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FACULTY WORKING
PAPER NO. 1401

The Generalized Rubinstein/Stein Covariance Operator and its Application to the Estimation of Real Systematic Risk

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October 1987

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The authors appreciate the very helpful comments from Steve Pruitt and an anonymous referee. We especially thank the referee for providing the shorter proof to the theorem. We alone remain responsible for any remaining errors.

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ABSTRACT

This paper generalizes Rubinstein's (1973, 1976), Stein's (1973) and Losq and Chateau's (1982) covariance operator to the case where both variables are functions of multivariate normal random variables. This resulting covariance operator is extremely useful for either implicit functions of or non-polynomials of multivariate normal random variables, such as exponential functions. An application of the use of the covariance operator to the estimation of real systematic risk is provided to illustrate the results. We also compare this covariance operator with the moment generating function method in this application. (COVARIANCE OPERATOR; MOMENT GENERATING FUNCTION; CAPITAL ASSET PRICING MODEL;SYSTEMATIC RISK)

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1. Introduction

If random variables x and y have a joint distribution which is bivariate normal and if f(.) is a continuously differentiable function of y, then Rubinstein (1973, 1976) and Stein (1973) have demonstrated that

$$\mathrm{Cov}(x,f(y)) = E[f'(y)]\mathrm{Cov}(x,y), \tag{1}$$

if E[f'(y)] exists, where E is the expectation operator, Cov is the covariance operator, and $f'(y) = df(y)/dy$. Later, Losq and Chateau (1982) generalize the above result from a function, f, of one random variable to n random variables as follows:

$$\mathrm{Cov}(x,f(y_i,\ldots,y_n)) = \sum_{i=1}^{n} E[f_i]\cdot\mathrm{Cov}(x,y_i), \tag{2}$$

provided that all expectation values exist, where f_i is the partial derivative of f with respect to y_i, $i = 1,\ldots,n$. Rubinstein (1976) has applied the covariance operator of equation (1) to derive the capital asset pricing model (CAPM) of Sharpe (1964) and the option pricing model of Black and Scholes (1973). Losq and Chateau (1982) employ the covariance operator of equation (2) to derive the multibeta CAPM.

Later work by Roll (1973) extends the CAPM of Sharpe to a world of stochastic inflation as follows:

$$E[R_iP] = R_fE[P] + \beta_i\cdot E[R_mP - R_fP], \tag{3}$$

where P is stochastic purchasing power, R_i is the nominal holding period return for asset i, f denotes the nominal riskless asset, m denotes the market portfolio, and β_i is defined as follows:

$$\beta_i = \mathrm{Cov}(R_iP,\ R_mP)/\mathrm{Var}(R_mP). \tag{4}$$

In deriving equation (3), it is necessary to assume either that investors possess quadratic utility functions of real wealth or that nominal return relatives and purchasing power are multivariate normal, or both. The latter case is, however, more common in the empirical literature.

The purpose of this paper is, first, to generalize the covariance operator of equations (1) and (2) to the case where both variables are functions of multivariate normal random variables (MNRVs), and, second, to apply this generalized covariance operator along with the moment generating function (MGF) method to estimate real systematic risk as defined in equation (4).

The covariance operator as defined in equations (1) and (2) is convenient for use in obtaining the covariance between one normal random variable and another variable which is a function of MNRVs. This is especially true when the latter variable is either an implicit function of or a non-polynomial of MNRVs. Unfortunately, the covariance operator fails when both variables are functions of MNRVs. In contrast, the MGF method can obtain the covariance between two variables which may be represented by polynomials of MNRVs (and non-MNRVs as well), such as in (4). However, when both variables are either implicit functions of or non-polynomials of MNRVs, the MGF method also fails.

In the next section, the covariance operator of equations (1) and (2) is generalized to the case where both variables are functions of MNRVs. In section 3, both the resulting covariance operator and the MGF methods are employed to estimate real systematic risk as defined in (4). Further examples of the application of this generalized Rubinstein/Stein (RS) covariance operator are shown in section 4. The final section summarizes the results.

2. The Generalized Rubinstein/Stein Covariance Operator

Suppose that $x_1,\ldots,x_n$, $y_1,\ldots,y_m$ are jointly MNRVs and that all the following indicated expectations exist. Since the proofs of the Theorem and

Corollary 2 are similar to that of Corollary 1 but great expense is required to carry out the algebraic complexities, therefore the proof is given in detail only for Corollary 1 and is sketched briefly for the Theorem but is omitted for Corollary 2.

