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The Generalized Fluctuation Test: A Unifying View

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Abstract

In this paper a general principle of constructing tests for parameter constancy without assuming a specific alternative is introduced. A unified asymptotic result is established to analyze this class of tests. As applications, tests based on the range of recursive and moving estimates are also considered, and their asymptotic distributions are characterized analytically. Our simulations show that different tests have quite different behavior under various alternatives and that no test uniformly dominates the other tests.

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1 Introduction

The topic of testing the goodness-of-fit of a probability model has a long history in the statistical literature, of which tests for the constancy of a mean function are a special case. In the linear regression context, this type of tests reduces to tests for constant regression coefficients. It is quite typical to construct tests against certain specific alternatives based on a prior belief. A popular alternative is a one-time structural change at known or unknown change point, e.g., Chow (1960), Quandt (1960), Hawkins (1987), and Andrews (1993). This alternative is convenient for deriving tests but may not describe many interesting phenomena, however. In the study of business cycle, for example, it is not uncommon to believe that a downswing of major aggregates takes place suddenly (Hicks (1950)), but there do not exist similar abrupt changes when the economy moves to a upturn period (e.g., Neftci (1979)). Another popular alternative is that parameters follow a random walk (or a martingale), e.g., Cooley & Prescott (1976), Lamotte & McWhorter (1978), Leybourne & McCabe (1989), and Nyblom (1989). This alternative is also somewhat restrictive. For example, suppose that a policy causes the economy shifting to a new regime, either suddenly or gradually, it is quite likely that, when rational expectation prevails, the economy will be returning to, instead of drifting away from, a "normal" level.

The specific tests can be extended in different ways. Andrews & Ploberger (1992) introduce a class of optimal tests against multiple structural changes. Another strategy is to construct tests without bearing any specific alternatives in mind. As one rarely knows how regression coefficients evolve over time, it would be desirable to construct tests with power against all possible mean functions. This class of tests is our primary interest in this paper, which includes estimates-based tests, such as the recursive-estimates (RE) test, also known as the fluctuation test, of Sen (1980) and Ploberger, Krämer, & Kontrus (1989), and the class of moving-estimates (ME) tests of Chu, Hornik, & Kuan (1992a), as special cases. The well known residual-based tests, such as the CUSUM tests of Brown, Durbin, & Evans (1975) and Ploberger & Krämer (1992) and the class of MOSUM tests of Bauer & Hackl (1978) and Chu, Hornik & Kuan (1992b), also belong to this class. Note, however, that the class of ME (MOSUM) tests differs from the RE (CUSUM) test in an important respect. Moving estimates (or moving sums of residuals) can be interpreted as non-parametric estimates of corresponding mean functions, whereas recursive estimates (or cumulated sums of residuals) do not have similar interpretation.

On the other hand, we observe that a common feature of the above "general" tests is that they are based on empirical processes consisting of two additive components, one satisfying a functional central limit theorem and one that is roughly a "straight line" under the null hypothesis. By suitable construction, this straight line component can be eliminated, for example, by applying a linear operator annihilating the straight line, so that the resulting empirical process under the null is essentially governed by the functional central limit theorem. Under the alternative, however, this empirical process will "fluctuate", in the sense that its behavior is not completely characterized by the functional central limit theorem. A test for parameter constancy can then be obtained by assigning an appropriate functional to measure the "fluctuation" of the empirical process; the null hypothesis is rejected if this process fluctuates too much. This class of tests will be referred to as the *generalized fluctuation* test. It includes the RE, ME, CUSUM, and MOSUM tests as special cases. Clearly, numerous tests can be constructed according to this general principle. As their power properties under different alternatives are far from obvious, it is extremely interesting to find out, by simulations, which combination of functional and operator can deliver "better" power results.

In this paper we first establish an asymptotic result for the generalized fluctuation test that can be written as $\lambda(\mathcal{L}_T Y_T)$, where λ is a functional and $\mathcal{L}_T$ is an operator annihilating the straight line component of an empirical process Y_T, from which many known results can be derived as corollaries. Our result greatly facilitates the analysis of these tests under the null and alternatives. In particular, we also consider tests based on the *range* functional, instead of the maximal functional typically adopted in existing tests. Specifically, the range of recursive-estimates (RR) and moving-estimates (RM) tests are investigated. The asymptotic null distribution of the RR test is well known in literature, but that of the RM tests is unknown. For certain bandwidths of moving windows, we derive a formula representing the asymptotic distribution of the RM test, from which critical values can be easily calculated; for other bandwidths of moving windows, critical values of the RM tests are obtained by simulations. Power simulations are also conducted to compare the performance of different tests.

This paper is organized as follows. We introduce the generalized fluctuation test in a simple location model and provide a unified asymptotic result in section 2. We then introduce range tests and derive their asymptotic null distributions in section 3. These results are extended to multiple regression in section 4. Power performance and simulation results are reported in section 5. Section 6 concludes the paper. Applications of the general result to known tests and mathematical proofs are summarized in the Appendix.

2 The Generalized Fluctuation Test

To illustrate the idea of a general class of tests for parameter constancy, first consider the data generating process (DGP):

$$y_i = \mu_i + \epsilon_i, \qquad i = 1, \ldots, T, \tag{1}$$

where $\{\epsilon_i\}$ is a sequence of i.i.d. random variables with mean zero and variance one. It is well known that ϵ_i satisfy a functional central limit theorem (FCLT):

$$\left(\frac{1}{\sqrt{T}} \sum_{i=1}^{[Tt]} \epsilon_i, \ 0 \le t \le 1 \right) \Rightarrow W, \tag{2}$$

as $T \to \infty$, where $[Tt]$ is the integer part of Tt, $\Rightarrow$ denotes weak convergence of associated probability measures, and W is a standard Wiener process. The null hypothesis of interest is $\mu_i = \mu_0$ for all i. In what follows, a function f is either in $C[0, \tau]$, the space of continuous functions on $[0, \tau]$, or in $D[0, \tau]$, the space of functions that are right continuous with left-hand limits on $[0, \tau]$. We always assume that the space C is endowed with the uniform topology and that the space D is endowed with the Skorohod topology. For more details about the spaces C and D we refer to Billingsley (1968). We also let $\to^p$ denote convergence in probability, and $=^d$ denote equality in distribution.

Consider the piecewise constant process Y_T on $[0, 1]$ with jump points

$$Y_T\left(\frac{k}{T}\right) = \frac{1}{\sqrt{T}} \sum_{i=1}^{k} y_i. \tag{3}$$

Under the null hypothesis,

$$Y_T(t) \ = \ \sqrt{T}\mu_0 \frac{[Tt]}{T} + E_T(t), \tag{4}$$

where E_T is also a piecewise constant process with jump points

$$E_T\left(\frac{k}{T}\right) \ = \ \frac{1}{\sqrt{T}} \sum_{i=1}^{[Tt]} \epsilon_i.$$

Observe that, apart from the factor $T^{1/2}$, the first term in (4) is roughly a "straight line" passing through the origin and that the second term satisfies the FCLT (2). When the straight line component is removed, the resulting empirical process is well behaved by the FCLT under the null hypothesis. If the null hypothesis is false, this empirical process will fluctuate, in the sense that its behavior is not completely characterized by the FCLT.

Hence, a test can be constructed by evaluating the fluctuation of an empirical process. This is the underlying idea of the generalized fluctuation (GF) test.

To fix the idea, consider the GF test that can be written as $\lambda(\mathcal{L}_T Y_T)$, where $\mathcal{L}_T$ is a linear operator in D which annihilates the straight line of (4), i.e., $\mathcal{L}_T Y_T = \mathcal{L}_T E_T$, and λ is a functional measuring the fluctuation of $\mathcal{L}_T Y_T$. If $\mathcal{L}_T E_T = \mathcal{L} E_T + o_p(1)$, then under the null, $\mathcal{L}_T Y_T \Rightarrow \mathcal{L} W$. When the null hypothesis is false, the deterministic component of Y_T is not a straight line so that $\mathcal{L}_T Y_T = \mathcal{L}_T E_T +$ something. For example, the operator $\mathcal{L}_T$ such that for f in $D[0,1]$

$$\mathcal{L}_T f(t) \;=\; f(t) - \frac{[Tt]}{T} f(1)$$

eliminates the straight line component of Y_T under the null. It follows that

$$\mathcal{L}_T Y_T \;=\; \mathcal{L}_T E_T \;\Rightarrow\; \mathcal{L} W,$$

where $\mathcal{L} f(t) = f(t) - t f(1)$. This class of tests includes many well known tests as special cases, as the examples below show.

In what follows, for functions f in $D[0,1]$, let

$$\max(f;\tau) \;=\; \max_{0\le t\le \tau} f(t), \qquad \min(f;\tau) \;=\; \min_{0\le t\le \tau} f(t),$$

be the maximum and minimum of f on $[0,\tau]$, and let

$$\mathrm{range}(f;\tau) \;=\; \max(f;\tau) - \min(f;\tau). \tag{5}$$

be the range of f on $[0,\tau]$. Finally, we write f^0 for the function $f^0(t) = f(t) - t f(1)$ such that W^0 is the familiar Brownian bridge ("tied-down Wiener process").

