.3 (16.4)

Here, again, an inspection of the final marks given in the table shows on the average no gain for the practiced group, while the differences, section by section, are too small to be significant and show (if anything) opposite tendencies.

In both of these studies the author gives tables (here omitted) showing that the gain through practice *within the practice medium itself* is very decided. Of course, if this had not been found to be the case, further experiment would have been worthless.

As a subsidiary question, in the first study, he compared the results of the preliminary tests in computation (these tests were given during the same period and under the same conditions as the preliminary tests in reasoning for correlation purposes, which are discussed below) with the computations made in the final tests in reasoning, to see if the practiced group carried over their improvement in numerical accuracy in pure computation to the numerical accuracy with which problems are worked. This is an open question. He found that such improvement did take place very decidedly except in the case of the poorest section. "The (four) children in this group showed no continuous improvement in the practice exercises in numerical accuracy; the continued practice at short intervals seems to have diminished their arithmetical capacity, which is not altogether an unexpected result with very weak pupils" (50, p. 572, footnote).

Conclusions.—1. Improvement in accuracy of arithmetical computation seems to have produced no improvement in the accuracy of arithmetical reasoning.

2. For the present, therefore, pending more conclusive experiments, numerical accuracy should be sought for because it is valuable in actual life, and not because we can feel confident that an improvement in it will transfer to accuracy of arithmetical reasoning.

3. Both practiced and unpracticed groups show a slight improvement in arithmetical reasoning, an improvement which is found to be practically equal in the two groups. The significance of this for experimental work with school children is considerable. The assumption which is made by the method of equal groups is as follows: *With equal groups of the same grades or class, and approximately of the same age, the growth effects in the mental function in relation to which the division is made are practically identical, at least for short periods.*

The assumption for groups of this size appears in this case to be corroborated by the results.

4. There exists high positive correlation between the two functions, numerical computation and arithmetical reasoning.

(The coefficients of correlation, Pearson formula, for the five classes experimented upon were as follows: + .68; + .79; + .69; + .85; + .74.)

The fact that these high correlations exist side by side with little or no empirical evidence of transfer is highly significant for the interpretation of correlation coefficients. It might be supposed that high correlation necessarily involves community of function. But, as a matter of fact, "finding correlation between two functions need not mean that improvement in one has brought about increased efficiency in the other; but the absence of correlation does mean the opposite" (Thorndike and Woodworth quoted by Heck, 77, p. 102).

At least three interpretations may be given to a high positive correlation coefficient: (1) one function causes the other, (2) the two functions have a common factor, (3) the two functions are coincident without being interdependent. Which of the three interpretations is correct must be determined upon the ground of inquiry on such lines as in the preceding experiments. No evidence of transfer being found, we must conclude that the correlation is a sign of mere "togetherness" due to natural or acquired ability in both functions.

In order to place Winch's pioneer experiments in the light in which he would have them appear, I append the following quotation from Winch's reply to L. Leland Locke (Journal of Educational Psychology, June, 1911, p. 335). "The strongest argument against definite conclusions on the question of transfer from experiments like those [described] is the comparative shortness of time over which the experiments extend. On page 572 (Journal of Educational Psychology, December, 1910) I said, 'It is not fair to conclude from this experiment that no amount of improvement in numerical accuracy would transfer to accuracy in numerical reasoning, but only that the improvement shown in this case was insufficient to transfer. There is a presumption that [any amount of improvement] might not, since there is a clear transfer of improvement in numerical accuracy itself from rule sums to problems.' But I am acutely conscious that improvements effected at high pressure for short periods are not improvements which we believe on general educational grounds to be very likely to transfer."

Also the following from Winch's article in the Journal of Educational Psychology, Dec., 1910, p. 589: "The value of the foregoing investigations appears to me to lie chiefly in the suggestion of a method by which a very important pedagogical question may be ultimately solved under school conditions. It is a method which teachers can easily use for themselves."

An experiment by Starch, 1911 (52), with adult subjects who added introspective testimony to objective results shows that such transfer as occurs is due to general abilities developed which are usable in both functions and not to any community of function. This is a confirmation of the general trend of thinking on the subject of transfer.

Eight observers practiced for fourteen days on mental multiplication, using problems composed of three digits in the multiplicand and one in the multiplier. Before and after the practice they were given six tests in arithmetical operations (computation) and two in auditory memory span. A different set of problems was used in the tests made after the training series, but of the same nature as those used before. The memory-span tests were (1) for numbers, (2) for words, using monosyllabic nouns. All calculation was done mentally; the subject looked at the problem long enough to get it in mind, then closed his eyes, or looked away, worked the problem, and set down the answer. The memory-span tests were made by reading to the subjects groups of words or numbers at the rate of one word or number per second. After each group had been read the subjects wrote down all they remembered.

For comparison, seven other observers were given the preliminary and final tests without the practice series. The purpose of this was to determine the amount of improvement in the performance of the end tests themselves, which must be deducted from the improvements made by those who took the training series. (This amount of improvement was found to be 20%; the importance of deducting this is apparent.)

The result of the experiment showed that the practiced observers improved from 20% to 40% more in the arithmetical tests than the unpracticed. There was little change in the memory span for either group. "In what way did the training in mental multiplication help in the other calculations? According to the introspective accounts, the largest factor was the ability acquired, in the training series of keeping the numbers better in mind" (52, p. 309). "Several individuals mentioned the acquisition of the method of calculating by simply visualizing the arithmetical operations" (52, p. 309). "At the beginning it was so difficult to bear the numbers of a problem in mind while calculating, and the problem often became so confused, that the calculating had to be started all over again" (52, p. 309).

"The improvement in the end tests was due, therefore, to the identical elements acquired in the training series and directly utilized in the other arithmetical operations. The two main factors were (a) the increased ability to apprehend and hold the numbers in mind, and (b) the acquisition of the ability to visualize arithmetical operation" (52, p. 310).

"The memory span showed practically no change, partly because it is too dissimilar to the functions involved in the training series and partly because it does not seem to be subject to much enlargement through training" (52, p. 310).

An interesting question, in this connection, is whether mathematical reasoning transfers to reasoning in everyday affairs; or, in other words, whether one may expect his training in mathematics to function in practical reasoning. The tradition is that the mathematical thinker is a strong reasoner in all situations. I quote from Heck (77) an account of experiments on this point (of which there are few) conducted by the Dartmouth Pedagogical Department under the direction of Lewis (published in School Review, April, 1905). "First, two test papers were prepared, one containing originals in geometry and the other questions in practical reasoning. (The second paper dealt with the value of high school education to the individuals and to the community.) These tests were submitted to twenty-four different groups of high school pupils. The students of each group belonged to the same class and were on an equality with respect to mathematical preparation. Each group took both tests. The results of these tests were carefully corrected and the pupils of each group arranged in two series, the first according to their ranking in mathematical and the second according to their ranking in practical reasoning" (77, p. 96).

"If we take the first five mathematical reasoners from each of the twenty-four groups, we have in all one hundred and twenty pupils most excellent in mathematical reasoning. Of this number seventy-six, or 63%, are at the foot of the practical reasoning series, conspicuous for their inefficiency in practical reasoning. Of the number of pupils at the foot of the mathematical reasoning series, fifty-seven, or 47%, are conspicuous for their positions at the head of the practical reasoning series" (77, p. 97).

"As a supplementary test, and one precisely the same in principle, one man examined the records of Dartmouth students who had taken mathematics and certain law courses which required a good deal of reasoning. The records for ten different classes were examined, and tables were formed as in the previous test" (77, p. 97).

"The results of this test were found to be strikingly parallel to those of the earlier test. Fifty per cent of the best students in law were conspicuous for their poor showing in mathematics; and 42% of those poorest in law stood at the head of the series in mathematics" (77, p. 97).

An investigation of Rietz and Shade at the University of Illinois, 1908, published at the University under the title "Correlation of Efficiency in Mathematics and Efficiency in Other Subjects" seems to the authors to point to conclusions opposite to those of the Dartmouth experiments. They found coefficients of correlation as follows: for mathematics and foreign languages .48; for mathematics and natural science .44; probable error in each case .015. Notwithstanding these low coefficients they conclude "that efficiency in mathematics and efficiency in foreign languages go together in general to a high degree, and that to substantially the same extent do efficiency in mathematics and in natural sciences go together" (quoted by Heck 77, p. 98).

But these coefficients guarantee nothing more than a certain (not very high) degree of togetherness, explicable upon the basis of common elements in the two subjects, or of coincidentally but independently acquired ability in each. To prove transfer of ability requires such a procedure as that used by Winch or Starch.

(d) *Hygiene.*

The fact that the several school subjects, aside from physiology, have considerations of the health of the child as one of their aspects is one that has received attention from students of education only recently. Studies in such matters as arrested development, arithmomania, and pathological conditions of various sorts among school children have brought a realization that dangers to mental and physical health hover about some of our most cherished school practices. The reference here is not to the unanalyzed and unspecified popular ascription of all the ills to which childish flesh is heir to an "overcrowded curriculum." The tendency, often fostered by the ill-advised opinions of pedagogically ignorant physicians, to blame on the schools the ill-health that is the result of excesses or indiscretions committed outside the school room, is a frequent cause of exasperation to teachers who know the truth. But school men are free to admit that there are several counts upon which from a hygienic standpoint they may be indicted legitimately. It is, however, upon the ground of careful and long-continued investigation by qualified experts that this legitimate indictment can be based, not upon popular clamor or merely medical assumption.

Little ground of complaint now remains in regard to the care exercised for physical and mental health of school children, generally speaking. Much has been accomplished in this respect and more is on the way. Sanitarily built and cared-for school buildings, school doctors and school nurses are doing their intelligent work. But it is the hygiene of *school instruction*, still very much in its infancy, that presents problems that will engage the best minds for some time to come, problems whose solutions will be welcomed and reduced to practice, it is to be hoped, by every intelligent school man at the earliest possible moment.

Prof. W. H. Burnham of Clark University must be regarded as the pioneer in this country in bringing attention to problems of the hygiene of instruction. His articles on the hygiene of the several school subjects have awakened an interest and contributed considerable illumination.

Many offenses have doubtless been committed against the well-being of children in the name of arithmetic. The overzealousness with which the study has been pursued in times past because of an overestimation of its educational as well as its practical value has been the fruitful cause of undue emphasis upon certain of its aspects without any thought of what were to be the effects upon the mental life in the long run. Proximate aims of efficiency in speed and accuracy of computation have led to heartless and almost mindless drills that often become heartrending and mind-wrecking. The overuse of the purely mechanical, untimely, unpsychologized drill, is the first of the sins to be noted. It is the "practical" sin. As Dr. Burnham (53) says, teachers as a rule take no thought of *secondary* results—the habit of thinking formed, the mental automatisms acquired, the possible arrests of development occasioned by processes that seem at first view to be the acme of pedagogical efficiency. While no one would seriously advocate the abandonment of drill we need to know more about when and how to use it. Along with drill goes the use of certain devices and methods whose abuse or unnecessary use have bad after effects.

Some cases cited by Triplett, Pedagogical Seminary, June, 1905 (quoted by Burnham), are the persistence in consciousness of red and green balls (from the abacus) when adding; of the fitting together of sliced birds (by which device the digits were learned) when thinking of the digits; when not considering number but paying close attention to a subject one person finds herself running up the scale of number by twos, threes or tens; another writes 1, 2, 3, 4 continually; another writes dollar signs by the hundreds; others note a set of mind by years of working for the answer; some who learned the multiplication table in concert are compelled to start at the beginning and go to the combination desired. These are examples of mental automatisms which, to say the least, are annoying, are utterly valueless to the unfortunate subject, and a constant source of interference and confusion in his mental functioning. Some of these automatisms are classed under the head of "arrests of development." Numerical relations fill the mind to the exclusion of more important relations; the subject is truly not sane on this subject, in short, he is a victim of arithmomania. He enumerates everything—counts telegraph poles, boards in the sidewalk, steps in the stairs, trees along the way, etc. And it is known that this mild form of not-saneness may develop in times of mental stress into pathological forms, not to be distinguished from true insanity. The case is cited, which is no doubt typical of many others, of a boy who excelled in speed and accuracy when in the elementary school. In the high school he was found to be incapable of thinking in the higher mathematics—a case of arrested development.

From an educational standpoint, arithmetic has been thought to have a tremendous value as a means of mental discipline. A mental power, especially of reasoning, was developed through its processes that could be applied, it was thought, to other than mathematical fields. This belief has led educators to give a large amount of time to the subject, and to devote much of that time to unessential processes, complex analyses, and obsolete topics. Much of the work in the lower grades has been too difficult. Mathematical puzzles brought down from the middle ages have been frequently given; and while much of this kind of work has been removed from the schools, it is still true that much of the work might be simplified or eliminated. Moreover, the doctrine that the mind may be trained on the basis of any content whatever, mathematical or otherwise, with the expectation that the power gained may be applied to data in another field, however dissimilar, has been found to be largely a myth. We have reviewed the results of the study of transfer elsewhere (pp. 129–141) and of the correlation of different abilities within the field of arithmetic itself, and find no evidence of such a thing as formal discipline in the old sense. One must agree then with Prof. Young (67) that the general training obtained from arithmetic is through its typical exemplification of certain modes of thought; one of these modes is the ability to grasp the facts in a situation. The child, of course, must have the experience necessary to do this, but, given the experience, he must grasp the essential fact in the situation, the guiding principle or, if you will, the major premise. For example, here is a situation: A man can do a piece of work in 6 days, how many days will it take 2 men to do it?

Major premise: 2 men require $\frac{1}{2}$ as many days as 1 man;
Minor premise: 1 man requires 6 days;
Conclusion: 2 men require $\frac{1}{2}$ of 6 days.

Here is another: A dog standing on 1 leg weighs 15 pounds, how much will he weigh when standing on 4 legs?

Major premise (false): A dog standing on 4 legs weighs 4 times as much as a dog standing on 1 leg;
Minor premise: A dog standing on 1 leg weighs 15 pounds;
Conclusion: A dog standing on 4 legs weighs 4 times 15 pounds.

The latter procedure is typical for the child who works without "gumption," in other words, who has no "concept of method." It is undoubtedly true that much practice in seeking and applying the major premise, without which little success can be hoped for, can be given in the problematical portions of arithmetic and the habit thus formed can be carried over. The application of this fact from the hygienic point of view is that children should not be worried with much problem work until they are sufficiently experienced to have acquired the apperceptive means of appreciating situations and sufficiently mature to have the power of grasping the core of the situation presented. It is not meant that any such formal, logical statement as that set forth above shall be used, but rather simple, direct means for getting at the heart of the matter at the start, from which point forward the child must be given the chance to work out his own salvation. The hygiene of instruction in arithmetic is also concerned with the following matters, among others, which can be touched upon only briefly. The time cost. Recent investigations of arithmetical efficiency (Rice, Stone), which we have considered in another place, show that efficiency is not a function of the time devoted to the subject. Hygiene demands that this subject be taught with the smallest time cost consistent with efficiency. All results from studies in the economy of learning are thus seen to be contributions to hygiene, since some of the time saved may be given to health exercises. The public needs to be educated, too, out of its exaggerated esteem for arithmetic growing out of a belief that it is the *sine qua non* of success in life, which has led to an unthinking concession on the part of pedagogues to the occupancy by arithmetic of a too large share of the curriculum.

Some attention should be given to differing mental types. Though this subject is shrouded in considerable mysticism, enough is known about it to show that some children learn arithmetic with ease while others learn it with difficulty. The latter are not on this account to be set down as stupid; they should be differentiated from the others as probably being of an extrospective type to whom quantitative aspects of things do not appeal and for which they have no ability. They should not be harassed with the subject, but allowed to pass with a very minimum requirement. Of more importance, however, are the discoveries of Courtis and others that few children can be ticketed as having arithmetical ability in general consequent upon a blanket type of mind. The ability myth is akin to the "faculty" myth. All sorts of variation of functioning in arithmetic are found in the same child. There are types within types, if we may so apply the term. It would be a matter of economy, then, and therefore of hygiene if he were given the work which he is diagnosed to need, and not the work which for him is time largely wasted.

Hygiene also notes that arithmetic should not be taught prematurely. Burnham says, "Formal instruction at an early age is liable to be injurious. The grotesque 'number forms' which so many children have and which originate in this period are evidence of the necessity which the child feels of giving some kind of bodily shape to these abstractions which he is compelled to study. Mathematics in every form is ill fitted to the childish mind; it involves comparison, analysis, abstraction. It calls for a fuller development of the association tracts and nerve-fibers (Patrick). Where arithmetic is not studied until 10 it is often found that there is no loss but rather an advantage" (53, p. 62). It is undoubtedly true that *formal* instruction should not be given at too early age, chiefly, however, because the time is more needed for other things. The reasons given by Burnham are not convincing. Comparison, analysis, abstraction are not characteristic of primary instruction, but perception and memory are in the pure number relations which only he may be called upon to deal with. Number forms are not "grotesque" except from the point of view of the layman. No harm has ever been shown to come from them, but on the contrary much good. In any event, as we have found in our study of the genesis of number, they are probably unavoidable, even without formal instruction, since they seem to take their rise in the prenaming stage when the counting impulse has an insufficient motor outlet.

That arithmetic may be deferred to the age of 10 or thereabouts without disadvantage is by no means proved. We may agree that problematical arithmetic may be deferred, but, as Hall concludes, it is probable that many kinds of work can profitably and safely be done early without transcending the natural interests and abilities of children.

As was intimated at the beginning, valuable results in hygiene can be obtained upon only an experimental basis. Dr. Burnham enumerates some investigations in pedagogy now going on which will contribute results of value in part to hygiene. The association processes involved in arithmetical operations; the ability of different mental types in such operations; the normal variations of ability with age and sex; the variations in individuals; the value of different methods with relation to different types and forms of ability, the value of discipline in this subject in relation to other school subjects; the value of drill; the secondary effects of different methods of learning and instruction. Some of these investigations we have reviewed elsewhere.

4. Didactical Studies.

The special problem in experimental didactics to which we must confine ourselves in this presentation of work that has been attempted is that of the material and method to be used in presenting to children things to be numbered. Psychological studies, such as we have reviewed (Messenger, Burnett, Cattell, Dietze, Warren, Nanu), as well as the observations of infants by Preyer, Fries, Jegi, and others, and the genetic studies of Phillips, to which we have referred, point to the conclusion that there is no original grasp of the number of impressions simultaneously or successively presented, but that apprehension is a matter of gradual growth and therefore may be facilitated or hindered by outer and inner conditions.

The apparently immediate apprehension of the number in a group, which children display even under the most unfavorable circumstances and which so greatly facilitates computation, is something worth developing; how best it may be developed is the question.

The beginning of the solution (by a trial and error method of experimentation) dates back to Pestalozzi. He had the insight that number in the percept is coördinate with *word* and *form*, that no percept is complete until the numerical relations of the *Anschauungdinge* are known. He, therefore, emphasized the objective aspect of beginning arithmetic and furthermore devised for the practice of his pupils his *Strichtabelle* (stroke-tables), which were the forerunner of the modern number pictures. To these we can take the space merely to refer, as they now possess only an historical interest. (For description and criticism of them, see

Meumann (57), pp. 335–338.) This artificial means of instruction presupposed, in Pestalozzi's thought, not only teaching with objects so far as that might be necessary, but also the unguided acquirement by intelligent children of number notions from their own experience with objects and events. "His art aims at bringing back the conceptual content into the artificially made perception material, in connection with which the artificial means have the advantage from a didactical point of view to be much more practically and extensively available than the natural objects by which the first number ideas were gained." (H. J. Walsemann's Pestalozzi's *Rechen-methode*, quoted by Meumann, Vol. 2 (57), p. 334.)

Since Pestalozzi's time a variety of artificial means has been evolved on the Continent, chiefly in Germany; number pictures arranging dots in two parallel horizontal rows by Born, Busse, and Boehme, arranging dots vertically by Hentschel, Beetz, Sobelewsky, and Kaselitz; also a variety of reckoning machines, largely with the vertical arrangement, but a few like the Russian with the horizontal arrangement. In connection with all of these there has been much discussion and speculation, with but little genuine experiment. The value of number pictures appealing to the spatial element of the number consciousness has been denied entirely by a school of pedagogues who emphasize the temporal element and would practice the child in counting exclusively, or at least in the numerical recognition of impressions presented successively. The classic controversy into the details of which we cannot go thus arising between the *Zählmethodiker* and *Zahlbildermethodiker* has been disposed of by Meumann, we think rightly, but has, further, been made a subject for experiment by Lay, who reaches the same conclusion. (See our review of Lay's *Führer durch den Rechenunterricht*, Leipzig, 1907, pp. 155–199.) Meumann (57, pp. 338–334) says substantially: There can be doubt that in the image and in the concept, both spatial and temporal elements are involved, so we conclude that both methods are one-sided and that with each of them the child has somewhat to supply from its own power. With a pure counting method the child generally would not attain to a simultaneous apprehension of a definite number of objects in space; on the other hand the simultaneous presentation involves for the little child a number of objects in space presented successively through a corresponding number of separate perception acts and the manifold presentation of these objects is for the childish mind in the beginning a repetition of equal or similar impressions in time. Witness the observations of Preyer and others that the first definite number concepts of the child develop seemingly from counting which has the form "one, yet one, yet one." Moreover, with a pure simultaneous method the understanding of the number order would be lacking; also the elementary and immediate opportunities for the attainment of concepts through simple temporal relationships, namely, the temporal occurrences of bodily and limb movements and the ability to use the temporal element when his simultaneous grasp fails. Therefore it is clear that only by the correct combination of both methods can the essence of number representation and processes be made clear to the child. Moreover, the ideational type plays a rôle here. A child of the visual type is favored by the number-picture method; on the other hand, the auditory type is suited to the counting method but handicapped by the pictures. *It would appear that to counting must be given a subsidiary place and one pertaining mainly in any good didactic to primary phases of apprehension. Through proper grouping, the perception of number pictures must be admitted, whatever may be one's explanation of it, to be capable of wide development.* Sufficient experimental work has been done and observations made to establish this. Even if a number group must be counted out, the simultaneous apprehension very soon follows and through practice the latter can be so established that the counting becomes eliminated; the number picture then replaces the counting and causes an entirely extraordinary ease and simplicity of perceptive reckoning (Meumann). (Lay would not entirely indorse this account of the growth into function of the number picture, but we will let his view appear in its proper place).

The exact province of perceptional apprehension and reckoning can perhaps best be indicated by quoting some further words of Meumann (57, p. 357). "The elementary arithmetical operations become to the adult completely familiar only if they take on an absolutely mechanical character, i. e., psychologically speaking, if they can be reproduced as pure successive word associations without any perceptional content. No merchant adds a long column of figures in his book while he has the visible group of just as many circles or strokes before his eyes; should he have them before his eyes he would progress like a snail; with small and large multiplications the adult calculates by pure mechanical series-association (reproduction) of number words. We add to this that all the larger numbers remain to people permanently unperceptional, pure, abstractly conceived large numbers. Therefore the perceptional reckoning can be overstepped. It had its special significance for the certainty and clearness of the first foundation of instruction in arithmetic; but should it be retained after the first operations become familiar to the child and extended even to operations which are developed out of this elementary work, then it will operate retardingly and hold back the natural development of arithmetic which moves toward work with the abstract number and with mechanical association and reproduction."

The value of the number picture being granted, the question remains (and this is our special problem) what form of it do children most readily apprehend and employ?

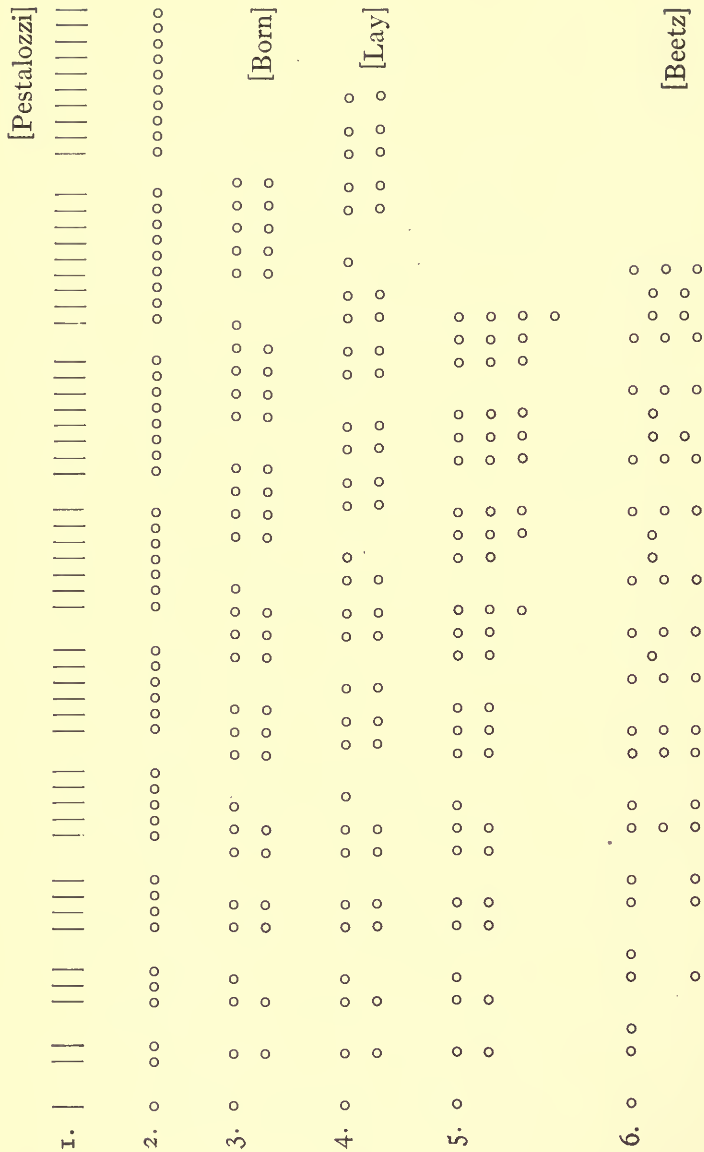
To the solution of this question the principal experimental contributions have been first by Lay (*Führer durch den ersten*

Rechenunterricht, Wiesbaden, 1898). With a much improved experimental method H. J. Walsemann repeated these experiments (*Anschauungslehre der Rechenkunst auf experimenteller Grundlage*, Schlesswig, 1907) and added others. Lay again presented a revised and enlarged edition of his *Führer* in 1907, incidentally replying to Walsemann, Knilling and some of his other critics. Lay's book is the most thoroughgoing presentation of the experimental didactics of elementary arithmetic extant and to it we shall give considerable space, including some experiments by Knilling, his critic (Knilling, *Kritik zu W. A. Lay's experimenteller Forschungsergebnissen. Päd.-psych. Studien III. 11.12*), and incidentally referring to some results of Pfeiffer, Schneider, and Walsemann.

Walsemann (56) tried the pictures in the table on page 154.

The table shows in the first and second row one-row groups, in the third and fourth two-row groups, in the fifth, from *seven* on, three-row groups, and in the sixth arbitrary groups. The one-row circle forms of the second row is the so-called Russian calculating machine essentially. The two-row material of the third row has, between the pairs of points, equal distances (normal number picture) which is exactly the same as the fourth row except that the latter, after two pairs of points, has a greater distance (quadratic number picture). These constructions of perceptional material Lay (1898) had tried with the exception of row 5. He had found that the one-row arrangement (rows 1 and 2) is unfavorable; also that the arbitrary grouping (row 6, Beetz number picture) was unfeasible. The most practical arrangement according to Lay (1898) was the two-row arrangement (a) with the equal distances (so-called normal number picture) and (b) with unlike distances (row 4, quadratic number picture); the latter slightly more practical.

Walsemann found next by a comparison of the stroke-formed elements (Pestalozzi) and the two-row circle forms a sixteen-fold


1.
[Pestalozzi]
2.
3.
[Born]
4.
[Lay]
5.
6.
[Beetz]

superiority of the latter. Then he compared the apprehension of the two-row and the three-row circle-form elements. The two-row (normal number picture) showed about a one and a half fold superiority. Next he tested with a new research arrangement (a machine of his own make) the normal and the quadratic number-pictures; first, whether the one-row picture of *four* and *five* can be grasped still more certainly [than in former tests] in two training classes and one school class. (Lay had claimed that this should be established more positively.) Walsemann found that the one-row dot-picture of *four* can be determined with a very high grade of certainty. The picture of *five* was less certainly determined. So then he investigated how the numbers from *five* to *twelve* can be grasped using normal and quadratic pictures (in all 1728 trials). It resulted that with the normal number picture 61 errors (6.8%) were made; with the quadratic 84 errors (9.4%). According to this the result of Lay should be changed in favor of the normal pictures. (Lay claims (*Führer*, 1907) that Walsemann did not use his quadratic pictures at all in this experiment; see page 178.) Still a greater superiority of the normal pictures shows itself in tests in division with the help of circle-form pictures. Through this result, Walsemann concludes, the small balance in favor of quadratic material is removed to the opposite side and not inconsiderably increased.

In the following account of the experiments of W. A. Lay, taken from his *Führer*, Leipzig, 1907, pp. 76–141, I have taken the liberty to make a few insertions for the sake of clearness, or correction [indicated by brackets], to correct some typographical errors, and to change the form of the tables so as to make them perhaps more quickly intelligible, but chiefly in order that percentages might be shown upon which to base comparisons with the results of my own experiments. I have also mentioned briefly what in a few instances seem to be mistaken inferences from his results.

Stufe der Experimentell-didaktischen Untersuchung.—
Die Experimente und die Deutung ihrer Ergebnisse.

1. **Concerning the numerical apprehension of temporal things through hearing and sight.**—Everyone knows from experience that both spatial and temporal things are accessible to numerical apprehension. This consideration brings us immediately to a problem, which the methodist instinctively has left aside, although it is not without significance: Is the succession or the simultaneity of things, are things in space or in time to be regarded as the most advantageous intuitive or counting material for the first number teaching?

In order that an insight might be obtained in the numerical apprehension of temporal things through hearing, an experiment was instituted with several kindergarten children in age from 4 to 6.

Experiment.—The experimenter knocked 2 or 3 or 4 times in a second on the table in such a manner that the children heard the knocks but could not see them.

The children were required to repeat the knocks as accurately as possible.

The research shows these results:

In the cases of 2 strokes, still more in the case of 3 and 4, mistakes occur, many more than when 2, 3 and 4 things, arranged in a spatial series, were apprehended through the eyes though a shorter time was used.

It was noticed that the apprehension of the sound series became easier if rhythm was brought into the series.

Observations on the teaching practice of the first school year and child psychology show that the apprehension of a temporal succession affords considerable difficulty to children. Lay on these and other grounds takes the position that in general the forming of the number image takes its rise not from temporal things and through hearing, but from spatial things and through sight; and ventures to assert that the first number teaching should not be based on the sense of hearing but preëminently on the sense of sight and touch.

Whereby it is not asserted that the numerical apprehension of temporal things and auditory perception should be neglected; systematic number teaching must give these things a fair share of attention.

2. **Concerning the limits of apprehension of things in rows.**— How many spatial and temporal things in a row, for instance, how many balls in the Russian machine can children of the first school year simultaneously visually grasp and represent? The methodist has sought to answer these questions on the ground of observations of adults or according to accounts of a racial psychology, and their answers have differed. At times they have set the limits at 2, 3, 4, 5, or 6. Recent accounts of primitive people show that their limit lies at 2 or 3.

"We must naturally expect that we can attain to no sharply defined answer through our experiment. Children with differing abilities will differ. We disclaim such an answer as our purpose."

Experiment.—The balls of the Russian machine were used.

Diameter = 4.5 cm.; color, yellow. After the 5th ball a space was left. Between the smaller numbers 2, 3, 4 the larger 5 and 6 were inserted, thus breaking up the succession.

TABLE

First School Year — 46 Pupils
Time of Exposure = ½ Sec.

Number of Balls	Number of Presentations	Average Number Mistakes
2	3	4
3	4	6
4	6	9
5	7	17
6	2	25

Three pupils had 0 mistakes; 8 pupils had 1 mistake.

[Note.—I have assumed that the averages given above are correct, but there are several errors in Lay's tabulation.]

In all research results a small number of mistakes due to defective attention occur. Lay sought to avoid these as much as possible, but it was impossible entirely to avoid the fluctuation of attention in individual pupils and in the class.

The attention is dependent on temporary physical condition and content of consciousness, which through inner and outer causes of an often insignificant and uncontrollable kind are changed every moment more or less. These mistakes are traceable to such "fluctuating errors;" the experiment was made after the numbers during half a year on the same machine had become "*veranschaulicht*," and it was observed that it was not always the same three pupils whose apprehension of two balls was not successful. Lay assumes that 3 mistakes in the experiment came from accidental circumstances, so the pupils are left 3, 6, 14, 22 (mistakes) [instead of 6, 9, 17, 25, see table] in the apprehension of 3, 4, 5, 6 balls respectively.

Lay regards the mistakes made with 2 balls as mistakes of fluctuating attention for reasons stated above, and so deducts 3 from his results (total mistakes), assuming that the remainder are genuine mistakes of apprehension. Since 3 of 46 pupils, after a half year's teaching, are able to apprehend in a row not more than 3 balls and to represent (the same) by means of dots after the perception; and since 6 pupils display the same inability relative to 4 balls, therefore, from the standpoint of teaching practice:

For these school classes the number 3 marks the limits of apprehension for rows of balls.

3. **Didactical experiments concerning counting and the counting method.**—On the basis of an extensive experimental research the *Führer* (1904) arrived at the following conclusion: The gen-

erally accepted and defended contention of most methodists, "the number comes about only through counting," is false. It is not unexpected that the counting methodists and their disciples should hesitate to admit that they are found in a grave error. But it is necessary here to emphasize that their attacks appear ill-advised since they put little confidence in the experimental-didactical research and did not give themselves the trouble to read the *Führer* attentively. Two counting methodists, Dr. Hartmann and Supervisor Knilling, have busied themselves with the *Führer* exhaustively. Knilling is the only counting methodist who has investigated the problem in an experimental way. Unfortunately his procedure was mistaken and we will show only briefly that his experiments are inadequate to refute the *Führer*.

[I quote here the first two of three experiments.]

Knilling. Exp. I.

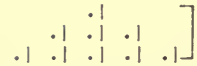
An adult was required to place next to a dot and under it (preserving on the whole a triangular form) in a horizontal direction first 2, then 3, then 4 dots [thus, ∴ .∴ ..∴.]. As a result, Knilling records: the person solves the question in regard to the total number of recorded dots with difficulty. This is easily conceivable:

(1) The dot groups used are wholly inappropriate for numerical apprehension. They are not at all didactically useful number pictures.

(2) Each was given in successive parts, not as a whole.

(3) The subject does not know that he must determine and reproduce the group as a total picture; the *einstellung* of the senses and of the consciousness for abstraction, analysis and synthesis, as the number apprehension demands them, are not at hand.

Knilling. Exp. II.

Six boys of the 3d school year were required to place strokes next to dots; first next to 1, then to 2, then to 3, then to 2, then to 1, [thus: ]. The pupils were required and were able "to mark again from memory" this group of 9 dots in the same way. Immediately, according to the tally, the total was asked for.

Result.—Knilling had to hear "a pair of false answers" and the abler pupils confessed besides that they had to reckon out the number in the head.

This experiment also missed entirely the already named conditions. The pupils said themselves that they had not comprehended the *Setzungen;* that to them the numerical comprehension 9 arrives subsequently out of the image speaks for, not against, the *Zahlbilder*.

Lay's experiments to show that the claim that number arises only through counting is false (p. 97 ff. book) follow:

In order to obtain information concerning the apprehension of quadratic pictures through children who are not yet school-bound, Lay instituted his experiments with children of the kindergarten. He saw beforehand that in this case it was best not merely on outer but also on inner grounds to use not mass (class) but individual experiments. He used in these experiments little sheets, which he could cover with the hand. On these were placed the quadratic pictures with ink. The circles had a diameter of $\frac{1}{2}$ cm.; the distance was equal to the diameter and the intervals between the quadrats $1\frac{1}{2}$ diameters. The pictures were exposed by removing the hand and after $\frac{1}{2}$ to 1 second were covered again. The children were told that they should draw with a pencil on the slate lying ready the dots which they came to see, and that the point at issue is "how many." (Four out of eight experiments are here given.)

Lay. Exp. I.

