A HIGH SCHOOL ALGEBRA

TWENTIETH CENTURY TEXT-BOOKS

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BY

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PREFACE

This volume presents a full high-school course in elementary algebra and contains all the topics given in the standard year-and-a-half courses. It is adapted to the prevailing practice of teaching elementary algebra in two courses — a full-year course followed by a half-year course. The first twenty-three chapters contain all the work required in any standard one-year course, and the remaining ten chapters comprise a subsequent half-year course, reviewing and extending the elementary processes, fractions, factoring, exponents, and methods of solving equations, before any new topics are given. The result is a single volume adapted to a continuous one-and-a-half-year course, or to a course in which geometry intervenes between first-year and second-year algebra. It is especially suited to the latter plan, because Chapters XXIV and XXV furnish the review necessary for those pupils who take the divided course. Moreover, the treatment of quadratic equations, radicals, exponents, ratio, proportion, variation, and graphs in the second as well as in the first year's work, gives the greatest flexibility to the use of the book. For example, if, for the purposes of a short course, one or more of the later chapters were omitted from the first year's work, the chapters in the second year's work would supply material on the subjects omitted.

In whatever manner the study of geometry and algebra is alternated, the student acquires little knowledge of the metrical properties of geometry during the first year. For this reason, the authors have used in their problems only the most obvious of these properties, and have given in a carefully prepared supplement the more difficult properties to which algebra may be applied.

Each important process of algebra is immediately applied to the solving of equations. This plan serves not only to secure the pupil's interest, but reveals to him the utility of algebra.

Great pains have been taken to supply ample practice work, and the authors have given under the more important topics, such as equations, factoring, highest common factor, fractions and exponents, a greater number of exercises than will be required by any one class. In fact, there have been included as many exercises and problems as a text-book of reasonable size will admit.

Particular attention has been given to the grading of the exercises and problems, and, for convenience in checking the results, the exercises have been so constructed that the answers are not more complex than the purpose of the exercises actually requires. The authors have followed the criterion that every principle should be exemplified with the minimum of calculation.

Among the features that contribute to the teachableness of the book are the Historical Notes. These brief sketches, describing the origin of some of the more important topics of algebra, tend to stimulate the pupil's interest, and the accompanying biographical notes and portraits of famous mathematicians serve further to humanize the subject. No attempt has been made to give a connected account of the development of algebra even in outline; these notes will serve their purpose if they create a desire to read some standard work on the history of mathematics.

Other aids which teachers will appreciate are the inductive developments, the cross references, illustrative problems, methods of testing results, careful statement of rules, topical and logical arrangement, definite classification, the frequent reviews, and the summaries of the theoretic chapters.

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THE AUTHORS.

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CHAPTER I

LITERAL NOTATION AND ITS USES

1. Numbers represented by Letters. In arithmetic, numbers are represented by means of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. But letters also may be used to stand for numbers.

For example:

p may stand for the number of *pounds* in the weight of a body;

d may stand for the number of *dollars* in a sum of money;

l may stand for the number of units in the *length* of an object, and the like.

2. The Use of Signs. The signs $+$, $-$, $=$, $\times$, $\div$, and $\sqrt{\ }$ have the same meaning in algebra as in arithmetic. But in algebra, multiplication is indicated also by the *absence of a sign of operation.* When a sign is needed, the *dot*, $\cdot$, is often used in preference to the symbol $\times$, which is likely to be mistaken for the letter x.

For example:

a plus b is written $a + b$, just as 3 plus 2 is written $3 + 2$.

a minus b is written $a - b$, just as 3 minus 2 is written $3 - 2$.

a divided by b is written $a \div b$ or $\frac{a}{b}$, just as 3 divided by 2 is written $3 \div 2$ or $\frac{3}{2}$.

2 times 5 is written 2×5 or $2 \cdot 5$.

a times b is written ab. 2 times a is written $2a$. And 2 times a plus b times c is written $2a + bc$.

The square root of a is written $\sqrt{a}$; the cube root of a, $\sqrt[3]{a}$; and so on.

3. The use of letters to represent numbers enables us to write statements in very brief form. This is an important feature of algebra.

For example:

1. The length of a lot diminished by $\frac{1}{5}$ of its length is 60 ft.

Using l for the number of feet in the length of the lot, this statement may be written:

l minus $\frac{1}{5}l$ is 60,

or, $l - \frac{1}{5}l = 60.$

2. A man's weight when increased by $\frac{1}{3}$ of itself is 200 lb.

Using w for the number of pounds in the man's weight, this statement may be written:

w plus $\frac{1}{3}w$ is 200,

or, $w + \frac{1}{3}w = 200.$

ORAL EXERCISES

1. If l represents the number of yards in the length of a street, what stands for the length of a street 75 yd. longer?

2. If w represents the number of rods in the width of a farm, what represents the width of a farm 20 rd. narrower?

3. One bank contains d dollars and another 3 times as many. How many dollars in the second bank? How many in both banks?

4. There are n pupils in a class and the same number increased by 13 in another. How many pupils in the second class? In both classes?

5. A merchant invested s dollars and lost $\frac{1}{10}$ of this the first year. How much had he left?

6. The line BC in the figure is 3 times as long as AB. If AB is l units long, how long is BC? AC?

7. In the following statements c stands for cost, s stands for selling price, and g for gain. Read each statement in words:

1. $s - c = g.$ 2. $c + g = s.$ 3. $s - g = c.$

4. Algebraic Symbols. Letters and other characters used as notations in algebra are called **algebraic symbols.**

5. Algebraic Expressions. Any expression representing a number by use of algebraic symbols is called an **algebraic expression.**

For example: $3b$ and $2a - bc + d$ are algebraic expressions.

The term *literal expression* is often used to denote an algebraic expression involving letters.

6. Value of Algebraic Expressions. The number represented by an algebraic expression is called its **value.** The value of an algebraic expression is found by substituting numbers for the letters.