Example I. Estimates-Based Tests:

1. The RE test: Sen (1980), Ploberger, Krämer, & Kontrus (1989).
   Let recursive estimates of μ_0 be $\hat{\mu}_k = k^{-1}\sum_{i=1}^{k} y_i$, $k = 1,\ldots,T$. The RE test is based on the fluctuation of recursive estimates in terms of the deviations $\hat{\mu}_k - \hat{\mu}_T$:

$$RE_T \;=\; \max_{k=1,\cdots,T} \frac{k}{\sqrt{T}} |\hat{\mu}_k - \hat{\mu}_T| \;=\; \max_{k=1,\cdots,T} \frac{1}{\sqrt{T}} \left| \sum_{i=1}^{k} y_i - \frac{k}{T} \sum_{i=1}^{T} y_i \right|. \tag{6}$$

   Hence, $RE_T = \max(|\mathcal{L}_T Y_T|; 1)$ with

$$\mathcal{L}_T f(t) \;=\; f\Big(\frac{[Tt]}{T}\Big) - \frac{[Tt]}{T} f(1) \;=\; f^0\Big(\frac{[Tt]}{T}\Big). \tag{7}$$

2. The ME test: Chu, Hornik, & Kuan (1992a).

Let moving estimates of μ_0 be $\tilde{\mu}_{k,h} = [Th]^{-1}\sum_{i=k+1}^{k+[Th]} y_i$, $k = 0,\ldots,T-[Th]$, where $[Th]$ is the bandwidth of moving windows and $0 < h < 1$. The ME test is based on the fluctuation of moving estimates in terms of the deviations $\tilde{\mu}_{k,h} - \hat{\mu}_T$:

$$\begin{aligned} ME_{T,h} &= \max_{k=0,\cdots,T-[Th]} \frac{[Th]}{\sqrt{T}} |\tilde{\mu}_{k,h} - \hat{\mu}_T| \\ &= \max_{k=0,\cdots,T-[Th]} \frac{1}{\sqrt{T}} \left| \sum_{i=k+1}^{k+[Th]} y_i - \frac{[Th]}{T} \sum_{i=1}^{T} y_i \right| \\ &= \max_{0 \le t \le 1-h_T} \left| Y_T\Big(\frac{[Tt]+[Th]}{T}\Big) - Y_T\Big(\frac{[Tt]}{T}\Big) - \frac{[Th]}{T} Y_T(1) \right|. \end{aligned} \tag{8}$$

Straightforward rescaling shows that $ME_{T,h} = \max(|\mathcal{L}_{T,h}Y_T|; 1-h)$ with

$$\begin{aligned} \mathcal{L}_{T,h}f(t) &= f(\kappa_T(t)+h_T) - f(\kappa_T(t)) - h_T f(1) \\ &= f^0(\kappa_T(t)+h_T) - f^0(\kappa_T(t)), \end{aligned} \tag{9}$$

where $h_T = [Th]/T$, $\kappa_T(t) = [N_T t]/T$, $N_T = (T-[Th])/(1-h)$.

Example II. Residual-Based Test:

1. The Recursive-CUSUM test: Brown, Durbin, & Evans (1975).

The recursive residuals are $v_i = y_i - \hat{\mu}_{i-1}$, $i = 2,\ldots,T$. The Recursive-CUSUM test is based on the fluctuation of cumulated sums of recursive residuals:

$$\begin{aligned} QS_T^r &= \max_{k=2,\cdots,T} \frac{1}{\sqrt{T}} \left| \sum_{i=2}^{k} v_i \right| \\ &= \max_{k=2,\cdots,T} \frac{1}{\sqrt{T}} \left| \sum_{i=2}^{k} \left(y_i - \frac{1}{i-1} \sum_{j=1}^{i-1} y_i \right) \right|. \end{aligned} \tag{10}$$

It is readily seen that $QS_T^r = \max(|\mathcal{L}_T Y_T|; 1)$ with

$$\mathcal{L}_T f(t) = f(t) - \int_0^t \frac{f(\tau)}{[T\tau]/T}\, d\tau. \tag{11}$$

2. The OLS-CUSUM test: Ploberger & Krämer (1992).

Let $e_i = y_i - \hat{\mu}_T$, $i = 1,\ldots,T$, be OLS residuals. Analogous to the Recursive-CUSUM test, the OLS-CUSUM test is based on the fluctuation of cumulated sums of OLS residuals:

$$QS_T^o = \max_{k=1,\cdots,T} \frac{1}{\sqrt{T}} \left| \sum_{i=1}^{k} e_i \right| = \max_{k=1,\cdots,T} \frac{1}{\sqrt{T}} \left| \sum_{i=1}^{k} y_i - \frac{k}{T} \sum_{i=1}^{T} y_i \right|. \tag{12}$$

Clearly, $QS_T^o = RE_T$, cf. (6).

3. The Recursive-MOSUM test: Bauer & Hackl (1978), Chu, Hornik, & Kuan (1992b). In contrast with the CUSUM-type of test, the Recursive-MOSUM test is based on moving sums (with bandwidth $[Th]$, $0 < h < 1$) of recursive residuals. Letting $T' = T - 1$, the statistic is

$$\begin{aligned} MS^r_{T,h} &= \max_{k=0,\cdots,T'-[T'h]} \frac{1}{\sqrt{T}} \left| \sum_{i=k+2}^{k+1+[T'h]} v_i \right| \\ &= \max_{k=0,\cdots,T'-[T'h]} \frac{1}{\sqrt{T}} \left| \sum_{i=k+2}^{k+1+[T'h]} \left(y_i - \frac{1}{i-1} \sum_{j=1}^{i-1} y_j \right) \right|. \end{aligned} \tag{13}$$

In view of (8)–(11), we can write $MS^r_{T,h} = \max(|\mathcal{L}_{T,h} Y_T|; 1-h)$ with

$$\mathcal{L}_{T,h} f(t) = f(\kappa_{T'}(t) + h_{T'}) - f(\kappa_{T'}(t)) - \int_{\kappa_{T'}(t)}^{\kappa_{T'}(t)+h_{T'}} \frac{f(\tau)}{[T'\tau]/T'} \, d\tau. \tag{14}$$

4. The OLS-MOSUM test: Chu, Hornik & Kuan (1992b).
Analogous to the Recursive-MOSUM test, the OLS-MOSUM test is based on moving sums of OLS residuals:

$$MS^o_{T,h} = \max_{k=0,\cdots,T-[Th]} \frac{1}{\sqrt{T}} \left| \sum_{i=k+1}^{k+[Th]} e_i \right|. \tag{15}$$

Clearly, $MS^o_{T,h} = ME_{T,h}$, cf. (8) and (9).

The tests above apply different operators to remove the straight line component but adopt the same maximal functional to evaluate the fluctuation of empirical processes. It is clear that numerous tests can be constructed by choosing different combinations of functionals and annihilators. For example, by applying the functional $\max(f;\tau)$ we obtain one-sided tests in the above examples, and by applying the range functional $\text{range}(f;1)$ we obtain range tests which will be discussed in details in next section. Therefore, a unified asymptotic result can facilitate the analysis of this class of tests.

More precisely, we assume the following conditions.

[G1] $\mathcal{L}_T$ and $\mathcal{L}$ are linear operators from $D[0,1]$ to $D[0,\tau]$ such that $\mathcal{L}_T \iota_T = 0$, where $\iota_T(t) = [Tt]/T$.

[G2] λ is a positively homogeneous functional on $D[0,\tau]$ which is continuous with respect to the Skorohod topology, i.e., $f_T \to f$ in the Skorohod metric implies $\lambda(f_T) \to \lambda(f)$.

In what follows the function Jf, the anti-derivative of f, is defined as

$$Jf(t) \quad = \quad \int_0^t f(u)\,du,$$

and the function $\Delta_h f$ is defined by $\Delta_h f(t) = f(t+h) - f(t)$ (for $h = 1$ we simply write $\Delta_1 = \Delta$). We then have the following.

Theorem 2.1 *Given the DGP* (1), *suppose that*

$$\mu_i = \mu_0 + T^{-\delta} g(i/T), \tag{16}$$

where $\delta \leq 1/2$ and g is a function of bounded variation on $[0,1]$. *If* [G1] *and* [G2] *hold with $\mathcal{L}_T Y_T = \mathcal{L} Y_T + o_p(1)$, then for $\delta = 1/2$,*

$$\lambda(\mathcal{L}_T Y_T) \Rightarrow \lambda(\mathcal{L}(W + Jg));$$

for $\delta < 1/2$,

$$T^{\delta - 1/2}\lambda(\mathcal{L}_T Y_T) \to^p \lambda(\mathcal{L}(Jg)).$$

Under the null hypothesis, g is identically zero so that this class of tests converges in distribution to $\lambda(\mathcal{L}W)$. The first result indicates that under local alternatives of order $T^{-1/2}$, $\lambda(\mathcal{L}_T Y_T)$ has non-trivial local power, provided that $\mathcal{L}Jg \neq 0$; the second conclusion says that the GF test diverges whenever $\lambda(\mathcal{L}Jg) > 0$, hence are consistent against the class of alternatives (16) with $\delta < 1/2$. Note that the term Jg characterizes the deviation of the limiting process under the alternative from the limiting process under the null. Note also that negative values of δ are allowed. Applying this theorem to tests discussed above we immediately obtain many known results in literature as special cases; these results are summarized in the Appendix.

