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THE

GENERATION OF SURFACES

DISSERTATION

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INTRODUCTION.

1. A surface may be defined by a property common to each of its points; in this case we obtain the equation of the surface by translating this property analytically. But more usually, however, a surface is defined by the movement of a line in space. Suppose

$$f(x, y, z, c)=0, \quad \varphi(x, y, z, c)=0,$$

the equations of a line containing an arbitrary parameter c; if c varies in a continuous manner, the line moves in space and generates a surface. The equation of the surface is obtained by eliminating c between the equations of the moving line. In this investigation, a surface is supposed to be generated by a variable plane curve which turns around a line of its plane. It will be convenient to define the motion of the plane curve, by subjecting it to the condition of moving on fixed curves, which may be either plane or twisted. The generating curve in any position will be defined by means of two rectangular coordinates (z, t). Any surface may be generated in this way, and surfaces generated by curves of the same kind have equations of the same form. Thus it will be shown, if the axis of z is taken for the axis of rotation, the generating curve being a straight line, that the equation of the surface will be of the form

$$z=x\varphi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right).$$

If the generating curve is a circle the equation of the surface takes the form

$$x^2+y^2+z^2+z\varphi_1\left(\frac{y}{x}\right)+x\varphi_2\left(\frac{y}{x}\right)+\varphi_3\left(\frac{y}{x}\right)=0.$$

2. As far as I know, only one special case of the method has been used heretofore and that is the case of surfaces of revolution. A surface of revolution

may be generated by the rotation of an invariable plane curve around a fixed right line lying in its plane. During the rotation the mutual positions of the line and curve remain unchanged. In order to find the equation of a surface of revolution, let us take for the axis of rotation the axis of z, and for the plane in which the generating curve was originally placed the (x, z) plane of a system of rectangular coordinates. Suppose that the equation of the generating curve is

$$z = f(x).$$

Now imagine the surface cut by any plane which passes through the axis of z and assume in this plane a system of rectangular coordinates (z, t). Since this plane must cut out from the surface one of the original congruent curves, the equation of the section must be

$$z = f(t).$$

Now for all points of an arbitrary meridian we have evidently

$$t^2 = x^2 + y^2,$$

therefore

$$z = f(\sqrt{x^2 + y^2})$$

is the equation of the surface of revolution.

3. The method depends upon the properties of a system of homothetic plane curves. The special case of surfaces of revolution depends upon the properties of a system of homothetic concentric circles, the axis of the surface being perpendicular to the plane of the circles and passing through their common centre.

4. The same method may be applied to the study of twisted curves. For if two plane curves lying in the same plane are referred to the (z, t) coordinates, their intersections are referred to the same coordinates and will trace out a curve as the plane rotates about a line as an axis.

I.

General Theory.

1. Being given any system of homothetic curves in the xy plane, it may be represented by the equation

$$f(kx, ky) = 0, \tag{1}$$

the homothetic centre being at the origin. The equation

$$f(x, y) = 0$$

represents a curve of the system which will be called the fundamental curve and the curve

$$f(kx, ky) = 0$$

will be referred to as the curve (k). If we denote by ρ the absolute value of the distance from the origin to a point P of the fundamental curve and by t the length of the radius vector (positive or negative according as it has the same sense as OP or the opposite sense) to the homologous point P' of the curve (k) we shall have

$$k \equiv \frac{\rho}{t}. \qquad (2)$$

Now suppose that at every point of these curves a perpendicular z is erected to the plane xy; the extremities of these perpendiculars trace out a surface. If z is determined for the point (x, y) of the curve (k) by

$$\psi\left(z, \frac{1}{k}, \frac{y}{x}\right) = 0, \qquad (3)$$

the equation of the surface will be obtained by eliminating k between equations (1) and (3). For a fixed value of $\frac{y}{x}$, that is for the points of the surface obtained along a line TOT' drawn through the origin, the points of the surface satisfy $\left(\text{since } \frac{1}{k} \equiv \frac{t}{\rho}\right)$ the equation

$$\psi\left(z, \frac{t}{\rho}, \frac{y}{x}\right) = 0, \qquad (4)$$

ρ and $\frac{y}{x}$ being constant. This equation, considering z and t as rectangular coordinates, is the equation of the section of the surface by the plane ZOT. If

$$f(x, y) = 0$$

is an algebraic curve of order n, then upon any line through the origin we get n points P and n values of ρ, say $\rho_1, \rho_2, \ldots \rho_n$. To each of these values will correspond a curve ψ, accordingly the section of the surface by a plane through the axis of z will consist of the n plane curves (which may or may not be different)

