HIGH MASONRY DAM DESIGN

BY

CHARLES E. MORRISON, C.E., PH.D.
Assoc. M Mem. Am. Soc. C. E.

AND

ORRIN L. BRODIE, C.E.
Assoc. Mem. Am. Soc. C. E., Asst. Eng. Board Water Supply, N. Y. City

FIRST EDITION

FIRST THOUSAND

NEW YORK
JOHN WILEY & SONS
LONDON: CHAPMAN & HALL, LIMITED
1910

The Scientific Press
Robert Drummond and Company
New York

PREFACE

It is the practice at Columbia University to require of the third-year students in the Department of Civil Engineering, the execution of the design of a masonry dam, and to aid them in this problem they have heretofore been furnished with "Notes on the Theory and Design of High Masonry Dams," prepared some years ago by Prof. Burr of the Department, and having for their basis the method as set forth by Mr. Edward Wegmann.

This procedure with which Wegmann is credited, and which was developed through the investigations undertaken in connection with the Aqueduct Commission of the city of New York, for the purpose of determining a correct cross-section for the Quaker Bridge dam, resulted in the first direct method for calculating the cross-section of such structures and is essentially a development of the Rankine theory.

The studies appeared first in the report made by Mr. A. Fteley to the chief engineer of the Aqueduct Commission of the city of New York, dated July 25, 1887, and later in Mr. Wegmann's treatise on "The Design and Construction of Dams."

Neither in the report nor in the treatise however, have the effects of uplift, due to water permeating the mass of masonry, and of ice thrust, acting at the surface of the water in the reservoir, been considered, and in consequence of this, objection might be legitimately raised that the series of equations determining the cross-section fails to account for these factors. Some difference of opinion may exist as to the relative importance of these considerations, but when a structure of great responsibility is projected, conservatism in design is essential.

The following presentation which aims to supply these omissions, has been prepared primarily that there may be had in convenient form a text, containing the general treatment and such consideration of these factors as more recent practice requires, together with a brief statement regarding the late investigations undertaken for the purpose of determining more accurately the variation of stress in masonry dams.

The formulæ relating to uplift, ice thrust, etc., were deduced by one of the authors and have been used in part in connection with the design of the large dams for the new water supply for the city of New York.

The computations for the design of a high masonry dam are appended to facilitate the ready comprehension and application of the formulæ.

It is hoped that the presentation may appeal to the practicing engineer as well as the student, and that there may be found therein enough to compensate him for the labor involved in its perusal.

C. E. M.
O. L. B.

Columbia University, 1910.

HIGH MASONRY DAM DESIGN

THE method of analysis by which an economical cross-section of a high masonry dam may be most directly calculated, and the one which is most generally adopted in engineering practice, was first devised by Mr. Edward Wegmann through studies made in connection with the Aqueduct Commissioners of New York city, and it is that method which will be employed here, though it will receive some modification in certain particulars and be elaborated in certain others.

In determining the cross-section by the series of equations developed in that analysis, no account is taken of the condition of uplift due to water penetrating the mass of masonry, nor of the ice thrust acting horizontally at the surface of the water in the reservoir against the up-stream face of the dam, though reference is made to it. Present practice requires however, that these two factors be considered where a structure of great responsibility is proposed, and in this respect at least, will the analysis be amplified.

While upward pressure in a masonry dam, either at the foundation or in joints higher up, should always be considered, the subject does not lend itself to a very exact treatment; in fact, it becomes necessary to make assumptions in regard to its presence and action which in the end depend principally upon the judgment of the engineer. It is not surprising therefore, that a wide range of opinion exists as to the method of dealing with this factor.

Such pressures may become effective from two causes: the percolation of water into small cracks either in the superstructure or the foundation, or by the presence of springs in the foundation itself.

In the best laid masonry it is undoubtedly true that small cracks exist into which the water gains entrance, but this should be guarded against as far as possible by the exercise of great care in the laying of the stone and the bonding of them, together with thorough inspection. "Temperature variations due to setting of concrete and also due to daily and seasonal changes, while inducing stresses that are indeterminate, thereby providing an argument for conservatism in design, in addition affect permeability to greater or less degree." Even in cyclopean masonry where no horizontal joints exist, except between the facing stones, the possibility of other small cracks being formed is always present, and therefore requires recognition here as well.

Where springs are found in the native rock of the foundation the effect of upward pressure from such a source must be overcome by a system of drains which shall lead the water below the downstream face of the dam. The difficulty here is from the possible formation of new springs as soon as the reservoir becomes filled, with which some connection will inevitably be formed, and thus cause an upward pressure due to the total hydrostatic head of the water back of the dam. It is evident therefore, that the foundation should be carefully examined and be specially prepared to receive the first course of masonry.

The method of allowing for upward pressure depends upon its properly assumed presence, and also upon some assumed law of variation. Present practice indicates that this may be considered as varying from a maximum at the heel to zero at the toe. But because it is hardly justified from experience with other dams to impose such severe conditions upon the masonry above, it is agreed to consider this pressure as acting over only a portion of the joint, or, in other words, to consider only a portion of the full hydrostatic head as acting at the upstream face of the joint under examination.

Although lack of exact data precludes the possibility of assigning a definite value to the force of expanding ice in its formation at the surface of a reservoir, yet it will be evident that a thrust from such a cause may greatly affect the dimensions of a profile, in cold climates especially. As this thrust is effective at the surface of the water, for a low structure it may become a very serious feature.

The studies involved in the determination of a cross-section demand an investigation along two general lines:

First, the direct calculation fixing the most economical cross-section under the imposed conditions, and

Second, studies in comparing cross-sections ranging between this one, which may be called the minimum, and one of an existing masonry dam, where the conditions and responsibility are practically the same as those under consideration. Before undertaking such an analysis however, it will be necessary to consider the manner in which water pressure is exerted against a submerged surface; its amount; the method of determining the point of application of the resultant; the assumed distribution of pressure in a masonry joint; and finally, the action of the forces in and upon the structure.

It may be stated as a general proposition that water pressure acts in all directions against a submerged object and that it depends for its value merely upon the "head," or depth of the center of gravity of the figure below the free surface of the liquid. In consequence of this principle it may be shown that the total normal pressure is represented by

$$P=\gamma Ah, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P = the total normal pressure;

γ = the weight of a unit volume of water;

A = the total area; and

h = the vertical distance of the figure's center of gravity beneath the free surface of water.

The demonstration * may be made by considering the surface divided into an infinite number of parts; the total pressure on each one of these elements, depending only upon the weight of water resting upon it, may be written,

$$p=\gamma ah_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* See Merriman's "Hydraulics."

in which p = the total normal pressure on the differential area;

a = the differential area;

h_1 = the head on a (practically constant over the differential area).

If, therefore, we take the sum of the pressures on all of these small areas, we shall obtain the previous equation, which is perfectly general and applies to any surface. In the case of a vertical, rectangular strip of the back of a dam, the application of the formula will give a total pressure of

$$P = \gamma b \int_0^H x dx = \frac{\gamma b H^2}{2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where b is the constant breadth of the strip, usually taken as one unit, x is the variable, and H is the total height of the rectangle.

The point on the submerged surface at which this resultant pressure acts may be determined by assuming arbitrarily any axis, taking the moment of inertia of the surface about this axis, and dividing the result by the *static* moment of the surface with reference to the same axis. Applying this to the strip referred to above and assuming the arbitrary axis to be the horizontal line in which the surface of the water cuts the plane of the back, there will result,

$$I = \frac{bH^3}{12},$$

$$I_1 = I + A\left(\frac{H}{2}\right)^2 = \frac{bH^3}{12} + \frac{bH^3}{4} = \frac{bH^3}{3},$$

which is the moment of inertia with regard to the assumed axis.

$$I_s = A \cdot \frac{H}{2} = \frac{bH^2}{2},$$

is the static moment of the surface about the same axis, hence,

$$y = \frac{I_1}{I_s} = \frac{2H}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is the distance of the center of pressure from the surface of the water.

In the investigation of the distribution of pressure in a masonry joint subjected to external forces, the material is assumed to be rigid, though in reality it is to a certain degree elastic. This elasticity gives the distribution of stress an indeterminate law, so that neither the direction nor the intensity is actually known at any point. It is certain, however, that the intensity must be zero at the edges, although it may increase with great rapidity to higher values very near the limits of the joint. Investigations have been made within the past few years to obtain more exact information as to this distribution of stress, but so far the results are not completely satisfactory. Reference will be made to this matter in the Appendix.

Inasmuch as the exact law of stress variation is not known, one of uniform variation of normal stress has been assumed in all practical treatments of masonry joints.

Fig. 1 represents the simplest case, in which the pressure is assumed to be uniformly distributed over the joint $a\ b$, with the constant intensity p; it might be taken as representing any horizontal joint with a superimposed load acting at its center.

To express this condition of uniform stress algebraically, l may be assumed to be the length of the joint from a to b, while the breadth, perpendicular to the plane of the paper, is taken as unity. The area of the joint will then be l, whence,

$$W = p\,l, \quad \ldots \ldots \quad (5)$$

or

$$p = \frac{W}{l}, \quad \ldots \ldots \quad (6)$$

the uniform intensity of stress over the entire joint.

It may be observed here that this pressure is uniform only because the total load represented by W, acts at the center of the joint, and that when the point of application is changed to some other position, there will be an increased stress in that direction toward which the load has been moved, and a corresponding decrease in the opposite direction.

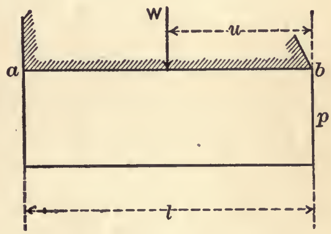


FIG. 1.

It will be necessary therefore, to consider this variation of pressure in eccentrically loaded joints and also the manner in which the eccentricity in the case of a dam is produced.

If $a\ b$ be any plane, horizontal joint in the dam at the distance H below the surface, OY the water surface, and ϕ the angle that the back makes with the vertical, then the total pressure on the back acting at a point one-third the distance up, will be

$$F' = \frac{\gamma H^2}{2} \sec \phi.$$

Combining this force with the weight of masonry W above the joint acting through the center of gravity of the section, the resultant R will intersect it at some point as e, on $a\,b$, other than the center of figure, called the center of resistance, and it is evident that with a variation of F' and W it may occupy any position along the joint.

Fig. 2, showing only the vertical component, exhibits such a case, where compression exists over the entire joint as in Fig. 1, but where the center of pressure is not at the center of figure.

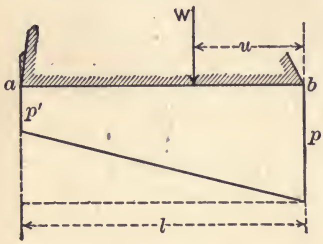


FIG. 2.

If the intensity of pressure at b may be represented by the vertical line p, and the intensity of pressure at a by the line p', then, since by the fundamental assumption the pressure varies uniformly over the entire joint, the vertical at any point, included between the horizontal $a\,b$ and the line joining the extremities of p and p' will indicate the intensity of pressure at that point, while the area of the trapezoid will represent the total pressure on the joint.

The former may be expressed algebraically thus:

$$p = p' + (p - p')\frac{x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the latter by,

$$\frac{(p+p')}{2}\,l, \quad \ldots\ldots\ldots \quad (8)$$

The determination of the maximum and minimum pressure p and p' may be made as follows:

Since the static moment of the rectangle $p\,l$ about a point $\frac{1}{3}l$ from p' is the same as the static moment of the trapezoid about the same point, because the moment of the triangle $p-p'$, l about that point is zero, that being the center of gravity of the triangle, there will result by taking moments

$$W(\tfrac{2}{3}l-u)=\frac{pl^2}{6}, \quad pe\cdot\frac{e}{l} \quad \ldots\ldots\ldots \quad (9)$$

whence,

$$p=\frac{2W}{l}\left(2-\frac{3u}{l}\right), \quad \ldots\ldots\ldots \quad (10)$$

which is an expression for the intensity of pressure at the point b, on the joint ab. To solve for the value of p', the intensity of the pressure at the point a, in a similar manner we may take moments about a point $\frac{1}{3}l$ from b, whence,

$$W\left(u-\frac{l}{3}\right)=\frac{p'l^2}{6}. \quad \ldots\ldots\ldots \quad (11)$$

or,

$$p'=\frac{2W}{l}\left(\frac{3u}{l}-1\right). \quad \ldots\ldots\ldots \quad (12)$$

When p' becomes zero, the trapezoid reduces to a triangle as shown in Fig. 3, with its center of gravity at a distance from b equal to $\frac{1}{3}l$, and, since the center of pressure of W must lie vertically above the center of gravity of the triangle graphically representing it, we shall have, $p'=O$, $u=\frac{1}{3}l$, and Eq. (9) reducing to

$$\frac{Wl}{3}=\frac{pl^2}{6}. \qquad (13)$$

whence,

$$p=\frac{2W}{l}. \qquad (14)$$

That is to say, the maximum pressure p is twice the value as obtained from Eq. (6).

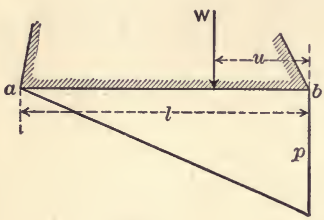


FIG. 3.

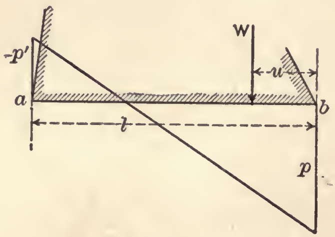


FIG. 4.

In Fig. 4 is represented a case in which tension exists over a portion of the joint. p' is here negative.

Although both masonry and the best hydraulic cement mortar have considerable tensile strength, running up to several hundred pounds per square inch in tests, the latter, together with the continued adhesion of the mortar to the aggregate in concrete, when used, is of uncertain value in this connection. The tensile strength is therefore always neglected in considering the stability of masonry dams or other similar structures, and is an omission which is the more justifiable since it leads to an error on the side of safety.

In the case represented by Fig. 4, the triangle, whose base is $3u$, and altitude p, is therefore alone considered, and by taking moments about b, there will result,

$$Wu = 3u \frac{p}{2} u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

whence,

$$p = \frac{2W}{3u}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If it is desirable to know what the tension in the joint is, it may be determined from Eq. (12). As $\frac{3u}{l} < 1.0$, the resulting value is negative, hence denoting a tension by that equation.

The pressures at *a* and *b* may also be determined as follows: Decomposing the resultant acting on any joint into its vertical and horizontal components, V will represent the total normal or vertical pressure, equal to W, the weight of masonry above the joint, *plus* the vertical component of the thrust from the water. The horizontal component of the resultant is disregarded, as its effect upon the joint is more or less indeterminate, and since too, it is assumed to be neutralized by the friction acting in the joint.

The vertical component V, acting through the point of application of the resultant R in the joint, is therefore the factor producing the difference in pressure between *a* and *b*, or the uniformly varying stress.

Assume that at the center of the joint, which is not necessarily vertically below the center of gravity of the mass above, two forces equal and opposite to each other, and of the same value V, are applied normal to the joint. The effect of each is to neutralize the other, but if we consider, apart from the other forces, the one acting downward, since it is applied at the center of figure it will produce a uniform stress p over the joint equal to $\frac{V}{l}$.

The two remaining and equal forces V and V, one acting downward at the point of application of R, and the other upward at the center, form a couple whose lever arm is v, and the moment of which is therefore $V \times v$. This moment produces a uniformly varying stress over the joint, increasing the intensity at b and decreasing it at a by an equal amount.

To determine its value we have but to consider the following:

$$M = Vv. \qquad \text{. . . . . . .} \qquad (17)$$

the moment caused by the couple and producing the varying stress. Also,

$$M = \frac{kI}{d_1}. \qquad \text{. . . . . . .} \qquad (18)$$

where k is the intensity of stress at the maximum distance from the neutral axis; I, the moment of inertia of the section about such an axis; and d_1 the normal distance from the neutral axis to that point where k exists.

Since the neutral axis passes through the center of figure of the joint, the value of d_1 is half the length of the joint, while I, the moment of inertia, equals $\frac{1}{12}l^3$, if we consider a horizontal section in the plane of the joint $a\,b$ extending back from the plane of the paper one unit's distance. Hence,

$$M = Vv = \frac{kI}{d_1} = \frac{k\frac{1}{12}l^3}{\frac{1}{2}l} = \frac{kl^2}{6}, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

or,

$$k = \frac{6\,Vv}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Here k represents the stress that must be added to the uniform stress $\frac{V}{l}$ to get the intensity of pressure at the toe b and the amount which must be subtracted from $\frac{V}{l}$ to get the intensity at the heel a. It is expressed in pounds per square inch, but if the distances are measured in feet and the forces in pounds, k will be designated in pounds per square foot.