THEOREM: Suppose that f is a p-order polynomial function of $x_1,\ldots,x_n$, and g is any p times continuously differentiable function of $y_1,\ldots,y_m$. Then

$$\begin{aligned}
&\mathrm{Cov}(f(x_1,\ldots,x_n),\ g(y_1,\ldots,y_m)) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} E[f_i]E[g_j]\cdot \mathrm{Cov}(x_i,y_j) \\
&\quad + (1/2!) \sum_{i1=1}^{n} \sum_{i2=1}^{n} \sum_{j1=1}^{m} \sum_{j2=1}^{m} E[f_{i1,i2}]E[g_{j1,j2}]\mathrm{Cov}(x_{i1},y_{j1})\mathrm{Cov}(x_{i2},y_{j2}) \\
&\quad + \ldots . + \\
&\quad (1/p!) \sum_{i1=1}^{n}\cdots\sum_{ip=1}^{n} \sum_{j1=1}^{m}\cdots\sum_{jp=1}^{m}\{E[f_{i1\ldots ip}]E[g_{j1\ldots jp}] \prod_{k=1}^{p} \mathrm{Cov}(x_{ik},y_{jk})\}, \qquad (5)
\end{aligned}$$

where the subscripts for functions f and g represent partial derivatives.

PROOF: The proof is briefly sketched as follows: First, the result is proven by using induction on the order $K = k_1 + k_2 + \ldots + k_n$ of the monomial $f(x_1,\ldots,x_n) = x_1^{k1}x_2^{k2}\ldots x_n^{kn}$, with Losq and Chateau's result for the case K = 1. Then the following additivity of the covariance operator is applied to the polynomial $f = a_1f_1 + \ldots + a_qf_q$ to complete the proof.

$$\mathrm{Cov}(a_1f_1 + \ldots + a_qf_q,\ g) = a_1\mathrm{Cov}(f_1,g) + \ldots + a_q\mathrm{Cov}(f_q,g).$$

Q.E.D.

Remark: The Theorem can be easily extended to the case where both functions f and g are continuously differentiable, with p perhaps infinite.

The next two corollaries are the special cases of the Theorem.

COROLLARY 1: Assume that g(y) is a p times continuously differential function of y. Then

$$\begin{aligned}\mathrm{Cov}(x^p,g(y)) = {} & p\cdot E[x^{p-1}]E[g^{(1)}]\cdot \mathrm{Cov}(x,y) + C^p_2\cdot E[x^{p-2}]E[g^{(2)}]\mathrm{Cov}^2(x,y) \\ & + \ldots + C^p_k\cdot E[x^{p-k}]E[g^{(k)}]\cdot \mathrm{Cov}^k(x,y) + \ldots \\ & + C^p_{p-1}\cdot E[x]E[g^{(p-1)}]\cdot \mathrm{Cov}^{p-1}(x,y) + E[g^{(p)}]\mathrm{Cov}^p(x,y), \qquad (6)\end{aligned}$$

where $2 < k < (p-1)$, $g^{(i)}$ is the i^{th} derivative of $g(y)$, and $C^p_q = p!/q!$.

PROOF: Define set $S = \{p \mid \text{equation (6) is true for the case of an integer } p\}$. We prove by the principle of strong induction on p that S is equal to the set of all positive integers. We first prove that $1 \varepsilon S$. From (1), we have

$$\mathrm{Cov}(x,g(y)) = E[g^{(1)}]\mathrm{Cov}(x,y), \qquad (7)$$

which is in S. In addition, since $g^{(1)}$ is continuously differentiable, we have the following relationship

$$\mathrm{Cov}(x,g^{(1)}) = E[g^{(2)}]\mathrm{Cov}(x,y). \qquad (8)$$

We now assume that $2, \ldots, n-1$ and $n \varepsilon S$. We will show that $n+1 \varepsilon S$. It can be shown that:

$$\begin{aligned}\mathrm{Cov}(x^{n+1},g(y)) = {} & E[x]\mathrm{Cov}(x^n,g(y)) + n\cdot \mathrm{Cov}(x^{n-1},g(y))\cdot \mathrm{Var}(x) \\ & + \mathrm{Cov}(x^n,g^{(1)})\cdot \mathrm{Cov}(x,y) + E[x^n]E[g^{(1)}]\cdot \mathrm{Cov}(x,y). \qquad (9)\end{aligned}$$

