FUNDAMENTAL PRINCIPLES
OF
ELECTRIC AND MAGNETIC CIRCUITS

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BY

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FIRST EDITION

McGRAW-HILL BOOK COMPANY, INC.
NEW YORK: 239 WEST 39TH STREET
LONDON: 6 & 8 BOUVERIE ST., E. C. 4
1920

PREFACE

This book has been written as an introduction to the study of electric power machinery and transmission. The material contained in it is what the author considers to be the vital fundamental principles. It is intended for undergraduate students and therefore does not go as deeply into the physical and mathematical theory of electricity and magnetism as would be required for graduate study, nor does it include all the possible variations in conditions which might affect the application of the principles as laid down. These may be brought out in discussion and the student taught to think out some of them for himself.

The author desires to thank Professors F. D. Paine and F. H. McClain for valuable suggestions in the preparation of the book.

F. A. Fish.

Ames, Iowa,
June, 1920.

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CHAPTER I

FUNDAMENTALS

1. There are certain fundamental principles and ideas concerning which the student of engineering must have a very clear understanding before he can possibly master the more complex relations and processes with which he must deal in following his profession. In this text, it is assumed that the ideas of length, time, and velocity are well understood. However, on account of their great importance, the topics of acceleration, mass, force, work, energy and power will be discussed in review. The treatment will be brief because it is understood that these subjects have been studied before, but no effort should be spared in fixing them firmly in mind.

2. Acceleration.—When at a given instant a body is moving at such a rate that it would traverse a distance of 120 ft. if it continued to move at the same rate for one second, its velocity is said to be 120 ft. per second. Velocity expressed in this way, does not necessarily mean that the body will travel 120 ft. during the next second, but that it is traveling at that rate at the given instant. The rate at which the velocity of a body changes with time is called its *acceleration.* If at a given instant its velocity is 120 ft.

per second, but is changing, and is changing at such a rate that at the end of one second its velocity would be 130 ft. per second, its acceleration is 10 ft. per second per second. Again, this does not necessarily mean that its velocity will be 10 ft. per second greater at the end of one second, but that it is changing at that rate at the given instant. Acceleration may be either positive, or negative; that is, the velocity may be either increasing or decreasing. One of the most common examples of acceleration is that due to the earth's attraction, or the acceleration of "gravity." This has been proven by experiment to be a constant for all kinds of bodies for any given place on the earth's surface, but varies slightly with latitude and altitude. Its value is approximately 981 cm. per second per second or 32.16 ft. per second per second. That is, the velocity of a falling body increases 32.16 ft. per second every second during its fall.

3. Mass.—The *mass* of a body is defined as the quantity of matter it contains. It is independent of volume, shape or chemical composition. It is also entirely independent of the force of gravity or weight, although the earth's attractive force is taken advantage of in comparing masses. If two bodies exactly balance each other when suspended one from each end of an equal arm balance, they are said to have equal mass, provided no forces act upon them besides that of gravity. This definition is entirely arbitrary, but it is found that mass as thus defined is one of the fundamental properties of matter. This method of comparing masses is based on the law, proven true only by experiment, that the earth's attractive force at a given point always produces the same acceleration on any body regardless of the quantity of matter it contains. There is no other reason for believing that bodies of equal weight have equal mass. Any arbitrary portion or kind of matter may be taken as the standard of mass, and in fact, an arbitrary piece of platinum-iridium *is* the International Standard. It is known as the International Kilogram. A more common standard or unit, is the gram, which is the 1/1000 part of a kilogram. In England and in the United States, the most common standard is the pound and is represented by a piece of platinum preserved in London.

4. Force.—Any push, pull, pressure, tension, attraction or repulsion which changes or tends to change the state of rest or motion of a body is a *force*. Strictly speaking, rest is a state of motion; however, the words, "rest or motion" are used here to avoid misunderstanding. A change in state of motion includes not only a decrease or an increase of linear velocity, but also a change of direction. We have learned, originally from Newton, that so long as a body is left to itself, that is, not acted upon by any outside influences, it will continue in the same state of rest or motion. The same law also holds when outside forces act upon the body provided the resultant of all such outside forces is zero. Under such conditions as these, a body is said to be in a state of equilibrium. The principle of equilibrium is one which should be thoroughly mastered. A body is in equilibrium when it is at rest, or when it is moving in a straight line with a constant velocity; because to start it from rest or to change its direction or velocity requires the application of a force which is not balanced by a force in the opposite direction. If a car be moving along a straight track at uniform speed, it is in equilibrium; for the force which drives it exactly balances the forces which tend to stop it; if the driving force were to exceed the restraining forces by ever so little, the car would be accelerated; if the driving forces were to become less than the restraining forces by ever so little, the speed of the car would decrease. If a man pushes against a body at rest, but is unable to move it, the force with which the body resists is equal to the force with which the man pushes, and they are in equilibrium; for if the force with which the body resisted were less than that with which the man pushed, the body would be moved, that is, accelerated; and if the force with which the body resisted were greater than that with which the man pushed, the man would be moved backward, that is, accelerated. We have learned that when a body is acted upon by an unbalanced outside force, it will be accelerated in direct proportion to the magnitude of the force so acting. Taking advantage of this fact, the magnitude of a force is measured fundamentally by the acceleration it will give to unit mass. If the gram is taken as the unit of mass and acceleration is measured in centimeters per second per second, then the unit of force is called the *dyne*, and it is that force which will give to a mass of 1 gm. an acceleration of 1 cm. per second per second. This unit is much used in developing the principles of electricity and magnetism. If the pound is taken as the unit of mass, and acceleration is measured in feet per second per second, then the unit of force is called the poundal. It is the force which will give to a mass of 1 lb. an acceleration of 1 ft. per second per second. This unit, however, is not much used. From the definition of force, it follows that the force required to give a body any desired acceleration is equal to the product of its mass and the acceleration.

The force with which the earth attracts a mass is called its weight; and since the acceleration due to gravity is a constant at any given point, it follows that the ratio of weight to mass is a constant at any one point. That is, the weight of a body at any point is equal to its mass times the value of the acceleration of gravity at that point. For this reason the force equal to the weight of unit mass is very commonly used in engineering as unit force. Since the acceleration due to gravity varies slightly over the earth's surface such a unit, in order to be invariable, must be based upon some standard value of acceleration. This has been agreed upon as 980.665 cms. per second per second, or 32.1739 ft. per second per second. Unfortunately the same name is given to this unit of force as to the unit of mass and this sometimes leads to confusion. For example, a most common unit of force in this country is the pound, which is also the English unit of mass. The use of the pound as a unit of force also leads, in Mechanics, to a different but unnamed unit of mass. The acceleration which a force of 1 lb. will give to a mass of 1 lb. is about 32.2 ft. per second per second; therefore a force of *one* pound would give an acceleration of *one* foot per second per second to a mass of 32.2 lbs.; that is, using the pound as a unit of force, the mass of a body is equal to its weight at any point divided by the acceleration of gravity at that point. This, of course, is a constant, as it should be, since the mass of a given body is invariable. The pound used as a unit of force is called the gravitational unit. There is nothing inconsistent about the two systems of units, provided one remembers that the force of gravity (i.e., weight) on a given mass varies slightly from place to place. The approximate value of a force of 1 lb. is $453.6 \times 981 = 445{,}000$ dynes.

5. Work.—When a force acting upon a body succeeds in moving it, work is said to be done on the body. The amount of work done upon a body is defined as the product of the distance moved through and the average value of the force which causes the motion. The unit of work is therefore that done by a force of one dyne acting through a distance of 1 cm., or that done by a force of 1 lb. acting through a distance of 1 ft., depending on the system of units employed. The first unit mentioned is sometimes called the centimeter-dyne, but is more commonly called an erg; that is, of course, a very small unit and a more common one is the *joule*, which is equal to 10,000,000 ergs. The second unit mentioned is known as the foot-pound; no single word has been coined to take the place of the compound word. Since there are 30.48 cm. in 1 ft. and 445,000 dynes in 1 lb., there are $30.48 \times 445{,}000 = 13{,}560{,}000$ ergs in 1 ft.-lb., or 1.356 joules in 1 ft.-lb.

6. Energy.—Nature has endowed the substances of the universe with certain properties by which, under suitable conditions, they are able to cause motion and thus to do what has been defined above as work. This ability to do work is given the name *energy*. A most important principle which the engineer must never forget is that of the *Conservation of Energy.* This principle is that the total amount of energy in the universe is constant, and can neither be added to nor subtracted from. This law is not susceptible of mathematical proof, but all experience leads to the conclusion that it is true, and it is to be accepted as one of the "Articles of Faith," for the scientist and the engineer.

However, it is an everyday observation that energy can be and repeatedly is transformed from one of several forms to others; these transformations are the means by which all processes are performed. Many of these transformations are relatively simple, and it is not difficult for us to form a mental picture of the processes by which they are accomplished; others are more obscure and we are obliged to accept the manifestations which our senses perceive and from these construct a more or less fictitious picture of the processes.

The definition of energy as the ability to do work implies the existence of force within the substance or system of substances possessing such energy. It should be noted, however, that force may be exerted without doing work and it is only when the force is great enough to overcome the opposition to it and cause motion that work is done. Work cannot be done without the transfer or transformation of energy, and the amount of work done represents a loss of energy at one point, or of some kind, and the gain of an equal amount at other points or of some other kind. The term "consumption of energy," therefore, does not mean that energy is destroyed, but that it is only changed to some other place or kind or both. It follows from this that energy is measured in the same units as work, that is, in ergs, joules, or foot-pounds; but distinction between work and energy must be carefully kept in mind.

7. Power.—The amount of work done by a force in overcoming resistance through a given distance is independent of the time required to do it. A force of 100 lbs. may move a body 200 ft. in one second or in ten seconds; the amount of work done is the same in both cases. But the rate at which work is done is generally a matter of great importance, and is defined as *power*. The two most important units of power are the *watt* and the *horse-power*. When work is done at the rate of 1 joule (10^7 ergs) per second, the power is 1 watt; when it is done at the rate of 550 ft.-lbs. per second or 33,000 ft.-lbs. per minute, the power is 1 h.-p. In engineering it is always much more convenient to measure power than to measure work. Therefore, when the work done or the energy "consumed" in a given time is required, the power is measured, the time recorded, and the work calculated as the product of power and time. If power is measured in watts or horse-power, and time in seconds, the work will be expressed in joules or foot-pounds respectively; but since the second is so small a unit, it is common to use the hour as a unit of time and to express work in watt-hours or horse-power-hours. One watt-hour does not mean that the power has been 1 watt and the time one hour, but that the product of the power in watts and the time in hours is 1 watt-hour. Thus, 100 watt-hours may be used in 0.5 hour at a rate of 200 watts, or in five hours at a rate of 20 watts. Since there are 3600 seconds in one hour and 1.356 joules (or, watt-seconds) in 1 ft.-lb., there will be $3600/1.3655 = 2655$ ft.-lbs. in 1 watt-hour.

CHAPTER II

ELECTRICITY AND MAGNETISM

8. Electricity.—Let a strip of zinc and a strip of copper be placed some distance apart in a dilute solution of sulphuric acid. It will be found that there exists between the two strips of metal a kind of force that did not exist before they were placed in the solution. For instance, if they are connected together outside of the solution by a piece of wire, the temperature of the wire will be increased; if the wire is sufficiently fine it will become so hot as to give off light. If a long piece of wire is used and it is wound up so as to form a coil, it will be found that a piece of iron will be attracted toward the center of the coil. It is of course impossible for us to see the mechanism by which these acts are performed. But the evidence is before us that out of the chemical energy of the solution a different kind of energy is produced which is transferred by some means out through or along the wire and some of it, at least, is transformed into heat energy. The energy into which the chemical energy is transformed is called *electrical energy*, the something by means of which this energy is transferred out through the wire is called *electricity*, and an electric current is said to flow in the wire. In the present state of our knowledge we are unable to define electricity any further than to say that it is the means by which electrical energy is carried from one place to another. A familiar example of this kind of a carrier is the water, which, flowing in pipes, carries with it the energy it possesses by virtue of having been put under a pressure higher than that at the point toward which it flows.

Whatever may be the real mechanism within the cell described above, it is evident that one of the terminals is at what may be called a higher electrical pressure than the other. In electrical terminology, one terminal is said to have a higher *potential* than the other, or there is said to be a *potential difference* between them. The cause of this potential difference, whatever its nature may be, is called *electromotive force;* it may be regarded as the force which causes or tends to cause electricity to move and is analogous to the difference in pressure which is set up between the intake and outlet of a pump. The unit of electromotive force is called a *volt;* the magnitude of this unit will be discussed in a later paragraph.

We have spoken of electricity as a carrier of electrical energy, and we should, therefore, expect to have to deal with it quantitatively. The unit quantity of electricity is called a *coulomb;* its magnitude will be taken up later. A coulomb of electricity may be thought of as analogous to a cubic foot of water. In hydraulics, we have occasion to deal not only with quantities of water, but also with the rate of flow of water, as for instance, in pumping machinery or in water-power developments. In electrical engineering we have much less occasion for dealing with the quantity of electricity than for dealing with its rate of flow, since electricity is used only for transmitting energy. However, the use of the idea of quantity is necessary for the development of the theory of certain phenomena and apparatus.

When electrical energy is being transmitted from one place to another by means of wires, electricity is said to be flowing through the wires. In hydraulics, the rate of doing work (that is, the power) is equal to the product of the rate of water flow and the difference in pressure between the upper and lower levels; similarly, in electrical circuits the power developed between two points is equal to the product of the rate at which electricity is flowing and the difference of potential between the points. In hydraulics, rate of water flow is expressed in cubic feet per second. In electric circuits, rate of flow is expressed in *amperes;* one ampere meaning that electricity is flowing past any given point at the rate of 1 coulomb per second. The use of the single word to express the idea of rate should be carefully noted; confusion often arises from thinking of the ampere as a quantity of electricity. We frequently use the word "*current*" in referring to a flow of water, and say that the current is strong or the current is weak as the case may be, meaning that the rate of flow is great or small; in like manner, the word current is used in connection with flow of electricity and means its rate of flow; the ampere is the unit of current, and not the unit of electricity.

The path through which an electric current flows is called an electric circuit; when the path is complete, leading out from the source, through wires, lamps, motors, and other apparatus and back to and through the source, the circuit is said to be closed. A steady current cannot flow unless a closed circuit is provided for it.

9. Magnetism.—Mention has been made of the effect on a piece of iron when it is brought near a coil of wire carrying an electric current. The region surrounding a wire which is carrying an electric current is found to be the seat of energy, and a mechanical force is found to be exerted upon certain kinds of material, particularly iron and steel, if they are brought into the neighborhood of such a wire. This kind of energy is called magnetic energy, and the force is called magnetic force. Magnetism is the general name given to the condition or state of affairs which exists in such a region, and the region itself is called a *magnetic field.*

10. Magnets.—If a bar of steel be placed inside a coil and current be sent through the coil, it will be found on removing the bar from the coil that the bar has acquired and retained the same property possessed by the coil, namely that of attracting toward it pieces of iron or steel. A bar, rod or needle of steel possessing this property is said to be magnetized and is called a magnet. The magnetic force exerted by a magnet is found to be by far the greatest at the ends of the axis which was parallel with the axis of the coil in which the bar or rod was magnetized. These ends are called the poles of the magnet. This concentration of the magnetic force at the ends of a magnet is much more pronounced in long, slim magnets than in short thick ones.

Either end of a bar of unmagnetized iron will be attracted to either pole of a magnet; but if the bar be magnetized one end will be attracted and the other end repelled. The earth is found to be in a permanently magnetized condition with one pole located near the geographic north pole and the other near the geographic south pole. If a magnetized bar or needle be suspended so as to swing freely in a horizontal plane, one pole will point north and one south; the north-pointing pole of a magnet is called its north pole and the south-pointing pole its south pole. The north pole of a magnet always attracts the south pole of another magnet while the north poles of any two magnets or their south poles always repel each other; that is, like poles repel each other and unlike poles attract each other. A north pole is frequently called a positive pole and a south pole a negative pole. The force exerted between the two poles is considered to be positive when it is a repulsive force.

11. Unit Pole.—It is found to be impossible to produce a north pole without producing a south pole of equal strength. Nevertheless, it is often convenient to imagine an isolated magnet pole, the other pole being thought of as so far away as to have no effect at the point under consideration. The strength of a pole is measured by the force with which it attracts or repels another pole. The force of attraction or repulsion between two poles has been experimentally proven to be inversely proportional to the square of their distance apart. *A magnet pole is said to have unit strength, or is said to be a unit pole, when it exerts a force of one dyne on another pole of equal strength at a distance of* 1 *cm.* The strength of any given pole in c.g.s. units is therefore numerically equal to the force in dynes which it exerts on a unit pole 1 cm. away. In general, the force exerted between any two poles is equal to the product of their strengths divided by the square of their distance apart.

In building up our system of electric and magnetic units this idea of a unit magnet pole was used as the starting point. This is generally regarded as having been unfortunate, as it involves some very difficult and delicate measurements, and some troublesome constants. However, these measurements have been made and legal values of the units established so that reference back to the fundamental measurements is not now required.

12. Magnetic Fields.—Any region in which a magnetic force would be exerted on a magnet pole is a magnetic field. The strength of a magnetic field is measured by the force exerted upon a unit magnet pole. A magnetic field is said to have *unit strength* at a given point when it exerts, or would exert, a force of 1 dyne on a unit pole placed at that point. The direction of a magnetic field at any point is the direction of the force which would be exerted upon an isolated north pole placed at the given point. A magnetic field is conveniently pictured in the mind by imagining lines drawn through the field, their direction at every point being the same as the direction of the force at that point, and their density at any point representing the value of the force, or the strength of the field at that point. Such lines are called *lines of magnetic force,* or in short, *lines of force.* A line of force therefore represents the path which would be taken by a magnet pole if it could be isolated and left free to move in a magnetic field. A field of unit strength at any point would be represented by one line of force per square centimeter of cross-sectional area at the given point, the area being taken in a plane at right angles to the direction of the line of force. Since the field may not be uniform, this does not mean that each square centimeter is to be thought of as containing one line of force, but that the density of the lines, at the given point, is equal to one line per square centimeter. The strength of a magnetic field is commonly called its *intensity*, and *unit intensity is one dyne* *per unit pole.* The product of the intensity and any given area, at right angles to the direction of the field and in which the given intensity is uniform, is called the *magnetic flux* across that area. Since intensity is represented by the number of lines of force per unit area, the flux is represented by the total number of lines of force crossing the given area. One line of force or unit magnetic flux, is called a *maxwell.* The path along which magnetic forces act is known as a magnetic circuit. Experiment has proven that every magnetic circuit is closed upon itself; that is, if an isolated pole be imagined to start moving from any point in a magnetic field and to keep moving always in the direction of the force action, the path followed will lead back to the starting point, coming up to it from the opposite direction from which the pole started, and passing, on the way, through the source of the magnetic force. Lines of force are therefore always to be thought of as closed lines, and all the lines which issue from one pole of a magnet must return to the other pole and pass through the magnet. The positive direction of a line of force is taken to be that in which a north magnet pole would be urged to move; that is, the lines are assumed to leave a magnet by its north pole and enter it at its south pole.

13. Properties of Magnetic Lines.—While it is probable that the magnetic flux has no motion around its circuit, yet it is very convenient to consider that it does move. In picturing this, a line of force is to be thought of as moving in a direction parallel with itself and at every point in the direction of the force action at that point. It is also necessary to think of this motion as without friction, since repeated experiments have proven that no energy is required to maintain a magnetic field. Imagine that a small tube is bent around, and following the direction of the force, is closed upon itself, and encloses within it a continuous stretched rubber band; suppose that by some means this band be made to move along through the tube *without friction;* this is the kind of motion a line of force may be pictured as having. The stretched rubber band is used in this illustration to typify the tension and the tendency to shorten themselves, which lines of force are experimentally proven to possess. A further important property which lines of force are found to possess is that they repel each other when in the same direction and attract each other when in opposite directions. This property is sometimes pictured by imagining that the lines are whirling with high velocity about their longitudinal axes, and, being of elastic material, they push against each other when whirling in the same direction and pull together when whirling in opposite directions. It is also to be noted that lines of force cannot cross each other.

14. Modern Theory.—The modern working theory of magnetism is that the molecules of all magnetizable substances are permanent magnets which in general point at random in all directions. When a number of these molecular magnets come under the influence of a magnetizing force, such as that produced by an electric current or by a magnet of appreciable size, it is supposed that they swing around under the influence of this force, so that their north poles point in one general direction and their south poles in the opposite direction. Inasmuch as a directed magnetic field can be produced in air or in a vacuum, it must be supposed that the molecules of the ether are likewise in a permanently magnetized condition. A considerable number of material substances are found to be magnetizable, but among them iron stands out as one which is magnetizable to a far greater degree than all others. While a magnetic field produced in air immediately disappears when the directive force is removed, it is found that under certain conditions as to its composition a piece of iron will retain magnetic properties to a marked degree after the directive force is removed. This is called residual magnetism and is accounted for by assuming a kind of molecular friction to exist in the iron which tends to prevent the molecules from returning to their random positions. In annealed iron this friction is very small, in cast iron and mild steel it is greater, and in hardened steel it is very great. Conversely, it requires a much greater magnetizing force to magnetize a piece of hardened steel to a given degree than to magnetize a piece of annealed iron to the same degree. When a large permanent magnet is brought near a small one, the north pole of the former will naturally attract the south pole of the latter, but if the north pole of the small one be forcibly held to the north pole of the large one, the small one may have its polarity entirely reversed by the stronger directive force of the large one. If either end of a bar of soft iron be placed near the north pole of a permanent magnet, all the molecular south poles of the soft bar will be attracted toward the north pole of the magnet, the ends of the bar as a whole will become polarized, and the bar is said to be magnetized by induction.

15. Further Laws Concerning Magnetic Fields.—Since lines of magnetic force are closed lines, and all lines must pass through the source of the magnetic field, it follows that the lines which pass through a magnet must return to it through the air outside. The cross-sectional area of the return path is infinite as compared with the cross-sectional area of the magnet, and as soon as the lines leave the north pole of a magnet, they spread out due to the repulsion between them and the intensity of the field becomes weaker and weaker as the distance from the pole increases. As has been stated, lines of force possess both longitudinal tension and transverse repulsion; the tension in the lines tends to shorten them and thus to bring them closer together, while the repulsion between lines tends to put them farther apart; therefore the configuration of a given set of lines of force will be determined by the position of equilibrium between these two forces. These ideas are based upon the very important principle that whenever two or more magnetic fields are brought within acting distance of each other, they will tend to place themselves in such a position that their paths will be parallel and in the same direction, and as short as possible. This law, like the law of gravity, is one of the fundamental principles of nature, and it is the principle on which all electric motors and meters operate. It applies whether the fields are produced by magnets, or coils carrying electric current, or a combination of the two. Its application to the simple cases of attraction and repulsion may be illustrated by the following examples: In Fig. 1, let *A* be a permanent magnet and *B* be either a

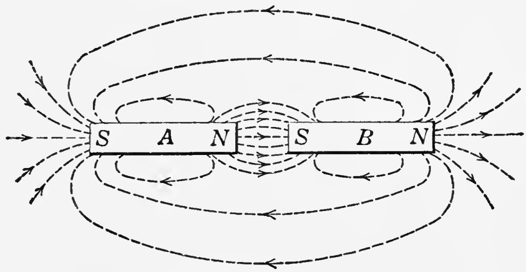

FIG. 1.

permanent magnet or a piece of soft iron; in the latter case the iron will be magnetized by induction and become a temporary magnet. It is readily seen that in this case the longitudinal tension of the lines which are common to the two magnets will draw the magnets toward each other and also the direction of the lines not common to the two magnets is such that they attract each other. In Fig. 2 where

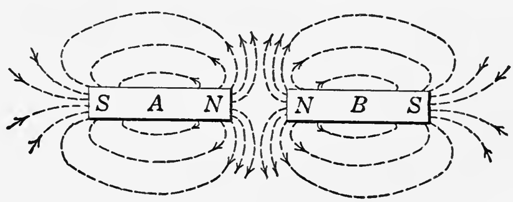

FIG. 2.

the N-pole of *A* is set opposite the N-pole of *B*, there are no lines common to the two magnets and therefore longitudinal tension has no effect in this case; however, the two sets of lines are in such direction with reference to each other that the resultant force is that of repulsion. In Fig. 3, where magnet *B* is set on a pivot above magnet *A*, the repulsion between the lines would cause *B* to swing around after which longitudinal tension would come into play and the

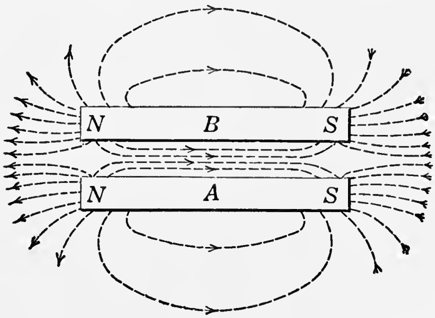

Fig. 3.

equilibrium position would be as shown in Fig. 5. Fig. 4 shows the condition (looking down on the magnets) after *B* has swung through 90°.

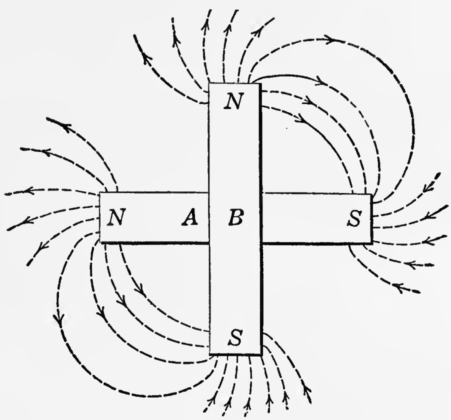

Fig. 4.

16. The Magnetic Field around a Wire.—When a wire is carrying a current, the direction of the magnetizing force which it exerts on a magnet pole depends upon the direction in which the current is flowing through the wire. Experiment has proven that the force action of the field due to a long straight wire is at right angles to the wire and at every point tangential to a circle drawn through the point, the center of the circle being at the center of the wire, and the plane of the circle being at right angles to the axis of the

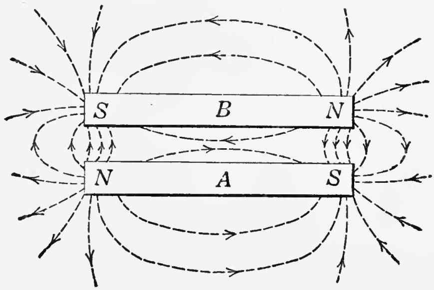

Fig. 5.

wire. This is shown in Fig. 6, where the wire is represented in cross-section at the center of the figure, the axis of the wire being at right angles to the plane of the paper, and the dotted circles representing the lines of force.

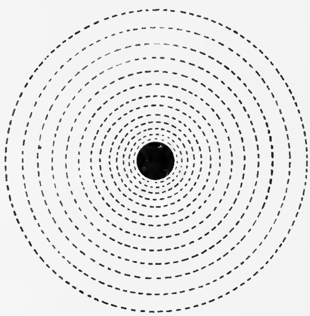

Fig. 6.

To illustrate further, suppose a current is flowing in the wire, WW', Fig. 7*a*, in such a direction that if a small compass needle be placed above the wire as at (*f*), the north pole will be deflected to the right; if placed below the wire as at (*g*) it will be deflected to the left; that is, if one looks along the wire in the direction from W toward W', the north pole of the needle will always point in a clockwise direction.

It is desirable to agree on which direction shall be taken as the positive direction of an electric current in a wire. The convention which has been universally agreed to is as follows: If when looking along the axis of a wire a north, or positive, magnet pole tends to move clockwise around the wire, the current is considered as flowing away from the observer, and this is taken as the positive direction. Conversely, if the current flows away from an observer looking along the axis of the wire the magnetic field is considered as directed clockwise around the wire. When looking at the cross-section of a wire as in Fig. 7*b*, a current flowing away from the reader is generally indicated by a cross (representing the tail of an arrow) in the small circle representing the cross-section; a current flowing toward the reader is indicated by a dot (representing the point of an arrow) in the circle, as shown in Fig. 7*c*.

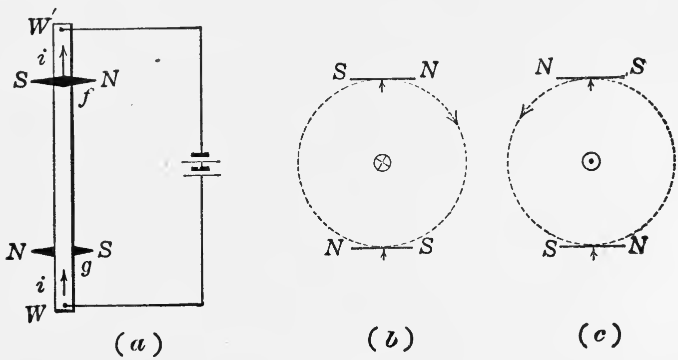

Fig. 7.

17. The Solenoid.—When a wire is wound in the form of a coil, and current is passed through it, it exhibits all of the properties of polarity, attraction and repulsion, that are shown by a magnet, but the magnetic field disappears when the current ceases to flow. Such a coil is called a *solenoid.* If it be suspended at the center so as to be able to move freely with its axis horizontal, it will assume a north and south direction. If a pole of a magnet be brought near it, one end of the coil will be attracted and the other repelled. If the end of a bar of soft iron be brought near one end of the solenoid, the bar will be magnetized by induction and drawn into the solenoid in accord with the principle stated in Article 13. If the axes of coil and bar are horizontal and there be no mechanical friction where the bar enters the coil, the center of the bar will go to the center of the coil, because this is the only position where the forces acting between bar and coil are in equilibrium. If the axes of coil and bar are vertical, the bar will be pulled up until the difference between the attractive forces just balances the force of gravity, and will be suspended at that point.

Fig. 8 illustrates the field associated with a solenoid. In accord with the rule given in the last article, the following rule may be used for determining the direction of the flux in a solenoid: If one looks along the axis of a coil and the current is flowing in a clockwise direction around the coil, the direction of the flux inside the coil will be away from the observer.

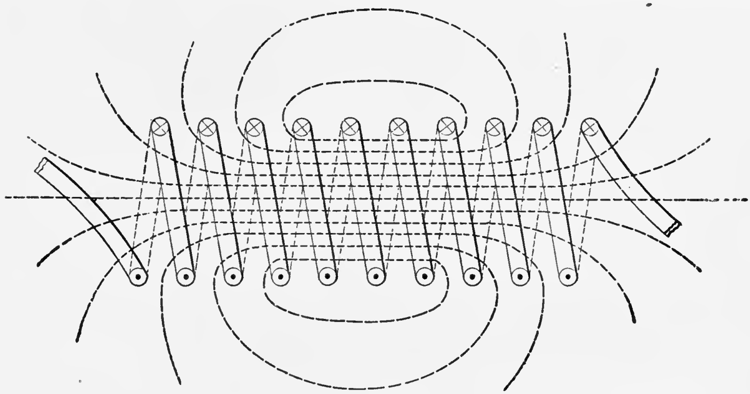

Fig. 8.

18. Action of a Magnetic Field on a Wire Carrying Current.—If a wire carrying current is placed across a magnetic field not its own, a force will be exerted upon the wire. Suppose the lines of force of the field in which the wire is placed pass from left to right as shown in Fig. 9, while the lines of force due to the wire pass clockwise around the wire. The result is that the intensity of the field is increased above the wire, and weakened below the wire, and a downward force will be exerted on the wire. This may be looked upon as the result of the natural tension in the lines of force and their tendency to shorten themselves. The direction of the force is always at right angles both to the field and to the wire and toward that side of the wire where the field is weakened. The value of this force (per unit length of wire) is greatest when the wire is at right angles to the field. A rule for determining the direction of the force in such a case is known as the "left-hand rule." If the forefinger of the left hand be pointed in the direction of the main field and the middle finger placed at right angles to the forefinger and pointed in the direction of the current, then the thumb, placed at right angles to both fore and middle fingers, will point in the direction of the force action upon the wire.

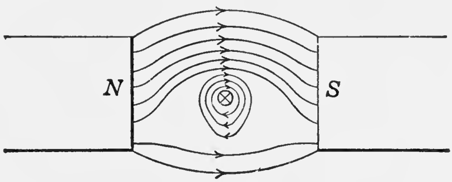

Fig. 9.

19. Unit Current.—The force action described above varies with the strength of the current, with the intensity of the magnetic field, with the length of the wire immersed in the field, and with the angle which the wire makes with the direction of the field. When the field and the wire are at right angles to each other the force is proportional to the product of the current, the field intensity, and the length of the wire. This fact is made the basis of the fundamental unit of current. This fundamental unit, known as the c.g.s. unit, or *abampere, is a current of such strength that the wire in which it flows has exerted upon it a force of one dyne per centimeter of length when the wire is at right angles to a magnetic field of unit intensity.* This is considered to be too large for practical purposes, and a unit one-tenth as large and called an *ampere* is used as the practical unit.

If the intensity of the field is H lines per square centimeter, the strength of the current I amperes, and the length of the wire l centimeters, then the force on the wire in dynes is

$$F = \frac{HlI}{10}. \tag{1}$$

When the wire is parallel with the field no force acts on the wire. When the angle between the wire and the direction of the field is θ, then the field may be thought of as having two components, one, $H \cos \theta$, parallel with the wire, and the other, $H \sin \theta$, at right angles to the wire. The former component has no action on the wire, while the latter produces a force,

$$F = \frac{HlI}{10} \sin \theta. \tag{2}$$

The exact determination of the value of a current by measuring this force action is an extremely difficult process. However, it has been done and the value of an ampere calculated in terms of an easier method of measurement, as explained in the next paragraph.

20. International Unit of Current.—It has been discovered that if an electric current be passed through certain chemical solutions, the substance will be decomposed. For example, if a current flows into a solution of copper sulphate by means of a rod or plate of solid conducting material, and out of it through another rod or plate, the copper will be deposited out of the solution; and if the rod by means of which the current leaves the solution is properly selected and prepared, the copper will adhere to it. This is called electrolytic deposition. The amount of material deposited in this way is found to be proportional to the quantity of electricity which passes through the solution; that is, to the product of the strength of the current which flows, and the time during which it flows. The electrolytic deposition of silver out of a solution of silver nitrate on to a plate of pure silver is found to be a most accurate way of measuring an electric current. It is found that 1 ampere measured by the fundamental method mentioned in the last paragraph will under certain carefully observed specifications deposit 1.118 milligrams of silver per second. For these specifications, see Circular No. 60, U. S. Bureau of Standards. The ampere, thus measured, was made the legal unit of current by Act of Congress in 1894, and was adopted internationally as the standard method of determining the unit of current by the International Electric Congress at London in 1908; the ampere measured in this way is called the International Unit of Current, or *International Ampere.*

21. The Coulomb.—The coulomb has been mentioned as the unit quantity of electricity. Since the ampere, or the unit rate of flow of electricity, is adopted as the primary unit, the value of the coulomb must be defined in terms of the ampere. Unit quantity of electricity, or the coulomb, is that quantity which flows past a given point in one second when the rate of flow is 1 ampere. When the current flowing in a circuit is 8 amperes, for instance, the quantity of electricity passing through the circuit is 8 coulombs per second. If 8 amperes flows through a circuit for twenty seconds, the total quantity of electricity passing through the circuit during that time is 160 coulombs. If 1 ampere flows for one hour, 3600 coulombs would pass through the circuit. In practical work the ampere-hour, equal to 3600 coulombs, is generally used as the unit of quantity. Thus, 15 amperes flowing for six hours would give 90 ampere-hours and would be called 90 ampere-hours instead of 324,000 coulombs.

22. The Galvanometer.—An instrument commonly used to *indicate* the flow of electricity is the galvanometer. It may also be used within certain limits to *measure* the value of a current or of the quantity of electricity passed through it in a very short interval of time. The principle on which such instruments operate is the force action between a magnetic field and a wire carrying an electric current. There are two general methods of construction: One, in which a small permanent magnet is suspended within a coil of wire, at right angles to the axis of the coil; and the other, in which a light coil of wire is suspended between the poles of a magnet with its axis at right angles to the direction of field due to the magnet. The movement of the suspended part may be indicated by a light pointer attached to it, or by means of a small mirror attached to it and from which a beam of light is reflected to a scale, or from which a scale is observed from a fixed position. In the case of a moving magnet the zero position is fixed by another magnet placed near the instrument or by the earth's magnetic field. The coil and the fixed field are so placed with respect to each other that the suspended magnet is held in the plane of the coil when in the zero position, that is, with no current in the coil. When current flows through the coil the needle takes up a position depending on a relative strength of the fixed field which tends to hold it in the zero position and the field produced by current in the coil which tends to turn it to a position parallel with its axis; that is, perpendicular to its plane. In the case of the suspended coil, the zero position is fixed by springs attached to it, and the current is led into and out of the coil through these springs. When current flows in the coil, it takes up a position depending on the relative strength of the spring and the combined strength of the fields which tends to turn the coil into a position with its axis parallel to the fixed field. A more complete description of these instruments may be found in any good hand-book for electrical engineers.

Within quite narrow limits the deflection of a galvanometer needle or coil is proportional to the current flowing, but such instruments are more commonly used for determining the condition of zero potential difference between two points. They may be constructed so as to indicate extremely small currents (as small as 10^{-12} amperes) and therefore for all practical purposes when no deflection is discernible there is no potential difference between the points where the instrument is connected. When the inertia of the moving part is sufficient to prevent its getting into motion until a momentary flow of electricity has passed through the coil, the swing which then takes place is, within narrow limits, proportional to the amount of electricity which has passed through. Such a galvanometer is called ballistic and is frequently used for measuring small quantities of electricity. Whenever a galvanometer is used to measure a current or a quantity of electricity, it must first be calibrated by reading its deflections when known current or quantities are passed through it. When used for determining the ratio of two currents or two quantities, this calibration is unnecessary.

The type of galvanometer using a movable coil suspended between permanent magnets is known as the D'Arsonval Galvanometer. The principle of this galvanometer is the most common one used in the construction of direct current ammeters and voltmeters for measuring current and voltage. See Figs. 28 and 29. In such meters, however, the coil is pivoted between jeweled bearings, and held in its zero position by spiral springs. A pointer is rigidly attached to the coil and swings over a graduated scale.

CHAPTER III

ELECTRIC CIRCUITS

23. Resistance.—Mention has been made of the fact that when an electric current flows in a wire, heat is generated therein. This fact leads immediately to the conclusion that the wire must possess some property by which it opposes the flow of current; if the wire offered no opposition, no effort would be required to send the current through the wire, no work would be done, and no heat generated. It has been proven experimentally that all materials possess an inherent property by which they oppose the flow of electricity—some offering very great opposition, some comparatively little. This property of opposition is called resistance. Those materials which offer comparatively small resistance are called conductors; the principal materials in this class are the metals, carbon, and solutions of mineral salts and acids. The materials which offer great opposition are called insulators; glass, porcelain, mica, rubber, silk, paper, paraffine, shellac and oil are examples of good insulators.