3 Range Tests

We have noted that a typical choice in the existing GF tests is the maximal functional. Other choices are possible; for example, the integral functional is used in the Cramér-von Mises test, and the weighted integral functional is used in the Anderson-Darling test. Following Feller (1951), we consider the range functional (5). Specifically, we consider the RR (range of recursive estimates) test:

$$RR_T = \max_{k=1,\cdots,T} \frac{k}{\sqrt{T}}(\hat{\mu}_k - \hat{\mu}_T) - \min_{\ell=1,\cdots,T} \frac{\ell}{\sqrt{T}}(\hat{\mu}_\ell - \hat{\mu}_T), \tag{17}$$

and the RM (range of moving estimates) test:

$$RM_T = \max_{k=0,\cdots,T-[Th]} \frac{[Th]}{\sqrt{T}}(\tilde{\mu}_{k,h} - \hat{\mu}_T) - \min_{\ell=0,\cdots,T-[Th]} \frac{[Th]}{\sqrt{T}}(\tilde{\mu}_{\ell,h} - \hat{\mu}_T). \tag{18}$$

That is, the RR and RM tests are based on the largest possible difference between the deviations $\hat{\mu}_k - \hat{\mu}_T$ and $\tilde{\mu}_{k,h} - \hat{\mu}_T$, respectively. Intuitively, the range functional can better pick up smaller fluctuations of a process which changes its signs, e.g., if $g(t) = \sin(2\pi t)$, $\max(|g|) = 1$, but $\text{range}(g) = 2$. Note that there is little problem of constructing tests with correct asymptotic size based on either the range or maximal functional. What matters is the behavior of tests under various alternatives. Comparison of tests is done by simulations and will be discussed in section 5.

It is easy to see from (17) and (18) that

$$\begin{aligned} RR_T &= \text{range}(\mathcal{L}_T Y_T; 1) = \text{range}(Y_T^0; 1), \\ RM_{T,h} &= \text{range}(\mathcal{L}_{T,h} Y_T^0; 1-h) = \text{range}(\Delta_{h_T} Y_T^0; 1-h), \end{aligned}$$

where $\mathcal{L}_T$ and $\mathcal{L}_{T,h}$ are defined in (7) and (9), respectively. We then obtain from Theorem 2.1 that:

Theorem 3.1 *Given the DGP* (1) *with* (16), *suppose that the FCLT* (2) *hold. Then for* $\delta = 1/2$, *we have*

$$\begin{aligned} RR_T &\Rightarrow \text{range}(\mathcal{L}(W + Jg); 1), \\ RM_{T,h} &\Rightarrow \text{range}(\mathcal{L}_h(W^0 + Jg); 1-h); \end{aligned}$$

for $\delta < 1/2$,

$$\begin{aligned} T^{\delta-1/2} RR_T &\to^p \text{range}(\mathcal{L}(Jg); 1), \\ T^{\delta-1/2} RM_{T,h} &\to^p \text{range}(\mathcal{L}_h(Jg); 1-h). \end{aligned}$$

where $\mathcal{L}$ *and* $\mathcal{L}_h$ *are such that* $\mathcal{L}f(t) = f(t) - tf(1)$ *and* $\mathcal{L}_h f(t) = \Delta_h f^0(t)$.

Under the null hypothesis, we thus have

$$\begin{aligned} RR_T &\Rightarrow \text{range}(W^0; 1), \\ RM_{T,h} &\Rightarrow \text{range}(\Delta_h W^0; 1-h). \end{aligned}$$

It is noted in Chu, Hornik, & Kuan (1992a) that, if g is periodic with period h and if $1/h$ is an integer, then $\mathcal{L}_h Jg = 0$. Consequently, the RM test has only trivial power (or is inconsistent) for local (or non-local) alternatives with this type of g function. As far as the asymptotic null distribution is concerned, it is well known that (see e.g., Shorack & Wellner (1986, p. 142)),

$$\mathbb{P}\{\text{range}(W^0; 1) \leq s\} = 1 - 2\sum_{k=1}^{\infty}(4k^2s^2 - 1)e^{-2k^2s^2}, \tag{19}$$

which is the distribution of the Kuiper (1960) statistic. A detailed table of this distribution can be found in Shorack & Wellner (1986, p. 144). We note that this distribution can be easily derived from Equation (4.3) of Feller (1951), cf. Dudley (1976). The distribution of the range of $\Delta_h W^0$ on $[0, 1-h]$ is unknown, but for $1/2 \leq h < 1$ it can be represented in terms of the range of a Wiener process on $[0, 1]$, as shown in the following theorem.

Theorem 3.2 *For* $1/2 \leq h < 1$,

$$\text{range}(\Delta_h W^0; 1-h) =^d \sqrt{2(1-h)}\ \text{range}(W; 1).$$

Let ϕ and Φ denote the density and distribution functions of the standard normal random variable, respectively. Feller (1951) shows that the density of range$(W; 1)$ at $w > 0$ is

$$8\sum_{k=1}^{\infty}(-1)^{k-1}k^2\phi(kw).$$

It follows that

$$\begin{aligned}
&\mathbb{P}\{\text{range}(W; 1) \leq s\} \\
&= 1 - 8\int_s^{\infty}\sum_{k=1}^{\infty}(-1)^{k-1}k^2\phi(kw)\,dw \\
&= 1 - 8\sum_{k=1}^{\infty}(-1)^{k-1}k\Phi(-ks).
\end{aligned}$$

With a little more effort we obtain an equivalent series representation of this probability.

Corollary 3.3 *Under the null hypothesis, for* $s > 0$ *and* $h \geq 1/2$,

$$\begin{aligned}
&\lim_{T\to\infty}\mathbb{P}\{RM_{T,h} \leq \sqrt{2(1-h)}s\} \\
&= 1 - 8\sum_{k=1}^{\infty}(-1)^{k-1}k\Phi(-ks) \\
&= 8\sum_{j=1}^{\infty}\left(\frac{1}{(2j-1)^2\pi^2} + \frac{1}{s^2}\right)e^{-(2j-1)^2\pi^2/2s^2}.
\end{aligned}$$

The asymptotic critical values for the RR and RM tests with $h \geq 1/2$ can then be solved from the formulae above. Table 1 summarizes some of these critical values; the critical values of $RM_{T,h}$ with $h > 1/2$ are not included because they are those of $RM_{T,1/2}$ times $(2(1-h))^{1/2}$. Asymptotic critical values of the RM test with $h < 1/2$ can be obtained by simulating the behavior of $\Delta_h W^0$ on $[0, 1-h]$. Simulated critical values for various h based on a sample of 2000 are summarized in Table 2. Note that the simulated critical values for the RM test with $h = 1/2$ are quite close to those in Table 1. Using a larger sample of 3000 or 5000 only results in a slight improvement, however.

4 Extension to Multiple Regression

The general approach of Section 2 can be extended to multiple regression models. Consider now the DGP:

$$y_i = x_i'\beta_i + \epsilon_i, \qquad i = 1, \cdots, T, \tag{20}$$

where x_i is the $n \times 1$ vector of explanatory variables. The null hypothesis is $\beta_i = \beta_0$ for all i. Following Krämer, Ploberger, & Alt (1988), we assume:

[M1] $\{\epsilon_i\}$ is a martingale difference sequence with respect to $\{\mathcal{F}^i\}$, the σ-algebra generated by $\{(x_{t+1}, \epsilon_t), t \leq i\}$ such that $\mathbb{E}(\epsilon_i^2 | \mathcal{F}^{i-1}) = \sigma^2$.

[M2] $\{x_i\}$ is such that $\limsup_{T\to\infty} T^{-1} \sum_{i=1}^{T} \mathbb{E}|x_i|^{2+\delta} < \infty$, and

$$Q_{[Tt]} = \frac{1}{[Tt]} \sum_{i=1}^{[Tt]} x_i x_i' \to^p Q, \tag{21}$$

uniformly in $c < t \leq 1$, $c > 0$, where Q is a non-stochastic, positive definite matrix.

Under these conditions, if $\hat{\sigma}_T^2$ is a consistent estimator for σ^2, we have

$$\left(\frac{1}{\hat{\sigma}_T \sqrt{T}} Q_T^{-1/2} \sum_{i=1}^{[Tt]} x_i \epsilon_i, \quad 0 \leq t \leq 1 \right) \Rightarrow \boldsymbol{W}, \tag{22}$$

where $\boldsymbol{W}$ is an n-dimensional, standard Wiener process. We also let $\boldsymbol{W}^0$ denote the n-dimensional Brownian bridge.