A girl 6 years old, 3 years in the kindergarten, after a single exposure, drew the exhibited pictures, 3, 4, 5, 6, 7, 6, 10 correctly from the image. Before drawing the 7 the child was asked concerning the number of dots. But the child knew not how many dots there were yet . . . drew them correctly. It had therefore the clearly defined image but not its name. Twelve after 5 exposures was not correctly symbolized; 8 and 10 dots were made, but many dots were placed one after another in varying ways.

Lay. Exp. II.

A boy 6 years old, after two exposures, grasped the number picture 7. He did not know the number name but found it by counting out of the image and then wrote it correctly under the picture. The picture 10 he could not count out of the image but he drew it correctly, not knowing afterward how many he had drawn until he counted with a loud voice the dots in his picture. The picture 12 required 3 exposures. The number name was unknown, nevertheless the boy recorded the 12 correctly. He had obtained a clear image without counting or number words. The picture 13 he recorded incorrectly after 7 exposures; he drew 12 or placed as many dots as possible one after the other.

Lay. Exp. III.

A boy 3–4 years old, able to count to 10, one year in the kindergarten, for 3 made the first time merely a dot, the second time 3 dots in a row; did not know the number name. Four and 5 were correctly represented, 6 after 5 exposures (1 to 2 sec.) was correctly recorded, without knowledge of the name.

Lay. Exp. IV.

A girl 4 years old, 3 weeks in the kindergarten, was able to record from memory 2, 3, and 5 after four exposures and 4 after two exposures. Five was at first counted as 6; at a second exposure she counts only 4; but at the third she counted 5 from the image.

These experiments with children who had received as yet no instruction are otherwise interesting and important; they show:

(a) That the numbers presented by means of the quadratic pictures can become easy for not yet school-bound children, firmly grasped without practice, impressed on the memory, imaged, and symbolized by drawing;

(b) That the number images embrace the entire first ten numbers, the foundation of elementary arithmetic, and even reach above the same, while the row images reach only 3;

(c) That clear and living images without counting in the usual sense come about, persist, and can be symbolized in drawing. Very clearly is this shown in a case where the child can only count to 4, but grasped 6 unities clearly, imaged and symbolized them in drawing from memory (Erinnerung).

4. **Comparison of row pictures and the number pictures of Born.**—The purpose of this experiment is to compare the visual apprehension of a number of circles arranged in a single row presented simultaneously, with that of an equal number of circles arranged in a group by double rows. [For example, 4 was presented at one time, thus: · · · ·; at another time, thus: ⁚ ⁚; five · · · · · was compared with ⁚ ⁚ ˙, etc.]

The conditions of the experiment for the comparative presentations were made the same in the following way. For each number a picture was presented on a separate sheet of paper, one picture of the number being according to the single row arrangement, the other the double row grouping of Born, the same kind of paper being used for all the sheets. The surfaces of the circles were blackened with India ink. The diameter of the circles was 46 mm. and the distance apart equal to the radius. In the single row pictures after 5 the distance between circles was made equal to two radii.

A wooden blackboard was placed on the desk, on which was fastened behind a screen a row or a group picture. With the ticks

TABLE.—REIHENBILDER VS. GRUPPENBILDER (BORN). THE PICTURES ARE RECORDED IN THE ORDER OF PRESENTATION

	First-year Pupils Time, 1 Sec. (1st presentation) Exp. I (44 pupils)				First-year Pupils Time, 1 Sec. (2d presentation) Exp. III (33 pupils)				First-year Pupils Training School Time, 1/11 Sec. Exp. IV (30 pupils)				Second-year Pupils Training School Time, 3/8 Sec. Exp. V (21 pupils)			
	Row Pictures	Mistakes	Born Pictures	Mistakes	Row Pictures	Mistakes	Born Pictures	Mistakes	Pictures	Mistakes	Born Pictures	Mistakes	Row Pictures	Mistakes	Born Pictures	Mistakes
	3	4					5	12			4	2			9	8
			4	5	9	26			5	1			10	5		
			3	4			3	0			5	1			10	2
	4	18					6	5	4	5			9	11		
			5	7	6	18			7	10			8	7		
	5	19					9	19			6	0			7	4
			6	9	5	17			6	15					8	1
	6	28					4	1			7	2	7	7		
	Exp. III (33 pupils)						8	10			8	1			5	1
			7	11	4	8			8	13			6	4		
			6	7	7	21					9	6			6	0
	6	22					7	15	9	14			5	1		
			8	7	8	27							3	0	4	0
	9	23			3	0										
	7	23											4	3		
			9	20											3	0
	8	22														
Total Presentations	208		208		231		231		180		180		168		168	
Total Mistakes		159		70		117		62		58		12		38		16
Per cent		.764		.336		.506		.268		.322		.067		.226		.095

of the metronome the operator counted twice "1, 2" beforehand. At the count of 1 following, the screen which covered the picture was dropped as far as necessary and at the count of 2 following, was raised again to cover the picture. The pupils now recorded what they had seen. After a pause, the same thing was repeated with another picture. The order of presentation is given in the accompanying table (page 163). The pupils were occupied at a time one hour at the most. A preliminary practice experiment was given to remove disturbing elements at the beginning of an experiment.

Although the children had had a half year's instruction, both kinds of pictures used in this experiment were unfamiliar to the pupils of the first school year, except that for the numbers 1 to 6 they had had presented to them the number pictures of Böhme in which 4 and 6 are the same as in the *Zahlbilder* of Born; their practice besides this was with the Russian machine and their fingers. [For number picture of Böhme, see Appendix VIII.]

Additional series of pictures were presented as follows:

A. 21 Second-year Training School Pupils

Time = 3/7 sec.

Numbers 6 to 10, each presented to class once.

Row pictures—Presentations...........	105	Mistakes..	28
Born pictures—Presentations...........	105	Mistakes..	9

B. 24 Second-year Training School Pupils

Time = 5/11 sec.

Numbers 6 and 7, each presented to class once

Row pictures—Presentations...........	48	Mistakes..	18
Born pictures—Presentations...........	48	Mistakes..	7

Summary:

From table: Row pictures—Presentations.	787	Mistakes..	372
Add: Row pictures—Presentations......	153	Mistakes..	46
	940		418[1]
From table: Born pictures—Presentations.	787	Mistakes..	160
Add: Born pictures—Presentations.....	153	Mistakes..	16
	940		176

[1] [Lay has incorrectly 408 mistakes.]

Conclusion.—The results of these experiments show undeniably that the Born group pictures are superior to the after-5-interrupted row pictures as *Anschauung* or counting material. Other experiments which follow and pursue other purposes will without exception directly or indirectly confirm these results.

5. **Comparison of Born pictures with the other pictures in common use.** The Born pictures will be used in the comparative experiments [of the next section (6)] because they are distinguished by great superiority in comparison with other pictures [in common use].

1. They are all without exception constructed according to a single fundamental plan, the second dot comes vertically under the first, the third to the right above near the second, the fourth vertically under the third, the fifth to the right above near the fourth, etc. 2. All smaller pictures appear in unchanged form as constituents of all the larger. 3. The process of apprehension is in all pictures equal and simple; the apprehension, the reproduction of the image, the written or bodily symbolization of all pictures are equally simple and easy. 4. Each picture illustrates the several calculations in similar, simple ways and the results, as already impressed pictures, can immediately be determined. The reader may convince himself of the correctness of these statements regarding the Born pictures [by an inspection] and after this compare with them the several pictures of Böhme, Hentschel, Sobelewsky, Kaselitz, Beetz on the basis of the points named, especially relative to the processes of division, addition, subtraction and multiplication. It is important that the reader carry out this comparison. [For the pictures, see Appendix VIII.]

6. **Comparison of the Born and quadratic Zahlbilder.**—The result of the research concerning the limits of the apprehension

for rows of balls, which agrees with the account of Beetz, leads to the following preliminary observations:

The Born number pictures are double rows, but still rows;

TABLE

VERGLEICHUNG DER BORNSCHEN UND

	First-year Pupils Time, 1 Sec. Exp. I (29 pupils)				First-year Pupils Training School Time, $\frac{5}{11}$ Sec. Exp. II. (31 pupils)				Second-year Pupils Training School Time, $\frac{5}{11}$ Sec. (1st presentation) Exp. III (16 pupils)			
	Born Pictures	Mistakes	Quadratic Pictures	Mistakes	Born Pictures	Mistakes	Quadratic Pictures	Mistakes	Born Pictures	Mistakes	Quadratic Pictures	Mistakes
	8	6			8	4			8	0		
			5	6			5	1			6	1
			9	17	10	0					8	0
	10	14					9	4	6	1		
	5	6			5	0			7	0		
			7	7	10	4			10	4		
	6	2					7	2			7	2
			10	14	6	1					10	4
	7	12					10	0	9	3		
			6	4	7	0			5	0		
	9	19					6	0			9	3
			8	3	9	4					5	1
							8	1				
Total Presentations ..	174		174		217		186		96		96	
Total Mistakes		59		51		13		8		8		11
Per Cent............		.339		.293		.060		.043				

since the apprehension of a row of balls extends to 3 only, so the Born pictures which show more than 3 pairs, that is to say, the pictures 7, 8, 9, 10 cannot be any more surely grasped, imaged and symbolized [than pictures above 3 in single rows]. An arrangement of double rows in groups of 3 pairs would be of doubtful value since the 3 cannot be grasped with facility by certain individuals even after they have had a half year of schooling. Much simpler and more reliable is the arrangement in groups of 2 pairs in quadratic pictures. The class can apprehend 3 balls, circles, squares, etc.; if now one imagines

der Quaratischen Zahlbilder

Second-year Pupils Training School Time, $\frac{3}{8}$ Sec. (2d presentation) Exp. IV (16 pupils)				Third-year Pupils Training School Time, $\frac{5}{12}$ Sec. (1st presentation) Exp. V (38 pupils)				Third-year Pupils Training School Time, $\frac{5}{12}$ Sec. 2d presentation Exp. VI (38 pupils)				Third-year Pupils Training School Time, $\frac{3}{8}$ Sec. (3d presentation) Exp. VII (38 pupils)			
Born Pictures	Mistakes	Quadratic Pictures	Mistakes	Born Pictures	Mistakes	Quadratic Pictures	Mistakes	Born Pictures	Mistakes	Quadratic Pictures	Mistakes	Born Pictures	Mistakes	Quadratic Pictures	Mistakes
8	1			5	2					8	1	7	5		
		6	1			9	4			6	1			10	4
		8	0			5	0	8	1					7	1
6	1			9	10			6	0			10	1		
7	1					10	9	7	3					9	3
10	1					7	10	10	12			8	2		
		7	1	10	13					7	5	9	4		
		10	0	7	5					10	9			8	0
9	1			6	0			9	7						
5	0					8	1	5	0						
		9	2			6	3			9	5				
		5	0	8	0					5	0				
96	5	96	4	228	30	228	27	228	23	228	21	152	12	152	8

Two additional series of pictures were presented to first-year pupils, not included in table above, with the following results:

B Presentation 150 Mistakes 94
Q Presentation 150 Mistakes 67

Summary

From table above.... Q Presentations....	1,160	Mistakes....	130	
Add.................Q Presentations....	150	Mistakes....	67	
	1,310		197	Per cent., .150.
From table above.....B Presentations....	1,191	Mistakes....	150	
Add.................B Presentations....	150	Mistakes....	94	
	1,341		244	Per cent., .182.

the corner dots (represented as balls) of 3 neighboring quadrats, it is conceivable that in this form 3 times 4 unities can be grasped.

These quadratic pictures maintain in contrast to the Born a wholly new character. The question is whether the foregoing *a priori* conclusion proves itself as a matter of fact correct. This can be decided only by experiment.

The *Auschauungsmittel* (materials) which were selected as the basis of the experiment were black circle surfaces on white drawing paper, 22 mm. in diameter and a radius apart, and the quadratic pictures prepared in corresponding ways; in their case the intervals between each quadrat was made equal to a double radius: It is necessary to admit that psychological conditions for the two kinds of apprehension in the experiment are different. The Born pictures are at a great advantage over the quadratic pictures. In a previous experiment they have been repeatedly apprehended by the pupils, besides they have a pronounced row character and the row apprehension has been much exercised; the apprehension of quadratic pictures above 4 is as yet entirely new and only above this does Lay extend comparative experiments. A comparatively small balance in favor of the quadratic pictures will therefore show a considerable superiority of the same. (For the quadratic number pictures of Lay, see Appendix VIII.)

Conclusion.—Considering the advantage which the Born pictures in the experiments possessed, the quadratic pictures are considerably superior to the Born. [See table with summary, pp. 166, 167.]

7. **Comparison of the Beetz and the quadratic pictures.**—Beetz did not use the customary number pictures but in his *Zahltypen* he constructed new pictures which according to his opinion answered all demands. But the comparison of the Born pictures with other pictures [in common use, see sec. 5 above] brought the conclusion that the Born pictures show great advantage over these, including those of Beetz. But Beetz is of the opinion that

his pictures surpass all others in usefulness. It was, therefore, important to show how the matter stands as to the relative apprehensibility of the Beetz and the quadratic pictures.

Beetz pictures were prepared in a way analogous with the already used quadratic pictures, both in a larger form with 33 cm., quadratic side, and in a smaller form 19 cm., quadratic side.

It should be noted that in the first experiment on second-year training school pupils, 19 (of the 28) incorrectly represented (pictured) the Beetz pictures. This was not counted a mistake (as long as the correct number of dots was represented), but before the second experiment was tried the several Beetz pictures were held up so that all could take a good look. These pupils confessed that after they had understood the composition of the pictures they no more sought to grasp the pictures as a whole, but having noted only the points in the middle, they then in their representation filled in the outline. Such a partial grasp has never taken place in the quadratic pictures; from the very first they produced much fewer mistakes than the Beetz pictures in this experiment.

Conclusion.—The quadratic pictures are significantly superior to the Beetz, indeed still more than the figures of the table (p. 170) show.

8. **Comparison of quadratic pictures, of Born pictures and the Russian machine.**—On the Russian machine first the yellow and then the black balls were used in experiments with second-year training school pupils. They had a diameter of 4.5 cm.; there was no background. After the fifth ball was left a small space. The Born pictures were presented on the Wendling apparatus. The gray pasteboard disks had a diameter of 14 cm. and were 10 cm. distant from one another in a horizontal direction and 16 cm. in a vertical. Since we know that the quadratic pictures are superior to the Born, the comparison of the Wendling apparatus with the Russian machine is an indirect comparison of the quadratic pictures with the latter.

TABLE

VERGLEICHUNG DER BEETZCHEN UND DER QUADRATISCHEN ZAHLBILDER

	First-year Pupils Training School Time, $\frac{5}{11}$ Sec. 1st Presentation (30 pupils)				First-year Pupils Training School Time, $\frac{5}{11}$ Sec. 2d Presentation (30 pupils)				Second-year Pupils Training School Time, $\frac{5}{11}$ Sec. 1st Presentation (28 pupils)				Second-year Pupils Training School Time, $\frac{5}{11}$ Sec. 2d Presentation (28 pupils)				First-year Pupils Elementary Schools Time, ? 1st Presentation (34 pupils)				First-year Pupils Elementary Schools Time, ? 2d Presentation (24 pupils)			
	Beetz Pictures Larger Form	Mistakes	Quadratic Pictures	Mistakes	Beetz Pictures Larger Form	Mistakes	Quadratic Pictures	Mistakes	Beetz Pictures Smaller Form	Mistakes	Quadratic Pictures	Mistakes	Beetz Pictures Smaller Form	Mistakes	Quadratic Pictures	Mistakes	Beetz Pictures Smaller Form	Mistakes	Quadratic Pictures	Mistakes	Beetz Pictures Smaller Form	Mistakes	Quadratic Pictures	Mistakes
			7	1	10	1					7	5			7	6	8	27			8	22		
	6	1					9	4	9	27			9	15					5	10			5	6
	7	1			5	1					8	0			8	2	10	29			10	21		
			6	1	8	3			6	1			6	0					6	9			6	3
			5	1			10	3			9	8			9	1	5	17			5	12		
	5	0			9	6			7	9			7	3					10	20			10	16
			8	1			8	0			10	9			10	5	7	26			7	16		
	9	10			7	4	5	0	5	2			5	0					9	22			9	19
			10	4			6	1			6	2			6	1	6	8			6	4		
	8	5			6	0			10	13			10	1					8	10			8	10
			9	5			7	4			5	0			5	0	9	27			9	19		
	10	3							8	5			8	3					7	17			7	11
Total Presentations..	180		180		180		180		168		168		168		168		204		204		144		144	
Total Mistakes.....		20		13		15		12		57		24		22		15		134		88		94		65
Per cent..........																								

Two additional series of pictures were presented to the first-year training school pupils, not included in the table above, with the following results:

SUMMARY		Be	Presentations 372		Mistakes 33		
		Q	Presentations 372		Mistakes 22		
	From table above....	Be	Presentations....	1,044	Mistakes....	342	
	Add...............	Be	Presentations....	372	Mistakes....	33	
				1,416		375	Per cent........ .265
	From table above....	Q	Presentations....	1,044	Mistakes....	217	
	Add...............	Q	Presentations....	372	Mistakes....	22	
				1,416		239	Per cent........ .169

Results.—Comparison of Russian machine with quadratic pictures:

Exp. 1.—First-year training school pupils (30), used as subjects. In this experiment the quadratic pictures were presented on the machine. Time of exposure not stated.

R–P 120	38 mistakes	(Numbers 6 to 9, presented once in each series, in irregular order.)
Q–P 120	8 mistakes	

Exp. 2.—Subjects, second-year training school pupils (22). Time of exposure, $\frac{3}{8}$ sec.

(a)	R–P 154	55 mistakes	(Numbers 4 to 10, presented once in each series in irregular order.)
	Q–P 154	12 mistakes	
(b)	R–P 110	yellow, 20 mistakes	(Numbers 4 to 8, presented once in each series, in irregular order.)
	–P 110	black, 26 mistakes	
	Av–P 110	average, 23 mistakes	
	Q–P 110	3 mistakes	

Summary:

R	Presentations 384	mistakes 116
Q	Presentations 384	mistakes 23

For results of comparison R and W see table on page 172.

Schneider's work (*Die Zahl im grundlegen Rechenunterricht*, Berlin, 1900) contains additional proof of this research. He finds,

R	460 mistakes
Born	71 mistakes

Conclusion.—The results show that the quadratic pictures surpass the Born and are vastly superior to the Russian machine; also that the apprehension of things in rows affords far more difficulty than the apprehension of things in groups.

TABLE

VERGLEICHUNG DER BORNCHEN ZAHLBILDER (WENDLENGSCHER APPARAT) UND DER RUSSISCHEN RECHENMASHINE

	First-year Pupils Elementary School Time. 1 Sec. (53 pupils)				First-year Pupils Training School Time, $\frac{5}{11}$ Sec. 1st Presentation (31 pupils)				First-year Pupils Training School Time, $\frac{5}{11}$ Sec. 2d Presentation (31 pupils)				Third-year Pupils Training School Time, $\frac{5}{12}$ Sec. 1st Presentation (38 pupils)				Third-year Pupils Training School Time, $\frac{5}{12}$ Sec. 2d Presentation (38 pupils)				Third-year Pupils Training School Time, $\frac{3}{8}$ Sec. 3d Presentation (38 pupils)			
	Russian Machine	Mistakes	Wendling Apparatus	Mistakes	Russian Machine	Mistakes	Wendling Apparatus	Mistakes	Russian Machine	Mistakes	Wendling Apparatus	Mistakes	Russian Machine	Mistakes	Wendling Apparatus	Mistakes	Russian Machine	Mistakes	Wendling Apparatus	Mistakes	Russian Machine	Mistakes	Wendling Apparatus	Mistakes
	7	24			7	3					5	0	5	2					8	1	7	5		
			10	25			10	0			8	0			9	4			6	1			10	4
	4	3			4	0					7	0			5	0	8	1					7	1
			6	5			6	0			6	0	9	10			6	0			10	1		
	8	21			8	3					4	0			10	9	7	3					9	3
			5	3			9	2	6	1					7	11	10	12			8	2		
	5	14			5	1			8	7			10	13					7	5	9	4		
			9	18			5	0	7	5			7	5					10	9			8	0
	9	19			9	5			5	1			6	0			9	7						
			8	13			8	0	4	0					8	1			5	0				
	6	15			6	1									6	3			9	5				
			4	8			4	0					8	0			5	0						
	10	17			10	2																		
			7	11			5	0																
	5	12																						
			9	19			7	0																
	9	19																						
			5	9																				
Total Presentations .	477		477		217		248		155		155		228		228		228		228		152		152	
Total Mistakes		144		111		15		2		14		0		30		28		23		21		12		8
Per cent...........																								

SUMMARY: R Presentations 1,457 Mistakes 238 Per cent. .163
W Presentations 1.488 Mistakes 170 Per cent. .115

TABLE

VERGLEICHUNG DER QUADRATISCHEN UND DER BORNSCHEN ZAHLBILDER MIT DEN FINGERN UND MIT DEN BILHARZ STRICHREIHEN

	First-year Pupils Training School Time, $\frac{5}{11}$ Sec. Exp. I				Third-year Pupils Training School Time, $\frac{3}{8}$ Sec. Exp. II				First-year Pupils Training School Time, $\frac{3}{8}$ Sec. 1st Presentation Exp. III				First-year Pupils Training School Time, $\frac{3}{8}$ Sec. 2d Presentation Exp. IV				Third-year Pupils Training School Time, $\frac{5}{18}$ Sec. Exp. V			
	With Fingers Two Presentations	Mistakes	Born Picture One Presentation	Mistakes	With Fingers Two Presentations	Mistakes	Born Pictures	Mistakes	Bilharz Strokes	Mistakes	Quadratic Pictures	Mistakes	Bilharz Strokes	Mistakes	Quadratic Pictures	Mistakes	Bilharz Strokes	Mistakes	Quadratic Pictures Two Presentations	Mistakes
	5	2	5	0	5	1	5	2												
	8	4	8	0	8	2	8	1	6	4			4	0			8	20	8	0
	3	1	3	0	3	2	3	0			8	0			7	1	7	16	7	0
	10	5	10	3	10	5	10	5			5	1	8	9			5	10	5	1
	4	4	4	0	9	2	9	2	7	7					4	0	6	16	6	0
	9	4	9	3	7	1	7	3			6	2	5	3						
	5	3	5	0	6	3	6	3	5	0					6	1				
	7	3	7	0	4	6	4	0			4	0	7	10						
	4	3	4	0					8	10					5	0				
											7	1	6	2						
									4	0					8	0				
Total Presentations	270		270		152		304		155		155		155		155		108		26	
Total Mistakes ...		29		6		22		16		21		4		24		2		62		1
Per cent.		.107		.022		.145		.053		.135		.026		.155		.013		.574		.005
	15 pupils each presentation		30 pupils		19 pupils		38 pupils													

SUMMARY

Bilharz—Presentations.....	418	Mistakes..	107	Per cent. ..	.256
Quadratic—Presentations...	526	Mistakes..	7	Per cent. ..	.013
Fingers—Presentations.....	422	Mistakes..	51	Per cent. ..	.121
Born—Presentations.......	574	Mistakes..	22	Per cent. ..	.038

9. **Comparison of quadratic and Born pictures with strokes, fingers and the Tillich apparatus.**—The numbers 4–10 were with black ink on seven sheets of drawing paper represented by strokes. In the size and distance the wall-tables of head-teacher Bilharz in Karlsruhe were used as a guide merely for the purpose of arriving at a numerically established judgment concerning the value of this teaching material which has been introduced in Karlsruhe and elsewhere. The surface of a stroke is $7\frac{1}{2}$ cm. long and 2 cm. wide. The distance between the strokes amounts to half the width of a stroke; after each group of five strokes, a space equal to the width of a stroke was left. The quadratic number pictures which were used were arranged to correspond. The circle surfaces and the strokes had equal areas and the distances were placed in similar relation. In the experiment with fingers a pupil covered his eyes with his hand or with a book, while his neighbor placed the number of fingers shown by the operator on the bench before him.

After the operator had counted 1, 2 twice with the beats of the metronome the number of fingers was looked at and at the next beat the neighbor drew the fingers back.

The necessary preliminary practice preceded the several experiments.

Dr. Walsemann (56) found the following results which he had recorded by the subjects numerically, not by drawings.

Stroke rows, after 5 a space 451 mistakes 59.7%
Born pictures, 28 mistakes 3.7%

Schneider (60) obtained the following results:

Tillich apparatus, 259.9 mistakes
Born pictures, 34.5 mistakes

The results show that the quadratic pictures are superior to the stroke rows, to the fingers and the Tillich apparatus.

These results also furnish new evidence that the apprehension, memory, and symbolizing of things offer more difficulty in row form than that of things which are arranged in group form.

The several experiments which have now been tried concerning the kind of spatial arrangement of things lead to the suspicion that the numerical grasp of things also depends on other spatial properties, their distance apart, their size, and the direction which their arrangement takes. To this conclusion physiological considerations also point, viz.: that the field of vision has certain limits and that the motor sensations of the eyes play a great rôle in the apprehension of size and form.

10. **The numerical apprehension of things, depending on their distance apart and marking.**—The purpose in the examination of this relation of dependency is practical. The question at first was the psychologically advantageous form of the quadratic pictures, and in this the theoretical interest predominated; now we come to the consideration of another point of view, viz.: the practical.

In order to learn the influence which the distances apart of the several things in rows or in groups have, the Russian machine with yellow balls was used. There were 5 to 8 balls arranged side by side, first close, and then with a distance of $1\frac{1}{2}$ cm.

Result:

31 First-year Training School pupils.

Time, $\frac{3}{8}$ sec. (Two presentations each of the numbers 5, 6, 7, 8.)

Without distance—69 mistakes

With distance—60 mistakes

Another experimental arrangement which tested fatigue showed corresponding results. The 10 yellow balls of the calculating machine were counted during 24 quarter-minutes.

Whoever became fatigued, as was shown by a flowing or swimming together [of the balls (objects counted)], made a note of the number [of the quarter-minute] which had been given out just before by a pupil who announced the quarter-minutes.

[The announcer, guided by the metronome, at the end of the first quarter-minute said *one*, at the end of the second quarter-minute said *two* and so on for as many quarter-minutes as might be required before fatigue for all subjects set in. (As a matter of fact in this experiment 16 of 31 pupils were not fatigued in the 24 quarter-minutes so that the announcer had to count through the whole 24).

The subjects began at the beginning of each quarter-minute and counted 1, 2, 3 up to 10, then paused until the end of the quarter-minute, then counted again 1, 2, 3–10.

The number of quarter-minutes counted through by each fatigued subject before fatigue set in, as heard from the announcer and noted by the subject, was added to the number noted by each of the other fatigued subjects; the total was the record of the experiment].

The Russian machine, which has no back, stood $2\frac{1}{2}$ m. distant from a bluish wall. In experiment 1 the balls had no distance; in experiment 2 a distance of 1 cm., and in experiment 3 a blackboard was placed directly behind the machine.

Thirty-one pupils of the second-year training school were subjects; of these 16 were not fatigued; the other 15 yielded the following results:

Exp. 1. 165 Viertelsminuten
Exp. 2. 206 Viertelsminuten
Exp. 3. 218 Viertelsminuten

In another experiment 17 second-year training school pupils counted 6 balls during 28 quarter-minutes at the distances apart (a) $1\frac{1}{2}$ cm. (b) 3 cm. The pupils who were fatigued before the 28th quarter-minute reached the following numbers:

(a) 114 Viertelsminuten
(b) 130 Viertelsminuten

Of special interest was the study of the influence which the distances apart of the circles in the quadratic pictures exerts on their apprehension. Pictures with three different distances were used. In the *narrow* pictures the circles were distant a half-diameter and the quadrats a diameter; in the *medium* pictures the distance of the circles was equal to the diameter and of the quadrats $1\frac{1}{2}$ diameters; and in the *wide* pictures the distance of the circles = $1\frac{1}{2}$ diameters and that of the quadrats = 2 diameters. The circles were cut out of white drawing paper and pasted on black paper; they had a diameter of 8 cm. There were eight experiments, two each in the first school year, in the first year of the training school and in the third year of the same. No clear reaction was shown with the narrow and medium pictures. The differences in the results was insignificant; but a clear reaction occurred in favor of the wide picture. The narrow and the medium pictures were also tested relative to fatigue [under the same conditions as described in the previous fatigue experiment]. Thirty second-year pupils of the training school counted the several circles in the number picture *nine* up to 24 quarter-minutes. The pupils who in this time-space became fatigued yielded the results:

Narrow 285 Viertelsminuten
Medium 298 Viertelsminuten

Since it is desirable that the operations be indicated in the pictures by a staff or a stroke, the medium pictures are to be preferred.

Results:

	Distance between the Circles	Distance between Two Quadrats	Mistakes
Narrow	1 half-diameter	1 diameter	93
Medium	1 diameter	$1\frac{1}{2}$ diameter	95
Wide	$1\frac{1}{2}$ diameter	2 diameters	101

Conclusion.—The distance of the quadrats from one another should not be too great in relation to the distance within the quadrats. The part at the right of the last of these distances will not be apprehended if these are too great.

In consideration of the fact that the parts are to be separated by a bar [rod, staff] [to indicate operations] the second of the foregoing arrangements recommends itself (1 to $1\frac{1}{2}$).

[Note.—Notwithstanding this recommendation the wide arrangement has a smaller ratio between the distances than the one recommended (the medium).

Narrow	Ratio 1 to 1 = 1
Medium	Ratio 2 to 3 = $1\frac{1}{2}$
Wide	Ratio 3 to 4 = $1\frac{1}{2}$

This seems to be a matter of threshold of the perception of the quadrats as entities and of the field of vision.

From the conclusion of Lay he seems to think that the medium presents the smallest ratio. His experiment tends to show, on the contrary, that a greater ratio would be still more favorable.]

Schneider and Dr. Walsemann have in their experiments with quadratic number pictures entirely neglected this important condition; therefore their experiments show nothing against the standard quadratic pictures.

11. **The numerical apprehension of things, depending on their size and distance.** This is also a practical question.

There were used for the solution of this question the already prepared medium pictures in which the circles had a diameter of 8 cm. and for comparison similar pictures were prepared in which the circles had only 5 cm. diameter, but as for the rest had the [same] middle distances as the larger pictures [viz., 8 cm. and 12 cm.].

Results:

Exp. I.—First-year training school. 27 pupils. Time = $\frac{6}{17}$ sec.

Four presentations each of numbers 5, 6, 7, 8, 9 in irregular order.

8 cm. pictures—6 mistakes
5 cm. pictures—9 mistakes

Exp. II.—Second-year training school. 34 pupils. Time — $\frac{10}{23}$ sec.

Two presentations each of numbers 5 to 9 in irregular order.

8 cm. pictures—4 mistakes
5 cm. pictures—2 mistakes

Exp. III.—Second-year training school. 34 pupils. Time = $\frac{3}{8}$ sec.

Two presentations each of numbers 5 to 9.

8 cm. pictures—7 mistakes
5 cm. pictures—3 mistakes

Exp. IV.—Third-year training school. 37 pupils. Time = $\frac{5}{16}$ sec.

One presentation of numbers 5 to 9.

8 cm. pictures—2 mistakes
5 cm. pictures—0 mistakes (2 in the *Führer*)[1]

Summary:

8 cm. pictures—19 mistakes (25 in the *Führer*)[1]
5 cm. pictures—14 mistakes (23 in the *Führer*)[1]

Conclusion.—The diameter of the circles or balls in the pictures between 8 cm. and 5 cm. can vary without influencing essentially the apprehension.

[1] Several mistakes occur in Lay's table.

This result for our practical purpose, as we shall see later, is of significance.

[Note.—Nevertheless, the difference between mistakes is greater than the summary in the *Führer* shows and is slightly in favor of the 5 cm. It is also in favor of greater distances between the dots ($1\frac{3}{5}$ diam.) and between the quadrats, $2\frac{2}{5}$ diam.]

Since some methodists in their number pictures or perception material (e. g., the Tillich apparatus or the Russian machine) arrange the unities vertically over one another, therefore it is of significance from the standpoint of method to ask whether the direction of the arrangement is indifferent, or, if this is not the case, whether the vertical or horizontal direction should be used. In order to gain through experiment an insight into this question the previously used (Section 4, Row Pictures and Born Pictures) rows of circles were used which were fastened now horizontally, now vertically.

Results:

Exp. I.—First-year training school. 31 pupils. Time = $\frac{3}{8}$ sec.

Two presentations each of numbers 4 to 8, arranged vertically in irregular order.

Average number mistakes for one presentation = 46

One presentation each of numbers 4 to 8, arranged horizontally.

Number mistakes = 6

Exp. II.—Second-year training school. 39 pupils. Time = $\frac{10}{23}$ sec.

One presentation each of numbers 4, 5, 6, in each arrangement, the two arrangements being mixed in the order of presentation.

Vertical rows —27 mistakes
Horizontal rows—12 mistakes

The experiments show:

The horizontal arrangement of the unities in the perception-material of number teaching is superior to the vertical.

The result speaks against the vertical "Zweierreihe" of Dienstbach, against the vertical "Zahlziffern" of Mayer, against the several "Zahlbilder" of Busse, Böhme, Hentschel, Sobelewsky, Kaselitz, Beetz.

12. **The numerical apprehension of things, depending on their form and brightness.**—It is clear that small strokes as individual things can come into use in teaching practice only in the form of rows. But the apprehension of groups was found to be superior to that of rows so that apparently there is not much value for teaching practice in further investigating the apprehension of circles or balls and that of strokes and rods; that is, the influence of the form of the individual things on numerical apprehension. But it will be shown that even these practical investigations are of significance and illuminate facts which we already know from a new side. According to certain observations it appears that short-sighted pupils must exert themselves more or become fatigued sooner in the apprehension of stroke rows than in the apprehension of circle-groups. At the same time a certain influence of the vermilion color of the strokes in the Bilharz reckoning table was occasionally observed. In order to discover somewhat concerning the influence of stroke- and circle-form and at the same time concerning the white and black colors Lay instituted the following research. There were fastened 10 black strokes on white paper, 10 white strokes on black paper, and 10 black circles on white paper. The strokes occur as in former experiments, $7\frac{1}{2}$ cm. long and 2 cm. wide, and the circles have the same area as the strokes. The distance apart of all strokes and circles was made equal to half the width or equal to half the diameter. The pupils now proceeded to count through the rows from 1 to 10 until definite appearance of fatigue set in, which

expressed itself in different individuals as delay, swimming together (of the objects), color-change of the strokes, etc.

(For further details of procedure and recording, see under section 10.)

Results:

17 first-year training school pupils. During 32 quarter-minutes.

(a)	Black strokes on white paper	238 Viertelsminuten
(b)	White strokes on black paper	340 Viertelsminuten
(c)	Black circles on white paper	419 Viertelsminuten

It is to be considered that in experiment *c* five pupils were not fatigued, and, besides that, this experiment was last conducted after already 16 minutes had been counted. The pupils asserted that they often could not easily get hold of a distinct stroke in the row on account of the little distance between the strokes (compare the experiments in section 4); that they confuse the stroke easily with the neighboring strokes on either side; that the confluence (of objects) and consequent strain are not present in the case of the circle rows. The fact is that counting the strokes in the Bilharz tables (charts) was fatiguing in a high degree; that, in other words, the apprehension is conjoined with relatively proportionately greater strain (effort) and therefore the apprehension is hindered in comparison with the circle rows.

It has been shown that the simultaneous apprehension of rows by children of the first school year in most favorable cases reaches to *Three*. It is now of interest to learn whether in the case of older pupils it naturally reaches further and how it acts in the apprehension of 5 to 10 things which are arranged in a row. In order to gain an insight into these relations the following experiments with stroke and circle rows were conducted with the training school pupils of all three years. A small apprehension time of scarcely $\frac{1}{3}$ sec. (160 to 192 pendulum strokes in a minute) was used so as to exclude counting.

Exp. I. 30 first-year training school pupils. Time = $\frac{3}{8}$ sec. Numbers from 4 to 8, presented as rows of strokes and as rows of circles in irregular order and with two kinds of presentation intermingled.