$$\psi\left(z, \frac{t}{\rho_i}, \frac{y}{x}\right)=0, \qquad i=1,\ 2,\ \ldots,\ n$$

t is not measured necessarily in the same direction in all these equations. For those of the points P which lie on OT, OT is the positive direction for t, for those which lie on OT', OT' is the positive direction. The surface obtained by eliminating k between

$$f(kx,\ ky)=0,$$

$$\psi\left(z, \frac{1}{k}, \frac{y}{x}\right)=0$$

is generated therefore by the motion of these n plane curves as their common plane rotates about the axis of z. The equation

$$\psi\left(z, \frac{t}{\rho}, \frac{y}{x}\right)=0$$

will be called the type equation of the generating plane curves. If the homothetic centre is at an ordinary point of the fundamental curve, k will enter the equation of the homothetic system to the $n-1$ degree and the section of the surface (leaving aside for the moment the special position of the cutting plane where TOT' is tangent to the fundamental curve at the origin) will consist of the $n-1$ curves

$$\psi\left(z, \frac{t}{\rho_i}, \frac{y}{x}\right)=0,\ \rho_i \neq 0, \qquad i=2,\ 3,\ \ldots,\ n$$

and the curve

$$\varphi\left(z, \frac{t}{\rho_1}, \frac{y}{x}\right)=0,\ \rho_1=0.$$

When TOT' is tangent to the fundamental curve at the origin, the section will consist of

$$\psi\left(z, \frac{t}{\rho_i}, \frac{y}{x}\right)=0,\ \rho_i \neq 0, \qquad i=3,\ 4,\ \ldots,\ n$$

and

$$\psi\left(z, \frac{t}{\rho_i}, \frac{y}{x}\right)=0, \ \rho_i=0. \qquad i=1, 2$$

If the homothetic centre is at a multiple point of order m the section will consist of the curves

$$\psi\left(z, \frac{t}{\rho_i}, \frac{y}{x}\right)=0, \ \rho_i \neq 0 \qquad i=m+1, m+2, \ldots, n$$

$$\psi\left(z, \frac{t}{\rho_i}, \frac{y}{x}\right)=0, \ \rho_i=0. \qquad i=1, 2, \ldots, m$$

When TOT' is tangent to the fundamental curve at the origin the section will consist of the curves

$$\psi\left(z, \frac{t}{\rho_i}, \frac{y}{x}\right)=0, \ \rho_i \neq 0, \qquad i=m+2, m+3, \ldots, n$$

$$\psi\left(z, \frac{t}{\rho_i}, \frac{y}{x}\right)=0, \ \rho_i=0. \qquad i=1, 2, \ldots, m+1$$

2. If we desire the line around which the plane of the generating curves rotates and which cuts the plane of xy in the homothetic centre to be parallel to the axis of z, it is sufficient to eliminate k between

$$f[kx+(1-k)x_0, ky+(1-k)y_0]=0,$$

$$\psi\left[z, \frac{1}{k}, \frac{y-y_0}{x-x_0}\right]=0,$$

x_0, y_0 being the coordinates of the homothetic centre.

3. Let us consider the circle

$$x^2+y^2=r^2$$

as the fundamental curve. The system of homothetic curves will be

$$k^2x^2+k^2y^2=r^2$$

from which we obtain

$$\frac{t^2}{\rho^2}=\frac{x^2+y^2}{r^2}.$$

Let the type equation be denoted by

$$\psi\left(z, \frac{t}{\rho}, \frac{y}{x}\right)=0.$$

Since the line TOT' cuts the fundamental circle in two points which are symmetrical with respect to the origin O, we have

$$\rho_1=\rho_2$$

whence the equations

$$\psi\left(z, \frac{t}{\rho_1}, \frac{y}{x}\right)=0,$$

$$\psi\left(z, \frac{t}{\rho_2}, \frac{y}{x}\right)=0,$$

represent two curves lying in the plane ZOT and these curves are symmetrical with respect to the axis of z. These two curves represent the section cut out by the plane ZOT from the surface obtained by eliminating $\frac{t}{\rho}$ between

$$\psi\left(z, \frac{t}{\rho}, \frac{y}{x}\right)=0,$$

$$\frac{t}{\rho}=\frac{\sqrt{x^2+y^2}}{r}.$$

If $\frac{y}{x}$ is actually present in the equation of the type curve, the section will vary with the cutting plane. If the surface is a surface of revolution, the section does not change with the cutting plane. This is indicated by the absence of $\frac{y}{x}$ from the type equation. Since the circle is a curve of the second order, there are two generating curves which may or may not coincide. If the type equation

$$\psi\left(z, \frac{t}{\rho}, \frac{y}{x}\right)=0$$

contains only even powers of t, the two generating curves will be coincident and this curve will be symmetrical with respect to the axis of z.