Thus, when $a = 3$, $2a = 2 \cdot 3$, or 6.
Similarly, when $n = 5$, $2n - 1 = 2 \cdot 5 - 1$, or 9.

ORAL EXERCISES

1. What is the value of $2n$ when n is 1? When n is 2?

2. What number is $\frac{1}{2}n$ when n is 2? When n is 6? When n is 10?

3. State the value of $2n$ when n is $\frac{1}{2}$. Also when n equals each of the following: $\frac{3}{5}$; $7\frac{1}{2}$; 10; .5; 1.5; 50.

4. State the value of $n + 1$ when n equals each of the following: 1; 2; 6; 5; $\frac{1}{2}$; .5; $8\frac{1}{2}$; 100; 0.

5. The length (l) of a box is twice its width (w). (l) is (?) w.

6. How many ounces are there in 5 lb.? In x lb.?

7. A pair of gloves costs c cents. What would the cost be if the price were raised 5 cents?

8. There were b books on a shelf and 2 were taken down. How many remained on the shelf?

9. A person is x years of age now. How old will he be a year hence? 5 years hence? How old was he 3 years ago? y years ago?

10. A merchant sold goods at 8 % above cost. If c was the number of dollars in the cost, the gain was 8 % of c, or $.08c$. What was the selling price?

11. Some goods costing x dollars were sold at a gain of 150 %. State the selling price.

WRITTEN EXERCISES

1. Write the sum of b and c, using the sign of addition.

2. Indicate the subtraction of c from b by using the sign of subtraction.

3. Write the product of a and b as it is expressed in algebra.

4. Write the product of 3 and b and c as it is expressed in algebra.

5. Indicate that a is to be divided by b by using the fractional form.

6. Indicate that the sum of a and b is to be divided by c.

7. Find the value of $2a+1$ when a equals each of the following: 4; 7; $\frac{7}{2}$; $11\frac{1}{2}$; 15; .5; 1.5; 100; 0.

Find the value of $a+b$, when a and b indicate in turn the following numbers:

8. $a=2, b=1$.
9. $a=14, b=6$.
10. $a=12, b=9$.
11. $a=\frac{1}{2}, b=\frac{1}{4}$.
12. $a=.8, b=.5$.
13. $a=1\frac{3}{4}, b=\frac{5}{8}$.

For each pair of values of a and b above, find the corresponding value of:

14. $a-b$.
15. ab.
16. $\frac{a}{b}$.
17. $\frac{a+b}{ab}$.

18. Draw a rectangle; write b for its base and a for its altitude. Express the area of the rectangle; its perimeter (sum of its four sides).

19. A rectangular bin is a ft. long, b ft. wide, and c ft. deep. How many cubic feet does it contain?

20. What number does $100a+10b+c$ represent when $a=1, b=2, c=3$? When $a=5, b=4, c=7$?

7. Tabulation of Values. In recording corresponding values it is convenient to use a table like that adjoining.

a	$2a+1$
5	11
0	1
3	7
$8\frac{1}{2}$	18
3.5	8

The values of a are written in the column under a, and the corresponding values of $2a+1$ are written opposite in the second column. The table records, for example, that when a is 5, $2a+1$ is 11; that is, $2 \times 5+1$. Verify the other values.

WRITTEN EXERCISES

Copy the following tables and supply the numbers to fill the blanks:

1.

n	$3n$
0	( )
1	( )
4	( )
5	( )
$4\frac{1}{3}$	( )
$\frac{10}{3}$	( )

2.

n	$2n-3$
2	( )
$2\frac{1}{2}$	( )
3	( )
$\frac{5}{2}$	( )
1.5	( )

3.

a	b	$a+b$
4	5	( )
$6\frac{1}{2}$	$2\frac{1}{2}$	( )
1.8	9.2	( )
2.5	8.3	( )

4.

n	$n-1$
1	( )
2	( )
7	( )
$6\frac{1}{2}$	( )
18	( )
25	( )

5.

a	$\frac{1}{2}a+1$
2	( )
12	( )
25	( )
1.8	( )
4.6	( )

6.

v	t	vt
$17\frac{1}{2}$	$6\frac{1}{2}$	( )
1.8	.7	( )
4	3.8	( )
12	$7\frac{1}{3}$	( )

7.

v	$10v$
.1	( )
1.2	( )
1.5	( )
6.3	( )
40.4	( )
.05	( )

8.

w	$\frac{3}{4}w$
0	( )
1	( )
24	( )
64	( )
100	( )

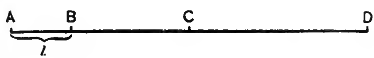
9.

l	b	t	lbt
6	8	3	( )
4	5	6	( )
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	( )
$\frac{2}{3}$	$\frac{3}{5}$	.8	( )

REVIEW

ORAL EXERCISES

1. A man invests d dollars in real estate, and 5 times as much in government bonds. How much does he invest altogether?



2. In the figure BC is twice as long as AB, and CD is three times as long as AB. If l denotes the length of AB, what denotes the length of BC? Of CD? Of the entire line?

Find the value of $A+B$, when A and B have the following values:

3. $A=7$, $B=12$.
4. $A=\frac{2}{3}$, $B=\frac{1}{6}$.
5. $A=3x$, $B=4x$.
6. $A=5y$, $B=7y$.

WRITTEN EXERCISES

1. Copy and fill the blanks:

		(1)	(2)	(3)	(4)	(5)
Given	$n=$	6	$\frac{2}{3}$	$\frac{3}{2}$	100	1.3
	$t=$	4	$\frac{1}{3}$	$\frac{5}{2}$	10	20.
Find	$3n-1=$	—	—	—	—	—
	$n+t=$	—	—	—	—	—
	$2nt=$	—	—	—	—	—

2. The length of one line is x and that of another is y. How long is the line formed by placing them end to end?

3. Line x is longer than line y. Express the difference between their lengths.

4. What number does $\frac{a}{b}$ represent when $a=29$, and $b=58$? When $a=35$, and $b=70$? When $a=127$, $b=210$?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What symbols may stand for numbers? Sec. 1.
2. Name the different *signs of operation* used in algebra. Sec. 2.
3. In what ways may multiplication be indicated in algebra? Sec. 2.
4. What is an *algebraic expression?* Sec. 5.
5. What is meant by the *value* of an algebraic expression? Sec. 6.

