By the Same Authors

NUMBERS, VARIABLES, AND MR. RUSSELL'S PHILOSOPHY
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FUNDAMENTAL CONCEPTIONS

OF

MODERN MATHEMATICS

Variables and Quantities

WITH A DISCUSSION OF THE GENERAL CONCEPTION
OF FUNCTIONAL RELATION

BY

ROBERT P. RICHARDSON

AND

EDWARD H. LANDIS

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PREFACE.

THE present work is essentially one of constructive criticism. It is, we believe, the first attempt made on any extensive scale to examine critically the fundamental conceptions of Mathematics as embodied in the current definitions. The purpose of our examination is not solely or even chiefly to show the presence of error, but to promote the development of a more scientific doctrine. In expounding our own views we have often been obliged to find fault with those of others; but we have not gone out of our way for the sake of mere criticism; we have merely cleared away false doctrine preparatory to replacing it with true. Our work, though in a sense dealing with definitions, does not have as its essential scope questions as to the words to be used in expressing something about which there is universal agreement; it really deals with the conceptions underlying the definitions where there is, as will be shown, a great diversity of view. Further than a discussion of definitions (in this sense) we do not go, and though we have at times occasion to enunciate axioms and theorems we never set down a demonstration. It is indeed undeniable that a discipline consisting of definitions alone would be perfectly futile, but this is no argument against deeming the definitions of a science worthy of a separate exposition. How far from being systematic is the treatment of the definitions of Mathematics in most mathematical writings will be appreciated by all who have given their attention to the matter. Definitions are laid down only as they are needed for the work in hand, and in their formulation attention is given, not to the needs of mathematical science as a whole, but to those of a single book—too often a book whose sole purpose is to enable more or less stupid youths to pose as graduates of a course in Mathematics. As to the articles of original research published in mathematical journals, definitions are hardly to be found in them at all. This state of affairs has reacted upon the demonstrations of Mathematics. When a systematic nomenclature and its concomitant, a clear and connected view of matters, are lacking, precision in statement cannot be expected. Nor is it to be found, and by far the most difficult task to the reader of a work on advanced Mathematics is not appreciating the cogency of the reasoning employed—or depreciating it, as one is sometimes compelled to do—but ascertaining what the author really means. This in no small number of cases is something very different from what he has said. Such a state of affairs does not rule in elementary Geometry; due in large measure to the Euclidean custom of beginning a demonstration with a precise statement of the fact about to be proven; this in turn necessitating more attention to matters of definition than modern mathematicians have thought fit to give. Mathematics to-day is indeed far behind most other sciences as regards lucidity of exposition. In a comparatively short time a young man of average ability can become so far familiar with Chemistry or Botany or Zoölogy, as to be able to read intelligently a work in any department of the science whatsoever. But this is not the case with Mathematics—a student far above mediocrity, who has taken the best University Course in Mathematics to be found, will come across mathematical works as unintelligible to him as Chinese or Choctaw. It is not merely that he finds himself unfamiliar with the theorems proven in such works: this would be neither surprising nor detrimental; but he will not even be able to understand what it is that the theorems are about. And to gain the knowledge requisite for this will not be a matter of consulting a lexicon; but one of hard study for several months.[1] This state of affairs is not, we hold, an unavoidable one due to the peculiar difficulties of Mathematics. It is due to the lack of systemization; and in particular to the failure of text-books to give any thorough exposition of the fundamental conceptions of Mathematics. The thirst for so-called "original research," and the credit attached to it, has led mathematicians to disregard such matters. The investigation, for example, of some particular differential equation not yet touched upon is classed as "original" work, while investigation of the current doctrine of differentiation is not. And by implication the impression is conveyed that work of the former type requires a higher degree of intellect than the latter—an impression very far from the truth. Thus the one is encouraged, the other discouraged; and in many quarters the impression prevails that there is nothing more to be done at the foundations of Mathematics; that the only object of a mathematician should be to rear

[1] As an illustration of the difficulties in the way of acquiring a thorough knowledge of a branch of mathematics, we may mention that Hamilton, assuredly no tyro in Vector Analysis, found the *Ausdehnungslehre* so obscure that he avowed himself unable to understand Grassmann's system in all its details. And Herschel, in turn, after reading three chapters of Hamilton's *Lectures on Quaternions*, was obliged "to give up in despair" his hope of mastering the subject. This was some years ago, but what change has since taken place in methods of mathematical exposition has not been a change for the better.

the superstructure still higher, leaving the old foundations alone. In fact, however, the great desideratum in Mathematics at the present day is, a rebuilding of the foundations, and a readjustment and systemization of what has been built upon them. There is needed a scientific exposition of the definitions, and a complete enumeration, with specific enunciation, of the axioms and postulates. After this (but not before) should come a systematic statement of the theorems, the conditions under which each is valid being stated with perfect precision. It is of little avail to have the theorem of some "original investigator" hidden away in a back number of some mathematical journal, and even there loosely stated or (as is more commonly the case) not explicitly stated at all.

This much-needed revision of Mathematics ought undoubtedly to be made from a philosophical standpoint, there being constantly maintained rigid adherence to the requirements of a sane Metaphysics in the best sense of the word and to the canons of a sound Logic. It is quite clear that unless our fundamental conceptions and principles accord with the one, and our processes of deduction with the other, we cannot develop anything worthy of the name of a deductive science. Unfortunately too many mathematicians look askance upon the application of philosophical doctrine to Mathematics. With but few exceptions, authors of mathematical works and teachers of the subject cultivate Mathematics as an art. They often show extraordinary ingenuity in the solution of problems and in the transformation of formulas, while giving little heed to the realities represented by their symbols and the processes of inference corresponding to their symbolic transformations. Were this all, no objection could be raised by those who wish to see Mathematics developed as a science. The bricklayer and carpenter are useful members of society, even though ignorant of the science of Mechanics. But too often the conventional mathematician arrogantly assumes, toward the philosophical side of the question, an attitude like that of the illiterate artisan toward physical science. He stigmatizes any attempt at logical precision as of no practical value; and is indeed in one respect worse than the carpenter or bricklayer, since the latter makes no claim to the title of scientist, while the artisan mathematician would arrogate this to himself to the exclusion of the philosophical investigator. Such an attitude is amusing, when one considers of how little bread-and-butter utility are many departments of Mathematics which find no lack of devotees. It is really remarkable how narrow many mathematicians are, not merely in their lack of knowledge, but in their ignorance of their own limitations. They are aware of these limitations only so far as the physical sciences are concerned. None of them would, for instance, venture to speak on a question of Botany without having studied the subject, and likewise a botanist who had never mastered the first book of Euclid would not dare to affirm it to be possible to square the circle; but a mathematician who has never even opened a book on Logic will calmly make a pronunciamento on logical doctrine as absurd as the paradoxes of modern circle squarers or the vagaries of the ignorant theologians who "refute" the theory of evolution. More excusable are those mathematicians who openly acknowledge their incompetence in the logical field; there is so much charlatanism, in Logic as well as in Metaphysics, that a person who has only seen certain works (not the least renowned) on philosophical matters, may be pardoned for giving up the whole subject in disgust. The *Logic* of Hegel, for example, has no more to do with the science of Logic or with any philosophical discipline than the speculations of the circle squarer have to do with true Mathematics. And even many works on not quite so low a level so intermingle truth and error that it is very doubtful whether their study is not more harmful than beneficial, so far as attaining an insight into correct philosophical doctrine is concerned. There are in fact cases of mathematicians of high order of intellect who, while not neglecting the study of philosophical works, have gone astray in selecting their masters. In some countries this was not to be wondered at, since conditions were such that hardly any sound works were likely to come into their hands. It was especially a pity that George Cantor—a really remarkable genius—should have been enveloped in that dense fog of Kantian philosophy which so perniciously pervades the intellectual atmosphere of Germany.[1]

The first portions of this treatise deal with Algebraic Mathematics, and it is of these portions alone that we shall give an account in our preliminary remarks here. It is to be noted that we have not attempted to take up all the conceptions relevant to Algebraic Mathematics. It would have been quite impracticable to do this and also give the more important the attention they merit. In particular we have almost always refrained from discussing such matters as are given a satisfactory exposition by the ordinary text-books. One result of so doing is to make it appear that there is greater diversity of our views from those ordinarily received than is actually the case. Authors who pass over in silence the thousand and one

[1] The work of the late Professor Mach of Vienna: *Analyse der Empfindungen* is, we hope, the promise of a new philosophical movement in the German speaking countries—a philosophical movement in which words will be counters instead of passing for money.

points on which they agree with the established doctrine, and dwell on the one or two points at which they disagree, are liable to produce on the unthinking reader an impression as to their general attitude which is wholly erroneous. This is regrettable, but we see no way of obviating it, save in so far as these remarks may tend to that end.

The keynote of our work is the distinction we find it necessary to make between quantities, values and variables on the one hand, and between symbols and the quantities or variables they denote or values they represent on the other. These distinctions though most important and obvious enough (one would think) have not heretofore been clearly brought to light; still less made the basis of a systematic exposition of the conceptions of Algebraic Mathematics. Mathematicians confuse values and quantities, and again quantities and variables, though not usually values and variables. And they also confuse symbols (and in general expressions) with the things these denote or represent. In saying that this last distinction has not been clearly brought to light, we do not mean that mathematicians when questioned would usually deny that such a distinction should be made. But a formal admission of its importance is very different from the actual enforcement of this distinction. And that in mathematical discussions (aside from Geometrical Mathematics) such enforcement is almost everywhere prominent by its absence, can, we think, be conclusively shown. The tendency to confusion instead of distinction would indeed seem to be growing, and certain mathematicians would avowedly make Mathematics entirely a matter of symbolism.

Keeping in view the distinction just mentioned, we proceed to examine into the nature of the so-called imaginary quantities, not only of Algebra but of the science of Quaternions. We are thus led to the consideration of the essential characteristics of quaternions and vectors, and are not able to accept as entirely satisfactory the expositions of the matter in the works now current. Our view of Quaternions is, we admit, not precisely that of Hamilton. Him we regard as one of the most scientific of mathematicians; but surely, great though he was, he can no more be accepted as an infallible authority in the philosophy of the science he founded, than Newton or Leibnitz can be followed as an unerring guide in the theory of Differential and Integral Calculus. In connection with Quaternions, we consider the essential characteristics of Arithmetical or Semi-Single Algebra, Single Algebra and Double Algebra, laying down what we esteem as distinguishing an n-tuple algebra. The ordinary algebra of the present day, the Double Algebra to which pertains the Argand scheme of representation of "imaginary" and complex quantities, ought, we hold, to be largely developed as a complanar vector analysis; and it is on such a basis that we deal, not merely with the imaginary quantities but also with the real negatives, the abstract as well as the applicate. The method of introducing the negative real abstract quantities by the sanction of the Principle of Permanence we are constrained to regard as especially unsatisfactory, though this method is used, in formulating Algebra as a system, by the most eminent mathematicians. Our own way of dealing with these and with the imaginary abstract quantities is the natural result of not confining our attention to the formal side of algebraic science, but taking into account its matter as well as its form. Those who would develop algebra as a purely formal science, and as nothing more, are satisfied to stop when they have ascribed the origin of the conception of a negative real abstract quantity to such equations as $x+1=0$, $x+2=0$, etc., and the conception of an imaginary abstract quantity to such equations as $x^2+1=0$, $x^2+2=0$, etc. But the conceptions attained when one does not look beyond the equations are nothing more than purely formal conceptions of quantities, and he who would master the matter as well as the form of the science must look deeper. He must attain what might be termed *entitative conceptions*. He must find classes of entities adequate to fulfill the conditions fixed by the formal conceptions. Even outside of Mathematics the distinction holds, and how important are the entitative conceptions and how trivial, relatively speaking, are the formal is readily seen. Thus the conception: *cause of yellow fever*, so long as it does not go beyond what is stated by these words, is a purely formal conception, and one of no great value; the only service it renders is to keep prominent the need of an etiological investigation. But when we conceive, as the cause of yellow fever, a microörganism carried to human beings by the mosquito, we arrive at an entitative conception, and one of some importance. Just so to conceive of a negative real abstract quantity or an imaginary quantity merely as something which will satisfy an equation of a certain type is only a preliminary step; what is really final and paramount is the finding of a class of entities that may be conceived to satisfy such equations. It is only thus that one can arrive at the entitative conceptions necessary for the development of algebra as an entitative (or "material" or "significant") discipline. And it is the search for the requisite entities that leads us to the field of vector analysis.

The incommensurable quantities we take up as soon as a sufficient foundation has been laid for their treatment, and here we find ourselves compelled to advocate the introduction of a new postulate. We also consider in due course the transfinites of Cantor (which we prefer not to designate as quantities), and here again we draw a distinction, ignored by Cantor and his successors, between the value of a transfinite and the transfinite itself.

In our discussion of variables we begin by the consideration of order, a most important attribute which has never been adequately treated of in connection with variables. Such a treatment we attempt to outline as a preliminary to our consideration of limits. Another preliminary is the discussion of domains and ranges, of regions, intervals, neighborhoods, etc. Here the utmost confusion prevails, and we have endeavored to introduce a much needed systemization. Of limits we treat at some length. The relation of a variable to a limit is undoubtedly the most important feature of the Theory of Variables, and we need make no apology for going into the question in detail. On so doing we find that certain matters connected with it, and by no means minor ones, have heretofore been entirely overlooked by mathematicians.

After discussing limits and various objects of mathematical inquiry that are called limits by mathematicians, but are certainly not true limits, we proceed to the consideration of continuity, and find that the investigations of Cantor, though perhaps the last word to be said as regards continuity of point aggregates, do not give adequate ground for the doctrine of continuity most serviceable in the theory of variables.

As essential to a discussion of equations we are compelled to give some attention to symbols, signs and other characters of Algebraic Mathematics. And naturally enough, we bring to light some facts that do not come to the knowledge of those who fail to distinguish between a character and what it denotes or represents or indicates. It seems somewhat amusing that those who do most toward making Mathematics a matter of mere symbolism really know the least about the symbols and other characters and expressions of the science. Of equations and other mathematical dictions we need say no more here than that we feel the results attained to have amply repaid our somewhat lengthy inquiries into the matter. We also take up transformations of equations. In making a provisional list of these we find that the treatment of Vieta was more truly scientific than that of the pygmies who followed him in this field. He made a classification of transformations that was very creditable, considering the state of Mathematics in his time, and provided a nomenclature. His successors sneer at his nomenclature, while they make no attempt at even a classification of their own. And certain most important distinctions drawn by Vieta are entirely ignored by mathematicians at the present day. After transformations of equations we find it advantageous to devote a few pages to consideration of the transformations of the Theory of Groups and the Theory of Quantics.

The next subdivision of our work is headed not *Functions* but *Functional Relations,* and this difference in heading marks a certain difference in point of view, though really, unless one ignores all progress made in Mathematics since the days of the Bernouillis and Euler, we do not see how he can refuse to speak of functional relations. In our definition we find ourselves unable to accept as adequate the view ordinarily attributed to Dirichlet (and which strange to say he does not appear to have held at all) but lay down in a way somewhat different from any that has ever before been followed, the conditions under which two or more variables are said to be in functional relation with each other. We take up the formulas pertaining to functional relations, and show that there are certain conditions not mentioned by mathematicians which such a formula must fulfill in order that it may be of any service in a mathematical investigation. We then give some attention to various types of functional relations and to certain formulas incorrectly called functions. In particular we discuss quantics and serials—so-called series; the true series and other sequences having already been discussed at an early stage in our work.

Of ordinary differentiation we give a short account, and then turn to the differentiation process of Quaternions of which we give an exposition from a point of view somewhat different from that of Hamilton. Attention is also paid to the Infinitesimal Method. Of ordinary integration we likewise give an account, followed by an exposition of Lebesgue integration in our own phraseology which is widely different from that of Lebesgue. Finally we consider continuity in reference to functional relations, and discuss the analytic and other monogenic functional relations.

In the course of our investigations we have been obliged to introduce a few new names in Mathematics and to revive (often in a modified sense) very many obsolete ones. It is only natural that, when a distinction is to be marked out, one should draw attention to the word he thinks best fitted to mark it. Improvement in classification and development of a technical nomenclature do in fact almost always go hand in hand.[1] The important thing, however, is not the nomenclature but the dis-

[1] Sylvester tells us (*Collected Mathematical Papers*, Vol. 2, p. 567, note) "To attain clearness of conception the first condition is *language*, the second *language*, the third *language*."

tinction it marks. Whether an object of mathematical inquiry shall be designated by a specific name or spoken of as belonging to the first, second, etc., species of its genus is immaterial, except in point of convenience. And if any onc, while rejecting our phraseology, shall accept the distinctions it embodies, we shall have gained our case.

We have often been compelled to dissent from and criticize men for whom we have the greatest respect and admiration: Hamilton, Cantor and Weierstrass may serve as examples. We hope our attitude toward such thinkers will not be misunderstood. And in general, as regards our treatment of those whom we criticize, we may in all sincerity echo the words of that great mathematician and logician, Wallis, who says in his preface: "And I have endeavored all along to represent the sentiments of others with candor, and to the best advantage. Not studiously seeking opportunities of caviling, or greedily catching at them if offered. . . . And have been careful to put the best construction upon their words and meanings."

CONTENTS.

Algebra, be complanar. The i, j, *and* k imaginaries of Quaternions are not, as mathematicians commonly imply, distinct from the imaginaries of ordinary Double Algebra. Every i or j or k imaginary may appear in Double Algebra where it will be an ordinary imaginary. Every quadrantal relation between vectors is an ordinary imaginary in Double Algebra (and *all* the imaginary abstract non-zeroes of ordinary Algebra are quadrantal relations between vectors), while in Quaternions it is an i-imaginary or a j-imaginary or a k-imaginary or a complex quantity. In Quaternions $+\sqrt{-1}$ and $-\sqrt{-1}$ cannot serve as value symbols as they do in Double Algebra. Entitative ("significant") and formal algebras. Classification of quantities into sorts. Definition of a sort. Classification of quantities into kinds and varieties. Definition of a kind and of a variety. Five things to be done in kind formation. Confluence. The positive non-zeroes of a kind. The zeroes of a kind. Whether a quantity is a non-zero or a zero is sometimes purely a matter of convention. Contrafluence. The negative non-zeroes of a kind and their relation to the positive non-zeroes. Kinds containing no positive quantities. Other possibilities in kind formation. No satisfactory exposition of the true theory of negative quantities and their relation to the positive quantities is given in current mathematical works, the *Encyclopaedie* included. The doctrine of "number" put forth by Schubert and others, and its absurdity. Kronecker's still more remarkable doctrine of "number." Schubert on the Principle of Permanence and his blunders on that subject. His doctrine of "negative numbers," and its inadequacy. Classification of the kinds of a sort as primitive and complex. Classification of primitive kinds as protomonic and neomonic. An abstract non-zero is protomonic ("real") if when applied as a multiplier it never changes the kind, but is neomonic ("imaginary") if when applied as a multiplier to a non-zero multiplicand it always changes the kind save when it gives as product a zero. With applicate quantities the distinction between protomonic and neomonic is a factitious one based upon mere symbolism. Classification of algebras in accordance with the number of their primitive kinds. Arithmetical or Semi-single Algebra. Single Algebra; this includes the ordinary Differential and Integral Calculus. Double Algebra; this includes the Theory of Monogenic Functions. Triple algebras of De Morgan and of J. and C. Graves. Quaternions. Hamilton's Icosian Algebra. Pluquaternions. Octonions. Multenions. Triquaternions. Vector sorts and Scalar sorts. Various senses in which "scalar" is used in

Mathematics. Hamilton's suggestion of admitting a fourth primitive kind to each vector sort. Quantuplicity of a vector. The "*h*" which Hamilton terms "the old and ordinary imaginary symbol of common algebra." Biquaternions and bivectors. Tessarines. Objects, entities, non-entities and chimeras. Hamilton's biquaternions and bivectors and Cockle's tessarines are not chimeras. None of the symbols of Mathematics are *mere* symbols. Clifford's biquaternions and McAulay's octonions. Combebiac's triquaternions. Examination of the conventional definitions of "variable." Baire, Genocchi. Bauer; his error of thinking that a constant must be known. Biermann. Burkhardt. Czuber. Pringsheim in the *Encyklopaedie der mathematischen Wissenschaften;* his definition can not be so interpreted as to be even approximately correct. Weber. Tannery. Pierpont. Harnack; his definition belongs to a transitional stage between the Newtonian definition and the symbol-definition. The symbolic logicians. Bertrand Russell as a representative of the Peano school. His definition of "number." Neither he nor Peano recognized the distinction between equality and identity. And both have failed to see the enormous difference that there is between the name of a variable and an ordinary class name. A variable is not a set of quantities. Philosophical considerations as to the possibilities in definition per genus et differentiam. The name "variable" cannot be advantageously defined in this way. The nearest one can come to such a definition is to say that a variable is a quantity aggregate, and this definition is by no means satisfactory. Progressions the simplest of variables. Terms. Series Summative series. Alternating summative series. Serial functional relations and the improper use of the name "series" in connection with them. Failure of mathematicians to recognize that series are variables. Erroneous definitions. Multiple series. Gradual numbers. Sequences. A sequence is a discrete unifarious variable having an immutable arrangement of its terms. A progression is a sequence with which the formula for the *n*th term lays down as essential operands *m* terms previous to the *n*th. A series is a sequence with which the formula for the *n*th term lays down *n* as essential operand.

ON THE GENERAL CONCEPTION OF FUNCTIONAL RELATION.

Quesitive and dative symbols. Essential characteristic of a functional relation. Consentaneous and dissentaneous functional

ERRATA

Page xvii,	line	24,	*for*	Monegenic *read* Monogenic.
xviii	"	3,	"	Inconvenience of distinguishing *read* Inconvenience of designating.
xviii	"	24,	"	ratios or vectors *read* ratios of vectors.
50	"	11,	"	changed *read* changed;
62	"	15,	"	quantitives *read* quantities.
100	"	8,	"	are then names *read* are the names.
107	"	7,	"	reductions *read* redactions.
110	"	9,	"	objects can be found *read* entities can be found.
154	"	4,	"	Disciples *read* Disciple.
167	"	17,	"	for the terms *read* from the terms.
177	"	8,	"	momentious *read* momentous.
199	"	6,	"	Bauer, 192 *read* Bauer, 146, 192.
210	"	5,	"	Peciprocal *read* Reciprocal.

FUNDAMENTAL CONCEPTIONS OF MODERN MATHEMATICS.

VARIABLES AND QUANTITIES.

No objects of mathematical inquiry are more remarkable than variables and limits. Our knowledge concerning them has been chiefly acquired in recent times, although they were not wholly unknown to mathematicians of antiquity. They were first drawn within the realm of inquiry by Archimedes and other Greek mathematicians in their endeavors to effect the rectification and quadrature of the circle. In the writings of these old pioneers of Mathematics neither "variable" nor "limit" nor any equivalent term can be found. Nevertheless, it must be acknowledged that in their aims, as well as in the methods they employed, these inquirers already had within their grasp the germ of the Integral Calculus of to-day. As to the name "variable," it appears to have found its way into mathematical language during the development of Analytical Geometry. Naturally it was adopted by the earliest writers upon the Calculus. Variable quantities and limits were spoken of by Newton in the *Principia,* in 1687, but without furnishing any precise definitions with respect to them. In his *Method of Fluxions,* he put forward what purports to be a fundamental classification of quantities which with some slight changes in terminology is still prevalent. He says: "Now those quantities which I consider as gradually and indefinitely increasing, I shall hereafter call fluents or flowing quantities and shall represent them by the final letters of the alphabet, that I may distinguish them from other quantities which in equations are to be considered as known and determinate, and which are therefore represented by the initial letters."[1] Of like import are the words of L'Hopital, who gave the first systematic exposition of the method of Leibnitz. "Those quantities are called variable which continually increase or diminish, and on the contrary those are called constant which remain the same while the others change."[2]

The works of subsequent mathematicians show that this view, which ascribes to a variable the character of a quantity, has apparently presented no difficulties, since it has gained an acceptance which is well-nigh universal. True it is, that in some recent works there has appeared what might seem to be a new view of the character of a variable. In these works, a variable is regarded merely as a symbol of a certain variety, and is even explicitly defined as a symbol. On its face, such a definition shows a puerile confusion of names with the objects they denote; there being exactly the same justification for calling a man, a word; or the King of England, a phrase; as for calling a variable, a symbol. We will bestow further consideration upon the symbol-definitions of the term "variable" hereafter. For our present purpose we will merely point out that the authors who define "variable" in this manner, do not seem to regard the adoption of the new definition as

[1] *Method of Fluxions*, translated from the Author's Latin Original not yet made publick, by John Colson, London 1738, p. 20, § 60. This was written about 1672.

[2] *Analyse des Infiniment Petits*, p. 1, § 1. The first edition of this work was published in 1696.

marking the eradication of an old and profound error of Mathematics; they give no intimation whatever that their view is more than a refinement upon the older one. In spite of this unanimity of opinion among those of high authority, we venture to declare that the accepted doctrines concerning the nature of variables are completely erroneous. And it is our purpose to furnish ample arguments to show that our charges of error are not mere frivolous objections. We shall produce abundant evidence in support of this proposition: that to conceive a variable as a quantity, or as a symbol, is to form a conception totally incongruous with the conception which must be formed in establishing a sound theory of variables.

Of all quantities the most primitive are the *natural numbers*. That a natural number is an attribute of a group of objects has long been more or less completely recognized. Indeed, for ascribing to a group of objects an attribute of number, we have a foundation perfectly similar to that for ascribing to a body an attribute of shape or an attribute of color—a quality. Every two bodies resemble or differ from each other in manifold ways. To treat of these various modes of resemblance and difference, we ascribe to them a distinct attribute for each mode of resemblance or difference. We have no other foundation for speaking of shape, of color, and of all the other attributes of bodies. It is upon this basis that the theory of these attributes rests. It is likewise with groups of objects. They, too, resemble and differ from each other in manifold ways. And one mode of resemblance and difference is in that respect which enables us to say that a group of objects possesses an attribute of number. It is not so obvious that a *natural fraction* is also an attribute. Nevertheless, this interpretation of the nature of a natural fraction is clearly tenable. Thus, if we are counting spherical bodies, a hemisphere may be regarded as having a number attribute of one-half, and any other segment of a sphere which when joined to n-1 other segments *exactly similar* to itself will produce an entire sphere, may be regarded as having a number attribute of one-nth.[1] Whenever we are treating of objects which can be decomposed into parts alike among themselves in certain respects, but different from the whole objects in these respects, we can take objects of two types into account, the whole objects and the part objects, and we can regard the objects of the second type as fractional parts of the objects of the first type. In counting objects of the first type, a group of objects of the second type can be regarded as having a fractional number attribute, where a more primitive system of computation would exclude it from recognition entirely. Whether there are attributes which constitute the remaining abstract quantities, we will not pause to consider here. But we must point out that certain attributes, which are not abstract quantities, constitute the *applicate quantities*, which may be very simple as the attribute length, or very complex as the quantities dealt with in the theory of electricity. For such an attribute to be an applicate quantity, it is necessary that it should be amenable to measurement. Whether or not this necessary condition is also sufficient; whether it may also be essential that attributes of this class be amenable to addition, in that we must be able to add two of them together, is a question that will be discussed later. We have further *concrete quantities*, which are not attributes at all. While an abstract quantity—a ten, in the case of a pile of ten

[1] Lodge advocates the introduction of the idea of fraction in arithmetical instruction by means of segments of apples and oranges. *Easy Mathematics*, London, 1906, p. 15.

oranges—is an attribute of a group; the corresponding concrete quantity—the ten oranges—is the group itself. In contradistinction to abstract quantities, the applicate and the concrete are called *denominate quantities.*[1]

It is customary when the number of objects in two groups are equal, to say that there are the same number of objects in the groups. This use of the word "same" to signify, not *identity,* but *complete similarity,* has crept into Mathematics, as well as into other branches of inquiry where precision and accuracy of thought are indispensable, and has been a fruitful source of confusion and error. It is a colloquial phraseology, rather than one suited to the requirements of an exact science. The question as to the propriety of this application of the word is far from being a mere verbal subtlety, for in any science it is of the utmost importance that every fundamental difference of fact be marked by a distinction of name. When we perceive an attribute of an object (or a group

[1] In view of the fact that in the theory of names the distinction between abstract and concrete names is that between the names of attributes and the names of substantive objects, it would seem more appropriate for mathematicians to designate as abstract all quantities not concrete. A still better course would be to entirely abandon the use of "abstract quantity" and "concrete quantity." Mathematical nomenclature on this subject (a nomenclature we merely attempt to systematize and not completely remodel) is full of confusion. Sometimes the name "concrete quantity" is applied to applicate quantities. Thus Napier (*De Arte Logistica,* Liber 3) would call the length of a single body a "concrete number," while if two or more bodies were laid down end to end, and the length taken, this he would term a "discrete number." Again "denominate number" is sometimes used in the less general sense of "applicate quantity." And the latter name itself, a term that we have resurrected from the terminology of the older mathematicians, was probably as often used in the broad sense given above to "denominate quantity" as in the narrower one.

of objects) at one time, and again perceive this attribute unchanged in the same object (or group) at another time, we speak of the attribute perceived upon the first occasion as being the *same* as the attribute perceived upon the second occasion. An entirely distinct case occurs when we have in view two *perfectly similar* attributes of two different objects, or two different groups of objects. Why should we say in such a case, when two objects are *exactly alike* in color, or two groups *exactly alike* in number, that they possess the same color attribute, or the same number attribute? The word "same" has already been chosen to express one set of facts concerning attributes. Why should we also employ it to express a totally different set of facts, when another word, "like" or "equal" is already at hand? To adopt the word "same" for both purposes is as absurd and misleading as to call two houses exactly alike, the same house.[1]

These considerations are sufficient to make it clear that the names "one," "two," "three," etc., are general names denoting the members of classes containing not one, but many objects. When we put forward the proposition: "Two plus three equals five," our proposition does not bear upon three classes, each of which contains only one object. We are enunciating, not a singular, but a general proposition concerning three classes containing many objects. The proposition concerns every group of two objects, every group of three objects, and every group of five objects—the affirmation is made that whenever a

[1] An interesting discussion regarding sameness of attributes will be found in Spencer's *Prin. of Psychology*, Ch. 5, § 32 and Mill's *System of Logic*, Book 2, Ch. 2, § 3, note. We may also mention in this connection the valuable monograph of G. S. Fullerton: *On Sameness and Identity* (Phila., 1890), a work that does not seem to have received the attention it deserves.

group of three objects is joined[1] to a group of two objects, no matter what groups are taken, the new group formed will be composed of five objects, and will be equal in number to every other group composed of five objects. Since the names "one," "two," "three," etc., are general, we may legitimately speak of a one, a two, a three, etc., although in present usage the article is omitted. Thus in speaking of the result of an operation—a sum for example—we may say "the sum is a four" instead of using the customary expression "the sum is four." The latter phraseology, while its entire rejection may not be necessary, has doubtless played its part in the birth of the error we have been discussing.

From the point of view just set forth it is plain that among the most elementary and fundamental classifications of Mathematics is the formation of classes each of which consists of all quantities that are mutually equal. These well-marked classes may be called *value-classes*. If two quantities are equal they belong to the same value-class, or, as it is more commonly put, are of the same value. With abstract quantities, "one," "two," "three," etc., are the general names belonging to these value-classes for the natural numbers; and "one-quarter," "one-half," "three-quarters," etc., are the general names belonging to these classes for the natural fractions. And likewise there are general names for the value-classes of abstract quantities other than the natural numbers and the natural fractions. With denominate quantities, to take only two sorts, "one centimeter," "two centimeters," "three centimeters," etc., are the general names belonging to these classes for lengths; and "one erg," "two ergs,"

[1] It should be observed that we do not here assert addition to include only cases in which there is a *physical* union of groups.

"three ergs," etc., are the general names belonging to these classes for quantities of work. In short, with every value-class of quantities, whether abstract, applicate or concrete, there is or should be a general name. Corresponding to the names just spoken of are the symbols, 1, 2, 3, etc., $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, etc., 1 cm., 2 cm., 3 cm., etc., 1 erg, 2 ergs, 3 ergs, etc., which are nothing more or less than substitutes for these names, adopted for convenience. They are general symbols belonging to the value-classes. They may be called symbols of value.

Our brief survey of various quantities of Mathematics involving an incidental consideration of attributes other than quantities has not drawn within its scope any case in which change takes place. Now suppose change does take place; suppose a bar of metal is heated, and its length changes from 1000 centimeters to 1020 centimeters, while its color changes from black to red. Are the facts well expressed by saying the bar possesses the "same" length and the "same" color as before? According to the manner of speaking of such changes in Mathematics, and in keeping with the long-accepted definition of "variable," it would be said that "the" length of the bar during the change is a variable, and hence is a quantity, that is, *one* quantity. Moreover, in harmony therewith, it must also be acknowledged that the manifold colors throughout the change are a color, that is, one color. Such a use of the word "same," implying identity where there is diversity, is even more repugnant than its application in the sense of "equal," and we are compelled to recognize, when an object initially possesses a certain attribute and undergoes a change with respect to this attribute, that we must regard as a separate and distinct attribute each stage of the process of change. Such a set of attributes may together constitute a variable and under this description come most of the variables of physical science. We may have a body with variable velocity—a type of variable which gave frequent opportunity for the exercise of Newton's genius—a body of gas with variable temperature, with variable pressure, and with variable volume. We cannot say that any of these variables are quantities unless we wish our language to be an impediment instead of an aid to exact thinking.

Even stronger do we find our case when we turn to Geometry and behold the type of variable which first came under the scrutiny of mathematicians. Antiphon, in his attempts to rectify the circle, inscribed in a circle, first a square, then a regular octagon, then a regular sixteen-sided polygon, etc. Archimedes inscribed first an equilateral triangle, then a regular hexagon, then a regular dodecagon, etc. In each case the perimeters of the inscribed polygons constitute a variable whose limit is the circumference of the circle, while the areas of the polygons constitute a variable whose limit is the area of the circle.[1] With what show of reason can we say that the perimeter of a triangle is one and the same quantity—one and the same perimeter—as the longer perimeter of a hexagon? How are we justified in saying that the area of a triangle and the area of a much larger ninety-six sided figure are the same quantity? And yet we must make these assertions if we acknowledge that a variable is a quantity. Nor is it only with the circle that such difficulties arise. The quadrature (or rectification) of

[1] Antiphon seems to have believed that a polygon could be obtained which would coincide with the circle—in modern phraseology, that the variable would reach its limit. The notion of variable is also apparent in the second proposition of the Twelfth Book of Euclid, which purports to prove that "circles are to one another as the squares on their diameters."

any curvilinear figure by the methods of the modern Integral Calculus present similar difficulties. We have a set of rectilinear figures, no two alike, the areas of which (or the lengths of certain lines on which) constitute a variable the limit of which is sought. Surely these considerations are overwhelming against the doctrine which ascribes to a variable the character of a quantity, and there is no recourse save to abandon that doctrine completely.

When a long-accepted and apparently well-established tenet of science has been shown to be untenable, it may justly be demanded, with the circumstances as they are here, that a more acceptable doctrine be put forward to replace that which has been found wanting. It is our purpose to meet this demand, but this involves as an essential requisite an inspection of some of the most striking characteristics of variables, of which not the least is their constitution. And the consideration of their constitution will lead us to a closer scrutiny of the various species of quantities of Mathematics than we have as yet undertaken.

The quantities contemplated together, when a variable is the object of inquiry, compose a class of quantities which we may call a variable-class. But a wide gulf separates the inquiries instituted with respect to variables and all other inquiries instituted with respect to the members of classes composed of quantities. This wide gulf cannot be more clearly pointed out than by a comparison of the propositions which are set forth concerning such classes as the value-classes already treated of and the propositions set forth concerning variables. This comparison will make it plain that propositions bearing upon a variable relate, in a peculiar and distinctive manner, to the members of the variable-class. When we form a value-class and give it a class name—for example, by grouping together all the twos and providing the class name "two"—our purpose is to state propositions concerning every two taken separately (or concerning each of some of the twos). When we form a variable-class we have a totally different purpose in view. The propositions enunciated concernng a variable do not, in the typical cases, treat of the members of the variable-class taken separately; they treat of the mutual relations between the members of the class. And among these relations, the most common is that state of affairs which exists when the variable possesses a limit—a limit being a quantity which may or may not belong to the variable. When we say "two plus three equals five" we assert that every two plus any three equals every other five. When, however, we say "the variable x approaches the limit l" we do not assert that any quantity of x approaches l as a limit—indeed such a proposition would be utterly meaningless.

In any inquiry concerning a variable, one of the most important, though one of the least regarded facts, is the arrangement in order of the quantities which compose the variable. It has just been pointed out that to inquire into the mutual relations of the quantities which are members of the variable-classes, and to set forth the laws pertaining to these relations is the purpose in forming those classes and in establishing a theory of variables. Now every proposition which can be framed bearing upon the mutual relations of the quantities of a variable, and therefore bearing upon those matters for which the variable has been expressly formed—every such proposition must have at least a tacit reference to the arrangement in order of the quantities which compose the variable. Hence no meaning whatever is conveyed by such a proposition unless an arrangement subsists among the quantities of the variable. Thus, if we say a variable throughout incessantly approaches a quantity *a*, either as a limit or otherwise, we assert among other facts[1] that every quantity of the variable is nearer *a* than every previous quantity. And we would be enunciating a proposition without meaning, if "previous" had no meaning with regard to the quantities of the variable, that is, if they were not arranged in order. Even in the simple affirmation that two variables are in functional relation, though it is not asserted that either of the variables is arranged for the moment, we hold there is necessarily in view arrangements of both at some time or other, past, present or future. At different times the variables may be arranged quite differently, but at any one time the two variables must have like arrangements. In the absence of such arrangement the purpose of a functional relation cannot be served at all. In the elementary Calculus the variables met with are never isolated, but are in functional relation with one or more other variables, and a procedure is more or less closely followed which is tantamount to arranging the quantities of the independent variable according to value; that is, each quantity of such a variable is regarded as subsequent to every lesser quantity. Even here, it is obvious that there is nothing fixed or immutable about the arrangement of the quantities, and that no necessity compels us to employ arrangement in order of value. A variable which in one functional relation is independent, may be dependent in a dozen or more other functional relations with as many distinct independent variables, giving rise to as many different

[1] Of course, when the proposition is false, we are asserting not a fact, but a fancy.

arrangements of its quantities. While it is hardly defensible not to speak of the order of the quantities of the variables in elementary Calculus, it is utterly inexcusable to follow this course in those disciplines which treat, not merely of quantities which are comparable, but also of quantities which are not comparable as a $+1$ and a $+\sqrt{-1}$. For in these disciplines arrangement in order of value fails completely.

Nevertheless, the present trend in Mathematics is not toward the recognition of order as an attribute whose consideration is requisite to an adequate theory of variables. That consideration of this attribute is indispensable to the theory of variables, has not gained recognition even through the labors of George Cantor, who has so conspicuously brought forward the attribute in his epoch-making researches upon the Transfinite Mengenlehre. The explanation of this anomaly is not far to seek. The Theory of Transfinite Aggregates deals with aggregates in general in a most abstruse way, and gives no specific consideration to aggregates composed of quantities. The earlier work of Cantor bearing upon point aggregates is more specific, and bestows particular attention to limiting points or points of accumulation. In establishing the theory of these aggregates, the attribute of order, of paramount importance with other aggregates, is not employed—or rather, we should say, an order in space, and this alone, is vaguely in view. It would appear from the writings of Cantor himself, and it is expressly stated by some who adopt the views of Cantor, that the theory of point aggregates and their limiting points is taken as the counterpart of the theory of variables and the limits of variables. The consequence of all this is that in recent works definitions with respect to limits of a variable have been laid down permitting us to say a variable possesses a limit in cases where no limit is possessed by the variable properly speaking.[1]

The word "order" most naturally brings to mind *order in time* and *order in space*. If a line be described from a point A to a point B, if a gaseous body expand in an enclosure, indeed, in any process of Nature, we have before us a set of events having an order in time. With the description of the line, the events are the coming into existence of its various points; with the expansion of the body, the events are the assumption of the various volumes from the initial to the final volume. In both of these sets of events, the relation between any two events of a set is that of *succession*. If the body does not expand isothermally, that is, if there is a different temperature of the body for each volume, we have another set of events, the assumption of the various temperatures from the initial to the final temperature, each of which is *simultaneous* with the assumption of a certain volume. Thus all processes in Nature are constituted of events between which subsist the relations of *simultaneity* and *succession*. And all order in time is founded upon these two relations. Order in space is far more intricate than order in time, since any spatial relations whatever, between objects, furnish a foundation for ascribing to them an order in space. Thus with any straight line AB, whether it has been described, or whether it has been generated in any other way whatsoever, the points of the line can have an order in space assigned to them starting at the point A, and another and quite different order in space

[1] For instance, Otto Stolz and J. Anton Gmeiner in their *Einleitung in die Functionentheorie*, p. 5, § 2, put forward a definition with regard to limit of a variable apparently expressly designed to correspond to Cantor's definition concerning limiting point. To these definitions we shall give full consideration later.