4. Assuming that the equations of the sections of any proposed surface by planes through the axis of z can be represented by one and the same equation

involving an arbitrary parameter determining the position of the section, there is no loss of generality in writing this equation

$$\psi\left(z, \frac{t}{\rho}, \frac{y}{x}\right) = 0,$$

since $\frac{y}{x}$ can be taken at that parameter. Accordingly by what precedes, any surface, whose sections by planes through the axis of z are symmetrical with respect to this axis, can be represented in the foregoing manner by means of a system of concentric circles and thus have its equation presented in the form

$$\psi\left(z, \frac{\sqrt{x^2+y^2}}{r}, \frac{y}{x}\right) = 0.$$

In the case of a surface in which the sections are not symmetrical with respect to the axis of z, the same method would be applicable, but would give not only the proposed surface, but also the symmetrical surface with respect to the axis of z. It may happen that apart from the occurrence of $\sqrt{x^2+y^2}$ through the value of t, it may be brought in also through the parameter $\frac{y}{x}$. In this event the rationalization of the equation may take place through cancellation of the radical and without raising to powers. Then since the double sign must be understood in the value of $t = \sqrt{x^2+y^2}$, one surface will be represented when one sign is taken, the symmetrical surface when the other sign is taken.

5. As an instance of this breaking up into two distinct surfaces let us generate the plane

$$x + y + z - 1 = 0,$$

by the motion of a straight line which passes through the point (0, 0, 1) on the axis of z. For the type equation of the generating curve let us take

$$z - 1 = mt,$$

m being the tangent of the angle which the line makes with the radius vector OT. We shall suppose m to vary with the position of the (z, t) plane, and accordingly to be a function of $\frac{y}{x}$, viz., we shall suppose

$$m = \frac{x + y}{\sqrt{x^2 + y^2}}$$

so that the type equation is

$$z - 1 = \frac{x + y}{\sqrt{x^2 + y^2}} t.$$

To obtain the equation of the surface we must substitute for t in the preceding equation its value $\mp \sqrt{x^2 + y^2}$. The equation then breaks up into the two equations

$$x + y + z - 1 = 0,$$

$$z - x - y - 1 = 0.$$

These two equations represent the proposed plane and its symmetrical plane. To generate a surface of revolution we must eliminate t between the equation of the type curve

$$\psi(z, t) = 0$$

and

$$t = \sqrt{x^2 + y^2}.$$

If only even powers of t occur in the type equation the two generating curves will coincide and only one surface will be generated. If odd powers of t occur there is nothing to make the irrationalities disappear without raising to powers, for the type equation does not contain $\frac{y}{x}$. We see therefore that the two sheets generated by invariable curves, whose positions with respect to the axis of z do not change, belong to the same surface.

6. A point on a twisted curve will be known if its z is known and if its projection on the plane of xy is known. We may therefore represent the equations of a twisted curve by

$$z = g\left(\frac{y}{x}\right),$$

$$\frac{t}{\rho} = h\left(\frac{y}{x}\right).$$

The first equation represents a right conoid having the axis of z as an axis, while the second represents a cylinder having its generators parallel to the axis of z. The second equation will be called the type equation. If for the fundamental curve we take the circle

$$x^2 + y^2 = \rho^2$$

the equations of the twisted curve become

$$z = g\left(\frac{y}{x}\right),$$

$$t = h\left(\frac{y}{x}\right).$$

The type equation

$$t = h\left(\frac{y}{x}\right)$$

may be considered as representing a curve in the plane of xy which is symmetrical with respect to the origin. In the same way as in the case of surfaces, we observe that this curve may break up into a curve which is not symmetrical with respect to the origin together with the symmetric of this curve. Assuming that the points of any twisted curve cut out by planes through the axis of z can be represented by the same equations involving an arbitrary parameter determining the position of the cutting plane, there is no loss of generality in writing these equations

$$z = g\left(\frac{y}{x}\right),$$

$$t = h\left(\frac{y}{x}\right),$$

since $\frac{y}{x}$ can be taken as that parameter. Accordingly, by what precedes, a curve whose points of section by planes through the axis of z are symmetrical with respect to this axis may be represented in the foregoing manner. In the case of a curve in which the points of section are not symmetrical with respect to the axis of z, the same method would be applicable, but would give not only the proposed curve, but also its symmetric with respect to the axis of z.