The energy consumed in overcoming resistance is transformed into heat. The rate at which heat is generated in a wire has been proven to be proportional to the square of the current flowing in the wire; that is, equal to a constant times the square of the current. This is known as Joule's Law. The value of the constant is called the resistance of the wire. The fundamental, or c.g.s. unit of resistance is that resistance in which 1 abampere of current will generate heat energy at the rate of 1 erg per second; it is called an abohm. *The practical unit of resistance is that resistance in which a current of* 1 *ampere will generate* 1 *joule of heat energy in one second.* This unit is called an *ohm,* and is equal to

10^9 abohms. Since the rate at which energy is transformed, or work is done, is power, and since work done at the rate of 1 joule per second is 1 watt, an ohm may be defined as that resistance in which 1 ampere develops power at the rate of 1 watt. If the power developed in a wire is 64 watts, when a current of 4 amperes is flowing in it, the resistance of the wire is $64/16=4$ ohms. That is, the resistance of the wire is equal to the power developed in it divided by the the square of the current flowing; or, the current flowing in a wire is equal to the square root of the quotient obtained by dividing the power by the resistance. Put into form of an equation, the law is

$$P=RI^2, \tag{3}$$

where P is the power in watts lost in heating a conductor of resistance R when a current I flows through it.

In comparing resistances, it is necessary, of course, to have a standard. By careful measurement, it has been found that a column of pure mercury measuring 106.3 cm. long at the temperature of melting ice, of uniform cross-sectional area, and weighing 14.4521 gm., has a resistance of 1 ohm as defined above. This is known as the International Ohm, and secondary standards are made by comparison with it.

24. Ohm's Law and Electromotive Force.—It has been proven by experiment that the current which flows in a conductor is directly proportional to the electromotive force which is applied to that conductor, and inversely proportional to the resistance of the conductor. This is known as Ohm's Law, and is the electrical application of the general law that the magnitude of any effect varies directly with the magnitude of the cause and inversely with the magnitude of the opposition. We have defined the practical units of current and of resistance and we can now define the *practical unit of electromotive force as that electromotive force which will cause a current of* 1 *ampere to flow through a resistance of* 1 *ohm.* As already stated, this unit is called a *volt.* The absolute unit, or abvolt, is that e.m.f. which will cause 1 abampere of current to flow in 1 abohm of resistance; the practical unit, or volt, is equal to 10^8 abvolts. Ohm's Law is one of the most important laws in Electrical Engineering and careful attention must be given to its operation. Expressed in the form of an equation the law is,

$$I=\frac{E}{R}, \quad \text{or} \quad E=RI, \quad \text{or} \quad R=\frac{E}{I}, \qquad (4)$$

where I is the current which an electromotive force E will cause to flow in a resistance R. This law also means that when a current is flowing through a given wire, the difference of potential between any two points is equal to the product of the current flowing and the resistance of the portion of the wire between the two points. An extended discussion of the application of Ohm's Law will be given in later articles.

When a current flows through a wire the difference of potential between two points is sometimes called the *potential drop* or the drop in potential and is often abbreviated as p.d. When the word " drop " is used it must be understood that it occurs in the direction of current flow, just as a drop in head or a drop in pressure in a pipe line carrying water means that the pressure is less in the direction of water flow. If the direction opposite to the flow is considered, there would be a *rise* of potential. The term " rise of potential " is also sometimes used to denote the increase in potential from one terminal to the other inside of a source of electromotive force. This rise of potential is caused by the unknown process which generates electromotive force and is in the direction of current flow since current is assumed to flow *out* from a source of e.m.f. by way of the terminal having the higher potential and *in* at the terminal having the lower potential. It should be clearly understood that *potential* refers to the condition at one point and that difference of potential, drop in potential or rise in potential refers to the difference in conditions between two points.

25. Chemical Sources of E.M.F.—While it is not within the scope of this text to enter into a discussion of details in regard to the theory of generation of electromotive force, there are certain principles relating to the action of sources of electromotive force which should be learned at this time.

When two plates or rods of different metals, or a piece of metal and a piece of carbon are placed in certain chemical solutions, an electromotive force is generated which establishes a difference of potential between the two *plates*. Such an arrangement is called a voltaic cell. The solution used in a voltaic cell is called the electrolyte and the two conducting rods or plates which are placed in it are called electrodes. The value of the electromotive force produced by a voltaic cell varies with different materials used. For example, copper and zinc in a solution of zinc sulphate develops an e.m.f. of about 1 volt; carbon and zinc in a solution of ammonium chloride develops an e.m.f. of about 1.4 volts; carbon and zinc in a solution of dilute sulphuric acid develops about 2 volts. The direction in which current will flow from a voltaic cell depends also upon the materials used for the electrodes. When copper and zinc are used, the current will flow out at the copper terminal; when carbon and zinc are used, it will flow out at the carbon terminal; that is, in the first case, the copper terminal is at a higher potential than the zinc, and in the second case the carbon terminal is at a higher potential than the zinc. The electrode at which the current flows out of a cell is called the positive electrode and the one at which the current flows into the cell is called the negative electrode.

In every voltaic cell when current is allowed to flow from it, the chemical composition of the electrolyte, or of the electrodes, or both, undergoes a change as the chemical energy gradually becomes exhausted. If a cell is connected to another source of electromotive force and current be sent through it in a direction opposite to that of its own e.m.f. the chemical action will be reversed and there will be a tendency to restore the cell to its original condition; but in most cases it is found to be impracticable by this means to restore the chemical energy in any considerable amount, owing to the fact that in the original operation of the cell, there are certain local chemical actions which are not reversible, and owing to the loss of certain constituents (principally gases) out of the electrolyte. There are, however, a few combinations into which from 75 per cent to 90 per cent of the energy can be restored by reversing the operation. These combinations are known as storage cells. The two most important of these are the lead cell, and the alkaline or Edison cell. The lead cell is made by filling two lead grids with a paste of lead sulphate and placing them in an electrolyte of dilute sulphuric acid. When a current is sent through the cell (which process is known as charging) the lead sulphate on the plate where the current enters is changed to lead peroxide and that on the plate where the current leaves is reduced to spongy lead. When all or most of the lead sulphate has been changed in this way to lead peroxide and spongy lead, the cell is said to be charged and may be used as an ordinary voltaic cell and will produce current until most of the lead and lead peroxide is changed back to lead sulphate. When the cell is charged the plate containing the lead peroxide is positive, or at the higher potential, and the one containing the spongy lead is negative, or at the lower potential. The e.m.f. of the cell is about 2.2 volts when fully charged and decreases at first slowly and then rapidly to zero as it is discharged by using it as a source of energy. If the discharge is continued beyond the point where the p.d. at the terminals is about 1.8 volts, the reversibility of the cell is greatly impaired, and for practical purposes a cell is considered to be discharged when its terminal p.d. has dropped to 1.8 volts.

When a cell is being charged the specific gravity of the electrolyte increases, and when fully charged it varies from about 1.2 to 1.3, depending on the service for which it is intended. When discharged the specific gravity will generally be from 1.13 to 1.18. The specific gravity is a better criterion as to the condition of a cell than is the e.m.f. The open-circuit voltage affords little, if any, indication as to the condition of a cell, until it is completely discharged.

The Edison cell consists of nickel peroxide for the positive plate and finely divided iron in a suitable container for the negative plate, with a solution of potassium hydrate as the electrolyte. This cell gives an initial open-circuit voltage of about 1.5 when charged, which falls to about 1.4 on closed circuit and gradually drops to about 1 volt when discharged. Results seem to show that this cell is much less liable to injury from overcharge or over-discharge than is the lead type.

26. Voltage Relations in Battery Circuits.—It must be understood that e.m.f. is generated in a cell whether or not the terminals are connected to a circuit. However, when the circuit is closed and current allowed to flow, the total e.m.f. developed in the cell becomes less as time goes on, on account of certain chemical changes which take place as the chemical energy of the cell is transformed into electrical energy. The complete circuit includes the solution within the cell which is, of course, a conductor and has resistance. The current which flows at any time is by Ohm's Law equal to the total e.m.f. divided by the total resistance of the circuit; a portion of the total e.m.f. is used in overcoming the resistance inside the cell, while the rest is used in overcoming the resistance connected between the terminals externally. The amount of e.m.f. used in overcoming the resistance of the cell is, again by Ohm's Law, equal to the product of the current flowing and the resistance of the cell, and the amount used on the outside resistance is equal to the product of the current and the outside resistance; that is, the potential difference at the terminals is less than the total e.m.f. of the cell by the amount of e.m.f. used in overcoming the internal resistance of the cell. This is an important principle and must be thoroughly mastered. If a cell has an e.m.f. of 1.4 volts and an internal resistance of 3 ohms, and is connected to an external resistance of 4 ohms, see Fig. 10, the total resistance of the circuit will be 3+4=7 ohms and the current will be 1.4/7=0.2 ampere. The portion of the e.m.f. which is used inside the cell to overcome its resistance is therefore, by Ohm's Law, 3×0.2=0.6 volt. The potential difference at the terminals is therefore 1.4−0.6=0.8 volt.

Stated in a different way, we may say that there is a rise of potential of 1.4 volts from the negative terminal, *a*, through the cell to the positive terminal *b*, due to the e.m.f. of the cell, but when a current of 0.2 ampere flows, there is a resistance drop of potential of 0.6 volt within the cell, leaving

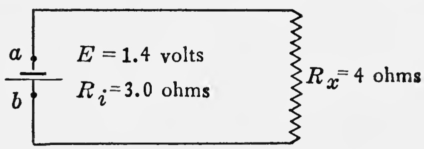

FIG. 10.

a net rise of potential of 0.8 volt. With reference to the external circuit, this 0.8 volt is a *fall* of potential. That portion of the e.m.f. which is used to overcome internal resistance is commonly called the *internal resistance drop*, meaning that the value of the potential difference at the terminals is less than the value of the e.m.f. of the cell by an amount equal to the product of the current and the internal resistance. Since *R* is the symbol always used for resistance, and *I* the symbol used for current, this drop is also very commonly called the internal *RI* drop. The expression *RI* drop, is also applied to the difference of potential between any two points in a wire when a current is flowing through it. When no current flows, the *RI* drop is zero, and the potential difference at the terminals is equal to the e.m.f. of the cell.

The word " voltage " is very commonly used to express both e.m.f. and potential difference; thus, when the e.m.f. of a cell is 1.4, it might be said that its voltage is 1.4, and when the potential difference between its terminals is 0.8, it might be said that the voltage at its terminals is 0.8.

27. Cells in Series.—When two or more cells are connected together in order to obtain more power, the combination is called a "battery." Sometimes the word battery is used to indicate a single cell, but this is an incorrect use of the word. When a number of cells are connected so that the same current passes through each of them one after another, they are said to be in series. If one terminal of each cell is connected to a common point, and the other

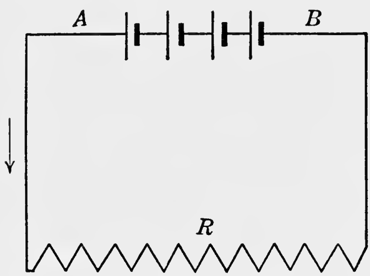

Fig. 11.

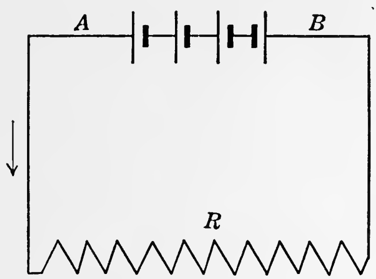

Fig. 12.

terminal of each cell connected to another common point, they are said to be in parallel. When connected in series, with the negative terminal of each cell connected to the positive terminal of its neighbor, the total e.m.f. of the battery is the sum of the e.m.f's of the individual cells. A common method of representing a cell is to draw two parallel lines as shown by the vertical lines in Fig. 11 where the longer thin line represents the positive terminal, and the shorter thick line represents the negative terminal. Fig. 11 represents a battery of four cells connected in series with all cells connected so that their e.m.f's act in the same way around the circuit. R represents a resistance connected to the terminals A and B of the battery. Suppose that each cell has an e.m.f. of 1.4 volts and an internal resistance of 3 ohms, while the resistance R is 16 ohms. The e.m.f.

of the battery is $4 \times 1.4 = 5.6$ volts and the total resistance of the battery is $4 \times 3 = 12$ ohms; therefore the total resistance of the circuit is $12 + 16 = 28$ ohms, and the current will be $5.6/28 = 0.2$ ampere. The e.m.f. used in overcoming the battery resistance is $0.2 \times 12 = 2.4$ volts, while that used in overcoming the external resistance, R, is $0.2 \times 16 = 3.2$ volts. The last result is, of course, the potential difference at the terminals of the battery and may also be calculated by subtracting 2.4 from 5.6. In Fig. 12 is shown a battery of four cells with one cell (the right-hand one) reversed, that is, it is so connected that its e.m.f. acts in a direction opposite to that of the other three. In this case, if each cell has an e.m.f. of 1.4 volts, the total e.m.f. acting on the circuit is $(3 \times 1.4) - 1.4 = 4.2 - 1.4 = 2.8$ volts. The two right-hand cells balance each other, and add nothing to the e.m.f. of the circuit. Note, however, that the internal resistance of a cell opposes the flow of current through the cell, no matter which way the current flows.

28. Cells in Parallel.—In any cell there is a limit to the current which can be allowed to flow through it, without

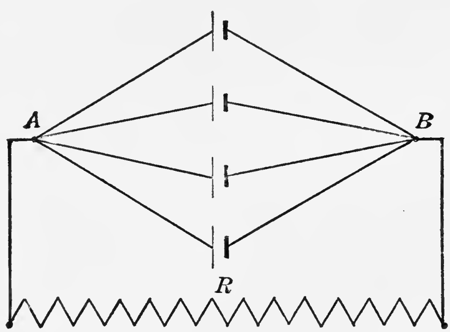

Fig. 13.

serious reduction of its e.m.f. and rapid deterioration from too rapid chemical action. When more current is desired than one cell can stand, it is common practice to connect them in parallel. Fig. 13 shows a battery of 4 cells in parallel connected to a resistance R. In this method of connection the total e.m.f. is only that of one cell, but the total current divides between the cells, each cell carrying, in the circuit shown, one-fourth of the current. If a battery be connected, as in Fig. 14, the total allowable current will be

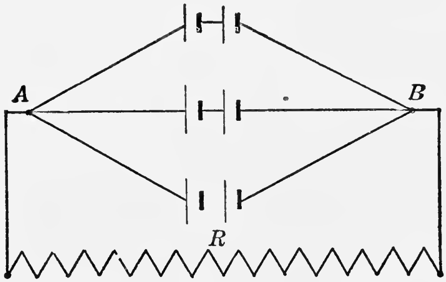

Fig. 14.

three times that for one cell, and the total e.m.f. will be twice that for one cell.

It is sometimes a matter of difficulty to understand why, when two or more cells are connected in parallel, the total e.m.f. acting on the circuit is equal only to the e.m.f. of one cell. Consider Fig. 15, which shows two cells in parallel between the points *A* and *B*. Evidently no current can flow around this circuit because the two cells act in opposite directions around the circuit and their e.m.f.'s are supposed to be equal. By Ohm's Law, the difference of potential between any two points along a given wire is equal to *RI*, where *R* is the resistance between the points, and *I* is the current flowing. Therefore, in this case, there is no difference of potential between the two positive terminals; that is, the two positive terminals and the point *B* are at the same potential. Similarly, the two negative terminals and the point *A* are at the same potential. Therefore, the difference of potential between *A* and *B*, considered either

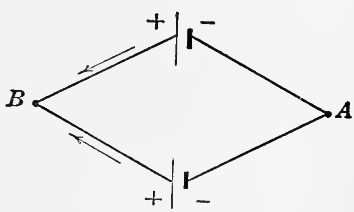

Fig. 15.

through the upper cell or through the lower cell, is equal to the e.m.f. of that cell.

Let A and B be connected through an external resistance R as in Fig. 16; consider that the two wires leading from A to the negative terminals of the cells have the same resistance and likewise the two wires leading from the two positive terminals to B. If the two cells have the same e.m.f. and the same internal resistance, the total current will divide equally between them. The drop in potential from each positive terminal to the point B is the same, and like-

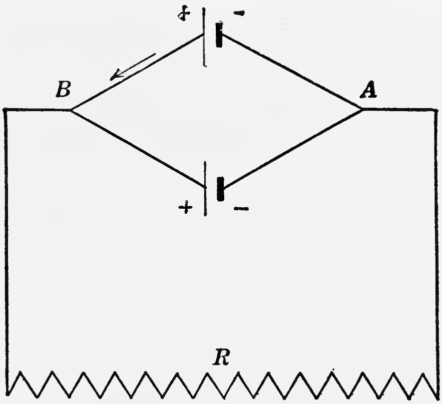

Fig. 16.

wise the drop from A to each negative terminal. Therefore, the potential difference between A and B is equal to the e.m.f. of either cell minus the drop in the wires leading to and from the cell and minus the drop within the cell. The p.d. between A and B is also equal to the product of the external resistance R and the current I.

29. Power and Energy in an Electric Circuit.—From Joule's Law we have the relation that the power developed in a resistance R due to a current I is equal to RI^2; that is,

$$P = RI^2 \tag{5}$$

We also have from Ohm's Law that

$$I = \frac{E}{R}. \tag{6}$$

Substituting (6) into (5), we get,

$$P = EI \tag{7}$$

which shows that the power in watts developed in an electric circuit is equal to the product of the current and the e.m.f. This equation must be understood to give the power only between the points having a potential difference E. The equation is true whether the voltage E is all used in overcoming resistance or partly used in overcoming a counter or back-e.m.f.; for in the latter case, the back e.m.f. can be replaced by a resistance r which will hold the current down to the same value that it has when the back e.m.f. is in circuit and the

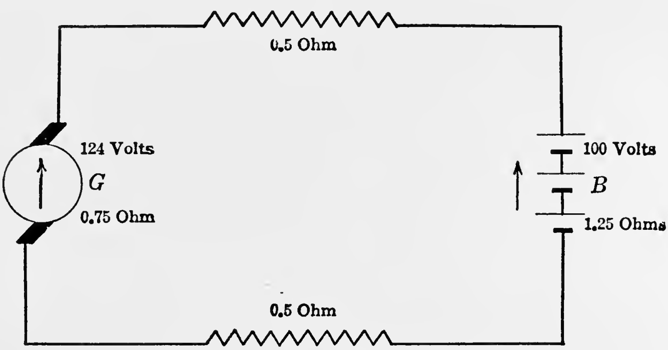

Fig. 17.

power used will therefore be the same in either case. Consider the circuit shown in Fig. 17, in which the generator G is charging the battery B. Suppose the generator develops an e.m.f. of 124 volts and the battery an e.m.f. of 100 volts; and suppose the resistance of the generator is 0.75 ohm, of the battery, 1.25 ohms, and of the wires leading from the generator to the battery, 1.0 ohm, making a total resistance of 3.0 ohms. The resultant e.m.f. acting around the circuit will be $124-100=24$ volts; the current flowing will therefore be $24/3=8$ amperes. Since 100 volts is used to send the current through the battery against its back e.m.f., this back e.m.f. may be represented by a resistance of $100/8=12.5$ ohms and the power developed will be $8^2 \times 12.5$

$=800$ watts; evidently, the same result would be obtained by multiplying the battery e.m.f. by the current, that is $8\times 100=800$ watts.

The total power developed in this circuit is $8\times 124=992$ watts, of which $8^2\times 1=64$ watts are used in the lead wires; $8^2\times 0.75=48$ watts are used in the resistance of the generator, and $8^2\times 1.25=80$ watts are used in the internal resistance of the battery. Since the current is flowing against the e.m.f. of the battery as well as against its resistance the potential difference at the terminals of the battery will be $100+(8\times 1.25)=110$ volts; the potential difference at the terminals of the generator is $124-(8\times 0.75)=118$ volts, and the drop in the two lead wires is 8 volts.

The work done or energy developed in any process during a given time is equal to the power, or rate of doing work, multiplied by the given time; that is, using the symbol W for work,

$$W=Pt=EIt \tag{8}$$

in an electrical circuit. But I is the rate of flow of electricity and therefore It is the quantity of electricity, Q, which flows through the circuit in the time t. We may therefore write

$$W=EQ, \tag{9}$$

which tells us that the work done in an electrical circuit is equal to the product of the electromotive force and the quantity of electricity Q, which is sent through the circuit either by or against the e.m.f., E. If the e.m.f. sends the quantity through the circuit, work is done *on* the circuit; if the quantity is sent through against the e.m.f., work is done *by* the circuit. Equation (9) may be written in the form

$$E=\frac{W}{Q}, \tag{10}$$

which, interpreted, means that electromotive force is equal to the work done in moving unit quantity of electricity. That is, when the potential difference between two points is E volts, the work done (in joules) in moving one coulomb of electricity from one of the points to the other is numerically equal to the voltage E.

30. The Circular Mil.—Most of the conductors used in electrical work are round and less than 1 inch in diameter. In order to avoid the use of decimals in expressing the diameter of such wires, a unit known as the *mil* has come into very general use. It is equal to 1/1000 of an inch. Instead of saying that a wire is 0.4 of an inch in diameter, it is said to be 400 mils in diameter.

The cross-sectional area, in square units, of a round wire is, of course, equal to the square of its diameter multiplied by $\pi/4$; thus the area of a wire 400 mils in diameter is 160,000 times $\pi/4$ or 125,700 square mils. In order to avoid the use of the factor $\pi/4$, another new unit, known as the *circular mil*, has come into general use. Its value is simply the area of a circle whose diameter is 1 mil; it is therefore equal to 0.7854 square mil, and the area of any circle expressed in circular mils is equal simply to the square of its diameter. That is, if the diameter of a circle is d mils, its area is equal to the sum of the areas of d^2 circles, each of whose diameter is 1 mil, or, d^2 circular mils; the area of a wire 400 mils in diameter is 160,000 circular mils. See Standard Handbook for Electrical Engineers, Section 4, Paragraphs 10 to 30 for information on wire gauges and tables.

31. Specific Resistance.—The resistance of a given conductor depends upon the material of which it is composed and upon its dimensions. It varies directly as the length and inversely as the cross-sectional area. Expressed in the form of an equation, the resistance, R, of wire of length, l, and area, a, is

$$R=\frac{\rho l}{a}, \tag{11}$$

where ρ is a constant expressing the value of the resistance of a piece of the given material 1 unit long and 1 unit in area. The constant ρ is called the *specific resistance,* or *resistivity* of the material. A wire 1 ft. long and 1 mil in diameter (i.e., 1 circular mil in area) is called a circular-mil-foot, or more commonly, a "mil-foot." The value of ρ varies somewhat with temperature, as will be discussed in the next paragraph. Its value for copper is 10.37 ohms per mil-foot at 20° C. This is known as the "International Annealed Copper Standard" and is for commercially pure electrolytic copper. It represents the most recent experimental determinations. The conductivity of copper varies with its purity and its physical condition. Copper having the above value of specific resistance is defined by international agreement as having 100 per cent conductivity. Copper of 98 per cent conductivity would have a specific resistance of $10.37/0.98 = 10.58$ ohms. The International Standard differs slightly from the older standard (known as Matthiessen's Standard), which was 10.35 ohms. The International Standard specific resistance for copper at 0° C. is 9.556 ohms per mil-foot. Values of specific resistance for other metals may be found in any Electrical Engineer's Handbook. For a complete table of the properties of copper wire, see Standard Handbook Section 4, Paragraph 50.

32. Effect of Temperature on Resistance.—The resistance of a conductor is found to depend upon its temperature as well as upon the material it is made of. The pure metals increase in resistance as the temperature increases. The resistance of certain alloys increases but very slightly with temperature, and in a few cases even decreases slightly. The resistance of salt and acid solutions and of carbon decreases with temperature.

The increase of resistance of a given wire, due to increase in temperature is proportional to the initial resistance of the wire and very nearly proportional to the rise in temperature. That is, if R_0 be the resistance of the wire at some standard temperature, such as zero Centigrade, the increase in resistance is equal, very closely, to $\alpha R_0 t$, where t is the rise of temperature above zero, and α is a constant known as the temperature coefficient of the given material. The temperature coefficient may be defined as the increase of resistance per degree Centigrade per ohm of resistance at the initial standard temperature. The resistance of the wire at temperature t is therefore

$$R = R_0 + R_0 \alpha t = R_0(1 + \alpha t) \tag{12}$$

The constant α, is not the same for different initial temperatures. The International Standard Temperature Coefficient for copper of 100 cent conductivity is as determined by experiment 0.00393 for an initial temperature of 20° C., and varies directly as the conductivity. For any initial temperature, t, and for 100 per cent conductivity the coefficient is $\alpha_t = 1/234.5 + t$, very nearly. It is convenient for many purposes to use the temperature coefficient corresponding to 0° C. as the initial temperature. The value of α_0 for copper at 0° C. is 0.00427.

The formula for finding the resistance at some temperature, t', when the resistance at some other temperature, t, is known, is

$$R_{t'} = R_t[1 + \alpha_t(t' - t)]. \tag{13}$$

For copper, this reduces, when $(1/234.5 + t)$ is substituted for α_t, to the form

$$R_{t'} = R_t\left(1 + \frac{t' - t}{234.5 + t}\right). \tag{14}$$

It is usual in electrical testing of machinery to determine the temperature, t', of windings from the measurement of their resistances at a known temperature, t, and at the unknown temperature, t'. For this calculation, equation (14) reduces to the form,

$$t' = \frac{R_{t'}}{R_t}(234.5 + t) - 234.5. \tag{15}$$

When a wire is carrying current, the heat generated in it due to its resistance raises its temperature, and the temperature will rise until the rate at which the heat is radiated and conducted away from the wire is equal to the rate at which heat is generated in it. The rate at which the heat is carried away depends upon the surroundings of the wire; that is, whether it is bare or insulated, and whether it is strung in open air or wound up in a coil, and if in a coil, whether the coil is enclosed or exposed or immersed in oil. See Standard Handbook for Electrical Engineers, Section 3, Paragraph 22, for table of carrying capacity of insulated wires as allowed by the National Board of Fire Underwriters. When the wire is wound in a coil, the carrying capacity may be estimated by the use of an experimentally determined constant which applies, as nearly as can be judged, to the conditions as to exposure, depth of winding, kind of insulation, etc. This constant is expressed as the number of "circular mils per ampere," and is the area in circular mils which the wire should have for each ampere it is to carry, in order that the temperature shall not exceed a safe value. For a temperature rise of 50° C. above 20° C., the required circular mils per ampere will vary from as low as 600 for shallow, well ventilated coils, to as high as 2500 for deep coils and little ventilation.

33. Kirchhoff's Laws.—Two laws of very great importance in connection with electric circuits were first clearly pointed out by Kirchhoff, and have been proven to be of universal application.

First Law.—The algebraic sum of all the e.m.f's acting in a chosen direction around any closed circuit is equal to the algebraic sum of all the resistance drops in the same direction around that circuit.

Second Law.—The sum of all the currents which flow up to any point in a circuit is equal to the sum of all the currents which flow away from that point.

The significance and application of these laws under different circuit conditions are shown in the following paragraphs.

34. Resistances in Series.—When a number of resistances are connected in series, the same current flows through all of them, and the total resistance is equal to the sum of the individual resistances. If a certain current is flowing through a series of resistances, the net e.m.f. acting in the circuit, by Kirchhoff's first law, is equal to the sum of the resistance drops, or to the product of the sum of the resistances, and the current. That is,

$$E = R_1 I + R_2 I + R_3 I = I(R_1 + R_2 + R_3). \qquad (16)$$

If there is more than one e.m.f. acting on the circuit, as, for example, the back e.m.f. of a battery or of a motor, then the net e.m.f., or the algebraic sum of the various e.m.f's, is equal to the sum of the various RI drops.

35. Resistances in Parallel.—When two or more resistances are connected in parallel between two points, the current flowing through each resistance is equal to the p.d. between the two points divided by that particular resistance, provided that there are no sources of e.m.f. connected in series with any of the resistances, and the total current is, by Kirchhoff's second law, the sum of the currents flowing in the various paths. It is many times convenient to determine the value of a single resistance which is equivalent to the several resistances in parallel. This equivalent resistance will evidently be equal to the p.d. divided by the total current which flows between the two points. Its value in terms of the individual resistances may be easily derived as follows: Let the total current be represented by I, the p.d. by E and the individual currents and resistances by I_1, I_2, I_3, etc., R_1, R_2, R_3, etc., respectively.
Then we may write

$$I = \frac{E}{R_1} + \frac{E}{R_2} + \frac{E}{R_3} = E\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right). \qquad (17)$$

Therefore

$$R = \frac{E}{I} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}} \qquad (18)$$

where R is the equivalent resistance of R_1, R_2, and R_3, when connected in parallel. That is, the equivalent resistance of a number of resistances in parallel is equal to the reciprocal of the sum of the reciprocals of the individual resistances. If the resistances are equal, the equivalent resistance will be equal to the value of one divided by the number in parallel.

The reciprocal of a resistance is called its conductance and in dealing with parallel circuits it is very common to use conductances instead of resistances. The unit of conductance is the mho (ohm spelled backward and pronounced *mo*). The symbol used for conductance is G and a circuit having say 5 ohms resistance, is said to have a conductance of 1/5, or 0.2 *mho*. By Ohm's Law, $I=E/R$; if we use G instead of R, it is $I=E\times G$. From the equation above, it is clear that the equivalent conductance of a number of conductances in parallel is equal to the sum of the individual conductances, and the equivalent resistance is the reciprocal of the equivalent conductance; that is,

$$I=G_1E+G_2E+G_3E=E(G_1+G_2+G_3) \tag{19}$$

or

$$\frac{E}{I}=\frac{1}{G_1+G_2+G_3}=R. \tag{20}$$

36. Series-parallel Circuits.—In the case of a mixed circuit, such as that shown in Fig. 18, it is necessary, first of all, to find the equivalent resistance of the parts in parallel, and then add the result to the resistances in series. For example, let $R_2=5$ ohms; $R_3=10$ ohms; $R_1=4$ ohms; r, the resistance of the battery $=3$ ohms; and E, the e.m.f. of the battery $=6.2$ volts; what will be the total current? The conductance of R_2 is 0.2 mho; of R_3 is 0.1; and of the combination, 0.3 mho; the equivalent resistance of R_2 and R_3 is therefore 3.33 ohms. The total resistance of the circuit is then $3.33+4+3=10.33$ ohms, and $I=6.2/10.33=0.6$ ampere. The p.d. between A and B will be $0.6\times3.33=2$ volts, the current through R_2 is $2/5=0.4$ ampere, and through R_3 is $2/10=0.2$ ampere. The p.d. in R_1 is $0.6\times 4=2.4$ volts, and in r is $0.6\times 3=1.8$ volts; the sum of the various p.d's is $2+2.4+1.8=6.2$, and is equal to the e.m.f. of the battery.

To further illustrate the principles used in solving a series-parallel circuit, the following example is given: Let the circuit be made up as shown in Fig. 19; required the p.d. at the terminals xy of R_9 when the p.d. at the terminals AB of the battery is 100 volts. The resistance of $KxyN$ is $25+3+3=31$ ohms, which is in parallel with 20 ohms.

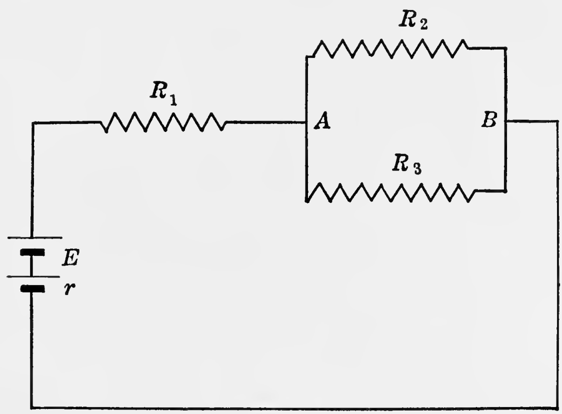

FIG. 18.

The conductance of $KxyN$ is $1/31=0.03226$; of R_6 it is $1/20=0.05$; the conductance of the parallel paths between K and N is therefore $0.03226+0.05=0.08226$ and the equivalent resistance of this path is $1/0.08226=12.15$ ohms; this equivalent resistance is in series with R_4 and R_5; the resistance of the path from D to F by way of R_6 and R_9 is therefore $12.15+2+2=16.15$ ohms, and its conductance is $1/16.15=0.06192$. The conductance of R_3 is $1/19=0.05263$ and of the parallel circuit between D and F, it is $0.06192+0.05263=0.11455$. The equivalent resistance of the parallel path between D and F is therefore $1/0.11455=8.73$ ohms; this is in series with R_1 and R_2 so that the equivalent resistance of the entire circuit is 8.73+1.5+1.5=11.73 ohms and the total current is 100/11.73=8.53 amperes.

This total current flows from A to D then divides; the two parts join again at F and flow from F to B. The drop of potential in R_1 is therefore $1.5\times 8.53=12.79$ volts and likewise the drop in R_2 is 12.79 volts; the p.d. between D and F is therefore $100-(2\times 12.79)=74.42$. The current which flows from D to F by way of R_6 and R_9 is therefore 74.42/16.15 =4.61 amperes. The current in R_3 is 74.42/ 19=3.92. It may be noted that 3.92+4.61 gives 8.53, which is the

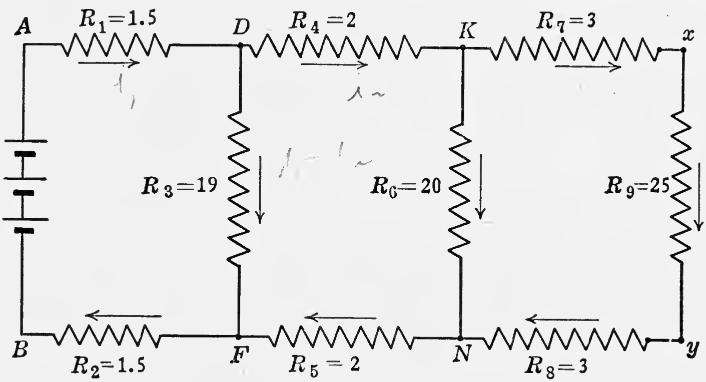

FIG. 19.

total current as previously found, and thus checks the arithmetical work. The drop in R_4 and also in R_5 is $4.61\times 2=9.22$ volts, so that the p.d. between K and N is $74.42-(2\times 9.22)=55.98$ volts. The current in $KxyN$ is $55.98/31=1.81$ and in R_6 it is $55.98/20=2.8$. The sum of these is 4.61 which again checks with the value found above for the circuit from D to F by way of R_6 and R_9. The drop in R_7 and R_8 is $2\times 3\times 1.81=10.86$ volts, so that the p.d. between x and y is $55.98-10.86=45.12$ volts.

If the above problem had been stated by giving the p.d. between x and y and requiring the p.d. between A and B, it would *not* have been necessary to calculate the equivalent resistance of the circuit. The current in R_9 would be found at once, then the drop in R_7 and R_8, then the p.d. between K and N by adding these drops to the p.d. between x and y. Then the current in R_6 would be found and added to that in R_9 giving the current in R_4 and R_5; then the drops in R_4 and in R_5 would be found and added to the p.d. between K and N, giving the p.d. between D and F. Then the current in R_3 would be found and added to that in R_4, giving the total current, then the drop in R_1 and R_2 would be found and added to the p.d. between D and F, giving the p.d. between A and B. This general problem may be met in practice in many forms; for example, instead of the resistances R_3, R_6, and R_9 being given, the currents may be given; or, the power in one or more may be given; or, the p.d's between A and B and between x and y may be given and the resistances, R_1, R_2, R_4, R_5, R_7, and R_8 required. The relations between power, p.d., current and resistance must all be kept continually in mind.

37. Complex Circuits.—There is a certain class of circuits which requires the application of Kirchhoff's and Ohm's Laws in a somewhat different manner from that used in the preceding problems. An example of this is shown in Fig. 20. The resistance of this combination cannot be found by the methods given above; but by applying Kirchhoff's two laws, equations can be written by which the unknowns can be found.

Let it be assumed that six resistances, r_1, r_2, r_3, r_4, r_5, and r_6 are known and also the p.d., E, between the terminals of the battery; the unknowns are the six currents in 1, 2, 3, 4, 5, and 6. It will be noted that there are four points in the circuit shown in Fig. 20, where the current divides. Writing the equations for the current at each of these points, we have:

$$I_6 = I_1 + I_2 \text{ (point } a\text{)} \qquad (21)$$

$$I_1 = I_3 + I_4 \text{ (point } b\text{)} \qquad (22)$$

$$I_2 + I_3 = I_5 \text{ (point } c\text{)} \qquad (23)$$

$$I_4 + I_5 = I_6 \text{ (point } d\text{)} \qquad (24)$$

It should be noted that the current in path r_3 is assumed to flow from b toward c; it is necessary to assume a direction for the current in all paths in order to write the equations; if, in the final solution, any current comes out with a negative sign, it means that the current in that path flows in the direction opposite to that assumed. It should be noted also that any one of the four equations written above may be derived mathematically from the other three and that therefore only three of the equations are independent of

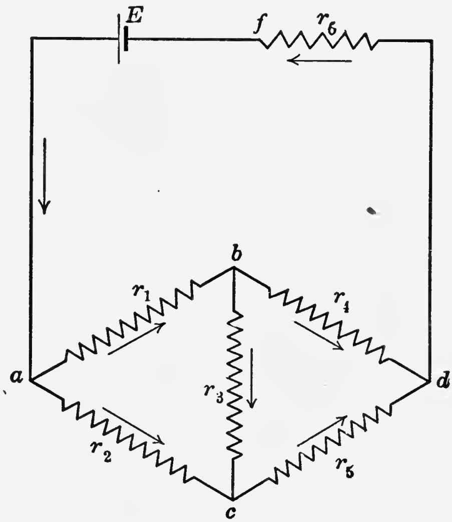

FIG. 20

each other and can be used for the solution of the unknown. To write the equations for the first law, note that there are also four paths for which equations may be written, namely, *abdfa*, *acdfa*, *abca*, and *bcdb*, but as in the case of the current equations, only three of them are independent. There are other paths which can be traced through, but they also are related to the rest so that if used they would reduce to identities. The equations for the four paths mentioned are,

$$r_1I_1 + r_4I_4 + r_6I_6 = E(abdfa) \tag{25}$$

$$r_2I_2+r_5I_5+r_6I_6=E(acdfa) \tag{26}$$

$$r_1I_1+r_3I_3-r_2I_2=0(abca) \tag{27}$$

$$r_3I_3+r_5I_5-r_4I_4=0(bcdb) \tag{28}$$

We thus have six equations from which to find six unknowns. Two of the equations, one from each set, are not to be used, and the solution of the problem is one of pure mathematics. The negative sign is used for r_2I_2 in equation (27) and for

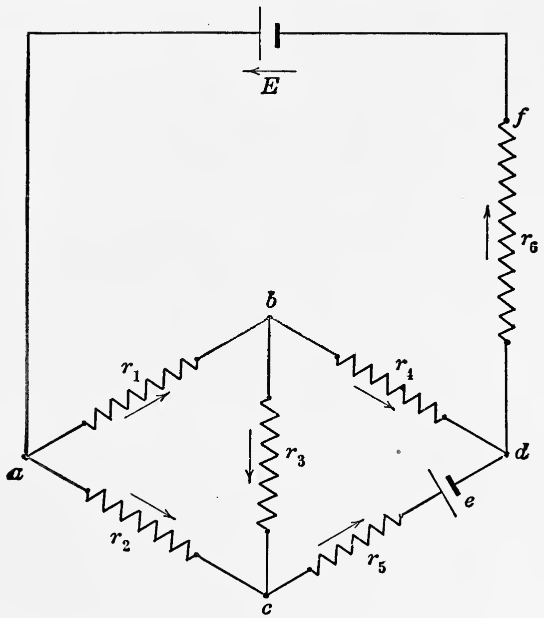

FIG. 21.

r_4I_4 in equation (28) because these drops are opposite in direction from the other two in their respective paths. In some cases, other e.m.f's may be placed in the network, as, for instance, in with r_5 as shown in Fig. 21. In such case the e.m.f. equation for the path *acdfa* would be

$$r_2I_2+r_5I_5+r_6I_6=E\pm e \tag{29}$$

the sign in front of e depending on whether the e.m.f. e is with or against the e.m.f. E. For the circuit shown, it would have a negative sign.

