HIGH SCHOOL ALGEBRA

ELEMENTARY COURSE

SLAUGHT & LENNES

To Professor Cajori,
with the compliments
and great esteem of
H. E. Slaught

HIGH SCHOOL ALGEBRA

Elementary Course

BY

H. E. SLAUGHT, Ph.D.

ASSISTANT PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CHICAGO

AND

N. J. LENNES, M.S.

INSTRUCTOR IN MATHEMATICS IN THE WENDELL PHILLIPS HIGH SCHOOL, CHICAGO

Boston

ALLYN AND BACON

1907

PREFACE

THE High School Algebra has been written in two parts, the one designed solely for the first year and the other for a complete review and advanced course at a later period.

The authors recognize the impossibility of combining in a single treatment the qualities necessary for beginners with the more mature point of view suitable for the third or fourth year student.

The important features of the Elementary Course are:

1. *Algebra is vitally and persistently connected with arithmetic.*

Each principle in the book is first studied in its application to numbers in the Arabic notation. The principles of Algebra are thus connected with those already known in arithmetic. Letters are introduced as *abbreviations* for "a number" or "any number." Literal expressions are called number expressions — the vague term "quantity" is not used. In the exercises, Arabic figures are constantly involved in the same manner as letters. Checking by the substitution of particular numbers for letters occurs throughout.

2. *The principles of Algebra used in the Elementary Course are enunciated in a small number of short statements — eighteen in all.*

The purpose of these principles is to furnish in simple form a codification of those operations of Algebra which are sufficiently different from the ones already familiar in arithmetic to require special emphasis. Such a codification has several important advantages:

By constant reference to these few fundamental statements they become an organic and hence permanent part of the learner's mental equipment.

By their systematic use he is made to realize that the processes of Algebra which seem so multifarious and heterogeneous are in reality few and simple. Invaluable training is thus afforded in connecting things which are essentially, though not apparently, related.

Such a body of principles furnishes a ready means for the correction of erroneous notions, a constant incitement to effective review, and a definite basis upon which to proceed at each stage of progress.

No attempt is made at formal demonstration of these principles. It is believed that conviction as to their validity is most effectively brought to a first year pupil by their proper empirical exhibition. *Formal* argumentation is reserved for the Advanced Course.

3. *The main purpose of the Elementary Course is the solution of problems rather than the construction of a purely theoretical doctrine as an end in itself.*

While the subject-matter of Algebra is developed in a logical sequence around the principles, the attempt is made to connect each principle in a vital manner with the learner's experience by using it in the solution of a large number and great variety of simple problems.

In making the problems the following criteria have been observed:

The subject-matter should be easily within the comprehension of the pupil, and, so far as possible, the problems should be such as one would actually need to solve in passing from known to unknown data by means of given relations. Such are problems on physical relations in Chapter IV, and those on geometrical relations in Chapters VI and VII.

However, with the pupil's meagre experience it is impossible that all problems should be of this kind. Hence a considerable body of problems has been introduced which involve artificial relations imposed upon numbers. Such problems are of two kinds:

(*a*) Those involving numbers related to concrete things, as, for example, the problems on pages 44 to 46. The data used in this class of problems (with very few exceptions) are such as have a distinct value or real interest in themselves. Every effort has been made to have the data entirely trustworthy.

(*b*) Those involving numbers not related to concrete things, as, for example, problems 12 to 18, page 43. Such problems are used as the basis for the development of formulas, pages 110 to 114, Chapter IV.

The utmost care has been taken in grading the problems, both as to difficulty and with reference to the principles upon which the solutions depend.

4. *The order of topics and the inclusion and exclusion of subject-matter have been determined by the main purpose of the Elementary Course as just stated.*

The equation, therefore, as the instrument for the solution of problems, occupies the leading place. New topics are introduced only as they are needed in extending the use of the equation. For example, the whole subject of linear equations, including those with two and with three unknown quantities, is treated before long division; and quadratic equations are placed before the formal treatment of literal fractions.

The subject of factoring is introduced when quadratic equations are to be solved, and is immediately used for that purpose in special cases.

Long division of polynomials is first found necessary as an introduction to square root, which together with radicals is essential to the solution of the general quadratic equation.

The topics excluded are: (*a*) complicated factoring, (*b*) H.C.F. by the long division process, (*c*) complicated fractions, (*d*) simultaneous equations in more than three unknowns, (*e*) cube root, (*f*) fractional and negative exponents, (*g*) equations containing complicated radicals, (*h*) simultaneous quadratics except the case of a quadratic and a linear equation.

These omissions have permitted the introduction of a much greater number and larger variety of problems than would otherwise be possible without diminishing the number of drill exercises.

For example, instead of an extended and purely theoretical discussion of radicals, only so much is given as is needed in treating simple quadratic equations; and this is immediately applied to the solution of a body of interesting problems involving such concrete geometrical relations as are freely used in modern grammar school arithmetics.

This larger space allotted to problems makes it possible to pass by slow gradation from extremely simple to more complicated cases which would otherwise be too difficult, and thus to secure the facility and power in interpreting and solving problems which is demanded in physics and other sciences.

Attention is further called to the following features:

The simple and scientific treatment of the solution of the equation as enunciated in Principle VIII, page 36.

The numerous illustrative solutions which have been given in full for the purpose of exhibiting the best methods of attack and the most effective arrangement of the work.

The development of negative numbers from concrete relations, immediately followed by a large number of problems showing their practical use and interpretation.

The introduction of graphs at the *beginning* of the chapter on simultaneous equations, thus making the graph the basis of the study of simultaneous equations.

The notions of a formula and of substitution in a formula are developed in connection with subjects already familiar, such as interest, areas, volumes, etc. (See page 101.) These notions are then applied to the study of other subjects. (See page 115.)

Literal equations are in each case introduced as a generalization of a series of concrete problems immediately preceding. (See page 111.) Such equations are then to be solved for each letter involved. (See page 127.)

Problems involving physical relations are in each case introduced by means of a series of carefully graded problems leading up to the general case. (See page 120.)

If in any case the problems are found to be too numerous, the later sections of Chapter IV may be omitted or postponed until after graphs and simultaneous equations have been studied. Many of the exercises are simple enough to be read in class as mental drill exercises.

Review questions and exercises are given at appropriate intervals throughout the book.

Ratio and proportion are treated in the chapter on fractions preparatory to their use in plane geometry.

The authors desire to express their thanks to many who have given useful suggestions, and especially to Mr. J. A. Dixon, Wendell Phillips High School, Chicago, and to Miss Mabel Sykes, South Chicago High School, Chicago, who have solved all the problems and exercises and whose criticisms have been of special value. Thanks are also due to Mr. Edwin Hand, Jr., for helping to prepare the cuts.

H. E. SLAUGHT.
N. J. LENNES.

CHICAGO, May, 1907.

CONTENTS

CHAPTER I

INTRODUCTION TO THE EQUATION

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

CHAPTER III

INVOLVED NUMBER EXPRESSIONS

CHAPTER IV

SOLUTION OF PROBLEMS

CHAPTER V

INTRODUCTION TO SIMULTANEOUS EQUATIONS

CHAPTER VI

SPECIAL PRODUCTS AND FACTORS

CHAPTER VII

QUOTIENTS AND SQUARE ROOTS

CHAPTER VIII

FRACTIONS WITH LITERAL DENOMINATORS

HIGH SCHOOL ALGEBRA

ELEMENTARY COURSE

CHAPTER I

INTRODUCTION TO THE EQUATION

ADDITION OF NUMBERS HAVING A COMMON FACTOR

1. Algebra like arithmetic deals with **numbers.** The numbers of algebra include those used in arithmetic, and also other numbers which are defined as need arises for them. The symbols employed to represent numbers, and the principles used in operating upon them, are introduced by means of illustrative problems as we proceed.

2. Illustrative Problem. The shortest railway route from Chicago to New York is 912 miles. How long does it take a train averaging 38 miles an hour to make the journey?

Solution in words. The product of the average number of miles per hour and the required number of hours equals the whole distance traveled. That is, 38 multiplied by "the required number of hours" equals 912. Hence "the required number of hours" is one thirty-eighth of 912, or 24.

Solution using abbreviations. If, instead of the expression "the required number of hours," we use the word "time," or simply the abbreviation t, the solution may be written:

$$38 \times t = 912.$$

Hence $$t = 912 \div 38 = 24.$$

Illustrative Problem. If in the above problem a train makes the journey in 18 hours, find the average number of miles per hour.

Solution. In words we have, as before, 18 multiplied by "the average number of miles per hour" equals 912. Hence "the average number of miles per hour" equals one-eighteenth of 912, or $50\frac{2}{3}$.

Using for the expression "the average number of miles per hour," the word "rate," or simply the abbreviation r, the solution reads:

$$18 \times r = 912.$$
$$r = 912 \div 18 = 50\tfrac{2}{3}.$$

It should be clearly understood that t and r represent *numbers* whose values are unknown at the outset, but which become known at the conclusion of the solution.

3. Abbreviations representing numbers and operations such as are used in these problems occur constantly in algebra. In fact, the systematic use of such abbreviations is one of the chief distinctions between arithmetic and algebra.

4. The signs $+$, $-$, $\times$, $\div$, and $=$ are used in algebra with the same meaning that they have in arithmetic. However, instead of $\times$ a point written above the line is often used. Thus $2 \cdot 3$ means 2×3.

The product of two numbers represented by letters is generally indicated by writing the letters consecutively with no sign between them.

Thus rt means the same as $r \cdot t$ or $r \times t$. For example, in such problems as those just given we would write $rt = d$, meaning "the number of miles per hour multiplied by the number of hours equals the distance or number of miles traveled."

Similarly $2\,r$ means $2 \cdot r$, but (as in arithmetic) 25 means $20 + 5$, not $2 \cdot 5$.

A fraction is often used to indicate division

Thus $$\frac{2}{3} = 2 \div 3;\ \frac{r}{t} = r \div t.$$

PROBLEMS

In the same manner as above solve the following problems, first by writing out the solution in words and then by using such abbreviations as may be found convenient.

1. Five times a certain number equals 80. What is the number? Use n for the number.

2. Twelve times a number equals 132. What is the number?

3. A tank holds 750 gallons. How long will it take a pipe discharging 15 gallons per minute to fill the tank?

4. The cost of paving a block on a certain street was $7 per front foot. How long was the block if the total cost was $4620?

5. A city lot sold for $7500. What was the frontage if the selling price was $225 per front foot?

6. An encyclopedia contains 18,000 pages. How many volumes are there if they average 750 pages to the volume?

7. What is the cost of fencing per rod if it requires $256 to fence a quarter section of land?

8. A square court 60 feet on a side is to be paved with square tiles. How many square feet in each tile if the pavement requires 1600 tiles?

9. An excavation for a building is to be 130 feet long, 80 feet wide, and 9 feet deep. If 900 cubic feet are removed each day, how long will it require to complete it?

10. A railway embankment which contains 48,900 cubic yards of earth was completed in 163 days. At what rate was it filled in?

11. Taking the length of the earth's orbit as 584 million miles, find how far the earth travels in one day; also in one hour.

5. **Definition.** If a number is the product of two or more numbers, then these numbers are called **factors** of the given number.

E.g. The integral factors of 10 are 1 and 10 or 2 and 5. $2\frac{1}{2}$ and 4 are also factors of 10.

6. Illustrative Problem. Divide the number 84 into three parts such that the second part is five times the first and the third part is eight times the first.

Solution. Using the abbreviation p for "the first part of the number," we have

$$1 \cdot p \text{ or } p = \text{the first part,}$$
$$5p = \text{the second part,}$$
$$8p = \text{the third part.}$$

Since the sum of the three parts is 84,

$$p + 5p + 8p = 84.$$

To complete the solution of this problem it is necessary to find the sum of the numbers p, $5p$, and $8p$ without knowing what number is represented by p itself. If we suppose this sum to be $14p$, then

$$14p = 84$$

and

$$p = 84 \div 14 = 6.$$

This supposition leads to the correct result since, if $p = 6$, then $p + 5p + 8p = 6 + 5 \cdot 6 + 8 \cdot 6 = 6 + 30 + 48 = 84$. Hence the three parts into which 84 is divided are 6, 30, and 48.

7. The process here used for adding p, $5p$, and $8p$ is a *new method of adding* which is of very great importance in algebra. This method is further exhibited by the following examples:

(1) To add 18, 42, 54, and 30, we first factor these numbers so as to show the common factor 6 and then add the remaining factors 3, 7, 9, and 5, and multiply this sum by the common factor 6. The result is $24 \cdot 6$, or 144. The work may be arranged as follows:

$$\begin{aligned} 18 &= 3 \cdot 6 \\ 42 &= 7 \cdot 6 \\ 54 &= 9 \cdot 6 \\ 30 &= 5 \cdot 6 \\ \hline 144 &= 24 \cdot 6 \end{aligned}$$

Similarly (2)

$$\begin{aligned} 16 &= 1 \cdot 16 = 2 \cdot 8 = 4 \cdot 4 \\ 64 &= 4 \cdot 16 = 8 \cdot 8 = 16 \cdot 4 \\ 32 &= 2 \cdot 16 = 4 \cdot 8 = 8 \cdot 4 \\ 48 &= 3 \cdot 16 = 6 \cdot 8 = 12 \cdot 4 \\ \hline 160 &= 10 \cdot 16 = 20 \cdot 8 = 40 \cdot 4 \end{aligned}$$

From example (2) we see that in case the numbers to be added have two or more common factors it does not matter which one is selected. If each number is separated into *two* factors one of which is a common factor, then it is evident that the sum in any case is found by multiplying the common factor by the sum of the other factors.

In the above manner add the following sets of numbers and show in each case by adding in the ordinary way that the result is correct:

14	36	32	17	39	32
28	72	64	34	52	48
35	48	128	68	78	40
70	12	256	85	91	72

8. **Definition.** If a number is the product of two factors, then either of these factors is called the **coefficient** of the other in that number.

E.g. In $2 \cdot 3$, 2 is the coefficient of 3, and 3 is the coefficient of 2. In $9\,rt$, 9 is the coefficient of rt, r is the coefficient of $9\,t$, and t is the coefficient of $9\,r$. In such expressions as $9\,rt$ the factor represented by Arabic figures is usually regarded as the coefficient.

The preceding examples illustrate the following principle:

9. **Principle I.** *To add numbers having a common factor, add the coefficients of the common factor and multiply the sum by the common factor.*

If n represents some number, then $4\,n$, in the same discussion, means four times that number, and $5\,n$ means five times that number. Hence, by Principle I, $4\,n + 5\,n = 9\,n$, which is true no matter what number is represented by n.

By means of this principle perform the following additions, understanding that each letter represents some number:

1. $8x + 7x + 16x + 2x$.
2. $13n + 8n + 7n + 9n$.
3. $3a + a + 2a + 6a$.
4. $7b + 8b + b + 6b$.
5. $8t + 7t + 5t$.
6. $3r + 5r + 11r$.

Show the correctness of the result in each of the above by letting $x=2$, $n=1$, $a=4$, $b=3$, $t=6$, and $r=7$. Try also other values for the letters.

Such a test for the correctness of an operation is called a **check.**

10. Definitions. Combinations of Arabic figures or letters, or both, by means of the signs of operation, $+$, $-$, etc., are called **number expressions.**

E.g. 38, $18r$, $p+5p+8p$, are number expressions.

Two number expressions representing the same number, when connected by the sign $=$, form an **equality.**

The expressions thus connected are called the **members** of the equality and are distinguished as the **right** and **left** members.

Equalities, such as $8t+7t=15t$, in which the letters may be any numbers whatever, are called **identities.** Equalities of the type $p+5p+8p=84$ are called **equations.** (See §§ 29–34.)

EXERCISES

1. Add $5t$, $11t$, $20t$, and $47t$. Check the result by letting $t=3$; also $t=50$, and $t=150$.

2. Add $7r$, $23r$, $28r$, $52r$, and $117r$. Check the result for $r=11$, $r=20$, and $r=1$.

3. Add $3rt$, $7rt$, $65rt$, and $16rt$. Check for $r=1$, $t=2$.

4. Add $1\frac{1}{2}n$, $2\frac{2}{3}n$, $3\frac{1}{6}n$, and check for $n=6$.

5. Add 66, 88, 99, and 121 by Principle I.

6. Add 144, 96, 120, and $50\cdot 12$ by Principle I.

7. Add $5\cdot 40$, $8\cdot 60$, and $6\cdot 20$ by Principle I.

8. Add $5ax$, $3ax$, and $7ax$. Check for $a=2$, $x=4$.

9. Add $7ax$, $3bx$, and $12cx$.

In this case the common factor is x. Hence using Principle I, $7ax+3bx+12cx=(7a+3b+12c)x$. The parenthesis here indicates that the numbers represented by x and the expression $7a+3b+12c$ are to be multiplied.

SOLUTION OF PROBLEMS

11. One great object in the study of algebra is to **simplify the solution of problems.** This is done by using letters to represent the unknown numbers, by stating the problem in the form of an equation, and by arranging the successive steps of the solution in an orderly manner.

Skill in translating problems into equations depends upon attention to the following points:

(1) *Read and understand* clearly the statement of the problem, as it is given in words.

(2) *Select the unknown number*, and represent it by a suitable letter, say the initial letter of a word which will keep its meaning in mind. If there are more unknown numbers than one, try to express the others in terms of the one first selected.

(3) Find two number expressions which, according to the problem, represent the same number, and set them equal to each other, *thus forming an equation.*

These steps are exhibited in the following solution:

A tree 108 feet high was broken off by the wind so that the part left above the first branch was three times as long as the part broken off, and the part below the first branch was twice as long as the part broken off. How long was the part broken off?

Solution. Let b represent the number of feet broken off.

Then $3b$ is the number of feet left above the first branch,

and $2b$ is the number of feet below the first branch.

Hence, $b + 3b + 2b$ and 108 are number expressions, each representing the total height of the tree.

Therefore $b + 3b + 2b = 108.$ (1)

By Principle I, $6b = 108.$ (2)

Then $b =$ one sixth of 108, or 18. (3)

Hence, the part broken off was 18 feet long.

Equation (3) is derived from (2) by dividing both members by 6.

PROBLEMS

1. The greater of two numbers is 5 times the less, and their sum is 180. What are the numbers?

2. A number increased by twice itself, 4 times itself, and 6 times itself, becomes 429. What is the number?

3. A father is 3 times as old as his son, and the sum of their ages is 48 years. How old is each?

4. In a company there are 39 persons. The number of children is twice the number of grown people. How many are there of each?

5. A and B receive $45 for doing a certain piece of work. If A gets 4 times as much as B, how much does each receive?

6. The population of Tokio is twice that of Canton, and the sum of their populations is 2,700,000. How many inhabitants in each city?

7. Find two consecutive integers whose sum is 133.

8. The area of Louisiana is (nearly) 4 times that of Maryland, and the sum of their areas is 60,930 square miles. Find the (approximate) area of each state.

9. The horse-power of a certain steam yacht is 12 times that of a motor boat. The sum of their horse-powers is 195. Find the horse-power of each.

10. There are three circles on the blackboard. The circumference of the second is 5 times that of the first, and the circumference of the third is 10 times that of the first. The sum of their circumferences is 16 feet. Find the circumference of each.

11. At a football game there were 2000 persons. The number of women was 3 times the number of children, and the number of men was 6 times the number of children. How many men, women, and children were there?

12. The population of Portland, Oregon (estimate of the Census Bureau, 1904), was twice that of Dallas, Texas, and the population of Toledo was 3 times that of Dallas. The three cities together had 300 thousand inhabitants. How many were there in each city?

Let n = number of thousands of inhabitants in Dallas.
Then $2n$ = number of thousands of inhabitants in Portland,
and $3n$ = number of thousands of inhabitants in Toledo.
Hence $n + 2n + 3n = 300.$

13. It is twice as far from New York to Syracuse as from New York to Albany, and it is 4 times as far from New York to Cleveland as from New York to Albany. The sum of the three distances is 1015 miles. Find each distance.

14. In Maryland there were (census of 1900) 4 times as many whites as negroes. The total population was 1185 thousand. How many of each were there?

If n equals the number of thousands of negroes, then the equation is $n + 4n = 1185$.

SUBTRACTION OF NUMBERS HAVING A COMMON FACTOR

12. Numbers having a common factor may be subtracted in a manner similar to the process exhibited under Principle I.

Thus, from	$64 = 8 \cdot 8$	From $84 = 12 \cdot 7$	From	$17n$
subtract	$48 = 6 \cdot 8$	subtract $49 = 7 \cdot 7$	subtract	$6n$
Remainder	$16 = 2 \cdot 8$	$35 = 5 \cdot 7$		$11n$

In like manner perform the following subtractions:

1. $9 \cdot 7 - 3 \cdot 7.$	5. $6 \cdot 99 - 5 \cdot 99.$	9. $6n - 2n.$
2. $10 \cdot 4 - 6 \cdot 4.$	6. $20 \cdot 19 - 13 \cdot 19.$	10. $6 \cdot 50 - 2 \cdot 50.$
3. $8 \cdot 8 - 2 \cdot 8.$	7. $8a - 3a.$	11. $10b - 4b.$
4. $5 \cdot 11 - 3 \cdot 11.$	8. $8 \cdot 5 - 3 \cdot 5.$	12. $7a - 4a.$

These examples illustrate the following principle:

13. Principle II. *To find the difference of two numbers having a common factor, subtract the coefficients of the common factor and multiply the result by the common factor.*

Illustrative Problem. If thirteen times a certain number diminished by eight times the number equals 75, what is the number?

Solution. Let n represent the required number.

Then $13n - 8n$ and 75 are expressions representing the same number.

Hence, $13n - 8n = 75$.

By Principle II, $5n = 75$.

Dividing each member by 5, $n = 15$, the required number.

Check. $13 \cdot 15 - 8 \cdot 15 = 195 - 120 = 75$.

EXERCISES AND PROBLEMS

1. By means of Principle II subtract 72 from 160; 50 from 300; 39 from 78; 34 from 85; 58 from 174; and 69 from 161.

2. Subtract $109 \cdot 87$ from $209 \cdot 87$ by Principle II. Check by first finding the products and then subtracting as in arithmetic.

Perform the following indicated operations and check those in which letters are involved by substituting convenient numbers:

3. $68t - 11t$.

4. $15n + 25n - 18n$.

5. $70x - 15x + 7x - 23x$.

6. $18 \cdot 7 - 3 \cdot 7 - 2 \cdot 7 + 6 \cdot 7$.

7. $3 \cdot 4n + 5 \cdot 4n + 11 \cdot 4n - 7 \cdot 4n$.

8. $13rt + 16rt + 3rt - 20rt$.

9. $144 - 96 + 50 \cdot 12 - 20 \cdot 12$.

10. $11ax - 3ax + 4ax$.

11. $11\,ax - 3\,bx + 4\,cx$.

12. $11 \cdot 9 - 6 \cdot 9 + 3 \cdot 9$.

13. $20\,n - 6\,n + 2\,n$.

14. $an - bn + cn$.

15. $5\,t + 20\,t - 3\,t$.

16. $8\,s - 3s + 20\,s$.

17. $6\,a - 4\,a + 3\,a - 2\,a$.

18. $11\,rs - 2\,rs + 4\,rs$.

19. $ar + br - cr$.

20. $3\,ry - 2\,sy - ty$.

21. $11 \cdot 17 + 47 \cdot 17 - 8 \cdot 17$.

22. $axy + bxy - 3\,xy$.

23. $3\,abc + 7\,abc - 2\,abc$.

24. $7 \cdot 5\,x - 3 \cdot 5\,x + 8 \cdot 5\,x$.

25. $a \cdot 2\,r + b \cdot 2\,r - c \cdot 2\,r$.

26. $2\,ar + 2\,br - 2\,cr$.

27. Four times a certain number plus 3 times the number minus 5 times the number equals 48. What is the number?

28. One number is 4 times another, and their difference is 9. What are the numbers?

29. Find a number such that when 4 times the number is subtracted from 12 times the number the remainder is 496.

30. The population of Ohio (1901) was twice that of Wisconsin. The difference of their populations was 2100 thousand. Find the population of each state.

31. The population of Illinois in 1903 was 5 times as great as that of West Virginia. The difference between their populations was 4080 thousand. What was the population of each?

32. There are three numbers such that the second is 11 times the first and the third is 27 times the first. The difference between the second and the third is 64. Find the numbers.

33. A cubic foot of asphaltum is twice as heavy as a cubic foot of light anthracite coal. Seven cubic feet of coal weigh 390 pounds more than 1 cubic foot of asphaltum. Find the weight per cubic foot of each.

34. Thirty-nine times a certain number, plus 19 times the number, minus 56 times the number, plus 22 times the number, equals 12. Find the number.

MULTIPLICATION OF A PRODUCT

14. Illustrative Problem. Three men, A, B, and C, invest together \$ 33,000. B puts in twice as much as A, and C 4 times as much as B. How much does each invest?

Solution. Let d represent the number of dollars invested by A. Then $2d$ represents B's investment, and $4(2d)$ represents C's investment.

Hence, $d + 2d + 4(2d) = 33000.$

To complete the solution of this problem it is necessary to multiply $2d$ by 4 without knowing what number is represented by d. If we suppose the product to be $8d$,

then $d + 2d + 8d = 33000.$

By Principle I, $11d = 33000.$

Dividing each member by 11, $d = 3000$, A's investment;

$2d = 6000$, B's investment;

$4 \cdot 2d = 24000$, C's investment.

The supposition that $4(2d) = 8d$ is justified by the fact that the numbers thus found satisfy the conditions of the problem.

That is, $d + 2d + 4(2d) = 3000 + 6000 + 24000 = 33000.$

15. The process here used for multiplying $2d$ by 4 is of great importance in algebra.

The following examples further exhibit this *new method of multiplying:*

1. $4(3 \cdot 5) = 4 \cdot 15 = 60$
Also $4(3 \cdot 5) = 12 \cdot 5 = 60$
And $4(3 \cdot 5) = 3 \cdot 20 = 60$

2. $2(3 \cdot 4 \cdot 5) = 2 \cdot 60 = 120$
Also $2(3 \cdot 4 \cdot 5) = 6 \cdot 4 \cdot 5 = 120$
And $2(3 \cdot 4 \cdot 5) = 3 \cdot 8 \cdot 5 = 120$
And $2(3 \cdot 4 \cdot 5) = 3 \cdot 4 \cdot 10 = 120$

In like manner find the following products in two or more ways:

3. $6(2 \cdot 3)$.
4. $4(25 \cdot 99)$.
5. $2(5 \cdot 149)$.
6. $4(19 \cdot 5)$.
7. $20(5a \cdot 4)$.
8. $16(4x \cdot 7)$.
9. $8(4x \cdot 3)$.
10. $9(xy \cdot 5)$.

These examples illustrate the following principle:

16. Principle III. *To multiply the product of several factors by a given number, multiply any one of the factors by that number, leaving the others unchanged.*

EXERCISES AND PROBLEMS

Multiply as many as possible of the following in two or more ways. Check where letters are involved.

1. $7(3 \cdot 4 \cdot 5)$.	7. $4(19 \cdot 25)$.	13. $2(rs \cdot 16)$.
2. $8(7 \cdot 2 \cdot 3)$.	8. $5(17 \cdot 20 \cdot 3)$.	14. $5(xy \cdot 5z)$.
3. $9(2 \cdot 3 \cdot 4)$.	9. $15(7ab)$.	15. $7(3 \cdot 4ab)$.
4. $5(2ab)$.	10. $3(4mn)$.	16. $9(a \cdot 5 \cdot xy)$.
5. $3(5xy)$.	11. $5(abc)$.	17. $4(125 \cdot 17)$.
6. $12(8 \cdot 4 \cdot 20)$.	12. $7(2xy)$.	18. $40(25 \cdot 29)$.

19. There are three numbers whose sum is 80. The second is 3 times the first and the third is twice the second. What are the numbers?

20. There are three numbers such that the second is 8 times the first and the third is 3 times the second. If the second is subtracted from the third the remainder is 48. Find the numbers.

21. The population of Bridgeport, Connecticut, is twice that of Butte, Montana. Three times the population of Bridgeport plus twice that of Butte equals 320 thousand. Find the population of each city.

22. It is 4 times as far from New York City to Cincinnati as from New York to Baltimore. Twice the distance from New York to Cincinnati minus 5 times that from New York to Baltimore equals 567 miles. How far is it from New York to each of the other cities?

23. The population of Hartford, Connecticut, is 3 times that of Oshkosh, Wisconsin. Four times the population of Hartford plus 5 times that of Oshkosh equals 510 thousand. Find the population of each city.

24. One cubic inch of emery weighs twice as much as 1 cubic inch of ivory. The combined weight of 10 cubic inches of each substance is 2.1 pounds. Find the weight per cubic inch of each.

25. It is twice as far from Boston to Quebec as from Boston to Albany and 3 times as far from Boston to Jacksonville, Florida, as from Boston to Quebec. How far is it from Boston to each of the other three cities, the sum of the distances being 1818 miles?

26. A cubic inch of porcelain china is twice as heavy as a cubic inch of ebony, and a cubic inch of rolled zinc is 3 times as heavy as a cubic inch of porcelain. The combined weight of 1 cubic inch of each substance is .387 pounds. Find the weight per cubic inch of each.

MULTIPLICATION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

17. Illustrative Problem. The length and width of a rectangle together equal 58 inches. If the width were 5 inches greater, the length of the rectangle would then be twice its width. Find its dimensions.

Solution. Let w represent the number of inches in the width. If it were 5 inches wider, it would then be $w + 5$ inches. Hence the length is $2(w + 5)$, the parenthesis indicating that the sum of w and 5 is to be found and the result multiplied by 2.

Hence the sum of the length and width is

$$w + 2(w + 5) = 58.$$

The solution of this problem involves the multiplication of the number $(w + 5)$ by 2 without knowing what number is represented by w.

If we suppose that $2(w+5) = 2w + 10$,

then we have $w + 2w + 10 = 58.$ (1)

By Principle I, $3w + 10 = 58.$ (2)

Hence $3w = 48,$ since 58 is 10 more than $3w$, (3)

and $w = 16,$ the width. (4)

Then $58 - 16 = 42,$ the length.

The correctness of the above process is shown by the fact that the numbers 42 and 16 fulfill the conditions of the problem;

that is, $2(16+5) = 2 \cdot 21 = 42,$

and $16 + 42 = 58.$

Equation (3) is derived from (2) by subtracting 10 from each member.

18. The multiplication of the number $(w+5)$ by 2 without knowing in advance what number is represented by w is a *new method of multiplying* which is constantly used in algebra. The following examples further exhibit this method:

(1) $4(2+7) = 4 \cdot 9 = 36,$

or $4(2+7) = 4 \cdot 2 + 4 \cdot 7 = 8 + 28 = 36.$

(2) $3(3+8+9) = 3 \cdot 20 = 60,$

or $3(3+8+9) = 3 \cdot 3 + 3 \cdot 8 + 3 \cdot 9 = 9 + 24 + 27 = 60.$

It is thus seen that in each case the same result is obtained whether we first add the numbers in the parenthesis and then multiply the sum or first multiply the numbers in the parenthesis one by one and then add the products.

Multiply each of the following in two ways where possible:

1. $3(2+7)$.
2. $5(3+4+5)$.
3. $8(5+9+7)$.
4. $7(6+11+9+9)$.
5. $3(a+6)$.
6. $11(h+k)$.
7. $4(5a+7b+c)$.
8. $a(5+4+7)$.
9. $x(3+7+10)$.
10. $15(x+y+z)$.
11. $20(m+n+p)$.
12. $9(2r+3s+t)$.

19. In a manner similar to the above the difference of two numbers represented by Arabic figures may be multiplied by a given number in either of two ways.

E.g. $6(8-3)=6\cdot5=30,$

or $6(8-3)=6\cdot8-6\cdot3=48-18=30.$

The same result is obtained whether we first perform the subtraction indicated in the parenthesis and then multiply the difference, or first multiply the numbers separately and then subtract the products. In the case of numbers represented by *letters* evidently the second process only is available.

E.g. $6(r-t)=6r-6t.$

Perform as many as possible of the following multiplications in two ways:

1. $7(9-2)$.	4. $17(18-11)$.	7. $5(x-1)$.	10. $m(r-s)$.
2. $12(17-7)$.	5. $9(a-2)$.	8. $3(y-2)$.	11. $x(y-z)$.
3. $5(12-8)$.	6. $8(h-4)$.	9. $a(c-d)$.	12. $t(u-v)$.

The second of the above methods is needed in problems like the following:

Illustrative Problem. The rate of an express train plus that of a freight is 70 miles per hour. If the rate of the freight were 7 miles less, the express would be going twice as fast as the freight. Find the rate of each.

Solution. Suppose the rate of the freight is now r miles per hour.

Then $r-7$ is the supposed rate of the freight,

and $2(r-7)$ is the present rate of the express.

Hence $r+2(r-7)=70$, the sum of the rates, (1)

and $r+2r-14=70.$ (2)

By Principle I, $3r-14=70.$ (3)

Then $3r = 84$, since 70 is 14 less than $3r$. (4)

Hence $r = 28$, the rate of the freight,

and $2(28 - 7) = 2 \cdot 21 = 42$, the rate of the express.

Check. $42 + 28 = 70$.

Equation (4) is derived from (3) by adding 14 to both members and observing that $14 - 14 = 0$.

The foregoing examples illustrate the following principle:

20. Principle IV. *To multiply the sum or difference of two numbers by a given number, multiply each of the numbers separately by the given number, and add or subtract the products.*

21. Principles III and IV should be carefully contrasted, as in the following example:

$$2(2 \cdot 3 \cdot 5) = 4 \cdot 3 \cdot 5 = 2 \cdot 6 \cdot 5 = 2 \cdot 3 \cdot 10,$$

but
$$2(2 + 3 + 5) = 4 + 6 + 10.$$

In multiplying the product of several numbers we operate upon *any one of them,* but in multiplying the sum or difference of numbers we operate upon *each of them.*

EXERCISES AND PROBLEMS

1. Multiply $5 + 7 + 11$ by 3 without first adding, and then check by performing the addition before multiplying.

2. Multiply $m + n$ by 4 and check for $m = 5$, $n = 7$.

$$4(m + n) = 4m + 4n$$

Check. $4(5 + 7) = 4 \cdot 12 = 48$, also

$$4 \cdot 5 + 4 \cdot 7 = 20 + 28 = 48.$$

3. Multiply $r + s + x$ by a and check for $r = s = x = a = 2$.

4. Multiply $a + b + c$ by m and check for $a = 1$, $b = 3$, $c = 5$, $m = 4$.

5. Multiply $x + y$ by r and check for $x = 2$, $y = 4$, $r = 6$.

Where letters are involved, check the results in the following by substituting convenient values:

6. $8(13-5)$.
7. $71(12+41-36)$.
8. $9(a-b+8)$.
9. $3(a+x+y-17)$.
10. $m(a+b-c)$.
11. $a(18-7)$.
12. $2(13+8+9-21)$.
13. $7(3+8+9-a)$.
14. $5(17+a-b)$.
15. $32(x-y+z)$.
16. $8(2a-3b+4c)$.
17. $37(3x-2y-z)$.
18. $13(5r-3x+t)$.
19. $20(2x+3x-5y)$.
20. $7(11s-2s+3t)$.
21. $35(x-2y+3z)$.
22. $78(10m+11n+2r)$.
23. $4(25x+32x-y)$.
24. $3(13x+14y-t)$.
25. $7(4a-3b+c)$.

26. Find two consecutive integers such that 3 times the first plus 7 times the second equals 217.

27. Find two consecutive integers such that 7 times the first plus 4 times the second equals 664.

28. There are three numbers such that the second is 17 less than the first, and the third is 8 times the second. The sum of the first and third is 89. What are the numbers?

29. The number of representatives and senators together in the United States Congress is 476. The number of representatives is 26 more than 4 times the number of senators. Find the number of each.

30. The area of Illinois is 6750 square miles more than 10 times that of Connecticut. The sum of their areas is 61,640 square miles. Find the area of each state.

31. The sum of the horse-powers of the steamships *Campania* and *Mauritania* is 102 thousand. The *Mauritania* has 12 thousand horse-power more than twice that of the *Campania*. What is the horse-power of each ship?

32. The population of Wyoming (census of 1900) was 50 thousand more than that of Nevada, and the population of Utah was 93 thousand more than twice that of Wyoming. The population of Utah was 277 thousand. Find the population of Nevada and Wyoming.

33. It is 73 miles farther from Newark to Philadelphia than from New York to Newark, and it is one mile more than 10 times as far from Philadelphia to Chicago as from Newark to Philadelphia. The sum of the distances from New York to Newark and from Philadelphia to Chicago is 830 miles. Find each of the three distances, and the total distance from New York to Chicago.

Let d = number of miles from New York to Newark.
Then $d + 73$ = number of miles from Newark to Philadelphia,
and $10(d + 73) + 1$ = number of miles from Philadelphia to Chicago.
Hence $d + 10(d + 73) + 1 = 830.$

34. In the championship season of 1906 the Chicago National League baseball team lost 20 games less than New York, and Pittsburg lost 12 less than twice as many games as Chicago. Pittsburg and New York together lost 116 games. How many did each of the three teams lose?

35. Pikes Peak is 3282 feet higher than Mt. Ætna, and Mt. Everest is 708 feet more than twice as high as Pikes Peak. The sum of the altitudes of Mt. Ætna and Mt. Everest is 39,867 feet. Find the altitude of each of the three mountains.

36. The altitude of Chimborazo is 22,820 feet less than 3 times that of Mt. Shasta. What is the altitude of each mountain if Chimborazo is 6060 feet higher than Mt. Shasta?

Let x = the number of feet in the altitude of Mt. Shasta. Then $3x - 22820$ = altitude of Chimborazo, and $3x - 22820 - x = 6060.$

37. The distance of Mars from the sun is 39 million miles less than 5 times as great as that of Mercury from the sun. Mars is 105 million miles farther from the sun than Mercury. What is the distance of each planet from the sun?

38. The population of New York City (estimate of Census Bureau, 1904) was 5 thousand more than 11 times that of Pittsburg. If 21 times the population of Pittsburg is subtracted from twice that of New York, the remainder is 363 thousand. Find the population of each city.

39. The standing army of France (1906) was 383 thousand less than twice that of Germany. If 7 times the number of men in the German army is subtracted from 6 times the number of men in the French army, the remainder is 177 thousand. Find the number of men in each army.

DIVISION OF A PRODUCT

22. The division of the product of several factors by a given number may be performed in various ways:

E.g. $(4 \cdot 6 \cdot 10) \div 2 = 240 \div 2 = 120.$

Also $(4 \cdot 6 \cdot 10) \div 2 = 2 \cdot 6 \cdot 10 = 120,$

$(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 3 \cdot 10 = 120,$

and $(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 6 \cdot 5 = 120.$

In each case only one factor is divided, and since any factor may be selected, we naturally choose one which exactly contains the divisor if possible.

Perform each of the following divisions in more than one way where possible:

1. $(5 \cdot 8 \cdot 3) \div 2.$
2. $20\,abc \div 4.$
3. $12\,abc \div 3.$
4. $(11 \cdot 20 \cdot 16) \div 4.$
5. $14\,xyz \div 7.$
6. $12\,abc \div c.$
7. $(10 \cdot 35 \cdot 3) \div 5.$
8. $14\,xyz \div x.$
9. $(12 \cdot 40 \cdot 13) \div 8.$

These examples illustrate the following principle:

23. Principle V. *To divide the product of several factors by a given number divide any one of the factors by that number, leaving the other factors unchanged.*

Principle V is already known in arithmetic in the process called cancellation.

Thus, in the fraction $\frac{2 \cdot 6 \cdot 9}{3}$, 3 may be canceled out of either 6 or 9, giving $\frac{2 \cdot 6 \cdot 9}{3} = 2 \cdot 2 \cdot 9$ or $2 \cdot 6 \cdot 3$.

Principle V is necessary in the solution of problems like the following:

Illustrative Problem. There are three numbers whose sum is 26. The second is 8 times the first, and the third is one-half the second. Find each of the numbers.

Solution. Let x represent the first number,

then $8x$ represents the second number,

and $\frac{8x}{2}$ represents the third number.

Hence, $x + 8x + \frac{8x}{2} = 26$, the sum of the numbers.

By Principle V, $x + 8x + 4x = 26$, since $8x \div 2 = 4x$.

By Principle I, $13x = 26$.

Hence, $x = 2$, the first number,

$8x = 16$, the second number,

and $\frac{8x}{2} = \frac{8 \cdot 2}{2} = 8$, the third number.

EXERCISES AND PROBLEMS

1. Multiply $2x + 3y$ by 5. Check for $x = 7$, $y = 11$.

2. Multiply $3a + y$ by 8. Check for $a = 2$, $y = 3$.

3. Divide $3 \cdot 7 \cdot 18$ by 6 by means of Principle V.

4. Multiply $7 \cdot 56 \cdot 5$ by 6 by means of Principle III.

5. Divide $21 \cdot 36 \cdot 42$ by 7, leaving the result in two different forms.

6. Multiply $5 \cdot 13 \cdot 27$ by 3, leaving the result in three different forms.

7. Divide $7\,a \cdot 14\,b \cdot 21\,c$ by 7 in three different ways.

8. Add $5\,a$, $\frac{12\,a}{2}$, $\frac{21\,a}{3}$, and $\frac{18\,a}{6}$, using Principles V and I.

9. From $\frac{28\,xy}{4}$ subtract $\frac{21\,xy}{7}$, using Principles V and II.

10. From $\frac{14\,a}{2}+\frac{10\,a}{5}$ subtract $\frac{6\,a}{3}$.

11. Find the sum of $\frac{16\,x}{8}$, $\frac{20\,x}{5}$, $\frac{16\,x}{4}$, $7\,x$, and $3\,x$.

12. Find the sum of $\frac{100\,rs}{10}$, $\frac{90\,rs}{9}$, and $\frac{25\,rs}{5}$.

13. From $25\,xy$ subtract $\frac{13\,xyz}{z}$.

14. Add $\frac{18\,abc}{a}$, $20\,bc$, and $\frac{30\,bc}{10}$.

15. Add $\frac{17\,rst}{s}$, $\frac{18\,art}{a}$, and $\frac{20\,rtu}{u}$.

16. From $\frac{150\,xy}{y}+17\,x$ subtract $\frac{79\,xz}{z}$.

17. From $179\,m+\frac{39\,mn}{n}$ subtract $\frac{25\,am}{a}$.

18. From $\frac{24\,(a+b)}{6}+\frac{14\,(a+b)}{2}$ subtract $7\,(a+b)$.

19. Cleveland had (estimate of Census Bureau, 1904) 8 times as many inhabitants as Portland, Maine. If twice the population of Portland is added to $\frac{1}{4}$ that of Cleveland, the sum is 212 thousand. Find the population of each city.

20. One cubic foot of a certain kind of brick weighs as much as 3 cubic feet of cedar. The combined weight of $\frac{1}{6}$ of a cubic foot of brick and 5 cubic feet of cedar is 187 pounds. Find the weight per cubic foot of each.

21. It is 8 times as far from Philadelphia to Louisville as from Philadelphia to Baltimore. If $\frac{1}{4}$ the distance from Philadelphia to Louisville is added to 3 times that from Philadelphia to Baltimore, the sum is 485 miles. Find each of the two distances.

22. Coinage silver weighs 4 times as much per cubic inch as feldspar. The combined weight of $\frac{1}{2}$ a cubic inch of silver and 7 cubic inches of feldspar is .846 pound. What is the weight of a cubic inch of each?

23. It is 3 times as far from New York to Washington, D.C., as from New York to New Haven, and it is 14 times as far from New York to Seattle as from New York to Washington. If $\frac{1}{7}$ the distance from New York to Seattle is added to 5 times that from New York to New Haven, the sum is 836 miles. Find the distance from New York to each of the other cities.

24. The population of Washington, D.C. (estimate of Census Bureau, 1904), was 9 times that of Galveston, Texas, and the population of Savannah, Georgia, was 33 thousand less than $\frac{1}{3}$ that of Washington. The combined population of Galveston and Savannah was 99 thousand. Find the population of each city.

25. A cubic foot of steel weighs 17 times as much as a cubic foot of yellow pine. The combined weight of 11 cubic feet of pine and 3 cubic feet of steel is 1773.2 pounds. Find the weight of 1 cubic foot of each.

DIVISION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

24. In dividing the sum or difference of two numbers by a given number, when these are represented by Arabic figures, the process may be carried out in two ways. Thus,

(1) $$(12 + 8) \div 2 = 20 \div 2 = 10,$$

or $$(12 + 8) \div 2 = 12 \div 2 + 8 \div 2 = 6 + 4 = 10.$$

(2) $$(20 - 12) \div 4 = 8 \div 4 = 2,$$

or $$(20 - 12) \div 4 = 20 \div 4 - 12 \div 4 = 5 - 3 = 2.$$

The same result is obtained in each case whether the numbers in the parenthesis are first added (or subtracted) and the result then divided by the given number; or the numbers in parenthesis are first divided separately and then the quotients added (or subtracted).

If the numbers in the dividend are represented by letters, the division can usually be carried out only in the second manner shown above.

E.g. $(r + t) \div 5 = r \div 5 + t \div 5$, or, $\frac{r+t}{5} = \frac{r}{5} + \frac{t}{5}$.

This is read: the sum of r and t divided by 5 equals r divided by 5 plus t divided by 5.

In this manner perform each of the following divisions in two ways when possible:

1. $(16 + 12) \div 4$.
2. $(20a - 10b) \div 5$.
3. $(36x - 24y) \div 6$.
4. $(108y - 72z) \div 12$.
5. $(50x + 75x) \div 25$.
6. $(x + y + z) \div 3$.
7. $(15t - 3t + 18t) \div 3$.
8. $(18x + 36y - 21t) \div 3$.
9. $(m - n - r) \div a$.
10. $(m + n + r) \div b$.

These examples illustrate the following principle:

25. Principle VI. *To divide the sum or difference of two numbers by a given number divide each number separately and find the sum or difference of the quotients.*

Principle VI is necessary in problems such as the following:

Illustrative Problem. The population of Iceland is 40,500 less than 10 times that of Greenland. The population of Greenland plus $\frac{1}{5}$ that of Iceland is 27,600. Find the population of each.

Solution. Let x be the number of inhabitants in Greenland.

Then $10x - 40500$ is the number in Iceland.

Hence $$x + \frac{10x - 40500}{5} = 27600.$$

By Principle VI, $$x + 2x - 8100 = 27600.$$

By Principle I, $$3x - 8100 = 27600.$$

Adding 8100 to each member and observing that $8100 - 8100 = 0$,

$$3x = 35700.$$

Therefore $x = 11900$, the population of Greenland,

and $10x - 40500 = 78500$, the population of Iceland.

Check. $$11900 + \frac{78500}{5} = 27600.$$

26. Principles V and VI should be carefully contrasted:

Thus: $$\frac{12 \cdot 18 \cdot 24}{3} = 4 \cdot 18 \cdot 24 = 12 \cdot 6 \cdot 24 = 12 \cdot 18 \cdot 8,$$

while $$\frac{12 + 18 + 24}{3} = 4 + 6 + 8.$$

That is, in dividing the product of several numbers we operate upon *any one of them* as found convenient, but in dividing the sum of several numbers we must operate upon *each of them.*

EXERCISES AND PROBLEMS

1. Divide $72 + 56$ by 8 without first adding.

2. Divide $144 - 36$ by 12 without first subtracting.

3. Divide $r + t$ by 5 and check the quotient when $r = 15$, $t = 25$; also when $r = 60$, $t = 75$.

4. Multiply $7 + 9$ by 3 without first adding 7 and 9.

5. Multiply $25 - 8$ by 5 without first subtracting.

6. Find the product of 12 and $a + b$, checking the result when $a = 5$, $b = 7$.

Perform the following indicated operations:

7. $3(a+b+c+d)$. Check for $a=1$, $b=2$, $c=3$, $d=4$.

8. $7(r-s+t-x)$. Check for $r=t=5$, $s=x=4$.

9. $(m+n+r)\div 4$. Check for $m=64$, $n=32$, $r=8$.

10. $(x+y+z)\div 5$. Check for $x=100$, $y=50$, and $z=25$.

11. Add $5(a+b+c)$ and $\frac{6a+12b+24c}{3}$.

12. From $25(x+y+z)$ subtract $\frac{100x+50y+25z}{25}$.

13. Add $\frac{6ab+7ac+ad}{a}$ and $\frac{12b+15c+9d}{3}$.

14. Perform the divisions: $\frac{33x-44y}{11}$ and $\frac{78t-39s}{13}$.

15. Divide $7mxy+3nxy-2mny$ by y, and check for $m=2$, $n=3$, $x=4$, $y=5$.

16. Add $\frac{21ax+49by+56ab}{7}$ and $\frac{24ax+56by+64ab}{8}$, and check for $a=b=c=2$, $x=y=3$.

17. If the sum of 4 times a number and 32 be divided by 2 the result is 30. Find the number.

18. The melting temperature of glass is 548 degrees (Centigrade) lower than 4 times that of zinc. One-half the number of degrees at which glass melts plus 7 times the number at which zinc melts equals 3434. Find the melting point of each.

19. The melting temperature of silver is 496 degrees (Centigrade) lower than that of nickel. Five times the number of degrees at which nickel melts plus 7 times the number at which silver melts equals 13,928. Find the melting point of each metal.

20. The population of Paris (1904) was 1360 thousand less than twice that of Berlin. The sum of their populations was 4730 thousand. Find the population of each city.

21. A cubic foot of nickel weighs 1288 pounds less than 4 cubic feet of tin. One-half a cubic foot of nickel plus 1 cubic foot of tin weighs 724 pounds. Find the weight per cubic foot of each metal.

22. A cubic foot of gold weighs 2730 pounds less than 6 cubic feet of silver. One-third of a cubic foot of gold together with 5 cubic feet of silver weighs 3675 pounds. Find the weight per cubic foot of each metal.

23. The population of Japan (1904) was 106 million less than 12 times that of Manchuria. If the population of Manchuria be subtracted from $\frac{1}{2}$ that of Japan, the remainder is 12 million. Find the population of each country.

24. The population of Panama (1900) was 1160 thousand less than 3 times that of Nicaragua (1905). Three times the population of Panama plus twice that of Nicaragua is 2020 thousand. Find the population of each country.

25. The population of the Philippines (1903) was 400 thousand less than 50 times that of Hawaii. One twenty-fifth of the population of the Philippines plus the population of Hawaii is 464 thousand. What was the population of each?

26. One gallon of benzine weighs 45.4 pounds less than 8 gallons of alcohol. The weight of $\frac{1}{2}$ a gallon of benzine and 2 gallons of alcohol is 16.9 pounds. Find the weight of one gallon of each liquid.

27. During the season 1906 the American League baseball team of Chicago won 201 games less than 6 times as many as Boston. One-third the number of games won by Chicago plus 7 times the number of games won by Boston equals 374. How many games did each team win?

28. The altitude of Aconcagua is 70,800 feet less than 6 times that of Mt. Blanc. One-sixth the altitude of Aconcagua added to that of Mt. Blanc equals 19,760 feet. Find the altitude of each mountain.

29. The area of Great Britain is 30,387 square miles less than 12 times that of the Netherlands, and the area of Japan is 42,065 square miles less than 15 times that of the Netherlands. One-third the area of Great Britain plus $\frac{1}{5}$ the area of Japan is 69,994 square miles. Find the area of each country.

30. The diameter of the earth is 1918 miles more than twice that of Mercury, and the diameter of Venus is 1700 miles more than twice that of Mercury. The diameter of the earth plus $\frac{1}{2}$ that of Venus equals 11,768 miles. Find the diameter of each planet.

31. The diameter of Jupiter is 500 miles more than 20 times that of Mars, and the diameter of Saturn is 4200 miles more than 16 times that of Mars. One-tenth the diameter of Jupiter plus $\frac{1}{2}$ that of Saturn is 45,150 miles. Find the diameter of each planet.

32. The diameter of Neptune is 29,000 miles less than twice that of Uranus. One-half the diameter of Neptune plus 4 times that of Uranus is 145,000 miles. Find the diameter of each planet.

From the last three problems make a table of the diameters of all the planets.

NUMBER EXPRESSIONS IN PARENTHESES

27. When a number expression is inclosed in a parenthesis, it is sometimes possible to perform the operations indicated within the parenthesis and thus to remove it.

E.g. $6 + (7 + 10 - 9) = 6 + 8 = 14$.

When, however, the operations within the parenthesis cannot be carried out, the parenthesis may sometimes be removed by Principle IV or VI.

E.g. $5(2n + 7r) = 10n + 35r$, and $(20x - 16y) \div 4 = 5x - 4y$.

In these examples the operations upon the parentheses are multiplication or division. The cases in which the operations on the parentheses are addition or subtraction are considered in the following examples:

(1) $5 + (3 + 4) = 5 + 7 = 12,$

also $5 + (3 + 4) = 5 + 3 + 4 = 8 + 4 = 12.$

(2) $10 + (7 - 3) = 10 + 4 = 14,$

also $10 + (7 - 3) = 10 + 7 - 3 = 17 - 3 = 14.$

(3) $20 - (7 + 2) = 20 - 9 = 11,$

also $20 - (7 + 2) = 20 - 7 - 2 = 13 - 2 = 11.$

(4) $20 - (7 - 2) = 20 - 5 = 15.$

also $20 - (7 - 2) = 20 - 7 + 2 = 13 + 2 = 15.$

From (1) it appears that the expression, $3 + 4$, may be added to 5 by first adding 3 and then adding 4.

From (2) it is seen that the expression, $7 - 3$, may be added to 10 by first adding 7 and then subtracting 3.

From (3) it is seen that the expression, $7 + 2$, may be subtracted from 20 by first subtracting 7 and then subtracting 2.

From (4) it is evident that the expression, $7 - 2$, may be subtracted from 20 by first *subtracting* 7 and then *adding* 2, since 7 is 2 more than the number which was to be subtracted.

In like manner perform each of the following operations in two ways:

1. $18 + (5 + 2 + 3)$.
2. $28 + (7 - 3 + 4)$.
3. $32 - (5 + 3)$.
4. $40 - (6 + 7 + 3)$.
5. $55 - (16 - 7)$.
6. $101 + (73 - 22)$.
7. $25n + (22n - 3n)$.
8. $18x - (6x - 2x)$.
9. $16r - (7r + 2r)$.

These examples illustrate the following principle:

28. Principle VII. *A number expression consisting of two or more numbers connected by the signs + or − may be added to another given number by adding each number with the plus sign and subtracting each number with the minus sign. Such a number expression may be subtracted from a given number by subtracting each number with the plus sign and adding each number with the minus sign.*

Principles IV, VI, and VII are useful in removing parentheses from number expressions when the values of the numbers involved are not known.

Instead of a parenthesis, a bracket [ ], or a brace { }, may be used. Thus, $2(6+4) = 2[6+4] = 2\{6+4\}$.

In expressions involving parentheses the operations within the parentheses should be performed first if possible. Then perform the indicated multiplications and divisions, and finally the remaining additions and subtractions.

E.g. Given $7 + 15 \div (9 - 4) - 4[7 + 11] \div (29 - 20)$.

Performing operations within the parentheses, this reduces to:

$$7 + 15 \div 5 - 4 \cdot 18 \div 9.$$

Performing multiplications and divisions, we have $7 + 3 - 8$.

Performing the remaining additions and subtractions, the given expression reduces to 2.

EXERCISES

In performing the following indicated operations, state in each case what principles are used.

1. $8(7+4+8+2)$.
2. $6(7-3) \div 2$.
3. $3(2a-4) - 4(a-6)$.
4. $14 + 4(8+4) \div (32-20)$.

The minus sign preceding $4(a-6)$ indicates that this whole product is to be subtracted. Hence, using Principle IV, we have $6a - 12 - (4a - 24)$. Then, using Principle VII, this becomes $6a - 12 - 4a + 24 = 2a + 12$.

5. $8-3(4-2)+6+3[6+1]$.

6. $16 \div [3+1]-\{12-3\} \div 3$.

7. $5(17-5)+18 \div (8-2)-(21+14) \div 7$.

8. $7(a+b)+6(a+b)$.

9. $5(a+b)+3(a+b+c)$.

10. $16(r+t)+11(r+t)$.

11. $15s-3(r+s)$.

12. $15(r+s)-3(r-s)$.

13. $20-(x+y)$.

14. $50x-25(x-y)$.

15. $12y-6(x-2y)$.

16. $11t+5(2t-1)-3(2+2)$. Check for $t=2$.

17. $(12t-6u) \div 2+3t-2u$. Check for $t=1$, $u=1$.

18. $3(5x-7y)+(21x-28y) \div 7$. Check for $x=2, y=1$.

19. $8(r-s)+5(2r+s)-3(r+s)$. Check for $r=1, s=1$.

20. $10(x+y) \div 5+6(x-y) \div 3$. Check for $x=2$, $y=1$.

21. $5(h+k)+2(h+k)+3(h+k)$.

Use Principle I, then IV; also IV, then I.

22. $5(7x-4y)+9(9x-5y)$.

23. $8(r-s)-2(r-s)$.

24. $9(3p-q)-8(3p-q)$.

Use Principle II, then IV; also reverse this order.

25. $(16x-12x) \div 4$.

26. $(18ab-12abx) \div a$.

Use first Principle VI, then II; also II, and then V.

27. $9(6+4)-3(6-4)$.

28. $(5axy-3cx) \div x$.

29. $9(6abc+4xyz)-3(6abc-4xyz)$.

30. $8(5x-y+2z)-11(3x+2y-7z)$.

31. $3[2(a+b+c)-3(a-b+c)]$.

First remove the parentheses, then the bracket.

32. $7\{8x-(2y-3x)+(2x-4y)\}$.

PROBLEMS

1. There are three numbers whose sum is 80. The second is 3 times the first, and the third twice the second. What are the numbers?

2. A man has three buildings whose total value is $46,800. The second building cost $800 less than the first, and the third cost twice as much as the second. What is the cost of each building?

3. The population of Connecticut (Census of 1900) was 50 thousand more than twice that of Rhode Island, and the population of Massachusetts was 81 thousand more than 3 times that of Connecticut. The population of Massachusetts minus that of Connecticut was 1897 thousand. Find the population of each state.

4. The population of Indiana (Census of 1900) exceeded that of Iowa by 284 thousand, and the population of Illinois was 210 thousand less than twice that of Indiana. The population of Illinois minus that of Indiana was 2306 thousand. Find the population of each state.

5. The melting point of copper is 250 degrees (Centigrade) lower than 4 times that of lead. Ten times the number of degrees at which lead melts minus twice the number at which copper melts equals 1152. What is the melting point of each metal?

6. The melting point of iron is 450 degrees higher than 5 times that of tin. Three times the number of degrees at which iron melts plus 7 times the number at which tin melts equals 6410. Find the melting point of each metal.

7. In 1904 the gold product of Africa was 11 million dollars more than 3 times that of Russia, and the gold product of Australia was 84 million less than twice that of Africa. Russia and Australia together produced 113 million. How much did each country produce?

8. In 1904 the value of the silver produced in the United States was 5 million dollars more than 14 times as much as that of Canada, and the product of Mexico was 1 million less than 16 times that of Canada. Twice the product of the United States minus that of Mexico equals 71 million. How much did each country produce?

9. The Nile is 100 miles more than twice as long as the Danube. Ten times the length of the Danube plus 4 times the length of the Nile equals 3400 miles. How long is each river?

10. The number of first-class battleships in the United States navy (1906) was 22 less than 5 times the number of protected cruisers. Twelve times the number of cruisers less twice the number of battleships equals 60. Find the number of each.

11. The number of torpedo boats in the United States navy (1906) was 77 less than 7 times as great as the number of torpedo boat destroyers. Three times the number of torpedo boats minus 4 times the number of destroyers equals 41. Find the number of each.

12. The weight of a cubic foot of spruce is 16 pounds more than that of a cubic foot of cork, and the weight of a cubic foot of dry live oak is 5 pounds more than twice that of spruce. One cubic foot of oak weighs 36 pounds more than one cubic foot of spruce. Find the weight of a cubic foot of each.

13. In 1904 Canada produced 3 million dollars more of gold than Mexico, and the United States produced 17 million more than 4 times as much as Canada. The combined product of Mexico and the United States was 94 million. How much did each country produce?

14. The value of the copper produced in the United States in 1904 was 5 million dollars more than the value of the crude petroleum, and the value of the bituminous coal was 94 million more than twice the value of the copper. Nine times the value of the copper minus twice the value of the coal equals 342 million. Find the value of each.

15. The pressure developed in the chamber of a modern twelve-inch gun is 21 thousand pounds per square inch more than that developed in an ordinary hunting rifle. The maximum pressure which gunpowder *can* develop is 18 thousand pounds per square inch less than 3 times as great as that developed in the twelve-inch gun. The sum of the three pressures is 151 thousand pounds. Find each pressure.

16. The standing army of Australia (1906) was 14 thousand more than that of Canada, and the standing army of Great Britain was 47 thousand more than 4 times that of Australia. Great Britain's army contained 287 thousand men. How many men were there in the armies of Canada and Australia?

IDENTITIES AND EQUATIONS

29. In the preceding pages equations have been freely used, and certain simple methods have been employed in solving problems by means of them. We now proceed to a more detailed study of these methods and of the equation itself.

30. Equalities in which letters are used as number symbols are of two kinds as shown by the following examples:

(1) $3n + 4n = 7n$ is a true statement no matter what number is represented by n. $3(5x + 6y) = 15x + 18y$ holds for all values which may be assigned to x and y.

(2) $w + 2(w+5) = 58$ is a true statement if, and only if, $w = 16$. If w is replaced by any number less than 16, the number expression on the left is less than 58, and if w is replaced by any number greater than 16, the result is greater than 58.

31. The equality $w + 2(w + 5) = 58$ is said to be **satisfied** by $w = 16$, because this value of w reduces both members to the same number, 58.

32. Definition. An equality which is satisfied, no matter what numbers are substituted for one or more of its letters, is called an **identity** with respect to those letters.

E.g. $3n + 4n = 7n$ is an identity with respect to n.
$a(b + c) = ab + ac$ is an identity with respect to a, b, and c.

When it is especially desired to distinguish an equality as an identity, the equality sign is written $\equiv$.

Each of the Principles I to VII may be thus stated in symbols as identities. Thus,

I, $4n + 6n \equiv 10n$; II, $12n - 5n \equiv 7n$;
III, $5(4ab) \equiv 20ab$; IV, $5(a \pm b) \equiv 5a \pm 5b$;
V, $30ab \div 6 \equiv 5ab$; VI, $(16a \pm 20b) \div 4 \equiv 4a \pm 5b$;
VII, $x + (a - b) - (c - d) \equiv x + a - b - c + d$.

33. Definition. An equality which is satisfied only when certain particular values are given to one or more of its letters is called a **conditional equality** with respect to those letters.

E.g. $3x + 5 = 35$ is an equality only on the condition that $x = 10$. $x + y = 10$ is an equality for certain pairs of letters like 1 and 9, 2 and 8, 3 and 7, 5 and 5, but certainly not for all values of x and y; for instance, not for 3 and 8.

34. A conditional equality is called an **equation**, and a letter whose particular value is sought to satisfy an equation is called an **unknown number** in that equation, or simply an **unknown.**

The equations at present considered contain only one unknown.

35. To solve an equation in one unknown is to find the value or values of the unknown which satisfy it.

The following example exhibits the process of solving an equation which contains one unknown and which is satisfied by one value only of the unknown:

Given	$w + 2(w + 5) = 58.$	(1)
By Principle III,	$w + 2w + 10 = 58.$	(2)
By Principle I,	$3w + 10 = 58.$	(3)
Subtracting 10 from both members,	$3w = 48.$	(4)
Dividing both sides by 3,	$w = 16.$	(5)

Check. Putting $w = 16$ in (1), $16 + 2(16 + 5) = 16 + 2 \cdot 21 = 58.$

Explanation. Any value of w which satisfies (1) also satisfies (2) and (3), since the expressions in the left members of (1), (2), and (3) represent the same number expressed in different forms. Any value of w which satisfies (3) also satisfies (4), for if 10 more than $3w$ is 58, then $3w$ must be 48. Any value of w which satisfies (4) also satisfies (5), for if $3w$ is 48, then w is $\frac{1}{3}$ of 48.

By similar considerations the value of w which satisfies (5) may be shown to satisfy (4), (3), (2), and (1). Hence $w = 16$ is the solution of the given equation.

The above solution illustrates the following principle:

36. Principle VIII. *An equation may be changed into another equation such that any value of the unknown which satisfies one also satisfies the other, by means of any of the following operations:*

(1) *Adding the same number to both members.*

(2) *Subtracting the same number from both members.*

(3) *Multiplying both members by the same number.*

(4) *Dividing both members by the same number.*

(5) *Changing the form of either member in any way which leaves its value unaltered.*

The operations under Principle VIII are hereafter referred to in detail by means of the initial letters, A for addition, S for subtraction, M for multiplication, D for division, and F for form changes.

NOTE.—It is not permissible to multiply or divide the members of an equation by any expression which is equal to zero. See Advanced Course.

37. The operations involved in Principles I to VII are all **form changes** which leave the value of the expression unaltered.

E.g. $4(m+n)$ by Principle IV has the same value as $4m+4n$.

There are other form changes which are already familiar in arithmetic.

E.g. $2+4=4+2$, or in general $a+b=b+a$. Likewise $2\cdot 7=7\cdot 2$, or in general $ab=ba$. That is, the order in which numbers are added or multiplied is immaterial.

Again, $3+4+6=(3+4)+6=3+(4+6)$, or in general $a+b+c=(a+b)+c=a+(b+c)$, and likewise $a\cdot b\cdot c=(a\cdot b)\cdot c=a\cdot(b\cdot c)$; *i.e.* numbers to be added or multiplied may be grouped in any manner desired. Still other form changes will be learned later.

38. Principle VIII is further illustrated as follows:

On the scale pans of a common balance are placed objects of uniform weight, say tenpenny nails. The scales balance only when the weights are the same in both pans; that is, when the number of nails is the same.

If now the scales are in balance, they will remain so under two kinds of changes in the weights:

(*a*) When the number of nails in the two pans is increased or diminished by the same number; corresponding to the operations *A, S, M, D*, on the members of an equation.

(*b*) When the number in each pan is left unaltered but the nails are rearranged in groups or piles in any manner; corresponding to the operations *F* on the members of an equation.

The equation, then, is like a balance, and its members are to be operated upon only in such ways as to preserve the balance.

DIRECTIONS FOR WRITTEN WORK

39. The solution of an equation consists in deducing, by means of Principles I to VIII, another equation whose first member contains the unknown only and whose second member does not contain the unknown. The successive steps should be written down as in the following example:

Solve $6(4n-3)+25(n+1)=50+31n+2(3-n)-9.$ (1)

By F, using III and IV, we obtain from (1)

$$24n-18+25n+25=50+31n+6-2n-9. \qquad (2)$$

By F, I and II, we obtain from (2)

$$49n+7=29n+47. \qquad (3)$$

Subtracting 7 and $29n$ from each member of (3) and using Principle II, we have $20n=40.$ (4)

Dividing each member of (4) by 20,

$$n=2. \qquad (5)$$

Check. Substitute $n=2$ in Equation (1).