HISTORICAL NOTE

If algebra appears more abstract than other school studies, if it seems less real than arithmetic, this is largely on account of its use of letters and other symbols of abbreviation. We have learned to express "any number" by a single letter, like a or x, to abbreviate the "square of a number" by a^2 or x^2, and to express operations with these numbers by signs like $+$, $-$, $\times$, $=$; all of this seems artificial and abstract, but this is one of the chief sources of algebra's power. With these symbols we can easily find in a few minutes results which the ancients sought in vain. Thus, through lack of a suitable notation, the Greeks made slow progress in studying the processes with numbers as well as those phases of geometry that require calculation, like the measuring of circles and polygons; and it was not until 300 A.D. that Diophantos, who taught in the University of Alexandria, Egypt, developed algebra into a science.

History tells little of the life of Diophantos. His birthplace, his parentage, his early education, and the steps which led to his great achievements are unknown. He died in 330 A.D., and his age at death is shown to have been 84 years by the following epitaph: "Diophantos passed one-sixth of his life in childhood, one-twelfth in youth, and one-seventh more as a bachelor. Five years after his marriage was born a son, who died four years before his father at half his father's age." It may seem strange to resort to this indirect means of telling the age of a famous man, but it was the custom of the ancients to inclose in the tombs of great men some possession or record associated with their lives; and this is one of the chief sources of early history.

Diophantos was the first to use abbreviations and symbols in algebra. On this account, and because of his simple solution of many problems then thought difficult, he has been called the Father of Algebra.

CHAPTER II

DEFINITIONS OF ELEMENTARY TERMS

8. Product and Factors. In algebra, as in arithmetic, the result of multiplication is called a **product**, and the numbers multiplied are called the **factors** of the product.

9. A number may be the product of different sets of factors.

10. Literal and Numerical Factors. Factors expressed by letters are called **literal** factors; factors expressed by numerals are called **numerical** factors.

For example:

> 10 is the product of the factors 2 and 5.
> $2\,ax$ is the product of the factors 2, a, and x, in which 2 is a numerical factor, but a and x are literal factors.

In a product it is customary to put the numerical factor (if any) first, and the literal factors in alphabetical order.

11. Commutative Law of Multiplication. *The product is the same in whatever order the factors are taken.*

For example: $3 \cdot ab \cdot x$ gives the same product as $ab \cdot 3 \cdot x$ and the same as $abx \cdot 3$.

12. Unity is a factor of every number, but it is not ordinarily mentioned in giving lists of factors.

Thus, a set of factors of abc are 1, a, b, and c; but 1 is usually not mentioned.

ORAL EXERCISES

Name a set of factors for each of the following products:

1. 10.	**3.** prt.	**5.** $\frac{1}{2}\,ab$.	**7.** gt.
2. $3\,a$.	**4.** lr.	**6.** mv.	**8.** $\frac{1}{2}\,sf$.

Name three sets of factors for each of the following and state which factors are literal and which are numerical:

9. 40. 11. $25\,ab$. 13. prt. 15. $75\,pq$.

10. $18\,a$. 12. $20\,hr$. 14. $25\,mv$. 16. $abcd$.

13. Power and Base. A product formed by using the same number one or more times as a factor is called a **power** of the repeated factor. The repeated factor is called the **base.**

14. Exponent. When a factor is to be repeated, it is usual to write the factor only once and place a small number above and to the right to show how many times the number is to be used as a factor. The small number is called an **exponent.**

Thus, $2 \cdot 2$ is the second power (or square) of 2, and is written 2^2; $a \cdot a \cdot a$ is the third power (or cube) of a, and is written a^3. Similarly, $3\,aabbb$ is written $3\,a^2b^3$.

15. A number without an exponent is understood to have the exponent 1, since the number is used once as a factor.

Thus, $5 = 5^1$; $a = a^1$; $7\,xyz^3 = 7^1x^1y^1z^3$.

16. The factors of numbers can be conveniently grouped by the use of exponents.

For example:

$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3. \qquad 144 = 12 \cdot 12 = 12^2.$$
$$225 = 3 \cdot 3 \cdot 5 \cdot 5 = 3^2 \cdot 5^2. \qquad 600 = 2^3 \cdot 3 \cdot 5^2.$$

17. Prime Numbers. As in arithmetic, so in algebra, an integer whose only integral factors are itself and unity is called a **prime number.**

18. Prime Factors. Prime numbers occurring as factors are called **prime factors.**

A number has only one set of *prime* factors.

ORAL EXERCISES

Name the exponents and tell what each means:

1. $2\,a^2x^2$. 2. $3\,x^3$. 3. $5\,xy^3$. 4. a^4b^4. 5. $2\,a^5y$.

WRITTEN EXERCISES

Regarding each letter as prime, indicate the prime factors of:

1. 18.
2. $75a$.
3. 96.
4. $12x^2$.
5. 640.
6. $24ab$.
7. 360.
8. $17a^2b$.
9. 1225.
10. $38x^3y$.

19. Coefficient. Any factor in a product is called the **coefficient** of the rest of the product.

Thus, in the product $3axy$, 3 is the coefficient of axy, $3a$ is the coefficient of xy, $3ax$ is the coefficient of y, $3ay$ is the coefficient of x, and the like.

20. Numerical Coefficient. A coefficient expressed in numerals is called a **numerical coefficient.**

Thus, 3 is the numerical coefficient in $3x$, and $\frac{1}{2}$ is the numerical coefficient in $\frac{1}{2}xy$.

The term "coefficient," used with no other indication, means the numerical coefficient.