The equation for path *bcdb* would be

$$r_3I_3+r_5I_5-r_4I_4=-e. \qquad (30)$$

When two e.m.f's are connected in parallel as in Fig. 22, the current will divide equally between the two sources

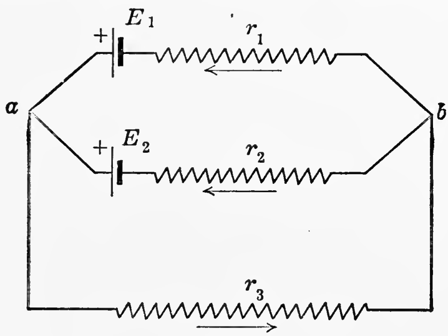

FIG. 22.

when the e.m.f's are equal and the resistances in the two paths are equal. If either or both of these are unequal, the current may not divide equally and may flow in either direction through one of the sources, depending on the relative

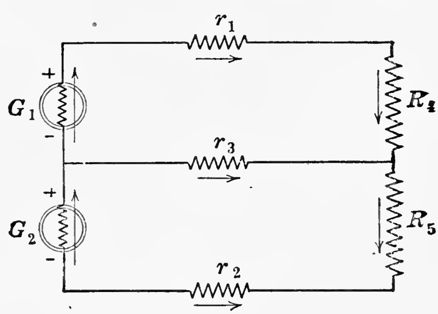

FIG. 23.

values of E_1, E_2, and the resistances in the three branches. The solution of such a problem involves the writing of the equations for Kirchhoff's Laws and solving them for the unknowns.

Another example of a complex circuit is the circuit frequently used in distributing electrical power, and known as the three-wire circuit. It is represented in simple form in Fig. 23. The middle wire (r_3) is frequently called the neutral wire and the current in it may flow in either direction, or, if the loads R_4 and R_5 are perfectly balanced, no current will flow in it.

38. The Wheatstone Bridge.—In Fig. 24 is shown a network similar to that of Fig. 20, but R_3 is replaced by a galvanometer and resistance r_g. If the resistances r_1, r_2, r_3

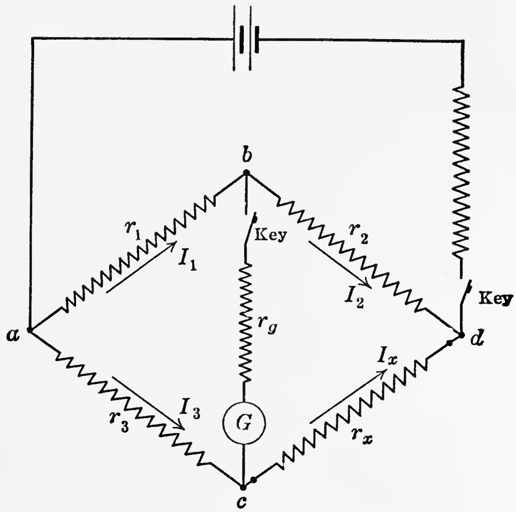

FIG. 24.

and r_x are given such values that no current flows through r_g and the galvanometer, then the points b and c must be at the same potential, and the p.d. from a to b must be the same as that from a to c; likewise, the p.d. from b to d must be the same as that from c to d. Also the current I_1 is the same as I_2 and I_3 is the same as I_x, since no current flows in r_g. Therefore we may write

$$r_1I_1 = r_3I_3 \tag{31}$$

and

$$r_2I_1 = r_xI_3. \tag{32}$$

From these equations, we get

$$\frac{r_2}{r_1}=\frac{r_x}{r_3}. \tag{33}$$

This relation is used in the measurement of resistance and the arrangement is called a Wheatstone Bridge. There are various forms of Wheatstone Bridges, among them being the Box Bridge and the Slide-wire Bridge. In the Box Bridge there are three sets of adjustable resistances within the box, which may be represented by r_1, r_2 and r_3 in Fig. 24. Binding posts are provided so that a battery may be connected to points a and d, a galvanometer between points b and c, and an unknown resistance between c and d. Sets r_1 and r_2 are usually called the ratio arms of the bridge and generally each one has a 10-ohm, a 100-ohm, and a 1000-ohm coil, any one of which may be connected in, so that the possible ratios of r_1 to r_2 are 0.01, 0.1, 1, 10, and 100. Sometimes a 1-ohm coil is also put in each set, thus giving possible ratios of 0.001 and 1000. Set r_3 consists of a number of coils, of values ranging from 1 to 500 or 1 to 5000, and so arranged that steps of 1 ohm can be made from 1 to 1111 or from 1 to 11111.

An unknown resistance being connected between c and d, the ratio arms and the resistance r_3 are adjusted until the galvanometer shows no deflection when its circuit is repeatedly opened and closed. Then the unknown resistance is calculated from the relation

$$r_x=r_3\frac{r_2}{r_1}. \tag{34}$$

In the slide-wire bridge (see Fig. 25) a piece of bare wire of uniform size is stretched between two points over a scale, usually one meter long. A known resistance being connected between a and b, and an unknown resistance between b and d, the sliding contact c, is moved along until there is no deflection in the galvanometer. When this condition is obtained the resistances are related to each other in the proportion $ab : bd :: ac : cd$; whence

$$bd = ab \times \frac{cd}{ac}. \tag{35}$$

But the resistances of ac and cd have the same ratio as the corresponding portions of the slide wire, which is therefore easily determined from the readings on the scale. Sometimes a telephone receiver is used in place of the galvanometer for determining the condition of balance, and answers very well for work not requiring great accuracy.

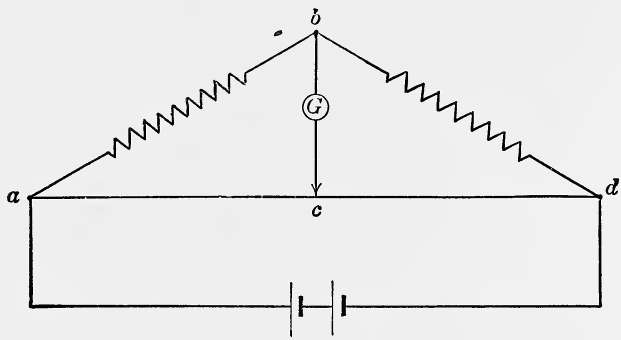

Fig. 25.

39. The Potentiometer.—A method of connection frequently useful in electrical testing work is known as the potentiometer connection. It consists of a resistance which is accessible at all points or at frequent intervals along its length, connected to a source of e.m.f. which is higher than the e.m.f. which is desired or which is to be measured. Fig. 26 illustrates the method of connection for obtaining any desired p.d. between the points a and c within the limits of the e.m.f. impressed on ab. A sliding contact is used at c and, as it is moved from a toward b, the p.d. increases from zero to the full value between a and b. The wire ab is called the potentiometer wire. Suppose the resistance of ab is 200 ohms and of R is 20 ohms; required a p.d. of 24 volts between a and c when the p.d. between a and b is 100 volts.

The current in R will be $24/20=1.2$ amperes; the p.d. between c and b will be $100-24=76$ volts. The current $I_{cb}=I_{ac}+1.2$; also $r_{ac}I_{ac}=24$; $r_{cb}I_{cb}=76$; and $r_{ac}+r_{cb}=200$. The solution of these equations will give the values of the four unknowns, I_{ac}, I_{cb}, r_{ac}, and r_{cb}.

If in the circuit aRc, a battery be placed so that its e.m.f. opposes the flow of current through that circuit, there will be some point c where the e.m.f. of this second battery will just balance the p.d. in the potentiometer wire between

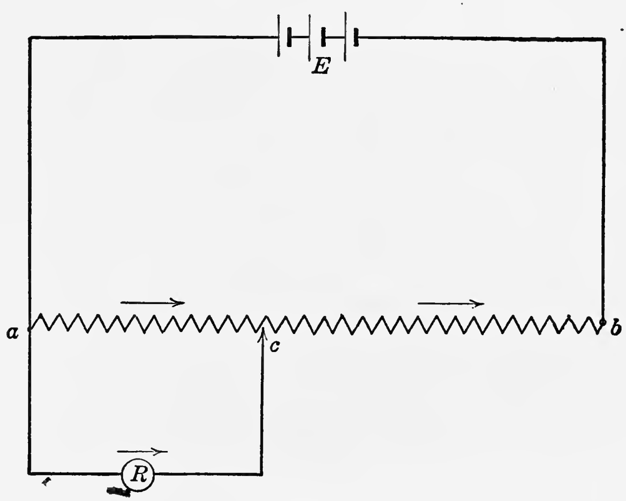

FIG. 26.

a and c and no current will flow in the circuit aRc. This principle is used for comparing the values of two e.m.f.'s. A known e.m.f. (a standard cell), is connected between a and c, Fig. 27, and the point c is found where no current flows through the galvanometer G_1; the e.m.f. to be measured is connected between a and d and the point d is found such that no current flows through the galvanometer G_2. In practice, switching arrangements are provided by means of which e is connected in the place of e_s, the same galvanometer is used, and a slider is used to find the positions c and d. The current has the same value throughout the wire and the p.d. between a and c is to the p.d. between a and d as the resistance of the potentiometer wire between a and c is to the resistance between a and d. These resistances may be known and their ratio is the ratio of e_s to e; or, the potentiometer wire may be of uniform cross-section so that the resistance of any portion is proportional to the

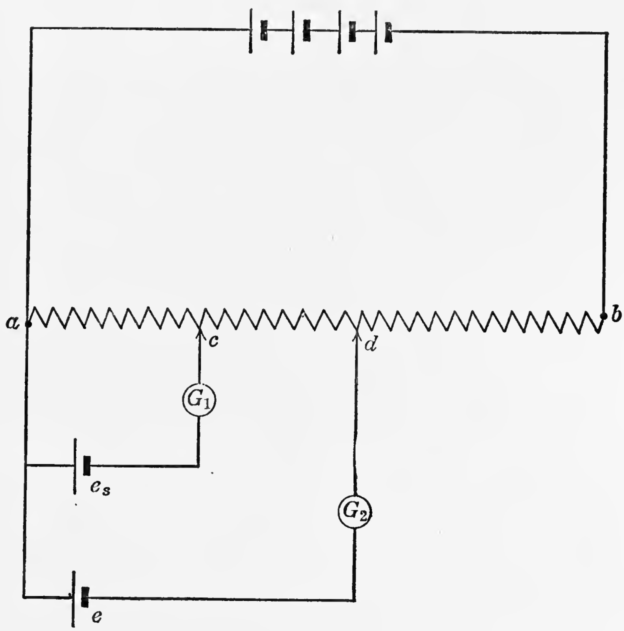

Fig. 27.

length of that portion; in this case the ratio of the lengths ac and ad is the required ratio of e_s to e.

40. Ammeters and Voltmeters for Direct Currents.—The D'Arsonval type of meter was briefly described in the last paragraph of Article 22. Fig. 28 shows a plan view of a commercial meter of this type. The coil, CC, is rectangular in shape as shown in Fig. 29. One end of the coil is connected to the inner end of the upper spiral spring, s'; and the other end of the coil is connected to the inner end of the lower spiral spring, s''. The outer ends of these spiral springs serve as terminals for leading the current to and from the coil. The coil is pivoted concentrically with the pole pieces, PP', Fig. 28, which are attached to the poles of the permanent magnet MM. A cylindrical soft iron core, I, occupies the space inside the coil, so that the air gaps, gg, in which the sides of the coil move, are uniform in length, thus producing a uniform magnetic flux density.

A typical value of the current required to produce enough torque to swing the needle over the full scale (80°

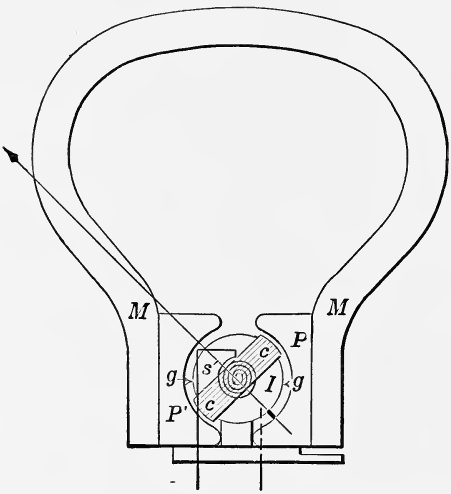

Fig. 28.

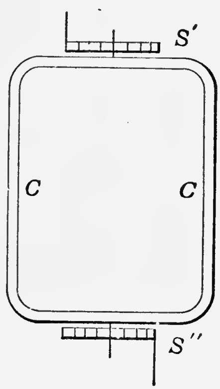

Fig. 29.

to 90°) is 0.01 ampere; the corresponding typical value of the voltage required to send this current through the resistance of the coil and springs is 0.05 volt (or, 50 millivolts). The resistance of the coil and springs in this case would be 5 ohms.

To adapt this arrangement to the measurement of large currents or voltages, shunts or multipliers must be used. To illustrate, suppose a meter is desired to measure 10 amperes at full scale deflection. This is 1000 times the current allowable through the coil; therefore a resistance must be connected in parallel with the coil, which will carry

9.99 amperes when the coil is carrying 0.01 ampere. The resistance of the shunt R_s would be, in this case 1/999 of 5 ohms. The connection is shown in Fig. 30. The scale of the meter would be marked to show the total current flowing, that is, 10 amperes at full deflection.

To use the same coil for measuring, say 150 volts at full deflection, it would be necessary to put in series with the coil

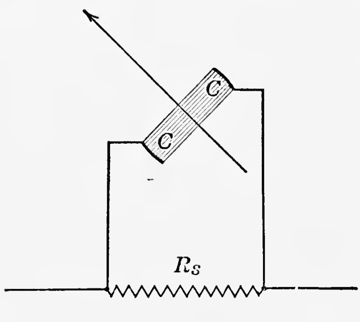

FIG. 30.

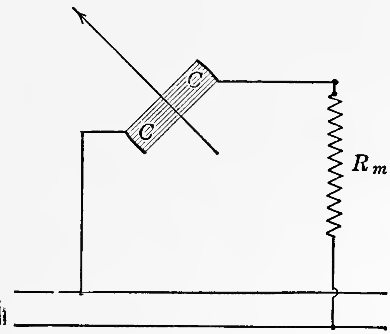

FIG. 31.

a resistance of such value that 150 volts would produce 0.01 ampere through the meter at full scale deflection. The total resistance would be 150/0.01, or 15,000 ohms; the multiplier, or additional resistance, R_m would have to be 14,995 ohms. This connection is shown in Fig. 31. In this case, the scale of the meter would be marked in volts, from 0 to 150.

CHAPTER IV

ELECTROMAGNETISM

41. Flux-linkage and Electromotive Force.—Whenever an electric circuit and a magnetic circuit are so related that the magnetic lines of force pass through and around the electric circuit like two adjacent links of a chain, the two circuits are said to be interlinked. When all the lines of force in a given field interlink with all the turns of a coil of wire, the total number of linkages is equal to the product of the number of turns in the coil and the number of lines of force. When the interlinkage is not complete, see Fig. 8, the total number of linkages is equal to the sum of all the lines that link each turn; that is, one linkage is one line linking with one turn.

It was discovered by Faraday that when from any cause the number of linkages in a circuit changes, there is induced in the circuit an electromotive force which is proportional to the rate at which the linkages change. The rate of change of linkage is frequently called rate of cutting lines of force, since, whenever the number of linkages changes, either lines of force must cut across the wire constituting the circuit, or the wire must cut across the lines of force. In this connection, it should be noted that lines of force may be cut by a wire without changing the linkage, and therefore, without producing an e.m.f. at the terminals of the wire. For example, if a coil of wire is moved within a uniform magnetic field in such a direction that its plane does not change in direction with respect to the field, both sides of the coil will cut lines of force, but the number of lines passing through the coil will not be changed so long as the whole coil remains in a field of uniform strength. The linkage with each half of the coil changes, or in other words, each half of the coil cuts lines of force, and equal e.m.f's are generated in the two halves but these two e.m.f's are found to be oppositely directed around the coil, so that they balance each other, and the resultant e.m.f. generated in the coil is zero.

When an e.m.f. is generated or induced in the manner mentioned above and the circuit is closed so that current flows, electric power will be developed in the circuit. Since power cannot be created out of nothing, the question at once arises, whence comes this power and from what kind of power is it transformed? The answer is, that when current flows in a wire, and that wire is in a magnetic field, a mechanical force is exerted on the wire tending to push it sideways through the field; this is the *reacting force against which the wire must be moved by mechanical means* through the field in order to produce the electric power. That is, mechanical power is expended in moving the wire through the field, and this mechanical power is transformed into electrical power by virtue of the e.m.f. generated and the current which flows. This is the principle of electric generators. Note that no power is required to generate an e.m.f. in a wire if no current is flowing in the wire.

Suppose, now, that a wire be placed in a magnetic field, and current be sent through it by an external source of e.m.f. The force exerted on the wire by the field will cause it to move and develop mechanical power; this power must be supplied by the electric circuit, and in order to do this there must be a *reacting force in the electric circuit against which the current is forced to flow.* When the wire moves through the magnetic field an e.m.f. is generated, which opposes the flow of current; this e.m.f. is called *back* or *counter e.m.f.* and is the reacting force against which the work is done. This is the fundamental principle of the action of an electric motor. Note that there will be some power required to supply the heat losses in the resistance of wires.

It is of extreme importance that this matter of the reacting forces which are exerted when electrical energy is transformed to mechanical energy, or vice versa, be thoroughly understood. Therefore, the principles discussed in the preceding paragraphs are here restated in somewhat different form. That there must be a reacting force follows directly from a broad interpretation of Newton's third law of motion, which is, that there can be no action without an equal and opposite reaction. The application of the law to electric circuits was discovered by Lenz, and the statement that an induced current always opposes the action which produces it, is known as Lenz's Law. In the case where any part of a closed electric circuit is moved across a magnetic field, or a magnetic field is moved across any part of a closed electric circuit, it has been discovered that an e.m.f. is generated, an electric current flows, and work is done. To do this work requires the application of a mechanical force to move the wire or the magnetic field and the reacting force is discovered to be an unseen force exerted by the magnetic field upon the wire. In the case where an electric current is sent through a wire which lies across a magnetic field, it has been discovered that the wire or the field will move and do mechanical work. To do this work requires the application of an e.m.f. to send the current through the wire and the reacting force is discovered to be an induced e.m.f. which opposes the flow of current.

42. Relation of Induced e.m.c. to Rate of Change of Linkage.—By the help of the principles just discussed, we may derive the fundamental relation between the value of an induced e.m.f. and the rate of change of linkages. Consider a simple case like that shown in Fig. 32 which represents a straight wire AB moving at right angles across a magnetic field. The wire $DKNG$ is supposed to be stationary and its plane is at right angles to a uniform magnetic field, the lines of force being perpendicular to the paper and represented in cross-section by the dots. The wire AB is supposed to slide toward the left along the wires DK and GN at a uniform velocity of v centimeters per second;

if the wires DK and GN are l centimeters from each other, the change in linkage will be Hlv lines per second where H is the intensity of the field, in lines per square centimeter. The movement of the wire AB will generate an e.m.f. of e volts, a current of i amperes will flow through the circuit and work will be done at the rate of ei joules per second, or $ei \times 10^7$ ergs per second. A mechanical force F_m, will therefore be required to move the wire. The reacting force F_r, will be that exerted by the magnetic field on the wire, and will be equal to $Hli/10$ dynes, as already shown in Article 19. The rate at which work is done against this force will be $Hliv/10$ ergs per second. The rate at which

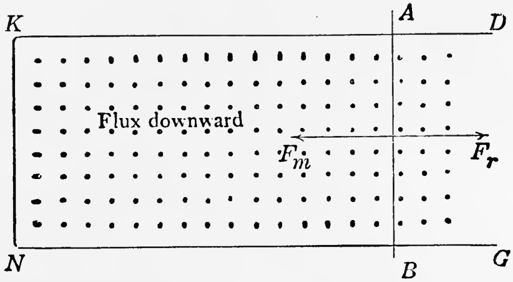

FIG. 32.

mechanical work is done on the wire in moving it must be equal to the rate at which electrical work is done by the e.m.f. which is generated as a result of moving it. Therefore, we may write the equation

$$ei10^7 = \frac{Hliv}{10} \tag{36}$$

or

$$e = \frac{Hlv}{10^8}. \tag{37}$$

That is, the e.m.f. in volts, is equal to the rate at which the linkages change, divided by 10^8. If the distance moved through in time t be called s, then $v = s/t$, and

$$e = \frac{Hls}{10^8 t} = \frac{\phi}{10^8 t} \tag{38}$$

where ϕ is the total change in linkage during the time t, and is equal to Hls, since ls is the area swept over, and H is the number of lines per square centimeter. The expression (ϕ/t) is evidently a rate of cutting lines of force; if this rate is not constant, due either to non-uniform field, or a non-uniform velocity, the e.m.f. will vary from instant to instant, and the expression must be put into differential form, as

$$e = \frac{d\phi}{10^8 dt}. \qquad (39)$$

If a coil of wire having N turns be moved in a magnetic field and the flux which passes through the coil changes by an amount $d\phi$, in the time dt, then the e.m.f. will be

$$e = N\frac{d\phi}{10^8 dt}. \qquad (40)$$

If the linkage is not complete, but the actual change of linkage can be represented by $d\phi' = kNd\phi$, where k is some factor less than unity, then

$$e = \frac{d\phi'}{10^8 dt}. \qquad (41)$$

This is the fundamental equation for an induced e.m.f. and is universal in its application. It should be noted, however, that it gives the instantaneous value and that the average value over an extended time will depend upon the average rate at which the linkages change with time, see equation (38). The application of these equations to electrical machinery will be taken up in later courses. The important things to learn here are that *at any instant an induced e.m.f. is equal to the time rate of change of flux linkage with the circuit at that instant,* and that *a rate of change of* 10^8 *linkages per second gives* 1 *volt of electromotive force.* The e.m.f. generated by a rate of change of 1 linkage per second is called an abvolt; 1 volt is therefore equal to 10^8 abvolts. It is desirable also at this time to learn a rule for determining the direction of an induced e.m.f. The so-called "Left-hand Rule" has already been given for determining the direction of the force action upon a wire in a magnetic field. Since the direction of motion required to produce an e.m.f. is opposite to that of the force action of the current produced by such e.m.f., it follows that if the forefinger of the *right* hand point in the direction of the flux, and the thumb at right angles to the forefinger, point in the direction of motion of the wire with respect to the flux, the middle finger, at right angles to both, will point in the direction of the induced e.m.f. This is known as the "Right-hand Rule." It must be noted that the thumb must point in the direction of the motion of the *wire with respect to the flux;* that is, if the flux is moving, say toward the left, the relative motion of the wire with respect to the flux is toward the right.

43. Work Done When an Electric Wire Cuts a Magnetic Field.—The force in dynes exerted on an electric wire l centimeters long and carrying a current, I amperes, when the axis of the wire makes an angle θ with the direction of a magnetic field of strength H, has been shown in Article 19 to be equal to $IHl \sin \theta/10$. If the wire is moved a distance s against this force and the direction of motion makes an angle α with the direction of the force, the work done will be $IHls \sin \theta \cos \alpha/10$ ergs; but $Hls \sin \theta \cos \alpha$ is equal to ϕ, the number of lines of force cut by the wire; therefore, the work W, done when a wire cuts a field, is equal to $\phi I/10$, the product of the flux cut and the current in the wire, divided by 10 when I is in amperes. If the electric circuit consists of a coil of wire of N turns, and the total amount of cutting of flux, or change in linkages, is ϕN, then the work done will be

$$W = \phi NI/10 \text{ ergs.} \tag{42}$$

44. Number of Lines of Force Issuing from a Unit Magnet Pole.—A unit magnet pole and unit strength of magnetic field have already been defined. It follows from these definitions that there is a field intensity of one line per square centimeter at 1 cm. distance from a unit magnet pole; therefore, since there are 4π square centimeters of surface in the sphere of unit radius surrounding a unit pole, there will be a total of 4π lines of force issuing from a unit pole.

45. The Field Intensity around a Long Straight Wire.—Consider a closed electric circuit, consisting in part of a long straight wire and with the rest of the circuit far enough removed from the straight part that the flux due to the rest of the circuit will be negligible in the vicinity of the straight part. The magnetic lines of force surrounding the straight part of the wire will then be concentric circles around the center of the wire and with their planes at right angles to the wire. Suppose a unit pole to be carried once around the wire along one of the concentric lines of force; each of the 4π lines of force from the pole will cut the circuit and the work done will be $0.4\pi I$, when I is in amperes, according to equation (42). The distance moved through by the unit pole is $2\pi x$, where x is the radius of the particular path followed. Therefore, the force, that is, the intensity of the field along this path, is

$$H = \frac{0.4\pi I}{2\pi x} = \frac{0.2I}{x}. \qquad (43)$$

This is an important law; it shows that the intensity of magnetic field produced at any given distance from a long straight wire by a current in it is directly proportional to the value of the current and inversely proportional to the radial distance from the center of the wire to the given point.

46. Force Exerted between Two Parallel Wires.—When two wires are parallel and current flows through them, there will be a force exerted between them; for each wire produces a field which is at right angles with the other wire and therefore exerts a force on it. Suppose the wires are d centimeters apart and the currents are I' and I'' amperes respectively, then the field intensity produced by the first wire at the second wire is $0.2I'/d$; this field, acting on the second wire produces a force of $0.02I'I''/d$ dynes per centimeter of length of wire. Likewise, the second wire produces a field of $0.2I''/d$ at the first wire and this field exerts a force of $0.02I'I''/d$ dynes per centimeter of wire. Each of these forces is the reacting force of the other, and the attraction or repulsion between the wires is therefore $0.02I'I''/d$ dynes per centimeter. An application of the "left-hand rule" will show that the force will tend to draw the wires together when the two currents are in the same direction and will tend to push them farther apart when the currents are opposite in direction. When the two currents are equal, then the force between the wires is proportional to the square of the current. An application of the law just discussed is to be found in the construction of Electrodynamometer instruments.

47. Field Intensity at the Center of a Coil of Large Radius.—It has been proven experimentally that the intensity of the magnetic field which emanates from a magnet pole varies inversely as the square of the distance from the pole; and since the field intensity is taken as unity at a distance of 1 cm. from a unit pole, it follows that the field intensity at a distance of r centimeters from a pole of m units strength is m/r^2. Since the intensity of a field H, is measured by the force it exerts on unit pole, the force exerted on a pole of m units strength by a field of intensity H, will be equal to mH. The force exerted on a wire at right angles to a uniform field has been shown to be equal to the product of the field intensity the strength of the current and length of wire. If a coil of wire has Z turns of radius r, the total length of wire is $2\pi rZ$, and if a current, I, be sent through it, a certain field intensity, H will be produced at the center of the coil. If now a magnet pole of strength m be placed at the center of the coil, the force exerted upon it by the field produced by the coil must equal the force exerted on the wire by the field produced by the pole; that is, if H is the field intensity produced at the center of the coil by the current in it and m/r^2 is the intensity of field produced at the wire by the pole, then

$mH = 0.2\pi rZIm/r^2$, since each of these forces is the reaction of the other. Therefore,

$$H = \frac{0.2\pi ZI}{r}. \tag{44}$$

It must be observed that this value of intensity holds only at the center of the coil, and also that it holds only for a coil whose radius is quite large in comparison with the radial depth and the breadth of the bundle of wires making up the coil. This is because the reasoning is based on the assumption that all of the wires may be considered as being equally distant from the pole which is located at the center of the coil.

48. Magnetomotive Force.—The ability of an electric circuit to produce magnetic flux is called its *magnetomotive force*, or m.m.f., just as the ability of a battery to produce an electric current is called its electromotive force, or e.m.f. The measure of a m.m.f. is taken as the work which would have to be done in moving a unit magnet pole from any point through any path which links the electric circuit back to the same point against the magnetic force produced by the current. This is analogous to the measure of an e.m.f. as expressed in equation (10). Suppose the electric circuit to consist of a coil of N turns of wire and to carry a current of I amperes; when a unit pole is moved from any point, through the coil, and back to the starting point, each of the 4π lines of force issuing from this unit pole will cut each of the N turns of the coil. Therefore the total cutting of flux will be $4\pi N$ and the work done will be $0.4\pi NI$ ergs. The product, $0.4\pi NI$, is the general expression for the magnetomotive force of an electric circuit, and is of the greatest importance in electrical engineering. The fundamental unit of m.m.f. is called a *gilbert; it is the m.m.f. which will produce a field intensity of one line per square centimeter in a path* 1 *cm. long.* Since the product of field intensity and length of path is the work which would be done on unit pole in moving it once around the path, unit m.m.f. will produce a field intensity of such a value that 1 erg of work would be done in moving unit pole once around the path. The longer the path, the smaller will be the value of the field intensity. It must be noted that m.m.f. is only the measure of the flux-producing power of a circuit and tells nothing as to the amount of flux produced; the latter depends upon the length and area and material of the magnetic circuit, as will be shown presently. The product, NI, is frequently called the ampere-turns of the circuit, and 1 *ampere-turn is very commonly used as a unit of magnetomotive force.* One ampere-turn is evidently equal to 0.4π (or 1.257) gilberts; 1 gilbert of m.m.f is produced when the expression $0.4\pi NI$ is equal to unity; that is, by 0.796 of an ampere-turn. A given number of ampere-turns, NI, does not require that either N or I shall have any particular value, but that the product shall have the specified value. That is, 600 ampere-turns will be given by 2 amperes through 300 turns or by 12 amperes through 50 turns; and the flux-producing power is the same in either case.

49. Field Intensity and Magnetizing Force.—Let Fig. 33 represent a coil of wire bent around so that its ends come together, forming a uniformly distributed winding. Let the mean length of the axis of the coil be l centimeters and let the intensity of the field along the axis be H; then the work done in moving a unit pole once around the path is Hl; but it has just been shown that this work is also equal to $0.4\pi NI$; therefore the value of the field intensity is

$$H = \frac{0.4\pi NI}{l}. \tag{45}$$

The term "*magnetizing force*" is frequently used to designate the intensity of a magnetic field, especially in connection with electromagnets. Since it is equal to the ratio of the magnetomotive force to the length of the path, the units very commonly used are "gilberts per centimeter," "ampere-turns per centimeter" or "ampere-turns per inch." A field intensity of 1 line per square centimeter corresponds to a magnetizing force of 1 gilbert per centimeter. One ampere-turn per centimeter is equal to 1.257 gilberts per centimeter and 1 ampere-turn per inch is equal to 0.495 gilbert per centimeter.

50. Flux Density; Permeability.—When the magnetic circuit consists of iron, the molecular magnets are swung into line by the magnetizing force of the coil and the lines of force due to these are added to the lines produced by the coil alone; that is, added to the field intensity H. If the combined strength of the poles of the molecular magnets is

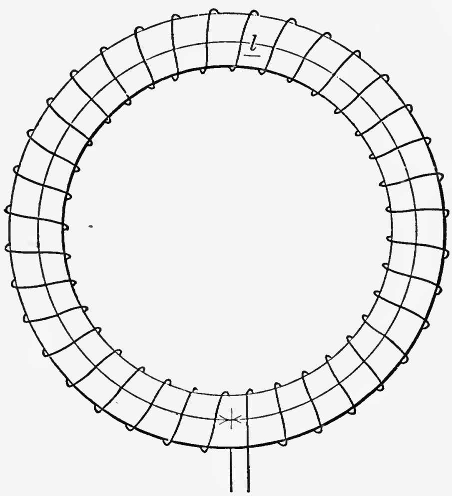

FIG. 33.

equal to J unit poles per unit of cross-sectional area, then the flux per unit area added by these poles will be $4\pi J$, since there are 4π lines issuing from a unit pole. The total number of lines of force per square centimeter is called the *density*, or sometimes the induction, and is generally represented by the symbol B and is expressed in *gausses*. We have, then, as one relation between B and H that $B = H + 4\pi J$. The number of lines per square centimeter added by the presence of iron depends on the value of the magnetizing force $0.4\pi NI/l$, and upon the composition of the iron. In any case, the ratio of the flux density, B, to the field intensity, H, is called the permeability of the iron and is generally represented by the symbol μ. We have then for the flux density, B, in a magnetic circuit of permeability, μ, when a magnetomotive force, $0.4\pi NI$, acts upon it, the equation,

$$B=\frac{0.4\pi NI\mu}{l}. \tag{46}$$

In the air, the density, B, is numerically equal to the field intensity, H, and the permeability of *air* is taken as equal to *unity*. It should be understood clearly that $H=0.4\pi NI/l$ is the equation for field intensity or magnetizing force, no matter what the magnetic circuit may consist of, and it is also the density of the flux when the magnetic circuit is of air, while $B=0.4\pi NI\mu/l$ is the equation for density under all conditions.

51. Total Flux Produced by a Coil.—The total flux, ϕ, through the circuit is, of course, equal to the product of the density, B, and the corresponding cross-sectional area, A. That is,

$$\phi=\frac{0.4\pi NI\mu A}{l}. \tag{47}$$

This equation may be written,

$$\phi=\frac{0.4\pi NI}{l/\mu A}=\frac{\text{M.M.F.}}{R}, \tag{48}$$

where R is written in place of $l/\mu A$. It will be seen that this equation is entirely similar to the equation for current in an electric circuit.

52. Reluctance.—The quantity, $l/\mu A$, is called the *reluctance* of the magnetic circuit, and stands in the same relation to the magnetic circuit as resistance stands to the electric circuit; likewise, flux and m.m.f. in the magnetic circuit have the same relation to each other as current and e.m.f. in the electric circuit. The reciprocal of reluctance is called *permeance* and corresponds to conductance in the electric circuit. The reluctance of a cubic centimeter of a substance is called its specific reluctance, or its *reluctivity.* Thus the reluctivity of air is unity. It should be noted that permeability is a ratio and that it corresponds to the

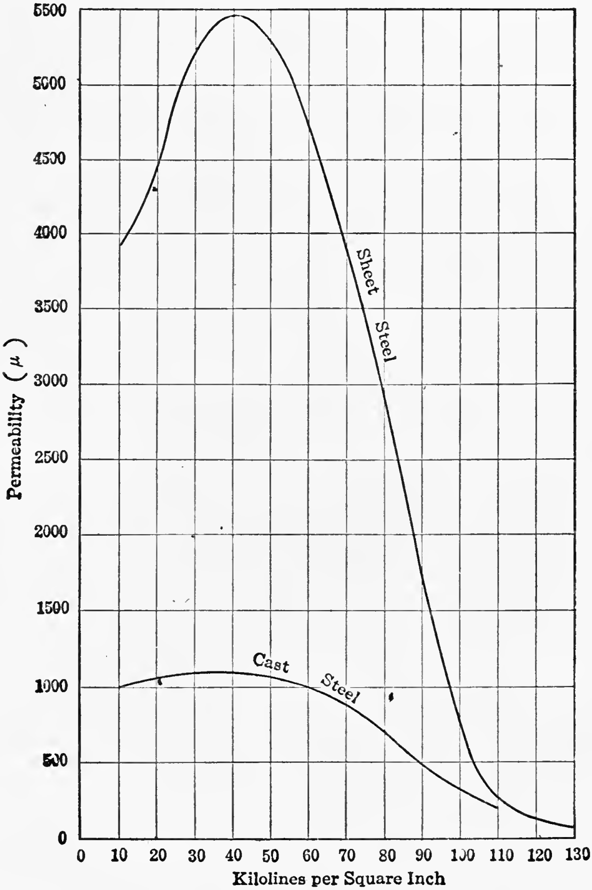

Fig. 34.—Permeability Curves.

reciprocal of relative resistance in electric circuits. The permeability of iron varies with the density and therefore the reluctivity of a magnetic circuit, which is equal to unity divided by μ, cannot be treated as a constant as is the specific resistance of electric circuits. See Fig. 34.

53. Solution of Magnetic Circuits.—In most cases, the solution of magnetic circuit problems relates to the determination of the ampere-turns required to produce a given flux density. Usually, a certain total flux is required and the cross-sectional area of the circuit is made such that the flux density will not exceed the limits set by saturation or hysteresis and eddy-current losses. It should be noted that the number of ampere-turns required is determined by the flux density, not by the total flux; that is, a large flux will be produced by the same number of ampere-turns as will a small flux, provided the ratio of total flux to cross-sectional area is not changed. It should also be noted that the flux density which will be produced in an iron circuit by a given number of ampere-turns cannot be determined directly because the permeability varies with density and there is no practicable mathematical relation between them.

Equation (42) may be written in the form

$$NI = Bl/0.4\pi\mu \qquad (49)$$

and the ampere-turns for a given density may be calculated from this formula provided data are at hand showing the values of μ for different values of B for the particular kind of iron used. Such data are secured from test pieces of known area and length, the flux being measured for various values of NI from zero up to the maximum practicable value. But $B = \phi/A$ and $\mu = B/H = Bl/0.4\pi NI$; therefore, corresponding values of B and μ can readily be calculated and plotted as a curve. See Fig. 34. Or, since $H = 0.4\pi NI/l$, the curve may be plotted between B and H. Or, as is most common for practical purposes, the curve may be plotted between B and (NI/l); such curves are called Magnetization Curves. See Fig. 35. Typical curves are also shown in the Standard Handbook for Electrical Engineers, pp. 288–290. Data are given in the accompanying table for various kinds of iron, from which the magnetization curves may be plotted. Using such curves, to find the ampere-turns required for any density in any length

TABLE I.—DATA FOR MAGNETIZATION CURVES

Ampere-turns per Inch.

Square Inch Density.	Standard Sheet Steel.	Wrought Iron.	Silicon Sheet Steel.	Soft Cast Steel.	Spec. Alloy Steel.	Cast Iron.
10,000	0.8	2.4	0.8	3.1	0.25	7.0
20,000	1.4	4.4	1.4	6.0	0.40	17.0
30,000	1.8	6.2	1.8	8.8	0.60	34.0
35,000	2.05	7.0	2.05	10.1	0.70	
40,000	2.3	7.7	2.3	11.6	0.85	
45,000	2.6	8.4	2.6	13.2	1.00	
50,000	2.95	9.0	3.0	14.9	1.25	
55,000	3.4	9.6	3.5	16.7	1.50	
60,000	4.0	10.4	4.3	18.8	1.85	
65,000	4.7	11.4	5.2	21.4	2.30	
70,000	5.55	12.9	6.4	24.8	2.90	
72,500	6.15	13.8	7.25	27.0	3.25	
75,000	6.8	14.9	8.4	29.5	3.70	
77,500	7.6	16.2	10.0	32.7	4.30	
80,000	8.5	17.7	12.3	36.2	5.1	
82,000	9.5	19.2	14.8	39.4	5.8	
84,000	10.7	21.0	18.0	43.0	6.6	
86,000	12.0	23.5	21.8	47.0	7.6	
88,000	13.8	26.4	26.5	52.0	9.0	
90,000	15.7	30.0	33.0	58.0	10.6	
92,000	18.4	34.1	41.0	64.0		
94,000	21.6	39.2	51.0	71.0		
96,000	26.0	47.0	67.0	80.0		
98,000	32.6	58.0	87.0	89.0		
100,000	41.0	70.0	111.0	100.0		
102,000	52.0	87.0	140.0	111.0		
104,000	68.0	107.0	170.0	125.0		
106,000	88.0	132.0	200 0	141.0		
108,000	112.0	162.0	240.0	160.0		
110,000	138.0	194.0	280.0	180.0		
112,000	168.0	225.0	320.0	208.0		
115,000	222.0	280.0	400.0	253.0		
120,000	340.0	395.0				
125,000	500.0					
130,000	700.0					
140,000	1200.0					
150,000	1700.0					

of iron, it is only necessary to multiply the given length of iron by the value of (NI/l) as found on the curve for the given kind of iron at the given density.