For convenience this work can be abbreviated as follows:

	$6(4n-3)+25(n+1)=50+31n+2(3-n)-9.$	(1)
By F, III, IV,	$24n-18+25n+25=50+31n+6-2n-9.$	(2)
By F, I, II,	$49n+7=29n+47.$	(3)
By $S\,\vert\,7, 29n$,	$20n=40.$	(4)
By $D\,\vert\,20$,	$n=2.$	(5)

$S\,|\,7, 29n$ means that 7 and $29n$ are each to be subtracted from both members of the preceding equation. $D\,|\,20$ means that the members of the preceding equation are to be divided by 20.

Similarly, in case we wish to indicate that 6 is to be added to each member of an equation, we would write $A\,|\,6$, and if each member is to be multiplied by 8, we would write $M\,|\,8$. It is important that the nature of each step be recorded in some such manner.

(2) Solve $17n + 4(2 + n) - 6 = 5(4 + n) - 5 + 3n.$ (1)

By F, IV, $17n + 8 + 4n - 6 = 20 + 5n - 5 + 3n.$ (2)

By F, I, $21n + 2 = 15 + 8n.$ (3)

By $S \mid 2, 8n$, $21n - 8n = 15 - 2.$ (4)

By F, II, $13n = 13.$ (5)

By $D \mid 13$, $n = 1.$ (6)

Check. Substitute $n = 1$ in equation (1);

then $17 + 12 - 6 = 25 - 5 + 3,$

or $23 = 23.$

An equation may be translated into a problem. For example, the equation $21x + 2 = 15 + 8x$ may be interpreted as follows: Find a number such that 21 times the number plus 2 is 15 greater than 8 times the number.

EXERCISES AND PROBLEMS

Solve the following equations, putting the work in a form similar to the above and checking each result. Translate the first twenty into problems.

1. $13x - 40 - x = 8.$
2. $3x + 9 + 2x + 6 = 18 + 4x.$
3. $5x + 3 - x = x + 18.$
4. $13y + 12 + 5y = 32 + 8y.$
5. $4m + 6m + 4 = 9m + 6.$
6. $7m - 18 + 3m = 12 + 2m + 2.$
7. $3y - 4 + 2y - 6 = y + 7 + y + 3 + 10.$
8. $5x + 3 + 2x + 3 = 2x + 5 + 3x + 3 + x.$
9. $2x + 4x + 9 + x + 6 = 20 + 3x + 5 + 2x.$
10. $18 + 6m + 30 + 6m = 4m + 8 + 12 + 3m + 3 + m + 29.$

11. $y + 72 + 45\,y = 106 + 12\,y.$

12. $42\,x - 56 = 20\,x + 10.$

13. $6\,x + 8 - 3\,x + 4 + 5\,x = 7\,x + 32 + x - 20.$

14. $32\,x - 4 + 7\,x = 58 + 3\,x + 5\,x.$

15. $12\,m - 3 - 3\,m = 32 + 2\,m.$

16. $15\,m + 3 - 2\,m + 7 = 3\,m + 60.$

17. $a + 7 + 3\,a = 2\,a + 45.$

18. $5\,b - 30 + 6\,b = 3\,b + 90.$

19. $3\,c + 18 + 14\,c = 6\,c + 51.$

20. $17\,x + 4 + 3\,x = 7\,x + 30.$

21. $7(m + 6) + 10\,m = 42 - 8(2\,m + 2) + 181.$

22. $20 - 3(x - 4) + 2\,x = 2\,x + 17.$

23. $(8\,x + 4) \div 2 + 7\,x = 4\,x + 9.$

24. $6\,x + 4(4\,x + 2) = 85 - 3(2\,x + 7).$

25. $8 + 7(6 + 6\,n) + 2\,n = 2(4\,n + 5) + 18\,n + 49.$

26. $5(9\,x + 3) + 6\,x = 24\,x - 4(3\,x + 2) + 36.$

27. $\frac{7(12\,x + 8)}{4} + 13 + 5\,x - 6 = 47.$ Apply V and VI.

28. $\frac{12(5 + 4\,x)}{6} - \frac{5(6 + 4\,x)}{2} + 50 = x + 18.$

29. $15 + \frac{21\,(3 + x)}{7} + \frac{2(6 + 18\,x)}{3} = \frac{3(9\,x + 12)}{3} + 28.$

30. $\frac{11(5\,x + 25)}{5} + \frac{3(6\,x - 2)}{2} = \frac{7(4\,x + 8)}{4} + \frac{12\,x + 36}{3} + 35.$

EQUATIONS INVOLVING FRACTIONS

40. Solve $n+\frac{n}{2}+\frac{n}{3}=88.$ (1)

First Solution. The coefficients of n are 1, $\frac{1}{2}$, and $\frac{1}{3}$.

Applying Principle I, $1\frac{5}{6}n = 88.$ (2)

By $D \mid 1\frac{5}{6}$, $n = 88 \div 1\frac{5}{6} = 48.$ (3)

Second Solution. Multiply both members of (1) by 6.

That is, by $M \mid 6$, $6n+\frac{6n}{2}+\frac{6n}{3}=528.$ (2)

By F, V, $6n+3n+2n=528.$ (3)

By F, I, $11n=528.$ (4)

By $D \mid 11$, $n=48$, as before. (5)

The object is to multiply both members of the equation by such a number as will cancel each denominator. Hence the multiplier must contain each denominator as a factor.

Evidently 12 or 18 might have been chosen for this purpose, but not 8 or 10. 6 is the smallest number which will cancel both 2 and 3, and hence this was chosen as the multiplier.

41. The process explained in the second solution above is called **clearing of fractions.**

As another illustration solve the equation

$$\frac{3x}{4}+\frac{x}{2}+\frac{5x}{9}=65. \quad (1)$$

Here the smallest multiplier available is 36.

Hence by $M \mid 36$, $\frac{36\cdot 3x}{4}+\frac{36\cdot x}{2}+\frac{36\cdot 5x}{9}=36\cdot 65.$ (2)

By V, each denominator is cancelled by the factor 36,

$$9\cdot 3x+18x+4\cdot 5x=36\cdot 65. \quad (3)$$

By III, $27x+18x+20x=36\cdot 65.$ (4)

By I, $65x=36\cdot 65.$ (5)

By D, V, $x=\frac{36\cdot 65}{65}=36.$ (6)

Check. Substitute $x=36$ in (1).

EXERCISES AND PROBLEMS

Solve the following equations, indicating the principles used at each step. Check each solution by substituting in the original equation the value obtained. Translate each into a problem.

For instance, from equation 2: Find a number such that when increased by its half, its third, and its fourth, the sum is 25.

1. $\frac{n}{2}+\frac{n}{3}=5.$

2. $n+\frac{n}{2}+\frac{n}{3}+\frac{n}{4}=25.$

3. $\frac{n}{4}+4n-\frac{5n}{3}=\frac{3n}{2}+26.$

4. $\frac{n}{2}+\frac{n}{3}-\frac{n}{4}+\frac{n}{10}=82.$

5. $7x+\frac{4x}{7}+\frac{3x}{5}+23=\frac{x}{5}+\frac{4x}{10}+5x+113.$

6. $4n+\frac{6n}{7}=\frac{3n+2}{2}+46.$

7. $12+\frac{4(9x+6)}{3}-\frac{2(3+11x)}{5}=\frac{5(4x+4)}{3}+14.$

8. $13b+7-\frac{9b+8}{7}=\frac{2b+9}{5}+38.$

9. $\frac{5a+7}{2}+\frac{2a+4}{3}=\frac{3a+9}{4}+5.$

10. $\frac{17a-5}{3}-\frac{10a+2}{4}=\frac{5a+7}{2}-5.$

11. $\frac{5x+3}{2}+\frac{5(2x+10)}{5}=2x+24.$

12. The sum of two numbers is 12, and the first number is $\frac{1}{3}$ as great as the second. What are the numbers?

13. The smaller of two numbers is $\frac{3}{8}$ of the larger. If their sum is 66, what are the numbers?

14. Find two consecutive integers such that 4 times the first minus 3 times the second equals 9.

15. Find three consecutive integers such that $\frac{1}{3}$ of the first plus the second minus $\frac{1}{2}$ the third equals 5.

16. Find three consecutive integers such that 3 times the first plus 9 times the second minus 4 times the third equals 73.

17. There are three numbers such that the second is 4 more than 9 times the first, and the third is 2 more than 6 times the first. If $\frac{1}{8}$ of the third is subtracted from $\frac{1}{7}$ of the second, the remainder is 3. Find the numbers.

18. There are three numbers such that the second is 2 more than 9 times the first and the third is 7 more than 11 times the first. The remainder when 4 times the third is subtracted from 13 times the second is 144. Find the numbers.

19. The population of Philadelphia (estimate of Census Bureau, 1904) was 508 thousand less than 20 times that of Dayton, Ohio. The population of St. Louis was 245 thousand more than 4 times that of Dayton. One-half the population of Philadelphia plus $\frac{1}{5}$ that of St. Louis was 851 thousand. Find the population of each city.

20. The shortest railway route from Boston to Chicago is 166 miles more than 4 times that from Boston to New York; and the shortest route from Boston to Atlanta is 196 miles less than 6 times that from Boston to New York. The distance from Boston to Chicago is 481 miles more than $\frac{1}{2}$ the distance from Boston to Atlanta. Find each of the three distances.

Solution. Let d = distance in miles from Boston to New York;

then $4d + 166$ = distance in miles from Boston to Chicago,

and $6d - 196$ = distance in miles from Boston to Atlanta,

and $4d + 166 = \frac{6d - 196}{2} + 481.$

By VI, $4d + 166 = 3d - 98 + 481.$

By *S*, II, $d = 481 - 98 - 166 = 217$ = distance from Boston to New York.

$4d + 166 = 1034$ = distance from Boston to Chicago,

and $6d - 196 = 1106$ = distance from Boston to Atlanta.

21. The railway mileage in the United States in 1900 was 8 thousand greater than twice that of 1880, and in 1904 it was 66 thousand less than 3 times that of 1880. One-half the mileage of 1900 plus $\frac{1}{3}$ that of 1904 equals 168 thousand. Find the railway mileage in 1880, 1900, and 1904.

22. The number of newspapers in the United States in 1880 was 381 less than 4 times that in 1850. The number in 1905 was 700 more than twice that of 1880 and 7990 more than 6 times that of 1850. Find the number of newspapers in 1850, 1880, and 1905.

23. The number of telephones in the United States in 1900 was 40 thousand less than 30 times as great as the number in 1880. Half the number in 1905 was 660 thousand more than that in 1900 and 80 thousand more than 40 times that in 1880. Find the number of telephones in use in 1880, 1900, and 1905.

24. The displacement of the battleship *Kearsarge* is 1232 tons greater than that of the *Oregon*, and the displacement of the *Connecticut* is 4480 tons greater than that of the *Kearsarge*. One-third the displacement of the *Kearsarge* minus $\frac{1}{5}$ that of the *Connecticut* equals 640. Find the displacement of each ship.

25. The distance of the fixed star Vega from the sun is 11.7 light-years greater than the distance from the sun to Sirius. The distance of Regulus is 8 light-years less than twice that of

Vega. What is the distance of each star from the sun if $\frac{1}{2}$ the distance of Vega minus $\frac{1}{4}$ that of Regulus is 2 light-years?

Light travels at the rate of 186,000 miles in one second. A light-year is the distance traveled by light in one year.

26. The distance from the sun to the fixed star 61 Cygni is 4 light-years more than that from the sun to Alpha Centauri (the star nearest the sun), and the distance from the sun to Polaris (the pole star) is 6 times that of 61 Cygni. One-eighth the distance of Polaris minus $\frac{1}{3}$ the sum of the distances of the other two equals 2 light-years. How many light-years is each star from the sun?

27. In 1905 the wheat crop of Russia was 47 million bushels less than 4 times that of Argentina, and the crop of the United States was 77 million bushels less than 5 times that of Argentina. The crop of the United States exceeded that of Russia by 124 million bushels. What was the wheat crop of each country?

28. The total imports of the United States in 1900 were 60 million dollars less than 10 times the imports in 1800. The imports in 1906 were 473 million less than twice the imports of 1900, and 135 million more than 12 times the imports of 1800. Find the imports of the United States in 1800, 1900, and 1906.

29. The total exports of the United States in 1900 were 26 million dollars less than 20 times the exports in 1800. The exports in 1906 exceeded those of 1900 by 350 million and were 31 million less than 25 times those of 1800. Find the exports of 1800, 1900, and 1906.

30. The gross tonnage of the German merchant marine was (1906) 548 thousand more than that of the United States, and the tonnage of the British marine was 2662 thousand greater than 4 times that of the German marine. The British marine was greater by 930 thousand tons than twice that of Germany and 3 times that of the United States combined. Find the tonnage of each marine.

31. The population of San Francisco (estimate of Census Bureau, 1904) was 289 thousand more than that of Springfield, Massachusetts, and the population of Chicago was 132 thousand more than 5 times that of San Francisco. The population of Springfield minus $\frac{1}{28}$ that of Chicago was 2 thousand. Find the population of each city.

32. The per capita tax of New York City (1905) was \$ .49 less than twice that of Chicago and the per capita tax of Boston was \$ 3.56 less than 3 times that of Chicago. The combined per capita tax of Boston and New York was \$ 52.15. What was the per capita tax of each city?

33. The number of pupils attending public high schools in Chicago (1905) was 1237 more than twice as great as in Philadelphia. One-third the number of pupils in Chicago minus $\frac{1}{4}$ the number in Philadelphia was 2524. How many pupils attended public high schools in each city?

34. The number of pupils in public high schools in Boston (1905) was 1574 less than 4 times as many as in Baltimore. Three times the number in Boston minus 6 times the number in Baltimore equals 5268. How many pupils in each city?

35. The distance from San Francisco to Yokohama is 2031 statute miles less than 3 times that from San Francisco to Honolulu, and the distance from San Francisco to Hongkong is 2219 more than twice that from San Francisco to Honolulu. One-fifth the distance to Hongkong plus $\frac{1}{3}$ the distance to Yokohama equals 3152. Find the distance from San Francisco to each of the other three cities.

36. The distance from New York to Singapore is 542 statute miles more than 3 times that from New York to London, and the distance from New York to Manila is 5342 miles less than 5 times that from New York to London. Three times the distance from New York to Manila plus 6974 miles equals 4 times the distance from New York to Singapore. Find the distance from New York to each of the other three cities.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

42. Thus far the numbers used have been precisely the same as in arithmetic, though their representation by means of letters and some of the methods used in operating upon them are peculiar to algebra.

43. A new kind of number will now be studied in connection with problems of the following type.

Illustrative Problems. 1. If a man gains $1500, and then loses $800, what is the net result? *Answer*, $700 gain.

2. The assets of a commercial house are $250,000, and the liabilities are $275,000. What is the net financial status of the house? *Answer*, $25,000 net liabilities.

3. The thermometer rises 18 degrees and then falls 28 degrees. What direct change in temperature would produce the same result? *Answer*, 10 degrees fall.

4. A man travels 700 miles east and then 400 miles west. What direct journey would bring him to the same final destination? *Answer*, 300 miles east.

In the statement of each of these problems the numbers are applied to things which are *opposite in quality;* namely, gain and loss, assets and liabilities, rise and fall, distances measured in two opposite directions, as east and west.

The answer in each case not only gives the proper arithmetical number, but also connects with this number one of the two opposite qualities involved in the problem.

E.g. The answer in (1) is $700 *gain* and not simply $700; in (2) it is not $25,000, but $25,000 *liabilities;* in (3) it is not 10°, but 10° *fall;* and in (4) it is not 300 miles, but 300 miles *east.*

These problems are illustrations of the sense in which the word "opposite" is here used.

E.g. Gain and loss are opposite in that they *annul* each other when taken together; rise and fall of temperature are opposite in that one *counteracts* the other.

44. In all problems involving one or both of two qualities which are opposite in this sense there is constant need to distinguish which one is meant.

E.g. On a day in winter it is not sufficient to say the temperature is 5°; we must specify whether it is *above* or *below* zero.

In the case of the thermometer we call the temperature positive if it is above zero and negative if it is below zero.

These words *positive* and *negative* are used to describe all pairs of qualities which are opposite in the sense here understood.

The signs $^+$ and $^-$ stand respectively for the words "positive" and "negative."

E.g. $\$^+700$ is read *positive* \$700, and means either \$700 gain or \$700 assets, according to the problem in which it occurs. Likewise $\$^-700$ is read *negative* \$700, and means either \$700 loss or \$700 liability.

Referring to the thermometer, $^+18°$ is read positive 18°, and means either a temperature of 18° *above* zero or a *rise* of the mercury 18° from any point. The latter is the meaning in Problem 3. Similarly $^-28°$ means either a temperature 28° *below* zero or a *fall* of the mercury 28° from any point.

It is commonly agreed to call gain positive and loss negative, assets positive and liabilities negative, above zero positive and below zero negative, motion upward positive and downward negative, motion to the right positive and to the left negative.

45. Numbers marked with the sign $^+$ are called **positive** and those marked with the sign $^-$ are called **negative**.

The positive sign may be omitted; that is, a number with neither sign written is understood to be positive.

ADDITION OF POSITIVE AND NEGATIVE NUMBERS

46. While in each of the problems on page 47 the result was obtained by subtracting one number from the other, yet they are not properly subtraction problems, but addition problems.

E.g. In Problem 1, we are not asking for the *difference* between \$1500 gain and \$800 loss, but for the *net result* when the gain and the loss are taken together; that is, the *sum of the profit and loss.* Hence we say \$1500 gain + \$800 loss = \$700 gain, or, using positive and negative signs,

$$^{+}1500 + {}^{-}800 = {}^{+}700.$$

Similarly in Problem 2,

\$250,000 assets + \$275,000 liabilities = \$25,000 net liabilities,

Or $$^{+}250{,}000 + {}^{-}275{,}000 = {}^{-}25{,}000.$$

In Problem 3, 18° rise + 28° fall = 10° fall,

Or $$^{+}18 + {}^{-}28 = {}^{-}10.$$

In Problem 4,

700 miles east + 400 miles west = 300 miles east,

Or $$^{+}700 + {}^{-}400 = {}^{+}300.$$

NOTE.—In the last case east is called $^{+}$ and west $^{-}$, since, as a map is usually held, east is to the right and west to the left. Likewise north or up is called $^{+}$ and down or south is called $^{-}$.

PROBLEMS

1. A balloon which exerts an upward pull of 460 pounds is attached to a car weighing 175 pounds. What is the net upward or downward pull? Express this as a problem in addition, using positive and negative numbers.

Solution. 460 lb. upward pull + 175 lb. downward pull = 285 lb. net upward pull. Using positive numbers to represent upward pull and negative numbers to represent downward pull, this equation becomes

$$^{+}460 + {}^{-}175 = {}^{+}285.$$

2. How would the morning paper print the following thermometer readings from the weather report? Jacksonville 38° above zero, Seattle 5° below zero, Nashville 28° above zero, Chicago 13° below zero.

3. The temperature rises 15° and then falls 24°. What direct change in temperature would produce the same final result? Express this as an example in addition.

4. A vessel on the equator sailed north 3° and was then forced south 5° by a hurricane. It then resumed its course northward 8°. Express this as a sum and show the final position with reference to the equator. (See note, § 46.)

In each of the following translate the solution into the language of algebra by means of positive and negative numbers, as in Problem 1.

5. A 450-pound weight is attached to a balloon which exerts an upward pull of 600 pounds. What is the net upward or downward pull?

6. A man's property amounts to $45,000 and his debts to $52,000. What is his net debt or property?

7. The assets of a bankrupt firm amount to $245,000 and the liabilities to $325,000. What are the net assets or liabilities?

8. A weight exerting a downward pull of 280 pounds when submerged in water is attached to a float which will just support a weight of 240 pounds. What is the net pressure upward or downward when both are submerged?

9. A man on the deck of a steamer is walking at the rate of 4 miles an hour toward the stern. If the boat is sailing eastward in a river at the rate of fifteen miles per hour, what is the actual motion of the man with respect to the bank of the river?

10. A man can row a boat at the rate of 6 miles per hour. How fast can he proceed against a stream flowing at the rate of $2\frac{1}{2}$ miles per hour; 7 miles per hour?

11. A steamer which can make 12 miles per hour in still water is running against a current flowing 15 miles per hour. How fast and in what direction does the steamer move?

12. A dove capable of flying 40 miles per hour in calm weather is flying against a hurricane blowing at the rate of 60 miles per hour. How fast and in what direction is the dove moving?

13. If of two partners, one loses $1400 and the other gains $3700, what is the net result to the firm?

14. A man's income is $2400 and his expenses $1500 per year. What is his saving?

15. A man loses $800 and then loses $600 more. What is the combined loss? Indicate the result as the sum of two negative numbers.

16. A man gains $500 and then gains $700 more. What is the combined gain? Express the result as the sum of two positive numbers.

17. A tug which can steam 9 miles per hour in still water is going down a stream whose current is 6 miles per hour. How fast is the tug moving?

18. The thermometer falls 8° and then 17° more. Express the combined result of the two changes as a negative number.

47. Definitions. Positive and negative numbers are sometimes called **signed numbers**, because each such number consists of an arithmetic part, together with a **sign of quality.**

The arithmetic part of a signed number is called its **absolute value.**

Thus, the absolute value of $^+3$ and also of $^-3$ is 3. Two signed numbers are of *like* quality when they have the same signs, and of *opposite* quality when one is positive and the other negative.

In the one case we say they have *like signs*, in the other opposite or *unlike signs*. The preceding exercises illustrate the following principle:

48. Principle IX. *To add two signed numbers of like quality, find the sum of their absolute values, and prefix to this the common sign of quality.*

To add two signed numbers of opposite quality, find the difference of their absolute values, and prefix to this the sign of that one whose absolute value is the greater.

In case their absolute values are equal, their sum is zero.

The sum of two signed numbers thus obtained is called their **algebraic sum.**

49. This principle may be further illustrated by means of the following **graphic representation** of signed numbers.

On an unlimited straight line call some starting point zero, and lay off from this point equal divisions of the line indefinitely both to the right and to the left, as shown in the figure.

In order to describe the position of any one of these division points, we require not only an integer of arithmetic, to specify *how far* to count from the starting point (the point marked zero) in order to reach the given point, but also a *quality sign*, to indicate the *direction* of the counting.

E.g. $^+7$ marks the division point 7 units to the right of zero, and $^-5$ marks the point 5 units to the left of zero. Such a diagram is called the **scale of signed numbers.**

The part of the scale to the right, taken alone, would need no signs, and would picture the integral numbers of arithmetic, while the two parts together require the distinguishing signs of quality, and picture the integral numbers of algebra.

Fractions would of course be pictured at points between the integral division points, on the right or the left of the scale, according as the fractions are positive or negative.

ADDITION BY COUNTING

50. In arithmetic two numbers are added by starting with either and counting forward as many units as there are in the other.

E.g. To add 3 and 5 we start with 5 on the number scale and count 6, 7, 8; or start with 3 and count 4, 5, 6, 7, 8.

51. In algebra two signed numbers are added in the same manner except that the *direction*, forward or backward, in which we count, is determined by the sign, $^+$ or $^-$, of the number which we are adding.

Thus, to add $^+7$ and $^+5$, begin at 7 to the right of the zero point in the scale of signed numbers and count 5 more toward the right, or begin at 5 to the right and count 7 more to the right, in either case arriving at $^+12$.

To add $^-7$ and $^-5$, we may begin at 7 to the left and count 5 more toward the left, or begin at 5 to the left and count 7 more in that direction, in either case arriving at $^-12$.

To add $^+7$ and $^-5$, begin at 7 to the right and count 5 toward the left, or begin at 5 to the left and count 7 toward the right, in either case arriving at $^+2$.

To add $^-7$ and $^+5$, begin at 7 to the left and count 5 toward the right, or begin at 5 to the right and count 7 toward the left, in either case reaching $^-2$.

I.e. $^+7 + {}^+5 = {}^+12$, $^-7 + {}^-5 = {}^-12$, $^+7 + {}^-5 = {}^+2$, $^-7 + {}^+5 = {}^-2$.

52. In adding several signed numbers, we reach the same result whether we add them in the order in which they happen to be given, or in any other order. Thus we may add first all the positive numbers and then all the negative numbers, and finally combine these two results.

E.g. $^+5 + {}^-8 + {}^+7 + {}^-6$, taken in order from left to right, gives $^-3$ as the sum of the first two, $^+4$ as the sum of the first three, and $^-2$ as the final result. But $^+5 + {}^+7 = {}^+12$, $^-8 + {}^-6 = {}^-14$, and $^+12 + {}^-14 = {}^-2$, the same result as before.

53. Addition by counting makes it clear that two numbers of opposite signs tend to cancel each other when added.

E.g. In adding $^-5$ to $^+8$ we start with $^+8$, which we reach by counting from zero eight units to the right, and then add $^-5$ by counting five units to the left, thus retracing five of the units just counted. That is, $^-5$ annuls five of the $^+8$ units, leaving the sum $^+3$.

In case two numbers have opposite signs and equal absolute values one completely cancels the other.

E.g. $^+8 + {}^-8 = 0$.

EXERCISES AND PROBLEMS

Perform the following indicated additions:

1. $^-31 + {}^+42$.
2. $^-17 + {}^-13$.
3. $^+68 + {}^-46$.
4. $^+34 + {}^-46$.
5. $^+104 + {}^-245$.
6. $^-11\frac{3}{4} + {}^+8\frac{1}{4}$.
7. $^+16 + {}^-3 + {}^+7 + {}^+4 + {}^-19$.
8. $^+42 + {}^+74 + {}^-92 + {}^-7 + {}^-3$.
9. $^-13\,n + {}^+7\,n = {}^+7\,n + {}^-13\,n = {}^-6\,n$.

Solution. By Principle IX, to add $^-13\,n$ to $^+7\,n$ or to add $^+7\,n$ to $^-13\,n$ is to "find the difference of their absolute values and prefix to this the sign of that one whose absolute value is the greater." That is, $^-13\,n + {}^+7\,n = {}^-(13\,n - 7\,n) = {}^-6\,n$, since by Principle II, $13\,n - 7\,n = 6\,n$.

10. $^+14\,t + {}^-8\,t + {}^-3\,t + {}^-45\,t$.
11. $^+68x + {}^+34x + {}^-16x + {}^-3x$.
12. $296\,rt + {}^-367\,rt + {}^-119\,rt$.
13. $^-3(a+b) + {}^+4(a+b)$.
14. $^+7(x-y) + {}^-5(x-y)$.
15. $^-81(r+s) + {}^-91(r+s)$.

16. A man travels 10 miles east, then 23 miles west, and finally 5 miles east. Express his final *distance* and *direction* from the starting point as the sum of three signed numbers.

17. The temperature falls 35°, rises 24°, falls 3°, rises 17°, and finally falls 5°. What is the net change in temperature between the last reading and the original reading?

18. A real estate firm gains $5000 on one transaction, loses $2500 on a second, loses $1400 on a third, and gains $200 on a fourth. What is the aggregate result of the four transactions?

AVERAGES OF SIGNED NUMBERS

54. Half the sum of two numbers is called their average. Thus 6 is the average of 4 and 8. Similarly, the average of three numbers is one-third of their sum, and in general the average of n numbers is the sum of the numbers divided by n.

Find the average of each of the following sets:

1. 10, 12, 14, 16, 18.
2. 5, 9, 20, 30, 3.
3. 11, 10, 4, 6, 5.
4. 7, 9, 11, 13, 15.
5. 7, 10, 21, 29, 30.
6. 13, 8, 9, 10, 4.

The average gain or loss per year for a given number of years is the *algebraic sum* of the yearly gains and losses divided by the number of years.

Illustrative Problem. A man lost $400 the first year, gained $300 the second, and gained $1000 the third. What was the average loss or gain?

Solution. $\frac{^-400 + {}^+300 + {}^+1000}{3} = \frac{^+900}{3} = {}^+300$

That is, the average gain is $300.

Illustrative Problem. During four years a certain business shows an average annual gain of $5000. What was the loss or gain the first year, if for the remaining years there were gains of $8000, $9000, and $7500 respectively?

Solution. Let x represent the number of dollars gained or lost the first year. Then the average for the four years is

$$\frac{x + {}^+8000 + {}^+9000 + {}^+7500}{4}.$$

But the average for the four years is given as \$5000.

Hence $\dfrac{x + {}^{+}8000 + {}^{+}9000 + {}^{+}7500}{4} = {}^{+}5000.$

By M, $x + {}^{+}8000 + {}^{+}9000 + {}^{+}7500 = {}^{+}20000.$

By IX, $x + {}^{+}24500 = {}^{+}20000.$

By A, $x + {}^{+}24500 + {}^{-}24500 = {}^{+}20000 + {}^{-}24500.$

By IX, $x = {}^{-}4500.$

Hence there was a loss of \$4500 the first year.

PROBLEMS

1. Find the average of \$1800 loss, \$3100 loss, \$6800 gain, \$10,800 loss, and \$31,700 gain.

2. Find the average of \$180 gain, \$360 loss, \$480 loss, \$100 gain, \$700 gain, \$400 gain, \$1300 loss, \$300 gain, \$4840 gain, and \$12,000 gain.

3. A merchant gained an average of \$2800 per year for 5 years. The first year he gained \$3000, the second \$1500, the third \$4000, and the fourth \$2400. Did he gain or lose and how much during the fifth year?

4. A certain business shows an average gain of \$4000 per year for 6 years. During the first 5 years the results were, \$8000 loss, \$10,000 gain, \$7000 gain, \$3000 gain, and \$12,000 gain. Find the loss or gain during the sixth year.

5. Find the average of the following temperatures: 7 A.M., $^{-}4°$; 8 A.M., $^{-}2°$; 9 A.M., $^{-}1°$; 10 A.M., $^{+}1°$; 11 A.M., $^{+}5°$; 12 M., $^{+}7°$.

6. During the 12 hours ending at 6 A.M., January 19, 1892, the U. S. Weather Bureau at Helena, Montana, recorded the following temperatures: $^{-}9°$, $^{-}8°$, $^{-}8°$, $^{-}9°$, $^{-}9°$, $^{-}9°$, $^{-}8°$, $^{+}36°$, $^{+}37°$, $^{+}40°$, $^{+}20°$, $^{+}16°$. Find the average temperature for the 12 hours.

Find the average yearly temperatures at the following places, the monthly averages having been recorded as given below:

7. For New York City: $^+29°$, $^+33°$, $^+39°$, $^+46°$, $^+53°$, $^+63°$, $^+67°$, $^+67°$, $^+61°$, $^+52°$, $^+47°$, $^+41°$.

8. For Singapore, Straits Settlement: $^+81°$, $^+84°$, $^+85°$, $^+85°$, $^+85°$, $^+86°$, $^+86°$, $^+87°$, $^+85°$, $^+84°$, $^+83°$, $^+81°$.

9. For St. Vincent, Minnesota: $^-5°$, $0°$, $^+15°$, $^+35°$, $^+55°$, $^+60°$, $^+66°$, $^+63°$, $^+55°$, $^+40°$, $^+22°$, $^+5°$.

10. For Nerchinsk, Siberia: $^-23°$, $^-13°$, $^-10°$, $^+35°$, $^+55°$, $^+70°$, $^+70°$, $^+64°$, $^+50$, $^+30°$, $^+5°$, $^-15°$.

If the latitude of a place is midway between the latitudes of two given places, then its latitude equals one-half the *algebraic sum* of the two given latitudes.

Thus, if the latitude of one place is $^+16°$ and that of the other $^-56°$, then the latitude of the place midway between them is $\frac{^-56° + ^+16°}{2} = \frac{^-40°}{2} = ^-20°$.

The data used in some of the following problems vary slightly from the published records, but in no case more than 5′.

11. The latitude of New Orleans, Louisiana, is $^+30°$, and of Toronto, Canada, $^+43°\ 40'$. Find the latitude of Norfolk, Virginia, which is midway between these latitudes. (Positive numbers represent north latitude and negative numbers represent south latitude.)

12. The latitude of Alexandria, Egypt, is $^+31°\ 10'$, and of Christiania, Norway, $^+59°\ 50'$. That of Venice, Italy, is midway between these latitudes. Find the latitude of Venice.

13. The longitude of Edinburgh, Scotland, is $^-3°\ 10'$, and of Warsaw, Poland, $^+21°$. Find the longitude of Bremen, Germany, which is midway between these. (Positive numbers represent east longitude and negative numbers west longitude.)

14. The longitude of Vienna, Austria, is $^{+}16°\ 20'$, and of Splügen Pass, Switzerland, $^{+}9°\ 20'$. The longitude of Splügen Pass is midway between the longitudes of Vienna and Paris, France. Find the longitude of Paris.

15. The longitude of Brussels, Belgium, is $^{+}4°\ 20'$, and of Jena, Germany, $^{+}11°\ 40'$. The longitude of Brussels is midway between those of Jena, and Liverpool, England. Find the longitude of Liverpool.

16. The longitude of Berlin, Germany, is $^{+}13°\ 20'$, and of St. Petersburg, Russia, $^{+}30°\ 20'$. The longitude of Berlin is midway between those of St. Petersburg and Madrid, Spain. Find the longitude of Madrid.

17. The latitude of Rome, Italy, is $^{+}41°\ 55'$, and of Cuxhaven, Germany, $^{+}53°\ 55'$. The latitude of Rome is midway between those of Cuxhaven and Cairo, Egypt. Find the latitude of Cairo.

18. The longitude of Calais, France, is $^{+}1°\ 55'$ and of Lucknow, India, $^{+}80°\ 55'$. The longitude of Calais is midway between those of Lucknow and Washington, D.C. What is the longitude of Washington?

SUBTRACTION OF SIGNED NUMBERS

55. In arithmetic the accuracy of subtraction is tested by showing that the remainder added to the subtrahend equals the minuend.

E.g. We say $8 - 5 = 3$ because $5 + 3 = 8$.

Indeed, we may, and often do perform subtraction by starting with the subtrahend and counting until we reach the minuend.

E.g. A clerk in changing a dollar bill after a purchase of 63 cents might count out two pennies, a dime, and a quarter, saying "65, 75, one dollar." That is, he counts from 63 cents (the subtrahend) up to 100 cents (the minuend).

56. In like manner, signed numbers also may be subtracted, but we must find not only the *number of units, but also the direction* from the subtrahend to the minuend. This is most easily done by reference to the number scale, page 52.

Ex. 1. $^+8 - {}^+5 = {}^+3$, since in passing from $^+5$ to $^+8$ on the number scale we count 3 units in the *positive* direction.

Ex. 2. $^-8 - {}^-5 = {}^-3$, since from $^-5$ to $^-8$ we count 3 units in the *negative* direction.

Ex. 3. $^-8 - {}^+5 = {}^-13$, since from $^+5$ to $^-8$ we count 13 units in the *negative* direction.

Ex. 4. $^+8 - {}^-5 = {}^+13$, since from $^-5$ to $^+8$ we count 13 units in the *positive* direction.

57. In each of these examples we see that the result fulfills the test of arithmetic; namely,

$$Subtrahend + Difference = Minuend$$

Hence $^-8 - {}^-5 = {}^-3$ is checked by finding that $^-5 + {}^-3 = {}^-8$.

$^+8 - {}^-5 = {}^+13$ is checked by finding that $^-5 + {}^+13 = {}^+8$.

EXERCISES AND PROBLEMS

Perform the following indicated subtractions by finding the *distance* and *direction* on the number scale from subtrahend to minuend, and apply the check to each result.

1. $^-10 - {}^-5$	6. $^-17 - {}^-20$	11. $^+93 - {}^+22$
2. $^-15 - {}^+5$	7. $^+6 - {}^-14$	12. $^+17 - {}^-13$
3. $^+20 - {}^-15$	8. $^+7 - {}^-9$	13. $^-78 - {}^-37$
4. $^+11 - {}^+3$	9. $^-11 - {}^+6$	14. $^+57 - {}^+84$
5. $^-11 - {}^+5$	10. $^-21 - {}^-6$	15. $^-48 - {}^-31$

16. How many degrees, and in what direction, must the temperature change in order to vary from 12° below zero to 38° above zero? This is an example in subtraction, since we are required to find a number which added to $^-12°$ gives $^+38°$. Hence we write it $^+38° - {}^-12° =$ what?

17. How many degrees, and in what direction, does the thermometer change in passing from 27° above zero to 3° below zero? That is, $^-3° - ^+27° =$ what?

18. What must be added to $35 loss to make the sum $30 gain? That is, $^+30 - ^-35 =$ what?

19. What must be added to $15 gain to make the sum $8 loss? That is, $^-8 - ^+15 =$ what?

58. Subtraction always possible. In arithmetic subtraction is possible only when the subtrahend is less than or equal to the minuend.

E.g. 5 from 8 leaves 3, 5 from 5 leaves 0; but we cannot take 5 from 2, since when we have subtracted 2 units from 2 there are no more to take away.

However, by means of negative numbers we can as easily perform the subtraction, 2 minus 5, as 5 minus 2.

Thus, $2 - 5 = ^-3$, since $^-3 + 5 = 2$; and $5 - 2 = ^+3$, since, $^+3 + 2 = 5$.

It thus appears that, in terms of signed numbers, $a - b$ has a meaning no matter what numbers are represented by a and b; that is, $a - b$ means the number which added to b gives a.

59. A short rule for subtraction. Since $^+8 - ^-5 = ^+13$, and since $^+8 + ^+5 = ^+13$, it follows that subtracting $^-5$ from $^+8$ gives the same result as adding $^+5$ to $^+8$. Similarly $^-8 - ^-5 = ^-3$ and $^-8 + ^+5 = ^-3$.

Hence *subtracting a negative* number is equivalent to *adding a positive* number of the same absolute value.

Since $^+8 - ^+5 = ^+3$, and since $^+8 + ^-5 = ^+3$, it follows that subtracting $^+5$ from $^+8$ gives the same result as adding $^-5$ to $^+8$. Similarly $^-8 - ^+5 = ^-13$ and $^-8 + ^-5 = ^-13$.

Hence *subtracting a positive* number is equivalent to *adding a negative* number of the same absolute value.

These statements are illustrated by such facts as: *Removing a debt* is equivalent to *adding property* and *removing property* is equivalent to *adding debt.*

Perform the following subtractions as explained in this paragraph, by changing the sign of the subtrahend and adding:

1. $^-5 - {}^-2$.	5. $^+57 - {}^-32$.	9. $^+37 - {}^+50$.
2. $^-4 - {}^+1$.	6. $^-32 - {}^+34$.	10. $^-23 - {}^+57$.
3. $^-5 - {}^+2$.	7. $^-52 - {}^-32$.	11. $^-16a - {}^+4a$
4. $^+3 - {}^-5$.	8. $^-16 - {}^-12$.	12. $^+13t - {}^-20t$

The preceding exercises illustrate the following principle:

60. **Principle X.** *To subtract one signed number from another signed number, add the subtrahend with its sign changed to the minuend.*

The change in the sign of the subtrahend may be made *mentally* without re-writing the problem. The results are to be checked by showing that the difference added to the subtrahend equals the minuend.

EXERCISES

1. From $^+6 + {}^-2$ subtract $^-14$.
2. From $^-6a + {}^+2a$ subtract ^-14a.
3. From $17ab + 8ab$ subtract ^-35ab.
4. From $5ax + 4ax$ subtract $7ax + 2ax$.
5. From $54abc + {}^-47abc + 36abc$ subtract $80abc$.
6. From $54 \cdot 13 + {}^-47 \cdot 13 + 36 \cdot 13$ subtract $80 \cdot 13$.
7. From $29 \cdot 3 \cdot 11 + 37 \cdot 3 \cdot 11$ subtract $^-34 \cdot 3 \cdot 11$.
8. From $29xy + 37xy$ subtract ^-34xy.
9. Solve $x + 8 = 4$.

Solution. Subtract $^+8$ from each member (which is equivalent to adding $^-8$).

Then $x = {}^{+}4 - {}^{+}8 = {}^{-}4$, which is correct, since ${}^{-}4 + {}^{+}8 = {}^{+}4$. This is a problem in subtraction, since one of two numbers, 8, and their sum, 4, are given, and we are to find the second number, which is represented by x.

Solve the following equations:

10. $x + {}^{-}3 = 7.$
11. $x + {}^{-}9 = 1.$
12. $3 + x = 0.$
13. $x + {}^{-}1 = 2.$
14. $x + 13 = 7.$
15. $x + 4 = 2.$
16. $x + 9 = 3.$
17. ${}^{-}4 + x = 7.$
18. ${}^{-}5 + a = 4.$
19. ${}^{-}20 + t = {}^{-}12.$
20. $8 + n = {}^{-}15.$
21. $k + {}^{-}40 = {}^{-}65.$

MULTIPLICATION OF SIGNED NUMBERS

61. The multiplication of signed numbers is illustrated by the following problems.

Illustrative Problem. A balloonist, just before starting, makes the following preparations: (a) He adds 9000 cubic feet of gas with a lifting power of 75 pounds per thousand cubic feet; (b) He takes on 8 bags of sand, each weighing 15 pounds. How does each of these operations affect the buoyancy of the balloon?

Solution. (a) A lifting power of 75 lbs. is indicated by ${}^{+}75$, and adding such a power 9 times is indicated by ${}^{+}9$. Hence ${}^{+}9 \cdot {}^{+}75 = {}^{+}675$, the total lifting power added.

(b) A weight of 15 lbs. is indicated by ${}^{-}15$, and adding 8 such weights is indicated by ${}^{+}8$. Since the total weight added is 120 lbs., we have ${}^{+}8 \cdot {}^{-}15 = {}^{-}120$.

Illustrative Problem. During the course of his journey this balloonist opens the valve and allows 2000 cubic feet of gas to escape, and later throws overboard 4 bags of sand. What effect does each of these operations produce on the balloon?

Solution. (*a*) The gas, being a lifting power, is positive, but the removal of 2000 cubic feet of it is indicated by $^-2$, and the result is a depression of the balloon by 150 lbs.; that is, $^-2 \cdot {}^+75 = {}^-150$.

(*b*) The removal of 4 weights is indicated by $^-4$, but the weights themselves have the negative quality of downward pull. Hence to remove 4 weights of 15 lbs. each is equivalent to increasing the buoyancy of the balloon by 60 lbs; that is, $^-4 \cdot {}^-15 = {}^+60$.

62. These illustrations of multiplying signed numbers are natural extensions of the process of multiplication in arithmetic.

E.g. Just as $3 \cdot 4 = 4 + 4 + 4 = 12$, so $3 \cdot {}^-4 = {}^-4 + {}^-4 + {}^-4 = {}^-12$, and since $3 \cdot 4$ is the same as $^+3 \cdot {}^+4$, we write $^+3 \cdot {}^+4 = {}^+12$.

Again, just as we take the multiplicand *additively* when the multiplier is a positive integer, so we take it *subtractively* when the multiplier is negative integer.

E.g. $^-3 \cdot {}^+4$ means to subtract $^+4$ three times; that is, to subtract $^+12$. But to subtract $^+12$ is the same as to add $^-12$. Hence $^-3 \cdot {}^+4 = {}^-12$. Again, $^-3 \cdot {}^-4$ means to subtract $^-4$ three times; that is, to subtract $^-12$. But to subtract $^-12$ is the same as to add $^+12$. Hence $^-3 \cdot {}^-4 = {}^+12$.

EXERCISES AND PROBLEMS

Explain the following indicated multiplications and find the product in each case:

1. $^-3 \cdot {}^-10$.	5. $43 \cdot {}^-192$.	9. $71 \cdot {}^-x$.
2. $^-3 \cdot {}^+10$.	6. $^-27 \cdot {}^-235$.	10. $^-112 \cdot {}^-t$.
3. $^-5 \cdot {}^+50$.	7. $^-5 \cdot {}^+r$.	11. $^-14 \cdot y$.
4. $^-75 \cdot {}^-89$.	8. $^+16 \cdot {}^-r$.	12. $^-20 \cdot {}^-v$.

13. A man gained \$212 each month for 5 months, then lost \$175 per month for 3 months. Express his net gain or loss as the sum of two products.

14. A man gained $2100 during a certain year. For the first 4 months he lost $125 per month. During the next 5 months he gained $500 per month. Find his gain or loss during the remaining 3 months of the year. Express the net gain as the sum of two products.

15. A raft is made of cork and iron. What effects are produced upon its floating qualities by the following changes? (*a*) Adding 4 braces, each weighing (under water) 5 lbs. (*b*) Removing 3 pieces of cork, each capable of sustaining 3 lbs. (*c*) Adding 10 pieces of cork, each capable of sustaining 7 lbs.

16. What are the effects on a shipwrecked man's ability to float? (*a*) If he holds fast to 3 bags of gold, each weighing 10 lbs. (*b*) If he ties on two life preservers, each capable of supporting 15 lbs. (*c*) If he throws away his two boots, each weighing 2 lbs.

The preceding exercises illustrate the following principle:

63. Principle XI. *If two signed numbers are of the same quality, their product is positive; if they are of opposite quality, their product is negative. The absolute value of the product is the product of the absolute values of the factors.*

In applying this principle observe that the sign of the product is obtained quite independently of the absolute value of the two factors.

E.g. $\frac{3}{4} \cdot {}^{-}5 = {}^{-}(\frac{15}{4}) = {}^{-}3\frac{3}{4}$; ${}^{-}12 \cdot {}^{-}3.5 = {}^{+}42$.

64. Principle XI is also stated in symbols as follows:

$${}^{+}a \cdot {}^{+}b = {}^{+}ab,\ {}^{-}a \cdot {}^{-}b = {}^{+}ab,\ {}^{+}a \cdot {}^{-}b = {}^{-}ab,\ {}^{-}a \cdot {}^{+}b = {}^{-}ab.$$

The product of several signed numbers is found as illustrated in the following:

${}^{-}2 \cdot {}^{+}5 \cdot {}^{-}3 \cdot {}^{-}4 \cdot {}^{+}6 = {}^{-}10 \cdot {}^{-}3 \cdot {}^{-}4 \cdot {}^{+}6 = {}^{+}30 \cdot {}^{-}4 \cdot {}^{+}6 = {}^{-}120 \cdot {}^{+}6 = {}^{-}720$. That is, any two factors are multiplied together, then this product by another factor, etc., until all the factors are multiplied.

65. Evidently the factors in such a product may be taken in any desired order. Let the student try other orders in the above example.

Since the product of all positive factors is positive, the final sign depends upon the number of negative factors. If this number is *even*, the product is positive; if it is *odd*, the product is negative.

E.g. If there are 5 negative factors, the product is negative; if there are 6, it is positive.

EXERCISES

In the following exercises determine the sign of the product before finding its absolute value. State each principle used in the reduction to the final form.

1. $^-4 \cdot {}^+3 \cdot {}^-6 \cdot {}^-7$.
2. $^-50 \cdot {}^-20 \cdot {}^-30 \cdot {}^-40$.
3. $^-a \cdot {}^-b \cdot {}^+c \cdot {}^+d \cdot {}^+e$.
4. $a \cdot {}^-b \cdot {}^+c \cdot {}^-d \cdot {}^-x$.
5. $^-5(^-3 + {}^-7)$.
6. $a(^-b - {}^-c)$.
7. $^-c(x - {}^-y)$.
8. $7a(x + {}^-y - {}^-z)$.
9. $n(3a + {}^-4a - {}^+5a)$.
10. $8(16x + {}^-20x) \div 4$.
11. $5x - 3(^-2x + {}^+3x - {}^-4x)$.
12. $6r + 4(3r - {}^-5r + {}^-7r)$.
13. $^-5(^-4 \cdot {}^+3 \cdot {}^-2)$.
14. $6x - {}^-14x - (^-5x + {}^-7x)$.
15. $8y - 16y + (4y + {}^-11y)$.
16. $11t - 20(t + {}^-3 \cdot {}^-5t)$.

DIVISION OF SIGNED NUMBERS

66. In arithmetic we test the correctness of division by showing that the quotient multiplied by the divisor equals the dividend.

E.g. $27 \div 9 = 3$, because $9 \cdot 3 = 27$.

Hence **division** may be defined as the process of finding one of two factors when their product and the other factor are given.

This definition also applies to the division of signed numbers. In dividing signed numbers, however, we must determine the *sign* of the quotient as well as its *absolute value*.

E.g. $^-42 \div {^+6} = {^-7}$, because $^-7 \cdot {^+6} = {^-42}$;

also $^-42 \div {^-6} = {^+7}$, because $^+7 \cdot {^-6} = {^-42}$.

So in every case the test is:

$$\textit{Quotient} \times \textit{Divisor} = \textit{Dividend}.$$

In like manner perform the following:

1. $^-25 \div 5$.
2. $^-ab \div a$.
3. $^+5xy \div {^-x}$.
4. $^-9\,rs \div {^+3}$.
5. $^+75y \div {^-15}$.
6. $^-121\,x \div {^+11}$.

The preceding exercises illustrate the following principle:

67. **Principle XII.** *The quotient of two signed numbers is positive if the dividend and divisor have the same sign, negative if they have opposite signs. The absolute value of the quotient is the quotient of the absolute values of dividend and divisor.*

EXERCISES

Perform the following indicated divisions. Check by multiplying quotient by divisor.

1. $^-28 \div {^+7}$.
2. $^-42 \div {^-6}$.
3. $51 \div {^-17}$.
4. $21\,xy \div 3$.
5. $^-16\,ab \div {^-4}$.
6. $^-15\,ax \div x$.
7. $^-32\,(a-b) \div {^-(a-b)}$.
8. $27\,(x+y) \div {^-9}$.
9. $4 \cdot {^-9x} \div {^-3}$.
10. $3 \cdot 8t \div {^-4}$.
11. $100 \cdot {^-99\,x} \div {^-25}$.
12. $1600 \cdot 87\,y \div {^-400}$.
13. $(8\,x + 4\,y) \div {^-2}$.
14. $(16\,a + {^-20\,b}) \div {^-4}$.
15. $({^-6\,r} + 9\,s - {^-12\,t}) \div {^-3}$.
16. $(7\,ax + {^-14\,ay} - {^-21\,bz}) \div {^+7}$.
17. $(12\,xy - {^-3\,ax}) \div {^-x}$.
18. $(27\,ab - 36\,ac) \div {^-9}$.
19. $(3 \cdot 4\,y + 6 \cdot 8\,x) \div {^-3}$.
20. $2\,a\,(4\,x - 3\,y - z) \div a$.

68. While Principles I–VIII were studied in connection with unsigned, or arithmetic numbers only, it is now very important to note that they all apply to *signed* numbers as well. The form changes described in § 37 also apply to signed numbers just the same as to arithmetic numbers.

In the statement of these principles the word *number* will from now on be understood to refer either to the ordinary numbers of arithmetic or to the signed numbers, as occasion may require. It should also be noticed that the numbers of arithmetic are used as freely in algebra as in arithmetic. It is only when we wish to distinguish them from negative numbers that they are called positive numbers.

The **number system of algebra**, as far as we have studied it, consists of the *numbers of arithmetic* together with the *negative numbers.*

INTERPRETATION AND USE OF NEGATIVE NUMBERS

69. Illustrative Problem. Divide 34 units into two parts such that one part is equal to the remainder when 3 times the other part is subtracted from 46.

Solution. Let the two numbers be represented by x and $34 - x$.

Then	$34 - x = 46 - 3x.$	(1)
$S \mid 34,$	$-x = 12 - 3x.$	(2)
$A \mid 3x,$	$3x - x = 12.$	(3)
Principle II,	$2x = 12.$	(4)
$D \mid 2,$	$x = 6.$	(5)

Substituting $x = 6$ in (1), $34 - 6 = 46 - 18$, or $28 = 28$.

In the above solution, equation (2) was obtained from (1) by subtracting 34 from each member. This would clearly be impossible without the use of negative numbers.

In this case the problem itself does not involve negative numbers, but in the course of its solution they naturally occur. If the negative number could not be used, we should be compelled to keep the members of each equation positive or zero. This would be impossible, since we do not know what numbers are represented by the letters involved, and hence cannot tell by inspection whether a given term is positive or not.

70. We have seen how naturally the use of signed numbers has arisen in problems where things of opposite qualities have to be distinguished.

In solving a problem, a negative result may have a natural interpretation or it may indicate that the conditions of the problem are impossible.

A similar statement holds in reference to fractional answers in arithmetic. For example, if we say there are twice as many girls as boys in a schoolroom and 35 pupils in all, the number of boys would be $35 \div 3 = 11\frac{2}{3}$, which indicates that the conditions of the problem are impossible.

71. **Illustrative Problem.** The crews on three steamers together number 94 men. The second has 40 more than the first, and the third 20 more than the second. How many men in each crew?

Solution. Let $n =$ number of men in first crew.

Then $n + 40 =$ number of men in second crew,

and $n + 40 + 20 =$ number of men in third crew.

Hence $n + n + 40 + n + 40 + 20 = 94,$

and

$$3\,n + 100 = 94.$$
$$3\,n = {}^{-}6.$$
$$n = {}^{-}2.$$

Here the negative result indicates that the conditions of the problem are *impossible.*

72. **Illustrative Problem.** A real estate agent gained \$8400 on four transactions. On the first he gained \$6400, on the second he lost \$2100, on the third he gained \$5000. Did he lose or gain on the fourth transaction?

Solution. Since we do not know whether he gained or lost on that transaction, we represent the unknown number by n, which may be positive or negative, as will be determined by the solution of the problem.

Thus we have	$6400 + {}^{-}2100 + 5000 + n = 8400.$	(1)
Hence by IX, *F*,	${}^{+}9300 + n = 8400.$	(2)
By S,	$n = 8400 - 9300.$	(3)
By X,	$n = {}^{-}900.$	(4)

In this case the negative result indicates that there was a *loss* on the fourth transaction.

PROBLEMS

In the following problems give the solutions in full and state all principles used, together with the interpretation of the results:

1. A man gains \$2100 during one year. During the first three months he loses \$125 per month, then gains \$500 per month during the next five months. What is the gain or loss per month during the remaining four months?

2. A man gained \$1250 during four months. During the second month he gained \$600 more than the first month, the third month he gained \$300 less than the second, and the fourth he gained \$200 more than the third. Find the gain or loss for the first month.

3. A box containing a Christmas toy weighed 25 oz. When the toy and the packing were removed, the box weighed 20 oz. The packing weighed 7 oz. What kind of a toy was it?

4. A man rowing against a swift current rows 8 miles in 5 hours. The second hour he rows one mile less than the first, the third two miles more than the second, and the fourth and fifth one mile more each than he rowed the third hour. How many miles did he row each hour?

5. There are three trees the sum of whose heights is 108 feet. The second is 40 feet taller than the first, and the third is 30 feet taller than the second. How tall is each tree?

Find the average yearly temperature at each of the following places, the average monthly temperatures being as here given:

6. Port Conger, off the northwest coast of Greenland; $^-37°$, $^-43°$, $^-32°$, $^-15°$, $^+14°$, $^+18°$, $^+35°$, $^+34°$, $^+25°$, $^+4°$, $^-17°$, $^-30°$.

7. Franz Joseph's Land; $^-20°$, $^-20°$, $^-10°$, 0°, 15°, 30°, 35°, 30°, 20°, 10°, 0°, $^-10°$.

8. Western Baffin Land; $^-30°$, $^-30°$, $^-20°$, 0°, 20°, 35°, 40°, 35°, 25°, 10°, $^-10°$, $^-20°$.

Find the average yearly loss or gain in each of the following:

9. \$1600 gain, \$8000 loss, \$24,000 gain, \$40,000 loss.

10. \$32,000 gain, \$45,000 loss, \$24,000 gain, \$42,000 loss.

11. The average yearly temperature of north central Siberia is $^-5°$. The average monthly temperatures beginning in February are: $^-50°$, $^-30°$, 0°, 15°, 40°, 40°, 35°, 30°, 0°, $^-30°$, $^-50°$. Find the temperature for January.

12. The business transactions of a certain firm averaged \$1500 loss for 4 years. For the first year there was a gain of \$800, the second year a loss of \$1800, the third year a loss of \$300. What was the loss or gain for the fourth year?

13. A commercial house averaged \$15,000 gain for 5 years. What was the loss or gain the first year if the remaining years show: \$8000 gain, \$24,000 gain, \$2000 loss, \$20,000 gain, and \$50,000 gain, respectively?

14. The longitude of Boston, Massachusetts, is $^-71°\ 10'$, and that of Chicago, Illinois, is $^-87°\ 35'$. Find the longitude of Lake Chautauqua, which is midway between these.

15. The longitude of New Haven, Connecticut, is $^-72°\ 58'$, and that of Bombay, India, is $^+72°\ 48'$. The longitude of St. Paul's Cathedral, London, is midway between these. Find the longitude of the cathedral.

16. The longitude of Cincinnati, Ohio, is $^{-}84° 30'$, and that of Indianapolis, Indiana, is $^{-}86° 5'$. The longitude of Cincinnati is midway between those of Indianapolis, and Columbus, Ohio. Find the longitude of Columbus.

17. The longitude of Bristol, England, is $^{-}2° 30'$, and that of Minneapolis, Minnesota, is $^{-}93° 20'$. The longitude of Bristol is midway between those of Minneapolis and Calcutta, India. Find the longitude of Calcutta.

18. The latitude of Columbus, Ohio, is $^{+}40°$ and that of Winnipeg, Canada, is $^{+}50°$. The latitude of Columbus is midway between those of Winnipeg and Houston, Texas. Find the latitude of Houston.

19. The longitude of Montreal, Canada, is $^{-}73° 40'$ and that of Baltimore, Maryland, $^{-}76° 40'$. Find the latitude of Philadelphia, which is midway between these.

20. The latitude of Lima, Peru, is $^{-}12°$ and that of Buenos Ayres, Argentina, $^{-}34° 35'$. The latitude of Lima is midway between those of Buenos Ayres and Caracas, Venezuela. Find the latitude of Caracas.

21. The longitude of Providence, Rhode Island, is $^{-}71° 75'$ and that of Fargo, North Dakota, $^{-}96° 50'$. The longitude of Fargo is midway between those of Providence and Seattle, Washington. Find the longitude of Seattle.

CHAPTER III

INVOLVED NUMBER EXPRESSIONS

73. Double Use of the Signs + and −. In the preceding chapter it has been found that the negative quality may be regarded as implying subtraction and the positive quality as implying addition. It was for this reason that $^+$ and $^-$ were selected as symbols for the words "positive" and "negative."

74. It is now possible to dispense with these special signs of quality. For, by Principle X, $a - {}^+b$ and $a + {}^-b$ are the same in effect, and likewise $a - {}^-b$ and $a + {}^+b$. Hence, omitting the positive signs (§ 45), we may write

$$a + {}^-b = a - b \text{ and } a - {}^-b = a + b.$$

One set of signs is, therefore, sufficient as symbols both of operation and of quality.

E.g. $5 - 7$ means either $5 + {}^-7$ or $5 - {}^+7$, and in either case equals ${}^-2$, which we now write -2. Thus $5 - 7$ equals -2, which is read, 5 *minus* 7 *equals negative* 2.

Ex. 1. ${}^-5 \cdot {}^-4 = {}^+20$ is now written $-5 \cdot -4$, or $(-5)(-4)$, $= +20$,

and ${}^-5 \cdot {}^+4 = {}^-20$ is written $-5 \cdot +4$ or, $(-5)\ (+4), = -20$.

Ex. 2. $5(9a + {}^-2a) = 5(9a - 2a)$.

By IV, $= 45a - 10a$.

By II, $= 35a$.

Ex. 3. $5a + {}^-4({}^-3a - {}^+7b) = 5a - 4(-3a - 7b)$.

By IV, XI, $= 5a + 12a + 28b$.

By I, $= 17a + 28b$.

EXERCISES

Rewrite the following, using one set of signs, and then perform the indicated operations.

1. $7 - {}^{+}3 + {}^{-}8$.
2. ${}^{-}9 - {}^{-}3 + {}^{-}12$.
3. ${}^{-}4\,a + {}^{-}5\,a + {}^{-}6\,a$.
4. ${}^{-}3 \cdot 5\,x + 4 \cdot 7\,x - 8 \cdot 8\,x$.
5. $27\,abc + {}^{-}35\,abc - 2\,abc$.
6. $5({}^{-}2 + {}^{-}3) + 4(5 - {}^{-}7)$.
7. ${}^{-}3\,(7 - 2) - 8(6 + {}^{-}9)$.
8. $3\,(4\,a - {}^{-}5\,b) - 11\,({}^{-}2\,a + {}^{-}3\,b)$

Rewrite the following expressions, using special signs of quality so that all signs of operation shall indicate addition:

9. $5 - 8 - 14 = 5 + {}^{-}8 + {}^{-}14$.
10. $-7 + 8 - 18$.
11. $-4\,a + 5\,a - 17\,a$.
12. $-7 \cdot 4\,x + 7 \cdot 4\,x - 8 \cdot 4\,x$.
13. $56\,ay - 72\,ay + 7ay$.
14. $3(2 - 5) + 5\,(3 - 7)$.
15. $-3\,(8 - 6) - 4(6 - 9)$.
16. $8\,(4\,t - 5\,n) - 5\,(-t + 4\,n)$.

POLYNOMIALS

75. We have found that the solution of problems leads us to build **involved number expressions** out of single number symbols.

E.g. If x is a number representing my age in years, then $2(x - 10)$ is double the number representing my age 10 years ago, and $2\,[(x - 10) + (x + 15)]$ is the number representing twice the sum of my ages 10 years ago and 15 years hence.

Number expressions are now to be studied more in detail.

76. Definition. A number expression composed of parts connected by the signs $+$ and $-$ is called a **polynomial**. Each of the parts thus connected together with the sign preceding it is called a **term**.

E.g. $5\,a - 3\,xy - \frac{2}{3}\,rt + 99$ is a polynomial whose terms are $5\,a$, $-3\,xy$, $-\frac{2}{3}\,rt$, and $+\,99$. The sign $+$ is understood before $5\,a$.

77. Definitions. A polynomial of two terms is called a **binomial**, one of three terms is called a **trinomial**. A term taken by itself is called a **monomial**.

E.g. $5a - 3xy$ is a binomial; $5a - 3xy - \frac{2}{3}rt$ is a trinomial; $5a$, $-3xy$, $-\frac{2}{3}rt$ are monomials.

According to the above definition $x + (b + c)$ may be called a binomial notwithstanding it is equivalent to the trinomial $x + b + c$.

In this case x is called a **simple** term and $(b + c)$ a **compound** term. Likewise we may call $3t + 4x - 5(a + b)y$ a trinomial having the simple terms $3t$, $+4x$, and the compound term $-5(a + b)y$.

It should be clearly understood that a negative or positive sign before a compound term (as well as before a simple term) applies to the number represented by the whole term.

78. Definition. Two terms which have a factor in common are called **similar with respect to that factor.**

E.g. $5a$ and $-3a$ are similar with respect to a; $-3xy$ and $-7x$ are similar with respect to x; $5a$ and $-5b$ are similar with respect to 5; $7abc$ and $-\frac{2}{3}abc$ are similar with respect to abc.

Similar terms may be combined by Principles I, II, and IX.

E.g. $5a - 3a = (5 - 3)a = 2a$; $-3xy - 7x = -x(3y + 7)$; $5a - 5b = 5(a - b)$.

ADDITION AND SUBTRACTION OF POLYNOMIALS

79. In adding or subtracting polynomials the work may be conveniently arranged by placing the terms in columns, each column consisting of terms which are similar.

Ex. Add $5x - 6y + 4z + 5at$, $-3x + 11y - 16z - 9bt$, and $-7y + 8z$.

Arranging as suggested and applying Principles I, II, and IX, we have

$$\begin{array}{r} 5x - \ \ 6y + \ \ 4z + 5at \\ -3x + 11y - 16z - 9bt \\ - \ \ 7y + \ \ 8z \qquad\quad \\ \hline 2x - \ \ 2y - \ \ 4z + t(5a - 9b) \end{array}$$

$5x$ and $-3x$ are similar with respect to their common factor x. Hence by Principle I we add the other factors 5 and -3, obtaining $(5-3)x = 2x$.

Likewise we add $+5at$ and $-9bt$ with respect to the common factor t, obtaining $(5a-9b)t$. In the second column the sum is $(-6+11-7)y = -2y$, and in the third column the sum is $(+4-16+8)z = -4z$.

Check by giving convenient values to x, y, z, t, a, and b.

EXERCISES

1. Add $7b-3c+2d$; $-2b+8c-13d$.

2. Add $6x-3y+4t-7z$; $x-5y-3t$; $4x-4y+8t$.

3. Add $5ac+3bc-4c+8b$; $2b+3c-2bc-3ac$; $4b+4c+bc-ac$; $2bc+4ac+c$; $3b-4c$.

4. Add $3\cdot4\cdot7-5x+5abx$; $3aby+3x-5\cdot4\cdot7$; $7x+2\cdot4\cdot7$; $5aby-3x-5\cdot4\cdot7$; $45x+abx-4\cdot7$.

5. Add $3(x-5)+a(c+b)+b(x-y)$; $b(c+b)-a(x-y)+8(x-5)$; $7(c+b)-4(x-y)$; $3(x-y)+(x-5)$.

6. Add $16(a+b-c)-3(x-y)+2(a-b)$; $2(x-y)-3(a-b)+(a+b-c)$; $7(a-b)+a(x-y)-b(a+b-c)$.

7. Add $a(a-b)-c(x+y)+d(x-z)-4abc$; $c(x-z)-d(x+y)+(a-b)+2abc$; $e(a-b)+mabc+3(x-z)+8(x+y)$.

8. Add $7a-4x+12z$; $ba-3x+cz$; $2ba+4x-3cz$.

9. Add $(a-b)-3(c-d)+m(a+d)$; $c(a-b)+a(c-d)$.

10. Add $34ax+4by-3z$; $2by+5z$; $3bx-7y+5dz$.

11. Add $3b+4cd-2ae$; $ab-3cd+3ae$; $3cd-2ab$.

12. Add $7ax-13by+5$; $9ax+8by-4$; $3b-12ax$.

13. Add $5(a+b)-3(c-d)$; $3(c-d)-8(a+b)$; $-2(a+b)$.

14. Add $3+4(c-d)-5(a-b-c)$; $4(a-b-c)+5(c-d)$; $3(a-b-c)-9(c-d)+12$.

15. Add $11(c-9)+3(x+y)+21wu$; $-71wu-5(x+y)-13(c-9)$.

16. Add $5ab-3\cdot 7\cdot 9+5(x-1)$; $5\cdot 9\cdot 7+3ab-2(x-1)$; $3(x-1)-4\cdot 7\cdot 9+2ab$.

17. Add $31\cdot 50-43\cdot 74+2\cdot 18$; $21\cdot 74+7\cdot 18-56\cdot 50$; $-12\cdot 18+42\cdot 50-6\cdot 74$.

18. Add $7(x-y)-4(x+y)+4\cdot 7$; $9(x+y)+3(x-y)-9\cdot 7$; $6(x-y)+2\cdot 7-3(x+y)$.

19. Add $16xy-13\cdot 64$; $15ab-2xy$; $34\cdot 64-3xy+2ab$; $14\cdot 64-3xy-2ab$.

80. The subtraction of polynomials is illustrated by the following example:

From $15ab-17xy+11rt$ subtract $-5ab+4xy-5nt$.

