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Generalized Geometric Structure of the Marginal Distributions of Congestion Processes

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Summary: The states of vector valued congestion processes with nearest neighbor transitions are divided into a sequence of sets A_i. The marginal distributions G_t are divided into a family of functions G_t^i so that $dG_t^{i+1} = L_i(dG_t^i)$. Recursive calculations provide [illegible] the linear functions L_i and the boundary function dG_t^0.

This paper continues the study of a general family of congestion processes [2]. The state of the system is defined as a vector $N(t), W(t)$. $N(t)$ is a finite dimensional vector with non negative integer coordinates. $W(t)$ is a k^* dimensional vector of elapsed times of each of k^* possible intervals which may be in process at time t. For each value n of $N(t)$ a k^* dimensional vector S_n is defined. It has a 1 in the i th coordinate if the i th type interval is in progress when $N(t) = n$ and a 0 otherwise. If $N(t) = n$ and no interval terminates in t to $t + \tau$, $W(t + \tau) = W(t) + \tau S_n$. Each of the i th type intervals is assumed to be a random variable with distribution function $F_i(t), F_i(0)=0$, continuous density $f_i(t)$, and a finite mean. If an event of type i occurs when $N(t) = n$ and $W(t) = w$ there is a change in the value of $N(t)$ from n to n' and w becomes w' according to a set of routing probabilities $p_i(n,w,n',w')$ which are continuous from the right in w and sum to 1 for each i, n, and w. The i th coordinate of w' becomes 0 indicating that either a new interval of the ith type is begun or that when $N(t) = n'$ no interval of type i is in process. In addition this change may cancel the j th type interval so that $w'_j = 0$ or it may interrupt the interval so that the j th coordinate of $S_{n'}$ is 0 and $w'_j = w_j$. The change to n' may be the start or continuation of any number of intervals which were either not in process or suspended while $N(t) = n$. Finally the routing probabilities are positive only for n' for which $|n'-n| = 1$ where the norm is the sum of the absolute values of the coordinates. This is the nearest neighbor assumption which is the basis of the analysis.

This paper will examine the marginal distributions of the process which often provide sufficient information about the nature of the process. Let these distributions be $G_t(n,w)$ with $dG_t(n,w)$ the probability that $N(t) = n$ and $W(t)$ is approximately w. The event structure provides a simple justification for a recursive relationship in n. This approach was first used in

queuing by Winsten [5] and then by Evans [1].

Sets of states

To use the nearest neighbor property, first the possible values of N(t), W(t) are divided into sets A_i, i = 0, ... These sets must have the property that if N(t), W(t) $\in A_i$ then the last event of the interval (o,t) must have occured at some time t at which N(t), W(t) was in $A_{i-1} \cup A_i \cup A_{i+1}$. The fact that except for a set of probability 0 there are a finite number of events in (o,t) guarantees that to the same accuracy there must be a last event in (o,t) [2]. This combined with the restriction that routing probabilities are only positive for pairs of values of N(t) for which |n-n´| [illegible] 1, makes the construction of a family of sets A_i feasible. Even in specific models there may not be an obvious unique partition. One possibility starts with A_0 containing the value n=0 and then defines A_i = {n,w| n-n´ [illegible] 1 for some n´ $\in A_{i-1}$}. Another possibility is A_0={n,w|at least one coordinate of n is 0} and then use the same inductive definition of A_i.

Iterative Relationship

The probability equation is based on decomposing a transition from n,w $\in A_0$ at time 0 to n´, w´ $\in A_i$ at time t according to the possible number of events j in the interval. Each appropriate sequence of j events is decomposed into two parts. The break point is the event which causes the last departure from some n*, w* $\in A_i$ [illegible] 1 st which occurs at time [illegible]. This gives

$$P_t^j(n,w,n',w') = \sum_{k=0} \sum_{n^*, w^* \in A_{i-1}} \int [illegible] P^k(n,w,n^*,w^*) Q_{t-}^{1,j-k}(n^*,w^*,n',w') d[illegible]$$

where

$P_t^j(n,w,n',w')$ is the probability of a j event transition from n, w at 0 to n´ and approximately w´ at t

and

$Q_{\tau}^{i,\alpha}(n^*,w^*,n',w')\,d\tau$ = is the probability of an [illegible] event transition from $n^*,w^* \in A_{i-1}$ in the interval $(0,d\tau)$ to n' and approximately $w' \in A_i$ at τ without entering state in A_{i-1} in $(d\tau,\tau)$