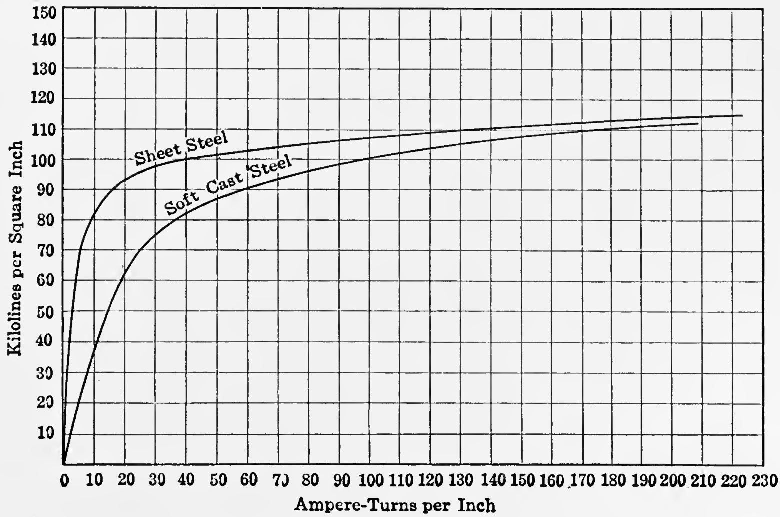

FIG. 35.—Magnetization Curves.

54. Series Magnetic Circuits.—When a series magnetic circuit is made up of different kinds of iron or has different densities in different parts, (NI) is found for each part separately and the total (NI) is the sum of these.

To determine the ampere-turns required for an air gap, it is only necessary to solve the equation

$$NI = Bl/0.4\pi = 0.796Bl, \tag{50}$$

where B is in gausses and l is in centimeters. If B is in lines per square inch and l is in inches, the formula is

$$NI = \left(\frac{B}{6.45}\right)(2.54l)/0.4\pi = 0.313Bl. \tag{51}$$

As an example of these calculations, consider the magnetic circuit shown in Fig. 36. Suppose the piece (c) is of sheet steel 4 in. long and has a cross-sectional area of 15 sq. in.; pieces (b), (b) are each of wrought iron 5 in. long and

14 sq. in. in cross-sectional area; piece (d) is of cast steel, with a magnetic path 45 ins. long and 20 sq. ins. in cross-sectional area; and the two air gaps (g), (g) are each 0.08 in. long and 14.5 sq. ins. in cross-sectional area. Required a flux of 900,000 lines through the circuit. The density in piece (c) will be 900,000/15 = 60,000 lines per square inch; in pieces b, b, 900,000/14 = 64,300 (results carried to the third significant figure only); in piece (d), 45,000; in the air gaps, 62,100. Magnetization curves show that sheet steel requires 4.0 ampere-turns per inch at 60,000 lines per square inch; that wrought iron requires 11.3 at 64,300; and that cast steel requires 13.2 at 45,000. The two air gaps will

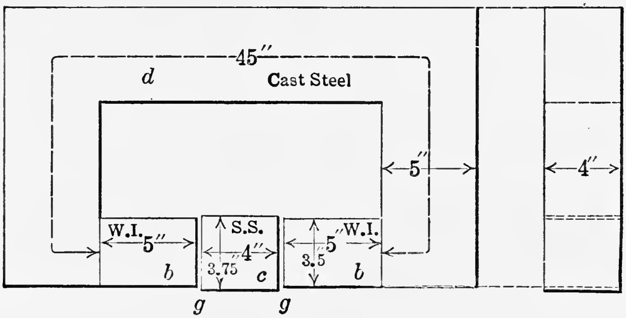

Fig. 36.

require (equation 47) $0.313 \times 62{,}100 \times 0.08 \times 2 = 3100$ ampere-turns; piece (c) will require $4.0 \times 4 = 16$ ampere-turns; pieces b, b, $11.3 \times 5 \times 2 = 113$ ampere-turns; and piece (d), $13.2 \times 45 = 594$ ampere-turns. The total ampere-turns required are then $3100 + 16 + 113 + 594 = 3823$. It should be particularly noted that more than 80 per cent of the m.m.f. is used in carrying the flux across the air gaps. This is because the reluctance of air is so much greater than that of iron. Therefore in all electrical apparatus where air gaps are required and it is important to keep the ampere-turns which produce the flux as small as possible, these air gaps are made as small as is practicable. The quantity, $\phi l/\mu A$, or, Bl/μ, for any part of a magnetic circuit, is frequently called the drop of magnetic potential in that part of the circuit, or the magnetic potential difference between the ends of that portion of the circuit. This is evidently equal to the m.m.f. used for that part of the circuit and the sum of these drops taken entirely around the circuit is always equal to the total m.m.f. acting on the circuit.

55. Parallel Magnetic Circuits.—When magnetic circuits are in parallel, and the different paths have the same reluctance, the m.m.f. calculated for one of the paths is the m.m.f. required for all the paths in parallel. For example, consider Fig. 37. Of the total flux in the middle leg (xy), one half will pass through path (a) and the other half through path (b). The densities in the corresponding parts of the

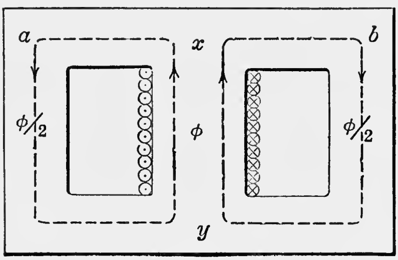

Fig. 37.

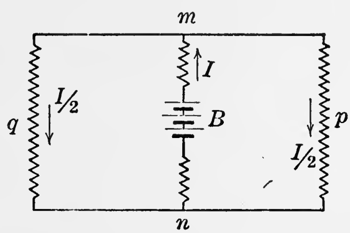

Fig. 38.

two paths will be the same. If 2000 ampere-turns are required to produce a given flux in path (a), the same 2000 ampere-turns will produce an equal flux in the path (b), if the coil is placed on the middle leg. However, if the winding is placed on the outside legs, 2000 ampere-turns will have to be placed on each of these legs. If the density in the middle leg is the same as in the outside legs, the perimeter of the middle leg will be greater than that of the outside legs; but more copper will be required to produce a given flux in the circuit, if coils are placed on the outside legs, than if one coil is placed on the middle leg.

Fig. 38 shows the corresponding electric circuit. With a constant e.m.f. between (m) and (m), the current through path (p) will be the same whether path (q) is open or closed.

No more e.m.f. will be required to send 5 amperes through (p) and 5 amperes through (q) than will be required to send 5 amperes through (p) with (q) open, if the current density in the path mBn is kept the same.

56. Size of Wire Necessary to Produce the M.M.F. Required for a Given Magnetic Circuit.—Let it be supposed that the m.m.f. has been calculated for a given magnetic circuit and (NI) ampere-turns are found to be required for it. If E is the voltage to be applied to the coil, then the resistance of the coil must be

$$R = E/I, \tag{52}$$

where I is the current which the coil will carry. If l_t is the mean length (in inches) of one turn, then the resistance must also be

$$R = \rho N l_t / 12A, \tag{53}$$

where ρ is the specific resistance of the wire per circular-mil-foot, N is the number of turns of wire which the coil will have and A is the cross-sectional area of the wire in circular mils (see equation 11, p. 39).

Equating these two expressions for R, we get

$$A = \rho N I l_t / 12E. \tag{54}$$

If the coil is of copper wire and the running temperature be assumed as 60° C., the value of ρ will be 12, and the equation becomes

$$A = (NI) l_t / E. \tag{55}$$

The area of the wire being thus determined, the current will be fixed by the carrying capacity of the wire, and thence the number of turns may be found.

57. The Field Intensity in a Solenoid.—From the preceding discussion, it should be understood that the m.m.f. of a coil is not used up at a uniform space rate around the magnetic circuit; that is, the m.m.f. used in sending the flux through 1 in. at one part of the circuit may not be the same as that used in 1 in. at some other part of the circuit.

The amount used in any given portion of the circuit depends upon the reluctance of that particular portion, just as the p.d. between different points of an electric circuit depends on the resistance between these points. The reluctance of any part of an air circuit is equal to its length divided by its area; if the reluctance of one part of the circuit is greater than that of another part, the former part will require the larger part of the m.m.f. In the solenoid, Fig. 39, the cross-sectional area, A', of the path within the coil is evidently very small as compared with the area, A'', of the return path outside the coil. The length l'', of the part of the path lying outside the coil is indefinitely greater than l', the part within the coil, but still the reluctance of the

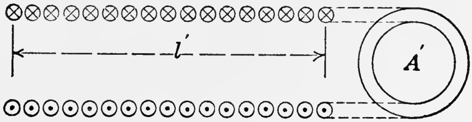

Fig. 39.

entire path is practically all within the solenoid. On account of this, the flux within a solenoid may be taken as equal to $0.4\pi NIA'/l'$, where l' is the length of the solenoid and not the length of the whole path. That is, in the formula

$$\phi = \frac{0.4\pi NI}{\frac{l'}{A'} + \frac{l''}{A''}} \tag{56}$$

the reluctance l''/A'' is negligibly small in comparison with l'/A' and may be neglected when l' is large compared with A'. This relation holds when l' is at least four times as great as the mean diameter of the coil. A discussion of the case for short solenoids is unnecessary here because it is seldom necessary in practice to calculate the flux in such coils except when dealing with alternating currents. The matter will be taken up again under the subject of inductance.

58. Magnetic Leakage.—If the winding which produces the flux in an iron magnetic circuit is completely distributed over the circuit as in Fig. 40, practically all the flux will be confined to the definite path in the iron within the winding. But if the winding is bunched so as to surround only a portion of the iron as in Fig. 41, some of the flux will take paths through the surrounding air. If there is a gap in the iron circuit, the flux will not pass straight across the gap but will spread out so that the density in the gap will be less than in the iron. When, for some definite purpose, a specified

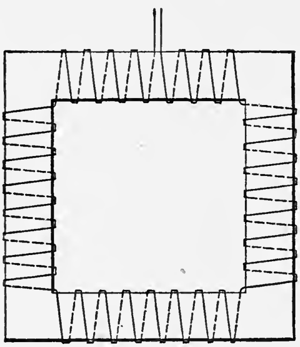

FIG. 40.

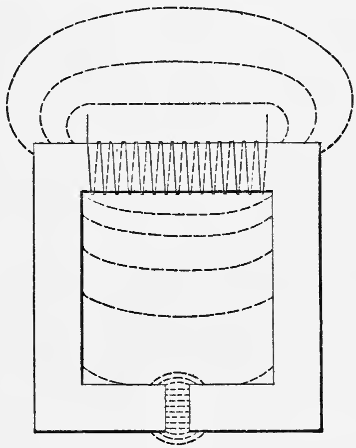

FIG. 41.

flux is required within a definite air gap area, the lines of force which do not so pass are called *leakage lines* and the ratio of the total flux produced by the coil to the useful flux passing across the definite gap, is called the *leakage coefficient.* This coefficient is always greater than unity. Its calculation from the dimensions of the magnetic circuit is generally difficult and complicated and it is usually determined by actual tests on different forms of magnetic circuit and when needed that coefficient is used which corresponds most closely to the type of circuit used.

59. Hysteresis.—The tendency of iron to retain magnetism is called *hysteresis.* When the magnetizing force applied to an iron core is increased the flux density in the iron also increases; when the magnetizing force is decreased, the flux density likewise decreases, but not to the same value it had with the same magnetizing force when increasing. To magnetize the iron to a certain density requires the expenditure of energy; when the magnetizing force is removed, some of the energy is returned to the electric circuit, but not all of it. The difference between the energy put into the magnetic circuit and that returned to the electric circuit represents the tendency of iron to retain the magnetism and is called *hysteresis loss*. When an iron core is magnetized by sending current through a coil surrounding the core, the flux through the core will increase from zero just before the circuit is closed to a certain maximum after the circuit is closed. The value of the final flux will depend on the ampere-turns of the coil and upon the reluctance of the core. During the time while the flux is growing an e.m.f. will be generated in the coil, the value of which will depend on the rate at which the linkage is changing. If there are N turns in the coil and the flux changes by an amount $d\phi$ in time dt, then the e.m.f. will be $10^{-8} N d\phi/dt$ in volts, and if the current at a given instant is i amperes, the power, or rate of doing work will be $10^{-8} N i d\phi/dt$ in watts. By Lenz's Law, which we have already discussed, this e.m.f. will be opposed to the growth of the current, and the work done in forcing the current, i, through the circuit against this e.m.f. will be the work required to magnetize the core by the amount $d\phi$. If dW represents the work done in the time dt, then we may write

$$\frac{dW}{dt} = \frac{N i d\phi}{10^8 dt}, \tag{57}$$

or

$$dW = N i d\phi / 10^8. \tag{58}$$

Now the flux $\phi = BA$ and $d\phi = AdB$, where B is the density in lines per square centimeter and A is the area of the core in square centimeters; also $Ni = Hl/0.4\pi$, where H is the field intensity and l is the length of the core in centimeters. Therefore,

$$dW = \frac{lA\,HdB}{0.4\pi 10^8} = \frac{V\,HdB}{0.4\pi 10^8} \tag{59}$$

since lA is the volume, V, of the core. Therefore the total work done in magnetizing a core up to a density B is

$$W = \frac{V}{0.4\pi 10^8}\int HdB \text{ joules.} \tag{60}$$

In magnetizing iron, it is found that at small values of field intensity, H, the increase in density is relatively rapid but as H becomes larger the iron approaches a saturated condition and the density increases less and less rapidly until finally the presence of the iron adds nothing to the flux and the increase in density becomes equal to dH. There is no known mathematical relation between B and H so that the integral of HdB cannot be mathematically determined. If the iron is without magnetism at the beginning, the manner in which B increases with H is shown by the curve Oa in Fig. 42. The area Oay' represents the work done in magnetizing the iron to a value of B represented by Oy'; to establish this fact, consider a narrow strip whose width is dy and whose length is Ox when the ordinate is Oy; the sum of all such strips included between the curve Oa and the y-axis is the area Oay'; but dy represents a certain change dB in the flux density and Ox represents the corresponding value of H. Therefore xdy represents a certain value of HdB and the entire area Oay' must represent the integral of HdB; that is, this area when multiplied by the scales used for H and for B and by the constant, $V/0.4\pi 10^8$, gives the work done in magnetizing the volume V to a density B. When the field intensity is decreased, the flux linkages decrease and again an e.m.f. is generated which opposes the change; that is, this induced e.m.f. tends to keep the current from decreasing and is therefore acting in the same direction through the electric circuit as the current.

It is therefore giving energy back to the electric circuit, and, in amount, it is, as before, equal to $V/0.4\pi 10^8 \int H dB$. However, it is found that when the field intensity is decreased, the density does not follow the same curve as it followed on increasing but decreases less rapidly and when H has been reduced to zero the iron will possess a certain amount of magnetism. This is known as residual magnetism and the

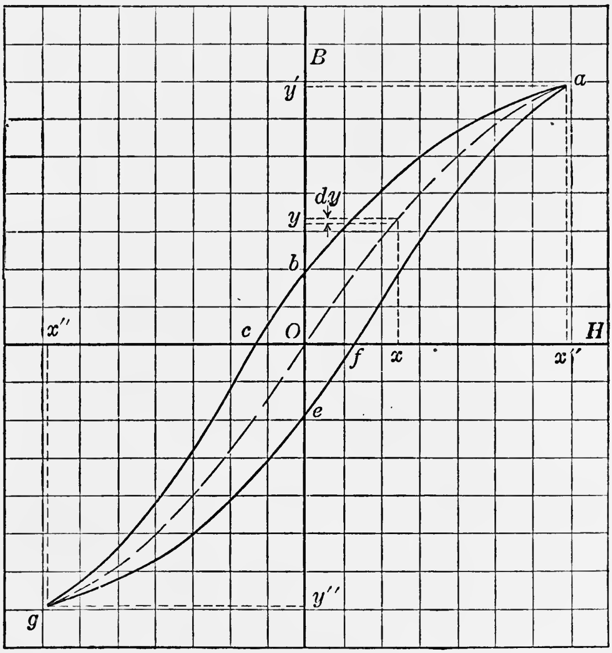

FIG. 42. Hysteresis Loop.

density corresponding to it is represented by Ob. By the same reasoning as before, the area bay' represents the energy which is returned to the circuit when the field intensity is reduced to zero. Therefore, the difference between the areas Oay' and bay', which is the area Oab, represents the energy which has been dissipated in the process. It is found when iron is magnetized and then demagnetized that its temperature is raised; therefore, heat energy must have been developed in the iron, and this heat energy is found to be accounted for by the energy represented by the area *Oab*. If now the magnetizing force be reversed, work must be done in reducing the flux to zero and building it up to a value y'' in the opposite direction; this work is represented by the area bgy''. When H is again reduced to zero, energy represented by the area egy'' will be regained by the circuit, and if H be reversed again and increased to Ox', the work done will be represented by the area eay'. The total energy dissipated by a complete cycle, *acgfa*, is therefore represented by the area included between the two curves *acg* and *gfa*. The phenomenon which causes these two curves to diverge is called *hysteresis* and the energy dissipated when a piece of iron is magnetized and demagnetized is called *hysteresis loss*. It is sometimes said to be due to molecular friction. Professor Steinmetz has found that when a piece of iron is subjected to repeated reversals of magnetism from a given maximum density, B, in one direction to an equal maximum value in the opposite direction, and so on, the power lost in hysteresis can be represented by the formula

$$P \text{ (watts)} = \frac{kfVB^{1.6}}{10^7}, \tag{61}$$

where f is the number of complete cycles per second, V is the volume (in cubic centimeters) of iron undergoing the reversal, B is the value of the maximum density (in lines per square centimeter) in either direction, and k is a constant depending on the quality of the iron. If V and B are expressed in inch units, the formula is

$$P_h = kf(16.38V)\left(\frac{B}{6.45}\right)^{1.6}/10^7 = 0.83kfVB^{1.6}/10^7. \tag{62}$$

Fair average values for k are 0.0012 for good annealed sheet steel, 0.001 for best annealed iron, 0.0008 for good silicon steel, and 0.0006 for best silicon steel. For typical hysteresis curves, see Standard Handbook for Electrical Engineers, pp. 290–292.

60. Eddy Currents.—If a mass of iron is in a magnetic field which is varying, the flux will cut across the iron in a direction at right angles with the direction of the flux, and thus generate electromotive forces in the iron, which in turn cause currents to flow in it. These currents are called *eddy currents*. Their energy is dissipated in heating the iron. Their paths are more or less indeterminate, but depend in general upon the shape of the iron with respect to the direction of the flux. The energy consumed by them can be greatly reduced by laminating the iron in the direction of the flux, assuming that the laminations are more or less completely insulated from each other by varnish or other insulating material; the layer of oxide formed in the process of annealing is often sufficient insulation. When a given volume of iron is subjected to an alternating magnetic flux, the power consumed by eddy currents can be represented by the formula, $P = KVt^2f^2B^2$, where K is a constant which depends upon the conductivity of the iron, the distribution of the flux, the manner of the variation of flux with time, and the units used; V is the volume of iron; B is the maximum value of the flux density in each direction; f is the number of cycles per second; and t is the thickness of the laminations. The above equation may be derived as follows: Let Fig. 43 represent a laminated core in which an alternating flux is produced by the coil as shown; the direction of the flux will be perpendicular to the paper. Let l represent the length of the core perpendicular to the paper, w the width of the laminations, and t their thickness. The paths of the eddy currents will be in the plane of the paper as shown by the dotted line in one of the laminations. The thickness of the laminations is assumed to be sufficiently small with respect to their width that the length of the current path can be represented by $2w$; the cross-sectional area of the current path is $tl/2$. The resistance of the path is therefore $r = 4K_1w/tl$, where K_1 is the specific resistance of the iron. The total flux in one lamination is wtB, and the flux cut per second will be $4fwtB$, since the flux wtB will be cut four times during each cycle. The effective e.m.f. will therefore be $e=4K_2fwtB$, where K_2 is a constant depending on the manner in which the flux varies with time. The value of the eddy current will be $e/r=K_2lt^2fB/K_1$, and the power loss per lamination will be I^2r, or

$$p=4(K_2^2/K_1)wlt^3f^2B^2. \quad (63)$$

But wlt is equal to v, the volume of the lamination, so that we can write (substituting K for $4K_2^2/K_1$),

$$p=Kvt^2B^2f^2. \quad (64)$$

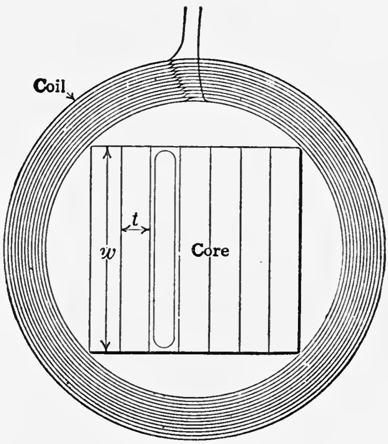

FIG. 43.

The total loss will be pn, where n is the number of laminations; but vn is the total volume V of the core, so that the total loss is

$$P=KVt^2f^2B^2 \quad (65)$$

as stated above.

This formula is useful as showing the general effect of the various factors on the eddy current loss, but it is not very reliable for actual calculations on account of the fact that certain other factors, such as incomplete insulation between laminations, variations in the flux distribution and its manner of change with time, make the proper selection of the constant K quite uncertain. In practice use is made of experimental curves secured from tests under the desired conditions, and showing total core loss (hysteresis and eddy current loss) per unit volume or weight. See Standard Handbook for Electrical Engineers, pp. 291–292.

The usual thickness of laminations range from 14 to 28 mils and the net volume of iron is taken as from 85 to 95 per cent of the stacked volume.

61. Pull of an Electromagnet.—It has been proven (equation 60) that the energy stored up in a magnetic field of volume V, that is, of length, l, and area A, is equal to

$$W = \frac{lA}{0.4\pi 10^8}\int HdB \text{ joules, or } \frac{lA}{4\pi}\int HdB \text{ ergs.} \quad (66)$$

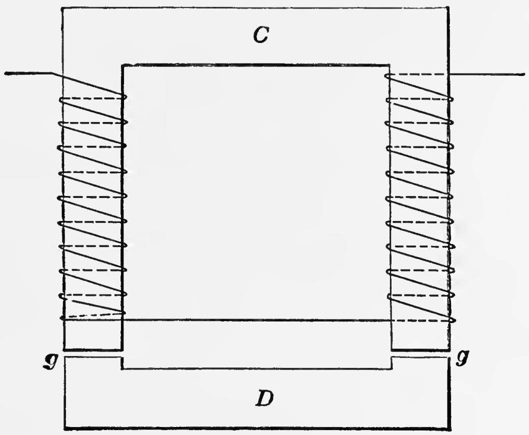

FIG. 44.

Now the energy stored in any particular portion of the circuit is proportional to the volume, lA, of that portion. Suppose the magnetic circuit consists of a core, C, Fig. 44, and an armature, D, which is separated from the core by gaps, gg, each of area A. Let the length of the gaps be increased by an infinitesimal distance, dx. Then the energy stored in each gap will be increased by an amount

$$dW = \frac{Adx}{4\pi}\int HdB. \quad (67)$$

But in the gap, the permeability is unity, and $B=H$; therefore,

$$dW=\frac{Adx}{4\pi}\int BdB=\frac{B^2Adx}{8\pi} \tag{68}$$

or

$$\frac{dW}{dx}=\frac{B^2A}{8\pi}. \tag{69}$$

But since dW is the increase in energy stored in the gap and is proportional to dx, it represents the mechanical work which would have to be done to separate the armature from the core by the additional distance dx; therefore dW/dx is the average value of the force, F, required to pull the armature down a distance dx. This force is in dynes; expressed in pounds, with A expressed in square inches, and B in lines per square inch, the force is

$$F=\left(\frac{B}{6.45}\right)^2(6.45A)\left(\frac{1}{8\pi\times 445{,}000}\right)=\frac{B^2A}{72{,}140{,}000}. \tag{70a}$$

If B is expressed in kilolines per square inch the formula becomes

$$F=0.01386B^2A. \tag{70b}$$

62. Inductance.—If an e.m.f. of 120 volts be connected to a circuit of 40 ohms resistance and consisting of incandescent lamps, the current will rise almost instantly to a value of 3 amperes as may be observed by placing an ammeter in the circuit; if, however, the circuit consists of a coil of wire of 40 ohms resistance and wound upon an iron core, the current may require several seconds to obtain its final value of 3 amperes. Evidently the latter circuit possesses to a much greater degree than the former some property which opposes the change in current. It shows that electricity, like matter, resists a change in its state of motion and thus has a property similar to that which we call inertia. Mention has already been made of the manner in which this opposition manifests itself, namely the generation of a back e.m.f. by the changing flux which is linked with the electric circuit. This property of an electric circuit, by virtue of which an opposing e.m.f. is generated whenever an attempt is made to change the current in it, is called *self-induction.* When the current in the wire is increased or decreased the magnetic flux surrounding the wire increases or decreases with it; the lines of force seem to expand out from the center of the wire when the current increases and to collapse toward the center of the wire when the current decreases. Thus they cut across the wire and generate an e.m.f. and this e.m.f. is always in such a direction as to oppose the change in current which produces it. The magnitude of the opposing e.m.f. depends, of course, upon the rate at which the flux linkage is changing, which, in turn, depends upon the value of the current and the rate at which it is changing. Let e_s represent the e.m.f. of self-induction; it is equal to $Nd\phi/10^8dt$, where N is the number of turns in the circuit and ϕ is the flux which links it. But it has been shown that $\phi = 0.4\pi\mu A\,Ni/l$; therefore, $d\phi = 0.4\pi\mu A\,Ndi/l$; substituting this value of $d\phi$, we get

$$e_s = \left(\frac{0.4\pi\mu A\,N^2}{10^8 l}\right)\frac{di}{dt}. \tag{71}$$

The quantity $\left(\frac{0.4\pi\mu A\,N^2}{10^8 l}\right)$ is called the *inductance* of the circuit, and is represented by the symbol L. Note that when there is iron in the magnetic circuit, the permeability varies with the flux and L is not constant; when the magnetic circuit consists of air and its length and area are perfectly definite, L is evidently a constant and easily calculated, but magnetic circuits of air are usually not very definite in length or area. The calculation of L is therefore generally not a simple matter. There are several ways in which inductance is defined. Equation (71) may be written, $e_s = Ldi/dt$, from which we get

$$L = \frac{e_s}{di/dt} \tag{72}$$

and from this equation L is defined as *the ratio of the e.m.f. induced in a circuit to the rate of change of current which induces it.* This should be taken as the most fundamental definition since it connects most closely with the property of self-induction as defined above. The practical unit of inductance is one *henry.* A circuit has an inductance of one henry when current in it changing at the rate of 1 ampere per second causes an e.m.f. of 1 volt to be generated. If a current changing at the rate of 1 ampere per second causes an e.m.f. of 5 volts, the inductance is 5 henries. A henry is a larger unit than is generally met in practice, and a smaller unit, the *millihenry* (1/1000 of a henry) is much used. If e_s is expressed in abvolts and i in abamperes, L is given in abhenries; the abhenry is therefore equal to 1 henry divided by 10^9, since 1 volt is 10^8 abvolts and 1 ampere is 1/10 abampere.

The factors which go to make up the values of L are, of course, those in the expression $\left(\frac{0.4\pi\mu A N^2}{10^8 l}\right)$, from which it should be noted that L varies directly as N^2, A, and μ, and inversely as the length l. This expression, however, will not give correct values for L except in the case of an air-cored coil closed on itself as shown in Fig. 33 and in which the cross-sectional area, A, is uniform and l is the mean length of the magnetic path inside the coil. It is approximately correct for a long solenoid.

Inductance is also defined as the rate of change of flux-linkages with current. In abvolts, $e_s = Nd\phi/dt$; if this expression is substituted for e_s in equation (72) we get

$$L = \frac{Nd\phi/dt}{di/dt} = Nd\phi/di, \qquad (73)$$

which is the mathematical expression for the definition just given. In this expression, i is in abamperes and L is in abhenries.

In the case of an air circuit, $d\phi/di$ is constant and equal to ϕ/i, and L may be expressed as $N\phi/i$, or defined as flux-linkages per unit current. This expression is the most convenient one for calculating the inductance of a transmission line as will be shown later.

For calculating the inductance of circular air-cored coils of any length, depth and diameter, Professor Brooks, of the University of Illinois, has worked out an empirical formula which will give results correct within 1 per cent. The formula is

$$L=\frac{25.07}{10^9}\times\frac{N^2(c+d)^2}{b+2c+0.5d}\times\frac{10b+14c+d}{10b+11.4c+0.7d}\times 0.5\log\left(100+\frac{14c+7d}{2b+3c}\right), \quad (74)$$

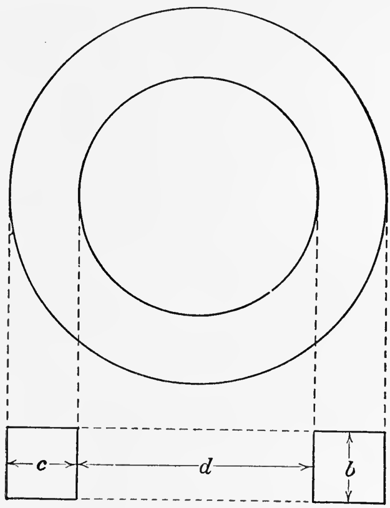

Fig. 45.

in henries, where N is the number of turns in the coil and b, c, and d, are the dimensions, in inches, shown in Fig. 45. It was found that the maximum inductance for a given number of feet of wire would be given when $b=0.6d$ and $c=0.5d$. Substituting these values in the above formula, we get

$$L=\frac{30.64\,N^2d}{10^9} \quad (75)$$

as the inductance of a coil when wound in such shape as to give the maximum value.

63. Growth of Current in an Inductive Circuit.—When an e.m.f. E is applied to a circuit containing a resistance r and an inductance L, the current will not instantly assume the value E/r, but will rise at a rate depending on the ratio of L to r. With large L and small r, it will grow slowly, while with small L and larger r, it will grow rapidly. At any instant during its growth, the e.m.f. used in overcoming the resistance will be equal to ri, while the remainder will be used in overcoming the e.m.f. of self-induction, which is equal to $L\frac{di}{dt}$; therefore, we can write the equation for the total e.m.f.

$$E = ri + L\frac{di}{dt}. \tag{76}$$

To solve this equation for i, it must be arranged in a form which can be integrated. To do this, write

$$(E - ri)dt = Ldi \tag{77}$$

$$\frac{dt}{L} = \frac{di}{E - ri}. \tag{78}$$

Then integrating,

$$\frac{t}{L} = -\frac{1}{r}\log_e (E - ri) + K. \tag{79}$$

The value of K is found by the condition that when $t = 0$, $i = 0$; whence

$$K = \frac{1}{r}\log_e E \tag{80}$$

and

$$-\frac{rt}{L} = \log_e \left(\frac{E - ri}{E}\right), \tag{81}$$

or

$$e^{-rt/L} = \frac{E - ri}{E}, \tag{82}$$

whence,

$$i = \frac{E}{r}(1 - e^{-rt/L}), \tag{83}$$

from which equation the value of the current can be calculated for any time, t, after the circuit is closed. When a sufficient time has elapsed that $e^{-rt/L}$ becomes sensibly equal to zero, then $i=E/r$.

The curve for equation (83) is shown in Fig. 46. The ratio r/L is generally so large that only a fraction of a second is required to meet this condition. When $t=L/r$, then $rt/L=1$ and

$$i=\frac{E}{r}\left(1-\frac{1}{2.718}\right)=0.632\frac{E}{r}. \tag{84}$$

The ratio L/r is called the *time constant* of the circuit and is the time required for the current to reach 63.2 per cent of its final steady value.

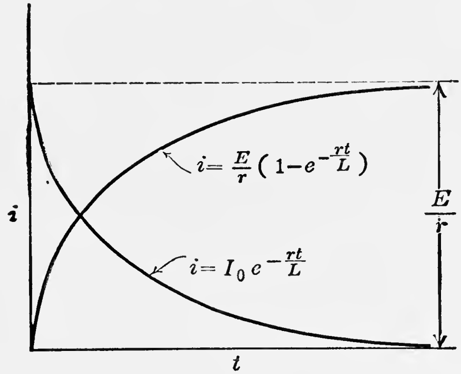

FIG. 46.—Growth and Decay of Current in an Inductive Circuit.

64. Decay of Current in an Inductive Circuit.—When a coil is short-circuited and the source of e.m.f. removed, a current will be found to continue to flow for a short time in the same direction as formerly and it can only be accounted for by the e.m.f. which is generated in the coil by the decreasing magnetic field. Since the impressed e.m.f. on the circuit is now zero, the equation for the circuit is

$$0=ri+L\frac{di}{dt}, \tag{85}$$

or

$$\frac{di}{i}=-\left(\frac{rdt}{L}\right). \tag{86}$$

Integrating, we get

$$\log_e i + K = -\left(\frac{rt}{L}\right). \tag{87}$$

Let i_0 represent the steady value (E/r) which the current had at the instant the circuit was short-circuited. Then the constant of integration, K, is found from the condition that when $t=0$, $i=I_0$; or,

$$K = -\log_e I_0, \tag{88}$$

therefore,

$$\log_e (i/I_0) = -(rt/L) \tag{89}$$

$$i/I_o = e^{-rt/L} \tag{90}$$

$$i = I_0 e^{-rt/L}. \tag{91}$$

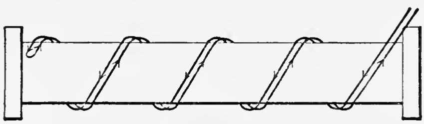

FIG. 47.

From this it is seen the current dies away according to the same law by which it increases when the circuit is first closed. The curve for equation (91) is shown in Fig. 46.

A circuit which has a sensibly large time constant, that is, one in which the current grows at a relatively slow rate, is called an *inductive circuit.* When an inductive circuit is opened quickly the magnetic field associated with it dies away very rapidly and may generate a very large e.m.f. The spark which occurs where such a circuit is opened is due to this e.m.f. and if the circuit be opened too quickly, the e.m.f. may be great enough to puncture the insulation of the circuit and cause considerable damage.

It is impossible for a circuit to be perfectly non-inductive, but if it is desired to wind a quantity of wire into a coil so that it shall be practically non-inductive, the wire may be wound back on itself as illustrated in Fig. 47. That is, two wires are wound side by side, their inside ends being connected together, and their outside ends forming the terminals of the coil. The current then flows in one direction around the core through half the turns, and in the opposite direction through the other half, so that the m.m.f. of one half is opposed to that of the other half and no flux is produced except what little may pass between adjacent wires. The coil is therefore practically non-inductive. Another method of making a non-inductive resistance is to wind the wire on a very thin flat card so that the area included within the turns is so small that an inappreciable amount of flux is linked with them.

65. Energy of a Magnetic Field.—Whenever a current flows against the resistance of a wire or against a counter-e.m.f., electric power is consumed. The power consumed in overcoming resistance is ri^2, the power required to overcome a counter-e.m.f. is equal to the product of the current and the counter-e.m.f. and the total power is equal to the sum of these, or to the product of the current and the total impressed e.m.f. Thus, in the case of an inductive circuit on which an e.m.f. E is impressed,

$$Ei = ri^2 + Li\frac{di}{dt}. \tag{92}$$

It is evident that the power, $Li\frac{di}{dt}$, will become zero as soon as the current reaches its steady value, and all the power supplied will be used in overcoming the resistance. When a steady current is flowing in a circuit, the magnetic field linked with it is steady and repeated experiments have proven that a magnetic field requires no energy to maintain it; it only requires energy to establish it or to increase it, and it returns this energy to the electric circuit when it decreases.

The work done, or the energy transformed, during any period, dt, will be,

$$Eidt = ri^2dt + Lidi. \tag{93}$$

That part of the energy represented by $Lidi$ is spent in building up the magnetic field. This point is sometimes rather difficult to comprehend, but it should be remembered that the opposition to the rising current comes from the fact that the increasing magnetic field generates the counter-e.m.f. and the counter-e.m.f. disappears as soon as the magnetic field becomes steady, so that the energy which was used *must* now be located in the magnetic field. If the last equation be integrated between $t=0$ and any later time T, we get

$$\int_0^T Eidt=\int_0^T ri^2dt+\tfrac{1}{2}LI^2, \tag{94}$$

where I is the value of the current at time T. The last term in this equation represents the energy, in joules, which is stored up in a magnetic field when the current which is producing it is I amperes and the inductance of the circuit is L henries. An idea of the amount of energy associated with a magnetic field may be obtained from an example. Suppose 2000 turns of wire be wound into a coil of 10 ins. inside diameter so as to give the maximum inductance; by the empirical formula (75) given above the inductance will be

$$L=\frac{30.64\times 2000^2\times 10}{10^9}=1.226 \text{ henries.}$$

If a current of 20 amperes flows in the coil, the energy of the field will be

$$0.5\times 1.226\times 400=245 \text{ joules,}$$

or $245/1.356=181$ ft.-lbs., that is, enough energy to raise 181 lbs. a distance of 1 ft. After the current becomes steady, the difference between the total energy which has been supplied and the energy which has been dissipated in heating the circuit is found to be constant and equal to $\frac{1}{2}LI^2$, thus showing that *no energy is required to maintain the magnetic field.* This may also be shown by measuring the power at any instant; all of the power supplied can be accounted for by the rate at which heat is dissipated in the coil.

66. Inductance of Two Long Parallel Wires.—An important factor in the design and operation of transmission lines is the inductance of such lines. The problem is to find the flux per unit current linking the circuit made up of two long parallel wires. Evidently, if we find this for unit length of circuit, we can calculate it for any length of circuit by simple multiplication. Let Fig. 48 represent two parallel wires, A and B, of radius r and distance D apart from center to center. We have seen (Article 45) that the field intensity, H, at a distance x from the center of a wire carrying I abamperes of current is $2I/x$. The total flux per centimeter length of wire A due to the current in wire A and passing between the surface of wire A and the center of wire B is therefore

$$\phi' = \int_r^D H dx = \int_r^D \frac{2I dx}{x} = 2I \log_e \frac{D}{r}. \tag{95}$$

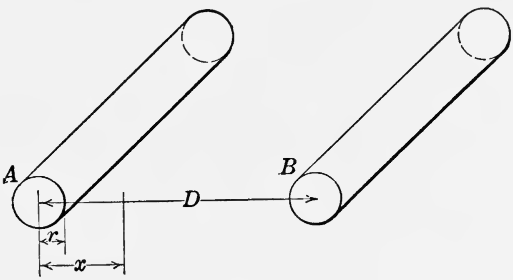

Fig. 48.

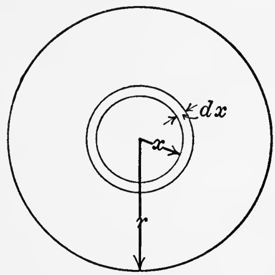

Fig. 49.