21. In any product whose numerical coefficient is not expressed, the coefficient 1 is understood.

Thus, ab, abx, bc^2y, are the same as $1ab$, $1abx$, $1bc^2y$; and the numerical coefficient in each is 1.

ORAL EXERCISES

1. In $6ab$ name the coefficient of ab. The coefficient of b.
2. In $\frac{2}{3}axy$ name the coefficient of xy. Of axy. Of y.

Name the numerical coefficient in each of the following:

3.	4.	5.	6.	7.	8.	9.	10.	11.
$2x$.	$3y$.	$\frac{2}{3}ax$.	by.	$.5cz$.	$\frac{4}{3}my$.	$\frac{1}{2}gt^2$.	xyz.	$\frac{1}{2}mr^2$.

22. Order of Operations. *In an expression containing a series of operations, multiplications and divisions are to be performed before additions and subtractions, unless otherwise indicated.*

Thus, $4 + 5 \cdot 3$ means $4 + 15$, or 19.
Similarly, $3 + 8 \div 2 - 5 = 3 + 4 - 5 = 7 - 5 = 2$.

ORAL EXERCISES

Perform the operations indicated:

1. $5 \cdot 4 - 3.$
2. $9 + 2 \cdot 6.$
3. $14 - 8 \div 4.$
4. $15 - 12 \div 4.$
5. $40 - 34 \div 2.$
6. $52 \div 26 - 1.$
7. $10 \cdot 3 - 3 \cdot 8.$
8. $3 \cdot 9 - 15 \div 5.$
9. $18 \div 2 + 3 \cdot 6.$
10. $3 + 4 \cdot 5 - 13.$
11. $a + 2a \div a - a.$
12. $b \cdot a + a \cdot b.$

23. The Use of the Parenthesis. In a series of operations the parenthesis may be used to indicate that certain additions and subtractions are to be performed first.

For example:

$6 + 4 \cdot 3$ means add 6 to 4 times 3, obtaining 18. But $(6 + 4) \cdot 3$ means add 6 and 4 and multiply the result by 3, obtaining 30. In other words, what is in the parenthesis is to be treated as a single number.

$(4 + 16) \div 2$ means $20 \div 2$, or 10, and not $4 + \frac{16}{2}$, or 12.

$8 - (12 - 7)$ means that 7 is first to be subtracted from 12 and then the result subtracted from 8. That is, $8 - (12 - 7) = 8 - 5 = 3$.

$14a - (7a + 3a) = 14a - 10a = 4a.$

24. When a number symbol is placed before or after a parenthesis with no intervening sign, multiplication is indicated.

For example:

$2a(c + 2c)$ means $2a \cdot 3c$, or $6ac$.
$(4x + 5x)4x$ means $9x \cdot 4x$ or $36x^2$.
$(a + 4a)(8b - 5b)$ means $5a \cdot 3b$, or $15ab$.

$3(b + 2c) - c$ means $3b + 6c - c$, the multiplication by 3 being performed before c is subtracted.

$5(100 - x) + 25$ means $500 - 5x + 25$, or $525 - 5x$, the multiplication by 5 being performed before 25 is added.

ORAL EXERCISES

Perform the operations indicated:

1. $(15 - 6) \div 3.$
2. $7(25 + 5).$
3. $(18 - 12) \div 6.$
4. $5(6 + 5 - 9).$
5. $(2 + 5) \cdot (5 - 3).$
6. $(24 + 6) \div (8 - 3).$
7. $(2a + a) \div 3.$
8. $(2a + 3a)b.$
9. $(2a + 3a) \div (3b + 2b).$
10. $a - (b + 2b - 3b).$

11. $ab(3c - 2c) + d$.
12. $m(m + 5m) - 2m$.
13. $(2a + a) \cdot (3c - c)$.
14. $x(5x - 2x) - y(y + 4y)$.
15. $19 - (4 + 7)$.
16. $8a - (7a - 3a)$.
17. $43x + (28x - 8x)$.
18. $86y - (10y - 4y) + 10y$.

25. Symbols of Grouping. The parenthesis is used to indicate that the number symbols grouped within it are to be taken as a single number. Other symbols of grouping are the brace, { }, the bracket, [ ], the bar, $\overline{\quad}$; these have the same meaning as the parenthesis. The bar of the fraction may also be a symbol of grouping.

Thus, in $\frac{a+b}{c+d}$, the bar groups $a + b$ into one number, and $c + d$ into one number. The fraction means $(a + b) \div (c + d)$.

26. Monomials. A **monomial** is an algebraic expression within which no operation of addition or subtraction is indicated, unless within a symbol of grouping.

Thus, a, ab, $a \div 2b$, $\frac{5ab}{c^2}$, $7(a + b)$, $\frac{2c-5}{x}$, are monomials.

27. Polynomials. An algebraic expression consisting of two or more monomials connected by the sign + or − is called a **polynomial.** The monomials are called the **terms** of the polynomial.

Thus, $a + 5b + c + \frac{d}{2}$ is a polynomial whose terms are a, $5b$, c, and $\frac{d}{2}$.

28. Binomials. A polynomial of two terms is called a **binomial.**

Thus, $a + b$, $b^2 - c$, $xy + m$, $3b^2 - a$, $\frac{cd}{2} - \frac{z}{w^2}$, are binomials.

29. Trinomials. A polynomial of three terms is called a **trinomial.**

Thus, $a + b + c$, $a + 2b - c$, $\frac{x}{y} - ab + 3c$, are trinomials.

30. Compound Terms. Expressions are sometimes grouped into **compound terms.**

Thus, $3a - 2b + c + d$ may be grouped into the trinomial $3a - 2b + (c + d)$, in which $(c + d)$ is a compound term.

ORAL EXERCISES

Name the monomials in the following list; the binomials; the trinomials:

1. $a + b - c$.	5. $gt^2 + a$.	9. $5x^2 + ay^3$.
2. $4x^2 + 7$.	6. mv^2.	10. $2hr$.
3. $a + b + c + d$.	7. $a + 5b - c$.	11. prt.
4. $\frac{1}{2}gt^2$.	8. $x - y + z + w$.	12. $2g - 5 + x^2$.