Denote that

$$\begin{aligned}A = {} & E[x]\mathrm{Cov}(x^n,g(y)) \\ = {} & n\cdot E[x]E[x^{n-1}]E[g^{(1)}]\cdot \mathrm{Cov}(x,y) + C^n_2\cdot E[x]E[x^{n-2}]E[g^{(2)}]\cdot \mathrm{Cov}^2(x,y) \\ & + \ldots + C^n_k\cdot E[x]E[x^{n-k}]E[g^{(k)}]\cdot \mathrm{Cov}^k(x,y) + \ldots \\ & + C^n_{n-1}\cdot E^2[x]E[g^{(n-1)}]\cdot \mathrm{Cov}^{n-1}(x,y) + E[x]E[g^{(n)}]\cdot \mathrm{Cov}^n(x,y) \\ = {} & n\{E[x^n] - (n-1)\cdot E[x^{n-2}]\cdot \mathrm{Var}(x)\}E[g^{(1)}]\cdot \mathrm{Cov}(x,y) \\ & + C^n_2\{E[x^{n-1}] - (n-2)\cdot E[x^{n-3}]\cdot \mathrm{Var}(x)\}E[g^{(2)}]\cdot \mathrm{Cov}^2(x,y) \\ & + \ldots + C^n_k\{E[x^{n-k+1}] - (n-k)\cdot E[x^{n-k-1}]\cdot \mathrm{Var}(x)\}E[g^{(k)}]\cdot \mathrm{Cov}^k(x,y) + \ldots \\ & + C^n_{n-1}\{E[x^2] - \mathrm{Var}(x)\}E[g^{(n-1)}]\cdot \mathrm{Cov}^{(n-1)}(x,y) + E[x]E[g^{(n)}]\mathrm{Cov}^n(x,y),\end{aligned}$$

from $n \varepsilon S$, and

$$\begin{aligned}B = {} & n\cdot \mathrm{Cov}(x^{n-1},g(y))\cdot \mathrm{Var}(x) \\ = {} & n(n-1)E[x^{n-2}]E[g^{(1)}]\mathrm{Var}(x)\mathrm{Cov}(x,y) + nC^{n-1}_2E[x^{n-3}]E[g^{(2)}]\mathrm{Var}(x)\mathrm{Cov}^2(x,y)\end{aligned}$$

$$+ \dots + nC^{n-1}_{k}E[x^{n-k-1}]E[g^{(k)}]\mathrm{Var}(x)\mathrm{Cov}^{k}(x,y) + \dots$$

$$+ nC^{n-1}_{n-2}E[x]E[g^{(n-2)}]\mathrm{Var}(x)\mathrm{Cov}^{n-2}(x,y) + n\cdot E[g^{(n-1)}]\cdot \mathrm{Var}(x)\mathrm{Cov}^{n-1}(x,y),$$

from $n-1 \varepsilon S$, and

$$\begin{aligned} C &= \mathrm{Cov}(x^{n},g^{(1)})\cdot \mathrm{Cov}(x,y) \\ &= n\cdot E[x^{n-1}]E[g^{(2)}]\cdot \mathrm{Cov}^{2}(x,y) + C^{n}_{2}\cdot [x^{n-2}]E[g^{(3)}]\cdot \mathrm{Cov}^{3}(x,y) \\ &\quad + \dots + C^{n}_{k-1}\cdot [x^{n-k+1}]E[g^{(k)}]\cdot \mathrm{Cov}^{k}(x,y) + \dots \\ &\quad + C^{n}_{n-1}\cdot E[x]E[g^{(n)}]\cdot \mathrm{Cov}^{n}(x) + E[g^{(n+1)}]\cdot \mathrm{Cov}^{n+1}(x,y), \end{aligned}$$

from $n \varepsilon S$ and (7). Recognizing that $(n-k)C^{n}_{k} = nC^{n-1}_{k}$ in A and B, then

$$\begin{aligned} \mathrm{Cov}(x^{n+1},f(x)) &= A + B + C + E[x^{n}]E[g^{(1)}]\cdot \mathrm{Cov}(x,y) \\ &= (n+1)\cdot E[x^{n}]E[g^{(1)}]\cdot \mathrm{Cov}(x,y) + (C^{n}_{2} + n)E[x^{n-1}]E[g^{(2)}]\cdot \mathrm{Cov}^{2}(x,y) \\ &\quad + \dots + (C^{n}_{k} + C^{n}_{k-1})E[x^{n-k+1}]E[g^{(k)}]\cdot \mathrm{Cov}^{k}(x,y) + \dots \\ &\quad + (1 + C^{n}_{n-1})\cdot E[x]E[g^{(n)}]\cdot \mathrm{Cov}^{n}(x,y) + E[g^{(n+1)}]\cdot \mathrm{Cov}^{n+1}(x,y). \end{aligned} \tag{10}$$