Number	Number of Times Presented as Stroke Rows	Mistakes	Number of Times Presented as Circle Rows	Mistakes
4	30 × 2 = 60	4	30 × 2 = 60	1
5	30 × 3 = 90	1	30 × 3 = 90	5
6	30 × 4 = 120	15	30 × 4 = 120	8
7	30 × 4 = 120	14	30 × 4 = 120	22
8	30 × 4 = 120	31	30 × 4 = 120	27
	30 × 17 = 510	65	30 × 17 = 510	63

Exp. II. 30 first-year training school pupils. Time = $\frac{5}{16}$ sec.

Numbers 4 to 8 presented as in Exp. I.

Number	Number of Times Presented as Stroke Rows	Mistakes	Number of Times Presented as Circle Rows	Mistakes
4	2	3	2	4
5	2	0	2	5
6	3	5	3	1
7	2	7	2	3
8	2	6	2	10
	30 × 11 = 330	21	30 × 11 = 330	23

Exp. III. 38 second-year training school pupils. Time = $\frac{5}{16}$ sec.

Numbers 4 to 8, presented in irregular order, first as rows of circles, then in the same order as rows of strokes.

Number	Number of Times Presented as Stroke Rows	Mistakes	Number of Times Presented as Circle Rows	Mistakes
4	3	3	3	4
5	2	8	2	7
6	1	8	1	10
7	1	11	1	18
8	1	15	1	20
	38 × 8 = 304	45	30 × 8 = 304	59

Exp. IV. 37 third-year training school pupils. Time = $\frac{5}{16}$ sec.

Numbers 4 to 8, presented as in Exp. III.

Number	Number of Times Presented as Stroke Rows	Mistakes	Number of Times Presented as Circle Rows	Mistakes
4	1	2	1	2
5	1	10	1	12
6	1	16	1	.18
7	1	16	1	23
8	1	20	1	32
	37 × 5 = 185	64	37 × 5 = 185	87

It turns out that the results of Experiments III and IV with second- and third-year pupils do not agree with Experiment II with first-year pupils, although the time of exposure is the same. This is explained in the following manner. With the first-year pupils other experiments preceded the above mentioned one in which the pupils were attentive to the intervals after 5. A counting of the units was impossible. The pupils therefore in the case of rows which exceeded 5 fastened their gaze only on the part (1 to 3) after the intervals (which was as easy to note in strokes as in circles), and constructed the picture *additively;* a perception or a memory or image of the whole is, that is to say, nowise present and the representation (symbolization) of the number is in no way the expression of total image of the units. The comparatively small number of mistakes in Experiment II came about in other words, through group-wise apprehension. Among the second- and third-year pupils, the intervals after 5 were entirely overlooked by a few pupils; since counting was impossible they merely estimated the number of strokes and circles, as they themselves asserted. Now these had in the stroke rows fewer mistakes than in the circle rows; the estimate of the stroke rows came easier to them than the estimate of the circle rows (and in the act of representing stroke rows by straight lines they are limited to the comparison and estimation of challenging surfaces, the judgment of whose interruptions is not so difficult as is the case with circle rows). It is also to be considered that all pupils previously had used the representation of numbers by strokes, not by circles or dots.[1]

The series of experiments has so far given this result: Ball and circle forms are superior to the rods of Tillich and strokes.

Exp. V. Ten vermilion strokes on green ground and ten black strokes on white ground were twice counted (that is, to 20) by 31 pupils of the first-year training school. The number of repetitions was recorded for each pupil. [The pupils were to continue counting from 1 to 20 until fatigue set in to see how many times they could repeat].

[1] This explantion really shows that the conditions of Experiments II-IV were such as to invalidate the results; but the fatigue experiment preceding is valid and is slightly confirmed by Experiment I, which in itself is not conclusive. We are perhaps warranted on the whole, however, in accepting Lay's conclusions.

At the discontinuance of the experiment, which took place when in the great majority the appearances of fatigue made themselves noticeable, it was shown that the red had shown only 272, but the black 428 repetitions; besides, 2 pupils could count the red strokes only three times and 4 pupils only four times.

Conclusion.—The red strokes cause an effort of vision and therefore of attention and hinders the counting of strokes in rows; (one may talk no longer of simultaneous apprehension in such a case). Black strokes on white are better. Thus, the influence of the color of the things to be grouped and the color of their backgrounds cannot be neglected.

Exp. VI. We therefore next raise the questions: How acts Black on White, White on Black, Red on Green, Yellow on Indigo, Blue on Orange? Furthermore, how act Red, Yellow, Green, Blue on Black ground?

[In these experiments 10 strokes with known dimensions and distances were always used. At a given signal, by a pupil whose duty it was to loudly count the minutes [guided by the metronome], the number *zero* was spoken and the other pupils began the exercise. At the end of each half minute [quarter or sixth-minute] the total number of half [quarter or sixth] minutes was announced and noted by those who, by reason of fatigue, directly after the announcement ceased to count. [In this fatigue-method, the pupils counted from 1 to 10 repeatedly, the speed of counting being regulated by the fraction of a minute allowed for each count (from 1 to 10). The greater the number of minutes or fractions of minutes counted through before fatigue set in, the more favorable the reaction.] The subjects were training school pupils. The following order was observed in the series of tests:

(a) 39 second-grade pupils counted through 12 half minutes

	Av. per pupil
Black strokes on White background 146 h. m.	1.9
White strokes on Black background 161 h. m.	2.1
(b) 17 first-grade pupils 32 quarter minutes	
Black on White 238 q. m.	3.5
White on Black 340 q. m.	5.0
(c) 30 first-grade pupils 20 quarter minutes; ½ min. pause after each trial	
Red on Green 338 q. m. 6 pupils not fatigued	3.5
Yellow on Indigo 406 q. m. 8 pupils not fatigued	4.6
Blue on Orange 298 q. m. 6 pupils not fatigued	3.1
[1]White on Black 411 q. m. 12 pupils not fatigued	5.7
(d) 39 second-grade pupils 36 sixth minutes	
Red on Black 649 s. m. (av. of two trials)	2.8
Yellow on Black 689 s. m. (av. of 4 trials)	2.9
Green on Black 639 s. m. (av. of 2 trials)	2.7
Blue on Black 735 s. m.	3.1
(e) 31 first-grade pupils 30 sixth minutes	
Red on Black 595 s. m. (av. of 2 trials)	3.2
Yellow on Black 578 s. m.	3.1
Green on Black 627 s. m.	3.4
Blue on Black 598 s. m.	3.2

1. Red, yellow, green and blue on black ground give no reactions which are clearly distinguished from one another. It appears that bright green deserves the preference.

2. The brighter the strokes and the darker the background the easier is the discrimination and grasp and the less the exertion and fatigue.

3. Not the kind of color and the color contrast but the difference in brightness between the counting objects and their back-

[1] [Counted beyond the limit.]

ground makes easier materially the distinguishableness and the grasp and lessens the effort and fatigue.

4. White counters on black ground works most favorably.

13. **Numerical apprehension of spatial things through the sense of touch.**—It is certain that the simple forms of quadratic number pictures are in themselves of advantage to the children; they are not monotonous and tiresome as the rows are; they afford much more variety, are simple, clearly arranged and therefore easily grasped. The little faces, says Lay, shone with joy when the apprehension and representation of pictures were successful but they appeared dissatisfied, fretful, disheartened and finally uninterested when time after time they failed in the apprehension of the rows. The intuitive apprehension of rows and of processes is, as has been shown sufficiently, above 3 or 4 simply impossible. Also the children find it difficult to conquer the tendency to illustrate by means of rows. If now the memory for the auditory word picture (Wortklangbild) and the speech motor image of the number words is not strong enough so that results hold fast purely externally, then necessarily must be shown loss of interest, dislike, taciturnity, antipathy, yea, insubordination, while on the contrary, in number work with the quadratic pictures, interest and pleasure are ever present.

The child finds pleasure in the mere placing and taking away of the buttons. Still more; since the composition of the quadratic pictures is simple, the children proceed of their own accord to the next number and seek the new numerical relations just as the child, who has found a great interest in reading, of himself goes further into a well-constructed primer.

The Shah of Persia, as is well known, pities us Europeans because we do not eat with the fingers; the pleasure which the sense of touch brings escapes us, if one does not feel the various foods with reference to the qualities of tenderness, softness, etc. He may be right; since we know that different qualities of touch work æsthetically, bring us satisfaction.

The child makes use of all the senses and in the incitement and in the functioning of the same lies a high source of pleasure and satisfaction. One needs think only of the pleasure which the child finds in movement; the motor sensations are but sensations of touch through which the qualities of roughness and smoothness, of angularity and roundness, etc., come into existence.

* * * * *

The experiments have repeatedly shown that the origination of the number image is bound up with spatial relations; therefore, it is certain that the touch sense plays a part therein.

But also on other grounds is the proposition maintained. The origination of the image presupposes the existence of things. In movement the child meets with resistance and through the sensation of resistance it knows somewhat of the foreign, something not itself—an object. The resistance against movement (even in touch) thus causes the child to realize objective things. The image does not actually enter the mind until the child can distinguish himself from things. In fact the teaching of blind children shows that number teaching can proceed entirely on this sense, and that with as little difficulty as with the seeing children.

Exp. I. An experiment was instituted with boy K., who attended the kindergarten and boy W., who had been in the first school year three quarters of a year. After they had been blindfolded they laid their hands on the number pictures of the button apparatus [an apparatus by means of which the unities of the quadrats can be represented by circular buttons]. They recognized through simultaneous touch sensations, without counting, comparatively quickly and certainly even the larger numbers.

Exp. II. Later the buttons both at a short distance (apart) and at no distance were arranged in rows. There was now shown an entirely similar relation as in the case of number apprehension through the sight sense: The direct simultaneous apprehension reached only 3; the other longer rows were not grasped clearly. The number was obtained only through counting, not through direct perception. The apprehension of the small rows and the counting of the larger took place more quickly if the buttons were placed a short distance apart.

Exp. III. These conditions on account of the need of making perception material which regards the touch sense were more exactly investigated with children of about 6 years.

The question was: at what distance will two touch impressions still as two impressions, etc., and no longer as one impression be perceived. The operator placed without regularity alternately now both points, now only one point of a compass on the skin of a 5-year and 6-year-old boy and (the boys looking to one side) asked them to state the number of impressions. On the finger tips at a distance of $1\frac{1}{2}$ mm., on the under side of the middle knuckle at a distance of 2 mm., and on the cheek at a distance of 20 mm., the number of impressions was still incorrectly determined. If the distance of the contiguous points on the underside of the knuckles amounted to 7 mm. it was rightly decided, and at a distance of about 12 mm. the decision for all parts of the under and upper hand surfaces was correctly given.

In the absence of sight, the touch sense alone can by means of the quadratic pictures obtain clear number-percepts up to 12. But we know that an image the better clings in memory the more it is associated with other images. Therefore, we should associate, in the apprehension of number, with the percept and image of the sight sense, those of the touch sense; thus the teaching result will be better and more certain, and this result can be reached through the use of a contrivance such as Lay's button apparatus. For this contrivance, see p. 179, *Führer*.

* * * * *

Perceptional instruction and number teaching should be based on the touch sense and give opportunity for the touch sense in large measure to manifest itself; the pictures to which convenience lends countenance, the strokes, the dots should only very sparingly appear as perception material. The pupils should use all the spatial things which come to perception and out of the perception and image by means of Plastima and simple drawings symbolize the same.

14. **Experiments concerning *operations* with rows and quadratic pictures.**—For these experiments the already known quadratic pictures made out of black circles on white drawing paper and the corresponding row pictures were used. The number picture was divided into two parts by a black stroke, thus: It was now the task of the pupil to grasp the whole picture and its two parts, or, conversely, the two parts and the whole picture, retain the same, and out of the memory represent the same on paper.

Exp. I. 31 first-year training school pupils. Time = $\frac{3}{8}$ sec.

Problems	Mistakes in Groups	Mistakes in Rows
6 + 3	3	13
4 + 5	1	11
2 + 7	2	9
5 + 3	7	10
	13	43

Exp. II. 39 second-year training school pupils. Time = $\frac{3}{8}$ sec.

Problems	Mistakes in Groups	Mistakes in Rows
4 + 3	1	21
6 + 1	9	27
2 + 7	8	12
6 + 3	5	19
4 + 5	5	7
	28	86

In the interpretation of these results in reckoning it is to be considered that to all pupils the separation and combination of row pictures from the first attendance at school till now were known and fully familiar, but that by all pupils the problems to distinguish the smaller in the larger pictures were attacked for the first time, and that several pupils who seldom took part in the experiments had seen the group pictures but seldom on the whole. Thus it is beyond doubt that the illustration of operations and the guidance to results on the ground of quadratic pictures proceed much more easily, quickly and certainly than is the case when rows are used.

Ludwig Pfeiffer (Exp. Pädagogik, 2. Bd. 1905) has undertaken the problem: Born or quadratic number pictures for reckoning?

His pupils had already been instructed a half-year according to the Born pictures; nevertheless, the following results were yielded:

(1) Apprehension
Born Pictures 108 mistakes
Quadratic Pictures 136 mistakes

(2) Addition and Subtraction

Born Pictures	254 mistakes
Quadratic Pictures	211 mistakes

(3) Division into Factors

Born Pictures	81 mistakes
Quadratic Pictures	70 mistakes

Pfeiffer's experiments confirm those of the *Führer* and oppose the views of Schneider, Walsemann, Troelltsch and other advocates of the Born pictures. He says rightly, "The didactical experimentation has hitherto consistently resulted (in this), that the illustration (presentation) in quadratic groups of four is better, both for the apprehension of individual numbers and for the representation of the operations, than all other pictures, especially better than the simple double rows of Born."

Conclusion.—All the experiments conducted with training school pupils and children of the first school year have shown that the quadratic pictures are superior to other known pictures and particularly to the rows.

Lay's conclusions from his investigations:

1. As the capability of the child to obtain clear and distinct object-images grows, the capability to grasp also grows.

2. Homogeneousness, arrangement, distance, size, color, position of the parts, influence the apprehension of the number.

3. Since nature offers no medium that furnishes all these facilities at once, in first number teaching a presentation medium must be made in which this is the case.

4. The presentation medium must make possible the simultaneous grasping of the objects not only through the sight sense, but also through the touch sense.

5. The nature of the number image consists in the cognition of the existence of things, which are united in place or in time.

6. In abstraction that aims at a number concept, attention must be directed to the existence of the things; this is done in the investigation, through a more closely determined homogeneity, arrangement, size, form and color of the objects to be grasped. The nature of the number, however, is independent of the arrangement of things in space or time; it is false when one says that the row form, which for arithmetical evidence for the purpose of logical deduction is proper, has anything to do with the nature of the number.

7. The consciousness of the existence of material things and with it the number images will be sharper when the material things are perceived not only through the sense of sight but importantly also through the sense of touch; the resistance leads us to the acquaintance with something strange, with something which we are not (ourselves), to the cognition of an external object. The resistance, which excitations find in the sense organs, is slight; the resistance against our movements (in the case of touch) is, however, palpable and convinces us best of the existence of things. What is grasped (touched) is even by the feeble-minded intelligible.

8. As every other image is clear and distinct, when not only the whole (as such) but also its individual characteristics are sharply grasped, so it is also with the number image; not simply the existence of the thing as a whole, but the existence of the individual groups, and in these the existence of each thing must be known and in consciousness.

9. This demand the quadratic pictures represented by objects answer the best; a presentation medium to represent rows cannot comply with the demands. Hence not counting, and content-void number names found through counting, but perception and clear number imaging (having content) wrought out through the simultaneously grasped sensation-complex is the principal thing.

V

10. As images that are connected with one another stick closer in memory, so it is of importance that the number apprehension through the sight sense should be connected with the apprehension through the touch sense.

11. The logical conviction through judgment has not the strength of the sensational conviction through perception. The sensational conviction of the individual existences can immediately, simultaneously and therefore the more efficaciously follow only when the material things are arranged in quadratic number pictures.

12. The feeling of certainty reached through the perception can be strengthened, if the number image of the material thing is supported not only by the sense of sight but also by the sense of touch.

13. The stronger the feeling of conviction, the stronger grows the interest, the energy, the pleasure, and love of reckoning, and the better is the result of instruction. When this feeling is lacking there is loss of interest, lack of energy, dejection, aversion, distaste and the instruction result is poor.

14. The true content-full number image is intimately connected with the "symbolical number image," for example, with many formal speech images, with the sound picture of the speech-motor image, with the written picture (figures) and the writing motor image of the number word and with many conceptual images which correspond to the properties of the numerically apprehended things. The total image of a number is, that is to say, put together from speech images, formal or symbolical images, and from conceptual or content-full partial images.

15. The partial speech images in the total image of a number support the memory of the number image and supply, by quick, ready reckoning, the conceptual image.

16. The properties of the number images are determined through the kind of illustration and instruction. Of these circum-

stances, which are of the greatest importance for the origin of the number, the nature of the number is independent; however, they influence considerably the view of the nature of the number.

17. The number images of the cardinal numbers are brought about through direct perception; the number images of all numbers which are above 10, through the imaging of the cardinal numbers in connection with the groups or sorts of ten systems.

18. Origin of the number and the nature of the number are to be sharply divided. The nature of electricity and the nature of number have primarily to be worked out by the theoretical physicist and the theoretical psychologist. Methodology of number instruction and technique of electricity can, however, exist, before the questions about the nature of the number and the nature of the electricity are solved. It is indispensable for the methodist, the psychological technologist, to learn to know the conditions of the origin of number and its influences, as it is necessary for the electrician, the practical technologist, to be able to make exact computation about the conditions of the origin of an electrical current. Until the present, the number methodists have proceeded from the various theories about number concepts and from a few untrustworthy observations and general psychological facts, if they have had any guide or basis for the teaching procedure in first number instruction. We have, however, shown that many of these theories and, therefore, also the resulting teaching procedure are false. Supported by much investigation I have presented (elsewhere) a theory about the nature of number, but I am careful about building the method upon this theory. That I accent with all force in order to avoid misunderstandings. As the instruction in spelling, so also is the first number instruction based not upon a theory but upon results of investigation with school classes. By the plan of our investigations nothing else is called for than that the first number instruction should be based upon the practical results of an

instruction in which the methodical steps and their quantitative results shall be closely controlled and compared.

It is possible in teaching spelling, as well as in teaching primary number, to know how to clarify the confusion of opinions concerning method only through foundations which have been built independently of untrustworthy experience and wavering theories, through objective, quantitatively determined facts.

Supplement to review of Lay's investigation.—In order to bring the comparative results of the investigation, regarding the best means of presentation of the cardinal numbers, together for convenience of reference, I have gathered up the data from the several experiments bearing on this point, especially with first-year children, and placed them in a single table (p. 198). For the Born and Lay pictures the training school pupils have also been included.

A comparative graph (p. 199) has been drawn for first-year children.

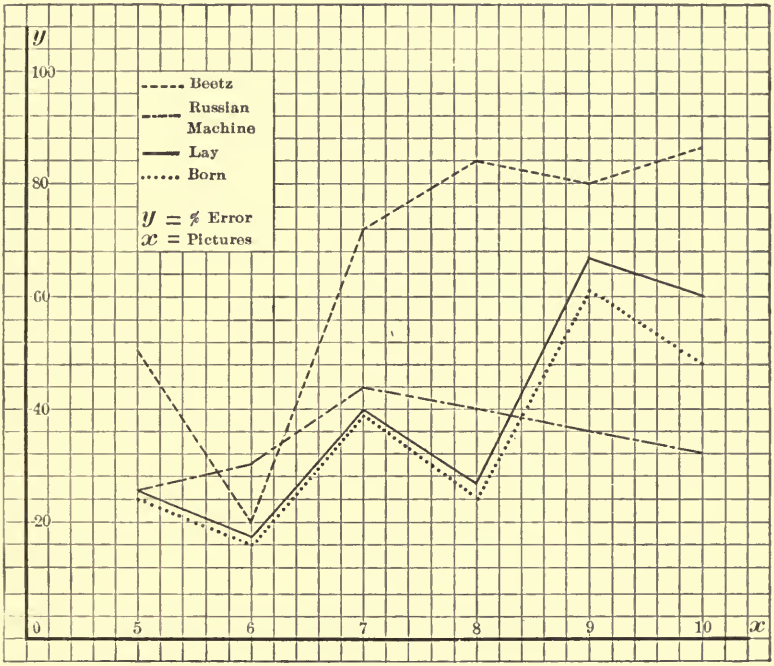
For first-year children, if one considers, as I have, all of the experiments yielding data regarding the apprehension of the Lay vs. the Born pictures (and not merely the experiment in section 6 in which they are directly compared) the Born pictures show slightly superior (35% of error as compared with 40%). But on the whole the rate or error is 15% for the Born and 13% for the quadratic (see table). The curves of relative apprehensibility are almost precisely the same. In computing these per cents, the data given by the use of the Wendling apparatus is excluded since the Lay pictures were not presented by a similar apparatus. The apparatus shows surprising superiority over the usual means of presentation, 24% as against 35% for first-year children, and 6% as against 8% for older ones. Doubtless a machine makes it possible to render the presentation more vivid and grips the attention better. Notice the good showing of the Russian machine. Lay's proposed apparatus for presenting the quadratic pictures demands, therefore, our very serious attention. (See page 179, Führer.)

TABLE

COMPARISON OF THE BORN, BEETZ, AND LAY PICTURES WITH ONE ANOTHER AND WITH THE RUSSIAN MACHINE IN THE APPREHENSION OF THE NUMBERS 5 TO 10. P—CHANCES FOR MISTAKES. M—NUMBER OF MISTAKES

Numbers	Lay Quadratic Pictures First-year Pupils			Born First-year Pupils			Wendling App. (Born) First-year Pupils			Beetz First-year Pupils			Russian Machine First-year Pupils		
	P	M	%	P	M	%	P	M	%	P	M	%	P	M	%
5	87	22	25	106	25	24	106	12	11	58	29	50	106	26	25
6	87	16	18	106	18	17	53	5	10	58	12	20	53	15	30
7	87	35	40	95	38	40	53	11	20	58	42	72	53	24	45
8	87	23	26	95	23	24	53	13	25	58	49	84	53	21	40
9	87	58	67	95	58	61	106	37	33	58	46	80	106	38	36
10	87	50	57	82	39	48	53	25	49	58	50	86	53	17	32
Total..................	522	204	40	579	201	35	424	103	24	348	228		424	141	
Total, Training School....	2,422	173	07	1,669	135	08	1,011	59	06						
Combined...............	2,944	377	13	2,248	336	15									

One wonders whether Lay presented his pictures with the strokes inserted. If so, he has probably added a handicap to the one mentioned by him in section 6 and so decreased the admittedly small measure of superiority of his pictures over those of Born, assuming that the Born pictures were nakedly presented.



By analyzing the mistakes made by the pupils in our own experiments (see part II, pp. 241–243) it has been found that the strokes exercise an influence in this particular at least, that the oblique strokes increase the difficulty of apprehension, the presence of more than two vertical strokes has a similar tendency, and the position of a single stroke, whether vertical or oblique, seems sometimes to help, sometimes to hinder.

PART II

EXPERIMENTS TO DETERMINE:

A. ABILITY IN APPREHENSION
B. ABILITY IN PROCESSES

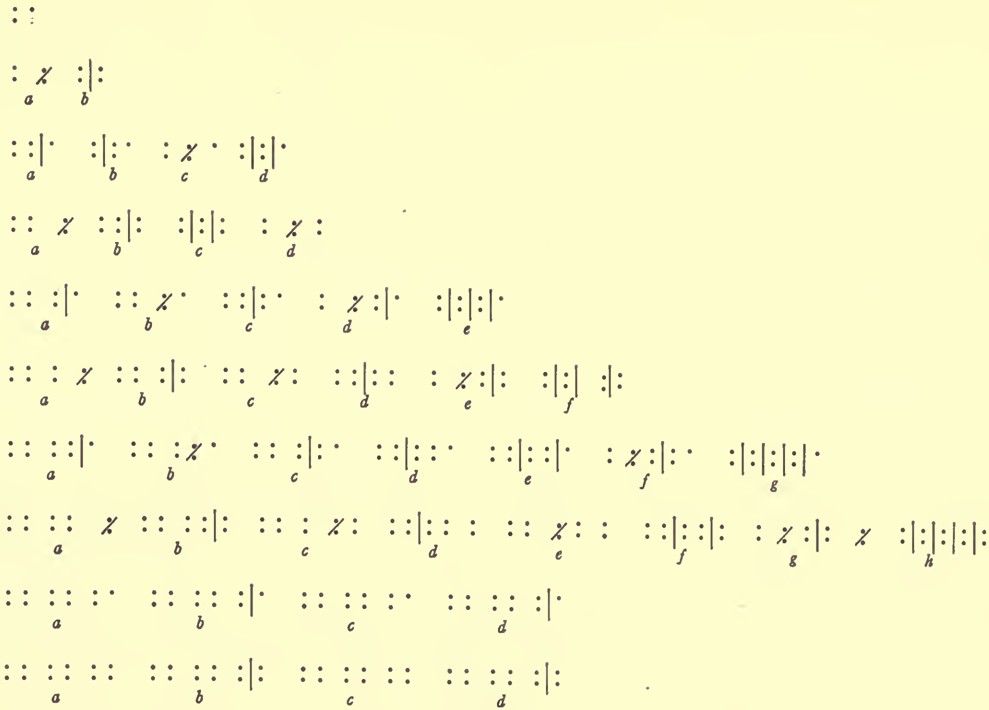
A. Experiments with School Children, First to Eighth Grades, to Determine Ability in Number Apprehension.

These experiments with children from the lowest primary grade to the last year of the elementary course were conducted in a city school by the writer personally in every instance, so that the conditions and procedure were carefully kept uniform throughout. Care was used to so distribute the exercises that none should be prolonged to the point of fatigue. In the lower grades particularly the exercises were so sectioned off that no section demanded more than 15 to 20 minutes. They were usually given in the morning, but not regularly so, since before proceeding very far with the work it was discovered that the pupils hailed the experiments as a welcome change from the regular routine, whatever the time of day.

Materials.—The materials used were number pictures (quadratische Zahlbilder) constructed after the manner of Lay's pictures. Black circles one inch in diameter were pasted on white cardboard, a separate card being prepared for each number from 3 to 12. The strokes between circles used by Lay to indicate operations were also reproduced, although they possess little or no significance for the investigation. The children were told to disregard them when viewing a card.

The series of cards is pictured on the next page.

In each square group (quadrat) of four circles, on the cards as constructed, the distance between the circles forming the sides of the square was 2 inches; and the distance between the quadrats, 3 inches: the ratio between the two distances was thus made: $2 : 3 = \frac{2}{3}$. This happens to be the same ratio as that recom-



3.	1 card
4.	2 cards
5.	4 cards
6.	4 cards
7.	5 cards
8.	6 cards
9.	7 cards
10.	8 cards
11.	4 cards
12.	4 cards
	Total, 45 cards

mended by Lay, although it was arrived at independently of him. (In this connection, see the comment on Lay's mistaken interpretation of the results of his experiment on these relative distances, p. 178.)

To determine the distance between the quadrats which would serve to render the quadrat a distinct entity, I called to my assistance several teachers, and, showing them, about 15 feet away, several cards already constructed with different distances, asked them to say whether the circles on each of the several cards were longitudinally at equal or unequal distances apart. By this means a card was finally selected which was judged rightly every time as containing unequal distances; furthermore, the observers placed the inequalities where they belonged, viz., between the quadrats. The selected card, then, was the one containing little more than a just noticeable difference between the quadrats (the next lower distance was $2\frac{1}{2}$ in., at which the inequality was *sometimes* noticed) and contained the measurements given above (2 : 3). The important point in this connection is that the distance between the quadrats shall be great enough to enable the observer to distinguish the quadrats as such, instead of perceiving the circles as mere double rows; on the other hand, the distance must not be greater than necessary for the purpose of such distinction, otherwise there is danger in the representation of the larger numbers of having a card so long that part of it will fall outside of the angle of vision. The same danger obviously obtains if the distance between the circles within the quadrat is too great, though otherwise this distance is immaterial. For experimental purposes I made this distance 2 inches, so as to be certain that the unities of the quadrat (and of the whole picture) should be very distinctly distinguished by any pupil present possessing normal vision. (Near-sighted pupils were brought near, as whole classes may be in regular work.) This distance occasioned cards of rather awkward length for regular class work, but not so long as to require a shifting of the eyes to bring the whole of the number within the field of vision. The number 12 occupied a longitudinal distance of 18 inches; the number 10, 15 inches. (Of course, the matter of angle of vision was partly governed by the distance of the chart from the observers.) For regular class work smaller distances between circles and quadrats (preserving a proper ratio) and perhaps somewhat smaller circles would be more convenient.

In order to regulate the time of exposure the forty-five cards were mounted on sheets of heavy manila paper, fastened together at one end in the slot of a wooden rod. The chart thus constructed was supported at any height desired (say 6 feet) by inserting in a hole halfway between the ends of the rod a vertically movable upright iron rod resting in the socket of a three-legged base and pinioned there by means of a screw-clutch. The alternate sheets of the chart were left blank. Grasping a sheet at the bottom and lifting it over the top, the operator allowed it to drop on the other side, exposing at will a card or a blank. The blank sheets served as curtains to shut off the view at the end of the time of exposure.

Purpose.—The purpose was to test *under schoolroom conditions* the numerical apprehension of the pictures of Lay. What is included under the term "apprehension" has been set forth in our review of Lay's experiments.

As is obvious, this is a test of how successfully a group of things can be grasped when simultaneously presented. Lay recognizes and quotes the pioneer investigations of Külpe, Cattell, Dietze, Warren, Messenger, Nanu, and others, which we have elsewhere (pp. 61, 62) reviewed; these were undertaken from the point of view of pure psychology, and while they have some pedagogical bearing they had no special didactical purpose. He has also been at considerable pains, as we have seen, to show experimentally with school children the superiority of the quadratic pictures over those of Pestalozzi, Busse, Born, Böhme, Hentschel, Sobelewsky, Kaselitz, and Beetz and over the rows of the Russian machine and other numeral apparatus. Accepting his conclusion on these points, the question remains, does counting (of successively presented things) play no part in the attainment of number ideas? Lay does not deny (nor can any one) that counting plays a large part. Our review of the investigation of Phillips (see pp. 8–11) has shown that the child enters school with the counting psychosis well established, even in cases to the point of obsession; it would be folly to ignore the fact or to do otherwise than wisely to take advantage of it. But that this is the only or most economical way in which number can be attained, Lay emphatically denies, and furthermore, as we have seen, he has shown that it can be obtained in another, more economical way.

It might be supposed that since Lay places so much stress on his quadratic pictures that he would have them take the place of dealing, visually or tactually, with things themselves. Such, however, is not his plan. He says (Führer durch den Rechenunterricht, p. 137), "Der Anschauungsunterricht und der Rechenunterricht müssen vor allem Körperliche Dinge zu Grunde legen und Gelegenheit geben, dass auch der Tastsinn in ausgedehntem Masse sich bethätigen kann; die Bilder, die Bequemlichkeit Vorschub leistern, die Striche, die Tupfen dürfen nur in sehr beschränktem Masse als Anschauungsmittel auftreten. Wohl aber muss der Schüler geübt werden, alle räumlichen Dinge, die zur Anschauung kommen, sowohl aus der Anschauung als auch aus der Vorstellung durch Plastima und einfache Zeichnungen darzustellen."

The use of the number pictures, then, follows or accompanies the seeing, counting, handling of objects; through practice with them the ability to grasp groups is strengthened; moreover, the thesis is that through them percept (Anschauung), image (Vorstellung) and symbolization (Darstellung) all are brought about with economy of time and effort. Furthermore, since the concept, if it is to be more than nominal, that is, *content-full*, must grow out of clear images, it is important that the images be of such a form that they may be returned to readily again and again, this growing into type-forms of a semi-abstract kind, working categories, under which may be subsumed by children all their early numerical experiences. It may be admitted that concepts of a certain sort would come *in time* to children confined to the more primitive and wasteful counting method, but the concept would be unrepresentable by any economical (groupwise) symbolization, would be slow in arriving, and would be undeniably less tangible and referrable to any clarifying content.

It may be assumed, on the basis of what has been said above, that numerical concepts develop, *pari passu*, with the growth of numerical apprehension (Anschauung und Vorstellung) so that the measure of the growth of apprehension may be taken as the measure of the growth of concepts. At this early stage, the child's thinking is fundamentally imagining and it would be difficult to draw the line between his images (Anschauliche Vorstellungen), his conceptual images (begriffliche Vorstellungen), and his concepts (Begriffe).

Procedure.—The purpose being to work under school conditions, 5 seconds was taken as the time of exposure for each card. This is about the time observed to be taken by several teachers in leafing over number or phonetic cards under ordinary circumstances. In some of the preliminary experiments 3 seconds was used, but with the apparatus at hand it was found difficult to keep the time uniformly at 3 seconds, and, also, the results differed scarcely at all from those of 5-second exposures.

Tests were also given in which, for the purpose of comparison with tests given with this uniform time, each card was given a different time, varying from 3 to 12 seconds.

Lay's exposures under laboratory conditions (using the metronome for timing) varied from $^1/_2$ sec. to $^5/_{16}$ sec. Of course, such short times are ordinarily impossible and impracticable in the school room. He wished to make sure that the group was grasped (without counting) as a "placing" (Setzung) both in its unitary and its total aspect, was clearly imaged and symbolically represented both as an Einheitensetzung and a Gesamtsetzung (but not necessarily numerically described); and this certainty was necessary for his purpose. But for our purpose it was only necessary to be reasonably sure that no counting or not much took place. As a matter of fact, it was observed that 5 seconds was too short a time in which satisfactorily to use a counting method, and while attempts at counting were noticed they were usually cut short by the dropping of the curtain and errors of record followed. Another variation from the procedure of Lay was the manner of recording. Lay, for the sake of his first-year children who knew neither written number words nor figures, required the drawing of the number picture after each presentation, but he also retains the same method with older children (training school pupils); "so," says he, "hatte ich durch die Darstellung des Zahlbildes die sichere Gewähr, dass die Zahl nicht durch Abschätzen bloss erraten wurde." For careful investigation perhaps this was necessary, but not for school practice. Our records, sufficiently accurate no doubt, were made in figures by all children except those in the first half of the first school year. The latter were instructed to draw as many circles as they saw, and drawings otherwise correct were accepted as without error no matter what form the drawing took. In fact, many of the drawings were in single rows, not always of circles, but sometimes of crosses or strokes. As a matter of mere apprehension numerically, this was regarded as eminently satisfactory, though not favorable to economical imaging and representing. Lay was quite willing to accept similar results, as he says (55, p. 102) "Ältere Schüler trotz Verbots, unbewusst, die gewonnene Zahlvorstellung zuweilen nicht bildich, sondern durch Ziffern oder in Reihen dargestellten."

The cards were invariably presented according to the following formula for each card: Ready! Now! Write! Pencils down! "Ready" was the signal that a card was to be exposed, pupils took pencils from top of desk; at the same time the operator grasped at the bottom the hindmost sheet of the chart and lifted it to the level of the top of the chart, holding it in such a position that the card which it contained was still out of sight; a moment later the operator said, "Now!" and at the same time dropped the sheet on the front side, exposing the card, and simultaneously with the dropping started a stop-watch; (grasping a moment later the next hindmost sheet (blank) he lifted it in the same manner as the previous one); at the end of the time of exposure, the operator dropped the blank sheet, which acted as a curtain to shut off the view, and said, "Write!"; the pupils then recorded on their papers (in the lowest primary grade by drawing in the other grades by figures) the number viewed by them; after a short interval the operator said "Pencils down!" at which all pupils were required instantly to place pencils on the desk.

The interval allowed for recording was but a few seconds, not long enough to admit copying from a neighbor; the pupils soon learned that this was not the time for gazing around, if they wished to write anything. After "Pencils down!" had been complied with, there was no restriction on gazing, a privilege which most availed themselves of keenly. Of course, a somewhat longer interval was allowed, for recording, to the pupils of the lowest primary grade, whose records took the form of drawings.

In order that no uncertainty might arise in marking the papers at the conclusion of the tests, the pupils were required to place the record of each presentation in a certain place on the paper according to a simple scheme which they easily understood and followed.