While it is customary to consider only the normal component of the resultant pressure acting in a horizontal joint and to assume it to vary uniformly, this is probably correct only for horizontal joints in rectangular walls vertically loaded and not subjected to lateral pressures. It will be shown later that the maximum stresses exist at or near the downstream face, and act in direction parallel to and on planes normal to that face. The fact also that acute edges do not crack off in the inclined faces of dams is in itself a partial confirmation of the statement.

Under these circumstances then, the maximum normal pressure in a horizontal joint must be much less than the actual maximum pressure in the dam, and it has been assumed to bear the ratio to the latter of about 9 to 13.

Development of Formulæ for Design.

Six series of formulæ, designated by the letters A, B, C, D, E, and F, will now be presented, in each of which a given set of conditions with respect to the external forces will be involved; but as the method of procedure is practically the same for all cases, only series A will be developed here.

The following nomenclature will be employed:

L = the width of the top of the dam cross-section;

l = length of a horizontal joint of masonry, *to be determined;*

l_0 = *known* length of the joint next above joint of length l;

h = depth of a course of masonry (vertical distance between l_0 and l);

P = line of pressure, reservoir full;

P' = line of pressure, reservoir empty;

u = distance from front edge of the joint l to the point of intersection of P with the joint l, measured parallel to joint l;

y = distance from back edge of the joint l to the point of intersection of P' with the joint l, measured parallel to joint l;

y_0 = distance from back edge of the joint l_0 to the point of intersection of P' with the joint l_0, measured parallel to joint l_0;

v = distance between P and P' at the joint l, measured parallel to joint l;

γ = weight in pounds of a cubic foot of water (62.5);

γ' = weight in pounds of a cubic foot of mud (75–90);

Δ = ratio of unit weight of masonry to unit weight of water (often assumed as $\frac{7}{3}$);

$\Delta\gamma$ = weight in pounds of a cubic foot of masonry;

H = head of water on joint l (vertical distance of joint l below water surface);

H' = depth of earth back fill over joint l on front;

H_1 = head of water on joint l when ice acts at surface of water;

$H - H_1$ = rise of water level, due to flood, wave, etc., above normal level for full reservoir;

h_1 = head of water above mud level (liquid mud of weight γ');

h_2 = head of liquid mud on joint l, on back;

a = vertical distance from the top of the dam to the surface of water (flood);

a_1 = vertical distance from the top of the dam to the surface of water when ice is considered (a_1 generally exceeds a);

b = vertical distance from water surface to top of dam when dam is overtopped;

c = ratio of upward thrust intensity, due to hydrostatic head H (or H_1, or $h_1 + h_2$), assumed to act at heel of joint l (usually assumed as $\frac{2}{3}$);

$T\gamma$ = horizontal ice thrust at water surface in pounds (47,000);

(The value here given, for example, was used in studies for design. Our present lack of exact data in regard to ice pressures prevents more than a speculation from being made as to a definite value to be assigned in any case);

$D\gamma$ = horizontal dynamic thrust of water in pounds;

$E\gamma$ = thrust of earth back fill in pounds (on front);

$W_v\gamma$ = vertical pressure on inclined upstream face above joint l, in pounds;

A_0 = total area of cross-section of dam above joint l_0;

A = total area of cross-section of dam above joint l.

t = batter of upstream face for vertical distance h;

s = distance of line of action of $W_v\gamma$ from upstream edge of joint l, measured parallel to joint l;

δ = angle that $E\gamma$ makes with horizontal;

a = angle of slope of downstream face of dam with horizontal;

β = angle R makes with the vertical;

p = maximum allowable pressure intensity at toe (in pounds per square foot);

q = maximum allowable pressure intensity at heel (in pounds per square foot) (p is assumed less than q) p and q may be used to signify the calculated, existent pressure intensities corresponding to P and P' respectively, for the joint l.

f = the coefficient of friction for masonry on masonry (usually 0.6 to 0.75);

S = the shearing resistance of the masonry per square unit;

$F=\frac{\gamma H^2}{2}$ = the horizontal static thrust of the water in pounds;

$M=\frac{\gamma H^3}{6}$ = the moment of F about any point in the joint l;

$W=A\Delta\gamma$ = the total weight, in pounds, of masonry resting on the joint l;

$W_0=A_0\Delta\gamma$ = the total weight, in pounds, of masonry resting on the joint l_0;

R = the resultant of F and W;

R' = the resultant of the reactions;

$\frac{cHl\gamma}{2}$ = upward thrust of water on base l.

In the figures, hydrostatic pressures are indicated by triangular and trapezoidal areas included within dotted lines, while ice pressure is shown to contrast H_1 with H.

As before, a unit length of one foot of dam will be considered. Then the letters T, D, E, W_v, A, A_0, and H^2 will signify volumes.

It will be observed that, where possible, the several equations will have been cleared of the term $\Delta\gamma$ thereby simplifying actual calculations.

In the above table c, in a manner, may be considered to provide for an assumption of a certain proportion of the joint's area being subjected to upward water pressure; and the distribution, as evidenced by $cHl\gamma/2$, varying from a maximum intensity at the heel to zero intensity at the toe, is assumed in view of the facts that the tendency to open the joint would begin at the heel and a zero intensity of upward pressure at the toe would presuppose an opening with consequent flow at that point. As the dam would then be failing in its chief function, i.e., to retain water, this flow is not considered to exceed a slight seepage.

In general four ways are recognized in which a masonry dam may fail:

1. By overturning about the edge of any joint, due to the line of action of the resultant passing beyond the limits of stability.

2. By the crushing of the masonry or foundation because of excessive pressure.

3. By the shearing or sliding on the foundation or any joint, due to the horizontal thrust exceeding the shearing and frictional stability of the material.

4. By the rupture of any joint due to tension in it.

An unsatisfactory foundation might also be mentioned as possibly leading to failure, and in view of this, the footing upon which the dam rests should always be most carefully scrutinized.

To preclude failure from any of the above mentioned causes, it is the practice to design the cross-section of the dam with the following conditions imposed:

1. The lines of pressure, both for the reservoir full and empty, must not pass outside the middle third of any horizontal joint.

2. The maximum normal working pressure on any horizontal joint must never exceed certain prescribed limits, either in the masonry itself or in the foundation.

3. The coefficient of friction in any plane horizontal joint, or between the dam and its foundation, must be less than the tangent of the angle which the resultant makes with a vertical.

As may be seen by referring to the figures showing the distribution of pressure on a joint, when the resultant lies within the middle third, tension can exist in no part of it, nor can the safety factor be less than two, if we neglect to consider the upward pressure of water percolating through any of the joints or beneath the dam.

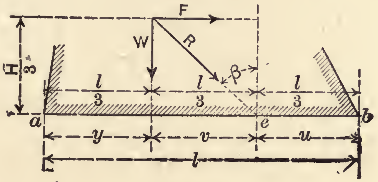


FIG. 5.

To illustrate the conditions that obtain and to derive the value of the safety factor when the resultant cuts the joint at the extremity of the middle third, we may take the case as shown in Fig. 5. Resolving R into its horizontal and vertical components, and taking moments about the center of resistance e, the following equation is obtained:

$$F\frac{H}{3}=W\frac{l}{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where F is the horizontal component of the thrust from the water behind the dam, acting at a point $\frac{1}{3}H$ above the plane of the joint, while W is the vertical component of the resultant, and as such, includes not only the weight of the masonry, but the vertical component of the thrust from the water as well, provided the latter is considered as acting normal to the back of the dam.

For the dam to be on the point of rotating about b, the downstream edge of the joint, it is obvious that the resultant R must pass through that point. Under these circumstances, since the lever arm of F is still $\frac{H}{3}$, and the lever arm of W has been increased to twice its former value or $2\left(\frac{l}{3}\right)$, for the above equation to still hold, F must also be increased to twice its former value. This would indicate that when R acts through the point e, the value of H is only one-half as great as is necessary to produce overturning; or, in other words, that the factor of safety is two as indicated by the ratio of $\frac{(u+V)}{V}$. It should be observed however, that the material near the edge of the joint will crush some time before the resultant has reached it, and that therefore the factor of safety against overturning with R at the limit of the middle third is something less than two.

When, however, the upward pressure of water acting over the joint due to percolation is taken into consideration, the factor of safety will be somewhat modified, as the following demonstration will make clear.

It is evident that the forces acting upon the joint are, neglecting the reaction, W, the weight of masonry, $\frac{\gamma H^2}{2}$ the horizontal thrust of the water normal to the back, and $\frac{cH\gamma l}{2}$, the uplift. Those tending to produce rotation about the downstream edge b are $\frac{\gamma H^2}{2}$ and $\frac{cH\gamma l}{2}$, the resultant of which may be represented by nm, and whose normal distance from b, the center of rotation is r: while the force resisting this is W, with a lever arm of $(u+V)$.

If nm represent the resultant of the overturning forces in direction and magnitude, and nd represent W, the final resultant will be found by combining the two and it will act through the point e, the center of resistance.

The overturning moment may be written,

$$M' = M_1 + M_2 = \frac{\gamma H^2}{2} z + \frac{cH\gamma l}{2}(u+V+V'),$$

and the resisting moment by

$$M_0 = W(u+V).$$

As the factor of safety is the ratio of the resisting to the overturning moment, it will be represented by,

$$\frac{M_0}{M'} = \frac{M_0}{M_1+M_2},$$

or, if the ratio of the resultant moment of the vertical components, to the resultant moment of the horizontal components be considered,

$$\frac{M_0 - M_2}{M_1}.$$

This latter implies that the horizontal thrust alone is instrumental in the case of overturning and that the effect of uplift is merely to reduce the resisting moment.

It is apparent that these two expressions will be equal when $M_2 = O$, i.e., when uplift is neglected, and also, when the factor of safety becomes unity; i.e., at the point of overturning. To secure this by means of the latter consideration M_0 must equal $M_1 + M_2$. Since W is constant, either or both of the other factors may be considered to vary; but as r has been shown to be constant also, both the pressure of the water on the back, and the uplift must be assumed to increase proportionately, if the resultant R is to pass through the point b. This seems justifiable, as the horizontal thrust from the water cannot increase without a corresponding increase in the uplift.

As was stated previously, the frictional and shearing resistance of a joint is assumed to withstand the tendency of the horizontal thrust to slide the upper portion over the lower, so that it is quite customary, even though it should be investigated, to neglect it.

For equilibrium in this regard,

$$F \gtreqless fW + Sl, \quad \ldots \ldots \quad (22)$$

where F is the horizontal component of the water's thrust, f the coefficient of friction, usually taken between 0.6 and 0.75 for masonry, and S is the shearing resistance per unit of area.

In spite of the fact that S has an appreciable value, and particularly so for monolithic masses of "cyclopean masonry," the value is practically indeterminate, and consequently usually ignored. Numerous attempts have been made however, to write expressions for it, the most rational of which depends upon the trapezoidal law of the distribution of normal stress; but this too is unsatisfactory from a practical standpoint.* We shall neglect S, therefore, in the previous equation, whence,

$$F \gtreqless fW, \qquad \qquad (23)$$

which gives at the limit,

$$f = \frac{F}{W} = \tan \beta. \qquad \qquad (24)$$

In every design the imposed conditions for equilibrium result in a cross-section in which the back has very much less of a batter than the front. It may be shown also,* that, as the shear along either face is zero, the greatest intensity of stress will act in a direction parallel to the face at, and near, the edge. Since the horizontal component of the pressure is ignored, this implies that the greatest vertical, or normal working intensity of pressure must be less at the downstream face where the inclination is greater than at the heel, in order that the components parallel to the respective faces shall be approximately equal. This is accomplished by using a smaller vertical normal working stress at the toe than at the heel.

As the upstream face of a masonry dam is vertical for a considerable distance from the top, and then becomes only slightly inclined to it, it is customary to consider the thrust from the water as acting horizontally. This is the more justifiable since the vertical component of the water resting upon the upstream face of the dam

---

* See Appendix.

causes an overturning moment about the center of resistance, opposite in direction to that induced by the horizontal thrust, and hence is an error on the side of safety.

It must be evident from the equation of pressure, $p = \gamma ah$, that where only this governs the resulting theoretical cross-section, it will be triangular in form with the apex at the surface of the water; but where it is intended there shall be no flow over the crest of the dam, it is customary to carry the masonry some distance above the elevation of the water in the reservoir, not only to allow for fluctuations, but because of economic conditions or to provide for a foot or carriage way. The superelevation and the width of top are therefore arbitrarily assumed and should be taken at about $\frac{1}{10}$ the height of the dam, with a minimum width of 5 feet and a maximum superelevation of 20 feet.

As no equation can be written simultaneously expressing the three conditions of stability, i.e., that the resultant lie within the middle third, that the maximum pressures shall not exceed certain limits, and that the horizontal components shall not cause sliding, it becomes necessary to determine the length of joints, usually taken vertically 10 feet apart for a depth of about 100 feet and increasing to 20 or 30 feet below, by the aid of that equation involving the limiting conditions known to apply, in order that the cross-section be a minimum, and then to test the joint, if necessary, by the other two. Generally speaking the third condition will be found to hold if the joint has been designed in accordance with the other two.

Considering Fig. 5, in which l is the length of joint, it is seen to be divided into three parts, u, v, and y, and from this what may be called the fundamental equation of the entire design can be written.

$$l=u+v+y. \qquad (25)$$

If M represents the overturning moment about e, then we have that at the limit of the middle third,

$$M=F\frac{H}{3}=Wv, \qquad (26)$$

or,

$$v=\frac{M}{W}. \qquad (27)$$

As the analysis will result in a cross-section polygonal in outline, composed of trapezoids with bases l and l_0 and altitudes h, we may write a general equation,

$$W=W_0+\left(\frac{l+l_0}{2}\right)h\Delta\gamma, \qquad (28)$$

or,

$$A\Delta\gamma=A_0\Delta\gamma+\left(\frac{l+l_0}{2}\right)h\Delta\gamma,$$

whence,

$$A=A_0+\left(\frac{l+l_0}{2}\right)h, \qquad (29)$$

and since,

$$\frac{W}{\Delta\gamma}=A_0+\left(\frac{l+l_0}{2}\right)h,$$

then,

$$v=\frac{\dfrac{M}{\Delta\gamma}}{A_0+\left(\dfrac{l+l_0}{2}\right)h}, \qquad (30)$$

which value of v, if substituted in Eq. (25) gives,

$$l=u+\frac{\frac{M}{\Delta\gamma}}{A_0+\left(\frac{l+l_0}{2}\right)h}+y, \quad . \quad . \quad . \quad (31)$$

The above Eq. (31) is a modification of Eq. (25) and, when proper values have been assigned to u and y, depending upon the existing conditions, is used throughout the entire design in the determination of the length of joints.

In the upper rectangular portion of the dam, where there is an excess of material above that required by the static pressure of the water, it will be found unnecessary to consider failure from crushing, as the maximum normal pressures are well below the allowed working pressure, and consequently the depth at which the section ceases to be rectangular will be fixed by the fact that the resultant may not pass outside the middle third. The algebraic expression for this condition is,

$$u \geqq \tfrac{1}{3}\, l \text{ for reservoir full,} \quad . \quad . \quad . \quad . \quad (32)$$

$$y \geqq \tfrac{1}{3}\, l \text{ for reservoir empty.} \quad . \quad . \quad . \quad (33)$$

Below this rectangular portion, trapezoidal sections will be found. At the base of the rectangle, $l=l_0=L$, $u=\frac{L}{3}$, and, since the center of gravity of the figure is vertically above the center of the joint, $y=\frac{L}{2}$.

If we wish to determine the depth to which the rectangular portion extends, we may do so by the use of Eq.

(31), which, as shown, must involve the condition that the resultant shall just touch the limit of the middle third, i.e., $u = \frac{L}{3}$. Substituting in Eq. (31), $u = \frac{L}{3}$, $y = \frac{L}{2}$, and remembering that $A_0 = o$

$$L = \frac{L}{3} + \frac{\frac{H^3}{6\Delta}}{L(H+a)} + \frac{L}{2},$$

whence, by dividing by L,

$$1 = \frac{5}{6} + \frac{H^3}{6\Delta(H+a)L^2} = \frac{H^3}{L^2(H+a)\Delta}.$$

or, solving for H,

$$H = \sqrt[3]{\Delta L^2 (H+a)}, \quad \ldots \ldots \quad (34)$$

If $H = h$ then $a = o$, and Eq. (34) reduces to,

$$H = h = L\sqrt{\Delta}. \quad \ldots \ldots \ldots \quad (35)$$

At this depth the rectangle ceases, the sections become trapezoidal, the back face is still vertical but the front face h', is inclined in order to increase the length of the successive joints and thus maintain the resultant for the reservoir full at the downstream limit of the middle third. For a considerable distance below the rectangular

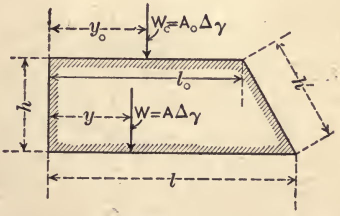


FIG. 6.

section therefore, Eq. (31) will be used with $u = \frac{l}{3}$ to determine the length of joint, and the back face will remain vertical, but for each new joint the resultant for the reservoir empty will approach nearer and nearer to the limits of the middle third, until finally $y=\frac{l}{3}$.