(the "inverse order") starting from the point B. In the first case it is laid down that A is *previous* to every point of the line other than A, and of any two of these other points, that point is *previous* which is nearer A, and that point is *subsequent* which is further from A. In the second case the point B plays a part similar to that of A in the first case. But these are only two of the many orders in space which can be assigned to the points of the line AB. Besides A and B, any point P of the line between A and B may be chosen as previous to every other point of the line; and the points other than P can be arranged in order by regarding any two points equidistant from P as *abreast* of each other, and, with any two points not equidistant from P, regarding as *previous* that which is nearer P, and as *subsequent* that which is further from P. The order of the points in space just specified will accord perfectly with their order of coming into existence in time if the line AB be generated by describing two portions starting from P, and going toward A and B, so that points equidistant from P come into existence simultaneously.

If a straight line AB be described on an Argand surface, beginning at the "origin" (or zero point of the surface), and *going thence in a positive direction,* the order in time, of the coming into existence of the points of the line, and the order in space of these points starting at A, both accord perfectly with the *order of the values* represented by the points of the line. But these three types of order ought not to be confused with each other, or with that fourth type of order with which we are concerned in dealing with variables. Order of this fourth type is just as distinct from order in time, or order in space, or order of value, as it is from the order of precedence of the English aristocracy. The order of the quantities in a variable may accord perfectly with the order of their values, and the order in space of the points of the Argand diagram representing these values, and the order in time in which these points came into existence; but on the other hand, it may not accord with these at all—such accord being in no way essential to the idea of order in a variable.

In the construction of a perfectly general theory of variables any set of relations between the quantities of a variable which will serve as a foundation for arranging them in an order must not be denied the mathematician as a means of establishing order among them. Moreover, the arrangement in order of the quantities of a variable must be regarded as entirely arbitrary, as amenable to change during an investigation,[1] and as determined solely by the fiat of the mathematician, who is permitted to bestow any order upon the quantities of a variable that his purposes may dictate.

We shall say that the quantities of a variable are arranged in order, when every quantity of the variable has had conferred upon it a relation of order with respect to at least one other quantity of the variable. The most simple arrangement is that in which with every two quantities of a variable a and b, a is either previous to or subsequent to b, and is not, in any case, *both* previous and subsequent to this quantity, while whenever, with three quantities, a, b and c, a is previous to b and b previous to c, then a is previous to c. Arrangements of this species are the most important, and will receive by far the largest share of our consideration. Among variables possessing this arrangement are the independent variables

[1] Amenable to change save in so far as it may be convenient to lay down in some exceptional cases that certain variables shall be immutable in this respect, any change of the order of their quantities producing what we agree to call a new variable.

in Calculus, which usually consist of one representative of every real abstract value arranged in order of value. In such a variable every quantity less than zero is previous to zero, and to every quantity greater than zero. The -1000 is previous to the -100, the -100 is previous to the -10; and all of these are previous to the zero, the $+10$, the $+100$ and the $+1000$, and so on. These arrangements with discrete variables may be roughly visualized by points in a row. In such a representation regard of the value of the quantities must be entirely subordinated to their order; and the points of the row must have an order in space, in perfect accord with the order, in the variable, of the quantities they represent. With continuous variables these arrangements would be visualized by lines.

For these arrangements in order of the quantities of a variable, we propose the name *unifarious,* and we shall speak of a variable as being unifarious when an arrangement of this character is under consideration.

The order under one unifarious arrangement is said to be the *inverse* of that under another (and the two arrangements are said to be inverse to each other) if with every pair of quantities a and b coming under the arrangements, a is previous to b under one arrangement, but subsequent to b under the other arrangement.

A unifarious arrangement is said to be *discrete* if there is consecution among the quantities arranged; that is, if every quantity having another quantity previous to it has a quantity next to it previously, and every quantity having another subsequent to it has one next to it subsequently. In other words, if b is a quantity of a variable so arranged, then when there are one or more quantities of the variable previous to b, there must be among these a unique quantity a such that there is no quantity in the variable both previous to b and subsequent to a, and when there are one or more quantities of the variable subsequent to b, there must be among these a unique quantity c such that there is no quantity in the variable both subsequent to b and previous to c. A variable then, under a unifarious arrangement of its quantities, may be discrete or non-discrete; the latter word is of course not synonymous with continuous, though it would not be difficult to cite authors who have expressly defined discrete as meaning not continuous.

Somewhat more complicated than the unifarious arrangements, are those which may be roughly visualized as looped or ring-like. In these, with every two quantities of a variable, one is either previous to or subsequent to the other; and besides, in one or more instances with two quantities, one is both previous to and subsequent to the other. The point of crossing of the loop represents the quantity which is both previous to and subsequent to the quantities represented by points on the loop. Thus, if in a variable composed of the quantities α, β, γ, . . . χ, ψ, ω; β is next subsequently to α, γ to β, . . . ψ next subsequently to χ, then γ next subsequently to ψ, and finally ω next subsequently to γ; the arrangement at hand is a looped one; γ, which is both previous to and subsequent to each of the quantities δ, ϵ, . . . ψ, corresponding to the crossing point of the loop. A ring-like arrangement would be obtained, if β were made next subsequently to α, γ to β, . . . ψ to χ, then ω next subsequently to ψ, and finally α, made next subsequently to ω. In such an arrangement, visualized by points lying in a ring, every quantity without exception is both previous to and subsequent to every other quantity.

To say, an arrangement may be roughly visualized by points in a row, in a line, in a loop, or in a ring, is not to say that the representation of the quantities of a variable upon an Argand surface will consist of points so placed. The primary purpose of the representation of quantities upon an Argand surface is to exhibit their relations as to value, by the position of points upon a plane. Therefore, only when the order in space on an Argand surface, of the points representing the quantities of a variable, *happens* to accord with the order of the quantities in the variable; then and then only, will the order of these quantities, as well as their values, be portrayed upon such a surface. It may often occur that these two orders do not accord at all; in which case a ring-like array of points, or a closed figure on the Argand surface may belong to a unifarious variable, while a straight row of points, or a straight line, may belong to a variable with ring-like arrangement of its quantities.

A fourth and still more complicated arrangement is that in which with one or more pairs of quantities of a variable, a and b, a is neither previous to nor subsequent to b. When this is the case with two quantities, they may be abreast of each other, or they may not have conferred upon them a relation as to order, or they may have conferred upon them some relation of order not included among the cases already mentioned. It is not necessary to dwell upon these arrangements, except to observe that some of the possible cases may be visualized by rows of points, which may be independent of each other, or may intersect, or interlace, or coalesce in a great variety of ways. We will call arrangements of this species *multifarious*.

The arrangements which have been discussed would seem to comprise all arrangements that could possible be conceived. But, if we turn to the Theory of Monogenic Functions, and observe how the quantities of an independent variable would seem to be arranged when the process of differentiation is in view, a most paradoxical state of affairs confronts us. Here, for a point on an Argand surface representing a quantity at which differentiation takes place, not only every radial path leading to the point, but every spiral or sinuous line leading thereto —in short every line whatsoever leading up to the point is a path of differentiation. And this is the case not merely with one point but with every point corresponding to a quantity of the independent variable at which differentiation can be performed. And the theory exacts that, for every point of this description and for each path leading up to it, there shall be at hand a unifarious arrangement of such quantities of the independent variable as are represented by points on the path. In these unifarious arrangements, the order of the quantities must accord with the order in space of the points along the lines leading up to the points of differentiation. Consider two quantities represented by points P_1 and P_2 lying in a straight line leading to a point of differentiation P. In the unifarious arrangement corresponding to this line, the quantity is previous which is represented by P_1, this point being further from P than the point P_2. Now consider a spiral line containing P_1 and P_2 and leading up to the point P. If this spiral has P as centre, in the corresponding unifarious arrangement the quantity represented by P_1 is again previous to that represented by P_2, but the intermediate quantities will not be the same— there will be some new intermediate quantities while some of those intermediate in the first case will not be taken into the second arrangement at all. Indeed, if the whorls of the spiral are far enough apart, none of the old intermediate quantities will enter into the second arrangement, and none of the new intermediates will be quantities that entered the first arrangement. Again we may have a sinuous line which passes first through P_2, then through P_1 and finally goes to P. Under the unifarious arrangement corresponding to this, P_2 will be previous to P_1. If we choose, we may regard the state of affairs embracing all these unifarious arrangements as itself an arrangement, into which enters every quantity that entered one of the unifarious arrangements. Such an arrangement we will call *multiplex,* while arrangements of the first four types we will call *simplex.* The distinguishing characteristic of a multiplex arrangement is that it has so to speak *subordinate* to it (as above described) at least two—usually innumerable—different simplex arrangements (not necessarily unifarious), each of the latter having as its scope part or all of the quantities reckoned as taken in by the multiplex arrangement.

In his treatment of the Theory of Aggregates, Cantor originally used "einfach geordnet," to signify what we here express by "unifariously arranged."[1] Later, however, he modified the meaning of this phrase, and said that an aggregate was einfach geordnet if with each pair of its elements A and B, A was either previous to, or subsequent to, *or abreast of* B.[2] Presumably he also intended to require that not more than one of these three relations be ever at hand, and that whenever A is previous to, or subsequent to, or abreast of B, and B likewise respectively previous to, or subsequent to, or abreast of C, then A must be respectively previous to, or subsequent to, or abreast of C. In this sense "einfach geordnet" would apply to all the cases we designate as unifariously arranged and moreover to some (but not all) of those we designate

[1] See *Math. Ann.*, Vol. 46, p. 496.

[2] See *Zeitsch. f. Phil.*, Vol. 92, p. 240 *et seq.*

as multifariously arranged. And "mehrfach geordnet" Cantor applied to those multiplex arrangements (to use our own terminology and not his) whose subordinate arrangements were all einfach in the second and broader sense, there being understood to enter in each of these unifarious arrangements *all* the elements within the scope of the mehrfach arrangement. Cantor's point of view is very different from that taken here. Of relations as to order between one element and another of an aggregate, he admits of only three:[1] that one is previous to the other, is subsequent to it, or is abreast of it. A literal translation of "geordnet" would be "ordered," were it admissible to use the verb "to order" as a passive. Some authors have gone so far as to take this liberty. "Ordinated" would be available as a substitute for "arranged," if the close connection of that verb to the noun "ordinate" of Analytical Geometry did not make the usage objectionable. We shall not venture to employ "ordinate" in this sense, but will content ourselves with "arrange." And when we speak of arranging or of the arrangement of a variable, we are to be understood to mean the arrangement in order of the quantities of this variable.

Our survey of the arrangement of the quantities of variables is completed. We must now enter upon an examination of the constitution of variables from another standpoint, which has for its purpose to discover what qualifications quantities must possess, with respect to number and character, to join in forming a variable.

Regarding number, it is plain that a variable must contain at least two quantities, since we can have no relation between quantities as to order or otherwise,

[1] *Op. cit.*, p. 241.

unless two are at hand. Although the inquiries which can be undertaken with a variable containing only two quantities are very trivial, we can see no good purpose in requiring more than two as the least number of quantities capable of forming a variable. As no other restriction upon the number of quantities is needful or desirable, we shall merely require that every variable be composed of at least two quantities. It is worthy of remark that in the Theory of Aggregates, which is closely related to that of variables, Cantor regards it as permissible to recognize a single element as constituting an aggregate.[1]

To discover what character quantities must possess to be amenable to those inquiries for which a variable is formed, and therefore to be eligible to join in forming a variable, we will inspect the most typical of all of the relations between the quantities of a variable—the relations which subsist when the variable incessantly approaches a quantity, either as a limit or otherwise. The qualifications hereby educed are conformed to by the quantities of every variable, as an exhaustive examination of the variables found in the mathematical sciences will show.

For a variable x to incessantly approach a quantity a, it is requisite: first, that there should be a unifarious arrangement of the quantities of the variable; second, that each quantity in the variable should be *nearer* a than is every previous quantity; third, that no quantity of the variable should be equal to or identical with a. The second condition is that relevant to our inquiry. In stating it we use the word *nearer* in a technical alge-

[1] *Beiträge zur Begründung der transfiniten Mengenlehre, Math. Annalen,* Vol. 46, p. 482.

braic sense. Of two quantities x_1 and x_2, the latter is said to be *nearer* the quantity a than is the former when the difference between x_2 and a is less numerically than the difference between x_1 and a. In such a case the distance on the Argand surface between the points representing x_2 and a will be shorter than that between the points representing x_1 and a. Hence, in order that the variable x shall incessantly approach the quantity a, it must be possible to find the difference between every quantity of x and a—in other words, with each quantity of x, it must be possible to either subtract this quantity from a or to subtract the latter from the former. And for these operations of subtraction to be possible, there must be a certain uniformity of character of a and the quantities of x. If a is a length, every quantity of x must be a length. If a is an area, every quantity of x must be an area. We cannot subtract a length from an area or an area from a length, and hence cannot have both lengths and areas among the quantities of the same variable. Similarly, if a is an abstract quantity, every quantity of x must be abstract. Of the genus quantity there are a number of species, and no variable can contain quantities of more than one of these species. Such species are of great importance in Mathematics, and it is desirable to denote them by a name of their own.

Unfortunately, here as elsewhere in Mathematics, we are confronted with a lamentable paucity of nomenclature pertaining to fundamental ideas. A vast and imposing superstructure has been raised with great labor, while the foundations have been slighted to a degree that verges close upon disrespect. Mathematicians have adopted no word to designate these sharply defined species. For this purpose we propose the word *sort*. Two quantities are of the same sort if it is possible to add either one of them to the other. When this condition is fulfilled as to the quantities x_1 and x_2, and as to either of these with respect to a, it can be proven that it is possible to find the difference between a and both x_1 and x_2.

It must be recognized that two quantities are sometimes of such a character, that it is desirable to include them in the same sort, although it is impossible to add either of them to the other. It is desirable to include in the same sort all lengths. But we cannot perform the operation of adding the two lengths of the same metallic bar at different temperatures which would consist in bringing them into juxtaposition. Likewise when we have two lengths a and b of two different bars, and we bring them into juxtaposition, thereby producing a new length—the quantity $a+b$ —this new length is of the same sort as a and b, but we cannot perform the operation of adding together either a and the new length, or b and the new length. For two such quantities to be of the same sort, we must not require that it be possible to add either one of them to the other, but merely that it be possible to add each of them to a third quantity.

Still another difficulty arises in the delineation of a sort of quantities, such as that under consideration consisting of all lengths, owing to the inability of human beings to carry out certain very easily-conceived processes. It is impossible to add physically a length on the earth to a length on the moon, and it is impossible to add them both physically to a third quantity. We must content ourselves with facing this difficulty without overcoming it, as we cannot here investigate how far the operations of Mathematics are physical; how far merely conceptual, and what is the character of an operation which is merely conceptual.

With quantities of some sorts which are not excluded as components of a variable, it is impossible to add physically any two of them whatsoever. We cannot add physically the temperatures of two bodies. It is impossible by bringing two bodies into juxtaposition, or by performing on them any physical operation having the semblance of addition, to produce a third body whose temperature is the sum of the temperatures of the bodies brought together. When the temperature of a body is said to be the sum of the temperatures of two other bodies, and a thermometer such as the ordinary mercury thermometer is used as the criterion of temperatures—in which the expansion of a liquid is indicated by the height of a column of that liquid—what is meant when positive temperatures alone are considered, is that the temperature of the first body is such that the length of the column of liquid above the zero, upon bringing the thermometer in contact with that body, is equal to the sum of the lengths of the columns of liquid above zero, upon bringing the thermometer successively in contact with the two other bodies.[1]

How far it is advisable to provide distinctive names for quantities of species of which some members at least are amenable to physical addition and for quantities of those species of which no members are amenable to physical addition, we need not dwell upon here. It is sufficient to point out that these two sharply distinguished types of quantities exist.

We shall also have occasion to use the word *kind* in reference to quantities. This is no innovation. We find it used by Wallis in his *Treatise of Algebra* published in

[1] A somewhat different account of the matter—an account too intricate to set down here—would have to be given on employing the normal constant volume hydrogen thermometer, adopted in 1887 as the ultimate criterion of temperatures by the International Committee upon Weights and Measures.

1685, and it is not infrequently met with in modern authors, though an explicit statement of its meaning is very uncommon. Two quantities are of the same kind if they are comparable, that is, can be compared as to whether one of three relations is borne by the first to the second, namely: *equality, major excess,* and *minor excess;* in other words, whether the first of these quantities is equal to, greater than, or less than the second, and moreover it can be determined that one of these relations does subsist between the quantities. Such a comparison is sometimes called a "comparison in magnitude," but this phraseology is misleading if it implies anything beyond what we have said. To avoid any misunderstanding, be it noted that we shall simply speak of two quantities as *comparable,* or *not comparable,* and of the *comparison* of two quantities; referring thereby to the comparison of the quantities as to equality or excess of one over the other. Quantities of the same kind are always of the same sort. A $+1$ and a $-2\frac{3}{4}$ are of the same kind, while a $+1$ and a $+\sqrt{-1}$ are of different kinds. Of still another kind is a $+(1+\sqrt{-1})$. The different kinds of the single sort: *abstract quantities,* are innumerable. The abstract zeroes belong to every one of these kinds. Therefore, in the classification of the abstract sort into kinds, the classes overlap. And in general, whenever any sort comprises more than one kind, the several kinds overlap by the inclusion in every kind of all the zeroes of the sort. The abstract sort of ordinary Algebra may be represented by the points of an Argand surface; here every interminable straight line passing through the origin serves primarily to represent a single kind of abstract quantity. In this case a point on the line represents, not a quantity of the kind, but a value; and hence represents every quantity of this value. It is only in a secondary sense that the point can be said to represent one or more, but not all quantities of this value.

A further classification, and one of no small importance in Algebraic Mathematics, is the classification of quantities in what we shall call *varieties*. A variety comprises either the zeroes and those non-zeroes of a kind greater than zero, or the zeroes and those non-zeroes less than zero. In the modern algebras a variety is usually a part of a kind, each kind commonly comprising two overlapping varieties; the one *positive* and containing the zeroes and all the positive non-zeroes of the kind, the other *negative* and containing the zeroes and all the negative non-zeroes of the kind. But this is not always the case, there being many sorts each of which includes only one kind and only one variety. All concrete sorts are of this character, and so are many applicate sorts, among which we mention those composed of lengths, of areas, of columes, of quantities of work, and of quantities of energy. Indeed in the primeval algebra—Arithmetical Algebra—no sort contained more than one kind, and no kind contained more than one variety. For the conventions of comparison first laid down were not such that a kind could contain both quantities greater than zero and quantities less than zero. And likewise the conventions of addition first accepted were such that each sort contained only one kind; so that in every case a variety, its kind and its sort comprised exactly the same set of quantities. In the passage to the algebras of more modern development, two varieties, that is, two different kinds and two different sorts of Arithmetical Algebra were taken, and supplemental conventions for comparison and addition laid down, under which these two varieties, while remaining different varieties, were comprised in a single kind. In the first instance (that is to say, in the first discipline developing out of Arithmetical Algebra) each sort comprised only one kind, and the only change as regards the sort, kind, variety classification made in the passage to the new discipline was that in this a kind may comprise two varieties. In the next step, the passage to the ordinary Algebra of to-day, by yet more supplemental conventions, several kinds, each in the earlier algebras of a different sort, while remaining different kinds were consolidated into a single sort of the modern algebraic discipline.

Among the various kinds of a sort, certain may be distinguished as *primitive*. Thus with the abstract quantities of what we have just designated as the modern algebraic discipline—*Double Algebra,* as De Morgan termed it—there are two primitive kinds: the *real* and the *imaginary* as they are very inappropriately called. A $+3$ and a -2, for example, are real; a $+7\sqrt{-1}$ and a $-13\sqrt{-1}$ are imaginary. These two kinds are primitive in this sense: that the value of each abstract non-zero of every other kind can be *and is* conventionally represented by conjoining together, with a plus or minus sign between (as in $3+7\sqrt{-1}$), two value symbols; one that of a real, the other that of an imaginary abstract value. These other kinds, which stand in antithesis to the primitive, are designated as *complex*. In the abstract sort of Double Algebra there are innumerable complex kinds, and every abstract zero belongs to all of them and to the real and to the imaginary kinds as well. And in general, every zero (whether it be abstract or applicate, whether it occur in Double Algebra or elsewhere) provided it belongs to a sort containing imaginary and complex as well as real quantities, is at once real, imaginary and complex. A non-zero, however, is never entitled to more than one of these three designations. Some of the applicate sorts of Double Algebra contain, like the abstract sort, two primitive and various complex kinds; others contain only one kind each (which is classed as real); but none have more than two primitive kinds. In certain other branches of Algebraic Mathematics, however, the case is more complicated. Thus, in the science of Quaternions, which we shall soon have occasion to take up, the abstract sort has four primitive and innumerable complex kinds, while an applicate sort can have at the most three primitive kinds. With some of the complex abstract non-zeroes of Quaternions, the value is represented by conjoining together four value symbols, each pertaining to a different primitive kind; with other complex abstract non-zeroes, three such symbols suffice; and with still others, only two need be taken. Analogously, with some complex applicate non-zeroes, the value must be represented by conjoining together three primitive value symbols; while with others, only two are needed for this purpose. A basis for the classification of the various disciplines coming under the head of Algebraic Mathematics is afforded by the number of primitive kinds that a sort may contain, and by the number of varieties (two or one) that may be comprised in a kind. Of this classification we shall say more later; at present we need merely remark that besides Double Algebra and Quaternions the most important algebras are *Arithmetical Algebra*, in which a kind can comprise only one variety, and a sort only one kind; and *Single Algebra*, in which likewise a sort can never contain more than a single kind, but in which a kind may comprehend either one variety or two varieties.

In Quaternions, of the four primitive kinds of the abstract sort, one is real and three are imaginary—there are *i*-imaginaries, *j*-imaginaries and *k*-imaginaries in Quaternions. And when an applicate sort of Quaternions has the full complement of three primitive kinds, all of these are imaginary; so that the applicate sorts most characteristic of Quaternions contain no "real" quantities at all, though the quantities of these sorts really exist. Precisely how the so-called "real" and "imaginary" quantities differ from each other, is an important question, and one that we shall soon attempt to answer. For the present it suffices to point out that, for mathematical use, the words "real" and "imaginary" labor under the disadvantage of colloquial associations which would lead one to think that the quantities designated by the latter adjective are to be found only in the realms of the imagination, and that real quantities are those which actually exist in our universe. As the technical mathematical application of "real" and "imaginary" is quite foreign to such implications it is fortunate that these are not the only adjectives available. In place of "imaginary" Halsted has proposed *neomonic;* and Lefevre has suggested *protomonic* as a substitute for "real."[1] This new terminology is so greatly superior to the old that it is difficult to see why the latter has not been entirely abandoned. We shall adopt it here, and ordinarily use the new adjectives, reverting to the synonymous "real" and "imaginary" only when the exigencies of the moment seem to require the use of the commonly accepted phraseology. It is to be noted that *non-protomonic* is not equivalent to *neomonic or complex;* this is due to the fact that a zero may be at once protomonic, neomonic, and complex.

It would seem perfectly obvious that, whichever adjec-

[1] See *Number and Its Algebra*, by Arthur Lefevre, Boston 1896. Napier (*De Arte Logistica*, Liber 3, Caput 1) used *nugacial*, or rather its Latin paronym *nugacia*, in reference to such expressions as $\sqrt{-1}$.

tives be adopted, precision and consistency is desirable in the application of the names taken to mark the distinction between the primitive and complex kinds, and that between the respective primitive kinds. Yet precision and consistency is very seldom found in the use made by mathematicians of "real," "imaginary," and "complex." At times we find in mathematical works "quantity" so used that it must be taken in the sense of protomonic quantity; while a "complex quantity," or an "imaginary quantity" with some authors, is a quantity which may, it would appear, be just as frequently protomonic as complex or neomonic! To us, such a way of using language seems utterly indefensible. Equally bad is a variorum, which, while using "complex quantity" as before, to include all three cases, uses "real quantity" for one of these cases; so that here the real quantities constitute a species of "complex quantities;" and the word "quantity" without an adjective is presumably regarded as having no definite meaning. Strange as it may seem, a terminology on such a basis was formally accepted by no less a mathematician than Gauss, who classified under the general head of "complex numbers" the "real numbers" and the "imaginary numbers." The "real numbers," in his view, included "zero" (*i. e.*, the zeroes), the "positive numbers" (*i. e.*, the positive protomonic non-zeroes) and the "negative numbers" (*i. e.*, the negative protomonic non-zeroes). "Imaginary numbers" Gauss regarded as including only non-zeroes, and he subdivided them into the "pure imaginaries" (*i. e.*, the neomonic non-zeroes) and the "mixed imaginaries" (*i. e.*, the complex non-zeroes).[1]

[1] See *Untersuchungen ueber hoehere Arithmetic* (translations from the Latin of Gauss by H. Maser), Berlin, 1889, p. 541.

Some modern mathematicians follow Gauss in refusing to call zeroes either positive or negative. Others regard them as *both* positive and negative. The latter course saves much circumlocution in the discussion of certain subjects; notably Budan's theorem. We adopt it here when the sort to which the zeroes belong contains both positive and negative non-zeroes. When, however, a sort contains only positive non-zeroes, we would regard the zeroes of this sort as positive, but not negative; and when it contains only negative non-zeroes, we would regard the zeroes as negative, but not positive. With neomonic and complex non-zeroes also we are constrained to differ from Gauss, since we draw a distinction between the positive and the negative. There is no difficulty in doing this with the neomonic quantities, but with the complex it is not always easy to tell which non-zeroes of a kind are best classified as positive and which as negative. Take a $1-\sqrt{-1}$ and a $-1+\sqrt{-1}$ in Double Algebra for example. Which should we call positive and which negative? To this question mathematical works afford no reply. However, in Double Algebra complex quantities of values represented by points in the upper right hand quadrant of an Argand diagram are undoubtedly to be classed as positive, and might be termed *positive complex quantities of the first family.* Those of values represented by points in the lower left hand Argand quadrant are certainly negative, and might be termed *negative complex quantities of the first family.* The remaining complex quantities of Double Algebra belong to the *second family,* and we might, provisionally at least, regard the lower right hand quadrant as representative of positive values, and the upper left hand as representative of negative. Then a $-1-\sqrt{-1}$ will be a positive complex quantity of the second family, while a $-1+\sqrt{-1}$ will be a negative.

We must now digress for a moment to inform the reader of the precise sense in which we shall use the name "ratio," a word that will frequently occur in the following pages. We draw a distinction, not ordinarily made by mathematicians, between *quotients* and *ratios*. And likewise we distinguish between the two operations—or rather, to be precise, the two *species of operations*—of which quotients and ratios are results; *division* and *finding the ratio* (or *ratiofication* if we may so speak). Each of these is an inverse of multiplication, to which species of operation we thus assign not one inverse but two. In view of the fact that mathematicians admit there to be two [species of] operations inverse to addition (*subtertraction* and *detraction* in Schubert's terminology, the title *subtraction* being by this author applied indifferently to both), it is somewhat singular that it does not appear to have been clearly recognized by any mathematician that a like course ought to be taken with multiplication. For just as with two operands b and s, there are distinguished the operation of subtertracting b from s, which consists in finding a quantity a such that $a+b=s$, and that of detracting b from s, which consists in finding a quantity c such that $b+c=s$; so likewise with m and n as operands, one may distinguish between division of m by n, which, as we define it, consists in finding a quantity q (represented by $m \div n$, and called the *quotient* of m to n) such that $n \times q = m$—such that q multiplied by n gives a product equal to or identical with m, and ratiofication of m by n, which we define as the finding of a quantity r (represented by $m:n$, and called the *ratio* of m to n) such that $r \times n = m$—such that n multiplied by r gives a product equal to or identical with m. The justification for drawing this distinction, between division and ratiofication, is assuredly as great as that for drawing one between subtertraction and detraction; and it is of especial importance to those who, like ourselves, look askance upon ascribing the role of multipliers to any quantities other than the abstract. We would not, for example, speak of dividing one volume by another; for this implies that we may multiply something by a volume and obtain another volume as product. And we would not say that there was a ratio of a volume to an abstract quantity; for this implies that upon multiplying the abstract quantity by something—which certainly cannot be another abstract quantity—there is obtained a volume as product. From our point of view the only quantities that can be used as divisors are the abstract (nothing but an abstract non-zero being capable of dividing a non-zero dividend); and the only ones that can serve as consequents of ratiofication are quantities of the same sort as their respective antecedents (nothing but a non-zero of the same sort being capable of acting as consequent to a non-zero antecedent). A consequent is thus not necessarily abstract; and any quantity whatsoever, abstract or denominate, zero or non-zero, may play the part of a dividend in division or of an antecedent in ratiofication. The result of an operation of ratiofication, a *ratio*, is always an abstract quantity, while the result of an operation of division, a *quotient*, may be abstract or denominate, but will in any event always be of the same sort as the dividend. The definition of ratiofication given by us is unquestionably nothing other than a distinct portrayal of that operation of finding a ratio which was employed in the *Geometry* of Euclid. The only mathematician that we recall as making a specific distinction between quotient and ratio is Hamilton,[1]

[1] See letters to De Morgan in Graves's *Life of Hamilton*, Vol. 3, p. 598.

and the distinction he would draw is so entirely foreign to the one accepted by us that we need not here consider it at all; we will merely remark that the word most commonly used by Hamilton is "quotient," and where he applied it in his treatment of Quaternions, as designating the result of an operation performed upon two vectors, it appears to mean what we prefer to express by "ratio."

Among the quantities of each sort are some that mathematicians term *units*. A distinction ought, we hold, be made between *units of the sort*, *units of each kind*, and *units of each variety*. Thus, with the abstract sort in Double Algebra, every quantity of value $+1$ is a unit of the sort, and is likewise a unit of the protomonic kind of that sort and of the positive variety of that kind. Every quantity of value $+\sqrt{-1}$ is a unit of the neomonic kind and of the positive neomonic variety, though it is not a unit of the sort. Each complex kind in Double Algebra has likewise a unit value, and every quantity of that value is a unit of its kind and of the positive variety of that kind, though not of the sort. Thus every quantity of value $\frac{4}{5}+\frac{3}{5}\sqrt{-1}$ is a unit of a certain complex abstract kind and of its positive variety. Again, each variety of Double Algebra has a unit value, and every quantity of that value is a unit of the variety. Every quantity of value -1, though not a unit of its sort and not a unit of its kind, is a unit of the negative protomonic variety. Likewise $-\sqrt{-1}$ and $-\frac{4}{5}-\frac{3}{5}\sqrt{-1}$ are unit values of negative abstract varieties.

For a quantity to be a unit of its sort, the necessary and sufficient condition is that it be a unit of the positive protomonic variety of that sort. For a quantity to be a unit of its kind, the necessary and sufficient condition is that it be a unit of the positive variety of that kind. The fundamental conception is thus that of unit of a variety; and the designation of a quantity as such a unit depends largely upon purely symbolic considerations—upon the type of value expression assigned to it. With the positive protomonic abstract variety, a quantity is a unit if it has as value expression the symbol "1" or "+1"; and with the negative protomonic abstract variety, a quantity is a unit if it has as value expression "−1". With each denominate protomonic variety, a quantity is a unit if its value expression arises from conjoining a denomination to "1" or "+1" when the variety is positive, and to "−1" when the variety is negative. It is customary with each neomonic kind to form the value expression of a non-zero by conjoining to a value expression which alone represents a protomonic value, a special character which marks the kind, *e. g.*, in Quaternions the characters "i," "j," and "k" mark the three abstract neomonic kinds. With each abstract neomonic variety, a quantity is a unit if it has as value expression the special character serving to mark the kind to which it belongs, or has as value expression this conjoined to the sign plus or to "1" or to "+1" or to the sign minus or to "−1". And for the denominate neomonic varieties the same rule will suffice to fix the unit values when we adhere to the notation now current, which omits to suffix a denomination in writing denominate value expressions. Under a more precise notation, however, we would have to say that a unit of a denominate neomonic kind has as value expression the character marking the kind, not alone, but conjoined with the denomination which serves to mark the sort, or has as value expression these two conjoined to the sign plus or to "+1" or to the sign minus or to "−1." As units of complex varieties play no part in value symbolism, notational practice does not afford a general rule for fixing upon a certain value of a complex variety as unit. With Double Algebra, however, it can be laid down that a complex quantity having as expression in the customary notation $A+B\sqrt{-1}$, with or without a denomination suffixed, is a unit of its variety provided $A^2+B^2=1$, and with Quaternions a complex quantity having as value expression $W+Xi+Yj+Zk$, with or without a denomination, is a unit of its variety provided $W^2+X^2+Y^2+Z^2=1$, an analogous rule holding when we have a trinomial or binomial instead of a quadrinomial.

Units of varieties are of no small importance, since on them depends the whole of the general theory of limits. That this is the case, is due to the fact that on the conception of unit of a variety rests that of unit of a kind, on which in turn rests the conception of modulus, on the latter resting the conceptions of numerical equality and numerical excess, these being absolutely essential to any consideration of limits where quantities of more than one kind are concerned. In view of this fact, it would be strange did the designation of a quantity as a unit of its variety depend entirely upon notational consideration; and indeed, in all the algebras coming within the field of the theory of limits, there are other considerations by which the mathematician finds himself more or less restricted when he comes to assign to a value the expression which makes it designable as the unit of its variety. Though there is an arbitrary element in the selection of values to serve as denominate units, there are, nevertheless, certain prerequisites which, in the algebras we refer to, must be taken into account in fixing the denominate unit values; while with the abstract quantities there is no arbitrary element at all in the selection of the units for these algebras. It is not the case that any positive protomonic abstract value whatsoever may, at the option of the mathematician, be represented by "1" or "+1"; it is essential that there be taken to be so represented the value of the number attribute of a group composed of a single object. Again, in these disciplines, with the negative protomonic abstract variety and the neomonic abstract varieties, not any value of the variety concerned is acceptable as the unit value; it is requisite that a quantity of the value taken as unit should, when multiplied into itself a sufficient number of times give a product of value +1; and with each variety there is thus only one value eligible as unit. And evidently, the values of the units of the primitive varieties once fixed, there is, in Double Algebra and Quaternions, left no opening for choice in the values to be taken as units of the complex varieties. Likewise, we apprehend, in any algebra with which the theory of limits in the present form is concerned, the abstract quantities apt to be made units by the assignment of certain expressions to them, must fulfill just such conditions as those that rule in Double Algebra and Quarternions.[1] But there are, it should be noted,

[1] Were all the algebras of such character it might be worth while to so formulate a definition of unit of an abstract variety as to leave symbolism entirely out of consideration. A definition of this description that at first sight seems irreproachable, might be formulated by saying that in order for an abstract quantity to be a unit of its variety, whether this variety be primitive or complex, the fulfillment of one or the other of two conditions is necessary and sufficient; these conditions being: First, the quantity multiplied into itself a sufficient number of times gives a product of value +1; or, if this condition is not satisfied, Second, that a being the quantity, we can with the multiplicative sequence $aaaa\ldots$, by proceeding to take a sufficient number of factors, reach a stage where the product is as near +1 as we wish. But this really has the vice of *circulus in definiendo*. For when we say "as near +1 as we wish," we assert that after predesignating any abstract non-zero, we can reach a stage where the product differs from +1 by less numerically than

branches of Algebraic Mathematics in which the case is quite otherwise—in which, for example, the square (and hence every higher power) of a neomonic abstract unit may be a zero.

Next turn to the units of the denominate varieties of Double Algebra, Quaternions and other cognate algebras. Here it is that the arbitrary element appears, though it is restricted by the following law: *The unit values of denominate varieties of such an algebra must be so chosen that the ratio of a unit of one variety to a unit of another variety of the same denominate sort shall always be a unit*

this non-zero. And if the difference between $+1$ and the product at any stage is of a variety whose units do not come under the first head, or if we have predesignated an abstract non-zero of a variety whose units do not come under this first head, *then* we are not as yet in a position to speak intelligibly of numerical equality or numerical excess between the two. We might amend the second stipulation by requiring that we can reach a stage where the difference between $+1$ and the product is of a variety whose units come under the first head, and is less numerically than any predesignated abstract non-zero of such a variety; here numerical excess may legitimately be spoken of. But some of the abstract quantities it is desirable to call units of their varieties might conceivably come under neither head, though with them (or with part of them) we might always be able to attain a product whose difference from $+1$ is of a variety whose units come under the *second* head, and is less numerically than any predesignated non-zero whose units come under either the first or the second head. In this event there would be a third head—a third alternative condition, the satisfaction of which was sufficient to give an abstract quantity the status of unit of its variety. On these lines one might proceed to formulate a fourth, fifth, etc., alternative condition, and it is conceivable that to insure each abstract variety without exception having its units, we might be compelled to leave open not merely four or five or n alternative conditions, but to never bring the set of alternative conditions to a close—it might be that no matter how many alternative conditions we laid down on the above lines there were always left varieties whose units fulfilled only a condition of a still higher order.

of an abstract variety. Under this restriction, with that variety of a denominate sort to which value expressions are first assigned, it lies wholly within the power of the mathematician to choose whatever non-zero value he wishes as unit value. But with all other varieties of the sort there is no choice open; the value to be taken as unit value is fixed by the law expressed by the words in italics.

The ratio of a quantity to a unit of its sort (any such unit being arbitrarily selected from the unit value class to fill the role of consequent) may be termed the *quantuplicity* of that quantity. An abstract quantity can always be taken as its own quantuplicity. The ratio of a quantity to a unit of its variety is termed the *modulus;* and the square of the modulus is termed the *norm.* A modulus is always a positive protomonic abstract quantity. If the value of a quantity in Double Algebra is represented by $A+B\sqrt{-1}$ or $(A+B\sqrt{-1})$ Den. (suffixing the denomination in the case of a denominate quantity) then in either case $A+B\sqrt{-1}$ represents the value of its quantuplicity, A^2+B^2 represents the value of its norm, and $+\sqrt{A^2+B^2}$ represents the value of its modulus; while with the value expression A or A Den. or $B\sqrt{-1}$ or $B\sqrt{-1}$ Den. for the original quantity, the quantuplicity, norm and modulus have as value expressions respectively A or $B\sqrt{-1}$, A^2 or B^2 and $+\sqrt{A^2}$ or $+\sqrt{B^2}$. In Quaternions, with the value of the original quantity represented by $W+Xi+Yj+Zk$ or by $(W+Xi+Yj+Zk)$ Den., the values of the quantuplicity, norm and modulus are respectively represented by $W+Xi+Yj+Zk$, $W^2+X^2+Y^2+Z^2$ and $+\sqrt{W^2+X^2+Y^2+Z^2}$, and the case is quite analogous when the original value expression reduces to a trinomial or to a binomial or to a monomial, or to one of these with a denomination suffixed. Usually this symbolic representation is taken as the basis of the definitions of norm and modulus, and the definitions we adopt in preference are of course innovations. Nor can we claim that any authority sanctions our definition of quantuplicity; a word that has not hitherto been used with any great precision. The name modulus, we need hardly say, is due to Argand. De Morgan makes use of the phrase "modulus of multiplication." Taking "function" in an improper sense, in which the result of an operation upon a quantity is called a function of that quantity, De Morgan designates, as the *modulus of multiplication,* that function "which in the product has the same value as the product of the functions of the factors."[1] While this modulus of multiplication has for formula in Double Algebra $+\sqrt{A^2+B^2}$ and in Quaternions $+\sqrt{W^2+X^2+Y^2+Z^2}$, it is not always of a formula perfectly analogous to these in certain triple algebras developed by De Morgan. In one case, for example, where the general value expression for an abstract quantity is $A\xi+B\eta+C\zeta$, the modulus of multiplication is $+\sqrt{A^2+(B-C)^2}$ instead of being $+\sqrt{A^2+B^2+C^2}$.