Arranging as on page 75 and applying Principles II and X,

$$\begin{array}{r} 15ab-17xy+11rt \\ -5ab+\ \ 4xy-\ \ 5nt \\ \hline 20ab-21xy+t(11r+5n) \end{array}$$

As suggested in § 60, it is sufficient to change the signs of the subtrahend *mentally*, rather than to rewrite them before adding to the minuend.

EXERCISES

1. From $9x+3y-11z$ subtract $-5x+8y-3z$.

2. From $12ab-3cd+12xy$ subtract $3ab+2cd-11xy$.

3. From $9xc+4ad-3cz+2y$ subtract $3y-3ad+5cz$.

4. From $13t+5mx-5cv$ subtract $2t-4mx+3cv$.

5. From $3v-2w+5mn-4xz$ subtract $-v+5w-3mn$.

6. From $31b+4xy+16ax-4$ subtract $8b-5xy-3ax$.

7. From $4-3a-5xz-3vy-x$ subtract $7a+2xz+4vy$.

8. From $8xy-3x+4y$ subtract $-2xy+13w+4x-2y$.

9. From $2ab-5+7v+13abc$ subtract $3ab+v+8abc$.

10. From $8cxa-4yb-3yc$ subtract $4bxa+2yb+4yc-49$.

11. From $31\cdot 45-7xy$ subtract $12\cdot 45+9xy$.

12. From $3abc-4\cdot 28+2(x+y)-3xy$ subtract $6\cdot 28+4xy-3(x+y)+8abc$.

13. From $7\cdot 3\cdot 5+9(xy-z)+4\cdot 3(a+b)$ subtract $8(xy-z)-8\cdot 3(a+b)+8\cdot 3\cdot 5$.

14. From $5ax-3by+4ax+5by$ subtract $5by-3ax+7by$.

15. From $19(r-5s)+13(5x-4)+7(x-y)$ subtract $17(5x-4)-5(x-y)-11(r-5s)$.

16. From $16-15\cdot 30+14(x-5yz)-13(5y-z)$ subtract $32-16\cdot 30+8(5y-z)$.

17. From $-41\cdot 3+13\cdot 4\cdot 16$ subtract $7\cdot 4\cdot 16-8\cdot 3$.

18. From $a(b+c)+4(m+n)-16c$ subtract $9(m+n)+31c-d(b+c)$.

19. From $5(7x-4)+3(5y-3x)+5\cdot 7$ subtract $8\cdot 7-9(7x-4)+8(5y-3x)$.

20. From $15\cdot 48+8ab+49x$ subtract $7\cdot 48-9ab-14x$.

EXERCISES IN ADDITION AND SUBTRACTION

1. Add $5x-3y-7r+8t$, $-7x+18y-4r-7t$, $-20x-24y+18r-15t$, and $13x+15y+11r+6t$.

Check the sum by substituting $x=1$, $y=1$, $r=1$, $t=1$.

2. Add $17a-9b$, $3c+14a$, $b-3a$, $a-17c$, and $a-3b+4c$. Check for $a=1$, $b=2$, $c=3$.

3. Add $2x+3y-t$, $-6y+8t$, $-x+y-t$, $-4t+7x$, and $3y$. Check for $x=2$, $y=3$, $t=1$.

4. Add $17r+4s-t$, $2t+3u$, $2r-3s+4t$, $5u-6t$, $7r-3s+8u$, and $8r-2t+6u$. Check by putting each letter equal to 1; also equal to 2.

5. Add $3h+2t+4u$ and $h+3t+3u$. Check by putting $h=100$, $t=10$, $u=1$; *i.e.* $324+133=457$.

6. Add $4h+3t+u$ and $3h+2t+7u$. Check as in 5.

7. Write 247, 323, 647, 239, and 41, as number expressions like those in 5 and 6 and then add them.

8. Add $4t-u$, $5t-u$, $6t-u$, $7t-u$, and $8t-u$. Check for $t=10$, $u=1$; also $t=1$, $u=1$.

9. Add 647, 391, 276, and 444 as in example 7.

10. Simplify: $3xyz-2xyz+5xyz-4xyz+xyz-xyz$.

11. Subtract $5a-3b+6c$ from $-8a+7b-11c$ and check.

12. From $7xy+8xz+9yz$ take $17xy-19xz-20yz$.

13. From $6x-3y$ take $8y-3z$.

14. From $3p-4q+8r$ take $7p-11r+11q$.

15. From $a+b+c$ take $x-y+z$. *Suggestion*: By Principle VII, $a+b+c-(x-y+z)=a+b+c-x+y-z$.

16. From $2x-3y$ take $5x+7y+2a-3b$.

17. From the sum of $18abc-27xyz+13rst$ and $-11abc+16xyz-52rst$ take $67rst-39abc$.

18. To the difference between the subtrahend $15x - 18y + 27z$ and the minuend $117x + 97y - 81z$ add $4x - 6y + 3z$.

19. Add $11(x-y) + 15(a-b)$ and $-20(x-y) - 37(a-b)$ and from the sum subtract $135(x-y) - 213(a-b)$.

20. Add $6ax + 7bx - 8cx$, $-11ax - 18bx + 25cx$, and $19ax - 16cx + 24bx$.

21. From $13mn - 25mp + 36mq$ subtract $18mn + 23mp$.

22. Add by Principle I, $6 \cdot 3 \cdot 9 - 11 \cdot 5 \cdot 7 + 16 \cdot 9 \cdot 11$ and $-8 \cdot 3 \cdot 9 + 24 \cdot 5 \cdot 7 - 23 \cdot 9 \cdot 11$.

23. From $83 \cdot 9 + 78 \cdot 13$ subtract $57 \cdot 9 - 93 \cdot 13 + 85 \cdot 17$.

24. From $3h + 4t + 2u$ subtract $h + 5t + 3u$. Check.

25. Subtract $7(a-x) - 10(b-y)$ from $13(a-x) + 5(b-y)$.

81. In solving problems it is often unnecessary to arrange the work of addition and subtraction in columns as above. In most cases the operations can be readily indicated by means of parentheses as illustrated in the following example:

From the sum of $3a + 4b$ and $5a - 8b$ subtract $7a - 6b$.

Indicate these operations thus: $3a + 4b + (5a - 8b) - (7a - 6b)$.
Applying Principle VII, we have $3a + 4b + 5a - 8b - 7a + 6b$.
Collecting similar terms, $3a + 5a - 7a + 4b - 8b + 6b$.
Finally, applying Principles I and II, we obtain $a + 2b$.
After a little practice the last two steps can be taken at once.

EXERCISES AND PROBLEMS

Perform the following indicated operations by collecting similar terms at once without arranging in columns:

1. $15 + (7 - 9x) - (-7x - 9) + 9$.
2. $7 + 5y - (3y + 2) + (8 - 4y)$.
3. $2a + 3 + (4a - 5) - (11a - 14)$.
4. $32b - (17b - 12) - (4b - 13)$.

5. $16c-(41-7c)+(15-8c)$.

6. $-(5a-3c)-(2c-8a)+3a$.

7. $-(-12x-7y-15x)-(-9y+8x+3y)$.

8. $(19x+4y-32x-17x)-12x-(49y+18x-70x)$.

9. $17a-3-(7a-2)+(6a-5)$.

10. $5x-(8-4x+7y)+(5x+3)-(5y+3x-99)$.

11. $-(3a+5b-7c)+(8a-4c)-(9c-4b+4a)-91a$.

12. $7-(4-4c+2d-2a)+31c-(4-2a-5d)-(-8c)$.

13. $(41ab-21c+4)-(36c+15-78ab)+(13c-90ab-8)$.

14. $9by-(4c-8by-13)-2c-16-(34by-12c+8by)$.

15. $6mn+(-9m-7n+14)-8n+(13mn-17m)+34mn$.

16. $34ax-(-17ax+42)+8x-(14a+24ax-7)$.

17. $19-(+2-7a-4b+11ab)-(-2b+8ab+4a)$.

18. $41by-(4b-13y+17by)-(-5b-17by+13y)$.

19. $39rs-20s-19r-(7rs+8s-19r)-(15r-5s-56)$.

20. $a(3x-2y-z)-(5ax-ay+3z)+az$.

21. $5(4h+3bk-7br)-b(15k-35r)-20h$.

22. The altitude of Popocatepetl is 1716 feet less than that of Mt. Logan, and the altitude of Mt. St. Elias is 316 feet greater than that of Popocatepetl. Find the altitude of each mountain, the sum of their altitudes being 55,384 feet.

23. The Ganges River is 1800 miles shorter than the Amazon, and the Orinoco is 300 miles shorter than the Ganges. The sum of their lengths is 6900 miles. How long is each?

24. A cubic foot of red oak weighs 35 pounds less than 2 cubic feet of cherry wood, and 21 pounds more than a cubic foot of chestnut; while a cubic foot of chestnut weighs 100 pounds less than 3 cubic feet of cherry. Find the weight of each kind of wood per cubic foot.

25. Lead weighs 259 pounds more per cubic foot than cast iron, and 166 pounds more than bronze; while a cubic foot of bronze weighs 807 pounds less than 3 cubic feet of iron. Find the weight per cubic foot of each metal.

26. Green glass weighs 60 pounds per cubic foot less than dense flint glass, and 8 pounds more than crown glass; while a cubic foot of crown glass weighs 293 pounds less than 2 cubic feet of flint glass. Find the weight per cubic foot of each.

27. Europe has 12 million inhabitants less than 10 times as many as South America, and North America has 29 million more than twice as many as South America. If 3 times the population of North America be subtracted from that of Europe, the remainder is 65 million. How many inhabitants has each continent?

28. The length of the Rio Grande River is $\frac{3}{4}$ that of the Volga, and the Mississippi is 600 miles less than twice as long as the Volga. If $\frac{1}{3}$ the length of the Mississippi be subtracted from that of the Rio Grande, the remainder is 400 miles. Find the length of each river.

29. In 1900 the total wealth of the United States was 1532 million dollars more than 13 times as great as in 1850, and 9016 million more than twice as great as in 1880. The total wealth in 1880 was 174 million less than 6 times as great as in 1850. What was the wealth in each of the three years?

30. The money circulation of the United States in 1880 was 13 million dollars more than 60 times that in 1800 and in 1905 it was 188 million more than 150 times that in 1800. One-seventh of the amount in 1880 plus $\frac{1}{4}$ the amount in 1900 was 786 million. Find the circulation for each year.

31. The total bank deposits in the United States in 1905 were 3127 million dollars less than twice as great as in 1900 and 681 million more than 5 times as great as in 1880. The deposits in 1880 were 5105 million less than in 1900. Find the deposits for each of the three years.

32. The amount of deposits in savings banks in the United States in 1905 was 703 million dollars greater than in 1900, and 183 million less than 4 times that in 1880. The amount in 1900 was 67 million less than 3 times as great as in 1880. Find the deposits for each of the three years.

33. The total value of the farms in the United States in 1880 was 280 million dollars more than 3 times their value in 1850, and 8333 million less than their value in 1900. The value of the farms in 1880 was 1924 million less than $\frac{1}{2}$ their value in 1900. Find the value in each of the three years.

34. The value of the manufactures in the United States in 1900 was 811 million dollars more than 12 times that in 1850, and 3071 million less than 3 times that of 1880; while the value in 1880 was 744 million less than 6 times that in 1850. Find the value of the manufactures for each year.

MULTIPLICATION OF POLYNOMIALS

82. Illustrative Problem. A rectangular field is 12 rods longer than it is wide; a second rectangular field which is 4 rods shorter and 2 rods wider than the first has an area of 80 square rods less. What are dimensions of the first field?

Solution. Let $w =$ number of rods in the width of first field.

Then $w + 12 =$ number of rods in the length of first field,

and $w(w+12) =$ width × length, or area of first field;

also $w + 2 =$ width of second field,

and $w + 8 =$ length of second field.

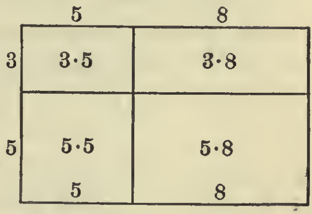
Hence $(w + 2)(w + 8) =$ width × length, or area of second field.

But the area of the second field is 80 square rods less than that of the first.

Hence $$(w + 2)(w + 8) = w(w + 12) - 80. \quad (1)$$

To solve this equation it is necessary to obtain the product of the two binominals $w + 2$ and $w + 8$ without first combining the terms of each binominal. In order to determine how these are to be multiplied, let us consider two binomials in each of which the terms can be combined if desired.

E.g. Consider the product of the binomials $5 + 3$ and $5 + 8$. This may be represented by the area of a rectangle 8 feet wide and 13 feet long. That is, $(5+3)(5+8) = 8 \cdot 13 = 104$ square feet. But such a rectangle may be divided into four rectangles, as in the figure.



Hence the area may be expressed as the sum of four areas. Thus, $(5 + 3)(5 + 8) = 5 \cdot 5 + 5 \cdot 8 + 3 \cdot 5 + 3 \cdot 8 = 104$, as before.

83. The product $5 \cdot 5$ is abbreviated to 5^2, the small figure indicating that 5 is to be used as a factor twice. It is read *the square of* 5, *or* 5 *squared*, since it represents the area of a square whose sides are each 5.

The second method here used for multiplying $(5+3)(5+8)$ is applicable to any similar case, and does not depend upon the possibility of first combining the terms of the binomials.

Applying this method to the binomials in equation (1) of the problem in § 82, we have

$$w^2 + 8w + 2w + 16 = w^2 + 12w - 80. \qquad (2)$$

Subtracting w^2 from both sides and applying Principle I,

$$10w + 16 = 12w - 80. \qquad (3)$$

Subtracting $10w$ from both sides and adding 80 to both sides,

$$96 = 2w. \qquad (4)$$

Hence by D, $48 = w$, the width of the field;

and $60 = w + 12$, the length of the field.

Check by substituting $w = 48$ in equation (1), and also by showing that the numbers 48 and 60 satisfy the conditions stated in the problem.

84. In a manner similar to that just illustrated we may multiply two trinomials.

E.g. The product of $a + b + c$ and $m + n + r$, in which the letters represent any positive numbers, may represent the area of a rectangle, divided into small rectangles as follows:

	m	n	r
a	am	an	ar
b	bm	bn	br
c	cm	cn	cr

Hence, the product is:

$(a + b + c)(m + n + r) = am + bm + cm + an + bn + cn + ar + br + cr,$
in which each term of one trinomial is multiplied by every term of the other, and the products are added.

Evidently the same process is applicable to the product of two such polynomials each containing any number of terms.

EXERCISES AND PROBLEMS

Find each of the following products in two ways:

1. $(3 + 7 + 10)(2 + 6)$.
2. $(5 + 11)(13 + 10 + 5)$.
3. $(6 + 11)(6 + 7)$.
4. $(15 + 8)(15 + 4)$.
5. $(7 + 13)(a + b)$.
6. $(m + n)(11 + 5 + 4)$.
7. $42 \cdot 36 = (40 + 2)(30 + 6)$.
8. $28 \cdot 73 = (20 + 8)(70 + 3)$.

Find as many as possible of the following indicated products in two ways:

9. $(a + b)(c + d)$.
10. $(x + 4)(x + 3)$.
11. $(x + y + z)(a + b + c)$.
12. $(11 + 13)(r + t)$.
13. $(5 + x)(5 + 7)$.
14. $(3 + 8)(2 + 4 + 6)$.
15. $(x + 7)(3x + 4)$.
16. $(a + 4)(3a + 1)$.
17. $(3 + x)(2 + 5x)$.
18. $(a + b)(3a + 7b)$.
19. $(x + y)(2x + 3y)$.
20. $(7x + 4x)(x + 8)$.

21. A rectangle is 7 feet longer than it is wide. If its length is increased by 3 feet and its width increased by 2 feet, its area is increased by 60 square feet. What are its dimensions?

22. A field is 10 rods longer than it is wide. If its length is increased by 10 rods and its width increased by 5 rods, the area is increased by 630 square rods. What are the dimensions of the field?

23. A farmer has a plan for a granary which is to be 12 feet longer than wide. He finds that if the length is increased 8 feet and the width increased 2 feet, the floor space will be increased by 160 square feet. What are the dimensions?

24. If the length of a rectangular flower bed is increased 3 feet and its width increased 1 foot, its area will be increased by 19 square feet. What are its present dimensions, if its length is 4 feet greater than its width?

85. **Polynomials with Negative Terms.** The polynomials multiplied in the foregoing exercises contain positive terms only. The same process is applicable to polynomials containing negative terms, as is seen in the following examples:

Ex. 1. Find the product of $(7-4)$ and $(3+5)$. This product, written out term by term, would give

$$\begin{aligned}(7+{}^-4)(3+5) &= 7\cdot 3+7\cdot 5+{}^-4\cdot 3+{}^-4\cdot 5\\ &= 21+35-12-20=24.\end{aligned}$$

Also $(7-4)(3+5)=3\cdot 8=24.$

Ex. 2. Multiply $7-4$ and $8-3$.

$$\begin{aligned}(7-4)(8-3)=(7+{}^-4)(8+{}^-3) &= 7\cdot 8+7\cdot {}^-3+{}^-4\cdot 8+{}^-4\cdot {}^-3\\ &= 56-21-32+12=15.\end{aligned}$$

Also $(7-4)(8-3)=3\cdot 5=15.$

EXERCISES

Find each of the following products in two ways, as in the above examples:

1. $(11-7)(6+5)$.
2. $(22-13)(3+7)$.
3. $(8-5)(7-3)$.
4. $(17-9)(29-4)$.
5. $(2-3)(4b-7b)$.
6. $(3+x)(7x-3x)$.
7. $(8-3)(8-2)$.
8. $(9-13)(9-17)$.

Perform the following indicated operations:

9. $(a-b)(c+d)$.
10. $(a-b)(c-d)$.
11. $(x-4)(x-5)$.
12. $(a+b-c)(m-n)$.
13. $(a-b)(7a+3b)$.
14. $(5-y)(5x+3y)$.
15. $(2a-3b+c)(m+n)$.
16. $(v-t)(7v-5t)$.

86. The preceding exercises illustrate the following principle:

Principle XIII. *The product of two polynomials is found by multiplying each term of one by every term of the other, and adding these products.*

In case some of the partial products are negative, these are combined by Principle IX.

If there are similar terms in either polynomial, these should usually be added first, thus putting each polynomial in as simple form as possible.

E.g. $(3x+2-2x)(4x+3-3x)=(x+2)(x+3)$
$$=x^2+2x+3x+6=x^2+5x+6.$$

EXERCISES AND PROBLEMS

Perform the following indicated operations:

1. $(x-7)(3x-4+8)$.
2. $(1-2x+x-3)(2x+4a+7x)$.
3. $(4a-x-3a)(2x+4a+7x)$.

4. $(5x + 3y - 4x - 2y)(6y + 3x - 2y + y)$.
5. $(13a - b - 12a)(2b - 3a)$.
6. $(xy - 5xy + 4y)(8y - 3 - 7y)$.
7. $(11ab + 3a)(2b - 3b + 5)$.
8. $(6 - 4x + 3x)(7x + y - 3x + 1)$.
9. $(13x - 12x - y + 3)(5x - 3y + xy + 5)$.
10. $(37 - 13n + a)(a - n + 8)$.
11. $(x - 2 + y)(4y - 3x)$.
12. $(11b - a - 10b)(6a - 3b - 2a)$.
13. $(7 + y - x)(2y + x - 1)$.
14. $(5x + 3y - 1)(x - 2)$.
15. $(-8a - 1 + 7a)(5a - 8 - 3a)$.

Solve the following equations, in each case verifying the solution by substituting in the given equation the result found for the unknown number.

16. $(x + 2)(x + 3) = (x - 3)(x + 10) + 10$.
17. $(5x - 4)(6 - x) - 97 = (x - 1)(6 - 5x)$.
18. $(3n - 1)(18 - n) = (n + 6)(16 - 3n)$.
19. $(7 - a)(9a - 8) = 31 + (36 - 9a)(a + 2)$.
20. $(4a + 4)(a - 3) = (4a + 1)(a + 7) - 13a + 221$.
21. $(n + 6)(3n - 4) - 14 = (n + 8)(3n - 3)$.
22. $(8n + 6)(10 - n) + 150 = (1 - n)(8n + 3)$.
23. $(a - 1)(13 - 6a) = (6a - 3)(8 - a) - 21$.
24. $(7x - 13)(6 - x) - (x + 4)(3 - 7x) = 70$.

25. A club makes an equal assessment on its members each year to raise a certain fixed sum. One year each member pays a number of dollars equal to the number of members of the club less 175. The following year, when the club has 50 more members, each member pays $5 less than the preceding year. What was the membership of the club the first year and how much did each pay?

26. There are two numbers whose difference is 6 and whose product is 180 greater than the square of the smaller. What are the numbers?

27. There are four consecutive even integers such that the product of the first and second is 40 less than the product of the third and fourth. What are the numbers?

28. There are four consecutive integers such that the product of the first and third is 223 less than the product of the second and fourth. What are the numbers?

29. There are four numbers such that the second is 5 greater than the first, the third 5 greater than the second, and the fourth 5 greater than the third. The product of the first and second is 250 less than the product of the third and fourth. What are the numbers?

30. Prove that for any four consecutive integers the product of the first and fourth is 2 less than the product of the second and third.

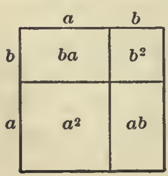
SQUARES OF BINOMIALS

87. Just as x^2 is written instead of $x \cdot x$, so $(a+b)^2$ is written instead of $(a+b)(a+b)$. The square of a binomial is found by multiplying the binomial by itself as in § 86.

E.g. $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ab + b^2$.

Hence $$\boldsymbol{(a+b)^2 \equiv a^2 + 2ab + b^2.}$$

This product is illustrated in the accompanying figure, and is evidently a special case of the type exhibited in the figure, page 83.



Translated into words this identity is: *The square of the sum of any two numbers is equal to the square of the first plus twice the product of the two numbers plus the square of the second.*

88. Similarly we obtain the square of the difference of two numbers.

$$(a-b)^2 \equiv a^2 - 2\,ab + b^2.$$

Translate this identity into words.

While these squares are ordinary products of binomials and can always be obtained according to Principle XIII, they are of special importance and should be studied until they can be reproduced from memory at any time.

EXERCISES AND PROBLEMS

Perform the following indicated operations:

1. $(a+b)^2$.
2. $(17-3)^2$.
3. $(6+a)^2$.
4. $(a-2)^2$.
5. $(21-b)^2$.
6. $(x-7)^2$.
7. $(4+9)^2$.
8. $(c-\frac{3}{7})^2$.
9. $(x-\frac{1}{3})^2$.
10. $(r-s)^2-(r+s)^2+(r-s)(r+s)$.

Check each of the above by substituting special values for the letters and combining the terms of the binomial before squaring.

11. Find the square of 42 by writing it as a binomial, $40+2$.

12. Square the following numbers by writing each as a binomial sum: 51, 53, 93, 91, 102, 202, 301.

13. Find the square of 29 by writing it as a binomial, $30-1$.

14. Square the following numbers by first writing each as a binomial difference: 28, 38, 89, 77, 99, 198, 499, 998, 999.

Solve the following equations, verifying each solution:

15. $(a+4)^2+(a-1)(2\,a+5)=(a+4)(3\,a+2)$.
16. $(a-1)(3\,a-1)-(a+1)^2=2\,a^2-18$.
17. $(6-a)^2+(a-3)(2\,a-5)=(3\,a+1)(a-3)+84$.
18. $(7\,a-18)(a+4)-(a-1)^2=6\,(a+1)^2-79$.

19. $(2b-30)(b-1)-5b^2=6b-3(b+5)^2+65.$

20. $(5-b)(6b+5)+4(b-3)^2=20-2(b+1)^2+3+16b.$

21. $(5-c)^2+(7-c)^2+(9-c)^2=(c-1)(3c-58)-93.$

22. $(5c-3)(2+c)-4(c-1)^2=(c+1)^2+54.$

23. $(8-4c)(5-c)=(c+1)^2+(c+3)(3c-8)+218.$

24. $(4y-9)(y-5)-5(y-4)^2=(8-y)(4+y)-82.$

25. $(y+6)^2-3(y-1)^2+(4-y)(5-2y)=25y+9.$

26. $(y-1)^2+4(y+1)^2+(1-y)(5y+6)=15y-29.$

27. $x(x+3)+(x+1)(x+2)=2x(x+5)+2.$

28. $x^2=(x-3)(x+6)-12.$

29. $(5+5x)(3-x)+2(x+1)^2+3(x+1)(x-7)=17(x+1).$

30. $(8+3x)(4-x)+(x-1)(x-2)+2(x+5)^2=105.$

31. There is a square field such that if its dimensions are increased by 5 rods its area is increased 625 square rods. How large is the field?

Suggestion: If a side of the original field is w, then its area is w^2, and the area of the enlarged field is $(w+5)^2$.

32. A rectangle is 9 feet longer than it is wide. A square whose side is 3 feet longer than the width of the rectangle is equal to the rectangle in area. What are the dimensions of the rectangle?

33. A boy has a certain number of pennies which he attempts to arrange in a solid square. With a certain number on each side of the square he has 10 left over. Making each side of the square 1 larger, he lacks 7 of completing it. How many pennies has he?

34. A room is 7 feet longer than it is wide. A square room whose side is 3 feet greater than the width of the first room is equal to it in area. What are the dimensions of the first room?

35. Find two consecutive integers whose squares differ by 51.

36. Find two consecutive integers whose squares differ by 97.

37. Find two consecutive integers whose squares differ by a. Show from the form of the equation obtained that a must be an *odd* integer.

38. There are four consecutive integers such that the sum of the squares of the last two exceeds the sum of the squares of the first two by 20. What are the numbers?

39. Two square pieces of land require together 360 rods of fence? If the difference in the area of the pieces is 900 square rods, how large is each piece? (*Hint:* $x^2 - (90 - x)^2 = 900$.)

40. There is a square such that if one side is increased by 12 feet and the other side decreased by 8 feet the resulting rectangle will have the same area as the square. Find the side of the square.

41. A regiment was drawn up in a solid square. After 50 men had been removed the officer attempted to draw up the square by putting one man less on each side, when he found he had 9 men left over. How many men in the regiment?

42. There is a rectangle whose length exceeds its width by 11 rods. A square whose side is 5 rods greater than the width of the rectangle is equal to it in area. What are the dimensions of the rectangle?

89. Thus far the processes of algebra have all been based upon thirteen **fundamental principles**, together with certain obvious *form changes* indicated in § 37. These latter are of general use in elementary arithmetic and need no special emphasis here. The principles, however, for the most part refer to methods not common in arithmetic. These should now be carefully reviewed and a list of them made for convenient reference.

REVIEW QUESTIONS

1. How would $3 \cdot 5$ and $7 \cdot 5$ be added in arithmetic? Why cannot $3n$ and $7n$ be added in the same manner? State in full the principle by which $3n$ and $7n$ are added. In this example what number is represented by n? Test the identity $3n + 7n \equiv 10n$ by substituting any convenient value for n.

2. How is $5 \cdot 9$ subtracted from $11 \cdot 9$ in arithmetic? In what different manner may this operation be performed? Why is it sometimes necessary to perform subtraction in the second way? State in full the principle by which $12x$ is subtracted from $31x$. In the identity $31x - 12x \equiv 19x$, what number is represented by x? Test the equality by substituting any convenient number for x.

3. How is the product $2 \cdot 3 \cdot 5$ multiplied by 4 in arithmetic? In what different way may this multiplication be performed? Why should it ever be performed in the second way? State in full the principle by which $2ax$ is multiplied by 3.

4. How is $11 + 3$ multiplied by 4 in arithmetic? In what different way may this operation be performed? Why is it sometimes necessary to multiply in the second way? State in full the principle by which $a + 8$ is multiplied by 7.

What principles are used in performing the following indicated operations?

$ax + bx + cx$, $aby + cby$, $c(4a + 2b)$,
$3y - by + cy$, $41a - 32b - 17a + 80b$.
$5(16a - 3b)$, $-3(6x - 7y)$, $a(11 - b)$.

5. Divide $2 \cdot 4 \cdot 6 \cdot 20$ by 2 without first performing the multiplication indicated in $2 \cdot 4 \cdot 6 \cdot 20$. Do this in several ways and show that all the quotients obtained are equal. State in full the principle used.

What principles are used in the following operations?

$3 \cdot 5ab = 15ab$, $c \cdot 5b = 5(bc)$, $20ab \div 4 = 5ab$,
$16xy \div x = 16y$, $3t + 15t = 18t$, $78h - 41h = 37h$.

6. How is $12 + 18$ divided by 6 in arithmetic? In what different way may this division be performed? Why is it sometimes necessary to perform division in the second way? State in full the principle used in performing the operation $(6x + 9y) \div 3$. What principle is used in performing each of the following indicated operations?

$$5(a + x), \quad 3y - 4y, \quad (24x + 9y) \div 3, \quad 5x + ax.$$

7. Define equality; equation; identity. State in detail how the equation and the identity differ. Give an example of each.

8. In what ways may an equation be changed into another equation such that any number which satisfies either also satisfies the other?

Describe some of the operations which change the *form* of the members of an equation, but not their *value*.

State Principle VIII in full.

9. Name several pairs of opposite qualities all of which are conveniently described by the words "positive" and "negative." What symbols are used to replace these words when applied to numbers?

10. When loss is added to profit, is the profit increased or decreased? What algebraic symbols are used to distinguish the numbers representing profit and loss?

11. Give an illustration by means of the number scale to show that a number may represent either a *change of position* in one direction or the other, or a *fixed position* with respect to the zero point.

12. Why do we call positive and negative numbers signed numbers? What is meant by the absolute value of a number?

13. State Principle IX in full.

14. By means of the number scale describe the "counting" method of adding signed numbers

15. Make a list of pairs of opposite qualities to which positive and negative numbers apply. State in each case what is represented by positive and what by negative numbers.

16. How is the correctness of subtraction tested in arithmetic? Is the same test applicable to subtraction in algebra?

17. Explain subtraction by counting on the number scale.

18. How do negative numbers make subtraction possible in cases where it is impossible in arithmetic?

19. What is a convenient rule for subtracting signed numbers? State Principle X.

20. Give examples of equations which could not be solved without negative numbers and show that such equations can be solved by means of negative numbers.

21. Give an example in which positive and negative numbers are multiplied. State Principle XI.

22. Define division. How do we obtain the law of signs in division? State Principle XII. What is the test of the correctness of division?

23. What may be the significance of a negative number when obtained as a result of solving a problem?

24. Explain how one set of signs + and − can be used to indicate both quality and operation.

Show that $a + {}^{-}b = a - b$ and $a - {}^{+}b = a - b$.

25. What is a polynomial? A term? How are polynomials classified? What are similar terms? By what principle are similar terms added? By what principle are they subtracted? In adding or subtracting polynomials, how may the terms be arranged for convenience? What is the principle for removing a parenthesis when preceded by the sign +? By the sign −? State Principle VII in full.

26. Make a diagram to show how to multiply $(7 + 4)$ by $(11 + 8)$ without first uniting the terms of the binomials. Multiply $(a + b)$ by $(c + d)$ in the same manner. Multiply $(12 - 3)$ by $(9 - 7)$ in two ways and compare results. State the principle by which two polynomials are multiplied.

27. Explain why x^2 is called the square of x or x squared. State in words what is the square of the binomial $(x + a)$; of the binomial $(x - a)$.

28. Show by an example how negative numbers may be used in solving a problem, even though the answer to the problem is positive and the statement of the problem does not involve negative numbers.

REVIEW EXERCISES

In the following exercises perform the indicated operations, remove all parentheses, solve all equations, verify the results, and in each case state the principles used.

1. Add $3x + 4y - 3z$, $5x - 2y - z$, and $3y - 5x + 7z$.

2. From $15a + 4b - 13bc$ subtract $3a - 8b + 2bc$.

3. Subtract $7x - 5y - 7a$ from $6x + 5y + 3a$.

4. $(3x - 4y - z)(x + y + z)$.
5. $(b - 5)(2a^2b - 3ab - b)$.

6. Add $11axy + 13x - 14y$, $2y - 4x$, and $3y + x - 8axy$.

7. $(5x - 3b) + (2x + b) - (4x - 2b - x + 5b)$.

8. Add $19b + 3c$, $2b - 7c$, $2c - 14b$, and $c + 8b$.

9. $-(a - 3b - c) - (2c - a - 5b) + (a - c + b)$.

10. Subtract $2x + 4y + z$ from $13x - 3y - 5z + 8$.

11. $\frac{x}{3} + \frac{x}{6} + \frac{x}{12} + \frac{15}{4} = x$.

12. $5(x - 7) - 3(14 - x) + 60 = 1 - 10x$.

13. $13(1 - x) - 6(2x - 5) = 80 + 12x$.

14. $\frac{x-3}{7}+\frac{x-4}{5}+\frac{x-8}{2}+\frac{x-6}{6}=18.$

15. Add $7x-3y-4$, $5x+2y+5$, and $3y-8x-6$.

16. Add $13a+4b-9c$, $2c-8b-16a$, and $8a-5b-8c$.

17. $(13-4-2)(5x-3+4x)$.

18. $5(x-y)^2 -5(y-x)^2+(x-y)(x+y)$.

19. Square 73, 79, 92, 98, 1003, and 995 as binomials.

20. From $17b-4a-2c-19$ subtract $8c-5a-8b+4$.

21. $3-(3-2+6+8-3)+8-(9-3+8)$.

22. $3(4-x)-2(5-6x)=8x+4$.

23. $\frac{x}{4}+\frac{x}{8}+\frac{x}{16}-\frac{x}{2}+\frac{x}{32}=-2.$

24. $\frac{y-3}{4}-\frac{y+9}{10}=\frac{y-11}{4}-2.$

25. $\frac{y-4}{3}+\frac{y+2}{3}+\frac{y+8}{3}=2y-20.$

26. $5x-(3x-2+2y+x)+13y-(6-3x+4)$.

27. From $3-4a-5c+8x^2$ subtract $2x^2-2a-4c+8$.

28. $12+(2a-3c-4b)-(3b-c-a-8)$.

29. $\frac{y}{2}+\frac{y+20}{4}+\frac{y+5}{5}=25.$

30. $\frac{y}{3}-\frac{y+20}{5}+\frac{y-5}{5}+\frac{y-10}{2}=15.$

31. Add $y-20$, $4y+6$, $2y+4x-13$, and $2x-8y-40$.

32. $(x-1)(2x-2)+(x-5)^2=(3-x)(24-3x)-7$.

33. Subtract $16-x+2z-4y$ from $3x-5z-8y$.

34. $19+(2x-7)-(31-4x-8-2x)=5x+7$.

35. $(4ab-6ac-5ad)(b-c+d)$.

36. $(17x+3)(x-1)+8=(2-x)(6-17x)+19$.

37. $5-(a+b-c-d+8)+(3+a+c-d)-5.$

38. $\frac{a}{10}+\frac{a+10}{10}+\frac{a-10}{10}=a-28.$

39. Add $6a+9$, $8a-13$, $46a-8$, and $6-54a$.

40. $(a-2)(6a-4)+2(a-1)^2=(6-a)(30-8a)+4.$

41. From $6(a+2)+3(c+4)-2(b-d)$ subtract $2(a+2)-2(c+4)+3(b-d)$.

42. $\frac{a}{3}+\frac{a+7}{4}-\frac{a-3}{3}=\frac{a+227}{5}-1.$

43. $(a-2-3c-8+2b)(6-a-c-b+8).$

44. $\frac{a-1}{2}+\frac{a+1}{2}+\frac{a-3}{12}=2+a.$

45. $a^2b-(3b-8a^2-7)+3ab^2-(4ab^2+8-2a^2).$

46. $\frac{a+1}{4}+\frac{a-3}{4}+\frac{a-7}{4}=2a-26.$

47. Add $12a^2b^2c+8ax$, $6ax-8a^2b^2c$, and $2ax+3a^2b^2c$.

48. Add $5xy^2+3x^2y+4xy$, $2x^2y-6xy^2-3xy$, and $4xy$.

49. Add $6ab-3c-2a$, $2c-4ab-5a$, $5c-a+ab$, and $3+5a-2c-3ab$.

50. $\frac{n+1}{3}+\frac{n+3}{4}+\frac{n-1}{4}=\frac{n+13}{3}+\frac{n-2}{3}.$

51. $(n-4)(6-3n)-(6-n)^2-10=-4n(n-4).$

52. From $35ab-8x-9z+13$ subtract $16ab-4z+5x+8$.

53. $(n+2)^2+(n-1)^2+(n+1)^2=3n(n+2)+60n+130.$

54. Subtract $5a-8x-6y$ from $13x+14y-15z-4a$.

55. From $9y-4x-6z-3b$ subtract $8-9y-3x-2z$.

56. $2x+4-6(5x-8-7x)+2-4x=6(2-3x)-42.$

57. $-(7+4x-8-2x)+4-2x=6x+25.$

58. $16+5x-(8x+9-4x+17)=8x-3$.

59. $6x-3-(4x+8-9x)-(5x-2)=x+11$.

60. $(a-1+b-c-d)(4a+5b+3c-2d)$.

61. $(4ax-3ay+5az-8)(x+y-z+2)$.

62. $(3ac-2ad+4ed)(2+3+4+5)$.

63. $7-(3a-2b-4a)+b+2a-(3b-2a-a)$.

64. $8x+(5y-5)+(2y-1)-(13y+8x-17)$.

65. $16ax+4-(8-8ax-a)-(12ax-13-ax)$.

66. Add $15ax^2+3bc^2$, $2bc^2-7ax^2$, and $5+2ax^2-5bc^2$.

67. Add $16-7ab-2a^2+5ab$, $4a^2-2ab$, and $5ab-8$.

68. Add $51x^2y-35+12a^2$, $41-17a^2-57x^2y$, and $3x^2y$.

69. Add $35b^2-13c^2$, $8c^2-3b^2c^2$, and $6b^2-8c^2-9b^2c^2$.

70. Add $19-2x+3x^2b+2b$, $4x^2b+x+5b+8$, and $4x$.

71. Subtract $2a-6x^2b-3x+21$ from $19-2x+3x^2b-7a$.

72. From $6a-45+8b-3c+82cb$ subtract $7b+18+6c$.

73. $(17+2a-3b-4bc)(2-a+b-c)$.

74. $(13c-4d+8e-3)(c-d)$.

75. $(4xy-2y-3x-2)(y-x+y+5)$.

76. $(9ax-3x-5a-2x+4)(5-x)$.

77. From $41a^2+7ab-5c^2-9ab$ subtract $8ab+7c^2+50a^2$.

78. From $15+2x-9xy-3y$ subtract $5xy-4x-2y$.

79. Subtract $12+8x-4a-6c-18abc$ from $5x-2a+3c$

80. $2-(71x+42y-15x-64)-(5-91x-2y-13xy)$.

81. $(31-2x+3y-5)(x+y)$.

82. $8x+(13+18x-6)-(5-6x)=16x+10$.

83. $(9x-3)(4-x)+(x-3)^2=-8(x+2)^2+94.$

84. $(x+1)^2+(x+2)^2+(x+3)^2=(3x-1)(x+12)-43.$

85. $(2x+5)(x-7)-(x-1)^2=(x+1)(x+2)-28.$

86. $3(5-x)^2-(2x-1)(x-1)=(x-7)(x+10)+17x+50.$

87. $5-(7-4x+2y-4b)-(8x-6y-9+2b)+8a.$

88. $8x-(-3-2-4-7)+5x+(2+6+4)-(-3x+2).$

89. $5y+2x-(6-4x-5x)-3y-(4x-2y)-(-7y+8).$

90. $35y-(41x-16-12y)+5x+(-6+46y-18x).$

91. $(x-1)^2-(x-8)(2x-1)=-x^2+98.$

92. $(32+x)(4x-1)+(5-x)^2+(x-1)^2=6(x+1)^2+194.$

93. $(2x-7)(5-x)-(2-5x)(1-x)=-x(7x-34)-17.$

94. $(7+x)(x-4)+(1-x)^2=-23+2x^2.$

95. $(12-4x)(2-x)-4(1+x)^2=5x+119.$

96. $(x-17)(59-2x)-(1-x)^2=(6-3x)(x-2)+384.$

97. $(3x-2)+(x-1)^2+(x-2)^2=2(x-1)(x-2)+5.$

98. $(6-3x)(2+x)+16(x-1)^2=13(x+4)^2+364.$

99. $\frac{x+8}{2}-\frac{x-9}{12}+\frac{x-17}{6}=\frac{4x-7}{2}+\frac{2x+6}{3}+\frac{5-31x}{12}.$

100. $\frac{3x-1}{6}-\frac{3x+3}{3}+\frac{x-1}{2}=\frac{x+5}{6}+4x-\frac{20}{3}.$

CHAPTER IV

THE SOLUTION OF PROBLEMS

90. Some of the advantages of algebra over arithmetic in solving problems have been pointed out in the preceding chapters. For instance, brevity and simplicity of statement secured through the use of letters to represent numbers; the translation of the words and sentences of problems into number expressions and equations; and the clear and logical solution of the equation, step by step.

91. A still greater advantage is set forth in the present chapter; namely, the opportunity offered by the symbols and processes of algebra to *summarize a whole class of problems* under one solution, called the **formula**, which is thereafter used to solve all problems of the class.

PROBLEMS INVOLVING INTEREST

92. A class of problems already within the pupil's experience will illustrate this point. The different cases of percentage or interest have been studied in arithmetic, and a large number of isolated problems have been solved according to the rules. In this instance, therefore, we proceed at once to summarize all of this work in a few short statements.

Let $p =$ any principal, *i.e.* a number of dollars at interest.
$i =$ the interest, *i.e.* the number of dollars accrued.
$r =$ the rate, to be expressed in hundredths.
$t =$ the time, to be expressed in years and fractions of a year.

Then the rule of arithmetic for finding the interest when the principal, rate, and time are given is

$$interest = principal \times rate \times time,$$

i.e. $$i = prt. \qquad (1)$$

If in this equation $i = 150$, $r = .05$, $t = 6$, find p. If $i = 190.5$, $p = 635$, $r = .03$, find t. If $i = 665$, $p = 1000$, $t = 17$, find r. Substitute other values for any three of these letters and find the value of the remaining letter.

Solve equation (1) for t in terms of i, p, and r; also for p in terms of i, r, and t, and for r in terms of p, t, and i.

It follows that if any three of the four numbers, p, r, i, and t, are given, the remaining one may be found.

Let the student state four rules of interest by translating these formulas into words.

Note the simplicity of these equations compared with the corresponding rules in arithmetic.

Solve each of the following problems by substituting in the proper formula:

1. What is the simple interest at 5 % on \$400 for 5 years and 9 months?

Solution. $i = p \cdot r \cdot t = 400 \cdot \frac{5}{100} \cdot 5\frac{3}{4} = 4 \cdot 5 \cdot \frac{23}{4} = 115.$

2. In what time will the simple interest on \$750 at 3 % amount to \$225? Substitute in the formula $t = \frac{i}{pr}$.

3. What is the semi-yearly income from an endowment of \$2,700,000, the rate being $4\frac{3}{4}$ % per annum? (Here $t = \frac{1}{2}$.)

4. A father invested \$1500 at $5\frac{1}{3}$ %, the simple interest on which was to go to his eldest son on his 21st birthday. The young man received \$1240. How old was the son when the investment was made?

5. What is the amount of money invested if it yields $787.50 interest per annum, the rate being $5\frac{1}{4}\%$? (Here $t = 1$.)

6. A certain investment yields $8160 in 8 years. What is the principal, if the rate of interest is 4%?

7. A $45,800 investment yielded $13,396.50 interest (simple) in $6\frac{1}{2}$ years. What was the rate of interest?

8. The endowment of a small college is $750,000, the yearly income from which is $45,000. What is the average rate at which the endowment is invested?

9. A capitalist has investments amounting to $360,000, the total income from which amounts to $1800 per month. What is the average rate at which the money is invested? ($t = \frac{1}{12}$.)

10. If $700 is invested at $5\frac{1}{2}\%$ simple interest, what is the amount at the end of 5 years 6 months? This problem calls for the amount, which is the sum of principal and interest.

If a = amount, then $\boldsymbol{a = p + i = p + prt}$.

11. Solve the equation $a = p + prt$ for p in terms of a, r, and t, and translate the result into words.

12. Solve $a = p + prt$ for r in terms of a, p, and t, and translate the result into words.

13. Solve $a = p + prt$ for t in terms of a, p, and r, and translate the result into words.

14. Seven years ago I invested a certain sum of money at 6% simple interest. The amount at present is $5680. How much did I invest? (Use the formula obtained in Problem 11.)

15. Six years ago A invested $470 at simple interest. The amount at present is $611. What is the rate per cent?

16. Some years ago I invested $500 at 7% simple interest, which now amounts to $815. How many years ago was the investment made?

17. A merchant bought goods for $250, and some months later sold the goods for $300, making a profit of 1% per month. How many months between the purchase and the sale?

18. A real estate dealer sold a house and lot for \$7500, for which he received a commission of 4%. What was the profit?

In this case the element of time does not enter. The word *commission* is then used for interest and the principal is called the *base*. Letting $c =$ commission, $b =$ base, and $r =$ rate, we have

$$c = br.$$

Applying this formula, $c = br = 7500 \cdot \frac{4}{100} = 75 \cdot 4 = 300$.

19. Solve the equation $c = br$ for b in terms of c and r, and translate the result into words.

20. Solve $c = br$ for r in terms of c and b, and translate the result into words.

21. A merchant's commission for selling a carload of peaches was \$18.75. What was the rate of commission if the peaches brought \$375? (Use the formula found in 19.)

22. A broker received \$25.50 commission for selling \$850 worth of bonds. What was his rate of commission?

23. An agent received 3% commission for buying and $3\frac{1}{2}$% for selling some property. He paid \$5750 for it and sold it for \$7200. What was his total commission?

24. How much must I remit to my broker in order that he may buy \$600 worth of bonds for me and reserve 5% for his commission?

I must send him both base and commission. Calling this the amount and representing it by a, we have

$$a = b + c = b + br = b(1 + r).$$

Hence $$a = 600 + 600 \cdot \frac{5}{100} = 600 + 6 \cdot 5 = 630.$$

25. Solve the equation $a = b + br$ for b and translate the result into words.

26. Solve $a = b + br$ for r and translate the result into words.

27. A merchant received \$918 with which to buy corn after deducting his commission of 2% on the price of corn. How much was his commission and how much was used to buy corn?

Here $a = 980$. Find b by the formula under 25, which gives the sum paid for corn.

28. A broker received \$790, of which he invested \$750 in stocks, reserving the balance as his commission. Find the rate of his commission by means of the formula obtained in 26.

29. An agent received \$945 with which to buy lumber after deducting his commission of 5% on the cost of the lumber. How much was his commission? (First find the base by substituting in the formula of 25.)

30. A dealer sold berries for \$18.95, and after deducting a commission of 2% sent the balance to the truck gardener. How much did he remit?

The sum he sent was the difference between the base and the commission; calling this d, we have

$$d = b - c = b - br = b(1 - r).$$

Hence in this case $d = 18.95\,(1 - \frac{2}{100}) = 18.57$.

31. Solve the equation $d = b(1 - r)$ for b in terms of d and r and translate the result into words.

32. Solve the equation $d = b - br$ for r in terms of b and d and translate the result into words.

33. After deducting a commission of 3% for selling bonds, a broker forwarded \$824.50. What was the selling price of the bonds? (Solve by means of the formula under Problem 31.)

34. A broker sold stocks for \$1728 and remitted \$1693.44 to his principal. What was the rate of his commission? (Solve by means of the formula under Problem 32.)

35. In how many years will \$200 double itself at 5 %?

In this case $i = p$. Hence, using formula (1), page 101, we have

$$200 = 200 \times \frac{5}{100} \times t.$$

Hence $$t = 20.$$

36. In how many years will any sum, p, double itself at any rate, r?

Here $p = prt$. Hence, solving, $t = 1 \div r$.

37. In what time will a sum of money double itself at 6 %? 7 %? $4\frac{1}{2}$ %? $3\frac{2}{5}$ %?

38. Collect all the formulas worked out in this set of problems. Translate each into words. Which were used as the original ones from which the others were deduced? How many and which ones are needed in order to derive all the others? Why are such formulas better than rules expressed in words?

PROBLEMS INVOLVING AREAS

93. Another class of problems already well known to the pupil in arithmetic concerns the areas of rectangles and triangles.

If in a rectangle we let the number of units of length be denoted by l, the number of units of width by w, and the number of square units in the area by a, then *area = length × width;*

i.e. $$a = lw. \quad (1)$$

If in equation (1) $a = 144$, $l = 16$, find w. If $a = 1116$, $w = 31$, find l. Substitute other values for any two of these letters and find the value of the remaining one.

Solve this equation for w in terms of l and a, and also for l in terms of w and a.

Also if b is the base of a triangle, h its altitude (height), and a its area, then *area* $= \frac{1}{2}$ (*base* $\times$ *altitude*);

i.e.
$$a = \frac{bh}{2}. \qquad (2)$$

Substitute particular values for any two of these letters and find the value of the remaining one.

Solve (2) for h in terms of a and b, and also for b in terms of a and h.

Let the student translate each of these equations into words. Use these formulas in the solution of the following problems:

1. How many tiles each 3 by 4 inches are needed to cover a rectangular floor 18 by 22 feet? Use formula (1).

2. How long a space 25 feet in width can be covered by 340 square feet of roofing?

3. The base and altitude of a triangle are 8 and 6 respectively. Find its area.

4. The base of a triangle is 12 and its area 72. Find its altitude.

5. The altitude of a triangle is 16 and its area 144. Find its base.

6. A rectangle is 5 feet longer than it is wide. If it were 3 feet wider and 2 feet shorter, it would contain 15 square feet more. Find the dimensions of the rectangle.

Let w equal the width; then construct number expressions for the length, width, and area under the supposed conditions.

7. A rectangle is 6 feet longer and 4 feet narrower than a square of equal area. Find the side of the square and the sides of the rectangle.

8. The base of a triangle is 8 inches greater than its altitude. If the base is increased by 4 inches and the altitude decreased by 2 inches, the area remains unchanged. Find the base and altitude of the triangle.

9. A rectangle is 14 inches longer than it is wide. If the width is increased by 5 inches and the length decreased by 4 inches, the area is increased by 70 square inches. Find the dimensions of the rectangle.

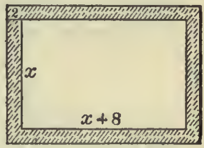
10. A rectangle is 15 rods longer and 10 rods narrower than an equivalent square. What are the dimensions of the rectangle?

11. The altitude of a triangle is 16 inches less than the base. If the altitude is increased by 3 inches and the base by 2 inches, the area is increased by 52 square inches. Find the base and altitude of the triangle.

12. The width of a rectangular field is 20 rods less than its length. If each side is decreased by 10 rods, the area is decreased by 900 square rods. What are the dimensions of the field?

13. A picture is 4 inches longer than it is wide. Another picture, which is 12 inches longer and 6 inches narrower, contains the same number of square inches. Find the dimensions of the pictures.

14. A picture, not including the frame, is 8 inches longer than it is wide. The area of the frame, which is 2 inches wide, is 176 square inches. Find the dimensions of the picture.



15. A picture, including the frame, is 10 inches longer than it is wide. The area of the frame, which is 3 inches wide, is 192 square inches. What are the dimensions of the picture?

16. The base of a triangle is 11 inches greater than its altitude. If the altitude and base are both decreased 7 inches, the area is decreased 119 square inches. Find the base and altitude of the triangle.

17. The base of a triangle is 3 inches less than its altitude. If the altitude and base are both increased by 5 inches, the area is increased by 155 square inches. Find the base and altitude of the triangle.

18. A commander in attempting to draw up his men in the form of a solid square finds that there are 80 men more than enough to complete the square. If he places 2 more men on each side of the square he needs 84 more men to complete it. How many men in his command?

PROBLEMS INVOLVING VOLUMES

94. Still another class of problems for which the fundamental formulas are already well known concerns the volumes of rectangular solids and of pyramids.

If the number of units of length, width, and height of a rectangular solid be denoted by l, w, and h respectively, and the number of units of volume by v, then

$$\textit{volume} = \textit{length} \times \textit{width} \times \textit{height};$$

i.e.
$$v = lwh. \tag{1}$$

If w is the width, l the length of the rectangular base of a pyramid, h its altitude, and v its volume, then

$$\textit{volume} = \tfrac{1}{3}\,(\textit{area of base} \times \textit{altitude});$$

i.e.
$$v = \frac{lwh}{3}. \tag{2}$$

In equations (1) and (2) substitute particular values for any three of the letters and find the value of the remaining one.

Use these formulas in solving the following problems:

1. Solve equations (1) and (2) for l, w, and h respectively and translate each equation into words.

2. How many cubic feet of earth are removed in digging a cellar 18 feet long, 12 feet wide, and 9 feet deep? Solve by substituting in formula (1).

3. If a cut in an embankment is 500 yards long and 4 yards deep, how wide is it if 18,760 cubic yards are removed?

4. How deep is a rectangular cistern which holds 500 cubic feet of water if it is 6 feet wide and 8 feet long?

5. The base of a pyramid is 16 inches long and 12 inches wide. Its altitude is 30 inches. Find its volume.

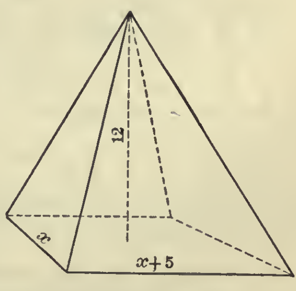
6. A pyramid whose volume is 72 cubic feet has a base whose area is 24 square feet. Find the altitude.

7. A pyramid whose volume is 91 cubic inches has an altitude of 21 inches. Find the area of its base.

8. A rectangular room which is 10 feet high is 4 feet longer than it is wide. If it were 5 feet longer and 2 feet wider, it would contain 950 cubic feet more than it does. Find its length and width.

9. A city building 50 feet high extends back 25 feet more than its frontage. If the building were 8 feet wider and extended 10 feet farther back, its capacity would be increased 59,000 cubic feet. What are the ground dimensions of the building?

10. A pyramid whose altitude is 12 inches has a rectangular base 5 inches longer than it is wide. If the length of the base is decreased by 1 inch and the width increased by 2 inches, the volume of the pyramid is increased by 72 cubic inches. Find the dimensions of the base of the pyramid.



11. The base of a rectangular pyramid whose altitude is 15 inches is 12 inches longer than it is wide. If the length and width of the base are both decreased by 3 inches, the volume is decreased by 675 cubic inches. Find the dimensions of the base of the pyramid.

12. The base of a rectangular pyramid is 15 inches wide. The altitude of the pyramid is 4 inches less than the length of the base. If the altitude is increased by 6 inches and the length of the base by 3 inches the volume is increased by 1155 cubic inches. Find the altitude of the pyramid and the length of its base.

13. The altitude of a pyramid is 7 inches less than the length of the base. The width of the base is 13 inches. If the altitude and the length of the base are both decreased by 3 inches, the volume is decreased by 364 cubic inches. Find the altitude of the pyramid and the length of its base.

14. The width of the base of a pyramid is 2 inches greater than its altitude. The length of the base is 34 inches. If the altitude and the width of the base are both increased by 6 inches, the volume is increased by 2992 cubic inches. Find the altitude and the width of the base.

PROBLEMS INVOLVING SIMPLE NUMBER RELATIONS

95. In the problems thus far considered in this chapter the formulas are already known from previous study or experience. Algebra affords a means of deriving such formulas in many cases where they are not already known.

1. The sum of two numbers is 35 and their difference is 5. What are the numbers?

Let $g =$ the greater number, then $35 - g =$ the lesser.

2. The sum of two numbers is 48 and their difference is 24. What are the numbers?

3. The sum of two numbers is $41\frac{1}{2}$ and their difference is $23\frac{1}{2}$. What are the numbers?

4. The sum of two numbers is 8590 and their difference is 3480. What are the numbers?

The four problems just preceding are similar in character. A simple rule can be found for solving all problems of this kind. Consider problem 1.

We have	$g-(35-g)=5.$	(1)
By VII,	$g-35+g=5.$	(2)
By I, *A*,	$2g=35+5.$	(3)
By *D*, VI,	$g=\frac{35}{2}+\frac{5}{2}.$	(4)
By *F*,	$g=20$, the greater number,	(5)
and	$l=35-g=15$, the lesser number.	(6)

The results in the final form tell nothing more than the answers to this particular problem; but the value of g in the form $g=\frac{35}{2}+\frac{5}{2}$, if examined closely, tells much more. Stated in full it means:

$$\text{the greater} = \frac{\text{the sum of the numbers}}{2} + \frac{\text{their difference}}{2}$$

Examine now the solution of the other three problems to see whether this same expression will give the greater number in each case.

The great advantage in not adding 35 and 5 before dividing by 2, is that the expression $\frac{35}{2}+\frac{5}{2}$ preserves the original numbers as given in the problem, so that we see how each enters into the result.

If the sum of the two numbers is called s and their difference d, then we are *compelled* to keep these letters separate to the end of the solution.

Thus, the lesser is	$l=s-g,$
and	$g-(s-g)=d.$
By VII,	$g-s+g=d.$
By I, *A*,	$2g=s+d.$

Hence by D, VI, the greater is $g = \frac{s}{2} + \frac{d}{2}$, and the lesser is

$$l = s - g = s - \left(\frac{s}{2} + \frac{d}{2}\right) = s - \frac{s}{2} - \frac{d}{2} = \frac{2s}{2} - \frac{s}{2} - \frac{d}{2} = \frac{s}{2} - \frac{d}{2}.$$

Hence $$g = \frac{s}{2} + \frac{d}{2}, \text{ and } l = \frac{s}{2} - \frac{d}{2}.$$

These results put into words are as follows:

The greater of any two numbers is half their sum plus half their difference, and the lesser is half their sum minus half their difference.

Any problem of this kind is solved by substituting in this formula the particular values given to s and d and simplifying the results.

E.g. In problem 4, page 110, $s = 8590$ and $d = 3480$. Hence the greater number is $\frac{8590}{2} + \frac{3480}{2} = 4295 + 1740 = 6035$, and the lesser number is $\frac{8590}{2} - \frac{3480}{2} = 4295 - 1740 = 2555$.

5. Find by this formula two numbers whose sum is 17,540 and whose difference is 11,240.

6. Find two numbers whose sum is 40 and whose difference is 52. Solve also without the formula.

Evidently one of these must be a negative number in order to make the difference more than the sum, but the formula applies even in such cases.

7. Find two numbers whose sum is 38 and whose difference is 50. Solve also without the formula.

8. The sum of two numbers is 48, and one is 3 times the other. Find the numbers.

9. The sum of two numbers is 168, and one is 6 times the other. Find the numbers.

10. The sum of two numbers is s, and one is k times the other. Find the numbers.

Let $n =$ one of the numbers.

Then $s - n =$ the other number,

and $kn = s - n.$

By A, $kn + n = s.$

By I, $(k + 1)n = s.$

By D, $n = \frac{s}{k+1}$, one of the numbers,

and $kn = k \cdot \frac{s}{k+1}$, the other number.

Problems 8 and 9 may be solved by substitution in this formula. State this formula in words.

11. The sum of two numbers is 195, and one is 14 times the other. Find the numbers. Solve also without the formula.

12. The sum of two numbers is 75, and one is $\frac{2}{3}$ of the other. Find the numbers. Solve also without the formula.

13. The sum of two numbers is -52, and one is 12 times the other. Find the numbers. Solve also without the formula.

14. If 9 be added to a number and the sum multiplied by 4, the product equals 7 times the number. What is the number?

15. If 21 be added to a number and the sum multiplied by 5, the product equals 12 times the number. What is the number?

16. If a be added to a number and the sum be multiplied by b, the product is c times the number. What is the number?

If $n =$ the number. Show that

$$n = \frac{ab}{c - b}.$$

17. If 24 be added to a number and the sum multiplied by 3, the product is 9 times the number. Find the number. Solve by use of the formula of 16, and also without it.

18. If 3 be added to a number and the sum multiplied by 16, the product is 10 times the number. Find the number by use of the formula, and also without it.

19. The sum of three numbers is 108. The second is 16 greater than the first, and the third 25 greater than the second. What are the numbers?

20. The sum of three numbers is 98. The second is 7 greater than the first, and the third is 9 greater than the second. What are the numbers?

21. The sum of three numbers is s. The second is a greater than the first, and the third is b greater than the second. What are the numbers?

If $n =$ the first number, show that $n = \frac{s - 2a - b}{3}$.

Translate the result of problem 21 into words. It should be realized that if in problems 19 and 20 the numbers had been kept uncombined, as they were necessarily in 21, those results would translate into words, exactly as in 21.

Use the formula derived in 21 to solve the following, and also solve each one without the formula.

22. The sum of three numbers is 198. The second is 28 larger than the first, and the third is 25 larger than the second. What are the numbers?

23. The sum of three numbers is 31. The second is 3 more than the first, and the third 2 less than the second. What are the numbers?

Since the third number is 2 less than the second, this means that b of the formula is negative; *i.e.* $b = -2$.

24. The sum of three numbers is 91. The second is 11 less than the first, and the third is 12 more than the second. What are the numbers?

25. The sum of three numbers is 69. The second is 9 less than the first, and the third is 6 less than the second. What are the numbers? (How does the formula apply in this case?)

26. Divide the number 248 into two parts, such that 7 times the first is 42 more than twice the second.

27. Divide the number 645 into two parts, such that 13 times the first part is 20 more than 6 times the other.

28. Divide the number a into two parts, such that b times the first part is c more than d times the second part.

If x is the first part, show that $x = \frac{ad + c}{b + d}$.

29. Translate the formula in 28 into words.

30. Divide the number 1240 into two parts such that 5 times the first is 200 more than the second. Solve by use of the formula, and also without it.

PROBLEMS INVOLVING MOTION

96. Problems like those in the preceding class are useful chiefly in cultivating skill in deducing formulas, and so making rules. Those, however, in this and most of the following classes are extremely important in themselves, because of the simple laws of nature which they exemplify, and of which they afford a wide range of application.

97. In scientific language the distance passed over by a moving body is called the **space**, and the number of units of space traversed is represented by s. The **rate** of motion, that is the number of units of space traversed in the unit of time, is called the **velocity**, and is represented by v. The number of units of time occupied is represented by t.

E.g. At a certain temperature sound travels 1080 feet per second. Hence, in 5 seconds it will travel $5 \cdot 1080 = 5400$ feet. In this case, $s = 5400$ feet, $v = 1080$ feet per second, $t = 5$ seconds.

We then have the formula

$$s = vt. \qquad (1)$$

Solve the equation $s = vt$ for t in terms of s and v, and for v in terms of s and t.

Translate each of these formulas into words.

In equation (1) give particular values to any two of the letters and find the value of the remaining one.

It is to be understood in all problems here considered that the velocity remains the same throughout the period of motion; *e.g.* sound travels just as far in any one second as in any other second of its passage.

1. If sound travels 1080 feet per second, how far does it travel in 6 seconds?

2. If a glacier moves 450 feet per year, how far does it move in 7 years?

3. If a transcontinental train averages 35 miles per hour, how far does it travel in $2\frac{1}{2}$ days? (Given $v = 35$, $t = 2\frac{1}{2} \cdot 24$, to find s.)

4. A hound runs 23 yards per second and a hare 21 yards per second. If the hound starts 79 yards behind the hare, how long will it require to overtake the hare?

If t is the number of seconds required, then by formula (1) during this time the hound runs $23\,t$ yards and the hare runs $21\,t$ yards. Since the hound must run 79 yards farther than the hare, we have: $23\,t = 21\,t + 79$.

5. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the liner overtake the freight?

6. A motor boat starts $7\frac{2}{3}$ miles behind a sailboat and runs 11 miles per hour while the sailboat makes $6\frac{1}{2}$ miles per hour. How long will it require the motor boat to overtake the sailboat?

7. A freight train running 25 miles an hour is 200 miles ahead of an express train running 45 miles an hour. How long before the express will overtake the freight?

8. A bicyclist averaging 12 miles an hour is 52 miles ahead of an automobile running 20 miles an hour. How soon will the automobile overtake him?

9. A and B run a mile race. A runs 18 feet per second and B $17\frac{1}{2}$ feet per second. B has a start of 30 yards. In how many seconds will A overtake B? Which will win the race?

10. Two objects, A and B, move in the same direction, A at v_1* feet per second and B at v_2 feet per second. If A has n feet the start, in how many seconds will B overtake him?

If t is the number of seconds, then during this time A moves v_1t feet and B moves v_2t feet. Since B must move n feet farther than A, we have

$$v_2t = v_1t + n. \quad (2)$$

The solution of (2) for t gives the time sought.

It is of the utmost importance that formulas (1) and (2) be clearly understood since they are fundamental in every motion problem in this chapter.

11. A fleet, making 11 knots per hour, is 1240 knots from port when a cruiser, making 19 knots per hour, starts out to overtake it. How long will it require?

12. In how many minutes does the minute hand of a clock gain 15 minute spaces on the hour hand?

Using one minute space for the unit of distance and 1 minute as the unit of time, the rates are 1 and $\frac{1}{12}$ respectively, since the hour hand goes $\frac{1}{12}$ of a minute space in 1 minute. Letting t be the number of minutes required, we have, just as in problem 10,

$1 \cdot t = \frac{1}{12} t + 15.$

13. In how many minutes after 4 o'clock will the hour and minute hands be together? (Here the minute hand must gain 20 minute spaces.)

* v_1 and v_2, read v *one* and v *two*, are used instead of two different letters to represent the velocities of the first and second respectively.

14. At what time between 5 and 6 o'clock is the minute hand 15 minute spaces behind the hour hand? At what time is it 15 minute spaces ahead?

Since, at 4 o'clock, it is 25 minute spaces behind the hour hand, in the first case it must gain $25 - 15 = 10$ minute spaces, and in the second case it must gain $25 + 15 = 40$ minute spaces. Make a diagram as in the preceding problem to show both cases.

15. At what time between 9 and 10 o'clock is the minute hand of a clock 30 minute spaces behind the hour hand? At what time are they together?

In each case, starting at 9 o'clock, how much has the minute hand to gain?

16. A fast freight leaves Chicago for New York at 8.30 A.M. averaging 32 miles per hour. At 2.30 P.M. a limited express leaves Chicago over the same road, averaging 55 miles per hour. In how many hours will the express overtake the freight?