The existence of these probabilities does not raise difficulties since they involve appropriate sequences of a finite number of intervals. Summing over j produces

$$\sum_{j=0}^{\infty} P_t^j(n,w,n',w') = \sum_{j=0}^{\infty}\sum_{k=0}^{j} \int \int_{w^*\,n^* \in A_{i-1}} P_\tau^k(n,w,n^*,w^*)\,Q_{t-\tau}^{i,j-k}(n^*,w^*,n',w')\,d\tau$$

Reversing the summation operations produces

$$P_t(n,w,n'w') = \int \int \int_{w^*\,n^* \in A_{i-1}} P_\tau(n,w,n^*,w^*)\;Q_{t-\tau}^{i}(n^*,w^*,n',w')\,d\tau$$

where $P_t = \sum_{j=0}^{\infty} P_t^j$ and $Q_{t-\tau}^{i} = \sum_{\alpha=0}^{\infty} Q_{t-\tau}^{i,\alpha}$

Assuming $dG_0(n,w) > 0$ only for $n,w \in A_0$ this result can be written as

$$dG_t(n',w') = \int \int \int_{w^*\,n^* \in A_{i-1}} dG_\tau(n^*,w^*)\;Q_{t-\tau}^{i}(n^*,w^*,n',w')\,d\tau$$

This means that there is a family of linear functions L_i which map the functions $dG_t(n^*,w^*)$ for $n^*,w^* \in A_{i-1}$ into $dG_t(n',w')$ for $n',w' \in A_i$.

Using this relationship requires more explicit forms for the taboo probabilities $Q_t^i\,dt$. These can be developed in terms of some less complicated probabilities. Define

$B_t^{i,k}(n,w,n',w')$ = probability that a sequence of k events starting from $n,w \in A_i$ at time 0 is completed in time t and that $N(t)$, $W(t) = n',w' \in A_i$ and $N(\tau)$, $W(\tau) \in A_i$ for $0 \leq \tau \leq t$.

$$B_t^i = \sum_{k=0}^{\infty} B_t^{i,k}$$

$Z_t^{i,k}(n,w,n',w')$ = probability that a sequence of k events starting from $n,w \in A_i$ at time 0 is completed in time t and that $N(t), W(t) = n',w' \in A_i$ and $N(\tau), W(\tau) \in A_j$, $j \geq i$ for $0 \leq \tau \leq t$

$U^i(n,w,n',w')\,dt$ = probability of an event occurring in dt causing the transition from $n,w \in A_i$ to $n',w' \in A_{i+1}$

$$U^i dt * Z_t^{i,\alpha} = Q_t^{i,\alpha+1} dt$$

$D^i(n,w,n',w')\,dt$ = probability of an event occurring in dt causing the transition from $n,w \in A_i$ to $n',w' \in A_{i-1}$

For each finite sequence of events the first exit from A_i and first reentry into A_i are well defined events. Thus

$$Q_t^{i,\alpha} dt = U^{i-1} dt * B_t^{i,\alpha-1} + \sum_{j=2}^{\alpha-3} \sum_{k=1}^{j-1} \int_0^t \int_0^\tau U^{i-1} d\tau * B_\xi^{i,k} * Z^{i+1,j-k} d\xi * D^{i-1} d\tau * Z_{t-\tau}^{i,\alpha-j}$$

For simplicity of notation, the sums and integrals over states have been surpressed in the operations. The limits on the event sums reflect the fact that it requires two events to exit A_{i+1} and return to A_i. All one and two event sequences whose probabilities must be included in $Q_t^{i,\alpha}$ never leave A_i after entering it. Now summing over the number of events.

$$Q_t^i dt = U^{i-1} dt * B_t^i + \int_0^t \int_0^\tau U^{i-1} d\tau * B_{\tau-\xi}^i * Q^{i+1} d\xi * D^{i-1} d\tau * Z_{t-\tau}^i$$

This equation can be solved iteratively using

$${}^0Z_t^i = B_t^i$$

$${}^{k}Q_{t}^{i} = U^{i-1} * Z_{t}^{i}$$

$${}^{k}Z_{t}^{i} = B_{t}^{i} + \int_{0}^{t} B^{i}_{t-\tau} * {}^{k-1}Q_{\tau}^{i+1} \ldots [illegible]$$

The validity of this iteration scheme is most [illegible] from its probabilistic interpretation. The parameter k measures the maximum number of entries into A_{i+1} of the event sequences considered in each approximation. This means that the sequence of approximations is monotone nondecreasing and converges because the number of events in t is finite with probability 1.