We have yet to find the flux inside the wire A and add it to ϕ'. Let Fig. 49 be an enlarged cross-section of wire A. Assume the current I to be uniformly distributed over the cross-section of the wire; then the current within the radius x is $I' = x^2I/r^2$. The flux density at distance x from the center is $2I'/x$ and the flux in the ring of width dx and length 1 cm. perpendicular to the paper is $2I'dx/x$ or $2Ixdx/r^2$. This flux links only with the current I' and the value of the flux which would link the current I is smaller than this in the ratio x^2/r^2 or is equal to $2Ix^3dx/r^4$. Integrating this between 0 and r, and we have the total equivalent flux within the wire as

$$\phi'' = \int_0^r \frac{2Ix^3dx}{r^4} = \frac{2I}{r^4}\left[\frac{x^4}{4}\right]_0^r = \frac{I}{2}. \tag{96}$$

The total flux then linking each wire per centimeter of length is $\phi' + \phi''$, or

$$\phi = \left(2I \log_e \frac{D}{r}\right) + \frac{I}{2}, \tag{97}$$

and the flux per unit current, or the inductance, is

$$L = \frac{\phi}{I} = \left(2 \log_e \frac{D}{r}\right) + \tfrac{1}{2} \tag{98}$$

in abhenries per centimeter, or,

$$L = 2.54 \times 12 \times 5280 \times \left(2 \times 2.303 \log \frac{D}{r} + 0.5\right) 10^{-9}$$

$$= \left(0.74 \log \frac{D}{r} + 0.0805\right) 10^{-3}, \tag{99}$$

in henries per mile of wire, using common logs instead of Napierian logs. Since there are 10^3 millihenries in 1 henry, the inductance in millihenries per mile is

$$L = 0.74 \log \frac{D}{r} + 0.0805. \tag{100}$$

In millihenries per 1000 ft. it is

$$L = 0.14 \log \frac{D}{r} + 0.015. \tag{101}$$

67. Skin Effect.—In Article 66, it was assumed, in arriving at the amount of flux within the wire, that the current was uniformly distributed throughout the wire. With varying or alternating current, the distribution will not be uniform. A solid wire may be considered as made up of a very small cylindrical core surrounded by very thin cylindrical shells of increasing diameter, and each fitting tightly over the one of next smaller diameter. The flux surrounding (that is, linking with) any shell is the flux produced by that shell and all the shells *outside* of it, but not including the flux produced by the shells inside of it; therefore the core is linked with more flux than the shell next to it, and each shell is linked with more flux than the next shell outside of it.

The result is that the inductance at the center of the wire is greater than at the surface, and the back-e.m.f. of self-induction is greater at the center and the current at the center of the wire is less than at the surface. The current may be thought of as crowded toward the surface of the wire; this phenomenon is known as *skin effect*. It has the apparent effect of increasing the resistance of the wire, because less current flows in the wire than would flow if it were a steady direct current. The deviation from uniformity of current distribution depends upon the size of the wire, the rapidity with which the current is changing, and the material of which the wire is made. With 60-cycle alternating current and copper or other non-magnetic wires of less than 400,000 circular mils area, the apparent resistance is less than 1 per cent greater than the direct current resistance. With iron or steel wire, however, the apparent resistance may be several times greater, even with wires as small as No. 6 B. & S., owing to the greater permeability of those materials.

68. Mutual Induction.—When two coils are so placed with reference to each other that the flux which links one coil will also link the other, either wholly or in part, then if the current in one, which we may call the primary, varies, the varying flux will not only produce an e.m.f. of self-induction in the primary coil, but will produce an e.m.f. of *mutual induction* in the other, or secondary coil. This is the principle of the alternating current transformer. The value of the e.m.f. induced in the secondary coil depends on how completely the flux produced by the primary links the turns of the secondary, and on the number of turns in the secondary. When the two coils are wound close together on an iron core, then the whole flux will very nearly completely link both coils. If the number of turns in the primary coil be N' and in the secondary coil be N'', and the flux be changing at the rate $d\phi/dt$, then the e.m.f. induced in the primary will be $N'd\phi/10^8dt$ and in the secondary it will be $N''d\phi/10^8dt$; that is, the ratio of the primary to secondary induced e.m.f's will be equal to the ratio of primary to secondary turns.

The ratio of the e.m.f. induced in circuit No. 2 to the rate of change of current in circuit No. 1 is called the *coefficient of mutual induction.* This coefficient depends upon the geometrical arrangement of the circuits and the permeability of the magnetic material surrounding them. If the permeability remains constant, the coefficient of mutual induction of circuit No. 1 with respect to circuit No. 2 is the same as that of circuit No. 2 with respect to circuit No. 1. Formulæ for calculating this coefficient are given in the Standard Handbook for Electrical Engineers, p. 74.

" Cross-talk " between parallel telephone lines and disturbances in telephone lines which are parallel to power transmission lines are frequently caused by mutual induction. These may be reduced and sometimes practically eliminated by proper " transposition." For methods of transposition, see Standard Handbook for Electrical Engineers, p. 1678.

CHAPTER V

ELECTROSTATICS

69. Electric Charges.—Take two pieces of metal *A* and *B* (that is, two conductors) and let them be insulated from each other and from all other conductors. They may be two wires of a transmission line or two pieces of tinfoil separated by a sheet of paper or a sheet of mica. Let a galvanometer be connected to each piece as shown in Fig. 50, each galvanometer being so connected that if current

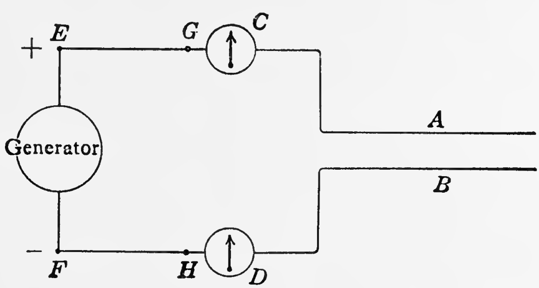

Fig. 50.

flows toward the right, the galvanometer needle will be deflected toward the right. Now let side *A* be connected to the positive terminal of a source of electromotive force and side *B* to the negative terminal as shown. At the instant the connection is made the needle of galvanometer *C* will be deflected to the right while that of *D* will be deflected to the left; both needles will then come back to rest at zero. The two wires are now said to be charged with electricity. Electricity is said to have flowed *into* wire *A* and charged it *positively;* electricity is said to have flowed *out* of wire *B* and charged it *negatively.* Every point on *A* is now at the same potential as the positive terminal of the generator, and every point on *B* is at the same potential as the negative terminal of the generator; from Ohm's Law, we have learned that there is no potential difference between two points in the same conductor unless there is a current flowing. The difference of potential between *A* and *B* is equal to the electromotive force of the generator. The circuit may now be opened and closed at will at *E* and *F* and no effect will be seen on the galvanometers. Assuming perfect insulation between the two wires, they will remain charged. That they do remain charged after being disconnected from the source of e.m.f. may be seen by disconnecting them from the generator at *E* and *F* and connecting *G* to *H*; the needle of *C* will be deflected to the left and that of *D* to the right; that is, electricity will flow out of *A* into *B* until the two are at the same potential.

70. The Electrostatic Field.—Let the wires *A* and *B* again be charged as before; with suitable apparatus, it would be found that there exists between them a force of attraction. This is known as electrostatic attraction and it shows that the region surrounding and between two bodies charged to different potentials is in a state of stress. Such a region is called an *electrostatic field* and the material separating two charged bodies is called a *dielectric.* It is found by experiment that when the dielectric is air (or vacuum) the magnitude of the stress, or the *intensity of the electrostatic field,* depends directly upon the potential difference between the conductors, and inversely upon the distance between them.

As electrostatic field can be conveniently pictured by imagining lines to be drawn from one of the charged bodies to the other; at points where the force is strong the lines should be thought of as crowded together and at points where the force is weak the lines should be thought of as spread apart. Fig. 51 illustrates the distribution of these lines in every plane at right angles to two equally charged straight parallel wires. The actual distribution will of course vary with the distance between the wires and the ratio of the diameter of the wires to the distance between their centers. The same figure also illustrates the distribution of the lines in every plane through the centers of two equally charged spheres. In the case of parallel conducting plates which are large in surface area relative to their distance apart, the lines will be uniformly distributed in the space between the plates, except near the edges. See Fig. 52. These lines are called electrostatic lines of force. The meas-

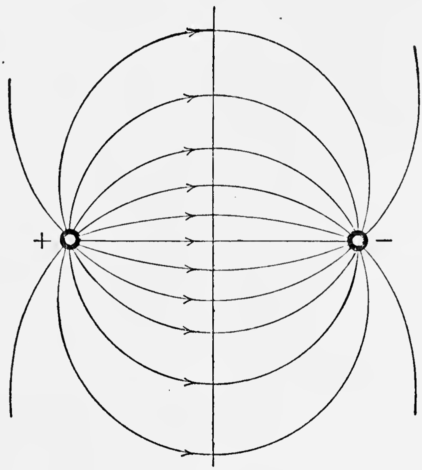

Fig. 51.

Fig. 52.

ure of the intensity of an electrostatic field at any point is taken as the force which would be exerted by it on a unit quantity of electricity placed at the given point when the dielectric is air. In the original development of the theory of electrostatics, the unit of quantity was taken to be that quantity which when placed 1 cm. from a similar and equal quantity would repel it with a force of 1 dyne. This unit is called the electrostatic unit. An electrostatic field then has an intensity of one line per square centimeter when it exerts a force of 1 dyne on an electrostatic unit of quantity. Therefore, since there are 4π square centimeters of area in the sphere of unit radius surrounding a quantity there will be 4π lines of force issuing from unit quantity, or charge.

When the dielectric is other than air, it is found that the charge which a given potential difference will produce on the conductors is increased and consequently the lines of force are increased in number. The sum of the original intensity (in air) and the added lines per square centimeter is called the *flux density*, and the ratio of the density to the intensity is known as the *dielectric constant* of the dielectric. It is also called *specific inductive capacity.*

Since the lines of force are proportional to the amount of charge on the conductors separating the dielectric, this constant may be determined by measuring the charge (by means of a ballistic galvanometer) on the conductors with the given dielectric between them and at a given potential difference, and again with air between the conductors and at the same potential difference. The charge on the conductors when air separates them may also be calculated mathematically, as will be shown later. The table in Article 81 gives the values of the dielectric constant for some of the more commonly used dielectrics.

71. Electrostatic Potential.—It has already been shown that the work done in an electric circuit is equal to QV, where Q is the quantity of electricity which passes between the points whose potential difference is V. Therefore the potential difference between two points may be defined as the work which would be done in moving unit quantity of electricity from one of the points to the other. Since work is equal to force times distance and since electrostatic intensity is numerically equal to the force exerted on unit quantity of electricity in a dielectric of air, it follows that electrostatic intensity is equal to the potential difference per unit distance measured in the direction of the electrostatic force. When the work is expressed in ergs and the charge in electrostatic units, the p.d. is expressed in what are called "electrostatic units." The intensity of an electrostatic field may therefore be expressed either in lines of force per square centimeter, or in electrostatic units of p.d. per centimeter of path, and they are numerically equal to each other. It has been determined experimentally that 1 coulomb is equal to 3×10^9 electrostatic units of quantity, and that 1 volt is equal to 1/300 of the electrostatic unit of potential. In practice, electrostatic intensity is generally expressed in volts per inch, volts per mil, or volts per centimeter, of distance between the two points under consideration.

72. Capacity.—When two conductors *A* and *B* are connected to a source of e.m.f. as in Fig. 50, the amount of electricity which flows out of one into the other (in other words the charge they receive) depends upon the surface area of the conductors, upon the distance between them, upon the nature of the dielectric which separate them, and upon the value of the e.m.f. to which they are connected. When the first three conditions are fixed, the charge depends only upon the e.m.f. and varies in direct proportion with it. In such a case, the charge which the circuit receives per volt of e.m.f. impressed upon it is known as the *capacity* of the circuit. Care should be taken to think of capacity *not* as the amount of electricity contained by a circuit, nor as the amount that it will stand without breaking down, but simply as the charge which each side of the circuit will have produced upon it by 1 volt of potential difference. The ratio of the capacity of a circuit of given dimensions when the dielectric is other than air to its capacity when the dielectric is air, is the *dielectric constant* of the given material, as defined in Article 70. The unit of capacity is called a *farad;* a circuit has a capacity of one farad when 1 volt between its terminals produces a charge of 1 coulomb on each of its sides. Speaking broadly, any two conductors separated by a dielectric is an electric condenser, but the term condenser is more commonly restricted to mean two parallel sets of plates or sheets of metal separated by a thin sheet of dielectric material.

73. Capacity of a Parallel Plate Condenser.—When a parallel plate condenser is connected to a source of potential, one plate or set of plates becomes charged positively and the other negatively and electrostatic flux is said to pass from one to the other. Since there are 4π lines of force from each electrostatic unit of quantity, there will be $4\pi \times 3 \times 10^9$ electrostatic lines from each coulomb of charge, or $12\pi \times 10^9$. If Q is the total number of coulombs on each conductor, and A is the surface area of one plate in square centimeters, then the flux density from that plate will be 12π times $10^9 Q/A$. If K is the dielectric constant of the material separating the plates, the electrostatic field intensity will be

$$F = \frac{12\pi \times 10^9 Q}{KA}. \tag{102}$$

This field intensity multiplied by the thickness, d, of the dielectric is the total p.d. between the conductors in electrostatic units. If the p.d. is expressed in volts then

$$Fd = \frac{12\pi \times 10^9 Qd}{KA} = \frac{V}{300}, \tag{103}$$

and the capacity in farads is

$$C = \frac{Q}{V} = \frac{884.2\,KA}{10^{16} d}. \tag{104}$$

This equation holds only when the distance d between plates is so small that the flux is distributed uniformly over the surface, A, and the flux passing from the edges of the plates can be neglected. If A is in square inches and d is in mils, then the equation becomes

$$C = \frac{884.2\,KA(6.45)}{10^{16} d(0.001 \times 2.54)} = \frac{225\,KA}{10^{12} d} \text{ farads.} \tag{105}$$

The farad is such a large unit that the *microfarad*, or 1/1,000,000 of a farad, is much more commonly used. Expressed in microfarads, the last formula becomes

$$C = \frac{225\,KA}{10^6 d}. \tag{106}$$

74. Condensers in Parallel and in Series.—When condensers are connected in parallel, the p.d. is the same across all of them and the total charge is the sum of the charges on the several condensers. Therefore, the equivalent capacity of several condensers in parallel is equal to the sum of the several capacities. When two or more condensers are connected in series, equal and opposite charges will be induced on each pair of plates connected together. Hence if the condensers have capacities C_1, C_2, C_3, etc., the total p.d. will be

$$V = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)Q. \tag{107}$$

Hence the equivalent capacity is

$$C_e = \frac{Q}{V} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}. \tag{108}$$

75. Capacity of a Transmission Line.—An important property of alternating current transmission lines is their capacity. The formula for such capacity is developed as follows: Let Fig. 53 represent two parallel wires of radius r

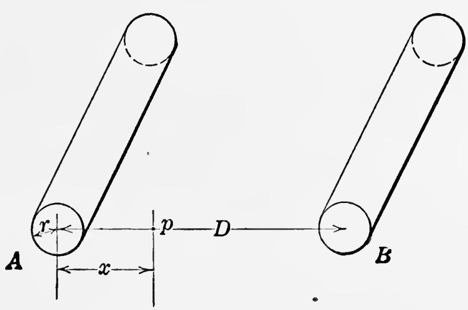

Fig. 53.

and distance D apart from center to center. The wires being long in comparison with their distance apart, and far apart in comparison with their radius, the charge on each wire may be considered as uniformly distributed over the surface of the wire. If there are Q units of charge per centimeter of length of wire the flux per unit of length will be $4\pi Q$. The area over which this flux is distributed at distance x from the center of the wire is $2\pi x$. The electrostatic intensity is therefore $4\pi Q/2\pi x = 2Q/x$ at point p due to wire A. That due to wire B is similarly $2Q/(D-x)$ The total intensity at p is therefore

$$H = 2Q\left(\frac{1}{x} + \frac{1}{D-x}\right). \tag{109}$$

It was shown in Art. 71 that electrostatic intensity is equal to drop in potential per unit distance, or

$$H = \frac{dv}{dx}. \tag{110}$$

Therefore the potential difference between A and B is

$$V = \int_r^{D-r} dv = \int_r^{D-r} H dx. \tag{111}$$

Substituting the value of H from equation (109) we have

$$V = 2Q\int_r^{D-r}\left(\frac{1}{x} + \frac{1}{D-x}\right)dx, \tag{112}$$

$$V = 2Q\int_r^{D-r}\frac{dx}{x} + 2Q\int_r^{D-r}\frac{dx}{D-x}, \tag{113}$$

$$V = 2Q \log_e x\Big]_r^{D-r} - 2Q \log_e (D-x)\Big]_r^{D-r}, \tag{114}$$

$$V = 2Q\,[\log_e (D-r) - 2 \log_e r + \log_e (D-r)], \tag{115}$$

$$V = 4Q \log_e \left(\frac{D-r}{r}\right). \tag{116}$$

Since capacity is the charge per unit potential difference we get

$$\frac{Q}{V} = C = \frac{1}{4 \log_e \left(\frac{D-r}{r}\right)} \text{ electrostatic units per cm.} \tag{117}$$

or, reducing to microfarads per 1000 ft.

$$C=\left[\frac{30.48\times10^3}{4\times2.3\log\left(\frac{D-r}{r}\right)}\right]\left[\frac{10^6}{300\times3\times10^9}\right]=\frac{0.00368}{\log\left(\frac{D-r}{r}\right)},\tag{118}$$

or, in microfarads per mile

$$C=\frac{0.0194}{\log\left(\frac{D-r}{r}\right)}.\tag{119}$$

76. Charging Current.—Since the charge on a condenser varies with the p.d. at its terminals, it follows that whenever the p.d. across a condenser is changing the charge on the condenser is changing; that is, electricity flows toward one side of the condenser and away from the other side; but electricity in motion is a current. The rate at which the charge changes, that is, the value of the current is proportional to the rate at which the p.d. is changing.

Since $i=dQ/dt$ and $Q=CV$, we get the value of the current which flows into and out of a condenser. due to a changing potential difference, as

$$i_d=C\frac{dV}{dt}.\tag{120}$$

This current is called the *displacement current,* or the *charging current,* to distinguish it from the steady current which we have in a closed electric circuit. From this relation, capacity is sometimes defined as the ratio of the displacement current of a condenser to the rate of change of p.d. which produces it.

77. Energy of a Condenser.—It follows from the fact that a current is produced when a p.d. is connected to a condenser, that energy is being transformed. The amount of energy which is stored up in a condenser when it is charged is $\frac{1}{2}CV^2$. Multiplying the equation $i=CdV/dt$, by V, we get

$$Vi=CVdV/dt,\tag{121}$$

or

$$Vidt=CVdV,\tag{122}$$

but $Vidt = dW$, which is the energy delivered to the condenser in the time dt, during which the p.d. changes by an amount dV. Therefore, the energy delivered when the p.d. is raised from 0 to V is

$$W = \int_0^V CVdV = \tfrac{1}{2}CV^2. \tag{123}$$

78. Distribution of Electrostatic Intensity.—Electrostatic lines of force are to be thought of as issuing from the charge on one conductor and passing to that on the opposite conductor. The density of these lines as they issue from a conductor will therefore depend upon the manner in which the electricity is distributed over the surface of the conductor. In the case of sheets of conducting material separated by sheets of insulating material, all of uniform thickness, the distribution will be uniform except at the edges. It has been found experimentally that, in distributing itself over a surface, electricity always piles up, so to speak, at edges, bends, and sharp points generally. Therefore the electrostatic intensity is always higher at such points than over the smooth portions of the surface separating the conductor from the dielectric. Immediately after leaving such points, however, the lines, spread out, or the intensity becomes less, and the general distribution becomes more nearly uniform.

79. Potential Gradient.—Since electrostatic intensity is proportional to the drop (or rise) in electrostatic potential per unit distance, the intensity at any point is very commonly expressed as the *potential gradient* at that point. The term potential gradient means the drop (or rise) of potential per unit distance. In uniform fields, it is a constant; in non-uniform fields, it is the derivative, dv/dx. Curves may be plotted with potential difference as ordinates and distances away from the conductor as abscissæ, or they may be plotted with potential gradient (dv/dx) as ordinates and distances as abscissæ. The potential difference referred to here is the potential difference between one conductor and different points along the shortest path between it and the other conductor. If this potential difference curve be plotted for the space between two plates of a condenser, the curve will start at zero and rise in a straight line to a value equal to the potential difference between the plates. The potential gradient curve would be a horizontal straight line. If the potential difference curve be plotted for the space between two long straight bare wires some distance apart in air, the curve will rise from zero at one wire, quite steeply at first, then slope upward less steeply until the other wire is approached when it will again rise steeply to a value equal to the p.d. between the wires. The potential gradient curve would start at a high value and slope steeply downward at first, then less and less steeply to a minimum midway between wires, then rise again more and more steeply as the other wire is approached. These curves are shown in Fig. 54 for the case of two parallel wires.

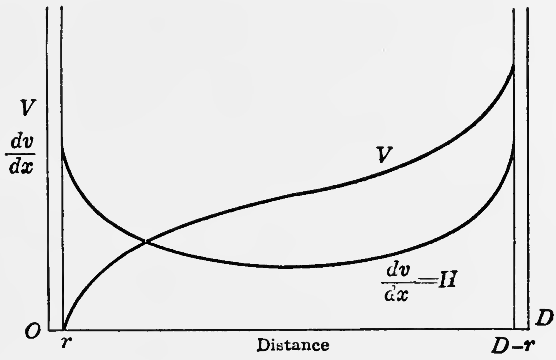

Fig. 54.—Potential and Potential Gradient between two Wires of Transmission Line.

The equation for the potential gradient curve in terms of charge on the conductors is equation (109). It may be expressed in terms of the potential difference (V) between the wires by substituting the value of Q from the equation (116) into equation (109). This gives

$$H=\frac{dv}{dx}=\left[\frac{V}{2\log_e\left(\frac{D-r}{r}\right)}\right]\left[\frac{D}{x(D-x)}\right]. \qquad (124)$$

The equation for potential difference between one wire and various points along the shortest path between it and the other wire will be found by integrating equation (124). The constant of integration will be found by applying the condition that $v=0$ when $x=r$. The solution is

$$v=\frac{V}{2}\left[1+\frac{\log_e x-\log_e (D-x)}{\log_e (D-r)-\log_e r}\right]. \tag{125}$$

This equation does not hold for values of x less than r or greater than $(D-r)$.

80. Losses in Dielectrics.—Since no material is a theoretically perfect insulator, there will always be an amount of current flowing through the substance of a dielectric, equal to the product of the impressed voltage, V, and the conductance, g, of the dielectric. That is, the *leakage current*, as it is called, is

$$i_g=gV. \tag{126}$$

The loss in the dielectric, due to this current, is of course equal to Vi_g or gV^2.

When a varying voltage is impressed on a dielectric, there is found by experiment to be an additional loss, that is, more loss than can be accounted for by the leakage current loss. This loss is called *dielectric hysteresis* loss and is probably caused by lack of homogeneity in the material and by the energy required to reverse the stresses when the voltage reverses.

81. Dielectric Strength.—It is found that when the intensity of the electrostatic field exceeds certain values (different for different dielectrics) the substance "breaks down" and allows the electricity to flow through it. The value of the volts per unit thickness at which a dielectric fails is known as its *dielectric strength.* In cases where the electrostatic intensity is not uniform, as, for example, between the wires of a transmission line, the dielectric strength of the air may be exceeded at points near the wires and not be exceeded in the rest of the space between wires. In such cases the air in the region where the dielectric strength is exceeded becomes a fairly good conductor. This change in its nature is accompanied by the appearance of a bluish light along the wire. This effect is called the "corona effect" or "brush discharge."

There are several factors which affect the dielectric strength of materials. In general a thick piece of insulation will break down at a lower pressure per unit thickness than will a thin one. A piece built up of several thin ones will generally stand more than an equally thick solid piece. The chief reason is that a solid thick piece is not likely to be so homogeneous nor so free from flaws as a thin one. An increase in temperature generally results in a decreased dielectric strength. A high frequency causes more loss and a higher temperature and consequently a lower strength than a low frequency. When time is taken in seconds or perhaps in minutes, a material will stand a high voltage for a short time but will break down if the same voltage is kept on for a longer time. Other conditions, such as the form of the conductors and the wave form of the voltage, also affect the breakdown value.

It should be noted that there is no definite relation between the dielectric strength of a material and its resistance as insulation. The following table gives characteristic values of the properties of dielectric and insulating materials:

TABLE 2.—CONSTANTS OF DIELECTRIC MATERIALS

Material.	Dielectric Constant.	Dielectric Strength, Volts per Mil.	Resistivity, Ohm-cm.
Air	1	97	
Glass	5 to 10	150 to 300	17×10^{9}
Mica	2.5 to 6	1000 to 3000	$(1 \text{ to } 120)\times10^{12}$
Paraffin	1.9 to 2.3	300	10^{15} to 10^{19}
Paper (dry) untreated	1.7 to 2.6	100 to 250	10^{15}
Petroleum	2 to 3	200 to 400	10^{12}
Rubber (para)	2 to 3	300 to 500	10^{14} to 10^{16}

82. Charging and Discharging a Condenser through a Resistance.—Attention has already been called to the fact that when a source of e.m.f. is first connected to two conductors separated by a dielectric (i.e., a condenser) a current will begin to flow but will cease as soon as the p.d. across the condenser becomes equal to the e.m.f. which has been connected to it. The time required to reach this condition depends upon the resistance in series with the circuit. When the source of e.m.f. is removed and condenser circuit is closed, the condenser will discharge; that is, a current will begin to flow but will cease as soon as the p.d. across the condenser becomes zero, provided there is no inductance in the circuit. (The case of capacity and inductance together will be discussed in the next article.) Again, the time required for the current to become zero depends upon the value of the resistance in series with the condenser. Following is the development of the mathematical relations between current and time for the two simple cases just mentioned.

Referring to Fig. 55, it will be seen that when the key K is in contact with b the condenser and resistance are in series with the source of e.m.f. E. With this connection, the drop in potential, v, through the condenser, plus the drop in potential, ri, through the resistance must be equal at every instant to the e.m.f. E. That this is true should be understood from the general principle that in any circuit the total e.m.f. in the circuit must be equal to the sum of the potential drops which consume this e.m.f. We have then as the e.m.f. equation for the circuit

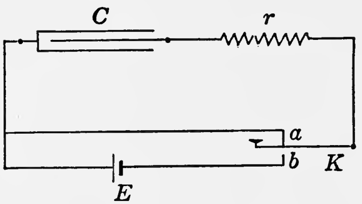

Fig. 55.

$$E = v + ri \tag{127}$$

But $v = q/C$ and $i = dq/dt$, where q is the charge on the condenser at voltage v, and dq/dt is the corresponding rate of charge of q. Therefore,

$$E=\frac{q}{C}+r\frac{dq}{dt}. \tag{128}$$

Separating tne variables in this equation, we get

$$\frac{dt}{rC}=\frac{dq}{CE-q}. \tag{129}$$

Integrating this equation

$$\frac{t}{rC}=-\log_e(CE-q)+K, \tag{130}$$

when K is the constant of integration. For the case under consideration, it is known that $q=0$ when $t=0$, if the zero of time is taken as the instant when the connection is made to E. Therefore. substituting these values in equation (130), we get

$$K=\log_e CE, \tag{131}$$

and changing signs in equation (130) and substituting the value of K

$$-\frac{t}{rC}=\log_e(CE-q)-\log_e CE=\log_e\left(\frac{CE-q}{CE}\right), \tag{132}$$

or

$$e^{-\frac{t}{rC}}=\frac{CE-q}{CE}, \tag{133}$$

where e is the base of the natural system of logarithms; whence

$$q=CE\left(1-e^{-\frac{t}{rC}}\right). \tag{134}$$

From this the value of i is at once found by differentiation with respect to t, as

$$i=\frac{dq}{dt}=\frac{E}{r}e^{-\frac{t}{rC}}. \tag{135}$$

The curve for this current is shown in Fig. 56. Of course, since no circuit can be completely without inductance, the current cannot rise instantly from zero to the initial value here shown as E/r, but the effect of inductance has been ignored in this discussion.

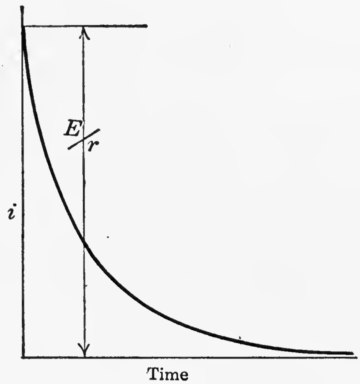

FIG. 56.

Now suppose after the condenser is charged to the potential difference, E, and contains a charge Q, the key in Fig. 56 is allowed to make contact at a. The condenser will begin to discharge and since the e.m.f. E has been removed, the equation for e.m.f's becomes

$$0=\frac{q}{C}+r\frac{dq}{dt}. \tag{136}$$

Again separating the variables and integrating, we get,

$$\frac{t}{rC}=-\log_e q+K. \tag{137}$$

In this case, the constant of integration is found from the condition that $q=Q$ at $t=0$, if the zero of time is taken as the instant when connection is made to a. We get, therefore, $K=\log Q$, and

$$\frac{t}{rC}=\log_e\frac{Q}{q}, \tag{138}$$

or

$$q=Qe^{-\frac{t}{rC}}, \tag{139}$$

whence

$$i=\frac{dq}{dt}=-\frac{Q}{rC}e^{-\frac{t}{rC}}, \tag{140}$$

or, since

$$Q=CE,$$

$$i=-\frac{E}{r}e^{-\frac{t}{rC}}. \tag{141}$$

The shape of the current curve is therefore the same as for the case of charging but it flows in the opposite direction, as is indicated by the minus sign.

83. Short-circuiting Inductance and Capacity in Series.—A mathematical study of transient phenomena, that is, those phenomena which occur when there is a change from a steady condition, is beyond the scope of this text. Two exceptions have been made to this; namely, the cases of building up a magnetic field in a simple inductive resistance, and of charging and discharging a condenser through a resistance. One additional case will be discussed briefly; that is the case of closing a circuit containing an inductance and a condenser, when there is either a current flowing in the inductance or a p.d. across the condenser. Such a circuit would be represented by a transmission line, if its capacity be considered as concentrated at some one point and a short circuit occurred at some other point, opening the circuit breakers at the power station. The effect of resistance in the circuit will at first be neglected. Assume that the short-circuit occurs at an instant when the line is charged to a potential difference of V volts and the current in the line is zero. The electrostatic energy of the line will be $\frac{1}{2}CV^2$ where C is its capacity. Current will begin to flow from one side of the line to the other through the short circuit, and by building up a magnetic field linking the circuit, the electrostatic energy will be transformed into electromagnetic energy. Since it is assumed that none of the energy is dissipated in heat losses, it follows that when the electrostatic energy has become zero, an amount of energy equal to the original value of the electrostatic energy must now be stored in the circuit in the form of electromagnetic energy. The energy stored in a magnetic field has been shown to be equal to $\frac{1}{2}LI^2$ where L is the inductance of the circuit. It follows therefore that the current in the circuit will reach such a value that $\frac{1}{2}LI^2 = \frac{1}{2}CV^2$, or $I = V\sqrt{C/L}$. This is not likely to result in a dangerous condition; but if the short circuit occurs when the current in the line is I and the p.d. across the line is zero, then the electromagnetic energy associated with the line will be equal to $\frac{1}{2}LI^2$, and as the current decreases, this energy will be transformed into electrostatic energy, and the p.d. across the line at the point where the capacity is located will rise to such a value that $\frac{1}{2}CV^2 = \frac{1}{2}LI^2$, of $V = I\sqrt{L/C}$. This may result in a dangerously high voltage, if the current and inductance are large and the capacity is small. In either one of the cases mentioned the energy associated with circuit at the time of short-circuit will continue to oscillate back and forth from one form to the other indefinitely. In any practical case, there will be resistance in the circuit which will consume a portion of the energy during each oscillation and thus the amplitude of the oscillations will gradually decrease until all of the energy has been dissipated in heat.

CHAPTER VI

SINE WAVE ALTERNATING CURRENTS

84. Definition of Alternating Current.—An alternating current (or e.m.f.) is one which alternates regularly in direction between the same positive and negative maximum values and whose average value is zero when taken over any whole number of cycles of values. In ordinary electric machinery, the positive sets of values are exactly like the negative sets of values. Furthermore, these values, if plotted against time as abscissæ, give a curve which approximates more or less closely to what is called a sine wave, which is a curve plotted with angles as abscissæ and the corresponding sines of those angles as ordinates. In this chapter, the entire discussion will be based on the assumption that the curves have the form of sine waves

85. The E.M.F. and Current Equations. Cycle. Frequency. Angular Velocity. Electrical Degrees. Phase.—A sine wave will be generated if a coil of wire is revolved at uniform speed in a uniform magnetic field as illustrated in Fig. 57. The fundamental e.m.f. equation has already been shown to be

$$e = -\frac{Nd\phi}{10^8 dt}. \qquad (142)$$

The negative sign is placed before the right-hand member because the current which this e.m.f. produces in the circuit will have a force exerted upon it by the field, which is in opposition to the force which causes the coil to turn. If ϕ_m is the maximum flux which can be enclosed by the coil, then the flux enclosed in any given position of the coil, such as $c-d$, is

$$\phi = \phi_m \cos \theta, \tag{143}$$

where θ is the angle between the plane of the coil and a plane at right angles to the direction of the field. But if the coil is rotating at uniform angular velocity of ω radians per second, and time, t, be counted from the instant of maximum enclosure, then $\theta = \omega t$ and

$$\phi = \phi_m \cos \omega t. \tag{143a}$$

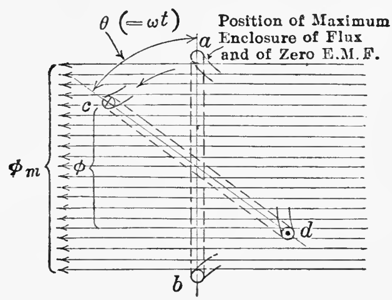

FIG. 57.

Substituting this value of ϕ into equation (142) gives

$$e = -\frac{Nd}{10^8 dt}(\phi_m \cos \omega t) = \frac{\omega N \phi_m}{10^8} \sin \omega t. \tag{144}$$

In equation (144) the part $\omega N \phi_m / 10^8$ is the maximum value of the e.m.f. and is the value at the instant when ωt is 90° or when the flux linking the coil is zero; at this instant the coil is cutting across the flux most rapidly, whereas, when the flux linking the coil is a maximum, the coil is sliding along the flux and the rate of cutting is zero. Denoting the maximum e.m.f. by E_m, we may write

$$e = E_m \sin \omega t. \tag{145a}$$

Similarly we may write the equation for the instantaneous value of current in a circuit as

$$i = I_m \sin \omega t. \tag{145b}$$

where $t=0$, when $i=0$. The successive values of the e.m.f. for one revolution of the coil, beginning at $t=0$, would be as shown in Fig. 58. One complete set of values is called a cycle. One cycle of values is passed through for each revolution of the coil in a 2-pole field such as is shown in Fig. 57. During the first half revolution the e.m.f. is in one direction in the wire and is indicated by ordinates above the x-axis in Fig. 58, while during the second half of the revolution, the e.m.f. is in the opposite direction in the wire and is indicated by ordinates below the x-axis. Either direction through the wire may be chosen as positive. The curve showing the

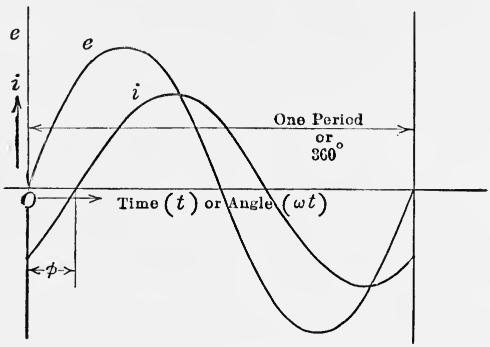

FIG. 58.

values of e.m.f. or current as a function of time is called the wave form. The time consumed in passing through one cycle is called the period, and the number of cycles passed through in one second is called the frequency; in a 2-pole field the frequency (f) is equal to the number of revolutions per second, and the angular velocity of the coil is equal to 2π times the frequency, or

$$\omega = 2\pi f. \tag{146}$$

In a 4-pole field such as is illustrated in Fig. 59, there will be two cycles of e.m.f. for one revolution of the coil, and the frequency will be equal to the number of pairs of poles times the number of revolutions per second, or, in general, if p is the number of poles and n is the number of revolutions per minute, the relation is

$$f=\frac{pn}{2\times 60}=\frac{pn}{120}. \tag{147}$$

Occasionally the term "alternations" is used in connection with an alternating current. This refers to the number of reversals per minute and is therefore equal to 120 times the frequency or to the number of poles times the revolutions per minute.

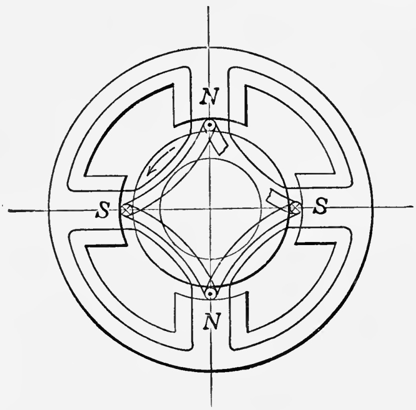

Fig. 59.

In equation (144) the quantity ωt is an angle and the e.m.f. passes through one cycle while the angle changes from 0 to 360°, or from 0 to 2π radians; and one cycle is always passed through for every pair of poles which is passed; therefore in electrical machinery the angle covered by each pair of poles is 2π (electrical) radians or 360 (electrical) degrees. The number of electrical degrees passed over in one revolution is $p/2$ times the mechanical degrees passed over. The angle swept through is a function of time. In the ground covered by this text, steady conditions of operation are assumed and therefore the angular velocity of the moving elements of generating apparatus is constant, and the angle swept over in any given time is equal to the product of that time and the angular velocity. Furthermore, under the conditions assumed, the recurring cycles of values are exactly alike. It is therefore permissible and convenient to express the instantaneous values of e.m.f. and current as a function of an angle, as has been done in the preceding discussion.

It will be proven in later articles that in circuits containing inductance or capacity the current and e.m.f. will not have their maximum values (nor any other corresponding values) occurring at the same instant. This condition is described by saying that the current and e.m.f. are "out of phase." If the e.m.f. equation is written as $e = E_m \sin \omega t$, then the instantaneous value of e is zero when $t = 0$, and if the current is out of phase with the e.m.f., the current will not be zero when $t = 0$. Consequently, in order to give correct corresponding values, the current equation must be written as

$$i = I_m \sin (\omega t - \phi) \qquad (148)$$

in which I_m is the maximum value of the current and ϕ is an angle whose value is $\omega(t' - t)$, where $(t' - t)$ is the constant difference in time between corresponding values of e and i. The angle ϕ is called the angle of phase difference and in practice the phase difference is expressed in terms of an angle instead of time. Note that when ϕ is a positive angle, the e.m.f. passes through zero in a positive direction before the current, or, as it is generally expressed the current "lags" the e.m.f., see Fig. 58. For instance, when $t = 0$, $e = 0$, and $i = I_m \sin (-\phi)$; that is, i is still negative and will not become zero until $\omega t = \phi$. On the other hand, if ϕ is negative, then when $t = 0$, $i = I_m \sin \phi$, and has passed through its zero value before the e.m.f. In such a case, the current is said to "lead" the e.m.f. If in the expression $(t' - t)$, which is defined above, we put $t = 0$, then $(t' - t) = t'$ and t' is the time that elapses between a zero value of e and the nearest zero value of i in the same direction and it may be either positive or negative, depending on whether the current has not yet reached its zero value or has already passed it.