Noticing that $C^{n}_{k} + C^{n}_{k-1} = C^{n+1}_{k}$, (10) becomes (6) with $p = n+1$, which is in S. The proof is complete. Q.E.D.

<u>COROLLARY 2</u>: Suppose that x, $y_1,\dots,y_n$ are multivariate normal, and that $g(.)$ is a p times continuously differentiable function of $y_1, \dots, y_n$. Then

$$\begin{aligned} \mathrm{Cov}(x^{p},g(y_1,\dots,y_n)) &= \sum_{i=1}^{n} p\cdot E[x^{p-1}]E[g_i]\mathrm{Cov}(x,y_i) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} C^{p}_{2}\cdot E[x^{p-2}]E[g_{ij}]\cdot \mathrm{Cov}(x,y_i)\cdot \mathrm{Cov}(x,y_j) + \dots \\ &\quad + \sum_{i1=1}^{n}\cdots\sum_{ik=1}^{n}\cdots\sum_{ip=1}^{n} \{E[g_{i1\dots ik\dots ip}] \prod_{k=1}^{p} \mathrm{Cov}(x,y_{ik})\}. \end{aligned} \tag{11}$$

3. Use of the Generalized RS Covariance Operator and Moment Generating Function Methods to Estimate Real Systematic Risk

3.1 The Generalized RS Covariance Operator

The result from the Theorem is now employed to obtain real systematic risk defined in (4) as follows:

$$\mathrm{Cov}(R_iP,R_mP) = E[R_i]E[R_m]\mathrm{Var}(P) + E[R_i]E[P]\mathrm{Cov}(P,R_m) + E[P]E[R_m]\mathrm{Cov}(R_i,P)$$
$$+ E^2[P]\mathrm{Cov}(R_i,R_m) + \mathrm{Cov}(R_i,P)\mathrm{Cov}(R_m,P) + \mathrm{Cov}(R_i,R_m)\cdot\mathrm{Var}(P) \quad (12)$$

$$\mathrm{Var}(R_mP) = E^2[R_m]\mathrm{Var}(P) + 2\cdot E[P]E[R_m]\mathrm{Cov}(P,R_m)$$
$$+ E^2[P]\cdot\mathrm{Var}(R_m) + \mathrm{Cov}^2(P,R_m) + \mathrm{Var}(R_m)\cdot\mathrm{Var}(P). \quad (13)$$

The real beta is given by the ratio of (12) over (13). If inflation is non-stochastic, the real beta is identical to the nominal beta and is reduced to $\mathrm{Cov}(R_i,R_m)/\mathrm{Var}(R_m)$. Q.E.D.

3.2 The Moment Generating Function Method

Given that R_i, R_m, and P are trivariate normally distributed, Hogg and Craig (1969, Ch. 13) show that the moment generating function of this distribution may be written as:

$$\varphi(t_1,t_2,t_3) = \exp\{t_1E[R_i] + t_2E[R_m] + t_3E[P] + (1/2)[(t_1)^2\cdot\mathrm{Var}(R_i)$$
$$+ (t_2)^2\cdot\mathrm{Var}(R_m) + (t_3)^2\cdot\mathrm{Var}(P) + 2\cdot t_1\cdot t_3\cdot\mathrm{Cov}(R_i,P)$$
$$+ 2\cdot t_2\cdot t_3\cdot\mathrm{Cov}(R_m,P) + 2\cdot t_1\cdot t_2\cdot\mathrm{Cov}(R_i,R_m)]\} \quad (14)$$