A more interesting version of the preceding equation results from taking Laplace transforms so that the convolutions become products. A related equation occurs if the interest is focused on stable distributions. Assuming that

$$\lim_{t \to \infty} dG_t(n,w) = dG_\infty(n,w)$$

Define

$$dG^{i} = dG_{\infty}(n,w) \text{ for } n,w \in A_i$$

The recursive relationship is

$${}_{k}G^{i+1} = G^{i} + \int_0^{\infty} Q_t^{i+1} dt$$

The integral must converge or

$$\lim_{t \to \infty} \int_0^t dG_\tau^{i-1} * Q_{t-\tau}^{i} d\tau = \lim_{t \to \infty} [illegible]$$

can converge only because dG_t^{i-1} converges to zero and states in A_{i-1} are transient.

The finiteness of the expected time spent in A_i before reentering A_{i-1}, $\int_0^\infty Q_t^i dt$ does not guarantee the existence of a limiting distribution. This requires that dG_t^0 converge and that the eigen values of these integrals as

functions of the dG^i be less than 1 for enough values of i so that ΣdG^i exists. Most queueing system models have event and routing sequences which effect transitions from any state n, w to any other state n', w' in finite time with positive probability. Usually this is sufficient to establish that for $i \geq 1$

1) $\int_0^\infty B_t^i\, dt * D^i * L^{i-1} + \int_0^\infty B_t^i * U^i * L^{i+1} \leq L^i$

and

2) $\int_0^\infty B_t^i dt * U^i * L^{i-1} \leq \beta L^i$

where $\beta < 1$ and $L^i(n,w) = 1$ for all $n, w \in A_i$.

The first requirement is that from an (n,w) in A_i the system eventually leaves A_i. Since D^i and U^i are finite this means that the expected length of stay in A_i must be finite; i.e. $\int_0^\infty B_t^i dt$ is finite for all (n,w) in A_i. The second requirement is that the exit is not always to a state A_{i+1}. These properties guarantee that $\int_0^\infty G_t^i dt$ converges. The argument is perhaps easiest to describe in terms of the functions Z_t^i.

Integrating the iterative definition of ${}^kZ_t^i$ with respect to time

$$\int_0^\infty {}^kZ_t^i dt = \int_0^\infty B_t^i dt + \int_0^\infty \int_0^t \int_0^\tau B^i_{\tau-s} * U^i ds * {}^{k-1}Z^{i+1} * D^{i+1} d\tau * {}^{k-1}Z^i_{t-\tau}$$

Rearranging the integration

$$\int_0^\infty {}^kZ_t^i\, dt = \int_0^\infty B_t^i dt + [\int_0^\infty B_t^i * U^i dt] * [\int_0^\infty {}^{k-1}Z_t^{i+1} * D^{i+1} dt] * [\int_0^\infty {}^{k-1}Z_t^i dt]$$

By induction on k for all i

$$\int_0^\infty {}^kZ_t^i\, dt * D^i * L^{i-1} = \int_0^\infty B_t^i\, dt * D^i * L^{i-1}$$

$$+ [\int_0^\infty B_t^i * U^i dt] * [\int_0^\infty {}^{k-1}Z_t^{i+1}] * [\int_0^\infty {}^{k-1}Z_t^i * D^i * L^{i-1}]$$

$$\leq \int_0^\infty B_t^i\, dt * D^i * L^{i-1} + \int_0^\infty B_t^i * U^i dt * L^{i+1}$$

$$\leq L^i$$

and

$$\int_0^\infty {}^kZ_t^i\, dt * L^i = [\int [illegible] B_t^i dr + [illegible] B^i * [illegible] dr] * [illegible] Z^{i+1} * D^{i+1} d\xi] * [\int_0^{\infty} {}^{k-1}Z_t^i dr]] * L^i$$

$$= \int_0^\infty B_t^i dt * L^i + [illegible] * [illegible] (\sum [illegible]) L^{i+1}$$

$$[illegible] \sum_{i=0}^{i+1} [illegible]$$

As k goes to infinity the bound [illegible]

The construction shows [illegible] solution to the equation. The solution found by the iterative procedure is also the only one which has the additional properties which are required for Q to have probabilistic meaning. First Z_t^i must be non negative. For any non negative measure W, $W * Z_t^i$ must be a non negative measure. Conversely, for any non negative function V, $Z_t^i * V$ must be a non negative function. In addition $\int_0^\infty Z_t^i * D^i * L^{i-1} \le L^i$ since the probability of returning to A_i must not exceed 1. Finally $Z_t^i(n,w,n',w') \ge B_t^i(n,w,n',w')$ since Z_t^i is less restricted than B_t^i. Let W_t^i satisfy these conditions and the equation. Compare W_t^i with the sequence ${}^kZ_t^i$ produced by the iterative process.