It is therefore expedient to determine the value of y under these conditions to learn exactly at what vertical depth or joint this value of y first equals $\frac{l}{3}$.

To do this, moments are taken about the vertical face, for both A_0 and the trapezoid, the latter being found by dividing the trapezoid into a rectangle and a triangle; its value is,

$$\frac{l^2{}_0 h}{2}+\frac{h}{2}(l-l_0)\left[l_0+\frac{l-l_0}{3}\right]=\frac{h}{6}(l^2+l\,l_0+l^2{}_0),$$

and hence,

$$Ay=A_0y_0+\frac{h}{6}(l^2+l\,l_0+l^2{}_0).$$

Substituting the value of A from Eq. (29) in the above, and solving for y there results,

$$y=\frac{A_0y_0+\frac{h}{6}(l^2+l\,l_0+l^2{}_0)}{A_0+\left(\frac{l+l_0}{2}\right)h}. \quad . \quad . \quad . \quad (36)$$

This gives a value of y to be substituted in Eq. (31) while the value of $u=\frac{1}{3}l$, which has been maintained since leaving the bottom of the rectangular section, is substituted also. There then results by reduction,

$$l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H^3}{\Delta}+6\,A_0y_0\right)+l^2{}_0, \quad . \quad . \quad . \quad (37)$$

which is the equation used in the determination of the length of joint from the foot of the rectangular section down to that joint where Eq. (36) first gives a value of $y=\frac{l}{3}$. At this point the back face must be made to slope, while $u=y=\frac{1}{3}l$ is substituted in Eq. (31) to obtain the following:

$$l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}, \quad . \quad . \quad . \quad . \quad . \quad (38)$$

which will determine the length of the joints.

The second condition will be a factor from here on, for below this section at some point, the intensities of the pressures at the toe will gradually approach and finally equal the allowable limit p, and the length of the joint will depend primarily upon this. It is therefore necessary, after each application of Eq. (38) to see if the limiting pressure p at the toe, which is smaller than q, at the heel, has been reached. Its value is derived from the equation $p=\frac{2W}{l}=\frac{2A\Delta\gamma}{l}$, and when the limiting value of p has been realized the value of u thereafter must be derived from,

$$u=\frac{2l}{3}-\frac{pl^2}{6A\Delta\gamma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

in which u, is seen to be dependent upon the normal working pressure p at the toe. (Eq. (39) follows directly from Eq. (9)).

There is some distance below this joint, however, where y still remains equal to $\frac{1}{3}l$, while the value of u is being determined from the above Eq. (39). Under these circumstances, l will be found from the following after substituting the values of $y=\frac{1}{3}l$, u from Eq. (39) and A from Eq. (29), all in Eq. (31).

$$l^2=\gamma\frac{H^3}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

This equation will be used until a joint has been reached where the application of $q=\frac{2A\Delta\gamma}{l}$ shows its value to be equal to or greater than that prescribed for q. Here y will be determined by

$$y=\frac{2l}{3}-\frac{ql^2}{6A\Delta\gamma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

in which it is seen to depend on q.

When this point has been reached, u will take its value from $u=\frac{2l}{3}-\frac{pl^2}{6A\Delta\gamma}$, and y from the equation $y=\frac{2l}{3}-\frac{ql^2}{6A\Delta\gamma}$, which must be substituted in Eq. (31) to determine l. This will give after reduction,

$$\left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}. \quad . \quad . \quad . \quad (42)$$

All joints below this point will be found by this last equation.

Summing up, we may say that Eqs. (34), (37), (38), (40), and (42), are the five equations to be used in determining the length of joints from the top down. Strictly speaking Eq. (34) gives the depth at which the rectangular portion ceases, while Eq. (37) gives the length of joints from the base of the rectangle down to where $y = \frac{1}{3}l$; Eq. (38) the length of joints from the point where $y = \frac{1}{3}l$ to where p reaches its limiting value; Eq. (40) the length of joints from the point where p equals its limiting value to where q equals its limiting value and Eq. (42) gives the length of all joints below.

Eqs. (34) and (37) involve the value of y, which is obtained with respect to the vertical back, but when that face begins to slope it is necessary to determine it with regard to the back edge of the joint in question.

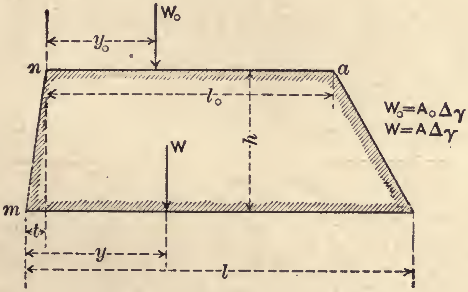


FIG. 7.

In Fig. 7, mn represents the back face of the dam and t is the batter to be determined by taking static moments of A and A_0 about the back edge, m, of the joint.

The trapezoid of the figure is composed of the triangles $ht/2$ and $(l-l_0-t)h/2$ and the rectangle hl_0.

By taking moments about the edge m,

$$Ay = A_0(y_0+t) + \frac{ht^2}{3} + l_0h\left(\frac{l_0}{2}+t\right) + \frac{h}{6}(l-l_0-t)(l+2l_0+2t). \quad (43)$$

For Eqs. (38) and (40) the value of y must be, as before, taken equal to $\frac{2}{3}l$, while A has the usual value of $A_0 + (l+l_0)h/2$. Substituting these in Eq. (43) and reducing:

$$t = \frac{2A_0(l-3y_0) - hl_0^2}{6A_0 + h(2l_0+l)}. \quad . \quad . \quad . \quad . \quad (44)$$

For the joints to which Eq. (42) applies the value of y is to be taken from Eq. (41) as was done before. In this case:

$$Ay = \frac{2}{3}l\left[A_0 + \left(\frac{l+l_0}{2}\right)h\right] - \frac{ql^2}{6\Delta\gamma}.$$

By substituting this value in the first member of Eq. (43) and reducing:

$$t = \frac{A_0(4l-6y_0) + l^2\left(h - \frac{q}{\Delta\gamma}\right) + l_0h(l-l_0)}{6A_0 + h(2l_0+l)}. \quad . \quad . \quad (45)$$

After the value of l is found by the use of Eqs. (38), (40) or (42), t can at once be determined for the same joint by either Eq. (44) or Eq. (45).

In this manner an entire theoretical cross-section can be determined. It will be noticed that the location of the center of pressure in the middle third of the joint is the governing condition in the upper part of the dam, while the lower portion is fixed by the limiting pressures p and q.

The difficulties preventing the forming of a simple working equation for the entire cross-section arise from the fact that the governing conditions are not introduced simultaneously nor in the same joint.

By taking h of the proper value, a polygonal cross-section may be determined by the preceding formulæ. This cross-section can be then modified by drawing what may be called "mean" lines, straight, broken or curved along the theoretical faces so as to adapt the latter to a practical arrangement and treatment of the joints and facing blocks, which may be of cut stone or concrete.

The conditions which have governed the analysis are essentially those of Rankine, i.e., the center of pressure has in all cases been kept within the middle third of the joint and the greatest intensity of pressure, either at the front face or back, has not been allowed to exceed the limit p or q.

Series A, B, C, D, E, and F, and Formulæ for Investigation.

As noted earlier six separate series of formulæ for investigation have been derived and they will be here set forth in suitable form for easy reference and use. As they have been developed by the method just outlined it is unnecessary to follow out the derivation of each series, although there exist some detailed differences in the treatment of each. These details however, would become evident to anyone following the deductions throughout.

Various conditions of "loading," with approved assumptions, such as pressure due to expanding ice at the water surface, upward water pressure on the base, etc., referred to in the table of nomenclature previously given, have been introduced and are specifically stated for each case.

It will be recalled that Eq. (31) is the fundamental expression for finding the length l, of any joint; and, as the several conditions are introduced, that the "M" must in each case signify the total overturning moment and not merely the moment of the static water pressure on the back.

The development of a cross-section, by any one of the following series, may comprise five stages, each stage representing the introduction of a governing condition. Hence, for each stage there obtains a main equation for finding the length of joint l, each main equation being supplemented by secondary equations for y, u, and t; p and q.

It may be necessary to employ more than one of the series of equations in determining a cross-section.

For ready reference, the five stages will be set forth and depicted in order as follows:

Stage I.—This stage, it will be remembered, extends from the top of the dam to the joint where the front face commences to batter. It is the rectangular section. $y > \frac{1}{3}L$; $u \geqq \frac{1}{3}L$ (see Fig. 8). (Ice pressure is purposely omitted in Fig. 8 to prevent confusion of letters in small space.)

Stage II.—This stage extends from the lower limit of Stage I to the point where the back face commences to batter. $u = \frac{1}{3}l$; $y \geqq \frac{1}{3}l$ (see Fig. 9).

Stage III.—This stage extends from the lower limit of Stage II to the point where the intensity of pressure on

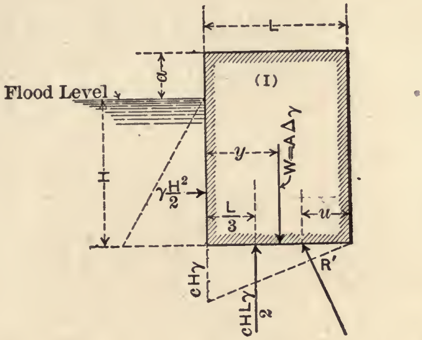


FIG. 8.

the toe has reached the maximum allowable intensity. In this stage $u=\frac{1}{3}l$; $y=\frac{1}{3}l$ (see Fig. 10).

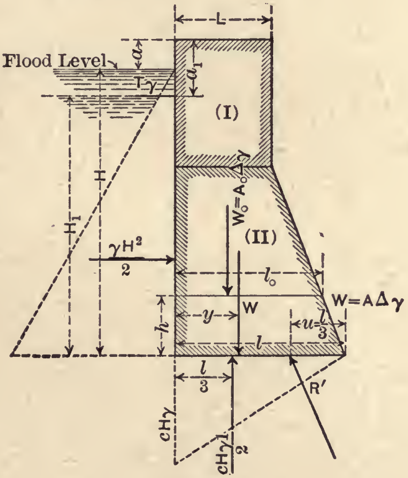


FIG. 9.

Stage IV.—This stage extends from the lower limit of Stage *III* to the point where the pressure intensity on the heel has reached the maximum allowable intensity. For this stage $u > \frac{1}{3}l$; $y = \frac{1}{3}l$ (see Fig. 10).

Stage V.—In this stage the limiting intensities of pressure at both toe and heel having been reached, $y > \frac{1}{3}l$; $u > \frac{1}{3}l$.

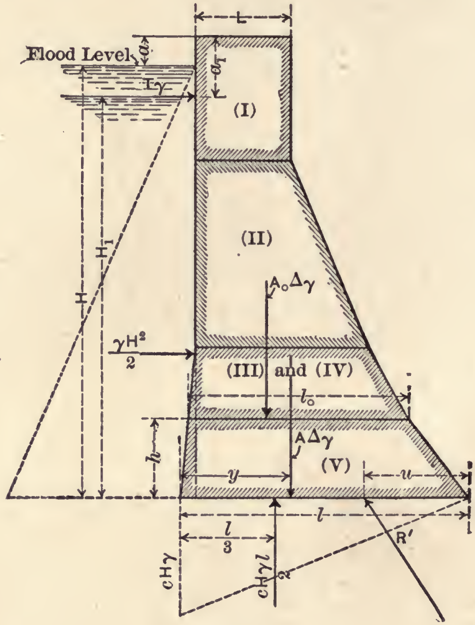


FIG. 10.

This stage extends from the lower limit of Stage IV downward. (See Fig. 10.)

The following secondary formulæ, supplementary to the main equations of all series, with substitutions as noted, are arranged in order corresponding with the preceding.

With the condition of hydrostatic upward pressure on the base obtaining, substitute the formulæ in this column in place of those corresponding, as indicated.

Stage I.

$$u \geqq \tfrac{1}{3}L$$

$$y = \tfrac{1}{2}L$$

$$t = 0$$

$$p = \frac{2\Delta\gamma A}{L}\left(2 - \frac{3u}{L}\right)$$

$$p = \gamma\left(\frac{2\Delta A}{L} - cH\right)\left(2 - \frac{3u}{L}\right)$$

$$q = \frac{\Delta\gamma A}{L}$$

Stage II.

$$u = \tfrac{1}{3}l$$

$$y = \frac{A_0 y_0 + (l^2 + l\,l_0 + l_0^2)\frac{h}{6}}{A_0 + \left(\frac{l + l_0}{2}\right)h}$$

$$t = 0$$

$$p = \frac{2\Delta\gamma A}{l}$$

$$p = \gamma\left(\frac{2\Delta A}{l} - cH\right)$$

$$q = \frac{2\Delta\gamma A}{l}\left(2 - \frac{3y}{l}\right)$$

Stage III.

$$u = \tfrac{1}{3}l$$

$$y = \tfrac{1}{3}l$$

$$t = \frac{2A_0(l - 3y_0) - hl_0^2}{6A_0 + h(2l_0 + l)}$$

$$p = \frac{2\Delta\gamma A}{l}$$

$$p = \gamma\left(\frac{2\Delta A}{l} - cH\right)$$

$$q = \frac{2\Delta\gamma A}{l}$$

Stage IV.

$$u = \tfrac{2}{3}l - \frac{pl^2}{6\Delta\gamma A}$$

$$y = \tfrac{1}{3}l$$

$$t = \frac{2A_0(l - 3y_0) - hl_0^2}{6A_0 + h(2l_0 + l)}$$

$$p = \frac{2\Delta\gamma A}{l}\left(2 - \frac{3u}{l}\right) \text{ (limiting intensity)}$$

$$q = \frac{2\Delta\gamma A}{l}$$

$$u = \tfrac{2}{3}l - \frac{pl^2}{3\gamma(2\Delta A - cHl)}$$

$$p = \gamma\left(\frac{2\Delta A}{l} - cH\right)\left(2 - \frac{3u}{l}\right)$$

Stage V.

$$u = \tfrac{2}{3}l - \frac{pl^2}{6\Delta\gamma A}$$

$$y = \tfrac{2}{3}l - \frac{ql^2}{6\Delta\gamma A}$$

$$u = \tfrac{2}{3}l - \frac{pl^2}{3\gamma(2\Delta A - cHl)}$$

$$t = \frac{A_0(4l - 6y_0) + \left(h - \dfrac{q}{\Delta\gamma}\right)l^2 + (l - l_0)hl_0}{6A_0 + h(2l_0 + l)}$$

$$p = \frac{3\Delta\gamma A}{l}\left(2 - \frac{3u}{l}\right) \text{ (limiting intensity)}$$

$$q = \frac{2\Delta\gamma A}{l}\left(2 - \frac{3y}{l}\right) \text{ (limiting intensity)}$$

$$p = \gamma\left(\frac{2\Delta A}{l} - cH\right)\left(2 - \frac{3u}{l}\right)$$

If T enter the following tormulæ, H above becomes H_1. (See Figs. 9 and 10).

The first column of formulæ just given would apply, with the condition of upward pressure on the base due to hydrostatic head, if a proper value of u corresponding, be taken, that is if the excursion of the force $A\Delta\gamma$, resulting from the effect of all other forces on $A\Delta\gamma$ be considered, rather than the effect of all the other forces on the resultant vertical force.

It will be found expeditious to design a section, where ice pressure at the level of full reservoir is to be considered in connection with the water surface at some higher flood level, first by series of formulæ containing T (cf. Series B and D) and then to investigate successive bases, or joints, thus obtained (beginning, for a high masonry dam, usually at a base, or joint, about 100 feet from the top of the dam) with series of formulæ lacking T, or the ice pressure condition (cf. Series A and C). A base will ultimately be obtained by these supplementary "Flood level" calculations greater than the base at its same elevation as previously determined by the "Ice Pressure" design.

Continuing with the design by means of the "Flood level" formulas to the bottom of maximum height required will determine the minimum cross-section area to meet the conditions both of "Flood" and of "Ice." It should be remarked in this connection that when a reservoir level is rising due to flood conditions prevailing, it is evident that ice formation cannot develop, or, in other words, the two conditions cannot be coexistent, hence the difference in designation of hydrostatic heads corresponding. (See Figs. 9 and 10.)

SERIES A.