In all of the experiments the numbers were presented in irregular order, as follows:

7_a 4_a 11_a 6_a 8_a 3_a 12_a 5_a 10_a 9_a 10_b 5_b 12_b 8_b 6_b 11_b 4_b 7_b 9_b 7_c 11_c 6_c 8_c 12_c 5_c 10_c 9_c 10_d 5_d 12_d 8_d 6_d 11_d 7_d 9_d 7_e 8_e 10_e 9_e 10_f 8_f 9_f 10_g 9_g 10_h

(Compare the pictures on page 204.)

It will be noticed that, when the whole series was presented, the number of presentations secured for each number was as follows:

For 3, one presentation;	8, six presentations;
4, two presentations;	9, seven presentations;
5, four presentations;	10, eight presentations;
6, four presentations;	11, four presentations;
7, five presentations;	12, four presentations.

Preliminary tests were given in all grades, using a different order, to accustom the pupils to the method of procedure. No record was kept of the results of these tests. During this practice the operator acquired a dexterity in the manipulation of the stop-watch and sheets of the chart that enabled him to maintain a sufficiently accurate time of exposure.

A.

The first series of tests, using forty-five cards (see page 204) presented in the order and under the conditions fully stated above, occupied a period of about two weeks and were repeated in full a little later except that the last ten cards (see order of presentation above) were not presented a second time to pupils of grade 1A. The following table sets forth the general results:

TABLE I

SHOWING MISTAKES BY GRADES IN THE APPREHENSION OF THE NUMBERS 5 TO 12, TAKEN COLLECTIVELY. (EXPOSURE TIME = 5 SEC.)

Grades	Average Number Pupils Present at Each Sitting	Number of Presentations, Both Series of Tests Included	Number of Chances for Mistakes	Mistakes	Per Cent
1A..........	23.0	74	1702	102	6.0
1B..........	37.6	84	3168	73	2.3
2A..........	42.9	84	3608	51	1.4
2B..........	50.8	84	4264	49	1.2
3A..........	50.0	84	4200	44	1.0
3B..........	49.5	84	4160	48	1.1
4A..........	49.5	84	3824	11	0.3
4B..........	48.0	84	4032	18	0.4
5A..........	52.0	84	4368	9	0.2
6A..........	48.0	84	4036	11	0.3
7A..........	45.2	84	3800	12	0.3
8A..........	43.5	84	3652	4	0.1
	536.0				

Notes on the Table:

1. Grade 1A means the first half of the first year, grade 1B the second half of the first year, etc.
2. The forty-five cards were presented in three "sittings" in the lower grades, in two in the upper grades.
3. The number of mistakes made in the apprehension of numbers 3 and 4 was so few that only the number space 5 to 12 is included in the table. 536 children made a total of only 5 mistakes on these numbers. All of these are doubtless errors of attention rather than of apprehension.
4. The number of chances for mistakes is computed by the formula C = N x n x n p, in which

   C = Chances for mistakes;

   N = Number of presentations of the several numbers;

   n = Number of repetitions.

   np = Average number of pupils present at a sitting

For example, grade 8A,

$$C = 42 \times 2 \times 43.5$$

$$C = 3652$$

The results shown in the table are graphically represented by the curve in Fig. 1.

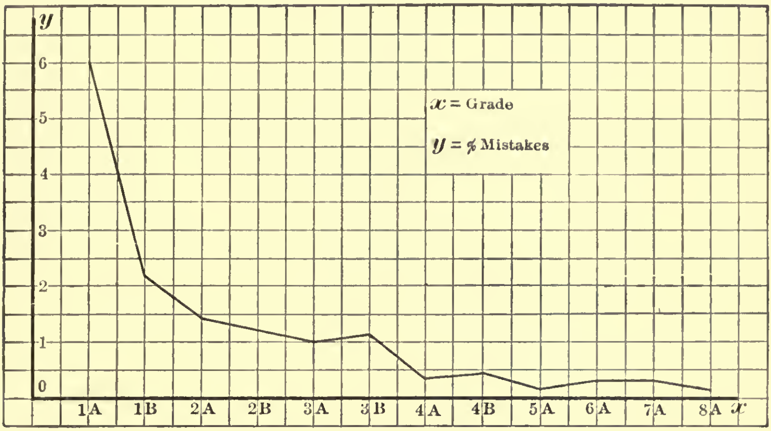


FIG. 1—See Table I.

Curve of Error for the Number Pictures 5 to 12 as a whole, Grades 1A to 8A, respectively.

The per cent column in the table and the general form of the curve shows a constant increase in sureness of grasp from the lowest grade to the highest. This showing, of course, was not unexpected considered as a general proposition. What we gain from a study of the curve is a knowledge of about where, so far as these children are concerned, uncertainty ceases and certainty takes its place. This is somewhere between 4A and 3B (age 8 to 10). Beginning with 4A the errors are so few that the curve approaches very close to the abscissa, and may safely be regarded, I think, as really the curve of attention; that is to say, in the absence of inattention these would be no errors and the curve would coincide with the abscissa. It should be said that the tests were given in the latter part of the term, so the 4A children, for example, were approaching the end of their work for that grade;

more importantly, perhaps, the children of 1A had had nearly a half-year's schooling; only the children of 1A present from the beginning of the term are included in the table.

The certainty ensuing to the children of 4A, it may be said, does not result in this case from continued practice with number pictures. Such practice is given in the school in grades 1A to 2A inclusive (using pictures of the domino pattern) but no longer. Whatever increase in power of apprehension of the numbers in this space may take place, therefore, in the subsequent year and a half or so, is the result of general number work or simply of growing maturity.

Grade 4A appears to be a rather late period for the mastery of the number space up to 12. It would seem that this certainty might have come about earlier if the children had been practiced with more favorable perception material (Anschauungsmittel) from the beginning. The comparative experiments of Lay and the fewness of errors made on the whole by even the youngest of our children with the quadratic pictures lead us to believe that with such material sure apprehension would bloom several seasons earlier. Still it is problematical; some factors may enter into the development of the number concept that postpone it to a certain period of the child's life regardless of favorable percepts. Only a thorough trying out can settle the matter.

It may occur to some that even the youngest pupils make a very high score; 6% failure, 94% accuracy (for 1A) would be regarded in most kinds of school work as a mark showing approximate perfection. However, that is not the case in these tests as our subsequent analysis of the results will show. The high score is occasioned by the comparative ease with which a few of the numbers are grasped. But other numbers present peaks of difficulty which show that, on the whole, the number space involved is far from being mastered.

It is possible to compare the results from our tests of 1A children with Lay's tests of first-year children with the quadratic pictures, as the two sets of children had been in school about the same length of time; neither set knew figures nor written number names; both had been taught perception and the addition and subtraction operations with objects and number pictures not quadratic.

Some results of Lay's tests, quadratic pictures.

(Drawn from tables, pp. 216, 219, 220, 221, 224.)

First-year Children			Training School Pupils		
Chances	Mistakes	Per Cent	Chances	Mistakes	Per Cent
174	51		186	8	
204	88		96	11	
144	65		86	4	
			228	27	
522	204	39.1	228	21	
			152	8	
			180	13	
			180	12	
			168	24	
			168	15	
			740	30	
			2,422	173	7.1

There is considerable difference between 39% of error, as shown by Lay's tests, and 6%. This is accounted for, of course, by Lay's short exposure time ($^1/_2$ to 1 second). Even the training school pupils have 7% of error. Now as these older pupils (age 16 to 18) certainly had at their mental stage clear concepts of the cardinal numbers, all their errors, aside from a few due to wandering attention, must have been due to the short exposure time ($^3/_8$ to $^5/_{11}$ sec.). If we compare their rate of error (7%) with the rate of the beginning children (39%), the ratio is approximately 1 to $5^1/_2$;

and this is about the ratio in the case of our third year and beginning children. A consideration of these facts leads us to suspect that Lay, in his anxiety to prevent counting, has somewhat overdone the matter of shortness of exposure time; in short, that his results would be more convincing pedagogically and quite as accurate relatively, if his exposures had been somewhat longer.

B.

Our first table has given us some general results, and, imporportantly, has, so to speak, cleared the decks for action. We may be permitted to drop the numbers 3 and 4 from all further consideration; and to give only incidental attention to school grades above 3B.

Confining our attention, then, to grades 1A to 3B and to the number space 5 to 12, we shall next make a study of the relative apprehensibility of the different numbers by all pupils of these grades, and by the several classes of the different grades.

The total number of mistakes and the percentage of error made by the pupils of grades 1A to 3B inclusive, taken collectively, in the apprehension of each number from 5 to 12 is given in Table II, following. Tables III, IV and V show the number of mistakes and percentage of error in each of the classes 1A to 3B for the same numbers. Using the data in these tables the graphs Figs. 2, 3, 4 and 5 have been constructed, Figs. 3, 4 and 5 being comparative graphs which will serve to make more specific the study of relative apprehensibility, both as between numbers and between grades.

From Table I and Fig. 1 we deduced the general result that the 250 pupils of grades 1A to 3B had not yet mastered the number space 5 to 12. By an examination of Table II and Fig. 2, we can discover in what particulars they have not mastered it. Numbers *eleven* and *seven* are the most prominent foci of error, followed

TABLE II

Showing Number of Mistakes by Grades 1A to 3B, taken Collectively, for each Number Picture, 5 to 12. (Time = 5 Sec.)

Number Pictures	Number of Pupils	No. of Presentations, Both Series Included	Number of Chances for Mistake	Mistakes	Per Cent
5	253	8	2024	22	1.1
6	253	8	2024	20	1.0
7	251.5	10	2515	63	2.5
8	250.5	12	3006	23	0.8
9	250	14	3500	67	1.9
10	249	16	3984	63	1.6
11	253	8	2024	77	3.8
12	253	8	2024	33	1.6
	252				

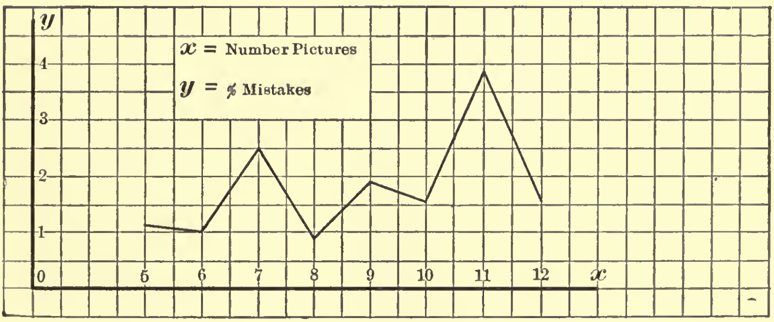


Fig. 2—See Table II.

252 Pupils, Grades, 1A to 3B; Number Pictures, 5 to 12 respectively.

at some distance by *nine*, *ten* and *twelve*, while *five* and *six* are points of comparative ease in the scale. At once it appears that the odd numbers are the peaks while the even numbers are the valleys with *eleven* as the most prominent peak. In our further analysis, it will appear that the *eleven* maintains its position as the chief source of error. It does not do this merely on the ground that it is a large number, for the larger *twelve* is apprehended more easily. *Seven* is *facile secundus*, in this group of pupils, as a source of error, but, as will appear later, does not maintain its position as such in every grade, though it is a prominent source throughout. *Nine* is only a shade more difficult than *ten* or *twelve*, but several degrees more so than *five*, *six* or *eight*. The surprises in the curve are, perhaps, that *twelve* is no more difficult than *ten* and that *eight*, near the middle of the scale, is easiest of all.

Tables III, IV and V, with their graphs, enable us to study more specifically: (a) the relative apprehensibility of the numbers in each grade, (b) the comparative apprehensibility of each number by the several grades.

In grade 1A, the curve (Fig. 3) corresponds remarkably with the general curve (Fig. 2) in general outline; but the contour is much

TABLE III

Showing Mistakes, Grades 1A and 1B, for each Number Picture, 5 to 12. (Time = 5 Sec.)

Number Pictures	1A				1B			
	Pupils	Chances	Mistakes	Per Cent	Pupils	Chances	Mistakes	Per Cent
5..........	23	184	7	.38	37	296	4	1.3
6..........	23	184	4	.22	37	296	1	0.3
7..........	23	207	18	.87	37.5	375	14	3.7
8..........	23	230	5	.22	38	456	4	0.9
9..........	23	253	17	.67	38.3	536	11	2.0
10..........	23	276	18	.65	38.5	616	8	1.3
11.........	23	184	25	.136	37	296	21	7.1
12..........	23	184	8	.43	37	296	11	3.7

more rugged, the peaks are higher and the valleys deeper; that is to say, the relatively great numbers of mistakes made in this lowest grade render the graph much more striking as a curve of error. In the order of difficulty of apprehension *eleven* ($13\frac{1}{2}$% error) again (as in the case of the whole group) occurs first, followed by *seven* (9% error), *nine* (7%), *ten* ($6\frac{1}{2}$%), *twelve* ($4\frac{1}{2}$%), *five* (4%), *six* (2%), *eight* (2%).

It is not contended that small differences in the percentages of error are of much significance. Fluctuations and distractions of attention on the part of the pupil (especially in the lowest grades) and unconscious variations of the conditions of presentation on the part of the operator are more or less unavoidable and doubtless account for the small differences in many cases and affect the percentages to a degree in all cases. The order stated should therefore not be taken too seriously as absolute even in the case

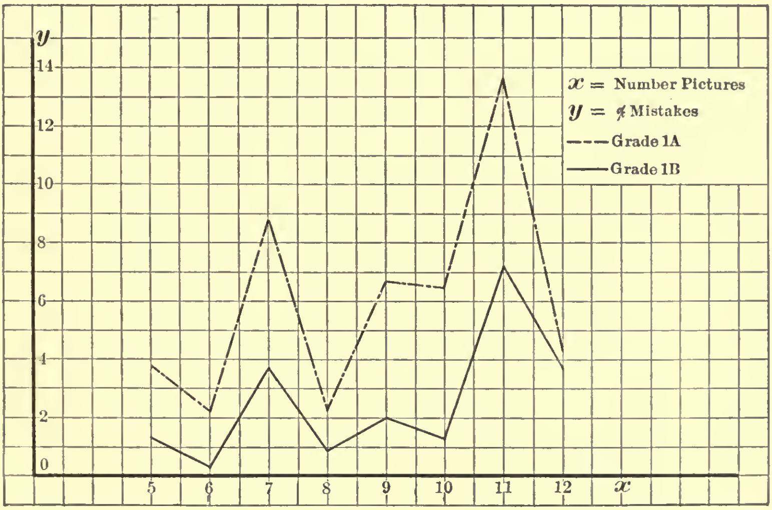


Fig. 3—See Table III.

Comparative Graph. Grades, 1A and 1B; Numbers 5 to 12.

of this group of 23 children. The result of the test can, however, be accepted without question in its larger aspects, viz., that *eleven*, *seven*, *nine* in the order named are the most difficult numbers for the children of this grade. The difference in per cent of error between *seven* and *nine* is small and might be considered insufficient to justify the relative ranking given to *seven* and *nine*, were it not that strikingly in the next higher grade (1B) the *seven* is more difficult than *nine;* and also in the group of pupils

TABLE IV

SHOWING MISTAKES, GRADES 1B, 2A, 2B, FOR EACH NUMBER PICTURE, 5 TO 12. (TIME = 5 SEC.)

No. picture	1A				2A				2B			
	Pupils	Chances	Mistakes	Per Cent	Pupils	Chances	Mistakes	Per Cent	Pupils	Chances	Mistakes	Per Cent
5	37	296	4	1.3	42	336	2	0.7	51	408	3	0.7
6	37	296	1	0.3	42	336	6	1.8	51	408	3	0.7
7	37.5	375	14	3.7	42.8	428	9	2.2	50.8	508	7	1.4
8	38	456	4	0.9	43.3	520	5	1.0	50.7	608	3	0.5
9	39	536	11	2.0	43.7	612	10	1.6	50.6	708	12	1.7
10	38.5	616	8	1.3	44	704	8	1.1	50.5	808	6	0.7
11	37	296	21	7.1	42	336	8	2.4	51	408	11	2.7
12	37	296	11	3.7	42	336	3	0.9	51	408	4	1.0

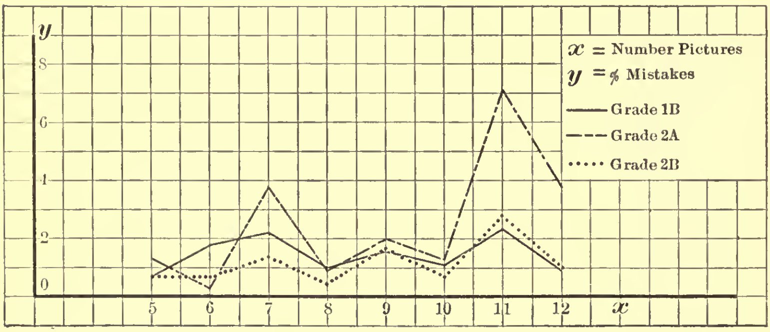


FIG. 4—See Table IV.

Comparative Graph. Grades, 1B, 2A, 2B; Numbers 5 to 12.

(1A to 3B) as a whole, although not in every class of the group (see Figs. 4 and 5).

We are led to believe also that the small difference between *ten* and *nine* is of significance, since the drop in the curve from *nine* to *ten* is constant in its appearance in each of the earlier grades (up to 2B inclusive). The number *twelve* appears, for some reason, to be relatively easy for the 1A children, not much more difficult than five. It will be remembered that *twelve* and *ten* stood on a par for the children of the whole group; here it appears

TABLE V

SHOWING MISTAKES, GRADES 2B, 3A, 3B, FOR EACH NUMBER PICTURE, 5 TO 12. (TIME = 5 SEC.)

No. picture	2B				3A				3B			
	Pupils	Chances	Mistakes	Per Cent	Pupils	Chances	Mistakes	Per Cent	Pupils	Chances	Mistakes	Per Cent
5	51	408	3	0.7	50	400	4	1.0	50	400	2	0.5
6	51	408	3	0.7	50	400	2	0.5	50	400	4	1.0
7	50.8	508	7	1.4	50	500	9	1.8	49.6	496	6	1.2
8	50.7	608	3	0.5	50	600	4	0.7	49.3	592	2	0.3
9	50.6	708	12	1.7	50	700	7	1.0	49.1	688	10	1.5
10	50.5	808	6	0.7	50	800	9	1.1	49	784	14	1.8
11	51	408	11	2.7	50	400	6	1.5	50	400	6	1.5
12	51	408	4	1.0	50	400	3	0.7	50	400	4	1.0

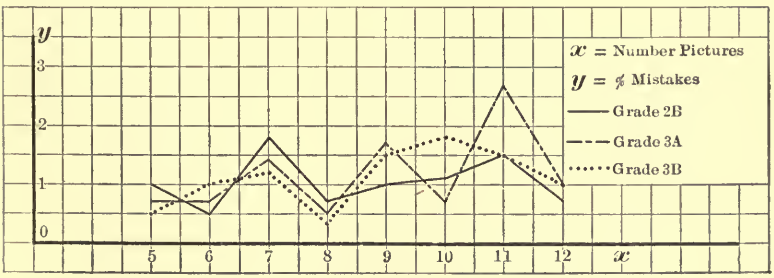


FIG. 5—See Table V.

Comparative Graph. Grades, 2B, 3A, 3B; Numbers 5 to 12.

easier, something that does not appear in any of the other grades; in fact, in the very next grade above (1B), the relative position of the two numbers is almost exactly reversed. However, I am inclined to believe in regard to *ten* that it is much more nearly on a par with *twelve* than it appears to be; we have seen that *ten* is probably much more difficult than *nine* than it appears to be, that is, *ten* is really easier than the curve shows it to be; of course, if this be true, the explanation of the position of *ten* is that it has received in the course of the tests more than its share of errors of attention or of manipulation, so that really its apprehension is no more difficult than that of *twelve*. I am confirmed in my view as to the true position of *ten* by the fact that Lay's first-year children found *ten* considerably more difficult than *nine*. I have gathered up the results of Lay's tests of first-year children on the quadratic number pictures in the following table. See also graph, Fig. 6.

TABLE VI

Number Pictures	Average Number Pupils	Number of Presentations	Chances for Mistakes	Mistakes	Per Cent
5	29	3	87	22	25
6	29	3	87	16	18
7	29	3	87	35	40
8	29	3	87	23	26
9	29	3	87	58	67
10	29	3	87	50	57

Lay's results have outstanding features of relativity that resemble our own. The odd numbers *seven* and *nine* are conspicuously more difficult than the neighboring even numbers; *five* is more difficult than *six*. But *nine* (and even *ten*) is more difficult than *seven;* and *eight*, which our children found so relatively easy, is not so relatively easy for Lay's children. All this is accounted for by his short exposure time, which, of course, affects the percentages all along the line, but in a greater degree proportionately in the case of the larger numbers. The effect is to raise *nine* (and even *ten*) above *seven*, and to give the easy *eight* an additional modicum of difficulty. It may be claimed also that our tests

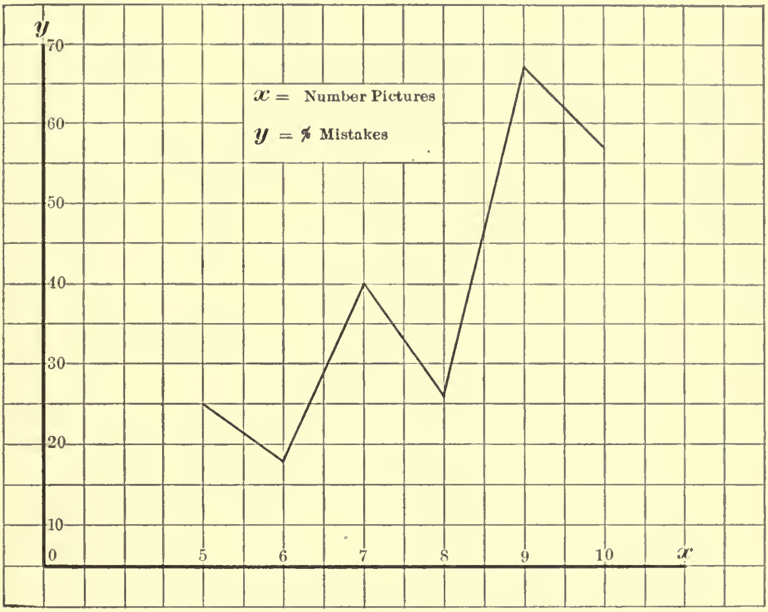


FIG. 6—See Table VI. Apprehension. Lay's First Year Pupils, Numbers 5 to 10.

were somewhat more thoroughgoing in that more chances for viewing each of the numbers were given. Thus each of Lay's pupils had only 3 chances on each number while ours had

on the number five,	8 chances;
on the number six,	8 chances;
on the number seven,	9 chances;
on the number eight,	10 chances;
on the number nine,	11 chances;
on the number ten,	12 chances.

On account of both of these facts, we are inclined to think that our graph reflects the performances of first-year children under ordinary conditions better than that of Lay, though neither, except possibly in their most outlying aspects, can afford a basis for absolute conclusions of universal validity; many more children need to be tested in order to establish a norm of absolute performance, and, at the last, the striking thing about this norm would be that in these days of common belief in a plurality of abilities wrapped up in a so-called single ability and in the variability among individuals, no one would expect any individual or even any class to conform to the norm. Nevertheless, if the number pictures prove to have sufficient value as teaching material to make it desirable to use them much more liberally than is at present the custom, doubtless the establishment of such a norm would be a distinctly valuable contribution to exact pedagogy; in that it would afford to teachers a standard which does not now exist for measuring the performances of their pupils with this material.

In grade 1B, while the curve (Fig. 3) closely resembles that of 1A in its general aspect, the following changes may be noted. *Ten* has become about as easy as *eight*, in fact has become so much easier than it was in 1A that the comparatively stationary *twelve* has its relation with it changed about; again *seven* has increased in ease so that it is no more difficult than *twelve* and *nine* coming closer to *eight* in ease of apprehension is no longer more difficult than *twelve*, though still, contrary to Lay's results, more difficult than *seven*. The greatest growth in apprehension is in the numbers *seven*, *nine*, *ten* and *eleven* as is shown by the following table of comparative percentage of error.

As has been heretofore pointed out, the position of *ten* in the curve for 1A is doubtful, and it is therefore doubtful whether the growth in apprehension is as much in the case of *ten* as it appears to be, since *ten* is probably easier for 1A than the curve indicates.

As might be expected, all of the numbers have suffered an increase in ease; *six* has become so easy as to be regarded as practically mastered. But this does not mean that we can conclude that *six* is mastered in grade 1B generally (the children of this class had been from 14 to 16 weeks in 1B and had been in school about 5 months longer than the 1A children). It means that *six* had been mastered by this particular group of 38 children; but that it is not fully grasped by all 1B children is shown by Table IV, where it is seen that, especially in grade 2A, it still affords considerable difficulty. The records of the individual children of this grade show that the mistakes on *six* are distributed among six children, three of whom make mistakes on no other number, and one of whom repeats a mistake on *six* when given an individual test.

TABLE

COMPARING APPREHENSION IN GRADES 1A AND 1B. SEE TABLE III AND FIG. 3.

Number Pictures	1A—% Error	1B—% Error	Decrease
5	3.8	1.3	2.5
6	2.2	0.3	1.9
7	8.7	3.7	5.0
8	2.2	0.9	1.3
9	6.7	2.0	4.7
10	6.5	1.3	5.2
11	13.6	7.1	6.5
12	4.3	3.7	0.6

On the number space as a whole, 1B shows great increase in accuracy, the rate of error decreasing from 6% in 1A to 2.3% in 1B (See Table I). In grade 1A the relative apprehensibility was in the following order, beginning with the most difficult (subject to qualifications as heretofore discussed): *eleven*, *seven*, *nine*, *ten*, *twelve*, *five*, *six* = *eight*; in 1B we find the order to be (subject to similar qualifications): *eleven*, *twelve* = *seven*, *nine*, *ten* = *five*, *eight*, *six;* or in tabular form:

$$1A \quad 11 - 7 - 9 - 10 - 12 - 5 - \mathit{6} = \mathit{8}$$
$$1B \quad 11 - \mathit{7} = \mathit{12} - 9 - \mathit{10} = \mathit{5} - 8 - 6$$

In grade 2A, with five months longer school experience, the pupils show a decided superiority over those of 1B, in the apprehension of the number space as a whole, but not so great as that of 1B over 1A. What increase the several numbers show is exhibited in the following comparative table of percentages of error:

TABLE

Comparing Apprehension in Grades 1B and 2A. See Table IV and Fig. 4.

Number Pictures	1B—% Error	2A—% Error	Decrease
5	1.3	0.7	0.6
6	0.3	1.8	...
7	3.7	2.2	1.5
8	0.9	1.0	...
9	2.0	1.6	0.4
10	1.3	1.1	0.2
11	7.1	2.4	4.7
12	3.7	0.9	2.8

Twelve is so much easier for this group of 43 pupils than for 1B, that it has been restored nearly to a parity with *ten* (which it has in the whole group of 250 pupils), and is no more difficult for these pupils than *eight; eleven*, showing the greatest increase in accuracy, is now on a par with *seven*, which shows a smaller increase; *six*, as has been noted before, is noticeably harder for this grade than for either of the lower grades; *five* shows some improvement while *nine* and *ten* are practically stationary. In order of relativity the numbers stand now in the following order:

$$\mathit{11} = \mathit{7} - \mathit{9} = \mathit{6} - \mathit{10} = \mathit{12} = \mathit{8} - 5$$

The pupils of 2B, in their apprehension of the numbers *five*, *eight*, *nine*, *ten*, *eleven*, and *twelve*, show differences so small from 2A that they are probably of no significance, but show a slight increase in ease in the case of *six* and *seven*. The increase in *six* is greater than ordinarily might be expected in passing from 2A to 2B on account of the anomalous position of *six* in the curve for 2A. Any attempt to differentiate 2B from 2A as to order of relativity would in view of these small differences be of doubtful value.

Reference to Table I shows on the whole little change for the better in apprehension after grade 2A. The pupils seem to have reached a plateau, which extends through grade 3B, and only five months later (in grade 4A) does it clearly appear that certainty of apprehension for the number pictures 5 to 12 has ensued (see Fig. 1). In this connection, the fact which has been referred to before that all drill with number pictures ceased in grade 2B may have considerable meaning. If this drill had been continued it is at least fair to presume, but by no means certain, that the pupils might have reached certainty prior to grade 4A.

An examination of Table V and the accompanying graph (Fig. 5) yields some anomalies which perhaps would not occur if this were a study of a single group of pupils progressing through the grades from 1A to 3B. The curves of 2B and 3A cross one another at several points, and there appears to be actual retrogression in grade 3A in the cases of the numbers *five*, *seven*, *eight*, and *ten*. However, the differences in accuracy on these numbers between the two classes is so small that the most that can be concluded is that at least there has been no progress. *Six* and *twelve* have remained practically stationary, while undoubted progress has been made in *nine* and *eleven*. Grade 3B presents what may be regarded as real anomalies; the curve cuts that of both 2B and 3A, showing that 3B is poorer than 3A on the number *ten* and still poorer than 2B, in fact finds *ten* the hardest of all the numbers;

showing also that 3B is poorer than 3A on *nine*, being but little, if any better on this number, as well as on *seven* and *eight* than 2B, while the same peculiar slump on *six* as was seen in 2A is noticeable. No decided progress over 3A is shown in the case of any number. These anomalies may be laid to the idiosyncrasies of a class differing, as all the classes have, from its predecessors in its composition. Indeed, the scores all along the line should be regarded as partially due to these varying individualities, so that the graphs for any class are not precisely what they would have been had the same group of individuals been studied through a succession of school terms. Any language used in the foregoing discussion that seems to imply that we are here studying the development of such a group has been used for convenience. Such a study, of course, would require several years and has so far as the writer knows nowhere as yet been attempted, but would be well worth while. With such a procedure, the anomalies in the later grades under consideration would probably not occur, though this is by no means certain, as the studies of Courtis and others have taught us that in arithmetic (and doubtless also in other subjects) classes and individuals do not show constant and steady progress but have their periods of no advance and of actual retrogression. However, it must be remembered that this is an objective statement of the matter and that subjectively the "levels," "plateaus," "retrogressions" may represent periods of incubation that are the potentials of actual progress in objective efficiency to be revealed, often in astonishing ways, later on.

Our main purpose in this part of the work has been to study growth in apprehension and in the concepts of the several numbers in the number space 5 to 12 when presented by means of the quadratic pictures. It has already been pointed out that the measure of apprehension is also the measure of at least conceptual images, if not of abstract concepts. The number pictures themselves constitute for the child a tremendous step toward complete abstraction, in that he is taken away from the specific numbered objects of various kinds to a representation that may stand for any of them. Many German students of pedagogy beyond all others have perceived this truth and are in substantial agreement in regard to it, regardless of what general theory of the formation of the concept they may severally hold. Sound or sight images of things presented in succession have been conclusively proved by the experiments of Lay to be inferior to any sort of group pictures; they hold the mind to particularities in a way which militates against (rather than helps) the formation of the concept, and consequently against apprehension; for while the concept grows out of apprehension (let us say) it in turn reacts on perception and imaging so that the growth of the former renders the latter *pari passu* more certain and adequate.

Proceeding upon the assumption, not unwarranted, that the earliest appearance of mastery of a number in any grade is, at least, the sign of the *possibility* of its mastery by the children of that grade, we offer as the tentative order of sure apprehension for children trained as ours were the following tabular statement, annexing the percentage of error.

Number 6	Grade 1B	Approximate Age $6\frac{1}{2}$–$7\frac{1}{2}$	0.3
Number 5	Grade 2A	Approximate Age 7 –8	0.7
Number 8	Grade 2B	Approximate Age $7\frac{1}{2}$–$8\frac{1}{2}$	0.5
Number 10	Grade 2B	Approximate Age 8 –9	0.7
Number 12	Grade 3A	Approximate Age $8\frac{1}{2}$–$9\frac{1}{2}$	0.7

In coming to this conclusion we have selected somewhat arbitrarily a rate of error, 0.7 as the permissible maximum of inaccuracy for any class which may fairly claim a condition of certainty with respect to any number. Our reason for this is that between this rate and the best record for the numbers *nine*, *seven* or *eleven* (not included above as among the mastered numbers) there is a considerable gap:

Number	Per Cent of error	
9	1.0	best record in 3A
7	1.2	best record in 3B
11	1.5	best record in 3A, 3B

That is to say, when a class makes 12 or more errors per thousand chances distributed among half a dozen or more pupils of fair to excellent numerical ability (as shown by teachers' marks in arithmetic), we consider that the class, as such, has not mastered the number occasioning the errors, though, of course, individuals in the class have done so. That this is the fact with respect to *seven* and *eleven* appears in our tables of individual records. Number *nine* seems well on the road to mastery in 3A, but falls back to .015 in 3B.

For the sake of completeness, the records of the numbers *nine*, *seven* and *eleven* are given in the higher grades (rate of error).

Number	4A	4B	5A	6A	7A	8A
9	0.1	0.0	0.1	0.0	0.3	0.2
7	0.2	0.2	0.0	0.6	0.4	0.5
11	1.1	1.0	0.1	0.2	0.3	0.0

There is nothing in these records that calls for further remark except that *eleven* appears still to be giving some trouble in 4A and 4B.

These results seem to bear out the opinion sometimes expressed that number concepts arise rather late in the child's life and that therefore it would be wise to postpone the study of number to a time considerably later than that which now is customary. Prof. Burnham expresses this opinion in his article on "Arithmetic and School Hygiene" (see our review of this article, pp. 143–148, and says that the agony of spirit in which German educators indulge over the matter of number pictures and other material for teaching beginning children might be wholly escaped provided number teaching were not begun earlier than the age of 9 or 10. It must be admitted that from our angle and conditions of approach to this subject, it seems to appear that any contentful concepts of the cardinal numbers must be denied to children under the age of 9 or 10.

But before reaching this conclusion it would be well to bear in mind the following points:

First. The limitations of our method which (necessarily in this study) has been to test children of several grades at about the same time and then to treat the results as if they had been obtained from a single group of children systematically trained through the several grades.

Second. A pertinent question to ask would be whether the concepts would not be deferred still longer without such training as even our children received, admittedly not of the most systematic or economical kind.

Third. No one is yet in a position to say just what would be the result of a systematic endeavor to build concepts with economical material. No one can say, to put the matter negatively, that the late rise of concepts observed is not the result of awkward and wasteful teaching procedure.

C.

As a supplementary proceeding, it was thought desirable to give the children another series of tests, using a varying time of exposure for the several number pictures from *three* to *twelve* to see what effect, if any, such variation might have on apprehension. In these tests the number of seconds of exposure given to each number was made equal to the number of units in the number. Thus the number *five* was given an exposure of five seconds, *six* of six seconds, *seven* of seven seconds, etc. By this means *three* and *four* were given a shorter time than in the tests with the uniform exposure, *five* the same time, and the rest a longer time in proportion to their size.

As the purpose was to discover only the general tendency of the results of such changes in the time, the presentations were limited to the first 21 cards, preserving the same order as in the uniform tests. (See order of presentation, page 211.) The number of presentations thus secured for each number was as follows:

For 3—one presentation
4—two presentations
5—two presentations
6—two presentations
7—three presentations
8—two presentations
9—two presentations
10—two presentations
11—three presentations
12—two presentations

The tests were given to grades 1A to 3A inclusive, at one sitting for each grade, shortly after the uniform tests were completed.

TABLE VII

SHOWING THE TOTAL NUMBER OF MISTAKES IN THE APPREHENSION OF THE NUMBERS 5 TO 12, IN EACH OF THE GRADES 1A, 1B, 2A, 3A; UNIFORM (5 SEC.) EXPOSURE COMPARED WITH VARIED EXPOSURE.

Grade	5 Seconds					Varied				
	No. Pupils Present	No. Presentations, 2 series	Chances	Mistakes	Per Cent	No. Pupils Present	No. Presentations, 1 series	Chances	Mistakes	Per Cent
1A	23	74	1,702	102	6.0	23	18	414	27	6.5
1B	37.6	84	3,168	73	2.3	37	18	666	17	2.6
2A	42.9	84	3,608	51	1.4	46	18	828	13	1.6
2B	50.8	84	4,264	49	1.2	51	18	918	15	1.6
3A	50	84	4,200	44	1.0	51	18	918	12	1.3

The procedure was in all points the same as already described for the previous tests. The comparisons between the two series have been made on the basis of rates of error, and are presented in the tabular and graph forms following.