$$W_t^i - {}^kZ_t^i = \int_0^t \int_0^{[illegible]} B_\xi^i * D^i d [illegible] * (W^{i+1}_{[illegible]} - {}^{k-1}Z^{i+1}_{[illegible]}) * D^{i+1} d\xi * (W^i_{t-\xi} - {}^{k-1}Z^i_{t-\xi})$$

By induction $W_t^i - {}^kZ_t^i \ge 0$ [illegible] ${}^1Z_t^i = B_t^i$ [illegible] assumption $W_t^i \ge B_t^i$. Let $(W_t - {}^1Z_t^i) * D^i * L^{i-1} \le B^i$. [illegible] Again by induction $(W_t^i - {}^kZ_t^i) * D^i$ [illegible] $\int_0^\infty B_\xi^i * D^i d\xi * L^{i+1} \le BL^i$. Repeating the induction $(W_t^i - {}^kZ_t^i) * L$ [illegible] L^i. Because of the non negativity this means that ${}^kZ_t^i$ converges to [illegible] is the only probabilistically meaningful solution of the equation.

Phase type intervals

In a previous unpublished version of this discussion, the author began with the infinitessimal generator of these processes. A reader remarked that congestion processes were derived from more elementary processes. That point

of view has been used here and has the advantage of [illegible] the arguments elementary. There is one questionable aspect of this. [illegible] the routing part of the process may be so inextricably meshed with the [illegible] process that its separation from the congestion process [illegible]. This is especially true if the [illegible] to optimize some measure of the future performance [illegible] simultaneously discover the optimal routing [illegible] of the congestion process.

There is another modification [illegible] which raises similar questions. Suppose that some or even all of the interval [illegible] are phase type random variables. Such variables are characterized by a finite number of exponential phases or sub intervals. At the end of each phase either the interval terminates or another phase is begun. The conditional probabilities of these alternatives depend only on the phase just completed. The phases need not be ordered and identically distributed [illegible]. For such interval distributions, the natural [illegible] uses a discrete supplementary variable which identifies the [illegible] the time since the start of the interval. If the routing [illegible] depends only on the congestion and not on the time since the start of [illegible] intervals, the congestion process can be studied in terms of [illegible] valued process N(t). This process may be [illegible] phases may have no physical identity. This process [illegible] conditioning on the number of transition [illegible] in [illegible] discrete supplementary variables increases the number of values of the integer valued variable in the sets A_i but also allows the fundamental [illegible] probability equation to become independent of the continuous supplementary variables. In this case the limiting behavior of the system can be analyzed in terms of vectors and matrices although they may have to be infinite dimensional. Such is the case when the number of states in the sets A_i is infinite. This method

may be the best method [illegible] making [illegible] problem discrete so that digital computers may be used [illegible]. One of the reasons for this discussion is to [illegible] discrete and continuous supplementary variable [illegible].

Boundary Set A_o

Piecewise [illegible] [3] is the general model [illegible] fundamental difference is that they assume A_o [illegible] system empty, which has no associated supplementary [illegible] imposes the assumption of some form of exponential [illegible] stationary probabilities are then developed as multiples of the probability that the system is empty. They exploit [illegible] developing probabilities recursively in terms of the number of events [illegible] system was empty. This is similar to the recursive [illegible] probabilities presented in [2]. The single system empty state [illegible] that its limiting probability can be found by first assigning it [illegible] value 1, then performing the recursive calculation, and [illegible] the probabilities sum to 1. [illegible] difference equation analysis of limiting probabilities in [illegible] A_o can contain a number of values of [illegible] associated supplementary variables. Thus it is [illegible] expression for dG^o_r. Again one can proceed through the [illegible] out the last departure from A_o. Summing over [illegible]

$$dG^o_t = dG^o_o * B^o_t + [illegible]$$

The limit of the expression gives

$$dG^0_\infty = dG^0_0 \, B^0_\infty + \int_0^\infty \int_0^t dG^0_\xi * U^0 d\xi * Z^1_{t-\xi} * D^1 dt * B^0_{t-\xi}$$

In interesting cases it is impossible to remain in A_0 indefinitely and B^0_∞ is the zero operator and $\int_0^\infty B^0_t dt$ is finite. In this case since Z^1_t must also be integratable and

$$dG^0_\infty = dG^0_\infty * U^0 * \int_0^\infty Z^1_\tau d\tau * D^1 * \int_0^\infty B^0_t dt$$

or

$$dG^0_\infty = dG^0_\infty * \int_0^\infty Q^1_\infty d\tau * D^1 * \int_0^\infty B^0_t dt$$