If the express requires t hours to overtake the freight, the latter had been on the way $t + 6$ hours. Then the distance covered by the express is $55t$, and the distance covered by the freight is $32(t + 6)$. As these must be equal, we have $55t = 32(t + 6)$.

17. In a century bicycle race one rider averages $19\frac{1}{2}$ miles per hour, while another, starting 40 minutes later, averages $22\frac{1}{4}$ miles per hour. In how long a time will the latter overtake the former?

18. A sparrow flies 135 feet per second and a hawk 149 feet per second. The hawk in pursuing the sparrow passes a certain point 7 seconds after the sparrow. In how many seconds from this time does the hawk overtake the sparrow?

19. A courier starts from a certain point traveling v_1 miles per hour, and a hours later a second courier starts, going at the rate of v_2 miles per hour. In how long a time will the second overtake the first, supposing v_2 greater than v_1?

If the second courier requires t hours to overtake the first the latter had been on the way $t + a$ hours. Thus the distance covered by the second courier is v_2t and by the first $v_1(t + a)$. As these numbers are equal we have

$$v_2t = v_1(t + a) \tag{3}$$

20. In an automobile race A drives his machine at an average rate of 53 miles per hour, while B, who starts $\frac{1}{4}$ hour later, averages 57 miles per hour. How long does it require B to overtake A? Use formula (3). Solve also by finding how far A has gone when B starts and then use formula (2).

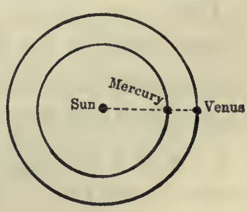
21. A freight steamer leaves New York for Liverpool averaging $10\frac{1}{2}$ knots per hour, and is followed 4 days later by an ocean greyhound averaging $20\frac{1}{2}$ knots per hour. In how long a time will the latter overtake the former?

22. One athlete makes a lap on an oval track in 26 seconds, another in 28 seconds. If they start together in the same direction, in how many seconds will the first gain one lap on the other? Two laps?

Let one lap be the unit of distance. Since the first covers one lap in 26 seconds his rate per second is $\frac{1}{26}$. Likewise the rate of the other is $\frac{1}{28}$. If t is the required number of seconds the distance covered by the first is $\frac{1}{26}t$ and by the second $\frac{1}{28}t$. If the first goes one lap farther than the second the equation is $\frac{1}{26}t = \frac{1}{28}t + 1$; if two laps farther it is $\frac{1}{26}t = \frac{1}{28}t + 2$.

23. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain 1 lap on the slower?

24. The planet Mercury makes a circuit around the sun in 3 months and Venus in $7\frac{1}{2}$ months. Starting in conjunction, as in the figure, how long before they will again be in this position?



25. Saturn goes around the sun in 29 years and Jupiter in 12 years. Starting in conjunction, how soon will they be in conjunction again?

26. Uranus makes the circuit of its orbit in 84 years and Neptune in 164 years. If they start in conjunction, how long before they will be in conjunction again?

27. The hour hand of a watch makes one revolution in 12 hours, and the minute hand in one hour. How long is it from the time when the hands are together until they are again together?

28. One object makes a complete circuit in a units of time and another in b units (of the same kind). In how many units of time will one overtake the other, supposing b to be greater than a?

29. At what times between 12 o'clock and 6 o'clock are the hands of a watch together? (Find the time required to gain one circuit, two circuits, etc.)

PROBLEMS INVOLVING THE LEVER

98. Two boys, A and B, play at teeter. They find that the teeter board will balance when equal products are obtained by multiplying the weight of each by his distance from the point of support.

Thus, if B weighs 80 pounds and is 5 feet from the point of support, then A, who weighs 100 lbs., must be 4 feet from this point, since $80 \times 5 = 100 \times 4$.

The teeter board is a certain kind of lever; the point of support is called the **fulcrum**.

In each of the following problems make a diagram similar to the above figure:

1. A and B weigh 90 and 105 lbs. respectively. If A is seated 7 feet from the fulcrum, how far is B from this point?

2. Using the same weights as in the preceding problem, if B is $6\frac{1}{2}$ feet from the fulcrum, how far is A from that point?

3. A and B are 5 and 7 feet respectively from the fulcrum. If B weighs 75 pounds, how much does A weigh?

4. A and B weigh 100 and 110 pounds respectively. A places a stone on the board with him so that they balance when B is 6 feet from the fulcrum and A $5\frac{1}{2}$ feet from this point. How heavy is the stone?

5. If the distances from the boys to the fulcrum are respectively d_1 and d_2, and their weights w_1 and w_2, then

$$d_1 w_1 = d_2 w_2. \qquad (1)$$

If any three of these four numbers are given, the fourth may be found by means of this equation. Solve $d_1 w_1 = d_2 w_2$ for each of the four numbers involved in terms of the other three. Problems 1 to 4 can be solved by substitution in the formulas thus obtained.

6. A and B are seated at the opposite ends of a 13-foot teeter board. Using the weights of problem 1, where must the fulcrum be located so that they shall balance?

If the fulcrum is the distance d from A then it is $(13 - d)$ from B. Hence $90d = 105(13 - d)$.

7. A and B together weigh $212\frac{1}{2}$ pounds. They balance when A is 6 feet, and B $6\frac{3}{4}$ feet, from the fulcrum. Find the weight of each.

8. A, who weighs 75 pounds, sits 7 feet from the fulcrum, and B, who weighs 105 pounds, sits on the other side. At what distance from the fulcrum should B sit in order to make a balance?

9. If in the preceding problem C, weighing 48 pounds, sits on the side with A and 4 feet from the fulcrum, where must D, who weighs 64 pounds, sit so as to maintain a balance?

From problem 8, when the teeter balanced it was found that A's weight acted like a lever with a downward force of $7 \cdot 75$ pounds and B's on the other side with a force of $5 \cdot 105$ pounds.

$$7 \cdot 75 = 5 \cdot 105.$$

Then in problem 9, C and D must sit so as to keep the board in balance, that is, so as to add the same downward force to both sides. Hence, as $4 \cdot 48$ is added on the side of A, $3 \cdot 64$ must be added on the side of B, since $4 \cdot 48 = 3 \cdot 64$. Therefore, adding these two equations, we still have the balance, namely,

$$7 \cdot 75 + 4 \cdot 48 = 5 \cdot 105 + 3 \cdot 64.$$

The weight of the boy multiplied by his distance from the fulcrum is called his **leverage.** The sum of the leverages on the two sides must be the same. Hence, if two boys, weighing respectively w_1 and w_2 pounds, are sitting at distances d_1 and d_2 on one side, and two boys, weighing w_3 and w_4 pounds, sitting at distances d_3, d_4 on the other side, then

$$w_1d_1 + w_2d_2 = w_3d_3 + w_4d_4. \qquad (2)$$

10. If two boys weighing 75 and 90 pounds sit at distances of 3 and 5 feet respectively on one side and one weighing 82 pounds sits at 3 feet on the other side, where should a boy weighing 100 pounds sit in order to make the board balance?

11. A beam carries a weight of 240 pounds $7\frac{1}{2}$ feet from the fulcrum and a weight of 265 pounds at the opposite end which is 10 feet from the fulcrum. On which side and how far from the fulcrum should a weight of 170 pounds be placed so as to make the beam balance?

PROBLEMS INVOLVING DENSITIES

99. If a cubic foot of a certain kind of rock weighs 2.5 times as much as a cubic foot of fresh water, the **density** of this rock is said to be 2.5. If a cubic foot of oak weighs .85 times as much as a cubic foot of fresh water, the density of the oak is .85. The density of water is taken as the standard with which the densities of other substances are compared.

Thus, when we say that the density of a certain kind of iron is 7.25, we mean that any given volume of the iron weighs 7.25 times as much as a like volume of fresh water; and when we say that the density of cork is .24, we mean that a given volume of cork weighs .24 times as much as a like volume of fresh water.

A cubic centimeter of distilled water at the freezing point which weighs one gram is used as a standard of comparison. We therefore say that the density of any substance is equal to the number of grams which a cubic centimeter of it weighs.

E.g. if a cubic centimeter (ccm.) of a certain kind of marble weighs 2.5 grams, then it weighs 2.5 times as much as the same volume of water, and hence its density is 2.5.

The weight of an object in grams is, therefore, the product of its volume in cubic centimeters multiplied by its density.

E.g. if the density of cork is .24, this means that a cubic centimeter of cork weighs .24 grams. Any volume of cork, say 10 ccm., weighs $10 \times .24 = 2.4$ grams. Hence, if we represent the weight of an object in grams by w, its volume in cubic centimeters by v, and its density by d, we have the relation,

$$w = vd. \qquad (1)$$

1. What is the weight of an object whose density is 4.3 and whose volume is 250 ccm.? (Here $v = 250$, $d = 4.3$. Find w.)

2. What is the density of an object whose weight is 23.5 and whose volume is 17 ccm.?

3. What is the volume of an object whose weight is 24 grams and whose density is .65?

4. If 500 ccm. of alcohol, density .79, is mixed with 300 ccm. of distilled water, what is the density of the mixture?

The volume of the mixture is the sum of the volumes, and the weight of the mixture is the sum of the weights, of the water and alcohol. Hence, from formula (1):

$$500 \times .79 + 300 \times 1 = d \times 800. \text{ To find } d.$$

5. If 1200 ccm. of cork, density .24, are combined with 64 ccm. of steel, density 7.8, what is the average density of the combined mass? Will it float or sink? (A substance sinks if its density is greater than that of water.)

In solving problems of this kind find the expressions representing weight, density, and volume, and substitute in the equation $w = dv$.

6. How many cubic centimeters of cork, density .24, must be combined with 75 ccm. of steel, density 7.8, in order that the average density shall be equal to that of water, *i.e.* so that the combined mass will just float?

Let v = volume of cork to be used. Then the total volume is $75 + v$, the total weight is $75 \times 7.8 + .24\,v$, and the density is 1. Hence, $75 \times 7.8 + .24\,v = 1 \cdot (75 + v)$. Solve this equation for v.

7. Brass is an alloy of copper and zinc. How many cubic centimeters of zinc, density 6.86, must be combined with 100 ccm. of copper, density 8.83, to form brass whose density is 8.31?

8. Coinage silver is an alloy of copper and silver. How many ccm. of copper, density 8.83, must be added to 10 ccm. of silver, density 10.57, to form coinage silver, whose density is 10.38?

9. The density of pure gold is 19.36 and of nickel 8.57. How many ccm. of nickel must be mixed with 10 ccm. of pure gold to form 14 karat gold whose density is 14.88.

10. How much mercury, density 13.6, must be added to 20 ccm. of gold, density 19.36, so that the density of the compound shall be 16.9?

11. What is the average density of 40 ccm. of water, density 1, and 180 ccm. of alcohol, density .79?

12. How many cubic centimeters of water must be mixed with 350 ccm. of alcohol, so that the density of the mixture shall be .97?

13. The density of copper is 8.83. 500 ccm. of copper is mixed with 700 ccm. of lead, whose density is 11.35. What is the density of the combined mass?

14. When 960 ccm. of iron, density 7.3, is fastened to 8400 ccm. of white pine, the combination just floats, *i.e.* has a density of 1. What is the density of white pine?

PROBLEMS ON MOMENTUM

100. The force with which a moving body strikes another depends both upon its weight and upon its rate of motion. The product of the weight and velocity of a moving body is called its **momentum.** The weight of a body is also commonly called its **mass.**

What is the momentum of a body whose weight is 10 pounds and which moves 15 feet per second?

What is the velocity of a body whose weight is 50 and whose momentum is 350 pounds?

What is the weight of a body whose momentum is 500 and whose velocity is 25 feet per second?

A bullet weighing $\frac{1}{5}$ of a pound, moving 2250 feet per second, has a momentum equal to that of a stone weighing 50 pounds, which is hurled at the rate of 9 feet per second, since $\frac{1}{5} \cdot 2250 = 50 \cdot 9$.

By careful experiment it has been found that when a moving body strikes a body at rest but free to move, the two will move on with a combined momentum equal to the momentum of the first body before the impact.

Thus, if a freight car, weighing 25 tons and moving at the rate of 12 miles per hour, strikes a standing car weighing 15 tons, the two will move on with the original momentum of $12 \cdot 25$. But as the combined weight is now $25 + 15$, the *rate* of motion has been decreased to $7\frac{1}{2}$ miles per hour, since $12 \cdot 25 = 7\frac{1}{2}\,(25 + 15)$. In this case the automatic coupler connects the cars and they move on together. Even if the two bodies after impact do not cling together, as two croquet balls, still the momentum of the one plus that of the other equals the original momentum.

E.g. if a croquet ball weighing 8 ounces and moving 20 feet per second, strikes another weighing 7 ounces and starts it off at the rate of 18 feet per second, then if the diminished velocity of the first ball is called v, we have

$$8 \cdot 20 = 7 \cdot 18 + 8v,$$

and solving, $v = 4.25.$

This indicates that the first ball is nearly stopped, which coincides with common observation.

1. In a switch yard a car weighing 40 tons and moving 8 miles per hour strikes a standing car weighing 24 tons. What is the velocity of the two after impact?

2. A billiard ball weighing 6 ounces and moving 16 feet per second strikes another ball which it sends off at the rate of 10 feet per second. The rate of the first ball is reduced to 9 feet per second by the impact. What is the weight of the second ball?

Since the momentum before impact equals the sum of the momentums after impact, we have $6 \cdot 16 = 9 \cdot 6 + 10w$, w being the unknown weight of the second ball.

3. A bowler uses a 16-ounce ball to take down the last pin. The ball sends the pin off at a velocity of 6 feet per second, the weight of the pin being 48 ounces, while the velocity of the ball is reduced to 4 feet per second. With what velocity did the ball strike the pin?

Since the momentum before impact equals the sum of the momentums after impact, we have $16v = 6 \cdot 48 + 4 \cdot 16$, v being the unknown velocity of the ball before impact.

4. In each of these problems we have considered the weight of two bodies, which we may call w_1 and w_2. If we call v_1 the velocity with which the first strikes the second, v_2 the velocity

imparted to the second, and v_1' the resulting decreased velocity of the first,* we have

$$w_1v_1 = w_1v_1' + w_2v_2 \quad (1)$$

Translate this equation into words. It contains 5 different numbers, w_1, w_2, v_1, v_1', and v_2. If any four of these are given, the fifth may be found by solving this equation for that one in terms of the other four.

5. Solve the equation (1) for v_1 in terms of w_1, w_2, v_1', and v_2. Translate into words.

6. Solve the equation (1) above for w_1 in terms of v_1, v_1', w_2, and v_2. Translate into words.

7. Solve the equation (1) above for v_1' in terms of w_1, w_2, v_1, and v_2. Translate into words.

8. Solve equation (1) for w_2 in terms of w_1, v_1, v_1', v_2, and translate the result into words.

9. Solve equation (1) for v_2 in terms of w_1, w_2, v_1, v_1', and translate the result into words.

10. In the result of the last exercise substitute $w_1 = 50$, $w_2 = 40$, $v_1 = 10$, $v_1' = 2$, and find the value of v_2. Make a problem to fit this case.

11. In the result of problem 8 substitute $w_1 = 1000$, $v_1 = 75$, $v_1' = 25$, $v_2 = 50$, and find the value of w_2. Make a problem to fit this case.

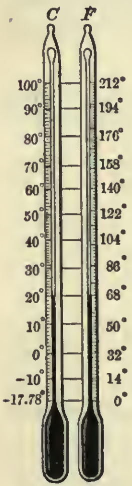
12. In the result of problem 6 substitute $v_1 = 60$, $v_1' = 10$, $w_2 = 80$, $v_2 = 25$, and find the value of w_1. Make a problem to fit this case.

13. In the result of problem 7 substitute $w_1 = 250$, $w_2 = 125$, $v_1 = 50$, and $v_2 = 50$, and find the value of v_1'. Make a problem to fit this case.

* v_1' is read *v one prime* and is used rather than a new letter for the purpose of recalling that this is *another* rate of the first body.

PROBLEMS ON THERMOMETER READINGS

101. There are two kinds of thermometers in use in this country, called the Fahrenheit and Centigrade, the former for common purposes, and the latter for scientific records and investigations. Hence it frequently becomes necessary to translate readings from one kind to the other.



The freezing and boiling points are two fixed temperatures by means of which the computations are made. On the Centigrade these are marked 0° and 100° respectively and on the Fahrenheit they are marked 32° and 212° respectively. See the cut. Hence between the two fixed points there are 100 degrees Centigrade and 180 degrees Fahrenheit.

That is, 100 degree spaces on the Centigrade correspond to 180 degree spaces on the Fahrenheit.

Hence 1° *Centigrade corresponds to* $\frac{9}{5}$° *Fahrenheit*, or 1° *Fahrenheit corresponds to* $\frac{5}{9}$° *Centigrade.*

All problems comparing the two thermometers are solved by reference to these fundamental relations.

1. If the temperature falls 15 degrees Centigrade, how many degrees Fahrenheit does it fall?

2. If the temperature rises 18 degrees Fahrenheit, how many degrees Centigrade does it rise?

3. Translate $+25°$ Centigrade into Fahrenheit reading.

25° Centigrade equals $\frac{9}{5} \cdot 25° = 45°$ Fahrenheit.

45° above the freezing point $= 45° + 32°$ above 0° Fahrenheit. Hence, calling the Fahrenheit reading F, we have $F = 32 + \frac{9}{5} \cdot 25$.

4. Translate $+14°$ Centigrade into Fahrenheit reading.

Reasoning as before, $F = 32 + \frac{9}{5} \cdot 14$.

From the two preceding problems we have the formula

$$F = 32 + \tfrac{9}{5}C. \qquad (1)$$

Translate this into words, understanding that F and C stand for readings on the respective thermometers.

5. Solve the above equation for C in terms of F and find

$$C = \tfrac{5}{9}(F - 32). \qquad (2)$$

Translate this into words. Verify the solution of each of the following by referring to the cut on page 128.

6. Translate + 41° Fahrenheit into Centigrade reading. Substitute in the proper formula.

7. Translate + 98° Fahrenheit, blood heat, into Centigrade reading.

8. Translate − 20° Fahrenheit into Centigrade reading.

9. Translate − 40° Centigrade, the freezing point of mercury, into Fahrenheit reading.

10. Translate 0° Fahrenheit into Centigrade by use of the formula. Also 0° Centigrade into Fahrenheit.

11. Translate + 212° Fahrenheit into Centigrade, and also + 100° Centigrade into Fahrenheit.

12. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is 102°?

13. What is the temperature Fahrenheit when the sum of the Centigrade and Fahrenheit readings is zero?

14. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is 140°?

15. What is the temperature in each reading when the Fahrenheit is 50° higher than the Centigrade?

16. The Fahrenheit reading at the temperature of liquid air is 128 degrees lower than the Centigrade reading. Find both the Centigrade and the Fahrenheit reading at this temperature.

PROBLEMS ON THE ARRANGEMENT AND VALUE OF DIGITS

102. If we speak of the number whose 3 digits, in order from left to right, are 5, 3, and 8, we mean $538 = 500 + 30 + 8$. Likewise, the number whose three digits are h, t, and u is written $100h + 10t + u$.

Hence, when letters stand for the digits of numbers written in the decimal notation, care must be taken to multiply each letter by 10, 100, 1000, etc., according to the position it occupies.

Illustrative Problem. A number is composed of two digits whose sum is 6. If the order of the digits is reversed, we obtain a number which is 18 greater than the first number. What is the number?

Solution. Let $x =$ the digit in tens' place.

Then $6 - x =$ the digit in units' place.

Hence, the number is $10x + 6 - x$. Reversing the order of the digits, we have as the new number $10(6 - x) + x$.

Hence $10(6 - x) + x = 18 + 10x + 6 - x.$

1. A number is composed of two digits, the digit in units' place being 2 greater than the digit in tens' place. If 4 is added to the number, it is then equal to 5 times the sum of the digits. What is the number?

2. A number is composed of two digits, the digit in tens' place being 3 greater than the digit in units' place. The number is one more than 8 times the sum of the digits. What is the number?

3. A number is composed of two digits whose sum is 9. If the order of the digits is reversed, we obtain a number which is equal to 12 times the remainder when the units' digit is taken from the tens' digit. What is the number?

4. A number is composed of two digits whose difference is 4. If the order of the digits is reversed, we obtain a number which is 3 less than 4 times the sum of the digits. What is the number?

5. The digit in tens' place is 1 more than twice the digit in units' place. If 36 is subtracted from the number, the order of the digits will be reversed. What is the number?

6. The digit in units' place is 2 less than twice the digit in tens' place. If the order of the digits is reversed, the number is unchanged. What is the number?

7. The digit in tens' place is 12 less than 5 times the digit in units' place. If the order of the digits is reversed, the number is equal to 4 times the sum of the digits. What is the number?

8. A number is composed of three digits. The digit in units' place is 3 greater than the digit in tens' place, which in turn is 2 greater than the digit in hundreds' place. The number is equal to 96 plus 4 times the sum of the digits. What is the number?

REVIEW QUESTIONS

1. In order that a problem may be solved by means of a formula, how many of the letters in the formula must be given by the problem? Illustrate this by the formula $i = prt$.

2. State the formulas involving areas and volumes which have been used in this chapter. Solve each formula for each of its letters in terms of all the others.

3. The area of a circle is found by squaring its radius and multiplying by 3.1416. State this rule as a formula, using the Greek letter π for 3.1416.

4. The volume of a circular column is found by multiplying the area of its base by its height. State this rule as a formula, using r for the radius of the base and h for the height.

5. State formulas for finding two numbers when their sum and difference are given.

Find a number such that if 16 be subtracted from it, 5 times this difference equals the number.

Construct other problems like this and then make a formula for the solution of all such problems. (See page 110.)

6. State three important formulas used in the solution of motion problems. Translate each into words. Solve each formula for each of its letters.

Of the motion problems, 4 to 29, which ones involve formula (1)? Which formula (2) and which formula (3)?

7. State the formulas used in this chapter in working problems on the lever. Solve each formula for each of its letters.

Using a teeter board 12 feet long with the fulcrum in the middle, how may three boys weighing 50, 75, and 100 lbs. respectively be seated on it so as to make the board balance? Is there more than one solution? Make a diagram for each of your results.

8. What is meant by the density of a substance? What is used as a standard (unit) of density with which the densities of other substances are compared?

What is the relation between the weight of a substance, its volume and its density?

9. Define momentum. When a moving car strikes a standing car, free to move, what can you say of the momentum of the two after impact?

10. Describe the Fahrenheit and Centigrade thermometers. State the formulas for the reading of each thermometer in terms of the other.

11. Write 347 as a trinomial. Write the number whose three digits in order are a, b, c; also the number whose three digits in order are c, b, a.

REVIEW PROBLEMS

103. Most of the following problems can be solved by substitution in some of the formulas developed in this chapter. Solve as many as possible in this way.

1. One boy runs around a circular track in 26 seconds, and another in 30 seconds. In how many seconds will they again be together, if they start at the same time and place and run in the same direction?

2. Divide the number 144 into two parts, so that $\frac{1}{4}$ of the greater is 9 more than $\frac{1}{5}$ of the smaller.

3. The sum of two numbers is 2890. Seven times one is 266 less than 5 times the other. What are the numbers?

4. What is the simple interest on \$400 at $6\frac{1}{2}\%$ for 7 years and 9 months?

5. The number of telegraph messages sent in the United States in 1905 was 5 million less than three times as great as in 1880, and 69 million less than twice that in 1900; while in 1900 it was 16 million more than twice as great as in 1880. Find the number of messages in each of these years.

6. The same number is added to each of the numbers 8, 9, 10, 12. What is this number if the product of the first and last sums is equal to the product of the second and third sums?

7. Find the time between 4 and 5 o'clock when the hands of the clock are 30 minute spaces apart.

8. A man buys a house for \$6500. His yearly tax on the property is \$57. The coal costs \$60 per year, repairs \$50, and janitor service \$108. To what monthly rental are his expenses equivalent if money is worth 5%?

9. A boatman rowing down a river makes 23 miles in 3 hours and returns at the rate of $3\frac{1}{2}$ miles per hour. How swift does the river flow?

10. A bird flying with the wind goes 65 miles per hour, and flying against a wind twice as strong it goes 20 miles per hour. What is the rate of the wind in each case?

11. A steamer going with the tide makes 19 miles per hour, and going against a current $\frac{1}{2}$ as strong it makes 13 miles per hour. What is the speed of the steamer in still water?

12. A boatman trying to row up a river drifts back at the rate of $1\frac{1}{2}$ miles per hour, while he can row down the river at the rate of 12 miles per hour. What is the rate of the current?

13. A boatman rowing with the tide makes n miles per hour; rowing against a tide k times as strong, he makes m miles per hour. At what rate does he row, and what is the velocity of the stream? State the result as a formula.

14. A beam is 16 feet long. At what point must it be supported if it is to carry, when balanced, 460 pounds at one end and 690 at the other, its weight being disregarded?

15. Two numbers differ by 2 and the difference of their squares is 100. What are the numbers?

16. A beam is 12 feet long. It carries a 40-pound weight at one end, a 60-pound weight 3 feet from this end, and a 70-pound weight at the other end. Where is the fulcrum if the beam is balanced?

17. There is a number composed of 2 digits whose tens' digit is 2 less than its units' digit. The number is 1 less than 5 times the sum of its digits. What is the number?

18. A farm containing 240 acres can be rented at $3 per acre. The renter finds that if he borrows money at 5 % to buy the farm he will save $125 per year. Find its value.

19. Two trains start from the same point at the same time and in the same direction, one making 25 miles per hour and the other 42 miles per hour. When will they be 238 miles apart?

20. The number of national banks in the United States on March 1, 1906, was 1322 less than twice as many as on March 1, 1900, and the number in 1903 was 1009 greater than in 1900. If the number in 1906 be subtracted from 3 times the number in 1903 the remainder is 7736. Find the number in each of the three years mentioned.

21. The earth and Mars were in conjunction July 12, 1907. When are they next in conjunction if the earth's period is 365 days and that of Mars 687 days? (See figure, page 119.)

22. $7500 is invested at 4 % simple interest. Seven years and three months later the amount is used to build a house. What is the cost of the house if $735 has to be added to complete it?

23. There is a number composed of two digits whose sum is 12. If the order of the digits is reversed, the number is increased by 18. Find the number.

24. A slow steamer sails from New York to Liverpool, making 9 knots per hour. A swift liner follows 62 hours later, making $20\frac{3}{4}$ knots per hour. In how many hours will the latter overtake the former?

25. A man invested a certain sum of money at 5 % simple interest. The amount $3\frac{3}{4}$ years later was $950. What was the investment?

26. The capital stock of the Bank of France is 35.6 million dollars less than that of the Bank of England and 6.3 million greater than that of the Imperial Bank of Germany. The combined capital stock of the three banks is 134.9 million dollars. Find the capital stock of each.

27. The capital stock of the Bank of Italy is 27.7 million dollars less than twice that of the Imperial Bank of Russia, and the capital of the Bank of Austria-Hungary is 13.6 million greater than that of the Bank of Russia. Their combined capital is 99.1 million dollars. Find the capital of each bank.

28. In a bicycle race A starts 32 minutes ahead of B. B rides at the rate of $20\frac{1}{3}$ miles per hour, while A rides $18\frac{2}{5}$ miles per hour. How many miles from the starting point does B overtake A?

29. A squadron of warships sails 13 knots per hour. A torpedo boat making $27\frac{2}{3}$ knots leaves port 19 hours later to overtake the squadron. In how many hours after leaving port will the torpedo boat overtake it?

30. A man takes out a life insurance policy for which he pays in a single payment. Thirteen years later he dies and the company pays \$12,600 to his estate. It was found that his investment yielded 2% simple interest. How much did he pay for the policy?

31. A merchant bought goods for \$600 and some months later sold them for \$648, making a profit of 2% per month. How many months elapsed between the purchase and the sale?

32. In a building there are at work 18 carpenters, 7 plumbers, 13 plasterers, and 6 hod carriers. Each plasterer gets \$1.90 per day more than the hod carriers, the carpenters get 35 cents per day more than the plasterers, and the plumbers 50 cents per day more than the carpenters. If one day's wages of all the men amount to \$183.45, how much does each get per day?

33. A train running 46 miles per hour leaves Chicago for New York at 7 A.M. Another train running 56 miles per hour leaves at 9.30 A.M. Find when the trains will be 15 miles apart. (Two answers.)

34. Divide the number 280 into two parts so that $\frac{2}{3}$ of one part is 48 less than $\frac{4}{5}$ of the other.

35. A merchant bought goods and sold them 5 months later for \$2687.50, making a gain on his investment of $1\frac{1}{2}$% per month. How much did he pay for the goods?

36. A bicyclist starts out riding 12 miles per hour, and is followed 40 minutes later by another riding 16 miles per hour. Find when they will be 5 miles apart. (Two answers.)

37. A father invested \$1000 at $6\frac{1}{2}$% interest. When the principal and simple interest amounted to \$2235, it was given to the son for the expenses of his college education. How long had the money been invested?

38. Two trains start west at the same time, one from New York and the other from Philadelphia. If the New York train runs 55 miles per hour and the Philadelphia train 47 miles per hour, how long before they are 15 miles apart, the distance from New York to Philadelphia being 90 miles?

39. A man bought a tract of coal land and sold it a month later for \$93,840. If his gain was at the rate of 24% per annum, what did he pay for the land?

40. There is a rectangle whose length is 60 feet more, and whose width is 20 feet less, than the side of a square of equal area. Find the dimensions of the square and the rectangle.

41. A number is composed of two digits whose sum is 14. If the digits are interchanged, the number is decreased by 18. What is the number?

42. What time between 7 and 8 o'clock are the hour and the minute hands of a clock together?

43. Change 104 degrees Fahrenheit (fever heat) to Centigrade reading.

44. A steamer leaves Liverpool for New York Saturday, 9 A.M., averaging 18 knots per hour. Seven hours later another steamer leaves New York for Liverpool, making $20\frac{1}{2}$ knots per hour. What time (Liverpool time) will the steamers meet if the trans-Atlantic distance by their course is 2940 knots?

CHAPTER V

INTRODUCTION TO SIMULTANEOUS EQUATIONS

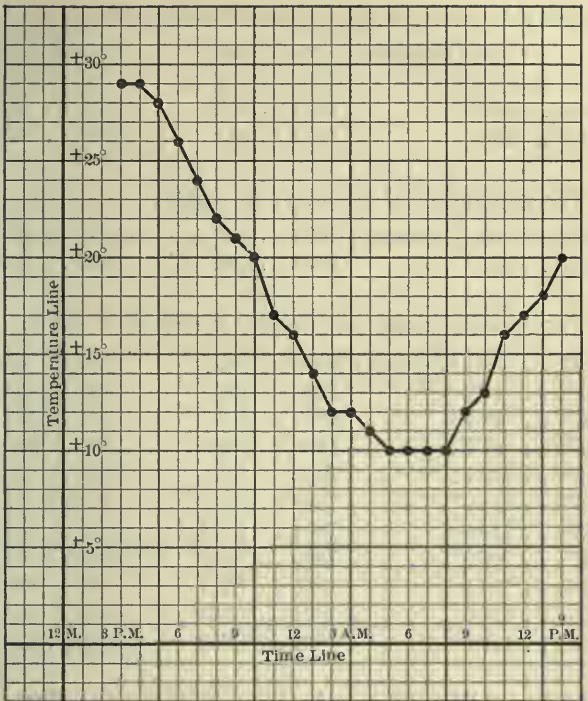
104. Graphic Representation of Statistics.

A graphic representation of the temperatures recorded by the U. S. Weather Bureau at Chicago on December 6 and 7, 1906, is shown on the next page. The readings were as follows:

3 P.M.	29°	9 P.M.	21°	3 A.M.	12°	9 A.M.	12°
4 P.M.	29°	10 P.M.	20°	4 A.M.	11°	10 A.M.	13°
5 P.M.	28°	11 P.M.	17°	5 A.M.	10°	11 A.M.	16°
6 P.M.	26°	12 M'T.	16°	6 A.M.	10°	12 Noon	17°
7 P.M.	24°	1 A.M.	14°	7 A.M.	10°	1 P.M.	18°
8 P.M.	22°	2 A.M.	12°	8 A.M.	10°	2 P.M.	20°

In the graph each heavy dot represents the temperature at a certain hour. The distance of a dot to the right of the heavy vertical line indicates the hour of the day counted from noon December 6th, and its distance above the heavy horizontal line indicates the thermometer reading at that hour. The lines joining these dots complete the picture representing the gradual changes of temperature from hour to hour.

Graphs of this kind are used in commercial houses to represent variations of sales, fluctuations of prices, etc. They are used by architects and engineers to show the comparative strength of materials, stresses to which they are subjected, and to exhibit multitudes of other data. They are used by the historian to represent changes in population, fluctuations in mineral productions, etc. In algebra they are used in solving many practical problems and in helping to understand many difficult processes. In the succeeding exercises the cross-ruled paper is essential.



EXERCISES

Make a graphic representation of each of the following tables of data:*

1. The population of the United States as given by the census reports from 1790 to 1900:

1790 . . 3.9 (million)	1830 . . 12.9	1870 . . 38.6
1800 . . 4.3	1840 . . 17.1	1880 . . 50.2
1810 . . 7.2	1850 . . 23.2	1890 . . 62.6
1820 . . 9.6	1860 . . 31.4	1900 . . 76.3

* In each case the number to be represented by one space on the cross-ruled paper should be chosen so as to make the graph go conveniently on a sheet. Thus in (1) let one small horizontal space represent two years and one vertical

2. The population of New York city since 1800:

1790	33 (thousand)	1830	202	1870	942
1800	60	1840	312	1880	1206
1810	96	1850	515	1890	1530
1820	123	1860	813	1900	1850

3. The population of Chicago since 1850:

1850	30 (thousand)	1870	306	1890	1100
1860	109	1880	503	1900	1698

4. The world's yearly production of gold since 1872:

1872	99.6 (million)	1883	102.4	1894	181.5
1873	96.2	1884	101.7	1895	194.0
1874	90.7	1885	108.4	1896	202.3
1875	97.5	1886	106.0	1897	236.1
1876	103.7	1887	105.8	1898	286.9
1877	114.0	1888	110.2	1899	306.7
1878	119.0	1889	123.5	1900	254.6
1879	109.0	1890	118.9	1901	262.5
1880	106.5	1891	130.7	1902	296.0
1881	103.0	1892	146.3	1903	325.5
1882	102.0	1893	157.2	1904	346.8

5. The world's yearly production of silver since 1872:

1872	65.0 (million)	1883	115.3	1894	214.5
1873	81.8	1884	105.5	1895	208.0
1874	71.5	1885	118.5	1896	203.0
1875	80.5	1886	120.6	1897	207.0
1876	87.6	1887	124.3	1898	218.6
1877	81.0	1888	140.7	1899	217.6
1878	95.0	1889	155.4	1900	224.0
1879	96.0	1890	163.0	1901	223.7
1880	96.7	1891	177.0	1902	208.6
1881	102.0	1892	197.7	1903	220.4
1882	111.8	1893	209.1	1904	217.7

space a million of population; and in (3) let one horizontal space represent one year and one large vertical space one hundred thousand of populatiou.

6. Temperature, at Port Conger, New York, and Singapore. The average temperature for each half month is given.

Month		Port Conger	New York	Singapore	Month		Port Conger	New York	Singapore
Jan.	1–15	−35°	30°	80°	July	1–15	+32°	66°	86°
	16–31	−40°	28°	82°		16–31	+37°	68°	86°
Feb.	1–15	−45°	32°	84°	Aug.	1–15	+37°	68°	87°
	16–28	−40°	35°	84°		16–31	+34°	66°	86°
Mar.	1–15	−35°	38°	85°	Sept.	1–15	+27°	62°	85°
	16–31	−30°	40°	85°		16–30	+20°	60°	85°
April	1–15	−20°	45°	85°	Oct.	1–15	+ 8°	55°	85°
	16–30	−10°	48°	85°		16–31	− 2°	50°	84°
May	1–15	0°	50°	85°	Nov.	1–15	−15°	48°	83°
	16–31	+ 8°	56°	86°		16–30	−20°	45°	82°
June	1–15	+16°	60°	86°	Dec.	1–15	−28°	42°	82°
	16–30	+20°	65°	86°		16–31	−32°	40°	81°

When the temperature is below zero, the distance is measured downward from the heavy horizontal line, as in the figure, page 149.

7. Average monthly rainfall at San Francisco, Valparaiso, Chili, and Quebec:

	(a) San Francisco	(b) Valparaiso	(c) Quebec
January . .	5.5 (inches)	0.2 (inches)	3.2 (inches)
February . .	4.5	0.3	6.4
March . . .	3.5	1	4.4
April . . .	2.5	2	6.6
May . . .	1.5	3	5.1
June . . .	0.5	4.2	2.5
July . . .	0.2	2.9	1.4
August . . .	0.2	1.8	1.8
September .	0.2	1.8	1.8
October . .	1.5	0.5	0.5
November . .	3.5	0.4	0.4
December . .	5.5	0.2	0.2

Use 10 small vertical spaces for 1 inch of rainfall and five horizontal spaces for 1 month.

8. After making the graphs in exercise 6, each on a separate sheet, put all three on the same sheet, using a different colored ink or pencil for each. In this way the relative average temperatures are simultaneously pictured.

9. After graphing (*a*), (*b*), and (*c*) of example 7, each on a separate sheet, combine all three, using different colors, as in 8.

10. Observe the weather reports in a daily paper and make a graph representing the hourly change of temperature for twenty-four hours.

GRAPHIC REPRESENTATION OF MOTION

105. A useful picture of the distance traversed by a moving body can be made by a graph similar to the preceding.

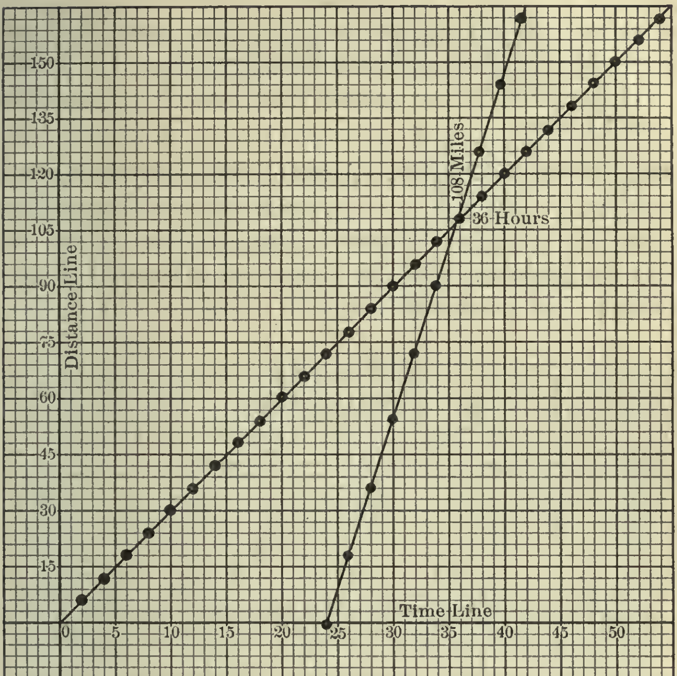
E.g. suppose a man is walking 3 miles per hour. We mark units of time from the starting point to the right along the horizontal reference line, and indicate miles traveled by the number of units measured vertically upward from this line. (See the figure on the opposite page.)

In the figure each horizontal space represents 1 hour, and each vertical space 3 miles. Then in 1 hour he goes 3 miles; in 5 hours, 15 miles; in 10 hours, 30 miles; etc. The dots representing the distances are found to lie on a straight line.

The graph shows at a glance the answers to such questions as: How many miles does he travel in 4 hours? in 13 hours? How long does it take him to go 18 miles? 23 miles?

Again, suppose 24 hours later a second man starts out on a bicycle to overtake the first man, and travels 9 miles an hour. The line drawn from the 24-hour point shows the distance the wheelman travels in any number of hours counting from his time of starting. The points marked in this line show how far he has gone in 1, 2, 3, 4, 5, 6 hours, etc., namely, 9, 18, 27, 36, 45, 54, etc.

The point where these two lines intersect shows in how many hours after starting the pedestrian is overtaken and also how far he has gone.



In like manner solve the following by means of graphs, and in each case suggest other questions which may be answered from the graph:

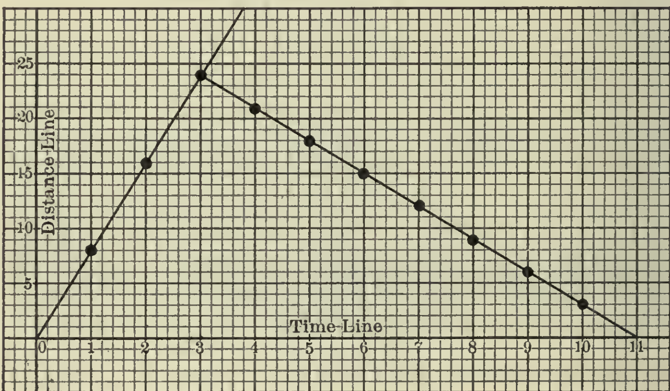
1. In a century bicycle race *A* averages 17 miles per hour; *B*, who starts 20 minutes later, averages 19 miles per hour. In how many hours will *B* overtake *A*? Who will win the race and where will the loser be when the winner finishes?

2. *A* starts for a town 12 miles distant, walking 3 miles per hour. $1\frac{1}{2}$ hours later *B* starts for the same place, driving $7\frac{1}{2}$ miles per hour. When does *B* overtake *A*? Where is *A* when *B* reaches town?

3. In a mile race A runs 6 yards per second and B 5 yards per second. B has a start of 250 yards. Who will win the race? How far in the lead is the winner at the finish?

Let 1 vertical small space represent 20 yards, and 1 horizontal space represent 5 seconds.

106. Illustrative Problem. A man rides a bicycle into the country at the rate of 8 miles per hour. After riding a certain distance the wheel breaks down and he walks back at the rate of 3 miles per hour. How far does he go if he reaches home 11 hours after starting?



In this graph each large horizontal space represents 1 hour, and each small vertical space represents 1 mile. The problem is solved as follows:

(1) Construct the line representing the outward journey at the rate of 8 miles per hour, extending this line indefinitely.

(2) Beginning at the point corresponding to 11 hours, find the points representing his position at each preceeding hour. The line connecting these points represents the homeward journey at the rate of 3 miles an hour. Extend this line until it meets the first line. The point where the lines meet represents 3 hours and 24 miles, which is the answer required in the problem.

PROBLEMS

Solve the following problems by means of graphs. In each case prepare a list of questions which may be answered from the graph.

1. A man rows 18 miles per hour down a river and 2 miles per hour returning. How far down the river can he go if he wishes to return in 10 hours?

2. A man goes from Chicago to Milwaukee on a train running $42\frac{1}{2}$ miles per hour, and returns immediately on a steamer going 17 miles per hour. Find the distance, if the round trip requires 7 hours.

3. A pleasure trip from New York to Atlanta by steamer and return by rail occupied 77 hours. Find the distance, if the rate going was 16 miles per hour and returning 40 miles per hour.

Let one small horizontal space represent one hour and one small vertical space 16 miles.

4. A invests \$1000 at 5 % and B invests \$5000 at 4 %. In how many years will the *amount* (principal and interest) of A's investment equal the *interest* on B's investment?

Let one large horizontal space represent one year and one small vertical space \$50. Then the line representing A's *amount* starts at the point marked \$1000, and rises one small vertical space each year. The line representing B's *interest* starts at the zero point and rises four small vertical spaces each year.

5. In how many years will the *interest* on \$6000 equal the *amount* on \$2000 if both are invested at 5 %?

6. A invests \$500 at 6 % and B invests \$1000 at 5 %. In how many years will A's interest differ by \$300 from B's?

Let one small vertical space represent \$20. In this case both lines start from the zero point. Find the point on one line which is three large spaces *vertically* above the corresponding point on the other line.

7. Construct a graph representing the relation between the Fahrenheit and Centigrade thermometer readings.

Let the line at the bottom of the sheet and the vertical line four large spaces from the left margin be the reference lines. Let one small horizontal space represent a degree C, and one small vertical space a degree F.

From $F = 32 + \frac{9}{5}C$ (page 129), we find that if $C = 0°$ $F = 32°$, if $C = 10°$ $F = 50°$, if $C = 20°$ $F = 68°$, if $C = 30°$ $F = 86°$, etc. Mark the point representing each pair of readings, and draw the straight line connecting these points. This is the required graph.

8. From this graph read the answers to the following questions:

Find C when $F = 41°$, $F = 59°$, $F = 79°$, $F = 14°$.

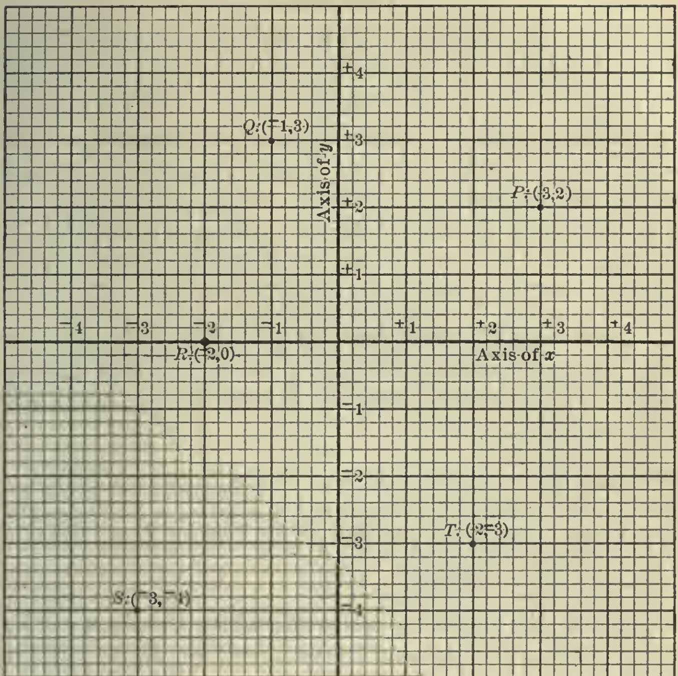
Find F when $C = 35°$, $C = 40°$, $C = -5°$, $C = -15°$.

From the graph it is possible to find any Fahrenheit reading when the corresponding Centigrade is given, and also to find any Centigrade reading when the corresponding Fahrenheit is given. Thus the graph shows to the eye all the information contained in the equation $F = 32 + \frac{9}{5}C$. In what follows we shall consider in detail how to represent equations in this manner.

GRAPHIC REPRESENTATION OF EQUATIONS

107. In all the graphs thus far constructed two lines at right angles to each other have been used as reference lines. These lines are called **axes**. The location of a point in the plane of such a pair of axes is completely described by giving its distance and direction from each of the axes. The direction to the right of the vertical axis is denoted by a positive sign, and to the left, by a negative sign; while direction upward from the horizontal axis is positive, and downward, negative.

The horizontal line is usually called the **x-axis** and the vertical line the **y-axis**. The perpendicular distance of any point P from the y-axis is called the **abscissa** of the point, and its distance from the x-axis is called its **ordinate**. The abscissa and ordinate of a point are together called its **coördinates**.



E.g. the abscissa of point P in the above figure is 3 and its ordinate 2, or we may say the coördinates of P are 3 and 2, and indicate it thus: $P:(3, 2)$, writing the abscissa first. In like manner for the other points we write $Q:(-1, 3)$, $R:(-2, 0)$, $S:(-3, -4)$, and $T:(2, -3)$. (For convenience *upper* signs are used in the figure.)

We see that in this manner every point in the plane corresponds to a pair of numbers and that every pair of numbers corresponds to a point. This scheme of locating points by two reference lines is already familiar to the pupil in geography, where cities are located by latitude and longitude, that is, by degrees north or south of the equator and east or west of the meridian of Greenwich.

EXERCISES

1. With any convenient scale, locate the following points: (2, 6), (– 3, 5), (0, 1), (1, 0), (0, 0), (0, –1), (0, – 5), (– 5, 0), ($2\frac{1}{2}$, $5\frac{1}{3}$), (–4, – 8), (3, – 10), (– 10, 3).

2. Calling north +, south –, east +, west –, so that (– 5°, 8°) means the point on a map whose longitude is 5° west and whose latitude is 8° north, verify the following on a map: New York (– 73° 57′, 40° 48′), Chicago (– 87° 35′, 41° 51′), Peking (116° 30′, 39° 50′), Sydney, New South Wales, (151° 15′, –33° 51′), Santiago, Chili, (– 70° 39′, – 33° 25′).

3. On a map of South America give approximately the location of the following cities, using the notation of the preceding exercise: Caracas, Rio de Janeiro, Bogota, Valparaiso, Lima, and Panama.

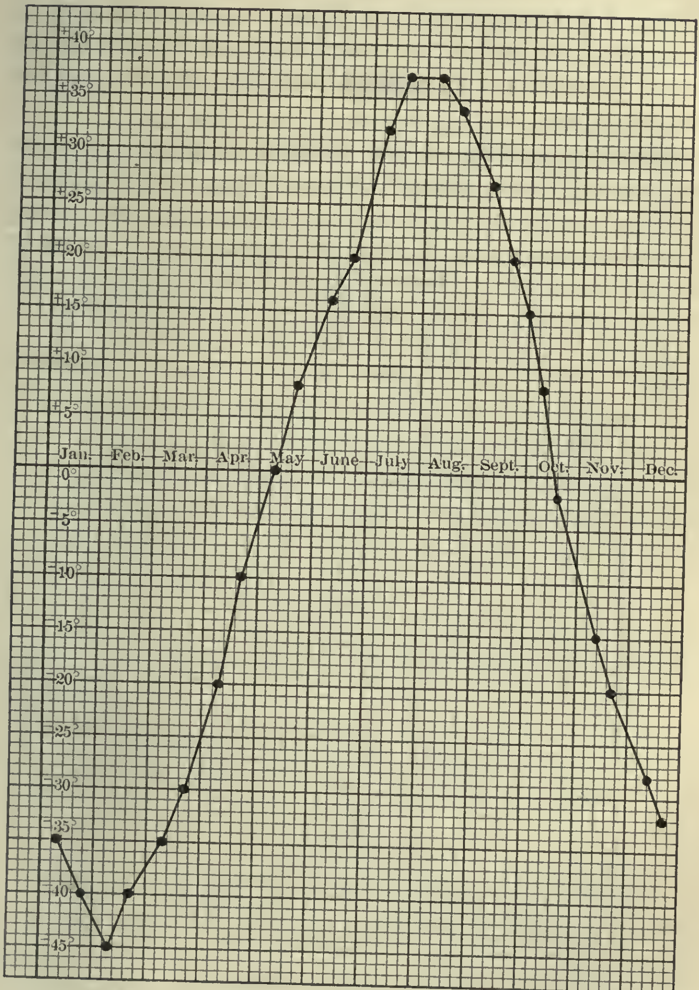
4. On a map of Africa locate in similar manner the cities or islands situated at the following points: (– 5° 40′, – 16°), (– 14° 30′, – 8°), (30° 18′, 30° 1′), (– 5° 40′, 36°), (40° 40′, – 15°), (18° 20′, – 33° 55′).

5. Locate the following series of points and then see if a straight line can be drawn through them: (0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (– 1, – 1), (– 2, – 2), (– 3, – 3). Name still other points lying on the same line.

6. Locate the following and connect them by a line: (1, 0), (1, 2), (1, 3), (1, 4), (1, 5), (1, – 2), (1, – 3), (1, – 4), (1, – 5). Name other points in this line.

7. Draw the line every one of whose points has its horizontal distance –2, also the line every one of whose points has its vertical distance + 3.

8. Locate the following points and see if a straight line can be passed through them: (1, 0), (0, 1), (2, – 1), (3, – 2), (4–3), (–1+2), (–2, 3), (–3, 4), (–4, 5), ($\frac{1}{2}$, $\frac{1}{2}$), ($\frac{1}{4}$, $\frac{3}{4}$), ($\frac{2}{3}$, $\frac{1}{3}$). Can you name other points on this line?



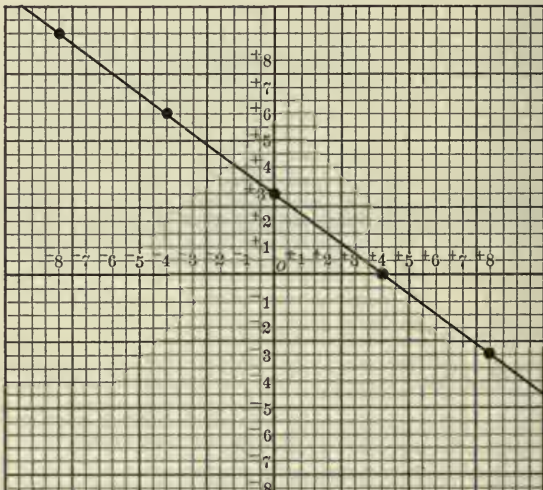
TEMPERATURE CHART FOR PORT CONGER. (See page 141.)

108. In the preceding exercises, in certain cases, a series of points has been found to **lie on a straight line** as in examples 6, 7, and 8. Evidently this could not happen unless the points were located according to some definite scheme or law.

Illustrative Problem. Locate a series of points whose coordinates are values of x and y which satisfy: $3x + 4y = 12$.

We see that $x = 0$, $y = 3$, also $x = 4$, $y = 0$ are pairs of such values. Evidently as many pairs of values as we please may be found by giving any value to x and then solving the equation to find the corresponding value of y. A table may thus be constructed as follows:

Let	$x = 0,\ 4,\ \ 8,\ 12,\ -4,\ -8$, etc.
Then	$y = 3,\ 0,\ -3,\ -6,\ \ 6,\ \ 9$, etc.



These pairs of values for x and y correspond to the points as plotted in the figure, and they are found to lie on a straight line. This line is called the **graph of the equation.**

Let the student find other pairs of numbers which satisfy this equation and see if the corresponding points lie on this line. Also find the numbers which correspond to any chosen point on this line and see whether they satisfy the equation.

109. Since an equation like $3x+4y=12$ is satisfied by indefinitely many pairs of values of x and y, it is common to call the unknowns in such an equation **variables.** Indeed if a point be thought of as moving along the graph of this equation the values of x and y corresponding to the moving point continually vary, *but always so that* $3x+4y=12$.

110. Definitions. An equation is said to be of the **first degree in x and y** if it contains each of these letters in such a way that neither x nor y is multiplied by itself or by the other.

E.g. $13x-5y=14$ is of the first degree, while $2xy-x=5$ and $3x-5y^2=13$ are not of the first degree in x and y.

Every equation of the first degree in two variables has for its graph a straight line; hence such an equation is commonly called a **linear equation.**

111. To graph an equation of the first degree it is only necessary to find two points on the graph and draw a straight line through them.

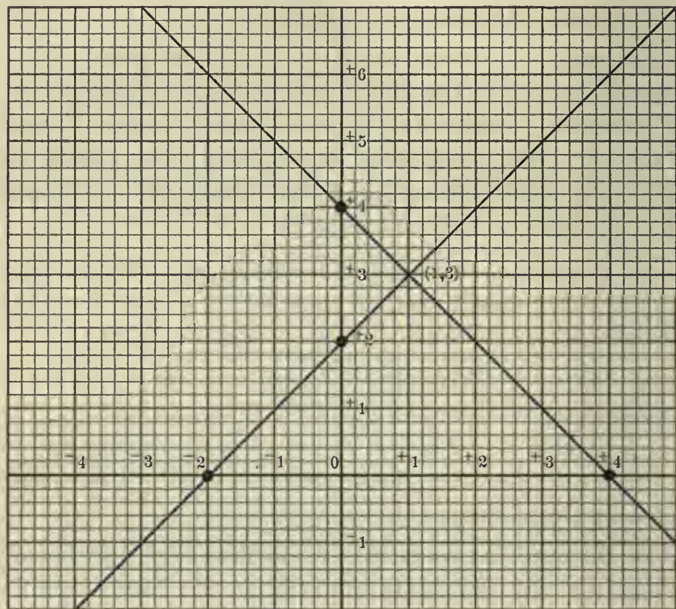
E.g. In graphing the equation $x-y=5$, we choose $x=0$ and find $y=-5$, and choose $y=0$ and find $x=5$ and plot the points $(0, -5)$ and $(5, 0)$. The line through these points is the one required.

EXERCISES

Construct the graph for each of the following equations:

1. $3x+2y=1$.	5. $5-2y=12$.	9. $3x-4y=-7$.
2. $5x-3y=-3$.	6. $3x+5y=-15$.	10. $3x-4y=-12$.
3. $7x+10y=2$.	7. $2x-y=0$.	11. $7y=9x-63$.
4. $x+2y=0$.	8. $3x-4y=7$.	12. $x=5y+3$.

112. Illustrative Problem. Graph on the same axes the two equations $x + y = 4$ and $y - x = 2$.



Solution. The two graphs are found to intersect in the point (1, 3). Since the point lies on both lines, its coördinates should satisfy both equations, as indeed they do. Since these lines have only one point in common, there is no other pair of numbers which, when substituted for the variables x and y, can satisfy both equations.

Hence $x = 1$, $y = 3$, which is written (1, 3), is called **the solution of this pair of equations.**

113. Definition. These two equations are called **independent** because their graphs are distinct. They are called **simultaneous** because there is at least one pair of values of x and y which satisfy both.

114. Since two straight lines intersect in but one point, it follows that two linear equations which are independent and simultaneous have one and only one solution.

EXERCISES

Graph the following and thus find the solution of each pair of equations:

1. $\begin{cases} 2x - 3y = 25, \\ x + y = 5. \end{cases}$

2. $\begin{cases} 5x + 6y = 7, \\ 2x - y = -4. \end{cases}$

3. $\begin{cases} 5x + 3y = 1, \\ 2x + y = -4. \end{cases}$

4. $\begin{cases} 6x + 8y = 16, \\ 2x - 3y = 11. \end{cases}$

5. $\begin{cases} 3x - 4y = 1, \\ 2x - 7y = 5. \end{cases}$

6. $\begin{cases} y + 3x = 7, \\ 2y + x = -6. \end{cases}$

7. $\begin{cases} x - 2y = 2, \\ 2x - y = -2. \end{cases}$

8. $\begin{cases} 5x - 7y = 21, \\ x - 4y = -1. \end{cases}$

9. $\begin{cases} 5x + 2y = 8, \\ 2x - 3y = -12. \end{cases}$

10. $\begin{cases} x + 3y = -6, \\ 2x - 4y = -12. \end{cases}$

SOLUTION OF SIMULTANEOUS EQUATIONS BY SUBSTITUTION

115. A pair of linear equations may be solved without constructing their graph.

Illustrative Problem. The sum of two numbers is 35 and their difference is 5. What are the numbers?

Solution. Let $x =$ the greater number and y the smaller.

Then $\begin{cases} x + y = 35, & (1) \\ x - y = 5. & (2) \end{cases}$ and

From (1) by *S*, $y = 35 - x$. (3)

Substituting $35 - x$ for y in (2), $x - (35 - x) = 5$. (4)

Solving, $x = 20$. (5)

Substituting $x = 20$ in (1), $y = 15$. (6)

116. In most problems solved heretofore there have been two or more unknown numbers. In forming the equations the object has been to express all but one of the unknowns in terms of that one. In the case of two unknowns this is now done more systematically as follows:

(*a*) State two equations involving the two unknowns, as (1) and (2) above.

(*b*) Solve one of these equations for one unknown in terms of the other as in (3) above.

(*c*) Substitute in the other equation the value of this unknown as thus expressed, obtaining as in (4) one equation in one unknown.

This equation (4) is the same as equation (1) on page 111, where this problem was solved by means of one unknown. The solution given here amounts to a *formal tabulation* of the steps there taken in obtaining the equation, $g-(35-g)=5$.

Since equation (4) contains but one of the unknowns, the other is said to be **eliminated.**

The process here used is called **elimination by substitution.**

Illustrative Problem. Solve the equations:

$$\begin{cases} 2x+3y=13. & (1) \\ 5x-6y=-8. & (2) \end{cases}$$

From (1) by S and D, $$y=\frac{13-2x}{3}. \quad (3)$$

Substituting in (2) $$5x-\frac{6(13-2x)}{3}=-8. \quad (4)$$

By F, V, $$5x-2(13-2x)=-8. \quad (5)$$

By IV, VII, $$5x-26+4x=-8. \quad (6)$$

By I, A, $$9x=18. \quad (7)$$

By D, $$x=2. \quad (8)$$

From (3), $$y=\frac{13-2\cdot 2}{3}=3. \quad (9)$$

Verify this by drawing the graphs and also by substituting these values of x and y in (1) and (2).

EXERCISES

Solve the following pairs of linear equations by eliminating one of the variables by substitution, and check by substituting the results in the original equations.

1. $\begin{cases} x + y = 4, \\ x - y = 10. \end{cases}$

2. $\begin{cases} x - y = -3, \\ x + 4y = 12. \end{cases}$

3. $\begin{cases} 2x + 3y = 5, \\ 7x - 5y = 33. \end{cases}$

4. $\begin{cases} 3x - 4y = 8, \\ 4x + 3y = -6. \end{cases}$

5. $\begin{cases} 2x - 4y = 8, \\ 3x + 2y = 4. \end{cases}$

6. $\begin{cases} x + 2y = 4, \\ 2x + y = -1. \end{cases}$

7. $\begin{cases} 3x - 4y = 8, \\ 2x + 3y = 11. \end{cases}$

8. $\begin{cases} 5x + 9y = 19, \\ 3x - y = 5. \end{cases}$

9. $\begin{cases} 4y - 2x = 3, \\ 2y + 5x = 6. \end{cases}$

10. $\begin{cases} 3x - 7y = -11, \\ 2x + y = 4. \end{cases}$

11. $\begin{cases} 5x - 3y = 4 - 2x + 7y, \\ 5y + x = 7. \end{cases}$

12. $\begin{cases} 3y + 5x = 12 + 2x, \\ 17x - y = 4y - 20. \end{cases}$

13. $\begin{cases} 6y - x = 7 + 4y, \\ 5x + 8y = 1. \end{cases}$

14. $\begin{cases} 6 + x + y = 2x - 1, \\ 3y + x = 6y + 9. \end{cases}$

15. $\begin{cases} x - y = 37, \\ 2x + 3y = 314 + 13y. \end{cases}$

16. $\begin{cases} 2x - 3y = y + 6, \\ x + 2y = 4y + 3. \end{cases}$

17. $\begin{cases} y + 5x = 2x + 5, \\ 2y - 3x = 19. \end{cases}$

18. $\begin{cases} 5x + 3y = 0, \\ 2x + y = 1. \end{cases}$

19. $\begin{cases} 2x + 3y = 6x - 1, \\ 3x - 2y = 3. \end{cases}$

20. $\begin{cases} 5x - 3y = 0, \\ 2x + 2 - 6y = 2 - x. \end{cases}$

21. $\begin{cases} 6x + 2y = 23, \\ 10x - 5y = 21. \end{cases}$

22. $\begin{cases} 3x - 7y = 15, \\ 5x + 4y = 11. \end{cases}$

Solve 21 and 22 by means of graphs and also by elimination by substitution. Notice that the latter method gives a more accurate solution than can be obtained from the graph.

SOLUTION OF SIMULTANEOUS EQUATIONS BY ADDITION OR SUBTRACTION

117. Illustrative Examples. Solve

$$\begin{cases} x + 2y = 7, & (1) \\ 3x - 2y = 5. & (2) \end{cases}$$

Adding the members of these equations, $+2y$ and $-2y$ cancel.

Hence, $4x = 12,$ (3)

$x = 3.$ (4)

Substituting in (1), $3 + 2y = 7,$ (5)

$y = 2.$ (6)

Verify this by drawing the graphs, and also by substituting $x = 3$, $y = 2$, in (1) and (2).

If one of the variables does not already have the same coefficient in both equations, the solution may be obtained as follows:

Solve the equations

$$\begin{cases} 7x + 3y = 4 - y + 4x, & (1) \\ 3x - y = 4y - 2 - x. & (2) \end{cases}$$

From (1) by *A*, *S*, $3x + 4y = 4.$ (3)

From (2) by *A*, *S*, $4x - 5y = -2.$ (4)

From (3) by *M*, $12x + 16y = 16.$ (5)

From (4) by *M*, $12x - 15y = -6.$ (6)

Subtracting (6) from (5), $31y = 22.$ (7)

From (7) by *D*, $y = \frac{22}{31}.$ (8)

Substituting in (3), $x = \frac{12}{31}.$ (9)

Hence, the solution is $\frac{12}{31}$, $\frac{22}{31}$.

118. The process used in the solution just given is called **elimination by addition or subtraction.**

This method is usually simpler than elimination by substitution, since the latter frequently involves fractions.

EXERCISES

Solve the following pairs of equations by addition or subtraction. Substitute the results in the given equations in each case to test the accuracy of the solution.

1. $\begin{cases} 2x + 3y = 22, \\ x - y = 1. \end{cases}$

2. $\begin{cases} 5x - 2y = 21, \\ x - y = 6. \end{cases}$

3. $\begin{cases} 6x + 30 = 8y, \\ 3y + 17 = 2 - 3x. \end{cases}$

4. $\begin{cases} 8x - 4y = 12x, \\ 4x + 2y = 3 + 4y. \end{cases}$

5. $\begin{cases} x + 6y = 2x - 16, \\ 3x - 2y = 24. \end{cases}$

6. $\begin{cases} 5x + 10y = -7, \\ 2x + 5y = -2. \end{cases}$

7. $\begin{cases} 5x + 3y = -2, \\ 3x + 2y = -1. \end{cases}$

8. $\begin{cases} 3a + 7b = 7, \\ 5a + 3b = 29. \end{cases}$

9. $\begin{cases} r = 3s - 19, \\ s = 3r - 23. \end{cases}$

10. $\begin{cases} 2p = 5q - 16, \\ 7q = -3p + 5. \end{cases}$

Solve the following by either process of elimination:

1. $\begin{cases} 7m = 2n - 3, \\ 19n = 6m + 89. \end{cases}$

2. $\begin{cases} 6c + 15d = -6, \\ 21d - 8c = -74. \end{cases}$

3. $\begin{cases} 2x - 3y = 4, \\ 2y - 3x = -21. \end{cases}$

4. $\begin{cases} u + v = 27, \\ \frac{2}{3}v = 19 - \frac{3}{4}u. \end{cases}$

5. $\begin{cases} 7a = 1 + 10y, \\ 16y = 10a - 1. \end{cases}$

6. $\begin{cases} 15k = 10 - 20l, \\ 25k - 30l = 80. \end{cases}$

7. $\begin{cases} 28x + 14y = 23, \\ 14x - 14y = 1. \end{cases}$

8. $\begin{cases} 5x + 2y = x + 18, \\ 2x + 3y = 3x + 27. \end{cases}$

9. $\begin{cases} 7y - x = x - 17, \\ 2y + 3x = 38. \end{cases}$

10. $\begin{cases} 6x + 2y = -2, \\ x - 4y = -35. \end{cases}$

11. $\begin{cases} 3x - y = 2x - 1, \\ 12x + y = 14. \end{cases}$

12. $\begin{cases} 2x + 3y = 5, \\ 6x + 14y = 0. \end{cases}$

13. $\begin{cases} 4x + 3y = 5, \\ 7x - 2y = 74. \end{cases}$

14. $\begin{cases} 6y + 2x = 11, \\ 3y + 12x = 18. \end{cases}$

15. $\begin{cases} 4y + 9x = -5, \\ x + y = -5. \end{cases}$

16. $\begin{cases} 7x - 4y = 3, \\ 5x + 8y = 6. \end{cases}$

17. $\begin{cases} 12y - 10x = -6, \\ 7y + x = 99. \end{cases}$

18. $\begin{cases} 7x - 3y = -7, \\ 5y - 9x = 1. \end{cases}$

19. $\begin{cases} 7x + 4y = 3, \\ 2x + 3y = 25. \end{cases}$

20. $\begin{cases} 34x + 70y = 4, \\ 5x - 8y = -36. \end{cases}$

119. The equations thus far given have for the most part been written in a standard form, $ax + by = c$, in which all the terms containing x are collected, likewise those containing y, and those which contain neither variable. When the equations are not given in this form, they should be so reduced, as in the second solution on page 156, before applying any method of elimination, and also before solving by means of graphs.