It may be shown from (14) (though somewhat time consuming) that

$$E[R_iR_mP^2] = [\partial^4\varphi(t_1,t_2,t_3)/\partial t_1\partial t_2(\partial t_3)^2]\big|_{t_1=t_2=t_3=0}$$
$$= \mathrm{Cov}(R_i,R_m)\cdot\mathrm{Var}(P) + E[R_i]E[R_m]\mathrm{Var}(P) + E^2[P]\cdot\mathrm{Cov}(R_i,R_m]$$
$$+ 2\cdot E[R_i]E[P]\cdot\mathrm{Cov}(R_m,P] + 2\cdot E[R_m]E[P]\cdot\mathrm{Cov}(R_i,P)$$
$$+ 2\cdot\mathrm{Cov}(R_m,P)\cdot\mathrm{Cov}(R_i,P) + E[R_i]E[R_m]E^2[P]. \quad (15)$$

$$E[R_m^2P^2] = [\partial^4\varphi(0,t_2,t_3)/(\partial t_2)^2(\partial t_3)^2]\big|_{t_2=t_3=0}$$
$$= \mathrm{Var}(R_m)\cdot\mathrm{Var}(P) + E^2[R_m]\cdot\mathrm{Var}(P) + E^2[P]\cdot\mathrm{Var}(R_m)$$
$$+ 2\cdot\mathrm{Cov}^2(R_m,P) + 4\cdot E[R_m]E[P]\mathrm{Cov}(R_m,P) + E^2[R_m]E^2[P]. \quad (16)$$

From the definition of covariance, we also have

$$E[R_iP] = \mathrm{Cov}(R_i,P) + E[R_i]E[P] \quad (17)$$

$$E[R_mP] = \mathrm{Cov}(R_m,P) + E[R_m]E[P] \quad (18)$$

$$E[R_iR_mP^2] = \mathrm{Cov}(R_iP,R_mP) + E[R_iP]E[R_mP] \quad (19)$$

$$E[(R_mP)^2] = \mathrm{Var}(R_mP) + \{E[R_mP]\}^2 \quad (20)$$

Similarly, it is possible to prove form equations (15)-(20) that $\mathrm{Cov}(R_iP, R_mP)$ and $\mathrm{Var}(R_mP)$ are equal to (12) and (13), respectively. Again, real beta is equal to the ratio of (12) over (13).

Obviously, the MGF method is significantly more time consuming than the generalized RS covariance operator method!

4. Further Examples

Example 1: If x and y are bivariate normal, what is $\mathrm{Cov}(x^3, y^4)$?

Solution: We may apply the MGF method to solve this problem, but it is tedious and time consuming. The answer is however easily obtained by applying Corollary 1 as follows:

$$\begin{aligned}\mathrm{Cov}(x^3,y^4) &= 12\cdot\{E^2[x] + \mathrm{Var}(x)\}\{E^3[y] + 3E(y)\cdot\mathrm{Var}(y)\}\mathrm{Cov}(x,y)\\ &\quad + 36\{E^2(y) + \mathrm{Var}(y)\}\cdot E(x)\cdot\mathrm{Cov}^2(x,y) + 24\cdot E(y)\cdot\mathrm{Cov}^3(x,y).\end{aligned}$$

Example 2: If x and y are bivariate normal, what is $\mathrm{Cov}(x^2y, e^y)$?

Solution: It is very difficult to apply the MGF method to solve this problem. Applying the Theorem to the problem yields

$$\begin{aligned}\mathrm{Cov}(x^2y,e^y) &= 2\cdot\{E[x]E[y] + \mathrm{Cov}(x,y)\}E[e^y]\cdot\mathrm{Cov}(x,y)\\ &\quad + \{E^2[x] + \mathrm{Var}(x)\}E[e^y]\cdot\mathrm{Var}(y) + E[y]E[e^y]\cdot\mathrm{Cov}^2(x,y)\\ &\quad + 2E[x]E[e^y]\mathrm{Cov}(x,y)\cdot\mathrm{Var}(y) + E[e^y]\cdot\mathrm{Cov}^2(x,y)\cdot\mathrm{Var}(y). \qquad (21)\end{aligned}$$

It is known from Aitchison and Brown (1957, p. 8) that $E(e^y) = \exp\{E(y) + (1/2)\cdot\mathrm{Var}(y)\}$. Consequently it is a relatively easy task to compute (21).

5. Conclusions

In this paper, the Rubinstein/Stein covariance operator is generalized to the case where both variables are functions of multivariate normal random variables. This new method is proven to be more powerful and convenient to employ than the moment generating function method.

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