Sometimes the value of the modulus of a quantity is termed the *numerical value* or *absolute value* ("absoluter Betrag" with Weierstrass) of that quantity. Two quantities of the same sort with equal moduli are said to be *numerically equal;* such quantities may or may not be equal. If with two quantities of the same sort, the modulus of one is greater than the modulus of the other the former is said to be *numerically greater* than the latter. Between quantities of different sorts, it is not, we think, advisable to ascribe relations of numerical equality or

[1] *Trans. Cambr. Phil. Soc.,* Vol. 8, p. 241.

numerical excess at all. We have seen that value-classes are formed by grouping together all quantities mutually equal. Analogously *numerical value-classes* may be formed on the basis of the relation of numerical equality, every two quantities belonging to such a class being numerically equal. And a numerical value might be regarded, not as the value of a modulus, but as something not a value at all, just as numerical equality is not a species of equality. That is, to say a quantity possesses a certain numerical value, means simply that it belongs to a certain numerical value-class. The same symbol could perhaps be used for a numerical value as for the corresponding positive protomonic value. And one could then follow the usual custom of saying that the numerical value of —2 and $+2\sqrt{-1}$ and $-2\sqrt{-1}$, etc., is $+2$. However, we ourselves are not enamored of the customary use of "numerical" in "numerical equality," "numerical value" etc.; though we do not see our way clear to avoid it, and regard the use of "absolute" here as still worse.

With the abstract quantities the *positive protomonic* variety is that which first comes under consideration. Among the quantities which it comprises are the natural numbers and natural fractions, which have already been given attention, and besides these there are included not only the *natural zeroes* and certain non-zeroes which may be called the *natural incommensurables* (and of which we shall have something to say in a later part of this work) but also various positive protomonic abstract quantities (zeroes and non-zeroes) which we shall designate as *relational* by antithesis to the former which we have called *natural.* It is not alone in the positive protomonic variety that relational quantities enter. In fact all the other abstract varieties owe their very existence to the acceptance of relational quantities in Mathematics, there being no complex or neomonic or negative protomonic nonzeroes among the natural abstract quantities, and all abstract quantities being either natural or relational. We ought to mention that some philosophers hold even the natural numbers to be relations, contending that the number attribute of a group of three oranges, for example, is a relation between this group and a single orange. It is argued that there is an essential difference between number and such an attribute as color; since the red color that a body may happen to have is independent of our volition, while the number of objects in a group is dependent on it, in so far that if we have before us (for instance) three pamphlets, composed respectively of twenty, thirty and fifty sheets of paper, there are in this group three obiects if we choose to count pamphlets, but a hundred if we choose to count sheets. That a distinction ought to be drawn between qualities and quantities, no one will deny. But this is a very different thing from saying all quantities should be classed as relations. We are inclined to demur at the assumption that the group of three pamphlets is the *same* as the group of a hundred objects obtained on considering their sheets; but of course this is largely a question of how it is most convenient to use "same" in reference to groups. To us it would seem most convenient to so use the word that the "same group" shall always have the same number attribute. In fixing upon the value of the number attribute of a group, one must of course have in view some standard by which the component objects are to be individualized—by which we may decide whether something at hand is to be counted as having a number attribute of unity, or whether several objects are to be discriminated in it, or whether it is to be ignored altogether (as are the wire fastenings of the pamphlets in a group composed of their sheets). But for our part we cannot see that this fact makes the number attribute of the group be a relation. On not entirely dissimilar grounds it might be contended that a color is a relation. For on any attempt at precise discrimination of colors, some standard must be taken for each color, and an object termed "red" or not "red" according as it does or does not accord in color with the standard red object. And it might be argued that the redness of an object was a relation between this object and the standard, depending entirely upon one's own volition—upon the choice made of a standard. It may also be mentioned, as having some bearing on the question of whether natural numbers are relations, that in comparing the number attributes of groups as to equality or excess, we do not compare the relations of the groups to a single object. The objects of the one group are put in correspondence with the objects of the other group, either mentally or otherwise, and it is noted whether the objects of the two groups are exhausted simultaneously or whether one group is exhausted before the other. It is by making such a comparison that number attributes are classified into value-classes. Certain groups are taken as standards, and a group whose objects are exhausted simultaneously with those of one of these standard groups has its number attribute put in the same value-class as that to which belongs the number attribute of the standard. As the standards for the value 2, are taken all groups owing their formation to the joining of one object to another object; as that for value 3, all owing their formation to the joining of one object to a group of two objects; as that for value 4, all owing their formation to the joining of one object to a group of three objects, etc. And thus the value-class 4 does not contain only all quantities which are sums of 3+1, but also all which, though not having come into existence by such an addition, are equal to a $3+1$. So that the process of ascribing a value to a number attribute can hardly be said to involve the consideration of the relation of this group to a single object.

Of the natural zeroes we shall need to say but little here. The name of such a quantity serves to express the want or absence of a number attribute. When there are no objects at all at hand, we can express this fact by saying that there are zero objects or that the number of objects is zero. But such cases are not the only ones in which we may speak of a natural zero. If we are counting objects of any particular type, a group composed of objects of another type not taken into consideration must be deemed to have a number attribute of zero. And in any computation there may be many distinct groups having no number attribute, so that there are many different natural zeroes. We may speak of adding a natural zero to a natural number, not only in the sense of doing nothing to the group of which the natural number is an attribute, but also in reference to joining to this group another group whose number attribute is zero.

Of the relational abstract quantities none stand forth more prominently than those brought into Mathematics by Sir William Rowan Hamilton, and called by him *quaternions*. We say that these quantities are prominent in Mathematics, but we must add that a clear recognition of their true nature is, in the conventional treatment of the subject—even in the treatment given by Hamilton himself—conspicuous by its absence. And hence it is by no means superfluous for us to insist upon the fact that a quaternion is nothing more or less than an abstract quantity. Of course not all abstract quantities are quaternions; and indeed of a single value there may be abstract quantities which are not quaternions and others which are, as we shall soon show.

Quaternions are closely connected with, and in fact depend for their existence on, certain applicate quantities known as *vectors*. Hamilton lays down that "A right line *A B*, considered as having not only *length*, but also direction, is said to be a Vector."[1] Whether he regarded this statement as furnishing an adequate definition of "vector" is not clear. His successors usually content themselves with a bare repetition of it.[2] As a definition, however, it is obviously inadequate, since a *linear velocity at a point*[3] is a vector, and of course a velocity is not a line. Further, not all straight lines possess the attributes requisite to vectors. With regard to their mode of generation, straight lines may be classed into those which have been described, and those which have not been described; and only a straight line of the first class possesses an attribute of direction of that type which is of utility

[1] *Elements of Quaternions*, London, 1866, p. 1.

[2] Thus Prof. C. J. Joly, the highest authority on Quaternions at the present day, does no more in his *Manual of Quaternions* (London, 1905) than to repeat textually on his first page this statement of Hamilton's.

[3] The adjective "linear" is employed to distinguish the velocities here spoken of, from *angular velocities;* and the phrase "at a point" to distinguish one species of linear velocities from two others. The other two species are *linear velocity from one point to another*, of value obtained by performing an operation usually called "dividing the space travelled by the time," and *average linear velocity from one point to another*, of value obtained by finding the average of the magnitudes of all the velocities at the various points of the path—"average" here referring, not to a result obtained by dividing a sum by the number of summands entering in it, but to a result obtained by integration. These last two species of velocity are usually not distinguished. They can hardly be said to possess direction, or currency as we will call it.

in Quaternions. With a straight line which has been described, each point further from the initial point has come into existence later than every point nearer the initial point. We can distinguish between two types of direction, which may be called respectively *geometrical direction* and *trigonometrical direction*. These cannot be more readily pointed out than by stating when two straight lines have like and when unlike directions. Two straight lines have like geometrical directions when they are either parallel or coaligned, and have unlike geometrical directions when they are neither parallel nor coaligned. Two straight lines have like trigonometrical directions when they satisfy a set of three requirements: first, they have been described; second, they are parallel or coaligned; third, they are not of contrary description. Two straight lines have unlike trigonometrical directions when they satisfy a set of two requirements: first, they have been described; and second, they are not parallel or coaligned; or when they satisfy a set of three requirements: first, they have been described; second, they are parallel or coaligned; third, they are of contrary description. We may speak of trigonometrical direction as *currency;* and say that two described straight lines, or in general two vectors, are *concurrent* if they are alike as to trigonometrical direction. The names "currency" and "concurrent" are due to Clifford. Some authors employ the term "sense" instead of currency, but this practice has no merits which justify it. "Current" and "currency" naturally suggest a running or flowing, but no such associations are called forth by the word "sense," which is used chiefly to refer to the meaning of a word, the mentality of a person, or the process of sensation. Two vectors parallel or coaligned, but not concurrent, are said to be *contrary*. With currencies distinctly delineated, we will turn to the other class of attributes, one of which, united with a currency, constitutes every vector not of value zero. These attributes are usually termed the *magnitudes* of the vectors. The magnitude of a line vector is simply its length; the magnitude of a force vector is its intensity, etc., etc. Wherever a currency and a magnitude subsist together, there a vector exists. For two vectors of the same sort to be of the same kind, it is necessary and sufficient that they should be parallel or coaligned. In order that they be of the same variety, it is necessary and sufficient for them to be concurrent. For two vectors of the same sort to be equal, it is necessary and sufficient that they be concurrent and of equal magnitude. The magnitudes of the vectors of each sort are themselves quantities, and together constitute a sort (which is a denominate though not a vector sort) containing only one kind and only one variety.

To recognize that a currency and a magnitude together constitute every non-zero vector does not appear to be sufficient to avoid confusion. Even so high an authority as Hamilton states that a right line is a vector; a view which is quite untenable.[1] The vector of a straight line,

[1] Occasionally an author gives a definition less open to reproach. Thus in *Vector Analysis, founded upon the lectures of J. Willard Gibbs*, by Edwin Bidwell Wilson, New York, 1902, p. 1, we find: "A vector is a quantity which is considered as possessing *direction* as well as *magnitude*." The typical vector, Gibbs tells us, "is the displacement of translation in space. Consider first a point P. Let P be displaced in a straight line and take a new position P'. This change of position is represented by the line PP'. The magnitude of the displacement is the length of PP'; the direction of it is the direction of the line PP' from P to P'." Translatory motion is not a species of motion *of a point*. It is a species of motion *of a body*. It occurs whenever all points of a body travel along paths exactly alike and along parts of these paths exactly alike during the same

which is composed of two attributes of the line, is as different from the line itself as the color of a red body is different from that body. Suppose that a straight line has been described from a point A to a point B, the length of the line being 150 cm. This line possesses the attributes of a vector, but it is not itself the vector. For let the line be turned about the point A to a new position without undergoing any change in length. It would still be regarded, and justly so, as the *same* line moved into a new position. But the vector of the line has changed *the line in the new position has a new vector which is not even equal to the vector of the line in its old position.* The magnitude of the vector of a line is, as we have already said, nothing more or less than the length of the line, and is expressed in centimeters, meters, etc. According to the requirements laid down by the founders of Vector Analysis, for two non-zero vectors to be equal, they must not only be equal in magnitude, but must also be concurrent. The vectors of the line AB in its two positions do not meet both these requirements. The two vectors are not

times. A sub-species of translatory motion is rectilinear translatory motion in which all points of a body move along straight lines of equal length. To this Gibbs refers when he speaks of "displacement of translation in space," the practice of referring to the sub-species by the name properly belonging to the species as a whole being as common as it is unjustified. We can, in any case of translatory motion, select one point of a body and employ its motion as descriptive of the motion of the body. Further, we can speak of the length of the line described by such a point as the magnitude of the displacement, and in the case of rectilinear translatory motion, *and in this case only,* we can speak of the direction of the line as the direction of the displacement. That the magnitude and displacement of a rectilinear translatory motion must be founded upon the length and direction of a straight line which has been described, appears to us conclusive against the statement: "The typical vector is the displacement of [rectilinear] translation in space."

concurrent. Moreover, if we construe "equality in magnitude," in the proper sense of equality which excludes identity—a distinction which was perhaps never plainly before the founders of Vector Analysis—then the two vectors are not equal in magnitude; the magnitude of the one vector is the same as the magnitude of the other vector. That the vectors of the line *AB* in its two positions have the same magnitude—not equal magnitudes—is an excellent example of identity, something so frequently improperly conceived, as to be in its misconception one of the most striking characteristics of the erroneous philosophical theories of Mathematics.

We have seen that every non-zero vector consists of two attributes: a currency and a magnitude. How is it with zero vectors or *null vectors* as they are sometimes called? With that class of vectors, a sort, whose non-zeroes are vectors of described straight lines, it is not unnatural to turn to points to see whether these cannot be said to possess vectors of zero value belonging to the same sort as vectors of lines. A little reflection suffices to show that such is indeed the case. Points can fulfill all the requirements which need be demanded of them for this purpose. The conventions can be laid down that all points have equal vectors, and that every vector of a point is greater than every negative vector of a line, but less than every positive vector of a line. These stipulations are not sufficient to enable us to ascribe to points vectors of the same sort as vectors of lines. There must also be laid down conventions for the addition of vectors of points to vectors of points, and for the addition to one another of vectors of points and vectors of lines. To add the vector of a point *B* to that of another point *A* would be to move *B* so as to bring it into coincidence with *A*, thereby forming a vector—a zero vector—of the same value as the operands. To add a vector of a point B to a vector of a line AC would be to move the point B so as to bring it into coincidence with the final point C of the line AC, thereby forming a vector of a line of the same value as the augend, the vector of the line AC. And finally to add the vector of a line AC to the vector of a point B would be to move the line AC, without altering the currency of its vector, so as to bring its initial point A into coincidence with the point B, thereby forming a vector of a line of the same value as the addend, the vector of the line AC. When these conventions have been adopted, it is perfectly proper to regard points as possessing attributes of magnitude. But it must be acknowledged that a point cannot possess an attribute of currency. Indeed, with all zero vectors whatsoever, attributes of currency are lacking.

We are now prepared to give an exposition of what we regard as the true doctrine of those remarkable objects of mathematical inquiry known as *quaternions*. Suppose there are two vectors a and b of lines running toward the East[1] of respective lengths 7 cm. and 21 cm. We may ascribe to this pair of vectors a relation—in fact two relations: that of b to a, and its inverse,[2] the relation of

[1] We use North, South, East, West, etc., in reference to horizontal currencies in a plane, supposed to be tangent to the spheroidal surface of the earth at a certain point, or in other planes parallel to this.

[2] As to the inverse relations, the question might be raised whether a relation and its inverse are really two relations—whether they are not merely different aspects of the same relation. It might be said, for instance, that paternity and filiation are not two relations but one and the same relation. But after all, since no one would contend that to be a father is the same thing as being a son, the question is largely one of how far it is most convenient to use the word "relation;" and it seems by far better to grant the claim of inverse relations to this title. For it is quite clear that there are certain impor-

a to *b*, classing these relations with abstract quantities, and assigning them the values 3 and $\frac{1}{3}$ respectively. If instead of *b* we have *c*, a vector likewise of magnitude 21 cm. but Westward instead of Eastward, the relation of *c* to *a* may be taken as a negative abstract quantity of value -3, and the inverse relation of *a* to *c* as another negative abstract quantity of value $-\frac{1}{3}$. Such abstract quantities as the four just described are quaternions in the broad (and most proper) sense of the word, though sometimes "quaternion" is used in a narrower sense as excluding the simplest cases—those of abstract quantities arising as relations of *concurrent or contrary* vectors. But at all events it is this point of view, of ascribing abstract quantities as relations between vectors, which characterizes Quaternions and certain other systems of vector analysis. It is taken broadly, and to every pair of non-zero vectors of the same sort is ascribed a relation and its inverse, which are classed as abstract quantities and in the discipline founded by Hamilton called *quaternions*. It is to this alone that is due the entrance into Mathematics of all abstract quantities save those which are positive as well as real.[1] Some of these relations between vectors

tant facts which, on making the distinction between such a relation as paternity and its inverse, can be well expressed by the use of the term "relation," and for the expression of which we must have recourse to some word other than "relation" if this distinction be not drawn.

[1] While it is quite clear that all complex or imaginary abstract non-zeroes and all negative real abstract non-zeroes appear in Mathematics as relations between applicate quantities, and in no other way, we must concede that the applicate quantities concerned need not in strictness be vectors. With any applicate kind containing two varieties, the relation of a non-zero of either variety to a non-zero of the other might be regarded as a negative real abstract quantity. And likewise with any applicate sort containing at least

are termed "real," others "complex" and still others "imaginary;" but all have an equal claim to be regarded as really existent.

In the *Elements*, Hamilton describes a quaternion as the quotient of one vector by another vector.[1] If we had deemed it proper to define a quaternion by means of an operation, we would have chosen ratiofication as we have expounded it—not division. For this was unquestionably the operation which Hamilton had in mind in speaking of "quotient" of one vector by another. We have not, however, been able to take this view, and have defined a quaternion as a relation. To some the use of the term "relation" may appear to be captious pedantry. It would be so; indeed it would be merely the idle substitution of one word for another, did we regard a ratio or a quotient as a relation; but we do not. The idea of a relation is entirely unessential to the idea of a ratio or to the idea of a quotient. Further, though it is true that each relational quantity may on occasions be a ratio or a quotient—that is, may be *arbitrarily selected* from among the quantities of a value-class to fill the role of result of a particular operation of ratiofication or division—it need not necessarily be so; a relational quantity may exist for millions and millions of years and yet never be chosen for such a role. And with the ratiofication of one vector in respect to another, it is not even invariably the case that the abstract quantity chosen as ratio need

two kinds of two varieties each, the relation of a non-zero of either kind to a non-zero of the other might be accepted as an imaginary abstract quantity. With applicate quantities other than vectors, however, the possibility of such a procedure is of purely speculative interest; utility it has none; and accordingly it has never yet been carried into mathematical practice.

[1] *Elements of Quaternions*, Book II, Chap. 1.

be a relational quantity. Thus consider the two concurrent vectors a and b specified above. Can we not find a ratio of b to a without resorting to quaternions or any other relational quantities? Most certainly we can; a natural number of value three will answer all the requirements exacted of the ratio. That is, if a be multiplied by a natural three, a process which consists in adding to a (viz., to a vector of a line Eastward in currency and 7 cm. in magnitude) another line vector equal to the former (in other words, also of Eastward currency and of 7 cm. magnitude) and to their sum a third line equal to each of the other two, then there is obtained a vector equal to or identical with b.[1] This much being admitted, we may ask whether any particular natural quantity of the value three ought to be taken as the result of the operation of finding the ratio of b to a. It might be thought that we ought to take that particular three which is the number attribute of the group of three vectors joined together. A moment's reflection, however, shows that the joining is a mere potentiality. That is, we can say the ratio is a three even though a joining of three vectors is never actually carried out. And, moreover, if it be in fact performed, there is obviously great room for choice in the selection of the second and third vector from among the many line vectors with Eastward currency and 7 cm. magnitude.

[1] Notice that the operation of multiplying a by a relational three *need not involve addition at all.* It consists merely in finding a vector whose relation to a is equal to (or identical with) the multiplier. To find such a vector we *may*, if we choose, *form* a new vector by joining together three vectors already at hand. But, though this can always be done, it is not a necessary part of the multiplication process. We need not form a new vector in multiplication by a relational abstract quantity if we can find already in existence a vector which satisfies the conditions required of the product.

Each selection gives a different group, and hence a different number attribute; but a mathematician is not bound to make such a selection and call the three belonging to it the ratio of b to a, any more than he is bound to select the relation of b to a as the ratio. In fact any relational quantity of value three or the number attribute of any group whatsoever containing three objects may be chosen as the ratio of the vector b to the vector a. We hold that a ratio, and likewise a quotient (and in general a result of any *inverse* operation) is a quantity arbitrarily selected from among the quantities of a value-class, entirely without regard to the operands of the operation, these having an influence only upon the value-class permissible.[1] And when a and b are non-zero vectors, the relation of b to a, or any equal relation between two other vectors may be chosen, or finally, when natural quantities equal to the relation of b to a are available, any one of these may be made the ratio.

It now behooves us to ask whether the ratio of one vector to another vector can be obtained when the vectors are non-concurrent without resorting to relational quantities. If it be acknowledged that there are entities other than relations which can serve as negative real abstract non-zeroes and as imaginary and complex abstract non-zeroes, then this question can be answered in the affirmative. In the *Elements of Quaternions*, Hamilton accepted without examination the existence of negative real abstract quan-

[1] Except that with some operations, a particular quantity or several particular quantities of the value-class may be barred when a certain set of operands are in question. Thus in subtraction, if b is a four and a a two, for choice of the remainder $b-a$ there is not open the whole value-class two, since we cannot accept a itself as remainder. To call a the remainder of $b-a$ would imply that a could be added to itself; and no quantity can be added to itself.

tities ("negative real numbers") as established entirely apart from any consideration of vectors or other applicate quantities. Thus if two line vectors were contrary, then a negative real abstract quantity, the existence of which was admitted by Hamilton *entirely apart from consideration of vectors,* was taken as their ratio. But with the imaginary and complex abstract quantities of Quaternions, he was not content to assume a preëstablished existence. Thus if two line vectors were neither concurrent nor contrary, an imaginary or complex quantity, the existence of which was based by Hamilton *entirely* upon the consideration of vectors, was taken as their ratio. These imaginary and complex quantities he regarded as totally different from the imaginary and complex quantities of ordinary Algebra, while he held that Quaternions and the older branch of Mathematics had in common the same set of real abstract quantities, positive and negative. He brought together and included under the title "quaternions," when using that name in its broadest sense; first, those imaginary and complex abstract quantities the existence of which he regarded as established entirely by consideration of vectors; second, such positive real abstract quantities and negative real abstract quantities of ordinary Algebra as were taken for ratios of vectors, though the existence of these real abstract quantities, called by Hamilton *scalars,* was regarded by him as established without any consideration of vectors at all. To us it appears far more proper to include among quaternions only those quantities which are relations between vectors. Thus we could not regard as a quaternion, the number attribute of a group of three objects, even if it were chosen as the ratio of a vector of 21 cm. magnitude to a concurrent vector of 7 cm. magnitude. And neither in ordinary Algebra nor in any other branch of Mathematics, are there, we hold, any entities other than relations, which can be properly designated as negative real abstract non-zeroes or as imaginary or complex abstract non-zeroes, all such quantities arising only as relations between applicate quantities.

In delineating the character of a quaternion, we have said that it is a relation between two vectors *of the same sort.* That the vectors giving rise to a quaternion must be of the same sort is never stipulated, though it must surely be acknowledged that neither Hamilton himself nor any other mathematician has ever conferred a standing as mathematical quantities upon such relations as that of a line vector to a linear velocity; that of a linear velocity to an acceleration; etc. In treating of that quaternion which is a relation of the vector b to the vector a, we shall speak of b as the *relate,* and of a as the *correlate* of the quaternion, which we shall represent by (b, a). When both relate and correlate vectors are non-zeroes and are parallel or coaligned (*i. e.,* are of the same kind), the quaternion (b, a) is a real abstract non-zero, and is positive or negative according as a and b are concurrent (are of the same variety) or contrary (are of different varieties). A quaternion whose relate is a zero, and whose correlate is a non-zero, is a *null quaternion;* that is to say, the value abstract zero is assigned to it. (As antonym to *null* Hamilton uses *actual.*) With the case in which both relate and correlate are zeroes, no decision has been reached by mathematicians as to the value to be ascribed the quaternion; in fact the question does not even seem to have presented itself to mathematicians in just this way. We shall not ourselves attempt a decision, but would remark that the case is widely different from that of the *ratio* of a zero to a zero; a matter which we will take up later on. There cannot well be ascribed more than one value to the relation of one vector to another, whether these be zeroes or not. But the operation of ratiofication can and, we hold, should in such cases have more than one result; namely a result of every value-class that contains quantities capable of satisfying the conditions required of a ratio. And as any abstract quantity whatsoever will satisfy the conditions exacted of a ratio of a zero to another zero of the same sort, there will be innumerable ratios in such a case, each abstract value being represented.

It has been previously pointed out that we may say a natural zero exists where there is an absence of number attribute. It has also been shown how points may be regarded as possessing vectors of zero value belonging to the same applicate sort as line vectors, and analogously with other denominate sorts, certain entities may play the role of zeroes. And finally we have shown that such a relation as that of the vector of a point to the vector of a line is a zero quaternion. A view of the nature of mathematical zeroes quite different from the one we take—a view we do not hesitate to characterize as most extraordinary—has been put forth by an author of some celebrity—G. Frege. In *Die Grundlagen der Arithmetic*[1] he says: "Since under the concept 'unequal to itself' nothing [nichts] falls, I declare: *O* is the number which belongs to 'unequal to itself.'" This view evidently rests upon two misapprehensions: First, that zero is simply nothing; Second, that every entity is equal to itself. The doctrine of Frege is that there is only one zero, this being nothing, and that the most suitable way of defining "nothing" is by means of the "concept" of unequal to itself. Surely the relation of a vector of a point to a vector of a line is very far from being nothing. And there are many

[1] Breslau, 1884, p. 87.

other zeroes which are not non-entities. If we turn to concrete quantities, we see that since on counting oranges, men are not taken into account, a man (a zero so far as oranges are concerned) is, according to Frege's doctrine, a mere nothing whose distinguishing characteristic is that he is unequal to himself! As to the second misapprehension, it is a part of that error to which we have to recur again and again in criticising the doctrines of current mathematical works—the confusion between *equality* (and other cases of perfect similarity) and *identity;* one of the most pernicious errors into which a mathematician can fall. And here we have an example of its consequences in leading so pitiably astray the thoughts of a man who aspires to be a philosopher. Many are the pitfalls which could be avoided if authors would only remember these maxims: No entity is equal to itself. Every entity is identical with itself, and may be perfectly similar to other entities in various respects. No writer is worthy of being taken as a guide in the Philosophy of Mathematics if he disregards them, and endeavors to rear a system involving a disregard of the fact that every sort contains many zeroes, each of which is equal to every other zero of this sort, and a repudiation of the truism that every quantity, non-zero as well as zero, though identical with is unequal to itself.

Of the services rendered by Hamilton to mathematical science, one of the most important has not been recognized by mathematicians. With the admission of the relations between vectors of the same sort to membership among the quantities of Mathematics, there is furnished ample argument to banish forever, to the limbo of doctrines outworn, the tenet so widely taught by mathematicians, even at the present day, that negative, imaginary and complex "numbers" are mere symbols.[1] That this doctrine now enjoys, more than half a century after the publication of Hamilton's immortal work, the support of mathematicians of the highest eminence—even men of no less authority than those brilliant mathematicians who contributed to the most authoritative mathematical work of modern times, the *Encyklopaedie der mathematischen Wissenschaften* —is owing in part to the attitude of Hamilton himself. His declarations on the nature of those quantities he called "quaternions" were not sufficiently consistent and unequivocal to prevent there arising on this subject an obscurity which still prevails. Though himself a man imbued with the true scientific spirit, he was not always able to resist the solicitations of his disciples, men rather of the artisan than of the scientific type, who cared much for the results attained by Quaternions but little for its principles, and who were continually urging Hamilton to abandon his work on the foundations of the science, and devote himself to extending its applications. His best work they disapproved of; and thus Tait, having the self-assurance to class as of the same caliber as himself all the students of Hamilton's works, tells us that Hamilton

[1] In teaching this doctrine mathematicians are not merely expounding the historical beginning of the use of such symbols as -1, $+\sqrt{-1}$, etc. They are enunciating the proposition that in the present state of development of Mathematics, we are under the necessity of regarding all inquiries concerning negative, imaginary and complex "numbers" as dealing solely with symbols representing no objects whatsoever. The view which we maintain throughout this work is that relations between vectors of the same sort fulfill all the requirements which can be exacted of them to deserve the title of complex, or of imaginary, of or negative real abstract quantities as the case may be. If this can be proven, it follows that it is wholly unjustifiable to lower the dignity of mathematical theory by treating these quantities as symbols, and making Mathematics a mere doctrine of symbolism.

"fatigued and often irritated his readers by constant excursions into metaphysics." Tait even calls the abstract imaginaries, *i*, *j* and *k*, of Hamilton "mere excrescences and blots on his improved method,"[1] and states that Hamilton had himself recognized this. That Hamilton might have taken this view in a moment of aberration, we cannot deny; as we have said, he was not always consistent, and often falls into the error of accepting vectors as multipliers. He has, however, the undeniable merit of being the only founder of an important system of Vector Analysis who made any attempt at all to distinguish between those quantities which can be properly used as multipliers—in his system, the abstract quantities called quaternions—and vectors, which are quantitives of an entirely different type, and ought only enter operations of multiplication as multiplicands. As has been well said, Grassmann and Hamilton's other rivals never even attained the conception of a quaternion. Indeed, a recent writer, E. Study, in the *Encyklopaedie* actually *reproaches* Hamilton for making with multiplication, the distinction between multiplier and multiplicand; a distinction which this author says "influences the clearness of the exposition" and "is not necessary for the application" of the theory of "complex quantities"[2]—a subject which Study treats from a purely symbolic standpoint. Such remarks would be impossible were mathematicians awake to the fact that Hamilton's achievement in conceiving the abstract *i*'s, *j*'s and *k*'s was not the mere introduction of a new set of symbols, and that it is a matter which concerns not Vector Analysis alone but the whole of Algebraic Mathe-

[1] *On the Intrinsic Nature of the Quaternion Method, Proc. Roy. Soc. Edin.*, Vol. 20, p. 276 *et seq.*

[2] *Op. cit., Theorie der gemeinen und hoeheren complexen Groessen,* Vol. 1 Part 1, p. 159 note.

matics. When this becomes recognized, then that day, October 16, 1843, when he cut on Brougham Bridge the fundamental formula he had just discovered: $i^2=j^2=k^2=ijk=-1$, will be esteemed one of the most significant dates in the history of mathematical science.

In his earlier articles Hamilton himself defined a quaternion as an expression: "Let an expression of the form $Q=w+ix+jy+kz$ be called a quaternion."[1] He never entirely abandoned this mode of speech, and like most mathematicians, seems at times quite unaware that a clear distinction ought always be drawn between names and what they denote. To define a quaternion in this manner is as futile as to define a man as a word of three letters "m," "a," "n." It would be unjust, however, to accuse Hamilton of not being aware of the insufficiency of this definition; particularly in view of many of his statements in the *Elements of Quaternions*. But we cannot absolve him completely of falling into those modes of thought and expression which have now become so predominant as to warrant speaking of the present as the Symbolic Period in Mathematics. In the *Elements*, Hamilton first mentions quaternions by informing us "that there is an important sense in which we can conceive a scalar to be *added* to a vector; and that the *sum* so obtained, or the combination, '*Scalar plus vector*,' is a quaternion."[2] This is plainly intended here merely as a comment upon one aspect of quaternions, since the conception of a quaternion is first unfolded by him through the operation of finding the quotient of one vector by another vector, as Hamilton calls the operation of ratiofication. But when he establishes the proposition that the general expression for a

[1] *Phil. Mag.*, July, 1844, p. 10.

[2] *Op. cit.*, Book 1, Chap. 1, p. 11.

quaternion is $w+ix+jy+kz$, we are told that "in fact, a quaternion may be *symbolically defined* to be a *quadrinomial expression* of the form, $q=w+ix+jy+kz$."[1] It is now perfectly clear that the proposition: "Scalar plus vector is a quaternion" is nothing more than the symbolic definition in disguise. It gives us no information concerning scalars themselves or vectors themselves; it gives us only information concerning their symbols of value: It informs us that if we put down the *symbol of value* of a scalar, say the value symbol 2, and after it a plus sign, and after these the *value expression* of a vector without the denomination, say $3i+4j+5k$, then we obtain the *value expression* of a quaternion; namely, $2+3i+4j+5k$.

When these aspects of quaternions are contemplated in conjunction with the statements made by prominent writers concerning them, it needs no argument to defend the assertion that obscurity now prevails concerning the nature of quaternions. Some of the workers in the field of Vector Analysis, and not the least renowned, regard them with suspicion and distrust. Oliver Heaviside, for instance, boasts that "there is not a ghost of a quaternion in any of my papers (except in one, for a special purpose)," and says: "A quaternion was, I think, defined by an American school-girl to be 'an ancient religious ceremony.' This was, however, a complete mistake. . . . . A quaternion is neither a scalar, nor a vector, but a sort of combination of both. It has no physical representatives, but is a highly abstract mathematical concept. It is the 'operator' which turns one vector into another. . . . The laws of vector algebra themselves are established through quaternions assisted by the imaginary $\sqrt{-1}$. But I am not sure that any one has quite understood this

[1] *Op. cit.*, p. V.

establishment. It is done in the second chapter of Tait's treatise. I never understood it, but had to pass on."[1] And one even finds a strange oblivion as to the nature of a quaternion with some of those authors who the most strenuously uphold the Vector Analysis of Hamilton in opposition to the rival systems in which quaternions play no part. Thus Prof. C. G. Knott, in a controversial article: *Recent Innovations in Vector Theory,* actually makes the plaint that "Of late years there has arisen a clique of vector analysts who refuse to admit the quaternion to the glorious company of vectors."[2] Can anyone conceive the relation of one vector to another vector as being itself a vector? As to formal definition of "quaternion," there has been no improvement since the time of Hamilton, and the latter at his worst compares favorably with some of the latest works. In the latest revision of the *Encyklopaedie,* for instance, we read: "A quaternion may be regarded as the complex (α_0, α) of an ordinary number or scalar α_0 and of a vector α . . . . . The quaternions include as particular cases the real numbers α_0, if α is null; and the vectors α, if α_0 is null."[3] In connection with the definitions of this type may be mentioned the use of *degraded quaternion* to designate quaternions of the four primitive kinds (which of course have monomials as value expressions) and sometimes apparently to designate these in common with quaternions of such complex kinds as have binomial or trinomial value expressions for their non-zeroes. The application of this title to a relation between vectors would seem to imply that the relation so

[1] *Electromagnetic Theory,* London, 1903, Vol. 1, p. 135.

[2] *Proc. Roy. Soc. Edin.,* Vol. 18, p. 212.

[3] French translation (*Encyclopædie des Sciences Mathematiques*) *Nombres complexes,* article by E. Study revised by E. Cartan, Tome I, Vol. I, p. 404.

designated was a quaternion only by courtesy. But no such implication should be taken; for in reality all relations between vectors of the same sort that appear in the science of Quaternions have an equal right to be called quaternions.

With quaternions, as with all other quantities, accession to membership among that large company, the quantities of Mathematics, grows out of the laying down of conventions, whereby any two quaternions can be compared and can be subjected to the operations of addition, multiplication, etc. Hamilton laid down such conventions, and then proceeded to bring to light the laws which hold sway over these quantities, thereby building the science of Quaternions for which his name is justly famous. We shall here give an outline of the conditions under which two quaternions are said to be equal, but first we shall have to give, as a preliminary, some account of angles and planes, of tensors and versors of quaternions.

When the two vectors of a quaternion are actual and concurrent, the quaternion is said to have an angle of 0°, while if they are actual but contrary, the quaternion is said to have an angle of 180°. The *angle of a quaternion* in other cases of actual vectors, is the minor of the two conjugate angles formed by them when they are coinitial; while if they are not coinitial, the minor angle formed by the correlate vector with a coinitial vector, taken as substitute for the relate and equal to the latter, is regarded as the angle of the quaternion.[1] The plane in which lies the angle of the quaternion is termed the *plane of the quaternion,* provided this angle is not 0° or 180°, and two

[1] Mathematicians usually say: *either* vector and a substitute for the other. But for the sake of clearness of thought, it would seem well to specify that in all cases the correlate must be taken or in all the relate. And we prefer to always take the correlate.

quaternions having the same or parallel planes are said to be *complanar*. When the angle is 0° or 180°, the quaternion is real, and then, in so far as "plane of the quaternion" and "complanar" are used at all, they are given a non-natural significance, as is also the case with a null quaternion, which properly speaking has no angle. In any of these last three cases, the quaternion is apparently regarded as complanar with every other quaternion (and in general, with every abstract quantity) whatsoever; entirely without regard to what is the plane of the latter, or whether it has anything that can properly be called a plane. It is usually stated that a quaternion can act as multiplier on a vector when and only the latter lies in the plane of the former. But, in fact, even with non-protomonic quaternions, the multiplicand vector may lie either in this plane or in any other plane parallel to it. And with protomonic quaternions, if the multiplicand be regarded as lying in the plane of the multiplier, then every plane in space, no matter how it lies, must be regarded as "the" plane of every protomonic quaternion.

Connected with every quaternion whose vectors are actual and neither concurrent nor equal in magnitude, are two other abstract quantities—themselves quaternions —called the *tensor* and the *versor* of the quaternion. We shall give an account of these, somewhat different from the formal definitions laid down by Hamilton, since the latter do not appear to us to bring forth with sufficient clearness adequate conceptions of tensor and versor. Suppose (b, a) is the quaternion, and let a' be a vector equal in magnitude to the correlate vector a, but concurrent with the relate vector b (a' being arbitrarily selected from the value-class thus fixed); then the relation of b to a' will be a new quaternion (b, a') which is called the tensor of the quaternion (b, a). Take, for example, the quaternion of two line vectors, with magnitudes 20 cm. and 10 cm. and with currencies Westward and Northeastward respectively. Here the tensor is the relation of the Westward 20 cm. vector to a vector also Westward but of 10 cm. magnitude; this tensor has the value 2. Next let a'' be a vector which, while concurrent with a, is equal in magnitude to b (and otherwise is arbitrarily selected). Then again we have a new quaternion (b, a''), and this is called the versor of the original quaternion (b, a). In the case of the two line vectors above, the versor is the relation of the Westward 20 cm. vector to a Northeastward vector also 20 cm. in magnitude. Tensors and versors are also ascribed to quaternions whose vectors, while actual, are concurrent or equal in magnitude; but here we must not speak of two *other* quantities. For though when the vectors are concurrent but not equal in magnitude, the above definition applies as to the versor, which is here of value 1, the quaternion is then its own tensor. Analogously when the vectors are equal in magnitude but not concurrent, the usual definitions applies for the tensor, which is of value 1, but the quaternion is its own versor. And when the two vectors are concurrent as well as equal in magnitude, the quaternion is at once its own tensor and its own versor. When the relate of a quaternion is a null vector, and the correlate actual, the tensor ought undoubtedly have the value zero; and probably here again the quaternion should be regarded as its own tensor. What value, however, to assign the versor, which will be a relation between two null vectors, and what tensors and versors should be attributed to quaternions with neither vector actual, remains a moot question. At all events we would take the view that a quaternion has only one tensor and only one versor which are always themselves quaternions.

It can be proven that a quaternion is always equal to (or identical with) the product of its tensor and its versor taken in either order. On multiplying the correlate vector of a quaternion by the tensor, there is obtained as product a vector equal in magnitude to (or identical with) the relate. Because of this the tensor of a quaternion is called the *stretching or shrinking factor*, the operation of multiplying the correlate vector by it being conceived, more or less improperly, as stretching or shrinking this vector until it becomes equal in magnitude to the relate. But a tensor does not always stretch or shrink, even when the correlate is a non-zero, since, as we have seen, it may be of value 1, the correlate vector being then already equal in magnitude to the relate. On multiplying the correlate vector by the versor, there is obtained a vector concurrent with the relate. On this account the versor is called the *turning factor* of its quaternion. A versor, however, does not always turn, even when both vectors are actual. It does not turn when the relate and correlate vectors are already concurrent, the versor then being of value 1. A tensor is always a positive protomonic abstract quantity, and is always equal to the modulus of its quaternion. A versor may be protomonic, neomonic or complex, positive or negative, but it is always a unit of its variety. If the vectors of the quaternion are concurrent, the versor is, as we have said, of value 1; if they are contrary, the versor is of value -1. A quaternion with an angle of $90°$ is said to be a *quadrantal* or *right* quaternion. The versor of such a quaternion is called a quadrantal or right versor. A quaternion of two actual vectors of equal magnitudes, and therefore with a tensor of value 1, is called a *radial* quaternion. And naturally a quadrantal radial or right radial quaternion is one of angle $90°$ with two actual vectors of equal magnitudes.