13. What is the coefficient of t^2 in Exercise 4?
14. What is the coefficient of v^2 in Exercise 6?
15. What is the numerical coefficient in Exercise 4?
16. Name the numerical coefficients in Exercise 7.
17. Name the coefficient of h in Exercise 10.
18. Read the numerical coefficients in Exercise 8.

WRITTEN EXERCISES

1. Write three monomials.
2. Write three binomials. Three trinomials.

Rewrite these expressions, using exponents where possible:

3. $a + bb$.	6. $ccc - bb$.	9. $15\,mvvq$.
4. $2\,aa + b$.	7. $3\,aayyy$.	10. $16\,xxy - cd$.
5. $aa - bbb$.	8. $2 \cdot 2 \cdot 2\,bbb$.	11. $100\,aabb - sss$.

Using $a = 1$, $b = 2$, and $c = 3$, find the value of each of the following polynomials:

12. $5a + 9b$.	21. $3a - 7b + 11c$.
13. $10a - 5b$.	22. $61b - 2c - 20a$.
14. $2a + b - c$.	23. $\frac{1}{2}a + \frac{2}{3}c - \frac{1}{6}b$.
15. $3a + 15b$.	24. $a^2 + b^2$.
16. $2a + 3b + 3c$.	25. $b^3 - a^2$.
17. $.9a + .3b - .1c$.	26. $ac^2 + 3b^3$.
18. $9b + 2a - c$.	27. $7ab^2 - c^3$.
19. $2a + 3b - 2c$.	28. $c^3 - b^3$.
20. $3a + 7b + 11c$.	29. $c^4 - 1 + 5b^3$.

31. Uses of Monomials. Monomials have various uses. For example:

1. *They are used as formulas in business arithmetic.*

Thus:

br is often used as a short way of stating *base times rate* in percentage.

prt is often used as a short way of stating *principal times rate times time* in interest.

When any particular value of r is substituted in the above formulas it must be expressed decimally. Thus, if r is 5%, it must be used as .05, or $\frac{5}{100}$.

2. *They are used as formulas of measurement.*

Thus:

ab is often used as a short way of stating *altitude times base* in finding areas.

abc is a short way of stating *length times breadth times thickness*, in finding volumes of rectangular solids, where a, b, and c are the edges.

π (read "pī") is used to denote *the number by which the length of the diameter of a circle must be multiplied to produce the circumference.*

The value of π is approximately 3.1416. Letting $c =$ circumference, and $d =$ diameter, we have $c = \pi d = 3.1416\,d$.

The word circumference, as used above, means the distance around the circle or the length of the curve. The conception of a circle as a curve is the one used in advanced mathematics, in other sciences, and in common parlance.

3. *They are used to express laws of physics.*

Thus, vt is often used as a short way of stating *product of velocity and time* in finding distance.

WRITTEN EXERCISES

1. Percentage $= br$. Find the percentage when $b = 400$ and $r = 30\,\%$. In substituting r use $\frac{30}{100}$.

2. Rate $= \frac{p}{b}$. Find the rate when $p = 15$ and $b = 750$.

3. Principal $= \frac{i}{rt}$. Find the principal when $i = \$500$, $r = 5\%$, and $t = 10$ yr.

4. Interest $= prt$. Find the interest when $p = \$100$, $r = 5\%$, and $t = 5$ yr.

5. Discount $= lr$. Find the discount when $l = \$820$ and $r = 12\frac{1}{2}\%$.

6. Rate of discount $= \frac{d}{l}$. Find the rate when $d = \$35$ and $l = \$875$.

7. The area of a rectangle $= ab$. Find the area when $a =$ 20 in. and $b = 17\frac{1}{2}$ in.

8. What is the area of a rectangle when $a = 4x$ inches and $b = 3x$ inches?

9. Copy the following table and fill out the blanks, using the value of π mentioned in Sec. 31. Answer from your table:

(*a*) What is the circumference of a circle whose diameter (d) is 3 in.?

(*b*) Of one whose diameter is 1.5 in.?

(*c*) Of one whose diameter is $\frac{1}{2}$ in.? 10 ft.?

d	πd
2 in.	6.2832 in.
3 in.	( )
$\frac{1}{2}$ in.	( )
1.5 in.	( )
10 ft.	( )

10. The distance (d) traveled by a body in time (t) moving with velocity (v) is vt. Copy the following table and fill out the blanks. Answer from your table:

(*a*) How far will a train moving 30 ft. per second go in 2 sec.?

(*b*) How far will a train moving 50 mi. per hour travel in 2 hr.?

(*c*) How far will a bullet traveling 400 ft. per second go in 5.5 sec.?

v	t	$vt = d$
30	2	60
50	2	( )
36	$1\frac{1}{4}$	( )
400	5.5	( )
150	17	( )

11. The number of square units in the area of a triangle is $\frac{1}{2}$ of the product of the numbers of linear units in its altitude (a) and base (b). Copy this table and fill out the blanks.

Answer these questions from your table:

(*a*) What is the area of a triangle of altitude 10.5 in. and base 8 in.?

(*b*) What is the area of a triangle of altitude 70 ft. and base 9 ft.?

(*c*) What is the area of a triangle of altitude 3.3 yd. and base 5 yd.?

a	b	$\frac{1}{2}ab$
6	5	15
10.5	8	( )
70	9	( )
3.3	5	( )
9	1.7	( )

32. Uses of Polynomials. Polynomials, like monomials, have various uses as formulas.

For example:

$l - lr$ may stand for *list price — discount*, or net price.

$c + rc$ may stand for *cost + rate of gain times the cost*, or the selling price.