It should be noted that two or more e.m.f.s or two or more currents may be out of phase with each other, and their phase differences would be shown in the same manner as the phase difference between an e.m.f. and a current.

86. Effective and Average Values of Current and E.M.F. —The power which is developed when a current flows in a resistance R is equal at each instant to i^2R, or e^2/R, where $e=Ri$. The average power is the average of the instantaneous values. The constant value of current or e.m.f. which would produce an amount of power in a given resistance equal to the average power produced by the alternating wave is called the *effective value* of the alternating current or e.m.f. That is, if the effective value is represented by I, the product RI^2 is equal to the average power developed by the alternating current, or,

$$RI^2=\text{average }(Ri^2)=\text{average }(RI_m^2\sin^2\omega t) \qquad (149)$$

or

$$I^2=I_m^2\ (\text{average }\sin^2\omega t). \qquad (150)$$

But $\sin^2\omega t=\frac{1}{2}-\frac{\cos 2\,\omega t}{2}$, and the average value of $\cos 2\,\omega t$ over any whole number of cycles is zero. Therefore, the average value of $\sin^2\omega t$ over any whole number of cycles is 1/2, and

$$I^2=\frac{I_m^2}{2}, \qquad (151)$$

or

$$I=\frac{I_m}{\sqrt{2}}=.707\,I_m. \qquad (152)$$

That is, the effective value of an alternating current is equal to its maximum value divided by $\sqrt{2}$.

Similarly, the effective value of an alternating e.m.f. is

$$E=.707\,E_m. \qquad (153)$$

In practical work effective values are nearly always of most importance and all measuring instruments are graduated to read effective values.

The average value of an alternating current is taken as the average ordinate of any half-cycle, since the average value over a whole cycle is evidently zero. Since the instantaneous value is

$$i = I_m \sin \omega t,$$

$$I_{av} = I_m \Big[\text{average value of } \sin(\omega t)\Big]_0^\pi. \qquad (154)$$

But the average value of the sine of an angle varying between the limits of *0* and π is

$$\text{average sine} = \frac{1}{\pi}\int_0^\pi \sin(\omega t)d(\omega t) = -\frac{1}{\pi}\cos(\omega t)\Big]_0^\pi = \frac{2}{\pi}. \qquad (155)$$

Therefore

$$I_{av} = \frac{2}{\pi} I_m = .637 I_m. \qquad (156)$$

The ratio of the effective value to the average value of an alternating current or e.m.f. is called form factor. This ratio is, for sine waves, $.707/.637 = 1.11$. The ratio of the maximum to the effective value is called the crest factor, or peak factor, and for sine waves it is $\sqrt{2} = 1.414$.

87. Current and E.M.F. Waves in Resistance Only.—If the instantaneous current in a resistance R is $i = I_m \sin \omega t$, the e.m.f. at each instant must be equal to Ri and therefore the e.m.f. wave is represented by the equation,

$$e = Ri = RI_m \sin \omega t = E_m \sin \omega t. \qquad (157)$$

The maximum e.m.f. is equal to RI_m and the effective value of e.m.f. is

$$E = RI \qquad (158)$$

or the effective current is

$$I = \frac{E}{R}. \qquad (159)$$

The current is said to be "in phase" with the e.m.f.; that is, they pass through zero in the same direction at the same instant. These relations are shown in Fig. 60.

88. Current and E.M.F. Waves in Inductance Only.—It has previously been shown that the e.m.f. produced by a varying current in an inductive circuit is $-Ldi/dt$. The negative sign indicates the fact that the e.m.f. of self-induction opposes the change in the value of the current. Hence, to maintain a current in an inductive circuit, there will be required an impressed e.m.f. equal and opposite to the e.m.f. of self-induction, and its equation will be

$$e=L\frac{di}{dt}. \tag{160}$$

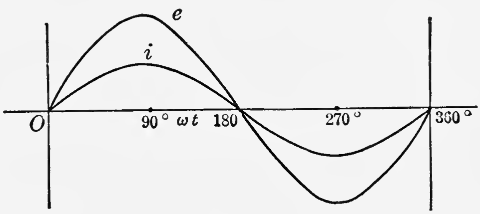

Fig. 60.

Therefore, if the equation for the current is $i=I_m \sin \omega t$ the equation for the e.m.f. will be

$$e=L\frac{d(I_m \sin \omega t)}{dt}, \tag{161}$$

$$e=\omega L I_m \cos \omega t, \tag{162}$$

or

$$e=\omega L I_m \sin (\omega t+90). \tag{163}$$

The interpretation of this equation is that the e.m.f. leads the current by 90°, or a quarter of a cycle; that is, the e.m.f. is a positive maximum when the current is passing through zero in the positive direction. This is also expressed by saying that the current is out of phase with the e.m.f. by 90°, or that the current lags the e.m.f. by 90°. More strictly, difference in phase means difference in time between successive corresponding values of the alternating quantities; but as already explained, time in connection with alternating quantities is usually expressed in terms of an angle, and therefore phase difference is usually expressed in degrees. In the case under consideration, this phase difference of 90° is caused by the back e.m.f. of self-induction.

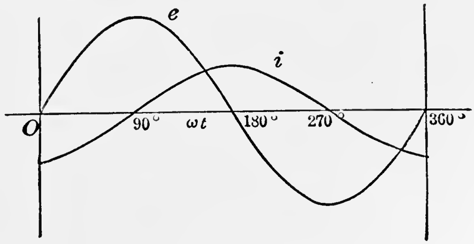

Fig. 61.

One-half the time, that is, while the current is rising, the current is in the same direction as the impressed e.m.f.; during the other half of the time, that is, while the current is decreasing, it flows in the opposite direction to that of the impressed e.m.f. See Fig. 61.

A clearer understanding of this matter may be had by studying an analogy. In Fig. 62 let it be considered that

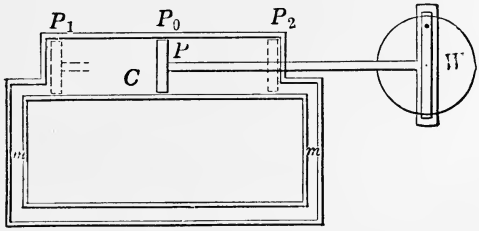

Fig. 62.—Analogy of Inductive Circuit.

the cylinder C and the pipe mm are full of water and that the water is made to flow back and forth through the circuit by the piston P which moves in simple harmonic motion if the drive wheel W revolves at uniform speed. Since the electric circuit which is being discussed is assumed to be without resistance, the frictional resistance to the flow of water can be neglected in the analogy. The inductance of the electric circuit is analogous to the inertia of the water in the water circuit. When the piston is in the position P_0, its speed and also that of the water is a maximum and at the instant is neither increasing nor decreasing. Therefore no pressure (corresponding to voltage in the electric circuit) will be exerted on the water, although the water flow (corresponding to the electric current) is a maximum. As the piston continues to move, say toward the position P_1, its speed decreases and the inertia of the water begins to exert a pressure upon it toward the left; the pressure of the piston upon the water will therefore be from left to right, although the water flow is in the opposite direction. The pressure will become a maximum when the piston reaches the position P_1 because at this position the rate of change of water flow is greatest and its inertia is therefore greatest. The actual rate of flow at this instant is zero, however, and as the piston begins to move to the right, its inertia will oppose the increase in rate of flow. The pressure of the piston will therefore be in the same direction as the flow of water until the piston again reaches the position P_0, when the conditions will repeat themselves as already described, while the piston moves to P_2 and back to P_0. A careful study of this analogy should make clear what is meant by the statement that an electric current is out of phase with its e.m.f., and how a current may at certain times be flowing in the opposite direction to the e.m.f. impressed on the circuit.

In an inductive circuit, resistance being ignored, there is no opposition to the flow of current, but there is opposition to a *change* in the *value* of the flow. This opposition is due to the back e.m.f. generated by the accompanying change in flux linking the circuit; more briefly, it is said to be due to inductance. Referring to Fig. 61, when the current is a maximum its rate of change is zero, the rate of change of flux is also zero and there is no back e.m.f. of self-induction; consequently at this instant no impressed e.m.f. is required.

As the current decreases in value, the flux also decreases and a back e.m.f. is generated which opposes the decrease and is consequently in the same direction as the current. Therefore there must be an impressed e.m.f. equal and opposite to the back e.m.f. at each instant and consequently in opposition to the direction of the current. The curve e in Fig. 61 is that of the impressed e.m.f. As the current decreases in value, its rate of change increases and the e.m.f. increases. When the current reaches zero, it is changing most rapidly, the back e.m.f. is therefore a maximum and the impressed e.m.f. must be a maximum. As the current increases from zero, the back e.m.f., since it opposes the rise, is opposite in direction and the impressed e.m.f. must be in the same direction as the current, but decreases in value until the current is again a maximum.

Returning to the discussion of equation (163), it should be noted that it shows that the maximum e.m.f. (E_m) is equal to $\omega L I_m$ and it follows that the effective e.m.f. is

$$E = \omega L I = 2\pi f L I. \tag{164}$$

The product ($2\pi f L$) is called *inductive reactance* and is measured in ohms just the same as resistance.

The value of the current I which will flow in a purely inductive circuit of inductance L, when an e.m.f., E, of frequency f is impressed upon it is, therefore

$$I = \frac{E}{2\pi f L}, \tag{165}$$

and it lags 90° behind the e.m.f.

89. Current and E.M.F. Waves in Capacity Only.—It has previously been shown that the current which flows through a condenser when the e.m.f. across it is varying is $i = dQ/dt = Cde/dt$; this is the instantaneous value of the current; hence to maintain a sine wave of current through the condenser, the relation must be

$$i = C\frac{de}{dt} = I_m \sin \omega t, \tag{166}$$

or

$$de = \frac{I_m}{C} \sin \omega t dt. \tag{167}$$

Integrating both sides of this equation we get,

$$e = -\frac{I_m}{\omega C} \cos \omega t, \tag{168}$$

or

$$e = \frac{I_m}{\omega C} \sin (\omega t - 90). \tag{169}$$

This equation shows that the current leads the e.m.f. by 90°, or a quarter of a cycle; that is, the e.m.f. is a negative maximum when the current is passing through zero in the positive direction. See Fig. 63. It also shows that the

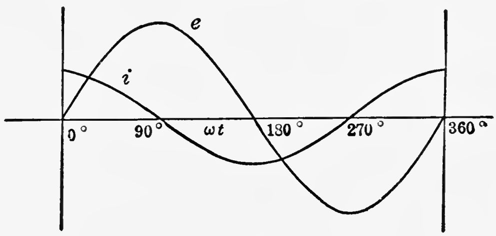

Fig. 63.

maximum e.m.f. is $I_m/\omega C$ and it follows that the effective e.m.f. is

$$E = I/\omega C = I/2\pi fC. \quad 170)$$

The quantity, $1/2\pi fC$ is called capacity reactance and is measured in ohms. The effective current in a condenser is therefore

$$I = 2\pi fCE, \tag{171}$$

and it leads the e.m.f. by 90°. Equations (169) and (163) should be compared and also equations (171) and (165) and the differences carefully noted and understood.

An analogy which may make the relations in a condenser (capacity) circuit clearer is illustrated in Fig. 64. Suppose that the cylinder C, the pipes mm, and the chamber B are full of water and that the water is made to flow back and forth through the circuit by the simple harmonic motion of the piston P, when the wheel W revolves at uniform speed. Let it be imagined that the water has no inertia and that there is no frictional resistance to its motion. Let D be an impervious elastic diaphragm stretched across the chamber B. The elasticity of this diaphragm is analogous to the capacity of a condenser. When the piston is in the position P_0, its speed and also that of the water is a maximum and at the instant is neither increasing nor decreasing. Therefore no pressure (corresponding to the voltage on a condenser) will be exerted on the diaphragm, although the flow of water

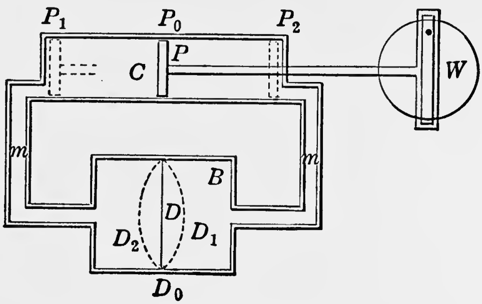

FIG. 64.—Analogy of Capacity Circuit.

(corresponding to the electric current) is a maximum. As the piston continues to move, say toward the position P_1, its speed, and with it the rate of flow of water, decreases, but an increasing pressure will be exerted on the diaphragm in the direction of the flow of water, and this pressure will reach its maximum when the piston reaches the end of its stroke and the rate of water flow (current) is zero. As the piston moves back toward the position P_0 the direction of water flow will be reversed and increase, but the pressure on the diaphragm will continue as before until the middle position is reached, when it will become zero and then increase in the same direction as the water flow. A study of these relations should make it clear that the rate of water flow in a given direction is one-quarter of a cycle ahead of the pressure in the same direction. This is analogous to the condition that exists when an alternating voltage is impressed on an electric condenser.

Referring to Fig. 63, when the e.m.f. impressed on the condenser is zero, it is changing most rapidly and therefore the current, that is, rate of change of charge, is a maximum. As the p.d. between the plates of the condenser increases, the flow of charge, that is, the current, is in the same direction as the e.m.f. across the condenser, but it decreases in value until the e.m.f. reaches a maximum when the rate of change of e.m.f., and therefore the current, is zero. As the e.m.f. decreases in value, the charge flows out of that side toward which the e.m.f. is directed and is therefore negative with respect to the e.m.f. It reaches its negative maximum when the e.m.f. has fallen to zero, and as the e.m.f. rises in the negative direction, the current decreases in value as before.

90. Vector Representation of Alternating Quantities.—The great majority of alternating current problems involve the addition or subtraction of alternating e.m.f's or currents. Much simpler than that of adding or subtracting the plotted waves is the method known as the vector method. In Fig. 65 let the waves (e_1) and e_2) represent two alternating e.m.f's differing in phase by the angle θ and the sum of which gives the wave e_s. Note that wave (e_2) lags behind wave (e_1). At the left, let the lines OE_1 and OE_2 represent the maximum values of waves (e_1) and (e_2), respectively, the line OE_2 being placed θ degrees behind (clockwise is behind) OE_1. Draw OE_s as the diagonal of a parallelogram on OE_1 and OE_2 as sides. These lines OE_1, OE_2 and OE_s are vectors, because they not only represent the magnitude of the e.m.f's but also by their positions represent the relative directions of the e.m.f's. Let all these vectors be supposed to revolve positively (counter-clockwise) at an angular velocity of ω. Then the ordinates of the waves (e_1), (e_2), and (e_s) at any point a which is ωt degrees to the right of the origin will be respectively,

$E_1 \sin \omega t$, $E_2 \sin(\omega t - \theta)$, and $E_s \sin(\omega t - \phi)$, where ϕ is the angle which OE_s makes with OE_1 and ωt is measured positively (counter-clockwise) from the axis Ox. That is, OE_s is the maximum value of the resultant wave, (e_s), and ϕ is its phase relation tc wave (e_1). These two quantities (E_s and ϕ) fully determine and locate the resultant wave. The value of OE_s is (see the triangle ObE_s).

$$E_s = \sqrt{(E_1 + E_2 \cos \theta)^2 + (E_2 \sin \theta)^2}, \qquad (172)$$

and

$$\phi = \tan^{-1} \frac{E_2 \sin \theta}{E_1 + E_2 \cos \theta}. \qquad (173)$$

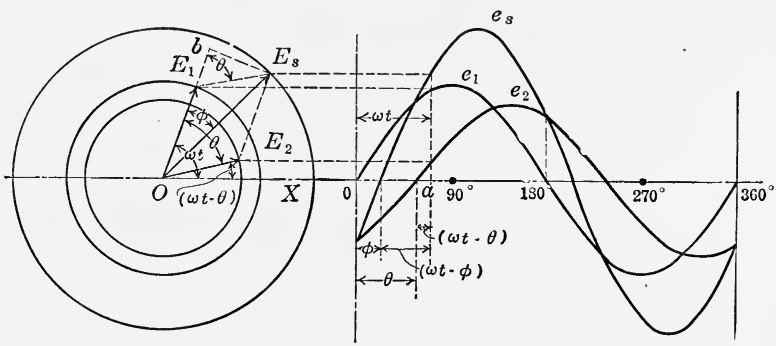

Fig. 65.

It is therefore unnecessary to plot and add the waves, but the resultant may be found by the use of the vector diagram and its mathematical solution. It should be noted that vectors cannot be used in the manner just described when the quantities have different frequencies, because they would then have to revolve at different speeds. Resultant values and phase angles can only be shown by vector diagrams when the vectors are stationary with respect to each other; that is, they must revolve at the same speed and therefore represent waves of the same frequency.

In the solution of problems dealing with quantities of the same frequency any set of vectors may be considered as actually as well as relatively stationary in any desired position, but the angles between the different vectors must correspond to their actual phase relation, and it must be kept in mind that the vectors represent quantities which are continuously changing in value and alternating in direction.

Since the relation of the maximum values of alternating quantities to their effective values is a constant ($\sqrt{2}$), and since effective values are generally of most importance, the latter values are nearly always used in vector diagrams.

Vector diagrams are used not only for finding resultant e.m.f.'s or currents, but also for showing the phase relations between e.m.f.'s and currents. Fig. 66, shows the vector

FIG. 66.

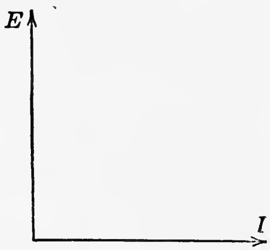

FIG. 67.

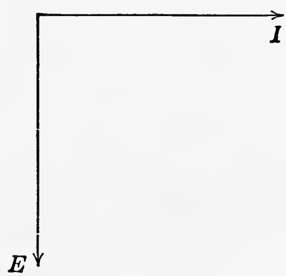

FIG. 68.

diagram for the case of Article 87, Fig. 67 for the case of Article 88, and Fig. 68 for the case of Article 89. These diagrams should be carefully observed and the significance of the relative positions of the E and I vectors for each case well understood.

91. Current and E.M.F. Relations in a Circuit Containing Resistance, Inductance and Capacity in Series.—The circuit will be as shown in Fig. 69. Let the current be $i = I_m \sin \omega t$. Then by equation (157)

$$e_R = RI_m \sin \omega t;$$

by equation (162)

$$e_L = \omega LI_m \cos \omega t;$$

and by equation (168)

$$e_C = -\frac{I_m}{\omega C}\cos\omega t.$$

The total required e.m.f. will therefore be

$$e = RI_m\sin\omega t + \omega LI_m\cos\omega t - \frac{I_m}{\omega C}\cos\omega t. \qquad (174)$$

It must be remembered that this equation gives the instantaneous values of the e.m.f. The effective value and its phase relation to the current can be most readily determined by means of a vector diagram. In Fig. 70, let the vector I represent the effective value of the current;

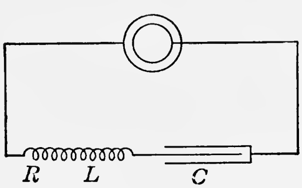

FIG. 69.

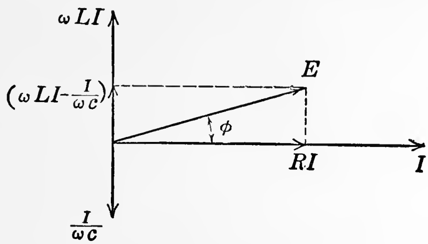

FIG. 70.

then the vector RI, drawn in phase with I, will represent the effective value of $RI_m\sin\omega t$; the vector ωLI, drawn 90° ahead of I, will represent the effective value of $\omega LI_m\cos\omega t$; and the vector $I/\omega C$ drawn 90° behind I, will represent the effective value of $-\frac{I}{\omega C}\cos\omega t$. From the geometry of the diagram, the resultant of these three vectors is readily found to be

$$E = \sqrt{(RI)^2 + \left(\omega LI - \frac{I}{\omega C}\right)^2}, \qquad (175)$$

or

$$E = I\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}. \qquad (176)$$

The terms ωL and $1/\omega C$ have already been defined as "inductive reactance" and "capacity reactance" respectively.

The quantity $(\omega L-1/\omega C)$ is the total reactance and is equivalent to a simple inductive reactance or a simple capacity reactance depending on whether ωL is larger or smaller numerically than $1/\omega C$. Let the quantity $(\omega L-1/\omega C)$ be represented by X. It may be called the "equivalent reactance" of the circuit. Equation (176) may then be written

$$E=I\sqrt{R^2+X^2}, \tag{177}$$

or

$$I=\frac{E}{\sqrt{R^2+X^2}}, \tag{178}$$

where X may be either positive or negative. The vector diagram shows that E leads I by an angle, ϕ, whose tangent is

$$\left(\omega LI-\frac{I}{\omega C}\right)/RI=XI/RI,$$

or

$$\phi=\tan^{-1}\frac{X}{R}. \tag{179}$$

When inductive reactance predominates, X is positive, ϕ is positive and E actually leads I; when capacity reactance predominates, X is negative, ϕ is negative and E leads I by a negative angle; that is, it lags behind I, or, what amounts to the same thing, I leads E. The quantity $\sqrt{R^2+X^2}$ is called the impedance of the circuit and is expressed in ohms as are resistance and reactance. The letter Z is universally used to represent impedance, and thus the simplest expression for the current in an a.c. circuit is

$$I=\frac{E}{Z}. \tag{180}$$

Evidently, if the circuit contains no capacity, $X=\omega L$, and if the circuit contains no inductance, $X=-1/\omega C$.

92. Effective Resistance.—When a direct current I flows in circuit of resistance R, the power dissipated in heat is RI^2. If the current is changed to an alternating one of the same effective value in the same circuit, the power dissipated in heat will in general be larger than that which was caused by the direct current. This may be due to one or more of several causes: (*a*) skin effect (see Article 67); (*b*) eddy currents in the conductors; (*c*) hysteresis in any magnetic material associated with the circuit; or, (*d*) eddy currents in any metallic material within the influence of the circuit. These additional heat losses are equivalent in their effect to an increase in the resistance of the circuit. The resistance to direct current is called ohmic resistance; that value of resistance which multiplied by I^2 gives the total heat loss when alternating current flows in the circuit is called the effective resistance of the circuit. Therefore the effective resistance of an alternating current circuit must be found by measuring the total power dissipated as heat and dividing this power by the square of the current. In all alternating current formulæ, the resistance, R, is to be understood to mean effective resistance. In certain cases, however, the hysteresis and eddy current losses in iron cores belonging to the circuit are measured and considered separately.

93. Power in A. C. Circuits.—Let an effective e.m.f. E be causing an effective current I to flow through a circuit of impedance

$$Z=\sqrt{R^2+X^2}.$$

The angle of phase difference between E and I will be $\phi=\tan^{-1}(X/R)$.

The average power expended in the circuit is

$$P=EI\cos\phi. \tag{181}$$

This may be proven as follows: Let the instantaneous e.m.f. be

$$e=E_m\sin\omega t=\sqrt{2}E\sin\omega t. \tag{182}$$

Then the instantaneous current will be

$$i=I_m\sin(\omega t-\phi)=\sqrt{2}I\sin(\omega t-\phi), \tag{183}$$

and the instantaneous power will be

$$p = ei = 2EI \sin \omega t \sin (\omega t - \phi). \tag{184}$$

Substituting $\omega t = a$, and $(\omega t - \phi) = b$, in the trigonometrical relation that

$$2 \sin a \sin b = \cos (a - b) - \cos (a + b), \tag{185}$$

equation (184) becomes

$$p = EI \cos \phi - EI \cos (2\omega t - \phi). \tag{186}$$

The average power is evidently equal to $EI \cos \phi$ minus the average value of $EI \cos (2\omega t - \phi)$; but the average value of any cosine or sine wave, over a whole number of cycles, is zero. Therefore the average power in the circuit is

$$P = EI \cos \phi. \tag{187}$$

In Figs. 71, 72, and 73 are shown the voltage, current and power waves for the three cases, (1) a circuit containing

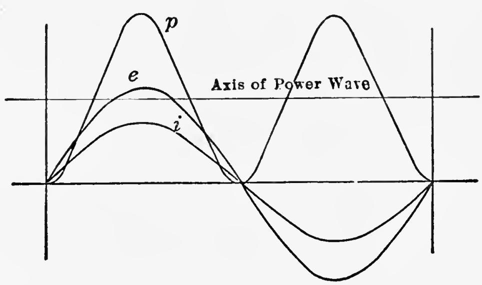

FIG. 71.

R only, (2) a circuit containing R and X, and (3) a circuit containing X only. In all cases the power wave is a sine wave of double frequency about an axis whose distance from the zero line is the distance representing the average power, $EI \cos \phi$.

In case (1) $\cos \phi$ is unity and the ordinate of the axis of the power wave is EI. The maximum power is $2EI$ and the minimum is zero. The instantaneous power is positive at all times; that is, the transfer of power is always in the same direction, namely from the source of power into the circuit.

In case (2) the ordinate of the axis of the power wave is $EI \cos \phi$. The maximum power is $EI \cos \phi + EI$. The minimum power is $EI \cos \phi - EI$ and is evidently negative;

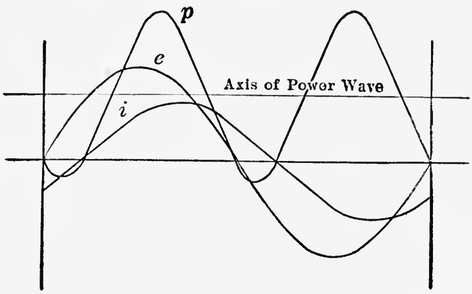

FIG. 72.

that is, during such times as the current and e.m.f. are in opposite directions, the flow of power is from the circuit back to the source. This means that during such times the energy of the circuit is being supplied by the magnetic field associated with the circuit.

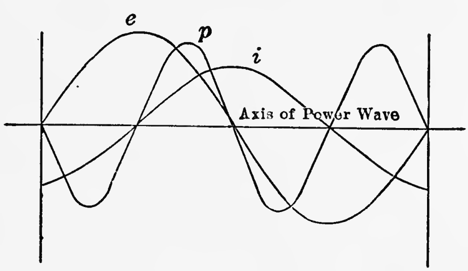

FIG. 73.

In case (3) $\cos \phi$ is zero, and the power wave axis is coincident with the zero line. The maximum power is EI and the minimum power is $-EI$; that is, the average power is zero. During one-half cycle all the energy goes to building up a magnetic field and during the next half cycle all that energy is returned by the collapsing field.

If the circuit contained resistance and capacity, the curves would be similar to those of case (2), but the current and power waves would be shifted along the time axis. A similar statement applies to a circuit of capacity alone with reference to case (3).

94. Power Factor. Apparent Power. Reactive Factor.—The ratio of the average power developed in a circuit to the product of the effective values of e.m.f. and current is defined as the power factor of the circuit. The product just mentioned is generally called the volt-amperes (or kilovolt-amperes, abbreviated *kv-a.*) of the circuit. The power (the word is generally understood to mean average power, unless otherwise specified) has just been proven to be equal to $EI \cos \phi$. Its ratio to the volt-amperes (EI) is evidently $\cos \phi$; that is, power factor is equal to $\cos \phi$ for the case under discussion, namely, sine waves of both e.m.f. and current. It should be particularly noted that the power factor is not defined as $\cos \phi$ but it is *equal* to $\cos \phi$ in the case of, and only in the case of, sine waves of both e.m.f. and current. Under these conditions, ϕ is frequently called the power factor angle of the circuit.

The product of the e.m.f. and current is sometimes called the apparent power. The rating of an electrical machine is based upon its apparent power rather than upon its real power for the following reasons: Primarily the rating depends upon the temperature rise, which in turn depends upon the losses in the machine. The losses which affect the rating are of two kinds: 1st, those caused by hysteresis and eddy currents, and 2d, those caused by the flow of current through the windings of the machine. The hysteresis and eddy current losses are a function of the magnetic flux and consequently of the voltage generated; the resistance losses are a function of the current which flows. A given voltage and a given current will cause the same total loss and consequently produce the same temperature rise whether they are in phase with each other or not. Therefore, the product of these two, rather than the actual power developed, determines the temperature rise of the machine. On this account, alternators and transformers are commonly rated in kilo-volt-amperes, or *KVA*, instead of in kilowatts, or *KW*.

The product of the volt-amperes and the sine of the angle of phase difference ($EI \sin \phi$), is called the reactive power of the circuit, and $\sin \phi$ is called the reactive factor. From the relation $\sin^2 \phi + \cos^2 \phi = 1$, we get

$$\text{Reactive Factor} = \sqrt{1 - (\text{Power Factor})^2}. \qquad (188)$$

95. Power and Reactive Components of E.M.F.—In the vector diagram, Fig. 74, the e.m.f. E may, in accordance with the principles explained in Article 90, be considered as

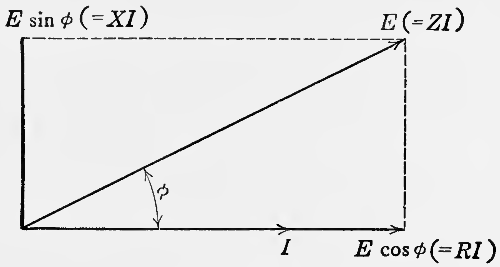

Fig. 74.

the vector sum of two e.m.f.'s ($E \cos \phi$) and ($E \sin \phi$), the former being in phase with I and the latter being 90° ahead of I. This resolution of E into two components is expressed mathematically by the equation:

$$E = \sqrt{(E \cos \phi)^2 + (E \sin \phi)^2}. \qquad (189)$$

The part, $E \cos \phi$, is called the power component of the e.m.f. because the product of it and the current gives the power developed in the circuit. The part, $E \sin \phi$, is called the reactive component of the e.m.f. It has already been proven (see equation 175) that the total e.m.f. required for an alternating current circuit containing resistance and reactance is $E = \sqrt{(RI)^2 + (XI)^2}$, where RI is in phase with I, and XI is 90° ahead of I. It is therefore evident that

$$E \cos \phi = RI, \tag{190}$$

and that

$$E \sin \phi = XI. \tag{191}$$

The power component of the e.m.f. is therefore that part of the e.m.f. required to overcome the resistance of the circuit, and the reactive component is that part required to overcome the reactance of the circuit.

From equations (190) and (180) we get the relation that

$$\cos \phi = RI/E = R/Z, \tag{192}$$

that is, the power factor of a circuit is equal to its resistance divided by its impedance.

From equations (191) and (180) we get the relation that

$$\sin \phi = XI/E = X/Z, \tag{193}$$

that is, the reactive factor of a circuit is equal to its reactance divided by its impedance. Dividing equation (193) by equation (192) gives

$$\tan \phi = X/R.$$

96. Power and Reactive Components of Current; Conductance, Susceptance and Admittance.—In the vector diagram, Fig. 75, the current is considered as the vector

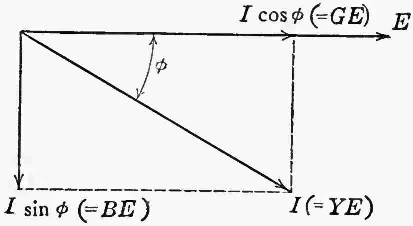

Fig. 75.

sum of two currents, ($I \cos \phi$) and ($I \sin \phi$), the former being in phase with E and the latter 90° behind E. The part $I \cos \phi$ is called the power component of the current and the part $I \sin \phi$ is called the reactive component of the current. The resultant of these two components is, of course, equal to I, or

$$I = \sqrt{(I \cos \phi)^2 + (I \sin \phi)^2}. \tag{194}$$

The values of $I \cos \phi$ and $I \sin \phi$ may be obtained in terms of E, R, and Z as follows:

$$I \cos \phi = (E/Z)\ (R/Z) = E(R/Z^2) = GE. \tag{195}$$

where G is a symbol for the expression (R/Z^2) and is called the conductance of the circuit.

$$I \sin \phi = (E/Z)\ (X/Z) = E(X/Z^2) = BE, \tag{196}$$

where B is a symbol for the expression (X/Z^2) and is called the susceptance of the circuit.

Substituting (GE) and (BE) for $(I \cos \phi)$ and $(I \sin \phi)$ in equation (194) we obtain

$$I = \sqrt{(GE)^2 + (BE)^2} = E\sqrt{G^2 + B^2}. \tag{197}$$

Since I has already been proven equal to E/Z, it follows that

$$\sqrt{G^2 + B^2} = 1/Z = Y, \tag{198}$$

where Y is a symbol for the reciprocal of Z and is called the admittance of the circuit.

From the relation that $G = R/Z^2$, it follows that

$$R = GZ^2 = G/Y^2 \tag{199}$$

and from the relation that $B = X/Z^2$, it follows that

$$X = BZ^2 = B/Y^2. \tag{200}$$

97. The Symbolic Method of Expressing Vector Quantities.—The manner of writing vector quantities, the calculation of combinations of vector quantities, and the interpretation of resulting vector quantities may be much simplified by the following simple convention: Let all vectors be considered as composed of two component vectors, one along any chosen axis of reference and the other at right angles to the chosen axis; and let the component at right angles to the axis of reference be designated by affixing to it the symbol j. Thus, the vector E in Fig. 74, would be written

$$E = E \cos \phi + jE \sin \phi. \tag{201}$$

and the vector I in Fig. 75, would be written

$$I = I \cos \phi - jI \sin \phi. \tag{202}$$

In the first case, the axis of reference is the vector I which would be written $I = I - j0$, and in the second case the axis of reference is the vector E, which would be written $E = E + j0$.

The algebraic sign in front of j indicates whether the component to which it is affixed is 90° ahead of or 90° behind the reference axis; the plus (+) sign is used when it it is ahead and the minus (−) sign when it is behind. In an algebraic sense the right angle component is multiplied by j and we may therefore say that multiplying a vector by j rotates it 90° from the position it would occupy if not multiplied by j. If a right angle vector, such as $jE \sin \phi$, be again multiplied by j, it becomes $j^2E \sin \phi$, and is rotated another 90°, or 180° altogether; but it is then equal to $-E \sin \phi$ and we may write the equation

$$j^2E \sin \phi = -E \sin \phi,$$

or

$$j^2 = -1,$$

or

$$j = \sqrt{-1}, \tag{203}$$

and thus j is seen to be mathematically equal to the so-called imaginary quantity, $\sqrt{-1}$.

Without giving the mathematical proof, it may be stated that when applied to vectors in one plane the symbol j obeys all the laws of ordinary algebra, while retaining the special significance already explained. It must be remembered, however, that an equation containing the symbol j

is a special vector equation, and its terms cannot be transposed from one side of the equality sign to the other. It should also be remembered that j^2 may always be replaced by -1.

The numerical value of any vector expressed in symbolic notation must always be found by extracting the square root of the sum of the squares of (the sum of the terms not containing j) and (the sum of the terms containing j).

98. Impedance and Admittance as Complex Numbers.— It has been shown that the e.m.f. required to send a current I through a resistance R and a reactance X is

$$E=\sqrt{(RI)^2+(XI)^2}.$$

Written in symbolic notation, this becomes

$$E=RI+jXI=(R+jX)I=ZI. \qquad (204)$$

Thus the symbolic expression for an impedance is $R+jX$, and expressed in this form it is called a complex number. The absolute value of the number is not the algebraic sum of its two parts but is equal to the square root of the sum of the squares of its two parts. The three factors, R, X, and Z are thus related to each other as the three sides of a right-angled triangle with Z as the hypothenuse. Such a triangle is called an impedance triangle. Note that R, X, and Z are not vectors, but are simple numerics, although the symbolic expression for Z is of the same mathematical form as that of a vector. The algebraic sign of an inductive reactance is positive and of a capacity reactance is negative. Thus, if $R=4$ ohms and $X=3$ ohms (the plus sign is understood), the e.m.f. required to send 20 amperes through the circuit will be

$$E=20(4+j3)=80+j60=100, \qquad (205)$$

and the fact that E leads I is indicated by the plus sign before $j60$. If $X=-3$, the e.m.f. will be

$$E=20(4-j3)=80-j60=100, \qquad (206)$$

and the fact that E lags behind I is indicated by the minus sign before $j60$.

In a similar manner, the admittance of a circuit is symbolically expressed as $G-jB$. The symbolic expression for equation (197) is

$$I=GE-jBE=(G-jB)E=YE, \qquad (207)$$

the minus sign being used here because, if E leads, then I lags, and if a positive sign is used to indicate a leading vector, than a negative sign must be used to indicate a lagging vector. The algebraic sign of an inductive susceptance is positive (same as inductive reactance) and of a capacity susceptance is negative (same as capacity reactance). Thus if $R=4$, $X=3$, $Z=\sqrt{4^2+3^2}=5$, then $G=4/25=0.16$ (see equation 195), and $B=3/25=0.12$ (see equation 196). The current that will flow under an impressed e.m.f. of $E=100$ will be

$$I=100\ (0.16-j0.12)=16-j12=20, \qquad (208)$$

and the fact that I lags is indicated by the minus sign before $j12$. If $X=-3$, then $B=-0.12$, and the current will be

$$I=100(0.16+j0.12)=16+j12=20, \qquad (209)$$

and the fact that I leads is indicated by the plus sign before $j12$. Thus equations (205) and (208) indicate the same phase relation between E and I, the former showing that E leads I, and latter showing that I lags behind E. Likewise equations (206) and (209) indicate that E lags behind I, or that I leads E.

99. Impedances in Series.—If two or more impedances are in series, the total e.m.f. is equal to the vector sum of the e.m.f's required for the separate impedances. Thus, if two impedances, $R_1+jX_1=Z_1$, and $R_2+jX_2=Z_2$, are in series, the e.m.f. on Z_1 is

$$E_1=I(R_1+jX_1)=IR_1+jIX_1,$$

and on Z_2 it is

$$E_2=I(R_2+jX_2)=IR_2+jIX_2;$$

but IR_1 will be in phase with IR_2 and thus may be added directly; and IX_1 is in phase with (or in opposition to) IX_2 and may also be added directly (algebraically). The total e.m.f. is therefore

$$E_0 = I(R_1 + R_2) + jI(X_1 + X_2) = (R_0 + jX_0)I = Z_0 I, \quad (210)$$

where R_0, X_0, and Z_0 are the equivalent resistance, reactance, and impedance, respectively, of the whole circuit. The equivalent impedance of a series circuit is therefore

$$Z_0 = (R_1 + R_2 + \ldots) + j(X_1 + X_2 + \ldots) = R_0 + jX_0, \quad (211)$$

that is, the equivalent resistance of a series circuit is the sum of the separate resistances, the equivalent reactance is the sum (algebraic) of the separate reactances, and the equivalent impedance is the square root of the sum of the squares of the equivalent resistance and equivalent reactance.