For two quaternions to be equal, there are, save in certain exceptional cases, four conditions that must be fulfilled, and this was one of the several facts taken by Hamilton as justifying his adoption of the name "Quaternions."[1] The exceptional cases are those of null quaternions and those of protomonic actual quaternions—in other words, of actual quaternions whose angles are 0° or 180°. As null quaternions are regarded as all equal to each other and to all other abstract zeroes as well, there is with them only one condition: that embodied in the very fact of both quaternions being zeroes. With protomonic actual quaternions, there are two conditions requisite for equality: First, the tensors of the two quaternions must be equal; Second the angles must be equal, being with both of 0° or with both of 180°. With all other quaternions, that is, with all neomonic or complex actual quaternions, there are, as we have said, four conditions. These are: First, that the tensors be equal; Second, that the angles be equal; Third, that the two quaternions be complanar; Fourth, it is requisite that (b, a) and (b', a') being the two quaternions, on looking from above toward the plane of the quaternion (b, a) with the vertex of the angle of the quaternion toward the observer, b must be to the right or to the left of a (or its substitute) according as b' lies to the right or to the left of a' (or its substitute) under an analogous view from above. If the first three conditions are fulfilled, but not the fourth, the quaternions are said to be *conjugate* to each other. A quaternion whose angle is 0° or 180° is

[1] From our point of view, there would seem to be *one* necessary and sufficient ground for the use of the name. And this is the fact expressed in our terminology by the statement that Quaternions is a quadruple algebra as it includes a sort with four primitive kinds but no sort with more than four.

its own conjugate, and a like view might perhaps be taken of the conjugate of a null quaternion. The product of a quaternion and its conjugate is equal to the square of its tensor. In Double Algebra "conjugate" is used in a quite analogous sense. Two complex non-zeroes of the same sort whose sum is protomonic and whose difference is neomonic are, in Double Algebra, said to be conjugate to each other. Thus a quantity of value $5+2\sqrt{-1}$ and one of value $5-2\sqrt{-1}$ are conjugates.[1] Two zeroes of the same sort are always conjugate, two neomonic quantities are conjugate when their sum is a zero, and two protomonic quantities are conjugate when their difference is a zero.

The consideration of vectors is not restricted to that special mathematical discipline known as Quaternions. There are other systems of vector analysis; and vectors and their relations can advantageously be considered in ordinary Double Algebra. Indeed we hold that in the latter discipline a scientific treatment of the neomonic and complex (to say nothing of the negative protomonic) abstract non-zeroes can only be attained by giving these quantities their footing as relations of vectors. The very same vectors treated of in Quaternions can be taken up in Double Algebra, but the grouping in sorts is not the same. Where Quaternions has one sort, Double Algebra will have several, in fact innumerable sorts.[2] In both dis-

[1] In Stevin's terminology, two such quantities were said to be *respondant*. See *Les Oeuvres Mathematiques de Simon Stevin*, ed. by Albert Girard, Leyden, 1634, p. 14.

[2] There is, however, the possibility of there being taken up in Quaternions a sort so narrow that Double Algebra is quite adequate for its treatment. In this case the sort would of course be the same in both. But a sort which can be treated by simpler methods is not usually taken up by Quaternions.

ciplines the relations between vectors of the same sort are regarded as quantities, excepting, of course, in the case in which the correlate vector is a zero. As has already been mentioned, all the vectors of Quaternions are either neomonic or complex; but in Double Algebra protomonic vectors also appear. And the self-same vectors may be classed as neomonic or complex in Quaternions and as protomonic in Double Algebra—a paradoxical fact that will be elucidated when we take up the question of the essential nature of the distinction between protomonic ("real") and neomonic ("imaginary") quantities. In Quaternions there are three primitive kinds of vectors to a sort, all neomonic. In Double Algebra there are two primitive kinds to a vector sort—one protomonic and the other neomonic. And the vectors of each sort in Double Algebra must be all complanar. In Double Algebra as well as in Quaternions, for two actual vectors of the same sort to be of the same kind, it is necessary and sufficient that they should be parallel or coaligned. For them to be of the same variety it is necessary and sufficient for them to be concurrent. In Double Algebra as in Quaternions any relation between two actual vectors of the same variety is a positive protomonic abstract non-zero, and any relation between two actual vectors of the same kind but of different varieties is a negative protomonic abstract non-zero. It can be proven that a relation of the latter type is always a square root of a positive protomonic abstract non-zero in this sense, that on multiplying a vector by such a relation and then multiplying the product again by the same multiplier the result attained will be what could be reached by multiplying once by a positive protomonic abstract non-zero—indeed by a *natural* abstract non-zero. A relation between two actual vectors of the same sort but of different kinds is always either a neomonic or a complex abstract quantity. In Double Algebra such a relation if quadrantal is invariably neomonic, but in Quaternions a quadrantal relation may be either neomonic or complex. It is quite commonly implied by mathematicians that the i, j and k "imaginaries" (neomonic abstract quantities) of Quaternions are distinct from the ordinary $\sqrt{-1}$ "imaginaries" of Double Algebra. Cayley indeed goes so far as to say that "the imaginary of ordinary algebra . . . has no relation whatever to the quaternion symbols i, j, k."[1] In fact, however, it is quite possible for identically the same abstract quantity —the relation between precisely the same pair of vectors —to come within the scope of both Quaternions and Double Algebra, and if it is an i, or a j, or a k "imaginary" in the former science, it will necessarily be an "ordinary imaginary" in the latter. Suppose there are three non-zero vectors a, b and c of three different kinds but of the same sort in Quaternions, and suppose that in this science the relation of a to b is an i–imaginary, the relation of b to c a j–imaginary, and the relation of c to a a k-imaginary. In Double Algebra a and b might enter the same sort while still being of different kinds—they would not, however, both be designated precisely as they were in Quaternions, since one of them would be called "real" in Double Algebra though it was classed as "imaginary" in Quaternions—and the relation of a to b would now be an "ordinary imaginary." Double Algebra could not now include in this same sort the vector c.[2] But it could,

[1] *Coördinates versus Quaternions, Proc. Roy. Soc. Edin.*, vol. 20, p. 271.

[2] True, according to the theories of Cantor, there are no more points, hence no more vector values, in three dimensional than in two dimensional space (*i.e.*, the vector values of the former can be put into one-to-one correspondence with those of the latter). Hence

after abandoning the first inquiry, take up another in which *b* and *c* are included in a single sort (a new and different sort), *a* being excluded therefore. And then the relation of *b* to *c* would be an "ordinary imaginary." Finally the second inquiry could be abandoned, and a third taken up in which *c* and *a* are put in a sort from which *b* is excluded. And now the relation of *c* to *a* would be an "ordinary imaginary." There are, moreover, abstract quantities of innumerable *complex* kinds in Quaternions, each of which may appear in Double Algebra as an "ordinary imaginary." Every quaternion with a quadrantal versor, if it appears in Double Algebra, appears as an "ordinary imaginary;" and there are innumerable complex abstract kinds made up of quadrantal quaternions. If radial, a quadrantal quaternion, whether it be an *i*-imaginary, a *j*-imaginary, a *k*-imaginary or a complex quantity, will always appear in Double Algebra as an "imaginary" abstract quantity of unit value (positive or negative); as a $+\sqrt{-1}$ or as a $-\sqrt{-1}$. The result attained by multiplying a vector by a quadrantal quaternion and then multiplying the product by the same quaternion will always be what could be attained by a single multiplication by -1. And hence the quadrantal radial quaternion, if positive is a $+\sqrt{-1}$; if negative, a $-\sqrt{-1}$. There are both positive quadrantal quaternions of innumerable different values and negative quad-

it is quite conceivable that a system of vector analysis might be developed which had only two primitive kinds to a sort, and yet included in the same sort all the line vectors and vectors of points of Quaternions, and in general had precisely the same sorts as that science. But this would be a discipline very different from ordinary Double Algebra, where all the vectors of a sort must be complanar. Under the conditions laid down above, of course, *a*, *b* and *c* cannot possibly be complanar.

rantal quaternions of innumerable different values; so neither $+\sqrt{-1}$ nor $-\sqrt{-1}$ can be used in Quaternions as a value symbol, though in ordinary Double Algebra this is substantially the role they (somewhat improperly) fill. If used in an analogous way in Quaternions, each must be understood to be a polyvalued expression; the one applicable as the name of all positive, the other as that of all negative quadrantal radial quaternions, just as in Double Algebra 2^{π} is a polyvalued expression applicable to quantities of innumerable different values.

We have seen that the classification of vectors into sorts is quite different in Double Algebra from what it is in Quaternions. It will not be without interest to survey the possibilities in the field of classifying quantities, both abstract and applicate, into sorts, kinds, and varieties. To examine the mode of laying down conventions which effectuate such classification is highly instructive, and reveals procedures of the greatest importance in Algebraic Mathematics. Indeed, it is not too much to say that an examination of this procedure leads to conclusions of a very remarkable character with respect to the foundations of mathematical science. There will be occasion in our discussion to distinguish between *entitative* and purely *formal* branches of Algebraic Mathematics. We shall designate an algebra as entitative if the symbols of this algebra denotes entities. *Significant* algebra has been used in this sense as an antonym to formal algebra; but as the symbols of every algebra must have meanings, even though the discipline be a merely formal one, the adjective "entitative" is preferable to "significant." A formal algebra is one whose symbols represent objects of thought which the mathematician has not as yet shown to be entities.

The formation of a class of objects which will constitute a sort of quantities grows out of the laying down of conventions for addition. The conjoining of objects into such a class is among the preliminary steps toward the admission of the objects to membership among quantities; and it is only when *all* the qualifications for such membership are possessed by the objects that the class constitutes a sort of quantities. From its very inception as a quantity, each quantity must belong to some sort, since the entrance into one of these classes is among the preliminary qualifications which must be demanded of an object for eligibility to membership among quantities. In view of what has already been said as to how sorts change on the passage from one branch of Algebraic Mathematics to another, and even from one investigation to another investigation in the same algebra, it is clear that the conventions of addition laid down are to a large extent arbitrary, depending on the fiat of the mathematician. A class of quantities is a sort, if and only if it includes *all* quantities that can be taken into it under the condition that every two quantities so taken shall be of the same sort. In Quaternions, vectors of described straight lines and of points constitute one sort of applicate quantities; linear velocities at points, another; linear accelerations at points, a third; and so on. In Double Algebra, as we have said, only complanar vectors can go to make up a sort. If vectors are admitted in Single Algebra, only parallel or concurrent vectors will be included in the same sort; while Arithmetical Algebra would accept in each of its sorts only concurrent vectors. As to abstract quantities, it would appear to us desirable to have the conventions of addition so laid down that they bring within a single sort all the abstract quantities that appear in a branch of Algebraic Mathematics. To attain this end, it is obviously not enough to lay down conventions for adding natural abstract quantities together, and others for adding together relations of vector pairs of the same sort. There must also be laid down conventions for adding natural abstract quantities and relational abstract quantities together, and conventions for adding a relation between two vectors of any sort to a relation between vectors of any other sort. Such conventions are easily formulated, once it is admitted that relations between vector pairs of different sorts may be equal, and that a natural abstract quantity may be equal to a relation between two vectors of any sort. For an addition which has a relational abstract quantity as summand, even when all the vectors concerned are of a single sort, is usually carried out mediately, not directly, there being taken in such a process instead of one of the summands, or instead of each of them, another quantity equal to it. Thus the addition of the quaternion (b, a) to the quaternion (d, c) may sometimes be carried out by taking instead of (b, a) an equal with c as correlate vector —the quaternion (b', c), while if c does not lie "in the plane" of (b, a) there must be taken *two* new quaternions, both with the same correlate vector: a vector lying in the intersection of the planes of the two original quaternions. The procedure of including within a single sort all the abstract quantities of an algebra is of course not the only one possible. It might be held that an abstract quantity appearing as a relation between vectors of one sort, and another appearing as a relation between vectors of a second sort, themselves belong to different abstract sorts. Every vector sort would then give rise to an abstract sort of its own, and we might further distinguish, as different from each of these, an abstract sort composed of the natural abstract quantities. A natural three then, the number attribute of a group of three objects, would be of a different sort from a relational three appearing as the relation of a vector to another vector of the same sort as itself and concurrent with it. This view is undoubtedly a tenable one, but it seems to us not as much in accord with the current use of such symbols as "3," "*i*," "*j*," "*k*," etc., as that which regards all abstract quantities as of a single sort.

The conventions of comparison which rule the classification of quantities into kinds are entirely independent of the conventions of addition from which arise the classification as to sorts. Nevertheless, be it noted, no mathematician has ever laid down conventions whereby two quantities, such as a length and an area, which are not of the same sort, can be compared and thereby become quantities of the same kind. It might indeed be possible to lay down, in some fantastic manner, such conventions for a length and an area, a velocity vector and an acceleration vector and so on. Then quantities not of the same sort would be of the same kind. But to follow this procedure would be a futile waste of time. Hence we say that only quantities of the same sort are of the same kind.

Since an object cannot properly be called a quantity until after a convention has been laid down by which it may be compared with other objects—also quantities—as to equality and excess, it follows that every quantity necessarily belongs to some kind. As with the conjoining of objects into a class which will form a sort of quantities, so the conjoining of objects into a class which will form a kind of quantities is among the preliminary steps toward the admission of the objects to membership among quantities; and of course it is only when all the qualifications for such membership are possessed by the objects that the class constitutes a kind of quantities. A class of quantities is a kind, if and only if it includes *all* quantities that can be taken into it under the condition that every two quantities so taken are comparable. And hence, in order that a class of quantities shall constitute a kind, it is necessary and sufficient that conventions of comparison shall have been laid down under which each quantity of the class is made comparable with every other quantity of the class, and that no conventions shall have been laid down under which any quantity not of the class is made comparable with *every* quantity of the class. Quantities outside the class may be comparable with some of the quantities in the class, but not with all. Thus in Double Algebra a kind may belong to a sort containing one or more other kinds, and then quantities not of the kind are comparable with every zero of the kind, though not with any of the non-zeroes.

In order that a set of quantities shall constitute a variety, they must all be of a single kind, there must be among them both zeroes and non-zeroes, and the former must comprise all the zeroes of the kind, while the latter must be either all the non-zeroes of the kind greater than the zeroes or all less than the zeroes. It is not obvious that every quantity must necessarily belong to some variety, since it is quite conceivable that a quantity might belong to a kind containing no zeroes. In the most rudimentary state of the mathematical activities of the human race, such kinds did indeed exist; men dealt with the natural numbers before they conceived of zeroes as quantities. But in anything worthy of the name of algebraic science, every kind contains its zeroes; and thus we may assert that each quantity, besides belonging to a sort and to a kind, also belongs to a variety.

Suppose there is under consideration a set of objects that it is desirable to put in the category of quantities. Let us assume that conventions of addition and in general all conventions, save those of comparison, requisite for giving these objects the footing of quantities have been already laid down. Then, some branch of Algebraic Mathematics being understood to take this set as the totality of the quantities with which it deals, a basis for marking out the various sorts of the discipline has been afforded, but with the kinds this yet remains to be furnished. What now must be done to form from among these objects a kind of quantities containing the full complement of two varieties?[1] Obviously five things: First (but not necessarily first in order of execution) there must be selected from among the set certain objects to fill the role of the positive non-zeroes of the kind, and conventions must be laid down making these positive non-zeroes comparable among themselves. There is nothing in the meaning of "kind" and nothing in the meaning of "positive variety," *in so far as the connotations of those terms have been heretofore set forth*, requiring these conventions to be laid down in any particular way. But in all ordinary cases (probably we may even say: in all the cases that occur in the entitative algebras of Mathematics) the quantities accepted as the positive non-zeroes of a kind are made comparable by conventions from which it follows, or by conventions which themselves expressly state, that any two quantities are *equal* if, and only if the ratio of each of them to the other is a one; and of two quantities, that

[1] As a matter of historical fact, an algebra usually grows, and is not brought to a state of completion by any single mathematician or by a group of mathematicians assembled in conclave. And thus it may never have been the case that a kind was formed by a mathematician or mathematicians contemplating the totality of quantities that enter an algebra, and deliberately selecting certain of them to constitute the kind. But for the purpose we have in view in the text, it is best to consider this purely ideal possibility, and leave for the moment the actual historical development of Mathematics entirely out of account.

and only that is *greater* which when taken as antecedent with the other as consequent gives as ratio an abstract quantity greater than one.[1] Moreover, there is never left out of the selection any object of the original set which has, to an object that is selected, a unique ratio of a protomonic value higher than zero. No quantities have ever (so far as we are aware) been brought together, in entitative Mathematics, to form the positive non-zeroes of a kind, whether this kind contains two varieties or only one, which are compared by conditions not of this character And the adoption of other conventions in our present algebras would evidently entail a very considerable alteration, in the form at least, of these disciplines. Of course, if precisely the same set of facts as before were dealt with, the alteration would be merely in phraseology; the facts of mathematical science are immutable, and do not in any way depend upon the volition of the mathematician. The doctrine of certain philosophers, that truth is merely a matter of convenience, is as erroneous in Mathematics as in other sciences. Such a change of conventions of comparison as we have reference to is in itself substantially a mere change in the application of "equal," "greater" and "less," which alone would necessitate solely a change in *the wording* of propositions involving "excess," "equality," "greater," "less," "equal," "kind," "positive variety," etc. But the change would almost inevitably be accompanied by a change in the scope of the disciplines; facts not formerly considered would be taken into account, and others that were originally given attention would be disregarded. So that in actual practice a change

[1] It is to be noted that in the present discussion we are leaving entirely out of account the *transfinites* of Cantor. These we prefer not to designate as quantities.

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in the conventions of comparison or in general a change in any of the important conventions of algebraic science, marks an alteration of algebraic methods. Just how far the conventions of comparison may be varied without overstepping the legitimate bounds of algebraic science—in other words, just how fantastic they may be made without it being advisable to bar the use of "excess" and "equality," of "greater," "less" and "equal" in connection with them—is a matter upon which we shall not presume to pronounce here. It will suffice to say that the application of these names must assuredly be restricted in one way or another; and that it would be a tenable position to hold that the meanings of "equal," "greater" and "less" and hence of "kind," "positive variety," "negative variety," etc., necessitates the laying down of the conventions of comparison in the manner we are now in the course of describing,[1] even though to take this stand may involve a rejection of some of the more or less fantastic algebras that have been devised by mathematicians. It is to be noted that, the conventions of comparison being as above described, the objects taken as positive non-zeroes of each kind must form a class of a certain character. Not every set of quantities is capable of having such conventions laid down for it. Only such sets as possess a property which we shall designate by the name of *confluence* can be made the positive non-zeroes of a kind. The quantities of a set may be said to be *confluent*, and the set may be called a *confluent class* of quantities, if this set comprises all quantities such that (1) each quantity of the set can be ratiofied with respect to every other quantity of the set, and (2)

[1] Aside, of course, from cases where transfinites are concerned.

gives in each case a unique ratio,[1] and (3) a natural abstract non-zero can always be taken as this ratio—a ratio being, as we may recall, a quantity *arbitrarily selected* from a value-class. Evidently confluence of the quantities of a set is a property dependent on the laws of addition that rule with these quantities. The quantities of a confluent class are necessarily all non-zeroes. As an example of a confluent class, we may take any class composed of all actual vectors of a sort that have a certain currency; *e. g.*, all line vectors with currency Eastward constitute such a class, and so do all line vectors with currency Westward.

The second thing to do is to select, from the set that comprises all objects designed to be quantities of the algebra, certain objects to fill the role of the zeroes of the kind, and to provide conventions of comparison by which each quantity of the previous class (the positive non-zeroes of the kind) is made greater than each of the zeroes. It may not be necessary to adopt a second convention for this; thus with temperatures, a single rule suffices for the comparison of all the quantities of the kind, positive or negative, zeroes or non-zeroes. But sometimes a specific convention is necessary. Thus the positive line vectors of a kind are made greater than the vectors of points by a fiat of the mathematician, entirely unconnected with the conventions for the comparison of line vectors of the same kind among themselves.

It may seem paradoxical to say that whether or not a quantity be classified as a zero is often purely a question of convenience. But that this really is the case may readily

[1] The stipulation that there be but one ratio bars from the set quantities whose laws of addition make them apt for designation as the zeroes of their sort. For a zero has, to a zero of the same sort, innumerable different ratios.

be seen by considering the type of quantities exemplified by temperatures. When a convention has been adopted for the comparison of temperatures, and all temperatures have been classed in a single kind, there is still necessary a convention for the selection of a value-class of the temperatures as that of zero value. There is nothing to prevent the choice of *any* value-class of temperatures whatsoever for this purpose, and in fact there are in common use zeroes of three different values. The [relative] Fahrenheit system of thermometry adopts as its zeroes temperatures of one value-class; the [relative] Centigrade and Reaumur take for this purpose temperatures of quite a different value; and finally, in the so-called Absolute systems, temperatures of a third value serve as the zeroes. Like remarks apply to all quantities of the same type as temperatures; that is, resembling the latter in incapability with respect to operations of physical addition. With quantities of the other type, however, necessity for compliance with the law (an axiom as we esteem it) that the sum of a zero a and any other quantity b (zero or non-zero) of the same sort, taken in either order is always equal to or identical with b, prevents us from selecting at our option whatever value-class of a kind we may wish to choose as that of zero value; the matter here is not one of convention at all. The law of course holds even with the pseudo-addition of quantities of the temperature type, but here what value is possessed by the sum of two quantities depends not only on the character of these quantities themselves but also on what value-class be made that of the zeroes. If a and b are the respective temperatures of two bodies—and by temperatures we of course do not mean figures or symbols, but actual attributes of the bodies—and the sum $a+b$ is equal to a or to b under the Centigrade system, this sum under the Fahrenheit or Absolute system will not be equal to either a or b but will be greater than either of them. And in general the addition process with quantities of the temperature type is such that if we choose to *call* the quantities of a value-class zeroes (of course excluding all other quantities of the sort from this title), the sum of a quantity of this class and any other quantity of the same sort is *by definition* equal to or identical with the latter quantity. We may remark that usually with quantities of this type there is but a single kind to a sort, but it would be quite possible to take temperatures, for instance, and another species of attributes that bodies may possess simultaneously with temperatures, and attain therewith the complex as well as the primitive kinds of a sort. And here we could still select at our option the zero value-class for temperatures, and select in entire independence the zero value-class for the other species of attributes (provided the latter were also of the type in question). The quantity thus considered in connection with a body would be a compound attribute, constituted by its temperature and its attribute of the other species; and this quantity would not be a zero unless the body were both at zero temperature and at zero in the other respect. If the temperature were taken as protomonic and the other attribute as neomonic, a body at zero temperature and not at zero as regards the other attribute, would have pertaining to it a neomonic non-zero quantity; while if not at zero temperature, but at zero in the other respect, the quantity would be a protomonic non-zero. And finally, if in neither respect the body was at zero, its quantity would be a complex non-zero.

The third thing to be done is to select objects from among the set to fill the role of the negative non-zeroes, and to make these comparable among themselves. In making this selection we must—that is to say, *must* if the present methods of our algebras are to remain unchanged—so proceed that the objects taken shall constitute a confluent class. But this is not all; it is necessary, unless we are willing to face a widespread change in algebraic methods, to so select those objects which are to form a second confluent class of quantities, that there shall be between the old class and the new a certain relation which we shall designate as *contrafluence*. Two classes of quatities may be said to be *contrafluent* to each other, and every quantity of each class said to be contrafluent to every quantity of the other, if, both classes being confluent, for every quantity in either class there can always be found a quantity in the other class which added to the former gives as sum a zero. Thus Eastward and Westward vectors are contrafluent classes; and so are Northward and Southward. Clearly then the eligibility of two classes of objects as positive and negative non-zeroes of the same kind is to some extent dependent on the conventions of addition that have been laid down for them. Suppose now that the question of eligibility of the objects we wish to make negative non-zeroes has been satisfactorily settled; we cannot (without changing algebraic methods) take at our will any convention of comparison for these objects; the condition must be fulfilled that any two quantities of the class are *equal* if the ratio of each of them to the other is a one, and of two quantities that is the *less* which when taken as antecedent with the other as consequent gives as ratio an abstract quantity greater than one. No quantities have ever been brought together, in entitative Mathematics, to form the negative non-zeroes of a kind which are compared by conventions not meeting the conditions laid down above, and conformity of the conventions of comparison to these conditions might reasonably be ascribed as part of the meaning of "negative variety."

The fourth thing to be done is to provide conventions under which every quantity of this second set of confluent quantities will be less than each of the quantities taken as the zeroes.

As fifth requisite, there must be provided conventions under which every quantity of the first confluent set (the positive non-zeroes) will be greater than each quantity of the second confluent set (the negative non-zeroes).

Finally to the five requisites of commission we must add one of omission; no conventions may be laid down which make any quantity not taken in forming the kind comparable to every quantity that is taken. And, of course, if there is to be adherence to the methods that rule in the present entitative algebras, no conventions may be laid down under which any quantity not taken is made comparable to even a single non-zero that is taken; and none may be laid down under which a quantity of the kind is made comparable with any quantity not included in that sort to which this kind belongs.

There are also, as we have seen, kinds embracing only one variety each. Indeed, all those first brought into service by mathematicians were of this character, containing only zeroes and positive non-zeroes. Such a kind is formed when the first two of the five things stipulated above have been done, but the other three have not. Kinds of one variety each might also be formed composed of zeroes and negative instead of positive non-zeroes. It is true that no such kind of quantities has ever been contemplated by mathematicians. The only kinds of quantities containing negatives of which they have treated are kinds containing positive as well as negative non-zeroes. But a kind comprising a negative variety alone is quite conceivable in the absence of any definite decree by mathematicians establishing that part of the meaning of "negative quantity" requires a quantity so designated to belong to a kind containing also positive non-zeroes. To form a kind including a negative but no positive variety we must first select objects to fill the role of negative non-zeroes, and lay down conventions making them comparable among themselves. The set of quantities thus formed must be confluent, but contrafluence to another set would not be a requisite. The conventions of comparison adopted must be such that two quantities of the class are equal when, and only when, the ratio of each to the other is a one; and that one quantity of the class is less than another when, and only when, the ratio of the former to the latter is greater than one. Second, we must select objects to fill the role of the zeroes of the kind, and lay down conventions by which each of these is made greater than every quantity of the former set.

To establish a set of objects in the position of zeroes or positive non-zeroes or negative non-zeroes of a kind (after the requisite conventions other than those of comparison have been provided) it is sufficient and, if we would hold fast to the methods that rule in our present entitative algebras, *necessary* to lay down for them conventions of comparison of the character stipulated above. As a finishing touch comes the representation of the values so formed. In all the primitive kinds each positive non-zero value represented by an expression having either a $+$ affix (a "plus sign") at its beginning or no affix at all, while each negative non-zero value is represented by an expression having at its beginning a $-$ affix (a "minus sign").

It is quite clear that, even with the above restrictions, whether an object, which is to play the part of a non-zero quantity in a mathematical investigation, shall be positive or negative is, to a large extent dependent on the volition of the mathematician. What is called a positive quantity might often just as well have been a negative. This is sufficiently obvious as regards a kind of vectors. Take, for example, the two classes composed respectively of all Eastward line vectors and of all Westward line vectors. Each of these classes is confluent, and the two are contrafluent to each other. We can make either one of these classes the positive non-zeroes of a kind, and the other the negative non-zeroes of the same kind. No one will deny that a mathematician may, as suits his convenience, either make Eastward line vectors positive and Westward line vectors negative or make the Westward ones positive and the Eastward ones negative. But a qualification must be made. The line vectors with currency Eastward and with currency Westward belong to a sort containing line vectors of all currencies. If a certain number of kinds and the positive and negative varieties of each of these kinds have been carved out of this sort, then with some of the remaining kinds capable of being carved from the same sort, there is left no freedom of choice between the two confluent (and mutually contrafluent) sets of line vectors, one of which must be taken as the positive non-zeroes of the kind, the other as the negative. If Northward and Eastward vectors are positive in their respective kinds, and Southward and Westward negative, we cannot in another kind of this same sort make Southwestward vectors positive and Northeastward vectors negative, though we can still deal as we please with vertically upward and downward vectors. Northeastward vectors must perforce be made positive and Southwestward negative, unless one is willing to infringe the rule that the sum of two positive quantities shall always be positive, and the sum of two negative quantities always be negative. In general, when a sort contains n primitive kinds, the choice as to positive and negative non-zeroes is free with n kinds of the sort, and not free with the remaining kinds. It will usually be convenient to make the choice with the n primitive kinds themselves. But obviously this is not absolutely necessary; the choice might just as well be made with n of the complex kinds. Thus with a vector sort of Quaternions, the choice can be made with any three kinds of vectors, whether these be primitive or complex, *provided the three kinds are not complanar.*

Turn now to the abstract quantities. How far is there here a freedom of choice with the primitive kinds as to positive and negative? (Since the choice, when there is one, can and, for the sake of convenience, always will be made with the n primitive kinds, we need not concern ourselves with the possibility of its being made with n kinds not primitive or not all primitive). With the neomonic kinds there is no doubt at all that the choice is entirely free—that, though we assume the protomonic positive and negative to be fixed, the fixing of the neomonic positive and negative depends solely on the fiat of the mathematician. In the Argand vector analysis of Double Algebra, with vectors in the plane of the paper before us, the relation of a vector with currency toward the right to a perpendicular horizontally upward vector is assigned a positive neomonic value while the relation of the vector with right hand currency to a perpendicular horizontally downward vector is assigned a negative neomonic value. But this could easily be reversed by mere convention, relations of the former type being, at the option of the mathematician, made positive and those of the latter made negative. In Quaternions the arbitrary nature of the classification of neomonic abstract quantities as positive and negative is even more obvious. For Hamilton adopted as positive direction of rotation that which is followed by the hands of a clock, while Tait and later quaternionists took the reverse direction of rotation as positive. And this amounts to saying that the relations (quaternions) which Hamilton designated as of value $+i$ and $+j$, and classed as positive, were by Tait classed as negative, and had their values represented by $-i$ and $-j$ respectively; and vice versa Hamilton's negative quaternions of value $-i$ and $-j$ were by Tait made positive, and given the value symbols $+i$ and $+j$.[1] In general, with each set of abstract non-zeroes that goes to make up a neomonic kind, the mathematician has, up to the time the conventions of comparison are laid down, merely two confluent and mutually contrafluent sets of quantities before him, and he can at his will so lay down these conventions as to make either set positive and the other set negative. As to the protomonic abstract varieties, the case is somewhat different. If we adhere to the rule of laying down the conventions of comparison between the non-zeroes of each positive variety in such manner that two of these quantities are equal when and only when the ratio of each to the other is a one, and one of them is greater than another when and

[1] Or rather *either* the Hamiltonian abstract varieties of the i and the j kinds, or those of the j and the k kinds, or those of the k and the i kinds were interchanged in Tait's scheme, the third primitive kind having its varieties unaltered. *Which* are altered and which remain unaltered depends on how the primitive kinds of vectors are taken. Of the three primitive kinds of Hamiltonian vectors in each sort (the I, J and K kinds) one kind, it is immaterial which, must have its varieties interchanged in the passage to Tait's system, and the others must remain unchanged. If the K vector kind has its varieties undergo interchange, the i and j abstract kinds likewise have their varieties undergo interchange, while the k abstract kind is unaltered etc.

only when the ratio of the former to the latter is greater than one; while we lay down for the non-zeroes of each negative variety a like stipulation as to equality, but stipulate that a non-zero of such a variety shall be less than another when and only when the ratio of the latter to the former is greater than one, then we are *assuming as preexistent* a convention of comparison concerning the natural abstract non-zeroes and the relational abstract quantities equal to natural abstract non-zeroes—a convention of comparison that obviously does not come under the negative head, and hence, if it be admissible at all, must necessarily come under the positive head. The very statement of the rule then implies that if it is adhered to and each confluent set of quantities subjected to it made positive (that is, has its quantities made greater than the zeroes of the sort) we are obliged to make *this set of protomonic abstract quantities* positive. And hence the contrafluent set of abstract non-zeroes, the relations between contrary vectors of the same sort, must, if it is to be included in the same kind, be made negative.

The foregoing considerations suffice to show how far the classification into positive and negative is arbitrary with a kind containing two varieties. Kinds exist, however, which contain only one variety each, there being no set of objects fit to fill the role of a class of quantities contrafluent to the non-zeroes of such a variety.[1] All concrete kinds are of this description, and among applicate kinds answering to it, we may mention lengths, areas and

[1] We leave out of account, as of no especial interest here, kinds which, though of only one variety, have contrafluent to the non-zeroes of that variety a suitable set of objects, so that the variety could at will be included in a two-variety kind. The conclusions reached in the text apply equally well to such kinds, whose natural habitat is Arithmetical Algebra.

volumes. These single-variety kinds were those first contemplated by mathematicians, who have always looked upon them as necessarily positive. With each of them the non-zeroes have invariably been held to be greater than the zeroes, and the conventions of comparison for the non-zeroes among themselves have always been laid down in the manner characteristic of the positive non-zeroes of a kind. It may be held that this is necessary; that part of the meaning of "negative non-zero" requires such a quantity to belong to a kind containing also positive non-zeroes, and to necessitate putting under the head of positive non-zeroes the quantities of any confluent set for which there cannot be found another set of confluent quantities contrafluent to the former. But if this contention be not made (and no mathematician has as yet put it forth), any class of denominate quantities which has hitherto been appointed to play the part of the positive non-zeroes of a kind could play that of the negative non-zeroes of a kind without doing any violence to the laws of Mathematics. It is quite possible, with the quantities of any single-variety kind, to agree that each non-zero shall be accepted, not as greater, but as *less* than the zeroes, and that conventions of the negative character shall be laid down for the comparison of the non-zeroes among themselves. We would then have a kind containing only negative quantities. Thus take areas; let it be decreed that each non-zero area shall be less than every zero area, and that two non-aero shall be esteemed equal if the ratio of each to the other is a one, but that one of them shall be esteemed less than the other if the ratio of the former to the latter is greater than one. Areas being then all negative, those of them which are non-zeroes would naturally be represented by value expressions beginning with a negative affix (a "minus sign"). An area which is at present represented by the value symbol 10 sq. cm. would then be represented by -10 sq. cm., and those areas which are now greater than this area would be less than it. For example, an area which is now represented by 20 sq. cm., and is greater than the former area, would be represented by -20 sq. cm., and would be less than the other area. And so with every other denominate kind comprising only one variety; this variety can just as well be made negative as positive.[1] No advantage at all would accrue on making it so, but likewise there would be no disadvantage. It is quite immaterial for the purposes of Mathematics whether the quantities of such a kind be made all positive or whether they be made all negative.

From the view here taken of the character of positive and negative quantities, the remarkable conclusion has been drawn with regard to applicate quantities such as lengths, areas, volumes and so on, and also with regard to concrete quantities: that they can serve either as positive or as negative quantities depending on the conventions adopted for comparing them. But a still more remarkable conclusion now confronts us. In the formation of kinds, mathematicians have never hitherto (aside from

[1] If this were done with a concrete kind, the question would naturally arise whether a group which was a concrete quantity of this kind could not be regarded as having a negative number attribute, and whether there could not in this manner appear negative abstract non-zeroes which were not relational. In the absence of any actual attempt by mathematicians to deal with concrete kinds in this way, we have not deemed it necessary to take this possibility into account when making the statement that all abstract quantities, save such as are positive and protomonic, exist only as relations. But we do not deny that natural negative abstract non-zeroes may *conceivably* be brought into Mathematics in the way just outlined *provided mathematicians are willing to accept negative non-zeroes which do not have contrafluent to them other quantities constituting the positive non-zeroes of the same kind.*

the including in a kind both "finite" quantities and Cantorian transfinites—a case that we need not as yet consider) gone further than to bring together with the zeroes two classes of confluent non-zeroes, each of which is contrafluent to the other. And, as we have already had occasion to say, if one of two confluent and mutually contrafluent sets of quantities be made positive, the other set, if it be included in the same kind, must be made negative. The "must" is a consequence of one's determination to adhere to the methods of our present entitative algebras. Suppose, however, we are willing to deviate from these methods, and adopt conventions of a character never before admitted into Mathematics. Then after making comparable in the same manner (for instance, that manner characteristic of the positive non-zeroes of a kind) within the set, the quantities of both sets, we might include both of these in the same kind without making them belong to different varieties, by merely putting forth the fiat that each quantity of one of the sets shall be deemed greater than every quantity of the other, and that those of both sets shall be alike deemed greater (or less) than the quantities taken as the zeroes of the kind. Thus we might take Eastward line vectors, and make them comparable by the positive method, under which, for example, an Eastward line vector of 20 cm. magnitude would be greater than one of 10 cm. magnitude. Then we might take the contrafluent set of Westward line vectors, and make these likewise comparable among themselves by the positive method, but decree that each Westward line vector shall be greater than every Eastward line vector, and that every Eastward and every Westward line vector shall alike be greater than every vector of a point. In this event a Westward line vector of 20 cm. magnitude would be greater than a Westward line vector of 10 cm. magnitude, *which would be greater than an Eastward line vector of* 20 *cm. magnitude,* which would be in turn greater than an Eastward line vector of 10 cm. magnitude, and finally this last vector and the three others would be greater than the vector of a point. And we might in like manner take a third confluent set, for example, Northward line vectors, compare its quantities among themselves by the positive method, and then make each of these quantities by our fiat greater than the zeroes and the quantities of the two previous sets. And we could proceed thus indefinitely, and include innumerable different sets of confluent quantities in a single positive variety, which need have no negative variety in its kind. Likewise innumerable different sets could be included in a single negative variety. This is one of the ways in which, at the option of the mathematician, negative quantities or positive applicate quantities could be entirely excluded from Mathematics. By still further deviating from present algebraic procedures, what we have termed the positive and the negative methods of fixing conventions of comparison could be abandoned, and sets of quantities, compared among themselves in some other fashion, might be made positive or negative; that is, greater than the zeroes of the sort or less than these. But into this possibility we need go no further.

To the view of positive and negative quantities which has just been set forth, any one will be driven who endeavors to attain an adequate view of their nature; not confining his attention to the protomonic abstract quantities alone, but bringing into consideration all types of quantities that enter Mathematics. So general a view of the matter is not ordinarily taken however. To-day the highest mathematical authorities, in contrasting negative quantities with positive, are satisfied to take into account negative protomonic ("real") abstract non-zeroes alone, or even solely a species of these: the negative protomonic abstract integers. In defense of this course, it may be argued that these quantities suffice to illustrate the distinguishing characteristics of negative quantities. That such is not the case, is in our opinion quite evident from the foregoing discussion. Indeed, even those negative quantities that mathematicians do purport to take into account, are not brought clearly into view; a delineation of the nature of the entities that come under the head of negative protomonic abstract non-zeroes being entirely lacking in mathematical text-books. Instead of speaking of negative real abstract non-zeroes, mathematicians ordinarily employ the name "negative numbers"—a crude term, quite unsuited to the present state of Mathematics—and most authors explicitly state that the negative numbers are mere symbols. Those who do not, clothe their ideas on the matter in language so vague as to make it almost impossible to determine what view they do accept as to the nature of such quantities. In the conventional treatment much stress is laid upon the Principle of Permanence, the laws that rule the negative real abstract quantities being derived by its sanction, instead of being derived from the properties possessed by those entities which enter Mathematics under that name. A relatively superior exposition of the prevailing views is to be found in the treatment given the matter by Schubert. One of his latest articles: *Grundlagen der Arithmetic,* as it appeared in the French version of the *Encyklopaedie der mathematischen Wissenschaften,* has had the revision and endorsement of Tannery and Molk (who, like Schubert, are mathematicians of the highest standing) to say nothing of the sanction of the other mathematicians who are responsible for this very authoritative work. From Schubert then we shall quote. In order to place his views of "negative numbers" in the proper light, it is necessary to put beside them his views on "number," which is a term he employs sometimes to denote the natural numbers alone, sometimes to denote these and the concrete non-zero integers, sometimes even to denote quantities of any character whatsoever. The term is also used by him—perhaps we should say: *most frequently* used by him—to denote the symbols of such quantities. For in his doctrine the symbols are given predominance over the quantities they represent, in so far at least as there is in evidence any distinction between the two. A clear distinction he never seems to attain. Indeed, obscurity of thought would appear to go hand in hand with the use of "number" which affords some of the worst examples of laxity in all of mathematical nomenclature. The following is from Schubert's essay on the *Notion and Definition of Number*[1] which sets forth his view of number more fully and lucidly than does his later articles in the *Encyklopaedie*, the two articles being, however, essentially in accord in their doctrine. "To count a group of things is to regard the things as the same in kind and to associate ordinally, accurately, and singly with them other things. In writing, we associate with the things to be counted simple signs, like points, strokes, and circles. . . . . We must add to the definition of counting given above a third factor or element which, though not absolutely necessary is yet very important, namely, that we must be able to express the results of the above-defined associating of certain other things with the things to be counted by some conventional sign or numeral word. . . . .

[1] *Mathematical Essays and Recreations*, trans. by Thos. J. McCormack, Chicago, 1903.