This suggests choosing ${}^0dG^0_\infty$ as an arbitrary measure and iteratively calculating

$${}^k dG^0_\infty = {}^{k-1} dG^0_\infty * U^0 * \int_0^\infty Z^1_\tau \, d\tau * D^1 * \int_0^\infty B^0_t dt$$

or

$${}^k dG^0_\infty = dG^0_\infty * U^0 * [\int_0^\infty Z^1_\tau d\tau * D^1 * \int_0^\infty B^0_t dt * U^0]^{(k-1)} \int_0^\infty Z^1_\tau d\tau * D^1 * \int_0^\infty B^0_t dt$$

where the power k-1 means repeat the * operation k-1 times. The expression $\int_0^\infty B^0_t dt * U^0 * I^1$ is the probability of leaving A_0 for the various starting states and thus must be I^0 if the limiting system results are to be strictly positive. Similarly $\int_0^\infty Z^1_\tau d\tau * D^1 * I^0$ is the probability of ever returning to A_0 from states in A_1. These also must be 1 for all states in A_1 or the system limit will be defective. Thus the operator in brackets must be strictly positive mapping all probability measures on the subset of n and values of the supplementary variables in A_1 which can be entered in a transition from A_0. From positive operator theory [ ] the repeated application of this to any non zero measure will produce a unique positive limit in the space of signed measures. Thus the entire iterative procedure will converge to such

a limit if ${}^0dC^U_\infty * U^0$ is positive at least for some states. Moreover applying $U^0 * L^1$ to both sides shows that ${}^1dC^U_\infty$ [illegible] $* L^1$ will be ${}^0dC^U_\infty * U^0 * L^1$. This number is immaterial since it is renormalized when the entire sum of probabilities is made equal to 1.

Degeneracy

So far this discussion has made little use of the restricted nature of the changes in the supplementary variables. Thus the arguments can be adapted to more general assumptions should a situation require this. The assumptions about changes in the supplementary variables imply that both U^i and D^i are degenerate. They assign positive probability only to subsets of states. Because entry into A_i coincides with the termination of some interval, some supplementary variables must be 0 at entry times. Thus both of the sets $A^U_i = \{(n', w') \mid (n', w') \in A_i, U^{i-1}(n, w, n', w') > 0$ for some $(n, w) \in A_{i+1}\}$ and $A^D_i = \{(n', w') \mid (n', w') \in A^i, D^{i+1}(n, w, n', w') > 0$ for some $(n, w) \in A_{i+1}\}$ are strict subsets of A_i. This strict inclusion is even true of $A^U_i \cup A^D_i$. Within A_i there is a Markov process for which the initial distribution is concentrated in $A^U_i \cup A^D_i$ and B^i_t is the transition operator. Exists from A_i to A_{i+1} and A_{i-1} have transition functions $B^i_t * U^i$ and $B^i_t * D^i$ respectively. Let $B^i_{U,t}$ be the restriction of B^i_t to transitions from A^U_i to A_i, $B^i_{D,t}$ the restriction of B^i_t to transitions from A^D_i to A_i. These operators are all that are needed to study the corresponding restrictions $Z^i_{U,t}$ and $Z^i_{D,t}$ of Z^i_t. The restricted operators satisfy the equations

$$Z^i_{U,t} = B^i_{U,t} + \int_0^t \int_0^\tau B^i_{U,\xi} * U^i * Z^{i+1}_{U,\tau-\xi} * D^{i+1} * Z^i_{D,t-\tau}\, d\xi\, d\tau$$

$$Z^i_{D,t} = B^i_{D,t} + \int_0^t \int_0^\tau B^i_{U,\xi} * U^i * Z^{i+1}_{U,\tau-\xi} * D^{i+1} * Z^i_{D,t-\tau}\, d\xi\, d\tau$$

From the solution to these

equations the complete function can be found [illegible]

$$Z_t^i = B_t^i + \int_0^t [illegible]$$

Further formulae may be developed using [illegible] the Laplace transform or concentrating on the [illegible] let the same symbols without the [illegible] integral of the operator with respect to time. [illegible] equations become

$$Z_U^i = B_U^i + B_U^i * U^i * Z^{i+1} * [illegible]$$

$$Z_D^i = B_D^i + B_D^i * U^i * Z_U^{i+1} * D^{i+1} * [illegible]$$

Use the previous assumptions $B^i * U^i * [illegible] \le L^i$ and $Z_u^{i+1} * D^{i+1} * L^i \le L^{i+1}$. These guarantee that measures G^i for which [illegible] are mapped into measures $G^{i\prime}$ for which $G^{i\prime} * L^i \le L^i$. Thus the largest eigen value of $B^i * U^i * Z_u^{i+1} * D^{i+1}$ is less than 1 which in turn guarantees an explicit solution to the second equation is

$$Z_D^i = \sum_{r=0}^{\infty} (B_D^i * U^i * Z_u^{i+1} * D^{i+1})^{(r)} * B_D^i$$

or

$$Z_D^i = (I - B_D^i * U^i * Z_u^{i+1} * D^{i+1})^{-1} * B_D^i$$

where the power is [illegible] respectively.