In Table VII and Fig 7, the data are given for the comparison of the performances of each of the grades on the numbers from 5 to 12 taken collectively. Again the numbers *three* and *four* are not considered, as only two errors were made on *three* and one on *four*.

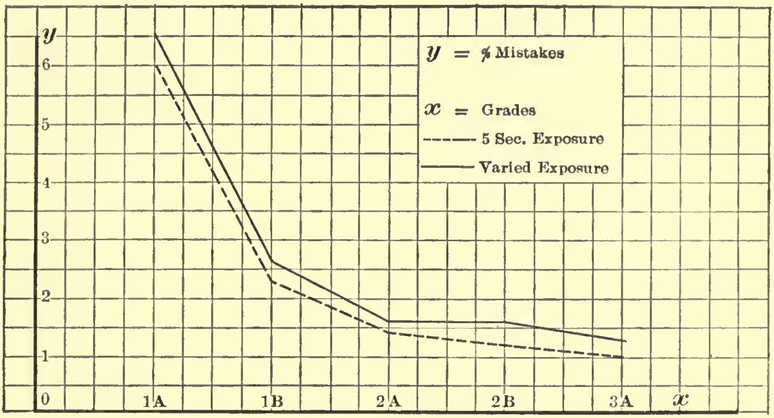


FIG. 7—See Table VII.
Comparative Graph, uniform and varied exposure, Grades 1A to 3A.

It appears from the table and the curve (Fig. 7) that there is a slight general tendency throughout the grades to make more mistakes on the number pictures as a whole when presented during different times (which except in the case of *five* are longer times) than when given a uniform five seconds exposure. It might be thought by the teacher who has given the matter only casual attention that at least the larger and (presumably) more difficult numbers would be helped by giving the pupils a longer look. Our general results (Table VII) seem to show, however, that such is not the case.

Whether this same tendency is shown in the case of every number, or only some of them, may appear from Table VIII, and the accompanying graph in Fig. 8.

TABLE VIII

SHOWING TOTAL NUMBER OF MISTAKES, GRADES 1A TO 3A, FOR EACH NUMBER PICTURE, 5 TO 12; UNIFORM (5 SEC.) EXPOSURE COMPARED WITH VARIED EXPOSURE.

	5 Seconds					Varied				
Number Pictures	Number Pupils	No. Presentations, 2 series	Chances	Mistakes	Per Cent	Number Pupils	No. Presentations, 1 series	Chances	Mistakes	Per Cent
5	203	8	1,624	20	1.2	208	2	416	4	1.0
6	203	8	1,624	16	1.0	208	2	416	3	0.7
7	201.8	10	2,018	57	2.8	208	3	624	15	2.4
8	201.2	12	2,414	21	0.9	208	2	416	6	1.5
9	200.6	14	2,809	57	2.0	208	2	416	12	2.8
10	200.2	16	3,204	49	1.5	208	2	416	6	1.5
11	203	8	1,624	71	4.4	208	3	624	30	4.8
12	203	8	1,624	29	1.8	208	2	416	8	2.0

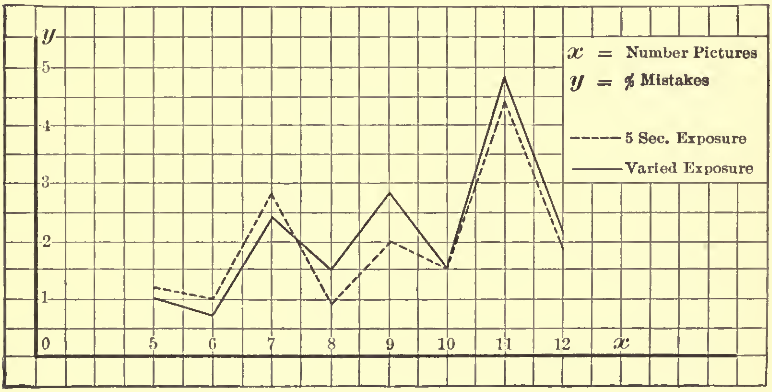


FIG. 8—See Table VIII.
Comparative Graph, uniform and varied exposure, Numbers 5 to 12.

Here the 200 pupils of grades 1A to 3A are taken as one group and the rate of error for the group computed for each number. At once we see that the numbers *eight*, *nine*, *eleven* and *twelve* seem to show the same tendency toward greater error when the exposure is varied, slight in the case of *eleven* and *twelve*, somewhat more decided in the case of *eight* and *nine;* while *five*, *six* and *seven* seem to show an opposite tendency and *ten* an equal score whether the exposure is 5 sec. or 10 sec. *Seven* and *nine* change their relative positions in comparative rate of error, but no other change in relativity takes place by reason of the longer exposures. In fact, the general similarity of the two curves testifies to the reliability of our data in both series of tests. It was thought worth while to make a further inquiry into the comparative apprehension of the numbers found in our previous study to be the most difficult, viz., *seven*, *nine* and *eleven*.

In Table IX, following, the comparative rates of error in each of the grades are given for the number *seven;* in Table X for the number *nine;* in Table XI for the number *eleven*.

In Table VIII and Fig. 8, a smaller tendency to error in the case of the number *seven* appeared when the time of exposure is seven seconds and the whole group of children in grades 1A to 3A considered. It now appears (in Fig. 9) that this tendency holds in every one of the grades in question except 2B, quite decided in the grades 1A and 1B, less so in 2A and 3A. The low average difference between scores on this number for the whole group of children is caused by the indifference of grade 2B. It would seem fair to conclude that at least in the first year the additional 2 sec. exposure is of advantage.

Table X and corresponding graph show that the number *nine*, in respect to which the whole group seemed to favor the uniform exposure somewhat decidedly, has grade 1A to thank for almost all of this seeming preference, though the tendency in all the other grades is in much smaller degree the same except in 2A, where

an opposite, but not decidedly opposite tendency, is shown.
With respect to number *eleven* (Table XI and Fig. 11) we may say that only decided tendencies appear in grades 2A and 3A and

TABLE IX. NUMBER 7

SHOWING THE NUMBER OF MISTAKES MADE BY EACH OF GRADES 1A TO 3A IN THE APPREHENSION OF THE NUMBER 7; UNIFORM COMPARED WITH VARIED EXPOSURE

	5 Seconds					Varied				
Grade	Pupils	Presentations	Chances	Mistakes	Per Cent	Pupils	Presentations	Chances	Mistakes	Per Cent
1A	23	9	207	18	8.7	23	3	69	5	7.2
1B	37.5	10	375	14	4.8	37	3	111	4	3.6
2A	42.8	10	478	9	2.2	46	3	138	2	1.4
2B	50.8	10	508	7	1.4	51	3	153	2	1.3
3A	50	10	500	9	1.8	51	3	153	2	1.3

TABLE X. NUMBER 9

	5 Seconds					Varied				
Grade	Pupils	Presentations	Chances	Mistakes	Per Cent	Pupils	Presentations	Chances	Mistakes	Per Cent
1A	23	11	253	17	6.7	23	2	46	5	10.9
1B	39.3	14	536	11	2.0	37	2	74	2	2.7
2A	43.7	14	612	10	1.6	46	2	92	1	1.1
2B	50.6	14	708	12	1.7	51	2	102	2	2.0
3A	50	14	700	7	1.0	51	2	102	2	2.0

TABLE XI. NUMBER 11

	5 Seconds					Varied				
Grade	Pupils	Presentations	Chances	Mistakes	Per Cent	Pupils	Presentations	Chances	Mistakes	Per Cent
1A	23	8	184	25	3.6	23	3	69	10	14.5
1B	37	8	296	21	7.1	37	3	111	7	6.3
2A	42	8	336	8	2.4	46	3	138	5	3.6
2B	51	8	408	11	2.7	51	3	153	4	2.6
3A	50	8	400	6	1.5	51	3	153	4	2.6

favor uniform exposure. Here again, as in case of *seven*, the small difference in the scores for number *eleven* in the general curve (Fig. 8) is accounted for by the indifference of 2B. So far, then, as decided differences in accuracy appear in the cases of the larger

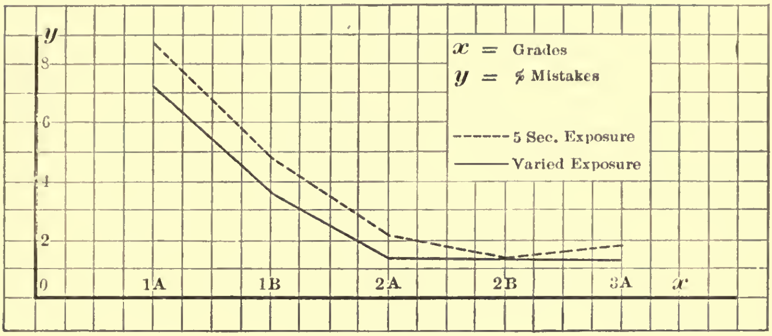


FIG. 9—See Table IX.
Comparative Graph, uniform and varied exposure, Number 7.

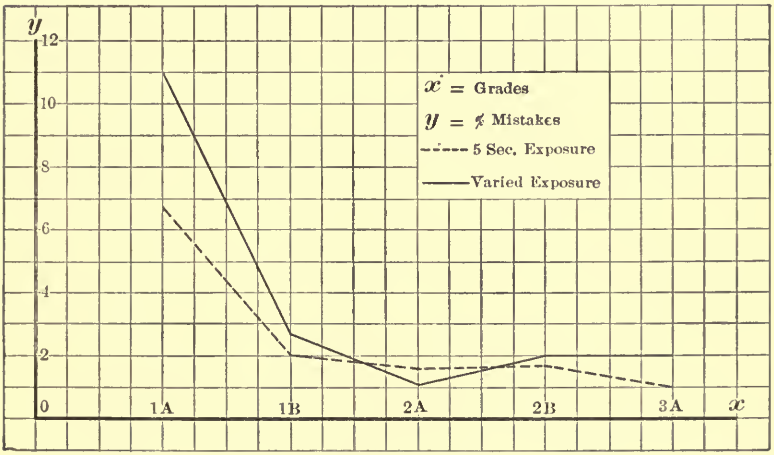


FIG. 10—See Table X.
Comparative Graph, uniform and varied exposure, Number 9.

numbers *nine* and *eleven*, they show against prolonging the exposures of these numbers to nine and eleven seconds respectively, and we may conclude in view of all our data on this point that apprehension gains nothing by varying the time of exposure according to the size of the numbers, except possibly in the case of

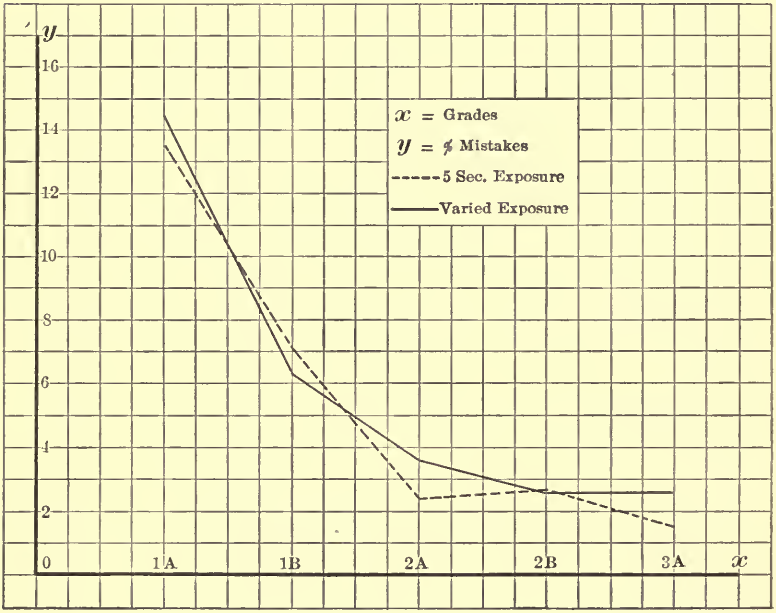


FIG. 11—See Table XI.
Comparative Graph, uniform and varied exposure, Number 11.

the number *seven* for first-year children; and even in the latter case the advantage is probably not great enough to warrant varying the time in the ordinary work of the school room, a procedure which would be somewhat difficult to manage under the usual conditions. A pertinent question here is, if the number *seven* gains in apprehensibility by reason of an additional 2 seconds exposure, why do not the numbers *nine* and *eleven* gain much more by reason of an additional four and six seconds respectively? The answer that I offer is that while two added seconds may help first-year children to get *seven*, nearly twice 5 seconds and more than twice five seconds are entirely more time than the children need to get *nine* and *eleven* respectively, and therefore the prolongation of the time gives opportunity for the entrance of elements of confusion, such as wandering attention, uncertainty due to reconsideration of first impressions, etc. The children during these long exposure-times were with difficulty restrained from seizing their pencils to write several seconds before they received the command "Write!", and it was a constant observation during the tests that the time after the first few seconds was time lost, if not worse than lost.

In conclusion, we may refer briefly (a) to the distribution of the pupils with reference to the mistakes made, (b) to the effect on apprehension, of the strokes introduced into Lay pictures to indicate operations, and lastly to two correlations, viz., apprehension with general intelligence and with teachers' marks in arithmetic.

In Table XII the distribution is shown. It indicates a considerable degree of variability, but this was to be expected in work of this character. From $\frac{1}{5}$ to $\frac{1}{3}$ of the pupils in the classes above 1A make no mistakes; but the number making mistakes is sufficiently large to warrant the acceptance of the class average as satisfactory measures for comparative purposes. This we have done in all our statistical work after changing our absolute number of mistakes to number of mistakes per 100 chances, or, in other words, finding the ratio of errors to a common base. A simpler tabulation, perhaps, for the comparative work might have been made by finding the average number of mistakes made by the class at each presentation of a number and changing this to the basis of a class

TABLE XII

DISTRIBUTION OF MISTAKES

Mistakes		0	1	2	3	4	5	6	7	8	9	10	11	Total Frequencies	Average Number of Mistakes	Average Deviation
1A	Boys	1	1	0	3	2	1	0	1		0	0	1	10	4.10	
	Girls	0	2	3	2	0	1	1	1		1	1	0	12	4.25	
	Total	1	3	3	5	2	2	1	2		1	1	1	22	4.2	-2.1
1B	Boys	7	2	3	6	0	1	0				1		20	2.0	
	Girls	8	4	1	0	1	0	2				0		16	1.4	
	Total	15	6	4	6	1	1	2				1		36	1.75	1.2
2A	Boys	4	8	7	0									19	1.2	
	Girls	4	11	5	1									21	1.1	
	Total	8	19	12	1									40	1.15	.5
2B	Boys	7	16	4	0									27	.9	
	Girls	8	9	4	1									22	.9	
	Total	15	25	8	1									49	.9	.5
3A	Boys	9	13	1	1	0								24	.75	
	Girls	7	10	5	0	1								23	1.1	
	Total	16	23	6	1	1								47	.9	.5
3B	Boys	13	5	5	1		1							25	.9	
	Girls	5	14	4	1		0							24	1.0	
	Total	18	19	9	2		1							49	1.0	.6

of 50. But the results would have been the same relatively and we believe that the tables as they stand are better exhibits of statistical completeness.

We found little difference, taking the group as a whole, between the boys and girls in variability, and, as Table XII shows, there is no difference worth mentioning in accuracy.

A question of some interest in connection with the presentation of the number pictures to pupils is whether the strokes have an influence on the perception of the group as a whole. It was Lay's purpose in using them to assist pupils to see small pictures in the larger ones, and so to render calculation *anschaulich.* But if they work as a hindrance to apprehension, as such, it might be better to postpone the introduction of the strokes until the pupil has considerably advanced in the stage of numeration. However that may be, it will be of interest and possibly of some value to know the precise effect, relatively speaking, which the strokes had on the apprehension of our children (grades 1A to 3B taken as a single group).

TABLE XIII

A—Effect of Oblique Lines

Number	Pictures with Vertical Lines. Presentations	Average Number of Mistakes. Each Presentation	Pictures with Oblique Lines. Presentations	Average Number of Mistakes. Each Presentation
5	3	4	1	8
6	2	3	2	6
7	3	8	2	15
8	3	3	3	4
9	5	6	2	16
10	5	5	3	10

B—Effect of More than One Vertical Line

Number	Mistakes One Line	Mistakes Two Lines	Mistakes Three Lines	Mistakes Four Lines
7	10	..	5	..
9	7	1	..	8
10	5	2	6	7

C—Effect of Location of a Line (Vertical)

Number	Mistakes	Mistakes	Mistakes
	: : │ ·	: │ : ·	
5	3	5	
	: : : │ ·	: : │ : ·	
7	13	7	
	: : : │ :	: : │ : :	
8	4	2	
	: : : : │ ·	: : : │ : ·	: : │ : : ·
9	5	6	10
	: : : : │ :	: : │ : : :	
10	5	5	

D—Effect of Location of a Line (Oblique)

Number	Mistakes	Mistakes	Mistakes
	: : %	: % :	
6	4	8	
	: : : %	: : % :	
8	6	2	
	: : : : %	: : : % :	: : % : :
10	11	14	4

The general result is that oblique lines seem to lend difficulty; two vertical lines help, but more than two do not; the location of the line sometimes helps, sometimes hinders, or is a matter of indifference. It is probable that the oblique lines, since they double the amount of mistakes, are a real menace to young children and that their use might well be postponed for a time.

Two groups of pupils, one of 58 first-year pupils and one of 89 second-year children were used to work out a possible correlation between the ability to apprehend numbers, as measured by number of mistakes made, and ability in arithmetic as indicated by teachers' marks. The degree of correlation was sought separately for each group.

The pupils were numbered in rotation (from 1 to 58 in the first group and from 1 to 89 in the second group) and then thrown into quartiles according to their ranking by teachers, the highest fourth of the class in Quartile I, the next highest in Quartile II, etc. Red ink was used for the numbers in Quartile I, and green, violet and blue (or black) for the other three quartiles respectively. Thus, in group No. I first-year pupils fall into the first quartile as follows (see quartile arrangement below); those numbered 8, 26, 48 having a mark of 90 (see marks below the quartiles), those numbered 9, 25, 40, 41, 58 having a mark of 85, etc. Now the pupils are again arranged in quartiles from high to low with respect to ability in apprehension, using the same numbers and the same colored ink that they had in the first arrangement. One can judge at a glance, in a general way, how the pupils of each quarter of the group in the first ability are distributed among the quartiles of the second arrangement. If the correlation were perfect all the reds of the first arrangement would appear in Quartile I of the second, all the greens in Quartile II, all the violets in III, all the blues in IV.

CORRELATION—APPREHENSION OF NUMBER PICTURES AND TEACHERS' MARKS IN ARITHMETIC

GROUP NO. 1.—FIRST-YEAR PUPILS (1A AND 1B)

Arithmetic

Q I			Q II		Q III		Q IV		
8	9	38	18	32	49	27	22	28	6
26	25	39	50	43	51	30	16	47	
48	40	46	17		56	20	45	55	
	41	54	12		23	24	53	57	
	58	52	4		21	37	36	31	
		1	2		29		19	44	
			7		33			42	
			14		34			15	
			10		35				
			5						
			11						
			13						
			3						
Teachers' Marks 90	85	80	80	75	75	70	70	65	60

Number Pictures

Q I	Q II				Q III			
8	56	41	1	50	17	12	33	2
26	28	52	18	29	20		34	4
48		23	21	27			35	7
25		47		30			24	14
40		55					37	22
58		57					31	
38								
39								
46								
54								
32								
43								
49								
51								
Mistakes 0	*Mistakes* 0	12	14	24	27	31	36	40

Q IV										
44	9	42	5	45	11	15	36	19	13	3
	10		16	53		6				
48	54	60	68	72	81	95	120	122	135	150

TABLE XIV

	Red	Per Cent	Green	Per Cent	Purple	Per Cent	Black	Per Cent	
Quartile I	10	71	2	13	2	14	0	0	
Quartile II	3	21	2	13	6	43	4	27	
Quartile III	0	0	6	40	6	43	2	13	
Quartile IV	1	7	5	33	0	0	9	60	
	—	—	—	—	—	—	—	—	
	14		15		14		15		58 Cases

Such, however, is not the case; some of the reds are in II and IV, the greens are scattered, etc.

Table XIV shows the exact distribution. Read each vertical column downward: Of the 14 cases in the first quartile (red) in arithmetic, 10 (71%) are in the first quartile in apprehension, 3 (21%) are in the second and 1 in the fourth, etc.

A study of the table reveals that pupils assigned by teachers' marks to the first and last quartile largely remain there (71% in the first, 60% in the last) in apprehension; but only 13% remain in the second and 43% in the third. Forty per cent of those in the second quartile in the first arrangement are in the third in the second arrangement and 33% in the third, while 43% of those at first in the third have moved to the second. However, of the 29 pupils assigned at first to the median rank (second and third quartiles) 20 are still to be found in that rank though 12 of the second quartile and 8 of the third have crossed the line between the two. Two reasons may be mentioned for this greater interchange in the middle section between quartiles; first, those in the extremes of the class have only one way to move, and thus their chances of moving are reduced; second, the teachers' marks are, doubtless, less carefully differentiated as between fair and good for the middle section, since the medium abilities are more difficult to discriminate than the extreme ones. Perhaps the clearest interpretation of the table may be given thus:

Number of cases true to form (assuming the marks in arithmetic as the "form"—27; number close (not more than one remove from the original quartile)—19; number off "form"—12. Now for a single numerical expression of these facts, we might say 27 points true plus 10 points ($\frac{1}{2}$ x 19) close equal 46 points out of a possible 58, or 64%. (It is not claimed, of course, that this is a mathematically correct coefficient of correlation; but it will do for comparative purposes; for example, a similar computation for group 2 (see next page) yields 76%, while everything about the table points to a higher degree of correlation than in the first group).

We may say, then, that the first-year children show a positive correlation between the two abilities. Turning now to the second group (second-year children) we find (Table XV) that again the pupils first assigned to the extreme quartiles remain there to a still larger extent than before (73% and 65%) while fewer cases are lost to the median quartiles. The number "true" is 56, close—25; off—8; per cent expressing this—76% showing a higher degree of correlation than the first group.

We tried another correlation by a somewhat different method, number apprehension with general intelligence, with each of the foregoing groups. The pupils were grouped into three or four ranks on the basis of general intelligence and also into three or four ranks on the basis of mistakes in apprehension. The object was to see what percentage of the pupils falling into each of the groups in general intelligence from high to low might fall into the same groups in the apprehension of number. Table XVI shows the grouping for first-year pupils and Table XVII for second-year. The general results for first-year pupils is that of the 33 pupils assigned to the upper half of the class 27 stick there in the second ability, but of the 25 pupils assigned to the lower half 16 are found to have moved to the upper half in apprehension. There is little relation shown between the two abilities on the whole, but our cases are too few in this as in our other correlations to enable us to reach anything but tentative conclusions.

As our fourth rank contained so few cases in the first group of pupils (Table XVI) and promised to contain as few in the second group, we used only three groups in the next relation table (Table XVII). Here again the relation is indifferent, less than half (41) remaining true to the original assignment. While we dare draw no general conclusion from 89 cases, the result agrees with the common as well as scientific observation that there is little

Correlation—Apprehension of Number Pictures and Teachers' Marks in Arithmetic

Group No. 2.—Second-year Pupils (2A and 2B)

Arithmetic

Q I			Q II			Q III			Q IV		
1	5	15	58	32	83	31	56	70	71	85	33
2	6	25	66	39	16	37	62	10	72	12	44
4	7	49	74	73	27	43	74	11	80	30	14
8	20	50	89	86		45	76	36	87	88	35
21	23	57	47	19		48	81	40	3	13	42
28	53		9	24		51	89	63	34	41	67
29	65		17	59		54	64	68	46		78
52			18	60		55			69		38
61			22	77					82		
75			26								
Teachers' Marks											
90	85	80	80	80	75	75	75	70	70	65	60

Number Pictures

Q I		Q II			Q III			Q IV				
1	52	70	4	86	55	40	64	85	2	46	13	38
7	53		21	16	56	63	67		28	69		41
8	57		5	27	62	68			6	82		
15	58		9	31	74	71			19	88		
20	61		17	37	76	72			24	12		
23	65		18	43	81	80			59	30		
25	66		45	22	89	87			60	14		
29	75		26	48	10	33			77	35		
47	79		32	51	11	44			3	42		
49	84		39	54	36	78			34			
50	83		73									
Mistakes												
0	0	0	12	12	12	12	14	16	24	24	27	36

TABLE XV

	Red	Per Cent	Green	Per Cent	Purple	Per Cent	Black	Per Cent	
Quartile I	16	73	6	27	0	0	0	0	
Quartile II	3	14	11	50	8	36	0	0	
Quartile III	0	0	0	0	14	64	8	36	
Quartile IV	3	13	5	22	0	0	15	65	
	—	—	—	—	—	—	—	—	
	22		22		22		23		89 Cases

connection between mathematical ability and general intelligence.

TABLE XVI

CORRELATION OF APPREHENSION WITH GENERAL INTELLIGENCE

First-year Pupils

	A	Per Cent	B	Per Cent	C	Per Cent	D	Per Cent	Cases
A........	6	55	3	27	2	18	0	0	11
B........	10	46	8	36	4	18	0	0	22
C........	9	39	7	30	2	9	5	22	23
D........	0	0	0	0	1	50	1	50	2
									58

Classification of pupils as A, B, C or D in apprehension was made on the basis of number of mistakes.

A = 0 to 1
B = 2 to 3
C = 4 to 6
D = 7 to 11

Read table as follows: Of the number of cases graded A in Intelligence, 55% grade A in Apprehension, 27% grade B, 18% grade C, etc.

TABLE XVII

Second-year Pupils

	A	Per Cent	B	Per Cent	C	Per Cent	Cases
A..............	15	35	18	42	10	23	43
B..............	8	22	21	58	7	20	36
C..............	0	0	5	50	5	50	10
							89

Classification of pupils as A, B, C in apprehension was made on the basis of the number of mistakes.

A = 0
B = 1
C = 2, 3

Commentary and Conclusions.—1. Our working hypotheses have been two: (a) The number concept develops *pari passu* with the growth of number apprehension which includes the perception and imagining through sight and touch (mainly) of a group of objects or their representatives; action between concept and apprehension is reciprocal; (b) The quadratic number pictures of Lay are the best perception material for spatial apprehension.

2. Hypothesis (a) does not preclude the notion that number contains temporal as well as spatial elements, but with that our experiments have nothing to do. Undoubtedly counting plays its part in preparing the way for spatial apprehension and so in building up concepts. For the auditory type it may play the largest part. Lay's notion that the number is grasped as a Setzung, and that the imaging of a group and its symbolization (with or) without counting becomes, through a Kantian reaction (construction), numerical apprehension may be an insight of great subtlety, but he is in profound disagreement with many other experimenters and psychologists (Meumann, Freeman, Phillips, Messenger, Burnett, Dewey, et al.). Lay in some passages seems to allow in a negative sort of way some value to counting for the attainment of number ideas (in addition to its value for numerical description which he freely concedes), but he minimizes this value to the extreme.

3. By accepting hypothesis (b) we do not mean that the matter of perception material is settled beyond peradventure. His own experiments show only a shade over the Born pictures, though, as I have intimated, Lay may have in his comparative experiments unconsciously handicapped his pictures (see page 254); also further experimentation may develop something better. I am not unaware of the results of Freeman (58) which go to show that a grouping by fives with .050 sec. exposure was best apprehended by children, while a grouping by fours was best

for adults. This needs confirmation. Granting that children grasp the five grouping as easily, at least, as the four, as far as mere perception is concerned, such grouping would still be inferior to Lay's pictures from the point of view of indicating by simple strokes the operations to be performed (see Appendix VII).

4. The first conclusion from our experiments is that there is no reason to abandon hypothesis (b). Although our research was not comparative on this point, the exceeding favorableness of the material was proved at every step. Fifteen first-year children (1B) made not a single mistake on the number space 3 to 12 with repeated exposures. It is quite probable that Lay's proposed apparatus (see Führer, pp. 179–181) will much enhance the usefulness of the quadratic arrangement, permitting as it does the use of touch as well as sight, and the introduction of other factors that have been found to influence numerical judgment, form, color, distance, direction, etc.

5. There is a certain order in which the cardinal numbers come to full consciousness. All of them cannot be said to be mastered until the end of the third year.

6. It is probable that with good perception material, systematically used, paying due regard to the order of development of the numbers as pointed out (tentatively) by these experiments, greater economy of time and effort in teaching and learning the cardinal numbers and their processes than now obtains can be brought about.

7. Therefore, a freer and better planned use of this material than has so far obtained in this country seems to be indicated. So far as the writer knows only one American text-book has so far systematically included the Lay number pictures in its pages. In this book[1] they are used quite ingeniously.

8. One economical aspect of the value of the pictures is as

[1] Name omitted, for obvious reasons.

concept-material. "The number pictures constitute for the child a tremendous step toward complete abstraction, in that he is taken away from the specific numbered objects of various kinds to a representation that may stand for any of them" (page 228). "Since the concept, if it is to be more than nominal, that is, contentful, must grow out of clear images, it is important that the images be of such a form that they may be returned to readily again and again, thus growing into type forms of a semi-abstract kind, working categories under which may be subsumed by children all their early numerical experiences" (page 208).

9. For the more particular results the reader must be referred to the body of the research account.

B. Experiments with School Children to Determine Ability in Fundamental Processes.

The Courtis tests in their present form are eight in number. A part of each test is given below to indicate their character, but the reader must be referred to the published texts of Mr. Courtis for the complete tests and the full account of the manner and purpose of giving them and of working up the results. The account that follows of the results obtained from testing about 300 third- to eighth-grade children will illustrate these points to a considerable extent and will perhaps serve to make the general plan more concrete than a review confined to Courtis's work with his own pupils. My purpose, in brief, is to present some of the results of the diagnosis of this group of pupils, made possible through the establishment of a norm of efficiency for individuals and classes of the several grades in the several abilities involved in arithmetical work, with the aim of showing the workableness and value of the tests even when applied in the limited way in which so far I have been able to use them, and of responding to a possible interest on the part of some in seeing how they work out in some one school other than that of Mr. Courtis.

Test No. 1. Speed Test—Addition

Write on this paper the answers to as many of these addition examples as possible in the time allowed (1 minute for this and the next five tests).

8 9 7 8 2	1 3 6 0 3	1 7 9 3 2
1 9 6 0 5	5 8 9 7 2	3 7 6 0 4
— — — — —	— — — — —	— — — — —

Note.—The sheet in this test as well as the next three contains 24 groups (units) or 120 examples, more than any pupil even of the highest elementary grade can complete in the time allowed.

TEST NO. 2. SPEED TEST—SUBTRACTION.

6	11	15	10	12	8	11	12	5	10	1	9	13	4	12
0	7	8	9	4	0	9	7	1	2	0	7	8	3	6

TEST NO. 3. SPEED TEST—MULTIPLICATION.

3	4	9	0	5	4	2	7	4	9	9	5	4	7	6
2	7	8	2	6	1	9	6	0	5	1	2	8	0	5

TEST NO 4. SPEED TEST—DIVISION

1)8 5)30 8)72 1)0 9)36 2)6 4)24 7)63 6)0 8)32

9)9 3)21 6)48 1)1 5)10

TEST NO. 5. SPEED TEST—COPYING FIGURES

Copy on this paper on the space between the lines as many of the printed figures as possible in the time allowed.

2 4 9 6 7 4 2 9 7 6 6 2 9 4 7 7 2 9 6 4

2 6 9 7 4 4 6 9 2 7 6 4 9 7 2 7 4 9 2 6

Note.—The sheet in this test contains 24 groups (units) or 120 figures. Time, 1 minute.

TEST NO. 6. SPEED TEST—REASONING.

Do not work the following examples. Read each example through, make up your mind what operation you would use if you were going to work it, then write the name of the operation selected in the blank space after the example. Use the following abbreviations: Add. for addition, Sub. for subtraction, Mul. for multiplication, Div. for division.

1. The children of a school gave a sleigh-ride party. There were 9 sleighs used, and each sleigh held 30 children. How many children were in the party?

2. Two school-girls played a number game. The score of the girl that lost was 57 points and she was beaten by 16 points. What was the score of the girl that won?

3. A girl counted the automobiles that passed a school. The total was 60 in two hours. If the girl saw 27 pass the first hour, how many did she see the second?

4. On a playground there were five equal groups of children each playing a different game. If there were 75 children playing altogether, how many were there in each group?

Note.—The sheet contains 16 examples. Time, 1 minute.

TEST NO 7. FUNDAMENTALS.

In the blank space below, work as many of these examples as possible in the time allowed. Work them in order as numbered, writing each answer in the "answer" column before commencing a new example. Do not work on any other paper.

Number	Operation	Example	Answer	Right
1	Addition.....	(a) 25 + 830 + 122 =		
		(b) 232 + 8021 + 703 + 3030 =		
2	Subtraction....	(a) 5496 — 163 =		
		(b) 943276 — 812102 =		
3	Multiplication.	2012 × 213 =		
4	Division......	158664 ÷ 132 =		
5	Addition.....	6134 + 213 + 4800 + 6005 + 3050 + 474		
6	Subtraction...	73210142 — 49676378		
7, 8	Multiplication.	46508 × 456 =		

Note.—The sheet contains 19 examples, more than any pupil can complete in the time allowed (twelve minutes). Blank space is left below the last example for the work in this test and the next.

Test No. 8. Reasoning.

In the blank space below, work as many of the following examples as possible in the time allowed. Work them in order as numbered, entering each answer in the "answer" column before commencing a new example. Do not work on any other paper.

	Answer
1. A party of children went from school to a woods to gather nuts. The number found was but 205 so they bought 1,955 nuts more from a farmer. The nuts were shared equally by the children and each received 45. How many children were there in the party?	
2. One summer a farmer hired 43 boys to work in an apple orchard. There were 35 trees loaded with fruit and in 57 minutes each boy had picked 49 apples. If in the beginning the total number of apples on the trees was 19,677, how many were there still to be picked?	

Note.—The sheet contains 8 problems, more than can be completed in the time allowed (six minutes).

In all of these tests the scoring is done by examples, one point for each example attempted and one for each right; note, however, the modification of this in test No. 7.

Nos. 1, 2, 3, 4 are tests of the control of knowledge of the elementary processes through 9's. It will be noted that an effort has been made to make the units of work as nearly equal objectively as possible (compare the structure of the several groups of five examples). The same effort has been made with considerable success, we think, in the other tests; for instance, in test No. 7, "each example is approximately the equal of every other example if the assumption is made that any step of the work as writing a figure, thinking a sum, carrying a figure, etc., is the equal of any other step; just what the relative difficulties are is not known, but the experimental results already obtained make it probable that while the assumption is not true of individuals it is true in general; an example with a double number is twice as long as a single example." (Manual of Instructions, p. 10.) The test might be devised, of course, in such a way as to render the "relative difficulties" of little moment. It would be a question of constructing units whose internal composition should comprise elements which in the aggregate cause one process unit to equal any other, objectively. If this be done, it is not necessary to assume that any step is the equal of any other step, but only to see that the steps (elements) are proportionately distributed in each unit. It seems possible to do this with considerable exactness even in the present state of limited knowledge of the psychology of fundamental processes. Fortunately we need only say: writing this figure equals writing that figure, thinking this sum, difference, product, quotient equals thinking that sum, etc., carrying this figure equals carrying that figure, etc., and make each unit cover as many of these elements as any other. True, there might be some room for differences of decision as to the relative difficulties of some of the corresponding elements, but there seems to be no doubt but that the unit structure could be so manipulated that in a series of units these differences would balance one another. Of course, an experimental determination of the relative difficulties on the whole would contribute to greater exactness and more satisfactory units. But it must be remembered that these norms of difficulty, even if exactly determined, would represent no individual and no single class of individuals. Such a general determination of the relative difficulty of elements of the same nature would be entirely possible and relatively easy. In this connection a question might be raised as to the advisability of scoring by units in all of the tests, rather than by elements as called for by the directions in tests Nos. 1, 2, 3, 4, 5. Thus, the units so carefully worked out by Mr. Courtis for the simple operations, it would seem, should not be broken up into their elements for scoring purposes, but logically should be scored thus: number of units attempted, number of units right (all parts of units being thrown out) instead of number of examples attempted, number of examples right.