Having thus established what counting or numbering means, we are in a position to define also the notion of *number*[1] which we do by simply saying that by number we understand the *results* of counting. These are naturally composed of two elements. First, the ordinary number-word or number-sign; and secondly, of the word standing for the specific thing counted. For example, eight men, seven trees, five cities. . . . We are ultimately led in our conception of number to abstract wholly from the nature of the things counted and to form the definition of unnamed number. . . . The preceding reflections have led us to the notion of unnamed number or abstract numbers. The arithmetician calls these numbers positive whole numbers, or positive integers, as he knows of other kinds of numbers, for example, negative numbers, irrational numbers, etc. Still, observation of the world of actual facts, as revealed to us by our senses, can naturally lead us only to positive whole numbers, such only, and no others, being results of actual counting. All other kinds of numbers are nothing but artificial inventions of mathematicians created for the purpose of giving to the chief tool of the mathematician, namely, arithmetical notation, a more convenient and more practical form, so that the solution of the problems which arise in mathematics may be simplified. All numbers excepting the results of counting above defined, are and remain mere symbols."[2]

[1] "Number" would here mean *concrete integral non-zero* if the author clearly distinguished between symbols and the quantities they represent.

[2] If Schubert, as is probably the case, intends here to give "number" the significance of *quantity* taken in the widest sense, the question arises: Does he forget lengths, areas, volumes, etc. (to say nothing of vectors and relations between vectors), or does he regard these as symbols, or does he cast them out from among quantities?

Any one who looks beyond words, and seeks to ascertain what an author means by his verbiage, will, upon the most cursory examination, find some remarkable blunders in these passages of Schubert's. "Number" Schubert first defines as the result of counting, and cites as the results such *names* as "eight men," "seven trees" and "five cities" which are then *names* of concrete integral quantities, or "numbers," and are not the *concrete quantities or numbers themselves.* Then he says that by abstraction in counting we are led to "unnamed" or "abstract numbers." To say, as Schubert does, that "the arithmetician calls these numbers positive whole numbers, or positive integers, as he knows of other kinds of numbers, for example, negative numbers" is at least to imply that positive whole number and abstract numbers are one and the same. The proper designation, however, is not "positive whole numbers," but "positive protomonic [non-zero] abstract whole numbers," as they are only one species of abstract quantities. Consistently with the statements concerning concrete integral quantities, it ought to be said that the positive abstract numbers are such *names* as "eight," "seven" and "five," which are the *names of the numbers,* and not the *numbers themselves.* Schubert finally declares, however, that the "positive abstract whole numbers" are *the only quantities which are not mere symbols,* since they are the only quantities resulting from counting. This appears to us a very insufficient argument to show that all quantities excepting the "positive abstract whole numbers" are mere symbols. We are unable to comprehend how the names "eight," "seven" and "five" are in any way better entitled than the names "eight men," "seven trees" and "five cities" to be regarded as more than mere symbols. Both sets of names can be attained in processes of counting. But in both cases we hold it to be erroneous to say these names are quantities; it is what these names denote that are quantities.

That any mathematician should speak of the quantities of the science as mere symbols, is really astounding. But those who take such a view seem never to have seriously considered what it means to say we are dealing with words or symbols merely. Such a statement can only mean that sentences containing the word or symbol in question are conversant solely with its *suppositio materialis*—with the spoken logophone or written logograph. This is the case when we say "Abracadabra is composed of five syllables" or "England is written with a capital." It is not the case with the proposition of any branch of Mathematics, whether this discipline be entitative or purely formal. The assertions made in Mathematics are not of the character of the assertions of Lexicology or Grammar. A confusion between two so widely separated types of inquiry would hardly seem excusable even in a very stupid school boy. And yet again and again one finds authors of good repute making, like Schubert, statements which are indubitable evidence of such confusion. Thus, Tannery gives the readers of his works the following misinformation as to the nature of an "integral number:" "Here is a bag of marbles, a flock of sheep, some letters forming a word, some words that form a phrase: how many marbles are there in this bag, how many sheep in this flock, letters in this word, words in this phrase? The reply to these questions is a *number*[1] or to be more precise an integral number. The idea of integral number is attained by

[1] A less slovenly wording would be "The *replies* to these questions *are numbers*," but we of course reproduce exactly what Tannery himself said.

abstraction from the idea of a collection of distinct objects."[1] As the reply to such a question will be a proposition, abbreviated on occasions into a word (the name of a natural number value), one might expect a consistent adhesion to the view just quoted to lead up to a definition of addition as an etymological operation somewhat like apheresis, syncope and apocope; while subtraction, one would think, would be analogous to prosthesis, epenthesis and paragoge. Mathematicians, however, do not always count consistency as a virtue. The last sentence quoted would seem to say that by abstraction from the idea of a flock of ten sheep (for example) one may attain the idea of the *word* "ten" or the idea of the *proposition* "There are ten sheep in this flock;" a view as to the nature of abstraction which has at least the merit of novelty. Another curious view of number, which may well find mention here, is that of Kronecker. This eminent mathematician actually holds that the number of a group of objects is a group of numeral words; that when there is a group of five apples before us, the number belonging to that group is another group composed of the five words: *first, second, third, fourth, fifth!*[2]

From this notice of the remarkable views as to the nature of "number" prevalent among mathematicians of the highest eminence, we pass on to consider Schubert's conception of the way "negative numbers" gain entrance into Mathematics. We quote first his formulation of the Principle of Permanence or the Principle of No Exception, as he prefers to call it. "Arithmetic follows a prin-

[1] *Leçons d'Arithmetic theorique et pratique,* by Jules Tannery, 2nd. Ed., Paris, 1900.

[2] See *Ueber den Zahlbegriff* by Leopold Kronecker, in *Philosophische Aufsaetze, E. Zeller . . . gewidmet,* Lpz., 1887, p. 265.

ciple, that is called the Principle of Permanence or of No Exception and consists of four parts: First, to impart to each combination of symbols, which represents none of the hitherto defined numbers, a meaning such that the combination can be treated by the same rules as could be applied to it if it represented one of the hitherto defined numbers: Second, to define such a combination as a number in the widened sense of the word, and thereby to widen the conception of number; Third, to prove that, for the numbers in the widened sense, the same laws hold as for the numbers in the not yet widened sense: Fourth, to define what equal, greater and less mean in the widened number realm."[1] This statement of the principle would be more lucid were it not marred by a confusion between symbols and what they represent. When Schubert speaks of a combination of symbols as represented or not representing a number, he would not seem to regard a number as being a symbol or combination of symbols. But when, in his next clause, he speaks of defining a combination of symbols as a number, he implies that a number is an expression of a certain character. He could have attained consistency by saying that the entities chosen to be represented by this combination are to be defined as numbers—or, if one wishes to be really correct on his language, *classified* as numbers. But nothing like this has any connection with the most important office of the Principle of Permanence. The fundamental applications of this principle are concerned with something that cannot reasonably be read into Schubert's words at all; they do not bear directly on the quantities represented by symbols and compound expressions; they concern the *operations* indicated by *signs* or indicated by a positional notation

[1] *Grundl. d. Arith.*, p. 12, in the *Encyklopaedie.*

as in a^n. To take a specific example of the workings of the principle, suppose we are at a stage in the development of Mathematics where a^n, whenever it has any meaning at all, is invariably synonymous with $aaaa. . . .$ to n factors; then the expression a^π indicates no operation at all, and is meaningless. Here the Principle of Permanence bids us find, *not a quantity to be denoted by* a^π, but *an operation suitable for indication by* a^π. And mathematicians have done this. Before it was done a^π had no meaning at all; afterward (a meaning any abstract quantity, and π having its usual meaning) a^π has a meaning even though it has no denotation—even though no entity fulfills the conditions requisite of a quantity to be accepted as a result of the operation indicated by a^π and to be hence denoted by this expression. It is quite incorrect to say that by such a process we introduced a *new number* into Mathematics; what we do is to introduce a new *type of operation* (commonly a compound, derived from the already known simple types; thus the operation indicated in a^π is a compound operation involving the performance of addition, multiplication, division and finding the limit), this new type being, however, in common with an older type or types, indicated by an old sign or an old positional notation. The Principle of Permanence, which Schubert says "created the negative numbers from $a-b$ where a is not greater than b,"[1] can be brought to bear upon the quantities represented by algebraic expressions only in virtue of its having a secondary office very different from the primary one. This second part of the principle, on attempting to put into precise language the more or less obscure locutions of the text-books, would seem to be a precept bidding the mathematician to turn a formal

[1] *Grundlagen*, p. 19.

algebra into an entitative ("significant") one whenever he is able to do so—to find entities that the symbols and compound expressions of the formal algebra can denote without infringing the laws of this algebra. Thus construed, the principle commands the mathematician (besides bringing new operations into play) to seek entities fit to be designated by 5–9, by $+\sqrt{-1}$, etc., and to so designate these entities, thereby giving each expression (which, be it noted, has already a *meaning*) what without this process it does not possess: namely a *denotation*. This secondary precept, and this alone, appears to be what Schubert has vaguely in view in the definition of the principle cited above, the primary precept being strangely ignored by him, though it assuredly is of equal if not greater importance than the secondary.[1] But whether both offices or only one be taken into account, the Principle of Permanence can properly play no part whatsoever in the scientific exposition of an entitative algebra as a deductive science. Nor indeed has it anything to do with the exposition of the laws of a formal algebra. Its proper use is as an instrument of discovery. Far too often, however, this Principle of Permanence plays the role of a mere pretentious set of words invoked by the symbolist

[1] In view of the fact that German mathematicians usually give Hankel credit for the first *adequate* formulation of the Principle of Permanence (Peacock's earlier Principle of Permanence of Equivalent Forms being, they hold, of merely historical importance) it may be of interest to note this formulation, which was put forth by Hankel under the name of the Principle of Permanence of Formal Laws, and reads as follows, the italics being ours, "when two forms expressed in general symbols of the arithmetica universalis are equal to each other they must also remain equal when the symbols cease to designate simple quantities and therefore *the operations* also bear some other significance." See *Vorlesungungen ueber die complexen Zahlen und ihre Functionen,* Part 1.

to save himself the trouble of inquiring what entities it is that his symbols can denote—in other words, as an excuse to justify his disregard of this very principle! Let us now see what use Schubert and his collaborators, Tannery and Molk, make of it in the *Encyclopedie.*

The Principle of Permanence, we are told, "permits the introduction of negative numbers. Let us designate, for this purpose, by A the class formed solely of the numbers 0, 1, 2, 3, ...; let us employ for the moment the letters a, b, . . . to signify only such numbers; let us suppose established the properties of addition and of subtraction for these numbers, and let us take again for the signs $+$, $-$ the significance relative to these operations. The symbol (*sic*) $a–b$ has meaning only if a is greater than or equal to b. Whether a is greater than, equal to, or less than b, we agree to call *number* the expression $a–b$ formed in reality by the two constituent numbers, a, b, which, it may be remarked, do not play the same role. *With respect to this new species of numbers all definitions must be reconstructed.* When the two symbols $a–b$, $a'–b'$ have a meaning, it is quite easy to derive from propositions concerning addition and subtraction that the equality $a+b'=a'+b$ is the necessary and sufficient condition of these two symbols representing the same number. It is natural to *define* in all cases by this condition the equality of the symbols $a–b$, $a'–b'$, noticing that this definition satisfies the conditions imposed by every condition of equality. It results from this, that if (a, a' being respectively smaller than b, b') $b–a$ and $b'–a'$ are equal to the same number c, the two new numbers $a–b$, $a'–b'$ are equal; they are equal to $0–c$. Instead of $0–c$ we agree to write $–c$, and it is under this abridged form that the new numbers which are called *negative numbers,* can be represented. . . . In antithesis to the negative numbers, we designate under the name *positive numbers*, the numbers such as c or $0+c$."[1]

Schubert's article in the original German *Encyklopaedie* did not treat the topic in quite so much detail as does the article of the French revision, but as to doctrine there is no essential difference between either of these reductions and Schubert's earlier article on *Monism in Arithmetic.* In the latter he begins by considering 5–9 as originally a "symbol" wholly destitute of meaning, and states that "As the form of the symbol 5–9 is the form of a difference, it will be obviously convenient to give it a meaning which will allow us to reckon with it as we reckon with every other real difference, that is with a difference in which the minuend is greater than the subtrahent." His idea of imparting a "meaning" to it does not, however, go beyond the laying down of conventions under which "all such symbols in which the number before the minus sign is less than the number behind it by the same amount may be put equal to one another," after which he tells us triumphantly that "we have invested thus, combinations of signs originally meaningless, in which a smaller number stood before than after a minus sign, with a meaning which enables us to reckon with such *apparent* differences exactly as we do with ordinary differences."[2] Likewise in the original German *Encyklopaedie*, after laying down the conditions under which a "difference form" $a–b$ with which $a<b$ can be equated to another such difference form, Schubert says: "Finally by calling such difference forms also numbers, one widens the number concept and attains the introduction of the *negative numbers*."[3]

This, then, is the doctrine of negative protomonic abstract quantities put forward under the auspices of an interna-

[1] *Op. cit.*, Tome I, Vol. 1, p. 34.
[2] P. 12 and 13. [3] P. 12.

tional body of mathematicians of the highest eminence. And it assuredly cannot be deemed a satisfactory one; indeed, it is so permeated with error as to be quite unworthy of those who stand sponsor for it. Taken, apparently as unfolding the nature of negative quantities in general, it in fact brings to light not one of their characteristics. From it one gets no inkling that a quantity is negative on account of the mathematician having more or less arbitrarily laid down conventions of comparison of a certain character for this quantity and its congeners, the other quantities of the same kind. Still less does the doctrine tell us that, of the various conceivable ways of laying down such conventions, a specific one is adopted in our present entitative algebras—one that might to advantage be made the basis of the very definition of "negative quantity"—and that the conventions of comparison cannot be laid down in this way unless certain conditions are fulfilled. To Schubert, anything like the conceptions of confluence and contrafluence seems wholly unknown. And the view, which he appears to hold, that negative quantities are brought into existence primarily to obviate the difficulties which arise in attempting to subtract a lesser quantity from a greater, is in no wise tenable. With the most characteristic of all sorts in which the distinction of positive and negative is of moment—the vector sorts—such difficulties have never arisen. The entire theory of the addition and subtraction of vectors of any sort can be and always is established without the slighest consideration of the distinction between positive and negative, and even without the slightest consideration of whether or not the operands are of the same kind. Whether the minuend is greater than, or equal to, or less than, or incomparable with the subtrahend is of no moment whatsoever in the subtraction of one vector from another of the same sort. And the like is true of subtraction applied to the relations between vectors. There is no need of calling in the aid of the Principle of Permanence when we classify as a negative quantity the relation of one actual vector to another of the same sort and contrary to the former. These relations are not classified as negative abstract quantities in order to render possible the performance of subtraction with minuend less than subtrahend, taking, as operands, relations between concurrent vectors of the same sort or natural abstract quantities. And how anyone can contend, even for a moment, that either relations between vectors or the vectors themselves are mere symbols, "artificial inventions of the mathematician, created to give arithmetical notation a more convenient form" we are unable to understand. But Schubert evidently never realized what the negative abstract quantities are; to conceive them as relations between applicate quantities seems never to have been dreamt of by him. His vision of negative quantities took in nothing further than what arose by the step he called investing with a meaning expressions originally meaningless—a step which is not in any proper sense the introduction of negative quantities into Mathematics. It is not even a carrying out of the behests of the Principle of Permanence, either in its primary precept or as extended to apply to quantities as well as to operations. It is *not* the finding of entities that the negative expressions of a formal algebra can denote, and the producing of an entitative algebra by giving these denotations to the expressions while also giving suitable denotations to the other expressions of the formal algebra. It is not even the investing of meaningless expressions with meanings. For *before* any such step is taken, 5–9, or in general a–b where a is not greater than b, has a meaning, that is to say, it has a connotation though it be as yet lacking a denotation. The expression 5–9 is the name of an object (of every object, we should say) which on having a nine added to it gives as sum a five; and $a-b$ is the name of an object which on having b added to it gives as sum a quantity equal to or identical with a. Whether there are already at hand quantities of this description; or whether, though there are not, objects can be found answering to it which may be put in the category of quantities at our option; or whether no such objects can exist, is quite immaterial. For in any event this *is a description;* the expression has a meaning. *And the step Schubert takes does not change the connotation of the expression one iota; neither does it give or take away or in any way change a denotation.* Why he should regard the making 5–9 equatable to 7–11, to 1–5, etc., as giving a meaningless expression a meaning, and as a process of any particular importance in the delineation of the negative quantities, we are utterly unable to fathom. Of one thing at least there is no doubt; namely, that many mathematicians would attain a much greater clearness of thought if they took the trouble to read carefully the first book of Mill's *Logic,* and to master the elements of the theory of definition.

The question of positives and negatives having been disposed of, we may next inquire into the manner in which kinds are classified as primitive or complex, and as protomonic primitive or neomonic primitive. The only denominate sorts we need here consider are vector sorts; no other denominate sorts containing complex kinds having yet found entrance into Mathematics. In each of these sorts there is absolute freedom of choice with the kind first selected as a primitive; but the selection of this decides the question as to which shall be the remaining primitives, and leaves the mathematician no option in the matter. That is to say, in each particular system of vector analysis, stipulations are laid down which make the selection of one primitive kind carry with it the selection of all the others. Thus, in Quaternions, Double Algebra and the cognate discipline it is stipulated that with every vector sort, each of the kinds selected as primitive must have its actual vectors at right angles to the actual vectors of all the other primitive kinds. Hence, if Eastward and Westward vectors of lines (together of course with the vectors of points, which enter into all the kinds of the sort) are selected in Quaternions as constituting one primitive kind of their sort, the two other primitive kinds must necessarily have as their components Northward and Southward vectors and vertically upward and downward vectors respectively. Were other primitive kinds selected, we could indeed have a system of vector analysis, but it would not be Quaternions. We might, for instance, while taking Eastward and Westward vectors for the first primitive kind, take Northeastward and Southwestward vectors for the second, and obliquely upward and downward vectors for the third. In Double Algebra, if Eastward and Westward vectors are taken for one primitive kind of a sort comprising vectors of points and horizontal line vectors, then Northward and Southward vectors must be taken for the other primitive kind. There might indeed be developed a vector analysis, dealing like Double Algebra only with sorts made up of complanar (and null) vectors, and having two primitive kinds which were taken obliquely to each other; *e. g.*, Eastward and Westward vectors for the first, Northeastward and Southwestward vectors for the second primitive kind. But this, though describable as a double algebra, would not be the ordinary Double Algebra of our present Mathematics.

With the abstract sort, in each of the various algebras, the law ruling the selection of primitive kinds is this: that when *a* and *b* are actual vectors belonging to the same kind (primitive or complex) or to different primitive kinds of the same sort, then the relation of *a* to *b* must always belong to a primitive abstract kind; and no abstract kind is to be designated as primitive that does not contain relations of this character. Hence one primitive abstract kind must, in Quaternions, etc., be that contained, among other quantities, the relational quantities equal to the natural abstract quantities; for under this head come all the relations between concurrent actual vectors of the same primitive kind. And, in these disciplines, all the other primitive abstract kinds must have quadrantal relations as their non-zeroes. In Double Algebra all quadrantal relations belong to a single primitive abstract kind. In Quaternions there are three different primitive abstract kinds having quadrantal relations as non-zeroes, but not every quadrantal relation belongs to a primitive kind.

The primitive kinds having been selected, which of them ought we to designate as protomonic and which as neomonic? What are the principles underlying the application of these adjectives and their synonyms "real and "imaginary" to kinds and to quantities of these kinds? We may dismiss from consideration the zeroes; they possess none of the characteristics which make imperative the distinction between protomonic, neomonic and complex non-zeroes. But convenience dictates that just as every zero is regarded as positive or as negative or as both, and never as neither, though the characteristics which mark the positive and negative non-zeroes are lacking with the zeroes; so every zero, according as the sort to which it belongs contains protomonic, neomonic and complex non-zeroes, or non-zeroes of only one or two of these types, ought to be designated as at once protomonic, neomonic and complex, or as entitled to have applied to it only one or two of the three adjectives. We may restrict then our attention to the non-zeroes of the various primitive kinds; and shall begin by seeking the essential characteristics of protomonic and neomonic abstract non-zeroes as contradistinguished from each other.

The neomonic abstract non-zeroes most prominent in elementary Mathematics are the square roots of -1 that appear in Double Algebra. Cognate to these, and constituting the other non-zeroes of the neomonic abstract kind, are other square roots of negative protomonic abstract non-zeroes. In some algebraic disciplines there appear neomonic abstract non-zeroes that are cube or fourth, etc., roots of negative protomonic abstract non-zeroes instead of square roots. And in still other cases neomonic abstract non-zeroes have been introduced which are the square, cube, fourth, etc., roots of *positive* protomonic abstract non-zeroes or even of abstract *zeroes*. Now the square, cube, etc., of a protomonic abstract non-zero is always itself a protomonic abstract non-zero in the ordinary entitative algebras, and is therefore either confluent or contrafluent to its base; while no protomonic abstract non-zero is (in these algebras) either confluent or contrafluent to any neomonic quantity. Hence, so far as Double Algebra, Quaternions, etc., are concerned, we might conveniently lay down that a primitive abstract non-zero is to be designated as *protomonic* if its square and its cube, etc., is always either confluent or contrafluent to the base; but is to be designated as *neomonic* if its square or its cube, etc., is neither confluent nor contrafluent to the base. A definition of this type, however, if taken as perfectly general for every algebra of mathematical science, obviously requires the formation of kinds to be invariably carried out in that particular manner adopted in Double Algebra, Quaternions, etc., where, among other things, the non-zeroes of each variety must be mutually confluent and must be contrafluent to the non-zeroes of the other variety of the same kind. As has been said, this mode of laying down conventions of comparison might be accepted as the only one admissible in Algebraic Mathematics; and we ourselves are somewhat inclined to favor taking this course. But there is no advantage in so laying down the definitions of "protomonic" and "neomonic" as to *necessitate* its being taken. The question may well be relegated entirely to that part of algebraic theory which deals with kind formation; and left quite out of account in the consideration of the distinction between protomonic and neomonic kinds. We shall then seek to so delineate this distinction as to allow it to apply not only in the ordinary algebras, but in all that may reasonably be considered as more than a mere play of the fancy. Now in all algebras, even those most remote from the disciplines commonly made use of by mathematicians, there is, we think, one principle that is never violated. It is this: the product of a protomonic abstract non-zero into a neomonic unit is always of the same kind as the latter. Whenever mathematicians elaborate an algebra which has a neomonic as well as a protomonic abstract kind, they adopt a notation for the neomonic non-zeroes which implies this principle; for they suffix, to each protomonic abstract value expression, an expression representing the unit value of the neomonic kind, and the result is always taken as denoting quantities *belonging to this neomonic kind* and of that value which is possessed by the product of a protomonic quantity having the value represented by the protomonic value expression into a unit of the neomonic kind. A natural complement of the principle, though not a logical consequence of it, would be the assertion that, even with quantities other than neomonic units as multiplicands, the kind is never changed on multiplying by a *protomonic* abstract non-zero; but that a change of kind is always effectuated on multiplying a non-zero by a *neomonic* abstract non-zero save when a zero is attained as product. And we shall, we think, risk nothing more than the possibility of barring the classification of primitive kinds into protomonic and neomonic from algebras too fantastic to be of any serious interest in mathematical science, if we lay down as distinguishing characteristics that a protomonic abstract non-zero when applied as multiplier to a non-zero (abstract or denominate) gives as product a *non-zero of the same kind as the multiplicand,* while a neomonic abstract non-zero gives a product of *different kind than the multiplicand or else gives as product a zero.* A primitive abstract kind is to be termed protomonic if all its non-zeroes are protomonic; but is to be termed neomonic if all its non-zeroes are neomonic. And were any mathematician to so form a primitive abstract kind that some of its non-zeroes when applied as multipliers to non-zeroes gave non-zero products of the same kinds as the respective multiplicands, while others gave products of different kinds, then this abstract kind ought not, we apprehend, to be termed either protomonic or neomonic; and a like remark would apply to a kind containing an abstract non-zero which when applied as multiplier to some non-zero multiplicands changes the kind, but applied to others leaves the kind unchanged while giving non-zero products.

The adoption of the definitions of protomonic abstract kind and neomonic abstract kind that have just been suggested will undoubtedly give the two terms and the classification they mark, a wider range of applicability than the definitions based upon the confluence and contrafluence of a power with its base, *provided Mathematics admits certain conceivable algebras in which the principle of confluence and contrafluence is completely set aside in kind formation.* It should be especially noted that, under the former definition, what are protomonic abstract quantities in the ordinary algebras (and in particular the natural abstract quantities and their equals among the relational abstract quantities) could in a large measure be excluded from the protomonic kind in other algebras —a fact which from some points of view may be regarded as a detriment rather than as a merit. Of this possibility we shall proceed to give an illustration. It will be convenient in so doing to make use of the graphic representation of values by the Argand scheme, and we shall designate as a *line of a kind,* any line so drawn on the Argand surface that every point in it serves to represent a value of the kind in question, and that every value of this kind is represented by a point of the line. In ordinary Single Algebra and Double Algebra every kind of a vector sort, and likewise every abstract kind, has as its line a *straight* line passing through the origin, and extending out "to infinity" in both directions. Suppose, however, we choose to devise quite another system of vector analysis which, though not the Single Algebra with which ordinary mathematical work is concerned, is yet a single algebra in that it takes into each of its sorts only a single kind. In this new system let the line of each vector kind, while passing through the origin (the zero point) and extending "to infinity" in both directions as usual, be *curvilinear,* having the part on one side of the origin congruent with the part on the other side, so that if the former be revolved 180° around the origin as axis, it will coincide with the former. And let the curve be such that every straight line through the origin which passes through another point of the curve intersects the latter in two points equidistant from the origin, and does not coincide with the curve elsewhere than in these two points and the origin. It is understood, of course, that the curve of each vector kind, and in general of each applicate kind that enters this system, is congruent with the curve of every other applicate kind of the discipline. Then the abstract sort of the algebra thus brought into existence contains quantities of value 0, +1 and −1; but contains no other quantities of the values designated as protomonic ("real") in the ordinary algebras. All the other abstract quantities of the science are what would be called *complex or neomonic* abstract quantities in Double Algebra, and do not enter our ordinary Single Algebra at all. These other quantities are, however, in the algebra under consideration, all included in a single kind, and ought all to be called protomonic if the definition advocated above be accepted. There are in the algebra no abstract quantities outside this protomonic kind, which includes abstract quantities, natural and relational, of values 0, +1 and −1 together with the other abstract quantities just mentioned *and nothing else.* These latter quantities are all relational, each being the relation of one actual vector to another that is complanar with but neither concurrent with nor contrary to the former. The product of any vector by an abstract quantity of this kind will be a vector of the same kind as the multiplicand. In order for multiplication of an abstract quantity by an abstract quantity to be possible, there must be attained as product an abstract quantity admissible in this algebra and, hence, of the same kind as the multiplier and multiplicand. Obviously this requirement might prevent it being always possible to multiply one abstract quantity of this algebra by another, or to square an abstract quantity. And thus such a system would be likely to have a set of postulates and axioms widely different from those of the ordinary algebras. We make no claim that a vector analysis of the character just described would be of any particular service in Mathematics; on the contrary we think it probable that no algebra will be found to be of any real advantage in mathematical practice unless the formation of its kinds be carried out on the basis of the confluence-contrafluence doctrine. And, of course, so far as the algebras founded on this principle are concerned, the definitions of protomonic and neomonic based on the confluence or contrafluence of a power with its base give the adjectives precisely the same denotations as the definitions based on the fact that multiplication by a protomonic abstract quantity does not change the kind.

From the abstract we now turn to the denominate primitive kinds. Here we find the distinction between protomonic and neomonic to be a wholly factitious one based upon mere symbolism. Whether the quantities of a denominate primitive kind be classed as "real" (protomonic) or as "imaginary" (neomonic) does not in any way depend on the quantities themselves; it depends entirely upon the volition of the mathematician. Primitive denominate non-zeroes have no specific attributes by which they can be distinguished as protomonic or neomonic; they are protomonic or neomonic according as the mathematician arbitrarily chooses protomonic or neomonic value expressions for them. And that is all there is to the matter. The three primitive kinds of each vector sort in Quaternions are "imaginary." Why? Merely because Hamilton chose to represent the values of the unit vectors of these kinds by the symbols "i," "j" and "k" which also do duty as symbols of "imaginary" abstract values, and in general used neomonic abstract value expressions to represent the values of the vectors of the primitive kinds. How far this procedure was an arbitrary one may be seen on considering the fact that any two of the three primitive kinds of vectors may enter Double Algebra as primitive kinds of a new vector sort, and there, while one (whichever one the mathematician chooses) will remain neomonic, the other will become (*i. e.*, must be made) protomonic. Hamilton ordinarily denotes a unit vector—with him the units referred to are always units *of the kind*—by that same value expression which denotes a unit quaternion whose plane is perpendicular to the vector. Thus a unit vector perpendicular to the plane of an i quaternion has itself "i" as value symbol; one perpendicular to the plane of a j quaternion has "j" as value symbol; one perpendicular to the plane of a k quaternion has "k," etc. Sometimes he improves this notation, and uses "I," "J" and "K" to denote the unit vectors perpendicular to the planes of the quaternions whose value symbols are respectively "i," "j" and "k." The making the three primitive vector kinds of Quaternions to be neomonic is not necessary; one or two or all three of them could have been made protomonic by the adoption of a suitable symbolism. But in that event the mathematician would be deprived of the very convenient analogy between the formulas $ij=k$ and $iJ=K$, between $jk=i$ and $jK=I$, etc. We need not say that in our view the best mode of deriving the value expressions for denominate quantities from abstract value expressions is usually to suffix denominations to the latter;[1] and

[1] Here also it is convenient (though not necessary) to take analogy as a guide. For example, it would be quite inconvenient in Quaternions when $ij=k$ and $i1=i$ hold, to have $i\times 1$ Den. $=k$ Den., as would be the case were the three unit values of each vector sort to be i Den., 1 Den. and k Den., there being then one protomonic and two neomonic kinds in each vector sort.

where this procedure is followed we may lay down that a denominate non-zero is protomonic if its value expression is formed by suffixing a denomination to a protomonic abstract value expression; but that it is neomonic if its value expression is formed by suffixing a denomination to a neomonic abstract value expression. A definition of this type cannot, however, be accepted with *complex* denominate quantities. For, though a complex denominate quantity may have its value represented by suffixing a denomination to a complex abstract value expression, it is not complex in virtue of this mode of representation. Thus in Double Algebra an applicate non-zero may have its value represented by $(A+B\sqrt{-1})$ Den., but it is classified as complex, not on this account, but on account of its being of a value representable by A Den.$+B\sqrt{-1}$ Den., and its belonging to a sort including two primitive kinds to which belong the values A Den. and $B\sqrt{-1}$ Den.

The algebras of mathematical science may be classified in accordance with the number of their primitive kinds. De Morgan, to whom the names Single Algebra, Double Algebra, etc., are due, lays down that: "A system of Algebra of the nth character is one in which there are n distinct symbols ξ_1, ξ_2, . . . ξ_n, each of which is a unit of its kind, of a difference from all other kinds such that $a_1\xi_1+a_2\xi_2+$ . . . cannot be equivalent to $b_1\xi_1+b_2\xi_2+$ . . . unless $a_1=a_2$, $b_1=b_2$, etc."[1] Mak-

[1] *Trans. Cambr. Phil. Soc.*, vol. 8, p. 241. De Morgan published four articles: *On the Foundations of Algebra* in vols. 7 and 8 of these *Transactions*, and we may also mention here his *Trigonometry and Double Algebra* (London, 1840) where he says, p. iii, that Double Algebra "means algebra in which each symbol stands for an object of thought having two distinct and independent qualities." The relative inaccessibility at the present day of such important writings is highly regrettable; other work by the same hand, and that of no

ing the slight variorum of terming "of the ½ character" an algebra having never more than one variety to a kind and never more than one kind to a sort, we are substantially following De Morgan in saying that Mathematics includes the following algebras:

Arithmetical or Semi-single Algebra, which takes within its scope solely such sorts as contain only a single variety each. This discipline includes, as a part of a much greater whole, the scientific principles underlying the art taught in the schools, and there called "Arithmetic." Mathematical practice has heretofore always assumed that the variety embraced by a sort of Arithmetical Algebra is necessarily the positive protomonic variety, but this assumption is justified only with the abstract sort. The abstract quantities of Arithmetical Algebra are necessarily positive and protomonic, comprising only the natural abstract quantities and such relational abstract quantities as are equal to natural abstract quantities. The denominate quantities that most obviously come within the scope of this algebra are the concrete quantities and such applicate quantities as lengths, areas, volumes, etc. Yet no denominate quantities at all need be excluded. Even vectors are not excluded from a science where the quantities of a sort are all of a single kind and of a single variety. But in order that a sort of vectors shall embrace vectors of one kind and one variety alone, the conventions of addition must be such that it is impossible to add two non-concurrent vectors together. A vector sort would, under these circumstances, consist of all actual vectors

small value lies buried in an obsolete encyclopedia. It is a crying shame that the University of Cambridge, which has recently stood sponser for so many treatises of dubious value, has not yet set her press to the work of issuing an edition of the collected works of Augustus De Morgan, one of the greatest of her sons.

of a certain type with a certain currency, and of all null vectors of the same type. Thus all line vectors with currency Eastward, together with all vectors of points would constitute a vector sort in Arithmetical Algebra. This mode of forming vector sorts would alone suffice to bar relations between non-concurrent vectors from being accepted as quantities in the science. Such relations could not be subjected to addition, for this would involve the addition of non-concurrent vectors. And therefore the only relational abstract quantities that can possibly enter this algebra are those which are relations between concurrent vectors—*i. e.*, those which are equal to natural abstract quantities. As the abstract quantities of Arithmetical Algebra are thus all positive and protomonic; and as the value expressions for the denominate quantities of the same algebra are most conveniently formed by merely suffixing a specific denomination for each denominate sort to the abstract value expressions, it is not surprising that such a notation has always been adopted in Arithmetical Algebra. And this it is, in conjunction with the customary adoption of the positive conventions of comparison in such cases, that occasions mathematicians incorrectly to regard the denominate as well as the abstract quantities of Arithmetical Algebra as *necessarily* positive and protomonic. In fact there is no necessity about it; any or every denominate variety of Arithmetical Algebra can, at the will of the mathematician, be made negative and protomonic or made neomonic (positive or negative) by the adoption of a suitable notation and suitable conventions of comparison. Among the teachings of Arithmetical Algebra is this proposition: that it is impossible (within this algebra) to subtract a greater from a lesser vector; just as it is impossible to subtract any abstract quantity from a lesser abstract quantity; and, in general, that it is impossible to subtract any quantity from a lesser quantity. And, of course, no actual vector can be either added to or subtracted from another unless the latter vector is concurrent with the former.

Single Algebra, which takes within its scope solely such sorts as contain only a single kind each, this kind embracing, in some cases, two varieties; in others, only one. In current mathematical practice the kind is always taken as protomonic. The abstract quantities that enter Single Algebra consist of first, the natural abstract quantities; second, such relational abstract quantities as are equal to these; third, such relational abstract quantities as are contrafluent to natural abstract quantities. Single Algebra thus accepts the protomonic abstract quantities, positive and negative, and no others. Its inclusion of the negative protomonic non-zeroes (and of course also its inclusion of the relational positive protomonic abstract quantities) requires for justification a treatment of vectors. Since a vector sort in Single Algebra can comprise only a single kind, the vector analysis of this discipline must be based on conventions of addition which make no provision for addition between two actual vectors that are neither concurrent nor contrary. A sort of vectors under these circumstances consists of all actual vectors of a certain type with a certain currency, all actual vectors of the same type with the contrary currency, and all null vectors of the same type. Thus all line vectors of Eastward currency, all of Westward currency, and all vectors of points can be made to form a sort which contains one kind of two varieties, and is suitable for inclusion in the subject matter of Single Algebra. Naturally one of the teachings of the science is that it is impossible (within Single Algebra) to add any actual vector to or subtract any actual vector from another unless the latter is concurrent with or contrary to the former. It is clear that the mode of forming sorts followed in Single Algebra bars all relations between vectors that are neither concurrent nor contrary from being accepted as quantities of this discipline—in other words bars from Single Algebra all abstract quantities save such as are protomonic.[1] And as it is convenient to form the value expressions for the denominate quantities of Single Algebra by merely suffixing denominations to the abstract value expressions, all the denominate quantities of the discipline have heretofore been invariably made protomonic by the adoption of such a notation. But of course there is no necessity to write the denominate value expressions of Single Algebra in this way, and hence no necessity for any denominate kind of Single Algebra to be taken as protomonic rather than as neomonic. The ordinary Differential and Integral Calculus is a part of Single Algebra; and we may here take occasion to point out that an entirely unnecessary restriction is laid upon these disciplines when it is required that all the quantities considered be "real." For, obviously enough, what is actually requisite in Calculus is that the quantities of a variable be of the same kind, though they may just as well be neomonic or complex as protomonic. Even with abstract quantities most of the theorems of Calculus hold for the neomonic and for each complex kind (and in this far go somewhat beyond the scope of Single Algebra), though here of course due regard must be paid to the fact that certain operations (such as squaring) may bring a second kind into play when the operands are neomonic

[1] Of course all non-protomonic abstract quantities are barred even without this, since the protomonic kind of the abstract sort is admitted, and there is no room for a second kind of that sort in Single Algebra.

or complex, notwithstanding there being only one kind of quantities concerned when the operands are protomonic.

Double Algebra, which admits sorts containing two primitive kinds, but no more, and innumerable complex kinds. It ought, we hold, to include a treatment of vectors —an Argand Vector Analysis, so to speak—to justify its use of neomonic and complex expressions, and make it a double algebra in an entitative instead of a merely formal use of such expressions. Indeed, as contradistinguished from Single Algebra on the one hand and from Quaternions on the other, it might be considered to be essentially a doctrine of complanar vectors. To it belongs the Theory of Monogenic Functions. Besides the ordinary Double Algebra of our present mathematical science, there are conceivable various other algebraic disciplines that might be elaborated into systems, each of which could be properly designated as a double algebra. But in view of the actual state of Mathematics, no inconvenience will be entailed by the use of "Double Algebra" without qualification in speaking of that discipline which by virtue of eminence is best entitled to the name. With Arithmetical Algebra and Single Algebra the case is somewhat different; for though other semi-single and single algebras could be devised, this could only be done by rejecting in these systems the usual method of comparison (and hence the usual method of forming kinds and varieties) based upon the doctrine of confluence and contrafluence —any system in which this method is adhered to not being a semi-single algebra distinct from Arithmetical Algebra or a single algebra distinct from ordinary Single Algebra. And, as we have said, adhesion to the principles of confluence and contrafluence might not unreasonably be held to be essential for the admission of a system to the ranks of the algebras.

Various triple algebras, which have been developed by several investigators, notably De Morgan. De Morgan's triple algebras grew out of vector analysis of tridimensional space, but more attention was paid in his investigations to the formal than to the entitative side of the matter. He distinguishes his triple algebras as "quadratic, cubic and biquadratic, according as the invented imaginary units represent square roots, cube roots, or fourth roots of the negative real unit."[1] Triple algebras quite different from those of De Morgan have been devised by other investigators. Thus, J. and C. Graves developed cubic triple algebras in which the units of one primitive abstract kind being as usual of value +1, those of the two other primitive abstract kinds were cube roots of +1, instead of cube roots of −1 as in De Morgan's cubic algebras. We may notice here that it would be possible to develop, as a complanar vector analysis, quite an interesting *semi-triple algebra*, that is, an algebra in which a sort may contain three primitive kinds of one variety each, there never being more than three primitive kinds to a sort, and never more than one variety to a kind. In this complanar vector analysis, the three primitive kinds taken for each vector sort have their lines straight and going out from the origin *on one side only*, extending of course out "to infinity" on that side. The three lines are spaced 120° apart. Let, for example, the line of one primitive kind go Eastward, and call this kind the E kind, using "E" as the value symbol for its units. Designate as the R kind that other primitive kind whose line is attained by 120° of right hand rotation of the E line about the origin as axis, and as the L kind the third primitive kind whose

[1] *Proc. Cambr. Phil. Soc.*, Vol. I, p. 14. (De Morgan's own abstract of his fourth article: *On the Foundations of Geometry* published in the *Transactions*.)

line will be attained by 120° of left hand rotation of the E line. And use "R" and "L" respectively as the value symbols for the units of these last two primitive kinds. There are no negative quantities at all in this algebra. We make *all* the vectors of each kind, whether this be primitive or complex, positive. Contrafluent to the positive non-zeroes of each vector kind are other actual vectors, but these, instead of being included in a negative variety of the same kind, constitute, together with the null vectors of the sort, quite another kind. Thus the vectors contrafluent to the actual vectors of a primitive kind belong to a complex kind. If a greater vector is subtracted from a lesser vector of the same kind, the difference is not a negative vector of that same kind, but a positive vector of another kind. There will be in this algebra three primitive abstract kinds, each comprising a positive variety alone. The first (the protomonic kind) will comprise the ordinary positive protomonic abstract quantities —the natural abstract quantities and such relations between vectors of the same sort as are equal to natural abstract quantities. The second or r kind (with unit value r) will be a neomonic kind whose units when used as multipliers upon actual vectors effect rotations of 120° to the right, so that $rE=R$, $rR=L$, $rL=E$. Every non-zero of this r kind is a relation of an actual vector to another actual vector of the same sort such that the latter vector could be brought into concurrence with the former by a rotation of 120° to the right. And conversely every relation of this description belongs to this abstract kind. The third or l abstract kind (with unit value l) will also be a neomonic kind, but its units effect rotation of 120° to the left, so that $lE=L$, $lL=R$, $lR=E$. And the non-zeroes of this l kind are those relations between actual vectors of the same sort with which the correlate could be brought into concurrence with the relate by rotations of 120° to the left. All the neomonic abstract non-zeroes of this algebra, the l quantities as well as the r quantities, are cube roots of positive protomonic abstract quantities. And we have as fundamental formulas $l^3 = r^3 = +1$, $l^2 = r$, $r^2 = l$. But, as far as we are aware, this semi-triple algebra would be of no particular utility in Mathematics, and we need dwell no further upon the matter. Nor need we consider the possibilities in the development of semi-double algebras etc.