7. A twisted curve may be represented also by means of the equations

$$\varphi\left(z, \frac{t}{\rho}, \frac{y}{x}\right) = 0,$$

$$\psi\left(z, \frac{t}{\rho}, \frac{y}{x}\right) = 0.$$

In the plane of ZOT these equations, taken together, will represent a certain number of points which are determined as the intersections of plane curves. As the plane rotates about the axis of z these points will trace out a curve.

II.

The Circle as the Fundamental Curve.

1. Problem. The common plane of three circles passes through the axis of z and each circle moves on three arbitrary curves. Determine the surfaces generated by these circles, the surfaces traced out by their radical axes, and the surface generated by the circle which cuts the three circles orthogonally. Determine also the curves on which the centres of the given circles move and the curves described by their radical centre. The three given circles will be referred to as the circles s_1, s_2, s_3. Suppose that the equation of s_1 is

$$z + t^2 + Az + Bt + C = 0,$$

where A, B, C are to be determined by the condition that the circle passes through the points (z_1, t_1), (z_2, t_2), (z_3, t_3). If these points lie on the curves

$$z = g_1\left(\frac{y}{x}\right), \quad t = h_1\left(\frac{y}{x}\right);$$
$$z = g_2\left(\frac{y}{x}\right), \quad t = h_2\left(\frac{y}{x}\right);$$
$$z = g_3\left(\frac{y}{x}\right), \quad t = h_3\left(\frac{y}{x}\right),$$

respectively, the equation of the circle s_1 is

$$\begin{vmatrix} z^2 + t^2 & z & t & 1 \\ g_1^2 + h_1^2 & g_1 & h_1 & 1 \\ g_2^2 + h_2^2 & g_2 & h_2 & 1 \\ g_3^2 + h_3^2 & g_3 & h_3 & 1 \end{vmatrix} = 0,$$

which may be written

$$z^2 + t + 2z D_1\left(\frac{y}{x}\right) + 2tE_1\left(\frac{y}{x}\right) + F_1\left(\frac{y}{x}\right) = 0.$$

The equations of the three circles may therefore be written

$$\left.\begin{array}{ll} s_1 & z^2 + t^2 + 2zD_1\left(\frac{y}{x}\right) + 2tE_1\left(\frac{y}{x}\right) + F_1\left(\frac{y}{x}\right) = 0, \\ s_2 & z^2 + t^2 + 2zD_2\left(\frac{y}{x}\right) + 2tE_2\left(\frac{y}{x}\right) + F_2\left(\frac{y}{x}\right) = 0, \\ s_3 & z^2 + t^2 + 2zD_3\left(\frac{y}{x}\right) + 2tE_3\left(\frac{y}{x}\right) + F_3\left(\frac{y}{x}\right) = 0. \end{array}\right\} \quad \text{(A)}$$

The ordinary method for the determination of a circle orthogonal to three given circles is now applicable and gives for its equation

$$\begin{vmatrix} D_1 + z & E_1 + t & zD_1 + tE_1 + F_1 \\ D_2 + z & E_2 + t & zD_2 + tE_2 + F_2 \\ D_3 + z & E + t & zD_3 + tE_3 + F_3 \end{vmatrix} = 0.$$

This circle will be called the circle O. To obtain the equations of the surfaces generated by the circles s_1, s_2, s_3, O, we must substitute $\sqrt{x^2 + y^2}$ for t in the equations of the circles. Doing this we get for the equations of the surfaces

$$\begin{array}{ll} s_1 & z^2 + x^2 + y^2 + 2zD_1 + 2E_1\sqrt{x^2 + y^2} + F_1 = 0, \\ s_2 & z^2 + x^2 + y^2 + 2zD_2 + 2E_2\sqrt{x^2 + y^2} + F_2 = 0, \\ s_3 & z^2 + x^2 + y^2 + 2zD_3 + 2E_3\sqrt{x^2 + y^2} + F_3 = 0, \end{array}$$

$$O \begin{vmatrix} D_1 + z & E_1 + \sqrt{x^2 + y^2} & zD_1 + \sqrt{x^2 + y^2}\, E_1 + F_1 \\ D_2 + z & E_2 + \sqrt{x^2 + y^2} & zD_2 + \sqrt{x^2 + y^2}\, E_2 + F_2 \\ D_3 + z & E_3 + \sqrt{x^2 + y^2} & zD_3 + \sqrt{x^2 + y^2}\, E_3 + F_3 \end{vmatrix} = 0.$$