Experience with the tests with our own children has shown, too, that, unless certain test papers are thrown out, scoring by elements in the first four tests at least is calculated to give misleading results. For instance, one pupil handed in a paper (test No. 2) which scored 20 attempted, 20 right (accuracy 100%), *but the pupil had skipped all the more difficult subtractions.*

A general statement of the matter as applied to all the tests would be that by enlarging the units in a test so as to include in each unit all the phases involved in the ability or abilities tested so that in quantity and quality of elements each unit would approximately equal any other, any assumption as to equality of steps would be largely eliminated. But while this plan, provided the scoring were done by units, might be desirable from the point of view of exact experimental method it is freely conceded that from the point of view of the giving, scoring and interpreting of such tests by comparatively untrained persons, Mr. Courtis has advisedly used a much more workable plan, after having duly considered more complicated methods. Scoring by units to yield entirely satisfactory data would require that at least a sufficient number of units be attempted to balance up any elemental inequalities within the units; for example, 20 units in each of the first four tests to cover all possible combinations through 9's. But this would require, further, a radical change in the manner of giving the tests; all pupils would now be required to work through the requisite number of units and have his individual time recorded, since time used is an important factor in determining *control* of knowledge. It is much simpler to keep the time uniform for all and to score by attempts and rights for each example and quite as satisfactory for practical purposes. The practical advantage would be the opportunity given for a more thoroughgoing analysis of mistakes (since chances would be offered for every kind of mistake), but this would scarcely compensate for the complications introduced into the conducting of the tests and the treatment of the gross scores.

No. 5 tests rate of motor activity in writing figures. It is obvious that this ability can conceivably be related to the number of examples attempted and might serve to explain in part the score in the latter from the point of view of mere muscular control. This point will be referred to again. It might be supposed, also, that some relation exists between motor ability and mental functioning. Bagley (72) has shown, however, that with his subjects there is no positive correlation between motor ability and intelligence.

No. 6 tests the ability to recognize a situation in a problem as calling for the use of a certain operation. Mr. Courtis says (Manual of Instructions, p. 9): "This ability is itself a complex, the ability mentioned representing merely the arithmetical phase of the mental activity. Speed in this test undoubtedly depends mainly upon speed of reading and of the comprehension of the meaning of printed words." Here again it would seem advisable to make the unit of performance a group of four and to score by such units. Without entering upon a discussion of the matter, the cues in one-step addition and subtraction problems are probably more directly and certainly suggestive of the operation than in multiplication and division. Any one who has tried the manufacture of problems has found it more difficult to "ring the changes" on cues for addition and subtraction. If this be true, the relative difficulty of the examples is not the same, but the relative difficulty of the groups of four examples would be approximately the same. However, this would make little difference in results for comparative purposes provided the examples are worked in the order in which they occur.

No. 7 involves all the abilities of the previous tests and tests knowledge of the symbols, processes and forms of the four operations; also ability to borrow and carry; also ability to attend to mechanical details. The first four examples involve no "borrowing and carrying;" but the examples from 5 to 11 do (as also those from 12 to 19) and make possible a measurement of the effect of the introduction of "borrowing and carrying." The ability to pay attention to details is tested by requiring that each of the examples be copied and that the answers be written in the answer column. Instead of scoring one point (attempted and right) for each example, in this test the exception is made, in order to keep the units equal, of scoring one for two in certain of the examples as the first and two for one in some of the others as in 7–8.

No. 8 tests the ability to handle the difficulties added by the organization of the previous material into two-step problems. Like test No. 6 it is also a language test—the ability to get meaning from the printed word.

The unconventional phrasing of the problems both in test No. 6 and No. 8 removes them as far as possible from appeal to mere memory (association with typical problems of the conventional sort, as with rules, which association may simulate reasoning). To further test out the rationality of procedure irrelevant numbers are introduced into examples 2, 4, 5 and 8.

Each problem is, in the abstract work called for, the equal of a single example in test No. 7. Each two examples involve the four operations.

Following the directions on page 21 of the Manual, for the reasons there stated, the tests were given in two sets in the following order: Set I—Nos. 5, 2, 8, 3; Set II—Nos. 4, 6, 1, 7; see the detailed instructions on pp. 21–25 for giving the tests. I copy the direction given to secure uniformity of timing (p. 15) (a stop watch was used by the conductor). Hold papers in left hand between thumb and finger. [Each pupil had before him a printed sheet containing the test and had been carefully instructed in all of its requirements.] Then turn them over and put them face down on your desks, but don't let go of them, so that you will be all ready to turn them over quickly when I say start. Take your pencils in your right hand, and when I say, "Get ready," raise your pencil hand in the air as if you were going to ask a question. Then when I say, "Start" you can bring your pencil down as you turn the papers over, and everyone will start at the same time. "Get ready." See that all hands are raised. "Start." At the end of the time allowed give the command "Stop."

All of the eight tests were given to six classes (3A, 4A, 5A, 6A, 7A, 8A) except that tests Nos. 7 and 8 were not given to 3A and 4A, and were personally conducted by the writer in all cases; the marking was done by the class teachers.

The following tables show the distribution of the pupils in each class according to frequencies in the several scores, the class averages, the average deviations and the coefficients of variability.

TABLE I.—Distribution—Tests 1 to 5

Test No. 1

Score	5 0 to 9	15 10 to 19	25 20 to 29	35	45	55	65	75	85	95	105	Av	A. D.	Var.
Freq. Att. 3A	0	4	18	16	6	1	1					32	8.0	.25
4A	0	2	20	21	5							30	6.9	.23
5A	0	0	2	12	19	11	4	3				47	8.6	.18
6A	0	0	0	8	17	13	10	1	1			52	9.6	.18
7A	0	0	0	1	15	19	8	2	2		1	56	8.0	.14
8A					4	7	19	13	4	1	1	58	8.8	.13

TEST NO. 2

SCORE	5	15	25	35	45	55	65	75	85	95	105	Av.	A. D.	Var.
Freq. Att. 3A	12	13	16	2	1	0	2					20	9.7	.49
4A	2	18	18	8	0	1	1					24	7.7	.33
5A	0	3	12	15	9	4	8					39	12.0	.30
6A	0	3	10	19	11	4	2					37	8.4	.23
7A	0	0	3	23	20	3						40	6.0	.15
8A				8	21	14	4	3				50	7.8	.16

TEST NO. 3

Freq. Att. 3A	10	30	5	1								16	5.2	.32
4A	1	12	15	17	1	0	1					27	7.8	.29
5A	0	2	11	18	8	6	4	2				39	12.0	.29
6A	0	1	8	17	12	5	5	1				41	11.0	.27
7A	0	0	2	26	13	2	5					41	8.6	.21
8A		1	7	7	10	11	7	2				47	12.0	.26

TEST NO. 4

Freq. Att. 3A	10	28	5	3								15	4.5	.29
4A	3	16	19	9	1							23	6.8	.30
5A	0	6	17	15	8	3	2					33	9.2	.28
6A	0	1	8	23	14	2	2	[illegible]				38	6.8	.18
7A	0	0	2	9	22	5	6	1	2			48	8.7	.18
8A			1	6	22	12	5	2	1			50	8.0	.16

TEST NO. 5

											(99 to 149)			
Freq. Att. 3A	1	0	1	1	12	12	8	2	9			60	12.1	.20
4A	0	1	1	0	10	11	13	5	4	1	2	62	13.7	.22
5A	0	1	1	1	1	4	3	9	6	12	13	85	20.0	.23
6A						1	1	5	15	7	21	99	16.8	.17
7A					1	2	2	13	16	6	9	86	11.8	.14
8A							1	1	3	5	38	115	15.4	.13

TABLE II.—Distribution—Tests 6 to 8

Test No. 6

Score	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	Av.	A. D.	Var.
Freq. Att. 3A	3	6	7	15	7	5	1	1	0	0	1									3.1	1.3	.42
Rts.	18	11	11	3	2	1														1.2	1.0	.83
Freq. Att. 4A	0	10	13	7	3	4	1	0	1	0	1									2.4	1.4	.58
Rts.	19	17	7	3	1	0	0	0	0	0	1									1.1	0.8	.73
Freq. Att. 5A	0	6	4	15	7	5	8	2	1	1	1	0	1							4.2	1.8	.43
Rts.	1	12	14	13	6	3	1	1												2.6	1.2	.46
Freq. Att. 6A	0	1	1	6	12	7	9	6	6	0	1	0	1							5.4	1.6	.30
Rts.	2	4	11	11	7	6	7	0	1	0	1									3.5	1.5	.43
Freq. Att. 7A			1	6	13	10	12	4	3	2										5.2	1.3	.25
Rts.		3	6	11	13	5	9	4												4.1	1.3	.32
Freq. Att. 8A					1	8	18	8	6	5	3	0	1							6.9	1.3	.19
Rts.			1	2	5	9	15	7	5	5	1									6.1	1.3	.21

Test No. 7

Score	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	Av.	A. D.	Var.
Freq. Att. 5A				2	2	5	5	0	13	14	0	6	0	2	2					8.2	1.8	.22
Rts.	4	2	2	5	12	6	4	8	5	1	1	0	1							4.9	2.0	.40
Freq. Att. 6A					1	0	6	0	5	20	2	9	0	5	2					9.4	1.6	.17
Rts.			1	4	3	6	16	5	9	4	2									6.3	1.4	.22
Freq. Att. 7A										7	1	22	2	8	7	0	4			11.7	2.6	.22
Rts.					1	3	5	5	6	7	11	5	5	1	0	1				8.8	2.0	.22
Freq. Att. 8A							1	0	0	1	0	5	1	6	12	1	6	9	8	14.7	2.2	.15
Rts.		2	1	4	0	3	5	2	4	4	9	8	0	4	3	0	1			8.6	3.0	.35

TEST NO. 8

SCORE	0	1	2	3	4	5	6	7	8	9	10	11	12	AV.	A. D.	VAR.
Freq. Att. 5A		9	21	10	7	3	1							2.5	1.0	.40
Rts.	34	14	3											.4	.4	1.00
Freq. Att. 6A		13	17	11	7									2.3	.8	.35
Rts.	15	23	8	2										1.0	.5	.50
Freq. Att. 7A		4	31	9	7									2.4	.5	.21
Rts.	3	18	22	8										1.7	.6	.35
Freq. Att. 8A		5	13	14	11	5	1							3.0	1.0	.33
Rts.	3	15	17	12	2									1.9	.7	.37

If we examine the distributions set forth in Table I, we find, as might be expected with such a small number of cases (about 50 to each distribution), that most of the distributions are "skewed." But five of the classes, three in test No. 1 (addition), viz., 3A, 4A, 5A, and two in test No. 2 (subtraction) viz., 6A and 7A, show a distribution surprisingly close to a normal one, that is to say, the greatest number of frequencies occur at the class measure (average) with fewer frequencies symmetrically disposed on either side of it and progressively decreasing in number. To show this more clearly distribution curves have been drawn for classes 3A, 5A and 6A, Figs. 1, 2, 3. The dotted lines on either side of the ordinate at M (representing the number of frequencies at the average) mark the limits of the zone of central tendency in which one-half the members of the class fall. In order to do this more accurately, I have computed and used (instead of the average deviation), to measure the distance on either side of M the probable error of the class measure (average). Thus, in class 3A (test No. 1) with an average of 32 (p. e. 7.0), 50% of the pupils are included in scores from 25 to 29, in class 5A (test No. 1) with an average of 47 (p. e. 7.0), 50% of the class are included in scores from 40 to 54, and in class 6A (test No. 2) with an average of 37 (p. e. 7.0) 50% are included in scores from 30 to 44. While the range of variation within the central zone happens to be the same

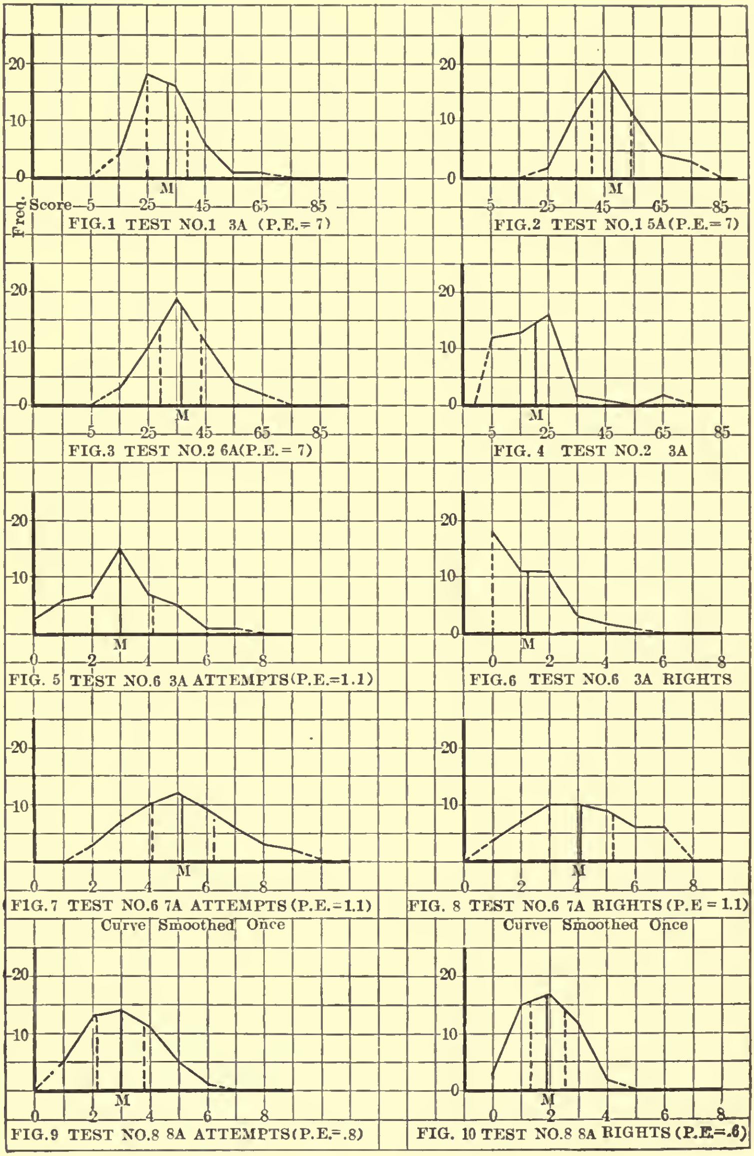


FIG. 1 TEST NO. 1 3A (P.E. = 7)

FIG. 2 TEST NO. 1 5A (P.E. = 7)

FIG. 3 TEST NO. 2 6A (P.E. = 7)

FIG. 4 TEST NO. 2 3A

FIG. 5 TEST NO. 6 3A ATTEMPTS (P.E. = 1.1)

FIG. 6 TEST NO. 6 3A RIGHTS

FIG. 7 TEST NO. 6 7A ATTEMPTS (P.E. = 1.1) Curve Smoothed Once

FIG. 8 TEST NO. 6 7A RIGHTS (P.E = 1.1) Curve Smoothed Once

FIG. 9 TEST NO. 8 8A ATTEMPTS (P.E. = .8)

FIG. 10 TEST NO. 8 8A RIGHTS (P.E. = .6)

Distribution Curves—See Tables I and II.

in these classes the coefficients of variability depending on the number and size of the frequencies that deviate from the average are not the same. Class 5A is least variable (18%); 6A shows 23% and 3A 25%. In class 3A, eight individuals fall above the central zone, fifteen below, while the entire range is from a score of 15 (in test No. 1) to 65; in class 5A twelve fall above and fourteen below, entire range (test No. 1) from 15 to 45; in class 6A, eleven fall above and thirteen below, entire range (test No. 2) from 15 to 65. Perhaps the most striking way to put the matter is to point out that in class 3A, fifteen out of 46, one-third of the class, do not know the addition tables, while at least two (of the eight above the central zone) know them too well; that is to say, the time spent with the rest of the class by them on tabular work is entirely wasted; again in class 5A, a grammar grade, fourteen out of 51 do not know the addition tables, and in class 6A, thirteen out of 49 are in the same fix. The conclusion cannot be escaped that there has been a failure here "to secure [early] complete all-around [class] development in fundamental abilities" and this, in turn, to failure to minister to individual needs. The question is also opened whether some individuals have not been given a kind of drill, perhaps through many wearisome class periods, which they do not need. An examination of the other coefficients in Table I shows similar results in regard to variability, and point to similar conclusions. Possible exceptions are classes 7A and 8A in all of the tests except No. 3 (multiplication). Just what degree of variability can be permitted within a grade from the point of view of efficiency is, of course, a moot point, but it would seem that these classes in addition, subtraction, and division had reached a fairly satisfactory degree of homogeneity. But a close examination of the distribution reveals that in 7A addition and 8A subtraction and division the distribution is much "skewed" toward the lower end of the class, giving 15 or 16 cases of pupils below the central zone. On the other hand in 8A addition there is a slight skew toward the upper end, giving 14 cases of pupils who are supernormal having scores of 77 to 105 (the norm is 63) and the upper limit of the central zone as determined by the probable error is 76. This fact, together with the relative fewness of cases at the lower end, brings the average of this class (68) above the standard fixed by Courtis for 8th grade speed addition (63). In 7A subtraction and division, the distribution approximates a normal one, but even here there are a dozen cases in each class below the central zone. It is evident from these considerations that even a coefficient as small as .13 indicates a larger range of variation than class efficiency can permit.

One of the worst examples of skewed distribution as well as extreme variability is afforded by class 3A in test No. 2. This is illustrated by the diagram Fig. 4, which shows a range of variation from 5 to 65 points. While the class average (20) is up to the standard, it really represents few individuals in the class, as the average deviation from it is nearly 10.

This average is further slightly vitiated as a representative measure by the fact that the individual scores (rights as well as attempts) show that the two highest scores in this distribution are spurious in a way, since in one of the cases the number of rights is 18 out of 60 attempts, and in the other 32 out of 65 attempts. An examination of the test papers of these two reveals that many of the more difficult subtractions are wrong, thus supplying additional instances in favor of scoring by units (group of five) rather than by single examples. (It will have been noted that only attempts are tabulated in the first five tests, in accordance with Mr. Courtis's belief, derived from his experience, that lack of control of knowledge of the tables is shown not by mistakes but by reduced speed; the examples in these tests are usually right, he says, except in the very lowest grades. This is borne out, on the whole, in our tests but there are occasional exceptions such as those noted above.) The fact is that the control of very many, at least half, of the class is feeble, a condition that doubtless could have been much improved, with the result of a reduced variability, by proper attention to individual cases.

Table II presents the distribution of frequencies in both attempts and rights in tests Nos. 6, 7, 8, viz., in speed reasoning, fundamental operations (organized use of the tables), and two-step reasoning. The scores are fairly satisfactory so far as class averages are concerned, except in test No. 8 where as compared with the Courtis standards they are very unsatisfactory.[1] But note the heterogeneity of the classes, as shown by the variability coefficients, especially in tests 6 and 8 (reasoning). The classes are much less compact in these tests than in the tabular tests; on the other hand, in test No. 7 (abstract work) they are about as homogeneous as in the tables, which is not saying much. The exception to this statement is class 8A in test No. 6, though the distribution is "skewed." Strange to say, this class is extremely variable in test No. 7, rights ranging from a score of 1 to a score of 16 points, though it had been comparatively compact in the tables. This points to a conclusion which will be brought out more fully later, one that Mr. Courtis has reached and which our results also show that there is indifferent correlation between knowledge of the tables, as such, and ability to work abstract examples involving these tables.

So far as *normality* of distribution is concerned, these tests show frequencies normally distributed in more instances than the previous ones. The curves drawn for class 7A, test No. 6, attempts

[1] Since this was written, new standards have been worked out by Mr. Courtis based on scores from a very much larger number of children than were available for the original standards. (See Bulletin Number Two, Courtis Standard Tests, Detroit, 1913, page 20.) The new standards differ scarcely at all from the old except for the tests in reasoning (Nos. 6 and 8). In these tests the old standards are found to have been set much too high. In view of this revision the statements in regard to comparative scores in reasoning deserve some qualification; the case for the local scores is made somewhat more favorable, though not sufficiently so to necessitate any radical change in the discussion.

and rights (Figs. 7 and 8); for class 3A, test No. 6, attempts (Fig. 5,); for class 8A attempts and rights, test No. 8 (Figs. 9 and 10) show clearly the approximately normal distribution, though they are more or less flattened on account of range of variation or size of frequencies outside the central zone. Four other tests approximate a normal distribution somewhat less closely, viz., 5A speed (one-step) reasoning rights; 5A fundamentals, attempts; 6A fundamentals, attempts; and 8A speed reasoning, rights.

The reason for these symmetrical distributions with so few cases is not clear. Only the very general statement can be made that such distributions imply the operation in the conditions of an indefinitely large number of individual factors each of which is equally present and effective.

The champion skew distribution among these tests is found in test No. 6, rights. The range is from no score, of which there are 18 cases, to 5, of which there is one case. The resulting curve is shown in Fig. 6. This 3A class is almost a complete failure, as a class, in speed reasoning as is shown by the distribution and variability (83%) and is brought up to a respectable average (1.2), respectable in view of the fact that it had advanced in school life only about four weeks beyond the second year (the tests were given near the beginning of the school year), by a talented few who make high scores.

On one point the conclusion from our data of distribution must differ from that of Mr. Courtis. He says, "In the early grades the classes are units, but as pupils pass from grade to grade, individuals react to training in different ways so that the unity is broken" (42a, p. 197). It does not follow from this, however, that these differently reacting pupils will be found to an increasing extent in class after class. Under any proper scheme of classifying and promoting pupils, differing pupils will find their level, so far as such a thing is possible, through the individual attention that is demanded to land them in their proper groups and the incentives that are afforded to both teacher and pupil to induce making the best of the individual regardless of time-tables and curricula. Our plan of promotion is the half-yearly one, perhaps the least flexible scheme now in use, but by dint of the awkward and somewhat dangerous maneuver of pulling up a pupil from one grade and planting him in another, the classes are kept measurably homogeneous. Our results would seem to show that in arithmetic, at least, they are increasingly so as the coefficients in Tables I and II show on the whole a progressive decline in size.

It is a fundamental question in connection with these comparative tests whether the results obtained from a single test are reliable measures of grade ability. The answer is that they are when properly analyzed and understood. Mr. Courtis early realized the importance of this point, and by repeated tests of his Detroit pupils of the same grade satisfied himself that where the conditions with reference to training are constant (as during a summer vacation) the grade averages for a given class are consistently the same. But this was not found to be true of the individuals composing the grade. Even when the grade average was perfectly constant, individual scores were found to vary. The constancy was due to the fact, as the records showed, that the variations of certain individuals in one direction are offset by the variations of others in the opposite direction. For the individual, therefore, repeated tests are needed to determine his most frequent score and the range of his variation from it.

It may be added that the class scores must always be considered in connection with the variabilities in order to reach a judgment of their value as representative of class performance and individual instruction.

The data for the working out of the Courtis standard scores was secured by the coöperation of some twenty superintendents, principals, heads of departments, etc., and of many teachers under their charge, from sixty or seventy schools in ten different states. Nearly 9,000 children were tested, using the tests already described under uniform conditions secured so far as possible by detailed instructions and precautions issued to examiners and scorers. The total results were distributed and the grade averages tabulated.

TABLE III.

(Elementary School Teacher, Chicago, Nov., 1911, page 133.)

GRADE AVERAGES FROM TOTAL DISTRIBUTIONS[1]

Grade	Average of scores for each test	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6		No. 7		No. 8	
							Att.	Rts.	Att.	Rts.	Att.	Rts.
3	525	26	19	16	11	63	2.8	2.1	5.4	1.7	2.7	0.6
4	1,222	33	25	23	21	70	3.7	2.5	6.6	3.6	2.6	0.8
5	1,177	40	32	30	28	80	4.4	3.4	9.0	5.3	2.8	1.2
6	1,282	46	37	34	35	88	5.1	4.4	10.3	6.9	3.4	1.7
7	1,432	51	40	38	38	98	5.9	5.2	11.5	7.6	3.7	2.2
8	1,370	57	45	43	44	102	6.8	6.1	13.1	8.9	4.1	2.7

[1] Grades 1, 2, 9, 10, 11, 12, 13, 14, are not reproduced here.

"The average for any grade represents simply a central tendency from which individuals vary widely. Yet in many ways the average scores of each group reveal the character of the development of each of the abilities through the school and show what should be true of the development of the individual. These data, then, in connection with the distributions from which the averages are derived furnish the desired basis for the selection of standard scores" (43, p. 133).

"Many will be content to use the average scores as standards, and if, indeed, no children in any grade had a lower score than the present average for that grade, the increase in efficiency of our schools in these abilities would be very great. But the writer feels that a higher score than the average is desirable for a standard and the scores given in the next table were derived as follows: For each test an eighth grade score was selected such that it was equaled or exceeded by thirty per cent of the highest grade children measured. This score was plotted and a smooth curve drawn, having the same general form as the average curve and coinciding with it in the lower grades. The scores for each of the other grades were then determined from the graph" (43, p. 135).

TABLE IV.

(Elementary School Teacher, Chicago, Nov., 1911, page 135.)

STANDARD SCORES

Grade	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6		No. 7		No. 8	
						Att.	Rts.	Att.	Rts.	Att.	Rts.
3	26	19	16	16	58	2.7	2.1	5.0	2.7	2.0	1.1
4	34	25	23	23	72	3.7	3.0	7.0	3.3	2.6	1.7
5	42	31	30	30	86	4.8	4.0	9.0	4.9	3.1	2.2
6	50	38	37	37	99	5.8	5.0	11.0	6.6	3.7	2.8
7	58	44	44	44	110	6.8	6.0	13.0	8.3	4.2	3.4
8	63	49	49	49	117	7.8	7.0	14.4	10.0	4.8	4.0

It will now be of interest perhaps to see how the local results compare with the standard scores. It will be borne in mind that the standard scores are set as ideals to be reached *at the end of the school year*, whereas our tests were given when the pupils were but a few weeks beyond the beginning of the year. We find no reason, however, to modify the standards in making comparisons on that account, except in tests No. 6 and No. 8; these, as already set forth, are in part reading tests and since a year's training in reading counts for so much, especially in the lower grades, it was thought only fair to use the standards next below in the case of each grade; for example, grade 3A (beginning the third year) is compared in these two tests with the standard scores of grade 2 (end of the second year), and the matter is arranged accordingly in the table below.[1] Even so, our classes will be seen to be weak in reasoning, especially so in two-step reasoning; on the other hand, in abstract work generally speaking, they are strong, in a class sense, even when compared with standards that are probably set too high, showing that much of their work in abstract arithmetic might better have been spent on problematic arithmetic.

TABLE V

SHOWING STANDARD SCORES AND LOCAL SCORES IN EACH OF THE TESTS

	Grade	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6		No. 7		No. 8	
							Att.	Rts.	Att.	Rts.	Att.	Rts.
Standard...	3	26	19	16	16	58	*1.8	*1.2	5.0	2.7		
Local......		32	20	16	15	60	3.1	1.2				
Standard...	4	34	25	23	23	72	2.7	2.1	7.0	3.3	2.0	1.1
Local......		30	24	27	23	62	2.4	1.1				
Standard...	5	42	31	30	30	86	3.7	3.0	9.0	4.9	2.6	1.7
Local......		47	39	39	33	85	4.2	2.6	8.2	4.9	2.5	0.4
Standard...	6	50	38	37	37	99	4.8	4.0	11.0	6.6	3.1	2.2
Local......		52	37	41	38	99	5.4	3.5	9.4	6.3	2.3	1.0
Standard...	7	58	44	44	44	110	5.8	5.0	13.0	8.3	3.7	2.8
Local......		56	40	41	48	86	5.2	4.1	11.7	8.8	2.4	1.7
Standard...	8	63	49	49	49	117	6.8	6.0	14.4	10.0	4.2	3.4
Local......		68	50	47	50	115	6.9	6.1	14.7	8.6	3.0	1.9

* Estimated.

From the data in this table two interesting studies may be made: (a) the development of each ability through the several grades, as compared with the standard development; (b) the performance of each grade in the several tests, as compared with the standard performance. The study will be greatly assisted by the comparative graphs, Figs. 11–18.

[1] The scores so selected correspond in a general way with the *average scores* for these tests (lower than the standard) in the Courtis table.

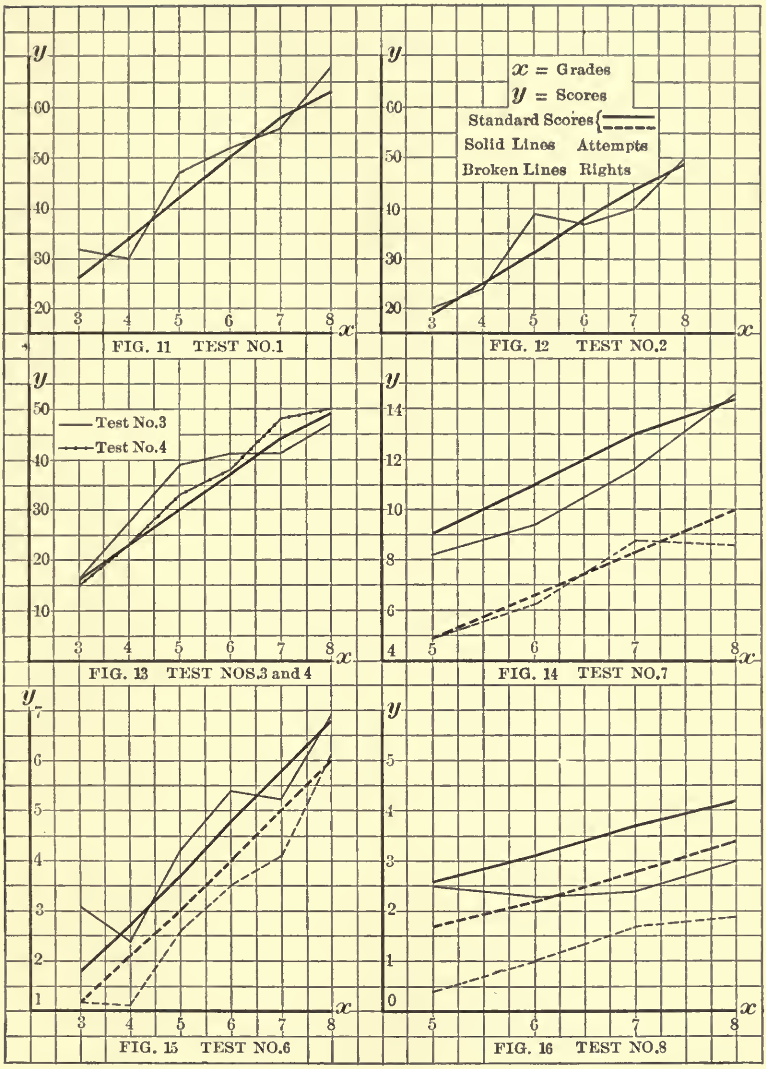


FIG. 11 TEST NO.1

FIG. 12 TEST NO.2

FIG. 13 TEST NOS.3 and 4

FIG. 14 TEST NO.7

FIG. 15 TEST NO.6

FIG. 16 TEST NO.8

Comparative Graphs—See Table V.

Range of scores through the several grades compared with the standard range in each test.

The graphs tell the story first of the development in the addition, subtraction, multiplication, and division tables (Figs. 11, 12, 13). The standard development, shown by the heavy unbroken line, is a smooth progression upward, while the local line falls now above, now below the standard. The most notable thing shown by these curves is that grade 5A is always above the standard, in three of the tests markedly above it, while grade 7A is slightly below in all except one. The fact of the matter is that grade 5A is too "smart" in the tables relatively speaking; contrast the work of this grade in two-step reasoning, shown in Fig. 16, where it falls far below the standard.

In Figs. 14, 15 and 16, the grades are compared with one another and with the standard in tests Nos. 7, 6 and 8, with respect both to number of examples attempted (solid line) and number of right answers (broken line). Fig. 14 shows that all of the grades tested measure fully up to the standard in number of right answers in fundamentals except grade 8A, but that all the grades make fewer than the standard number of attempts except grade 8A. On the whole this showing is favorable to the local classes; they are somewhat lacking in speed, but they get there just the same, performing with greater accuracy than the standard calls for. (The smaller the distance between the solid and broken line on any ordinate the greater the accuracy.) Grade 8A has the lowest comparative number of right answers and is least accurate. In contrast with this grade 8A is the only class to reach the (modified) standard number of rights in the speed reasoning test (Fig. 15) and has also the (modified) standard number of attempts and therefore the standard percentage of accuracy. All the other grades from 3A to 7A inclusive fall below in right answers though they approximate the standard quite closely except grade 4A, which also makes relatively fewer attempts than any other grade except 7A; the latter, however, has 90% accuracy (4:5), being slow but sure. Fig. 16 makes a sorry showing for the school efficiency in two-step reasoning, grades 5A to 8A reaching on the whole about 50% of the efficiency called for. The only comfort to be extracted from the situation is the wholesome and very obvious lesson that is to be gathered from such results. It is questionable if any one was fully aware of this condition prior to the tests.

The comparative graph (Fig. 17) shows the performances of each grade from 3A to 6A, as compared with the stand-

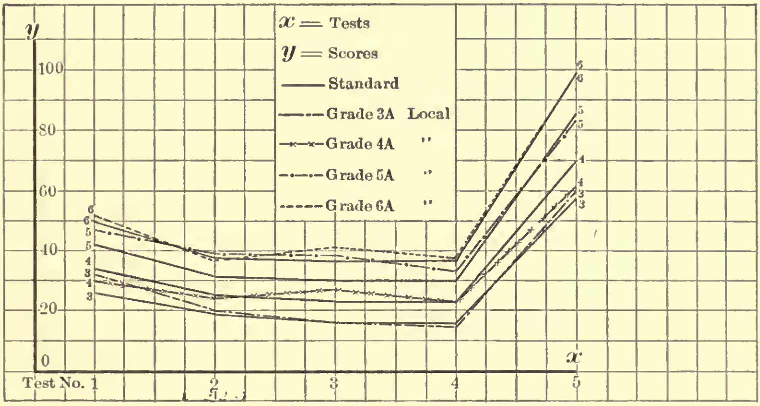


FIG. 17—Comparative Graph—See Table V.

Grade scores (attempts), compared with standard scores.
Tests Nos. 1, 2, 3, 4, 5.—Grades, 3A, 4A, 5A, 6A.

ards, in the first five tests (attempts only). The graph yields its meaning most readily perhaps if one reads along the ordinate for each test. In test No. 1 (speed addition), grade 3A approaches the 4A standard, grade 4A is on the dividing line between 4A and 3A, grade 5A approaches 6A efficiency, and grade 6A is above the standard. In test No. 2 (speed subtraction) grade 3A is slightly above and 4A slightly below while 5A is startlingly above even the 6A standard (too smart, by half); and so on. To summarize we may regard each solid line as the equator of a zone of standard performance and imagine the limits of each zone as set midway between the solid lines. Reading then the graph for signs of overlapping of the grades, we find that grade 3A keeps close to the equator of its zone, except in addition where it over-

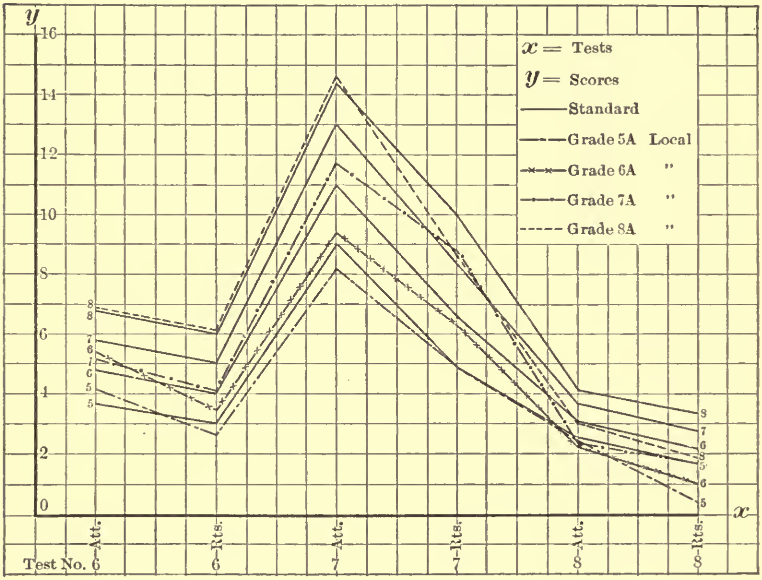


FIG. 18—Comparative Graph—See Table V.
Grade scores in attempts and rights compared with standard scores.
Tests Nos. 6, 7, 8—Grades, 5A, 6A, 7A, 8A.

laps 4A; grade 4A is at the lower limits of its zone in addition, overlaps 5A in multiplication and 3A in copying figures; grade 5A is in 6A's zone in every ability except in division, where it is on the border, and in copying figures; grade 6A gets out of its zone in multiplication only where it trenches on 7A. From Table V we gather similarly that grade 7A barely holds its own in subtraction and multiplication, does 8th grade work in division and 5th grade work in copying figures, while grade 8A is above the standard in addition, subtraction and division.