Define now the piecewise constant process Y_T on $[0, 1]$ with jump points:

$$Y_T\left(\frac{k}{T}\right) = \frac{1}{\hat{\sigma}_T \sqrt{T}} Q_T^{1/2} Q_k^{-1} \sum_{i=1}^{k} x_i y_i, \tag{23}$$

so that under the null hypothesis

$$Y_T(t) \;=\; \sqrt{T}\frac{[Tt]}{\hat\sigma_T T}Q_T^{1/2}\beta_0 + \frac{1}{\hat\sigma_T\sqrt{T}}Q_T^{1/2}Q_{[Tt]}^{-1}\sum_{i=1}^{[Tt]} x_i\epsilon_i.$$

The first term on the right-hand side is the "straight line" component to be removed by an operator $\mathcal{L}_T$; the second term is the component satisfying the FCLT. In the present context, σ^2 and Q must be estimated suitably to ensure proper FCLT effect. Now $\mathcal{L}_T$ and $\mathcal{L}$ in [G1] are linear operators from $D[0,1]^n$ to $D[0,\tau]^n$, and λ in [G2] is a functional in $D[0,1]^n$. For f in $D[0,1]^n$ with elements f_i, define

$$\text{range}(f;\tau) \;=\; \max_{i=1,\cdots,n}(\max(f_i;\tau) - \min(f_i;\tau)),$$

and let $\|.\|$ denote the maximal norm.

Let the recursive OLS estimates be

$$\hat\beta_k = \left(\sum_{i=1}^{k} x_i x_i'\right)^{-1}\sum_{i=1}^{k} x_i y_i, \qquad k = n,\cdots,T,$$

and the moving OLS estimates be

$$\tilde\beta_{k,h} = \left(\sum_{i=k+1}^{k+[Th]} x_i x_i'\right)^{-1}\sum_{i=k+1}^{k+[Th]} x_i y_i, \qquad k = 0,\cdots,T-[Th].$$

It can be easily verified that

$$RE_T \;=\; \max_{k=n,\cdots,T}\frac{k}{\hat\sigma_T\sqrt{T}}\|Q_T^{1/2}(\hat\beta_k - \hat\beta_T)\| \;=\; \max(\|\mathcal{L}_T Y_T\|;1)$$

with $\mathcal{L}_T$ defined in (7) and that

$$ME_{T,h} \;=\; \max_{k=0,\cdots,T-[Th]}\frac{[Th]}{\sqrt{T}}\|\hat D_T^{-1/2}(\tilde\beta_{k,h} - \hat\beta_T)\| \;=\; \max(\|\mathcal{L}_{T,h}Y_T\|;1-h)$$

with $\mathcal{L}_{T,h}$ defined in (9). We also have

$$\begin{aligned} RR_T \;&=\; \max_{i=1,\cdots,n}\Bigg(\max_{k=n,\cdots,T}\frac{k}{\sqrt{T}}[\hat D_T^{-1/2}(\hat\beta_k - \hat\beta_T)]_i - \\ &\qquad \min_{\ell=n,\cdots,T}\frac{\ell}{\sqrt{T}}[\hat D_T^{-1/2}(\hat\beta_\ell - \hat\beta_T)]_i\Bigg) \\ &=\; \text{range}(\mathcal{L}_T Y_T;1) \end{aligned} \tag{24}$$

$$\begin{aligned} RM_{T,h} \;&=\; \max_{i=1,\cdots,n}\Bigg(\max_{k=0,\cdots,T-[Th]}\frac{[Th]}{\sqrt{T}}[\hat D_T^{-1/2}(\tilde\beta_{k,h} - \hat\beta_T)]_i - \\ &\qquad \min_{\ell=0,\cdots,T-[Th]}\frac{[Th]}{\sqrt{T}}[\hat D_T - 1/2(\tilde\beta_{\ell,h} - \hat\beta_T)]_i\Bigg) \\ &=\; \text{range}(\mathcal{L}_{T,h}Y_T;1-h). \end{aligned} \tag{25}$$

Let $\hat{\sigma}_T^2 = T^{-1}\sum_{i=1}^T (y_i - x_i'\hat{\beta}_T)^2$ be the estimate of σ^2. Then under the alternative

$$\beta_i = \beta_0 + T^{-\delta} g(i/T), \tag{26}$$

where $\delta \le 1/2$ and g is a vector-valued function of bounded variation on $[0,1]$, we have $\hat{\sigma}_T^2 \to^p \sigma_\delta^2$, where

$$\sigma_\delta^2 = \begin{cases} \sigma^2, & 0 < \delta \le \frac{1}{2}, \\ \sigma^2 + \int_0^1 \left(g(u) - \int_0^1 g(v)\,dv\right)' Q \left(g(u) - \int_0^1 g(v)\,dv\right) du, & \delta = 0; \end{cases}$$

see e.g., Chu, Hornik, & Kuan (1992a). The result below is an extension of Theorem 2.1:

Theorem 4.1 *Given the DGP* (20) *with* (26), *suppose that* [M1] *and* [M2] *hold. If* $\mathcal{L}_T Y_T = \mathcal{L} Y_T + o_p(1)$ *for some* $\mathcal{L}$, *then for* $\delta = 1/2$,

$$\lambda(\mathcal{L}_T Y_T) \Rightarrow \lambda(\mathcal{L}(W + \sigma^{-1} Q^{1/2} J g));$$

for $\delta < 1/2$,

$$T^{\delta - 1/2} \lambda(\mathcal{L}_T Y_T) \to^p \lambda(\mathcal{L}(\sigma_\delta^{-1} Q^{1/2} J g)).$$

It is now straightforward to verify that Theorem 2 of Ploberger, Krämer, & Kontrus (1989) and Corollary 4.4 of Chu, Hornik, & Kuan (1992a) can be obtained from this theorem. For range tests we have, analogous to Theorem 3.1:

Corollary 4.2 *Given the DGP* (20) *with* (26), *suppose that the conditions* [M1] *and* [M2] *hold. Then for* $\delta = 1/2$, *we have*

$$\begin{aligned} RR_T &\Rightarrow \text{range}(\mathcal{L}(W + \sigma^{-1} Q^{1/2} J g); 1), \\ RM_{T,h} &\Rightarrow \text{range}(\mathcal{L}_h(W^0 + \sigma^{-1} Q^{1/2} J g); 1 - h); \end{aligned}$$

for $\delta < 1/2$,

$$\begin{aligned} T^{\delta - 1/2} RR_T &\to^p \text{range}(\mathcal{L}(\sigma_\delta^{-1} Q^{1/2} J g); 1), \\ T^{\delta - 1/2} RM_{T,h} &\to^p \text{range}(\mathcal{L}_h(\sigma_\delta^{-1} Q^{1/2} J g); 1 - h), \end{aligned}$$

where $\mathcal{L}$ *and* $\mathcal{L}_h$ *are such that* $\mathcal{L} f(t) = f(t) - t f(1)$ *and* $\mathcal{L}_h f(t) = \Delta_h f^0(t)$.

Corollary 4.2 implies that under the null hypothesis,

$$\begin{aligned} RR_T &\Rightarrow \text{range}(W^0; 1), \\ RM_{T,h} &\Rightarrow \text{range}(\Delta_h W^0; 1 - h). \end{aligned}$$

Then by (19) and Corollary 3.3, we have the following distributions.

Corollary 4.3 *Under the null hypothesis, for $s > 0$,*

$$\lim_{T\to\infty} \mathbb{P}\{RR_T \le s\} = \left(1 - 2\sum_{k=1}^{\infty}(4k^2s^2 - 1)e^{-2k^2s^2}\right)^n,$$

and for $h \ge 1/2$,

$$\lim_{T\to\infty} \mathbb{P}\{RM_{T,h} \le \sqrt{2(1-h)s}\} = \left(1 - 8\sum_{k=1}^{\infty}(-1)^{k-1}k\Phi(-ks)\right)^n.$$

Simulated critical values of the RM test with various h and n up to 5 are summarized in Table 2. Other critical values for $n = 6, \ldots, 10$ are available upon request.

For residual-based tests, consider the empirical process Y_T with jump points:

$$Y_T(k/T) = \frac{1}{\hat{\sigma}\sqrt{T}}\sum_{i=1}^{k} y_i.$$

It is readily seen that the straight line component of Y_T can be removed exactly if $x_i'\hat{\beta}_{k-1}$ or $x_i'\hat{\beta}_T$ is subtracted from y_i. Hence, the CUSUM- and MOSUM-type of tests are GF tests. Additional structures are needed to incorporate residual-based tests into the functional-operator framework, however. To reduce excessive notations, we do not pursue this possibility here.

5 Simulations

In this section we evaluate finite-sample performance of different tests by simulations. Size simulations are based on the location model

$$y_i = 2 + \epsilon_i,$$

where ϵ_i are i.i.d. $N(0, 1)$. We consider the RR test and RM tests with $h = 0.1, \ldots, 0.5$ and samples T = 100, 200, 300, and 500. The number of replications is 10000. These results are summarized in Table 3. It can be seen that all tests are conservative but not very different from nominal sizes; in particular, the RR test has the largest size distortion in different finite samples, and the RM tests with smaller window bandwidth h has larger size distortion.