EXERCISES

After reducing each of the following pairs of equations to the standard form, solve by graphing, or by means of either process of elimination, as seems best available.

1. $\begin{cases} x - 14 = 7y, \\ 6y + 1 = x. \end{cases}$

2. $\begin{cases} 16x - 3y = 7x, \\ 4y = 7x + 5. \end{cases}$

3. $\begin{cases} r + 1 = -4s, \\ 2s = 13 - 5r. \end{cases}$

4. $\begin{cases} m = \dfrac{1 - 8n}{5}, \\ 3m + 5n = 1. \end{cases}$

5. $\begin{cases} m - n = 16, \\ 3m = 8 - 2n. \end{cases}$

6. $\begin{cases} \dfrac{7x - 15}{3} = y, \\ 2x - y = 3. \end{cases}$

7. $\begin{cases} \dfrac{x - 3}{5y} = -2, \\ x + 7y = 6. \end{cases}$

8. $\begin{cases} 3x - 5 = -y, \\ 8y + 76 = 5x. \end{cases}$

9. $\begin{cases} a + 4b = 14, \\ 3a - b = 14. \end{cases}$

10. $\begin{cases} \dfrac{x + y}{2} + \dfrac{x - y}{2} = 10, \\ 2x - y = 16. \end{cases}$

11. $\begin{cases} \dfrac{x - y}{5} + \dfrac{x + y}{3} = 8, \\ \dfrac{2x - y}{2} - \dfrac{3y - x}{4} = 12\frac{1}{2}. \end{cases}$

12. $\begin{cases} \dfrac{7m + 8}{5} - \dfrac{7n - 1}{4} = -2, \\ \dfrac{2m - 4}{2} + \dfrac{n - 1}{3} = -\frac{1}{3}. \end{cases}$

13. $\begin{cases} \dfrac{x - 3}{4} + \dfrac{y + 8}{5} = 2, \\ \dfrac{x + 7}{2} + \dfrac{2y - 4}{7} = 5. \end{cases}$

14. $\begin{cases} \dfrac{8a - 3}{9} + \dfrac{5b - 2}{3} = 13, \\ \dfrac{2a + 7}{5} - \dfrac{3b + 10}{10} = -3\frac{4}{5}. \end{cases}$

15. $\begin{cases} \dfrac{7y - 4}{5} + \dfrac{2x - 3}{2} = 7, \\ \dfrac{6x - 3}{5} + \dfrac{2y + 1}{5} = 7. \end{cases}$

16. $\begin{cases} \dfrac{3y + 7}{2} - \dfrac{5x - 7}{3} = 10, \\ \dfrac{2x - 4}{3} - \dfrac{2y - 1}{4} = -2\frac{1}{4}. \end{cases}$

17. $\begin{cases} \dfrac{5 + 3p}{7} - \dfrac{5q - 2}{4} = -2, \\ 6p + 8q = 108. \end{cases}$

18. $\begin{cases} 3x - 2y = 4, \\ \dfrac{2x - 1}{5} - \dfrac{7y - 4}{3} = -19. \end{cases}$

19. $\begin{cases} 5x + 7y = 89\frac{1}{2}, \\ \dfrac{2x - 4}{3} + \dfrac{6y - 1}{5} = 13\frac{1}{3}. \end{cases}$

20. $\begin{cases} 32x - 9y = 299, \\ \dfrac{2x - 5}{7} - \dfrac{3y - 1}{2} = -16. \end{cases}$

PROBLEMS

Solve the following problems, using two variables in each case.

1. A rectangular field is 32 rods longer than it is wide. The length of the fence around it is 308 rods. Find the dimensions of the field.

2. Find two numbers such that 7 times the first plus four times the second equals 37; while 3 times the first plus 9 times the second equals 45.

3. A certain sum of money was invested at 5% interest and another sum at 6%, the two investments yielding $980 per annum. If the first sum had been invested at 6% and the second at 5%, the annual income would be $1000. Find each sum invested.

4. The combined weight of 3 cubic centimeters of platinum and 50 cubic centimeters of poplar is 84 grams, and the weight of 1 cubic centimeter of platinum and 150 cubic centimeters of poplar is 80 grams. Find the weight of 1 cubic centimeter of each.

5. The combined distance from the sun to Jupiter and from the sun to Saturn is 1369 million miles. Saturn is 403 million miles farther from the sun than Jupiter. Find the distance from the sun to each planet.

6. The sum of the distances from the sun to the fixed stars Altair and Capella is 45.3 light-years. Twice the distance of Altair plus 3 times that of Capella is 119.6 light-years. Find the distance from each star to the sun.

7. Find two numbers such that 7 times the first plus 9 times the second equals 116, and 8 times the first minus 4 times the second equals 4.

8. The sum of two numbers is 108. 8 times one of the numbers is 9 greater than the other number. Find the numbers.

9. Two investments of \$24,000 and \$16,000 respectively yield a combined income of \$840. The rate of interest on the larger investment is 1% greater than that on the other. Find the two rates of interest.

10. A father is twice as old as his son. Twenty years ago the father was six times as old as his son. How old is each now?

11. If the length of a rectangle is increased by 3 feet and its width decreased by 1 foot, its area is increased by 3 square feet. If the length is increased by 4 feet and the width decreased by 2 feet, the area is decreased by 3 square feet. What are the dimensions of the rectangle?

Let $l =$ the original length and w the width,

then $$(l+3)(w-1) = lw + 3, \quad (1)$$

and $$(l+4)(w-2) = lw - 3. \quad (2)$$

From (1) by XIII, $$lw + 3w - l - 3 = lw + 3. \quad (3)$$

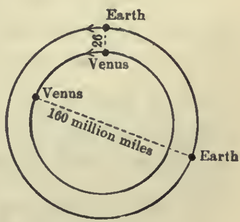
From (2) by XIII, $$lw + 4w - 2l - 8 = lw - 3. \quad (4)$$

From (3) and (4) $$3w - l = 6, \quad (5)$$

and $$4w - 2l = 5. \quad (6)$$

(5) and (6) may now be solved in the ordinary manner.

12. The greatest distance from the earth to Venus is 160 million miles and the shortest distance is 26 million miles. How far from the sun are Venus and the earth, assuming that they move around the sun in concentric circles with the sun at the center?



13. Two weights, 35 and 40 pounds respectively, balance when resting on a beam at certain unknown distances from the fulcrum. If 15 pounds is added to the 35-lb. weight, the 40-lb. weight must be moved 2 feet farther from the fulcrum in order to maintain the balance. What

was the original distance from the fulcrum to each of the weights?

If the distances from the fulcrum to the weights are d_1 and d_2 respectively, then by the formula, $w_1d_1 = w_2d_2$, given on page 121, we have $35\,d_1 = 40\,d_2$ and $50\,d_1 = 40\,(d_2 + 2)$.

14. A steamer on the Mississippi makes 6 miles per hour going against the current and $19\frac{1}{2}$ miles per hour going with the current. What is the rate of the current and at what rate can the steamer go in still water?

15. A man starts at 7 A.M. for a walk in the country. At 10 A.M. another man starts on horseback to overtake the pedestrian, which he does at 1 P.M. If the rate of the horseman had been two miles per hour less, he would have overtaken the pedestrian at 4 P.M. At what rate does each travel?

16. A camping party sends a messenger with mail to the nearest post office at 5 A.M. At 8 A.M. another messenger is sent out to overtake the first, which he does in $2\frac{1}{4}$ hours. If the second messenger travels 5 miles per hour faster than the first, what is the rate of each?

17. There are two numbers such that 3 times the greater is 18 times their difference, and 4 times the smaller is 4 less than twice the sum of the two. What are the numbers?

18. Roy is three times as old as Fred was 8 years ago. Five years from now Roy will be 16 years less than twice as old as Fred. How old is each now?

19. A picture is 3 inches longer than it is wide. The frame 4 inches wide has an area of 360 square inches. What are the dimensions of the picture?

20. The difference between 2 sides of a rectangular wheat field is 30 rods. A farmer cuts a strip 5 rods wide around the field, and finds the area of this strip to be $7\frac{1}{2}$ acres. What are the dimensions of the field?

21. The sum of the length and width of a certain field is 260 rods. If 20 rods are added to the length and 10 rods to the width, the area will be increased by 3800 square rods. What are the dimensions of the field?

22. In a number consisting of two digits the sum of the digits is 12. If the order of the digits is reversed the number is decreased by 36. What is the number?

23. A bird attempting to fly against the wind is blown backward at the rate of $7\frac{1}{2}$ miles per hour. Flying with a wind $\frac{1}{4}$ as strong, the bird makes 48 miles an hour. Find the rate of the wind and the rate at which the bird can fly in calm weather.

24. There is a number whose two digits differ by 2. If the digit in units' place is multiplied by 3 and the digit in tens' place is multiplied by 2, the number is increased by 44. Find the number, the tens' digit being the larger.

25. In a number consisting of two digits one digit is equal to twice their difference. If the order of the digits is reversed, the number is increased by 18. Find the number, the units' digit being the larger.

26. If the length of a rectangle is doubled and 8 inches added to the width, the area of the resulting rectangle is 180 square inches greater than twice the original area. If the length and width of the rectangle differ by 10, what are its dimensions?

27. The Centigrade reading at the boiling point of alcohol is 96° lower than the Fahrenheit reading. Find both the Centigrade and the Fahrenheit reading at this temperature.

Use C and F as the unknowns. Then one of the equations is the formula connecting Fahrenheit and Centigrade readings obtained on page 129, and the other is $C + 96 = F$.

28. The Centigrade reading at the boiling point of mercury is 312° lower than the Fahrenheit reading. Find both the Fahrenheit and the Centigrade reading at this temperature.

29. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

30. A man rowing against a tidal current drifts back $2\frac{1}{4}$ miles per hour. Rowing with this current, he can make $12\frac{1}{3}$ miles per hour. How fast does he row in still water and how swift is the current?

31. Flying against a wind a bird makes 28 miles per hour, and flying with a wind whose velocity is $2\frac{2}{3}$ times as great, the bird makes 46 miles per hour. What is the velocity of the wind and at what rate does the bird fly in calm weather?

32. A freight train leaves Chicago for St. Paul at 11 A.M. At 3 and 5 P.M. respectively of the same day two passenger trains leave Chicago over the same road. The first overtakes the freight at 7 P.M. the same day, and the other, which runs 10 miles per hour slower, at 3 A.M. the next day. What is the speed of each?

33. Two boys, A and B, having a 30-lb. weight and a teeter board, proceed to determine their respective weights as follows: They find that they balance when B is 6 feet and A 5 feet from the fulcrum. If B places the 30-lb. weight on the board beside him, they balance when B is 4 and A 5 feet from the fulcrum. How heavy is each boy?

34. C is $6\frac{1}{2}$ feet from the point of support and balances D, who is at an unknown distance from this point. C places a 33-lb. weight beside himself on the board and when $4\frac{2}{3}$ feet from the fulcrum, balances D, who remains at the same point as before. C's weight is 84 pounds. What is D's weight and how far is he from the fulcrum?

35. E weighs 95 pounds and F 110 pounds. They balance at certain unknown distances from the fulcrum. E then takes a 30-lb. weight on the board, which compels F to move 3 feet farther from the fulcrum. How far from the fulcrum was each of the boys at first?

36. A fast freight leaves New York for Chicago at 8 A.M. At 4 P.M. the same day an express train leaves New York for Chicago and passes the freight 12 hours later. Another express leaving New York at 6 P.M. of the same day overtakes the freight 10 hours after starting. Find the rate of each train if the second express goes 8 miles per hour faster than the first.

37. The Centigrade reading at the melting point of silver is 796° lower than the Fahrenheit reading. Find both Centigrade and Fahrenheit readings at this temperature.

38. The Fahrenheit reading at the melting point of gold is 992° higher than the Centigrade reading. Find both Centigrade and Fahrenheit readings at this temperature.

39. $10,000 and $8000 are invested at different rates of interest, yielding together an annual income of $820. If the first investment were $12,000 and the second $6000, the yearly income would be $840. Find the rates of interest.

40. In a switch yard a car weighing 50 tons and going at a certain rate strikes a standing car, whereupon both cars move off at the rate of 4 miles per hour. If the second car, moving at the same rate as the first before impact, were to strike the first car when standing still, they would move off at the rate of 2 miles per hour. How fast did the first car move before impact and what is the weight of the second car? (Set up the equations by means of the formula $w_1v_1 = w_1v_1' + w_2v_2$, obtained on page 127.)

41. 200 ccm. of white oak is fastened to 25 ccm. of steel, making a combination whose average density is 1.56. If 250 ccm. of oak is fastened to 20 ccm. of steel, the average density of the combination is 1.3. Find the density of white oak and also of steel.

SIMULTANEOUS EQUATIONS IN THREE VARIABLES

120. Illustrative Problem. Three men were discussing their ages and found that the sum of their ages was 90 years. If the age of the first were doubled and that of the second trebled, the aggregate of the three ages would then be 170. If the ages of the second and third were each doubled, the sum of the three would be 160. Find the age of each?

Solution. Let x, y, and z represent the number of years in their ages in the order named.

Then, $$x + y + z = 90, \quad (1)$$
$$2x + 3y + z = 170, \quad (2)$$
and $$x + 2y + 2z = 160. \quad (3)$$

Since by supposition x represents the same number in all three equations, and likewise y and z, if we subtract (1) from (2), we obtain a new equation from which z is eliminated.

I.e. $$x + 2y = 80. \quad (4)$$

Again, multiplying (2) by 2 and subtracting (3),
$$3x + 4y = 180. \quad (5)$$

(4) and (5) are two equations in the two variables x and y. Solving these by eliminating y, we find $x = 20$. (6)

Substituting $x = 20$ in (4), $y = 30$. (7)

Substituting x and y in (1), $z = 40$. (8)

Check by substituting the values of x, y, and z in all three given equations and also by showing that they satisfy the conditions of the problem.

The values of x, y, and z as thus found constitute the **solution of the given system** of equations.

Evidently x could have been eliminated first, using (1), (2) and (1), (3), giving a new set of two equations in y and z. Let the student find the solution in this manner.

Also find the solution by first eliminating y, using (1), (2) and (2), (3), getting two equations in x and z, from which the values of x and z can be found.

121. Definition. An equation is said to be of the **first degree in three variables** if no one of the variables is multiplied by itself or by one of the others (§ 110).

The fact that the solutions are found to be the same no matter in what order the equations are combined, indicates that *a system of three independent and simultaneous equations of the first degree in three variables has one and only one solution.*

As in the case of two equations, each should be first reduced to a standard form in which all the terms containing a given variable are collected and united and all fractions removed by *M*, Principle VIII.

EXERCISES

Solve the following systems of equations, and check the results by substituting the values found for each variable in the given equations:

1. $\begin{cases} 2x - y + z = 18, \\ x - 2y + 3z = 10, \\ 3x + y - 4z = 20. \end{cases}$

2. $\begin{cases} 5x - 3y + z = 15, \\ x + 3y - z = 3, \\ 2x - y + z = 8. \end{cases}$

3. $\begin{cases} 4x + 2y + z = 13, \\ x - y + z = 4, \\ x + 2y - z = 1. \end{cases}$

4. $\begin{cases} 6x + 4y - 4z = -4, \\ 4x - 2y + 8z = 0, \\ x + y + z = 4. \end{cases}$

5. $\begin{cases} x + 2y + 3z = 5, \\ 4x - 3y - z = 5, \\ x + y + z = 2. \end{cases}$

6. $\begin{cases} 2x - 8y + 3z = 2, \\ x - 4y + 5z = 1, \\ 3x - 10y - z = 5. \end{cases}$

7. $\begin{cases} x + y + z = 1, \\ x + 3y + 2z = 8, \\ 2x + 8y - 3z = 15. \end{cases}$

8. $\begin{cases} 2x - 3y + z = 5, \\ 3x + 2y - z = 5, \\ x + y + z = 3. \end{cases}$

9. $\begin{cases} x + y + z = 6, \\ 3x - 2y - z = 13, \\ 2x - y + 3z = 26. \end{cases}$

10. $\begin{cases} x + y + z = 6, \\ 4x - y - z = -1, \\ 2x + y - 3z = -6. \end{cases}$

PROBLEMS INVOLVING THREE VARIABLES

122. Illustrative Problem. A broker invested a total of \$15,000 in the street railway bonds of three cities, the first investment yielding 3 %, the second $3\frac{1}{2}$ %, and the third 4 %, thus securing an income of \$535 per year. If the second investment was one-half the sum of the other two, what was the amount of each?

Solution. Suppose x dollars were invested at 3 %, y dollars at $3\frac{1}{2}$ %, and z dollars at 4 %.

Then, $x + y + z = 15000,$ (1)

$.03\,x + .035\,y + .04\,z = 535,$ (2)

and $x + z = 2\,y.$ (3)

From (3), $x - 2\,y + z = 0.$ (4)

Subtract (4) from (1), $3\,y = 15000,$ (5)

and $y = 5000.$ (6)

From (1), by M, $.035\,x + .035\,y + .035\,z = 525.$ (7)

Subtract (7) from (2), $-.005\,x + .005\,z = 10.$ (8)

Divide (8) by .005, $-x + z = 2000.$ (9)

Substitute (6) in (4), $x + z = 10000.$ (10)

Add (9) and (10), $2\,z = 12000.$ (11)

$z = 6000.$ (12)

Substitute (5) and (12) in (1), $x = 4000.$ (13)

Hence, \$4000, \$5000, and \$6000 were the sums invested.

Solve the following problems, using three unknowns:

1. The sum of three angles A, B, and C of a triangle is 180 degrees. $\frac{1}{3}$ of $A + \frac{1}{4}$ of $B + \frac{1}{5}$ of C is 48 degrees, while $\frac{1}{6}$ of $A + \frac{1}{8}$ of $B + \frac{1}{4}$ of C is 30 degrees. How many degrees in each angle?

2. The combined weight of 1 cubic foot each of compact limestone, granite, and marble is 535 pounds. 1 cubic foot of limestone, 2 of granite, and 3 of marble weigh together 1041 pounds, while 1 cubic foot of limestone and 1 of granite together weigh 195 pounds more than 1 cubic foot of marble. Find the weight per cubic foot of each kind of stone.

3. A number is composed of 3 digits whose sum is 7. If the digits in tens' and hundreds' places are interchanged, the number is increased by 180; and if the order of the digits is reversed, the number is decreased by 99. What is the number?

4. The sum of the angles A, B, and C of a triangle is 180 degrees. If B is subtracted from $\frac{1}{2}$ of A the remainder is $\frac{1}{4}$ of C, and when C is subtracted from twice A the remainder is 4 times B. How many degrees in each angle?

5. The sum of the three sides a, b, c of a certain triangle is 35, and twice a is 5 less than the sum of b and c, and twice c is 4 more than the sum of a and b. What is the length of each side?

6. The combined number of students at Harvard, Yale, and Columbia during the year 1905–1906 was 13,390. The number at Harvard minus the number at Columbia plus twice the number at Yale was 6893, and the number at Columbia plus 4 times the number at Yale minus twice the number at Harvard was 7258. What was the number at each university?

7. The total number of students at the universities of Illinois, Michigan, and Wisconsin during the year 1905–1906 was 12,216. Twice the number at Illinois plus 3 times that at Michigan plus 4 times that at Wisconsin was 36,145. If the number at Michigan is subtracted from the sum of the numbers at Illinois and Wisconsin, the remainder is 3074. Find the number at each university.

8. The total number attending schools in the United States in 1904–1905 was 17,953,844. If the number in secondary schools and colleges combined be subtracted from the number in elementary schools, the remainder is 15,924,656; while if twice the number in colleges be added to the number in elementary and secondary schools combined, the sum is 18,092,388. Find the number of students of each kind.

9. The combined foreign trade in 1905 at the three ports, London, Liverpool, and New York, was 3598 million dollars. If the amount at London is subtracted from the combined amounts at New York and Liverpool, the remainder is 988 million; and if the amount at New York is subtracted from the combined amounts at London and Liverpool, the remainder is 1384 million. Find the amount of foreign trade at each port.

In the following three examples find the number of seconds in each record and reduce the results to minutes and seconds.

10. If x is the number of seconds in the Eastern intercollegiate record for a mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} x + y + z = 781.15, \\ -x + 2y + z = 519.35, \\ 2x - y + z = 514.55. \end{cases}$$

11. If x is the number of seconds in the Eastern intercollegiate record for a half mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} 2x + 3y + z = 697.7, \\ 3x + 2y + 2z = 809.8, \\ 2x - y + z = 228.1. \end{cases}$$

12. If $x =$ number of seconds in the world's mile trotting record in 1806, $y =$ number of seconds in the world's record in 1885, and $z =$ number of seconds in the world's record in 1903, then

$$\begin{cases} x + y + z = 426.25, \\ 2x + 4y + 6z = 1584, \\ -x + y + 2z = 186.75. \end{cases}$$

REVIEW QUESTIONS

1. How may a point in a plane be located by reference to two fixed lines? What are these fixed lines called? What names are given to the distances from the point to the fixed lines? Why are negative numbers needed in order to locate all points in this manner?

2. Draw a pair of axes in a plane and locate the following points: (5,0), (−2,0), (0,3), (0,−1), (0,0).

3. State a problem involving motion and solve it by means of a graph.

4. How many pairs of numbers can be found which satisfy the equation $x - 2y = 6$? State five such pairs and plot the corresponding points. How are these points situated with respect to each other? What can you say of all points corresponding to pairs of numbers which satisfy this equation? What is meant by the graph of an equation?

5. How many pairs of numbers will simultaneously satisfy the two equations $3x + 2y = 7$ and $x + y = 3$? Show by means of a graph that your answer is correct.

6. Describe elimination by the process of substitution; also by the process addition or subtraction. Under what conditions is one or the other of these methods preferable?

7. Why is the solution by elimination in some cases preferable to the solution by means of the graph?

8. Describe the solution of a system of three linear equations in three unknowns. Is it immaterial which of the three variables is eliminated first?

9. Can you find a definite solution for two equations each containing three unknowns? Illustrate this by means of the equations $4x - 3y - z = 5$ and $x + y + z = 2$.

CHAPTER VI

SPECIAL PRODUCTS AND FACTORS

123. Repeated Factors. Number expressions containing repeated factors have already been considered in Chapter III. $x \cdot x$ was written x^2 and called *the square of x*, or *x square;* similarly, $(a+b)(a+b)$ was written $(a+b)^2$ and read *the square of the binomial* $a+b$ or *the binomial* $a+b$ *squared.*

124. Definitions. Any number written over and to the right of a number expression is called an **index** or **exponent** and, if a *positive integer*, shows how many times that expression is to be taken as a factor.

A product consisting entirely of equal factors is called a **power** of the repeated factor. The repeated factor is called the **base** of the power.

E.g. x^3 means $x \cdot x \cdot x$ and is read *the third power of x* or *x cube;* x^5 means $x \cdot x \cdot x \cdot x \cdot x$, and is read *the fifth power of x* or briefly *x fifth.* $(x-y)^3 = (x-y)(x-y)(x-y)$ and is read $(x-y)$ *cubed* or *the cube of the binomial* $(x-y)$.

Notice two important differences between an exponent and a coefficient.

(1) $5a = a + a + a + a + a$, while $a^5 = a \cdot a \cdot a \cdot a \cdot a$.

(2) In $5abc$ the coefficient 5 applies to the product abc, while in abc^5, the exponent 5 applies only to the factor c. In order to make it apply to the product it is necessary to use a parenthesis, thus, $(abc)^5$ means the product abc taken five times as a factor.

EXERCISES

Perform the following indicated multiplications:

1. $2^3, 2^4, 2^5, 2^6$.
2. $3^2, 3^3, 3^4, 3^5$.
3. $4^2, 4^3, 4^4, 4^5$.
4. $5^2, 5^3, 5^4, 5^5$.
5. $6^2, 6^3, 6^4$.
6. $7^2, 7^3, 7^4$.
7. $8^2, 8^3, 8^4$.
8. $9^2, 9^3, 9^4$.
9. $10^2, 10^3, 10^4, 10^5$.
10. $(a+b)^2$.
11. $(c-d)^2$.
12. $98^2=(100-2)^2$.
13. $(a+b+c)^2$.
14. $(a+b-c)^2$.
15. $(3-a)^2$.
16. $(3-b-c)^2$.
17. $(x-y-5)^2$.
18. $(w+z-4)^2$.
19. $(2-z-x+w)^2$.
20. $(x-y-5)^2$.
21. $(x-y)^3$.
22. $(x+y)^3$.
23. $(a+b)^4$.
24. $(c-d)^4$.

125. In the case of factors expressed in Arabic figures multiplications like the following may be carried out in either of two ways.

E.g. $3^2 \cdot 3^4 = 9 \cdot 81 = 729.$

or $3^2 \cdot 3^4 = (3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3) = 3^{2+4} = 3^6 = 729.$

But with literal factors the second process only is possible.

E.g. $a^2 \cdot a^4 = (a \cdot a)(a \cdot a \cdot a \cdot a) = a^{2+4} = a^6.$

The process in which the exponents are added applies only when the factors are powers of the same base.

E.g. $2^3 \cdot 3^2 = 8 \cdot 9 = 72$ cannot be found by adding the exponents.

2^1 or 2 is called the first power of 2.

Thus $2 \cdot 2^3 = 2^1 \cdot 2^3 = 2^{1+3} = 2^4.$

EXERCISES

In the following exercises carry out each indicated multiplication in two ways in case this is possible:

1. $2^5 \cdot 2^6$.
2. $2^2 \cdot 2^3$.
3. $2 \cdot 2^4$.
4. $3^2 \cdot 3^3$.
5. $3 \cdot 3^4$.
6. $3^2 \cdot 3^5$.
7. $4^2 \cdot 4^3$.
8. $4 \cdot 4^4$.

9. $5 \cdot 5^2$.	12. $7 \cdot 7^3$.	15. $x^7 \cdot x^4$.	18. $2^3 \cdot 2^2 \cdot 2^4$.
10. $5^2 \cdot 5^3$.	13. $a^2 \cdot a^3$.	16. $t^3 \cdot t^4$.	19. $3 \cdot 3^2 \cdot 3^3$.
11. $6^2 \cdot 6^2$.	14. $x^3 \cdot x^2$.	17. $t^2 \cdot t^3 \cdot t^4$.	20. $2^2 \cdot 2^3 \cdot 2^2 \cdot 2$.

126. **Illustrative Problem.** To multiply 2^k by 2^n, k and n being any two positive integers.

Solution. 2^k means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to k factors,

and 2^n means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to n factors.

Hence $2^k \cdot 2^n = (2 \cdot 2 \cdot 2 \cdots \text{ to } n \text{ factors})\ (2 \cdot 2 \cdots \text{ to } k \text{ factors})$

$= 2 \cdot 2 \cdot 2 \cdot 2 \cdots \text{ to } k + n \text{ factors in all.}$

That is, $$2^k \cdot 2^n = 2^{k+n}.$$

The preceding examples illustrate the following principle:

127. **Principle XIV.** *The product of two powers of the same base is found by adding the exponents of the powers and making this sum the exponent of the common base.*

EXERCISES

Perform the following indicated multiplications by means of Principle XIV:

1. $2^3 \cdot 2^7$.	8. $3^{2a} \cdot 3^{2b}$.	15. $w^x \cdot w^{y+3x}$.
2. $a^3 \cdot a^7$.	9. $5^{2+n} \cdot 5^{2-n}$.	16. $n^2 \cdot n^{3c+4b}$.
3. $3^4 \cdot 3^5$.	10. $7^{2x+y} \cdot 7^{2x-y}$.	17. $c^x \cdot c^{2-x}$.
4. $x^4 \cdot x^5$.	11. $a^m \cdot a^n$.	18. $x^{ac} \cdot x^{bc}$.
5. $3^k \cdot 3^n$.	12. $t^{2a} \cdot t^{3b+a}$.	19. $r^{3c} \cdot r^{4c}$.
6. $x^k \cdot x^n$.	13. $y^{4a} \cdot y^{3a}$.	20. $s^{ax} \cdot s^{cx}$.
7. $4^a \cdot 4^b$	14. $x^{3b} \cdot x^{2a+4b}$.	21. $v^{2x-1} \cdot v^{3x+2}$.

Perform the following multiplications by means of Principles IV, III, and XIV:

22. $2^3(2^2 + 2^4)$.	25. $a^2(a^3b - ab^3)$.
23. $2^2(3 \cdot 2^4 + 5 \cdot 2^3)$.	26. $x^{2a}(4x^a + 3x^{3a})$.
24. $4^4(3 \cdot 4^3 - 5 \cdot 4^2)$.	27. $r^s(5r^{2s} - 3r^s)$.

Multiply the following and state all principles used:

28. $a^2m^2 - b^2m^3$ by m^4.

29. $4 \cdot 3^2 - 5 \cdot 7 \cdot 2$ by 2^4.

30. $2 \cdot 3 - 4 \cdot 3^2 + 7 \cdot 3$ by 3.

31. $4x^2 - 3x^3 + 6x^4$ by x^2.

32. $5x - 3x^2 + 2x^4$ by x^4.

33. $3y^2 + 4y^4 - y^3$ by y.

34. $7x^4 - 5x^3 - 2x$ by x^3.

35. $3a^2b^4 + 2a^2b - 4a^2b^2$ by a^3.

36. $4a^{2b} - a^{3c} + a^{2d}$ by a^{2b}.

37. $3^2 \cdot 4^{2n} - 6^3 \cdot 4^{4n}$ by 4^{3n}.

38. $a^{2c}b^{3c} - a^cb^{2c}$ by a^{4c}.

39. $6^2 \cdot 4^3 - 7^4 \cdot 4^2 + 4^4$ by 4^3.

40. $12x^2y^3 - 6x^3y^2 + 3xy$ by x^3.

41. $2a^{3b} - 3a^{2c} - 4a^{4b}$ by a^{2c}.

42. $6x^{3a-2b} + 8x^{2b-3a}$ by x^{2a+2b}.

43. $y^{3m+2n} - y^{2n-3m+1}$ by y^{3m+3n}.

44. $x^{5c-3a} + x^{3c+2a} + x^{2a}$ by x^{4a-2c}.

45. $a^{nx-bx} - a^{bx-nx} - a^x$ by a^{bx+nx}.

PRODUCTS OF MONOMIALS

128. In multiplying two numbers each of which is in the factored form, if the factors are all expressed in Arabic figures, the operation may be carried out in two ways.

$$E.g.\ (2 \cdot 3)(2 \cdot 3 \cdot 5) = 6 \cdot 30 = 180.$$
$$\text{Also } (2 \cdot 3)(2 \cdot 3 \cdot 5) = 2^2 \cdot 3^2 \cdot 5 = 4 \cdot 9 \cdot 5 = 180.$$

In the second process one of the factors, 2 or 3, is multiplied in and this product is then multiplied by the other factor, 2 being combined with the factor 2 and 3 with 3 by Principles III and XIV. In the case of literal factors the second process only is available.

$$E.g.\ (a^2b^2)(5a^4b^3c) = 5a^{4+2}b^{2+3}c = 5a^6b^5c.$$

EXERCISES

Perform the following indicated multiplications, each in two ways where possible:

1. $(5 \cdot 7^2)(5^3 \cdot 7^3)$.
2. $(3^2 \cdot 5)(3 \cdot 2^3 \cdot 5^2)$.
3. $(4 \cdot 5^3)(5 \cdot 4^3)$.
4. $(3 \cdot 4^2)(4 \cdot 3^2)$.
5. $3ab(5a^2b^2)$.
6. $4x(3xy)$.
7. $5t(3st^2)$.
8. $7m(4m^2n)$.
9. $6k^2(6^2k^3)$.

By definition (§ 77), expressions in the factored form, such as $3ab$, $5a^2b^2$, $3 \cdot 2^3 \cdot 5^2$, etc., are monomials.

The preceding examples illustrate the following principle:

129. Principle XV. *The product of two monomials is found by multiplying either one by each factor of the other in succession.*

Each factor of the multiplier is associated with any desired factor of the multiplicand according to Principle III, and where the bases are the same the exponents are added according to Principle XIV.

If there are no factors common to multiplier and multiplicand, the product can only be indicated by writing all the factors of both in succession.

EXERCISES

Multiply:

1. $4 \cdot 7 \cdot 8 \cdot 9$ by $2 \cdot 3 \cdot 5$.
2. $3 \cdot 8 \cdot 2$ by $2 \cdot 5 \cdot 6$.
3. $2\,xyz$ by $3\,x^2yz$.
4. $6^3 \cdot 2^4 \cdot x^5$ by $6 \cdot 2^3 \cdot x$.
5. $3\,x^4y^2$ by $4\,xy^3$.
6. $5\,a^2b^3c$ by ab^4c.
7. $2\,x^3b^4c^2$ by $5\,xb^4c$.
8. $9\,x^ay^bz^c$ by $4\,x^{2a}y^{2b}z^c$.
9. $6\,x^{4n+1}y^{2n-4}z^n$ by $3\,x^{1-4n}y^{5-n}z^{6-n}$.
10. $3\,n^{2x+4}m^{y-5}r^{2x-1}$ by $n^{2-x}m^{y-1}r^x$.

In each of the following exercises state which of the Principles I–XV are used:

11. $3^2 \cdot 2^3(2^4 . 3^2 - 4 \cdot 2^2 - 2 \cdot 3^2)$.
12. $6 \cdot 3^4(4^2 \cdot 2^2 \cdot 3^2 + 4^3 \cdot 2^4 - 5 \cdot 3^4)$.
13. $2\,x(4 + 7\,x^4 - 3\,x^2y)$.
14. $4\,yx(3\,y^2x^3 - y^3x^2 + y^4x^4)$.
15. $5\,a^2b^2(a^3 - b^3 + a^2b^2)$.
16. $4\,x^4y^3(3\,ax - 4\,by + 2\,xy)$.
17. $2\,x^a(x^a - xb - 3\,c)$.
18. $2\,yx^c(y - x^{2c} + 3\,y)$.
19. $4\,x^{2n+1}(x^{3n+1} - 4\,y^{2n+5} + 5\,x^{2n}y^4)$.
20. $4\,x^{2n+1}y^{n+2m}(x^ny^m - x^my^n + x^m + y^n)$.
21. $(a+b)^3(a-b)$.
22. $(a-b)^3(a+b)$.
23. $(a+b)^2(a-b)^2$.
24. $(a^3+a^2b+ab^2+b^3)(a-b)$.
25. $(a+b-c)(a+b+c)$.
26. $(3\,x-2\,y-1)(2\,x+y)$.

27. $(5a-3b)(6a^2+2b^2-1)$.
28. $(3a^2-2b^2-3)(4a+3b^3)$.
29. $(1+a+a^2)(1-a)$.
30. $(1-a+a^2-a^3)(1+a)$.
31. $(a+b)(a-b)(a^2+b^2)$.
32. $(a+b)(a^2-ab+b^2)$.
33. $(a-b)(a^2+ab+b^2)$.
34. $(x+y)(x-y)$.
35. $(100+1)(100-1)$.
36. $(100+2)(100-2)$.
37. $(x+3)(x+5)$.
38. $(x-3)(x-5)$.
39. $(t+8)(t-3)$.
40. $(x-2y)(x+3y)$.
41. $(x^2+x+1)(x^2-x+1)$.
42. $(x+y)(x^3-x^2y+xy^2-y^3)$.
43. $(x-y)(x^3+x^2y+xy^2+y^3)$.
44. $(x^2-xy+y^2)(x^2+xy+y^2)$.
45. $(x^2+2xy+y^2)(x-y)^2$.
46. $(x^2-y^2)(x^4+x^2y^2+y^4)$.
47. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
48. $(x^2+y^2)(x+y)(x-y)$.
49. $(x-y)(x^2+xy+y^2)(x^3+y^3)$.
50. $(a^n-b^n)(a^{2n}+a^nb^n+b^{2n})$.

FACTORS OF NUMBER EXPRESSIONS

130. The factors of numbers are of great importance in arithmetic. For instance, the multiplication table consists of pairs of factors whose products are committed to memory for constant use. Likewise in algebra the factors of certain special forms of number expressions are so important that they must be known at sight.

An expression containing no fractions is said to be prime if it has no integral factors except itself and 1.

Thus $2, 3, x, x+2, a^2+b^2$, are prime expressions.

A prime expression may be factored by using fractions or radicals. See p. 214.

Thus $5=2\cdot 2\frac{1}{2}$ and $2=\sqrt{2}\cdot\sqrt{2}$.

Such factors are not included in what are here called prime factors.

131. Monomial Factors. If the terms of a polynomial contain a common factor, they may be combined with respect to this factor according to Principles I and II. The expression is thus changed into a product of a monomial and a polynomial.

Illustrative Examples.

1. $ax + ay = a(x + y)$, by Principle I.

2. $a^2 - ab = a(a - b)$, by Principle II.

3. $9 \cdot 8 + 3 \cdot 4 \cdot 5 = 3 \cdot 4(3 \cdot 2 + 5)$, by Principle I.

4. $6a^2b - 4ab^2 = 2ab(3a - 2b)$, by Principle II.

5. $5xy - 3x^2y + 4x^3y = xy(5 - 3x + 4x^2)$, by I and II.

Observe that factoring each term of a polynomial *does not factor the polynomial.*

E.g. $330 - 210 - 60$ is not factored by writing it $2 \cdot 3 \cdot 5 \cdot 11 - 2 \cdot 3 \cdot 5 \cdot 7 + 2^2 \cdot 3 \cdot 5$; but by adding the coefficients of the common factor $2 \cdot 3 \cdot 5$, thus,

$$2 \cdot 3 \cdot 5 \cdot 11 - 2 \cdot 3 \cdot 5 \cdot 7 + 2^2 \cdot 3 \cdot 5 = 2 \cdot 3 \cdot 5(11 - 7 + 2).$$

Likewise $10a^2bc - 15ab^2c + 20abc^2$ is not factored, although *each term* is in the factored form. But if $10a^2bc - 15ab^2c + 20abc^2$ is written in the form $5abc\ (2a - 3b + 4c)$, it is then factored.

These examples illustrate the factoring of a polynomial when it contains a monomial factor common to every term.

When such a common factor has been found the whole expression is then written in the form of a product by means of Principles I and II. The result may be checked by Principles IV and XV. If the terms of a polynomial which is to be factored contain a common factor, this should always be removed at the outset.

EXERCISES

Factor the following polynomials:

1. $8-12-18+48.$
2. $3\cdot 4\cdot 5-15-20+35.$
3. $3\cdot 11\cdot 4-22\cdot 2+44\cdot 6.$
4. $3^4\cdot 2^4-3\cdot 2^3-5\cdot 2^2.$
5. $6\cdot 5^4\cdot 7+3\cdot 5^3\cdot 2-5^4\cdot 9\cdot 2^2.$
6. $13\,a^4b-16\,a^3b^4-2\,a^3b^2.$
7. $15\,xy^4-20\,x^3y+x^2y^2.$
8. $9\,v^3w^4+21\,v^4w^3-18\,v^2w^2.$
9. $12\,a^4b^3-8\,a^3b^4-6\,a^2b^2.$
10. $32\cdot 3^4-64\cdot 3^3-16\cdot 2^3\cdot 3^2.$
11. $27\cdot 2^3+54\cdot 2^4-36\cdot 2^3.$
12. $11\,a^4x^2-44\,a^3x^4+33\,ax.$
13. $1\cdot 2\cdot 3\cdot 4+2\cdot 3\cdot 4\cdot 5-3\cdot 4\cdot 5\cdot 6+4\cdot 5\cdot 6.$
14. $72\,x^4b^3a-36\,x^4b^2a^2-48\,x^3b^3a^4.$
15. $84\,x^9b^7y^4+18\,x^3b^5y^4+12\,x^2b^3y^2.$
16. $19\cdot 3^4\cdot 2^4\cdot 5^3-13\cdot 3^4\cdot 2^3\cdot 5^2-29\cdot 3^2\cdot 2^2\cdot 5^4.$
17. $17\,a^4b^3c^2+51\,a^3b^4c^3-34\,a^2b^2c^4.$
18. $38\,a^{12}b^{14}c^4-76\,a^{11}b^{12}c^3-76\,a^{13}b^{11}c^2.$
19. $4\,x^{2a}y^{3b}+6\,x^{3a}y^{2b}-8\,x^{5a}y^{4b}.$
20. $3\,a^{4n+2}b^{3n+4}+6\,a^{6n+4}b^{3n+3}-12\,a^{5n+3}b^{5n+6}.$

TRINOMIAL SQUARES

132. In §§ 87 and 88 we found by multiplication:

$$(a+b)^2=a^2+2\,ab+b^2, \tag{1}$$

and

$$(a-b)^2=a^2-2\,ab+b^2. \tag{2}$$

By means of these formulas we may square the sum or difference of any two number expressions.

E.g. $(3\,x+2\,y)^2=(3\,x)^2+2\cdot 3\,x\cdot 2\,y+(2\,y)^2=9\,x^2+12\,xy+4\,y^2.$
Also $[(5+r)-(s-1)]^2=(5+r)^2-2(5+r)(s-1)+(s-1)^2.$

The last expression may now be reduced by performing the indicated operations.

In this manner write the squares of the following binomials. Verify the first ten results by actual multiplication.

1. $t+8$.
2. $r-12$.
3. $3x-5y$.
4. $6a-7$.
5. m^5+3n.
6. $7x^2+3x$.
7. $3y^3-2y^2$.
8. $8m^2n-7mn^2$.
9. $5a^3-3b^3$.
10. $4a^2x^2y^3-3a^3xy^4$.
11. $7(a-3)-2(b+c)$.
12. $4(x-y)+2(z+y^2)$.
13. $(3a-2b)+5$.
14. $7x-(4r-s)$.
15. $7(m^2-3)-6(m^3+n)$.
16. $3(z+y)-2(3+x)$.
17. $4(x^3-3y)+(x^2+4y^2)$.
18. $2x^2y-5(x-y)$.
19. $3(5x^2-z)-4x^2z^2$.
20. $7(r-s)+3(r^2s^2-2rs)$.

133. The binomial $a+b$ is one of the two equal factors of $a^2+2ab+b^2$, and is called the **square root** of this trinomial.

Likewise $a-b$ is the square root of $a^2-2ab+b^2$.

In each case a is the square root of a^2 and b of b^2. Hence $2ab$ is twice the product of the square roots of a^2 and b^2.

From the squares obtained in the last article, we learn to distinguish whether any given trinomial is a perfect square, as in the following examples:

1. x^2+4x+4 is in the form of (1), since x^2 and 4 are squares each with the sign $+$, and $4x$ is twice the product of the square roots of x^2 and 4. Hence

$$x^2+4x+4=x^2+2(2x)+2^2=(x+2)(x+2)=(x+2)^2.$$

2. x^2-4x+4 is in the form of (2), since it differs from (1) only in the sign of the middle term. Thus

$$x^2-4x+4=x^2-2(2x)+2^2=(x-2)(x-2)=(x-2)^2.$$

A trinomial which is the product of two equal factors is called a **trinomial square.**

EXERCISES

Determine whether the following are trinomial squares, and if so indicate the two equal factors. If any trinomial is not a square, make it so by modifying one of its terms.

1. $x^2 + 2xy + y^2$.
2. $x^2 - 2xy + y^2$.
3. $x^4 + 2x^2y^2 + y^4$.
4. $x^4 - 2x^2y^2 + y^4$.
5. $m^2 + n^2 - 2mn$.
6. $r^2 + s^2 + 2rs$.
7. $4x^2 - 8xy + 4y^2$.
8. $a^6 + b^6 + 2a^3b^3$.
9. $a^8 + b^8 - 2a^4b^4$.
10. $64 + t^2 - 16t$.
11. $16 + x^2 - 8x$.
12. $9 - 6y + y^2$.
13. $25x^2 + 16y^2 + 40xy$.
14. $4m^2 + n^2 + 2mn$.
15. $100 + s^2 + 20s$.
16. $64 + 49 + 112$.
17. $16a^2 + 25b^2 - 50ab$.
18. $16a^2 + 25b^2 + 40ab$.
19. $81 - 270b + 225b^2$.

134. From the foregoing examples we see that a trinomial is a perfect square if it contains two terms which are squares each with the sign +, while the third term, whose sign is either + or −, is twice the product of the square roots of the other two. Then the square root of the trinomial is the sum or the difference of these square roots according as the sign of the third term is + or −.

Since on multiplying we find $(a - b)^2$ and $(b - a)^2$ give the same result, we may write the factors of $a^2 - 2ab + b^2$ either $(a - b)(a - b)$ or $(b - a)(b - a)$. See page 214.

EXERCISES

Factor the following. If any one of the trinomials is found not to be a square, make it so by modifying one of its terms.

1. $9 + 2 \cdot 3 \cdot 4 + 16$.
2. $x^2 + 4y^2 + 4xy$.
3. $9x^2 + 18xy + 9y^2$.
4. $4x^2 + 4xy + y^2$.

5. $4x^2 + 8xy + 4y^2$.

6. $25x^2 + 12xy + 4y^2$.

7. $16x^2 + 16xy + 4y^2$.

8. $9r^2 + 36rs + 25s^2$.

9. $16x^6 + 8x^3y + y^2$.

10. $4x^8 + 12x^4a^3 + 9a^4$.

11. $a^{10} + 6a^5b + 9b^2$.

12. $(a+1)^2 + 2(a+1)b + b^2$.

13. $(x+3)^2 + 4(x+3)y + 4y^2$.

14. $x^6 + 12x^3 + 36$.

15. $a^4 + 18a^2 + 12$.

16. $121 + 4x^8 - 44x^4$.

17. $16x^4 + 64y^4 - 64x^2y^2$.

18. $81a^2 - 216a + 144$.

19. $4a^2 + 8ab^2 + 4b^2$.

20. $9b^4 + 18b^2c^4 + 9c^8$.

21. $4x^2 + 4y^2 - 8xy$.

22. $9a^2 - 16ab + 4b^2$.

23. $9x^4 - 24x^2b + 16b^2$.

24. $25 + 49x^2 - 70x$.

25. $-30ab^2 + 9a^2 + 25b^4$.

26. $16a^2 - 24ab + 9b^2$.

27. $36x^2 - 84x + 49$.

28. $25 - 90 + 81$.

29. $64x^2 - 32x + 9$.

30. $(3+a)^2 + b^2 - 2b(3+a)$.

31. $(2-x)^2 - 2(2-x)(x-1) + (x-1)^2$.

32. $(2+y)^2 + 2(2+y)(1+4) + (1+4)^2$.

33. $(3a-2b)^2 - 10(3a-2b) + 25$.

34. $(6a-b)^2 + (2a+1)^2 - 2(6a-b)(2a+1)$.

35. $25(a+b)^2 + 50(a+b)(a-b) + 25(a-b)^2$.

36. $x^2 + 12x(a+b+c) + 36(a+b+c)^2$.

37. $49(m-3)^4 + 36(m+1)^6 - 84(m-3)^2(m+1)^3$.

38. $16(x-y)^2 - 16(x-y)(x+y) + 4(x+y)^2$.

39. $-30(a+b)(a-b)^2 + 25(a-b)^4 + 9(a+b)^2$.

THE DIFFERENCE OF TWO SQUARES

135. By multiplication,

$$(a+b)(a-b)=a^2-b^2.$$

Translate this formula into words. By means of this formula the product of the sum and difference of any two number expressions may be found.

E.g. $(3a+2b)(3a-2b)=(3a)^2-(2b)^2=9a^2-4b^2.$

Also $(x+y-z)(x+y+z)=[(x+y)-z][(x+y)+z]$
$=(x+y)^2-z^2.$

The last expression may now be simplified by performing the operations indicated.

In this manner form the following products. Verify the first ten by actual multiplication.

1. $(4a+5b)(4a-5b)$.
2. $(24x+12y)(24x-12y)$.
3. $(16a^2b^3-3c)(16a^2b^3+3c)$.
4. $(5-6t^2)(5+6t^2)$.
5. $(3x-2y)(3x+2y)$.
6. $(x^3-y^3)(x^3+y^3)$.
7. $(3x^4-5y^4)(3x^4+5y^4)$.
8. $[x+(y-z)][x-(y-z)]$.
9. $(x^m+y^n)(x^m-y^n)$.
10. $(a^{2n+1}+b^{2n-1})(a^{2n+1}-b^{2n-1})$.
11. $[c-(a-b)][c+(a-b)]$.
12. $[x-(y+z)][x+(y+z)]$.
13. $(a+b+c)(a-b-c)$.
14. $(a-b+c)(a-b-c)$.
15. $(r-y-z)(r-y+z)$.
16. $(a+b+c)(a+b-c)$.
17. $[a+b+(c-d)][a+b-(c-d)]$.
18. $[x+y+(u+v)][x+y-(u+v)]$.
19. $[4x-(a-2b)][4x+(a-2b)]$.
20. $[a+2b-(x-y^2)][a+2b+(x-y^2)]$.
21. $(11b^3x-3bx^3)(11b^3x+3bx^3)$.

From the preceding examples we see that every binomial which is the difference between two perfect squares is composed of two binomial factors; namely, the sum and the difference of the square roots of these squares.

E.g. $16\,x^2 - 9\,y^2$ is the difference between two squares, $(4\,x)^2$ and $(3\,y)^2$. Hence we have

$$16\,x^2 - 9\,y^2 = (4\,x)^2 - (3\,y)^2 = (4\,x + 3\,y)\,(4\,x - 3\,y).$$

EXERCISES

Factor each of the following:

1. $x^2 - 4\,y^2$.
2. $9\,x^2 - 36\,y^2$.
3. $x^4 - b^2$.
4. $4\,x^2 - 9\,b^8$.
5. $16\,a^4 - 9\,b^4$.
6. $64 - b^2$.
7. $1 - b^2$.
8. $a^2 - 1$.
9. $1 - 9\,x^4$.
10. $4 - 36\,a^2$.
11. $1 - 64\,a^8$.
12. $144\,x^2b^4 - 1$.
13. $256\,a^4b^6 - c^2$.
14. $1 - (x + y)^2$.
15. $4 - (x - 2\,y)^2$.
16. $16\,a - 25\,ab^2$.
17. $49\,x - 4\,xy^2$.
18. $225 - 64\,x^2y^4$.
19. $576\,a - 144\,ay^2$.
20. $5^8 - 3^8$.
21. $x^4 - 81\,y^2$.

136. It is important to determine whether a given expression can be written as the difference of two squares.

E.g. $a^2 + b^2 + 2\,ab - c^2 = (a + b)^2 - c^2 = (a + b + c)\,(a + b - c)$.
Also, $c^2 - a^2 + 2\,ab - b^2 = c^2 - (a^2 - 2\,ab + b^2) = c^2 - (a - b)^2$
$= (c - a + b)(c + a - b)$.

EXERCISES

In the following determine whether each is the difference of two squares, and if so, factor it accordingly; if not, make it so by modifying one of its terms.

1. $x^2 - (y - z)^2$.
2. $(x - y)^2 - z^2$.
3. $a^2 + b^2 - 2\,ab - 4$.
4. $x^2 + y^2 - 2\,xy - z^2$.
5. $4\,a^2b^2 - (a^2 + b^2 - c^2)^2$.
6. $a^2 - (b^2 + c^2 + 2\,bc)$.
7. $(2\,a - 5)^2 - (3\,a + 1)^2$.
8. $(3\,x^2 - y)^2 - (x + y)^2$.

9. $(3a-2b)^2-(8a+5b)^2$.
10. $(3m-4)^2-(2m+3)^2$.
11. $(2r+s)^2-(3r-s)^2$.
12. $81-(a+b+c)^2$.
13. $x^4+x^2+1-4a^2$.
14. $a^2-(x+2y)^2$.
15. $9x^2-(a-b)^2$.
16. $25m^2-(3r+2s)^2$.
17. $4c^2-(4a^2+12ab-9b^2)$.
18. $9a^2+6ab+b^2-c^2$.
19. $16x^2-a^2+4ab-4b^2$.
20. $a^2-2ab+b^2-c^2$.
21. $c^2-(a^2+2ab+b^2)$.
22. $c^2-(a^2-2ab+b^2)$.
23. $(a+b)^2-(a-b)^2$.
24. $a^2+4ab+b^2-x^2$.
25. $a^2+4ab+4b^2-x^2$.
26. $9a^2+16b^2-32ab-x^4$.
27. $a^2+4ab+4b^2-(x^2-2xy+y^4)$.
28. $(3x-2)^2-(4x^2+9y^2-12xy)$.
29. $x^2+4xy-y^2-(a^2+2ab+b^2)$.
30. $16x^4y^2-(4x^2+9y^2+12xy)$.
31. $(a+b)^2-(4a^2+9b^2-12bc)$.

THE SUM OF TWO CUBES

137. By multiplication we find

$$(a+b)(a^2-ab+b^2)=a^3+b^3.$$

The pupil should perform the multiplication indicated in this formula and carefully note how the terms cancel in the product.

By means of this formula obtain the following products:

1. $(x+y)(x^2-xy+y^2)$.
2. $(a+2)(a^2-2a+4)$.
3. $(3a+b)(9a^2-3ab+b^2)$.
4. $(1+4x)(1-4x+16x^2)$.
5. $(w^2+3a^2)(w^4-3a^2w^2+9a^4)$.
6. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
7. $(5a+b)(25a^2-5ab+b^2)$.
8. $(1+5x^2)(1-5x^2+25x^4)$.
9. $(1+2xy)(1-2xy+4x^2y^2)$.
10. $(2x+3y)(4x^2-6xy+9y^2)$.

These products show that every binomial which is the sum of the cubes of two numbers is the product of two factors, one of which is the sum of the numbers, and the other is the sum of their squares minus their product.

E.g. (1) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

(2) $8a^3 + 27b^3 = (2a)^3 + (3b)^3$
$= (2a + 3b)(4a^2 - 2a \cdot 3b + 9b^2)$.

(3) $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

Notice the difference between the trinomial $x^2 - xy + y^2$ and the trinomial square $x^2 - 2xy + y^2$.

EXERCISES

Determine whether each of the following is the sum of two cubes, and if so, find the factors; if not, make it so by modifying one of its terms. Check the results by multiplication.

1. $x^3 + y^3$.
2. $a^3 + 8b^3$.
3. $27a^3 + b^3$.
4. $8a^3 + 1$.
5. $1 + 64x^3$.
6. $2^3 + 3^3$.
7. $125 + 729$.
8. $1 + 125x^6$.
9. $27x^6 + 1$.
10. $8a^3 + 27b^3$.
11. $8a^3 + 64b^3$.
12. $w^3x^6 + x^9a^3$.
13. $1 + 8a^3b^3$.
14. $64x^3 + 343$.
15. $1 + a^3$.
16. $a^3 + 9b^3$.
17. $125x^3 + y^6$.
18. $1 + x^8$.
19. $64x^2 + 27y^3$.
20. $8^3 + 10^3$.
21. $1 + 729x^6$.
22. $x^6 + y^{12}$.
23. $a^9 + b^3$.
24. $27r^3 + 125s^3$.

THE DIFFERENCE OF TWO CUBES

138. By multiplication, we find

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3.$$

In performing this multiplication, note carefully how the terms cancel in the product.

By means of this formula obtain the following products.

1. $(x-y)(x^2+xy+y^2)$.
2. $(b-3)(b^2+3b+9)$.
3. $(2a-b)(4a^2+2ab+b^2)$.
4. $(1-4x)(1+4x+16x^2)$.
5. $(a^2-b^2)(b^4+a^2b^2+b^4)$.
6. $(w^2-2a^2)(w^4+2w^2a^2+4a^4)$.
7. $(3a-2b)(9a^2+6ab+4b^2)$.
8. $(3x^2-1)(9x^4+3x^2+1)$.
9. $(1-2xy)(1+2xy+4x^2y^2)$.
10. $(2x-3y)(4x^2+6xy+9y^2)$.

These products show that the difference of the cubes of two numbers is the product of two factors, one of which is the difference of the numbers, and the other the sum of the squares of the numbers plus their product.

E.g. (1) $x^3-y^3=(x-y)(x^2+xy+y^2)$.
(2) $8a^3-64b^3=(2a)^3-(4b)^3$
$=(2a-4b)(4a^2-2a\cdot 4b+16b^2)$.
(3) $a^9-b^9=(a^3)^3-(b^3)^3=(a^3-b^3)(a^6+a^3b^3+b^6)$.

Notice the difference between the factor x^2+xy+y^2, and the trinomial square $x^2+2xy+y^2$.

EXERCISES

Determine whether each of the following is the difference of two cubes, and if so, find the factors; if not, make it so by changing one of its terms. Check the results by multiplication.

1. a^3-b^3.
2. $8a^3-b^3$.
3. a^3-8b^3.
4. $8a^3-8b^3$.
5. 3^3-2^3.
6. $1-a^{10}$.
7. $1-8a^3$.
8. $64x^3-y^3$.
9. $27-125a^3$.
10. x^6-y^6.
11. $1-x^6$.
12. x^9-8.
13. $1-125x^3$.
14. $8-27x^3$.
15. $27x^3-64$.
16. 2^3x^3-1.
17. $8x^4-y^3$.
18. $64a^3-27b^3$.
19. $1-729x^6$.
20. x^6-y^{12}.
21. $27r^3-125s^3$.

22. Also factor 10, 11, 19, and 20 as the difference of two squares.

TRINOMIALS OF THE FORM $x^2 + (a + b)x + ab$.

139. In §§ 82 to 85 were found such products as

(1) $(x + 5)(x + 2) = x^2 + 7x + 10$.

(2) $(x - 5)(x - 2) = x^2 - 7x + 10$.

(3) $(x + 5)(x - 2) = x^2 + 3x - 10$.

(4) $(x - 5)(x + 2) = x^2 - 3x - 10$.

In each case the binomials to be multiplied have one term x in common, and the other two terms unlike. The trinomial product in each case is the square of the term common to the binomials, the algebraic sum of the unlike terms times the common term, and the product of the unlike terms.

Thus, the coefficient of x in (1) is $5 + 2$, in (2) it is $(-5) + (-2)$, in (3) it is $(+5) + (-2)$, and in (4) it is $(-5) + (+2)$.

The third term of the product in (1) is $(+5)(+2)$, in (2) it is $(-5)(-2)$, in (3) it is $(+5)(-2)$, and in (4) it is $(-5)(+2)$.

With these points clearly in mind, such products may be written out at once, without the formal work of multiplication.

EXERCISES

Find the following products by inspection:

1. $(a + 7)(a + 9)$.
2. $(b - 5)(b + 7)$.
3. $(x - 7)(x - 17)$.
4. $(y + 8)(y - 3)$.
5. $(y + 7)(y - 9)$.
6. $(y - 7)(y - 1)$.
7. $(x - 3)(x - 4)$.
8. $(x - 5)(x - 9)$.
9. $(x - 1)(x + 11)$.
10. $(c - 3)(c + 12)$.
11. $(c^2 - 1)(c^2 - 5)$.
12. $(x^4 - 5)(x^4 - 3)$.
13. $(a^2 - 2)(a^2 - 4)$.
14. $(a^3 - 1)(a^3 + 4)$.
15. $(2a - 1)(2a - 3)$.
16. $(2a + 3)(2a - 4)$.

17. $(x-5)(x+2)$.
18. $(3a-7)(3a+2)$.
19. $(a^2-8)(a^2-4)$.
20. $(b^3-3)(b^3-4)$.
21. $(2x+a)(2x+b)$.
22. $(5c-4)(5c+5)$.
23. $(x^2y^2+9)(x^2y^2-7)$.
24. $(1-3a)(1-2a)$.
25. $(3bc+4)(3bc-7)$.
26. $(100+3)(100+5)$.
27. $(2a^u+b)(2a^u+c)$.
28. $(x+a)(x+b)$.
29. $(x+a)(x-b)$.
30. $(x-a)(x-b)$.
31. $(3x^k+2y)(3x^k-5y)$.

140. It is possible to recognize such products at sight, and thus to find the factors by inspection.

Illustrative Examples. Determine whether the following trinomials are of the kind just considered:

1. $x^2+7x+12$. The question is whether two numbers can be found such that their sum is $+7$ and their product 12. The numbers 3 and 4 answer these conditions. Hence,

$$x^2+7x+12=(x+3)(x+4).$$

2. $x^2-5x-14$. Since the product of the numbers sought is -14, one number must have the sign $-$ and the other $+$; and since their sum is -5, the one having the greater absolute value must have the sign $-$. Hence, the numbers are -7 and $+2$, and we have $x^2-5x-14=(x-7)(x+2)$.

3. $x^2-7x+12=(x-3)(x-4)$. Since $(-3)(-4)=+12$ and $(-3)+(-4)=-7$.

4. $x^2+4x-12=(x+6)(x-2)$. Since $(+6)(-2)=-12$ and $(+6)+(-2)=+4$.

It is to be noted that it is not always possible to find integers to fulfill these two conditions.

E.g. given x^2+5x+3. By inspection, it is easily seen that there are no two integers such that their sum is $+5$ and their product $+3$.

EXERCISES

Determine whether each of the following trinomials can be factored by inspection, and if so, find the factors; if not, modify one term so as to make such factoring possible.

1. x^2+3x+2.
2. x^2+x-6.
3. x^2-x-6.
4. x^2-6x+8.
5. x^2+6x+8.
6. x^2-3x-8.
7. x^2+2x-8.
8. $a^2-4a-32$.
9. $a^2+4a-32$.
10. $b^2+15b+56$.
11. $b^2+8b+15$.
12. b^2-b-56.
13. b^2+b-56.
14. $c^2-3c-15$.
15. $x^2-15x+56$.
16. $x^2+15x-54$.
17. $x^2-14x-95$.
18. $y^2+21y+98$.
19. $y^2-7y-98$.
20. $x^2-19x+78$.
21. x^4+18x^2+77.
22. x^4-5a^2-104.
23. $a^2+32a+240$.
24. a^4-11a^2+28.
25. a^4-11a^2-60.
26. $a^2-14a-51$.
27. $a^2-3a-54$.
28. x^4-8x^2-32.
29. a^6-3a^3-154.
30. $x^2-10x+25$.
31. $a^2b^6-13ab^3-30$.
32. $x^2-17xyz+72y^2z^2$.
33. $r^8+6r^4s-91s^2$.
34. $a^4c^4+9a^2c^2-162$.
35. $a^2+11a-210$.
36. $m^4+4m^2n^2+4n^2$.
37. $s^2t^2-15st-54$.
38. $a^2b^2-27ab+26$.
39. $l^2+13l+42$.
40. $x^2y^2-11xy-180$.
41. $9a^2+24a+16$.
42. $81a^2-99a+30$.
43. $g^2+26g+133$.
44. $x^2+5xy-84y^2$.
45. $r^2+3r-154$.
46. $u^2+38uv+165v^2$.
47. $(a+b)^2-19(a+b)+88$.
48. $(x-y)^2-14(x-y)+40$.
49. $(r-s)^2-17(r-s)+60$.
50. $x^2+(a+b)x+ab$.

141. Trinomials of the form ax^2+bx+c.

Ex. Find the product of $2x+5$ and $3x+2$.

$$\begin{array}{r} 2x+5 \\ 3x+2 \\ \hline 6x^2+15x \\ 4x+10 \\ \hline 6x^2+19x+10 \end{array}$$

The products $3x \cdot 2x$ and $2 \cdot 5$ are called *end products* and $2 \cdot 2x$ and $5 \cdot 3x$ are called *cross products*. In this case the cross products are similar with respect to x and are added. Hence the final product is a trinomial two of whose terms are the end products and the third term is the sum of the cross products.

This fact enables us to write such products at once.

E.g. $(3a+4)(5a-7)=15a^2-a-28$.

In this case $15a^2$ is the first end product and -28 the second, while $-a$ is the sum of the two cross products, $20a$ and $-21a$.

In this manner obtain the following products:

1. $(2a+3)(a+3)$.
2. $(4a-1)(3a+2)$.
3. $(2x+5)(x-7)$.
4. $(7r+8)(3r-6)$.
5. $(2x+8)(9x-4)$.
6. $(3m-1)(4m+3)$.
7. $(5s-7)(2s-4)$.
8. $(2x-1)(7x+4)$.
9. $(4n-9)(5n-7)$.
10. $(8y-1)(5y+11)$.
11. $(3t-5)(t+4)$.
12. $(5x-y)(2x+3y)$.
13. $(3x-2y)(x+3y)$.
14. $(4a-3y)(a+y)$.
15. $(3r-2s)(2r+s)$.
16. $(5m-n)(2m+n)$.
17. $(5a+3x)(3a-4x)$.
18. $(4a-5b)(a+3b)$.
19. $(3a+5b)(a-b)$.
20. $(3c-7d)(2c+3d)$.

142. Trinomials in the form of the above products may sometimes be factored by inspection.

Ex. 1. Factor: $5x^2 + 16x + 3$.

If this is the product of two binomials they must be such that the end products are $5x^2$ and 3 and the sum of the cross products $16x$.

One pair of binomials having the required end products is $5x + 3$ and $x + 1$, another is $5x - 1$ and $x - 3$, and still another $5x + 1$ and $x + 3$.

The sum of the cross products in the first pair is $8x$, in the second pair $-16x$, and in the third pair $16x$. Since the sum of the cross products in the last pair is the one required, the factors are $5x + 1$ and $x + 3$.

Ex. 2. Factor: $6a^2 - 5a - 4$.

In this case as in the one preceding there are several pairs of binomials whose end products are $6a^2$ and -4, such as $2a - 2$ and $3a + 2$, $6a - 1$ and $a + 4$, etc. By trial we find that among these $3a - 4$ and $2a + 1$ is the only pair the sum of whose cross products is $-5a$. Hence $6a^2 - 5a - 4 = (3a - 4)(2a + 1)$.