For the surfaces traced out by the radical axes of the three circles we have

$$\begin{array}{ll} s_1s_2 & 2z(D_1 - D_2) + 2\sqrt{x^2 + y^2}\,(E_1 - E_2) + F_1 - F_2 = 0, \\ s_2s_3 & 2z(D_2 - D_3) + 2\sqrt{x^2 + y^2}\,(E_2 - E_3) + F_2 - F_3 = 0, \\ s_3s_1 & 2z(D_3 - D_1) + 2\sqrt{x^2 + y^2}\,(E_3 - E_1) + F_3 - F_1 = 0. \end{array}$$

Writing the equations of the three circles in the form

$$\begin{array}{ll} s_1 & (z + D_1)^2 + (t + E_1)^2 = D_1^2 + E_{13} - F_1, \\ s_2 & (z + D_2)^2 + (t + E_2)^2 = D_2^2 + E_2^2 - F_2, \\ s_3 & (z + D_3)^2 + (t + E_3)^2 = D^2 + E_3^2 - F_3, \end{array}$$

we see that their centres will trace out the curves

$$\begin{array}{lll} s_1 & z=-D_1, & t=-E_1; \\ s_2 & z=-D_2, & t=-E_2; \\ s_3 & z=-D_3, & t=-E_3. \end{array}$$

Using equations (A) we obtain for the equations of the curve traced out by their radical centre

$$z=-\tfrac{1}{2}\frac{\begin{vmatrix} F_2-F_1 & E_2-E_1 \\ F_3-F_2 & E_3-E_2 \end{vmatrix}}{\begin{vmatrix} D_1-D_2 & E_1-E_2 \\ D_2-D_3 & E_2-E_3 \end{vmatrix}}, \quad t=-\tfrac{1}{2}\frac{\begin{vmatrix} D_1-D_2 & F_1-F_2 \\ D_2-D_3 & F_2-F_3 \end{vmatrix}}{\begin{vmatrix} D_1-D_2 & E_1-E_2 \\ D_2-D_3 & E_2-E_3 \end{vmatrix}}.$$

2. Ruled Surfaces. Determine the equation of the surface generated by the motion of a straight line which meets the axis of z and moves on two arbitrary curves.

The type equations may be written

$$\frac{z-z_0}{t-t_0}=\frac{z_1-z_0}{t_1-t_0}.$$

Suppose that the point (z_0, t_0) lies on the curve

$$z=g_0\left(\frac{y}{x}\right), \quad t=h_0\left(\frac{y}{x}\right),$$

the point (z_1, t_1) on

$$z=g_1\left(\frac{y}{x}\right), \quad t=h_1\left(\frac{y}{x}\right).$$

The type equation now becomes

$$\frac{z-g_0}{t-h_0}=\frac{g_1-g_0}{h_1-h_0}=F\left(\frac{y}{x}\right) \text{ say.}$$

Substituting $\sqrt{x^2+y^2}$ for t, the equation of the surface is

$$z-F\left(\frac{y}{x}\right)\sqrt{x^2+y^2}-g_0\left(\frac{y}{x}\right)+h_0F\left(\frac{y}{x}\right)=0,$$

which may be written

$$z = x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right),$$

or in the equivalent form

$$z = yf\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right).$$

Either of these forms is the equation of the surface. The problem may also be worded, find the equation of the surface generated by a straight line which meets the axis of z, moves on an arbitrary curve and makes an angle varying according to a given law with the axis of z.

The type equation is

$$t - t_0 = \lambda(z - z_0).$$

Writing

$$z_0 = g_0\left(\frac{y}{x}\right), \quad t_0 = h_0\left(\frac{y}{x}\right), \quad \lambda = F\left(\frac{y}{x}\right),$$

we obtain readily the same equation as before.

3. The Cone. If the type equation is

$$z - z_0 = \lambda t,$$

where

$$z_0 = c, \quad \lambda = F\left(\frac{y}{x}\right),$$

the surface is generated by the motion of a right line which passes through the fixed point c on the axis of z, and makes an angle varying according to a given law with the axis of t. The surface is therefore a cone whose equation is

$$z = F\left(\frac{y}{x}\right)\sqrt{x^2 + y^2} + c,$$

which may be written

$$z = x\varphi\left(\frac{y}{x}\right) + c.$$

4. The Conoid. Suppose that the type equation is

$$z - z_0 = 0,$$

where

$$z_0 = \varphi\left(\frac{y}{x}\right).$$

The surface is generated by the motion of a straight line which passes through the axis of z and remains parallel to the plane of xy. The surface is therefore a right conoid whose equation is