Substituting this into the [illegible] gives

$$Z_U^i = B_U^i + B_U^i * U^i * Z_U^{i+1} * D^{i+1} * \sum_{r=0}^{\infty} (B_D^i * U^i * Z^{i+1} * D^{i+1})^{(r)} * B_D^i$$

this can be rearranged as

$$Z_U^i = B_U^i * \sum_{r=0}^{\infty} (U^i * Z_u^{i+1} * D^{i+1} * B_D^i)^{(r)}$$

The operator series must converge and thus

$$Z_U^i = B_U^i (I - U^i * Z_U^{i+1} * D^{i+1} * B_D^i)^{-1}$$

or

$$Z_U^i = B_U^i + \ldots$$

This equation [illegible] ... holds the last exit from [illegible]

$$Z^i = B^i \ldots$$

Restricting [illegible] ... gives the previous [illegible]

$${}^0Z_U^i = B_U^i$$

$${}^kZ_U^i = B_U^i + {}^{k-1}Z_U^i * \ldots$$

For all k, $Z_U^i \geq {}^kZ_U^i$ since

$$Z_U^i - {}^kZ_U^i = \ldots * B^i$$

$$= [(Z_U^i - {}^{k-1}Z_U^i) * \ldots (Z_U^{i+1} - {}^{k-1}Z_U^{i+1})] * D^{i+1} * B^i$$

$$\geq 0$$

Also induction shows [illegible] ... is monotone non decreasing and bounded [illegible] ... which satisfies the equation [illegible]

$$Z_D^i = \sum \ldots$$

is well defined and [illegible] ... uniqueness of Z_U^i, Z_D^i satisfying the pair of [illegible] ... the expression for Z_U^i was derived. Thus the [illegible] converge to Z_U^i

M/M/1

Although successive approximations often provide feasible calculations for numerical analysis, it is instructive to examine at least one example in which complete analytic solution is possible. Obvious candidates for discussion are either M/G/1 or G/M/1 using only the single necessary supplementary variable. In these cases D^i and U^i are extremely degenerate since $A_i^D = (i,o)$ and $A_i^U = (i,o)$ respectively. These special funneling states make it easy to solve for the Z^i which are all identical. The analysis easily provides limiting results in probability rather than the more familiar transform form.

In many ways more interesting here is the less general system M/M/1 in which the analysis uses supplementary variables w_1 and w_2 for both elapsed interarrival and service times respectively. The state space is partitioned according to sets $A_i = \{(i,w_1,w_2) \mid w_1 \geq 0,\ w_2 \geq 0\}$. The set $A_o = \{(o,w_1,w_2) \mid w_1 \geq 0, w_2 \equiv 0\}$ is the obvious exception. When there is only one customer in the system, it is his service which is in process and this must have begun after his arrival which is also the last arrival. Thus $A^1 = \{(1,w_1,w_2) \mid w_1 \geq w_2 \geq 0\}$ is also an exception. The degeneracy occurs because $A_i^U = \{(i,o,w_2) \mid w_2 \geq 0\}$. $A_i^D = \{(i,w_1,0) \mid w_1 \geq 0\}$. It is obviously important but not the extreme of a funneling state.

The operators which define the process are

$$B_t^o\ (w_1,0,w_1',0) = e^{-\lambda(w_1'-w_1)}dw_1'$$

$$B^o = \int_o^\infty B_t^o dt = e^{-\lambda(w_1'-w_1)}dw_1'$$

For all supplementary variable values for which it is defined and all i including i = 1

$$B_t^i(w_1,w_2,w_1',w_2') = e^{-(\lambda+\mu)(w_1'-w_1)}\delta(w_1'-w_1,w_2'-w_2)dt$$

$$B^i = \int_o^\infty B_t^i dt = e^{-(\lambda+\mu)(w_1'-w_1)}\delta(w_1'-w_1,w_2'-w_2)dw_1'$$

where

$$\delta(x,y) = \begin{cases} 1 & x=y \\ 0 & x\neq y \end{cases}$$

For $i \geq 0$

$$U^i(w_1, w_2, w_1', w_2') = \begin{cases} \lambda & \text{if } w_1' = 0, w_2 = w_2' \\ 0 & \text{otherwise} \end{cases}$$