The graph (Fig. 18) and Table V presents even a more serious story of interlapping of the grades in tests Nos. 6, 7, 8. Here both attempts and rights are entered. Confining our attention to grade scores in right answers, and summarizing as above (remembering that our standards for tests Nos. 6 and 8 are modified ones), grade 5A does 5th grade work in one-step reasoning and fundamentals, and 3d grade work (or worse) in two-step reasoning; grade 6A is on the border of the 5th grade zone in one-step reasoning (many of the individual members are, of course, well within it), does 6th grade work in fundamentals but only 4th grade work (or worse) in two-step reasoning; grade 7A does 6th grade work in one-step reasoning, approximates 8th grade work in fundamentals, does 5th grade work in two-step reasoning; grade 8A does 8th grade work in one-step reasoning, 7th grade work in fundamentals, and is not quite up to the 6th grade standard in two-step reasoning.

This overlapping of classes was a constant feature of the returns from the seventy schools concerned in furnishing data for the standard scores, and, as Mr. Courtis says, utterly breaks down the notion that our schools are graded as they are at present constituted.

One of the most striking things that emerges from a study of these graphs is that surpassing ability in the tables does not guarantee extraordinary performance in organized abstract work. Grades 5A and 8A showed supernormal ability in the tables, but 5A precisely reaches the standard in fundamentals and 8A falls below.

It would seem that there is a certain minimum ability in the tables beyond which any increase of proficiency does not count for increased ability in the fundamental operations. Furthermore, test No. 7 is a better test of practical efficiency in abstract work than tests Nos. 1, 2, 3, 4. Grade 7A is an illustration of this. While the class was slightly below the standard in two of the tables it proved to be very strong in fundamentals. It must, therefore, have possessed the necessary minimum efficiency in the tables. It is highly desirable, to prevent waste,that teachers should know when this minimum efficiency has been reached and this knowledge can be obtained only by frequent testing of the matter.

A study of the individual records of Grade 5A, the class having the best record in the tables, with respect to this relation points to the same conclusion. Eight members of the class have an 8th grade total score in the tables; only two of these have a better than 5th grade score in fundamentals, and three have a poorer; two members have a 7th grade score in the tables, both are below 5th in fundamentals; seven have a 6th grade in the former, of these, two have a better than 5th grade score in the latter, three a poorer. On the other hand, several cases of pupils who are below the standard (133) in the tables or barely reach it are fully up to the standard (5) in fundamentals or above it. The highest girl in the class in the latter (score 10) makes a score of 132 in the former; the next highest (score 9) makes a score of only 115 in the tables. But again, several who are extremely accurate in the tables, though somewhat slow, make either no score or a very low one in fundamentals.

These evidences of lack of correlation between knowledge of the tables and control of the same in organized work led us to make a more systematic grouping of the 190 pupils in all of four grammar grades on the basis of right answers to see what percentage of the pupils falling into each of four groups, A, B, C, D (best, good, fair, poor) in test No. 7 might fall into the same groups according to their total record in the tables. The following grouping was the result:

TABLE VI

Relation of the scores in Test No. 7 with the total scores of the same individuals in Tests Nos. 1, 2, 3, 4 (right answers). Grades 5A to 8A.

	A	Per cent	B	Per cent	C	Per cent	D	Per cent	Total
A.........	10	(20)	19	(36)	19	(36)	4	(8)	52
B.........	9	(16)	22	(40)	17	(30)	8	(14)	56
C.........	12	(24)	16	(31)	16	(31)	7	(14)	51
D.........	7	(22)	6	(20)	11	(36)	7	(22)	31
									190

Read: Of the 52 pupils in rank A in test No. 7 (fundamentals) 10 (20%) rank A in the tables, 19 (36%) rank B, 19 (36%) rank C, 4 (8%) rank D, etc.

Without dwelling on all of the details of this table the outstanding inferences from the data therein are that there is a slight tendency for those in the upper half of the class in fundamentals to be found in the upper half in the tables, but it is very slight. Forty-four per cent of those in group A in test No. 7 fall into the lower half of the class in the tables; likewise forty-four per cent of those in group B. On the other hand, there is a slight tendency on the part of those grouped in the third quarter (C) in test No. 7 to be found in the upper half in the table (fifty-five per cent). But of the 31 in the lowest quarter, those having the poorest minds for fundamental examples, fifty-eight per cent are found in the lower half in the tables.

The correlation may, then, on the whole be roughly stated as neither decidedly positive nor negative, but indifferent; and this conclusion has an important moral for teachers, viz., the necessity of recognizing the heretofore not clearly understood fact, already mentioned, that there is a certain minimum efficiency in tabular work which so far as it goes insures satisfactory work in examples; but, further, also the presence of other factors that enter into the complex called a subtraction or multiplication example, for instance; factors that practically cannot be eliminated. Among these additional abilities that are demanded are ability to copy correctly, to borrow and carry, to correctly place partial results, and to give the prolonged attention needed to arrive at complete results. It is probable that if these abilities and their attendant habits (or lack of habits) could be eliminated (as, of course, they cannot be) the correlation would be much closer.

In order to illustrate with more definiteness the condition of the classes with reference to standard performances we have selected two classes for test No. 7 (fundamentals) and two for test No. 6 (one-step reasoning), in each case one with a relatively high and one with a relatively low variability, according to the coefficients in Table II, p. 262.

The distribution of the individuals of a class from the standpoint of standards will serve to make still clearer the situation as to variability, the critical factor in this whole discussion. The table shows in what grades the pupils of the four classes fall according to efficiency standards on the basis of right answers.

TABLE VII

Grade	Test	Var.	Below Grade 3	Grade 3	Grade 4	Grade 5	Grade 6	Grade 7	Grade 8	Above Grade 8	Total
5A	No. 7	.40	6	2	5	22	8	6	1	1	51
7A	No. 7	.22				9	5	13	16	7	50
6A	No. 6	.46	2	4	11	11	7	6	7	2	50
8A	No. 6	.21			1	2	5	9	22	11	50

Thus in grade 5A only 22 (43%) fall within the grade, in grade 7A 13 (25%), in grade 6A, 7 (14%), and in grade 8A, 22 (44%). That some of the pupils should be placed a grade above or below their proper one would not be so very alarming, but it is more startling to find 8 pupils of 3d grade and below 3d grade ability in the 5th grade, 9 pupils of 5th grade ability in the 7th grade, 17 pupils of 4th grade ability and below in the 6th grade, and 8 pupils of 6th grade ability and below in the 8th grade.

The pupils who rank above their respective grades also call for comment and some detailed attention. Accordingly the scores (right answers), in fundamentals and in reasoning, of the individuals in the extreme upper end of each of these classes are tabulated below to discover whether the high record in the one was maintained in the other. At the same time, as a matter of further interest, the teachers' marks in arithmetic given to the same individuals have been transferred from our records to the table.

At once it appears that there is little or no relation indicated between the abilities called upon in the two tests. The individuals who are good in both tests are the exception and it would seem, therefore, that these function equally well in abstract work and in problems not because of psychic community between the two kinds of work but because of facility gained in some way independently in each. In grade 5A, the individuals selected are all above the sixth grade in ability in abstract work, only one shows better than 5th grade in reasoning, four are below 5th grade; in grade 7A all the individuals fall below 7th grade in reasoning except one (they were above the 8th grade in fundamentals). In grades 6A and 8A the eight best pupils in reasoning in the former class (ranking above the 7th year) fall below the 6th grade in fundamentals with two exceptions and of the eleven best in reasoning in 8A, five fall below in fundamentals.

The evidences of lack of correlation between problematic and abstract work in this select lot of pupils opened up the question with reference to the whole group of grammar department pupils. To settle this matter more thoroughly so far as this group was concerned the correlation was worked out, this time on the basis of accuracy (ratio of right answers to number of attempts), by the Pearson formula. The coefficient of correlation was found to be +.15 (probable error .046). A rough interpretation of this small coefficient would be to say that with a small percentage of the children accuracy in speed reasoning and accuracy in abstract work occur together, but with a large percentage of the children they do not occur together. Even in the cases where they occur together it is a question whether there is a real community of function, but anything more than an idiosyncratic juxtaposition

TABLE

COMPARING SCORES OF CERTAIN INDIVIDUALS IN FUNDAMENTALS

5A					7A				
Individual	F. 5	R. 3	Rank in R	T Mark	Individual	F. 8	R. 5	Rank in R	T Mark
1. P. C........	12	4	6th Gr.	Good	1. G. A......	15	4	6th Gr.	Fair
2. L. F........	8	3	5th Gr.	Fair	2. H. B......	12	4	6th Gr.	Poor
3. J. W.......	8	0	Below 3d Gr.	Good	3. J. D.......	12	4	6th Gr.	Fair
4. B. A.......	10	1	3d Gr.	Good	4. P. L.......	12	6	8th Gr.	Good
5. L. A.......	8	2	4th Gr.	Fair	5. B. H......	12	2	4th Gr.	Fair
6. D. I........	9	2	4th Gr.	Good	6. R. R......	12	4	6th Gr.	Poor
7. A. N.......	8	3	5th Gr.	Fair	7. L. S.......	13	3	5th Gr.	Fair

of abilities or contemporaneousness of functioning, due to the peculiar inner organization of the minds in question that has permitted or potentialized the growth of simultaneous abilities independently of one another. It has been suggested that a more thoroughgoing test of this matter would be afforded by using the general method of "transfer" experimentation, such as we describe on page 131; that is to say, use three groups of pupils, one for control, another for special work in accuracy in fundamentals, another for special work in accuracy in reasoning; by giving all the same tests, preliminary and final, one could determine whether the condition of the individuals at the close in each were due to natural growth in each or to the transfer of special training in the one ability to the other. However, it is doubtful if this elaborate procedure would be worth while unless experiments with a large number of children should result in a much higher coefficient. It

VIII

AND REASONING, WITH TEACHERS' MARKS OF THE SAME

6A					8A				
Individual	R. 4	F. 7	Rank in F	T Mark	Individual	R. 6	F. 10	Rank in F.	T Mark
1. M. A......	6	5	5th Gr.	Fair	1. R. B......	9	13	Above 8th Gr.	Fair
2. A. H......	6	6	5th Gr.	Poor	2. E. H......	8	10	8th Gr.	Fair
3. E. S......	7	5	5th Gr.	Good	3. E. K......	8	11	8th Gr.	Exc.
4. F. D......	6	8	7th Gr.	Good	4. M. W......	8	9	7th Gr.	Fair
5. D. H......	6	6	5th Gr.	Good	5. A. A......	9	12	Above 9th Gr.	Exc.
6. C. H......	6	5	5th Gr.	Fair	6. P. B......	9	6	5th Gr.	Good
7. E. K......	6	5	5th Gr.	Poor	7. C. F......	8	13	Above 8th Gr.	Fair
8. J. W......	10	9	7th Gr.	Good	8. C. M......	9	7	6th Gr.	Fair
					9. F. O......	10	7	6th Gr.	Fair
					10. A. O......	9	13	Above 8th Gr.	Good
					11. F. P......	8	2	3d Gr.	Exc.

would then still be necessary to test the validity of this high coefficient as showing true correlation in the way indicated; but in the event of the low coefficient being still in evidence, the absence of correlation can be safely inferred.

The teachers' marks in arithmetic as compared with the records made in tests in the case of the individuals included in Table VIII afford considerable interest. For convenience of reference these have been tabulated again (Table IX); in the table are also set down the marks that would seem to be proper, in view of a balancing of the scores of the two tests for each individual, had the teacher known all that the tests brought forth or had the pupils performed regularly for the teacher in the manner in which they did under the stimulus of the tests. The latter consideration is important in view of the fact that the teachers' marks are on the whole relatively low. It is probable that some lazy and indifferent pupils were stimulated and some latent abilities, not revealed by ordinary school-room procedure (though, it may be said, they should be), were brought to light. In fact my acquaintance with the pupils has led me to observe such facts in the case of a considerable number of the pupils. But such revelations form part of the value of the tests.

TABLE IX

COMPARING THE TEACHERS' MARKS OF CERTAIN INDIVIDUALS WITH MARKS ESTIMATED BY BALANCING THE SCORES OF EACH INDIVIDUAL IN FUNDAMENTALS AND REASONING (SEE TABLE VIII)

INDIVIDUAL NO.	1	2	3	4	5	6	7	8	9	10	11
5A Teachers' Marks	G	F	G	G	F	G	F				
Estimated Marks	E	E	P	P	G	G	E				
7A Teachers' Marks	F	P	F	G	F	P	F				
Estimated Marks	G	G	G	E	P	G	F				
6A Teachers' Marks	F	P	G	G	G	F	P	G			
Estimated Marks	G	G	G	E	G	G	G	E			
8A Teachers' Marks	F	F	E	F	E	G	F	F	F	G	E
Estimated Marks	E	E	E	G	E	P	E	F	F	E	P

Number of marks that agree 8
Number of marks that practically agree .... 12 (not more than one remove apart).
Number of marks that disagree 13

Total 33

Number of teachers' marks distinctly higher 4
Number of teachers' marks distinctly lower 9

Or, taking the twenty-five marks outside of the eight that exactly agree, five of the teachers' marks are higher and twenty lower than the estimated marks.

The result, 20 out of 33 marks agreeing or practically agreeing, on the whole speaks well for the accuracy of the teachers' marks or for the validity of the tests as a measure of ability, depending on the point of view. One must incline, generally speaking, in cases of difference of marking to give greater weight to the teachers' marks; the estimated marks are based on the results of single tests, and the individual cannot be finally judged on the basis of single tests; the teacher, on the other hand, has known the pupil for months. But the facts that the teachers' marks show a decided tendency to be the lower ones and that pupils rarely surpass themselves in a test lead us to believe that, fundamentally speaking, at least in the nine cases that are marked lower, the estimated rather than the teachers' marks reflect the real ability. There were some things about these pupils, and to their credit, which remained undiscovered prior to the tests.

The number of studies that might be made on the basis of the data supplied by the Courtis tests is practically endless. It must not be supposed that it is necessary or expected that in the practical use of the tests these detailed studies, interesting to the inquirer who has leisure for them, are demanded. The minimum, and highly valuable, procedure is to discover by single tests the measure of class ability and variability and, tentatively, individual ability; to give frequent tests, if desired, to make sure of individual ability (one test has been shown to be sufficient for the determination of the class measure); and to analyze out of the complex of mistakes that are made the kind of mistakes that prevailingly cause failure; this much for diagnostic purposes. But, if reform is to be secured, the same tests must be given a stated repetition at intervals of five months or a year under the same conditions to the same classes. This is really the most important feature of all, as it affords data for the measurement of growth in the several abilities and fundamentally an opportunity to determine the value of different drill methods for securing growth, the class measures and analysis having pointed the way to the drill-topics. The matter of drill is one of the many questions in school practice which only time and experimentation can hope to solve. We have space for only the bare mention of this feature; we have done no work on it ourselves and must refer any one interested to Mr. Courtis's account of his work in the Detroit school bearing on this point of growth (Elementary School Teacher, Chicago, March, 1911, 360–370; June, 1911, 528–539); also to his Manual of Instructions and accompanying folders.

This practically concludes our review of the Courtis tests by the method of giving an account of their application to a local group of children. We had expected to make a study of the effect of the introduction of irrelevant numbers into problematical work, but the character of the work of our pupils in test No. 8 precludes this; also of the effect of borrowing and carrying, copying, etc. on abstract work, but the analysis of the mistakes in test No. 7 was unfortunately confined by the scorers to "initial" mistakes. We have the data but have not had the time for the analysis of the mistakes in test No. 6. But we have made a complete tabulation of the mistakes in the subtraction tables and in the division tables, largely for the purpose, in the case of subtraction, of discovering evidences, if any, of the working of the Ebbinghaus law discussed in our review of Browne's The Psychology of Simple Arithmetical Processes (see pp. 81, 82) and, in the case of division, to note how far the mistakes made fall under the same rubrics as those made by Browne's subjects (see p. 86).

Complementary subtractions are inserted in parentheses. Nothing is plainer in Table X than that knowing a subtraction does not insure knowledge of its complement.

TABLE X

Showing the Most Frequent Mistakes in Each Table Above 3's in Subtraction. Tables Below 4's Not Included as the Mistakes Are Very Few

Subtractions	Grades 3	4	5	6	7	8	Total Mistakes
12—4	3	10	1	0	0	0	14
(12—8)	0	0	0	0	0	0	0
13—5	2	2	1	1	0	0	8
(13—8)	3	8	2	1	1	0	15
11—6	3	2	0	1	2	2	10
(11—5)	0	0	0	0	0	0	0
11—7	8	9	2	1	0	2	22
(11—4)	2	1	0	0	0	1	4
12—7	4	7	0	5	1	0	17
(12—5)	0	0	0	0	0	0	0
13—8	3	8	2	1	1	1	15
(13—5)	2	2	1	1	0	0	8
15—8	6	6	1	3	0	1	17
(15—7)	0	0	0	0	0	0	0
11—9	4	5	1	3	1	1	15
(11—2)	1	1	1	1	0	0	4
17—9	3	1	0	4	2	2	12
(17—8)	1	0	0	0	0	0	1

TABLE XI

Showing Subtractions in which Mistakes Might Be Expected to Occur but in which Few Occur

Subtractions	Number of Mistakes	Subtractions	Number of Mistakes
13—4	0	16—7	2
14—5	0	17—8	1
13—6	0	13—9	1
14—6	0	14—9	0
15—6	1	15—9	6
15—6	0	16—9	0

It was not expected that evidences of the working of the Ebbinghaus law would be found among the practiced subtracters of the upper grades. We repeat here what has been written on another page (p. 79): "Short cuts and well-memorized direct associations tend to obscure the working of the law . . . so that any attempts to verify the law, say, through comparative studies of the efficiency of school children in the several combinations would need to be carefully scrutinized before their results could be held for or against the law. It would seem, for instance, that the more expert adders [also, subtracters] must be ruled out and attention given to those who are still in the barbaric stage of their mathematical evolution, that is to say, to the comparatively inefficient."

It was thought possible, however, that in the earlier grades and with the poorer pupils, some confirmation of the law might be found. It must be remembered that it applies only to those individuals who are in the additive stage of subtraction, and, furthermore, to those whose addition processes are still allied to the count. The evidence sought, therefore, concerns the developmental, not the finished stage of the knowledge product. Those who have read our review of Browne's article will understand that according to the law the more difficult subtractions, generally speaking, will be exemplified by 12 — 4 as against 12 — 8, that is to say, those cases in which the largest difference exists between minuend and subtrahend. Curiously enough, every case (but one) cited in our table seems to contradict the law. The mistakes of two individuals who made an exceptionally large number, a boy (C. K.) of grade 3A and a girl (L. K.) of grade 5A, were carefully gone over, but the mistakes were found to be of all sorts and no law could be perceived to run through them. Incidentally both these pupils were above the average in addition, both in speed and accuracy. This latter fact really threw them out of the class of pupils who are sufficiently uncivilized to show the marks of primitive culture. We must conclude, therefore, that either there is no such law at work or that our pupils are too practiced in direct associations to show it. The two pupils who failed so utterly in subtraction did so because they had no control over direct associations and were not sufficiently barbarous (as shown by their work in addition) to subtract by associations allied to the count. If they had been in this stage, the existence of the law would have been evidenced, if at all, by the slow but accurate performance of such examples as 12 — 8, while errors would freely appear in such as 12 — 4.

The experiment, as will have been gathered, really requires for a successful outcome the testing of a large and peculiarly constituted group of children, children who are in a very early stage in *both* addition and subtraction.

The data given in Table XI tell their own story. The conclusion is either that teachers are mistaken in considering these subtractions difficult for children or that they have been the ones selected for special drill. The latter seems more likely to be the case.

We have similarly tabulated below the most frequent mistakes in the division tables.

TABLE XII

Divisions Grades	3	4	5	6	7	8	Total Mistakes
1 ÷ 1	5	10	7	12	8	13	55
2 ÷ 1	0	2	2	11	2	7	24
2 ÷ 2	0	0	3	1	7	18	29
21 ÷ 3	3	5	1	1	2	1	13
24 ÷ 4	13	13	0	4	3	1	34
5 ÷ 5	2	3	4	10	0	14	33
36 ÷ 6	2	4	6	1	2	2	17
63 ÷ 7	9	10	2	3	0	1	25
9 ÷ 9	0	15	12	7	0	14	48
54 ÷ 9	2	3	1	9	3	4	22

Again we find, as in subtractions, that many of the operations usually regarded as the most difficult are absent, and probably for the same reason. One of the most peculiar things in the tables is the performance of certain simple divisions. Note the 13 mistakes on $1 \div 1$, 7 on $2 \div 1$, 18 on $2 \div 2$, 14 on $5 \div 5$ and 14 on $9 \div 9$ in the 8th grade, and similar results on these divisions all through the grades. Such results must be in part the effect of a mistaken impression on the part of teachers that such operations are too simple to need attention. Not only our results but those of Browne show that such divisions are a most frequent source of error. Just what substitutions are made for true quotients in such cases will appear from our further analysis. An anomalous condition in this connection must be mentioned, in passing. While mistakes are frequent on $1 \div 1$, $2 \div 1$, $2 \div 2$, $5 \div 5$, $8 \div 8$, $9 \div 9$, no mistakes (or very few) are made, as our complete records show, on $3 \div 3$, $4 \div 4$, $6 \div 6$, $7 \div 7$. Why children's minds should thus discriminate is what "no fellow can find out."

A further analysis of the mistakes seemed desirable to see how far Browne's classification of mistakes might be confirmed, especially as his attempt at classification was only tentative, and was based on introspective results from a few adult subjects. It seemed that it might be possible to work out a more thoroughgoing classification, based on our objective results, which would be of some theoretical as well as practical significance. A study of the mistakes made in all the grades tested yielded the following rubrics:

1. Making the quotient figure the same as the divisor,
   - (a) When a difference of only one exists between the divisor and quotient;
   - (b) When the quotient is commonly used as the divisor of the given dividend.

2. Making some factor (other than the divisor), commonly used as the divisor of a given dividend, the quotient figure.

3. When dividing a digit by itself, making the quotient figure *the same.*

4. When dividing a digit by itself, making the quotient figure *zero*.

5. When dividing by *one*, making the quotient *one*.

6. When the dividend is *zero*, making the quotient the same as the divisor.

7. Pupils whose associations are as yet feeble or become so through fatigue or distraction are commonly observed to resort to running up the table.

They frequently miss count and get a quotient figure one remove (say) from the right one.

8. Making one of the quotient figures the quotient.

9. Substituting multiplication for division.

10. Unclassified.

When a mistake might be classified indifferently, which it is but proper to say happened but seldom, preference was given to the class nearest the head of the list.

TABLE XIII

SHOWING THE NUMBER OF MISTAKES IN THE DIVISION TABLES FALLING INTO EACH CLASS

Grade	Kinds of Mistakes										Total Mistakes
	1	2	3	4	5	6	7	8	9	10	
	a & b										
3	16	12	2	14	1	2	26	6	2	43	124
4	13	7	2	20	4	4	23	3	8	49	133
5	5	7	3	23	3	1	14	0	2	4	62
6	4	8	0	32	16	4	16	2	0	17	99
7	10	6	1	27	4	2	7	2	0	19	78
8	9	9	4	65	11	11	5	3	0	3	120
Total........	57	49	12	181	39	24	91	16	12	135	616

Rubrics 1, 2 and 3 correspond with Browne's classification and comprise the extent of it. They account for 118 (19%) of the mistakes. Under rubric 1 two conditions are named above as giving rise to mistakenly making the quotient figure the same as the divisor, the first of which Browne's subjects seem to regard as a frequent source of error (1a). He says, "When only a difference of one exists between the divisor and the quotient digits, the subject frequently has difficulty in selecting the required quotient. In $72 \div 8 = 9$, 8 and 9 are contiguous members of the counting series; as one comes into consciousness it tends to bring the other along with it and this creates a doubt as to which is really right" (p. 86).

Our results do not confirm this as a frequent source of error. Only 8 mistakes in all were of this character, so few that they were not tabulated separately but included in the total (57) under b. But 49 mistakes answer to the second condition (1b), also named by Browne. Forty-nine mistakes also answer to his second condition of error (rubric 2). But rubric 3 (Browne's) covers only 12 mistakes, while 4 accounts for 181 (30%) of the mistakes. In fact this is the largest single source; that is to say, instead of its being common, when dividing a digit by itself, to make the quotient figure the same, it is exceedingly uncommon if we may judge by our results; on the contrary, the common thing is to make the quotient figure *zero*. No. 7 accounts for a large number of mistakes and indicates for the most part imperfectly formed direct associations and a crude but somewhat intelligent effort to overcome the handicap. No. 10 stands for pure guessing. To summarize in a word we should say that rubrics 1, 2, 3, 4, 5, 6, 8, 9 designate for the most part errors of inadvertence, while 7 and 10 may be regarded as indices of knowledge of the tables. The latter part of this statement will be borne out if one reads columns 7 and 10 from the top downward.

Another question of considerable interest arising in connection with the tables is how far knowledge of one table contributes to knowledge of another. In other words, do these knowledges "transfer?" It is a common opinion among teachers that at least addition has a close relation to subtraction and multiplication to division. Our results tend to bear out this opinion to a considerable extent. A glance over the individual records of the pupils, comparing their scores in the four tables, shows some relationship. To test this matter somewhat more thoroughly, we threw the 275 pupils, grades 3A to 8A, who had been present at all the four speed tests into eleven groups of 25 pupils each, ranging them from low to high, regardless of grade in the order of their scores (right answers) (a) in addition, (b) in multiplication; we then computed the averages of the several groups, also the averages of the corresponding groups containing the same individuals, (a) in subtraction, (b) in division. Two series of averages were thus obtained for each relation to be explored, viz., addition with subtraction and multiplication with division, and relation curves were drawn, one series being plotted as the function of the other in each case.

TABLE XIV

SHOWING GROUP AVERAGES OF 275 PUPILS, GRADES 3A TO 8A

	Group										
	1	2	3	4	5	6	7	8	9	10	11
Addition..........	20	26	31	35	40	44	48	52	58	62	78
Subtraction.......	15	13	22	26	32	35	36	39	42	45	54
Multiplication.....	8	13	19	24	28	31	34	36	40	46	59
Division..........	11	15	19	24	32	35	36	34	40	46	43

"If the first series ranges upward for values from low to high and the second from left to right for values from low to high, a positive correlation will be indicated by a line running in a southwest-northeast direction, inverse correlation by a line running in a northwest-southwest direction, and zero correlation by a horizontal line [in these cases]. In proportion as the correlation is complete the line assumes an oblique position." (Whipple, Manual of Mental and Physical Tests, page 38.)

Our curves, Figs. 19, 20, if we read them correctly, show on the whole a positive correlation between the abilities explored. Figure 19 indicates a high positive correlation between addition and subtraction in groups 3 to 6, and lower but positive correlation in

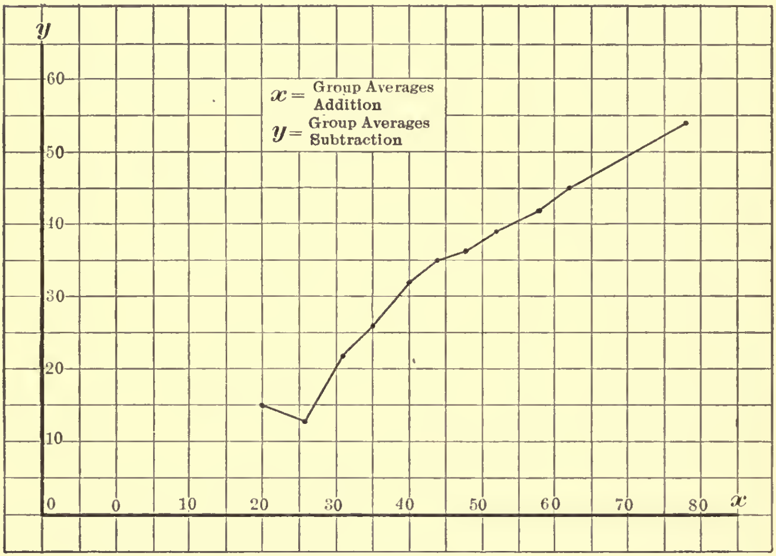


FIG. 19—See Table XIV.
Correlation—Addition with subtraction speed tests.

the higher groups. The lower groups show a zero or slightly inverse relation. Figure 20, also, indicates a high positive relation between multiplication and division, except apparently in the very highest group. We have reason to believe, however, that the downward drop in the curve at the upper end is largely caused by the number of trivial mistakes in division made in the upper grades (see Table XII), which tend to obscure the real relation. If we assume that the high degree of correlation signifies transfer and express the facts shown accordingly we should say that subtraction is a function of addition and division of multiplication.

An interesting correlation was made by the foregoing method by Mr. Courtis with his Detroit children between rate of motor activity (test No. 5) and total score in the speed tests (tests Nos. 1, 2, 3, 4). He found a very close connection between the two

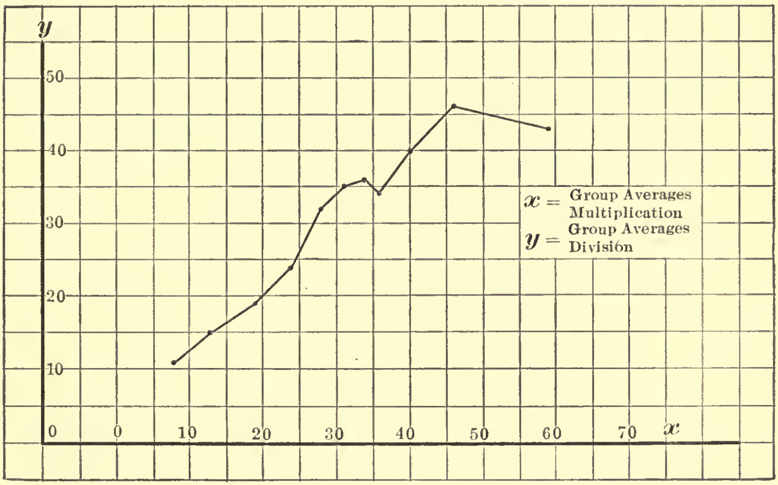


Fig. 20—See Table XIV.
Correlation—Multiplication with divison speed tests.

types of scores up to a certain point, but that beyond this point the ability to copy figures with extreme rapidity does not carry with it the corresponding ability to obtain a high score in the other speed tests. The critical score in copying (115 figures in a minute) was, therefore, taken by him as the standard to be reached before the work in Arithmetic is completed (8th grade).

Little, so far, has been said about accuracy. The writer feels that in speed tests, a percentage of accuracy, computed as the ratio of number of right answers to number of examples attempted, has little significance and may be positively misleading. Records like the following impelled this conclusion: Number attempts 2, number rights 2, accuracy 100%; number attempts 5, number rights 4, accuracy 80%; query—which record shows the greater efficiency?

The number of attempts may be regarded as an index of ambition in many cases, but a survey of our records indicates that it is also an index of self-delusion. Some pupils, by dint of making a large number of attempts, succeed in making a good score of rights, though making also a respectable number of mistakes; others seem to think they are doing great stunts and succeed only in making a great number of mistakes. In either case, the *number of rights* is a good index of efficiency both for speed and accuracy.

Percentage of accuracy can only serve as a factor in such index when the number of attempts is a denominator common to all individuals compared.

In order to lend a measure of completeness to this review, however, the percentages of class accuracy in tests No. 6 and No. 7 have been computed and compared with standard percentages of accuracy computed from table. As an additional note of interest the boys and girls have been separated.

On the whole the boys and girls do not differ in abstract work either in number of right answers or percentage; but in reasoning, as soon as they may be said to perform at all, the boys are decidedly superior. In reasoning the percentages, except in 8A, fall below the standard (compare column 6 with 13); in abstract work, except in 8A, they fall above (compare column 12 with 14). The last column is the standard percentage of accuracy in fundamentals and reasoning combined and suggests, at least, a point of some importance to principals and superintendents, that in tests including both kinds of work, classes may fairly be expected to make 65% in the 5th grade, 69% in the 6th, 72% in the 7th, and 77% in the 8th.

As before intimated, we place little stress on these per cents and believe that the scoring, tabulation and interpretation of results of the tests could be further simplified by disregarding number of attempts altogether.

For the teacher the practical considerations from this whole discussion are two: (1) how to raise the standard of the class if it is

TABLE XV

SEXES COMPARED IN RESPECT TO SPEED AND ACCURACY IN FUNDAMENTALS AND IN REASONING, AND CLASS ACCURACIES COMPARED TO STANDARD ACCURACIES

GRADE	TEST NO. 6						TEST NO. 7						STANDARD ACC.		
	BOYS		GIRLS		CLASS		BOYS		GIRLS		CLASS		Test No. 6	Test No. 7	Combined
	Avg. No. Rts.	%	Avg. No. Rts.	%	Avg. No. Rts.	%	Avg. No. Rts.	%	Avg. No. Rts.	%	Avg. No. Rts.	%			
Col.	1	2	3	4	5	6	7	8	9	10	11	12	%	%	%
3A	1.1	35	1.3	41	1.2	39	[1]18	[1]93	[1]16	[1]97	[1]17	[1]89	78		
4A	1.2	48	1.1	48	1.2	48	[1]22	[1]91	[1]23	[1]89	[1]22	[1]90	81		
5A	2.9	66	2.2	56	2.6	61	4.5	57	4.9	57	4.7	57	83	54	65
6A	3.7	64	3.1	62	3.4	63	6.3	64	6.0	67	6.2	65	86	60	69
7A	4.6	84	3.3	7c	4.1	78	9.1	75	8.4	74	8.8	74	88	64	72
8A	6.5	90	5.7	90	6.2	90	8.4	56	9.2	60	8.7	58	90	70	77

[1] Scores in the tables. Grades 3 and 4 were not given Test No. 7.

below the standard, (2) how to reduce the variability of the individuals from the standard as far as possible. The discovery of these variations through the tests is really the crux of the whole matter, the key to a controllable situation.

For both of these purposes drill is necessary, both individual and class. The question of drill methods has been studied very little, and in fact drill has fallen into disfavor. We can see, however, that it is plainly indicated, and that more light is needed on the subject. One of the best studies is that of J. C. Brown—An Investigation of the Value of Drill Work, Jour. Ed. Psychol. II, 1911, 81–88. But his results are in such shape that they are not available here. The suggestions of Mr. Courtis are so good that we cannot do better than quote them. He says, "Inordinate drill is surely harmful. Drill out of place is also harmful. But drill in its proper place (after not *before* understanding control), and in proper amount, enough to insure the minimum ability necessary to effective work but not enough to involve waste of teaching effort, such drill is one of the vital factors in efficient teaching" (Elementary School Teacher, Nov., 1911, p. 136). Some drill exercises used with his own classes follow:

The first five minutes of each class period was devoted to drill work on the tables. Again the spiral method was used, the same table, the same operation, or the same type of practice (visual, motor, oral, etc.) never being followed two days in succession; that is, if the table of 6's in multiplication was written one day, the table of 3's in addition might be practiced orally the next. In another class the drill on one operation was continued weeks at a time, although the type of practice was frequently changed. In another the drill was on one topic for a short period with frequent changes of practice. Just what the results of these devices might be has not been determined with exactness. "If the schools of a city were divided into districts of, say, ten schools each and definite methods of drill assigned to each district, appropriate records of the work actually done being kept, comparative tests would determine in a single year, with scientific exactness, the relative effectiveness of the various methods" (Elementary School Teacher, March, 1911, p. 370). We also quote from folder H, Courtis Standard Tests: "If the time at the disposal of the teachers will not permit special drill periods, emphasize the special points in the regular lesson, slightly, if necessary, the kinds of work in which the grade excels. Uniform development is the thing to be desired." "Drill work should be rapid, free, pleasant *play* periods of short duration. Skill is not acquired by mere repetition but by unusual exertion." "Drill work for a group must be broad." "Narrow drill produces a specific ability *which does not transfer.*"

"Education is an individual matter and no grade instruction nor group work can possibly be devised to minister adequately or efficiently to the widely differing needs of any group selected *under the present system.* . . . The necessity [however] for handling large numbers of children and the many benefits that accrue from group work make its continuance a certainty. At the same time the differences in individuals make it imperative that each child receive the kind and amount of instruction that his particular makeup demands. These two conditions can be met by having for each subject and each grade standard scores and standard growths in all those component abilities that are fundamental. Individual measurement by standard tests and constant checking of the results of classwork, supplemented by individual prescriptions of study and work, would insure the attainment of any desired degree of minimum ability, and still leave room in the nonessentials of the subject for all those permissible individual variations which make for personality. If education is to become either scientific or efficient, more attention must be given to accurate determinations of both the material to be acted upon and the effects produced." (Courtis—Elementary School Teacher, Chicago, June, 1911, p. 356.)