$$z = \varphi\left(\frac{y}{x}\right).$$

5. The Cylinder. If the type equation is

$$t - t_0 = 0,$$

where

$$t_0 = \varphi\left(\frac{y}{x}\right),$$

the surface is generated by the motion of a line which moves parallel to the axis of z. The surface is a right cylinder and has for its equation

$$\sqrt{x^2 + y^2} = \varphi\left(\frac{y}{x}\right),$$

which may be written

$$z = \psi\left(\frac{y}{x}\right).$$

6. Let us write the type equation in the form

$$t - \varphi_1\left(\frac{y}{x}\right) = \varphi_2\left(\frac{y}{x}\right)\left[z - \varphi_3\left(\frac{y}{x}\right)\right].$$

If ϕ_1, ϕ_2, ϕ_3, are uniform functions, the type equation will represent two lines in the plane of ZOT which are symmetrical with respect to the axis of z. As an example suppose that the line passes through the circle

$$x^2 + y^2 = R^2,$$

and makes an angle with the axis of z equal to that made by the tangent to the circle

$$x^2 + y^2 = R^2$$

at the point where the generating line meets this circle. We have

$$\varphi_1\left(\frac{y}{x}\right) = R, \quad \varphi_2\left(\frac{x}{y}\right) = -\frac{x}{y}, \quad \varphi_3\left(\frac{y}{x}\right) = 0,$$

whence the equation of the surface is

$$(x^2+y^2)\,y^2=(Ry-xz)^2.$$

which represents a quartic scroll. As an example of the case where ϕ_1, ϕ_2, ϕ_3 are not all uniform let us put

$$t_0=\varphi_1\left(\frac{y}{x}\right)=R, \quad z_0=\varphi_3\left(\frac{y}{x}\right)=\tan^{-1}\frac{y}{x}.$$

These equations represent two helices which are symmetrical with respect to the axis of z. The equation of the surface is

$$\sqrt{x^2+y^2}-R=\varphi_2\left(\frac{y}{x}\right)\left[z-\tan^{-1}\frac{y}{x}\right].$$

The section of the surface by any plane through the axis of z divides the plane up into equal rhombuses. For the section consists of two systems of lines symmetrical with respect to the axis of z and each system consists of equidistant parallel lines. If

$$\varphi_2\left(\frac{y}{x}\right)=1,$$

the section of the surface by a plane through the axis of z divides the plane up into equal squares and the size of the squares does not vary with the cutting plane. The equation of this surface is

$$z-\tan^{-1}\frac{y}{x}-\sqrt{x^2+y^2}+R=0.$$

If

$$\varphi_2\left(\frac{y}{x}\right)=\infty,$$

the two systems of lines will coincide and the surface is the helicoid

$$z=\tan^{-1}\frac{y}{x}.$$

7. The Circle as the Generating Curve.

Determine the equation of the surface generated by the motion of a circle, whose plane passes through the axis of z while its centre moves on one arbitrary curve and its circumference passes through another. Let z_1, t_1 be the coordinates of the centre, z_2, t_2 the coordinates of a point on the circumference. The equation of the circle is

$$(z-z_1)^2+(t-t_1)^2=(z_2-z_1)^2+(t_2-t_1)^2.$$

Suppose that

$$z=\varphi_1\left(\frac{y}{x}\right),\quad t=\psi_1\left(\frac{y}{x}\right),$$

is the curve on which the centre moves, and

$$z=\varphi_2\left(\frac{y}{x}\right),\quad t=\psi_2\left(\frac{y}{x}\right),$$

the curve through which the circle passes. The equation of the circle now becomes

$$(z-\varphi_1)^2+(t-\psi_1)^2=(\psi_2-\psi_1)^2+(\varphi_2-\varphi_1)^2.$$

Substituting $\sqrt{x^2+y^2}$ for t in this equation we get the equation of the surface which is easily put in the form

$$z^2+x^2+y^2+zf_1\left(\frac{y}{x}\right)+yf_2\left(\frac{y}{x}\right)+f_3\left(\frac{y}{x}\right)=0,$$

or in the equivalent form

$$z^2+x^2+y^2+zF_1\left(\frac{y}{x}\right)+yF_2\left(\frac{y}{x}\right)+F_3\left(\frac{y}{x}\right)=0.$$

Let us write the equation of the circle in the form

$$\left[z-\varphi\left(\frac{y}{x}\right)\right]^2+\left[t-\psi\left(\frac{y}{x}\right)\right]^2=\left[\theta\left(\frac{y}{x}\right)\right]^2.$$

If θ is constant the surface is generated by an invariable circle. If ϕ is constant the centre moves on a plane curve. If ψ is constant the centre moves on a curve which lies on a right circular cylinder. If

$$\varphi=0,\ \psi=c,\ \theta=R,$$

the surface is the anchor ring.