Quaternions, which is a quadruple algebra. De Morgan sometimes so terms it, but sometimes on the other hand he speaks of it as "one of the triple algebras." On one occasion he says: "in my view of the subject, it is not *quadruple* but *triple*, since every symbol is explicable by a line drawn in space."[1] We cannot quite see the relevancy of the clause beginning with "since," and from our own point of view it is undoubtedly quadruple; for though Quaternions has only three primitive kinds of vectors, it has four primitive kinds of abstract quantities; and we hold an n-tuple algebra to be one which with some sorts admits n primitive kinds of two varieties each, never more than n, though with some sorts there may be less than n kinds. We may also mention here Hamilton's Icosian Algebra, in which the units of the four primitive kinds are of values $+1$, ι, $\varkappa$, λ; where $\iota^2 = +1$, $\varkappa^3 = +1$ and $\lambda^5 = +1$; ι, $\varkappa$ and λ being respectively a "new square root, cube root, and fifth root of positive unity; the latter root being the product of the two former when taken in an *order* assigned but *not* in the opposite order." The ι's, $\varkappa$'s and λ's, like the i's, j's and k's of Quaternions, are associative but not commutative.

[1] *Op. cit.*, Vol. 8, p. 254.

And so we might go on to pentuple algebras, etc. The name *Pluquaternions*[1] serves to designate any n-tuple algebras, with n higher than 4, analogous in its laws to Quaternions. It has been proven that a formal octuple algebra analogous to Quaternions could be developed, and also proven that no 16–tuple algebra possessing the requisite analogy is possible. And in general, even the formal development, with n higher than 8, of an n-tuple algebra analogous to Quaternions does not appear to be possible. Besides Octonions, which seems to be put forth as an entitative octuple algebra of the Quaternion type, we may mention Multenions and Triquaternions as algebras which have been more or less thoroughly developed by mathematicians.

In connection with the discussion of sorts, kinds and varieties, we may notice a distinction sometimes made between *vector sorts* and *scalar sorts*. In saying this distinction is made, we of course do not mean that they who make it use the word "sort." What we refer to, as tantamount to such a distinction, is the fact that physicists not infrequently classify the denominate quantities with which they deal, as vector quantities and scalar quantities. "Vector quantity" here means simply vector; the term being applied to forces, accelerations, etc. Examples of scalar quantities are masses, volumes, and temperatures. And what physicists have in view, as basis for the distinction they make, is the fact that while vector quantities are "directional" (that is, while vector

[1] The word "Pluquaternions" is due to Rev. Thomas P. Kirkman: *On Pluquaternions, Phil. Mag.*, 1848, Vol. 33, p. 447 and 494. This author, however, used it, not so much to designate branches of Mathematics, as in application to the expressions representing the values of pluquaternions; the abstract quantities characteristic of these branches.

non-zeroes possess *currencies*), scalar quantities are not (no scalar quantity possesses an attribute of currency). This application of "scalar" is assuredly *not* an extension of Hamilton's original use of it as denoting the protomonic abstract quantities of Single Algebra. What would be a natural extension of this, is the use of the term to designate a quantity belonging to a sort which includes only one kind—a *scalar sort*. In this sense, however, "scalar sort" is not an antonym to "vector sort." For Single Algebra may take within its scope a sort of vectors comprising only a single kind, and being in this sense a scalar sort, though according to the usage of physicists the quantities composing it would not be termed scalars. We may incidentally mention here that half the sum of a quaternion and its conjugate—a sum that will always be a protomonic abstract quantity, and thus be a scalar in the original sense of that term—was called by Hamilton the *scalar of that quaternion*. Quite different from the use of "scalar" in reference to denominate quantities, is an extension of its original application sometimes made, under which it is applied to any abstract quantity with which the commutative law of multiplication holds universally (so far as is concerned the algebra in which the discussion arises). That is, an abstract quantity a is called a scalar if it be such that whenever b is another abstract quantity, we invariably have $ab=ba$. Even though b is not a scalar at all (if for example it is an i or a j or a k), when a is a scalar we must have $ab=ba$. In this sense all the abstract quantities of Double Algebra, whether protomonic, neomonic or complex, are scalars *in that branch of Mathematics*. In still another sense "scalar" is used by De Morgan in the phrase "scalar function."[1]

[1] See his *Trigonometry and Double Algebra*, p. 162.

In the classification of the vectors of Quaternions into kinds one thing especially noticeable is the inconvenience of the customary use of "real" and "imaginary" to designate primitive kinds, and the advantage of adopting "protomonic" and "neomonic" in their place. Even the most eminent mathematicians fail to keep distinct the technical and the colloquial meanings of the former adjectives. Thus, Tait tells us that: "Hamilton's system makes all directions in space equally imaginary or rather equally real,"[1] in which sentence "imaginary" is used in its technical sense, while "real" is not. Surely this is not an especially lucid way of stating the fact that in Quaternions there are three primitive kinds of vectors to each sort, all imaginary, technically speaking, and various complex kinds; and that no actual vector is real in the technical sense of Mathematics though it really exists. Hamilton himself wavered in his use of the words "real" and "imaginary." The *I*, *J*, and *K* quantities he sometimes called "imaginaries" and sometimes "geometrical reals." Even more paradoxical would have been the nomenclature of Quaternions had Hamilton adopted throughout his system the procedure suggested by him of admitting a fourth primitive kind of vectors to each sort, or rather to use his own phraseology, admitting "a species of Fourth unit in Geometry." This course, as Hamilton points out, will obviate the use of vectors as multipliers, any notation apparently involving such multiplication being construed as a mere abbreviation which really refers to multiplication by the quantuplicities of these vectors. The quantuplicity of a vector is its ratio to a unit of its sort, and without the acceptance

[1] *Elementary Treatise on Quaternions,* by P. G. Tait, 3rd Ed., Cambridge, 1890, p. 5.

of this new kind, a vector has no quantuplicity in Quaternions,[1] there being in our three dimensional space no unit of any vector sort of that science. Another change brought about by the admission of a fourth primitive kind is that the operations of Quaternions are made more widely applicable. Quaternions is defective in that it is not always possible to multiply a vector by a quaternion. Thus, if the relation of one vector to another is the quaternion q, it does not necessarily follow that we can multiply a third vector by q and obtain a vector as product. In such cases an attempt to apply the ordinary rules to the expressions representing the multiplicand and multiplier may give a product expression which does not fit into our vector system at all. The result may be a symbol of "real" quantity not a zero, or it may be complex of quadrinomial form, involving a "real" term as well as

[1] Unless, indeed, we choose to define "quantuplicity of an applicate quantity" on a symbolic basis, and say that when the value of an applicate quantity is represented by suffixing a denomination to an abstract value expression (or, *improperly*, by this abstract value expression alone) then a quantity arbitrarily selected from the abstract value-class pertaining to this expression is the quantuplicity of the applicate quantity. With the notation current in those of our present entitative algebras that have protomonic kinds in each sort, this definition will in every case give the same value for the quantuplicity as the ratio definition. And with the vectors of Quaternions, it can be proven that the value fixed for a quantuplicity by the definition based on symbolism is necessarily that which would be possessed by the ratio of the vector to a unit of the fourth primitive kind mentioned in the text *if such a kind existed*. And the theorem to this effect affords in fact the only possible means of calculating the value of such a supposititious ratio. So really, so far as calculating the value of the quantuplicity of a vector is concerned, it makes no difference whether we make the vectors in Quaternions have quantuplicities by adopting the definition based on symbolism, or do this by accepting a fourth dimension of space while retaining the ratio definition.

three terms of the I, J and K character respectively, or it may be a trinomial or a binomial involving a "real" term. Aside from the case in which the multiplier is a protomonic quaternion (in which case multiplication is invariably possible), we can multiply a vector c by a quaternion q and obtain a vector as product when and only when c lies in the plane of q—that is, the plane determined by the two vectors of which q is the relation —or is parallel to that plane. The current treatment of Quaternions can usually gloss over such difficulties, since very loose definitions are given, and no distinction at all is made between abstract and applicate quantities. Multiplication by vectors is admitted, and the products obtained may be vectors or may not. Thus no scruple is made of multiplying one vector by another parallel to it, and obtaining as product a "number"—that is, a protomonic abstract quantity. On the other hand it is equally admissible to multiply together two perpendicular vectors, and the product is a third vector perpendicular to both. This procedure may well awaken the wonder of the philosophical inquirer if not his admiration; it certainly far surpasses the ancient doctrine that the product of two lengths was an area, which De Morgan so aptly characterized as the mysterious notion that multiplication placed two straight lines at right angles to each other, and drew parallels through their extremities. If, however, we admit of multiplication by abstract quantities only, and require the product to be of the same sort as the multiplicand, multiplication of a vector by an abstract quantity can be made invariably possible in only one way—by assuming hypothetical vectors of a fourth primitive kind which shall be real in the technical sense. We would then have an assumed fourth dimension of space; vectors in this direction perpendicular to the three I, J, K dimensions would be "real;" a vector oblique to this direction, and having only its origin in ordinary space, would be a complex quantity involving one "real" and one or more "imaginary" terms in its value expression. Under such circumstances, to say a non-zero vector was imaginary would be to say it really existed, while to say it was real would mean that it did not exist—except in the four dimensional space of our imagination. We may remark that this increased applicability of the multiplication process entails an abandonment of the condition requiring two equal quaternions, if non-protomonic to have the vectors of which they are the relations in the same or parallel planes.

The procedure outlined above is the only way in which multiplication of a vector by a quaternion can be made universally possible, but it is not the only method which multiplication by vectors may be obviated. For when one vector is "multiplied" by another, and a third vector obtained as "product," we may demur at the application of the names "multiplication" and "product," though we accept as a valid and useful mathematical operation this process of finding from two given vectors (of the same sort) a third vector having a certain connection with them.

We have seen that i, j, and k quaternions (the imaginary abstract quantities commonly regarded as peculiar to the science of Quaternions) are in fact imaginary abstract quantities of ordinary Double Algebra. There appear, however, in Quaternions certain expressions, which, while denoting neither i–quantities, nor j–quantities, nor k–quantities, are nevertheless regarded as imaginary; such an expression being termed by Hamilton "the old and ordinary imaginary symbol of common algebra." Hamilton often has occasion to formulate an equation in which occurs "q," the symbol of an abstract quantity of unknown value, the equation expressing certain conditions (*not necessarily realizable*) which this unknown quantity must fulfill. On solving such an equation, in some cases he attained this result: that q, the unknown abstract quantity must be of the value fixed by the equation:

$$q = w + xi + yj + zk + h(w' + x'i + y'j + z'k)$$[1]

"where "w," "x," "y," "z," ""w'," "x'," "y'" and "z'" denote protomonic abstract quantities, and where "i," "j" and "k" denote units of the three neomonic abstract kinds, while "h" must denote a quantity of the following description: First, h must be a square root of -1; that is, on multiplying a vector by it, then again multiplying h into the product, multiplying h into this second product and finally multiplying h into the third product, there must be attained as ultimate product a vector equal to (or identical with) the original vector. Second, it must be commutative with every abstract quantity; so that (among other analogous equations) we have:

$$ih = hi;\ jh = hj;\ kh = hk.$$

Now the first condition is satisfied by the units of the i, j and k imaginary abstract kinds and of various complex abstract kinds—every quadrantal radial quaternion satisfies this condition. But none of these (in fact no non-protomonic quantity of Quaternions) satisfies the second. With the non-protomonic quantities of Quaternions multiplications is commutative only when multiplier and

[1] Of course, there may vanish from the right hand member some or all of its first four terms and any one or two or three of the four terms of the expression in parenthesis.

multiplicand are complanar or when one factor alone is non-protomonic—the commutability thus not being universal as must be the case with h. The second condition is satisfied by all the protomonic abstract quantities, but of course these do not satisfy the first. It is clear that within the bounds of the science of Quaternions there are no quantities answering to the description of h. And in fact it is found, when "h" appears in the expression finally equated to "q," that the original equation involving "h" laid down conditions that could not be realized; "q" denoting from the start, not a quaternion, but a quantity that, so far as Quaternions is concerned, could not possibly exist. Hamilton then regarded

$$w+xi+yj+zk+h(w'+x'i+y'j+z'k)$$

as denoting what he called a *biquaternion*; and

$$xi+yj+zk+h(x'i+y'j+z'k)$$

which may arise in an analogous way (and which, since it denotes an applicate quantity, ought to have a denomination suffixed) he regarded as denoting what he called a *bivector*.

Hamilton sometimes used the symbol "h,"[1] but usually he wrote $\sqrt{-1}$; a procedure which we regard as improper as would be using the name "animal" in place of "centaur," or in general putting any generic name where the name of one of the species properly belongs. The i, j and k imaginary abstract units and the complex quadrantal radial quaternions are just as much entitled to the desig-

[1] See *Lectures on Quaternions* by Sir W. R. Hamilton, Dublin, 1853, p. 730 note.

nation $\sqrt{-1}$ as is an *h;* and he who wishes to make himself clear ought not use a name applying to all of these species when he is really speaking of only one. It is also to be noticed that, though the interpretation of an equation in which "*h*" appears in Quaternions is analogous to that of an equation of Arithmetical Algebra or of Single Algebra in which $\sqrt{-1}$ appears—on which fact is doubtless based Hamilton's view that his $\sqrt{-1}$ is the "old and ordinary imaginary symbol of common algebra"—there is no analogy at all with the appearance of $\sqrt{-1}$ in an equation of Double Algebra: that science which developed from the Argand scheme. In Double Algebra $\sqrt{-1}$ is not essentially a mark of impossibility, and its appearance is quite analogous to the appearance of "*i*," "*j*" and "*k*" in Quaternions. True it is that with certain sorts $\sqrt{-1}$ may mark impossibility in Double Algebra; but so may negative or incommensurable or fractional expressions —in considering men, for example, $\frac{1}{2}$ will mark impossibility—and under a scientific view of Quaternions it cannot be denied that this must be regarded as having within its scope all the quantities of Double Algebra, and hence, with certain sorts "*i*" or "*j*" or "*k*" will mark impossibility in Quaternions. As to the commutative property of the *h*'s, which is also referred to by Hamilton as a reason, this would seem to suggest something quite different from the view that an *h* is an ordinary imaginary of Double Algebra. For commutability is at hand with *all* abstract quantities whenever both multiplier and multiplicand are complanar, and it is in virtue of this and the fact of the applicate sorts of Double Algebra being so restricted that all vectors of a sort are complanar (all their relations being hence complanar) that a $\sqrt{-1}$ of Double Algebra has the commutative property—rather we should say: *has it in Double Algebra.* It follows that an analogous case in Quaternions would be a relation between vectors whose plane is parallel with every other plane, so that the relation (a biquaternion, for of course an h itself comes under this head) would be complanar with every quaternion and with every biquaternion. And considering how mathematicians use or, one had better say, *misuse* the word "infinity," it is surprising that no one has suggested that a biquaternion might be a relation between vectors located at "infinity," where so many wonderful things happen, and where all the planes of space might be said to become parallel or perpendicular or anything else one's fancy may suggest.

A case in Double Algebra more truly analogous than that of $\sqrt{-1}$ to the "h" of Quaternions is given in the equations

$$1+(+\sqrt{x})=0 \text{ and } 1+(+A)=0.$$

where the second + sign is understood to signify that the *positive* square root is to be taken, or, in the last equation, that the quantity whose symbol is enclosed in parentheses is positive. For such an equation to be true, we need a new type of positive abstract quantities of such a character that the sum of a one (itself positive of course) and a positive quantity of the new type is a zero. Though no such quantities have yet been brought into service,[1] there is

[1] The late Sir James Cockle of Trinity College (and incidentally chief justice of Queensland) investigated the formal laws of combinations of symbols in a branch of Mathematics where such expressions as the $+\sqrt{x}$ above would be admissible. But, owing to the lack of entities that these expressions can conveniently denote, his investigations remain in the realms of formal as distinguished from entitative Mathematics. The formal algebra Cockle developed he called Tessarines. A *tessarine* would seem to be an abstract quantity (though Cockle himself defines it as an *expression*, just as Hamilton originally defined a quaternion to be an expression) which, unless it

no real impossibility about it unless we accept the contention that part of the *meaning* of "positive quantity" requires the sum of two positive non-zeroes to be always itself a positive non-zero. If that contention be denied (or if it be denied that part of the meaning of "zero" requires the sum of every quantity and a zero of the same sort to be equal to or identical with the former), quantities of this type are conceivable objects, even though inexistent in the mathematical universe of Double Algebra. Their symbols are not on a par with the words "blictri" and "abracadabra," which have not the shadow of a meaning; but are analogous to such words as "centaur" and "satyr;" and so is Hamilton's "*h*." When a symbol or other name is defined as one applicable to an object (or to each of several objects) possessing certain properties, these properties may, so to speak, be consistent with each other or they may not. Thus to say: an X is an object that is at one time both white and non-white, is to assign to the name "X" inconsistent properties. Such a name represents, not an object of thought, but a *chimera;* for example,

degrades, has a value expression of the form: $w+x\alpha+y\beta+z\gamma$. Here "α" represents the unit value of what Cockle would term the *unreal* kind, this value being fixed by the equation: $1+\alpha^2=0$. The kind of which "β" represents the unit value is termed by Cockle *impossible,* the value for β being fixed by the equation: $1+(+\sqrt{\beta})=0$. As for γ, which should probably be regarded as the unit value of still another primitive kind (though Cockle himself does not regard his system as including "three independent imaginaries)," its equation of definition is $\gamma=\alpha\beta$. Cockle classifies quantities as *algebraic,* under which genus comes the three species: *real, unreal* and *impossible,* and *hyper-algebraic,* this second genus including as species the *typal* and the *ideal* quantities. See *On Certain Functions resembling Quaternions and on a new Imaginary in Algebra, Phil. Mag.,* 1848, Vol. 33, p. 435. See also Vol. 34, p. 37, p. 132, p. 402; Vol. 35, p. 434.

an honest thief is a chimera. Now when the definition of a name is perfectly self-consistent, assigning no inconsistent properties to what that name is to denote, even when nothing answering the description exists in our universe, and the name has no denotation, this name nevertheless represents, not a chimera, but a true object of thought although a nonentity. Of this character are biquaternions; they are not chimeras, but objects, which, however, so long as nothing answering their description has been found, cannot be regarded as entities. Of a like character are centaurs, satyrs, and unicorns. We may define a certain symbol as denoting a quantity satisfying certain conditions which transgress the laws of that branch of Mathematics in which we are working. The conditions laid down may nevertheless be self-consistent, and then the symbol does not represent a chimera. When it thus represents an object of thought, the question will naturally arise: Can we find entities satisfying such conditions, and fit to be the operands of a new branch of Mathematics? It was this question that arose in connection with the expression $\sqrt{-1}$ in Single Algebra, and the movement initiated by Argand has answered it in the affirmative, and given us our modern Double Algebra.

Digressing for a moment to consider the historical development of Double Algebra, we may notice that Argand was anticipated by Wessel, not to speak of Truel, but that the investigations of these precursors remained unnoticed, and played no part in the subsequent development of vector theory. All the early investigators (Wessel, Argand, Buée, Francais, Gergonne, Servois and Gauss) seem to have confused abstract value expressions with vector value expressions, and to have regarded $+\sqrt{-1}$ as denoting a "directed line" (*i. e.*, the vector of a directed line) perpendicular to another directed line which $+1$

was supposed to denote, instead of recognizing that to an abstract value symbol a denomination ought to be suffixed, if it is to be rendered suitable to represent a vector value. Recent writers, purporting to give expositions of the Argand scheme, assign to the perpendicular "directed lines" the roles of being mere geometrical representations of real and imaginary quantities. A more legitimate procedure is to distinguish between the *Argand Vector Analysis* and the *Argand Diagrams*, the *points* of each such diagram serving to represent protomonic, neomonic and complex values (primarily, and *quantities* secondarily) of a sort (abstract or applicate) of Double Algebra. Mathematical text-books do not make such a distinction, and do not tell us where are to be found the imaginary and complex abstract quantities which they say are "represented" in the Argand scheme. A conception of these quantities as relations between vectors does not seem to have entered the minds of the authors of such works. As to the early writers, Argand and Francais spoke of relations and ratios ("rapports"), but not in a way which exhibited any conception of such relations in the light of quantities. So we certainly cannot agree with Prof. Hardy when he says that in the articles in the *Annales de Mathematiques* by Argand, Francais, Gergonne and Servois, "the true theory of the so-called *imaginary* quantities. . . . . was so exhaustively treated that nothing new has since been found to add to them."[1] Gauss spoke of relations as quantities; but these were not relations between vectors; they were relations between points, and were themselves vectors. That is to say, he regarded a line vector beginning at the point A and ending at the point B, as a relation

[1] See *Imaginary Quantities; their geometrical interpretation*, (a translation from Argand with comments) by A. S. Hardy, New York, 1881.

between the point A and the point B.[1] The recognition of *relations between vectors* as abstract quantities owes its inception to Hamilton, though even he was not as clear on this matter as would have been desirable. Still it must be admitted that by recognizing quaternions as entities distinct from vectors—as operators upon vectors—he showed a genius that puts him far above both his predecessors and his contemporaries.

Returning now to the question of symbols which have no denotation, we may say that the "$+\sqrt{x}$" and "$+A$" mentioned above, Hamilton's "h" and the expressions of his other biquaternions, the expression "$\sqrt{-1}$" in Single Algebra and "-1" in Arithmetical Algebra come as near being "mere symbols" as anything in Mathematics. But we would hold that nothing ought to be called a "mere symbol" whose habitual use is in discourse where what is referred to is not the symbol itself but what that symbol represents. And thus none of the symbols of Mathematics are, properly speaking, *mere* symbols.

It may be of interest to note that Hamilton regarded as a "fault of Double Algebra that it interprets too much. There ought to be uninterpreted symbols; and if you insist on treating $\pm\sqrt{-1}$ as representing two real unit lines, at right angles to the two lines ± 1, in a given plane, you will just find yourself obliged to invest some new signs for the old imaginaries of algebra, in order to meet the necessities of geometry, even within the plane."[2] And

[1] See *Werke*, Goettingen, 1863, Vol. 2, p. 175 and 176. In this discussion Gauss does not restrict his consideration of relations to relations between points, but he nowhere appears to consider relations between vectors as quantities.

[2] Letter to De Morgan in *Life of Sir W. R. Hamilton*, by R. P. Graves, Vol. 3, p. 603. See also letter p. 298.

Salmon held that the appearance in Quaternions of the uninterpreted imaginary symbol "*h*" was a great advantage, and that the lack of such a mark of impossibility would have made Quaternions "good for nothing."[1] As an example of what the appearance of a biquaternion may signify, there may be taken the so-called imaginary intersections. When the question at issue is where a certain intersection takes place, the arising of a biquaternion, in the equation purporting to give the solution, is an indication that intersecting is in the present case impossible. We may remark here that among other peculiarities Hamilton's biquaternions possess the interesting property that the product of two non-zero biquaternions may be a zero.

The word "biquaternions," originally introduced by Hamilton, has also been used in a quite different sense by Clifford,[2] who attempted to develop a new branch of

[1] Letter to De Morgan in *Life of Sir W. R. Hamilton*, by R. P. Graves, Vol. 3, p. 603. See also letter p. 298.

[2] The first use of "biquaternions" in a sense deviating from that of Hamilton is due to the Rev. Thomas P. Kirkman, *Phil. Mag.*, 1848. He used (p. 449) "biquaternion or octad" nominally to designate certain octonomial expressions involving symbols of imaginary units of seven different kinds. Probably he actually meant to apply the term to what such expressions can denote. In the latter sense the conditions required of Kirkman's biquaternions would seem to be fulfilled by those abstract quantities which appear as relations between Clifford's motors. The impropriety of Kirkman's use of "biquaternion" is shown by Hamilton: *Lectures*, p. 730 note. We may also mention the use of "biquaternion" by Gaston Combebiac in his *Calcul des Triquaternions*, Paris, 1902. In his terminology, "A biquaternion is a complex quantity of the form $q+\omega q_1$, where q and q_1 are quaternions and ω is a new complex unit commutative with the four quaternion units and having its square zero" (p. 10). Triquaternions is a "complex number system with twelve units" and a *triquaternion* is a complex quantity of the form $q+\omega q_1+\mu q_2$,

Mathematics called by him Biquaternions.[1] The applicate quantities peculiar to Clifford's system (and taking a role analogous to that of vectors in Quaternions) are designated as *motors*, and the abstract quantities appearing as relations of motors are called *biquaternions*. The most important characteristic of motors as contrasted with vectors is that in comparison of motors as to equality, the positions of straight lines called *axes* pertaining to the respective motors are taken into account, while for two vectors to be esteemed equal, it is sufficient for them to be alike in currency and magnitude. A velocity of rotation about a certain axis is a motor of one of the simplest types, and motors of such simple types are designated *rotors*. More recently the subject has been taken up by McAulay[2] who replaced the word "biquaternion" by "octonion." An *octonion* ought by analogy with Quaternions be an abstract quantity: a relation between two motors of the same sort; and likewise by analogy there ought to be seven primitive imaginary kinds and one primitive real kind in the abstract sort of Octonions. But neither Clifford (who from the difference between his *Lectures and Essays*, which are a model of clearness of thought and exposition, even where Mathematics is concerned, and his not very lucid *Mathematical Papers*, seems to have been philosophically something of a Dr. Jeckyll and Mr. Hyde) nor McAulay have paid much attention to relations between motors; both seem to prefer

where q, q_1 and q_2 are quaternions, and ω and μ units commutative with the quaternion units and following the laws: $\omega^2=0$, $\mu^2=1, \omega\mu=-\mu\omega=\omega$.

[1] See *Mathematical Papers of W. K. Clifford* ed. by Robert Tucker, London, 1882.

[2] See *Octonions, a development of Clifford's Biquaternions*, by Alexander McAulay, Cambridge, 1898.

to treat the subject on the lines followed in vector analysis by Grassmann, Gibbs, et al, rather than to adopt a more scientific method.[1]

A stage has now been reached where we may with advantage examine the conventional definition of variable in its more modern form. Practically all the mathematical text-books now in use in England and the United States, either give no definitions at all of variable and constant, or reproduce almost verbatim the definitions of Newton, As, however, such text-books are brought forth almost invariably by mere compilers, rather than mathematicians of authority, we turn to continental Europe, where we find equally bad definitions from more authoritative sources. For instance, in *Leçons sur les Theories Generales de l'Analyse* by Professor René Baire of Dijon, a most eminent mathematician, we are told that: "In Mathematics, one represents by a letter a number susceptible of taking different values. We say then that we have a *variable*."[2] At another place (p. 43) Baire speaks of a

[1] Any one with a philosphical cast of mind who looks into the *Ausdehnungslehre* will be inclined to agree with Hamilton's estimate of Grassmann (written not for publication but in a private letter to De Morgan, see *Life* vol. 3, p. 442): "Grassmann . . . . has not anticipated, nor attained the conception of, the *quaternion*, even so nearly as I guessed that he might have done, from a notion hastily taken up, of what might have been his meaning (and what it *was, I very* dimly know even *now*), in his doctrine of 'eingewandte multiplication.' I quote from memory. His *outer* products (auessere) I think that I *do* understand; and that is saying something for a person who has not learned to smoke. And even his *inner* products . . . . I can swallow pretty well. . . . I think that my own researches, or speculations, would have a better chance of being *appreciated* in these countries, if readers had first been put through a sufficient course or dose of Grassmann. I must say that I should not fear the comparison."

[2] *Op. cit.*, vol. 1, p. 20. This work is dated 1907.

variable quantity, while again he takes occasion (p. 20) to refer to "several variables; that is to say, several letters each of which is susceptible of representing different numbers." The word *number* itself, he does not seem to define.

For another example of the doctrine in vogue in the Latin countries, it will surely not be unfair to turn to a work written under the supervision of Professor Giuseppe Peano of the University of Turin, who has attained an international celebrity from his application of Symbolic Logic to Mathematics. In *Calcolo Differenziale e principii di Calcolo Integrale* by A. Genocchi "publicato con aggiunto dal Dr. Giuseppe Peano (Roma, 1884)" we find the following (p. 3): "In the questions considered there may appear quantities to which determinate and fixed values are supposed to be attributed, and these are called constants, and other quantities supposed to be able to assume diverse values, and these are called variables."

In Germany those eminent mathematicians who, during the last fifty years, made such noteworthy investigations of the Theory of Functions, usually fail to give any definition at all of variable, and here again we are compelled to consult University text-books, rather than memoirs of original investigations. In a very recent work; the *Vorlesungen ueber Algebra* of Dr. Gustav Bauer, "Geheimrat, Ordentliche Professor an der Universitaet Muenchen" (2nd Ed., Leipzig, 1910), we are told (p. 1) that: "One distinguishes between constant and variable quantities. The former have a fixed given value, the latter can take different arbitrary [beliebige] values." The only innovation here apparent over the old Newtonian definition is a new error; the stipulation that the value of a "constant quantity" must be "given." As will be shown in our subsequent discussion of the known (or "given") and unknown quantities of Algebra, the distinction between these has nothing whatever to do with the distinction between variables and "constants." A constant may be unknown just as well as it may be known, and a variable may be composed of known quantities, or of unknown, or of both.

In the *Theorie der analytischen Functionen* by O. Biermann (Leipzig, 1887) occurs the following: "In the formation of named expressions we will posit [festsetzen] that certain elements retain values fixed once for all, other elements take in turn various values of our system of quantities [Groessensystem]. The former quantities are called unalterable or constant, the latter alterable or variable." (p. 64).

Professor Heinrich Burkhardt of Munich is a mathematician of some prominence. In his *Algebraische Analysis* (Leipzig, 1903) he gives the following definition (p. 37): "A number is said to be alterable or variable when in the course of an investigation there is always assigned to it one value after another [immer andere und andere Werte beigelegt], and to be constant when the value first assigned to it is retained throughout the whole investigation (which does not preclude its changing during another investigation.)"

In E. Czuber's *Vorlesungen ueber Differential und Integral Rechnung* (2nd Ed., Leipzig, 1906) we find the following (p. 13): "By a real variable is to be understood a symbol for a variable quantity, which, in accordance with the problem in which the variable quantity appears, can be assigned several or unlimitedly many number-values [Zahlenwerte]."[1] To distinguish between "vari-

[1] We translate *Zahlenwerte by number-values,* as numerical values would obviously not be what is meant. Since "real variables" are in question, *protomonic* values are probably what are referred to.

able" and "variable quantity" in this way, does not, of course, surmount any of the difficulties that arise in connection with the Newtonian definition, since the question at once comes up whether or not the variable quantity, of which we are told a variable is the symbol, is really a quantity.

Of the various encyclopedic works on Mathematics, the latest and most authoritative is the *Encyklopaedie der mathematischen Wissenschaften* edited (so far as Division 1, *Pure Mathematics* is concerned) by Burkhardt and Meyer. The section on the *Grundlagen den allgemeinen Functionenlehre* (Vol. 2, Part 1, Leipzig, 1889) was written by Professor Alfred Pringsheim of Munich, who is without question one of the very highest authorities on the subject. On page 8 we find this definition: "By a real variable is to be understood a *symbol,* usually one of the last letters of the alphabet, to which is assigned successive different number-values (for example, all possible between two fixed number-values, all rational, all integral)." Quite similar statements are to be found with various other authors, and for some strange reason this symbol modification of the old definition would seem to be growing in favor. We speak of it as a mere modification, and not as something entirely new, since those authors who adopted it do not bring it forward as expressing a new conception of a variable, and have no objections to offer against the old definition, even though (like Pringsheim) they purport to give a historical account of the fundamental conceptions of the Theory of Functions. The obvious interpretations of such a silence would be that these authorities do not deny that a variable is a quantity, but class variable and quantities alike as symbols, and thus regard Mathematics as conversant with words and symbols merely, We have already, in discussing the

i, j, k quantities of Quaternions, shown how preposterous is such a view when closely examined, and we would be loath to believe that a man like Pringsheim could have fallen into this error. One would suppose it must have been clear to him that Algebra is more than mere symbol juggling, and no more has for its subject matter the words and symbols it makes use of, than Botany has as its subject matter such words as "coniferæ" and "cruciferæ." But it avails little to attempt to absolve Pringsheim from this absurd blunder, since no matter how we construe his words, there cannot be read into them any meaning that is even approximately correct.

Like remarks apply to the various formulations of the symbol definition to be found in mathematical works. Among such we may cite that of Weber, who after defining an integral function as an expression of a certain character involving x, says that x is "a symbol for which any number value we choose can be put. We then call x the variable."[1] Tannery says: "the notion of a variable, of a letter which can take any number-value [valeurs numeriques] whatsoever has been difficult to bring to light . . . it is well to modify the notion of the variable: this is not, as I have said provisionally, a letter which can take any values whatsoever, but any values whatsoever belonging to a certain ensemble. There are functions which are only defined for the integral and positive values of the variable."[2] We regret to also find a definition of this type in the recent work of Professor Pierpont of Yale: *Lectures on the Theory of Functions of Real Variables* (Boston, 1905), a work of really great merit. On page 118 of Vol. 1,

[1] *Encyklopaedie der elementaren Algebra und Analysis* by H. Weber, Leipzig, 1906, p. 185.

[2] *Mathèmatiques Pure* by Jules Tannery (in *De La Methode dans les Sciences*, Paris, 1909).

Pierpont says: "A symbol which takes on more than one value, in general an infinity of values is called a variable." Here, besides the symbol fallacy, there is another error in the restriction requiring there to be more than one value represented in a variable. This is entirely needless, since Mathematics is not bound to defer to etymological prejudices. To require that there be variation of value in the quantities of a variable, is a cumbersome and useless restriction, however it may appear to the etymologist. In Analytical Geometry, for instance, there is no reason why the ordinates of a line drawn parallel to the axis of abscissas should not be regarded as constituting a variable. In such a case all the quantities of the variable are of the same value, but it is a variable for all that. A variable-class differs from a value-class not by what it includes, but by the purpose for which it is formed.

The *Elements der Differential und Integralrechnung* by Professor Axel Harnack is a work of some repute. We quote the following from the English translation of 1897 (p. 15): "A quantity is said to be variable when it is able to assume different numerical values.[1] As in purely arithmetical investigation we no longer consider what are the things given in number, so in the conception of 'variable quantity' we have also to free ourselves entirely from considering what this quantity represents. The distance of a movable point, the temperature, the tension of vapor, in a word everything measurable in nature can enter into calculation as variable quantity." These remarks evidently belong to a transitional stage in the change from the Newtonian quantity-formulation of the

[1] In the original: *Zahlenwerte,* which we ourselves would render by number-values not by numerical values. The English translation above quoted is entitled *Introduction to the Study of Calculus,* and is said to be by G. L. Cathcart.

definition of variable to the symbol-formulation. Harnack appears to take the untenable ground that "variable" is an ordinary class name. When he speaks of freeing ourselves entirely from considering what a variable quantity represents, he probably means the leaving out of considerable certain differences in value when forming classes. He seems to have a faint and confused idea of the process of forming value-classes and variable-classes, though he is unable to see that they are composed of quantities. Moreover, he is unaware of the essential difference in purpose of these two class forming processes, and fails to see that while "two," for example, the name corresponding to a value-class, is an ordinary class name, the name assigned a variable is not an ordinary class name, and in its characteristic use does not occur in propositions concerning the members of the variable-class taken individually. The proposition: "Two plus three equal five" is in this respect quite analogous to the ordinary logical proposition: "Every man is mortal," and "two" and "man" are both ordinary class names. But this analogy does not hold with the proposition: "The variable x approaches the limit a;" and the name "x" or "variable x" does not bear to the quantities of this variable the relation a class name bears to the objects of its class.

To confuse names of variable with ordinary class names in this way would be far more reprehensible in a logician than in a mathematician, who can seldom lay claim to more than a narrow technical education. But we do find this very error with authors who deem themselves logicians. There are logicians and "logicians," however; there are mathematicians who devote themselves to Logic, but who instead of introducing logical methods into their mathematical investigations, find it more congenial to introduce "mathematical" methods into Logic. In this treatment of Logic, manipulation of symbols plays a preponderant part, while attention to the meaning of the symbols, and to the processes of inference represented by symbolic transformations is reduced to a minimum. In so speaking we do not have in mind *all* works on Symbolic Logic (that of Venn, for instance, is of true scientific value); but we do not hesitate to say that such treatises as those of Schroeder and Peano are a hindrance rather than a help to precision of thought and speech. If, after developing Logic on a "mathematical" basis, the symbol juggling "logician" again turns his attention to Mathematics, we might well expect to see, not a clearing away of old errors, but an aggravation of them; to say nothing of the introduction of new, and this is usually what actually happens. Among English mathematicians of the Peano School the Honorable Bertrand Russell stands preëminent. He is the author of a ponderous and pretentious treatise entitled *Principles of Mathematics*[1] from which we shall give a few citations bearing on quantities and variables.

Let us begin with Mr. Russell's views concerning natural numbers. Mr. Russell, using "class" where we would prefer to say "group of objects," and taking "similar" in a sense under which a thing may be said to be similar to itself, defines "as the number of a class the class of all classes similar to the given class,"[2] which is such as though

[1] Cambridge, at the University Press. All our citations are from the first volume (1903).

[2] *Op. cit.*, p. 115. A definition, not wholly unlike this and clearly of equal value is formulated by Frege (*Op. cit.*, p. 79): "Ich definire demnach: die Anzahl, welche dem Begriffe F zukommt, ist der Umfang des Begriffes 'gleichzahlig dem Begriffe F.'" This definition and that of zero previously cited are not unfair specimens of what Frege puts forth. As a logician he cannot be ranked above the level of Schroeder and Peano.

one were to define whiteness as the class of all white objects. This absurd definition, which Mr. Russell calls "an irreproachable definition of the number of a class in purely logical terms," is not due to Peano, but is claimed with pride by Mr. Russell as all his own. Both master and disciple, however, agree that such similar classes all have the same number, which is, Peano contends, a property *common to all* the classes. And thus Peano, as well as Russell, has failed to recognize the important distinction between equality and identity—a distinction which must form the corner-stone of any really scientific treatment of Mathematics.

According to Mr. Russell: "The variable is, from the formal standpoint, the characteristic notion of Mathematics. Moreover, it is the method of stating general theorems. . . . That the variable characterizes Mathematics will be generally admitted, though it is not generally perceived to be present in elementary Arithmetic. Elementary Arithmetic, as taught to children, is characterized by the fact that the *numbers* occurring in it are constants: the answer to any schoolboy's sum is obtainable without propositions concerning *any* number. But the fact that this is the case can only be proved by the help of propositions about any number, and thus we are led from schoolboy's Arithmetic to the Arithmetic which uses letters for numbers and proves general theorems. . . . Now the difference consists simply in this, that our numbers have become variables instead of being constants. We now prove theorems concerning *n*, not concerning 3 or 4 or any other particular number."[1] It is clear from these passages that Mr. Russell (like his master Peano we must add) utterly fails to see the essen-

[1] *Op. cit.*, p. 90.

tial difference between the name of a variable and an ordinary class name. It is certainly not very creditable to the Peano School that both Master and Disciples fail to perceive the enormous difference between the name of a variable, "x," in such a characteristic proposition as "The variable x approaches the limit a," and a true class name "x" in a proposition like "Every x is greater than zero."