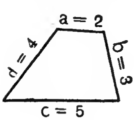
$2a + 2b$ may stand for *the perimeter of a rectangle of sides a and b.*

$2ab + 2ac + 2bc$ may stand for *the surface of a rectangular solid of edges a, b, c.*

ORAL EXERCISES

1. What is the value of $l - rl$ when $l = \$100$ and $r = 5\%$? What is the net price of goods listed at \$100 and bought at a discount of 5%?

2. What is the value of $c + cr$, when $c = \$200$ and $r = 10\%$? What is the selling price of goods which cost \$200 and are sold at a gain of 10%?



3. When $a = 3$, $b = 4$, $c = 5$, $d = 6$, what is the value of $a + b + c + d$?

4. What is the value of $a + b + c + d$ in the figure?

5. What is the value of $2a + 2b$ when $a = 3$, $b = 5$? What is the perimeter of a rectangle whose sides are 3 yd. and 5 yd.?

WRITTEN EXERCISES

1. When $a = 10$, $b = 15$, $c = 24$, find the value of $2ab + 2ac + 2bc$. What is the area of the surface of a rectangular prism whose edges are 20 in., 15 in., 50 in.?

2. Find the value of $2ab + 2bc + 2ac$, when $a = 20$, $b = 25$, $c = 50$.

3. Find the value of $a^2 + b^2$, when $a = 13$, $b = 210$; also when $a = 75$, $b = 100$.

4. Find the value of $a^2 + b^2 + c^2$, when $a = 35$, $b = 20$, $c = 65$; also when $a = 100$, $b = 75$, $c = 150$.

5. Find the value of $ut + \frac{1}{2}at^2$, when $u = 1500$, $a = 200$, $t = 10$.

33. Degree of a Monomial. The degree of a monomial is the sum of the exponents of its literal factors.

Thus: a^2 is of the second degree.
$3\,ab$ is of the second degree.
$2\,a^3b$ is of the fourth degree.

But the degree is often expressed with respect to some letter or letters.

Thus, $3\,ax^2y^3$ is of the first degree with respect to a, of the second degree with respect to x, of the third degree with respect to y, and of the fifth degree with respect to x and y.

34. Degree of a Polynomial. The degree of a polynomial is that of its term of highest degree; its degree with respect to a letter is the highest degree of that letter in the polynomial.

Thus, $a^3x^2 - 5\,by + xy^4z$ is of the sixth degree; it is of the third degree in a, the first in b and in z, the second in x, the fourth in y, and the sixth in x, y, and z.

NOTE. It is not necessary in elementary algebra to define the degree of expressions containing radicals or fractions.

ORAL EXERCISES

State the degree of each of the following monomials:

1. a^2b.
2. ax.
3. $2\,ax$.
4. $3\,a^2x$.
5. a^3b.
6. $4\,a^2b^2$.
7. $5\,mn^2$.
8. $9\,xyz$.
9. $9\,x^2bz$.
10. x^2yz^2.
11. $\frac{1}{2}\,mv$.
12. $\frac{1}{2}\,gt^2$.

13. State the degree of the expressions in Exercises 1–6 with respect to a; in Exercises 8–10 with respect to x.

14. State the degree of the expressions in Exercises 4–9 with respect to each letter involved.

State the degree of each polynomial; also its degree with respect to each letter:

15. $ab^2 + b$.
16. $a^2b + a^2c + d$.
17. $3\,x^2 + 2\,x + 1$.
18. $abc + c^3 + bcd$.
19. $3\,ax + 3\,x^2y + y^3$.
20. $\frac{1}{2}\,m^2 + n^2 + 3\,pq$.

REVIEW

ORAL EXERCISES

State the product of each set of factors:

1. $8, 6, a$. **2.** $b, x, 3$. **3.** $a, y, 3, 4$. **4.** $2, x, a, x$.

Name three sets of factors for each of the following:

5. $24\,mx$. **6.** ax^2y. **7.** $\frac{1}{2}gt^2$. **8.** $21\,abc$.

In each of the following name (1) the coefficient of x; (2) the numerical coefficient:

9. $4\,ax$. **10.** $12\,bx$. **11.** acx. **12.** $5\,mxy$. **13.** $12\,c^2x$.

State the value of:

14. 2^4. **15.** 7^2. **16.** 5^3. **17.** $3^2 \cdot 2^4$. **18.** $2^3 \cdot 5^2$.

From the following list select by number the binomials; the trinomials; the monomials:

19. ax^2.

20. $a - x$.

21. $2\,x^3 - 5\,a$.

22. $6\,a + 7\,xy$.

23. $3\,ay^2 - 4\,x + 1$.

24. $2\,ab + 7\,x^3 - 5\,x^2$.

25. $a^2 - 2\,ab + b^2$.

26. $x^3 + 3\,x^2 + 3\,x + 1$.

27. State the degree of each expression in Exercises 19–26 with respect to x. With respect to a.

28. The sides of a triangle are $3\,a$, $2\,b$, $5\,c$. What is its perimeter? What kind of polynomial is this?

29. What is the value of $c + cr$ when $c = \$500$ and $r = 20\,\%$?

30. What is the value of $x^2 + 5\,a$ when $x = 3$ and $a = 2$?

State the results of the indicated operations:

31. $39 \div 13 + 1$.

32. $8 + 5 \cdot 2$.

33. $7 \cdot 8 - 6$.

34. $39 \div (13 + 1)$.

35. $(8 + 5)2$.

36. $7(8 - 6)$.

37. $8 - 5 + 2$.

38. $7\,a - 3\,a + 2\,a$.

39. $8\,x + 5\,x - x$.

40. $8 - (5 + 2)$.

41. $7\,a - (3\,a + 2\,a)$.

42. $8\,x + (5\,x - x)$.

WRITTEN EXERCISES

Indicate the prime factors of the following, using exponents where possible:

1. 88. 2. 144. 3. 200. 4. 525.

Factor so that one factor of each is a power of 10:

5. 1000. 6. 900. 7. 23,000. 8. 3,000,000.