For $i \geq 1$

$$D^i(w_1, w_2, w_1', w_2') = \begin{cases} \mu & \text{if } w_2' = 0, w_1' = w_1 \\ 0 & \text{otherwise} \end{cases}$$

For the limiting distribution the solution to

$$Z_U^i = B_U^i + Z_U^i * U^i * Z_U^{i+1} * D^{i+1} * B_D^i$$

is the same for all $i \geq 1$. It is

$$Z_U = e^{-(\lambda+\mu)w_1'} \delta(w_1', w_2'-w_2)\, dw_1' + \lambda e^{-\lambda w_2' - u w_1'} \Gamma(w_2', w_1')\, dw_1' dw_2'$$

for

$$\Gamma(x,y) = \begin{cases} 1 & x \leq y \\ 0 & x > y \end{cases}$$

From this

$$Z_U * U = \lambda e^{-(\lambda+\mu)(w_2'-w_2)} dw_2' \; \Gamma(w_2, w_2') + \frac{\lambda^2}{\mu} e^{-(\lambda+\mu)w_2'} dw_2'$$

$$Z_U * D = \mu e^{-\mu w_1'} dw_1'$$

These combine to give

$$Z_U * U * Z_U * D = \lambda e^{-\mu w_1'} dw_1'$$

from which

$$Z_U * U * Z_U * D * B_D = \lambda e^{-\mu w_1' - \lambda w_2'} \Gamma(w_2', w_1')\, dw_1' dw_2'$$

For the distribution at the boundary the solution to

$$dG_\infty^o = dG_\infty^o * U^o * Z_U^1 * D^1 * B^o$$

is

$$dG_\infty^o = \lambda(e^{-\lambda w_1'} - e^{-\mu w_1'})dw_1'$$

where the arbitrary scale factor has already been set so that

$$\int_0^\infty dG_\infty^o = 1 - \lambda/\mu$$

In this way the recursive definition

$$dG_\infty^n = dG_\infty^{n-1} * U * Z$$

produces functions which sum to 1 as required for $\lambda/\mu < 1$, when the limiting distribution exists. For $\lambda/\mu > 1$ the ΣdG^n does not converge. The explicit for m is

$$dG_n^\infty = (1 - \frac{\lambda}{\mu}) \left[\sum_{i=0}^{n-3} \frac{\lambda^{n+1}}{\mu^{n-i-1}} \frac{(w_2'-w_1')}{i!} e^{-(\lambda+\mu)w_2'} + \right.$$

$$\left. \frac{\lambda^n}{\mu}(\lambda+\mu) \frac{(w_2'-w_1')^{n-2}}{(n-2)!} e^{-(\lambda+\mu)w_2'} \right] \Gamma(w_1,w_2)dw_1dw_2 +$$

$$\frac{\lambda^{n+1}}{\mu^{n-1}} e^{-\lambda w_2'-\mu w_1'} \Gamma(w_2',w_1')dw_1'dw_2'$$

From this the marginal distributions in n and one supplementary variable are

$$\int_{w_2'=0}^\infty dG_\infty^n = (1 - \frac{\lambda}{\mu}) (\lambda^n/\mu^{n+1})e^{-\mu w_1'}dw_1'$$

$$\int_{w_1'=0}^\infty dG_\infty^n = (1 - \frac{\lambda}{\mu}) \left(\frac{\lambda^{n+1}}{\mu^n} + \frac{\lambda^{n+1}}{\mu^{n-1}} \sum_{i=1}^{n-2} \frac{\mu^{i-1}w_2'^i}{i!} + \frac{\lambda^n(\lambda+\mu)}{\mu} \frac{w_2'^{n-1}}{(n-1)!}\right)e^{-(\lambda+\mu)w_2'}dw_2'$$

As anticipated marginal probabilities for the integer variable are

$$\int_0^\infty \int_0^\infty dG_\infty^n = (1 - \frac{\lambda}{\mu})\frac{\lambda^n}{\mu^n}$$

and for the supplementary variables they are

$$\sum_{n=0}^{\infty} \int_{w_2'=0}^{\infty} dG_{\infty}^{n} = \lambda e^{-\lambda w_1'} \, dw_1'$$

and

$$\sum_{n=0}^{\infty} \int_{w_1'=0}^{\infty} dG_{\infty}^{n} = \mu e^{-\mu w_2'} \, dw_2'$$