In power simulations competing tests we consider are the ME, RE, MAX-F (Andrews (1993)), AVG-F and EXP-F (Andrews & Ploberger (1992) and Andrews, Lee, &

Ploberger (1992)) tests. Note that the AVG-F and EXP-F tests are optimal in the sense of Andrews & Ploberger (1992). For moving-estimates based tests, we compute tests with $h = 0.1$, 0.2 and 0.5. All power results are based on empirical critical values simulated from a sample of 100 observations with 10000 replications. In what follows we shall write moving-estimates based tests as ME(h) or RM(h). The empirical critical values are RM(0.1)=1.602, RM(0.2) = 2.005, RM(0.5)=2.065, ME(0.1)=0.910, ME(0.2)=1.149, ME(0.5)=1.289, RR=1.472, RE=1.176. The MAX-F, AVG-F and EXP-F tests are computed specifically for the alternative in simulations. For the alternative of a single structural change:

$$y_i = \begin{cases} 2 + \epsilon_i, & i = 1, \cdots, [T\lambda], \\ 2 + \Delta + \epsilon_i, & i = [T\lambda] + 1, \cdots, T, \end{cases} \tag{27}$$

empirical critical values are MAX-F=7.328, AVG-F=2.157, EXP-F=1.60, which are computed for treating each observation $[Ts]$, $s \in [0.1, 0.9]$, as a hypothetical change point. For the alternative of double structural changes:

$$y_i = \begin{cases} 2 + \epsilon_i, & i = 1, \cdots, [T\lambda_1], \\ 2 + \Delta_1 + \epsilon_i, & i = [T\lambda_1] + 1, \cdots, [T\lambda_2], \\ 2 + \Delta_2 + \epsilon_i, & i = [T\lambda_2] + 1, \cdots, T, \end{cases} \tag{28}$$

empirical critical values are MAX-F=5.718, AVG-F=1.861, EXP-F=2.756, which are computed by treating each pair of observations $([Ts_1], [Ts_2])$, $s_1 \in [0.1, 0.85]$ and $s_2 = s_1 + 0.05, \ldots, 0.9$, as a pair of two change points. Note that the trimming of observations is arbitrary; see Andrews (1993), Andrews & Ploberger (1992), and Andrews, Lee & Ploberger (1992).

For the alternative of a single structural change (27), we consider two cases: $\Delta = 0.5$ and 0.25. The number of replications is 5000. Because these tests have symmetric performance, we only report results for $\lambda = 0.1, \ldots, 0.5$ in Table 4. We can ignore the ME tests in this case because Chu, Hornik, & Kuan (1992a) have shown that under a single change the RE test dominates the ME test for every possible change point. We observe from Table 4A that:

1. $\lambda = 0.1$, the MAX-F test is the best;

2. $\lambda = 0.2$, the AVG-F and EXP-F tests are the best;

3. $\lambda = 0.3$, the RE, AVG-F and EXP-F tests are the best;

4. $\lambda = 0.4$, the RM(1/2), RE, AVG-F and EXP-F tests are the best;

5. $\lambda = 0.5$, the RM(1/2) test is the best.

When the parameter changes becomes smaller, the differences between these tests are less significant. It is interesting to note that it is possible to find some tests outperforming the AVG-F and EXP-F tests which are optimal.

For the alternative of double structural changes (28), we consider four cases: $\Delta_1 = 0.5$ with $\Delta_2 = 0.75, 0.25, 0, -0.25$. The first change points λ_1 are 0.2, 0.4, 0.6 and 0.8, the second change points are $\lambda_1 + 0.1, \ldots, 0.9$, and the number of replications is 5000. These results are summarized in Table 5. The results are quite mixed; for example:

1. $\lambda_1 = 0.2$ and $\lambda_2 = 0.5$: the best tests are AVG-F in Table 5A, RR in Table 5B, RM(0.5) and RR in Table 5C, and RM(0.5) in Table 5D. In this case, the RM(0.2) test performs similarly to the AVG-F or EXP-F test in Tables 5B, 5C and 5D.

2. $\lambda_1 = 0.4$ and $\lambda_2 = 0.9$: the best tests are RE and AVG-F in Table 5A, RM(0.5) in Table 5B, and ME(0.5) and RR in Tables 5C and 5D.

3. $\lambda_1 = 0.6$ and $\lambda_2 = 0.9$: the best tests are RE and AVG-F in Table 5A, RM(0.5) in Tables 5B and 5C, and RR in Table 5D.

In particular, there is no test uniformly better than the other tests.

6 Conclusions

In this paper we provide a unifying view of the tests for parameter constancy which are determined by the fluctuation of empirical processes. We establish a unified asymptotic result which allows us to analyze the behavior of these tests quite easily. As applications we also consider tests based on the range functional, rather than the typical maximal functional, and characterize their asymptotic null distributions. Our simulation results show that tests may have very different power performance under different alternatives and that it is possible to find tests outperforming tests that are optimal in the sense of Andrews & Ploberger (1992). What we want to convey here is that if one is uncertain about the behavior of parameter changes, it would be better to conduct a family of tests to safeguard various directions of alternatives. For this purpose, different estimates-based tests can be easily computed and complement other likelihood-based tests.

Appendix

Proof of Theorem 2.1: Let V_g and M_g be the variation of g on $[0,1]$ and $\max(|g|;1)$, respectively. Clearly,

$$Y_T(t) = \frac{1}{\sqrt{T}}\left([Tt]\mu_0 + T^{1-\delta}\frac{1}{T}\sum_{i=1}^{[Tt]} g(t_i) + \sum_{i=1}^{[Tt]} \epsilon_i\right),$$

where $t_i = i/T$. Hence, as $|[Tt]/T - t| = |[Tt] - Tt|/T \leq 1/T$ and

$$\begin{aligned}
&\left|\frac{1}{T}\sum_{i=1}^{[Tt]} g(t_i) - \int_0^t g(s)\,ds\right| \\
&\quad = \left|\sum_{i=1}^{[Tt]} \int_{t_{i-1}}^{t_i} (g(t_i) - g(s))\,ds - \int_{[Tt]/T}^{t} g(s)\,ds\right| \\
&\quad = \left|\int_0^{1/T} \sum_{i=1}^{[Tt]} (g(t_i) - g(t_{i-1}+s))\,ds - \int_{[Tt]/T}^{t} g(s)\,ds\right| \\
&\quad \leq (V_g + M_g)/T,
\end{aligned}$$

we have

$$Y_T = T^{1/2}\iota_T\mu_0 + T^{1/2-\delta}Jg + E_T + R_T,$$

where $|R_T(t)| \leq T^{-1/2-\delta}(V_g + M_g)$. As $\mathcal{L}_T$ annihilates ι_T,

$$\lambda(\mathcal{L}_T Y_T) = \lambda(T^{1/2-\delta}\mathcal{L}Jg + \mathcal{L}E_T + \mathcal{L}R_T + o_p(1)).$$

We immediately conclude that for $\delta = 1/2$, $\lambda(\mathcal{L}_T Y_T) \Rightarrow \lambda(\mathcal{L}Jg + \mathcal{L}W)$, and that for $\delta < 1/2$, $T^{\delta-1/2}\lambda(\mathcal{L}_T Y_T) \to^p \lambda(\mathcal{L}Jg)$ as asserted. □

Applications of Theorem 2.1: It is easily verified that for $\mathcal{L}_T$ in the RE test, the corresponding $\mathcal{L}$ is such that

$$\mathcal{L}f(t) = f(t) - tf(1).$$

For $\mathcal{L}_{T,h}$ in the ME test, the corresponding $\mathcal{L}_h$ is such that

$$\mathcal{L}_h f(t) = f(t+h) - f(t) - hf(1) = \Delta_h f(t) - hf(1);$$

see also Chu, Hornik, & Kuan (1992a). For $\mathcal{L}_T$ in the Recursive-CUSUM test, the corresponding $\mathcal{L}$ is such that

$$\mathcal{L}f(t) = f(t) - \int_0^t \frac{f(\tau)}{\tau}\,d\tau;$$

for $\mathcal{L}_{T,h}$ in the Recursive-MOSUM test, the corresponding $\mathcal{L}_h$ is such that

$$\begin{aligned}\mathcal{L}_h f(t) &= f(t+h) - f(t) - \int_t^{t+h} \frac{f(\tau)}{\tau}\, d\tau;\\ &= \Delta_h f(t) - \Delta_h \int_0^t \frac{f(\tau)}{\tau}\, d\tau.\end{aligned}$$

Given the DGP (1), the results of the RE, ME, CUSUM, and MOSUM tests now follow straightforwardly from Theorem 2.1. For the Recursive-CUSUM test, note that

$$Z(t) := W(t) - \int_0^t \frac{W(\tau)}{\tau}\, d\tau$$

is a Gaussian process with continuous sample paths, mean zero, and covariance function $\min(t,s)$, hence a Wiener process. □

Proof of Theorem 3.1: Straightforward application of Theorem 2.1. □

To prove Theorem 3.2, we utilize the following two lemmas.