8. In general, five points determine a conic, so a surface may be generated by allowing a conic, whose plane passes through the axis of z, to move on five arbitrary curves.

Suppose that the equation of the conic is

$$at^2 + 2htz + bz^2 + 2gt + 2fz + c = 0,$$

and that the conic moves on the five curves

$$t = \varphi_i\left(\frac{y}{x}\right), \quad z = \psi_i\left(\frac{y}{x}\right), \qquad i = 1, 2 \ldots 5.$$

Consider the curve in any one of its positions and let the corresponding points for the five curves be

$$(t_i, z_i), \qquad i = 1, 2 \ldots 5$$

respectively.

The equation of the conic now takes the form

$$\begin{vmatrix} t^2 & tz & z^2 & t & z & 1 \\ t_1^2 & t_1z_1 & z_1^2 & t_1 & z_1 & 1 \\ t_2^2 & t_2z_2 & z_2^2 & t_2 & z_2 & 1 \\ t_3^2 & t_3z_3 & z_3^2 & t_3 & z_3 & 1 \\ t_4^2 & t_4z_4 & z_4^2 & t_4 & z_4 & 1 \\ t_5^2 & t_5z_5 & z_5^2 & t_5 & z_5 & 1 \end{vmatrix} = 0.$$

Substituting for t_i, z_i their values, this equation becomes

$$\begin{vmatrix} t^2 & tz & z^2 & t & z & 1 \\ \varphi_1^2 & \varphi_1\psi_1 & \psi_1^2 & \varphi_1 & \psi_1 & 1 \\ \varphi_2^2 & \varphi_2\psi_2 & \psi_2^2 & \varphi_2 & \psi_2 & 1 \\ \varphi_3^2 & \varphi_3\psi_3 & \psi_3^2 & \varphi_3 & \psi_3 & 1 \\ \varphi_4^2 & \varphi_4\psi_4 & \psi_4^2 & \varphi_4 & \psi_4 & 1 \\ \varphi_5^2 & \varphi_5\psi_5 & \psi_5^2 & \varphi_5 & \psi_5 & 1 \end{vmatrix} = 0.$$

The coefficients a, h, b, g, f, c are therefore functions of $\frac{y}{x}$ and the equation may be written

$$\theta_1\left(\frac{y}{x}\right)t^2 + \theta_2\left(\frac{y}{x}\right)tz + \theta_3\left(\frac{y}{x}\right)z^2 + \theta_4\left(\frac{y}{x}\right)t + \theta_5\left(\frac{y}{x}\right)z + \theta_6\left(\frac{y}{x}\right) = 0.$$

Substituting $\sqrt{x^2 + y^2}$ for t in this equation we get the equation of the surface which is easily put in the form

$$x^2 f_1\left(\frac{y}{x}\right) + xz f_2\left(\frac{y}{x}\right) + z^2 f_3\left(\frac{y}{x}\right) + x f_4\left(\frac{y}{x}\right) + z f_5\left(\frac{y}{x}\right) + f_6\left(\frac{y}{x}\right) = 0.$$

If the guiding curves are algebraic the surface will be algebraic. In general, if they are transcendental the surface will be transcendental.

9. In like manner we may find the general equation of surfaces generated by the motion of a plane curve of order n which turns about a line of its plane as an axis.

III.

Homothetic Systems in Which ρ is Variable.

1. In the case of homothetic concentric circles we have seen that if $\frac{y}{x}$ does not occur in the type equation, the section of the surface does not change with the cutting plane. This is the only system which possesses this property, for in all other systems $\rho_1, \rho_2, \rho_3, \ldots \rho_n$ vary with the position of the cutting plane.

2. Suppose that the straight line

$$ax + by = 1$$

is the fundamental curve. We find

$$\frac{t}{\rho} = ax + by.$$

Since ρ has only one value there is only one generating curve.
Let

$$\rho z - t = 0$$

be the type equation. We have to find the equation of the surface generated by the motion of a straight line which passes through the origin, the tangent of its angle with the axis of z being numerically equal to the distance from the origin to the line

$$ax + by = 1$$

measured along the radius vector TOT'. We get for the equation of this surface

$$ax + by - z = 0.$$

If the type equation is

$$\frac{z^2}{c^2} + \frac{t^2}{\rho^2} = 1,$$

the surface is generated by the motion of an ellipse which has one axis constant, the end of the other axis moving on the line

$$ax + by = 1.$$

The equation of this surface is

$$\frac{z^2}{c^2} + (ax + by)^2 = 1,$$

which represents a cylinder.