Note particularly that the total impedance is not the arithmetical sum of the separate impedances, but can be

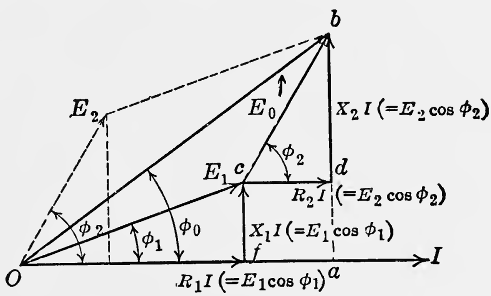

FIG. 76.

calculated only when the separate resistances and reactances are known. Note also that the total e.m.f. is not the arithmetical sum of the e.m.f's on the separate impedances, unless the separate e.m.f's are in phase with each other. A study of Fig. 76 should make this clear. If $X_1/R_1 = X_2/R_2$ then $\phi_1 = \phi_2$ and E_1 will be in phase with E_2 and $E_0 = E_1 + E_2$ but otherwise and generally E_0 must be calculated by equation (210).

100. Electromotive Forces in Series.—In most practical problems, the circuits consist not only of resistance and reactance, but also of power-consuming devices other than resistance, such as motors. Of course, a motor can be represented by equivalent values of resistance and reactance, the resistance being of such value that when multiplied by the current taken by the motor it gives the value of the active component, $E \cos \phi$, of the voltage impressed on the motor, and the reactance being of such a value that when multiplied by the current it gives the value of the reactive component, $E \sin \phi$. However, it is generally unnecessary to calculate these equivalent values of resistance and reactance. The data generally given for power consuming devices other than resistance, are the E.M.F., Current, and Power, or the E.M.F., Power, and Power Factor, or the E.M.F., Current, and Power Factor. In series circuits consisting of resistances, reactances, and power-consuming devices other than resistance, the total active E.M.F. will be $RI+E \cos \phi$, and the total reactive E.M.F. will be $XI+E \sin \phi$. For example, consider the circuit represented by Fig. 77, in which the impedance at the right represents

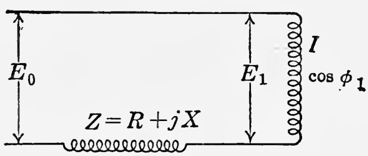

FIG. 77.

the load at the end of a single phase transmission line and Z is the impedance of the line itself. The equation for the voltage at the station is

$$E_0=(E_1 \cos \phi_1+RI)+j(E_1 \sin \phi_1+XI), \qquad (212)$$

or

$$E_0=\sqrt{(E_1 \cos \phi_1+RI)^2+(E_1 \sin \phi_1+XI)^2}, \qquad (213)$$

and the vector diagram is shown in Fig. 78. The triangle *Oab* is the one represented by equation (213).

Equation (213) gives the best form for the calculation of E_0 when E_1 is given; but it frequently happens that E_0 is given and E_1 must be found. This is a case of subtracting vectorially the impedance drop in Z from the e.m.f. E_0. An inspection of the diagram shows that if a right triangle be constructed on E_1 extended to the point c,

$$E_0 = (E_1 + RI\cos\phi_1 + XI\sin\phi_1) + j(XI\cos\phi_1 - RI\sin\phi_1). \quad (214)$$

The solution of this equation for E_1 gives

$$E_1 = \sqrt{E_0^2 - (XI\cos\phi_1 - RI\sin\phi_1)^2} - (RI\cos\phi_1 + XI\sin\phi_1). \quad (215)$$

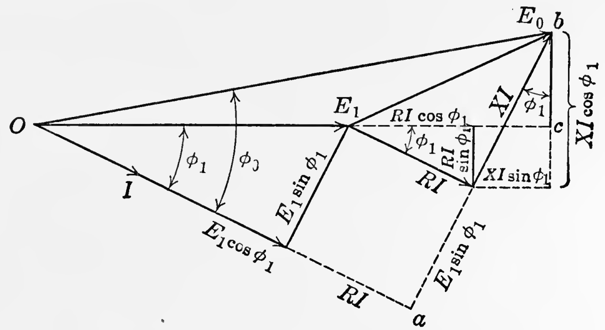

FIG. 78.

This equation (215) may also be readily deduced directly from the vector diagram.

The power factor at the station is

$$\cos\phi_0 = \frac{E_1\cos\phi_1 + RI}{E_0}. \quad (216)$$

101. Resonance in Series Circuits.—In a series circuit, Fig. 79*a* if R is the total resistance, L is the total inductance, and C is the total capacity, then the total impedance is (see equation 176)

$$Z = R + j\left(2\pi fL - \frac{1}{2\pi fC}\right) = R + j(X_1 - X_2), \quad (217)$$

where X_1 = the inductive reactance and X_2 = the capacity reactance. Suppose a certain circuit has $R = 3$, $X_1 = 16$, and $X_2 = 12$; the negative sign for X_2 is already written into equation (217). The impedance will be

$$Z = 3 + j(16 - 12) = 3 + j4 = 5, \qquad (218)$$

and if an e.m.f. of 100 volts is impressed on the circuit, the current will be

$$I = 100/5 = 20, \qquad (219)$$

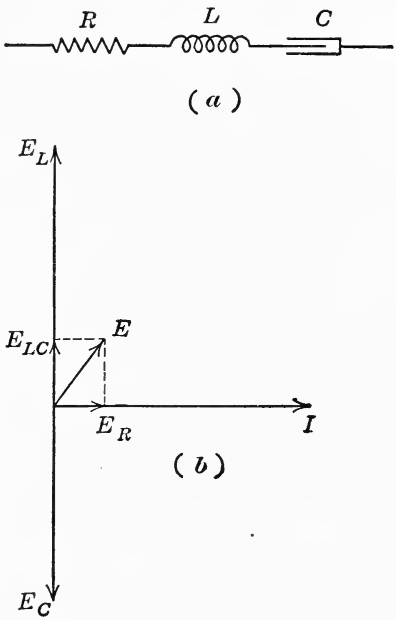

Fig. 79.

and the symbolic equation for the e.m.f. will be

$$E = 20[3 + j(16 - 12)] = 60 + j(320 - 240)$$
$$= 60 + j80 = 100. \qquad (220)$$

The vector diagram for the circuit is shown in Fig. 79 *b*. Note that the e.m.f. across the inductive reactance is 320 volts, and that across the capacity is 240, both of which are higher than that across the entire circuit. If the capacity reactance is increased to 16 ohms, then the impedance becomes $Z=3+j0=3$, and $I=100/3=33.3$ while the voltage across each reactance will rise to $33.3\times16=533$ volts. This condition of equality between the inductive reactance and the capacity reactance is known as resonance, and in general is to be guarded against, lest the voltage across the inductance or the capacity or both, shall rise to a dangerous value.

An inspection of equation (217) shows that with constant values of R, L, and C, the impedance varies with the frequency of the circuit and will be a minimum when $2\pi fL=1/2\pi fC$, or when

$$f=1/(2\pi\sqrt{LC}). \tag{221}$$

This value of frequency is known as the resonant frequency. It is worth noting that the current in such a circuit, and with it the potential across the inductance and across the capacity, may rise very rapidly as the frequency approaches its resonant value. This is especially so when R is small in comparison with X_1 and X_2 at resonant frequency.

102. Impedances in Parallel.—If two or more impedances are in parallel, the total current is equal to the vector sum of the separate currents. Thus, if two impedances, $R_1+jX_1=Z_1$, and $R_2+jX_2,=Z_2$, are in parallel, the current in Z_1 is $I_1=E(G_1-jB_1)$ and in Z_2 is $I_2=E(G_2-jB_2)$. But EG_1 and EG_2 are the two power components of the current, are therefore in phase with each other and may be added directly; EB_1 and EB_2 are the two reactive components of the current and these may also be added directly (algebraically). The total current is therefore,

$$I=E(G_1+G_2)-jE(B_1+B_2)=(G-jB)E=YE, \tag{222}$$

where G, B, and Y are the equivalent conductance, susceptance and admittance, respectively, of the whole circuit. The equivalent admittance may therefore be written as

$$Y=(G_1+G_2+\ldots)-j(B_1+B_2+\ldots)=G-jB, \tag{223}$$

and the equivalent impedance of the whole circuit is

$$Z=(G/Y^2)+j(B/Y^2)=R+jX, \tag{224}$$

where $G/Y^2 = R$, the equivalent resistance of the parallel circuit, and $B/Y^2 = X$, the equivalent reactance of the parallel circuit. Note especially that the resistance is not the reciprocal of conductance, nor reactance the reciprocal of susceptance, but that impedance is the reciprocal of admittance. Note also that the currents can not be added arithmetically, but must be added vectorially; and that the separate admittances cannot be added arithmetically but must be combined as indicated in equation (223).

103. Currents in Parallel.—In Fig. 80, let Z_1 and Z_2 represent two impedances or other loads in parallel, and let

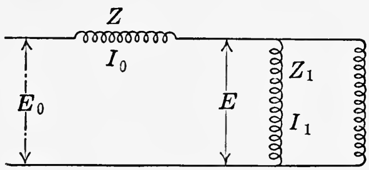

FIG. 80.

the currents be I_1 and I_2 respectively. It is not necessary to calculate the admittance of the combined circuit; the total current can be found most readily by calculating the active and reactive components of each current and combining them by the equation,

$$I_0 = (I_1 \cos \phi_1 + I_2 \cos \phi_2) - j(I_1 \sin \phi_1 + I_2 \sin \phi_2), \qquad (225)$$

or

$$I_0 = \sqrt{(I_1 \cos \phi_1 + I_2 \cos \phi_2)^2 + (I_1 \sin \phi_1 + I_2 \sin \phi_2)^2}. \qquad (226)$$

The resultant power factor is

$$\cos \phi_0 = (I_1 \cos \phi_1 + I_2 \cos \phi_2)/I_0. \qquad (227)$$

The total current lags if $(I_1 \sin \phi_1 + I_2 \sin \phi_2)$ is positive and leads if it is negative. Fig. 81 (a) shows the vector diagram for a case where ϕ_1 is positive and ϕ_2 is negative. Fig. 81 (b) shows the diagram for a case where ϕ_1 and ϕ_2 are both positive.

104. Mixed Circuits.—In calculations on combined series and parallel circuits, great caution must be exercised to be sure that in adding e.m.f's or currents, the different components are expressed in terms of the same reference

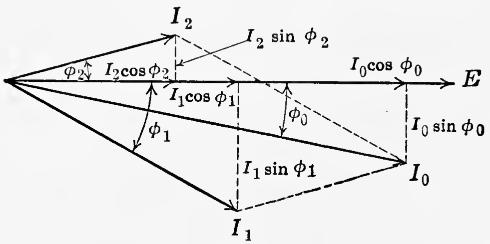

Fig. 81*a*.

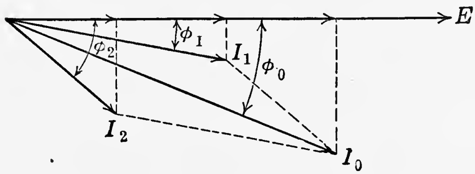

Fig. 81*b*.

vector. It is frequently necessary to change the vector with reference to which a current or e.m.f. is expressed. A careful study of the following problem should give an understanding of the application of these principles. · Let Fig. 82

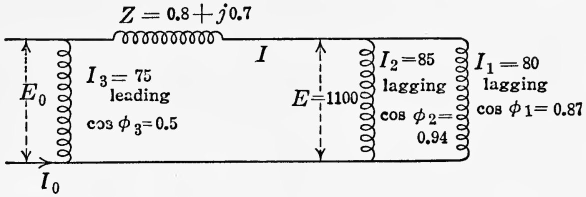

Fig. 82.

represent a circuit in which it is desired to find the values of I, I_0, E_0, and $\cos \phi_0$. To find I it is first necessary to resolve I_1 and I_2 into components with reference to E.

From the data given, we find $\sin\phi_1 = 0.493$ and $\sin\phi_2 = 0.3412$. Therefore

$$I_1 = 80(0.87 - j0.493) = 69.6 - j39.44 \qquad (a)$$

$$I_2 = 85(0.94 - j0.3412) = 79.9 - j29.00 \qquad (b)$$

$$\text{and } I = 149.5 - j68.44 = 164.4. \qquad (c)$$

The components of I in equation (c) are with reference the voltage E. To find E_0 we must add to E, the drop in Z due to I; to do this we must get the components of E with respect to I, and add RI and XI to the active and reactive components, respectively. Letting ϕ represent the angle between E and I, we get $\cos\phi = 149.5/164.4 = 0.909$ and $\sin\phi = 68.44/164.4 = 0.4164$.

Therefore, with respect to I

$$E = 1100 \quad (0.909 + j0.4164) = 1000 + j458 \qquad (d)$$

$$ZI = 164.4\ (0.8 + j0.7) = 132 + j115 \qquad (e)$$

$$\text{and } E_0 = 1132 + j573 = 1269. \qquad (f)$$

Note that since the sign of j is negative in equation (c) it must be positive in equation (d). To get I_0, we must add I_3 to I; from the data given we get $\sin\phi_3 = -0.866$; therefore

$$I_3 = 75(0.5 + j0.866) = 37.5 + j64.95, \qquad (g)$$

with respect to E_0. Equation (c) gives I with respect to E; therefore the components of I in equation (c) cannot be added to those of I_3 in equation (g); the components of I must be found with respect to E_0. Note that equation (f) gives E_0 with respect to I; therefore, letting ϕ' represent the angle between E_0 and I, we get

$$\cos\phi' = 1132/1269 = 0.892 \text{ and } \sin\phi' = 573/1269 = 0.4515.$$

Therefore, with respect to E_0

$$I = 164.4\ (0.892 - j0.4515) = 146.6 - j74.23 \qquad (h)$$

$$I_3 = 75.0\ (0.5 + j0.866) = 37.5 + j68.95 \qquad (i)$$

$$\text{and } I_0 = 184.1 - j\ 5.28 = 184.2 \qquad (j)$$

The power factor of the combined circuit is 184.1/184.2 = 1.0 (practically). The total current lags the total voltage by a very small angle.

CHAPTER VII

NON-HARMONIC WAVES

105. Composition of Non-harmonic Waves.—In Chapter VI our attention was confined entirely to the phenomena of sine waves and the methods of dealing with them. Although in many practical engineering problems the waves are close approximations to pure sine waves, it is also true that there are many cases where the waves depart so far from sinusoidal form as to make a knowledge of the mathematics as well as of the physical phenomena of non-harmonic waves a necessity to the engineer.

The fundamental proposition in dealing with non-harmonic waves is that all such waves may be represented mathematically by a series of harmonic terms, called Fourier's Series. This series is of the form

$$e = E1 \sin(\omega t + \theta 1) + E2 \sin(2\omega t + \theta 2) + E3 \sin(3\omega t + \theta 3). \quad (228)$$

NOTE.—In this chapter, a numeric following a letter is to be interpreted as a subscript, not as a multiplier.

Each term in the above series evidently represents a sine wave. The first term is called the first harmonic, or fundamental; its maximum value is $E1$ and the angular velocity of the vector $E1$ is ω; the phase relation of the fundamental to the resultant wave is represented by $\theta 1$, the origin being taken at the zero value of the original or resultant wave, when $\omega t = 0$. The second term is called the second harmonic; its maximum value is $E2$, its angular velocity is twice that of the fundamental, or 2ω, and its phase relation to the resultant wave is $\theta 2$. Similarly, the third term is called the third harmonic and so on. In plotting a wave and its components it must be noted that the scale of angles for any harmonic, say the nth, is n times the scale of the fundamental, since there are n complete harmonic cycles to one fundamental cycle. $\theta 2$ and $\theta 3$ as used in the above equation are expressed in terms of their respective harmonic scales; for example, if $\theta 2$ is 30°, its measure on the scale of the fundamental wave would be 15°.

The algebraic sum of the values of all the harmonics and the fundamental at any instant is equal to the corresponding instantaneous value of the original or resultant wave.

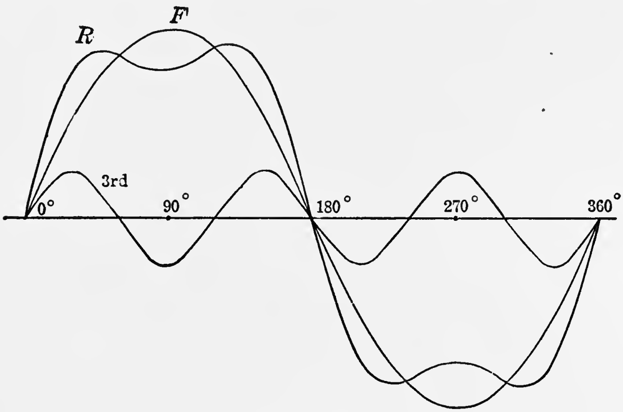

Fig. 83.

In Fig. 83 is shown a wave (R) whose equation is $e=160 \sin \omega t+40 \sin 3\omega t$; it would be said to contain a 25 per cent third harmonic. In Fig. 84, the equation for R is $e=160 \sin \omega t+40 \sin 2\omega t$; it contains a 25 per cent second harmonic. It will be noticed that in Fig. 83 the positive and negative halves of wave R are alike, while in Fig. 84 they are unlike. A little study will disclose the fact that any even harmonic will produce dissimilar positive and negative halves in the resultant wave; for, the values of the fundamental wave at any two points 180° apart will be equal and opposite in sign, while the values of any even harmonic at two points 180° apart on the fundamental scale will be equal and alike in sign; the sum of the fundamental and the harmonic will therefore be different at points 180° apart (except at the points where they pass through zero) and the two halves will not be similar. In all ordinary electrical machines, the positive and negative waves are similar, and it follows that even harmonics are not generally to be found.

In actual waves, the 3d and 5th harmonics are generally the ones of most importance, although harmonics of a higher order are by no means uncommon and are sometimes of

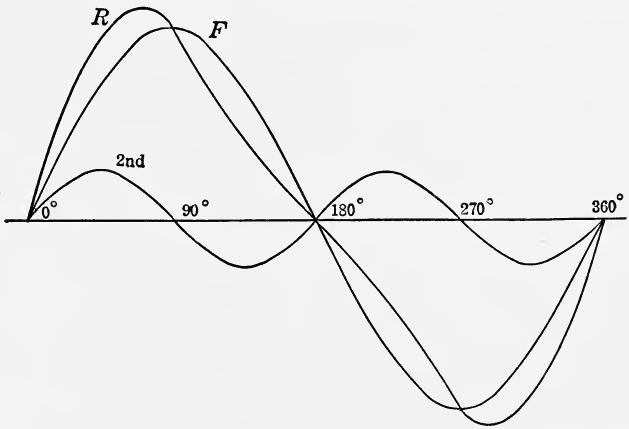

Fig. 84.

considerable magnitude. Fig. 85 shows a wave containing a 25 per cent 3d harmonic and a $16\frac{2}{3}$ per cent 5th harmonic.

106. The Oscillograph.—The most common method of determining the actual form of an alternating e.m.f. or current wave is by means of an oscillograph. This instrument is essentially a mirror galvanometer of very high natural frequency, so as to be able to follow accurately the variations in the wave form. For voltage waves the galvanometer coil is connected in series with a high non-inductive resistance, as in the case of a voltmeter; for current waves, the coil is shunted by a low non-inductive resistance. If a beam of light is thrown onto the mirror when it is oscillating due to an alternating current in the coil, the reflected beam will also oscillate and if a photographic plate or film be moved with sufficient speed across the path of this reflected beam and at right angles to its direction of motion, the position of the beam will be photographed in all its positions with respect to a time axis and this will constitute a photograph of the wave form of the current. If, instead of the plate or film, a second mirror be placed in the path of the reflected beam and be oscillated about an axis at right angles to the

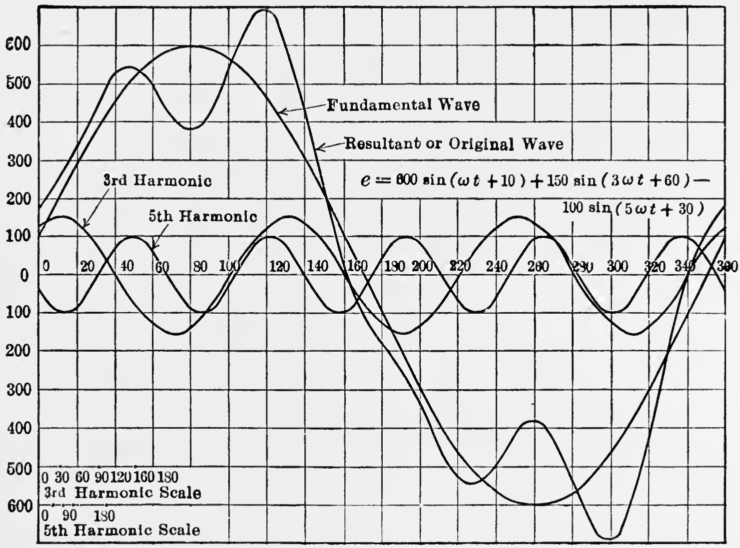

FIG. 85.

motion of the beam and in synchronism with that motion, the second reflection will show the positions of the first reflection with respect to a time axis and may be viewed upon a suitable screen as a standing wave.

107. Analysis of a Non-harmonic Wave.—It is frequently desirable to know the magnitudes and phase relations of the harmonic components of a non-harmonic wave. Having obtained a photograph or a tracing of the original wave, the analysis involves the determination of the maximum values of the existing harmonics, the magnitude and algebraic signs of the phase angles of the harmonics and the algebraic signs of the harmonic terms as a whole. That is, in the equation

$$e = E1 \sin(\omega t \pm \theta 1) \pm E3 \sin(3\omega t \pm \theta 3) \pm E5 \sin(5\omega t \pm \theta 5) \pm \text{etc.} \quad (229)$$

the values of $E1$, $E3$, $E5$, $\theta 1$, $\theta 3$, $\theta 5$, etc., must be found and also whether the algebraic signs are plus or minus. Several analytical and two or three mechanical methods have been devised for this purpose. The method to be explained here is one of the simplest analytical methods and at the same time affords a good insight into the make-up of a non-harmonic wave.

The method is based on the following four propositions:

(1) The algebraic sum of any n equally spaced ordinates of a sine wave is zero when these ordinates are so spaced as to divide k complete wave lengths into n equal parts and k is *not* a multiple of n. For example, the algebraic sum of 3 ordinates which divide 1, 5, or 7 wave lengths into 3 equal parts, will be zero. The mathematical proof of this proposition will not be given here, but it is based on the general principle that the resultant of any number of equal vectors, equally spaced over an angle of 360°, is zero. These may be considered as independent vectors, or as different positions of the same vector revolving at uniform angular velocity. The vertical projections of such a system of vector positions, when plotted as ordinates against the corresponding angular positions of the vector as abscissæ, become the ordinates of a sine wave, and the sum of any one set of such ordinates, corresponding to any one set of vector positions, equally spaced over 360°, is zero. To illustrate, in Fig. 86 the sum of the ordinates $u0$, $u120$ and $u240$ would be zero because these ordinates are the projections of 3 vectors (or 3 vector positions) equally spaced over 360°. Similarly $z0 + z120 + z240 = 0$ for the same reason. Also, $u0 + u72 + u144 + u216 + u288 = 0$ because these ordinates are the projections of 5 vector positions equally spaced over 360°; and for the same reason, $t0 + t72 + t144 + t216 + t288 = 0$.

(2) The algebraic sum of any n equally spaced ordinate of a sine wave is equal to n times the value of the ordinates at any one of the points, when these ordinates divide k wave lengths into n equal parts, and k *is* a multiple of n. That is, the ordinates will be all equal and of the same sign. For example, 3 ordinates which divide 3, 6, or 9 wave lengths into 3 equal parts will have a sum equal to 3 times the value of any one of the ordinates. In other words, if the number of equally spaced ordinates is divisible a whole number of times into the number of wave lengths used, the ordinates will be one or more whole wave lengths apart and their sum will be n times the value of any one of the ordinates. For example in Fig. 86,

$$t0+t120+t240=3t0;$$

also,

$$z0+z72+z144+z216+z288=5z0.$$

(3) The maximum value (E) of a sine wave is equal to the square root of the sum of the squares of any two ordinates (e_1) and (e_2) 90° apart. For if

$$e_1=E \sin \theta \text{ and } e_2=E \sin (\theta \pm 90)=E \cos \theta,$$

then

$$e_1^2+e_2^2=E^2 (\sin^2\theta+\cos^2\theta)=E^2.$$

(4) The tangent of the angle between any ordinate (e_1) of a sine wave and the nearest zero point on the wave is equal to e_1/e_2 where e_2 is the ordinate of the wave, 90° to the right of e_1. For $e_1=E \sin \theta$ and $e_2=E \cos \theta$ and therefore $e_1/e_2=E \sin \theta/E \cos \theta=\tan \theta$. If e_1 and e_2 are both positive, the wave passes through zero to the left of e_1 and upward; if e_1 and e_2 are both negative, the wave passes through zero to the left of e_1 and downward; if e_1 is positive and e_2 is negative the wave passes through zero to the right of e_1 and downward; if e_1 is negative and e_2 is positive, the wave passes through zero to the right of e_1 and upward. The algebraic sign of the numerical expression for the wave is the same as the algebraic sign of e_2. We have then the following four cases: if e_1 and e_2 are positive, the wave is

represented by $e = +E \sin(\omega t + \theta)$; if e_1 and e_2 are negative, $e = -E \sin(\omega t + \theta)$; if e_1 is positive and e_2 is negative, $e = -E \sin(\omega t - \theta)$; if e_1 is negative and e_2 is positive, $e = +E \sin(\omega t - \theta)$.

These facts regarding the relation of the signs of the ordinates and the position of the wave and the signs in the term expressing it, can be most easily established by inspection, and the student should make such an inspection of the accompanying curves.

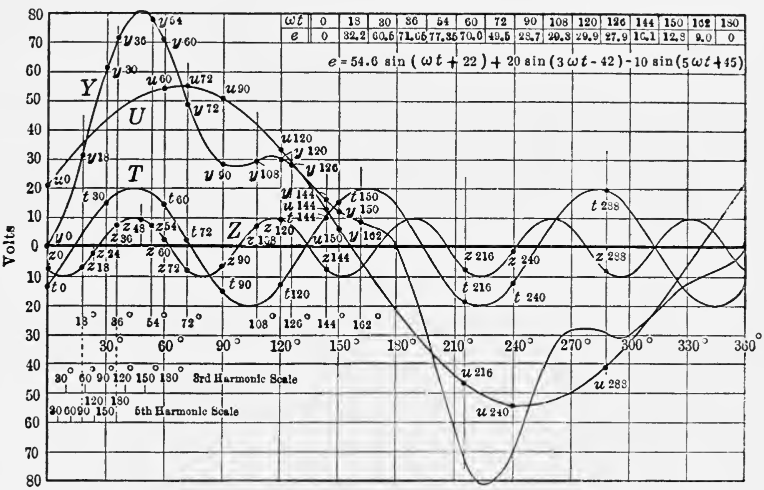

FIG. 86.

Before proceeding with the application of these principles to the analysis of a wave, the following fact should also be carefully noted: All ordinates 180° apart on a sine wave are equal in value but opposite in sign; therefore if n ordinates be equally spaced over a whole number of half wave lengths, their sum will be the same as if spaced over an equal number of whole wave lengths, provided the algebraic signs of the alternate ordinates be reversed, beginning with the second ordinate. For example, referring to Fig. 86, $u0 + u120 + u240 = u0 - u60 + u120 = 0$; or, again $z0 + z72 + z144 + z216 + z288 = z0 - z36 + z72 - z108 + z144 = 5z0$; and $t0 + t120$

$+t240 = t0 - t60 + t120 = 3t0$. The significance of this fact is that only one-half of a complex wave need be drawn for its analysis.

108. Example.—On Fig. 86, let the curve Y be a non-harmonic wave which has been found by the oscillograph or other method and is to be analyzed. Briefly, the process is to determine the harmonics, one after another, then add them together and subtract their sum from the original wave to determine the fundamental.

The formulæ given below are general, but the numerical values for the accompanying wave are added for illustration.

First, take three ordinates, $y0$, $y60$, and $y120$. The value of the third harmonic at 0° is

$$t0 = (y0 - y60 + y120)/3 = -13.4. \qquad (230)$$

(Note that $y0 = 0$, since the first ordinate is taken at the origin.) To prove equation (230), note that whatever may be the values of the fundamental and other harmonics,

$$y0 = u0 + t0 + z0,$$

also,

$$y60 = u60 + t60 + z60$$

and

$$y120 = u120 + t120 + z120,$$

attention being given to algebraic signs in each case. But from propositions (1) and (2)

$$u0 - u60 + u120 = z0 - z60 + z120 = 0,$$

and

$$t0 - t60 + t120 = 3t0.$$

Therefore,

$$y0 - y60 + y120 = 3t0.$$

(If the wave contained a 9th harmonic, or other odd multiple of 3, it also would be included in the sum $y0 - y60 + y120$. To determine the presence or absence of a 9th, for example, find the sum of 9 ordinates ($y0 - y20 +$ etc., up to $y160$), and 1/9 of this sum is $n0$, the value of the 9th harmonic at 0°. The value of $t0$ is then found by subtracting $n0$ from $(y0-y60+y120)/3$. The wave shown contains only a 3d and 5th.

Second, take three ordinates, $y30$, $y90$ and $y150$. These ordinates are respectively 90° (on the 3d harmonic scale) to the right of $y0$, $y60$ and $y120$. The value of the 3d harmonic at 30° is, by the same reasoning as above,

$$t30=(y30-y90+y150)/3=14.9. \tag{231}$$

(If the wave contained a 9th harmonic, this value would also have to be corrected in the same manner as for $t0$, namely by subtracting $(y10-y30+\text{etc., to } y170)/9$.)

Third, by proposition (3), the maximum value of the 3d harmonic is

$$E3=\sqrt{(t0)^2+(t30)^2}=20. \tag{232}$$

Fourth, by proposition (4), the tangent of the angle ($\theta 3$) between the origin and the point where the 3d harmonic crosses the X-axis is

$$\tan \theta 3=t0/t30=-13.4/14.9=-0.9. \tag{233}$$

The angle is therefore $-42°$ ($-14°$ on the fundamental scale) and the crossing is upward 14° to the right of the origin, since $t0$ is negative and $t30$ is positive. Also, since $t30$ is positive, the entire third harmonic term takes the positive sign; that is, it is $+20 \sin (3\omega t-42)$. This is drawn in as wave T on the diagram.

Fifth, take five ordinates, $y0$, $y36$, $y72$, $y108$, and $y144$. The value of the fifth harmonic at 0° is (if no multiple of the 5th is present)

$$z0=(-y36+y72-y108+y144)/5=-7.07. \tag{234}$$

Sixth, take five ordinates, $y18$, $y54$, $y90$, $y126$, and $y162$. The value of the fifth harmonic at 18° (which on the 5th harmonic scale is 90° to the right of 0) is

$$z18=(y18-y54+y90-y126+y162)/5=-7.07. \tag{235}$$

Seventh, find the maximum value of the fifth harmonic, as

$$E5 = \sqrt{(z0)^2 + (z18)^2} = 10. \qquad (236)$$

Eighth, find the position of the wave;

$$\tan \theta 5 = z0/z18 = -7.07/-7.07 = 1. \qquad (237)$$

The angle is therefore 45° (9° on the fundamental scale) and since $z0$ and $z18$ are both negative, the nearest crossing is downward 9° to the left of the origin. Since $z18$ is negative, the expression for the fifth harmonic is $-10 \sin (5\omega t + 45)$. This is wave Z on the diagram.

Ninth, find the value of the fundamental at 0° as

$$u0 = y0 - t0 - z0 = 20.45. \qquad (238)$$

Tenth, find the value of the fundamental at 90° as

$$u90 = y90 - t90 - z90 = y90 + t30 - z18 = 50.7. \qquad (239)$$

Note that $t90 = -t30$, since they are 180° apart on the 3d harmonic scale; also that $z90 = z18$, since they are 360° apart on the 5th harmonic scale.

Eleventh, find the maximum value of the fundamental as

$$E1 = \sqrt{(u0)^2 + u90^2} = 54.6. \qquad (240)$$

Twelfth, find the position of the wave;

$$\tan \theta 1 = u0/u90 = 20.45/50.6 = 0.404. \qquad (241)$$

The angle is therefore 22° and the wave crosses the X-axis upward 22° to the left of 0. This is drawn in as wave U on the diagram.

The equation for the original wave (Y) may now be written as

$$e = 54.6 \sin (\omega t + 22) + 20 \sin (3\omega t - 42) - 10 \sin (5\omega t + 45). \qquad (242)$$

If harmonics of higher order than those mentioned here are present, the process of determining them is the same as given above that is, for a 7th harmonic, 7 ordinates would be used, 25.7° apart, and so on. In each case, where necessary, correction must be made for the higher multiples of any harmonic, as explained in connection with the 3d harmonic and these higher multiples must also be completely determined and included in the final equation for the wave.

109. Effective Value of a Non-harmonic Wave.—The effective value of an alternating wave has already been shown to be equal to the square root of the mean square of the instantaneous values taken over any whole number of cycles. If the equation for a non-harmonic wave is

$$e = E1 \sin(\omega t + \theta 1) + E3 \sin(3\omega + \theta 3) + E5 \sin(5\omega t + \theta 5), \quad (243)$$

the equation for the curve of squared values is

$$\begin{aligned} e^2 = E1^2 \sin^2(\omega t + \theta 1) + E3^2 \sin^2(3\omega + \theta 3) + E5^2 \sin^2(5\omega + \theta 5) \\ + 2E1 \sin(\omega t + \theta 1)\ E3 \sin(3\omega t + \theta 3) \\ + 2E3 \sin(3\omega t + \theta 3)\ E5 \sin(5\omega t + \theta 5) \\ + 2E1 \sin(\omega t + \theta 1)\ E5 \sin(5\omega t + \theta 5). \quad (244) \end{aligned}$$

The mean value of e^2 is equal to the sum of the average values of each term on the right-hand side of equation (244), each average being taken over one complete cycle of fundamental frequency. Each of the first three terms of equation (244) may be expanded into the form

$$\frac{E_m^2}{2}[1 - \cos(2mx + 2\theta_m)],$$

and each of the last three terms may be expanded in the form

$$E_m E_n[\cos(x(m-n) + \theta_m - \theta_n) - \cos(x(m+n) + \theta_m + \theta_n)],$$

where $x = \omega t$ and m and n are the order of the respective harmonics. Using these expansions, multiplying the equation by dx, integrating between the limits of o and 2π, and dividing the result by 2π, we get

$$(\text{average } e^2 = E1^2 + E3^2 + E5^2)/2, \quad (245)$$

and the effective value of e is

$$E=\sqrt{\frac{E1^2}{2}+\frac{E3^2}{2}+\frac{E5^2}{2}}=.707\sqrt{E1^2+E3^2+E5^2}. \quad (246)$$

That is, the effective value of a non-harmonic e.m.f. or current is equal to the square root of the sum of the squares of the effective values of its component sine waves, or equal to .707 times the square root of the sum of the squares of the maximum values of its component sine waves.

110. Peak Factor.—The ratio of the maximum value of an e.m.f. or current to its effective value is defined as its peak factor, or crest factor. The peak factor of a sine wave is $\sqrt{2}=1.414$. The peak factor of commercial waves may be greater or less than this. When it is greater the wave is called a peaked wave and the insulation of the circuit is subjected to a greater strain than with a sine wave. When the peak factor is less than 1.414 the wave is called a flat-topped wave.

111. Average Value of a Non-harmonic Wave.—The average value is found by multiplying equation (243) by $d(\omega t)$ and integrating between the limits o and π and dividing the result by π. This gives

$$E\text{ (average)}=.637\left(E1\cos\theta1+\frac{E3}{3}\cos\theta3+\frac{E5}{5}\cos\theta5\right), \quad (247)$$

where $.637=2/\pi$.

112. Form Factor.—The ratio of the effective value of an e.m.f. or current to its average value is defined as the form factor of the wave. For a sine wave the form factor is $.707/.637=1.11$.

113. Power in Circuits Carrying Non-harmonic Waves.—Let equation (248) be the e.m.f. equation for a circuit and (249) the corresponding current equation:

$$e=E1\sin(\omega t+\theta1)+E3\sin(3\omega t+\theta3)$$
$$+E5\sin(5\omega t+\theta5). \quad (248)$$

$$i=I1\sin(\omega t+\theta1')+I3\sin(3\omega t+\theta3')$$
$$+E5\sin(5\omega t+\theta5'). \quad (249)$$

The instantaneous power in the circuit will be the product of equations (248) and (249). This product is

$$
\begin{aligned}
p = & E1I1 \sin (\omega t+\theta 1) \sin (\omega t+\theta 1') \\
& +E3I3 \sin (3\omega t+\theta 3) \sin (3\omega t+\theta 3') \\
& +E5I5 \sin (5\omega t+\theta 5) \sin (5\omega t+\theta 5') \\
& +E1I3 \sin (\omega t+\theta 1) \sin (3\omega t+\theta 3') \\
& +E3I1 \sin (\omega t+\theta 1') \sin (3\omega t+\theta 3) \\
& +E5I1 \sin (\omega t+\theta 1') \sin (5\omega t+\theta 5) \\
& +E1I5 \sin (\omega t+\theta 1) \sin (5\omega t+\theta 5') \\
& +E3I5 \sin (3\omega t+\theta 3) \sin (5\omega t+\theta 5') \\
& +E5I3 \sin (5\omega t+\theta 5) \sin (3\omega t+\theta 3'). \qquad (250)
\end{aligned}
$$

The average power is found by integrating this product multiplied by $d(\omega t)$ between the limits o and 2π and dividing the result by 2π; that is, by finding the sum of the average values of each term over a complete fundamental cycle. By the same process as that used in Article (109) it will be found that the average value of each of the last six terms of the equation is zero. By the same process as that used in Article (93) the average value of the first three terms will be found to be

$$P=\frac{E1I1}{2} \cos \phi 1+\frac{E3I3}{2} \cos \phi 3+\frac{E5I5}{2} \cos \phi 5, \qquad (251)$$

where $\phi 1=\theta 1-\theta 1'$ and $\phi 3=\theta 3-\theta 3'$ and $\phi 5=\theta 5-\theta 5'$. Note that $\phi 1$, $\phi 3$, and $\phi 5$ are, respectively, the phase angles between the fundamental e.m.f. and current, the 3d harmonic e.m.f. and current, and the 5th harmonic e.m.f. and current. Note also that if $E1$, $E3$, $E5$, $I1$, $I3$ and $I5$ be taken to represent effective values instead of maximum values, the equation for power becomes

$$P=E1I1 \cos \phi 1+E3I3 \cos \phi 3+E5I5 \cos \phi 5. \qquad (252)$$

From this it is seen that the total average power is equal to the sum of the powers which would be produced by each component e.m.f. wave and its corresponding current wave if they were acting independently of the other harmonics.

114. Equivalent Sine Waves and Phase Difference.—Equation (246) gives the effective value for the non-harmonic wave represented by equation (243). These equations hold, of course, for either an e.m.f. wave or a current wave. If E be the effective value of a non-harmonic e.m.f., the equation for the sine wave which would be exactly equivalent to the non-harmonic wave is

$$e = \sqrt{2}E \sin \omega t. \tag{253}$$

If I be the effective value of the corresponding current, the equation for the equivalent sine wave of current is

$$i = \sqrt{2}I \sin (\omega t - \phi). \tag{254}$$

The power in the circuit is represented by equation (252) and the power factor is P/EI; the angle ϕ in equation (254) is therefore the angle whose cosine is P/EI and is called the equivalent angle of phase difference. It bears no definite relation to any of the angles connected with the non-harmonic waves, but is the angle which would have to exist between two sine waves of the same effective values as the given non-harmonic waves, in order that the power shall be the same as that given by the non-harmonic waves.