PART III. Conclusion

CONCLUSION

Our account of the genetic studies of primitive men, children, prodigies, and persons having number forms, pointed to certain psychological and pedagogical conclusions, which are stated in their appropriate places. The same is true of the psychological studies of perception, counting, processes and reasoning; of the statistical studies of efficiency, transfer, hygiene, and ideation; of the didactical studies of apprehension; as well as of our inquiry into the ability of school children (a) to apprehend the cardinal numbers, and (b) to deal with the processes.

It is not proposed to repeat in detail these conclusions here. It may be of some assistance, however, if some of the points arrived at are summarized, though the conclusions in detail are not repeated (see index under Conclusions).

1—The number sense (including apprehension and calculation) arises primitively from counting. We have found this to be the case in our studies of primitive men, of prodigies, of children and of adults (when, in times of fatigue or confusion, they respond to primitive impulses). All the fundamental processes are, primitively speaking, counting. These processes are what they are, because man is what he is, viz., from this point of view, a counting animal. (The lower animals probably do not count; all their mathematical performances can be explained as *quantitative* discrimination.)

2—Counting assumes several phases in the child's history: (a) Counting is a motor response to an inner rhythmical series without reference to objects, a subjective ordering of experiences which the child can not resist. The responses at first are certain muscular movements, as repeating strokes with the clock, rolling mud balls, nodding the head, arranging pebbles, moving toes and fingers, etc.; later articulating, using number names. It is in this stage that number forms often originate. In the absence of number names which later serve to free the motor mechanism by giving it adequate and easily pursued avenues of discharge, the field of response is much restricted. The number forms serve as the means of bursting the bounds of these restrictions, and as a kind of concrete symbolism analogous to the sticks, pebbles, etc., of the prenaming period. The motor impulses find their outlet and their relief in a geometrizing psychosis (doubtless instinctive), which leads to localization in space or imaginary movement along imagined lines, which movement, of course, must have a spatial background. It might pay to cultivate in children good number forms as aids to memory and reproduction and thus to elementary operations;

(b) Counting is a mechanical application of number names to objects. From a psychophysical point of view, it is the matching up or approximate synchronization of the terms of two series of events in consciousness; the one being a series of innervations tending to motor discharge, in other words to articulatory response or naming, the other a series of impressions from sense-stimuli;

(c) Counting becomes rational, i. e., a conceptual process. This involves the putting of units (parts) in a certain ordered relation to one another, as well as marking them off or discriminating them. "The concept two [for example] involves the act of putting together and holding together two discriminated ones." It is in this sense of the word that counting may be said truly to require "a considerable power of intellectual abstraction." At this point a new element from the pedagogical standpoint enters in, viz., motivation; the motive is the need for exact measurement felt by the child in adjusting himself to his environment. The measurement motive runs as well through his later calculations and is partly social in its nature, partly mathematical *per se*.

3—The child enters school with the counting psychosis well established (at least, in its first stage); it is the part of wisdom not to ignore it, but wisely to take advantage of it. But, for the *economical* development of number concepts, the use of number pictures to follow or accompany the seeing, counting, and handling of objects is indicated. Through practice with them the ability to grasp groups is strengthened and percept, image and symbolization all are brought about with economy of time and effort. Furthermore, since the concept, if it is to be more than nominal, that is contentful, must grow out of clear images, it is important that the images be of such a form that they may be returned to readily again and again, thus growing into type-forms of a semi-abstract kind, working categories under which may be subsumed by children all their early numerical experiences. Variety of pictures for a given number should be avoided. The number pictures constitute for the child a tremendous step toward complete abstraction in that he is taken away from the specific numbered objects of various kinds to a representation that may stand for any of them. Concepts of a certain sort would doubtless come *in time* to children confined to the more primitive and wasteful counting method, but the concept would be unrepresentable by any economical (group-wide) symbolization, would be slow in arriving, and would be undeniably less tangible and referable to any clarifying content.

4—The instantaneous grasp of a group of objects visually presented is not intuitive. The perception of number is a *judgment* mediated in part by spatial data, in part by other qualitative characteristics. This judgment is made possible by repeated prior experiences of association of certain qualitative differences in the unit impression with certain numbers of parts derived by actual count. When the association has become established the experiences that have led up to the fixation of association have dropped out of consciousness and the apprehension seems to be immediate. Lay's notion that the apprehension is immediate cannot be sustained. It is true that his kindergarten children with no knowledge of number names made numerically correct drawing of number pictures exposed for a second or two and that they were not simply reproducing a geometrical form, since the drawings often differed greatly in form from the copy. But the weight of testimony from other experimenters and observers being in favor of mediated apprehension, the burden is on him to show that these children did not practice a primitive method of numeration, such as has been so often observed, and punctuate the image which in each case they were *about to symbolize* with nods of the head, wriggling of the toes or what not.

5—Lay's number pictures may be said, on the basis of his experimental results, to be on the whole the best perception-material. Our own experiments on apprehension with his pictures under school room conditions tend to confirm this; and also showed, among other things, that there is a certain order in which the cardinal numbers come to full consciousness. (For complete statement see pp. 249–251.)

6—Psychophysically the simple fundamental processes (carried on orally usually for tabular drill, as when a series of single digits are added, subtracted, etc.) may be analyzed into four factors, stated in order of functioning: (a) The recognition of a result; (b) the motorizing of the result; (c) the recognition of the digit to be added, subtracted, etc.; (d) the associative process leading to the sum, difference, product, quotient. The first is conscious; the second, subconscious; the third, conscious usually not motorized; the fourth subconscious. The points of attention are (a) and (c). Any imperfection in either phase of attention or in (d) results in slowness or mistakes or in both. Any retardation or difficulty in (b) in some cases leads to the reinstatement of the full verbalism of the tables.

The organization of these simple processes into written examples is more complex, rendered so by borrowing, carrying, etc. Some of the added details of functioning are mentioned in the text.

From a pedagogical standpoint, other factors must be considered besides the psychological, e. g., choice and organization of subject matter from a utilitarian or a mathematical viewpoint; the former is an external factor having little influence on methodology, the latter, internal, involving a choice between two aims, (a) that of conferring mathematical insight or (b) of mere calculating skill. We have tried to show that the former is the true aim and that the teaching of processes can be made rational without sacrificing the time or the skill of pupils.

7—Calculation *per se* is of importance. It would probably pay, therefore, to study the methods of prodigies to discover feasible short cuts now not much noted.

8—Reasoning in both problematic and process work is deductive. Pupils can probably be improved in problematic work by practice in attending to and clearly formulating the "major premise." There is an order in which the several operations in one-step and two-step problems can probably be best introduced.

Tests of efficiency in reasoning await the determination of standards in reading before their results can be interpreted at their real value as measures of reasoning.

9—Transfer and efficiency experiments seem to show that the mind instead of being a unity is extremely pluralistic. Little evidence is found of community of function between one ability and another, e. g., between computation and reasoning, or between addition in tables and addition in columns, multiplication in tables and multiplication in examples. It seems that each ability must be dealt with specifically, and that learning tables as such is not of much value. To confer skill particular acts must be repeated in a number of situations likely to occur. Knowledge of $6+3 = 9$ as an isolated fact will not necessarily function in the complex $4+2+3$. Hence it is probable that economical learning of the simple processes involves learning them in complex situations, that is, in organized examples.

10—There are no types in number imagery in the sense that children think exclusively in one kind of imagery (Lay, Springer). Although most children are of the mixed type, yet certain methods of presentation will produce better results than certain other methods. We can therefore classify children into types according to their results. Neglect to do this is a cause of failure and retardation.

11—Hygiene counsels against mechanical, untimely, unpsychologized drill, leading to automatisms and arrested development; against the too early introduction of problematic work and the use of number puzzles; and against giving too much time to the subject under an exaggerated notion of its value as mental discipline or as practical training or the mistaken impression that efficiency is a function of the time devoted to a subject.

12—Our experiments with children to determine ability in the processes and reasoning point to the value of such tests for the diagnosis of present conditions and the measurement of growth. The striking result was the variability within the class and within the individual.

Incidentally, the data from the tests in subtraction were analyzed to find confirmation, if possible, of the Ebbinghaus law according to which the more difficult subtractions, generally speaking, would be exemplified by 12—4 as against 12—8, that is to say, those cases in which the largest difference exists between minuend and subtrahend. No traces of the working of such a law were discovered. Also an analysis of the mistakes made in division yields a much larger number of rubrics under which such mistakes must be classified than has heretofore been recognized.

Finally, it is hardly necessary to state that no attempt has been made to say the final word on any of the topics in this book, but only to blaze a few of the paths along which experimenters must travel before the pedagogy of arithmetic can approach scientific completeness.

IV. BIBLIOGRAPHY

1. L. L. Conant. The Number Concept, New York, 1896.
2. W J McGee. The Beginning of Mathematics. American Anthropologist I (new series), 1899, 646–674.
3. C. H. Judd. Studies in Number Consciousness. Psychological Bulletin XI, 1909, 42–43.
4. R. E. Marsden. A Child's Appreciation of Number. Paidologist V, 1903, 91–93.
5. D. E. Phillips. Number and its Applications. Ped. Seminary V, 1897–98, 221–281, 290–298.
6. G. Stanley Hall. Educational Problems, Vol. II, chap. 18; "The Pedagogy of Elementary Mathematics."
7. D. E. Phillips. Genesis of Number Forms. Amer. Jour. Psychol. VIII, 1896–97, 506–527.
8. A. R. Hornbrook. Pedagogical Value of Number Forms. Educational Review, V, 467–480.
9. M. W. Calkins. A Statistical Study of Pseudo-chromesthesia and Mental Forms. Amer. Jour. Psychol. V, 1893, 439–464.
10. G. J. W. Patrick. Number Forms. Popular Science Monthly XLII, 1893, 504–514.
11. F. Galton. Inquiries into the Human Faculty. Everyman's Library, 79–105.
12. F. D. Mitchell. Mathematical Prodigies. Amer. Jour. Psychol. XVIII, 1907, 61–143.
13. E. W. Scripture. Arithmetical Prodigies. Amer. Jour. Psychol. IV, 1891, 1–59.
14. I. Ioteyko. Les Calculateurs prodigies. La Revue Psychologique III, 1910, 320–328.
15. V. Kipiani. Visualization colorée et Sens chromatique chez Mlle. Uranie Diamandi, III, 1910, 329–325, La Revue Psychologique.
16. W. B. Carpenter. Mental Physiology. New York, 1891.
17. H. H. Goddard. A Sidelight on the Development of the Number Concept. Supplement No. 1, The Training School, No. 46, 20–24.
18. C. J. Burnett. The Estimation of Number. Harvard Psychological Studies II, 1906, 349–404.
19. J. F. Messenger. The Perception of Number. Psychol. Review, Monograph Supp. V, 1903, 123–144.

20. G. Dietze. Über den Bewusstseinsumfang, daselbst, Leipzig, II, 1885.
21. W. A. Lay. Führer durch den ersten Rechenunterricht, Wiesbaden, 1898.
22. H. C. Warren. The Reaction Time of Counting. Psychol. Review, Monograph Supp. Nov. 1897, 569–591.
23. J. M. Cattell. Über die Zeit des Erkennens, u s w. Philos. Studien III, 1886, 94–127.
24. H. A. Nanu. Zur Psychologie der Zahl-Auffassung, Diss. Würzburg, 1904.
25. L. A. Arnett. Counting and Adding. Amer. Jour. Psychol. XVI, 1905, 327–336.
26. H. Leuba. The Personifying Passion in Youth, etc. The Monist X, 1900, 536–548.
27. M. C. Whiting. Individuality in Numerals. Ped. Seminary II, 1892, 107–110.
28. E. C. Sanford. On the Guessing of Numbers. Amer. Jour. Psychol. XIV, 1903, 383–401.
29. C. H. Judd. Studies in Number Consciousness. Psychol. Bulletin XI, 1909, 42–43.
30. C. E. Browne. The Psychology of the Simple Arithmetical Processes. Amer. Jour. Psychol. XVII, 1906, 1–37.
51. J. Dewey. Some Remarks on the Psychology of Number. Ped. Seminary V, 1897–98, 426–434.
32. McLellan and Dewey. The Psychology of Number. New York, 1898.
33. H. Suzallo. Reasoning in Primary Arithmetic. California Education, June, 1906.
34. F. G. Bonser. The Reasoning Ability of Children of the Fourth, Fifth and Sixth Grades. Teachers' College Contributions to Education, 1910.
35. I. Springer. Mental Reproductive Types in Arithmetic. Thesis, N. Y. U., 1911.
36. T. L. Bolton. Growth of Memory in School Children. Amer. Jour. Psychol. IV, 1892, 362–380.
37. M. W. Meyerhardt. Economical Learning. Ped. Seminary. XIII, 1906, 145–184.
38. K. Eckhardt. Visuelle Erinnerungsbilder beim Rechnen. Zeitschrift f. Exp. Pädagogik V, 1907, 1–22.
39. C. H. Judd. Genetic Psychology for Teachers. New York, 1903.
40. W. Wundt. Outlines of Psychology, 2d English edition.
41. W. Brown. An Objective Study of Mathematical Intelligence. Biometrica VII, 1910, 352–367.

42. S. A. Courtis. Measurement of Efficiency and Growth in Arithmetic. The Elementary School Teacher, Chicago: (a) X, 1909, 58–74, 177–199; (b) XI, 1910, 171–185; (c) XI, 1911, 360–370; (d) XI, 1911, 528–539.

43. " " Standard Scores in Arithmetic. Elementary School Teacher, XII, 127–137.

44. " " Manual of Instructions. Detroit, 1910.

45. W. Scott. Social Education, Chapter 8, 170–198.

46. C. W. Stone. Arithmetical Abilities. New York Teachers College Record, 1908.

47. J. M. Rice. Educational Research: A Test in Arithmetic, The Forum, XXXIV, 1902, 281–297; Causes of Success and Failure in Arithmetic, The Forum, XXXIV, 192, 437–452.

48. Fox and Thorndike. The Relationship between the different Abilities Involved in the Study of Arithmetic, etc. Columbia University Contributions to Philos., Psychol., and Education, Feb., 1903, 32–40.

49. J. C. Brown. An Investigation of the Value of Drill Work. Jour. Educational Psychol. II, 1911, 81–88.

50. W. H. Winch. Accuracy in School Children. Does Improvement in Numerical Accuracy Transfer? Jour. Educational Psychol. I, 1910, 557–589.

51. " " Further Work in Numerical Accuracy. Jour. Educational Psychol. II, 1911, 262–271.

52. D. Starch. Transfer of Training in Arithmetical Operations. Jour. Educational Psychol. II, 1911, 306–310.

53. W. H. Burnham. Arithmetic and School Hygiene. Ped. Seminary, XVIII, 1911, 54–73.

54. " " Hygiene of Arithmetic. Encyclopedia of Education, 1911, 207–209.

55. W. A. Lay. Führer durch den Rechenunterricht der Unterstufe. Leipzig, 1907.

56. H. J. Walsemann. Anschauungslehre der Rechenkunst. Schleswig, 1907.

57. E. Meumann. Vorlesungen zur Einführung in die experimentelle Pädagogik, Leipzig, 1907, vol. II, 329–360; "Das Rechnen."

58. F. N. Freeman. Untersuchungen über den Aufmerksamkeitsumfang und die Zahlauffassung. Pädagogische-Psychologische Arbeiten, vol. I, 88–168.

59. K. Knilling. Kritik zu W. A. Lay's experimenteller Forschungsergebnissen, III, 11–12.

60. G. Schneider. Die Zahl im Grundlegen Rechenunterricht, Berlin, 1900.

61. L. Pfeiffer. Experimentelle Bewehrtung der Rechen-apparate, die auf die Bornschen und die quadratischen Zahlbilder gegründet sind. Zeitchr. f. experim. Pädag. II, 1905, 133–146.
62. H. J. Walsemann. Pestalozzi's Rechenmethode, Hamburg, 1901.
63. R. Knilling. Pädagogische-psychologische Untersuchungen über die Natur der Zahlen. Pädagog.-psycholog. Studien II, 1901, 11–13, 19–23; III, 1902, 65–70.
64. J. C. Gray. Number by Development. Phila., 1910.
65. D. E. Smith. Mathematics in the Elementary School. Teachers' College Record, March, 1903.
66. D. E. Smith. The Teaching of Arithmetic. Teachers' College Record, 1909.
67. J. W. A. Young. The Teaching of Mathematics in the Elementary and Secondary School. New York, 1907.
68. B. Branford. Study of Mathematical Education. Oxford, Clarendon Press, 1908, Chap. 7.
69. A. D. Yocum. An Inquiry into the Teaching of Addition and Subtraction. Phila., 1901.
70. S. Newcomb. The Teaching of Arithmetic. N. E. A., 1906.
71. C. W. Stone. Problems in the Scientific Study of the Teaching of Arithmetic. Jour. Educational Psychol. IV, i–16.
72. W. C. Bagley. On the Correlation of Mental and Motor Ability in School Children. Amer. Jour. Psychol. XII, 1901, 193–205.
73. B. Russell. On the Relation of Number and Quantity. Mind VI, 326–341.
74. M. A. Bailey. The Teaching of Arithmetic in Elementary Schools. N. E. A. Proc. 1905.
75. H. Suzallo. Teaching of Primary Arithmetic. Teachers' College Record, March, 1911.
76. S. Badanes. Grube Method. Thesis N. Y. U., 1895.
77. W. H. Heck. Mental Discipline and Educational Values. New York, 1911.
78. P. R. Radosavljevich. Apperception in the Light of Experiments Made on School Children. Educational Foundations, XXIV, 1912, 132–140, 210–220.

V. APPENDICES

APPENDIX I

Materials and Procedure Used in Experiments on the Estimation of Number (C. T. Burnett, Harvard Psychological Studies, Vol. 2).

Influence of Group Area.

Four sets of experiments were given.

A. Squares, 1 cm., neutral gray, no. 1 (Bradley), were arranged irregularly in two groups with irregular outlines on a background of black cardboard. One group was large in area, the other small, the attempt being made to fill each square homogeneously. Groups were not proportional in shape of area.

B. As above, save that circles 11 mm. approximately in diameter were substituted for squares in the interest of distinctiness for the several objects.

C. The area of the groups was oblong and regular and the sides proportional (Compact—72.5 mm.: 58 mm.; scattered—110 mm.: 88 mm.). These relations were determined by the size of the frame that had already been used, and by the desire to make the difference in area as marked as the other necessary conditions would admit. The color of the compact group was the deepest shade of normal gray; of scattered group of the next higher shade.

These dark grays were used to reduce to a minimum the tendency to produce after-images. The difference in the shades of the two groups was in the interest of avoiding the greater brightness due to the mass effect of the compact group.

D. As in C, except that India ink outline circles $^1/_3$ to $^1/_2$ mm. line were used on a background of granite cardboard. This change was made to avoid as far as possible the greater mass stimulation due to the reinforcing effect of the compact arrangement. The size of the circles remained as before.

Care was taken to eliminate all influence of numerical judgment, since the experiments were to determine the influence of other factors. The numerical factor was eliminated by equalizing the numbers in the two groups, but the fact of their being equal was, of course, not known to the observers. Furthermore, to give the impression of inequality some of the pairs exposed together were as a matter of fact not equal, though nearly so. Conclusions were to be drawn from the erroneous judgments, not the correct ones, since the correct ones would be the result of numerical judgment as such.

The standard number of objects in each group was 20. The individual cards in seven pairs were presented for comparison as follows; 3 pairs, 20 to 20; 1 pair, 20 to 19; 1 pair, 19 to 20; 1 pair, 17 to 23; 1 pair, 23 to 17.

Three kinds of errors are likely to arise in such experimentation: (a) distribution errors, space errors, and time errors. The distribution error (the tendency to give more wrong judgments in favor of one kind of irregular distribution than of the other kind with which, in a given pair, it is mated) was removed by giving to each group an irregular arrangement, each having its peculiar irregularity; also by presenting each of its several pairs with the order of the individual groups composing it reversed (see presentation above); space errors (the tendency to favor one side [right vs. left] more than the other when two groups are presented side by side) and time errors were eliminated by performing an equal number of experiments with the groups in reversed arrangement.

(The simultaneous exposures in these and the other experiments on relative number were given by a screen apparatus electrically controlled, enabling an exposure of 1.2 to 1.6 sec. The successive exposures for the judgment of absolute number were given by a similar apparatus, with an interval of 1.6 sec.)

The following proportion was kept among the numbers of observations upon each kind of cards: one-half upon the group objectively equal; five-twelfths upon those differing by one from each other, where half each went to 20 to 19 and 19 to 20, one-twelth upon those showing the maximum objective difference of 6, half to 17 to 23 and half to 23 to 17.

Subjects were asked to declare which was the larger group. Quite a large number of experiments were performed with each subject, from 88 (44 in a few cases) to 274 and the results recorded as *average per cent of judgments as larger in favor of* (a) the smaller group (b), the larger group; judgments upon objectively equal groups not recorded. Judgments in favor of the larger group are of course correct; as has been said before, it is in the incorrect judgments that interest centers (first line of table below). When average per cent of judgments was below 10% the record was made under the rubric "no tendency."

Judgments of equality upon objectively unequal groups are entered as overestimations of the smaller groups.

The per cent of correct judgments of unequal groups is equally divided between the two other classes, because interest centers in the difference between the tendency of error in one dirrection and that in the other direction.

An illustration of the manner of tabulation follows:

TABLE

INFLUENCE OF GROUP AREA—RESULTS OF EXPERIMENTS, A, B, C, D.

	A		B		C		D	
	No. Subjects	*Avge. Per cent*	*No. Subjects*	*Avge. Per cent*	*No. Subjects*	*Avge. Per cent*	*No. Subjects*	*Avge. Per cent*
(a) Small........	5	31.5	10	34.6	7	44.1	10	46
(b) Large........	3	35.5	4	44.1	5	41.5	3	26.3
No Tendency.	1	8.8	2	4.2	4	5.7	3	6.6

APPENDIX II

C. E. BROWNE—THE PSYCHOLOGY OF SIMPLE NUMBER PROCESSES—AM. JOUR. PSYCHOL., XVII, 1906

Procedure in Experimentation

The subjects were eight students in Clark University, none of whom was a practical arithmetician.

Single-digit adding (six subjects).

A pack of twenty cards, a single digit written upon each with a cover card, was held by the subject in his left hand. At a signal, he removed the cover card with his right hand and saw the first figure, removed the card carrying that, saw the second figure, added it; and so on to the end of the series, handling the cards as a player would to see that all were there. The cards were highly glazed so as to be easy of manipulation.

The figures were added silently, the total sum being announced by the subject as a signal that the series was completed. The experimenter then recorded the introspections and the time. The digits 1 to 9 inclusive, twenty of each evenly distributed in ten packs, differently arranged at each sitting, were added five times over in five different sittings by each subject.

The removal of the cards was practically automatic from the first and much more rapid than the adding. The figure to be added always appeared with the readiness to add and without conscious effort on the part of the subject.

Simple subtraction (four subjects).

The same pack of cards was used as in addition. No pack had digits adding 100, 99 being the highest sum, 79 the lowest. The subject always started at 100, subtracting each digit as it appeared, from the previous remainder. Other conditions as in single-digit adding.

Written subtraction (four subjects).

Fifty examples written upon white cards, having each digit in both subtrahend and minuend, but differently arranged each time, were performed by each subject. The card held in place by the left hand rested across a tin plate so arranged that the writing of the digits on the card pressed down the plate sufficiently to bring it into contact with a row of brass screws directly under it, thus completing an electrical circuit and recording the time of writing each digit upon a rotating drum. The example was covered by a piece of paper, removed simultaneously with the signal "go."

Simple multiplication (four subjects).

The numbers upon the cards used in single digit adding were multiplied digit by digit consecutively by each of the digits 2 to 9 inclusive by each subject. The multiplier was announced an instant before the "ready" and "go" signals. Other conditions were the same as in single digit adding.

Written multiplication (four subjects).

Eight examples were written upon white cards each containing all the digits differently arranged. These multiplicands were multiplied by each digit 2 to 9 inclusive by each subject. Otherwise the method was the same as in written subtraction.

Simple dividing (four subjects).

Eight packs of 18 cards each were used. Each pack contained all the nine multiples of the digits except one yielding a quotient 1 to 9, each multiple appearing twice. The subject divided through each pack, each time differently arranged, ten times. The divisor was announced just before the "ready" and "go" signals.

Written short division (four subjects).

The method and examples used were the same as described in written multiplication except that numbers were divided instead of multiplied.

APPENDIX III

Reinstatement of the Details of Mitchell's Scheme for Raising 3 to the 8th Power.

(See Mitchell's Mathematical Prodigies, Ped. Sem. XVIII, p. 61–143.)

3	
3 + 3 = 6	
6 + 3 = 9	2d power
9 + 9 = 18	
18 + 9 = 27	3d power
27 + 27 = 54	
54 + 27 = 81	4th power
81 + 81 = (1)62	
62 + 81 = (1)43	5th power
43 + 43 = 86	
86 + 43 = (1)29	6th power
29 + 29 = 58	
58 + 29 = 87	7th power
87 + 87 = (1)74	
74 + 87 = (1)61	8th power

It is especially to be noted that this is essentially an addition or counting method.

APPENDIX IV

Mitchell's Method of Converting Odd Numbers into Even Numbers for Calculating Purposes.

(Taken from Mitchell's Mathematical Prodigies, Ped. Sem., XVIII.)
The following is a table of the endings of certain products.

	07	32	57	82
23	61	36	11	86
48	36	36	36	36
73	11	36	61	86
98	86	36	86	36

The numbers at the left (23, 48, 73, 98) differ in pairs by 25; and so with the numbers at the top. In each set of factors there is one multiple of 4—48 in first set, 32 in second set; also, in each set there is an odd multiple of 2—98 ($= 2 \times 49$), 82 ($= 2 \times 41$); again, in each set there is one number of the form $40 + 1$ (73, 57) and one number of the form $40 - 1$ (23, 07).

The sixteen numbers, in the body of the table, all belong to a similar series—11, 36, 61, 86. If either of the factors is a multiple of 4, the product has the ending 36 as shown by the second line, second column; if both are odd multiples of 2 (98, 82), the product again ends with 36; if one is an odd multiple of 2 (98, 82) and the other an odd number, the product has the ending 86 ($= 36 + 50$). Finally, if both numbers are odd the ending of the products is $36 + 25$, i.e., 11 or 61; 61 (a number of the form $40 + 1$), if the numbers multiplied are either both of the form $40 + 1$ (73×57) or both of the form $40 - 1$ (23×07); 11 (a number of the form $40 - 1$), if one of the factors is of the form $40 + 1$ and the other of the form $40 - 1$ (73×07, 23×57).

Thus, by applying a few simple rules any one of the 16 products in the table can be made to depend on the single product, 48×32, of the two multiples of 4 in the table. Hence to find the ending of the product of two odd numbers, change each into a multiple of 4 by adding or subtracting 25, multiply these multiples of 4 together and then add or subtract 25 as the case may require to get the answer. A similar principle applies to the power series of any odd number; simply find the required power of the corresponding even number, and then either add or subtract 25.

Leaving out of account multiples of 5, in a class by themselves and very easy to multiply, the whole of multiplication so far as the endings are concerned may be reduced to the 200 possible products of any two of the 20 numbers 04, 08, 12, 16, 24, 28, 32, 36, etc., instead of 3200 combinations which would otherwise be required; and the problem of finding the last two figures of any power may be reduced to less than 400 simple cases.

APPENDIX V

The Application of the Method of Two-Figure Endings to Factoring.

(Adapted from Mitchell's Mathematical Prodigies, Ped. Sem. XVIII.)

If the number to be factored is not already odd and prime to 3 and 5, it may be made so by division in order to limit the possible number of two-figure endings in the number to be factored.

Take the case of an odd number prime to 5. If the last two figures of one of its divisors are known and the division is exact, the last two figures of the other can have only one value; a table may be constructed, if desired, showing the different pairs of endings in the factors which will produce a given ending in the product.

The procedure would be as follows: Having rendered the number to be factored prime to 3 or 5, if not so already, observe the two-figure ending; consult the table (the table in Appendix IV shows the kind of table indicated, though the actual table would be more exhaustive of two-figure endings) for the two-figure endings of each of an initial pair of factors; assume that a certain number with one of these endings is a factor; carry a division far enough to decide whether the last two figures of the other factor can result, and as soon as this is seen to be impossible, abandon the work; try again.

For example, factor the number 487305.

$$487305 \div 5 = 97461$$

Since the two-figure ending is 61, the two-figure endings of the initial factors are 73 and 57 (table, Appendix IV).

Try 373 as one factor; it is known that the other factor will end in 57.

```
373) 97461 (26
     746
     ----
     2286
```

Abandoned, because 6 appears in the quotient instead of 5.

Try 273

```
273) 97461 (357
     819
     ----
     1556
     1365
     ----
     1911
     1911
     ----
```

Initial factors are 273 and 357

$$273 = 3 \times 81$$
$$357 = 7 \times 51$$

The factors of $487305 = 5 \times 3 \times 7 \times 51 \times 91$.

It is thus possible to save much work, especially where the numbers involved are not very large; a factor may often be rejected almost at a glance which would otherwise have to be divided through to the end.

Note.—I have slightly modified and somewhat changed the wording of Mitchell's account, in order to make the scheme more workable by the ordinary person. For the sake of clearness, an example has been worked out.

APPENDIX VI

George P. Bidder, Sr.'s Method of Multiplication

From Bidder's own account quoted by Scripture (13). Bidder, Sr. did not use cross-multiplication, as his son did, but a "natural" method.

Problem—Multiply 397 by 173

$100 \times 397 = 39700$							
$70 \times 300 =$	$21000 = 60700$						
$70 \times \ \ 90$		$6300 = 67000$					
$70 \times \ \ \ 7$			$490 = 67490$				
3×300				$900 = 68390$			
$3 \times \ \ 90$					$270 = 68660$		
$3 \times \ \ \ 7$						$21 = 68681$	

"The last result in each operation being alone registered by the memory, all the previous results are consecutively obliterated until a total product is obtained." —*Bidder.*

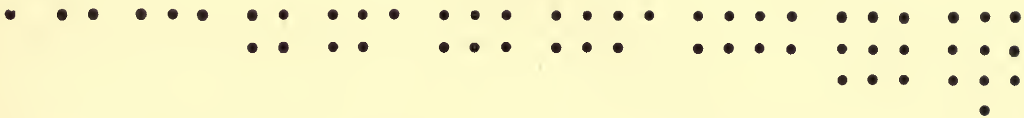
Appendix VII

Showing Operations with 8 and 9

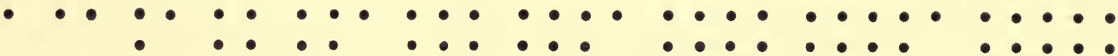
: : : ⁄	: : : | :	: : ⁄ :	: : | : :
$7 + 1 = 8$	$6 + 2 = 8$	$5 + 3 = 8$	$4 + 4 = 8$
$1 + 7 = 8$	$2 + 6 = 8$	$3 + 5 = 8$	$8 - 4 = 4$
$8 - 1 = 7$	$8 - 2 = 6$	$8 - 3 = 5$	$8 = 2 \times 4$
$8 - 7 = 1$	$8 - 6 = 2$	$8 - 5 = 3$	$8 \div 4 = 2$
$8 = 1 \times 7\ (+ 1)$	$8 = 1 \times 6\ (+ 2)$	$8 = 1 \times 5\ (+ 3)$	
$8 \div 7 = 1$ (R1)	$8 \div 6 = 1$ (R2)	$8 \div 5 = 1$ (R3)	
: ⁄ : | :	: | : | : | :		
$8 = 2 \times 3\ (+ 2)$	$8 = 4 \times 2$		
$8 \div 3 = 2$ (R2)	$8 \div 2 = 4$		
: : : : | ·	: : : ⁄ ·	: : : | : ·	: : | : : ·
$8 + 1 = 9$	$7 + 2 = 9$	$6 + 3 = 9$	$5 + 4 = 9$
$1 + 8 = 9$	$2 + 7 = 9$	$3 + 6 = 9$	$4 + 5 = 9$
$9 - 1 = 8$	$9 - 2 = 7$	$9 - 3 = 6$	$9 - 4 = 5$
$9 - 8 = 1$	$9 - 7 = 2$	$9 - 6 = 3$	$9 - 5 = 4$
$9 = 1 \times 8\ (+ 1)$	$9 = 1 \times 7\ (+ 2)$	$9 = 1 \times 6\ (+ 3)$	$9 = 1 \times 5\ (+ 4)$
$9 \div 8 = 1$ (R1)	$9 \div 7 = 1$ (R2)	$9 \div 6 = 1$ (R3)	$9 \div 5 = 1$ (R4)
: : | : : | ·	: ⁄ : | : ·	: | : | : | : | ·	
$9 = 2 \times 4\ (+ 1)$	$9 = 3 \times 3$	$9 = 4 \times 2\ (+ 1)$	
$9 \div 4 = 2$ (R1)	$9 \div 3 = 3$	$9 \div 2 = 4$ (R1)	

APPENDIX VIII.
NUMBER PICTURES

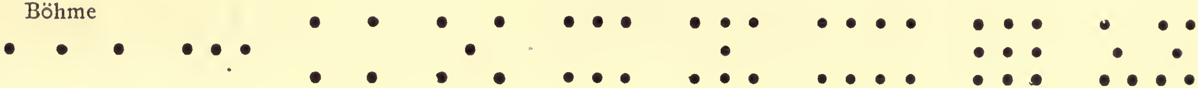
Busse

1. 

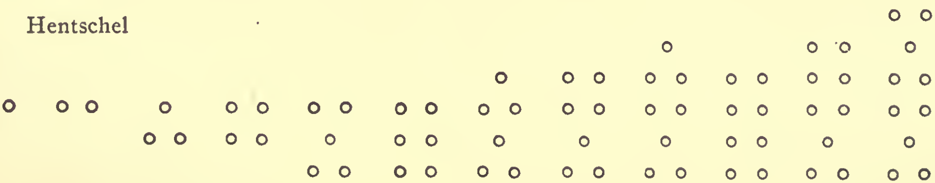
Born

2. 

Böhme

3. 

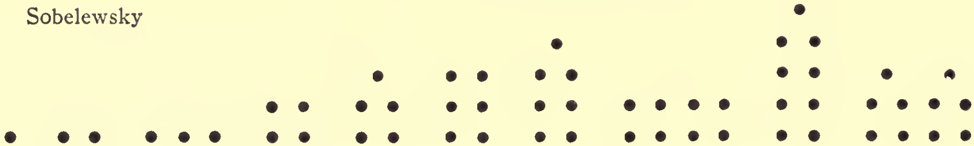
Hentschel

4. 

APPENDIX VIII.

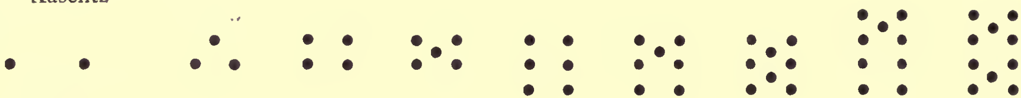
Sobelewsky

5.



Kaselitz

6.



Beetz

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Lay

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Freeman

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