3. Quadric Surfaces.

1°. The ellipsoid. Let the type equation be

$$\frac{z^2}{c^2} + \frac{t^2}{\rho^2} = 1,$$

the fundamental curve being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Then

$$\frac{t^2}{\rho^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Substituting this value of $\frac{t}{\rho}$ in the type equation we get for the equation of the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The type equation shows that the surface is generated by the motion of an ellipse which has one axis constant and equal to $2c$, while the extremity of the other moves on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

2°. The hyperboloid of one sheet. Let the type equation be

$$\frac{z^2}{c^2}+\frac{t^2}{\rho^2}=1,$$

the fundamental curve being

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Then

$$\frac{t^2}{\rho^2}=\frac{x^2}{a^2}-\frac{y^2}{b^2}.$$

Substituting this value of $\frac{t}{\rho}$ in the type equation we get for the equation of the surface

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The surface is generated by the motion of an ellipse which has one axis equal to $2c$, while the extremity of the other moves on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

3°. The hyperboloid of two sheets. Let the type equation be

$$\frac{t^2}{\rho^2}-\frac{z^2}{c^2}=1,$$

the fundamental curve being

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Then

$$\frac{t^2}{\rho^2}=\frac{x^2}{a^2}-\frac{y^2}{b^2}.$$

Substituting this value of $\frac{t}{\rho}$ in the type equation we get for the equation of the surface

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

The surface is generated by the motion of a hyperbola which has one axis equal to $2c$, while the extremity of the other moves on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

4°. The elliptic paraboloid. Let the type equation be

$$t^2=4\rho^2 z,$$

the fundamental curve being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then

$$\frac{t^2}{\rho^2}=\frac{x}{a^2}+\frac{y^2}{b^2}.$$

Substituting this value of $\frac{t}{\rho}$ in the type equation we get for the equation of the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=4z.$$

The surface is generated by the motion of a parabola whose vertex is at the origin and whose axis is the axis of z. The distance of the focus from the origin is equal to the square of the distance from the origin to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the distance being measured along the radius vector TOT'.

5°. The hyperbolic paraboloid. Let the type equation be

$$t^2=4\rho^2 z,$$

the fundamental curve being

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The equation of the surface is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=4z.$$

The surface is generated by the motion of a parabola whose vertex is at the origin and whose axis is the axis of z. The distance of the focus from the origin is equal to the square of the distance from the origin to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

the distance being measured along the radius vector TOT'.

4. The method may be used for the generation of surfaces by the motion of a plane curve in a space of n dimensions. At present, however, I shall not dwell upon this subject. The surface is obtained by eliminating k between

$$f(kx_1, kx_2, kx_3, \ldots, kx_{n-1})=0,$$

and

$$\psi\left(x_n, \frac{t}{\rho}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_{n-1}}{x_1}\right)=0, \quad k\equiv\frac{\rho}{t}$$

where the first equation represents a homothetic system, and the second equation a plane curve lying in the plane X_nOT, the axes being supposed rectangular.

Thus if the homothetic system is

$$k^2x^2+k^2y^2+k^2z^2=\rho^2,$$

the type equation being

$$w^2+t^2=R^2,$$

the equation of the surface is

$$x^2+y^2+z^2+w^2=R^2,$$

which represents the hypersphere. Consider the generating circle

$$w^2+t^2=R^2$$

in any one of its positions, say when it lies in the plane determined by the axis of w and the right line TOT' lying in the space

$$w = 0.$$

Now imagine the line TOT' to coincide successively with every right line drawn through the origin in the space

$$w = 0.$$

In this motion the circle will generate the hypersphere.

BIOGRAPHICAL SKETCH.

The author, H. A. Sayre, was born in Montgomery, Ala., August 2, 1866. He entered the University of Alabama in October, 1884, but left at the end of the session to engage in business. In February, 1892, he entered the University of Virginia as a student in mathematics and astronomy. At the same time he was employed as an assistant in the McCormick Observatory. In October, 1892, he entered Johns Hopkins University, electing for his subjects, mathematics, astronomy and physics. In June, 1894, he received from the University of Alabama the degree of B. E. extra ordinem.