It will be observed that, while we have made plain what a variable is not, and have described in what manner it is constituted, care has been taken to avoid any statement as to what a variable is. We have shown of what a variable class consists, but have not defined "variable." A variable as we have seen is neither a quantity nor a symbol, and reflection shows that we cannot in consistency with the phraseology of Mathematics say categorically that a variable is the variable-class or set of quantities which constitute the variable, When y is a variable in functional relation with another variable x, the functional formula being $y=x^2$, we cannot well replace the names "y" and "x" by "a set of quantities" and "another set of quantities," for we cannot intelligibly speak of the square of a set of quantities at all. Of course the interpretation of the proposition $y=x^2$ presents no great difficulty; it clearly refers to the operation of squaring *each* of the quantities of the set in question; but when this is once admitted we cannot, if the rules of language have any sway at all, say categorically that the variable x is the set of quantities—we must content ourselves with saying the variable is *constituted* by this set of quantities. Were the names of variables to occur only in such propositions as $y=x^2$ or only in such as "x approaches a as a limit," there would be comparatively little difficulty in defining "variable," but the use of propositions of both types side by side renders the name variable anomalous, and though, on the one hand, it would seem we cannot call "x," the symbol of a variable, a class name,[1] on the other, after once accepting the use of "x" in such a proposition as $y=x^2$, one can hardly be expected to refrain from passing on to the phrases "an x," "two of the x's," "all of the x's," etc., etc., where quite clearly "x" is used to denote quantities of the variable, just as though it were nothing more or less than an ordinary class name belonging to these quantities. We must thus recognize that, with a variable, one and the same name is used in denoting the variable itself and the quantities which constitute it.

Any attempt to give a precise account of the definition of the term "variable" would require a somewhat lengthy consideration of the philosophical theory of the categories, which cannot be given in this place. We may, however, make a few remarks on this matter; remarks which owing to their brevity cannot constitute an entirely satisfactory treatment of the subject. In Metaphysics the point of view which is perhaps the most prevalent is Realism in its various modifications, a doctrine which upholds the existence of things in themselves or substrata underlying our perceptions. Opposed to this we have the School of

[1] Or rather let us say: "It would seem we cannot call 'x' a class name of a variable-class." Of course if there be included under class names *individual names* (*e. g.*, "Socrates," "Plato") put forward each as the name of a single object, the class in such a case containing only this one member, the name of a variable is a class name by virtue of its denoting the variable itself. But this is quite a different matter from its being a non-individual class name and being the class name of the class of quantities constituting the variable. The wording of our text is not we apprehend incorrect from the point of view of current usage, which hardly sanctions calling an individual name a class name. However, it seemed best to add this note, as we ourselves are inclined to include individual names under class names.

Empiricism, which starting in the Philosophy of Locke found further development with Berkeley and reached its culmination with Mill. In antithesis to *Realism* we have the word *Idealism* to denote the views of the Philosophers last mentioned; Berkeley having denied the existence of one class of "things in themselves" (having denied the "existence of matter") while Mill without reservation denied the existence of substrata altogether.[1] Besides its application in reference to the denial of such existences, the term "Idealism" has frequently been applied to the point of view which Sir William Hamilton (of Edinburgh) called "Cosmothetic Idealism," and which consists in a philosopher "denying an immediate or intuitive knowledge of the external reality whose existence he maintains." Those who, like Kant, hold this view obviously stand on a very different footing from that of the true Idealist, who utterly refuses to admit the existence of an "external reality." The most extreme development of Empiricism in this respect is the denial that we can intelligibly speak of things in themselves, as distinguished from things as they appear to some sentient being. Where the moderate Idealist would deny the propositions of the Realist concerning existence and thus admit them to be intelligible,[1]

[1] It will be observed that we make no mention of Hume, an omission not without its reasons, though they cannot be given here. We need hardly remark that Empiricism in Metaphysics has nothing in common with the "Positivism" of Auguste Comte—a "Philosophy" characterized by a total absence of any attempt to analyze mental phenomena. As far as very recent works are concerned, the only one for which an Empiricist can feel any enthusiasm is Mach's *Analyse der Empfindungen*.

[1] A Philosopher, however, who classes meaningless propositions with false, instead of making a threefold classification of propositions into true, false and meaningless, can consistently deny the existence of substrata, and yet be an extreme rather than a moderate Idealist.

the more extreme Empiricist would regard the propositions in question as mere gibberish without a shadow of meaning. This latter view needs a specific name to distinguish it from moderate as well as Cosmothetic Idealism, and as it will usually be held in conjunction with the Nominalistic view concerning the nature of general terms and has some connection with the latter, we may provisionally denote the standpoint of the more extreme Empiricist (which we ourselves uphold) by the title *Nominalism*. Now according as the system accepted be Realism or Nominalism, the question of the categories appears under one of two varying aspects. The Realist will think that every object worthy of consideration comes under one of several *summa genera*, it being regarded as possible to completely enumerate these *summa genera* or highest classes. Any name not belonging to an object coming under one of these heads is, he holds, of no importance, being the name not of a really existing object but of a fictitious entity. The Realist then thinks it always possible to give a definition per genus et differentiam. The Nominalist makes of course no distinction between the name of an object possessing a substratum and the name of an object devoid of this. To him every name is alike a device to describe a state of affairs which on analysis ultimately reduces to certain perceptions[1] or groups of perceptions of sentient beings. Thus to say: "An orange lies before me" means simply that, if my senses be in a normal state, by proper attention I can have certain perceptions of sight, touch, taste, etc. Take now the more complicated statement: "The English constitution exists." This of course does

[1] The word *perception*, as here and elsewhere used by us, includes sensations and in general all states of consciousness whatever. Its use in this sense, though by no means an innovation, is of course not the one most prevalent.

not refer to a roll of parchment covered with writing; in fact England has no written constitution. But the English constitution is a reality for all that, even though it cannot be put under any of Aristotle's categories. To say the English have a constitution merely means that under the proper circumstances certain facts would be observed by a sentient being or beings, and the observations thus made would amount ultimately to nothing more or less than very complicated groups of perceptions.

The Nominalist then, unlike the Realist, regards the English constitution as a really existing object, just as much as the orange; neither are fictitious, but the former is at a much greater remove from its elemental perceptions than is the latter. It is futile, from the standpoint of Nominalism, to attempt a complete enumeration of summa genera; if a complete list be made to-day, to-morrow a new phraseology may spring up which groups perceptions in a way never hitherto attempted and thus brings into existence a new summum genus. Definition per genus et differentiam is available when the name to be defined is that of an object belonging to a well known summum genus, but when the object concerned is of an entirely new type—when we are concerned with an entirely novel grouping of perceptions—any attempt to define in this way would result in failure. To tell us an X is a Y possessing the differentia Z does not give us any information of service unless we are already familiar with the genus Y. When this difficulty arises a new mode of definition must be adopted. We must gather together the various propositions that can be asserted concerning the objects in question, and say under what state of affairs we would assert each of them to be true and under what false. Thus, instead of attempting to say a variable is an object of a certain genus with certain differentia, the aim should be to gather together the most important propositions in which the word "variable" is used, and to say under what conditions they are true or false. The name "variable" was not coined to denote members of a class carved out of one of Aristotle's categories, it was brought into use to describe certain important facts concerning sets of quantities. If it can be stated what these facts are, and the reader be enabled to interpret every proposition involving the word "variable," he being thus put into a position to tell what state of affairs corresponds to the affirmation of the proposition and what to its denial, more will have been done than if a variable had been defined by means of a genus utterly unknown to him. When the student of Mathematics reaches the subject of variable it is utterly impossible to give him a definition per genus et differentiam based upon the familiar conceptions of elementary Mathematics. Variables are not special cases of a class of objects already familiar to the student; they are objects of an entirely new type, though, as we shall see, he usually deals with variables of a simple character long before he becomes acquainted with the name. One might, indeed, it would seem, include variables in the summum genus *aggregates*, and say a variable is a *quantity aggregate*, but this definition sheds more light on the meaning of "aggregate" than on that of "variable." Moreover, the Theory of Aggregates exhibits a marked difference from the Theory of Variables, in that the former takes not the slightest account of the individual characteristics of the elements of an aggregate (*e. g.*, their values in the case of an aggregate of quantities) while in the latter the values of quantities of a variable are of supreme importance. And as a natural consequence of this, there are in the Theory of Aggregates no propositions made use of analogous to those expressing functional relations between variables.

So that here the anomaly does not arise of having the same symbol denote both an aggregate and the elements of that aggregate. Aside from this fact (to which mathematicians have hitherto paid no attention) it would seem a quite obvious step, after the Theory of Aggregates had once been developed, to define a variable as an aggregate composed of quantities (or "numbers);" and this has occasionally been done, but the authors who take such a course show no indications in what they say elsewhere of any insight into the true doctrine of variables. A definition of this type is given by Durège, who says: "Instead of the designation 'number' we use also the designation 'number quantity' [Zahlgroesse] or 'quantity.' We use in particular the designation 'variable quantity' or 'variable' when we represent to ourselves that the symbol x shall denote in succession the different numbers of a number aggregate M."[1] Durège evidently failed to perceive the important difference between the name or symbol of a variable and an ordinary class name used to denote the component quantites of that variable, and nowhere in his works can be found traces of the widespread changes in other definitions which are necessitated by a recognition of the true distinction between quantities, variables, and symbols.

The most simple of all variables are the ordinary progressions of Arithmetic: arithmetical, geometrical, harmonical, etc. That these are not given a more prominent place in the current treatment of the theory of variables is highly regrettable. The four distinct and somewhat difficult conceptions of *variable, limit, algebraic continuity*

[1] *Elemente der Theorie der Functionen einer komplexen veraenderlichen Groesse* by H. Durège (5th Ed; edited by L. Maurer, Leipzig, 1906), p. 23.

and *function* are best introduced one by one, and progressions are par excellence the variables whose theory can be developed to a considerable extent without bringing into play the three remaining conceptions.

A progression is a discrete unifarious variable of a certain character. The quantities of a progression are called its *terms*.[1] It is more common to speak of the symbols representing the quantities as the terms, but if this be accepted, it is difficult to see how we can intelligibly speak of the sum of n terms, of the mathematical relation borne by the second term to the first, etc.[2] If to a closed vessel filled with air we apply a piston pump, and remove air without leakage, stroke by stroke, the weights of the bodies of air successively drawn out will consitute a variable, and this variable will be a decreasing geometrical progression. With arithmetical and geometrical progressions, to obtain the value of any term from the progressional formula, it is necessary to have given besides the formula, merely the value of the preceding term. Thus with such formulas as $a_n=a_{n-1}+5$ (which is that of an arithmetical progression) and as $a_n=5a_{n-1}$ (which is that of a geometrical progression) we are able to do this. With other more complicated progressions, the progressional formula is such that to find the value of any term from the formula, it is necessary to have given, besides the

[1] The name term may be applied to the quantities of a variable whenever the variable is discrete.

[2] What we take as the mathematical significance of "term" is in sharp contrast to its meaning in Logic, where a term is a name and is thus not what this name denotes. We prefer to use "term" in these two so different senses, in the two different sciences, rather than use it in both Logic and Mathematics as referring, not to what is spoken or written of, but to a part of the speech or writing itself, while making the requirements under which an expression may be called a term different in the two disciplines.

formula, the value of two or even more of its immediate predecessors. But, however this may be, if the value of n preceding terms is sufficient with one term of a progression, the values of a like number of predecessors will also be sufficient with every other term. That is to say, with all progressions, to obtain the value of any term by means of the progressional formula, it is necessary and sufficient to have given, besides this formula, the values of a certain number of preceding terms. The problems met with in Arithmetic concerning progressions are: given certain data, to find the value of the nth term of a progression, the sum of n consecutive terms, or the formula; there is here no consideration of approach to a limit.

We will next consider a class of discrete unifarious variables differing from progressions solely in the character of their formulas. The formula of a variable of this class enables us to find the value of any term when we are given no data except the ordinal rank of the term in its variable. That is to say, to find the value of the nth term by means of the formula, it is necessary and sufficient to have given, besides this formula. the value of n. Here then n is an *essential operand*, while this is not the case in the application of the formula of a progression. We should prefer to apply to variables of this character the name *series;* but "series" as commonly used in Algebra does not include all such cases. If we have a variable of the class just mentioned whose formula is of the simple type exemplified in $a_n=5^n$ or $a_n=3^n+10^{n+1}$ we can evidently form another less simple variable of the same class by taking as first term, the first term of the original variable; as second term, the sum of the first and second terms of the original; as third term, the sum of the first, second and third terms of the original and so on. In the new variable there is clearly not a great deviation from the original type, the only difference in character being that the formula is more complicated. If the original formula be $f(n)$, the new formula will be $f(1)+f(2)+ \ . \ . \ . +f(n)$. In the present usage of mathematicians, the name "series" is given to variables whose terms are derived from the terms of other variables by such processes of addition, but it is not granted to these other variables if they are of the simple type (*i. e.*, if they do not themselves originate by such summation). We, however, prefer to use the name *summative series* for variables produced by the processes of summation described above, widening the name series to include the original variables as well as the new variable derived from it. The adjective "summative" may seem hardly suitable when, as in the case of $a-b+c-d+ \ . \ . \ .$ for instance, there seem to be involved in the formula of the series one or more minus signs. But in treating such a series mathematicians follow a procedure which amounts to putting in the place of $-b$, $+(-b)$; in the place of $-d$, $+(-d)$; etc., so that we have as formula for the series $a+(-b)+c+(-d)+ \ . \ . \ .$; and hence there can be really no objections to the use of "summative" here, provided it be understood that the first formula is merely an abbreviation for the second. In $a+(-b)$, etc., the sign $-$ does not indicate subtraction, and of the operation it does indicate there will be more to say later. If in the primitive series, by the addition of whose terms a summative series is derived, the terms are all of a single kind and alternately positive non-zeroes and negative non-zeroes, the summative is said to be an *alternating series.*

Completely irreconcilable, not merely with the view we have taken of series, but likewise with that which would restrict the title to the summative series, is a practice which by its prevalence makes the current use of the term not even self-consistent. When an expression is said to represent a series, what is frequently meant is that it represents a dependent variable which is not a series, though it bears to another variable what might be termed a *serial functional relation*. Thus take the expression:

$$1+x+x^2+x^3+ \quad . \quad . \quad .$$

where "x" represents an independent variable ranging in value continuously from 0 to +1 (both exclusive). The expression in question represents, not a series, but a non-discrete variable, the quantities of which are the *limits* of the innumerable summative series whose formulas can be obtained by substituting for "x" in the above expression an individual symbol representing a quantity of the variable x. If "a" is such an individual symbol, the formula for the nth term of the corresponding series will be:

$$1+a+a^2+a^3+ \quad . \quad . \quad . \quad +a^{n-1}.$$

In a case like this, then, although there are innumerable series concerned, neither the independent nor the dependent variable of the discussion is a series.

The application of "series" which we have advocated and intend to adopt in the present work is not entirely an innovation. Cayley sanctions its use in a sense which is certainly not less broad. He says: "A series is a set of terms considered as arranged in order. Usually the terms are or represent numerical magnitudes and we are concerned with the sum of the series."[1] Cauchy thus defines series: "an indefinite succession of quantities u_0, u_1, u_2, u_3, . . . which are derived from each other according

[1] *Collected Math. Papers*, Vol. 11, p. 617 (Article *Series*).

to a determined law. The quantities themselves are the different terms of the series."[1] We may also invoke the authority of Wallis, a mathematician and logician of repute, who deserves the chief credit for the introduction of infinite series into mathematical investigations. In the terminology of Wallis, series are simply "certain Progressions or Ranks of Quantities orderly proceeding."[2] None of these authors, however, distinguish, as we do, between a primitive series and its summative; between the terms of the primitive and the terms of the summative; between the convergence of the former and the convergence of the latter. With all mathematicians heretofore the two series are confused; when terms are spoken of, the terms of the primitive series are always meant, while the convergence or divergence considered is always the convergence or divergence of the summative. Another distinction which is not properly drawn is that between series and progressions. Not all mathematicians go so far as Wallis, who would seem to regard the two names as synonymous, but it would be difficult to find an author that even attempts to precisely delineate a distinction between them. The distinction we have laid down is based it should be noted, not upon the constitution of the variable itself, but upon the character of the formula assigned to it. The same set of quantities, arranged in the same way, may be made to constitute either a progression or a

[1] *Oeuvres*, Series 2, Vol. 3, Paris 1897, *Ánal. algebr*. p. 114.

[2] His definition has reference more particularly to *convergent* series. In his *Treatise of Algebra*, London 1685, Ch. 73, he says: "There is yet another thing to be spoken of which I look upon as a great improvement; which is that of *Infinite Series* (as they are wont to be called); That is certain, Progressions or Ranks of Quantities, orderly proceeding, which make continual approaches, and if infinitely continued would become equal to what is inquired after." Note the recognition that a series is composed of *quantities*.

series, according to the formula laid down as belonging to this variable. Thus a variable whose successive terms are of values, 5, 10, 15, 20, etc., is a progression if $a_n = a_{n-1} + 5$ is laid down as its formula; while if $a_n = 5n$ is laid down, it is a series. Every progression can at will be changed into a series, but some series cannot be changed into progressions, there being no progressional formula adequate to express the law that rules such a series.

That the series of Algebra are variables is, one would think, sufficiently clear from the mere fact that nearly all the discussions of series in mathematical works bear upon their convergence or want of convergence—that is, their tending or failure to tend to a limit. And yet it would be difficult to find anywhere an explicit statement to the effect that a series is a variable. Some definitions of "series" are indeed glaringly erroneous. Thus, in a comprehensive work of high repute by Professor G. Chrystal,[1] we are told that "By a series is meant the sum of a number of terms formed according to some common law. . . . An Arithmetic Series or an Arithmetic Progression, as it is often called, is a series in which each term exceeds the preceding by a fixed quantity." That with the series to which mathematicians have paid most attention, summation plays an essential part, affords not the slightest excuse for defining a summative series as a sum; while there is lacking even the shadow of a pretext to justify speaking of a sum whose summands constitute a progression as the progression itself. Incidentally we may add that any one who defines a series as a sum, must in consistency deny the name to many of what mathematicians now call series. With a convergent summative series of innumerable terms, it is customary

[1] *Algebra*, 5th Ed., London, 1904, Part 1, p. 480 and 482.

to speak of the limit toward which the series converges as the "sum of an infinite number of terms,"[1] and when an "infinite" series is divergent the sum is said to be plus infinity or minus infinity. But, even admitting for the moment such a use of the word "sum," there is nothing which is or can be called a sum in the case of an "infinite" series neither convergent nor divergent. And yet no one would propose to abandon the use of the name series in such cases.

We find the following in a work of well deserved repute: "When the terms of a sum are to be added in the order in which they are written, the operation is called a series . . . for the succession of numbers . . . we retain the word sequence, and we observe that the series is the process or series of simple operations by which we build the sequence for the terms . . . the act (whether of thought, writing or speech) by which we pass from the sequence of terms to the sequence of sums—this is the infinite series."[2] The advantages of this use of the word "series" we must profess ourselves utterly unable to apprehend. In the *Encyklopaedie der mathematischen Wissenschaften* no definition of series is given, so far as we can find, but a procedure is followed which implies some such view as the above. Infinite series are put under the heading "Irrationalzahlen und Konvergenz unendlicher Prozesse." Do the eminent mathematicians responsible for this work really regard a series as a process?

Besides the series of the character so far considered there are also variables which mathematicians call *mul-*

[1] The terms referred to in this phrase are not terms of the summative series itself (properly speaking), but are terms of the primitive series from which is derived the summative series that converges.

[2] *Introduction to the Theory of Analytic Functions* by James Harkness and Frank Morley, London 1898, p. 97.

tiple series, these being multiplex instead of simplex; and if they be admitted to be series, the ordinary series should perhaps be called simple series. But it is very questionable whether the use of "series" as a generic name to include multiple or better *multiplex* series together with simple or *simplex* series is attended with any advantage. We shall have no occasion to discuss multiple series and will merely remark that the extension of "series," to cover series not derived by summation as well as summative series, applies here also.

In some of the older works the first term of a progression was said to have 1 as *gradual number,* the second to have 2 as gradual number, the third to have 3, etc. This nomenclature originally arose with geometrical progressions, where, as in the case of a, a^2, a^3, etc., there occur in the representations of the successive terms the exponent symbols "1," "2," "3," etc. (though, of course, "1" is ordinarily not written). There might perhaps, on occasions, be some advantage in using the name gradual number in this way, but without restriction to the case where the terms concerned are in geometrical progression, and in general making use of the designation in connection with a progression or series of any character.

The term *sequence* which has been mentioned above is in common use in modern Mathematics, and is there given a denotation broader than that of either series (even in the broad sense that we have taken for it) or progression. A sequence is simply a discrete unifarious variable of any character whatever, or rather, of any character in so far that its terms may follow any law that can be designated (for example: the law of a progression or the law of a series) or may even be entirely unamenable to law. This at least is the meaning that must be inferred from the current use of "sequence," though we do not recall having ever seen it explicitly stated that either a series or a sequence is a variable.

When in Mathematics we deal with simple series (or with progressions) there is most frequently considered, in the discussion, only one arrangement of the quantities of the variable in question, and this arrangement is necessarily unifarious. There is nothing, however, to prevent our rearranging the quantities of such a variable in quite a different order, which may be either unifarious or non-unifarious, and we might even give them a non-discrete unifarious arrangement, under which the variable in question would necessarily cease to be a series. It would really seem best to include as a part of the meaning of "sequence" (and hence of "series" and "progression)" the fact that the arrangement of the terms in the order originally given them is always thereafter adhered to (which is, of course, not the case with all variables), and that a change in order is not a change in the sequence, but its destruction and the creation of an entirely new sequence. Taking this view, we must define a sequence as a discrete unifarious variable having an immutable arrangement of its terms. A progression is a sequence with which the formula for the nth term lays down as essential operands m terms previous to the nth; while a series is a sequence with which the formula lays down n itself as essential operand.

ON THE GENERAL CONCEPTION OF FUNCTIONAL RELATION.

The following discussion of the general conception of functional relation is a portion of Part X (Functional Relations) of the first division (Algebraic Mathematics) of this work. Since the publication of Part X as a whole will probably not take place for some time to come, we have thought it best to here insert a few excerpts from that part which form a natural complement to what has been said in the foregoing pages concerning variables. One name that occurs in these passages, "quesitive symbol" will be new to the reader, for it belongs to the terminology of our Part VII (Symbols, Signs and Sigla). As we use the term, it denotes those symbols used to represent the unknown quantities of a mathematical investigation. The antonym to "quesitive symbol" is "dative symbol."

The essential characteristic of a functional relation between variables we hold to be the like order of corresponding quantities in these variables. For two variables, y and x, to be in functional relation, it is necessary and sufficient that there be two or more quantities of x, x_1, x_2, etc., which respectively correspond to y_1, y_2, etc., quantities of y, and that with every two pairs of corresponding quantities, x_m and y_m, x_n and y_n, y_m is subsequent to y_n when x_m is subsequent to x_n and vice versa; y_m is previous to y_n when x_m is previous to x_n and vice versa; and finally when x_m is neither previous nor subsequent to x_n (*e. g.*, is abreast of it, as may be the case under a multifarious arrangement) y_m bears a like relation of order to y_n, and is neither previous nor subsequent to the latter and vice versa. In the case of three or more variables, x, y, z, etc., the sufficient and necessary conditions are quite analogous. Thus there must be two or more quantities of x, x_1, x_2, etc., which respectively correspond to y_1, y_2, etc., quantities of y, to z_1, z_2, etc., quantities of z, etc., etc. And, with every two sets of corresponding quantities, x_m, y_m, z_m, etc., x_n, y_n, z_n, etc., whenever a quantity of the first set is subsequent to the cognate quantity of the second, every other quantity of the first must likewise be subsequent to its cognate in the second; whenever a quantity of the first is previous to the cognate quantity of the second set, every other quantity of the first must be previous to its cognate in the second; and whenever a quantity of the first set is neither previous nor subsequent to the cognate quantity of the second, every other quantity of the first must bear a like relation of order to its cognate in the second set.

The simplest case of a functional relation is that in which, with all the variables, every quantity without exception has a unique corresponding quantity in each of the other variables. Such a case we propose to designate as a *consentaneous* functional relation. The functional relations which do not answer this description may be termed *dissentaneous*. If y is a dissentaneous function of x, there may be quantities of x having no corresponding quantities in y, or quantities of y having no corresponding quantities in x, or both; moreover there may be quantities in x each of which has two or more quantities of y corresponding to it, and the like may be true of some of the quantities of y. Still more complicated cases of dissentaneous correspondence may be conceived, all of which are covered by our definition, but as these will hardly be likely to arise in mathematical investigations, it is needless to go into further details. Certain dissentaneous functional relations do however frequently occur, and failure to recognize this is a grave defect in the current definitions (founded on Dirichlet's definition of the phrase "y is a continuous function of x") as we shall show in our discussion of the latter, which also fail to specify likeness of order as an essential feature in a functional relation. Since the fixing of a correspondence between the quantities of two or more variables, and the giving them a like order of arrangement, is entirely a matter of convention, several variables are in functional relation whenever we choose to make them so—any variable without exception can be made a function of any other or any others whatsoever, by the mere adoption of a convention which puts the quantities of the variable into correspondence and makes them alike in order. It is not however always possible to put a set of variables into *consentaneous* functional relation, as here it is requisite that there be a like number of component quantities to each variable. In our view two (or more) variables, no matter how closely they may be connected in the natural occurrence of their quantities, are not functions of each other until made so by a convention, while on the other hand two (or more) variables which show no natural connection at all can be put into functional relation whenever it is so desired. There however, is not usually any practical advantage in thus putting into functional relation variables not connected naturally. Obviously a set of variables can be put into functional relation in more than one way, as there is always room for choice in deciding what quantity or quantities of one variable shall be made to correspond to a given quantity (or quantities) of another. Thus suppose y and x are capable of entering into consentaneous functional relation of such character that the functional formula is $y-2x=0$. Let no two quantities of y be of the same value, and let x be likewise panvariant. Then there is only a single quantity of y to which a particular quantity of x can be made to correspond if $y-2x=0$ is to be true. If a y is a 6, the corresponding quantity of x must be a 3, and ex hypothesi there cannot be more than one 3 in x. Hence any convention, which makes the 6 of the y correspond to any other quantity, puts y and x into an entirely different functional relation, not having as formula $y-2x=0$. As regards the likeness of order of arrangement of the quantities of variables in functional relation, it is to be noted that this does not preclude a change in arrangement, the manner of arranging the quantities being still entirely arbitrary and according to our option. We can take any one of the variables and change its arrangement to what we choose, but, since the variables are in functional relation, such a change in one of them necessitates an analogous change in arrangement of the quantities of each of the others, unless the functional relation is dissentaneous and the change first made is restricted to quantities not concerned in the correspondence between the variables. We need hardly say that "y is a function of x," "x is a function of y," and "y and x are in functional relation" have exactly the same meaning; and this is likewise true of "y is a consentaneous function of x," "x is a consentaneous function of y," and "y and x are in consentaneous functional relation;" and of "y is a dissentaneous function of x," "x is a dissentaneous function of y," and "y and x are in dissentaneous functional relation." And similarly when three or more variables are concerned.

The distinction we purpose to mark by means of the words consentaneous and dissentaneous is in no way tantamount to that already recognized in the Theory of Functions in connection with the adjectives one-valued and multivalued. The former refer to quantities, not to values; and y may be a consentaneous multivalued function of x, or a dissentaneous one-valued function of it, just as well as a consentaneous one-valued or a dissentaneous multivalued function. With a functional relation between the variables y and x, y is said to "be a one-valued function of x at a certain value represented in x" (and as it might also, we think, be put: "the functional relation is one-valued per y at this value of x") whether there be one quantity of the value in x or more than one, and whether corresponding to this quantity or quantities there be in y one quantity or several; it is merely necessary that these corresponding quantities be all of a single value, and there be at least one such corresponding quantity.

A functional relation may or may not have pertaining to it a functional formula. We shall say more about this later on, but in the meantime let it suffice to say that if a functional formula has once been laid down, we hold it must be adhered to, and that though what are obtainable from it by various transformations are formulas *concerning* the functional relation in question, they should not be confused with the true functional formula or given the title "functional." It is in connection with functional formulas and the formulas deduced from them that first arises the distinction customarily made between the *independent* and the *dependent* variables of a functional relation. The lines upon which this distinction is drawn are not very precisely fixed, the distinction made when one matter is under consideration not being at all the same as is made when another aspect of the same functional relation is given attention. First, when there is under consideration a functional formula of the type:

$$y=\varphi(x, z, \quad . \quad . \quad . \quad); \qquad e.\,g.,\ y=x+a+z^2,$$

belonging to a functional relation between two or more variables, $y, x,$ . . . , the variables $x, z,$ . . . are here said to be the independent variables, and y to be the dependent variable. And when reverting we pass to

$$x=\psi(y, \ z, \quad . \quad . \quad . \quad); \qquad x=y-a-z^2$$

x is now understood to have become dependent, the independent variables being $y, z,$ . . . The chief purpose of such a transformation would be to facilitate the finding what value of the variable here made dependent corresponds to a given set of values for the respective independent variables, so that, in this sense, to say a variable is dependent, verges on regarding its symbol as quesitive for the time being. Similarly, with implicit formulas and with the explicit $y=x$, the distinction to be drawn between the independent and the dependent variables is that, at the stage in question of the discussion, one or more sets of values taken by the former are laid down, and there is to be found by means of the formula the corresponding value or values taken by the latter variable. In this first sense, among all the variables of a functional relation, only one (at any particular stage of the discussion) can be a dependent variable, all the rest being necessarily independent. A second distinction between the variables of a functional relation arises in connection with their arrangement in order. In ordinary protomonic Calculus a procedure is adopted, when there are only two variables entering the functional relation, which amounts to first giving one of these an arrangement of its quantities in order of value, and then letting the dependent variable take such an arrangement as will give likeness in order of corresponding quantities. The variable first arranged is that with symbol "x," which is usually the independent one in the sense above, and it would seem sometimes that in speaking of independent and dependent variables the distinction sought to be made is at bottom nothing more or less than that between the variable (the independent in this second sense) whose arrangement is taken as pattern for the arrangement of the others, and these other variables of the functional relation (the dependent variables). In this sense, in every functional relation, under each particular arrangement, there is necessarily one and only one independent variable, all the rest being necessarily dependent. Thirdly, a distinction between independent and dependent variables arises in connection with differentiation. If we differentiate a functional relation, we must do so in respect to one of the variables, which is termed the "independent." Here again all the other variables are necessarily dependent. Besides these three quite distinct and different senses in which "independent" (and "dependent") is used in reference to variables there may possibly be still others. Of the existence of three at least there can however be no doubt at all. Mathematical text-books we need hardly say give a most inadequate account of the matter. Indeed Molk, in the *Encyclopedie des Sciences Mathématiques,* dismisses the matter in a single sentence, telling us that "When the variable x is not regarded as a function of another variable, x is said to be an independent variable."[1] In another

[1] Tome II, Vol. 1 §5, p. 22. The article in the original German edition (by Pringsheim) does not contain this statement.

paragraph of the same article there is mentioned "the most general notion of a function of an independent variable;"[1] so apparently in the sentence first cited Molk does not mean to deny that an independent variable belongs to a functional relation, but is merely using "function" in the sense of "dependent variable," and telling us the momentious fact that an independent variable is one not dependent!

The word "function" is said to have been used by the older analysts as synonymous with "power." The first step towards its use is the modern sense is commonly ascribed to Leibnitz, who in an article: *Considerations sur le difference qu'il y a entre l'Analyse ordinaire et le nouveau Calcul des Transcendantes,* after making some remarks concerning a problem that had been proposed by John Bernouilli, says: "Here is a more general one which includes it [Bernouilli's] with an infinity of others. Given the ratio as M to N between any two functions whatever of the line ACC to find the line. I call functions all the portions of straight lines that are made in constructing the indefinite straight lines which pass between [repondant au] a fixed point and the points of the curve, as are AB or $A\beta$ abscissa, BC or βC ordinate, AC chord, CT or $C\theta$ tangent, CP or $C\pi$ perpendicular, BT or $\beta\theta$ sub-tangent, BP or $B\pi$ sub-perpendicular, AT or $A\theta$ 'resectae' or cut off [retranchée] by the tangent, AP or $A\pi$ cut off by the perpendicular, $T\theta$ and $P\pi$ 'sous retranchée sub-resectae a tangente, vel perpendicular,' TP or $\theta\pi$ corresectae and an infinity of others that can be imagined of a more complicated construction. The problem can always be solved and there is a method of constructing the line, at least

[1] *Op. cit.* §3 p. 13.

by the quadratures or by the rectifications."[1] The meaning given to "function" by this definition is evidently exceedingly remote from its present application; even more so than the earlier meaning of power. No mathematician would now term a tangent (or anything else) a function of a curve, and, if we pass from curves to the equations representing them, we find nothing in modern Mathematics that it is customary to call a function of an equation, and nothing that may be called a function of the fact asserted by an equation. In view of this it is hardly worth while inquiring whether Leibnitz's usage was in strict accord with his definition. Nor need we investigate the usage of other mathematicians up to the time, twenty four years later, at which was formulated the first authoritative definition that really comes in touch with the modern use of the name "function." This definition is that of John Bernouilli. It reads: "We name a quantity composed in any manner whatever, of a variable magnitude and constants, a function of the variable magnitude."[2] This definition would seem to imply that quantities (or "magnitudes)" are of three species: constant quantities, variable quantities and functions—a classification that we need hardly say is entirely erroneous. Even at the present day however it is quite common for writers to use the name "function" in designating a dependent variable of a functional relation, and to put in antithesis to this, "variable" or "argument" as designating an independent variable.

[1] *Opera Omnia*, Geneva 1768, Vol. 3, p. 302, from *Journal des Savans* 1694. In Latin in the *Acta Eruditorum* 1694. See *Opera Omnia*, Vol. 3, p. 300. Two years previously in the *Acta Eruditorum* of 1692 (See *Opera Omnia* Vol. 3, p. 265), Leibnitz had spoken of "tangens, vel aliæ non-nullæ functiones ab ea [the curve] pendentes, verb. gr. perpendiculares ad tangentem ab axe ad curvam ductæ."

[2] *Par. Mem.*, 1718, p. 106.

Such a manner of speaking is quite inexcusable in one who admits that whenever one variable is a function of another the latter is necessarily also a function of the former, and this is now we believe universally admitted. It would be well not to apply the name "function" to any of the variables of a functional relation, but to regard "is a function of" as a convenient though misleading phrase which would be more precisely rendered by "is in functional relation with." It is perfectly justifiable to do this, as in fixing the most suitable scientific use of "function" (or any other word) we need not be bound by its original application.

In 1748, thirty years later, appeared Euler's definition. He says: "A function of a variable quantity is an analytical expression composed in some way of that variable quantity and of numbers or constant quantities."[1] From this definition it would seem that in Euler's mind quantities and variables were not clearly distinguished from the symbols and compound expressions that represent them while otherwise his definition appears to agree with Bernouilli's. The latter indeed by its use of "composed" might be construed to refer to expressions, rather than to what the expressions represent. The most obvious interpretation of the expression-definition is that a functional relation is a relation between a compound expression, which is called the function and a symbol or set of symbols; the relation being that the latter is contained in the former. Thus the expression $3x^2$ would be called a function of the symbol "x," for this symbol is contained in the compound expression $3x^2$. Moreover, although the definitions do not expressly state it, those who accept them would hold

[1] *Introductio in analysin infinitorum* by Leonard Euler, Lausanne 1748, Vol. 1, p. 4.

that in the equation $y=3x^2$, the symbol "y" is a function of the symbol "x," in that the former symbol can be substituted for the expression $3x^2$. Still more remotely removed from the definition is the case in which "y" is said to be an implicit function of "x" through the two symbols being involved in a compound expression which is one number of an equation having as other member a cipher.

Definitions of this type have become entirely obsolete in authoritive works on the Theory of Functions, but when we turn to other branches of Mathematics we sometimes find even an author of repute laying down a definition embodying this antiquated error, and making no reference whatever to the modern definition of functional relation. Thus Burnside and Panton, who, if we may judge by their remarks on functions, are in serene omninescience of the progress made since the middle of the eighteenth century, give the following definition:[1] "A mathematical expression involving a quantity is called a function of that quantity." Similarly, in the *Encyklopaedie der mathematischen Wissenschaften,* Netto defines an integral function as an *expression,* noting that the use of the name function in this and a more general sense dates from Leibnitz, and quoting John Bernouilli's definition as "ganz

[1] *Theory of Equations,* Vol. 1, p. 1. Chrystal, in his *Text Book of Algebra,* gives no general definition of "function," but he says, Part 1, p. 30, "The result of multiplying or dividing any number of letters or numbers one by another addition and subtraction being excluded, for example, $3\times a\times x\times b\div c\div y\times d$ is called a (rational) *monomial algebraic function* of the numbers and letters involved, or simply a *term,*" and on p. 281: "any intelligible concatenation of operations in which the operands selected for notice and called the variables are involved in no other ways than by addition, subtraction, multiplication, division and root extraction is called an Ordinary Algebraical Function of these variables."

modern!"[1] We would not, of course, contend that a doctrine now current is necessarily better than one in vogue a hundred and fifty years ago, but a writer who prefers the antiquated one may at least be expected to justify his choice by some discussion of the merits of the two. In the elementary text-books of Algebra used in schools and colleges the old error is quite common, and here we may fairly attribute it to the ignorance of the authors, who are probably not aware that there once lived a mathematician named Dirichlet.

As might have been expected, the modern doctrine did not arise among those mathematicians who devoted themselves to the solution of general equations and other purely symbolic work, but is due to men who, by their investigations in physical science, came into contact with facts as well as formulas. It developed out of the investigations begun by D'Alembert in the study of the oscillation of cords, in which he was joined by various other scientists. These investigations led to the discovery of the extraordinary properties of the so-called trigonometric series (which are really serial limit expressions and not series at all) with which will always be associated the name of Fourier. It was now made manifest that what at the time of Euler would have been considered as two or more functional relations between as many pairs of variables could be united, in many cases, by the use of such expressions into a single functional relation between two such variables, and finally a definition was framed that entirely cast aside the old trammels under which a functional rela-

[1] Vol. 1, Pt. 1, p. 222: "Ein Ausdruck von der Form . . . . heisst eine ganze Funktion der Variablen z . . . . Der Ausdruck Funktion ruehrt in diesem und in allgemeinerem Sinne von G. W. Leibniz her. Ganz modern definiert schon Joh. Bernouilli 'On appelle, etc.' "

tion was dependent for existence on a mathematical formula.

This definition is due to P. G. Lejeune-Dirichlet who says: "Let by a and b be understood two fixed values, and by x a variable quantity which gradually assumes all values lying between a and b. If now a single finite y corresponds to every x and does this in such manner that while x continuously [stetig] passes through the interval from a to b, $y=f(x)$ likewise varies gradually, then y is called a continuous [stetig oder continuirliche] function of x for this interval. (As in what follows, the discussion will be of continuous functions alone, the adjective may be omitted without disadvantage.) It is thereby quite unnecessary that y in this entire interval should be dependent upon x according to the same law; indeed we need not once think of a dependence expressible by means of mathematical operations. Geometrically represented; that is, considering x and y as abscissas and ordinates, a continuous function appears as a coherent [susammenhaengende][1] curve in which only one point corresponds to every abscissa contained between a and b. This definition ascribes to the several [einzelnen] parts of the curve no common law. We can think of the curve as made up of the most heterogeneous parts or as delineated entirely without law. It follows that such a function is only to be looked upon as completely determined for an interval if it is either given graphically[2] for the entire extent of the

[1] We need hardly remind the reader that Dirichlet's "zusammenhaengend" has not the significance given the word by Cantor some fifty years later, and hence should not be translated "concatenated."