Obviously not every trinomial of this form can be factored in this manner. Thus, for example, in $6a^2 + 5a + 4$, no pair of binomials whose end products are $6a^2$ and 4 has the sum of its cross products $5a$.

In this manner factor the following:

1. $3x^2 + 5x + 2$.
2. $9a^2 + 9a + 2$.
3. $2x^2 + 11x + 12$.
4. $9x^2 + 36x + 32$.
5. $2x^2 - x - 28$.
6. $12s^2 + 11s + 2$.
7. $6t^2 + 7t - 3$.
8. $6x^2 - x - 2$.
9. $5r^2 + 18r - 8$.
10. $14a^2 - 39a + 10$.
11. $5x^2 + 26x - 24$.
12. $2x^2 - 5x + 2$.
13. $2m^2 - m - 3$.
14. $7c^2 - 3c - 4$.
15. $5x^4 + 9x^2 - 18$.
16. $7a^4 + 123a^2 - 54$.

17. $6c^2 - 19c + 15$.
18. $3a^2 - 21a + 30$.
19. $6d^2 + 4d - 2$.
20. $20a^2 - a - 99$.
21. $12c^2 + 25c + 12$.
22. $8 + 6a - 5a^2$.
23. $15 - 5x - 10x^2$.
24. $6 - 5x - 4x^2$.
25. $3h^2 - 13h + 14$.
26. $15r^2 - r - 2$.
27. $2t^2 + 11t + 5$.
28. $10 - 5x - 15x^2$.
29. $5x^2 - 33x + 18$.
30. $20 - 9x - 20x^2$.

FACTORS FOUND BY GROUPING

143. Besides the methods of factoring which have been applied to the types of expressions thus far considered, there are various other processes which will be considered in the Advanced Course. One other method of general application will suffice here.

Ex. 1. Find the factors of $ax + ay + bx + by$.

By Principle I, the first two terms may be added and also the last two.

Thus, $$ax + ay + bx + by = a(x + y) + b(x + y).$$

The given expression has thus been changed into the sum of two *compound terms* which have a common factor, $(x + y)$, and may be added with respect to this factor by Principle I.

Thus, $$a(x + y) + b(x + y) = (a + b)(x + y).$$

Hence, $$ax + ay + bx + by = (a + b)(x + y).$$

Ex. 2. Factor $ax - ay - bx + by$.

Combining the first two terms with respect to a and the second two with respect to $-b$, we have,

$$ax - ay - bx + by = a(x - y) - b(x - y).$$

Again combining with respect to the factor $x-y$,

$$ax - ay - bx + by = (a-b)(x-y).$$

The success of this method depends upon the possibility of so grouping and combining the terms as to reveal a *common compound term*, § 77. The order of the terms in the given polynomial may be changed as found desirable.

EXERCISES

Factor the following expressions by means of Principles I and II:

1. $ab^2 + ac^2 - db^2 - dc^2$.
2. $6\,ms - 15\,nt + 9\,ns - 10\,mt$.
3. $8\,ax - 10\,ay + 4\,bx - 5\,by$.
4. $2\,a^2 + 3\,ak - 14\,an - 21\,nk$.
5. $ac + bc + ad + bd$.
6. $ax^2 - bx^2 - ay^2 + by^2$.
7. $8\,ac - 20\,ad - 6\,bc + 15\,bd$.
8. $2\,ax - 6\,bx + 3\,by - ay$.
9. $5 + 4\,a - 15\,c - 12\,ac$.
10. $15\,b - 6 - 20\,bc + 8c$.
11. $2\,n^2 - cn + 2\,nd - cd$.
12. $5\,ax - 15\,ay - 3\,bx + 9\,by$.
13. $3\,xa - 12\,xc - a + 4\,c$.
14. $3\,xy - 4\,mn + 6\,my - 2\,xn$.
15. $7\,mn + 7\,mr - 2\,n^2 - 2\,nr$.
16. $a - 1 + a^3 - a^2$.
17. $3\,s + 2 + 6\,s^4 + 4\,s^3$.
18. $as^2 - 3\,bst - ast + 3\,bt^2$.
19. $3\,mn + 6\,m^2 - 2\,am - an$.
20. $2\,ar + 2\,as + 2\,br + 2\,bs$.

MISCELLANEOUS EXERCISES

Classify the following expressions according to the foregoing types for factoring, and indicate the factors:

1. $x^2 + 5\,x + 6$.
2. $1 - x^3$.
3. $x^2 + 11\,x + 30$.
4. $4\,x^2 + 9\,y^2 + 12\,xy$.
5. $4\,x^2 + 9\,y^2 - 12\,xy$.
6. $5\,x^2 + 4\,ax + 7\,xy$.
7. $2\,n^2 - 6\,nc - 3\,ny + 9\,cy$.
8. $4\,x^2 - y^2$.

9. $a^3 + b^3$.
10. $9x^2 + y^4 + 6xy^2$.
11. $2y^2a^3 + 4ya^2 - 8ya$.
12. $x^2 + 7x + 6$.
13. $9x^2 + 36y^4 + 36xy^2$.
14. $9y - 9z - 2xy + 2xz$.
15. $a^3 - 1$.
16. $a^4 + b^4 + 2a^2b^2$.
17. $a^4 - 25$.
18. $27a^3 - 125$.
19. $4a^2 + 4ab + ab^2$.
20. $4a^2 + 9x^4 - 12ax^2$.
21. $1 + x^3$.
22. $2x^2 + 5x + 3$.
23. $36 + 4x^6 + 24x^3$.
24. $(x-1)^2 - (x+1)^2$.
25. $8 + 64a^6$.
26. $ac - ax - 4bc + 4bx$.
27. $27 - 216a^3$.
28. $3^3 + 6^3a^3$.
29. $25(x+1)^2 - 4$.
30. $5cx - 10c + 4dx - 8d$.
31. $4(x+2)^2 + y^2 + 4(x+2)y$.
32. $ra + 2rh - 5sa - 10sh$.
33. $-2a^2b + a^4 + b^2$.
34. $2ha - hb + 6a - 3b$.
35. $3(a+1)^3 + 4(a+1)^2 + a + 1$.
36. $(x+a)^2 - (x-a)^2$.
37. $15m^2 + 224m - 15$.
38. $3x^2 + 27x + 42$.
39. $x^4 + 49a^2 + 14ax^2$.
40. $27a^6 - a^3x^3$.
41. $3^3a^6 + a^3x^3$.
42. $8bd - 40be + 3cd - 15ce$.
43. $x^2 - 11x + 30$.
44. $(x+2)^2 - 4(x-2)^2$.
45. $x^4 + 9y^2 - 6yx^2$.
46. $4a^2 - 7ca^2 - 4d^2 + 7cd^2$.
47. $a^2 + 15a - 16$.
48. $18 - 27c + 16b - 24cb$.
49. $4 - (a^2 + b^2 - 2ab)^2$.
50. $10r + 3bs - 6br - 5s$.
51. $25 + 64x^6 + 80x^3$.
52. $1000 - x^3$.
53. $10^3 + x^3$.
54. $8a^3 + a^3b^3 + b^2a^2$.
55. $100 - 49x^4$.
56. $100 + 625 + 500$.
57. $a^2 - 17a + 72$.
58. $a^2 + 17a + 72$.
59. $a^2 + 16b^2 - 8ab$.
60. $x^6 - y^9$.

61. $24a^2 + 37a - 72$.

62. $x^4 + 15x^2 - 100$.

63. $9^6 + 8^3$.

64. $9x^4 + 16y^2 + 24x^2y$.

65. $1 - 1000$.

66. $16a^2b^2 + 24ab + 36b^3$.

67. $64 + 8$.

68. $16a^2b^2 + 9a^2c^2 + 24a^2bc$.

69. $a^2 + 4b^2 + 4ab - 4x^2$.

70. $a^3b^6 + c^3$.

71. $5x^3 + 10x^3y^2 + 30x^3y^4$.

72. $16a^2c^2 + 4c^2x^2 + 16ac^2x$.

73. $a^6y^3 - z^3$.

74. $x^4 - 7x^2 - 120$.

75. $9a^4b^2 - 12a^3b + 4a^2$.

76. $8ab + 27ab^7$.

77. $x^4 + 4x^2 + 4 - x^6$.

78. $1 - 125a^3b^6$.

79. $16 + 16ab + 4a^2b^2$.

80. $64a^3 + 8a^2b^3$.

81. $36a^2b^4 + c^2b^4 + 12ab^4c$.

82. $4a^2 + 9b^4 + 12ab^2 - 16a^4$.

83. $x^6 + 17x^3 + 30$.

84. $25 - (a^4 - 2a^2b^3 + b^6)$.

85. $-112a^2c^3 + 49a^4 + 64c^6$.

86. $a^2 - a - 380$.

87. $24a^4c^4 + a^6 + 144c^8a^2$.

88. $9x^2 + 4y^4 - 12xy^2 - 16$.

89. $81 + 100x^8 - 180x^4$.

90. $a^4 + 27a^2 + 180$.

91. $a^4 + 3a^2 - 180$.

92. $a^4 - 3a^2 - 180$.

93. $144 - (a^4 + b^2 - 2a^2b)$.

94. $81a^2b^4 + 49c^2 - 126ab^2c$.

95. $12s^2 - 23st + 10t^2$.

96. $36x^4 + 12x^2y^4 + y^8$.

97. $16x^6 + 9y^4 + 24x^3y^2 - 49$.

98. $y^2 + 35y + 300$.

99. $5y^2 - 80y + 300$.

100. $100 - (16x^2 + y^6 - 8xy^3)$.

101. $ac - bc + ad - bd$.

102. $6rd - 15re + 22cd - 55ce$.

103. $z^3 + ya - y^3z^3 - ay^4$.

104. $x^4 + 2x^2 + 1 - x^2$.

105. $60x^2 + 7xy - y^2$.

106. $x^2 - 20xy + 75y^2$.

107. $x^2 - 17x - 60$.

108. $65r^2 + 8r - 1$.

109. $a^2 - 13a - 140$.

110. $39x^4 - 16x^2 + 1$.

111. $625 - (31 - 4a^2)^2$.

112. $36a^2 - 29ab + 5b^2$.

113. $c^4 - 31c^2 + 220.$

114. $ac + d^5a - b^4c - b^4d^5.$

115. $26 + 39n - 22m - 33mn.$

116. $12x^2 + 11x - 56.$

117. $a^2 + 4ab + 4b^2 - (a^2 - 4ab + 4b^2).$

118. $(3x - 1)^2 - (x^2 + 4y^2 - 4xy)^2.$

119. $(x + 3y)^2 + (x - 2y)^2 + 2(x + 3y)(x - 2y).$

120. $16(a + b)^2 - 8(a - b)(a + b) + (a - b)^2.$

121. $256x^2 - (49x^2 + 4y^4 - 28xy^2).$

122. $(2x - a)^2 + 100(a - 3x)^2 + 20(2x - a)(a - 3x).$

123. $-48(a - b)(a + b) + 36(a - b)^2 + 16(a + b)^2.$

EQUATIONS SOLVED BY FACTORING

144. Illustrative Problem. There are two consecutive numbers the sum of whose squares is 61. What are the numbers?

Solution. Let $x =$ one of the numbers, then $x + 1$ is the other.

Hence, $x^2 + (x + 1)^2 = 61$ (1)

By F, $x^2 + x^2 + 2x + 1 = 61$ (2)

By F, I, S, $2x^2 + 2x = 60$ (3)

By D, $x^2 + x = 30$ (4)

These equations differ from any which we have studied heretofore in that they contain the squares of the unknown number, which cannot be removed by addition or subtraction.

It is evident on inspection that $x = 5$ satisfies equation (4).

That is, $5^2 + 5 = 30.$

Hence 5 is one of the numbers sought, and $5 + 1$ is the other, and these numbers satisfy the conditions of the problem,

Since $5^2 + 6^2 = 25 + 36 = 61.$

145. Definition. Equations which involve the second but no higher degree of the unknown number are called **quadratic equations.**

One method of solving quadratic equations is now to be considered.

By S, equation (4) above may be written

$$x^2 + x - 30 = 0. \qquad (5)$$

Factoring the left member,

$$(x + 6)(x - 5) = 0. \qquad (6)$$

Equation (6) is satisfied by any value of x which, substituted in the left member, reduces it to zero; and, since the product of two factors is zero if either factor is zero, we seek values of x which make $x - 5 = 0$ and also $x + 6 = 0$. Hence the equation is satisfied by $x = 5$ and also by $x = -6$. Thus,

$$(5 + 6)(5 - 5) = 11 \cdot 0 = 0,$$

and also $$(-6 + 6)(-6 - 5) = 0 \cdot -11 = 0.$$

Therefore -6 and $-6 + 1 = -5$ are two numbers which meet the condition of the problem.

That is, $$(-6)^2 + (-5)^2 = 36 + 25 = 61.$$

Hence this problem has two solutions, namely, the numbers 5 and 6 and the numbers -6 and -5.

Illustrative Problem. A rectangular flower bed 10 feet long and 6 feet wide is surrounded by a gravel walk whose area is 192 square feet. How wide is the walk?

Solution. Let the width of the walk be x feet, then

$2 \cdot 10x$ is the area of the sides,
$2 \cdot 6x$ is the area of the ends,
$4x^2$ is the area of the corners.

Hence the total area is

$$4x^2 + 2 \cdot 10x + 2 \cdot 6x = 192. \qquad (1)$$

By F, $$4x^2 + 32x = 192. \qquad (2)$$

By D, $$x^2 + 8x = 48. \qquad (3)$$

By S, $$x^2 + 8x - 48 = 0. \quad (4)$$

Factoring, $$(x + 12)(x - 4) = 0. \quad (5)$$

But (5) is satisfied if $x + 12 = 0$, and also if $x - 4 = 0$. (6)

Hence $$x = -12 \text{ and also } x = 4. \quad (7)$$

Check by substituting in equation (3):

$$(-12)^2 + 8(-12) = 144 - 96 = 48.$$

Also $$4^2 + 8 \cdot 4 = 16 + 32 = 48.$$

Hence the quadratic equation to which this problem gives rise has two solutions, but since the width of a walk cannot be a negative number, only the number 4 satisfies the conditions of the problem.

Ex. 1. Solve the quadratic equation

$$5x^2 + 30x + 3 = 3 - 5x. \quad (1)$$

By A, S, $$5x^2 + 35x = 3 - 3 = 0. \quad (2)$$

Factoring, $$5x(x + 7) = 0. \quad (3)$$

But (3) is satisfied if $5x = 0$, and also if $x + 7 = 0$. (4)

Hence $$x = 0, \text{ and also } x = -7. \quad (5)$$

Check. Substitute $x = 0$ and also $x = -7$ in (1).

Ex. 2. Solve the quadratic equation

$$6x^2 + 11x = 10. \quad (1)$$

By S, $$6x^2 + 11x - 10 = 0. \quad (2)$$

Factoring, $$(3x - 2)(2x + 5) = 0. \quad (3)$$

But (3) is satisfied if $3x - 2 = 0$ and also if $2x + 5 = 0$.

Hence $$x = \tfrac{2}{3} \text{ and } x = -\tfrac{5}{2}.$$

Check by substituting each of these values in (1).

Ex. 3. Solve the quadratic equation

$$3x^2 - 5x - 7 = 2x^2 - x - 11. \quad (1)$$

By A and S, $$x^2 - 4x + 4 = 0. \quad (2)$$

Factoring, $$(x-2)(x-2) = 0. \quad (3)$$

It follows from the first factor and also from the second that (3) is satisfied if $x = 2$.

In this case the two solutions turn out to be identical, while in Example 1 one solution was zero and the other was -7.

If we count each of these results as two solutions, then for every quadratic equation thus far solved we have found two values of the unknown number.

146. The method of solution above explained consists of three steps:

(1) Transform the equation so that all terms are collected in one member, with similar terms united, leaving the other member zero. This can always be done by Principle VIII. It is convenient to make the right member zero.

(2) Factor the expression on the left.

(3) Find the value of x which makes each of these factors zero. This is readily done by setting each factor equal to zero and solving it for the unknown.

EXERCISES

Find two solutions for each of the following quadratic equations:

1. $x^2 - 3x + 2 = 0$.
2. $x^2 + 7x = 30$.
3. $a^2 - 11a = -30$.
4. $a^2 + 13a = 30$.
5. $a^2 + 10a + 8 = -3a - 34$.
6. $a^2 + 3a = 10a + 18$.
7. $a^2 + 10a = -24 - 4a$.
8. $2x^2 - 6x = -40 + 12x$.
9. $3x + x^2 = 20x - 72$.
10. $17x + 30 = -x^2 - 40$.

11. $7x^2 + 2x = 30x - 21.$

12. $11x + 3x^2 = 20.$

13. $16 - 5x + x^2 = -2x^2 - 20x - 2.$

14. $x^2 - 16 = 0.$

15. $x^2 - 1 = 0.$

16. $x^2 - x = 0.$

17. $x^2 + x = 0.$

18. $4x^2 = 25.$

19. $x^2 + 4x + 4 = 0.$

20. $x^2 + 8x + 16 = 0.$

21. $x^2 + 12x + 6 = 5x - 4.$

22. $2x^2 - 7x = 60 + 7x.$

23. $60x + 4x^2 + 144 = 8x.$

24. $18x = 63 - x^2.$

25. $24x^2 = 12x + 12.$

26. $2x = 63 - x^2.$

27. $22x + x^2 = 363.$

28. $3x^2 + 7x = 6.$

29. $2x^2 = 2 - 3x.$

30. $x - 2 = -3x^2.$

147. It is sometimes possible to solve other equations than quadratics by the above process.

Ex. 1. Solve the equation:

$$x^3 + 30x = 11x^2. \quad (1)$$

By *S*, $$x^3 - 11x^2 + 30x = 0. \quad (2)$$

By § 131, $$x(x^2 - 11x + 30) = 0. \quad (3)$$

By § 139, $$x(x-5)(x-6) = 0. \quad (4)$$

(4) is satisfied if $x = 0$, if $x - 5 = 0$, and if $x - 6 = 0$.
Hence the solutions are $x = 0$, $x = 5$, $x = 6$.

Ex. 2. $x(x+1)(x-2)(x+3) = 0.$

Any value of x which makes one of these factors zero reduces the product to zero and hence satisfies the equation. Hence the solutions of the equation are found from

$$x = 0,\ x + 1 = 0,\ x - 2 = 0, \text{ and } x + 3 = 0.$$

That is $x = 0$, $x = -1$, $x = 2$, $x = -3$ are the values of x which satisfy the equation.

Notice that this process is applicable only when one member of the equation is *zero* and the other member is *factored*.

EXERCISES

Solve the following equations by factoring:

1. $x^3 - x^2 = 6x.$
2. $5x = 4x^2 + x^3.$
3. $x^3 - 25x = 0.$
4. $x^3 - 3x^2 = -2x.$
5. $3x^3 = 15x^2 + 42x.$
6. $5x^3 + 315x = 80x^2.$
7. $x^2 + ax + bx + ab = 0.$
8. $x^2 - ax - bx + ab = 0.$
9. $4(x-2)^2 - (x+3)^2 = 0.$
10. $x^2 - ax + bx - ab = 0.$
11. $x^2 + ax - bx - ab = 0.$
12. $9(x+2)^2 - 4(x-3)^2 = 0.$
13. $x^3 - x - 3x^2 + 3 = 0.$
14. $x^3 - 4x - 8x^2 + 32 = 0.$
15. $5x^3 + 120x^2 = 119x - 2x^3 + 8x^2.$
16. $(x^2 + 6x - 16)(x^2 - 7x + 12) = 0.$
17. $(x+7)(x^2 - 17x + 72) = 0.$
18. $(x-15)(x^3 + 11x^2 + 30x) = 0.$

148. Illustrative Problem. The paving of a square court costs 40 ¢ per square yard and the fence around it costs \$ 1.50 per linear yard. If the total cost of the pavement and the fence is \$ 100, what is the size of the court?

Solution. Let $x =$ the length of one side in yards.

Then $40x^2 =$ cost in cents of paving the court,

$150 \cdot 4x = 600x =$ cost of the fence in cents.

$$40x^2 + 600x = 10000. \quad (1)$$

By D, $$x^2 + 15x = 250. \quad (2)$$

By S, $$x^2 + 15x - 250 = 0. \quad (3)$$

Factoring, $$(x-10)(x+25) = 0. \quad (4)$$

Whence, $$x = 10, \text{ and also } x = -25. \quad (5)$$

It is clear that the length of a side of the court cannot be -25 yards. Hence 10 is the only one of these two results which has a meaning in this problem.

It happens frequently when a quadratic equation is used to solve a problem that one of the two numbers which satisfy this equation will not satisfy the conditions of the problem.

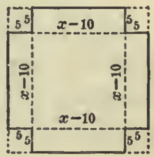
PROBLEMS

In each of the following problems find all the solutions possible for the equations and then determine whether or not each solution has a reasonable interpretation in the problem.

1. The dimensions of a picture inside the frame are 12 by 16 inches. What is the width of the frame if its area is 288 square inches?

2. There are two consecutive integers such that the sum of their squares is 3961. What are the numbers?

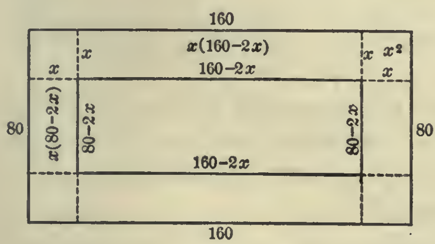
3. An open box is made from a square piece of tin by cutting out a 5-inch square from each corner and turning up the sides. How large is the original square if the box contains 180 cubic inches?



If x = length of a side of the tin, then the volume of the box is: $5(x-10)(x-10)=180$. (See the figure.)

4. A rectangular piece of tin is 4 inches longer than it is wide. An open box containing 840 cubic inches is made by cutting a 6-inch square from each corner and turning up the ends and sides. What are the dimensions of the box?

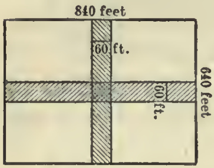
5. A farmer has a rectangular wheat field 160 rods long by 80 rods wide. In cutting the grain, he cuts a strip of equal width around the field. How many acres has he cut when the width of the strip is 8 rods?



6. How wide is the strip around the field of problem 5, if it contains $27\frac{1}{2}$ acres?

7. In the northwest a farmer using a steam plow starts plowing around a rectangular field 640 by 320 rods. If the strip plowed the first day lacks 16 square rods of being 24 acres, how wide is it?

8. A rectangular piece of ground 840 by 640 feet is divided into 4 city blocks by two streets 60 feet wide running through it at right angles. How many square feet are contained in the streets?



9. A farmer lays out two roads through the middle of his farm, one running lengthwise of the farm and the other crosswise. How wide are the roads if the farm is 320 by 240 rods, and the area occupied by the roads is 1671 square rods?

QUADRATIC AND LINEAR EQUATIONS

149. When two simultaneous equations are given, one quadratic and one linear, they may be solved by the process of substitution, which was used (§ 116) in the case of two linear equations.

Illustrative Example. Solve the equations:

$$\begin{cases} x^2 - y^2 = -16. & (1) \\ x - 3y = -12. & (2) \end{cases}$$

From (2) by *S*, $x = 3y - 12.$ (3)

Substituting (3) in (1), $(3y - 12)^2 - y^2 = -16.$ (4)

From (4) by *F*, $9y^2 - 72y + 144 - y^2 = -16.$ (5)

From (5) by *F*, *A*, $8y^2 - 72y + 160 = 0.$ (6)

By *D*, $y^2 - 9y + 20 = 0.$ (7)

Factoring, $(y - 5)(y - 4) = 0.$ (8)

Hence, $y = 5$, and $y = 4.$ (9)

Substitute $y = 5$ in (2) and find $x = 3.$

Substitute $y = 4$ in (2) and find $x = 0.$

Therefore (1) and (2) are satisfied by the two pairs of values,

$$x = 3,\ y = 5, \text{ and } x = 0,\ y = 4.$$

Check by substituting these pairs of values in (1) and (2).

EXERCISES

In the manner just illustrated solve the following:

1. $\begin{cases} x + 2y = 8, \\ 5x^2 + 12y^2 = 128. \end{cases}$

2. $\begin{cases} x + y = 1, \\ x^2 + y^2 = 1. \end{cases}$

3. $\begin{cases} 2x - y = 6, \\ 4x^2 + 5y^2 = 36. \end{cases}$

4. $\begin{cases} x + 3y = 6, \\ x^2 + 3y^2 = 12. \end{cases}$

5. $\begin{cases} x - 2y = -2, \\ x^2 - 6y^2 = 10. \end{cases}$

6. $\begin{cases} 8x - 16y = -120, \\ 7x^2 + 2y^2 = 585. \end{cases}$

7. $\begin{cases} 7x + 9y = 88, \\ 7x^2 + 9y^2 = 736. \end{cases}$

8. $\begin{cases} x - y = 6, \\ x^2 - 7y^2 = 36. \end{cases}$

9. $\begin{cases} 3x + 2y = 7, \\ 3x^2 + 8y^2 = 35. \end{cases}$

10. $\begin{cases} x - 8y = -11, \\ 3x^2 - 16y^2 = 11. \end{cases}$

11. $\begin{cases} x - y = -7, \\ 4x^2 + 3y^2 = 147. \end{cases}$

12. $\begin{cases} x - y = 2, \\ x^2 - 5y^2 = 4. \end{cases}$

13. $\begin{cases} x - y = 1, \\ 3x^2 - 2y^2 = -5. \end{cases}$

14. $\begin{cases} 5x - 7y = -28, \\ 15x^2 + 49y^2 = 784. \end{cases}$

15. $\begin{cases} 6x - 7y = 18, \\ 36x^2 - 7y^2 = 324. \end{cases}$

16. $\begin{cases} x - 9y = 2, \\ x^2 - 45y^2 = 4. \end{cases}$

17. $\begin{cases} x + y = 8, \\ 13x^2 + 3y^2 = 160. \end{cases}$

18. $\begin{cases} 2x - 5y = -16, \\ 4x^2 + 15y^2 = 256. \end{cases}$

19. $\begin{cases} 7x + 4y = 7, \\ 49x^2 - 8y^2 = 49. \end{cases}$

20. $\begin{cases} x - 3y = -12, \\ x^2 - y^2 = -16. \end{cases}$

150. Illustrative Problem. The fence around a rectangular field is 280 rods long. What are the dimensions of the field, if its area is 30 acres?

Solution. Let $w =$ the width of the field in rods,

and $l =$ length of field in rods. (1)

Then $2l + 2w = 280,$

and $lw = 4800$ (one acre = 160 square rods). (2)

From (1), $l + w = 140$, and $w = 140 - l$.

Substituting this value of w in (2),

$$l(140 - l) = 4800,$$

or,

$$l^2 - 140l + 4800 = 0.$$

Then,

$$(l - 60)(l - 80) = 0.$$

Whence 60 and 80 both satisfy the equation.

If $l = 60$, then from equation (1) $w = 80$, and if $l = 80$, then $w = 60$.

We group these pairs of numbers as follows:

$$\begin{cases} l = 60, \\ w = 80, \end{cases} \text{and} \begin{cases} l = 80, \\ w = 60. \end{cases}$$

Substituting these pairs of values in both (1) and (2), we have

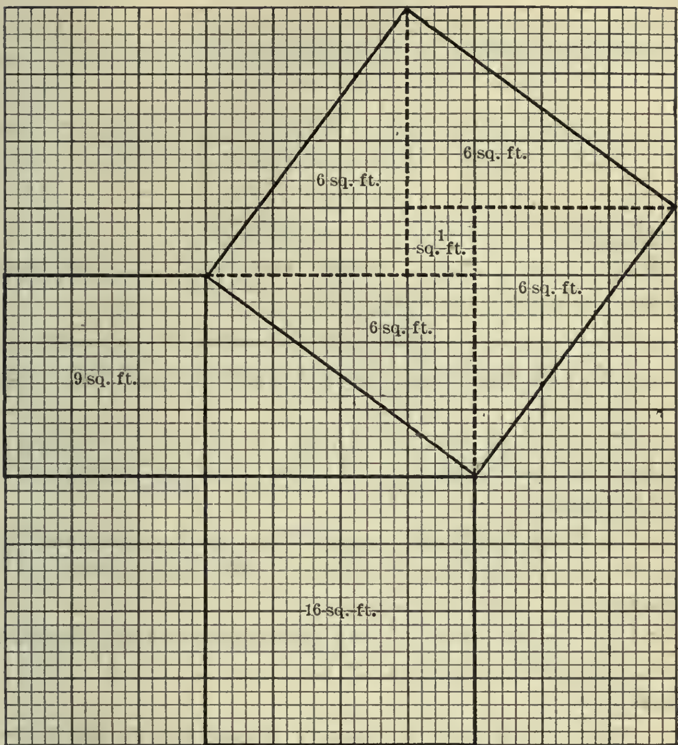
$$\begin{cases} 2 \cdot 60 + 2 \cdot 80 = 280, \\ \qquad 60 \cdot 80 = 4800, \end{cases} \text{and} \begin{cases} 2 \cdot 80 + 2 \cdot 60 = 280, \\ \qquad 80 \cdot 60 = 4800. \end{cases}$$

Hence, we obtain two pairs of numbers which satisfy both of these equations. The solution $l = 60$ and $w = 80$ is applicable to this problem only by calling the greater side the *width* instead of the length. We have seen before that solving a quadratic often results in one solution which is without meaning in the problem that gives rise to the equation.

In any case each solution should be carefully examined to ascertain whether under any reasonable interpretation they are both applicable to the problem.

151. If squares are constructed on the two sides, and also on the hypotenuse of a right-angled triangle, then the sum of the squares on the sides is equal to the square on the hypotenuse. This is proved in geometry, but may be verified by counting squares in the accompanying figure. This proposition was first discovered by the great philosopher and mathematician Pythagoras, who lived about 550 B.C. Hence it is called the Pythagorean proposition.

We now proceed to solve some problems by this proposition.



PROBLEMS

1. The sum of the sides about the right angle of a right triangle is 35 inches, and the hypotenuse is 25 inches. Find the sides of the triangle.

Solution. Let $a =$ the length of one side in inches,

and $b =$ the length of the other.

Then $a + b = 35,$ (1)

and $a^2 + b^2 = 25^2 = 625$ (Pythagorean proposition). (2)

From (1), $a = 35 - b.$

Substituting in (2), $(35 - b)^2 + b^2 = 625,$

or $1225 - 70b + b^2 + b^2 = 625,$

$$2b^2 - 70b + 600 = 0,$$

$$b^2 - 35b + 300 = 0,$$

$$(b - 20)(b - 15) = 0.$$

Whence $b = 20$, and $b = 15$.

From (1), if $b = 20$, $a = 15$, and if $b = 15$, $a = 20$; that is, the sides of the triangle are 15 and 20.

2. The difference between the two sides of a right triangle is 2 feet, and the length of the hypotenuse is 10 feet. Find the two sides.

3. The sum of the length and width of a rectangle is 17 rods, and the diagonal is 13 rods. Find the dimensions of the rectangle.

4. A room is 3 feet longer than it is wide, and the length of the diagonal is 15 feet. Find the dimensions of the room.

5. The length of the molding around a rectangular room is 46 feet, and the diagonal of the room is 17 feet. Find its dimensions.

6. The longest rod that can be placed flat on the bottom of a certain trunk is 45 inches. The trunk is 9 inches longer than it is wide. What are the dimensions of the bottom?

7. The floor space of a rectangular room is 180 square feet, and the length of the molding around the room is 56 feet. What are the dimensions of the room?

8. A rectangular field is 20 rods longer than it is wide, and its area is 2400 square rods. What are its dimensions?

9. A ceiling requires 24 square yards of paper, and the border is 20 yards long. What are the dimensions of the ceiling?

10. The area of a certain triangle is 18 square inches, and the sum of the base and altitude is 12. Find the base and altitude.

11. The altitude of a certain triangle is 7 inches less than the base, and the area is 130 inches. Find the base and altitude.

12. The sum of two numbers is 17, and the sum of their squares is 145. Find the numbers.

13. The difference of two numbers is 8, and the sum of their squares is 274. Find the numbers.

14. The difference of two numbers is 13, and the difference of their squares is 481. Find the numbers.

15. The sum of two numbers is 40, and the difference of their squares is 320. Find the numbers.

16. The sum of two numbers is 45, and their product is 450. Find the numbers.

17. The difference of two numbers is 32, and their product is 833. What are the numbers?

CHAPTER VII

QUOTIENTS AND SQUARE ROOTS

QUOTIENT OF TWO POWERS OF THE SAME BASE

152. Illustrative Problem. To divide x^6 by x^4.

Since by § 66 the quotient times the divisor equals the dividend, we seek an expression which multiplied by x^4 equals x^6.

Since by Principle XIV two powers of the same base are multiplied by adding their exponents, the expression sought must be that power of x whose exponent added to 4 equals 6. Hence the exponent of the quotient is $6 - 4 = 2$. That is, $x^6 \div x^4 = x^{6-4} = x^2$.

EXERCISES

Perform the following indicated divisions:

1. $2^4 \div 2^2$.
2. $2^3 \div 2^2$.
3. $2^4 \div 2$.
4. $3^3 \div 3^2$.
5. $3^4 \div 3$.
6. $3^4 \div 3^2$.
7. $9^{11} \div 9^2$.
8. $5^{13} \div 5^{12}$.
9. $7^{24} \div 7^{22}$.
10. $8^3 \div 8$.
11. $6^4 \div 6^2$.
12. $a^3 \div a^2$.
13. $a^4 \div a^3$.
14. $m^4 \div m^2$.
15. $x^4 \div x^2$.
16. $t^{14} \div t^4$.
17. $m^3 \div m$.
18. $n^6 \div n^2$.
19. $(20)^4 \div (20)$.
20. $(101)^{14} \div (101)^{13}$.
21. $41^7 \div 41^6$.

153. The process of division by subtracting exponents leads in certain cases to strange results.

Thus, according to this process, $x^4 \div x^4 = x^{4-4} = x^0$, which is as yet without meaning, since an exponent has been defined only when it is a *positive integer*. It cannot indicate, as in the case of a positive integral exponent, how many times the base is used as a factor. We know, however, that $x^4 \div x^4 = 1$, since any number divided by itself equals unity. Hence if we use the symbol x^0 it must be interpreted to mean 1, no matter what number x represents. It is sometimes convenient in algebraic work to use it in this way.

Again by this process $x^2 \div x^4 = x^{2-4} = x^{-2}$, which is as yet without meaning, since negative exponents have not been defined. Cases of this kind are considered in the Advanced Course.

The preceding exercises illustrate the following principle:

154. Principle XVI. *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

For the present only those cases are considered in which the exponent of the dividend is greater than or equal to that of the divisor.

Notice that Principle XVI does not apply to powers of different bases.

E.g. $3^7 \div 2^4$ does not equal any integral base to the power, $7 - 4$. This division can be performed only by first multiplying out both dividend and divisor.

EXERCISES

Perform the following indicated divisions by means of Principle XVI:

1. $2^7 \div 2^3$.
2. $a^7 \div a^3$.
3. $3^4 \div 3^2$.
4. $x^4 \div x^2$.
5. $3^{3n} \div 3^{2n}$.
6. $x^{4n} \div x^{2n}$.
7. $3^{2a-1} \div 3^{a-2}$.
8. $5^{n+5} \div 5^{n+2}$.
9. $x^{a+4} \div x^{a+2}$.
10. $t^{4a} \div t^a$.
11. $x^{2a+b} \div x^{a+b}$.
12. $w^{2x} \div w^x$.
13. $(17)^{14} \div (17)^{13}$.
14. $4^3 \div 4$.
15. $(12)^4 \div (12)^3$.

In the following use Principles V, VI, and XVI:

16. $(2^4 + 2^3) \div 2^3$.
17. $(3 \cdot 2^4 + 5 \cdot 2^3) \div 2^2$.
18. $(3 \cdot 4^3 - 5 \cdot 4^4) \div 4^2$.
19. $(a^3b - a^4b^2) \div a^2$.
20. $(4x^3 + 3x^4) \div x^2$.
21. $(a^2m^4 - b^2m^3) \div m^3$.
22. $(4 \cdot 3^2 - 3^3 \cdot 5 \cdot 7) \div 3^2$.
23. $(2^3 \cdot 3 + 2^4 \cdot 3^3 - 2^3) \div 2^3$.
24. $(12x^3y - 11x^2y^2 + 5x^4) \div x^2$.
25. $(x^{3m+4} + x^{2m+3} - 5x^{m+2}) \div x^{m+1}$.

DIVISION OF MONOMIALS

155. In finding the quotient of two numbers each in the factored form, if the factors are represented by Arabic figures, the operation may be carried out in either of two ways.

E.g. $2^3 \cdot 3^3 \cdot 5 \div 2^2 \cdot 3 = 1080 \div 12 = 90.$

Also $2^3 \cdot 3^3 \cdot 5 \div 2^2 \cdot 3 = 2 \cdot 3^2 \cdot 5 = 90.$

In the second process we divide by one of the factors, 2^2 or 3, and divide this result by the other. $2^3 \cdot 3^3 \cdot 5$ divided by 2^2 gives, according to Principle V, $2 \cdot 3^3 \cdot 5$, and this result divided by 3 gives by the same principle $2 \cdot 3^2 \cdot 5$. In practice such operations may readily be performed simultaneously.

In the case of literal factors the second process only is available.

E.g. $5\,a^4b^3c \div a^2b^2 = 5\,a^{4-2}b^{3-2}c = 5\,a^2b^1c = 5\,a^2bc.$

EXERCISES

Perform the following indicated divisions in two ways when possible:

1. $5^8 \cdot 7^9 \div 5 \cdot 7^2$.
2. $2^3 \cdot 3^3 \cdot 5^3 \div 2 \cdot 3^2 \cdot 5$.
3. $4^4 \cdot 5^4 \div 4 \cdot 5^3$.
4. $3^4 \cdot 5^2 \div 3 \cdot 5$.
5. $15\,a^3b^4 \div 5\,ab$.
6. $12\,x^2y \div 4\,x$.
7. $18\,st^3 \div 3\,t^2$.
8. $28\,m^3n \div 7m$.

The preceding examples illustrate the following principle:

156. Principle XVII. *The quotient of two monomials is found by dividing the dividend by each factor of the divisor in succession.*

Each factor of the divisor is associated with any desired factor of the dividend according to Principle V, and when the bases are the same the exponents are subtracted according to Principle XVI.

157. If there are factors in the divisor not found in the dividend, this process terminates before the operation is completed. The remaining steps of the division must then be indicated, which is usually done in the form of a fraction.

E.g. $$x^2 \div 3\,x^3y = \frac{x^2}{3\,x^2 \cdot xy} = \frac{1}{3\,xy}.$$

Again, $$15\,a^3b^2c \div 3\,a^2bx^2y = \frac{15\,a^3b^2c}{3\,a^2bx^2y} = \frac{5\,abc}{x^2y}.$$

By this process all factors common to dividend and divisor have been canceled. Notice that in the first example the factor 1 remains when x^2 of the dividend is divided by x^2 of the divisor.

EXERCISES

Divide:

1. $4 \cdot 7 \cdot 9$ by $2 \cdot 3$.
2. $12 \cdot 8 \cdot 20$ by $2 \cdot 4 \cdot 5$.
3. $6\,x^3y^2z$ by $2\,xyz$.
4. $6^4 \cdot 3^4 \cdot x^3$ by $6^2 \cdot 5^2 \cdot x^2$.
5. $12\,x^{12}y^{13}$ by $4\,xy^3z$.
6. $5\,a^4b^{11}c$ by ab^4c^2.
7. $10\,x^4b^{14}c^3$ by $2xb^4c$.
8. $36\,x^4y^3$ by $6\,x^3y^5$.
9. $35\,x^{2a-1}y^{2x+1}$ by $5\,x^{a-1}y^{x+1}$.
10. $2\,m^{2a+4}n^{3a-2}$ by $m^{a+2}n^{a-2}$.

In each of the following exercises state which of the Principles I–XVII are used:

Divide:

11. $2^3 \cdot 3^2 - 2^4 \cdot 3^3$ by $2^3 \cdot 3^2$.
12. $5 \cdot 2^7 \cdot 3^8 + 7 \cdot 2^5 \cdot 3^4$ by $2^5 \cdot 3^4$.
13. $4\,x^2y^3 - 3\,x^3y^2$ by x^2y^2.
14. $18\,x^4y^4 - 12\,x^3y^3 + 6\,x^2y^2$ by $6\,x^2y^2$.
15. $49\,a^4 + 21\,a^3 - 7\,a$ by $7\,a$.
16. $12\,ax^4y^3 - 16\,a^2x^3y^2 + 8\,a^3xy$ by $4\,axy$.
17. $2\,x^{3a} + 4\,x^{4a} - 8\,x^{2a}$ by $2\,x^a$.
18. $6\,x^{2n+1} + 12\,x^{3n+1} - 10\,x^{n+1}$ by $2\,x^{n+1}$.
19. $4\,x^{13} - 6\,x^{11}b - 10\,x^4c$ by $2\,x^4$.
20. $10\,a^3b^2 - a^2b^3 + 15\,a^4b^4$ by $5\,a^2b^2$.

SQUARE ROOTS OF MONOMIALS

158. Definition. The **radical sign**, $\surd$, indicates that we are to find one of the two equal factors of the number expression which follows it, and the **vinculum** is attached to it, $\sqrt{}$, to show how far its effect is to extend.

E.g. $\sqrt{9}$ is read the *square root of* 9.
Similarly, $\sqrt{a^2+2ab+b^2}$ is read the *square root* of $a^2+2ab+b^2$.

The square root of any number is at once evident if we can resolve it into two equal groups of factors.

E.g.

$$\sqrt{576}=\sqrt{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot3\cdot3}=\sqrt{(2^3\cdot3)(2^3\cdot3)}=\sqrt{24\cdot24}=24.$$

It should be noted that every square has *two* square roots.

E.g. $\sqrt{9}=-3$ as well as $+3$, since $(-3)^2=9$ and $3^2=9$.

In obtaining a square root the two results should always be indicated. This is usually done by attaching the double sign $\pm$ to the square root, *e.g.* $\sqrt{9}=\pm3$.

EXERCISES

Find by inspection the following square roots:

1. $\sqrt{4}$.	8. $\sqrt{121}$.	15. $\sqrt{324}$.	22. $\sqrt{2^4}$.
2. $\sqrt{9}$.	9. $\sqrt{169}$.	16. $\sqrt{289}$.	23. $\sqrt{5^{12}}$.
3. $\sqrt{16}$.	10. $\sqrt{225}$.	17. $\sqrt{625}$.	24. $\sqrt{7^8}$.
4. $\sqrt{25}$.	11. $\sqrt{196}$.	18. $\sqrt{900}$.	25. $\sqrt{a^{12}}$.
5. $\sqrt{36}$.	12. $\sqrt{256}$.	19. $\sqrt{10000}$.	26. $\sqrt{3^{14}}$.
6. $\sqrt{49}$.	13. $\sqrt{576}$.	20. $\sqrt{a^4}$.	27. $\sqrt{a^{24}}$.
7. $\sqrt{81}$.	14. $\sqrt{400}$.	21. $\sqrt{x^6}$.	28. $\sqrt{3^4}$.

159. The square root of the product of several factors, each of which is a square, may be found in two ways if the factors are expressed in Arabic figures.

E.g. $\sqrt{4 \cdot 16 \cdot 25} = \sqrt{1600} = \sqrt{40 \cdot 40} = \pm 40,$

or $\sqrt{4 \cdot 16 \cdot 25} = \sqrt{2^2 \cdot 4^2 \cdot 5^2} = \pm 2 \cdot 4 \cdot 5 = \pm 40.$

But with literal factors, the second process only is available.

E.g. $\sqrt{16\, a^2b^4c^2} = \sqrt{4^2\, a^2b^4c^2} = \pm 4\, ab^2c.$

EXERCISES

Find the following indicated square roots, keeping the results in the factored form:

1. $\sqrt{2^2 \cdot 3^2}$.
2. $\sqrt{81 \cdot 121}$.
3. $\sqrt{49 \cdot 25 \cdot 169}$.
4. $\sqrt{8^2 \cdot 5^2 \cdot 3^2}$.
5. $\sqrt{5^4 \cdot 3^2 \cdot 4^4}$.
6. $\sqrt{25 \cdot 36}$.
7. $\sqrt{3^{12} \cdot 5^{14}}$.
8. $\sqrt{2^{22} \cdot 3^{12}}$.
9. $\sqrt{16\, a^2b^2c^2}$.
10. $\sqrt{64\, a^4x^4}$.
11. $\sqrt{4^4\, a^2b^4}$.
12. $\sqrt{3^2\, x^2y^2}$.
13. $\sqrt{9\, x^4y^{12}}$.
14. $\sqrt{121\, a^2x^4}$.
15. $\sqrt{7^4\, a^4b^2}$.
16. $\sqrt{625\, x^4y^2}$.
17. $\sqrt{1225\, a^{2r}}$.
18. $\sqrt{36\, b^{4m}c^{2n}}$.

Notice that the square root of a *sum* is *not* obtained by taking the square roots of the terms separately. Thus, $\sqrt{9+16}$ is not equal to $\sqrt{9}+\sqrt{16}$.

The preceding exercises illustrate the following principle:

160. Principle XVIII. *The square root of a product is obtained by finding the square root of each factor separately and then taking the product of these roots.*

In order that a factor may be a perfect square it must be a power whose exponent is *even*. Its square root is then a power of the same base whose exponent is equal to one-half the given exponent.

Thus, $\sqrt{x^6} = \sqrt{x^3 \cdot x^3} = x^3$. The exponent 3 of the root is one-half the exponent 6 of the power. Hence to find the square root of a monomial we divide the exponent of each factor by 2.

EXERCISES

Find the following square roots:

1. $\sqrt{4\,a^6b^8}$.
2. $\sqrt{3^2x^{12}y^{14}}$.
3. $\sqrt{5^2 \cdot 3^{22}\,t^{24}}$.
4. $\sqrt{121\,x^4y^{12}}$.
5. $\sqrt{576\,a^2b^4}$.
6. $\sqrt{10^4\,a^4b^4}$.
7. $\sqrt{5^4\,m^{14}}$.
8. $\sqrt{5^4 \cdot 3^8 \cdot 7^2}$.
9. $\sqrt{3^{14} \cdot 7^{12}\,a^4}$.
10. $\sqrt{25\,a^2b^4c^{12}}$.
11. $\sqrt{81\,x^4y^8c^{10}}$.
12. $\sqrt{729\,a^6y^{10}z^{14}}$.
13. $\sqrt{64 \cdot 625\,a^2b^4}$.
14. $\sqrt{216\,x^{2k}y^{4n}}$.
15. $\sqrt{3^{2x} \cdot 5^{2y}}$.

DIVISION BY A POLYNOMIAL

161. The simplest case of division by a polynomial is that in which the dividend can be resolved into two factors, one being the given polynomial divisor and the other a monomial.

E.g. To divide $4\,x^3 + 4\,x^2y$ by $x + y$, factor the dividend and we have

$$4\,x^2\,(x+y) \div (x+y) = 4\,x^2.$$

In case the dividend cannot be factored in this manner, then, if the division is possible, the quotient must be a polynomial. The process of finding the quotient under such circumstances is best shown by studying a particular case.

Illustrative Example 1. Consider the product

$$(x^2 + 2\,xy + y^2)(x+y) = x^2(x+y) + 2\,xy\,(x+y) + y^2\,(x+y).$$

The products, $x^2(x+y)$, $2\,xy(x+y)$, and $y^2(x+y)$ are called **partial products**, and their sum, $x^3 + 3\,x^2y + 3\,xy^2 + y^3$, the **complete product.**

In dividing $x^3 + 3\,x^2y + 3\,xy^2 + y^3$ by $x + y$ the quotient must be such a polynomial that when its terms are multiplied by $x + y$ the results are these partial products, which in the solution are called 1st, 2d, and 3d products.

The work may be arranged as follows:

Dividend or product:	$x^3 + 3x^2y + 3xy^2 + y^3$	$x + y$, divisor.
1st product, $x^2(x+y)$:	$x^3 + x^2y$	$x^2 + 2xy + y^2$, [quotient.
Dividend minus 1st product:	$2x^2y + 3xy^2 + y^3$	
2d product, $2xy(x+y)$:	$2x^2y + 2xy^2$	
Dividend minus 1st and 2d products:	$xy^2 + y^3$	
3d product, $y^2(x+y)$:	$xy^2 + y^3$	
Dividend minus 1st, 2d, and 3d products:	0	

Explanation. Since the dividend or product contains the term x^3, and since one of the factors, the divisor, contains the term x, the other factor, the quotient, must contain the term x^2. Multiplying this term of the quotient by the divisor, we obtain the first partial product, $x^3 + x^2y$.

Subtracting the first partial product from the whole product $x^3 + 3x^2y + 3xy^2 + y^3$, the remainder is $2x^2y + 3xy^2 + y^3$, which is the product of the divisor and that part of the quotient which has not yet been found. Since this product contains the term $2x^2y$ and the divisor contains the term x, the quotient must contain the term $2xy$. The product of $2xy$ and $x + y$ is the second partial product.

Subtracting this second partial product from $2x^2y + 3xy^2 + y^3$, we have $xy^2 + y^3$ still remaining after the first and second partial products have been subtracted from the whole product. This remainder is the product of the divisor and the part of the quotient not yet found. Since the product contains the term xy^2 and the divisor contains the term x, the quotient must contain the term y^2; hence, the third partial product is $xy^2 + y^3$.

Subtracting the third partial product the remainder is zero. Hence the sum of the three partial products thus obtained is equal to the whole product, and it follows that $x^2 + 2xy + y^2$ is the required quotient.

162. Problems in division may be checked by substituting any convenient values for the letters. For example, in this case, $x=1$, $y=1$, reduces the dividend to 8, the divisor to 2, and the quotient to 4, which verifies the correctness of the result.

Since division by zero is impossible (see Advanced Course), care must be taken not to select such values for the letters as will reduce the divisor to zero.

Illustrative Example 2. Divide $2x^4+x^3-7x^2+5x-1$ by x^2+2x-1.

Solution.

		[divisor.
Dividend or product:	$2x^4+x^3-7x^2+5x-1$	x^2+2x-1,
1st product, $2x^2(x^2+2x-1)$:	$2x^4+4x^3-2x^2$	$2x^2-3x+1$,
Dividend minus 1st product:	$-3x^3-5x^2+5x-1$	[quotient.
2d product, $-3x(x^2+2x-1)$:	$-3x^3-6x^2+3x$	
Dividend minus 1st and 2d products:	$+x^2+2x-1$	
3d product, $1\cdot(x^2+2x-1)$:	x^2+2x-1	
Dividend minus 1st, 2d, and 3d products:	0	

Check. Substitute $x=2$ in dividend, divisor, and quotient.

Illustrative Example 3. Divide $20a^2-8+18a^4+22a-19a^3$ by $2a^2-3a+4$.

Solution. Arranging dividend and divisor according to the descending powers of a, we have

		[divisor.
Dividend or product:	$18a^4-19a^3+20a^2+22a-8$	$2a^2-3a+4$,
1st product:	$18a^4-27a^3+36a^2$	$9a^2+4a-2$,
Dividend minus 1st product:	$8a^3-16a^2+22a-8$	[quotient.
2d product:	$8a^3-12a^2+16a$	
Dividend minus 1st and 2d products:	$-4a^2+6a-8$	
3d product:	$-4a^2+6a-8$	
Dividend minus all products:	0	

Check. Substitute $a=1$ in dividend, divisor, and quotient.

163. From a consideration of the preceding examples the process of dividing by a polynomial is described as follows:

1. Arrange the terms of dividend and divisor according to descending (or ascending) powers of some common letter.

2. Divide the first term of the dividend by the first term of the divisor. This quotient is the first term of the quotient.

3. Multiply the first term of the quotient by the divisor and subtract the product from the dividend.

4. Divide the first term of this remainder by the first term of the divisor, obtaining the second term of the quotient. Multiply the divisor by the second term of the quotient and subtract, obtaining a second remainder.

5. Continue in this manner until the last remainder is zero, or until a remainder is found whose first term does not contain as a factor the first term of the divisor. In case no remainder is zero, the division is not exact.

EXERCISES

Check the result in each case, being careful to substitute such numbers for the letters as do not make the divisor zero.

Divide the following:

1. $a^2 + 2\,ab + b^2$ by $a + b$.

2. $a^2 - 2\,ab + b^2$ by $a - b$.

3. $a^3 - 3\,a^2b + 3\,ab^2 - b^3$ by $a - b$.

4. $2\,x^3 + 2\,x^2y - 4\,x^2 - x - 4\,xy - y$ by $x + y$.

5. $x^3 + xy^2 - x^2y - y^3$ by $x - y$.

6. $x^3 + 4\,x^2 + x - 6$ by $x + 3$.

7. $x^3 + 4\,x^2 + x - 6$ by $x - 1$.

8. $x^4 - 6\,x^3 + 2\,x^2 - 3\,x + 6$ by $x - 1$.

9. $x^3 + 3x^2y + 3xy^2 + y^3$ by $x^2 + 2xy + y^2$.

10. $x^3 - 8x^2 + 75$ by $x - 5$.

11. $2a^3 + 19a^2b + 9ab^2$ by $2a + b$.

12. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x - y$.

13. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ by $x^2 + 2xy + y^2$.

14. $x^4 + x^3y + xy^3 + y^4$ by $x + y$.

15. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

16. $x^4 - y^4$ by $x - y$.

17. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

18. $2x^4 + 11x^3 - 26x^2 + 16x - 3$ by $x^2 + 7x - 3$.

19. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.

20. $x^5 - x^4 - 27x^3 + 10x^2 - 30x - 200$ by $x^2 - 4x - 10$.

21. $3x^2 - 4xy + 8xz - 4y^2 + 8yz - 3z^2$ by $x - 2y + 3z$.

22. $9r^2s^2 - 4r^2t^2 + 4rst^2 - s^2t^2$ by $3rs - 2rt + st$.

23. $9a^2b^2 + 16x^2 - 4a^2 - 36b^2x^2$ by $3ab + 6bx - 2a - 4x$.

24. $x^3 + x^2y + x^2z - xyz - y^2z - yz^2$ by $x^2 - yz$.

25. $a^5 + a^4b + a^3 - a^3b^2 - 2ab^2 + b^3$ by $a^2 + ab - b^2$.

26. $x^3 + y^3 - z^3 + 3xyz$ by $x + y - z$.

27. $a^3 + b^3 + 3ab - 1$ by $a + b - 1$.

28. $6x^{5k} - 11x^{4k} + 23x^{3k} + 13x^{2k} - 3x^k + 2$ by $3x^k + 2$.

29. $a^{3k} - 3a^{2k}b^k + 3a^kb^{2k} - b^{3k}$ by $a^k - b^k$.

30. $32s^{4a} - 9s^{3a}t^b + 12s^{2a}t^{2b} - 18s^at^{3b} - 17t^{4b}$ by $s^a - t^b$.

164. Since the dividend is the product of the divisor and quotient, it follows that if one factor of an expression is given, the other factor may be found by division.

EXERCISES

Obtain the factors of each of the following products, one factor being given in each case:

	Product	*Given factor*
1.	$x^3 - y^3$.	$x - y$.
2.	$x^3 + y^3$.	$x + y$.
3.	$a^5 - b^5$.	$a - b$.
4.	$a^5 + b^5$.	$a + b$.
5.	$a^6 + b^6$.	$a^2 + b^2$.
6.	$a^9 + b^9$.	$a^3 + b^3$.
7.	$r^3 - s^3$.	$r^2 + rs + s^2$.
8.	$r^4 - s^4$.	$r^3 + r^2s + rs^2 + s^3$.
9.	$r^5 + s^5$.	$r^4 - r^3s + r^2s^2 - rs^3 + s^4$.
10.	$a^3 - 12\,a^2 + 27\,a + 40$.	$a - 5$.
11.	$x^5 - 5\,x^4y + 11\,x^3y^2 - 14\,x^2y^3 - 51\,xy^4 + 54\,y^5$.	$x^2 - 3\,xy + 2\,y^2$.
12.	$x^4 + x^2y^2 + y^4$.	$x^2 - xy + y^2$.
13.	$a^3 + 5\,a^2 - 2\,a - 24$.	$a^2 + 7\,a + 12$.
14.	$a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 5\,ab^4 - b^5$.	$a^2 - 2\,ab + b^2$.
15.	$x^5 - 5\,x^3y^2 - 5\,x^2y^3 + y^5$.	$x^2 - 3\,xy + y^2$.

SQUARE ROOTS OF POLYNOMIALS

165. In §§ 87, 88, we found certain trinomials which were perfect squares, namely,

$$a^2 + 2\,ab + b^2 = (a + b)^2, \tag{1}$$

$$a^2 - 2\,ab + b^2 = (a - b)^2. \tag{2}$$

Hence we know the square roots of all trinomials which are in either of these forms. These trinomial squares may be used to discover a process for finding the square root of any polynomial which is a perfect square.

Finding a square root may be regarded as a process of division in which divisor and quotient are equal and both are to be found simultaneously.

Illustrative Example. Find the square root of $4x^2+12xy+9y^2$.

Considering the formula (1) we are to pass from the square $a^2+2ab+b^2$ to the square root $a+b$, and for this purpose we write $a^2+2ab+b^2$ in the form $a^2+b(2a+b)$ and arrange the work as follows:

Square or product,		$4x^2+12xy+9y^2 \mid \underline{2x+3y}$,	square root.
		$4x^2$	1st par'l product.
1st par'l divisor,	$4x$	$12xy+9y^2$,	square minus 1st par'l prod.
1st compl. divisor,	$4x+3y$	$\underline{12xy+9y^2}$,	2d par'l product.
		0	

Supposing that $4x^2$ is the a^2 of the formula, a is then $2x$, which is the first term of the root. Squaring $2x$ gives $4x^2$, the first partial product. Subtracting $4x^2$ from the total product leaves $12xy+9y^2$, which is the $b(2a+b)$ of the formula.

Since b is not yet known, we cannot find completely either of the factors of $b(2a+b)$; but since a has been found, we can get the first term of the factor $2a+b$, viz. $2a$ or $2\cdot 2x=4x$, which is the first partial divisor. Dividing $12xy$ by $4x$ we have $3y$, which is the b of the formula. Then $2a+b=4x+3y$ the first complete divisor.

To obtain the second partial product, $b(2a+b)$ or $12xy+9y^2$, we multiply $4x+3y$ by $3y$. On subtracting, the remainder is zero and the process ends, whence the required root is $2x+3y$.

It should be clearly understood that the sum of the first and second partial products is a square, viz., the square of $2x+3y$, because it has been constructed just as $a^2+b(2a+b)$ was formed from $a+b$.

EXERCISES

Find in the manner just described the square roots of the following trinomials:

1. $36x^2 - 84xy + 49y^2$.
2. $16a^2 - 40ab + 25b^2$.
3. $121t^2 + 264ta + 144a^2$.
4. $81m^4 + 144m^2n^2 + 64n^4$.
5. $49s^6 - 84s^3 + 36$.
6. $1 - 20x^4 + 100x^8$.

166. The process just applied to trinomials is applicable to polynomial squares having a larger number of terms, by regarding the a of the formula at each step as representing the part of the root already found and b as the term of the root about to be found.

Illustrative Example. Find the square root of $x^2 + 2xy + y^2 + 6x + 6y + 9$. The work may be arranged as follows:

Square or product,	$x^2+2xy+y^2+6x+6y+9$		$x+y+3$,	square root.
	x^2			1st par'l product.
1st par'l div'r,	$2x$	$2xy+y^2+6x+6y+9$,		1st remainder.
1st compl. div'r,	$2x+y$	$2xy+y^2$,		2d par'l product.
2d par'l div'r,	$2x+2y$	$6x+6y+9$,		2d remainder.
2d compl. div'r,	$2x+2y+3$	$6x+6y+9$,		3d par'l product.
		0		

After the first two terms of the root have been found, namely, x and y, then we consider $x + y$ as the a of the formula and call it a'. The new b, which we call b', is then found to be 3. Subtracting the first and second partial products is the same as subtracting $(x+y)^2$, that is, the square of a'. Hence, the second partial divisor, which is twice a', is $2(x+y)$.

In case there are four terms in the root, then the sum of the first three, when found as above, is regarded as the new a, called a''. The remaining term of the root is the new b, and is called b''.

The formula $(a-b)^2 = a^2 - 2ab + b^2$ is not needed, since this may be written $(a+(-b))^2 = a^2 + 2a(-b) + (-b)^2$, which is in the form of $(a+b)^2 = a^2 + 2ab + b^2$.

EXERCISES

Find the square roots of the following polynomials:

1. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz.$
2. $9x^2 + 4y^2 + z^2 - 12xy + 6xz - 4yz.$
3. $a^2 - 8ab + 16b^2 - 2ac + c^2 + 8bc.$
4. $a^4 + 4a^3 + 6a^2 + 4a + 1.$
5. $c^4 - 4c^3 + 6c^2 - 4c + 1.$
6. $y^4 - 4y^2 - 8xy^2 + 16x + 16x^2 + 4.$
7. $x^4 - 2x^3 + 3x^2 - 4x + 4.$
8. $a^2 + a^2b^2 - 2a^2b + 2abc - 2ab^2c + b^2c^2.$
9. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$
10. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.$
11. $x^6 - 4x^5 + 4x^4 + 6x^3 - 12x^2 + 9.$
12. $a^4 + 53a^2 + 14a^3 + 28a + 4.$
13. $x^2 + 16x^2y^2 + 289 + 8x^2y + 34x + 136xy.$
14. $9a^4 + 4a^2 + 256 - 12a^3 - 96a^2 + 64a.$
15. $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1.$
16. $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1.$
17. $16x^6 + 4y^2 + 1 - 16x^3y + 8x^3 - 4y.$
18. $25 + 49x^2 + 4x^4 - 70x - 20x^2 + 28x^3.$
19. $64x^6 + 192x^5 + 240x^4 + 160x^3 + 60x^2 + 12x + 1.$
20. $4x^6 - 12x^5 + 13x^4 - 14x^3 + 13x^2 - 4x + 4.$
21. $a^4b^4 - 2a^3b^2 + a^2 - 2a^2b^3 + 2ab + b^2.$
22. $16a^6 + 24a^5 + 25a^4 + 20a^3 + 10a^2 + 4a + 1.$
23. $x^6y^6 + 2x^5y^5 + 3x^4y^4 + 4x^3y^3 + 3x^2y^2 + 2xy + 1.$
24. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 4x^5 + 3x^6 + 2x^7 + x^8.$
25. $9a^2 - 6ab + 30ac + 6ad + b^2 - 10bc - 2bd + 25c^2$
    $+ 10cd + d^2.$

SQUARE ROOTS OF NUMBERS EXPRESSED IN ARABIC FIGURES

167. The square root of a number expressed in Arabic figures may be found by the process just used for polynomials.

Illustrative Example. Find the square root of 5329.

In order to decide how many digits there are in the root we observe that $10^2 = 1000$ and $100^2 = 10000$; hence the root lies between 10 and 100, *i.e.* it contains two digits. Since $80^2 = 6400$ and $70^2 = 4900$, we see that 7 is the largest number possible in tens' place of the root. The work is arranged as follows:

The given square,	5329	$70 + 3$, square root.
$a^2 = 70^2$	4900,	1st partial product.
$2a = 2 \times 70 = 140$	429,	1st remainder.
$b = 3$		
$2a + b = 143$	429	$= b(2a + b)$.
	0	

Having decided as above that the a of the formula is 7 tens, we square this and subtract, obtaining 429 as the remaining part of the power.

The first partial divisor, $2a = 140$, is divided into 429 giving a quotient 3, which is the b of the formula. Hence the first complete divisor, $2a + b$, is 143, and the second partial product, $b(2a + b)$, is 429. Since the remainder is zero, the process is exact and 73 is the square root sought.

EXERCISES

Find the square roots of the following:

1. 3249.	5. 6241.	9. 1849.
2. 8836.	6. 7056.	10. 7225.
3. 7569.	7. 9409.	11. 3025.
4. 8281.	8. 9801.	12. 9216.

168. The square of any integer from 1 to 9 contains one or two digits; the square of any integer from 10 to 99 contains three or four digits; the square of any integer from 100 to 999 contains five or six digits; etc.

Hence it is evident that, if the digits of a number which is a perfect square be separated into groups of two each, counting from units' place toward the left, the number of groups thus formed is the same as the number of digits in the square root.

Illustrative Example. Find the square root of 120,409.

Since this number is divided into three groups, the first digit is in hundreds' place. The work is arranged as follows:

Square,	12 04 09	$300 + 40 + 7 = 347$, square root.
$a^2 = 300^2$	9 00 00,	1st partial product.
$2a = 2 \times 300 = 600$	3 04 09,	1st remainder.
$b = 40$		
$2a + b = 640$	2 56 00	$= b(2a + b)$.
$2a' = 2 \times 340 = 680$	48 09,	2d remainder.
$b' = 7$		
$2a' + b' = 687$	48 09	$= b'(2a' + b')$.
	0	

The first partial divisor, 2×300, is completed by adding the second term of the root, 40, that is, $2a + b = 600 + 40$. At first glance it might be supposed that the second digit is 5 instead of 4, since 600 is contained in 30,409 50 times, but account must be taken of the addition to be made to the partial divisor, and when this is done the quotient is 40, not 50.

In the second partial divisor, $2a'$ stands for 2 times $(300+40)=680$, and b' stands for the third digit of the root.

In case a square contains an odd number of digits the last group at the left will have one instead of two digits.

E.g. 3 47 21 has three places in its square root, of which the first, hundreds' digit, is the largest square in 3, namely, 1.

EXERCISES

Find the square root of each of the following:

1. 294,849.	**5.** 3481.	**9.** 100,489.	**13.** 35,721.
2. 37,636.	**6.** 7569.	**10.** 26,569.	**14.** 16,641.
3. 872,356.	**7.** 1849.	**11.** 874,225.	**15.** 32,761.
4. 599,076.	**8.** 73,441.	**12.** 170,569.	**16.** 223,729.

169. Since the square of any decimal fraction has twice as many places as the given decimal, it is evident that the square root of a decimal fraction contains one decimal place for every two in the square.

E.g. $(.15)^2 = .0225$; $(.012)^2 = .000144$.

Hence, for the purpose of determining the decimal places in the root, the decimal part of a square should be divided into groups of two digits each, counting from the decimal point toward the right.

Illustrative Example. Find the square root of 4.6225. According to §§ 168, 169 the root contains one digit in the integral part and two in the decimal part. The work is as follows:

	4.62 25	$2 + .1 + .05 = 2.15$
$a^2 = 2^2$	4	
$2a = 2 \times 2 = 4.0$	.62 25,	first remainder.
$b = .1$		
$2a + b = 4.1$	.41	$= b(2a + b)$.
$2a' = 2 \times 2.1 = 4.2$	.21 25,	second remainder.
$b' = .05$		
$2a' + b' = 4.25$	.21 25	$= b'(2a' + b')$.
	0	

This process is also applicable for the purpose of approximating the square root of a number which is not a perfect square.