Conditions: Overturning moment due to *horizontal static water pressure on back of dam only.*

Stage I.

$$H=\sqrt[3]{\Delta L^2(H+a)}.$$

Stage II.

$$l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H^3}{\Delta}+6A_0y_0\right)+l_0^2.$$

Stage III.

$$l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}.$$

Stage IV.

$$l^2=\frac{\gamma H^3}{p}.$$

Stage V.

$$\left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}.$$

SERIES B.

Conditions: Overturning moment due to:

(*a*) *Horizontal static water pressure on back* and

(*b*) *Ice pressure applied at distance* (a_1) *from top.*

Stage I.

$$H_1=\sqrt[3]{\Delta(H_1+a_1)L^2-6TH_1}.$$

Stage II.

$$l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H_1^3+6TH_1}{\Delta}+6A_0y_0\right)+l_0^2.$$

Stage III.

$$l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{1}{\Delta h}(H_1^3+6TH_1).$$

Stage IV.

$$l^2=(H_1^3+6TH_1)\frac{\gamma}{p}.$$

Stage V.

$$\left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{1}{\Delta h}(H_1^3+6TH_1).$$

SERIES C.

Conditions: Overturning moment due to:

(*a*) *Horizontal static water pressure on back.*

(*b*) *Upward water pressure on base.* Pressure intensity decreasing uniformly from $cH\gamma$ at heel to zero intensity at toe.

Stage I.

$$H=\sqrt[3]{L^2[\Delta(H+a)-cH]}.$$

Stage II.

$$\left(1-\frac{cH}{\Delta h}\right)l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H^3}{\Delta}+6A_0y_0\right)+l_0^2.$$

Stage III.

$$\left(1-\frac{cH}{\Delta h}\right)l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}.$$

Stage IV.

(a) $$\left(1-\frac{cH}{\Delta h}\right)l^3+\left(\frac{2A_0}{h}+l_0\right)l^2-\frac{\gamma H^3}{p}\left(1-\frac{cH}{\Delta h}\right)l$$

$$=\frac{\gamma H^3}{p}\left(\frac{2A_0}{h}+l_0\right),$$ which reduces to

(b) $$l^2=\frac{\gamma H^3}{p}.$$

Stage V.

(a) $$\left(1-\frac{cH}{\Delta h}\right)\left(1-\frac{p+q}{h\Delta\gamma}\right)l^3+\left(\frac{2A_0}{h}+l_0\right)\left(2-\frac{p+q}{h\Delta\gamma}-\frac{cH}{h\Delta}\right)l^2$$

$$+\left[\frac{2A_0}{h}\left(\frac{2A_0}{h}+2l_0\right)+\frac{H^3}{h\Delta}\left(1-\frac{cH}{h\Delta}\right)+l_0{}^2\right]l$$

$$=-\frac{H^3}{h\Delta}\left(\frac{2A_0}{h}+l_0\right),$$ which reduces to

(b) $$\left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h}.$$

SERIES D.

Conditions: Overturning moment due to:

(a) *Horizontal static water pressure on back* (head $=H_1$).

(b) *Ice pressure applied at distance* (a_1) *from top.*

(c) *Upward water pressure on base.* Pressure decreasing uniformly from $cH_1\gamma$ at heel to zero intensity at toe.

Stage I.

$$H_1=\sqrt[3]{L^2[(H_1+a_1)\Delta-cH_1]-6TH_1}.$$

Stage II.

$$\left(1-\frac{cH_1}{\Delta h}\right)l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H_1{}^3+6TH_1}{\Delta}+6A_0y_0\right)+l_0{}^2.$$

Stage III.

$$\left(1-\frac{cH_1}{\Delta h}\right)l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{1}{\Delta h}(H_1{}^3+6TH_1).$$

Stage IV.

$$(a)\quad \left(1-\frac{cH_1}{\Delta h}\right)l^3+\left(\frac{2A_0}{h}+l_0\right)l^2-\frac{\gamma(H_1{}^3+6TH_1)}{p}\left(1-\frac{cH_1}{\Delta h}\right)l$$

$$=\frac{\gamma(H_1{}^3+6TH_1)}{p}\left(\frac{2A_0}{h}+l_0\right), \text{ which reduces to}$$

$$(b)\quad l^2=\frac{\gamma(H_1{}^3+6TH_1)}{p}.$$

Stage V.

$$(a)\quad \left(1-\frac{cH_1}{\Delta h}\right)\left(1-\frac{p+q}{h\Delta\gamma}\right)l^3+\left(\frac{2A_0}{h}+l_0\right)\left(2-\frac{p+q}{h\Delta\gamma}-\frac{cH_1}{h\Delta}\right)l^2$$

$$+\left[\frac{2A_0}{h}\left(\frac{2A_0}{h}+2l_0\right)+\frac{(H_1{}^3+6TH_1)}{h\Delta}\left(1-\frac{cH_1}{h\Delta}\right)+l_0{}^2\right]l$$

$$=-\frac{(H_1{}^3+6TH_1)}{\Delta h}\left(\frac{2A_0}{h}+l_0\right), \text{ which reduces to}$$

$$(b)\quad \left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{H_1{}^3+6TH_1}{\Delta h}.$$

In the preceding series of equations it will be observed that the final expressions for l in stages IV and V are very similar, and that the quantity c in equations (a) of these stages disappears in equations (b). Equations (b) of course, are to be used for purposes of calculation of cross-sections.

SERIES E.

Conditions: (See Fig. 11) Ice pressure neglected in Fig. 11. Overturning moment due to:

(*a*) *Horizontal static water pressure on back* (head $=h_1$).

(*b*) *Ice pressure applied at distance* (a_1) *from top.*

(*c*) *Upward water pressure on base;* pressure intensity decreasing uniformly from $c(h_1+h_2)\gamma$ at heel to zero intensity at toe.

(*d*) *Mud (liquid) pressure on back* (head h_2), commencing at distance h_2 above joint in question. Weight of mud $=\gamma'$. As before, if T be equated to zero, a_1 becomes equal to a, in the formulas.

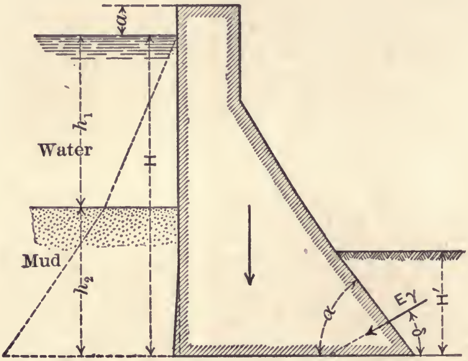


FIG. 11.

Stage I. (h_1, of known value, h_2 to be determined.)

$$\frac{h_2{}^3\gamma'}{\gamma}+(h_1+h_2)[3h_1h_2+6T+L^2(c-\varDelta)]=L^2a_1\varDelta-h_1{}^3.$$

Stage II.

$$\left[1-\frac{c(h_1+h_2)}{\varDelta h}\right]l^2+\left(\frac{4A_0}{h}+l_0\right)l$$

$$=\frac{1}{\varDelta h}\left[(h_1+h_2)(3h_1h_2+6T)+h_1{}^3+\frac{h_2{}^3\gamma'}{\gamma}+6A_0y_0\varDelta\right]+l_0{}^2.$$

For trapezoidal section at top, make $A_0 = 0$ and $y_0 = 0$ and $l_0 = L$ in Stage II. This applies generally.

Stage III.

$$\left[1 - \frac{c(h_1 + h_2)}{\Delta h}\right] l^2 + \left(\frac{2A_0}{h} + l_0\right) l$$
$$= \frac{1}{\Delta h}\left[(h_1 + h_2)(3h_1h_2 + 6T) + h_1^3 + \frac{h_2^3 \gamma'}{\gamma}\right].$$

Stage IV.

$$l^2 = \frac{\gamma}{p}\left[(h_1 + h_2)(3h_1h_2 + 6T) + h_1^3 + \frac{h_2^3 \gamma'}{\gamma}\right].$$

Stage V.

$$\left(\frac{p+q}{h\Delta\gamma} - 1\right) l^2 - \left(\frac{2A_0}{h} + l_0\right) l$$
$$= \frac{1}{\Delta h}\left[(h_1 + h_2)(3h_1h_2 + 6T) + h_1^3 + \frac{h_2^3 \gamma'}{\gamma}\right].$$

From a study of the formulæ thus far developed it will be observed that by reducing certain conditions to zero, with their corresponding quantities, the main equations of a given series reduce to those of a simpler series.

For instance—

In Series B make T (for ice pressure condition of loading) equal to zero and $H_1 = H$ and $a_1 = a$ and the main equations of that series reduce to Series A equations.

In Series C, by making c (for upward water pressure condition) equal to zero in main equations of Stages I, II, and III and also in equations (a) of Stages IV and V, the equations of Series C reduce to those of Series A.

Likewise, by making the proper eliminations and substitutions, Series E will reduce to Series D, C, B or A.

SERIES F.

This series consists of general formulæ for a number of imposed conditions of loading. For any given case, the terms of factors expressing those conditions not appertaining must be eliminated by equating them to zero. (See Fig. 12.)

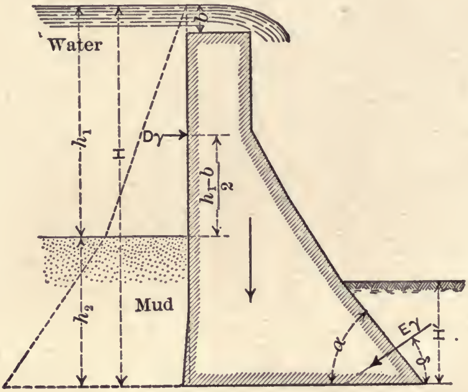


FIG. 12.

Conditions for General Formulæ.

Overturning moment due to:

(*a*) *Horizontal static water pressure on back* (head $=h_1$).

(*b*) *Upward water pressure on base;* pressure intensity decreasing uniformly from $cH\gamma$ or $c(h_1+h_2)\gamma$, at heel to zero intensity at toe.

(*c*) *Mud* (*liquid*) *pressure on back* (head h_2) as before.

(*d*) *Dynamic pressure of water, $D\gamma$.*

(*e*) *Water flowing over top of dam,* weight of water, of depth b, on top of dam being neglected.

For condition of water *not* overtopping dam, $b=0$ and $D=0$.

For condition of no dynamic pressure, $D=0$.

For condition of no upward water pressure, $c=0$.

For condition of no mud (i.e., mud being replaced by water) make $h_2=0$, $h_1=H$.

Stage I.

Rectangular cross-section at top or rectangular dam, $l=l_0=L$.

This may fall under either of two cases, viz.—

Case (1)

Condition: $h_1=H$; $h_2=0$.

$$H^3+H[+3D-3b^2+L^2(c-\Delta)]=b(3D-2b^2-L^2\Delta).$$

Case (2)

Condition: h_1 of known value; h_2 to be determined.

$$\frac{h_2{}^3\gamma'}{\gamma}+3Dh_2+(h_1+h_2)[3h_1h_2+3D-3b^2+L^2(c-\Delta)]$$
$$=b(3D-2b^2-L^2\Delta)-h_1{}^3.$$

As in the preceding series, the value of H or h_2, of Stage I may be determined by several successive trial substitutions.

Stage II.

(a) Trapezoidal cross-section at top of dam or trapezoidal dam (spillway) front face battered. ($A_0=0$, $l_0=L$ and $y_0=0$.) Note: For a triangular dam $l_0=0$, also.

$$\left[1-\frac{c(h_1+h_2)}{\Delta h}\right]l^2+Ll=\frac{1}{\Delta h}\Big[(h_1+h_2)(3h_1h_2-3b^2)+h_1^3+\frac{h_2^3\gamma'}{\gamma}+2b^3+3D(h_1+2h_2-b)\Big]+L^2.$$

(*b*) Trapezoidal section continued (front face battered).

$$\left[1-\frac{c(h_1+h_2)}{\Delta h}\right]l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{\Delta h}\Big[(h_1+h_2)(3h_1h_2-3b^2)+h_1^3+\frac{h_2^3\gamma'}{\gamma}+2b^3+3D(h_1+2h_2-b)+6A_0y_0\Delta\Big]+l_0^2.$$

Stage III.—Both faces battered.

$$\left[1-\frac{c(h_1+h_2)}{\Delta h}\right]l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{1}{\Delta h}\Big[(h_1+h_2)(3h_1h_2-3b^2)+h_1^3+\frac{h_2^3\gamma'}{\gamma}+2b^3+3D(h_1+2h_2-b)\Big].$$

Stage IV.—Limiting intensity of pressure, p, introduced.

$$l^2=\frac{\gamma}{p}\Big[(h_1+h_2)(3h_1h_2-3b^2)+h_1^3+\frac{h_2^3\gamma'}{\gamma}+2b^3+3D(h_1+2h_2-b)\Big].$$

Stage V.—Limiting intensities, p and q.

$$\left(\frac{p+q}{h\Delta\gamma}-1\right)l^2-\left(\frac{2A_0}{h}+l_0\right)l=\frac{1}{\Delta h}\Big[(h_1+h_2)(3h_1h_2-3b^2)+h_1^3+\frac{h_2^3\gamma'}{\gamma}+2b^3+3D(h_1+2h_2-b)\Big].$$

The increased number of overturning loads then, tend to render the right-hand members of the various equations more involved; though after a little practice one may easily carry through a design with surprising rapidity. The slide rule may be used to great advantage and it is suggested that the results be tabulated as determined, in some such form as the following:

TABLE OF RESULTS.

$c=$, $T=$, etc.

Joint No.	H	H^3	h	$H+a$	A_0	A	l_0	l_0^2	l	y_0	y	u	t	p	q	A_0y_0	Etc.
1																	
2																	
3																	
4																	
Etc.																	

The effect upon the calculation of a cross-section, of backfill on the downstream face could of course be cared for by introducing that condition into the preceding series of equations; but as this effect as computed, would be, in any case, largely dependent upon assumptions which may vary widely and as the placing of backfill is generally a later consideration with respect to construction, the propriety of such introduction at that stage of design is questionable.

In the following formulæ for *investigation* therefore, the general conditions of an earth thrust acting at the downstream face and of a vertical component of thrust of material on the upstream, inclined face of the dam, are introduced.

By any of these formulæ the position of the line of resistance for any given cross-section and respective conditions may be determined with regard to any horizontal joint and its downstream edge; the value of u being the quantity to be sought.

Any condition may be disregarded by equating its term to zero.

The first expression below contains all of the conditions heretofore considered with the additional ones just stated; and from it follow the succeeding expressions for u. It should be remembered that the term T cannot be coexistent in any expression for stability with b and therefore with D. Nevertheless all of these terms are written with the understanding that the proper eliminations be always made. Three general group equations will be written.

Formulæ for Investigation.

First, Conditions of retained mud, water, overtopping, etc. (see Fig. 12).

$$u = l - y - \frac{\begin{array}{l}(h_1+h_2)[3h_1h_2+6T-3b^2+cl(3y-l)]+h_1^3 \\ +\dfrac{h_2^3\gamma'}{\gamma}+2b^3+3D(h_1+2h_2-b)-6W_v(y-s) \\ +6E\left[(l-y)\sin\delta-\dfrac{H'}{3}\dfrac{\sin(\delta+a)}{\sin a}\right]\end{array}}{6(W_v+E\sin\delta+A\varDelta)-3c(h_1+h_2)l}.$$

Whence, for conditions of *retained mud, water, etc.*, but *no overtopping*, by making $b=0$ and $D=0$, there follows (see Fig. 11):

$$u = l - y - \frac{\begin{array}{l}(h_1+h_2)[3h_1h_2+6T+cl(3y-l)]+h_1^3+h_2^3\dfrac{\gamma'}{\gamma} \\ -6W_v(y-s)+6E\left[(l-y)\sin\delta-\dfrac{H'}{3}\dfrac{\sin(\delta+a)}{\sin a}\right]\end{array}}{6(W_v+E\sin\delta+A\varDelta)-3c(h_1+h_2)l}.$$

From this last expression for u, for conditions of *retained water, etc.*, but *neither mud nor overtopping*, by making $h_1 = H_1$; $h_2 = 0$, there is obtained:

$$u = l - y - \frac{6T + H_1^2 + cl(3y - l) - \frac{6W_v}{H_1}(y - s) + \frac{6E}{H_1}\left[(l-y)\sin\delta - \frac{H'}{3}\frac{\sin(\delta + a)}{\sin a}\right]}{6\left(\frac{W_v + E\sin\delta + A\Delta}{H_1}\right) - 3cl}.$$

As in the equations for design, when $T = 0$, $H_1 = H$. (See Figs. 9 and 10.) If H' is of such depth that the downstream batter of the cross-section varies considerably, an approximate solution is possible by assuming some average batter for the lower portion. The expression for earth thrust is general, as is evidenced. After u is determined for each joint, the intensities of maxima pressures can be determined for the given cross-section, the general expression for p, corresponding to above expressions for u, being:

$$p = \frac{2\gamma}{l}\left[W_v + E\sin\delta + A\Delta - \frac{c(h_1 + h_2)}{2}l\right]\left(2 - \frac{3u}{l}\right).$$

In connection with the computation for the value of y in an investigation, as indicated above, it is necessary to obtain the position of the centroid of a trapezoid with respect to the back, or upstream, edge of the joint in question. The following expression for x, in connection with Fig. 13, may prove convenient:

$$x = \frac{(l^2 + l\,l_0 + l_0^2) + t(l + 2l_0)}{3(l + l_0)}.$$

In the studies for design, referred to before, the analytic work should be checked throughout by the graphic method wherever possible. This should always be done both in designing and investigating cross-sections.