Find the value of each of the following if $a=2$, $b=3$, $x=1$:

9. $4ax+1$. 10. $\frac{3ax^2-1}{2b}$. 11. $\frac{b^4-a^4}{x+1}$. 12. $\frac{a^4+4}{b+2x}$.

13. The area of a triangle is $\frac{1}{2}ab$. What is the area of a triangle in which $a=3n$ feet and $b=14n$ feet?

14. Find the value of $a^2+2ab+b^2$ when $a=7$, $b=3$.

15. Find the value of $a^3-3a^2x+3ax^2$ when $a=7$, $x=2$.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define *product*. Also *factor*. Sec. 8.
2. What is a *numerical* factor? A *literal* factor? Sec. 10.
3. Explain the *order* of the factors in a product. Sec. 10.
4. State the *Commutative Law of Multiplication*. Sec. 11.
5. Define *exponent*; also *base*; also *power*. Secs. 13, 14.
6. Define *coefficient*; also *numerical coefficient*. Secs. 19, 20.
7. Name the usual symbols of grouping. Secs. 23, 25.
8. What process is indicated by a number standing next to a parenthesis with no intervening sign? Sec. 24.
9. Define *monomial*; also *polynomial*. Secs. 26, 27.
10. Define *binomial*; also *trinomial*. Secs. 28, 29.
11. What is a *compound term*? Sec. 30.
12. How is the *degree of a monomial* found? Also the degree with respect to a given letter? Sec. 33.
13. What is the *degree of a polynomial*? Sec. 34.

HISTORICAL NOTE

To understand how the symbols of algebra came to be what they are, we may again refer to Diophantos, for in his work, *Arithmetica*, he made the first approach to algebraic notation by using certain signs and abbreviations. Since these characters are formed from Greek letters and words, it would be confusing to illustrate more than a few of them, and we will confine ourselves to showing how Diophantos would have written the polynomial, $3x^2 + x - 2$. His coefficients were placed after the literal factors; hence, $3x^2$ would be $x^2 3$, and x would be $x 1$. We must also remember that the coefficients were expressed by the Greek numerals: $\bar{\alpha}$ for 1, $\bar{\beta}$ for 2, $\bar{\gamma}$ for 3, and so on. For x he used the symbol ς, probably a contraction of the first two letters of the Greek word meaning "unknown number." Diophantos did not know of exponents, but for x^2 he used $\delta^{\hat{v}}$, an abbreviation of the word meaning "square." His sign for minus was ⋔, derived from the letters of the word meaning "minus." He had no sign for plus, but placed numbers next to each other to indicate addition. Thus, $3x^2 + x - 2$, if written by Diophantos, would be arranged thus, $x^2\ 3\ x\ 1 - 2$
and look like $\delta^{\hat{v}}\ \bar{\gamma}\ ς\ \bar{\alpha}$ ⋔ $\bar{\beta}$

The notation of algebra, as we now use it, was developed about the sixteenth century. By that time the processes came to have signs to indicate them; for example, addition and subtraction were denoted by $\bar{p}$, $\bar{m}$, probably abbreviations for the words "plus" and "minus," and these were finally supplanted by + and −. The latter forms may have resulted from the rapid writing of the letters p and m, but a more likely origin is found in the commercial arithmetic of that time. In recording weights of merchandise as marked in the warerooms, the sign + was used to denote overweight, and the sign − to denote short weight. Thus, if a bale supposed to weigh 100 pounds weighed 103 pounds, it was recorded as 100 + 3, or if it weighed 97 pounds, it was recorded as 100 − 3. From this practice the signs + and − probably came to be generally used as signs of addition and subtraction. The sign × was first used by the English mathematician Oughtred in 1631, and the simpler sign, · , for multiplication was used by the famous French mathematician, René Descartes. For the square of a he used aa, which later became a^2. The symbol for square root became ℞, the initial letter of the Latin word *radix* ("root"); this was afterward changed to $\sqrt{}$ by the German algebraist, Stifel (1554). (The abbreviation ℞, now used in physicians' prescriptions, denotes similarly the first letter of the Latin word *recipe* — "take.") The sign of equality (=) was first used by Robert Recorde, an Englishman, 1542. His explanation of this sign was that "Noe. 2. thynges can be moare equalle" than these parallel lines.

CHAPTER III

THE EQUATION

35. PREPARATORY.

1. What weight, w, will balance the package of rice in the figure? What must w be to balance two such packages?

2. If half of the rice be taken from the package, what must w be?

3. What must w be to balance a 16-ounce package and an 8-ounce package?

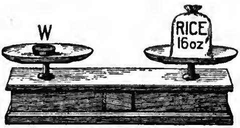


FIG. 1.

4. What must w be to balance the pans in each of the following figures?

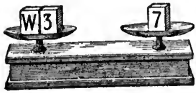


FIG. 2.

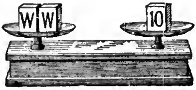


FIG. 3.

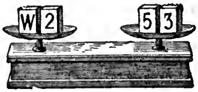


FIG. 4.

5. In Fig. 2 the fact that the weights balance is expressed by $w + 3 = 7$. Express the condition that the weights balance in Fig. 3. In Fig. 4.

36. The Equation. If two expressions represent the same number, their equality may be indicated by the sign $=$; such a statement of equality is called an **equation**.

Thus, $w + 3 = 7$, $2w = 8$, $y = 3 + 4$, and $2x + 1 = 9$ are equations.

37. Members of an Equation. The two expressions connected by the sign of equality are called the **members** of the equation.

Thus, in $w + 3 = 7$, $w + 3$ and 7 are members; they are called respectively the *left* member and the *right* member, or, also, the *first* member and the *second* member.

38. Identical Equation. An equation that either involves no letters, or that is true for any values whatever that may be given to the letters involved, is called an **identical equation** or, briefly, an **identity.**

Thus, $3 + 4 = 7$, $a + b = b + a$, $5x = 8x - 3x$, are identities

39. Conditional Equation. An equation that is true only on condition that the letter or letters involved have particular values is called a **conditional equation** or, briefly, an **equation.**

Thus, $w + 3 = 7$ is an equation that is true only on condition that w represents 4.