Lemma A.1 *For* $0 < h < 1$,

$$\text{range}(\Delta_h W^0; 1-h) =^d \sqrt{h}\,\text{range}(\Delta W; (1-h)/h).$$

Proof: Note that

$$\begin{aligned}\text{range}(\Delta_h W^0; 1-h) &= \max_{0\le s,t\le 1-h} |\Delta_h W^0(t) - \Delta_h W^0(s)|\\ &= \max_{0\le s,t\le 1-h} |\Delta_h W(t) - \Delta_h W(s)|\\ &= \text{range}(\Delta_h W; 1-h).\end{aligned}$$

As $W_h(u) = h^{-1/2} W(hu)$ is a Wiener process,

$$\begin{aligned}&(h^{-1/2}\Delta_h W(t), 0 \le t \le 1-h)\\ &=^d (h^{-1/2}\Delta_h W(hu), 0 \le u \le (1-h)/h)\\ &=^d (\Delta W_h(u), 0 \le u \le (1-h)/h)\\ &=^d (\Delta W(u), 0 \le u \le (1-h)/h). \quad \square\end{aligned}$$

Lemma A.2 *For* $0 < \tau \le 1$,

$$\text{range}(\Delta W; \tau) =^d \sqrt{2\tau}\,\text{range}(W; 1).$$

Proof: Let C_τ be the space of continuous functions on $[0, \tau]$, and let $\bar{\mu}_x$, μ_x and μ_W be the measures on C_τ induced by ΔW conditional on $\Delta W(0) = x$, by $x + \sqrt{2}W$, and by W, respectively. By (16.11) of Shepp (1966),

$$\frac{d\bar{\mu}_x}{d\mu_x}(f) = (2/(2-\tau))^{1/2} e^{x^2/2} e^{-(x+f(\tau))^2/4(2-\tau)};$$

hence, as under μ_x, the functions $g(t) = (f(t) - x)/\sqrt{2}$ are distributed according to μ_W, we have

$$\begin{aligned}
&\mathbb{P}\{\text{range}(\Delta W;\tau) \le s | \Delta W(0) = x\} \\
&= \int_{\text{range}(f;\tau)\le s} (2/(2-\tau))^{1/2} e^{x^2/2} e^{-(x+f(\tau))^2/4(2-\tau)}\, d\mu_x(f) \\
&= \int_{\text{range}(g;\tau)\le s/\sqrt{2}} (2/(2-\tau))^{1/2} e^{x^2/2} e^{-(2x+\sqrt{2}g(\tau))^2/4(2-\tau)}\, d\mu_W(g) \\
&= \int_{\mathbb{R}} (2/(2-\tau))^{1/2} e^{x^2/2} e^{-(2x+\sqrt{2}y))^2/4(2-\tau)} \\
&\qquad \times \quad d\mathbb{P}\{\text{range}(W;\tau) \le s/\sqrt{2}, W(\tau) \le y\}
\end{aligned}$$

and thus

$$\begin{aligned}
&\mathbb{P}\{\text{range}(\Delta W;\tau) \le s\} \\
&= \int_{\mathbb{R}} \mathbb{P}\{\text{range}(\Delta W;\tau) \le s | \Delta W(0) = x\}\, \phi(x)\, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} (\pi(2-\tau))^{-1/2} e^{-(2x+\sqrt{2}y)^2/4(2-\tau)} \\
&\qquad \times \quad d\mathbb{P}\{\text{range}(W;\tau) \le s/\sqrt{2}, W(\tau) \le y\}\, dx \\
&= \int_{\mathbb{R}} d\mathbb{P}\{\text{range}(W;\tau) \le s/\sqrt{2}, W(\tau) \le y\} \\
&= \mathbb{P}\{\text{range}(W;\tau) \le s/\sqrt{2}\} \\
&= \mathbb{P}\{\text{range}(W;1) \le s/\sqrt{2\tau}\},
\end{aligned}$$

where the last equation again follows by rescaling. □

Proof of Theorem 3.2: By successively putting together the previous lemmas, we have

$$\begin{aligned}
RM_{T;h} &\Rightarrow \text{range}(\Delta_h W^0; 1-h) \\
&=^d \sqrt{h}\, \text{range}(\Delta W; (1-h)/h) \\
&=^d \sqrt{h}\sqrt{2(1-h)/h}\, \text{range}(W;1). \quad \square
\end{aligned}$$

Proof of Corollary 3.3: It remains to show that two expressions of the distribution of the RM test are the same. By the extended Poisson summation formula (see e.g. Feller,

1971) applied to the standard normal density ϕ with characteristic function $\psi(a) = e^{-a^2/2}$, we find that for $w \neq 0$,

$$\sum_{k=-\infty}^{\infty} \phi(kw)e^{ikz} = \frac{1}{w}\sum_{j=-\infty}^{\infty} \psi\left(\frac{z+2j\pi}{w}\right).$$

Differentiating this identity twice with respect to z, we obtain

$$-\sum_{k=-\infty}^{\infty} k^2\phi(kw)e^{ikz} = \frac{1}{w^5}\sum_{j=-\infty}^{\infty} ((z+2j\pi)^2 - w^2)e^{-(z+2j\pi)^2/2w^2},$$

which upon letting $z = \pi$ then gives

$$\sum_{k=-\infty}^{\infty} (-1)^{k-1}k^2\phi(kw) = \frac{1}{w^5}\sum_{j=-\infty}^{\infty} ((2j+1)^2\pi^2 - w^2)e^{-(2j+1)^2\pi^2/2w^2}.$$

Thus, by substituting $u = 1/w$, we have

$$\begin{aligned}
&\mathbb{P}\{\mathrm{range}(W;1) \leq s\} \\
&= 4\int_0^s \sum_{j=-\infty}^{\infty} w^{-5}((2j+1)^2\pi^2 - w^2)e^{-(2j+1)^2\pi^2/2w^2}\,dw \\
&= 4\sum_{j=-\infty}^{\infty} \int_{1/s}^{\infty} ((2j+1)^2\pi^2u^3 - u)e^{-(2j+1)^2\pi^2u^2/2}\,du \\
&= 4\sum_{j=-\infty}^{\infty} \left(-u^2e^{-(2j+1)^2\pi^2u^2/2}\Big|_{1/s}^{\infty} + \int_{1/s}^{\infty} ue^{-(2j+1)^2\pi^2u^2/2}\,du \right) \\
&= 4\sum_{j=-\infty}^{\infty} -\left(u^2 + \frac{1}{(2j+1)^2\pi^2}\right)e^{-(2j+1)^2\pi^2u^2/2}\Big|_{1/s}^{\infty} \\
&= 4\sum_{j=-\infty}^{\infty} \left(\frac{1}{s^2} + \frac{1}{(2j+1)^2\pi^2}\right)e^{-(2j+1)^2\pi^2/2s^2},
\end{aligned}$$

as asserted. □

Proof of Theorem 4.1: The proof is essentially the same of that of Theorem 2.1. Here,

$$\begin{aligned}
Y_T(t) &= \frac{1}{\hat{\sigma}_T\sqrt{T}}\Bigg([Tt]Q_T^{1/2}\beta_0 + T^{1-\delta}Q_T^{1/2}Q_{[Tt]}^{-1}\frac{1}{T}\sum_{i=1}^{[Tt]} x_ix_i'g(t_i) + \\
&\qquad Q_T^{1/2}Q_{[Tt]}^{-1}\sum_{i=1}^{[Tt]} x_i\epsilon_i \Bigg).
\end{aligned}$$

We observe that for some M^*,

$$\begin{aligned}
&\left\| \frac{1}{T} \sum_{i=1}^{[Tt]} x_i x_i' g(t_i) - Q \int_0^t g(s)\, ds \right\| \\
&\quad \leq \left\| \frac{1}{T} \sum_{i=1}^{[Tt]} (x_i x_i' - Q) g(t_i) \right\| + \left\| Q \left(\frac{1}{T} \sum_{i=1}^{[Tt]} g(t_i) - \int_0^t g(s)\, ds \right) \right\| \\
&\quad \leq M^*/T.
\end{aligned}$$

Hence,

$$Y_T = \frac{1}{\hat{\sigma}_T} \left(T^{1/2} \iota_T Q^{1/2} \beta_0 + T^{1/2-\delta} Q^{1/2}\, Jg + Q^{-1/2} \sum_{i=1}^{[Tt]} x_i \epsilon_i + R_T + o_p(1) \right),$$

where $\|R_T(t)\| \leq T^{-1/2-\delta} M^*$. Thus, for $\delta = 1/2$, $\hat{\sigma}_T^2 \rightarrow^p \sigma^2$ and

$$\lambda(\mathcal{L}_T Y_T) \Rightarrow \lambda(\mathcal{L}W + \sigma^{-1} \mathcal{L} Q^{1/2}\, Jg);$$

and for $\delta < 1/2$, $\hat{\sigma}_T^2 \rightarrow^p \sigma_\delta^2$ and $T^{\delta-1/2} \lambda(\mathcal{L}_T Y_T) \rightarrow^p \lambda(\mathcal{L} \sigma_\delta Q^{1/2}\, Jg)$. □

Proof of Corollary 4.2: Straightforward application of Theorem 4.1. □

Proof of Corollary 4.3: It is easy to see that

$$\begin{aligned}
&\mathbb{P}\{\text{range}(\boldsymbol{W}^0; 1) \leq s\} \\
&\quad = \mathbb{P}\{\text{range}(\boldsymbol{W}_i^0; 1) \leq s \text{ for all } i = 1, \cdots, n\} \\
&\quad = \left(\mathbb{P}\{\text{range}(\boldsymbol{W}_i^0; 1) \leq s\} \right)^n.
\end{aligned}$$

The first assertion follows from (19). Similarly, the second assertion follows from Corollary 3.3. □

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Table 1: Asymptotic Critical Values of Range Tests.