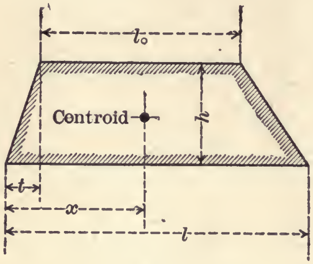


FIG. 13.

It should be stated here that, after a cross-section has been fixed upon for a given dam and the faces drawn to chosen batters and curves, the entire cross-section should be investigated as just outlined so as to give the actual values for this final cross-section.

Again, in comparing different cross-sections, especially of different dams, by superimposing, their *water lines* should be made to coincide and not their tops for a fair comparison.

APPENDIX I

RECENT CONSIDERATIONS OF THE CONDITION OF STRESS IN A MASONRY DAM

Considerable discussion has been raised within the past few years, by criticisms being leveled at the present general procedure in the design of high masonry dams. This has properly perhaps, been more pronounced abroad than in this country, since the matter may be said to have been precipitated by the publication of a paper by Mr. L. W. Atcherley of London University, "On Some Disregarded Points in the Stability of Masonry Dams."*

It is the purpose to outline the analysis as presented there, and to call attention to some of the discussion which followed, in order to indicate the status of the theory involved in the design of such structures.

The paper referred to takes exception to current practice in regard to the matter of design and indicates a need for both revision and extension in the analysis, and then, supplementing the generally accepted ideas as to the distribution of normal stress on horizontal planes, by an assumption as to the shear on these planes, proceeds to show that peculiar and unexpected conditions arise.

---

* Dept. of Applied Mathematics, University College, University of London. Drapers' Company Research Memoirs. Technical Series II.

It is a fact that, owing 1st, to the manner in which masonry structures are built, i.e., of a mixture of stone and cement, and 2d, to the nature of the sections at the springings or areas of support, it is practically impossible to apply to them the general theory of elastic bodies. Consequently, the treatment as it is employed to-day has been developed, but only by the use of certain assumptions which it may be shown are not precisely exact.

The basis of the present investigation rests upon the four following formulæ, in which the usual distribution of normal unit stress on horizontal planes is accepted, but to which is added an assumed condition as to the distribution of horizontal shear.

$$cd = \frac{1\zeta^2}{12}, \quad \ldots \quad (1)$$

$$C_{max} = \frac{Q}{A}\left(1 + \frac{6c}{\zeta}\right), \quad \ldots \quad (2)$$

$$T_{max} = \frac{Q}{A}\left(\frac{6c}{\zeta} - 1\right), \quad \ldots \quad (3)$$

$$S = \frac{3}{2}\frac{P}{A}\left(1 - \frac{4y^2}{\zeta}\right). \quad \ldots \quad (4)$$

c = distance along the horizontal joint from the centroid to the point of application of the resultant.

d = distance along the horizontal joint from the centroid to the point locating the neutral axis.

ζ = length of the horizontal joint.

C_{max} = maximum compressive stress on the joint.

T_{max} = maximum tensile stress on the joint.

Q = vertical component of the resultant force acting on the joint.

A = area of the joint.

S = shear at any point y in the joint.

P = total shear on the joint.

y = the distance from the centroid to any point on the joint.

With regard to Eq. (4) it may be stated that it has not heretofore been customary to consider the distribution of shearing stress on horizontal joints. But, if the distribution of normal stresses may be assumed to be represented by Eqs. (1), (2), and (3), with equal validity for the usual types of dam, may the shear at any point be assumed to be represented by Eq. (4). It is believed by Mr. Atcherley that these equations more nearly express the conditions of equilibrium in a dam than the usual ones do, even though the latter tacitly assume the first three by imposing the condition of the middle third, and use a friction condition, instead of one for shear as expressed by Eq. (4).

In reference to this friction factor there may be some question of doubt, since M. Levy* prescribes an angle of 30° for masonry on masonry, while Rankine gives 36°; on the other hand, examination of dams actually built frequently shows the angle to lie somewhere between the above values.

But whatever its exact value, the friction condition leaves some doubt as to the actual distribution of shear

* "La Statique graphique." IVe Partie, 'Ouvrages en Maçonnerie," page 92.

over a horizontal joint, the variation of which must be known, in order to determine the tensile and compressive stresses *on the vertical sections of the tail* (*i.e., downstream portion*) *of the dam.* In consequence of this the parabolic law as expressed by Eq. (4) has been assumed and will later be shown to be more nearly correct than any other hypothesis.

According to the author there is no reason whatever why dams should be tested solely by taking horizontal cross-sections, and asserting that the line of resistance must lie in the middle third, while the stresses across the vertical sections of the tail are absolutely neglected. If the former condition is valid, then no dam ought to be passed unless it can be shown also that there is no tension of any serious value across vertical cross-sections of the tail, parallel to the length of the structure. It is believed that a great number of dams as now designed will be found to have very substantial tension in these sections and this, in the opinion of the author, is a source of weakness in dam construction which has not been properly considered and allowed for.

If the problem is to be solved on the assumption that a dam is an "isotropic and homogeneous" structure, the general equations for the stresses can be determined only by the following considerations:

(*a*) The normal and shearing stresses on the horizontal top and curved flank, i.e., downstream face, are both zero.

(*b*) The normal stress on the battered front or upstream face is equal to the water pressure, and the shear is zero, and (*c*) Either the stresses or the shifts must be supposed given over the base.

It follows at once from this that Eqs. (1), (2), and (3) are not *absolutely* true, but that the shear is fairly closely represented by Eq. (4).

As far as the present investigation is concerned, however, the enquiry is not as to the validity of the usual treatment; it is obviously faulty. But it is the purpose to try to indicate that, supposing it to be correct, its present *partial* application, i.e., to horizontal joints only, involves the serious, and, it is believed, often dangerous, neglect of large tension across the vertical sections.

To justify the above statement, two model dams of wood were employed for experimental purposes, the cross-sections being identical, and agreeing with that of a dam actually constructed. One of these models was subdivided into horizontal strata to study the effect on such planes, and the other into vertical longitudinal strata, for a similar purpose. The application of the loading was such that it approximated as closely as possible the conditions obtaining in an actual dam. The general conclusions from these experiments were that:

(*a*) The current idea that the critical sections of a dam are the horizontal ones is entirely erroneous. A dam collapses first by the tension on the vertical sections of the tail.

(*b*) The shearing of the vertical sections over each other follows immediately on this opening up by tension.

(*c*) It is probable that the shear on the horizontal sections is also a far more important matter than is usually supposed.

It follows consequently, that keeping the line of resistance within the middle third of the horizontal sections is by no means the hardest part of dam design. It would be surprising if, with all the labor spent on this point, the bulk of existing dam constructions are not, for masonry, under very considerable tension, i.e., a tension across the vertical sections which has been hitherto disregarded.

It is proposed therefore to lay it down as a rule for the construction of future dams that the stability of the dam from the standpoint of the vertical sections must be considered in the *first* place. If this be satisfactory, it is believed that the horizontal sections will be found to be stable, but of course the latter must be independently investigated.

The above conclusions were apparently verified by a combined analytical and graphical treatment in which the algebraical analysis will here be considered first.

Denoting the total vertical force acting on a horizontal joint by Q_0, and the total horizontal force acting over the same by P_0, under the assumption that the reservoir is full, the variation of the normal pressure on the joint may be represented by the straight line of Eq. (1).

If the resultant pressure on the joint be assumed to cut it at the extremity of the middle third, then according to the previous notation, d will have a value of $\frac{1}{3}\frac{b^2}{c}$, provided $2b$ is the length of the joint. This indicates that the line representing the variation of normal pressure over the joint intersects it at the upstream edge, and any vertical between it and the joint itself will represent the normal pressure at that point where the vertical is erected.

Denoting this by y, it may be termed "the vertical height-giving pressure," and may also be expressed in terms of height of masonry, if the factors upon which it depends are expressed in cubic feet of masonry.

Again, we may write an equation of the downstream face, with respect to the same joint so long as that face is a straight line, by making $y' = mx$.

Evidently then if this latter line, and the one indicating the variation of pressure over the base, be referred to the same origin, the tip of the tail, the difference in areas included between each and the base will represent the total upward force, in cubic feet of masonry, acting over any assumed portion, "x" of the joint, measured from the tail.

Representing this upward force by F_1 its point of application may be easily determined, while the shear may be written as F_2, being regulated by Eq. (4).

As F_1 and F_2 thus give all the external forces, considering a wedge-shaped piece of dam bounded by the downstream face, a vertical and a horizontal plane, the total shear on the vertical plane must equal F_1 and the total thrust F_2, since these internal stresses are held in equilibrium by external forces. Thus F_1 equals the total shear on the vertical section, at a distance x from the tip of the tail, while F_2 equals the total horizontal thrust over the same.

If y be expressed in terms of x, and locate the point on the successive vertical planes through which the resultant acts, then the equation will represent the line of resistance on these vertical planes. It is found to be an hyperbola.

Considering the stresses on the vertical sections, it is found: First, that the maximum shear may be properly represented by $\frac{3}{2}$ the mean value, and may be so arranged as to be expressed in terms of F_1 and mx. Such an equation, representing a straight line, immediately shows the necessity of thickening the tip of the tail which, as a matter of fact, is the usual procedure in actual design. Second, the line representing the maximum tensile stress may be shown to vary as a parabola whose axis is vertical.

When the downstream face ceases to be linear, it becomes necessary to apply a graphical solution for the determination of the stresses. This it is unnecessary to reproduce here, but the curves may be said to indicate the following results:

(1) That the line of resistance for the vertical sections lies outside the middle third for rather more than half the vertical sections. In other words, these sections are subjected to tension.

(2) That the tensile stresses in the tail are, for masonry, very serious, amounting to nearly 10 tons per square foot at the extreme tip, and to 6 tons per square foot after we have passed the vertical section, where the strengthening of the tail has ceased.

(3) That the maximum shearing stresses amount to 6 tons per square foot at the tip of the tail and 5 tons per square foot after we have passed the vertical section, where the strengthening of the tail has ceased. No undue importance should be laid on the actual values of these "maximum" shears on the vertical sections however, as they are obtained from the mean shears by using the round multiplier 1.5. This round number is assumed because the maximum is certainly greater than the mean shear. The

actual distribution of shear on the vertical sections has not been discussed. It could, of course, be found from that on the horizontal sections, if the latter were really known with sufficient accuracy, by the equality of the shears on two planes at right angles. It is sufficient to show that the *mean* shears on the vertical sections appear to be higher than those on the horizontal section, and thus indicate that the parabolic distribution applied to sections some way above the base, probably *under-estimates* the maximum shearing in the dam.

In other words: *Whether the test is made by the line of resistance lying outside the middle third, or by the existence of serious tensile stresses, or by the magnitude of the mean shearing stresses, the vertical sections are critical for the stability in a far higher degree than the horizontal sections.*

In a well-designed dam, all the conditions for stability of the horizontal sections may have been satisfied, yet if the very same conditions be applied to the vertical sections not one of them will be found to be satisfied. It seems accordingly very unsatisfactory that the current tests for stability should, if they are legitimate, be applied to the horizontal instead of to the far more critical vertical sections. In the case of the latter they fail completely; and if higher tension and shear are to be allowed in the vertical sections, then it is absurd to exclude them in the case of the horizontal sections. It is maintained by the author that the current treatment of dams is fallacious, for it screens entirely the real source of weakness, namely, in the first place the tension, and in the second place the substantial shear, in the vertical sections, and this at distances from the tail far beyond the usual tail-strengthening range.

Nor do these theoretical results stand unverified by experiment; they are absolutely in accord with the experiments on the model dams. These collapsed precisely as might have been expected from the above investigation, i.e., the dam with vertical sections gave long before the dam with horizontal sections. The former collapsed by opening up of the joints by tension towards the tail, followed almost immediately by a shear of the whole structure. In the case of the horizontally stratified dam, the collapse, which occurred much later, was by shear of the base, followed almost simultaneously by a shear of one or more of the horizontal sections.

The question then arises as to how far the previously assumed distribution of shear affects the main features of the results, and so the other extreme was taken, i.e., uniform shear, and the effect determined.

This distribution must be further from the actual than the first hypothesis, yet it is still found:

(1) That the line of resistance falls well outside the middle third for about half the dam.

(2) That there exist considerable tensions, 3 to 4 tons per square foot, in the masonry.

(3) That the average shearing stresses on the vertical sections are greater than on the horizontal sections. As a result of this extreme case, it is believed that the real distribution of shear over the base, whatever it may be, must lead us to a line of resistance lying well outside the middle third, and to tensions amounting to something between 5 and 10 tons per square foot.

From these investigations the author concludes as follows:

(1) The current theory of the stability of dams is both theoretically and experimentally erroneous, because:

(*a*) Theory shows that the vertical and not the horizontal sections are the critical sections.

(*b*) Experiment shows that a dam first gives by tension of the vertical sections near the tail.

(2) An accepted form of cross-section is shown to be stable as far as the horizontal sections are concerned, but unstable by applying the same conditions of stability to the vertical sections.

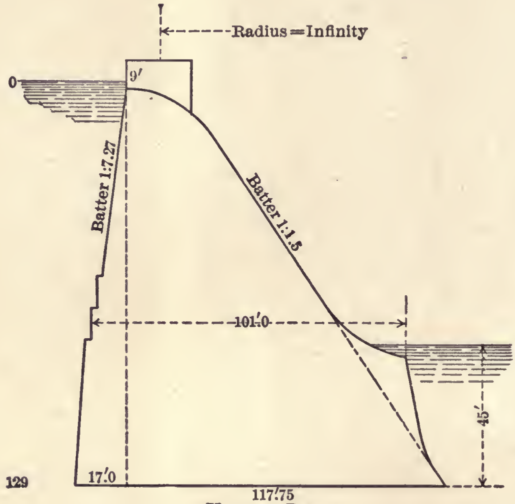
(3) The distribution of shear over the base must be more nearly parabolic than uniform, but as no reversal of the statements follows in passing from the former to the latter extreme hypothesis, it is not unreasonable to assume the former distribution will describe fairly closely the facts until we have greater knowledge.

(4) In future it is held that in the first place masonry dams must be investigated for the stability of their vertical sections. If this be done it is believed that most existing dams will be found to fail, if the criteria of stability usually adopted for their horizontal sections be accepted. This failure can be met in two ways:

(*a*) By a modification of the customary cross-section. It is probable that a cross-section like that of the Vyrnwy dam would give better results than more usual forms.

(*b*) By a frank acceptance that masonry, if carefully built, may be trusted to stand a definite amount of tensile stress. It is perfectly idle to assert that it is absolutely necessary that the line of resistance shall lie in the middle third for a horizontal treatment, when it lies well outside the middle third for at least half the dam for a vertical treatment.

Immediately upon the publication of the preceding results, Sir Benjamin Baker undertook some experiments of a like nature.* The models employed by him were



Vyrnwy Dam

Fig. 1.

made of ordinary jelly however, and included not only the transverse section of the dam itself, but the rock upon which it rested as well. It is shown in the figure.

The horizontal and vertical lines drawn on the sides of the model were for the purpose of detecting any distortion that might result through the application of

---

* Vol. 162, page 120. Minutes of Proceedings of the Institution of Civil Engineers.

pressure. These pressures were applied against both the upstream face and the floor of the reservoir, as it was believed that, while according to the theory of the middle third there could be no tension in the heel, nevertheless for the case of reservoir full, fairly severe tension in the masonry might thus be caused.