115. Inductive Reactance with Non-harmonic Wave.—The inductive reactance due to an inductance L has been shown to be equal to $2\pi fL$. This expression applies to each harmonic of a complex wave, when f is the frequency of that harmonic. If f is the fundamental frequency, the fundamental reactance is $2\pi fL$, the reactance against a 3d harmonic e.m.f. will be $6\pi fL$, and against a 5th harmonic, it will be $10\pi fL$. Or, if the reactance to the fundamental be called $x_1 = (2\pi fL)$, then the 3d harmonic reactance is $3x_1$ and the 5th harmonic reactance is $5x_1$. These facts will be evident if it is recalled that the e.m.f. to overcome inductance is $e = Ldi/dt$; then if $i = I \sin 2\pi ft$, $e = (2\pi fL)I \cos 2\pi ft$; if $i = I \sin 6\pi ft$, $e = (6\pi fL)I \cos 6\pi ft$.

It follows from this, that if a circuit contains a resistance r and a reactance x_1 at fundamental frequency, the fundamental current will be $I1 = E1/(r + jx_1)$, the 3d harmonic current will be $I3 = E3/(r + j3x_1)$ and the 5th harmonic current will be $I5 = E5/(r + j5x_1)$.

The equation of the resultant effective current is therefore

$$I = .707\sqrt{\frac{(E1)^2}{r^2 + x_1^2} + \frac{(E3)^2}{r^2 + 9x_1^2} + \frac{(E5)^2}{r^2 + 25x_1^2}}. \qquad (255)$$

(See equation (246).)

It is evident that the ratio of $I3$ to $I1$ will be less than the ratio of $E3$ and $E1$ and the ratio of $I5$ to $I1$ will be still smaller. That is, the harmonic currents will be smaller in proportion to the fundamental current than are the corresponding harmonic e.m.f.'s to the fundamental e.m.f. Inductive reactance is therefore said to dampen out the harmonics in the current wave. The amount of damping will of course depend on the relative values of r and x_1. If x_1 is negligibly small there will be no damping; if r is negligibly small, the ration of the 3d harmonic current to its fundamental will be 1/3 of the ratio of 3d harmonic e.m.f. to its fundamental.

Furthermore, the phase difference between the component e.m.f. and current waves will increase with the order of the harmonic; the tangent of the angle of lag for the fundamental will be x_1/r; for the 3d harmonic, it will be $3x_1/r$; and so on. Therefore both on account of the damping and on account of the change in phase relations, the shape of the current wave will be different from that of the e.m.f. wave.

A method frequently used for determining the reactance (and sometimes the inductance) of a circuit is to impress a known e.m.f. on the circuit and measure the current. This is called the impedance method. The ratio of the e.m.f. to the current is the impedance (z) of the circuit. Then, the resistance of the circuit having been measured, the reactance (x) is $\sqrt{z^2 - r^2}$, and the inductance L is $x/2\pi f$, where f is the frequency of the e.m.f. The use of this expression assumes the e.m.f. wave to be sinusoidal, and it should be noted that this value of L will not be correct unless the e.m.f. wave is a sine wave, and also noted that the reactance of a given coil or circuit as determined for one wave form will not be the same for any other wave form. This can be most easily shown by an example: Suppose a 60-cycle sine wave of 100 volts (effective) be impressed on a circuit of negligible resistance and an inductance of .01061 henry. The reactance at 60 cycles will be $2\pi \times 60 \times .01061 = 377 \times .01061 =$ 4 ohms. The effective current will therefore be $100/4 = 25$ amperes. Now suppose the e.m.f. wave is non-harmonic with a fundamental of 95.4 volts (effective) and a 3d harmonic of 30 volts (effective). The effective value of the wave will still be 100 volts, since $\sqrt{95.4^2 + 30^2} = 100$. The fundamental current will be $95.4/4 = 23.85$ amperes (effective) and the 3d harmonic current will be $30/(3 \times 4) = 2.5$ amperes (effective). The total effective current will be $\sqrt{23.85^2 + 2.5^2} = 23.98$ amperes. The reactance for this wave form is therefore $100/23.98 = 4.17$ ohms, and if the inductance were calculated from this value on the assumption that the wave is a sine wave, the result would be $4.17/377 = .01105$, which is in error by over 4 per cent.

If the components of the e.m.f. wave are known, the value of x_1 could, of course, be calculated from equation (255) but this is obviously a rather tedious solution and is impractical. The points of this discussion are (1st) that it is unsafe to assume that a wave is sinusoidal in calculating the value of inductance, and (2d) that the reactance of a given circuit varies with the wave form of impressed e.m.f.

116. Capacity Reactance with Non-harmonic Waves.— The effect of capacity reactance with non-harmonic waves is just opposite to that of inductive reactance. Capacity reactance has been shown to be equal to $1/2\pi fC$. This expression applies to each harmonic of a complex wave, when f is the frequency of that harmonic. If f is the fundamental frequency, the expression as given is the fundamental reactance, the reactance to a 3d harmonic e.m.f. is $1/6\pi fC$, and to a 5th harmonic it is $1/10\pi fC$. If the fundamental reactance be called x_1, the 3d harmonic reactance is $x_1/3$ and the 5th harmonic reactance is $x_1/5$. The fundamental current will be $E1/x_1$; the 3d harmonic current will be $3E3/x_1$, and the 5th harmonic current will be $5E5/x_1$. From this it is evident that the harmonic currents are *increased* in their ratio to the fundamental, instead of diminished as in the case of inductance. The wave form of the current will therefore be different from that of the e.m.f. and will differ more widely from a sine wave than does the e.m.f. wave.

The relation $I=2\pi fCE$ may be used to determine the capacity of a condenser, if the impressed e.m.f. wave is sinusoidal, but will give incorrect results if the e.m.f. wave is non-harmonic. The formula for the total effective current is

$$I=\sqrt{(2\pi fCE1)^2+(6\pi fCE3)^2+(10\pi fCE5)^2}, \qquad (256)$$

and this formula may be used for determining C if the components of the e.m.f. wave are known.

If a resistance r, and inductive reactance x_1 at fundamental frequency and a capacity reactance x_2 at fundamental frequency are in series the fundamental current is

$$I1=E1/r+j(x_1-x_2), \qquad (257)$$

the 3d harmonic current is

$$I3=E3/r+j(3x_1-x_2/3), \qquad (258)$$

and the 5th harmonic current is

$$I5=E5/r+j(5x_1-x_2/5). \qquad (259)$$

From these equations, it will be seen that the phase relations between the current components are different from the phase relations between the e.m.f. components, that the ratios of the current components are different from the ratios of the e.m.f. components, and therefore the current wave form will be different from the e.m.f. wave form.

CHAPTER VIII

POLYPHASE CURRENTS

117. Kirchhoff's Laws Applied to Alternating Currents.—These laws as stated with respect to direct currents of course apply also to instantaneous values of alternating currents. It should be noted, however, that these laws also apply to vector quantities, and therefore to effective values, *provided* the laws are so stated as to take into account the vectorial nature of the quantities, and *provided* that in applying them, all vectors are resolved into their components with respect to some one vector as a reference line. As applying to alternating currents and e.m.f's, the laws may be stated as follows:

I. The vector sum of all externally induced e.m.f's in a given direction around a closed circuit is equal to the vector sum of all impedance drops in the same direction around the circuit. By externally induced e.m.f's are meant those e.m.f's generated by relative motion between inductors and a magnetic field where the field is not produced by the current in the circuit under consideration. A self-induced e.m.f. is included in the impedance drop as that part of the impedance drop which is consumed as reactance drop.

II. The vector sum of all currents flowing toward a point is equal to the vector sum of all currents flowing away from the point.

In both these laws, where direction is referred to, it must be understood that the chosen positive direction is meant, and that all phase relations must be expressed with reference to these chosen positive directions.

118. Two-phase Connections.—In Fig. 87 are represented the essential features of a two-phase four-pole alternator. The poles are represented as revolving, as is usually the case with alternators. They are excited by direct current, fed into the field coils through two slip rings. The full lines connecting the armature inductors represent the connections at the front end of the armature and the dotted lines the connections at the rear end. The terminals of phase (1) are a and b; of phase 2 are c and d. It will be noted that the inductors of phase (1) are 90 electrical degrees from the inductor of phase (2); therefore the e.m.f's in the two phases will be 90° out of phase. Since the e.m.f's are alternating in direction, either direction may be thought

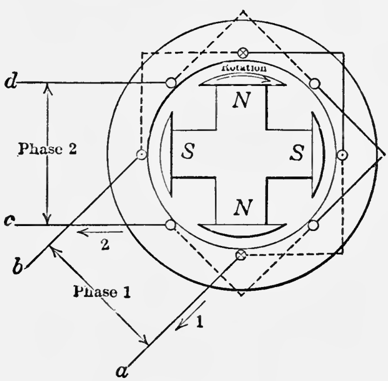

Fig. 87.

of as the positive direction, and it is therefore impossible to say which e.m.f. is ahead of the other, unless one direction or the other is arbitrarily chosen as the positive direction. Let the arrows marked (1) and (2) show the arbitrarily chosen positive direction in phases (1) and (2) respectively: then we can say that, with the particular choice made, the e.m.f. in phase (1) is 90° ahead of that in phase (2). A study of the figure will show that the e.m.f. in phase (1) reaches its positive maximum 90° before that in phase (2) reaches its positive maximum. The usual convention for representing the windings of an alternator is shown in Fig. 88. Positive directions must be chosen for all windings and other parts of the circuits, before vector relations can be be expressed either graphically or mathematically. These chosen positive directions should always be shown by properly placed arrows. It must be remembered that these arrows do not represent the direction of e.m.f. or current at any particular instant, but the direction which is considered as the positive direction. It is also necessary to know (or assume to be known) the phase relations between the induced e.m.f's in the different windings, with respect to the chosen positive directions. For example, to draw the vector diagram, Fig. 89, for the e.m.f's in Fig. 88, it is necessary to know that E_2 in its chosen positive direction lags 90° behind E_1 in its chosen positive direction.

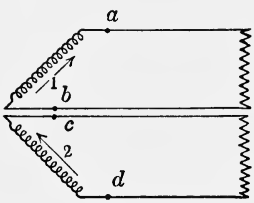

FIG. 88.

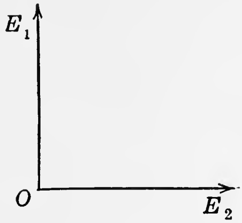

FIG. 89.

If the two circuits of a two-phase alternator are kept separate, the vector diagrams for the two circuits will be distinct single-phase diagrams, and no combined diagram is possible, except to indicate the relative positions of the vectors as in Fig. 89. It is usual, however, to join one terminal of each phase to a common wire in the case of a two-phase machine and use three wires for the line instead of four, as shown in Fig. 90. In this case, with the positive directions shown, the voltage between the outside wires will be the vector sum of the two e.m.f's and the current in the common wire will be the vector difference between the two currents in the outside wires. The vector diagram is shown in Fig. 91. Assuming that the two e.m.f's are equal, it is seen that the voltage V_{ad} is equal to $\sqrt{2}$ times the phase e.m.f. and leads E_2 by 45°. Mathematically this result may be found as follows: Taking E_2 as the reference vector, the e.m.f. E_1 would be expressed in symbolic notation as jE_1; and the vector sum of the two would be

$$V_{ad} = E_2 + jE_1 = \sqrt{E_2^2 + E_1^2} = \sqrt{2}E_2, \qquad (260)$$

and the sine of the angle between E_2 and V_{ad} is .707; therefore V_{ad} leads E_2 by 45°.

Assume that the two currents are equal and in phase with their respective e.m.f.'s. As positive direction of current in the common wire (m) choose the direction indicated by the arrow. Then, since the positive direction of I_1 is away from the junction and the positive direction of I_2 is towards the junction, I_m must be the vector difference of I_2 and I_1; or

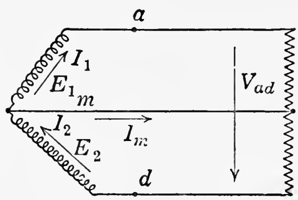

FIG. 90.

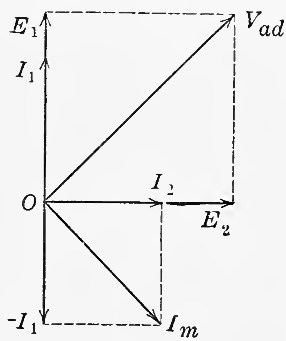

FIG. 91.

$$I_m = I_2 - jI_1 = \sqrt{2}I_2, \qquad (261)$$

and the sine of the angle between I_2 and I_m is −.707; therefore I_m lags 45° behind I_2.

Graphically, subtracting a vector means that it is to be reversed in direction and then added.

When a receiving circuit is fed over a 3-wire 2-phase line as shown diagrammatically in Fig. 92, the system becomes more or less unbalanced owing to the effect of the drop in the middle wire. This is shown by the vector diagram in

Fig. 93. The load currents, I_4 and I_6 are assumed to be equal and to lag by equal angles behind the generator volt-

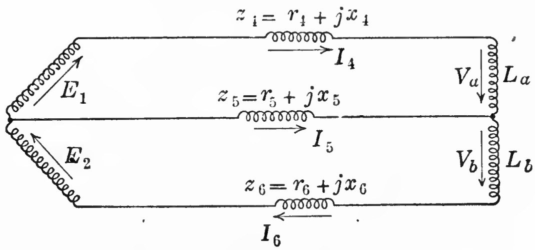

FIG. 92.

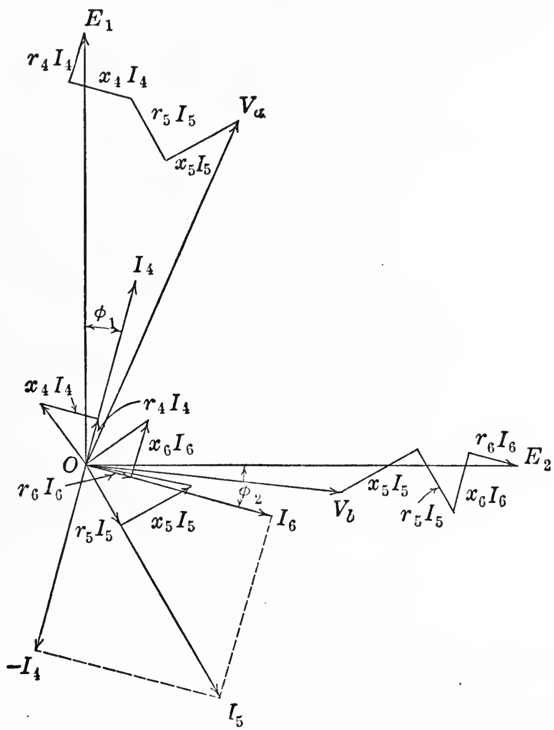

FIG. 93.

ages E_1 and E_2. The voltages across L_a and L_b are respectively

$$V_a = E_1 - z_4 I_4 + z_5 I_5, \qquad 262)$$

and

$$V_b = E_2 - z_6 I_6 - z_5 I \qquad (263)$$

Note that the drop in the middle wire adds vectorially to E_1 because the positive direction chosen in it is opposite to that chosen for E_1, while this same drop subtracts vectorially from E_2. Observe carefully the statements concerning Kirchhoff's Laws, Article 117. Equations (262) and (263) are vector equations and all quantities must be resolved into their components with respect to some one vector as a reference axis, before the equations can be solved numerically. Using E_1 as the reference vector, the current I_4 must be expressed as I_4 ($\cos\phi_1 - j \sin \phi_1$) and I_5 as I_5 ($\cos (\phi_1+135) - j \sin (\phi_1+135)$. Substituting these values in equation (262) and expanding, remembering that $z_4 = r_4 + jx_4$ and $z_5 = r_5 + jx_5$, the value of V_a is

$$V_a = E_1 - r_4 I_4 \cos \phi_1 - x_4 I_4 \sin \phi_1 + r_5 I_5 \cos (\phi_1 + 135) + x_5 I_5 \sin (\phi_1 + 135) + j[r_4 I_4 \sin \phi_1 - x_4 I_4 \cos \phi_1 - r_5 I_5 \sin (\phi_1 + 135) + x_5 I_5 \cos (\phi_1 + 135)]. \quad (264)$$

Using E_2 as the reference vector, I_6 is I_6 ($\cos \phi_2 - j \sin \phi_2$) and I_5 is $I_5 \cos (\phi_2+45) - j \sin (\phi_2+45)$. Substituting these values in equation (263) and expanding, the value of V_b is

$$V_b = E_2 - r_6 I_6 \cos \phi_2 - x_6 I_6 \sin \phi_2 - r_5 I_5 \cos (\phi_2 + 45) - x_5 I_5 \sin (\phi_2 + 45) + j[r_6 I_6 \sin \phi_2 - x_6 I_6 \cos \phi_2 + r_5 I_5 \sin (\phi_2 + 45) - x_5 I_5 \cos (\phi_2 + 45)]. \quad (265)$$

In Fig. 93, the vectors representing the drops are exaggerated for the sake of clearness, and it is evident that V_a and V_b are not equal nor are they 90° apart.

119. Three-phase Connections.—In a three-phase machine, there are three identical armature windings with corresponding inductors in each winding spaced 120 electrical degrees apart. The e.m.f's in these windings are therefore 120° apart in phase when the positive directions are so chosen as to be in the same direction across the armature face in each of three corresponding inductors, one in each winding, spaced 120° apart. In practice, these windings are connected according to one of two different schemes.

One is known as the "delta" connection, generally pictured as shown in Fig. 94, and the windings are so connected that the positive directions in the three windings are in the same direction around the delta. The other is known as the "star" or "Y" connection, generally pictured as shown in Fig. 97 and the windings are so connected that the positive directions in the three windings are in the same direction with respect to the common junction. In the delta connection the three-line wires are connected to the three points where the windings join each other, and in the Y connection, the line wires are connected to the three free ends of the windings.

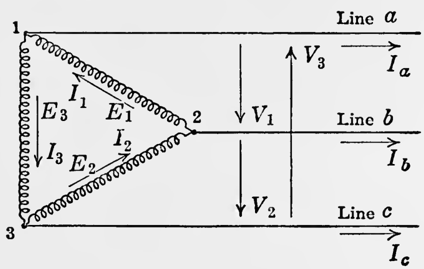

FIG. 94.

120. Relation of Line Voltages to Phase Voltages in Three-phase Delta-connected Systems.—In Fig. 94, the arrows E_1, E_2, and E_3 represent the chosen positive directions of the voltages generated in phase 1, 2, and 3 respectively. It is evident that the line voltages are numerically the same as the phase voltages. If the positive directions for the line voltages, V_1, V_2 and V_3 are chosen as shown in Fig. 94 then the vector diagram of all voltages will be as shown in Fig. 95*a*. Note that the same vector which represents the voltage from terminal (2) to terminal (1) through the winding, also represents the voltage from the terminal (1) to terminal (2) across the line; if the positive direction across the line had been chosen in the opposite direction from that shown in Fig. 94, then the vector diagram would have been as shown in Fig. 95*b*.

The vector sum of the three voltages, as may be seen from the vector diagram, is zero and therefore the resultant voltage around the delta is zero. That this relation holds at every instant may be shown mathematically as follows: Let e_1, e_2 and e_3 be the instantaneous values of the three e.m.f's then

$$e_1 = E'_1 \sin \omega t, \tag{266}$$

$$e_2 = E'_2 \sin (\omega t - 120) \tag{267}$$

$$e_3 = E'_3 \sin (\omega t - 240) \tag{268}$$

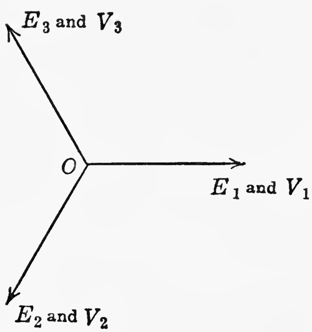

FIG. 95*a*.

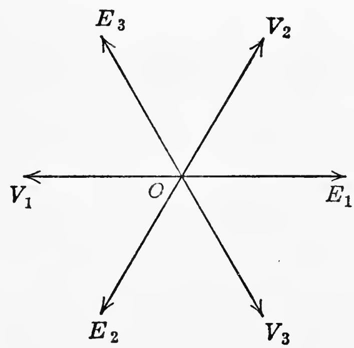

FIG. 95*b*.

where the prime (′) indicates maximum values. The sum of these is, assuming the maximum values to be equal,

$$e_1 + e_2 + e_3 = E' [\sin \omega t + \sin (\omega t - 120) + \sin (\omega t - 240)]. \tag{269}$$

Expanding the quantity in brackets, we get

$$\sin \omega t + \sin \omega t \cos 120 - \cos \omega t \sin 120 + \sin \omega t \cos 240 - \cos \omega t \sin 240 = \sin \omega t - \frac{1}{2} \sin \omega t - \frac{\sqrt{3}}{2} \cos \omega t - \frac{1}{2} \sin \omega t + \frac{\sqrt{3}}{2} \cos \omega t = 0. \tag{270}$$

121. Relation of Line Currents to Phase Currents in Three-Phase Systems.—(*a*) *Delta Connection.*—Assume that the loads are balanced; that is, that the three currents in the phases are equal in value and are 120° apart in phase relation. Choose positive directions as shown in Fig. 94. Then the vector diagram will be as shown in Fig. 96. The current in line *a* is the vector sum of I_1 and $-I_3$; in line *b* it is the vector sum of I_2 and $-I_1$; and in line *c* it is the vector sum of I_3 and $-I_2$. The statements made in the preceding sentence are true whether the circuits are balanced or not. In the case of balanced circuits, however, it is evident from the vector diagram that each line current is 30° and 150° respectively behind the two currents

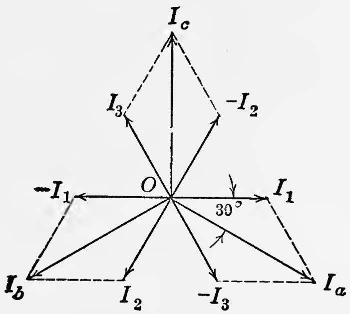

FIG. 96.

which enter that line. It is also evident that the value of the line currents is equal to 2 cos 30° times the value of the phase currents; that is, the line currents are $\sqrt{3}$ times the phase currents.

These relations are shown mathematically as follows: Let i_1, i_2 and i_3 be the instantaneous values of the currents and let the maximum values be indicated by I_1', I_2', and I_3'; then

$$i_1 = I_1' \sin \omega t \tag{271}$$

$$i_2 = I_2' \sin (\omega t - 120), \tag{272}$$

$$i_3 = I_3' \sin (\omega t - 240). \tag{273}$$

The instantaneous value of the current in line *a* is then

$$i_a = i_1 - i_3 = I_1' \sin \omega t - I_3' \sin (\omega t - 240). \tag{274}$$

Assuming the maximum values to be equal

$$i_a = I'[(\sin \omega t - \sin (\omega t - 240)], \tag{275}$$

$$= I' (\sin \omega t - \sin \omega t \cos 240 + \cos \omega t \sin 240) \tag{275a}$$

$$= I' \left(\sin \omega t + \frac{1}{2} \sin \omega t - \frac{\sqrt{3}}{2} \cos \omega t\right), \tag{275b}$$

$$= I' \left(\frac{3}{2} \sin \omega t - \frac{\sqrt{3}}{2} \cos \omega t\right), \tag{275c}$$

$$= \sqrt{3} I' \left(\frac{\sqrt{3}}{2} \sin \omega t - \frac{1}{2} \cos \omega t\right), \tag{275d}$$

$$= \sqrt{3} I' (\sin \omega t \cos 30 - \cos \omega t \sin 30), \tag{275e}$$

$$= \sqrt{3} I' \sin (\omega t - 30). \tag{276}$$

The value of the line current is therefore equal to $\sqrt{3}$ times the value of the phase current, and the line current I_a lags 30° behind the phase current I_1. The values of the other line currents can be found similarly and the relations will be the same.

In the case of unbalanced currents the line currents would be calculated as follows, where θ_1 is the angle between I_1 and I_3, θ_2 is the angle between I_2 and I_1 and θ_3 is the angle between I_3 and I_2:

$$I_a = I_1 + I_3 \cos (180 - \theta_1) - jI_3 \sin (180 - \theta_1), \tag{277}$$

$$I_b = I_2 + I_1 \cos (180 - \theta_2) - jI_1 \sin (180 - \theta_2), \tag{278}$$

$$I_c = I_3 + I_2 \cos (180 - \theta_3) - jI_2 \sin (180 - \theta_3). \tag{279}$$

(*b*) **Y-Connection.**—Evidently the line current and the phase current are identical in this method of connection. See Fig. 97.

122. Relation of Line Voltages to Phase Voltages in Three-phase Y-connected Systems.—Assume that the voltages are balanced; that is, equal in value and 120° apart in phase. Choose the positive direction shown in Fig. 97.

Then the vector diagram will be as shown in Fig. 98. The line voltage V_1 is the vector sum of E_1 and $-E_2$; V_2 is the vector sum of E_2 and $-E_3$; and V_3 is the vector sum of E_3 and $-E_1$. This statement is true whether the voltages are balanced or not; but in the case of balanced voltages, it is evident from the vector diagram that the line voltages are 30° and 150° respectively ahead of the two voltages which combine to make them. It is also evident that the line voltages are equal to $\sqrt{3}$ times the phase voltages. These relations may be found mathematically by the

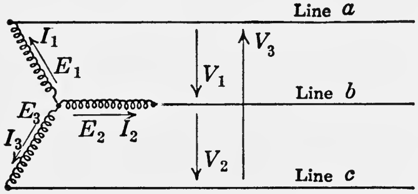

FIG. 97.

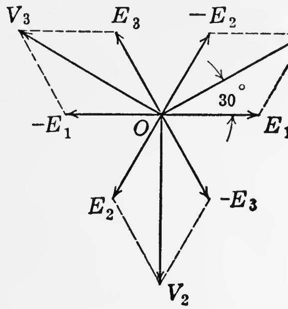

FIG. 98.

same method as was used for finding the line currents in the case of the delta connection.

123. Power in Three-phase Circuits.—The total power in any three-phase circuit is

$$P = E_1 I_1 \cos \phi_1 + E_2 I_2 \cos \phi_2 + E_3 I_3 \cos \phi_3, \qquad (280)$$

where E_1, E_2, and E_3 are the phase voltages, I_1, I_2, and I_3 are the phase currents, and $\cos \phi_1$, $\cos \phi_2$ and $\cos \phi_3$ are the corresponding power factors in the three phases. If the circuits are balanced, however, then $E_1 = E_2 = E_3$, $I_1 = I_2 = I_3$ and $\cos \phi_1 = \cos \phi_2 = \cos \phi_3$ and

$$P = 3EI \cos \phi, \qquad (281)$$

where E, I, and $\cos \phi$ are the common values of the phase voltage, current and power factor. The power may be expressed in terms of line voltage, V, and line current I', by substituting $V/\sqrt{3}$ for E in the case of a Y-connection, or $I'/\sqrt{3}$ for I in the case of a delta connection. The power would then be expressed as

$$P = \sqrt{3}VI' \cos \phi. \tag{282}$$

It is to be particularly noted that in this expression the angle ϕ is the power factor angle of the phases and is not the angle between line voltage and line current, although V and I' are line voltage and line current respectively. In the delta connection $V = E$ and in the Y-connection $I' = I$, so that the expression is correct for either method of connection.

124. Power Measurement in Three-phase Circuits.—The total power delivered to a polyphase system may of course be measured by connecting a single-phase wattmeter in each phase of the system and taking the sum of the readings. The connections for this method applied to a three-phase system are shown in Fig. 99. The sum of the three

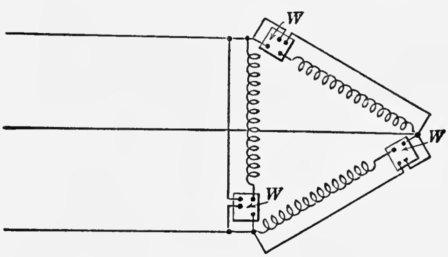

Fig. 99.

wattmeter readings will give the true power regardless of condition as to balance or wave form. If the load is known to be balanced, one wattmeter is sufficient and the total power is three times this wattmeter reading; in general, however, load is not well enough balanced to permit this. In practice the most common method of measuring three-phase power consists of using two wattmeters connected as shown in Fig. 100, and it will now be shown that the algebraic sum of the readings of these two wattmeters gives the correct total power regardless of balance, wave form or power factor. A wattmeter registers the average value of the product of the instantaneous values of the current through its current coil and the p.d. across its potential circuit; the average value of this product is the average power developed in the circuit in which the wattmeter is connected. (See Article 93.) If the positive directions of current in the current coil and p.d. in the pressure circuit are both chosen in the same direction with reference to the common point of connection, (point p in Fig. 100), the wattmeter will read positively (that is, forward on its scale) when the equivalent sine waves of current and p.d. are less than 90° out of phase; it will read negatively (that is, backward)

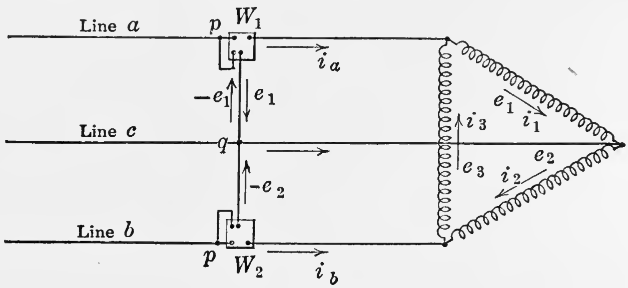

Fig. 100.

when these waves are more than 90° out of phase. In the latter case, it may be made to read positively by reversing the connections of either coil; in practice, the potential coil is reversed under such circumstances. In Fig. 100 let the positive directions be taken as shown by the arrows and let the small letters indicate the instantaneous values of current and p.d. Then the reading of wattmeter No. 1 will be

$$W_1 = \text{average } (e_1 i_a), \tag{283}$$

and the reading of wattmeter No. 2 will be

$$W_2 = \text{average } (-e_2 i_b). \tag{284}$$

The sign of e_2 is negative because the positive direction of e_2 is chosen to be from line c to line b but a positive reading on the wattmeter requires that the positive direction through the pressure circuit shall be from line b to line c. The sum of these two readings is

$$W_1+W_2=\text{average }(e_1i_a)+\text{average }(-e_2i_b). \quad (285)$$

But

$$i_a=i_1-i_3, \text{ and } i_b=i_3-i_2;$$

therefore

$$W_1+W_2=\text{average }(e_1i_1)-\text{average }(e_2i_3) -\text{average }(e_1i_3)+\text{average }(e_2i_2). \quad (286)$$

But

$$e_1+e_2=-e_3;$$

therefore

$$W_1+W_2=\text{average }(e_1i_1)+\text{average }(e_2i_2) +\text{average }(e_3i_3), \quad (287)$$

which is the total average power delivered to the three phases.

If the effective values of current and p.d. be used, the wattmeter readings will be

$$W_1=E_1I_a\cos\alpha, \quad (288)$$

and

$$W_2=E_2I_b\cos\beta, \quad (289)$$

where α is the phase angle between E_1 and I_a, and β is the phase angle between $-E_2$ and I_b. These phase relations are shown in Fig. 101, which is the vector diagram for Fig. 100. It will readily be seen that if ϕ_1 or ϕ_3 or both are increased, α will be increased and may become equal to or greater than 90°. See Fig. 102. If α becomes equal to 90°, the reading of wattmeter W_1 will become zero; if α becomes greater than 90°, then the wattmeter will read backward or negatively, and its connections must be reversed in order to get the value of this negative reading. The total power will then be the numerical difference of the two wattmeter readings. Therefore when two wattmeters are connected in a three-phase circuit in which the power factors are unknown, or are known to be low, and the pressure coils are connected so that both meters read up on their scales, there will be uncertainty as to whether the readings should be added or subtracted. To determine which to do, the load may be switched off and a load which is known to be non-inductive (incandescent lamps, for example) put in its place; then if both meters read up on their scales, their readings on the original load are additive, but if one wattmeter reads backward, the original readings must be subtracted. Generally, unless the loads are considerably unbalanced, the wattmeter giving the smaller reading is the

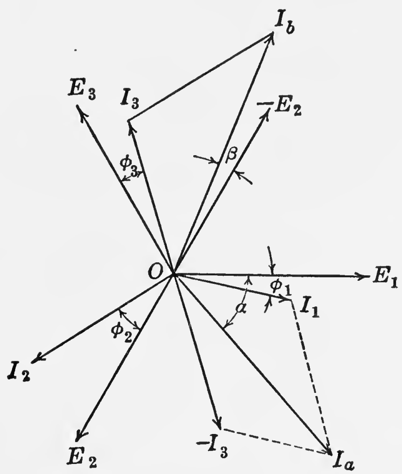

Fig. 101.

one in doubt and the following more simple method will show whether its reading is to be added or subtracted; disconnect the potential terminal from the common wire (line *c* in Fig. 100) and connect it to the line in which the other meter is connected. If the wattmeter reads backward its original reading is negative. For example, let Fig. 102 be the vector diagram of a certain load, connections being as shown in Fig. 100. I_a is more than 90° behind E_1 and to get a forward reading on W_1 its potential circuit must be connected so that the voltage on it is $-E_1$; but this fact is not known until the test has been made. If the terminal

q of W_1 be taken from line c and connected to line b, the voltage on the potential circuit will be E_3 and the angle

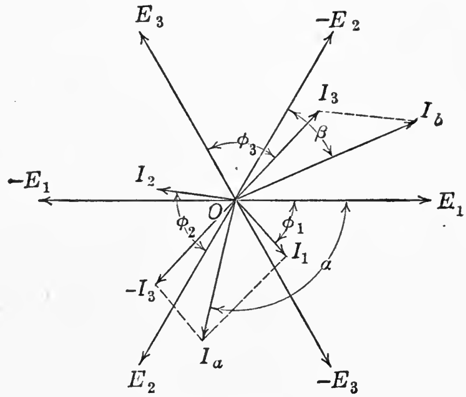

FIG. 102.

between E_3 and I_a is more than 90° so that the reading will be backward. If α had been less than 90° the voltage on the potential circuit of W_1 (for a forward reading) would have

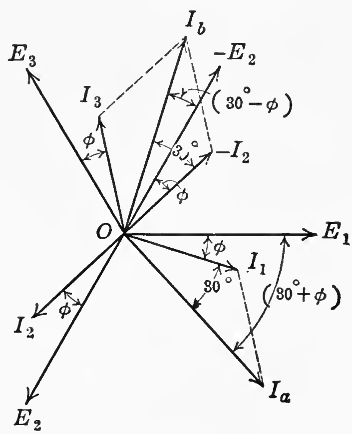

FIG. 103.

been E_1 and when q was carried to line b, the voltage on the potential circuit would have been $-E_3$ and the reading would not have reversed. Therefore, a reversed reading when this test is made indicates that the difference of the two wattmeter readings is the true power.

In the special case of balanced load and power factor, the vector diagram of Fig. 103 applies. The reading of one wattmeter will be

$$W_1 = E_1 I_a \cos (30+\phi), \tag{290}$$

and the other

$$W_2 = E_2 I_b \cos (30-\phi). \tag{291}$$

When the power factor becomes 0.5, $\phi=60°$ and $W_1=0$, while

$$W_2 = \frac{\sqrt{3}}{2} E_2 I_b = 3EI \cos 60° = \frac{3}{2} EI,$$

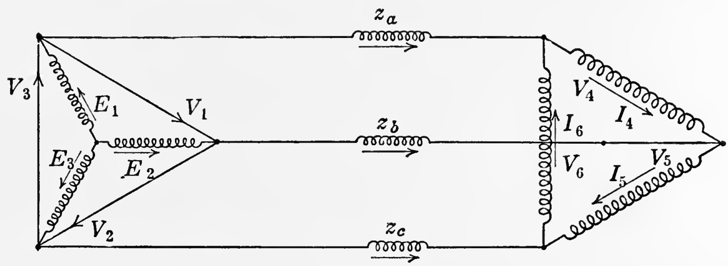

FIG. 104.

where E and I are the phase voltage and current, respectively. With power factors less than 0.5 wattmeter W_1 will read negative.

In the case of balanced circuits the power factor of the load may be computed from the two wattmeter readings as follows:

$$W_1+W_2 = E_1 I_a [\cos (30+\phi) + \cos (30-\phi)] = \sqrt{3} E_1 I_a \cos \phi, \tag{292}$$

$$W_2-W_1 = E_1 I_a [\cos (30-\phi) - \cos (30+\phi)] = E_1 I_a \sin \phi. \tag{293}$$

Therefore

$$\frac{W_2-W_1}{W_1+W_2} = \frac{1}{\sqrt{3}} \tan \phi, \quad \text{or} \quad \tan \phi = \sqrt{3}\left(\frac{W_2-W_1}{W_1+W_2}\right). \tag{294}$$

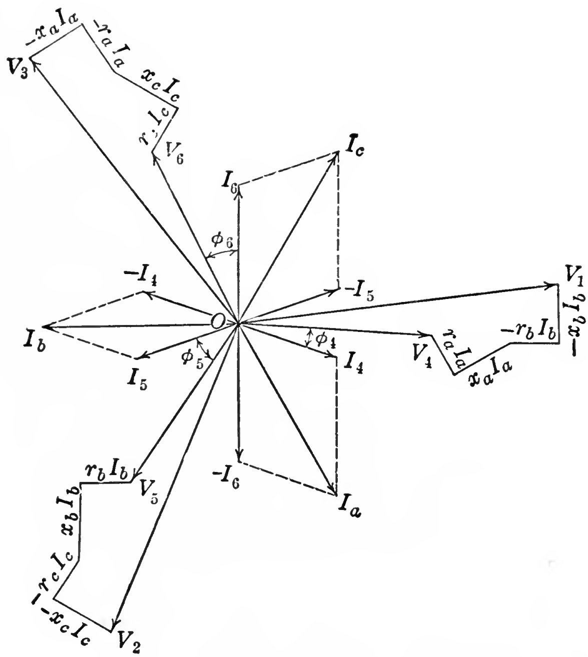

Fig. 105.

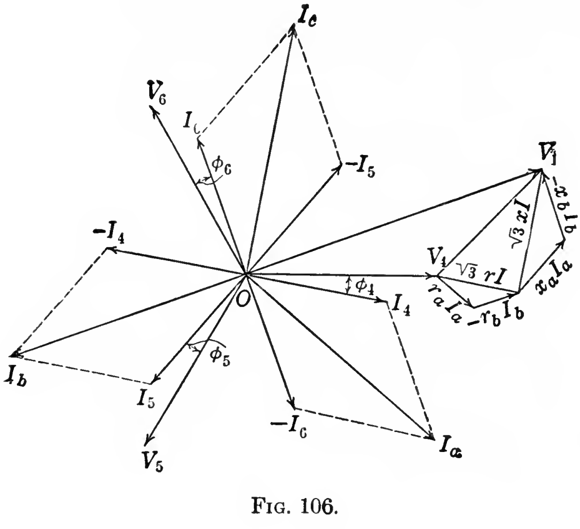

Fig. 106.

125. Line Drop in Three-phase Circuits.—Fig. 105 is the vector diagram showing the relations of the line drops to the voltages at the two ends of a transmission line as represented in Fig. 104. To the student is left the problem of formulating the equations for computing the voltages at the generating end when the constants of the line and the voltages, currents and power factors of the load are given. The principles are the same as those used in connection with the two-phase problem at the end of Article 118. In the special case of balanced load and power factor, the generator voltage is

$$V_1 = V_4 \cos \phi + \sqrt{3} r_a I_a + j\,(V_4 \sin \phi + \sqrt{3} x_a I_a). \quad (295).$$

This formula may be deduced directly from the vector diagram in Fig. 106.

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