Tests	n	Tail Probability					
		0.20	0.15	0.10	0.05	0.025	0.01
RR	1	1.47337	1.53692	1.61960	1.74726	1.86243	2.00090
	2	1.60894	1.66698	1.74272	1.86040	1.96747	2.09740
	3	1.68277	1.73796	1.81017	1.92280	2.02578	2.15134
	4	1.73294	1.78629	1.85620	1.96558	2.06590	2.18862
	5	1.77069	1.82270	1.89096	1.99797	2.09637	2.21700
	6	1.80083	1.85179	1.91877	2.02396	2.12085	2.23985
	7	1.82583	1.87596	1.94190	2.04560	2.14128	2.25896
	8	1.84715	1.89658	1.96166	2.06413	2.15878	2.27535
	9	1.86571	1.91454	1.97889	2.08029	2.17407	2.28969
	10	1.88211	1.93043	1.99413	2.09462	2.18764	3.30242
$RM_{1/2}$	1	1.95843	2.07958	2.24117	2.49767	2.73436	3.02334
	2	2.22011	2.33550	2.48844	2.73017	2.95322	3.22650
	3	2.36716	2.47875	2.62645	2.85987	3.07558	3.34055
	4	2.46855	2.57741	2.72147	2.94926	3.16006	3.41950
	5	2.54549	2.65226	2.79357	3.01716	3.22433	3.47968
	6	2.60725	2.71235	2.85147	3.07175	3.27605	3.52819
	7	2.65871	2.76242	2.89974	3.11729	3.31926	3.56875
	8	2.70274	2.80527	2.94106	3.15631	3.35630	3.60357
	9	2.74116	2.84266	2.97713	3.19040	3.38869	3.63404
	10	2.77519	2.87579	3.00910	3.22064	3.41743	3.66110

Notes: Critical values are solved from the formulae in Corollary 4.3 with 5 terms in the summation; n is the number of parameters in a linear regression model.

Table 2: Simulated Asymptotic Critical Values of RM Tests.

n	h	Tail Probability					
		0.20	0.15	0.10	0.05	0.025	0.01
	0.05	1.2758	1.3101	1.3533	1.4208	1.4811	1.5569
	0.10	1.6224	1.6752	1.7418	1.8433	1.9398	2.0514
	0.15	1.8300	1.8986	1.9866	2.1199	2.2409	2.3788
	0.20	1.9600	2.0409	2.1472	2.3100	2.4577	2.6354
1	0.25	2.0421	2.1354	2.2604	2.4450	2.6068	2.8068
	0.30	2.0816	2.1877	2.3230	2.5329	2.7171	2.9385
	0.35	2.0826	2.1976	2.3468	2.5781	2.7922	3.0376
	0.40	2.0648	2.1830	2.3401	2.5885	2.8140	3.0834
	0.45	2.0074	2.1298	2.2941	2.5553	2.7808	3.0715
	0.50	1.9193	2.0407	2.2037	2.4628	2.7023	3.0016
	0.05	1.3464	1.3775	1.4161	1.4784	1.5350	1.6026
	0.10	1.7330	1.7805	1.8410	1.9368	2.0260	2.1282
	0.15	1.9754	2.0372	2.1181	2.2416	2.3495	2.4836
	0.20	2.1324	2.2058	2.3015	2.4501	2.5840	2.7467
2	0.25	2.2432	2.3277	2.4384	2.6072	2.7571	2.9458
	0.30	2.3077	2.4018	2.5285	2.7216	2.8974	3.1107
	0.35	2.3307	2.4357	2.5695	2.7816	2.9755	3.2081
	0.40	2.3207	2.4343	2.5808	2.8103	3.0191	3.2791
	0.45	2.2704	2.3857	2.5437	2.7842	3.0050	3.2872
	0.50	2.1839	2.2997	2.4533	2.6961	2.9243	3.2103
	0.05	1.3857	1.4143	1.4524	1.5128	1.5690	1.6350
	0.10	1.7933	1.8387	1.8974	1.9897	2.0736	2.1748
	0.15	2.0516	2.1098	2.1874	2.3043	2.4080	2.5411
	0.20	2.2313	2.3021	2.3928	2.5326	2.6571	2.8086
3	0.25	2.3506	2.4296	2.5337	2.6996	2.8463	3.0344
	0.30	2.4253	2.5166	2.6345	2.8162	2.9843	3.1839
	0.35	2.4657	2.5648	2.6985	2.9015	3.0861	3.3173
	0.40	2.4620	2.5730	2.7142	2.9326	3.1412	3.3894
	0.45	2.4239	2.5369	2.6831	2.9120	3.1310	3.3938
	0.50	2.3283	2.4406	2.5889	2.8243	3.0355	3.2950
	0.05	1.4115	1.4395	1.4763	1.5345	1.5894	1.6543
	0.10	1.8333	1.8765	1.9341	2.0242	2.1058	2.2074
	0.15	2.1048	2.1605	2.2342	2.3460	2.4534	2.5785
	0.20	2.2919	2.3600	2.4466	2.5871	2.7098	2.8665
4	0.25	2.4232	2.5000	2.6024	2.7609	2.9012	3.0802
	0.30	2.5084	2.5956	2.7105	2.8896	3.0495	3.2457
	0.35	2.5592	2.6527	2.7778	2.9761	3.1576	3.3762
	0.40	2.5610	2.6649	2.8007	3.0145	3.2079	3.4414
	0.45	2.5257	2.6379	2.7805	3.0030	3.2107	3.4561
	0.50	2.4286	2.5387	2.6850	2.9099	3.1216	3.3740
	0.05	1.4322	1.4594	1.4956	1.5536	1.6063	1.6707
	0.10	1.8646	1.9082	1.9629	2.0497	2.1273	2.2281
	0.15	2.1452	2.2004	2.2728	2.3829	2.4855	2.6057
	0.20	2.3430	2.4074	2.4917	2.6258	2.7462	2.8959
5	0.25	2.4800	2.5541	2.6530	2.8074	2.9483	3.1270
	0.30	2.5717	2.6568	2.7705	2.9455	3.1049	3.3065
	0.35	2.6230	2.7165	2.8371	3.0300	3.2068	3.4197
	0.40	2.6367	2.7399	2.8725	3.0892	3.2801	3.5177
	0.45	2.5979	2.7069	2.8463	3.0632	3.2653	3.5153
	0.50	2.5099	2.6158	2.7573	2.9813	3.1916	3.4545

Table 3: Size Simulation at 10% Level.

Sample Size	RM Tests						RR Test
	$h = 0.5^*$	$h = 0.5$	$h = 0.4$	$h = 0.3$	$h = 0.2$	$h = 0.1$	
100	6.5	7.4	7.3	6.7	5.3	4.0	4.2
200	7.6	8.3	8.6	8.0	7.2	5.8	5.9
300	8.0	8.9	8.3	7.9	7.8	6.6	6.1
500	8.4	9.3	9.4	9.0	8.6	7.5	6.7

Notes: All numbers are in percentage. The first column of the RM test size is based on asymptotic critical value from Table 1; other columns of the RM tests are based on simulated asymptotic critical values from Table 2.

Table 4A: Power Simulation under a Single Structural Change: $\Delta = 0.5$.

λ	RM Tests			ME Tests			RR Test	RE Test	Tests for a Single Change		
	$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$			MAX-F	AVG-F	EXP-F
0.1	14.3	14.2	16.6	14.5	12.7	16.0	16.0	16.4	26.0	19.7	23.0
0.2	26.9	36.0	27.4	28.4	32.5	25.5	33.5	39.1	45.2	42.6	45.4
0.3	47.3	46.3	32.9	42.7	42.4	29.7	47.2	57.3	55.9	58.0	58.2
0.4	65.5	50.8	35.2	55.0	45.0	30.6	55.5	67.2	61.1	65.9	64.4
0.5	76.4	54.0	37.3	62.7	46.6	31.5	60.0	70.9	64.2	69.4	67.6

Table 4B: Power Simulation under a Single Structural Change: $\Delta = 0.25$.

λ	RM Tests			ME Tests			RR Test	RE Test	Tests for a Single Change		
	$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$			MAX-F	AVG-F	EXP-F
0.1	11.0	11.2	11.4	10.8	10.8	11.5	11.3	11.7	13.9	12.3	12.8
0.2	14.3	15.6	13.8	15.1	14.3	13.3	16.2	16.4	18.4	18.0	18.4
0.3	21.9	19.4	16.1	19.8	18.4	14.4	21.1	24.1	22.8	25.0	24.4
0.4	27.2	21.8	17.0	22.3	19.9	15.4	22.4	26.6	24.2	26.8	25.5
0.5	31.8	21.7	16.0	23.5	19.8	14.7	22.1	27.3	24.4	28.2	26.8

Table 5A: Power Simulation under Double Structural Changes: $\Delta_1 = 0.5$ and $\Delta_2 = 0.75$.