The experiments indicated that the distribution of shearing stress in the plane of the base, i.e., where the

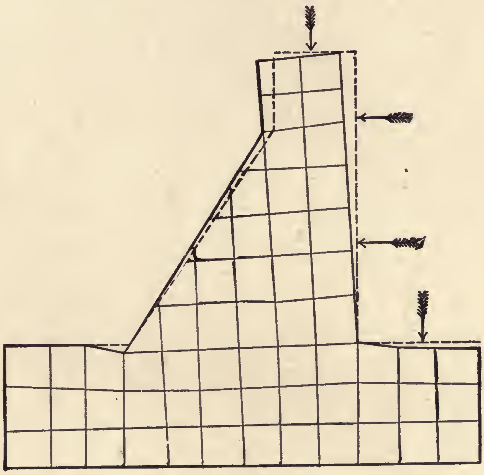


FIG. 2.

dam met the rock, was more nearly uniform than parabolic, and that the strain extended into the rock for a distance equal to about half the height of the dam before it became undetectable. To solve the complete problem, therefore, it would be necessary to consider the elasticity of the rock on which the dam rested. Partially as a result of these and the previous experiments, it may be pointed out in passing, the proposed increase in elevation of the Assouan dam, whereby the capacity of the reservoir would have been considerably augmented, was indefinitely postponed.

The new feature in Atcherley's analysis is that, even though the condition of "no tension in a horizontal joint" is satisfied, dangerous tensions may be shown to exist across vertical planes.* In connection with this consider, for example, a section of the dam *ABC*, which is triangular in profile, and construct *BEC* so that the ordinates represent the variation of the unit normal stress over the horizontal joint *BC*.

Taking a vertical section *IK* in which *J* locates the centroid, the forces to the left are the upward pressure acting over *BK*, tending to cause rotation in a clock-wise manner and thus produce tension at *K*, and two counteracting forces tending to neutralize this pressure: the weight of the portion *BKH* and the horizontal shearing force acting along *BK*. The resultant effect of all three will be tension at *K*, provided the rotation is right-handed, with a consequent splitting along the vertical plane *HK*.

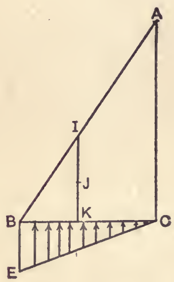


FIG. 3.

In view of the fact that the horizontal shear is present as a factor, it is necessary to determine its distribution, and this Prof. W. C. Unwin undertook to do.† Instead however, of accepting the distribution in accordance with Atcherley's assumptions, an analysis was attempted by

---

* "Engineering," Vol. 79, page 414.

† "Engineering," Vol. 79, page 513. "Note on the Theory of Unsymmetrical Masonry Dams," by W. C. Unwin.

which the shear might be actually calculated, and in doing so attention was called to the fact that the accepted theory of dam design is incomplete in just that feature, since it fails to consider the rate of change in the horizontal shear.

In any analysis the fundamental assumption must be made that a masonry dam is a homogeneous-elastic solid, and, while it is not absolutely essential that no tension exist at any point in the cross-section, yet it seems desirable that there should be none at the upstream face of horizontal joints.

It may be said therefore, that for a more exact analysis the problem resolves itself into one of the determination of shear on horizontal planes, and Prof. Unwin suggests as follows, a method of procedure by which this may be accomplished:

If, as in the figure, we assume a dam of triangular section, in which AB is some horizontal joint, other than the base, and C its centroid, then Q will represent the water thrust, P the weight of masonry, and R their resultant.

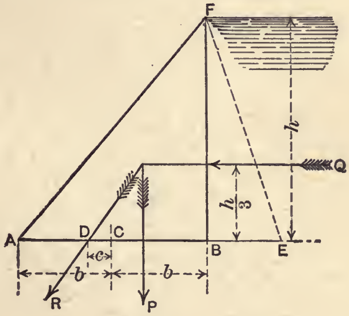


FIG. 4.

In agreement with the ordinary theory we may write the well-known formula for the unit normal pressure on a horizontal joint, at any point x, measured from A, as follows:

$$P_n = \frac{r'}{2b}\left(1 + \frac{3c(b-x)}{b^2}\right). \quad \ldots \ldots \quad (5)$$

For the horizontal shear we must proceed further. Consider therefore, the forces to the left of HK in Fig. 1 we have (1) the vertical pressure on AK, (2) the weight of AHK, and (3) the shear acting along AK. It is evident that the difference between (1) and (2) represents the total vertical shear on HK.

If, therefore, the figure $ALMB$ represent, in masonry units, the distribution of normal stress on AB, as given by Eq. (1), then $ALTH$ will, in like manner, represent the above-mentioned total vertical shear on HK.

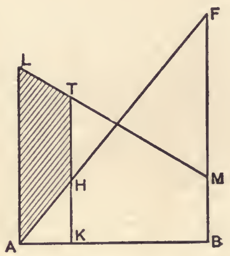


FIG. 5.

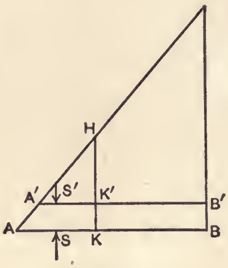


FIG. 6.

Consider now a second section $A'B'$, a small distance z above AB; the total shear on HK' may then be found as before. Denoting the former by S, and the latter by S', then $S-S'$ equals the total shear on KK', which, when divided by z, will give the intensity of vertical shear at K, and consequently the intensity of horizontal shear at the same point.

Since all the forces to the left of HK are now known, the normal stress on that plane may be found, and from it we may readily determine whether tension or compression exists at K.

At the base these results would be much modified, because of the discontinuity of form, which, in the opinion of Prof. Unwin, places the exact determination of the stresses beyond the power of mathematics. The author believes the effect of the rock into which the dam is built is to reduce the variation of stress which would otherwise exist.

In a subsequent paper,* giving a complete demonstration of the preceding analysis as applied to a masonry dam of triangular cross-section, it is found that the distribution of shear on a plane horizontal joint may be represented by a right triangle whose base is the length of the joint and whose vertex is perpendicularly below the downstream edge. The figure illustrates the variation of normal stress and shear on AB; the lines of resistance for both vertical and horizontal planes; and the centers of gravity of the sections above the successive horizontal joints.

FIG. 7.

Consequently the total normal or shearing stress on any part of AB is equal to the area between that part and the line of normal stress or the line of shearing stress.

If the upward reactions and the weights of the dam to

---

* "Engineering," Vol. 79, page 593. "Further Note on the Theory of Unsymmetrical Masonry Dams." W. C. Unwin.

the left of each vertical section be combined with the shears T, acting along AB, the resultants will cut the vertical sections at points shown on the line of resistance for these vertical sections. As this line lies wholly within the middle third, there can be no tension on any vertical section.

The total compressive stress on any vertical section at its lower edge will therefore be:

$$\frac{T}{y}\left(1+\frac{6z}{y}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where T is the shear on the horizontal plane from the toe to the vertical section taken, y the height of the vertical section, and z the distance from the center of the vertical section to the point of application of the resultant forces on that section.

Near the upstream toe the plane on which the greater principal stress acts is found to be vertical while near the downstream toe it approaches the horizontal. The stresses are all compressive and on the water face the compressive stress it at all points equal to the water pressure at that point.

The above analysis is simply an application to vertical sections of the method now accepted as applicable to the horizontal planes and is a possible solution, since the distribution of shear is known. It differs from Atcherly's method in the fact that the latter assumes the usual distribution of *normal* stress, together with *a parabolic variation for the horizontal shear*. This latter hypothesis the author thinks inconsistent with the previous one.

Further investigations by Prof. Unwin* on dams of various sections lead to the following conclusions:

(1) For a rectangular dam the distribution of shearing stress on horizontal planes may be represented by the ordinates of a parabola.

(2) For a triangular dam, the distribution may be represented by the ordinates of a triangle with the apex below the downstream toe.

(3) For a dam with vertical upstream face and curved downstream face the distribution may be represented by a figure consisting of a parabola superposed on a triangle.

(4) For a dam with rectangular base the distribution is represented by a parabola.

Following the results of the experimental investigations of Atcherley and Baker, several other papers of a like nature appeared in the Minutes of Proceedings of the Institute of Civil Engineers, Vol. 162. The first of these to be considered here is that by Sir John Walter Ottley and Arthur William Brightmore, entitled, "Experimental Investigations of the Stresses in Masonry Dams subjected to Water-Pressure."

In presenting this paper, the authors drew attention to the fact that until the publication of Mr. Atcherley's results, the question of dam design had been accepted as settled, and that his memoir had had the effect of reopening the entire subject of the distribution of stress in structures of this class.

---

*"Engineering," Vol. 79, page 825. "On the Distribution of Shearing Stress in Masonry Dams." Prof. W. C. Unwin.

It was also pointed out that tension was found by him to exist on vertical planes near the outer toe, whether the distribution of shearing stress over the base was assumed to be uniform or to vary according to the parabolic law.

Considering a transverse section of a dam, the authors argued that, whatever the distribution of shear over the base might be, it must follow some other law near the top, since the conditions in these higher levels are radically different from those existing in the lower, where the dam is fixed to the foundation, and where the water pressure ceases abruptly.

The investigation was therefore undertaken, at least in part, to determine the distribution of shear on horizontal planes in the higher levels of the dam and to see how it varied from that at the base; and it might be stated here that it was found to be uniform in the latter plane but to vary uniformly from zero at the heel to a maximum at the toe in the higher levels, the change from the one condition to the other being gradual. It will be shown that it is near the inner toe rather than near the outer toe that tension may be anticipated.

The model dams were triangular in section, made from a kind of modeling clay called "plasticine," and so proportioned that the resultant pressure on the base cut that plane at the downstream extremity of the middle third.

For purposes of observation the sections were placed between vertical sides of plate glass, upon which vertical and horizontal lines had been etched, corresponding to similar lines on the model, so that any displacement in the latter might be noted by comparison with the former. Pressure was applied, by means of a thin rubber bag containing water which was made to fit the frame. Though the water was allowed to act over a period of 33 days, after the elapse of one week a crack was noticed at the upstream toe, running downward and at an angle of about 45°. At the end of the longer period an examination showed that in the neighborhood of the base the displacement of the vertical lines was such as to make them all about equally inclined, thus indicating a uniform intensity of shear on that section, while in the higher levels and near the outer portion of the dam the lines became more inclined as the elevation increased, indicating that the intensity of shear increased also as the top was approached.

Turning to the horizontal lines in the model for the purpose of discovering the method of distribution of normal stress, it was found that they were curves at the base, sloping downward from the inner toe to a point about two-thirds the distance to the outer toe, then remaining fairly level until almost reaching the downstream face, when they finally bent up slightly. In the higher levels, however, these lines gradually developed a uniform slope running from the inner to the outer toe.

An investigation of the shearing stresses on vertical planes requires that, to draw the line representing the intensity of normal reaction at the base the following facts must be considered:

(1) The total normal reaction equals the weight of the dam.

(2) Since the resultant pressure on the base acts at

one-third the width from the outer toe, the moment of the reaction stresses about this point must be zero.

(3) The intensity of the reaction at the outer toe must equal the intensity of the shearing stress in the vertical plane multiplied by the ratio of the height to the base of the dam.

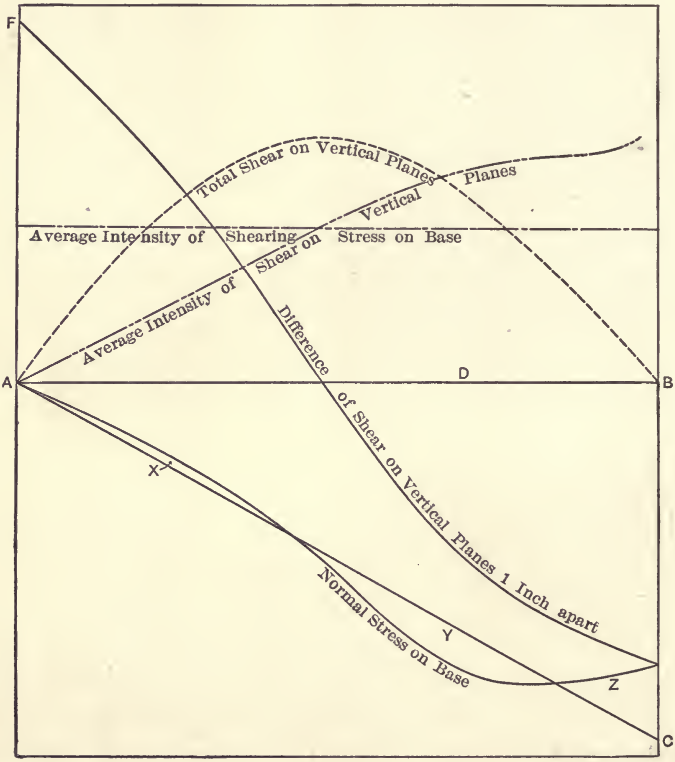


Fig. 8.

Referring to the figure: *AB* represents the base of the dam, and *BC* twice the average intensity of normal stress on *AB*. *AC* is then drawn; consequently *ABC* represents the total normal stress on *AB*, or the weight of the structure.

If AE, on the other hand, represents the actual intensity of normal reaction over AB, then for (1) to hold true the area Y must equal the areas $(x+z)$ and if (2) is to hold, the moments of x, y, and z, about D (equal to $\frac{1}{3}AB$ from B), must be zero; also for (3) to be satisfied, BE must equal the limiting value of shearing stress in a vertical plane near the toe, multiplied by the height and divided by the base of the dam.

From these considerations AE may be fitted in by trial till it is found to satisfy all of the above conditions.

Dividing the cross-sections into strips 1 inch wide we may properly consider the equilibrium of each such strip. Evidently the difference between the weight of each strip and the normal reaction on the base is equal to the difference in shear on the two adjacent vertical planes, and if in the figure these weights be plotted upward from AE, the curve FE will result. Furthermore, both the curves for "total shear on vertical planes" and "average intensity of shear on vertical planes" may now be drawn, whereupon it is evident to what extent the average intensity of shear on vertical planes varies, and how it compares with the average intensity on the base.

Since the shear on horizontal and vertical planes at any one point is equal, and the shear on the base is practically constant, it follows that above the base the shear on horizontal or vertical planes is small near the heel while in the outer half above the base it increases as the outer edge is approached; in fact it increases from zero at the heel to a maximum at the toe. These facts show that the shearing stresses to be provided for are those existing in the higher levels and near the toe, and not those at the base.

In considering the effect of shear on the base, neglecting the "fixing" at that level, we may assume that the reaction stress and that due to the weight of a strip, is constant over each inch. They then act at the middle of each strip; and, taking these points successively as centers, the difference of the moments of the horizontal pressures on the vertical sides of the strip, it is evident, will equal the sum of the shearing stresses on the same vertical sides multiplied by ½ inch.

This makes possible the determination of the moment of the horizontal pressures on each vertical strip.

The horizontal shear on each inch of base being the difference between the horizontal pressures acting on the two vertical sides, the latter may be determined as soon as their points of application are given. As these points are known for the innermost and outermost strip, an easy curve may be drawn which will approximately locate the other points and thus give the desired heights. From these results it may be shown that the shearing stress on the base increases from practically zero at the inner toe to a point near the center of the base and then remains fairly constant.

The modification of this distribution, due to the fixing of the dam to its base, must, on the other hand, be considered. The water tends to cause a maximum pressure and displacement at the inner face, which diminishes to zero at the outer. As the dam is fixed, this displacement is prevented, thus inducing corresponding shears, and the effect of this conflicting condition, with that previously shown to exist, causes a nearly uniform shear over the base.

Further evidence of uniform shear on the base was obtained as follows: The models, after being subjected to water pressure, showed cracks which appeared at the inner toe, the angles which these made with the horizontal steadily diminishing as the base was decreased in width from a maximum of 45° for the widest base used to 25° for the narrowest.

The variation of these inclinations corresponded closely with the computed directions, on the assumption that the shear was uniform over the base and the experiments therefore strongly support the inference that shear over the base is uniformly distributed.

It was shown by means of the models that there are tensile stresses on other than horizontal planes passing through the inner toe. The models indicated this by cracking, even when the back was sloped away from the vertical so as to cause vertical pressure and hence compression on the upstream face.

The impossibility of tension on vertical planes near the outer toe may be shown by means of the following equation for principal stress:

$$f = \frac{p + p' \pm \sqrt{(p+p')^2 - 4(pp' - q^2)}}{2} \quad . \quad . \quad . \quad (7)$$

where compressions are plus and tensions are minus. When $pp' > q^2$ at any point, there can be no tension at that point, since under the above conditions both principal stresses will be compression and hence stresses on all other planes passing through that point will be compression also. This condition may be shown to exist near the outer toe, and hence no tension can act across any vertical plane in that position.

For example consider the equilibrium of a wedge of unit length cut off by a vertical plane near the toe.

p' = intensity of pressure normal to a vertical plane at base.

p = intensity of reaction normal to the base.

q = intensity of shearing stress.

Then $p'h = qb$ or,

$$p' = q\frac{b}{h} \quad \ldots \quad (8)$$

The weight of the particle is negligible because it varies with h^2.