Illustrative Example. Find the approximate square root of 582 to three decimal places. The solution below shows all the steps of the work.

	5 82	$20 + 4 + .1 + .02 + .004 = 24.124$
$a^2 = 20^2$	4 00	
$2a = 2 \times 20 = 40.$	1 82,	first remainder.
$b = 4.$		
$2a + b = 44.$	1 76	$= b(2a + b).$
	6.00,	second remainder.
$2a' = 2 \times 24 = 48.$		
$b' = .1$		
$2a' + b' = 48.1$	4.81	$= b'(2a' + b').$
	1.1900,	third remainder.
$2a'' = 2 \times 24.1 = 48.2$		
$b'' = .02$		
$2a'' + b'' = 48.22$	.9644	$= b''(2a'' + b'').$
	.225600,	fourth remainder.
$2a''' = 2 \times 24.12 = 48.24$		
$b''' = .004$		
$2a''' + b''' = 48.244$	.192976	$= b'''(2a''' + b''').$
	.032624	

The decimal points are handled exactly as in division of decimals in arithmetic, the chief care being needed in forming the divisors.

170. Evidently the process in this example may be carried on indefinitely. 24.124 is an **approximation** to the square root of 582. In fact, the square of 24.124 differs from 582 by the small fraction .032624. 24.12 is the nearest approximation using two decimal places. If the third figure were 5 or any digit greater than 5, then 24.13 would be the nearest approximation using two decimal places. Hence three places must be found in order to be sure of the nearest approximation to two places.

EXERCISES

Find the square roots of the following, correct to two decimal places:

1. 387.	7. 2.	13. .02.
2. 5276.	8. 3.	14. .003.
3. 2.92.	9. 5.	15. .5.
4. 27.29.	10. 7.	16. .005.
5. 51.	11. 8.	17. .307.
6. 3.824.	12. 11.	18. 200.002.

SQUARE ROOTS OF FRACTIONS EXPRESSED IN ARABIC FIGURES

171. Since in arithmetic the product of two fractions is found by multiplying their numerators and their denominators, a fraction is squared by squaring its numerator and its denominator separately.

Hence, to extract the square root of a fraction, we find the square root of its numerator and its denominator separately.

E.g. $\sqrt{\frac{16}{25}} = \frac{4}{5}$, since $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$.

However, in approximating the square root of a fraction whose denominator is not a perfect square, if the final result is to be reduced to a decimal, the direct use of this method is cumbersome since it usually involves the approximation of two square roots and always necessitates dividing by a long decimal fraction.

E.g. $\sqrt{\frac{2}{3}} = \sqrt{2} \div \sqrt{3} = 1.4142 \div 1.7321$, in which we are now obliged to divide by 1.7321.

This difficulty may be avoided in either of two ways:

1. The fraction may be reduced to a decimal before the root is approximated.

E.g. $\sqrt{\frac{2}{3}} = \sqrt{.666\cdots} = .8165.$

2. The denominator of the fraction may be made a perfect square before approximating the root.

E.g. $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3} = \frac{2.4495}{3} = .8165.$

By either of these two processes only one square root is approximated, and there is only a short division by 3 instead of a long division by 1.7321.

It is clear that any fraction can be changed into an equal fraction whose denominator is a perfect square by multiplying numerator and denominator by the proper number.

E.g. $\frac{1}{2} = \frac{2}{4}$, $\frac{3}{5} = \frac{15}{25}$, $\frac{7}{8} = \frac{14}{16}$, etc. If a fraction is given in the form $\frac{1}{\sqrt{3}}$, it may be written $\sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}} = \frac{1}{3}\sqrt{3}$. In like manner,

$$\frac{7}{\sqrt{11}} = 7 \cdot \frac{1}{\sqrt{11}} = 7\sqrt{\frac{1}{11}} = 7\sqrt{\frac{11}{121}} = \frac{7}{11}\sqrt{11}.$$

EXERCISES

Find approximately correct to two decimal places the following square roots.

In the first ten obtain the results in three different ways: (*a*) Find the root of each numerator and denominator separately; (*b*) reduce each fraction to a decimal; (*c*) reduce each fraction so as to make its denominator a perfect square.

In the remaining exercises use methods (*b*) and (*c*) only. In each case compare the results obtained.

1. $\sqrt{\frac{3}{5}}$.
2. $\sqrt{\frac{4}{8}}$.
3. $\sqrt{\frac{5}{7}}$.
4. $\sqrt{\frac{11}{13}}$.
5. $\sqrt{\frac{5}{3}}$.
6. $\sqrt{\frac{4}{5}}$.
7. $\sqrt{\frac{31}{8}}$.
8. $\sqrt{\frac{42}{51}}$.
9. $\sqrt{\frac{6}{7}}$.
10. $\sqrt{\frac{5}{6}}$.
11. $\sqrt{\frac{25}{4}}$.
12. $\sqrt{\frac{16}{7}}$.
13. $\sqrt{\frac{1}{2}}$.
14. $\sqrt{\frac{3}{7}}$.
15. $\sqrt{\frac{1}{7}}$.
16. $\sqrt{\frac{2}{5}}$.
17. $\frac{1}{\sqrt{5}}$.
18. $\frac{5}{\sqrt{13}}$.
19. $\frac{3}{\sqrt{7}}$.
20. $\frac{7}{\sqrt{17}}$.
21. $\sqrt{\frac{3}{8}}$.

172. Principle XVIII may be used to advantage in approximating the square roots of certain integral numbers.

E.g. suppose $\sqrt{2}$ has been computed, and $\sqrt{8}$ is desired. It is unnecessary to compute the $\sqrt{8}$ directly, for by XVIII,

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$$

This sort of simplification is possible whenever the number under the radical sign can be resolved into two factors, one of which is a perfect square.

E.g. suppose the $\sqrt{5}$ to have been computed, then

$$\sqrt{125} = \sqrt{25 \cdot 5} = \sqrt{25} \cdot \sqrt{5} = 5\sqrt{5}.$$

In like manner, $\sqrt{a^5b^3}$ may be written

$$\sqrt{a^4b^2 \cdot ab} = \sqrt{a^4b^2} \cdot \sqrt{ab} = a^2b\sqrt{ab}.$$

Definition. A radical expression is said to be **simplified** when the number under the radical sign is in the integral form and contains no factor which is a perfect square.

E.g. the simplified forms of $\sqrt{125}$, $\sqrt{a^5b^3}$, $\sqrt{\frac{1}{5}}$, $\frac{1}{\sqrt{3}}$, are respectively,

$$5\sqrt{5},\ a^2b\sqrt{ab},\ \tfrac{1}{5}\sqrt{5},\ \tfrac{1}{3}\sqrt{3}.$$

EXERCISES

Given $\sqrt{2} = 1.4142$, $\sqrt{3} = 1.7321$, $\sqrt{5} = 2.2361$, compute the following, correct to three places of decimals, without further extraction of roots:

1. $\sqrt{80}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $\sqrt{48}$.
5. $\sqrt{75}$.
6. $\sqrt{2 \cdot 3}$.
7. $\sqrt{72}$.
8. $\sqrt{98}$.
9. $\sqrt{363}$.
10. $\sqrt{125}$.
11. $\sqrt{27} + \sqrt{\frac{1}{3}}$.
12. $\sqrt{45} + \sqrt{\frac{1}{5}}$.
13. $\sqrt{50} - \sqrt{\frac{1}{2}} + \sqrt{8}$.
14. $\sqrt{48} + \sqrt{75} - \sqrt{3}$.
15. $\sqrt{32} + \sqrt{72} - \sqrt{18}$.

Simplify the following:

16. $\sqrt{32\,a^2b}$. **19.** $\sqrt{45x^3y^5b^3}$. **22.** $\sqrt{500\,x^7\,a^3b}$.

17. $\sqrt{81\,x^3b^2}$. **20.** $\sqrt{63\,bc^5d^4}$. **23.** $\sqrt{3\,x^2+6\,xy+3\,y^2}$.

18. $\sqrt{50\,a^5b^4c^2}$. **21.** $\sqrt{900\,ab^4c^5}$. **24.** $\sqrt{8\,x^2-12\,y^2}$.

25. $\sqrt{32\,a^2-64\,ab+32\,b^2}$. **26.** $\sqrt{125\,x^2+250\,xy+125\,y^2}$.

27. Find approximately to four decimal places the sides of a square whose area is 120.

28. Approximate to four decimals the side of a square having an area equal to that of a rectangle whose sides are 15 and 20.

29. How many rods of fence are required to fence a square piece of land containing 50 acres, each acre containing 160 square rods?

30. A square checkerboard has an area of 324 square inches. What are its dimensions?

173. In adding or subtracting expressions containing radicals it is always best to first reduce each radical expression to its simplest form, since this often gives opportunity to combine terms which are similar with respect to some radical expression.

Ex. 1. $\sqrt{32}+\sqrt{72}-\sqrt{18}=4\sqrt{2}+6\sqrt{2}-3\sqrt{2}=7\sqrt{2}$ by Principles XIII, I, and II.

Ex. 2. $\sqrt{\frac{1}{3}}+\sqrt{12}-\sqrt{\frac{3}{4}}=\frac{1}{3}\sqrt{3}+2\sqrt{3}-\frac{1}{2}\sqrt{3}$
$$=(\tfrac{1}{3}+2-\tfrac{1}{2})\sqrt{3}=1\tfrac{5}{6}\sqrt{3}.$$

EXERCISES

Simplify each of the following as far as possible without approximating roots.

1. $\sqrt{27}+2\sqrt{48}-3\sqrt{75}$.
2. $\sqrt{20}+\sqrt{125}-\sqrt{180}$.
3. $3\sqrt{432}-4\sqrt{3}+\sqrt{147}$.

4. $3\sqrt{2450}-25\sqrt{2}+4\sqrt{13122}.$

5. $3y^2\sqrt{x^3z}+2\sqrt{x^5z^3}-yz^4\sqrt{\frac{xz}{y^2}}.$

6. $\sqrt{4x^3y}+\sqrt{25xy^3}-x\sqrt{xy}.$

7. $\sqrt{ax^2-bx^2}+\sqrt{4ar^2s^2-4br^2s^2}.$

8. $4\sqrt{\frac{3}{4}}-\frac{3}{4}\sqrt{\frac{3}{16}}-3\sqrt{27}.$

9. $2\sqrt{\frac{5}{3}}+\sqrt{60}+\sqrt{\frac{3}{5}}.$

10. $5\sqrt{3}-2\sqrt{48}+7\sqrt{108}.$

11. $\sqrt{a^3-a^2b}-\sqrt{ab^2-b^3}-\sqrt{(a+b)(a^2-b^2)}.$

12. $\sqrt{a}+3\sqrt{2a}-2\sqrt{3a}+\sqrt{4a}-\sqrt{8a}+\sqrt{12a}.$

13. $\sqrt{x^3+2x^2y+xy^2}-\sqrt{x^3-2x^2y+xy^2}-\sqrt{4xy^2}.$

14. $\sqrt{r-s}+\sqrt{16r-16s}+\sqrt{rt^2-st^2}-\sqrt{9(r-s)}.$

15. $\sqrt{(m-n)^2a}+\sqrt{(m+n)^2a}-\sqrt{am^2}+\sqrt{a(1-m)^2}-\sqrt{a}.$

16. $\sqrt{32x^2y^4}+\sqrt{162x^2y^4}-\sqrt{512x^2y^4}+\sqrt{1250x^2y^4}.$

APPLICATIONS OF SQUARE ROOT

174. Some of the most interesting and useful applications of the square root process are concerned with the sides and areas of triangles.

The fact that the sum of the squares on the two sides of a right triangle equals the square on the hypotenuse was used in Chapter VI. (Pythagorean Proposition, page 206.)

If a and b are the lengths of the sides, and c the length of the hypotenuse, all measured in the same unit, this proposition says:

$$c^2=a^2+b^2. \tag{1}$$

Hence, by S,

$$a^2=c^2-b^2, \tag{2}$$

and

$$b^2=c^2-a^2. \tag{3}$$

Taking the square root of both sides in each of these equations,

$$c = \sqrt{a^2 + b^2}. \quad (4)$$

$$a = \sqrt{c^2 - b^2}. \quad (5)$$

$$b = \sqrt{c^2 - a^2}. \quad (6)$$

The negative square root is omitted here, as a negative length cannot apply to the side of a triangle. By these formulas, if any two sides of a right triangle are given, the other may be found.

E.g. if $a = 4$, $b = 3$, then, by (4),

$$c = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

If $c = 5$, $b = 3$, then, by (5),

$$a = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4.$$

If $c = 5$, $a = 4$, then, by (6),

$$b = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3.$$

Illustrative Problem. If the two sides of a right triangle are 8 and 12, find the hypotenuse correct to two decimal places.

Solution. We have $c = \sqrt{a^2 + b^2} = \sqrt{64 + 144} = \sqrt{208}$,

$$\sqrt{208} = \sqrt{16 \cdot 13} = \sqrt{16} \cdot \sqrt{13} = 4\sqrt{13} = 4(3.605) = 14.420.$$

PROBLEMS

In solving the following problems, simplify each expression under the radical sign before extracting the root. Find all results correct to two decimal places. In each case construct a figure.

1. The sides about the right angle of a right triangle are each 15 inches. Find the hypotenuse.

2. The hypotenuse of a right triangle is 9 inches and one of the sides is 6 inches. Find the other side.

3. The hypotenuse of a right triangle is 25 feet and one of the sides is 15 feet. Find the other side.

4. The hypotenuse of a right triangle is 7 rods and one of the sides is 5 rods. Find the other side.

5. The hypotenuse of a right triangle is 12 inches and the two sides are equal. Find their length.

Let s equal the length of one of the equal sides.

Then
$$s^2 + s^2 = 144.$$
$$2s^2 = 144.$$
$$s^2 = 72.$$
$$s = \sqrt{72} = 6\sqrt{2} = 6 \times 1.414 = 8.484.$$

6. The hypotenuse of a right triangle is 30 feet and the sides are equal. Find their length.

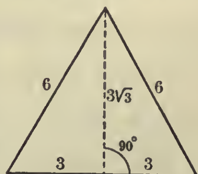
7. The hypotenuse of a right triangle is h and the sides are equal. Find their length. Solve 5 and 6 by means of the formula here obtained.

8. The diagonal of a square is 8 feet. Find its area.

9. The diagonal of a square is d. Find an expression in terms of d representing its area.

10. The side of an equilateral triangle is 6 inches. Find the altitude.

A line drawn from a vertex of an equilateral triangle perpendicular to the base meets the base at its middle point. Hence this problem becomes: the hypotenuse of a right triangle is 6 and one side is 3. Find the remaining side.



11. The side of an equilateral triangle is 10. Find the altitude.

12. The side of an equilateral triangle is s. Find the altitude.

This is equivalent to finding a side of a right triangle whose hypotenuse is s, the other side being $\frac{s}{2}$. Let a equal altitude.

Then
$$a = \sqrt{s^2 - \left(\frac{s}{2}\right)^2} = \sqrt{s^2 - \frac{s^2}{4}}$$
$$= \sqrt{\frac{4s^2 - s^2}{4}} = \sqrt{\frac{3s^2}{4}} = \sqrt{\frac{s^2}{4} \cdot 3}$$
$$= \sqrt{\frac{s^2}{4}} \cdot \sqrt{3} = \frac{s}{2}\sqrt{3}.$$

This formula gives the altitude of any equilateral triangle in terms of the side. By means of this formula solve 11 and 12. It is interesting to notice that the square root of 3 is the only root required in finding the altitude of any equilateral triangle whatever.

13. Find the altitude of an equilateral triangle whose side is $4\frac{1}{2}$. Substitute in the formula under 12.

14. Find the area of an equilateral triangle whose side is 5.

Since the area of a triangle is $\frac{1}{2}$ the product of the base and altitude, we first find the altitude by means of the formula under 12, and then multiply by $\frac{1}{2}$ the base.

15. Find the area of the equilateral triangle whose side is s. Show the result to be $\frac{s^2}{4}\sqrt{3}$.

16. If the area of an equilateral triangle is 16 square inches, find the length of the side.

Let s equal the length of the side. Then by the formula derived under 15, $16 = \frac{s^2}{4}\sqrt{3}$.

Hence (§§ 171, 172), $s^2 = \frac{64}{\sqrt{3}} = \frac{64}{3}\sqrt{3} = 21.33 \times 1.732$.

17. The area of an equilateral triangle is 50 square inches. Find its side and altitude.

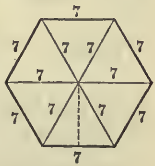
18. The area of an equilateral triangle is a square inches. Find the side.

Solve the equation $a = \frac{s^2}{4}\sqrt{3}$ for s, and simplify the expression,

finding $s^2 = \frac{4a}{\sqrt{3}}$, and $s = \sqrt{\frac{4a\sqrt{3}}{3}} = \frac{2}{3}\sqrt{3a\sqrt{3}}$.

19. The area of an equilateral triangle is 240 square inches. Find its side. (Substitute in the formula obtained under 18).

20. Find the area of a regular hexagon whose side is 7.



A regular hexagon is composed of 6 equal equilateral triangles, whose sides are each equal to the side of the hexagon (see figure). Hence this problem may be solved by finding the area of an equilateral triangle whose side is 7, and multiplying the result by 6.

21. Find the area of a regular hexagon whose side is s. (Solve 20 by substituting in the formula obtained here.)

22. The area of a regular hexagon is 108 square inches. Find its side.

If the area of the hexagon is 108 square inches, the area of one of the equilateral triangles is 18 square inches. Hence this problem can be solved like 18.

23. The area of a regular hexagon is a square inches. Find its side. (Solve 22 by substituting in the formula obtained here.)

24. Find the radius of a circle whose area is 9 square inches.

The area of a circle is found by squaring the radius and multiplying by 3.1416. The number 3.1416 is approximately the quotient obtained by dividing the length of the circumference by the diameter of the circle. This quotient is represented by the Greek letter π

(pronounced pi). In this chapter we use $3\frac{1}{7}$ as an approximation to π. This differs from the real value of π by less than .0013, and hence is accurate enough for most purposes. If a represents the area of a circle, the above rule may be written

$$a = \pi r^2.$$

Hence if $a = 9$, $$r^2 = \frac{9}{\pi} = \frac{9}{3\frac{1}{7}} = \frac{63}{22} = 2.863,$$

and $$r = \sqrt{2.863}.$$

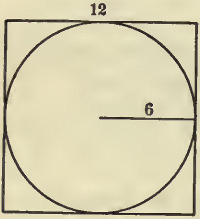
25. Find the radius of a circle whose area is 68 square feet.

26. Find the radius of a circle whose area is a square feet.

We have $$a = \pi r^2, \text{ or } r^2 = \frac{a}{\pi},$$

Hence $$r = \sqrt{\frac{a}{\pi}} = \sqrt{\frac{a\pi}{\pi^2}} = \frac{1}{\pi}\sqrt{a\pi}.$$

In problems stated in terms of letters, the results, of course, cannot be reduced to a decimal. In such formulas it is best not to replace the letter π by any of its approximations.



27. Find the sum of the areas of a circle of radius 6 and the square circumscribed about the circle.

The area of the circle is $6^2\pi = 36\pi$, and the area of the square is $4 \cdot 6^2 = 4 \cdot 36$; *i.e.* the square contains 4 squares whose sides are 6. The sum of the areas is

$$4 \cdot 36 + 36\pi = (4 + \pi)\,36 = (4 + 3\tfrac{1}{7})\,36.$$

28. Find an expression for the sum of the areas of a circle of radius r and the circumscribed square. (Solve 27 by substituting in the formula here obtained.)

29. If the sum of the areas of a circle and the circumscribed square is 64, find the radius of the circle.

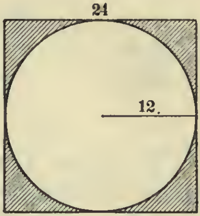
By the formula obtained under 28,

$$64 = (4 + \pi)\,r^2 = \tfrac{50}{7} r^2.$$
$$r^2 = 64 \cdot \tfrac{7}{50} = 8.96,$$

Hence, $$r = \sqrt{8.96} = 2.99.$$

30. If the sum of the areas of a circle and the circumscribed square is 640 square feet, find the radius of the circle.

31. The sum of the areas of a circle and the circumscribed square is a. Find an expression representing the radius of the circle. (Replace π by $3\frac{1}{7}$ before simplifying.)

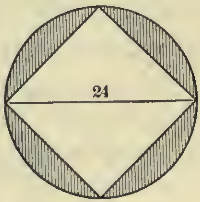


32. If the radius of a circle is 12, find the difference between the area of a circle and the circumscribed square.

33. If the radius of a circle is r, find the difference between the area of the circle and the circumscribed square. (Solve 32 by substituting in the formula obtained here.)

34. If the radius of a circle is 16, find the area of the inscribed square. (This is the same problem as finding the area of a square whose diagonal is 32. See problems 8 and 9.)

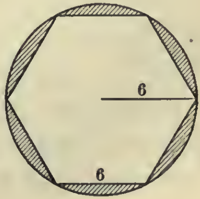
35. If the radius of a circle is r, find an expression representing the area of the inscribed square. (This is problem 9, the hypotenuse being $2r$.)



36. If the radius of a circle is 12, find the difference between the area of the circle and the area of the inscribed square.

37. If the radius of a circle is r, find an expression representing the differences between the areas of the circle and the inscribed square.

38. The radius of a circle is 10. Find the area of an inscribed hexagon. See note, problem 20.



39. The radius of a circle is 6. Find the difference between the areas of the circle and the inscribed hexagon.

40. Find an expression representing the difference between the areas of a circle with radius r and the inscribed regular hexagon.

SOLUTION OF QUADRATIC EQUATIONS BY MEANS OF SQUARE ROOT

175. Illustrative Problem. A rectangular field is 18 rods longer than it is wide and its area is 50 acres. What are its dimensions?

Let $x =$ the width of the field.

Then, $x + 18 =$ the length of the field.

And $x(x + 18) =$ the number of square rods in the field.

Hence, $$x^2 + 18x = 8000, \quad (1)$$

or $$x^2 + 18x - 8000 = 0. \quad (2)$$

We are not able to factor the left member of the equation by any method thus far known, and hence we cannot solve the equation as in § 145. We therefore proceed to study a *general method* of solving quadratic equations.

Consider the equation in the form (1) above. We seek a number k^2 such that $x^2 + 18x + k^2$ shall be a perfect square. By § 134 the middle term of a trinomial square is twice the product of the square roots of the two square terms. Hence, $18x = 2kx$, that is, k must equal 9.

Hence, adding $9^2 (= k^2)$ to each member of (1),

$$x^2 + 18x + 9^2 = 8000 + 9^2, \quad (3)$$

or, $$x^2 + 18x + 81 = 8081. \quad (4)$$

Since the left member is now a perfect square, we may extract the square root of both sides, approximating the root on the right.

Hence, $$x + 9 = \pm\sqrt{8081} = \pm 89.89,$$

giving $$x = -9 + 89.89 = 80.89,$$

and also, $$x = -9 - 89.89 = -98.98.$$

In this case the negative result is not applicable to the problem. Hence the width of the field is 80.89 rods, which is correct to two decimal places.

Illustrative Example. Solve the equation:

$$x^2 - 12x + 42 = 56. \quad (1)$$

By S, $x^2 - 12x = 14$. (2)

By A, $x^2 - 12x + k^2 = 14 + k^2$. (3)

By § 175, $-12x = 2kx$ or $k = -6$.

Hence, $x^2 - 12x + (-6)^2 = 14 + 36 = 50$. (4)

Taking square roots, $x - 6 = \pm\sqrt{50} = \pm 5\sqrt{2}$. (5)

By A, $x = 6 \pm 7.071$. (6)

Hence $x = 6 + 7.071 = 13.071$,

and also $x = 6 - 7.071 = -1.071$.

176. This process is called solving the quadratic equation by **completing the square,** since in each case a number is added to both sides which makes the left member a trinomial square.

Since the process always involves extracting the square root in order to find the value of the unknown, the two solutions of a quadratic equation are commonly called the **roots of the equation.** By analogy the solutions of any equation are sometimes called its roots.

EXERCISES

In solving the following quadratic equations the result may in each case be reduced so that the number remaining under the radical sign shall be 2, 3, or 5. (§ 172.) Use these square roots only.

1. $x^2 - 4x = 8$.
2. $x^2 = 3 - 6x$.
3. $4x = 16 - x^2$.
4. $x^2 + 6x = 9$.
5. $x^2 + 6x = 11$.
6. $x^2 - 12x = 12$.
7. $x^2 - 8x = -14$.
8. $x^2 = 2x + 1$.
9. $x^2 - 4x = 16$.
10. $2x = 14 + 4x$.
11. $24 = 3x^2 + 12x$.
12. $69 - 18x = 3x^2$.
13. $84 + 24x = 12x^2$.
14. $25 - x^2 = 5x$.
15. $x^2 + \frac{7}{3}x = 2$.
16. $x^2 - \frac{3}{2}x = \frac{27}{2}$.

177. In case the coefficient of x^2 is not unity, both members may be divided by this coefficient.

EXAMPLE. Solve $3x^2 + 8x = 4.$ (1)

By D, $x^2 + \frac{8}{3}x = \frac{4}{3}.$ (2)

By A, $x^2 + \frac{8}{3}x + k^2 = \frac{4}{3} + k^2.$ (3)

By § 175, $\frac{8}{3}x = 2kx$ or $k = \frac{4}{3}.$

Hence, $x^2 + \frac{8}{3}x + (\frac{4}{3})^2 = \frac{4}{3} + \frac{16}{9} = \frac{28}{9}.$ (4)

Taking square roots, $x + \frac{4}{3} = \pm\sqrt{\frac{28}{9}}.$ (5)

Hence, $x = -\frac{4}{3} \pm \frac{2}{3}\sqrt{7},$ (6)

and the two roots are $x = 0.43,$

and $x = -3.10.$

The preceding example may also be solved as follows:

Multiplying each member of (1) by $4 \cdot 3 = 12,$

then, $36x^2 + 96x = 48.$

By A, $36x^2 + 96x + k^2 = 48 + k^2,$

where $12kx = 96x$ or $k = 8.$

Hence, $36x^2 + 96x + (8)^2 = 48 + 64 = 112,$

and $6x + 8 = \pm\sqrt{112} = \pm 4\sqrt{7}.$

Therefore $x = -\frac{4}{3} \pm \frac{2}{3}\sqrt{7}.$

178. The advantage of this solution is that fractions are avoided until the last step, and the value of k is found to be the *same as the coefficient of* x in the given equation. This may always be accomplished by multiplying the members of the given equation by *four times the coefficient of* x^2.

In the solution of the following equations only the square roots $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, need be used. In all cases where the roots are integers or exact fractions the solution may also be obtained by factoring as in § 145.

1. $2x^2 + 3x = 2.$
2. $3x^2 + 5x = 2.$
3. $3x = 9 - 2x^2.$
4. $6x + 1 = -3x^2.$
5. $2x^2 = 5x + 3.$
6. $4x = 2x^2 - 1.$
7. $2x^2 - 3x = 14.$
8. $3x^2 = 9 + 2x.$
9. $4x^2 = 2x + 1.$

10. $6x - 1 = 3x^2$.
11. $2x^2 + 4x = 23$.
12. $3x^2 - 7 = 4x$.
13. $2x^2 - 5 = 3x$.
14. $4x^2 = 6x - 1$.
15. $2x = 1 - 5x^2$.
16. $3x - 20 = -2x^2$.
17. $2x + 3x^2 = 9$.
18. $4x^2 - 1 = 3x$.
19. $4x = 7 - 2x^2$.
20. $2x + 1 = 5x^2$.
21. $3x^2 + 4x = 7$.
22. $3x + 9 = 2x^2$.
23. $2x - 1 = -4x^2$.
24. $5x^2 + 16x = -2$.
25. $4x^2 + 1 = 8x$.
26. $2x^2 - 3x = 20$.
27. $2x^2 - 3 = -5x$.
28. $3x^2 + 4x = 8$.
29. $10 - 4x = 5x^2$.
30. $1 + 4x^2 = -6x$.
31. $5 - 3x = 2x^2$.
32. $7 + 4x = 2x^2$.
33. $6x^2 + 12x = 2$.
34. $6x^2 - 12x = -2$.
35. $6x^2 + 12x = -2$.
36. $6x^2 - 12x = 2$.
37. $3x^2 + 2x = 5$.
38. $2 + 3x = 2x^2$.
39. $8x + 1 = -4x^2$.
40. $8 + 4x = 3x^2$.
41. $10 + 4x = 5x^2$.
42. $2 + 5x = 3x^2$.
43. $3x - 14 = 2x^2$.
44. $3x^2 - 2x = 5$.
45. $2x^2 + 4x = 1$.
46. $3x - 1 = -4x^2$.
47. $23 + 4x = 2x^2$.
48. $3x - 1 = 2x^2$.

179. **Solution by Formula.** Solve the equation

$$ax^2 + bx + c = 0. \qquad (1)$$

By *S, M*, $4a^2x^2 + 4abx = -4ac$. (2)

By *A*, $4a^2x^2 + 4abx + k^2 = -4ac + k^2$, (3)

where $4akx = 4abx$ or $k = b$.

Hence $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$. (4)

Taking square roots, $2ax + b = \pm\sqrt{b^2 - 4ac}$. (5)

By *S, D*, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. (6)

Calling the two values of x in the result x_1 and x_2 we have,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Any quadratic equation may be reduced to the form of (1) by simplifying and collecting the coefficients of x^2 and x. Hence any quadratic equation may be solved by substituting in the formulas just obtained.

Ex. 1. Solve $x^2 - 4x + 1 = 0.$

In this case $a = 1, b = -4, c = 1.$

Hence $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}.$

From which $x_1 = 2 + \sqrt{3}$

and $x_2 = 2 - \sqrt{3}.$

Ex. 2. Solve $3x^2 + 16x - 12 = 0.$

Here $a = 3, b = 16, c = -12.$

Here $x = \frac{-16 \pm \sqrt{(16)^2 - 4 \cdot 3(-12)}}{2 \cdot 3}.$

From which $x_1 = \frac{2}{3}$, and $x_2 = -6.$

180. A quadratic equation may be proposed for solution which has no roots expressible in terms of the numbers of arithmetic or algebra so far as yet studied.

EXAMPLE. Solve $x^2 + 4x = -8.$ (1)

By A, $x^2 + 4x + 4 = -4.$ (2)

Taking square roots, $x + 2 = \pm\sqrt{-4}.$ (3)

$\sqrt{-4}$ is unknown to us as a number symbol, since there is no number thus far considered whose square equals -4. (See Principle XI.) Such symbols are defined and used in the Advanced Course, and are called **imaginary numbers.** For us any quadratic equation which gives rise to such a solution is to be interpreted as stating some impossible condition.

EXERCISES

1. $7 - 3x = 5x^2.$
2. $51x - 33 = 3x^2.$
3. $14x + 8 - x^2 = 52 - 3x^2.$
4. $12 - 51x = 36 + 6x^2.$
5. $5x + x^2 + 8 = 0.$
6. $5x^2 - 31x = -6.$

7. $x^2 - 8 + 3x = -15x$.
8. $11x^2 - 49x + 57 = 0$.
9. $3x^2 + 18 - 16x = 5$.
10. $37 - 4x^2 - 12x = 79 - 5x^2$.
11. $10x^2 + 41 + 7x = 44$.
12. $45 + 3x^2 - 85 - 2x = 0$.

MISCELLANEOUS QUADRATICS

Solve as many as possible of the following equations by factoring. When this is not convenient, use the formula of § 179, or complete the square independently in each case. Show which equations state impossible conditions. Approximate all square roots to two decimal places.

1. $x^2 + 11x = 210$.
2. $5x^2 - 3x = 4$.
3. $7x + 3x^2 - 18 = 0$.
4. $2 = 5x + 7x^2$.
5. $6x - 11x^2 = -7$.
6. $-51 + 42x - 3x^2 = 0$.
7. $3x^2 + 3x = 2x + 4$.
8. $13 - 8x + 3x^2 = 0$.
9. $2x^2 + 11x = 32x - x^2 - 27$.
10. $176 + 3x - x^2 = 2x$.
11. $x^2 + 6x - 54 = 0$.
12. $5x^2 + 9x + 12 = 4x^2 + x$.
13. $2x^2 - 4x - 25 = 0$.
14. $7x^2 + 11x = 6$.
15. $2x^2 - 11x + 5 = 0$.
16. $2x^2 - 11x = 6$.
17. $25x - 95 = x^2$.
18. $11x^2 - 42x = 2$.
19. $x^2 - 8x - 4 = x - 22$.
20. $8x^2 + 5x = -8$.
21. $2x^2 + 3x - 3 = 12x + 2$.
22. $3x^2 - 7x = 10$.
23. $17x + 31 + 2x^2 = 0$.
24. $18 - 41x = 3 + x^2$.
25. $10x + 25 = 5 - 2x - x^2$.
26. $3x - 59 + x^2 = 0$.
27. $5x^2 + 7x - 6 = 0$.
28. $x^2 + 12 = 7x$.
29. $8x - 5x^2 = 2$.
30. $5x + 3x^2 - 22 = 0$.
31. $50 + 20x + x^2 = 5x$.
32. $x^2 + x + 4 = 0$.
33. $20x + 2x^2 + 42 = 33x + x^2$.
34. $17x - 3x^2 = -6$.
35. $8x + 5x^2 = -2$.
36. $10 + 15x + x^2 = 26x$.
37. $3x^2 - 2x - 7 = 0$.
38. $5x^2 - 9x - 18 = 0$.
39. $7x - 7x^2 + 24 = 0$.
40. $31 + 2x + x^2 = 0$.
41. $7x^2 + 7x - 5x^2 + 20 = x^2 - 2x + 2$.

181. The solution of two equations in two variables, one of which is linear and the other quadratic, can be reduced to the solution of a quadratic equation in one variable. (See § 149.)

EXAMPLE. Solve
$$\begin{cases} x + y = 3, & (1) \\ 3x^2 - y^2 = 14. & (2) \end{cases}$$

From (1),
$$y = 3 - x. \qquad (3)$$

Substituting in (2) and reducing,
$$2x^2 + 6x - 23 = 0. \qquad (4)$$

Substituting in formula § 179, $x = \dfrac{-6 \pm \sqrt{36 - 4\cdot 2(-23)}}{4}$. (5)

Hence
$$x_1 = 2.21 \text{ and } x_2 = -5.21.$$

Substituting these values of x in (1) we have as the approximate roots,
$$\left.\begin{matrix} x_1 = 2.21 \\ y_1 = 0.79 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x_2 = -5.21 \\ y_2 = 8.21 \end{matrix}\right\}.$$

y_1 and y_2 are here used to designate the values of y which correspond to x_1 and x_2 respectively.

In this manner solve the following equations simultaneously, finding in each case two pairs of roots. In the case of roots which are neither integers nor exact fractions, find the approximate results to two places of decimals.

1. $\begin{cases} x - y = 1. \\ x^2 + y^2 = 13. \end{cases}$

2. $\begin{cases} x + y = 9. \\ x^2 + y^2 = 41. \end{cases}$

3. $\begin{cases} x + y = 13. \\ xy = 42. \end{cases}$

4. $\begin{cases} 3x - y = 5. \\ x^2 + y^2 = 25. \end{cases}$

5. $\begin{cases} x + 4y = 26. \\ x^2 - y^2 = 11. \end{cases}$

6. $\begin{cases} \dfrac{x^2}{9} - \dfrac{y^2}{4} = 3. \\ x - y = 4. \end{cases}$

7. $\begin{cases} x - y = 1. \\ \dfrac{x^2}{36} + \dfrac{y^2}{16} = \frac{1}{2}. \end{cases}$

8. $\begin{cases} 2x + y = 5. \\ 3x^2 - 5y^2 = 7. \end{cases}$

9. $\begin{cases} x - y = 3. \\ x^2 - 3y^2 = 13. \end{cases}$

10. $\begin{cases} x - 3y = 1. \\ y^2 + 2x^2 = 33. \end{cases}$

11. $\begin{cases} 3x - 4y = 1. \\ x^2 - y^2 = 24. \end{cases}$

12. $\begin{cases} x + y = 4. \\ 2x^2 - 3xy + y^2 = 8. \end{cases}$

13. $\begin{cases} x - y = 1. \\ 4x^2 + 2xy - y^2 = 19. \end{cases}$

14. $\begin{cases} 5x + y = 12. \\ 2x^2 - 3xy + y^2 = 0. \end{cases}$

15. $\begin{cases} x - 2y = 3. \\ 2y^2 - x^2 = 4. \end{cases}$

16. $\begin{cases} x + y = 9. \\ x^2 - 2y^2 = -7. \end{cases}$

17. $\begin{cases} y - 2x = 5. \\ y^2 - 3xy = 16. \end{cases}$

18. $\begin{cases} 2y - 3x = 0. \\ y^2 + x^2 = 52. \end{cases}$

19. $\begin{cases} y - 2x = 5. \\ x^2 + y^2 = 40. \end{cases}$

20. $\begin{cases} x - 4y = 12. \\ 3x^2 + 2xy - 6y = 44. \end{cases}$

21. $\begin{cases} y - 3x = 7. \\ 2x^2 - 3xy = 4. \end{cases}$

PROBLEMS

In each problem find the two roots of the quadratic equation and determine whether both are applicable to the problem:

1. The area of a window is 2016 square inches and the length of the frame is 180 inches. Find the dimensions of the window.

2. The area of a rectangular city block, including the sidewalk, is 19,200 square yards. The length of the sidewalk when measured on the side next the street is 560 yards. Find the dimensions of the block.

3. A farmer starts to plow around a rectangular field which contains 48 acres. The length of the first furrow is 376 rods. Find the dimensions of the field.

4. A rectangular blackboard contains 38 square feet and the length of the molding is 27 feet. Find the dimensions of the board.

5. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park, still keeping it rectangular, by adding strips of equal width to one end and one side. Find the width of the strips.

6. A fancy quilt is 72 inches long and 56 inches wide. It is decided to increase its area 10 square feet by adding a border. Find the width of the border.

7. A city block is 400 by 480 feet when measured to the outer edge of the sidewalk. At 4 cents per square foot it costs $416.64 to lay a sidewalk around the block. Find the width of the walk.

8. A farmer starts cutting grain around a field 120 rods long and 70 rods wide. How wide a strip must he cut to make 12 acres?

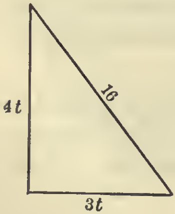
9. The sides of a right triangle are 6 and 8 inches respectively. How much must be added to each side so as to increase the hypotenuse 10 inches?

10. A rectangular lot is 16 by 12 rods. How wide a strip must be added to one end and one side to obtain a rectangular lot whose diagonal is 1 rod greater?

11. A picture is 15 inches by 20 inches. How wide a frame must be added to increase the diagonal 3 inches?

12. An athletic field is 800 feet long and 600 feet wide. The field is to be extended by the same amount in length and width so that the longest possible straight course (the diagonal) shall be increased by 100 feet. What will be the dimensions of the new field?

13. *A* starts north from a certain place going 4 miles per hour and *B* starts east from the same place at the same time going 3 miles per hour. In how many hours will they be 16 miles apart, the earth's surface being considered as a plane?



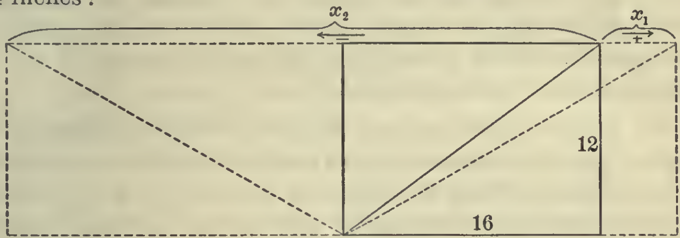
Let t equal the required number of hours.

Then $$(4t)^2 + (3t)^2 = 16^2 = 256.$$
$$16t^2 + 9t^2 = 256.$$
$$25t^2 = 256.$$
$$5t = \pm 16.$$
$$t = \pm 3\tfrac{1}{5}.$$

The solution $t = -3\frac{1}{5}$ is not applicable to this problem.

14. In the preceding problem if A goes 5 miles per hour and B 4 miles per hour, in how many hours will they be 24 miles apart?

15. A rectangle is 12 inches wide and 16 inches long. How much must be added to the length to increase the diagonal 4 inches?



Let x = number of inches to be added to the length. The diagonal of the original rectangle is $\sqrt{12^2 + 16^2} = 20$. Hence the diagonal of the required rectangle is 24.

Then $$12^2 + (16 + x)^2 = 24^2,$$
or $$x^2 + 32x - 176 = 0.$$
Solving, $$x_1 = -16 + 12\sqrt{3} = 4.78,$$
and $$x_2 = -16 - 12\sqrt{3} = -36.78.$$

The negative solution obtained here may be taken to mean that if the rectangle is extended in the *opposite* direction from the fixed corner, we shall get a rectangle which has the required diagonal. See the figure.

16. How much must the *width* of the rectangle in problem 15 be extended so as to increase the diagonal by 4?

17. A trunk 30 inches long is just large enough to permit an umbrella 36 inches long to lie diagonally on the bottom. How much must the length of the trunk be increased if it is to accommodate a gun 4 inches longer than the umbrella?

18. A rectangle is 21 inches long and 20 inches wide. The length of the rectangle is decreased twice as much as the width, thereby decreasing the length of the diagonal 4 inches. Find the dimensions of the new rectangle.

19. In a rectangular table cover 24 by 30 inches there are two strips of drawn work of equal width running at right angles through the center of the piece. What is the width of these strips if the drawn work covers one-tenth of the whole piece?

20. A certain university campus is 100 rods long and 80 rods wide. There are two driveways running through the center of the campus at right angles to each other and parallel to the sides. What is the width of these driveways if their combined area is 356 square rods?

21. A farm is 320 rods long and 280 rods wide. There is a road 2 rods wide running around the boundary of the farm and lying entirely within it. There is also a road 2 rods wide running across the farm parallel to the ends. What is the area of the farm exclusive of the roads?

22. A rectangular park is 480 rods long and 360 rods wide. A walk is laid out completely around the park and a drive through the length of the park parallel to the sides. What is the width of the walk if the drive is three times as wide as the walk and the combined area of the walk and the drive is 3110 square rods?

23. The sum of the sides of a right triangle is 18 and the length of the hypotenuse is 16. Find the length of each side.

24. The length of a fence around a rectangular athletic field is 1400 feet, and the longest straight track possible on the field is 500 feet. Find the dimensions of the field.

Using 100 feet for the unit of measure the equations are

$$\begin{cases} x + y = 7, \\ x^2 + y^2 = 25. \end{cases}$$

25. The difference between the sides of a right triangle is 8 and the hypotenuse is 42. Find the lengths of the sides.

26. A room is 5 feet longer than it is wide and the distance between two opposite corners is 25 feet. Find the length and width of the room.

27. One side of a right triangle is 8 feet, and the hypotenuse is 2 feet more than twice the other side. Find the length of its hypotenuse and of the remaining side.

28. A vacant corner lot has a 50-foot frontage on one street. What is the frontage on the other street if the distance between opposite corners along the diagonal is 110 feet less than twice this frontage.

In an old Chinese arithmetic said to have been written about 2600 B.C., we find the following two problems in each of which the square of the unknown occurs but cancels.

29. In the middle of a square pond whose sides are 10 feet there grows a reed which reaches 1 foot above the water. When the reed is bent over to the side of the pond, it just reaches the top of the water. How deep is the water?

30. A bamboo reed 10 feet high is broken off so that the top reaches the ground just three feet from its base. How far from the ground is the reed broken off?

31. The sum of the squares of two consecutive integers is 13,945. Find the numbers.

32. The product of two consecutive integers is 4422. Find the numbers.

33. A square piece of tin is made into an open box, containing 864 cubic inches, by cutting out a 6-inch square from each corner of the tin and then turning up the sides. Find the dimensions of the original piece of tin.

34. A rectangular piece of tin is 8 inches longer than it is wide. By cutting out a 7-inch square from each corner and turning up the sides, an open box containing 1260 cubic inches is formed. Find the dimensions of the original piece of tin.

35. By cutting out a square 8 inches on a side from each corner of a sheet of metal and turning up the sides, we obtain an open box such that the area of the sides and ends is 4 times the area of the bottom. Find the dimensions of the original sheet if it is twice as long as it is wide.

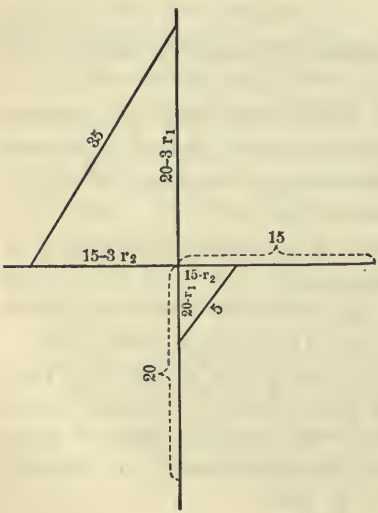
36. An open box whose bottom is a square has a lateral area which is 400 square inches more than the area of the bottom. Find the other dimensions of the box if it is 10 inches high. (By lateral area is meant the sum of the areas of the four sides.)

37. A box whose bottom is 4 times as long as it is wide has a lateral area 600 square inches less than 4 times the area of the bottom. Find the dimensions of the bottom if the box is 6 inches high.

38. A train approaching Chicago from the south at the rate of 50 miles per hour is 75 miles away when a train starts west from Chicago at the rate of 25 miles per hour. How long after the second train starts will they be 50 miles apart measured diagonally across the country?

If t is the number of hours required, then $(75-50t)^2+(25t)^2=(50)^2$. This may be written: $(25)^2\cdot(3-2t)^2+(25)^2\cdot t^2=2^2\cdot(25)^2$.

Hence dividing both members by $(25)^2$, we have $(3-2t)^2+t^2=4$.



39. An automobile running northward at the rate of 15 miles per hour is 20 miles south of the intersection with an east and west road. At the same time another automobile running westward on the cross-road at the rate of 20 miles per hour is 15 miles east of the crossing. How far apart (diagonally) will they be 15 minutes later? One hour later?

40. Under the conditions of problem 39 how long after the time specified will the automobiles be 10 miles apart? Is there more than one such position?

41. What are the rates of motion of the automobiles in problem 39 if one hour later they are 5 miles apart and 3 hours later they are 35 miles apart? (See the figure.)

If the rates of the automobiles are r_1 and r_2, then after 1 hour we have

$$(20-r_1)^2+(15-r_2)^2=5^2, \quad (1)$$

and after 3 hours we have $(20-3r_1)^2+(15-3r_2)^2=35^2$. (2)

Simplifying (1) and (2), $r_1^2+r_2^2-40r_1-30r_2+600=0$, (3)

and $9r_1^2+9r_2^2-120r_1-90r_2-600=0$. (4)

Multiplying (3) by 9, subtracting from (4), and simplifying,

$$4r_1+3r_2=100. \quad (5)$$

The solution of this problem may now be completed by solving (5) and (1) simultaneously.

NOTE. In equation (2), $20-3r_1$ and $15-3r_2$ are both negative numbers. This means that in this problem the distances north and west of the crossing are *negative*, while those south and east are *positive*.

REVIEW QUESTIONS

1. Define exponent. Explain the difference between an exponent and a coefficient.

2. Explain why and under what circumstances exponents are added in multiplication.

Show that $(a^2)^3=a^6$, $(a^3)^4=a^{12}$.

3. Explain the method of multiplying two monomials.

How is Principle III involved in Principle XV?

4. What is meant by factoring? Is the following expression factored? $x(a+b)+y(a+b)$. Why?

5. What are the characteristics of a trinomial square?

Are the following trinomials squares? If not, change one term in each so as to make it a square. x^2+xy+y^2; $x^4+x^2y^2+y^4$; $a^2-2ab-b^2$; $4a^2+4ab+4b^2$.

6. What are the factors of the difference of two squares? Factor $x^6 - y^6$ as the difference of two squares.

7. What are factors of the difference of two cubes? Factor $x^6 - y^6$ as the difference of two cubes.

8. What are the factors of the sum of two cubes? Factor $x^6 + y^6$ as the sum of two cubes.

9. Explain how to factor a trinomial by inspecting the end products and cross-products of two binomials.

By this method factor,

$$3x^2 - 7x - 10;\ x^2 - 9x + 18;\ 3x^2 + 5x - 12.$$

10. By means of the following examples explain the process of factoring by grouping.

$$x^2 + ax + bx + ab;\ x^3 - x - 3x^2 + 3.$$

11. How may a quadratic equation be solved by factoring? Why is it essential that one member of the equation be zero when the other member is factored?

12. Explain the method of solving the quadratic equation by completing the square.

13. How many roots has a quadratic equation? When does the solution of a quadratic equation indicate that the conditions of a problem are impossible?

14. Explain why and under what circumstances exponents are subtracted in division.

15. Explain the method of finding the quotient of two monomials. Show how Principle V is involved in Principle XVII.

16. State Principle XVIII. Given $\sqrt{7} = 2.645$, find $\sqrt{28}$ by means of this principle.

17. Explain how a number in Arabic figures is divided into groups for the purpose of finding its square root.

18. Show how the value of the following may be approximated by finding only one square root.

$$5\sqrt{20} + 2\sqrt{45} - 3\sqrt{80} + 2\sqrt{\tfrac{1}{5}}.$$

CHAPTER VIII

LITERAL FRACTIONS

COMMON FACTORS

182. If a number is a factor of each of two or more numbers, it is said to be a **common factor** of these numbers.

Thus, 8 is a common factor of 16 and 48, and 12 is a common factor of 12, 36, and 48.

If each of a given set of numbers is factored into prime factors, any common factor which they may have is at once apparent.

Illustrative Example. Find the common factors of 72, 96, 120, and 288.

Factoring, we have

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2.$$
$$96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 2^5 \cdot 3.$$
$$120 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 2^4 \cdot 3 \cdot 5.$$
$$288 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^5 \cdot 3^2.$$

The common **prime** factors are 2, 2, 2, and 3. The other common factors are the various combinations of these, namely, $2 \cdot 2 = 4$, $2 \cdot 2 \cdot 2 = 8$, $2 \cdot 3 = 6$, $4 \cdot 3 = 12$, and $8 \cdot 3 = 24$. 24 is, therefore, the **greatest common factor** of these numbers.

To find common factors of literal expressions we proceed in the same manner.

Illustrative Example. Find the common factors of $x + y$, $x^2 - y^2$, and $x^2 + 2xy + y^2$.

Factoring, we have

$$x + y = x + y.$$
$$x^2 - y^2 = (x + y)(x - y).$$
$$x^2 + 2xy + y^2 = (x + y)(x + y).$$

Hence, $x + y$ is the only common factor of these expressions.

Illustrative Example. Find the common factors of

$10x^2 + 20xy + 10y^2$, $5(x+y)(x^2-y^2)$, and $15(x+y)(x^3+y^3)$.

Factoring, $10x^2 + 20xy + 10y^2 = 2 \cdot 5(x+y)(x+y)$.

$$5(x+y)(x^2-y^2) = 5(x+y)(x+y)(x-y).$$

$$15(x+y)(x^3+y^3) = 3 \cdot 5(x+y)(x+y)(x^2-xy+y^2).$$

The common prime factors are 5, $x+y$, and $x+y$. The other factors, obtained by combining these, are $5(x+y)$, $(x+y)^2$, and $5(x+y)^2$. The last factor, $5(x^2+2xy+y^2)$, is called the highest common factor.

The name *highest* instead of *greatest* is used in algebra referring to the degree of the factor. Thus x^2 is of higher degree than x, although if $x = \frac{1}{2}$, x^2 is not greater than x.

183. **Definition.** In general that common factor which contains the greatest number of prime factors is the **highest common factor.** This is usually abbreviated to H. C. F. (See § 130.)

EXERCISES

Find the H. C. F. of the following sets of expressions:

1. $x-y$, x^2-y^2, $x^2-2xy+y^2$.
2. x^2+2x+1, $3x+6x^2+3x^3$.
3. x^2+4x+4, $x^2-6x-16$.
4. $x^2-8x+16$, $x^2+10x-56$.
5. a^3-b^3, $a^2-2ab+b^2$.
6. x^3+y^3, x^2-y^2, $x^2+2xy+y^2$.
7. $x^2-7x+12$, $ax-3a-bx+3b$.
8. $a^2-13a+42$, a^3-216, a^2-a-30.
9. $27+y^3$, $y^2+9y+18$, y^2-9.
10. $b^2+7b-30$, $b^2+11b-42$, b^2-b-6.

11. $a^3 + 2a^2 + a,\ a^2 + a,\ a^3 + 5a^2 + 4a.$

12. $x^3 + y^3,\ x^3 + x^2y + xy^2 + y^3.$

13. $x^4 + 3x^3 + 2x^2,\ x^3 + x^2,\ x^4 + 7x^3 + 6x^2.$

14. $x^2 - 11x + 30,\ xz - 5z + x^2 - 5x.$

15. $m^3 - n^3,\ 2x^2m^2 + 2x^2mn + 2x^2n^2.$

16. $x^2 - 1,\ x^3 - 1,\ x^2 - 13x + 12.$

17. $1 - 64x^3,\ 1 - 16x^2,\ 5 - 2z - 20x + 8xz.$

18. $1 + 125a^3,\ 1 + 10a + 25a^2,\ 1 - 25a^2.$

19. $ac - ax + 3bc - 3bx,\ a^3 + 27b^3.$

20. $5c - 2,\ 5ac + 20c - 2a - 8.$

21. $4x^4 - x^2,\ 2x^4 + x^3 - x^2,\ 2x^4 - 3x^3 + x^2.$

22. $3a^3 - 3a,\ 3a^3 - 6a^2 + 3a,\ 6a^3 + 12a^2 - 15a.$

23. $6x - 10xy + 4xy^2,\ 18x - 8xy^2,\ 54x - 16xy^3.$

24. $3x^5 + 9x^4 - 3x^3,\ 5x^2y^2 + 15xy^2 - 5y^2,\ 7ax^2 + 21ax - 7a.$

25. $18x^3 - 57x^2 + 30x,\ 9x^3 - 15x^2 + 6x,\ 18x^3 - 39x^2 + 18x.$

COMMON MULTIPLES

184. A number is said to be a **multiple of any of its factors.** In particular any number is a multiple of itself and of one.

Thus, 18 is a multiple of 1, 2, 3, 6, 9, and 18, but not of 12. $3a^2x^2$ is a multiple of 3, $3x$, $3x^2$, etc.

Since a multiple of a number is divisible by that number, it must contain as a factor every factor of that number.

E.g. 108 is a multiple of 54 and contains as factors all the factors of 54, namely 3, 3, 3, and 2, and also 2, 6, 9, 18, and 54.

Definition. A number is a **common multiple** of two or more numbers if it is a multiple of each of the numbers.

Thus, 18 is a common multiple of 6, 9 and 18. Evidently $3 \cdot 18$, $4 \cdot 18$, $5 \cdot 18$, etc. are also common multiples of 6, 9 and 18. Of all these common multiples 18 is the *smallest* and is called the *least* or *lowest* common multiple.

185. The process of finding the least common multiple of a set of numbers in Arabic figures is shown as follows:

Illustrative Example. Find the least common multiple of 16, 24, and 98.

Finding the prime factors of each number,

$$16 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$$
$$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3$$
$$98 = 2 \cdot 7 \cdot 7 \quad = 2 \cdot 7^2$$

Any multiple of 16 contains all its prime factors, namely, 2, 2, 2, and 2. Any common multiple of 16 and 24 contains in addition to the prime factors of 16 any prime factors of 24 not in 16, namely 3. Any common multiple of 16, 24, and 98 contains in addition to 2, 2, 2, 2 and 3 those prime factors of 98 not in 16 or 24, namely 7 and 7. Hence $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 \cdot 7 = 1176$ is a *common* multiple of 16, 24, and 98. Moreover, it is the *least common multiple* because no unnecessary factor has been included.

We proceed with a set of literal expressions in the same manner as above.

Illustrative Example. Find the L. C. M. of

$$x^2 - y^2;\ x^2 + 2xy + y^2;\ x^2 - 2xy + y^2.$$

Factoring,

$$x^2 - y^2 = (x - y)(x + y). \quad (1)$$
$$x^2 + 2xy + y^2 = (x + y)(x + y). \quad (2)$$
$$x^2 - 2xy + y^2 = (x - y)(x - y). \quad (3)$$

In order that an expression may be a multiple of (1) it must contain the factors $x - y$ and $x + y$. To be a common multiple of (1) and (2) it must contain a second factor $x + y$, giving $(x - y)$, $(x + y)$, $(x + y)$. To be a common multiple of (1), (2) and (3) it must contain a second factor $x - y$, giving $(x - y)$, $(x + y)$, $(x + y)$, $(x - y)$. The product thus found contains the fewest prime factors possible in order to be a common multiple of (1), (2), and (3). Hence $(x - y)(x + y)(x + y)(x - y) = (x - y)^2 (x + y)^2$ is called the lowest common multiple of (1), (2), and (3), since it is the common multiple of lowest degree.

In general, the process may be described as follows: to obtain the lowest common multiple of a set of expressions, factor each expression into prime factors; use all factors of the first expression together with those factors of the second which are not in the first, those of the third which are not in the first and second, etc. It is evident that in this manner we obtain a product which is a common multiple of the given expressions, but such that if any one of these factors is omitted, it will cease to be a multiple of some one of the expressions; that is, it will no longer be a common multiple of them *all*.

Thus, if in the example above either of the factors $x - y$ is omitted, the product will no longer be a multiple of $x^2 - 2xy + y^2$.

186. Definition. We now define the **lowest common multiple** of a set of expressions as that common multiple which contains the smallest number of prime factors. The lowest common multiple is usually abbreviated to L. C. M.

EXERCISES

Find the L. C. M. of the following expressions:

1. $2 \cdot 3 \cdot 4;\ 3 \cdot 7 \cdot 8;\ 2^3 \cdot 3 \cdot 4.$
2. $5x^2y^4,\ 10x^3y,\ 25x^2y.$
3. $2ab,\ 6a^2,\ 4b^2c.$
4. $x^2 - y^2,\ x^2 - 2xy + y^2.$
5. $x - y,\ x + y,\ x^2 - y^2.$
6. $4 - x^2,\ 2 - x,\ 2 + x.$
7. $a^2 + 2ab + b^2,\ a^2 - 2ab + b^2.$
8. $x^2 + 3x + 2,\ x^2 - 4,\ x^2 - 1.$
9. $25x^2 - 1,\ 125x^3 - 1.$
10. $2x^2 - 7x + 6,\ 4x^2 - 11x + 6.$
11. $x^3 - y^3,\ x - y,\ x^2 + xy + y^2.$
12. $x^3 - y^3,\ x^3 + y^3,\ x^2 - y^2.$
13. $5x^2 + 7x - 6,\ x^2 - 15x - 34.$
14. $x^3 + y^3,\ x^2 - y^2,\ (x - y)^2.$
15. $3abc,\ a^2 - 4ac + 4c^2,\ a - 2c.$
16. $x^2 - 1,\ x + 1,\ x^2 + 8x + 7.$
17. $4x^3y - 44x^2y + 120xy,\ 3a^3x^2 - 22a^3x + 35a^3.$
18. $x^2 + 2xy + y^2,\ 2ax^2 - 10ax + 12a.$

19. $3bx^2 - 21bx + 36b,\ x^2 - 5x + 4.$

20. $5a^2b^2 - 5a^2c^2,\ b^2 + 2bc + b + c + c^2.$

21. $15c^2ax^2 + 16c^2ax + c^2a,\ 2cax^2 + 10cax + 8ca.$

22. $ac - 4a + bc - 4b,\ ax + ac + bx + bc.$

23. $x^2 + 5x + 4,\ x^2 + 7x + 12,\ x^2 + 5x + 6.$

24. $x^2 + 2x + 1,\ x^2 - 2x + 1,\ x^2 - 1.$

REDUCTION OF FRACTIONS TO LOWEST TERMS

187. In arithmetic the denominator of a fraction is usually regarded as indicating the number of equal parts into which a unit is divided, while the numerator designates a certain number of these parts.

Thus, $\frac{2}{3}$ means 2 of the 3 equal parts of a unit.

However, a fraction such as $\dfrac{5}{3\frac{1}{2}}$ cannot be regarded in this way, since a unit cannot be divided into $3\frac{1}{2}$ equal parts. $\dfrac{5}{3\frac{1}{2}}$ indicates that 5 is to be divided by $3\frac{1}{2}$. *I.e.* $\dfrac{5}{3\frac{1}{2}} = 5 \div 3\frac{1}{2}$.

In algebra any fraction is usually regarded as an **indicated division** in which the numerator is the dividend and the denominator is the divisor.

Thus, $\dfrac{a}{b}$ is understood to mean $a \div b$.

The numerator and denominator are together called the **terms of the fraction.**

In case the numerator and denominator have common factors, these may be removed, without changing the value of the fraction, by means of Principle XVII, as applied in § 157.

Thus, $\dfrac{2 \cdot 3 \cdot 4 \cdot 5}{3 \cdot 7 \cdot 11} = \dfrac{2 \cdot 4 \cdot 5}{7 \cdot 11}$, where the common factor 3 is canceled.

Similarly $\frac{2^3 \cdot 3^2 \cdot 4x}{2^4 \cdot 3 \cdot 4^2} = \frac{3x}{2 \cdot 4}$, the factors 2^3, 3, and 4 being canceled; and $\frac{(x^2 - 7x + 12)}{(x^2 - 5x + 6)} = \frac{(x-3)(x-4)}{(x-2)(x-3)} = \frac{(x-4)}{(x-2)}$.

If the terms of a fraction have no common factor, the fraction is said to be in its **lowest terms**. Evidently the process of canceling common factors in the numerator and denominator may always be continued until the fraction is reduced to its lowest terms.

EXERCISES

Reduce the following fractions to lowest terms:

1. $\frac{3 \cdot 9^2 \cdot 2^6}{2^4 \cdot 5^3 \cdot 9^4}$.
2. $\frac{4a^4b^{12}c^3}{8a^3b^4c^4}$.
3. $\frac{x^2y^3z^4}{xy^2z^3}$.
4. $\frac{a^4b^3}{a^2b^2}$.
5. $\frac{x^2 + 2xy + y^2}{x^2 - y^2}$.
6. $\frac{x^2 + 7x - 30}{x^2 - 7x + 12}$.
7. $\frac{x^3 - y^3}{2x^2 - 3xy + y^2}$.
8. $\frac{64 - b^3}{16 - 8b + b^2}$.
9. $\frac{x^3 + 27z^3}{xy - 5x + 3yz - 15z}$.
10. $\frac{1 - 216c^3}{x - 4y - 6cx + 24cy}$.
11. $\frac{14bz - 2bx + ax - 7az}{x^2 - 49z^2}$.
12. $\frac{3a^2 - 29a + 56}{63 - 9a - 7m + ma}$.
13. $\frac{a(x-y)^3}{(x^2 - y^2)(x - y)}$.
14. $\frac{x^3 + 27}{4x^2 + 24x + 36}$.
15. $\frac{a^2 - 3a - 3b + ab}{(a^2 - b^2)(a - 3)}$.
16. $\frac{2^7 \cdot 3^5 \cdot 5^4 - 2^6 \cdot 3^4 \cdot 5^7}{2^4 \cdot 3^2 \cdot 5^5 - 2^5 \cdot 3^2 \cdot 5^2}$.
17. $\frac{5x^3y^4 - 12x^2y^5 + 7x^3y^2}{6x^5y^2 + 3x^2y^2}$.
18. $\frac{5c + 10b - bc - 2b^2}{8c^3 + 64b^3}$.
19. $\frac{4x^4 - 28x^3 + 48x^2}{2x^4 - 8x^3 + 6x^2}$.
20. $\frac{9a^2b^4 + 18a^2b^3c + 9a^2b^2c^2}{3ab^3 - 3abc^2}$.
21. $\frac{7xy^2 - 133xy + 126x}{15xy^2 - 36xy + 21x}$.

22. $\dfrac{20x^3+20x^2y+5xy^2}{60x^5-15x^3y^2}$.

23. $\dfrac{3ab^4-3ab^2c^2}{27a^3b^2+27a^3bc}$.

24. $\dfrac{4a^3-42a^2+20a}{2a^4b^6-20a^3b^6}$.

25. $\dfrac{(x-1)(x-2)(x-3)(x-4)}{(x-1)(x-3)(x-3)(x-4)}$.

26. $\dfrac{(x^2-y^2)(x^2+2xy+y^2)}{(x^2-2xy+y^2)(x+y)}$.

27. $\dfrac{(x^2-1)(x^2+1)(3x^2+3)}{3(x^4-1)}$.

28. $\dfrac{(3a^4-3a^2b^2)(a^2+13a+42)}{(a^2+3a+2)(a^2+5a-6)}$.

29. $\dfrac{2^3\cdot 4^3\cdot 5^4(x^2-b^2)(x^2+19x+90)}{2^2\cdot 4\cdot 5^3(x^2+9x-10)(x^2+10x+9)}$.

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR

188. Since we have just seen that a common factor may be *removed* from the numerator and denominator of a fraction without changing the value of the fraction, it is evident that a factor may be *introduced* into both terms without changing the value of the fraction.

Thus since $\dfrac{3\cdot 4}{3\cdot 5}$ is reduced to $\dfrac{4}{5}$ by removing the factor 3 from both numerator and denominator, so $\dfrac{4}{5}$ is changed to $\dfrac{3\cdot 4}{3\cdot 5}$ by introducing the factor 3 in both the terms. Likewise any other factor may be introduced. *E.g.* $\dfrac{4}{5}=\dfrac{4\cdot 7}{5\cdot 7}$, $\dfrac{a-b}{a+b}=\dfrac{(a-b)(a+b)}{(a+b)(a+b)}=\dfrac{a^2-b^2}{(a+b)^2}$.

In this manner any fraction may be changed into an equal fraction whose denominator is any given multiple of the denominator of the given fraction.

Thus, $\frac{3}{4}$ can be changed into a fraction whose denominator is 72 (a multiple of 4) by multiplying both terms by 18, *i.e.* $\dfrac{3}{4}=\dfrac{3\cdot 18}{4\cdot 18}$;

and $\frac{3a-2b}{x-y}$ can be changed into a fraction whose denominator is x^2-y^2 by multiplying both of its terms by $x+y$, *i.e.*,

$$\frac{3a-2b}{x-y}=\frac{(3a-2b)(x+y)}{x^2-y^2}.$$

Any two or more fractions may therefore be changed into respectively equal fractions which shall have a *common denominator*, namely, a common multiple of the denominators of the given fractions.

Illustrative Example. Reduce $\frac{x-1}{x+1}, \frac{x+1}{x-1}, \frac{2x+3}{x^2-1}$ to fractions having a common denominator.