The infintessimal generator for this process is found from

$$dG_{t+\Delta t}^{n}(w_1,w_2) = (1-\lambda\Delta t-\mu\Delta t)dG_{t}^{n}(w_1-\Delta t,w_2-\Delta t) \text{ for } w_1 > 0 \text{ and } w_2 > 0$$

$$dG_{t+\Delta t}^{n}(0,w_2) = \int_{w_1 o}^{\infty} \lambda\Delta t dG_{t}^{n-1}(w_1,w_2-\Delta t)$$

$$dG_{t+\Delta t}^{n}(w_1,0) = \int_{w_2=0}^{\infty} \mu\Delta t \; dG_{t}^{n+1}(w_1-\Delta t,w_2)$$

$$dG_{t+\Delta t}^{o}(w_1,o) = (1-\lambda\Delta t)dG_{t}^{o}(w_1-\Delta t,o) + \int_{w_2=0}^{\infty} \mu\Delta t dG_{t}^{1}(w_1-\Delta t,w_2)$$

$$dG_{t+\Delta t}^{1}(o,0) = \int_{w_1=0}^{\infty} \lambda\Delta t dG_{t}^{o}(w_1,o)$$

Substituting the functional forms found for dG_{∞}^{n} into these relationships for small Δt also verifies that the stationary distribution has been found.

Truncated Processes

Another interpretation of this process allows it to be used more generally or perhaps suggests making A_o relatively large. First $\int_o^{\infty} Q_{\tau}^{i} d\tau * D^i \, dt$ can be interpreted as the conditional probability of a transition from states in A_{i-1} to states in A_{i-1} in dt in the stachastic process derived from the original by ignoring time spent in states in A_j for $j > i$. The truncated process produces probabilities which are conditional probabilities of the original

process. In the same vein $\int_0^\infty Q_\tau^i d\tau * D^i * \int_0^\infty B_\tau^{i-1} dt$ can be considered as conditional probabilities for transitions from states in A_{i-1} through A_i in a truncated process in which time is discrete and measures the exits from the sets A_j for $j = 0, i-1$. The analysis for A_0 alone is special because all exits from A_o lead to A_1. In general there are also transitions from A_j to A_{j-1} in the state just before exit process.

The use of truncated processes is very appropriate in numerical analysis when for $j > i$ the sets A_j are identical and the transition structure does not depend on j, i.e. $U^j = U^i$, $B_t^j = B_t^i$ and $D^j = D^i$ for $j \geq i$. In this case all $Q_t^j = Q_t^{i+1}$ for $j > i$. Thus only a single equation need be solved for these functions. Once $\int_0^\infty Q_t^{i+1} dt$ is known then the truncated process can be analyzed to produce $d\tilde{G}_\infty^k$ for $k = o,i$. The recursive relation $d\tilde{G}_\infty^j = d\tilde{G}_\infty^{j-1} * \int_0^\infty Q_\tau^{i+1} d\tau$ starting from $j = i+1$ and $d\tilde{G}_\infty^i$ produces a complete set of $d\tilde{G}_\infty^i$. These may now be normalized to sum to 1 and the result is the limiting distribution dG_∞^i for the original process.

This was the approach used by Winsten [5] in discussing some problems in which the upper tail of the limiting distributions are geometric. He also allowed transitions from A_i to and A_j for $j \leq i + 1$ with probabilities which depend only of i-j. This is enough to prove the recursive relationship $dG_\infty^{i+1} = dG_\infty^o * R$ even in the more general context of this paper. The problem is that the equation for R becomes extremely complicated. The suggestion that single event transitions from A_i to A_j for $j \leq i + k$ produces a k term recursive structure can also be persued. The expressions for the coefficients in this relationship also become complicated as k becomes large and the structure on the A_i becomes complex. The introduction of groups of states by the author makes this last generalization unnecessary for treating Erland k distributions as Winsten suggests.

Applications

So far, analytic application of this structure has been restricted to systems in which the A_i are identical and discrete from some point on. The result is that the upper tail of the limiting congestion distribution is geometric with a matrix for the term ratio. Although not explicitly used in derivations this approach can provide relatively easy access to limiting distributions for systems such as $E_j/E_k/s$ and many priority models. The intimate relation between convergent iterative calculations and the theoretical analysis make this approach useable even when it is difficult to proceed further analytically.

In developing piecewise linear processes, Gnedenko and Kovalenko [3] used the remaining length of the intervals in process as supplementary variables. The arguments presented here can easily be revised to use this representation especially if one uses rates of progress toward termination which depend on the congestion. In terms of functional forms, there seems to be no strong preference at the moment. Although this approach raises questions because (N(t), W(t)) may not be observable, it does provide an analytic structure which matches that used in computer simulations. From both the philosophical and practical points of view it is important to think of simulation as one form of numerical analysis for complicated stochastic processes.

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