Since the resultant stress must be parallel to the outer face it follows that,

$$p = q\frac{h}{b} \quad \ldots \quad (9)$$

Multiplying (8) by (9) there results

$pp' = q^2$ at outer toe. But p has been shown to increase for some distance from outer toe and the point of application of p' becomes relatively lower as the inner toe is approached and since the average pressure is constant it follows that p increases as the distance from the outer toe increases and hence in the vicinity of the outer toe pp' is greater than q^2 and consequently there can be no tension in that neighborhood. This was checked by the behavior of the models.

It is a fact that in dam work the normal stress is the only one specified, whereas the absolute maximum is about 50 per cent greater.

The conclusions reached from this set of experiments follow:

(1) If a masonry dam be designed on the assumption that the stresses on the base are uniformly varying and that the stresses are parallel to the resultant force acting on the base, the actual normal and shearing stresses on both horizontal and vertical planes would be less than those provided for.

(2) There can be no tension on any planes near the outer toe.

(3) There will be tension on certain planes other than the horizontal near the inner toe, and the maximum intensity of such tension in the foundation being generally equal to the average intensity of shearing stress on the base, and the inclination of its plane of action being about 45°; and its maximum intensity in the dam above the base about ½ the above amount and acting on a plane less inclined to the horizontal.

The investigation undertaken by Mr. Hill * for "The Determination of the Stresses on any Small Element of Mass in a Masonry Dam," are on the other hand purely analytical in character, being directed toward a solution of (1) the vertical, (2) horizontal, and (3) tangential shearing forces acting on the faces and along the edges of such an element.

---

* Minutes of Proceedings of the Inst. of C. E., 72.

In this analysis, there is first expressed a perfectly general formula for C (the distance of the load point from the center of the joint), and two other general formulæ for the pressures p_1 and p_2 in terms of the total load and C from its above value, where p_1 is the minimum and p_2 the maximum pressure. For the pressure p at any point x on the joint of length b the following equation is used:

$$p = p_1 + \frac{x}{b}(p_2 - p_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Up to this point the analysis is identical with the general procedure of investigation, which assumes that the horizontal pressures are proportional to the vertical, and does not analyze the shear.

Citing Prof. Unwin, the author states tnat the former "suggested that the shearing stress at any point might be found by considering the difference between the total net vertical reactions (between that point and either face) along two horizontal planes a unit's distance apart, and has applied the principle by the use of algebraical methods." Mr. Hill, on the contrary, employs the calculus to obtain more rigorous results.

The procedure follows: Consider any point distant x from the inner toe and on the lower of two horizontal planes, a unit's distance apart. The total vertical reaction is then $\int_0^x pdx$. Subtracting the weight of masonry resting on this portion of the horizontal joint, and denoting the difference by r we have an expression for the "net vertical reaction." If this value of r be differentiated with respect to h, the distance between the two horizontal planes, the change in the reaction will be obtained, and this change or difference is the vertical shearing stress at the point located by x. It is also, therefore, the horizontal shear at the same point, which we may denote by q.

If q be integrated with respect to x, between the limits of x and b, the resulting expression will give the entire horizontal shear between such limits on the joints in question. Represent this by Qx.

To find the horizontal pressure intensity, we have but to consider the above integration. This shear must be resisted by the material along the vertical section at x. Similarly the total shear on a plane a differential distance below the last must be resisted by the vertical section at x, differing in height from the former by dh. Consequently the differential of Qx with respect to $h=p'$ will represent the horizontal pressure intensity at point x. These expressions for p, p' and q therefore give respectively the values of the vertical pressure intensity, horizontal pressure intensity, and shearing force acting on a unit element of mass.

Cain * presents a treatment of this matter, which, while presenting no new features, is strictly arithmetical in character, and in that respect at least differs from the preceding. Its purpose, as Hill's, is to determine the amount and distribution of stress at any point in a masonry

* Wm. Cain, M. Am. Soc. C. E., Trans. Am. Soc. C. E., Vol. 64, page 208.

dam, on the assumption that the law of the trapezoid represents the variation of pressure on horizontal joints.

The analysis finally establishes formulæ for (1) the normal unit stress at any point in a horizontal joint, (2) the normal unit stress on a vertical plane at any point of a horizontal joint, (3) the unit shear on either horizontal or vertical planes at any point of a horizontal joint, and at the same time indicate the method of determining the maximum and minimum normal stresses and the planes on which they act.

The solutions are only approximate, but the results are found to be close enough for the purpose.

Before proceeding it may be advisable to review certain features involved in a consideration of the stresses in a masonry dam which Prof. Cain presents in a very satisfactory manner.

1. It will be evident from an examination of the figure that the intensities of shear on two planes at right angles to each other are equal. For, in the elementary cube under consideration, the weight may be neglected, since it is an infinitesimal of the third order, while the opposing normal forces balance as the cube is reduced in size.

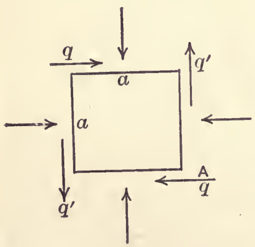


Fig. 9.

For equilibrium then, $q \cdot a \cdot a = q' \cdot a \cdot a$, or $q = q'$ and, because each side is a differential quantity, it may be assumed that the values q and q' represent the average unit shear on the respective faces. As a consequence they are equal to the shear at any point, for example A, of the particle.

2. In a triangular element of the dam, at the down-stream edge, and of unit's length, the force sacting are those shown. Because it is an element we may neglect the weight, and therefore, if p' is the normal intensity of stress on a vertical plane, p the normal intensity of stress on a horizontal plane, and q the shear intensity, for

$$\Sigma V = O,\ pb = qa \quad \text{or} \quad p = q\frac{a}{b} \quad \text{and} \quad q = p \tan\phi. \quad . \quad (11)$$

for

$$\Sigma H = 0,\ p'a = qb \quad \text{or} \quad p' = q\frac{b}{a}. \quad . \quad . \quad (12)$$

then

$$p' = p \tan^2 \phi,$$

or,

$$p, p = q^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

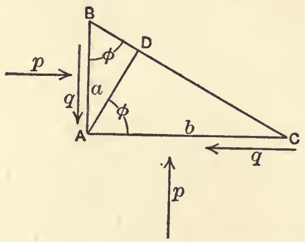


FIG. 10.

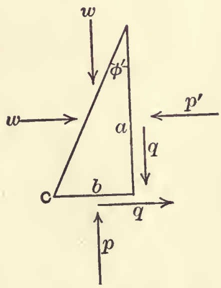


FIG. 11.

3. The same analysis may be applied to an element at the inner face, where ϕ is the inclination to the vertical; but, for the reservoir full, the intensity of water pressure, horizontal or vertical, at c, and in this case represented by w, must be taken into account.

Under these circumstances,

$$pb = qa + wb, \quad \ldots \ldots \quad (14)$$

and

$$p'a = qb + wa, \quad \ldots \ldots \quad (15)$$

$$\therefore \quad p = q \cot \phi' + w, \quad \ldots \ldots \quad (16)$$

and

$$p' = q \tan \phi' + w. \quad \ldots \ldots \quad (17)$$

When, as is usually the case, the vertical component of water pressure acting along the back is neglected, the above equations become,

$$p = q \cot \phi', \quad \ldots \ldots \quad (18)$$

$$p' = q \tan \phi' + w, \quad \ldots \ldots \quad (19)$$

$$\therefore \quad q = p \tan \phi', \quad \ldots \ldots \quad (20)$$

and

$$p' = \tan^2 \phi' + w. \quad \ldots \ldots \quad (21)$$

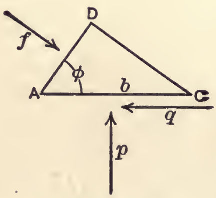


FIG. 12.

4. If an element at the down stream face be again considered, since the shear on the outer face DC is zero, that on a plane AD perpendicular to DC, must be zero also, and hence the stress d on AD is wholly normal.

The total pressure on AD is therefore,

$$f \cdot AD = f \cdot b \cos \phi. \quad \ldots \ldots \quad (22)$$

The vertical component of this is $f \cdot b \cos^2 \phi$, because $EV = O$,

$$pb = f \cdot b \cos^2 \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or,

$$f = \frac{p}{\cos^2 \phi} = p \sec^2 \phi, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

which is the maximum intensity of normal stress at the outer face.

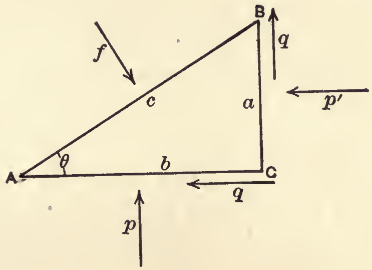


FIG. 13.

5. To determine the planes of principal stress, i.e., planes upon which the stress is wholly normal, and also the intensity of that stress, we may assume the conditions indicated in the figure.

The total shear on c then is fc; its vertical component $fc. \cos = fb$, and its horizontal component $fc \sin = fa$.

When $EV = 0$ and $EH = 0$,

$$fb = pb + qa; \qquad \therefore \quad f - p = q \tan \theta, \quad . \quad . \quad (25)$$

$$fa = qb + p'a; \qquad \therefore \quad f - p = q \cot \theta, \quad . \quad (26)$$

The difference of these two equations gives,

$$p-p'=q\,(\cot\theta-\tan\theta)=q\,\frac{1-\tan^2\theta}{\tan\theta} \quad . \quad (27)$$

$$\therefore\quad \tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2q}{p-p'}. \quad . \quad . \quad (28)$$

This equation gives a plane upon which there is none but normal stress.

To determine f, multiply equation (25) by (26).

$$(f-p)\,(f-p')=q^2, \quad . \quad . \quad . \quad . \quad (29)$$

whence,

$$f=\tfrac{1}{2}p+p'\pm(p+p')^2-4\,(p,p'-q^2). \quad . \quad . \quad (30)$$

This will give two values of f, which correspond to the two principal planes of stress, the stress being compressive when f is positive, and tensile when f is negative. There can be no tension when $p, p'>q^2$.

Determination of the vertical unit stress at any point of a horizontal plane joint: From the law of the trapezoid, we have the pressures at the upstream and downstream toes represented respectively as follows:

$$p_2=\frac{4b-6EC}{b^2}W, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$p_1=\frac{4b-6CB}{b^2}W. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The resultant is supposed to act within the middle third. If x' represent any point along EB, measured from E, then p, the pressure at x, is given by,

$$p=p_2+\frac{p_1-p_2}{b}x', \quad . \quad . \quad . \quad . \quad (33)$$

while the total normal stress from E to x' is, by integration,

$$P = p_2 x' + \frac{p_1 - p_2}{2b} x'^2. \qquad (34)$$

To find the unit shear on vertical or horizontal planes, we have but to consider a slice of dam between two horizontal joints one foot apart, extending from the inner to the outer face, a distance x along the lower joint. (The back is supposed to slope .02 feet for each foot in height).

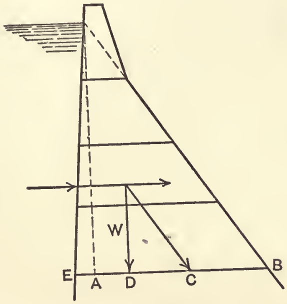


FIG. 14.

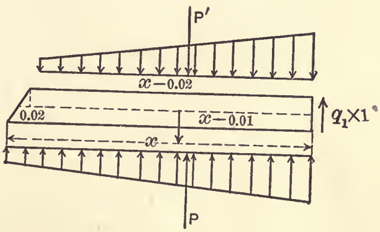


FIG. 15.

The vertical forces acting are:

(1) A uniformly varying stress on the upper joint acting downward.

(2) The same on the lower joint acting upward.

(3) The weight of the strip.

(4) The shear on the vertical face at x.

For equilibrium,

$$q_1 = P' - P + (x - 0.01). \qquad (35)$$

P' and P may be obtained as indicated in the previous demonstration.

The above value of q_1 is the average unit shear at the depth taken, but a similar value q_2 may be determined at a depth one foot below. Under these circumstances $\frac{q_1+q_2}{2}$ is the average of the two, and may be said to be approximately equal to the shear at the depth of the joint between the two slices.

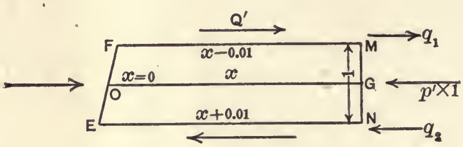


FIG. 16.

To find the normal unit stress on a vertical plane, a similar section to that just used may be employed; but the horizontal components are now to be equated for equilibrium.

Let h = the horizontal water pressure at the assumed depth.

Q' = total shear on upper face.

Q = total shear on lower face.

p' = average normal stress.

q_1 and q_2 = the intensities of horizontal shear at the points indicated.

Q' and Q may be found by integrating q_1 and q_2 between the proper limits.

$$\therefore \quad p' = h + Q' - Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

This value of p' is assumed as the average intensity on the vertical plane and as the unit intensity on the vertical plane at a point midway between the two horizontal planes.

Three general formulæ may be written for p, q, and p' which, it has been suggested, be put in the following form:

$$p = a + bx, \quad \ldots \ldots \quad (37)$$

$$q = c + dx + ex^2, \quad \ldots \ldots \quad (38)$$

$$p' = f + gx + hx^2 + jx^3. \quad \ldots \ldots \quad (39)$$

APPENDIX II

THE DESIGN OF A HIGH MASONRY DAM

The following calculations illustrate the method employed in the determination of the theoretical cross-section of a high masonry dam.

These conditions were assumed:

Height of free water surface above foundation, 250′–0″.
Width of top, 23′–0″.
Working pressures: $p=14$ tons per square foot.
$q=18$ tons per square foot.
Weight of masonry 146 pounds per cubic foot.
Weight of water 62.5 pounds per cubic foot.
Coefficient of friction $f=0.7$.

Joint J_1.

Generally speaking it may be assumed that the top of the dam is about one-tenth of the height above the water, but in the case under consideration, a superelevation of only 20 feet will be employed. While the choice in this respect is purely arbitrary, the above ratio is the one usually prescribed if there are no other governing conditions.

Since the length of a joint depends upon its depth below the water, it is evident that at the surface this dimension should be zero. For various reasons however, such as the desirability of a footwalk or a driveway on the crest, a top width is chosen which will satisfy these demands.

As 23 feet has been decided upon in this problem, for a considerable distance below the water level the rectangular section will more than satisfy the only condition for stability that applies in this portion and which requires that the resultant of all the external forces lie within the middle third of the cross-section. It becomes necessary therefore, to determine the depth at which a modification of this dimension should take place, and for this purpose Eq. (34) is employed.

(Series A, Stage I.)

$$H=\sqrt[3]{\varDelta L^2(H+a)},$$

$$\varDelta=2.333,\ L=23'.0,\ a=20'.0.$$

This equation may be solved by successive substitutions until such a value of H is found that equality results. In the present instance it is found that $H=42.6$ feet satisfies the equation, and hence it is necessary to carry the rectangular cross-section of the dam down to a depth of 62.6 feet below the top.

The solution may be expedited by the use of the graphic method. Thus, assume at least three values for H, say 30, 40, and 50 feet in the present case, substitute successively in the right-hand member of the above equation and solve. Plot these resulting values as abscissæ and the assumed values of H corresponding as ordinates. A smooth curve drawn through the points thus obtained will give a point on the line where ordinate and abscissa are equal, and this will be the desired value.

At no point in this portion of the dam does the length of a horizontal joint change; but below this elevation the dimension will have to be increased in order to comply with the requirement that the resultant shall not pass outside the middle third, and it is accomplished by giving a batter to the downstream face of the dam, while the upstream face still remains vertical.

Joint J_2.

The investigation for the purpose of determining the length of J_2 involves the use of an equation in which u shall have a value of $\frac{l}{3}$, since the resultant of the external forces for the reservoir full reached the limit of the middle third at J_1 and since it may not pass outside that limit. This is expressed by Eq. (37). (Series A, Stage 2.)

$$l^2+\left(\frac{4A_0}{h}+l_0\right)l=\frac{1}{h}\left(\frac{H^3}{\Delta}+6A_0y_0\right)+l_0^2.$$

As the joints are taken every 10 feet apart, at least in the upper levels of the dam, the factors in the above expression take the following values:

$$A_0=1440, \quad H=52'.6, \quad h=10', \quad l_0=23'$$

$$y_0=11.5, \quad \text{and} \quad \Delta=2.333 \text{ (or } \tfrac{7}{3}).$$

Substituting these values therefore in the above equation, completing the square and solving for l, we obtain

$$l = 26.8 \text{ feet.}$$

Now Eq. (37) is dependent upon y_0; it is therefore necessary after each application of that equation to employ Eq. (36) in order to solve for y, which in turn becomes y_0, at the next joint. (See Supplementary Equations, Stage II.)

$$y = \frac{A_0 y_0 + \frac{h}{6}(l^2 + l\,l_0 + l_0^2)}{A_0 + \left(\frac{l + l_0}{2}\right)h}.$$

As all of these factors are already known, it merely requires that they be substituted in the above.