The L.C.M. of the denominators is $(x-1)(x+1)$. Multiply the numerator and denominator of each fraction by an expression which will make the denominator of each new fraction $(x-1)(x+1)$.

Thus,
$$\frac{x-1}{x+1}=\frac{(x-1)(x-1)}{(x+1)(x-1)}=\frac{x^2-2x+1}{(x+1)(x-1)};$$
$$\frac{x+1}{x-1}=\frac{(x+1)(x+1)}{(x-1)(x+1)}=\frac{(x+1)^2}{(x+1)(x-1)};$$
$$\frac{2x+3}{x^2-1}=\frac{2x+3}{(x+1)(x-1)}.$$

It is best to *indicate* the multiplication in the common denominator, since this makes it more easily apparent by what expression the numerator and denominator of a fraction must be multiplied in order to reduce it to a fraction with the required denominator.

It should be noticed that there are three signs in connection with a fraction, the sign of the fraction itself, the sign of the numerator, and the sign of the denominator. It follows from the law of signs in division (Principle XII) that any two of these signs may be changed simultaneously without changing the value of the fraction.

E.g.
$$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b}.$$

This is useful in cases like the following:

Reduce $\frac{x+1}{1-x}$, $\frac{x}{x^2-1}$, and $\frac{1}{x+1}$ to fractions having a common denominator.

$$\frac{x+1}{1-x}=\frac{-x-1}{x-1}=\frac{(x+1)(-x-1)}{(x+1)(x-1)}=\frac{-x^2-2x-1}{x^2-1},$$

$$\frac{x}{x^2-1}=\frac{x}{x^2-1}, \text{ and } \frac{1}{x+1}=\frac{x-1}{(x+1)(x-1)}=\frac{x-1}{x^2-1}.$$

EXERCISES

Reduce each of the following sets of fractions to equivalent fractions having a common denominator.

1. $\frac{x+3}{x-y}$, $\frac{4}{x^2-2xy+y^2}$.

2. $\frac{a-1}{a^2-b^2}$, $\frac{a+1}{a^2+2ab+b^2}$.

3. $\frac{3-x}{x^2-9x+20}$, $\frac{x+4}{7x^2-26x-8}$.

4. $\frac{2}{3-x}$, $\frac{x-1}{x+1}$, $\frac{x+1}{x-3}$.

5. $\frac{a+b}{b-a}$, $\frac{a-b}{(a+b)^2}$, $\frac{a}{a^2-b^2}$.

6. $\frac{1}{x^2-3x-4}$, $\frac{1}{x^2+3x+2}$.

7. $\frac{a+1}{a^2-2ab+b^2}$, $\frac{a-1}{a^2+2ab+b^2}$.

8. $\frac{1}{a^3-b^3}$, $\frac{1}{b-a}$, $\frac{1}{a^2+ab+b^2}$.

9. $\frac{a}{5a^2-4a-12}$, $\frac{b}{a^2+4a-12}$, $\frac{c}{a-2}$.

10. $\frac{x-2}{x^2-5x-6}$, $\frac{x+2}{x^2+12x-108}$, $\frac{x-1}{x^2+19x+18}$.

In each of the following exercises reduce the given fractions to a common denominator and check by substituting a convenient number for each letter, taking care that no denominator becomes zero:

11. $\frac{mr}{m-1}, \frac{d}{1+n}$.

12. $\frac{Rr}{(R+r)(m-1)}, \frac{1}{R+r}$.

13. $\frac{CS}{R+RS}, \frac{Rr+Sr+SR}{RSs}$.

14. $\frac{a}{n-a}, \frac{b}{n-b}$.

15. $\frac{W(T-Q)}{w(Q-t)}, \frac{V}{w}$.

16. $\frac{x(T-t)}{W(Q-T)}, \frac{W(T-Q)}{w(Q-t)}$.

17. $\frac{P}{W(p-2t+1)}, \frac{d(V-w)}{W-w}$.

18. $\frac{V}{V-v}, \frac{V}{V+v}, \frac{1}{V^2-v^2}$.

19. $\frac{RA}{a-A}, \frac{1}{R+r}, \frac{1}{A-a}$.

20. $\frac{R}{x}, \frac{r}{y}, \frac{1}{x-y}, \frac{Rx}{x+y}$.

189. Since any number may be written as a fraction with the denominator 1, the above process may be used to reduce an integral expression to the form of a fraction having any desired denominator.

Thus, $3 = \frac{3 \cdot 5}{5}$; $x - y = \frac{(x-y)(x^2-1)}{x^2-1}$, etc.

It is sometimes convenient to reduce expressions, some of which are not fractions, to the form of fractions having a common denominator.

Illustrative Example. Reduce $5x$, $\frac{5x-1}{x^2-1}$, $\frac{2x-y}{x-1}$, to fractions having a common denominator. The lowest common denominator is x^2-1.

Thus, $$5x = \frac{5x(x^2-1)}{x^2-1} = \frac{5x^3-5x}{x^2-1},$$

$$\frac{5x-1}{x^2-1} = \frac{5x-1}{x^2-1},$$

$$\frac{2x-y}{x-1} = \frac{(2x-y)(x+1)}{(x-1)(x+1)} = \frac{2x^2+2x-yx-y}{x^2-1}.$$

EXERCISES

Reduce the following mixed expressions to fractions having a common denominator:

1. $5,\ x-y,\ \frac{5x-3}{x^2+2xy+y^2}$.

2. $\frac{3a-c}{x-y},\ \frac{2b-c}{x+y},\ 2c+2$.

3. $1+a+a^2,\ \frac{a+1}{a-1}$.

4. $x^2+xy+y^2,\ \frac{x+y}{x-y}$.

5. $x^2-xy+y^2,\ x^2-y^2,\ \frac{1}{x+y}$.

6. $x^4+x^2y^2+y^4,\ \frac{x}{x-y},\ \frac{y}{x+y}$.

7. $3a-2b-c,\ \frac{5}{a-b},\ \frac{2}{b-c}$.

8. $x^4-1,\ x^2-1,\ \frac{x+1}{x-1}$.

9. $x^2+2xy+y^2,\ \frac{1}{x+y},\ \frac{1}{1-x}$.

10. $x+y,\ x-y,\ \frac{x-y}{x^2+y^2},\ \frac{x+1}{x-y}$.

In each of the following reduce the expressions to the form of fractions having a common denominator and check the results by substituting convenient numbers for the letters:

11. $\frac{RA}{a-A},\ r$.

12. $\frac{RA}{a-A},\ R-r$.

13. $T,\ \frac{Q+t}{2}$.

14. $\frac{W(q-t)}{w},\ sT,\ \frac{s(t-q)}{2}$.

15. $H,\ \frac{hd}{D}$.

ADDITION AND SUBTRACTION OF FRACTIONS

190. Fractions having a common denominator may be added or subtracted, exactly as in arithmetic, by adding or subtracting the numerators and dividing the result by the common denominator.

For we have, by Principle VI, $\frac{16+20}{4}=\frac{16}{4}+\frac{20}{4}=4+5$. Likewise $\frac{3+5}{4}=\frac{3}{4}+\frac{5}{4}$, the division being indicated in this case. Hence, reading this identity from right to left, we have $\frac{3}{4}+\frac{5}{4}=\frac{3+5}{4}$.

Likewise $\frac{6}{x-y}-\frac{4}{x-y}=\frac{6-4}{x-y}=\frac{2}{x-y}$. In general,

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}, \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

If fractions which are to be added or subtracted do not have a common denominator, they must be reduced to this form

Example. Add $\frac{a-b}{a+b}$ and $\frac{a^2+2ab+b^2}{a^2-2ab+b^2}$.

Reducing the fractions to the common denominator

$$(a-b)(a-b)(a+b),$$

we have

$$\frac{a-b}{a+b}=\frac{(a-b)(a-b)(a-b)}{(a+b)(a-b)(a-b)}=\frac{a^3-3a^2b+3ab^2-b^3}{(a+b)(a-b)(a-b)},$$

and $$\frac{a^2+2ab+b^2}{a^2-2ab+b^2}=\frac{(a+b)(a^2+2ab+b^2)}{(a+b)(a-b)(a-b)}=\frac{a^3+3a^2b+3ab^2+b^3}{(a+b)(a-b)(a-b)}.$$

Adding the numerators, we have $2a^3+6ab^2$; whence the sum of the fractions is

$$\frac{2a^3+6ab^2}{(a+b)(a-b)(a-b)}.$$

EXERCISES

Perform the following additions and subtractions:

1. $\frac{3}{4}+\frac{2}{7}+\frac{a+b}{3a-b}.$

2. $\frac{x-y}{(x+y)^2}-\frac{x-y}{x^2-y^2}.$

3. $\frac{x^2-9x+18}{x^2-13x+36}+\frac{x}{4-x}.$

4. $\frac{2}{3}+\frac{a}{a+b}+\frac{b}{a-b}.$

5. $\frac{3}{2^3\cdot 3^2}+\frac{5}{2^2\cdot 3^4}-\frac{2}{2^4\cdot 3^3}.$

6. $\frac{a^2-9b^2}{a^2+6b+9b^2}-\frac{a^2-6ab}{a^2-9b^2}.$

7. $\frac{x+y}{x-y}+\frac{x-y}{x+y}-2.$

8. $\frac{2x^2}{x^2-y^2}-\frac{y}{x-y}-\frac{y}{x+y}.$

9. $\frac{a+1}{a^2+a+1}-\frac{a-1}{a^2-a+1}.$

10. $\frac{b}{1-b}+\frac{b}{1+b^2}-\frac{b^2}{1-b^2}.$

11. $\frac{x+1}{x-2}+\frac{x+1}{x+2}+\frac{3x+2}{x^2-4}.$

12. $\frac{x-1}{x+1}-\frac{x+1}{x-1}+\frac{x^2-5}{x^2-1}.$

13. $\frac{y^2}{y^2-1}+\frac{y}{y+1}-\frac{y}{1-y}.$

14. $\frac{1}{x}-\frac{1}{y}-\frac{1}{x-y}+\frac{-1}{x+y}.$

15. $\frac{1}{2}-\frac{b}{b-a}+\frac{2a^2}{b^2-a^2}+\frac{-a}{b+a}.$

16. $\frac{x^2+4xy}{x^3+y^3}+\frac{1}{x+y}-\frac{x}{x^2-xy+y^2}.$

17. $\frac{1}{1-x^3}+\frac{1}{1-x}-\frac{1}{1+x+x^2}.$

18. $\frac{a-3}{a^2-3a+2}-\frac{a-1}{a^2-5a+6}+\frac{a}{a^2-4a+3}.$

19. $\frac{x}{x^2-5x-14}+\frac{2}{x-7}-\frac{x}{x^2-9x+14}$.

20. $\frac{a}{ac+ad-bc-bd}-\frac{b}{a^2-2ab+b^2}$.

21. $\frac{2}{a-5}+\frac{39}{a^2+3a-40}+\frac{3}{a+8}$.

22. $\frac{1}{y^2+8y+16}-\frac{1}{y(y+4)}+\frac{4}{y^2(y+4)}$.

23. $\frac{x}{x^2+4x-60}+\frac{x}{x^2-4x-12}$.

24. $\frac{a-c}{a^2-c^2}+\frac{c-a}{a^2+2ac+c^2}-\frac{2}{a-c}$.

25. $\frac{9}{x^2+7x-18}-\frac{8}{x^2+6x-16}$.

26. $\frac{a+2}{a^2-a-6}+\frac{a-4}{a^2-7a+12}-\frac{a+2}{a^2-2a-8}$.

27. $\frac{2}{x^2-11x+30}-\frac{1}{x^2-36}+\frac{1}{x^2-25}$.

28. $\frac{1}{(x-1)(x+2)}+\frac{1}{(x+2)(x-3)}+\frac{1}{(x-1)(3-x)}$.

29. $\frac{1}{(a-b)(b-c)}-\frac{1}{(b-a)(c-d)}+\frac{1}{(b-c)(c-d)}$.

30. $\frac{4}{a-3}-\frac{a-1}{a^2+3a+9}+\frac{a^2-38a-3}{a^3-27}$.

MULTIPLICATION AND DIVISION OF FRACTIONS

191. The product of a fraction and an integer. In arithmetic the product of a fraction and an integer is obtained by multiplying the numerator of the fraction by the integer.

Thus, $\frac{2}{3} \times 4 = \frac{2 \cdot 4}{3}$ and $7 \times \frac{4}{5} = \frac{7 \cdot 4}{5}$.

Since in algebra a fraction is an indicated quotient, and since multiplying the dividend multiplies the quotient, it follows that in algebra also the product of a fraction and an integral expression is obtained by multiplying the numerator by the integral expression.

That is, in general, $\boldsymbol{a \cdot \frac{b}{c} = \frac{b \cdot a}{c} = \frac{ab}{c}}$.

It is best to factor completely the expressions to be multiplied and keep them in the factored form until all possible cancellations have been made.

EXERCISES

Find the following indicated products and reduce the fractions to the simplest form.

1. $(1 - a) \times \frac{1 + a + a^2}{a - 1}$.
2. $(x^3 - y^3) \times \frac{x + y}{x - y}$.
3. $(x^2 - 2\,xa + a^2) \times \frac{x + a}{x - a}$.
4. $\frac{3\,x - 1}{x^2 - 5\,x + 6} \times (x^2 - 11\,x + 18)$.
5. $\frac{a^2 - 4\,a - 3}{a^2 - 8\,a + 16} \times (a^2 - 5\,a + 4)$.
6. $(x^2 + 9\,x + 18) \times \frac{x - 5}{x^2 - 2\,x - 15}$.
7. $(1 - x^3) \times \frac{1 - x}{1 + x + x^2}$.
8. $(27\,a^3 - 1) \times \frac{a + 1}{9\,a^2 + 3\,a + 1}$.
9. $(a^2 + ab + b^2) \times \frac{a - b}{a^3 - b^3}$.
10. $(1 - a + a^2) \times \frac{a + 1}{a^3 + 1}$.

192. In arithmetic a fraction is divided by an integer by multiplying its denominator or dividing its numerator by the integer.

Thus, $\frac{1}{3} \div 7 = \frac{1}{3 \cdot 7}$; $\frac{14}{5} \div 7 = \frac{14 \div 7}{5} = \frac{2}{5}$.

Since multiplying the divisor or dividing the dividend divides the quotient, it follows that an algebraic fraction is divided by an integral expression by dividing the numerator or multiplying the denominator by the integral expression.

That is, in general, $\frac{a}{b} \div c = \frac{a}{bc} = \frac{a \div c}{b}$.

EXERCISES

Find the following indicated quotients and reduce the fractions to their lowest terms:

1. $\frac{x^3 - y^3}{x + y} \div (x^2 + xy + y^2)$.
2. $\frac{x^3 + y^3}{x - y} \div (x^2 - y^2)$.
3. $\frac{x^2 + 4x + 4}{x^2 - 2x + 1} \div (x^2 - 4)$.
4. $\frac{1 - 27x^3}{1 - 9x^2} \div (1 + 3x + 9x^2)$.
5. $\frac{x^2 + 2x - 35}{x^2 + 10x + 21} \div (x^2 - 4x - 5)$.
6. $\frac{x^2 - 16x + 39}{x^2 - 8x + 15} \div (x^2 - x - 156)$.
7. $\frac{ac + bc - ad - bd}{xy - 4x - 3y + 12} \div (cx - 3c - dx + 3d)$.
8. $\frac{x^2 + ax + bx + ab}{x^2 + ax - 3x - 3b} \div (x^2 + ax - 5x - 5a)$.
9. $\frac{mr + ms - nr - ns}{mx - m - nx + n} \div (3r - xr + 3s - xs)$.
10. $\frac{x^2 - 3x - 88}{x^2 - 9x - 22} \div (x^2 + 9x + 2)$.

TO MULTIPLY A FRACTION BY A FRACTION

193. In arithmetic, to multiply a number by the quotient of two numbers is the same as to multiply by the dividend and then divide the product by the divisor.

E.g. $10 \cdot \frac{18}{3} = 10 \cdot 6 = 60$; or, $10 \cdot \frac{18}{3} = (10 \cdot 18) \div 3 = 60.$

Likewise, $\frac{2}{3} \cdot \frac{5}{7} = \left(\frac{2}{3} \cdot 5\right) \div 7 = \frac{2 \cdot 5}{3} \div 7 = \frac{2 \cdot 5}{3 \cdot 7}.$

Since in algebra a fraction is an indicated quotient, to multiply a number by an algebraic fraction we multiply by the numerator and divide the product by the denominator.

Thus, $$\frac{a}{b} \cdot \frac{c}{d} = \left(\frac{a}{b} \cdot c\right) \div d = \frac{ac}{b} \div d = \frac{ac}{bd}.$$

Hence, the product of two algebraic fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.

Illustrative Example. Multiply $\frac{x^2 - 1}{x^2 - 7x + 10}$ by $\frac{x^2 - 3x + 2}{x^2 + 2x + 1}$ and reduce the resulting fraction to its lowest terms.

$$\frac{x^2 - 1}{x^2 - 7x + 10} \times \frac{x^2 - 3x + 2}{x^2 + 2x + 1} = \frac{(x-1)(x+1)(x-2)(x-1)}{(x-2)(x-5)(x+1)(x+1)}$$

$$= \frac{(x-1)(x-1)}{(x-5)(x+1)} = \frac{x^2 - 2x + 1}{x^2 - 4x - 5}.$$

It is desirable to resolve each numerator and denominator into prime factors, and then cancel all common factors before performing any multiplication.

EXERCISES

Find the following indicated products and reduce each fraction to its lowest terms:

1. $\frac{3x^2y^2}{2yz^2} \times \frac{6az}{9x^3}$.

2. $\frac{5a(a-b)}{3c(a+b)} \times \frac{9(a+b)^2}{15(a^2-b^2)}$.

3. $\frac{12c^3b}{5(c^3-b^3)} \times \frac{35(c^2+cb+b^2)}{14c^2b^2}$.

4. $\frac{y^2+3y+2}{y^2-5y+6} \times \frac{y^2-7y+12}{y^2+8y+7}$.

5. $\frac{3^2 \cdot 4^3}{5^2 \cdot 2^4} \times \frac{10 \cdot 2}{3^4}$.

6. $\frac{3^2 \cdot 2^3}{5^2} \times \frac{5^4 \cdot 7^2}{3^4 \cdot 2^3} \times \frac{6 \cdot 3^2}{5^4 \cdot 7^3}$.

7. $\frac{x^2-x}{x^2-1} \times \frac{2x^2+4x+2}{3x^2+6x}$.

8. $\frac{a^2-10a+16}{a^2+6a+9} \times \frac{a+3}{a^2-4}$.

9. $\frac{(x+y)^2-z^2}{x^2+xy-xz} \times \frac{x}{(x-z)^2-y^2} \times \frac{(x-y)^2-z^2}{xy-y^2-yz}$.

10. $\frac{a^2+7a+12}{a^3+5a^2+6a} \times \frac{3a^3+27a^2+42a}{6a^2+66a+168}$.

11. $\frac{3^4(a-5)(a+2)}{2^3(a-3)(a-2)} \times \frac{2^2(a-3)(a-5)}{3^3(a-5)^2}$.

12. $\frac{3(x+4)^2}{4(x+4)(x-7)} \times \frac{(x-7)^2}{3(x+4)(x-7)}$.

13. $\frac{a^3+5a^2-36a}{a^2-7a-144} \times \frac{(a-16)(a-3)}{a(a-4)(a+2)}$.

14. $\frac{3a(a-7)(a-5)}{7b(a-3)(a-7)} \times \frac{b(a-3)(a+10)}{a(a-5)(a-10)}$.

15. $\frac{42(b-3)(b-4)}{3(b-4)(b+7)} \times \frac{6(b+7)(b-5)}{14(b-3)(b-6)}$.

16. $\frac{a(b^3-a^3)}{b(b+a)} \times \frac{(b^2-a^2)^2}{b^2+ba+a^2} \times \frac{(b+a)^2}{(b-a)^2}$.

17. $\frac{a^2-4a+3}{a^2-5a+4}\times\frac{a^2-9a+20}{a^2-10a+21}\times\frac{a^2-7a}{a^2-5a}.$

18. $\frac{3a(a-7)(a-5)}{7b(a-3)(a-7)}\times\frac{b(a-3)(a+10)}{a(a-5)(a-10)}.$

19. $\frac{a^2-12a+35}{c(a^2-10a+21)}\times\frac{a^2-8a+15}{a^2+3a-70}.$

20. $\frac{c^2-10c+21}{c^2-18c+77}\times\frac{c^2-3c-88}{c^2-8c+15}.$

21. $\frac{x^2(x^2-16)}{y^2(y^2-3y-28)}\times\frac{y^2-12y+35}{x^2-9x+20}\times\frac{y^2(x+5)}{x^3(x-4)}.$

22. $\frac{3a^2b^2(c^2-14c+33)}{4ab(c^2-10c+21)}\times\frac{2(c^2+c-56)}{a^2(c^2-16c+55)}\times\frac{c^2-2c-15}{b^2(c^2+12c+32)}.$

23. $\frac{3t^2-2t-1}{2t^2+t-1}\times\frac{2t^2+5t-3}{3t^2+7t+2}\times\frac{4t^2+10t+4}{4t^2-2t-2}.$

24. $\frac{6x^2-7x+2}{10x^2-7x+1}\times\frac{6x^2-5x-1}{6x^2+x-1}\times\frac{10x^2+3x-1}{5x^2-4x-1}.$

25. $\frac{4b^2-17b+4}{6b^2-7b+2}\times\frac{10b^2-21b+9}{5b^2-23b+12}\times\frac{3b^2-5b+2}{4b^2-5b+1}.$

TO DIVIDE A FRACTION BY A FRACTION

194. In arithmetic, to divide a number by the quotient of two numbers is the same as to divide by the dividend and multiply the result by the divisor.

E.g. $$72\div\frac{18}{3}=72\div 6=12;$$

or, $$72\div\frac{18}{3}=(72\div 18)\cdot 3=4\cdot 3=12.$$

Likewise, $$\frac{2}{3}\div\frac{5}{7}=\left(\frac{2}{3}\div 5\right)\cdot 7=\left(\frac{2}{3\cdot 5}\right)\cdot 7=\frac{2\cdot 7}{3\cdot 5}.$$

Since in algebra a fraction is an indicated quotient, to divide a number by an algebraic fraction we divide by the numerator and multiply this result by the denominator.

E.g. $$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \div c\right) \times d = \left(\frac{a}{b \cdot c}\right) \times d = \frac{ad}{bc}.$$

Hence, in algebra, as in arithmetic, a number is divided by a fraction by inverting the fraction and multiplying by the new fraction thus obtained.

Thus, $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

EXERCISES

Perform the following indicated divisions, and reduce the resulting fractions to their lowest terms:

1. $\frac{a^3+b^3}{a^2-9b^2} \div \frac{a+b}{a+3b}$.

2. $\frac{x^2+x-2}{x^2-3x} \div \frac{x^3+2x^2}{x^2+9x-36}$.

3. $\frac{x^2-6x-16}{x^2+4x-21} \div \frac{x^2+9x+14}{x^2-8x+15}$.

4. $\frac{x^2-1}{x^2-4x-5} \div \frac{x^2+2x-3}{x^2-25}$.

5. $\frac{a^2-11a-26}{a^2-3a-18} \div \frac{a^2-18a+65}{a^2-9a+18}$.

6. $\frac{x^2+9xy+18y^2}{x^2-9xy+20y^2} \div \frac{x^2+6xy+9y^2}{xy^2-4y^3}$.

7. $\frac{x^2+mx+nx+mn}{x^2-mx-nx+mn} \div \frac{x^2-n^2}{x^2-m^2}$.

8. $\frac{3a^4-9a^3-54a^2}{9a^3-117a^2+378a} \div \frac{a^3+8a^2+15a}{3a^2-33a+84}$.

9. $\frac{a^2-11a+30}{a^3-6a^2+9a} \times \frac{a^2-3a}{a^2-25} \div \frac{a^2-9}{a^2+2a-15}$.

10. $\dfrac{x^2-10x+21}{x^2+x-56} \div \dfrac{x^2-8x+15}{x^2+4x-32}$.

11. $\dfrac{a^2-3a+2}{a^2-7a+12} \times \dfrac{a^2-8a+15}{a^2-6a+5} \div \dfrac{a^2-9a+14}{a^2-12a+32}$.

12. $\dfrac{n^2-11n+18}{m^2-7m-18} \times \dfrac{n^2-8n-9}{m^2-5m-14} \div \dfrac{6n-12}{an+a}$.

13. $\dfrac{a^2-b^2}{ab^2x} \times \dfrac{b(a-b)}{a^2+2ab+b^2} \div \dfrac{b(a+b)}{a^2-2ab+b^2}$.

14. $\dfrac{a+b}{ab} \times \dfrac{a^2-b^2}{3(a^2+b^2)} \div \dfrac{(a-b)^2(a+b)^2}{3a^3y+3ay^3}$.

15. $\dfrac{5x^4-5x^3}{7x^2-56x-63} \div \dfrac{x^4-9x^3+8x^2}{14x^2+14x-1260}$.

16. $\dfrac{8y^2(y+4)(y+5)}{2^4(y+5)(y-7)} \div \dfrac{y(y+4)(y+8)}{2^3(y-7)(y+11)}$.

17. $\dfrac{(c+4)(c-4)}{(c-3)(c-2)} \times \dfrac{(c+2)(c-5)}{(c-5)(c-6)} \div \dfrac{(c+4)(c-13)}{(c-6)(c+11)}$.

18. $\dfrac{2a^2b(b-4)(b+4)}{b(b+4)(b+6)} \times \dfrac{(b+6)(b+8)}{(b+8)(b-9)} \div \dfrac{a(b+8)(b-4)}{(b-9)(b-5)}$.

19. $\dfrac{x^3(x-2)^2}{(x+2)^2} \times \dfrac{(x+2)(x-3)}{(x-2)(x-7)} \div \dfrac{x^2(x-3)(x-5)}{(x-5)(x-7)}$.

20. $\dfrac{a^2b^2(c+5)(c-4)}{(c-4)(c-8)} \times \dfrac{(c-8)(c+9)}{(c+4)(c+7)} \div \dfrac{ab(c+9)(c-1)}{(c+7)(c+1)}$.

21. $\dfrac{21x^2+23x-20}{10x^2-27x+5} \times \dfrac{6x^2-11x-10}{3x^2+2x-5} \div \dfrac{7x^2+17x-12}{5x^2+9x-2}$.

22. $\dfrac{x^2+6x+9}{9x^2-4}\times\dfrac{9x^2+9x-10}{7x^2+20x-3}\div\dfrac{3x^2-7x-20}{35x^2-12x+1}$.

23. $\dfrac{(a+4)^2-b^2}{12x^3-16x^2}\times\dfrac{3x-4}{ax^2+4x^2-bx^2}\div\dfrac{ab^2+4b^2+b^3}{x^3}$.

24. $\dfrac{12yx^2-18yx}{15x^2+32x-7}\times\dfrac{3bx^2+7bx}{6xy+15y}\div\dfrac{3bx^3+7bx^2}{10x^2+23x-5}$.

25. $\dfrac{ac+bc-ar-br}{3a^2-8a-3}\times\dfrac{6a^2+23a+7}{cb-rb+c^2-rc}\div\dfrac{3a^2+a+b+3ab}{ab-3b+ac-3c}$.

COMPLEX FRACTIONS

195. Sometimes fractions occur whose numerators or denominators, or both, contain fractions.

E.g.
$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}} \text{ and } \frac{\frac{1}{x+1}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x+1}}.$$

Such fractions are called **complex fractions.** A complex fraction is said to be **simplified** when it is reduced to an equal fraction whose numerator and denominator are in the integral form.

Ex. 1.
$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}}=\frac{\frac{a}{a}+\frac{1}{a}}{\frac{a}{a}-\frac{1}{a}}=\frac{a+1}{a}\div\frac{a-1}{a}$$
$$=\frac{a+1}{a}\times\frac{a}{a-1}=\frac{a+1}{a-1}.$$

This result may also be obtained directly by multiplying both terms of the given fraction by a, finding at once $\frac{a+1}{a-1}$.

Ex. 2. $$\frac{\dfrac{1}{x+1}+\dfrac{1}{x-1}}{\dfrac{1}{x-1}-\dfrac{1}{x+1}}=\frac{\dfrac{x-1+x+1}{x^2-1}}{\dfrac{x+1-x+1}{x^2-1}}$$

$$=\frac{2x}{x^2-1}\times\frac{x^2-1}{2}=x.$$

By multiplying the terms of the given fraction by $(x+1)(x-1)$ we may also get directly $\frac{x-1+x+1}{x+1-x+1}=x.$

Such fractions seldom occur in the problems of elementary algebra. A more extended discussion is given in the Advanced Course.

EXERCISES

Reduce each of the following complex fractions to its simplest form:

1. $\dfrac{1+x}{1+\dfrac{1}{x}}.$

2. $\dfrac{1-\dfrac{a}{b}}{1+a}.$

3. $\dfrac{x+\dfrac{x}{2}}{x-\dfrac{x}{2}}.$

4. $\dfrac{\dfrac{m+n+1}{3}}{\dfrac{m-n-1}{2}}.$

5. $\dfrac{4+\dfrac{a+b}{2}}{4-\dfrac{a-b}{2}}.$

6. $\dfrac{\dfrac{x^2-y^2}{4}}{\dfrac{x+y}{2}}.$

7. $\dfrac{\dfrac{a^3-8b^3}{27}}{3a-2b}.$

8. $\dfrac{\dfrac{M}{D^3}}{\dfrac{1+d^2}{D^2}}.$

9. $\dfrac{\dfrac{H}{c}-\dfrac{hd}{cD}}{\dfrac{1+t}{c}}.$

RATIO AND PROPORTION

196. Definitions. A fraction is often called a **ratio**. Thus $\frac{a}{b}$ may be read *the ratio of a to b*, and is also written $a : b$.

The numerator is called the **antecedent** of the ratio, and the denominator the **consequent**. The antecedent and consequent are called the **terms** of the ratio.

An equation, each of whose members is a ratio, is called a **proportion**.

Thus, $\frac{a}{b} = \frac{c}{d}$ is a proportion, and is also written $a : b = c : d$.

It is read *the ratio of a to b equals the ratio of c to d*, or briefly, *a is to b as c is to d*.

The four numbers a, b, c, and d, are said to be **in proportion**. a and d are called the **extremes** of the proportion, and b and c the **means**.

IMPORTANT PROPERTIES OF A PROPORTION

1. If, in the proportion $\frac{a}{b} = \frac{c}{d}$, both members of the equation be multiplied by bd, we have, $ad = bc$.

That is: *If four numbers are in proportion, the product of the means equals the product of the extremes.*

2. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

Hint. Divide $1 = 1$ by the members of the given equation.

This process is called taking the proportion by **inversion**.

3. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Hint. Multiply both members of the given equation by $\frac{b}{c}$.

This process is called taking the proportion by **alternation**.

4. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

Hint. Add 1 to both members of the given equation.

This process is called taking the proportion by **composition.**

5. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$.

Hint. Subtract 1 from each member of the given equation.

This process is called taking the proportion by **division.**

6. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Hint. Divide the members of the equation obtained under 4 by the members of the one obtained under 5.

7. Show that if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$; then $a=bk,\ c=dk,\ e=fk$.

Hence, $a+c+e=bk+dk+fk=(b+d+f)k$,

and $$\frac{a+c+e}{b+d+f}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$$

That is, *If several ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

EXERCISES

1. If $ad=bc$, show that $\frac{a}{b}=\frac{c}{d}$. *Hint.* Divide by bd.

2. If $ad=bc$, show that $\frac{a}{c}=\frac{b}{d}$.

3. If $ad=bc$, show that $\frac{d}{c}=\frac{b}{a}$.

4. If $ad = bc$, show that $\frac{d}{b} = \frac{c}{a}$.

5. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{a} = \frac{c+d}{c}$.

6. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a} = \frac{c-d}{c}$.

7. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a+b} = \frac{c-d}{c+d}$.

8. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a-b}{c-d}$.

9. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a}{c}$.

10. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+c}{b+d} = \frac{a}{b}$.

11. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{c-d} = \frac{a}{c}$.

12. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-c}{b-d} = \frac{a}{b}$.

13. Solve the equation $\frac{a}{b} = \frac{c}{d}$ for each letter in terms of all the others. If $a = 3$, $b = 5$, $c = 8$, find d. If $b = 7$, $c = 9$, $d = 3$, find a. If $c = 13$, $d = 2$, $a = 5$, find b. If $d = 50$, $a = 3$, $b = -7$, find c.

14. If $\frac{a}{b} = \frac{c}{x}$, then x is said to be a **fourth proportional** to a, b, and c.

Find a fourth proportional to 3, 5, and 7; also to 9, 5, and 1, and to 3, -2, and -5.

15. If $\frac{a}{x}=\frac{x}{b}$, then x is called a **mean proportional** between a and b.

Solve the equation $\frac{a}{x}=\frac{x}{b}$ for x in terms of a and b. Show that there are two solutions, each of which is a mean proportional between a and b.

Find two mean proportionals between 4 and 9; also between 5 and 125, and between -4 and -36.

16. Which is the greater ratio $\frac{5+3d}{5+4d}$ or $\frac{5+4d}{5+5d}$.

Hint. Reduce the fractions to a common denominator and compare numerators. (d is a positive number.)

17. Which is the greater ratio, $\frac{a+7b}{a+8b}$ or $\frac{a+9b}{a+10b}$.

18. Which is the greater ratio, $\frac{a}{b}$ or $\frac{a+c}{b+c}$, if b and c are positive, and a less than b? a equal to b? a greater than b?

19. Find two numbers in the ratio of 3 to 5 whose sum is 160.

20. Find two numbers in the ratio of 2 to 7 whose sum is -108.

21. Find two numbers in the ratio of 3 to -4 whose sum is -15.

22. What number added to each of the terms of the ratio $\frac{5}{7}$ makes it equal to $\frac{65}{75}$?

23. What number must be added to each term of the ratio $\frac{1}{11}$ to make it equal to the ratio $\frac{3}{5}$?

24. What number added to each of the numbers 3, 5, 7, 10, will make the sums in proportion, when taken in the given order?

25. Two numbers are in the ratio of 2 to 3, and the sum of their squares is 325. Find the numbers.

EQUATIONS INVOLVING FRACTIONS

197. We have already seen, § 40, that in solving an equation involving fractions the first step is to multiply both members of the equation by a number which will cancel all the denominators. Evidently, any common multiple of the denominators is such a multiplier. For the sake of simplicity, and for reasons which are fully discussed in the Advanced Course, the *lowest common multiple is always used for this purpose.*

Illustrative Example. Solve the equation:

$$\frac{x}{4}+\frac{x-6}{6}-\frac{1-2x}{8}=\frac{55}{8}. \qquad (1)$$

Solution. The L.C.M. of 4, 6, and 8 is 24. Multiplying both members of the equation by 24,

$$\frac{24x}{4}+\frac{24(x-6)}{6}-\frac{24(1-2x)}{8}=\frac{24\cdot 55}{8}. \qquad (2)$$

By F, V, $$6x+4(x-6)-3(1-2x)=3\cdot 55=165. \qquad (3)$$

By F, $$6x+4x-24-3+6x=165. \qquad (4)$$

By F, A, $$16x=192. \qquad (5)$$

By D, $$x=12. \qquad (6)$$

As in this solution, so in general, each denominator is canceled by Principle V, after multiplying by the L. C. M. In practice, however, equation (2) may be omitted, the canceling being done mentally, and equation (3) may be written down at once.

The dividing line of the fraction acts like a parenthesis with respect to the sign preceding the fraction.

EXERCISES

Solve the following equations, and check each solution by substituting in the original equation:

1. $\frac{4x-5}{3}-\frac{2x+5}{5}=\frac{x-3}{2}+\frac{7x+5}{8}-4.$

2. $\frac{x+3}{2}-\frac{5x-3}{2x}+\frac{5x-3}{4}=4,$

3. $\frac{2x+5}{3}=\frac{5x+20}{15}+\frac{7x+13}{2}+\frac{15x+3}{4}.$

4. $\frac{3x+10}{5}-\frac{7x+15}{3}+\frac{5x-14}{7}=\frac{3x-12}{4}-2.$

5. $x+\frac{3x+5}{4x}+\frac{7-3x}{2x}=\frac{15-x}{2x}+\frac{3-11x}{4}.$

6. $\frac{7x-2}{6}-\frac{3x-24}{9}=\frac{3x+4}{2}-\frac{5-4x}{3}.$

7. $\frac{2x}{3}-\frac{5x}{12}+\frac{7x}{8}-\frac{x}{2}=45.$

8. $\frac{2x}{3}-\frac{3x}{4}+\frac{5x}{7}-\frac{11x}{12}=-24.$

9. $\frac{s-3}{2s}+\frac{2s+3}{3}-\frac{5s-3}{6}=\frac{3s-1}{2}-3.$

10. $\frac{2t+1}{5}-\frac{t-1}{8}+\frac{4t-8}{15}=\frac{3t-1}{10}+3.$

11. $\frac{x+3}{2}-\frac{4x+5}{7}+\frac{3x-5}{4}=\frac{5x-7}{12}+4.$

12. $\frac{8k-6}{5}+\frac{13k}{2}-\frac{21k-12}{5}=\frac{k-2}{10k}+\frac{14k-3}{5}+4.$

13. $\frac{5x-1}{2}-\frac{2x+3}{3}-\frac{5x+1}{4}=\frac{13x+5}{11}-4.$

14. $2-\frac{h-15}{2}+\frac{h-20}{3}-\frac{4h+2}{11}=0.$

15. $-\frac{17+m}{4}-\frac{2m-7}{3}+\frac{5m-3}{2}+2=\frac{3}{m}.$

16. $\frac{3r+4}{5}-\frac{5r+1}{12}-\frac{8r+4}{10}=2-\frac{6r}{7}.$

17. $\frac{9+2y}{3}-\frac{4y+3}{3y}=\frac{27+3y}{6}-3.$

18. $\frac{2x+5}{3}-\frac{5x+2}{7}+\frac{4x-5}{9}=\frac{3x+4}{7}.$

19. $\frac{z+3}{2}-\frac{2z-15}{z}-\frac{5z-11}{8}-\frac{11}{2}=0.$

20. $\frac{3x+20}{2}+\frac{5x-3}{11}-\frac{4x-1}{5}=3.$

198. Illustrative Example. Solve the following equation:

$$\frac{2x-1}{x-1}+\frac{4}{x+1}-\frac{3x}{x^2-1}=2. \qquad (1)$$

Solution. The L. C. M. of the denominators is x^2-1. In multiplying both members of the equation by x^2-1, $x-1$ is canceled in the first fraction, $x+1$ in the second, and x^2-1 in the third, giving

$$(2x-1)(x+1)+4(x-1)-3x=2(x^2-1) \qquad (2)$$

Solving, $$2x^2+x-1+4x-4-3x=2x^2-2, \qquad (3)$$

$$2x=3, \qquad (4)$$

and $$x=\tfrac{3}{2}. \qquad (5)$$

Check by substituting $x=\frac{3}{2}$ in (1).

EXERCISES

Solve the following equations and check each solution by substituting in the original equation:

1. $\frac{3x-1}{x+1}-\frac{4x+3}{x-1}+\frac{x^2}{x^2-1}=-\frac{27}{x^2-1}+1.$

2. $\frac{3x+5}{x-9}+\frac{2x+1}{x+2}=\frac{x-1}{x^2-7x-18}.$

3. $\frac{3x-4}{x+5}-\frac{4x-1}{x+4}+\frac{x^2+44}{x^2+9x+20}=0.$

4. $\frac{4-x^2}{x^2-5x-14}-\frac{3x+6}{x+2}=-\frac{2x+1}{x-7}.$

5. $\frac{x-4}{2x-10}-\frac{3x-15}{2x-6}=-\frac{3x^2-114}{4x^2-32x+60}.$

6. $\frac{6(x+4)}{x+5}-\frac{3(2x-1)}{x+1}=\frac{7}{2}.$

7. $\frac{3x-4}{x-4}+\frac{5x-7}{2x-2}=\frac{9x^2-38}{2x^2-10x+8}.$

8. $\frac{x+17}{x+5}-\frac{2(x+6)}{x+3}=-\frac{x-1}{x+3}.$

9. $\frac{x+2}{x-5}+\frac{3x-15}{x-3}=\frac{3x-21}{x-3}.$

10. $\frac{2x-3}{-4x}+\frac{3x+1}{x-2}=\frac{4x+17}{x-2}.$

11. $\dfrac{3x-2}{2x+3}=\dfrac{2x^2+15x+28}{2x^2+5x+3}+\dfrac{2x-1}{x+1}.$

12. $\dfrac{2x-3}{2x+2}-\dfrac{x-8}{5x+2}=\dfrac{x+2}{2x+2}.$

13. $\dfrac{20x^2+7x-3}{9x^2-1}-\dfrac{3x+1}{3x-1}=1.$

14. $\dfrac{7x^2+11x+4}{6x^2+13x+5}+\dfrac{x+3}{2x+1}=\dfrac{7x+11}{3x+5}.$

15. $\dfrac{3x+1}{5x-7}-\dfrac{x-3}{2x-7}=\dfrac{2x^2-10x+12}{10x^2-49x+49}.$

199. Sometimes it is best to *add fractions before multiplying* by the L. C. M. and in other cases to multiply by the L. C. M. of *part of the denominators first*, and, after simplifying, multiply by the L. C. M. of the remaining denominators.

Ex. 1. Solve the equation

$$\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{x-4}-\frac{1}{x-3}. \qquad (1)$$

Adding fractions on the right and left,

$$\frac{1}{(x-2)(x-1)}=\frac{1}{(x-4)(x-3)}. \qquad (2)$$

Multiplying by L. C. M., $(x-4)(x-3)=(x-2)(x-1)$. (3)

Hence $4x=10,$ (4)

and $x=2\frac{1}{2}.$ (5)

Check by substituting $x=2\frac{1}{2}$ in equation (1).

Ex. 2. Solve the equation:

$$\frac{4t-3}{16}-\frac{t-2}{4}=\frac{2t-2}{5t+2}. \qquad (1)$$

Multiplying by 16, $$4t-3-4t+8=\frac{32t-32}{5t+2}. \qquad (2)$$

Hence, $$5=\frac{32t-32}{5t+2}. \qquad (3)$$

Multiplying by $5t+2$, $$25t+10=32t-32. \qquad (4)$$

Hence, $$7t=42, \qquad (5)$$

and $$t=6. \qquad (6)$$

Check by substituting $t=6$ in equation (1).

EXERCISES

1. $\dfrac{3x+6}{5}-\dfrac{9x+3}{15}=\dfrac{x+7}{6x-8}+4.$

2. $\dfrac{7x+1}{12}-\dfrac{14x-22}{24}=\dfrac{11x+5}{8x-28}.$

3. $\dfrac{3x+4}{2}-\dfrac{12x+1}{8}=\dfrac{5x-1}{3x+2}.$

4. $\dfrac{7t+3}{5}-\dfrac{21t+9}{15}=\dfrac{17t-3}{3t+11}+2.$

5. $\dfrac{11v-15}{10}-\dfrac{33v+15}{30}=\dfrac{5v+5}{v-5}.$

6. $\dfrac{x-1}{x-2}+\dfrac{x-2}{3-x}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}.$

7. $\dfrac{1}{x-1}-\dfrac{2}{2x+1}=\dfrac{1}{x-2}-\dfrac{4}{4x+1}.$

8. $\dfrac{x-2}{x-3}-\dfrac{x-3}{x-4}=\dfrac{x-4}{x-5}+\dfrac{x-5}{6-x}.$

9. $\dfrac{9}{x-7}-\dfrac{9}{x-2}=\dfrac{5}{x-8}-\dfrac{5}{x+1}.$

10. $\dfrac{x-1}{x-2}+\dfrac{x-2}{3-x}=\dfrac{x-3}{x-4}-\dfrac{x-5}{x-6}.$

SIMULTANEOUS FRACTIONAL EQUATIONS

200. When pairs of fractional equations are given, each should be reduced to the integral form before eliminating, except in special cases like those in the second following illustrative example.

Ex. 1. Solve the equations:

$$\begin{cases} \dfrac{4}{x-y}+\dfrac{6}{x+y}=\dfrac{36}{x^2-y^2}. & (1) \\ \dfrac{3}{2x-y}-\dfrac{2}{x-3y}=\dfrac{-18}{(2x-y)(x-3y)}. & (2) \end{cases}$$

From (1) by M, $4(x+y)+6(x-y)=36.$ (3)

By F, D, $5x-y=18.$ (4)

From (2) by M, $3(x-3y)-2(2x-y)=-18.$ (5)

By F, D, $x+7y=18.$ (6)

From (4) by M, $35x-7y=126.$ (7)

Adding (6) and (7), $36x=144.$ (8)

By D, $x=4.$ (9)

Substitute $x=4$ in (6), $y=2.$ (10)

Check by substituting $x=4$, $y=2$ in (1) and (2).

Ex. 2. Solve the equations:

$$\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=2, & (1) \\ \dfrac{20}{x}-\dfrac{21}{y}=3. & (2) \end{cases}$$

In this case it is best to solve the equations for $\frac{1}{x}$ and $\frac{1}{y}$ instead of for x and y.

From (1) by M, $$\frac{14}{x}+\frac{21}{y}=14. \quad (3)$$

Adding (2) and (3), $$\frac{34}{x}=17. \quad (4)$$

Hence by D, $$\frac{1}{x}=\frac{1}{2}. \quad (5)$$

Substituting $\frac{1}{x}=\frac{1}{2}$ in (1), $$\frac{1}{y}=\frac{1}{3}. \quad (6)$$

From (5) and (6) by M, $$x=2,\ y=3. \quad (7)$$

EXERCISES

Solve the following equations:

1. $$\begin{cases} \dfrac{2x-1}{x+1}-\dfrac{3y-1}{y+1}=\dfrac{-xy}{(x+1)(y+1)}, \\ \dfrac{x+2}{2y-1}+\dfrac{2x-1}{y+1}=\dfrac{5xy}{(2y-1)(y+1)}. \end{cases}$$

2. $$\begin{cases} \dfrac{3x+2}{3y-5}=\dfrac{x+1}{y-1}, \\ \dfrac{3x-2}{y+1}=\dfrac{3x-1}{y-1}-\dfrac{2}{(y-1)(y+1)}. \end{cases}$$

3. $\begin{cases} \frac{5}{t} - \frac{6}{v} = 2, \\ \frac{17}{v} + \frac{4}{t} = 67. \end{cases}$

4. $\begin{cases} \frac{3}{x} + \frac{1}{y} = 21, \\ \frac{7}{x} - \frac{9}{y} = -19. \end{cases}$

5. $\begin{cases} \frac{7}{a} - \frac{1}{b} = 12\frac{1}{2}, \\ \frac{3}{a} + \frac{12}{b} = 24. \end{cases}$

6. $\begin{cases} \frac{3}{x} - \frac{2}{y} = -4, \\ \frac{6}{x} + \frac{11}{y} = 52. \end{cases}$

7. $\begin{cases} \frac{12}{x} - \frac{10}{y} = 1, \\ \frac{9}{x} + \frac{2}{y} = 15. \end{cases}$

8. $\begin{cases} \frac{a}{x} + \frac{b}{y} = 1, \\ \frac{c}{x} + \frac{d}{y} = 1. \end{cases}$

9. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 16, \\ \frac{1}{y} + \frac{1}{z} = 14, \\ \frac{1}{z} + \frac{1}{x} = 12. \end{cases}$

10. $\begin{cases} \frac{1}{x} + \frac{1}{y} = a, \\ \frac{1}{y} + \frac{1}{z} = b, \\ \frac{1}{z} + \frac{1}{x} = c. \end{cases}$

In 9 first add all three equations, and from half the sum subtract each equation separately. Likewise in 10.

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

In solving the following problems use one or two unknowns as may be found most convenient.

1. There are two numbers whose sum is 51 such that if the greater is divided by their difference, the quotient is $3\frac{1}{3}$. Find the numbers.

2. There are two numbers whose sum is 91 such that if the greater is divided by their difference, the quotient is 7. Find the numbers.

3. There are two numbers whose sum is s such that if the greater is divided by their difference, the quotient is q. Find an expression in terms of s and q representing each number. Solve 1 and 2 by substituting in the formula just obtained.

4. What number must be subtracted from each term of the fraction $\frac{11}{17}$ so that the result shall be equal to $\frac{1}{2}$?

5. What number must be subtracted from each term of the fraction $\frac{21}{35}$ so that the result shall be equal to $\frac{2}{9}$?

6. What number must be subtracted from each term of the fraction $\frac{a}{b}$ so that the result shall be equal to $\frac{c}{d}$? Solve 4 and 5 by substituting in the formula obtained under 6.

7. What number must be added to each term of the fraction $\frac{1}{2}$ to obtain a fraction equal to $\frac{16}{17}$?

8. What number must be added to each term of the fraction $\frac{a}{b}$ to obtain a fraction equal to $\frac{c}{d}$?

By means of the formula thus obtained, solve problem 7, and also 4, 5, and 6. (See work under Problem 23, p. 114.)

9. There are two numbers whose difference is 153. If their sum is divided by the smaller, the quotient is equal to $\frac{47}{15}$. Find the numbers.

10. There are two numbers whose difference is b. If their sum is divided by the smaller, the quotient is q. Find the numbers. Solve 9 by substituting in this formula.

11. Divide 548 into 2 parts, such that 7 times the first shall exceed 3 times the second by 474.

12. There are two numbers whose sum is 48 such that 3 times the first is 8 more than 5 times the second. Find both numbers.

13. There are two numbers whose sum is s such that a times the first is b more than c times the second. Find both numbers.

14. What number must be subtracted from each of the numbers 12, 15, 19, and 25 in order that the remainders may form a proportion when taken in the given order?

15. What number must be added to each of the numbers 13, 21, 3, and 8 so that the sums shall be in proportion when taken in the order given?

16. What number must be added to each of the numbers a, b, c, d so that the sums shall be in proportion when taken in the given order?

17. What number must be subtracted from each of the numbers a, b, c, d so that the remainders shall be in proportion when taken in the given order?

Compare the results in 16 and 17 and explain the relation between them. (See remark under Problem 23, page 114.)

Solve 14 and 15 by substituting in the formulas obtained in 16 and 17.

18. There is a number composed of two digits whose sum is 11. If the number is divided by the difference between the digits, the quotient is $16\frac{4}{5}$. Find the number, the tens' digit being the larger.

19. There is a number composed of two digits whose sum is s. If the number is divided by the difference between the digits, the quotient is q. Find the number, the tens' digit being the larger.

20. **Illustrative Problem.** A can do a piece of work in 8 days, B can do it in 10 days. In how many days can they do it together?

Since A can do the work in 8 days, in one day he can do $\frac{1}{8}$ of it, and since B can do it in 10 days, in one day he can do $\frac{1}{10}$ of it. If x is the number of days required when both work together, in one day they can do $\frac{1}{x}$ of it. Hence we have the equation,

$$\frac{1}{8}+\frac{1}{10}=\frac{1}{x}.$$

21. A can do a piece of work in 12 days and B can do it in 9 days. How long will it take both working together to do it?

22. A pipe can fill a cistern in 11 hours and another in 13 hours. How long will it require both pipes to fill it?

23. A can do a piece of work in a days and B can do it in b days. How long will it take both together to do it?

24. A cistern can be filled by one pipe in 20 minutes and emptied by another in 30 minutes. How long will it take to fill the cistern when both are running together?

25. A pipe can fill a cistern in 12 hours, another in 10 hours, and a third can empty it in 8 hours. How long will it require to fill the cistern when they are all running?

26. A man can do a piece of work in 18 days, another in 21 days, a third in 24 days, and a fourth in 10 days. How long will it require them when all are working together?

27. A and B working together can do a piece of work in 12 days. B and C working together can do it in 13 days, and A and C working together can do it in 10 days. How long will it require each to do it when working alone?

Suggestion: Let $a =$ the fraction of the work A can do in one day, $b =$ the fraction of the work B can do in one day, and $c =$ the fraction of the work C can do in one day.

Then $$a+b=\tfrac{1}{12},\quad b+c=\tfrac{1}{13},\quad c+a=\tfrac{1}{10}.$$

28. A and B working together can do a piece of work in l days. B and C can do it in m days and C and A can do it in n days. How long will it require each working alone?

29. The circumference of the rear wheel of a carriage is 1.8 feet more than that of the front wheel. In running one mile the front wheel makes 48 revolutions more than the rear wheel. Find the circumference of each wheel.

If x is the number of feet in the circumference of the front wheel, then $\frac{5280}{x}$ is the number of revolutions in going one mile.

30. The circumference of the rear wheel of a carriage is 1 foot more than that of the front wheel. In going one mile the two wheels together make 920 revolutions. Find the circumference of each.

31. The distance from Chicago to Minneapolis is 420 miles. By increasing the speed of a certain train 7 miles per hour the running time is decreased by 2 hours. Find the speed of the train.

If r is the original rate of the train, then $\frac{420}{r}$ is the running time.

32. The distance from New York to Buffalo is 442 miles. By decreasing speed of a fast freight 8 miles per hour the running time is increased 4 hours. Find the speed of the freight.

33. A motor boat goes 10 miles per hour in still water. In 10 hours the boat goes 42 miles up a river and back again. What is rate of the current?

34. A train leaving New York over the Pennsylvania Road requires 9 hours to overtake a train leaving Philadelphia westward at the same time. If the Philadelphia train had started toward New York, they would have met in one hour. Find the rate of each train, the distance from New York to Philadelphia being 90 miles.

35. The average of the numbers x, 34, 0, -58, -19, 0, -20, and y is 12; while the average of $2x$, $3y$, -18, 50, and -30 is -4. Find x and y.

36. The length of a rectangle is 8 feet greater, and its width is 4 feet greater, than the side of a certain square. The sum of the areas of the square and rectangle is 736 square feet. Find the dimensions of each.

37. The difference between the areas of a circle and its circumscribed square is 12 square inches. Find the radius of the circle. (See problem 33, p. 239.)

38. The difference between the areas of a circle and its inscribed square is 12 square inches. Find the radius of the circle.

39. The difference between the areas of a circle and the regular inscribed hexagon is 12 square inches. Find the radius of the circle.

40. The altitude of an equilateral triangle is 6. Find its side and also its area. Find the side and area, if the altitude is h.

41. The radius of a circle is 3 feet. Find the area of the regular circumscribed hexagon. Find the area if the radius is r feet.

42. The radius of a circle is r. Find the difference between the areas of the circle and the regular circumscribed hexagon.

43. The difference between the areas of a circle and the regular circumscribed hexagon is 9 square inches. Find the radius of the circle.

44. A circle is inscribed in a square and another circumscribed about it. The area of the ring formed by the two circles is 25 square inches. How long is the side of each square?

45. A square is inscribed in a circle and another circumscribed about it. The area of the strip inclosed by the two squares is 25 square inches. Find the radius of the circle.

In the following five problems solve the resulting literal equations for each letter in terms of the others, and for each solution state a corresponding problem.

46. A hound pursuing a deer gains 400 yards in 25 minutes. If the deer runs 1300 yards a minute, how fast does the hound run? If the hound gains v_1 yards in t minutes and the deer runs v_2 yards per minute, find the speed of the hound.

47. A disabled steamer 240 knots from port is making only 4 knots an hour. By wireless telegraphy she signals a tug, which comes out to meet her at 17 knots an hour. In how long a time will they meet? If the steamer is s knots from port and making v_1 knots per hour, and if the tug makes v_2 knots per hour, find how long before they will meet.

48. A motor boat starts $7\frac{2}{3}$ miles behind a sailboat and runs 11 miles per hour while the sailboat makes $6\frac{1}{2}$ miles per hour. How far apart will they be after sailing $1\frac{1}{3}$ hours? If the motor boat starts s miles behind the sailboat and runs v_1 miles per hour, while the sailboat runs v_2 miles per hour, how far apart will they be in t hours?

49. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the two boats be 280 knots apart? Is there more than one such position? If the liner makes v_1 knots per hour and the freight boat, which is s_1 knots out, makes v_2 knots per hour, how long before they will be s_2 knots apart?

50. A passenger train running 45 miles per hour leaves one terminal of a railroad at the same time that a freight running 18 miles per hour leaves the other. If the distance is 500 miles, in how many hours will they meet? If they meet in 8 hours, how long is the road? If the rates of the trains are v_1 and v_2 and the road is s miles long, find the time.

51. In going 1200 yards the rear wheel of a carriage makes 60 revolutions less than the front wheel. If the circumference of each wheel be increased by 3 feet, the rear wheel will make only 40 revolutions less than the front wheel. Find the circumference of each wheel.

High School Physics.

By Professor HENRY S. CARHART, of the University of Michigan, and H. N. CHUTE, of the Ann Arbor High School. New edition, thoroughly revised. 12mo, cloth, 440 pages. Price, $1.25.

NO other text-book on Physics, published in this country, has ever enjoyed the popularity or the success that has, from the first year of its publication, attended Carhart and Chute's High School Physics. Throughout the country the demand for the book has been far in excess of that for any other manual covering the same field, while in many states it has been used more widely than all its competitors combined. The new edition of the book is a distinct improvement on its predecessor. A comparison of the two books will show numerous changes in details and a smoothing down of the rougher spots. Physics is not an easy subject; but the authors have made an honest effort to relieve the difficulties for immature students without such an emasculation of the subject as to diminish its value for either discipline or scientific information.

The problems have all been replaced by entirely new ones, in which the purpose kept in view is the concrete illustration of principles, with a minimum of arithmetical computation. Attention has been given also to the careful grading of the problems. None of them have been inserted as puzzles to test the student's intellectual skill, while, as a whole, they are distinctly easier than were those of the preceding edition.

The book remains, as it was before, the most attractive manual for the study of Physics that has been published. Its method cannot be improved. The principles of the science are stated in a simple, clear, and direct manner; their application is illustrated by apt experiments; finally, numerous practical problems test the pupil's mastery of the subject.

Its arrangement of material, its completeness along the lines of the most recent developments of physical science, its excellent woodcuts, its wealth of problems, and the ease and precision of its style deserve and have received unstinted praise.

First Principles of Chemistry.

By RAYMOND B. BROWNLEE, Far Rockaway High School; ROBERT W. FULLER, Stuyvesant High School; WILLIAM J. HANCOCK, Erasmus Hall High School; MICHAEL D. SOHON, Morris High School; and JESSE E. WHITSIT, DeWitt Clinton High School; all of New York City.

THIS manual includes a treatment of the common elements and their important compounds, together with a full discussion of the fundamental theories of chemistry. These theories are simply, clearly, and accurately stated without being buried in a confusing mass of detail. The recent developments in chemical theory are given due recognition.

In the development of laws and hypotheses, the historical order is followed, the experimental facts in each case being described before stating the discovery that resulted from them. Theoretical topics are introduced as soon as the pupil is able to take in their full significance; this, on the ground that to delay their presentation is to deprive the pupil of a very useful tool in his acquisition of chemical ideas.

This book marks a step in advance by its omission of much material that has found a place in other elementary manuals through tradition rather than for its real value. On the other hand, a number of the metallic elements, sometimes neglected, are given the thorough treatment that their industrial importance deserves. One of the chief aims of the descriptive matter is to show the student the many points of contact between the life about him and the science he is studying.

An important feature of the book is the brief summary and the test exercises given at the end of each chapter. The summary is a series of pithy statements emphasizing the essentials and affording systematic review.

This book has been prepared by five teachers of long experience in both college and secondary school work. Three years of careful discussion and revision by the authors have produced a unified text-book especially planned for the beginner in chemistry.

Practical Physiography.

By Dr. Harold Wellman Fairbanks, of Berkeley, California. 8vo, cloth, 570 pages. 403 Illustrations. Price, $1.60.

IN this volume the author has tried to work out a practical, concrete treatment of the subject of Physical Geography. The book is intended as an aid to study, not as a compendium of information; consequently a description of the world as a whole is omitted. Attention is devoted specifically to the region of the United States, and the typical examples afforded by it are studied as representatives of universal processes.

The purely descriptive method has been discarded as far as practicable, the object being to lead the student to investigate and find out for himself. Instead of being told everything, he is asked to use his observing and reasoning powers.

No separate chapters are devoted to the relation between physical nature and life, but, instead, this relation is brought out in its appropriate place in connection with each topic throughout the book. Such an arrangement, it is believed, will make the whole matter much more vital.

Another feature which the author trusts will meet with favor from the practical teacher is the distribution of the questions and exercises throughout each chapter, in close connection with the descriptive portions to which they refer. The placing of questions and exercises by themselves at the close of each chapter, as is done in many text-books on the subject, puts a premium upon mere memorizing of the text and the omission of all practical work. It is not expected, however, that the pupils will be able to answer all the questions without aid and direction from the teacher.

The illustrations are a marked feature of the book. Photographs have been used wherever possible, as they appeal with much more force to pupils of high school age than do diagrams or sketches. Most of the views are from the author's own negatives.

Lessons in Elementary Botany for Secondary Schools.

By Professor THOMAS H. MACBRIDE, Iowa State University. 16mo, cloth, 244 pages. Price, 80 cents.

THIS is a new edition of Professor Macbride's well-known text-book, enlarged and revised. This edition contains ten new full-page plates, and while it calls for no material but such as is easily accessible to every teacher, its method is nevertheless in perfect harmony with the best instruction of the time. The student is not asked to study illustrations or text; he is sent directly to the plants themselves and shown how to study these and observe for himself the various problems of vegetable life. The plants passed in review are those which are more or less familiar to every one, and the life history of which every child should know.

Austin C. Apgar, *State Normal School, Trenton, N.J.:* There are many points in the book which please me. Not the least of these is that the author has not been misled by the craze for that "logical order" which begins with protoplasm and, some time or other, if the subject be pursued long enough, reaches such things as can easily be found and examined. He begins with the known and gradually advances to the unknown,—the only order in which successful teaching can be accomplished.

A Key to Native Plants.

To accompany Macbride's Elementary Botany.

IN response to a desire expressed by many teachers, Professor Macbride has prepared a Key to the More Common Species of Native and Cultivated Plants occurring in the Northern United States. The material thus furnished has never before been offered in such convenient form for class-room work. Its use will in no way change the original plan of the Elementary Botany, but it offers many advantages for additional study.

The Key will be furnished without charge to those ordering the author's Botany. Copies ordered without the Botany will be furnished at twenty-five cents each.

A Practical Course in Bookkeeping.

By CARLOS B. ELLIS, of the Central High School, Springfield, Mass. 8vo, cloth, 000 pages. Price, $0.00.

THIS book is the outgrowth of long experience in teaching and in actual business. It is designed to give pupils the knowledge of the principles of bookkeeping and of the customs of business that every man needs, together with sufficient practice in the application of these principles and customs to meet the requirements of the best business offices.

A feature of the book is its logical development of principles whereby the pupil makes a careful study of the essential accounts instead of merely memorizing rules. The pupil who understands the uses of the principal accounts and the relations which each account sustains to the others, will not need to remember rules for journalizing. In a word, the book is designed to teach the pupil to think.

A large amount of work is provided for supplementary drill, and there is abundant practice in the use of business forms and of the variations in method adapted to special requirements.

Introduction to Geometry.

By WILLIAM SCHOCH, of the Crane Manual Training High School, Chicago. 12mo, cloth, 142 pages. Price, 60 cents.

THE aim of the author is to furnish a series of exercises and problems that will enable the pupil to pass easily from the individual notions of his old experience to a firm knowledge of the elementary facts and concepts of Geometry. The pupil is to convince himself of geometric truths through measuring and drawing, and thus gradually to come into possession of the material with which formal Geometry deals. Along with this, the practical side of the subject is taken into account, the pupil's knowledge being tested constantly by his power to apply it.

Plane and Spherical Trigonometry.

By President ELMER A. LYMAN, Michigan State Normal College, and Professor EDWIN C. GODDARD, University of Michigan. Cloth, 146 pages. Price, 90 cents.

THIS text-book aims to furnish sufficient material in analytical trigonometry, and also in the solution of the triangle, both of which have been adequately met heretofore by no single book.

Particular attention has also been given to the proofs of the formulæ for the functions of $\alpha \pm \beta$. Nearly all other text-books treat the same line as both positive and negative in the same discussion, thus vitiating the proof, and in many cases proofs are given for acute angles and are then supposed to be established without further discussion for all angles. These difficulties have been avoided by so stating the proofs that the language applies to figures involving any angles and proving the general case algebraically to avoid drawing an indefinite number of such figures.

Computation Tables.

By LYMAN and GODDARD. Price, 50 cents.

Plane and Spherical Trigonometry, with Computation Tables.

By LYMAN and GODDARD. Complete edition. Cloth, 214 pages. Price, $1.20.

Principles of Plane Geometry.

By J. W. MACDONALD, Agent of the Massachusetts Board of Education. 16mo, paper, 70 pages. Price, 30 cents.

A Primary Algebra.

By J. W. MACDONALD, Agent of the Massachusetts Board of Education. 16mo, cloth. The Complete Edition, 218 pages (containing the Teacher's Guide, the Student's Manual, and Answers to Problems). Price, 75 cents. The Student's Manual, cloth, 92 pages. Price, 30 cents.

Plane Trigonometry.

By Professor R. D. BOHANNAN, of the State University, Columbus, Ohio. Cloth, 379 pages. Price, $2.50.

SOME of the features of this book are the use of triangles having variety of form, but always such form as they should have if they had been actually measured in the field; the suggestion of simple laboratory exercises; problems from related subjects, such as Physics, Analytical Geometry, and Surveying.

With a view to minimize the difficulties of the subject, the author has ventured to present the subjects in the following order: the sine, its inverse and reciprocal; the cosine, its inverse and reciprocal; the sine and the cosine family in union; the tangent, its inverse and reciprocal; all the functions in union.

A pamphlet containing four-place mathematical tables is furnished free with the book.

Calculus with Applications

By ELLEN HAYES, Professor of Mathematics at Wellesley College. 12mo, cloth, 170 pages. Price, $1.20.

THIS book is a reading lesson in applied mathematics, intended for persons who wish, without taking long courses in mathematics, to know what the calculus is and how to use it, either as applied to other sciences, or for purposes of general culture. Nothing is included in the book that is not a means to this end. All fancy exercises are avoided, and the problems are for the most part real ones from mechanics or astronomy.

Logarithmic and Other Mathematical Tables.

By WILLIAM J. HUSSEY, Professor of Astronomy in the Leland Stanford Junior University, California. 8vo, cloth, 148 pages. Price, $1.00.

VARIOUS mechanical devices make this work specially easy to consult; and the large, clear, open page enables one readily to find the numbers sought. It commends itself at once to the eye, as a piece of careful and successful book-making.

Difficult explanation p. 62
p. 64 Principle too complex in word
Not struck with name "signed nos"
p. 72 a desirable notation
p. 125 weight & mass
Multiplier 14