[2] A discussion of the possibility of arbitrarily fixing a functional relation graphically will be found in the *Mathematische Annalen* for 1883, Vol. 22, p. 249 et seq. *Ueber den allgemeinen Functions-*

interval or is subjected mathematically to laws valid for the several parts of the interval. As long as we have determined a function for only a part of the interval, the character of its continuation for the remainder of the interval is entirely arbitrary."[1]

It will be seen that Dirichlet restricts his attention to an interval in which y is a one-valued and continuous function of x, and thus gives a definition relating, not to functional relations in general, but to continuous functional relations. A more general definition is however current to which the name of Dirichlet is usually attached. In Hankel's wording it is as follows: "y is said to be a function of x, if there corresponds to every value of the variable quantity x within a certain interval a fixed value of y, it being immaterial whether throughout the whole interval y is dependent on x according to the same law or not: whether the dependence can be expressed by mathematical operations or not. This purely nominal definition, which I shall in future call Dirichlet's, since it lies at the base of his researches concerning Fourier's series which have conclusively shown the old conception of functional relation to be untenable does not suffice however for the needs of Analysis, as functions of this kind possess no general properties and hence all relations of values of the function for different values of the argument are wanting."[2] Later authors, not content with giving

begriff und dessen Darstellung durch eine willkuerliche Curve by F. Klein. There are also some interesting remarks bearing, in a sense, on this topic by Jourdain: *J. reine & angew. Math.*, Vol. 123 (1905), p. 185 note.

[1] *Ueber die Darstellung ganz wilkuerlichen Functionen durch sinus und cosinusreihen; Werke*, Vol. 1, p. 135. From *Repertorium der Physik*, edited by H. W. Dove and L. Moser, 1837, Vol. 1, p. 152.

[2] H. Hankel: *Untersuchungen ueber die unendlichen oft oszillierenden und unstetigen Funktionen*, Tuebingen, 1870, p. 5.

to this general definition of function the name of "Dirichlet's definition," actually assert he formulated it, and refer in so doing to the place in which he defined, not functional relation, but continuous functional relation. Thus Dini says: "In earlier times the expression function served exclusively for the denotation of the powers of one and the same quantity. . . . In this century Dirichlet (*Dove's Rep. d. Physik,* Vol. 1, p. 152; *Journ. f. Math.,* Vol. 4, p. 157) gave the word function a significance independent of any assumption of the possibility of an analytical representation, and called a function of a real variable x any quantity y which for every special value of x within the interval the frontiers [Grenzen] included has a single fixed value which is known or can be found no matter whether our knowledge of this value of y is attained by analytical operations upon the variable x or is attained in any way whatever."[1] The passage by Dirichlet in the *Rep. d. Physik* to which Dini refers has just been quoted by us. As to the fourth volume of the *Journ. f. Math.,* there is on page 157 an article by Dirichlet: *Sur la convergence des series trigonometrique qui servent à representer une fonction arbitraire entre les limites donnees,* but this contains no definition bearing on functional relation. Harkness and Morley say:[2] "Dirichlet defined y as a function of x in the following manner—Let x take certain values in an interval (x_0 to x_1); if y possess a definite value or definite values for each of these, y is said to be a function of x. It is not necessary that y should be related to x by any law or arithmetic expression. Moreover

[1] *Grundlagen fuer eine Theorie der Functionen einer veraenderlichen reelen Groesse,* by Ulisse Dini, translated into German by J. Lueroth and A. Schepp, Lpz., 1892, p. 48.

[2] *Treatise on the Theory of Functions,* New York and London, 1893, p. 53.

the function may be defined for certain values only of x within the interval (x_0 to x_1) *e. g.*, for all rational numbers; but usually it is defined for all the values within the interval (*Repertorium der Physik* her. v. Dove t. 1, p. 152). According to this definition the values of y when $x=a$, may be entirely unrelated to the value of y for any other value of x, $x=b$. This definition in contrast to those used before Dirichlet's time, errs on the side of excessive generality; for it does not of itself confer properties on the function. The functions so defined must be subject to restrictive conditions before they can be used in analysis. Nevertheless, this definition forms and must continue to form the basis for researches upon discontinuous functions of a real variable. In Dirichlet's sense $f(x)$ is a function of x throughout an interval when, to every value of x within the interval belongs a definite value of $f(x)$. A value of x which makes the function infinity is excluded."

Professor W. F. Osgood in his *Lehrbuch der Functionentheorie* tells us that "according to Dirichlet $f(x)$ is called a function of x when for every value of x belonging to an interval $a \leqq x \leqq b$ there is assigned according to a fixed law a second value $f(x)$."[1] Pringsheim makes the usual blunder, but accentuates it by prefixing "one-valued" [eindeutige] to "function" in the definition he ascribes to Dirichlet, while omitting to prefix "continuous." He says: "Dirichlet defined y as a (one-valued) function of x in the interval (a, b) when to every value $a \leqq x \leqq b$ there corresponds a fixed value of y without regard to the manner in which be effectuated the assignment of the y values to the separate x's."[2]

In these formulations of the so-called definition of

[1] Vol. 1, p. 1.

[2] *Encyk. math. Wiss.*, Vol. 2, Part 1, p. 7.

Dirichlet it is not intended to preclude y from having the same value for two different values of x, though a reader not already acquainted with the subject might well so construe them. They would all however seem to exact a requirement which may be expressed in our terminology by saying that when x has a quantity of a certain value, a corresponding value must be represented in y by a quantity of the latter variable; or, in a more conventional phraseology, when x takes a certain value y must take a corresponding value. All consentaneous functional relations fulfill this requirement and some dissentaneous ones can be found which do so. Thus with a dissentaneous functional relation there might be several quantities in x of each of certain values, and one of these quantities, in the case of each value, might have a corresponding quantity in y, while the duplicate x's did not, there being moreover a unique y corresponding to every x not of one of these values. But aside from this and a few other cases, a relation between variables that we would express by saying "y is a dissentaneous function of x" would seem to be excluded from the title "functional relation" by the definition under discussion. The stipulation that "the function may be defined for certain values only of x" does not, if we interpret it rightly, have dissentaneous functional relations in view. No specific explanation of its meaning is given by the mathematicians who use this and analogous phrases involving "defined for a value," but we apprehend the idea to be that when x is for a moment hypothetically assigned a value between x_0 and x_1—the interval between these being merely a range of values including all possible values intermediate between x_0 and x_1, without regard to whether x contains quantities of each such value or not—there may be found to be no value of y corresponding to this value of x, in which case the hypothesis that x takes the value in question becomes untenable. And the way this would be ascertained would usually be by the functional formula no longer indicating a possible process. Thus another author,[1] whose definition we shall quote later on, says in a note to this definition: "For example, the function $f(x)$ can be defined for only the case in which x is an integral positive number. This occurs when we put $f(x)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ .\ \ .\ \ .+\frac{1}{x^2}$." We think the interpretation just set forth of the clause "the function may be defined for certain values only of x" is more favorable to the authors who use it than any other. For if we take it to mean that x may possess a value within the interval without y possessing a corresponding definite value or values, this flatly contradicts the previous statement that y must possess a definite value or definite values for each value of x within the interval. And it is no compliment to an author to suppose he uses two contradictory statements to assert what any one not absolutely illiterate would express by replacing "each of these" in the first statement by "each of some or all of these," and omitting the second. Moreover, it must be remembered that the authors who say "y may be defined for certain values only of x" say nothing from which we can infer they regard their formulation of the "Dirichlet definition" as fundamentally different from that of such authors as Hankel and Dini.

We are not then unfair in holding all four of the quotations above (from Dini, from Harkness and Morley, from Osgood and from Pringsheim) to so formulate the definition attributed to Dirichlet that if y is a function of x "in an interval," and if a value comprised in this interval is represented in x, there must necessarily be in y a corre-

[1] Durège.

sponding quantity of some value or other. Now with this restriction, the definition cannot be regarded as the most general possible definition of functional relation; it cannot even be regarded as general enough to accord with ordinary usage, and as Dirichlet may have perceived this and on that account used the phrase "stetige oder continuirliche Function" instead of the word "function" without an adjective, we think it unjust to ascribe to him an erroneous definition that he did not give. That the definition is erroneous can be made plain to the veriest tyro in Mathematics. For if the variable y is a function of the variable x, the functional formula being $y=\psi(x)$, and we differentiate and obtain $\frac{dy}{dx}=\psi'(x)$, $\frac{dy}{dx}$ or y' is here a third variable, which like y is said by mathematicians to be a function of x. Suppose x comprises quantities of certain values, for instance suppose x contains a unique quantity of every protomonic abstract value, and for each quantity of x there is a corresponding quantity in y. In this case it would be said that whenever x "takes" a protomonic abstract value, y possesses a definite value corresponding. Will any mathematician contend that $\frac{dy}{dx}$ must likewise "possess a definite value" for every value of x that is abstract and protomonic? Or will any one say that if $\frac{dy}{dx}$ fails to "possess a definite value" for every protomonic abstract value of x, usage does not justify us in terming $\frac{dy}{dx}$ a function of x? What *can* be said is that in such cases $\frac{dy}{dx}$ would not be called a *continuous* function of x,[1] on which

[1] It might of course be a continuous function of x throughout an interval *smaller* than that comprising all protomonic abstract values. In other words, in such cases $\frac{dy}{dx}$ is not a continuous function of x throughout the whole of the original interval, though it may be this throughout one or several sub-intervals.

account Dirichlet does not stand convicted of the errors of his successors. It may also be noted that while in defining continuous functional relation the introduction of the phrase "in an interval" has a purpose, since for one interval y may be a *continuous* function of x and for another not, its use in connection with a general definition of functional relation is quite absurd. A variable y cannot be a function of a second variable x in one interval and not be a function of x in another interval. Of course, when y is a function of x, for a given interval of values there may or may not be representatives of these values in x, and in the former case there may or may not be corresponding quantities in y. But to say, that y, though a function of x, is not a function of x in a certain interval, because no values of this interval are represented in x by quantities having corresponding quantities in y, is a clumsy and misleading manner of speaking, and has no compensating advantages that we are able to perceive.

To acquit Dirichlet of the errors made by his commentators is one thing; to say he furnished a satisfactory definition of functional relation in general is quite another. In reality, as we have seen, he did not even attempt to formulate a general definition. Dirichlet's definition has played an important part in broadening the ideas of mathematicians concerning functional relations, but it must not on that account be looked upon as something it is not, and was never intended to be. We might indeed suppose from this definition of continuous functional relation that Dirichlet was aware how functional relation in general should be defined; in fact when he speaks of "each x" he almost seems to have reached the point of realizing that a variable, instead of being a quantity (or a symbol), is composed of quantities. If however we assume he knew all this, the question at once arises why he omitted to disclose his knowledge to the world? Why did he not define "variable," and refute the prevalent errors as to the nature of variables? Why did he fail to give a general definition of functional relation? If he held the correct views he could have had no possible reason for failing to give an exposition of them in his works; and the most natural conclusion is that the doctrines one would read into his words are not really there: that, though he had attained a point of view which enabled him to avoid giving an erroneous definition of functional relation, he was not in a position to formulate a correct one. Similarly we may infer that his ideas as to variables had not become sufficiently clarified to enable him to recognize their true nature.

In no work have we been able to find a discussion of the difference between Dirichlet's definition and that which was designated as his by Hankel, or any recognition of the fact that suppression of the word "continuous" from Dirichlet's definition necessitates any other change besides a change in the phrase "varies gradually." In none of the general definitions of functional relation that we have seen is it clearly stated that there may be values represented in x without there being corresponding values represented in y, and as an author who recognized this might be expected to make an unambiguous statement to that effect, and distinguish his definition from those usually laid down, we are compelled to conclude that mathematicians have completely overlooked the glaring defect of the so-called Dirichlet definition.

Durège, as we have seen, regards a variable as an aggregate, without however thereby attaining an insight into the true nature of variables. His definition of function does not avoid the usual errors, and indeed the aggregate phraseology makes them if anything more apparent. What he says is: "The most general conception of function . . . When a fixed number-value y is put into correspondence [zugeordnet] with every member x of an aggregate of numbers W, then y is called a function of the quantity x, and the relation is expressed by the notation $y=f(x)$. It is herewith immaterial whether the aggregate of numbers W is discrete or whether it is, at least in an interval, pantachisch [ueberalldicht]. (The function $f(x)$ can for example be defined only for the case of x being an integral positive number. This occurs when we put $f(x)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ .\ \ .\ \ .+\frac{1}{x^2}$). It is further immaterial in what manner the correspondence is defined, it is only essential that this shall be fixed unambiguously."[1] We even find this author on occasions deliberately going back to the antediluvian error of speaking of a function as an expression. Thus he says: "By an integral rational function of the complex variable z we understand—in harmony with the definition made use of in the domain of the real variables—an expression of the form

$$w=f(x)=c_0+c_1z+c_2z^2+c_3z^3+.\ \ .\ \ .+c_nz^n$$

where c_0, c_1, c_2, . . . signify any [beliebige] complex constants. The quotient of two integral rational functions we designate as a fractional function."[2]

Equally plain in his promulgation of error is another mathematician of high standing, Thomae, "The quantity y is said to be a function of x in the interval from a to b when a number y is assigned [zugeordnet] to every number x between a and b. Hence a function of x is a kind of table in which for every number x a corresponding number y is set down . . . In general such a function is not

[1] *Op. cit.*, p. 23, §3.
[2] *Op. cit.*, p. 75.

reversible, for, on the one hand, the value of y need not coherently [? zusammenhangende] fill out an interval, on the other, to one y there may belong an innumerable aggregate of values of x so that for a y the x is not unambiguously determined."[1] If we construe literally the second sentence of this, we must understand that, when Thomae says y is a function of x, he means by y not a variable, but a table written or printed on paper. Of course he means nothing of the sort, and his language seems due to the too common impression that the only way to explain the meaning of a name is to append to it a phrase beginning with "is."

In the *Vorlesungen ueber Algebra* of Dr. Gustav Bauer ("Geheimrat, O. Professor an der Universitaet Muenchen"), published in 1910, we find: "A quantity which depends on one or more variables and hence is itself a variable, since it alters its value when the value of the variable alters, is called a function of this variable."[2] The blunder of thinking that in a functional relation between two variables the one variable *necessarily* alters its value when the value of the other alters is, we hope, so far obsolescent as to be peculiar at the present day to the learned ordentliche Professor of the University of Munich. It may be a propos here to recall that even a specific name "invariability stretch" [Tratta d'invariabilita; Invariantszuege] has been coined by mathematicians for reference to intervals where there is no such alteration in value of the dependent variable, though there is the usual alteration in value of the independent.

The true author of the definition ordinarily attributed to Dirichlet, in so far as it is erroneous is probably Riemann,

[1] *Einleitung in die Theorie der bestimmten Integrale*, Halle, 1875, p. 4.

[2] P. 1, §1.

though the latter may possibly have been anticipated in error by some earlier writer we have overlooked. In Riemann's *Grundlagen fuer eine allgemeine Theorie der Functionen einer veraenderlichen complexen Groesse,* his inaugural dissertation at Goettingen in 1851, we find a definition which, though not purporting to have as wide a scope as the general definitions of functional relation that have just been examined, exhibits all the vices of these. Riemann says: "Let by *z* be understood a variable quantity which can in turn take all possible real values, if to each of its values there corresponds a single value of the undetermined quantity [unbestimmte Groesse] *w*, *w* is then called a function of *z*, and if while *z* continuously [stetig] passes through all values lying between two fixed values *w* also varies continuously, then this function within the interval is said to be continuous [stetig oder continuirlich]."[1] Riemann then goes on to make the usual remarks, that under this definition when *w* has been fixed for a certain interval, nothing is thereby implied as to the manner of its continuation beyond this interval, and that it is immaterial whether one defines the dependence of *w* on *z* given arbitrarily, or as conditioned through fixed quantitative operations. The two concepts are "congruent," since there have been found to be analytical expressions by means of which there can be represented for a given interval any continuous function whatever [eine jede stetige Function]. Riemann designated as a dependence expressible by *quantitative operations* [Groessenoperationen] "any dependence expressible by means of the four most simple operations of calculation [Rechnungsoperationen] addition and subtraction, multiplication and division. The expression 'quantitative operation'

[1] *Gesammelte mathematische Werke,* 2nd. Ed. Lpz., 1892, p. 1.

designates (in opposition to 'numerical operation' [Zahlen-operation] such operations of calculation with which the commensurability of the quantities does not come into consideration."

Riemann however, holds that "matters are quite different when the variation of the quantity z is not restricted to real values, but is allowed to take complex ones of the form $x+yi$." Here it is true that "no matter in what way w may be defined as a function of z by combinations of the simple quantitative operations the value of the differential quotient $\frac{dw}{dz}$ will always be independent of the particular value of the differential dz" but "if the dependence of the quantity w on z is fixed arbitrarily" the relation $\frac{du+dvi}{dx+dyi}$ [$u+vi$ being put for w, and $x+yi$ for z] will generally speaking alter with the values of dx and dy." And evidently then one cannot take "any dependence he chooses of the complex quantity w on the complex quantity z and express it by combinations of the simple quantitative operations." On this account Riemann lays down the following definition: "A variable complex quantity w is called a function of another variable complex quantity z when the former so varies with the latter that the value of the differential quotient $\frac{dw}{dz}$ is independent of the value of the differential dz."

In this last definition "variable complex quantity" is used, not in reference to a variable composed of complex quantities, as one might suppose on merely reading the definition, but to designate a variable which *may* take complex values (and here probably Riemann understood by "complex value" any value not protomonic, whether it be neomonic or truly complex). And, as to say that complex values are *admissible* [zugelassen] with a variable does not *necessitate* this variable actually taking such values, a complex variable in Riemann's sense may be composed entirely of protomonic quantities, or composed entirely of neomonic quantities, or composed entirely of complex quantities, or composed of both complex and neomonic quantities, or of both complex and protomonic quantities, or of both protomonic and neomonic quantities, or finally may contain complex quantities, neomonic quantities and protomonic quantities. In short "variable complex quantity" or "complex variable" means nothing more or less than the single word "variable." The phraseology can hardly be regarded as conducive to clearness, but inability to use language with precision seems to be a failing endemic among mathematicians, and Riemann was not immune. Of course if fellowship in error be accepted as a justification, Riemann can be wholly absolved, since this absurd use of "complex variable" is with mathematicians the rule rather than the exception. The definition in which Riemann lays down the conditions under which one "complex variable" is to be termed a function of another cannot however be admitted to be a general definition of functional relation, since, though z and w be two variables, the expression $\frac{dw}{dz}$ (and the name "differential quotient)" is utterly without meaning unless w be a function of z (that is, unless there be a like order of corresponding quantities of w and z).[1] The definition in question then simply specifies a particular type of functional relation (in designating which the adjective "mono-

[1] Any one who takes the step of denying that consideration of functional relation is necessary in the theory of differentiation might as well go further, and deny the conception of functional relation to be of any service to Mathematics. For in no part of mathematical science is this conception more indispensable than in the theory of differentiation.

genic" is sometimes used); a type that is to be the subject of the investigations of Riemann's dissertation, and is to be there (by another wanton misuse of language) called by the name "function" to the exclusion of other types. We might hence expect to find a general definition given by Riemann elsewhere in the dissertation, but none is at hand. One might almost think that he failed to see his definition of a function of a complex variable would involve the fallacy of *circulus in definiendo* unless it rested on a previous definition of functional relation in general, and that he regarded his two definitions, this one pertaining to cases in which complex values are admissible with z, and the definition previously given pertaining to cases in which z can take only real values, as together constituting a satisfactory general definition of functional relation, which would thus be defined as anything belonging to either of the two species. We shall not however impute this gross absurdity to Riemann, but he cannot be acquitted of the use of language slovenly enough to imply it. Differentiation having not yet been taken up, we cannot now fully discuss Riemann's remarks concerning $\frac{dw}{dz}$ and its independence of or dependence on dz, but we may say here that such a dependence of $\frac{dw}{dz}$ on dz can only be said to exist by giving to the symbol "dz" a non-natural significance. In the cases where such a dependence is said to exist, the value of $\frac{dw}{dz}$ at z_1, a specified quantity of z, depends in fact on the *path of differentiation*. That is, if z_1 has adjacent to it more than one domain throughout which it is approached or inversely approached as a limit, each such domain (each path) usually gives rise to a different $\frac{dw}{dz}$ at z_1, the difference being not merely one as to identity but also as to *value*. By *arbitrarily* assigning to each of the different $\frac{dw}{dz}$'s (of course not merely at this particular z but everywhere) a corresponding quantity (which is usually most conveniently made a non-zero) and including all the latter quantities in a variable which is *improperly called "dz,"* one may speak of the variables $\frac{dw}{dz}$, z and dz as being in functional relation, and hence of $\frac{dw}{dz}$ as dependent on dz as well as on z. But evidently this is giving a most unnatural significance to "*dz*," which would naturally signify a variable composed of zeroes, of the limits of the Δz variables. We hardly think that the average student would be likely, from Riemann's remarks, to attain a knowledge of the actual facts in the case, and he would even be very liable to get a confused idea that $\frac{dw}{dz}$ was dependent on (was in functional relation with) a Δz variable.

It is possible that Riemann regarded as the general definition of functional relation what one obtains by omitting from his first sentence the clause "which can in turn take all possible real values." We have then substantially the definition usually attributed to Dirichlet, and in what is thus obtained all that is good must be credited to Dirichlet, the part that seems original with Riemann being hopelessly, irredeemably bad. We need not however take this liberty with Riemann's text as even without so doing his definition can be seen to contain all the vices of the so-called Dirichlet definition. For even of we restrict our attention to variables containing only protomonic quantities, it is not true that mathematical usage permits us to call the variable w a function of the variable z only when there is a value of w corresponding to every value possessed by z. And the fact that in the next breath Riemann proceeds to define continuous functional relation (in reference to protomonic variables) only aggravates the blunder, since any one of moderate intellectual capacity might be reasonably expected to perceive the role played by "continuous" in Dirichlet's definition when he had once given attention to the matter.

One more definition we shall quote: that of Tannery. "This is the most general notion that one can have of the function of a variable: it consists in the correspondence of the numbers of another aggregate (the values of the variable) and the formula $y=f(x)$ indicates no other thing than this determined correspondence."[1] This definition fails to plainly state that two variables may be in functional relation and yet there be components of the one variable having nothing corresponding in the other; moreover the inference may be drawn that this was not in the mind of the author, as there is a total absence of any remark to indicate that his definition was put forward as essentially different from the conventional one. We may sum up then our examination of definitions of functional relations by saying that in none of them hitherto given is there to be found a distinct recognition of any functional relations save those we have termed consentaneous, and in none is anything at all stated as to the arrangement in order of the variables concerned, a matter of paramount importance with functional relations. For it will hardly be contended that to speak of putting the objects of two sets into correspondence is to say anything about a likeness or unlikeness in order. Likeness in order implies correspondence, but correspondence does not imply likeness in order, and may subsist where the latter does not.

[1] *Mathematiques Pure* (in *De la Methode dans les Sciences*), p. 56.

INDEX OF SURNAMES.

INDEX OF TOPICS.

FUTURE PARTS OF FUNDAMENTAL CONCEPTIONS OF MODERN MATHEMATICS.

That portion of *Fundamental Conceptions of Modern Mathematics* dealing with Algebraic Mathematics will consist of thirteen parts. A synopsis of the contents of the last twelve of these parts is here given. The authors would be glad to have any suggestions toward improving the present redaction of these later parts that may occur to a reader of this synopsis and would also appreciate comments in criticism of Part I.[1]

PART II.—DOMAINS AND RANGES.

Need of a name to denote a portion of a variable. Domain the most suitable. Current use of domain [Bereich]. Sub-domains. Adjacent previously and subsequently. Invariant domains. Monotonous and univariant domains. Increasingly and decreasingly univariant. Panvariant domains. Ranges. Intervals. Domains covered by an interval. Collinear and tralinear. Invariability stretches and points of invariability. Diagrams. Coherent diagrams Closed diagrams. Contours. Linear and superficial figures. Closes. Regions. T-regions [Open regions]. B-regions [Closed regions, Complete regions]. Office of the adjective in the phrase "complete region." Determinatives, restrictives, amplifiers, explicatives, alienatives, dilitatives. Alienatives, explicatives and dilitatives quite common in Mathematics, and liable to give rise to confusion of thought if the true offices of such adjectives are not clearly recognized. Partially closed region. Other species of regions. Superficial ranges. Diagrammatic ranges. Lacunary spaces. Realms. Neighborhoods. Sinistro-lateral, dextro-lateral, bilateral and panlateral neighborhoods. Environments and vicinities.

[1] Communications may be addressed to either R. P. Richardson, 5010 Parkside Ave., Philadelphia, Pa., or E. H. Landis, Cynwyd, Pa.

PART III.—LIMITS, BOUNDS AND APPANAGES.

Use of "innumerable" in preferance to "infinite." Difficulties in accepting, as existent, variables containing innumerable quantities. Conceivability of such cases. Mathematics takes all conceivable cases into account, and must begin by considering those ruled by the simplest laws. Such cases, however, are not what usually come to hand in our actual universe. Two types of hypothesis should be recognized by science; those of one type being capable of verification or refutation; those of the other (examplified by any hypothesis concerning the existence of innumerable objects) being capable of refutation but never of verification. Incoherent use of "infinite" by certain mathematicians. Approach to a limit. This is not necessarily incessant. Definition of incessant approach to a limit. Necessity of speaking of domains of approach. A variable may approach the same limit throughout several different domains. Number of quantities in a domain of approach. Limits are not included in their domains of approach. In reference to limits, predesignated non-zero should be used where most authors use assignable quantity. Two species of limits: terminals, which are limits present in their variables; and confines, which are not present. Limiting values. In the quadrature of curvilinear figures certain limiting values are, not *inferentially* but by *definition*, the values of the areas of these figures. The like holds as to lengths in the rectification of curved lines. Definition of approach to a limit. Inverse approach and inverse limits. Incessant increase and decrease to a limit. Oscillation to a limit. Gyration to a limit. Incessant increase without upper and incessant decrease without lower bound. Increase without upper and decrease without lower bound. Numerical approach and numerical limits. Recession from an anti-limit. Approach not the only relation of a domain to its limit. Conformity to a limit. Vacillation to a limit. Inverse conformity and vacillation. Erroneous view that so-called limiting points are perfectly analogous to limits. Definitions based on this view found in current mathematical works, and what they improperly sanction including under the title limit. Convergence to a limit includes approach, vacillation and conformity. Convergent sequences. Non-convergent sequences are divergent or aberrant. Wanton misuse by mathematicians of divergent in sense of non-convergent as well as in its proper sense. Convergence with multiplicative sequences. Sequences of numerable terms are not convergent or divergent or aberrant. Backward sequences. Numerical convergence, divergence and aberrance. Pesudo-divergence. Absolute and non-absolute convergence. Conditional and unconditional convergence. Consideration of the so-called upper and lower Limes and Grenze is necessary in a discussion of limit. Improper use of upper and lower limit for upper and lower Grenze. Most suitable names are upper and lower bound. Maxima and minima. Extremes. Maximants and minimants. Use of maximum and minimum in Differential Calculus. Summit and immit preferable for such cases. Definition of Du Bois Reymond. Sub-domains as summits and immits. Definitions of upper bound and lower bound. Pringsheim's treatment of limits and upper and lower Limes. Upper and lower Limes with domains not necessarily discrete. Inceptive and desitive sub-domains. Sinistro-lateral and dextro-lateral upper and lower Limes at a. Deviation of these definitions from those now current. Inadequacy of the treatment given limits in the *Encyklopaedie der mathematischen Wissenschaften*. The process which Pringsheim regards as generalization of the concept Limes, and its relation to true generalization. Defects of the limit and Limes terminology accepted in the *Encyklopaedie*. As names, upper and lower appanage are preferable to upper and lower Limes. Use of limit in sense of frontier—frontiers of integration. So-called limits of the roots of an equation, more suitably called diorisms.

Part IV.—Continuity and the Incommensurable Quantities.

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Part V.—The Transfinites.

Articulate [wohlgeordnet] aggregates. Inversely articulate. Cantor regards articulations [Ordnungszahlen] and other arrangements as quantities—Ordnungstypen. Cantor's failure to distinguish between likeness and identity with articulations. Articulations of the same type. Commutation and permutation. Improper use by Cantor of same symbols for articulation values as for abstract values. Transfinite articulations. Cantor's erroneous belief that there are fractional articulations. Impletions [Maechtigkeiten]; these are abstract. Abzahlbar. The transfinite impletion values: Alef-null, Alef-eins, etc. Transfinites should not be called quantities. Cantor's transfinites do not justify designating increase without upper bound as increase towards infinity as limit.

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Transformations of dictions. Mutation. Metathesis. Dissociation. Association. Transposition [Antithesis]. Avulsion. Paraplerosis. Egestion. Ingestion. Abbreviation. Inversion. Absolute inversion. Distribution. Composition. Transversion. Vieta's parabolism and hypobibasm; neglect by modern mathematicians of the important distinction marked by these words. Cancellation. Enantiosis. Climacticism. Climactic descent, ascent and interscent. Nomenclature with transformations of proportions. Alternation. Inversion. Syllepsis. Dialepsis. Commixion. Transformations concerned with more than one diction. Libration, perturbed and unperturbed. Syncrisis; Vieta's use of the term. Correlate equations with Vieta. Substitution. Metastasis. Hypostasis Duplicate hypostasis and parabolic hypostasis with Vieta. Transmutatio per alterationem radicis. Expurgatio per uncias. Uncia. Proton-eschaton. Anastrophe. Consideration of transformations in light of their purpose. Abridgment. Isomeria [Conversion]. Elevation. Depression. Plasma. Expansion. Development. Involopment. Binomial expansion. Diagonals and complements with Oughtred. Gnomons. Contraction. Aggregation. Dissemination. Elimination. Eliminants. The vanishing of the eliminant, usually regarded by mathematicians as a necessary and sufficient condition for the consistency of the equations from which the eliminant is derived, is in truth neither a necessary nor a sufficient condition. Limitative [Restrictive] coefficient as used by Cauchy. Dialytic method of Sylvester. Dialysis. Canonization. Harriot's original, canonical and common equations. Use of canonical form in the Theory of Quantics. Epanorthosis. Segregation. Reversion. Transformations of the Theory of Quantics and of the Theory of Groups. Operations of the latter theory. Transformations. Transform. Transformee [Primitive]. Spurious [Identical] transformation. Improper use of "unchanged" and "same transformation" by mathematicians. Equiergic transformations. Inverse. Conjugate. Transforming a transformation. Linear, quadratic, cubic, etc., transformations. Primitives. Modulus of transformation. Unimodular. Singular. Like [Similar, Concurrent] transformations. Impropriety of using inverse in this case. Cogredient. Contragredient. Groups of linear transformations. Automorphic expressions. Interchanges [Substitutions]. Monocyclic [Circular] and polycyclic interchanges. Cycles. Transpositions. Regular. Similar. Product of interchanges. Commutable [Permutable]. Commutative group. Commutators. Odd and even interchanges. Symmetric expressions. Symmetric groups. Alternating groups. Alternating expressions. Homotypical expressions. Requirement exacted of transformations constituting a group. Equivalent expressions. Degree and order of a group. Simply isomorphic groups. Holohedrally [Multiply] and merohedrally isomorphic. Cogredient and contragredient isomorphism of a group with itself.

Part IX.—Functional Relations.

Essential characteristics of a functional relation. Consentaneous and dissentaneous functional relations. One-valued and multivalued. Use of per. Monotropic and polytropic. Branch of a variable of a functional relation. Monodromic and polydromic. Independent and dependent variables. Definitions of "Function" given by Leibnitz, Bernouilli, Euler. Dirichlet's definition. The definition called Dirichlet's by Hankel. Dini, Harkness and Morley, Osgood, Pringsheim on Dirichlet's definition. Error of the so-called Dirichlet definition. Durege, Thomae and Bauer on the definition of "Function." Rieman. Definition given by Tannery. Functional dictions. Ostensive and inostensive functional relations. Partitively ostensive. Ostensible and inostensible. Functional relations explicit per a variable. Implicit per a variable. Implicit functional relations. Euler's use of "implicit" as an alienative and of "explicit" as an explicative. Changing the formula of a functional relation should be held to change the functional relation itself into a new one. Inverse functional relations. Notation $y=\varphi(x)$ and $\varphi(y,x)=0$. An equation, to render a functional relation ostensive, must not be recusant ["identical"] as regards the symbol of any of the variables. And with at least one variable the recusant and discursive instances of the equation must all be evaluable. The equation may be redditive ["determinate conditional"] as regards the symbols of all the variables. Range of ostensibility of a formula. Indeterminate forms. The form $\frac{0}{0}$ is not discursive ["indeterminate"], but recusant ["identical"]. Ought such forms as 2^{π} be called indeterminate? Evaluation. "Function" improperly defined. Limitary functional relations. Direct, indirect and indirect limiting functional values. Symbols of constants; these sometimes denote variables. Hyperfunctional formulas. Parameters. Other use of "parameter." Parametric equations. Unicursal curves. Unicursal surfaces. Transcendental and algorithmetic [algebraic] functional relations. Stirpal [rational] and radical [irrational]. Fractional and integral misnomers as used in reference to "functions." Serials and serial functional relations. The latter are not necessarily limit functional relations; and serials (so-called series) may be polynomials or simple monomials. A serial may represent series but it does not denote them. A limit serial denotes the limits of the series it represents. Power serials. In ascending and decending powers. Positive integral and negative integral power serials. Laurent serials. Fractional serials. Recurrent serials. Determinants are improperly called "functions." They are not necessarily even functional expressions. Quantics likewise are not "functions." Use of designations "differential coefficient" and "variable" with quantics. Cossic a more suitable name than "variable." So-called numerical and literal coefficients. The use of "literal" especially bad. Intactual a suitable substitute. Uncial coefficients. Inuncials. Uncial, inuncial and pan-intactual coefficients. Intactual degree [Degree] and cossic degree [Order]. Degorder. Unipartite and multipartite quantics. Tantipartite. Binary, ternary, quaternary, etc. Syzygetic, quadric, cubic, etc. Definition of unipartite quantic. Algebraic form. Equation of a quantic. Nilfactum. Linear and quadratic, and the distinction between these and syzygetic and quadric.

Part X.—Differentiation.

Consideration properly due first to specific functional relations and their differentiation rather than to general formulas (which are all that the text-books consider). Δx_1 variables. Δy_1 variables. $\frac{\Delta y_1}{\Delta x_1}$ variables. Limits of these variables. First derived quantity at x_1. First derivative. Primitive [Anti-derivative]. Second, etc., derived quantities and derivatives. Differential equations. Definite or quantitative differentiation and formula differentiation. Differential coefficients. The conventional mathematical works deal with hyperfunctional formulas; not with specific functional relations. Absurdity of terming $\frac{dy}{dx}$ the limit of $\frac{\Delta y}{\Delta x}$. Even when only a single functional relation is in question, $\frac{\Delta y}{\Delta x}$ is usually a class name for a host of variables. Cases of dissentaneous functional relations. Cases in which y and x are not abstract and are of different sorts. Here what are really dealt with are the corresponding quantuplicity variables. Derivative quantities. Derivate variables: total, sinistro-lateral, dextro-lateral. Du Bois Reymond's middle derivate quantity. Agnate quantities and agnate variables. Differentiation in Quaternions. Here the variable obtained by differentiation is a derivate, not a derivative, and a third variable enters the derived functional relation. This third variable is masked by the conventional formula. Only collinear domains taken into account in Quaternion differentiation. Advantage in leaving other domains out of account. Differentials. Their use a minor matter and not really characteristic of Quaternion differentiation. When used to denote a differential dx is taken in a non-natural sense, $\frac{dy}{dx}$ not being dependent on dx properly speaking, but on the path of differentiation. The Method of Infinitesimals. Berkeley's exposure of the errors of Newton and Leibnitz. Merits of Berkeley's work seldom recognized by mathematicians. No systematic account of the Infinitesimal Method to be found. Infinitesimal and infinite quantities of the first order, second order, etc. Relative greatness of infinitesimal finite and infinite quantities of no real moment in the Infinitesimal Method. The real essentials are certain laws of addition and multiplication. Requirements by this method for the existence of a first derived quantity. Unjustified assumption by Infinitesimalists that these conditions are fulfilled when those of the Method of Limits are fulfilled. Absurd introduction of limits in the Infinitesimal Method. Error of thinking that in the Method of Limits Δx_1 and expressions involving it cannot be dropped from the equations at an early stage. Those expressions can be dropped which represent variables whose quotients by Δx_1 are comminuent with the latter, provided later on nothing analogous to "division" by a higher "power" of Δx_1 than the first is allowed to take place. Neglect to stipulate that such conditions must be fulfilled, in order for an expression to be legitimately dropped, is a typical error of the Infinitesimalist. Use of name infinitesimal to denote variables.

PART XI.—INTEGRATION.

Quadrature of a functional relation over a continuous unifarious domain. Numerical maximum of a set of links to which such a quadrature pertains. Use of "norm" in a sense approximating to this. Quadrature variables and their limits. Integrals. Functional relation integrable over a domain. Quantitative [Definite] and formula [indefinite] integration. The fundamental law of Integral Calculus. Frontiers of integration. Upper integral. Lower integral. Darboux's theorem. Integration over a unifarious domain whose continuity is broken by the absence of frontier quantities: and by the absence of quantities in one or more but not innumerable other isolated places. Integration over a continuous unifarious domain having no frontier at beginning or at end or at both on account of its "extending to infinity." Principal value of an integral. The integration process of Lebesgue. Lebesgue quadrature variables. Lebesgue integrals. Young's method of integrating. The Borel-Lebesgue measure of a set of points. Descriptive and constructive definition of this according to Lebesgue. Exterior measure. Complement of a set of points. Interior measure.

PART XII.—CONTINUITY WITH FUNCTIONAL RELATIONS.

Conventional definition of function continuous at a point. Of function continuous throughout a region and throughout an interval. Partitively continuous. Forwardly and backwardly continuous. The definition of function continuous throughout an interval not based on any doctrine of continuity connected with Cantor's investigations. In the light of these the so-called definition is a mere criterion. And even as such it is not entirely satisfactory on its face. Riemann's definition still worse. Fluctuation of a domain. Leap of a unifarious domain at a quantity. Sinistro-lateral and dextro-lateral leap. Leap values. Vibration of a unifarious domain at a quantity. The leap definition of continuity of a function throughout an interval. As a criterion it is as defective as Riemann's definition. The usual definition even is, at most, a criterion for joint continuity of x and y where x is panvariant. Precise meaning of "y is a continuous function of x throughout an interval." Functional relation continuous throughout an interval of one of its variables. Definition of continuity at a point is a criterion sometimes of continuity around a quantity, but sometimes of what might be called continuousability around a quantity. Encircling domain of a quantity of a variable. Radial domains of an encircling domain. Definitions of continuity and continuousability around a quantity, at the right hand side of a quantity, etc. Hankel's continuity of a function in the immediate vicinity and up to the immediate vicinity of a point.

Part XIII.—The Monogenic and Analytic Functional Relations.

Definition of continuity based on derivability. Mixed curves and the functional relations to which they correspond. Monogenic functional relations. Polygenic. Cauchy's original definition. Type. Monotypic. Want of accord among matematicians in the use of "monogenic," etc. Monogenic at a quantity. Derivable. Differentiable. Orthoid. Anorthoid. Ordinary. Synectic. Regular. Holomorphic. The original definition of holomorphic function given by Briot and Bouquet differed from that laid down later by Briot. An integral functional relation is not necessarily holomorphic. Meromorphic. Poles and essential singularities. The definition of Briot and Bouquet and that of Weierstrass. Geometrical methods associated with the monogenic function nomenclature; algebraic associated with the analytic function nomenclature. Term analytic function originally due to Lagrange. He gave no definition. The assumptions made by him as to the functions that occur in analysis. What this amounts to if transformed into a definition, taking, not Lagrange's doctrine of differentiation, but the limit doctrine. Conditions stipulated by such a definition are held to be sufficient, but not necessary, for a functional relation to be analytic in the modern sense. Conditions under which a functional relation should be termed analytic. Necessity of a stipulation as to continuity when specific functional relations are taken into account. This stipulation not necessary as far as the conventional hyperfunctional formulas are concerned. An analytic functional relation may be ostensible by a polynomial or simple monomial involving the symbol of the independent variable or even by a value symbol. A single serial may cover several distinct analytic functional relations. Customary method of defining analytic function. Elements of an analytic functional formula. Functional relation analytic at a quantity. Natural boundaries of the range of ostensibility of a serial. Continuation of an analytic function to the "point of infinity." Weierstrass's definition of monogenic analytic function. Error of supposing an analytic function is monogenic in Cauchy's sense. Other definitions of analytic function. Legitimate and illegitimate functional relations. Suppositionless functional relation.