This results in

$$y = 11'.65.$$

Joint J_3.

Again Eq. (37) will be used to determine the length of the joint, but the factors will now have the following values:

$$A_0 = 1689, \quad H = 62'.6, \quad h = 10', \quad l_0 = 26'.8,$$

$$y_0 = 11'.65, \quad \Delta = 2.333.$$

These substituted in that equation give

$$l = 31.4 \text{ feet.}$$

In like manner we will use Eq. (36) to determine y. This gives

$$y = 12.08 \text{ feet.}$$

Joint J_4.

Using Eq. (37) with the following values:

$$A_0 = 1980, \quad h = 10', \quad l_0 = 31'.4, \quad H = 72'.6,$$
$$y_0 = 12\,.08, \quad \Delta = 2.333.$$

Using Eq. (36) to determine y, we find,

$$y = 12.82 \text{ feet.}$$

Joint J_5.

While the formulæ 37 and 36 will still be employed at this joint to determine the values of l and y respectively, h will be taken equal to 4.4 feet, thus making $H = 77.0$ feet instead of 82.6 feet. The reason for this is that if the latter value of H be used the resulting value of y would be less than $\frac{l}{3}$, bringing the resultant for reservoir empty outside the prescribed limit. To keep it just within this limit it is found by trial that h should not exceed 4.4 feet.

Using Eq. (37) therefore to determine l, with the following values inserted,

$$A_0 = 2321, \quad h = 4'.4, \quad l_0 = 36'.9, \quad H = 77'.0,$$
$$y_0 = 12'.82, \quad \Delta = 2.333.$$
$$l = 39.5 \text{ feet.}$$

Solving for y in Eq. (36),

$$y = 13.2 \text{ feet,}$$

in which it is seen that y is practically $\frac{1}{3}$ of l.

In the employment of Eq. (37) for the determination of the length of the successive joints we used the value $u=\frac{1}{3}l$, thus indicating that for reservoir full the resultant acted at the downstream limit of the middle third. On the other hand, it is now found that for reservoir empty the limiting condition is reached, and hence at this point it is necessary to batter the back face while u and y still remain equal to $\frac{l}{3}$. These conditions require the use of Eq. (38), (Series A, Stage III), for the determination of the length of joint, and of Eq. (44), Supplimentary Equations, Stage III, for the determination of the batter of the upstream face.

Joint J_6.

Using Eq. (38),

$$l^2+\left(\frac{2A_0}{h}+l_0\right)l=\frac{H^3}{\Delta h},$$

with the following values inserted:

$$A_0=2489, \quad h=10', \quad l_0=39'.5, \quad H=87'.0,$$
$$\Delta=2.333, \quad y_0=13'.23,$$

whence,

$$l=48.5 \text{ feet.}$$

To determine the batter, we employ Eq. (44),

$$t=\frac{2A_0\,(l-3y_0)-hl_0^2}{6A_0+h\,(2l_0+l)},$$

and substituting the quantities from above,

$$t=1.8 \text{ feet.}$$

In view of the fact that as the lower levels of the dam are approached the unit pressures increase and finally become controlling factors, it will be desirable here to determine the intensity for the *downstream* edge of this joint, where the permissible maximum normal unit stress is only 14 tons per square foot.

For this purpose Eq. (14) will be employed, since $u=\frac{l}{3}$ and $p'=0$. This application gives to p the following value:

$$p=\frac{2W}{l}=8.8 \text{ tons.}$$

The unit pressure, it is thus seen, is far below the prescribed limit, so that it will be unnecessary to consider it yet as a feature in the design, although it will be necessary to examine each joint to determine at just what point it does begin to control.

Joint J_7.

At this elevation we may properly assume a depth h of 20 feet for the succeeding joints, since the larger value will reduce the number of applications of the formulæ and at the same time will not affect the analysis.

Using Eq. (38) again with the following values:

$$A_0=2929,\quad h=20',\quad l_0=48'.5,\quad H=107'.0,$$
$$\Delta=2.333,\quad y_0=16'.07,$$
$$l=64'.6 \text{ feet.}$$

Solving for t, the batter of the upstream face, by Eq. (44),

$$t=2.3 \text{ feet.}$$

Likewise solving for p, from Eq. (14),

$$p = 9.1 \text{ tons.}$$

Joint J_8.

Applying Eqs. (38), (44) and (14) for the determination of l, t, and p, respectively, with the following values substituted:

$$A_0 = 4060, \quad h = 20', \quad l_0 = 64'.6, \quad H = 127'.0,$$
$$y_0 = 21'.6, \quad \Delta = 2.333.$$

there results,

$$l = 79'.7, \quad t = 1'.4, \quad \text{and} \quad p = 10.1 \text{ tons.}$$

Joint J_9.

Using the same equations for l, t, and p, with the following values inserted:

$$A_0 = 5503, \quad h = 20', \quad l_0 = 79'.7, \quad H = 147'.0,$$
$$\Delta = 2.333, \quad y_0 = 26'.6,$$

there results,

$$l = 93'.9, \quad t = 0'.8, \quad \text{and} \quad p = \frac{2W}{l} = 11.3 \text{ tons.}$$

Joint J_{10}.

Applying the same formulæ with the following values inserted:

$$A_0 = 7239, \quad h = 20', \quad l_0 = 93'.9, \quad H = 167'.0,$$
$$\Delta = 2.333, \quad y_0 = 31'.3,$$

there results,

$$l = 107'.7, \quad t = 0'.5, \quad \text{and} \quad p = 12.6 \text{ tons.}$$

Joint J_{11}.

Using the same formulæ as before with the following values inserted:

$$A_0 = 9255, \quad h = 20', \quad l_0 = 107'.7, \quad H = 187'.0,$$
$$y_0 = 35'.9, \quad \Delta = 2.330,$$
$$l = 121'.2, \quad t = 0'.3, \quad \text{and} \quad p = 13.9 \text{ tons.}$$

It is noticed here that the unit normal pressure at the downstream edge of the joint has practically reached the limit prescribed of 14 tons per square foot, and it could be shown that an investigation of joint J_{12}, on the same lines as for J_{11} would give a pressure at that point considerably in excess of this prescribed value. It is therefore necessary to use such an equation that this condition of the pressure at the downstream edge may be involved in it.

This is expressed by Eq. (40) (Series A, Stage IV.)

$$l^2 = \frac{\gamma H^3}{p}.$$

Joint J_{12}.

Using the following values in the above equation, $H = 207'.0$, $\gamma = 62.5$ and $p = 14$ tons per square foot, there results,

$$l = 140.7 \text{ feet.}$$

To find the batter of the back we must use Eq. (44) as heretofore, with the following values inserted:

$$A_0 = 11544, \quad l = 140'.7, \quad y_0 = 40'.4,$$
$$l_0 = 121'.2, \quad h = 20',$$

whence,

$$t = 2'.0.$$

It is now also necessary to determine u, since in the above equation it has no influence, and for this purpose Eq. (39) must be used. (Stage IV, Supplementary Equation.).

$$u = \frac{2l}{3} - \frac{pl^2}{6\Delta\gamma A},$$

whence, substituting the values,

$$u = 49.1 \text{ feet.}$$

It is also necessary to determine the pressure at the upstream edge of each joint since it is gradually approaching the limit of 18 tons per square foot.

Here Eq. (14) will have to be used,

$$p = \frac{2W}{l} = 14.7.$$

As this value is well inside the limit it is unnecessary to use any equation in which the pressure intensity at the upstream face is the controlling feature.

Joint J_{13}.

Here again we must use Eq. (40) with $H = 227'$, $p = 14$, and $\gamma = 62.5$ whence,

$$l = 161.6 \text{ feet.}$$

Solving for t,

$$t = 2.1 \text{ feet.}$$

Solving for u,

$$u = 59.1 \text{ feet.}$$

Solving for q,

$$q = 15.5 \text{ tons.}$$

Investigating this last joint for stability as to friction we find,

$$\frac{H^2}{2}=\frac{227^2}{2}=25765,$$

$$\Delta A=40111,$$

$$\frac{H^2}{2A\Delta}=f=0.64.$$

Hence stability is assured.

Joint No.	H	H^3	h	$h+a$	A_0	A	l_0	l_0^2
1	42.6	77,309		62.6		1,441		
2	52.6	145,532	10	72.6	1,441	1,690	23.0	529
3	62.6	245,314	10	82.6	1,690	1,982	26.7	713
4	72.6	382,657	10	92.6	1,982	2,324	31.6	1,000
5	77.0	456,533	4.4	97.0	2,324	2,492	36.8	1,356
6	87.0	658,503	10	107.0	2,492	2,930	39.5	1,560
7	107.0	1,225,040	20	127.0	2,930	4,059	48.2	2,323
8	127.0	2,048,380	20	147.0	4,059	5,503	64.7	4,186
9	147.0	3,176,520	20	167.0	5,503	7,240	79.7	6,352
10	167.0	4,657,460	20	187.0	7,240	9,255	94.0	8,836
11	187.0	6,539,200	20	207.0	9,255	11,544	107.7	11,449
12	207.0	8,869,740	20	227.0	11,548	14,169	121.4	14,738
13	227.0	11,697,100	20	247.0	14,169	17,192	140.7	19,800

Joint No.	l	y_0	y	u	t	p	q	A_0y_0
1	23.0		11.5	$\frac{7}{8}$				
2	26.7	11.5	11.64	$\frac{7}{8}$				16,560
3	31.6	11.64	12.08	$\frac{7}{8}$				19,653
4	36.8	12.08	12.80	$\frac{7}{8}$				23,913
5	39.5	12.80	13.23	$\frac{7}{8}$		9.1		29,773
6	48.2	13.23	16.07	$\frac{7}{8}$	1.64	8.9		
7	64.7	16.1	21.6	$\frac{7}{8}$	2.4	9.2		
8	79.7	21.6	26.6	$\frac{7}{8}$	1.2	10.1		
9	94.0	26.6	31.3	$\frac{7}{8}$	0.8	11.2		
10	107.7	31.3	35.9	$\frac{7}{8}$	0.5	12.6		
11	121.4	35.9	40.4	$\frac{7}{8}$	0.8	13.9		
12	140.7	40.4	46.9	49.0	2.0	14.0		
13	161.6	$\frac{l}{3}$	$\frac{l}{3}$	59.2	2.9	14.0	15.5	

DATA

Wt. of Masonry = 146#/cu. ft.

Wt. of Water = 62.5#/cu. ft.

p = 14 Tons

q = 18 Tons

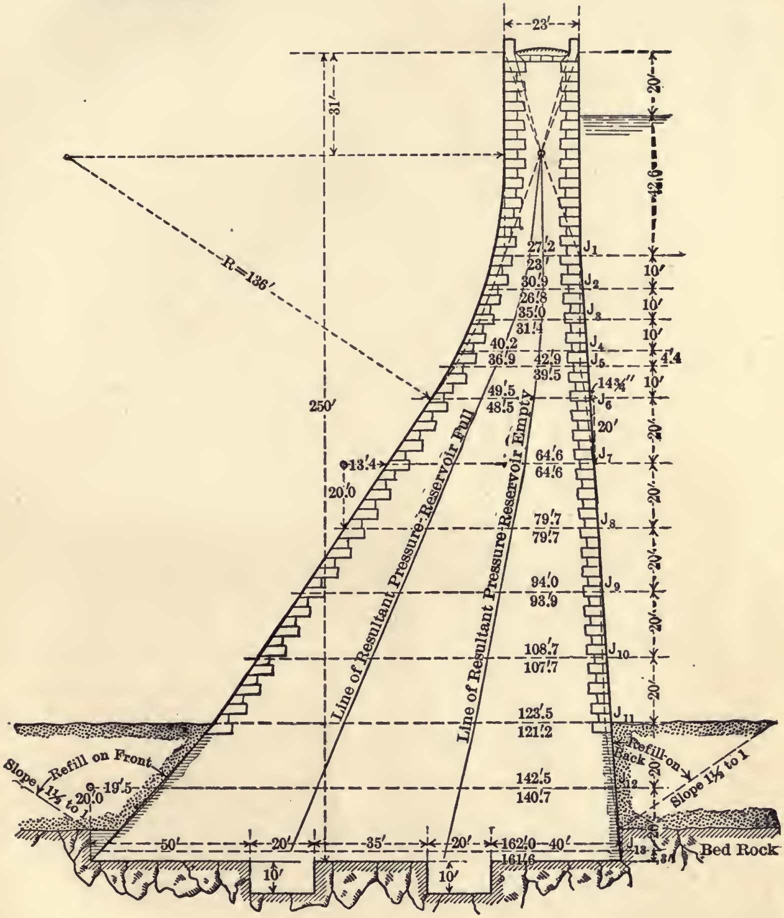


FIG 17.

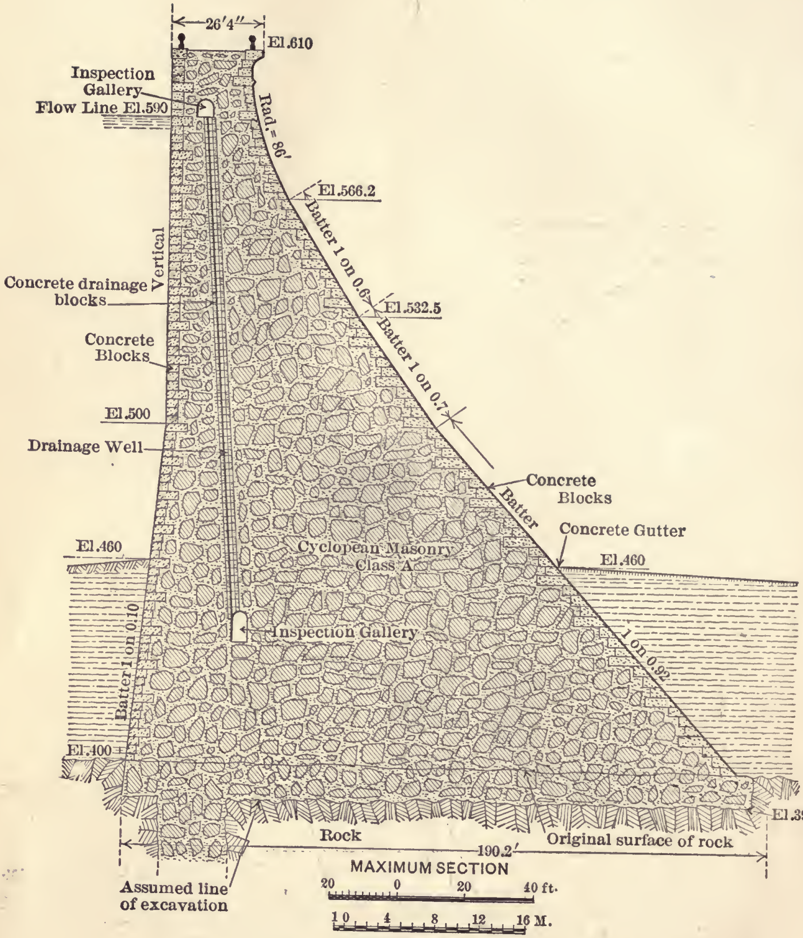


FIG. 18.—Olive Bridge Dam.

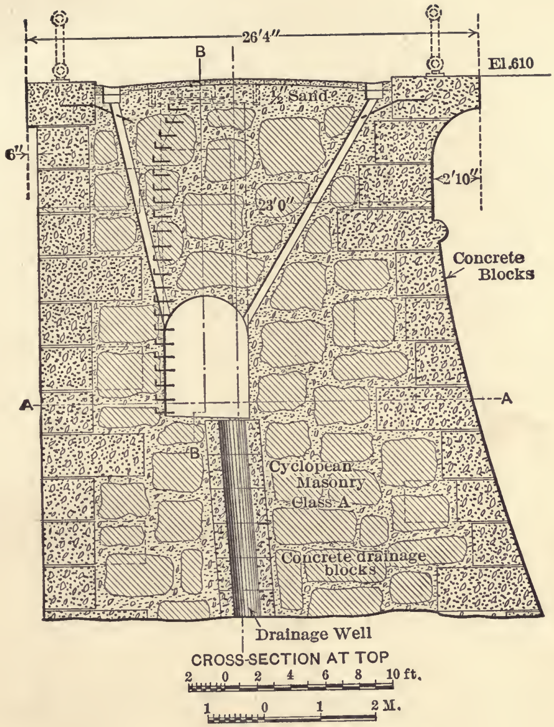

26′4″
B
El.610
½″ Sand
6″
2′10″
23′0″
Concrete
Blocks
A
A
B
Cyclopean
Masonry
Class A
Concrete drainage
blocks
Drainage Well
CROSS-SECTION AT TOP
2 0 2 4 6 8 10 ft.
1 0 1 2 M.

FIG. 19.

pes net N
150