λ_1	λ_2	RM Tests			ME Tests			RR	RE	Tests for Double Changes		
		$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	Test	Test	MAX-F	AVG-F	EXP-F
0.2	0.3	58.4	68.9	51.5	56.0	65.1	47.5	65.5	76.6	69.0	80.9	78.3
	0.4	64.5	65.1	48.9	57.0	60.5	44.1	63.5	75.9	65.8	80.9	76.1
	0.5	69.3	62.4	46.6	57.1	58.1	41.6	60.3	75.1	63.1	78.6	73.9
	0.6	60.2	58.6	43.1	47.1	54.4	38.8	54.7	72.2	60.1	77.1	71.4
	0.7	49.6	53.6	39.2	37.5	49.3	35.6	46.3	66.2	55.3	71.6	66.3
	0.8	39.6	47.4	36.5	29.5	43.7	33.6	36.7	57.3	48.8	64.5	59.1
	0.9	30.6	38.9	31.4	24.4	36.6	28.8	31.1	47.0	42.1	53.3	50.2
0.4	0.5	94.5	83.5	63.7	88.7	76.3	53.2	88.7	94.5	85.1	94.0	91.5
	0.6	91.0	78.5	57.5	83.0	71.1	48.4	84.1	92.1	81.5	92.1	88.3
	0.7	85.7	73.4	53.4	75.3	66.4	45.0	76.9	88.6	74.9	88.7	83.9
	0.8	78.0	65.8	47.3	65.7	58.3	39.4	67.5	83.0	68.4	83.7	78.6
	0.9	70.0	55.8	41.9	56.5	49.4	35.9	58.8	75.3	60.4	76.1	70.7
0.6	0.7	88.3	79.5	60.1	82.0	73.5	51.5	84.3	92.4	82.0	91.7	89.0
	0.8	82.0	72.1	53.4	73.9	65.7	45.7	75.7	86.4	75.3	87.4	83.3
	0.9	73.2	58.2	44.3	63.9	50.8	37.6	64.4	78.8	64.5	79.2	74.7
0.8	0.9	36.3	49.2	41.0	36.5	45.4	38.2	41.7	55.6	53.6	64.6	62.9

Table 5B: Power Simulation under Double Structural Changes: $\Delta_1 = 0.5$ and $\Delta_2 = 0.25$.

λ_1	λ_2	RM Tests			ME Tests			RR	RE	Tests for Double Changes		
		$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	Test	Test	MAX-F	AVG-F	EXP-F
0.2	0.3	12.3	16.2	16.0	13.4	15.4	14.8	15.4	13.3	18.4	17.0	18.7
	0.4	13.4	23.6	19.9	17.7	22.6	18.8	21.6	15.0	24.7	19.2	23.9
	0.5	15.1	26.9	21.3	24.2	24.6	19.8	29.0	16.9	27.1	23.1	27.8
	0.6	18.0	31.4	24.7	32.5	28.0	23.1	36.3	21.1	32.6	29.3	34.3
	0.7	21.7	32.8	26.1	39.6	29.5	22.9	40.1	22.9	34.2	34.1	36.5
	0.8	23.1	34.0	26.9	39.0	29.5	24.0	42.5	26.7	35.9	37.0	39.5
	0.9	26.0	34.0	26.9	34.1	31.3	23.8	39.3	30.9	36.4	40.4	41.4
0.4	0.5	23.0	24.6	19.7	23.4	22.0	19.1	25.0	27.5	24.6	27.9	27.1
	0.6	30.3	34.4	25.6	31.9	31.2	22.5	34.0	33.2	32.1	33.1	34.0
	0.7	40.5	40.2	28.6	41.1	36.9	26.2	44.1	41.0	38.6	41.2	42.0
	0.8	52.3	44.5	31.4	51.0	40.0	27.2	52.2	49.3	43.6	49.5	48.8
	0.9	61.7	49.1	34.4	57.5	44.2	30.0	57.2	58.0	49.0	57.0	55.3
0.6	0.7	35.9	26.7	21.2	27.6	24.4	19.8	27.1	33.1	26.3	31.4	30.1
	0.8	47.0	39.1	28.6	37.2	35.8	26.0	39.6	43.9	35.7	41.3	39.7
	0.9	58.0	47.9	33.3	47.7	43.0	29.1	50.9	55.7	44.7	54.1	50.9
0.8	0.9	21.7	25.5	21.1	20.3	23.0	20.3	22.3	25.9	24.6	28.8	27.5

Table 5C: Power Simulation under Double Structural Changes: $\Delta_1 = 0.5$ and $\Delta_2 = 0$.

λ_1	λ_2	RM Tests			ME Tests			RR	RE	Tests for Double Changes		
		$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	Test	Test	MAX-F	AVG-F	EXP-F
0.2	0.3	14.9	17.7	18.3	15.1	16.6	17.6	15.9	14.3	19.4	14.3	17.4
	0.4	32.3	36.4	28.7	27.8	35.9	26.1	32.8	24.3	32.1	25.8	31.0
	0.5	47.3	44.4	31.9	42.7	42.7	28.9	48.2	32.2	42.3	37.6	42.7
	0.6	43.1	48.3	35.7	55.2	42.9	31.3	57.4	35.5	47.6	45.9	49.4
	0.7	38.4	48.6	35.8	63.3	42.3	29.8	61.7	33.0	50.6	49.7	52.9
	0.8	29.0	42.5	32.8	55.7	36.7	27.8	59.1	27.3	47.6	48.2	50.5
	0.9	25.2	34.2	29.4	42.5	30.4	26.0	49.6	25.6	41.9	42.1	45.1
0.4	0.5	14.4	18.5	18.1	14.8	18.1	18.2	16.5	12.4	18.8	13.3	16.5
	0.6	16.1	36.7	28.6	28.3	36.3	27.2	33.3	20.1	32.1	23.5	30.1
	0.7	26.9	46.0	32.6	43.8	43.6	29.4	50.2	27.7	43.3	37.2	43.5
	0.8	42.6	47.9	35.6	56.0	44.6	31.3	59.1	35.1	48.0	45.5	50.1
	0.9	59.0	48.0	34.3	62.8	42.3	28.8	62.0	48.9	50.8	54.3	55.1
0.6	0.7	15.8	18.6	18.5	15.0	17.4	18.3	17.3	13.0	19.2	13.1	16.9
	0.8	31.7	35.4	27.2	27.6	35.8	25.8	32.1	23.0	31.4	25.4	30.4
	0.9	51.7	44.9	33.0	43.0	42.9	30.4	48.0	43.5	43.4	44.1	45.5
0.8	0.9	15.8	18.9	18.7	15.3	17.1	18.7	16.2	15.4	18.4	16.8	17.5

Table 5D: Power Simulation under Double Structural Changes: $\Delta_1 = 0.5$ and $\Delta_2 = -0.25$.

λ_1	λ_2	RM Tests			ME Tests			RR	RE	Tests for Double Changes		
		$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	$h=\frac{1}{2}$	$h=\frac{1}{5}$	$h=\frac{1}{10}$	Test	Test	MAX-F	AVG-F	EXP-F
0.2	0.3	39.5	40.2	35.6	33.8	36.6	33.8	37.6	42.8	40.2	40.5	42.6
	0.4	68.3	67.1	50.1	57.8	64.6	45.7	62.8	63.9	61.7	61.4	64.9
	0.5	86.4	74.6	54.9	77.1	70.7	47.9	79.9	74.4	73.5	74.6	76.7
	0.6	79.4	75.7	57.1	83.2	70.5	47.9	84.2	74.7	76.8	79.2	80.3
	0.7	69.9	73.6	54.9	85.0	66.4	45.2	83.6	67.9	75.9	78.9	79.6
	0.8	44.6	64.2	49.1	72.7	56.2	41.8	77.3	47.8	69.4	69.2	72.5
	0.9	28.0	36.6	38.0	52.3	31.6	32.3	60.8	25.2	54.3	50.9	57.2
0.4	0.5	48.6	36.5	33.0	36.6	32.9	30.5	37.4	39.6	39.1	36.3	39.8
	0.6	35.6	58.3	44.1	48.1	55.7	40.2	54.6	44.9	54.7	47.7	54.8
	0.7	29.9	63.4	48.3	58.0	60.7	42.1	67.2	44.1	63.6	56.1	64.4
	0.8	39.0	61.3	46.5	66.9	56.2	38.6	72.4	38.3	64.9	59.2	66.0
	0.9	58.1	47.3	39.7	68.9	43.7	33.2	68.6	40.8	59.2	55.4	60.9
0.6	0.7	14.1	26.9	26.9	20.6	25.1	25.5	24.1	21.6	30.9	24.6	30.2
	0.8	23.3	43.3	35.1	28.0	43.2	33.0	37.3	18.2	42.9	28.0	40.6
	0.9	46.4	43.3	34.6	40.9	43.9	30.9	51.1	32.2	47.8	38.1	47.4
0.8	0.9	13.1	15.1	18.7	12.7	15.0	18.4	14.8	10.4	20.5	12.4	18.1