The equation is an abbreviated sentence. The identical equation is a declarative sentence and states that the two members are necessarily the same — differing, at most, in form. The conditional equation is an interrogative sentence, and asks what numbers must be used in place of the letters in order that the two members may be equal.

ORAL EXERCISES

1. What must be added to 3 to make 10? State the number for which the question mark stands in $3 + ? = 10$.

2. State the number denoted by the question mark in each case: $5 + ? = 12$; $? + 15 = 25$; $60 = 45 + ?$

3. State the number for which x stands in each of the following: $5 + x = 12$; $x + 15 = 25$; $60 = 45 + x$; $25 + x = 40$.

State the number denoted by the question mark in each case:

4. 2 times $? = 12$.
5. 5 times $? = 30$.
6. $\frac{1}{2}$ of $? = 8$.
7. $\frac{3}{2}$ of $? = 12$.
8. 2 times $? + 1 = 9$.
9. 3 times $? + 5 = 11$.

10. State the number represented by t in each case: $2t = 8$; $4t = 32$; $\frac{1}{2}t = 10$; $\frac{3}{2}t = 30$; $2t + 1 = 11$.

11. A certain number less 4 is 20. What is the number?

12. A certain number less 5 is 15. What is the number?

13. x less 4 is 20. What is x? $2x$ less 5 is 25. What is $2x$?

14. In $3x + 5 = 17$, what is $3x$? What is x?

15. In $4p + 2 = 10$, what is $4p$? What is p?

40. In equations we need to express relations between numbers by algebraic symbols.

For example, a number that is greater than n by 4 may be written $n+4$.

Or, if the sum of two numbers is 11 and one of them is n, the other is expressed by $11-n$.

Or, if the difference of two numbers is 15 and the larger of them is x, the other is expressed by $x-15$.

WRITTEN EXERCISES

Represent a number:

1. Greater than n by 7.
2. Greater than n by 3.
3. Less than n by 5.
4. Less than n by a.
5. Greater than $2n$ by b.
6. $2b$ less than 5 times n.
7. 3 times n less one.
8. 3 greater than $a-b$.
9. a greater than $b+c$.
10. a less than $2c-b$.
11. 6 greater than 4 times x.
12. $2a-1$ greater than $b+2$.

Write the other part if:

13. One part of 12 is 7.
14. One part of x is 3.
15. One part of 10 is y.
16. One part of n is a.
17. One part of a is n.
18. One part of $a-b$ is x.

19. The difference between two numbers is 8. The smaller of the two is 12. Write the larger.

20. The difference between two numbers is d. The smaller of them is a. Write the larger.

21. The difference between two numbers is 10. The larger is 25. Write the smaller.

22. The difference between two numbers is 21. The larger is y. Write the smaller.

23. The difference between two numbers is d. The larger is $2c$. Write the smaller.

24. $4n$ plus a number is 26. Write the number in symbols.

25. The sum of three numbers is 45. One of them is 6, another is a. Write the third.

26. The sum of three numbers is s. One of them is a, another $2b$. Write the third.

Express by how much:

27. 18 exceeds a.
28. 17 exceeds $2c$.
29. 24 exceeds $a-b$.
30. 36 exceeds $2a-c$.
31. 12 is less than n.
32. 15 is less than $2a$.
33. 44 is less than $b+c$.
34. $35+a$ exceeds $b+c$.

If A is n years old, express his age:

35. 5 years ago.
36. x years ago.
37. 5 years hence.
38. x years hence.
39. $a-3$ years ago.
40. $2a-5$ years hence.

41. The translation of algebraic expressions into words assists in interpreting equations.

EXAMPLES

1. $n-3=7$ is translated "a number less three is seven."
2. $2x-1=5$ is translated "twice a number less one is five."
3. $3x-2=5x-8$ is translated "three times a number less two equals five times the number less eight."
4. $2n-m=4$ is translated "twice a number less a second number is four."
5. $A=\frac{1}{2}ab$, referring to a triangle, is translated "the area of a triangle is one half the product of the base and altitude."
6. $V=\pi r^2a$, referring to a circular cylinder, is translated "the volume of a cylinder is *pi* times the square of the radius times the altitude."

ORAL EXERCISES

Translate into words:

1. $n+5=8$.
2. $x-1=10$.
3. $3x-2=12$.
4. $5x-1=x+4$.
5. $m+n=17$.
6. $m-n=1$.
7. $2x+3y=15$.
8. $n, n+1, n+2$.
9. $2n, 2n+2, 2n+4$.
10. $n, n-1, n-2$.
11. $2n, 2n-1, 2n-3$.
12. $2x-1=2b+1$.
13. $3p+6=4q-6$.
14. $pq+1=2pq-10$.
15. Taking A to mean the area of a rectangle, read $A=ab$.
16. Taking V to mean the volume of a rectangular solid, of dimensions a, b, and c, read $V=abc$.

17. Taking A to mean the area of a circle, read $A = \pi r^2$.

18. Taking C to mean the circumference of a circle, read $C = 2\pi r$.

19. Taking I to mean simple interest, read $I = prt$. Also read $t = \frac{I}{pr}$.

42. Substitution. A number symbol put in place of another is said to be **substituted** for it.

For example:
$5a + 2$ becomes $5 \cdot 3 + 2$ when 3 is substituted for a; and $ax + 7$ becomes $ab + 7$ when b is substituted for x.

43. Unknown. A number symbol whose value is not known is called an **unknown number,** or simply an **unknown.**

44. Satisfying an Equation. If an equation becomes an identity when certain numbers are substituted for the unknowns, the numbers substituted are said to **satisfy** the equation.

Thus, 5 is said to *satisfy* the equation $3x = 15$, because $3 \times 5 = 15$. The equation is not satisfied by any other number, because 3 times any other number is not 15. Also, 7 and 5 are said to satisfy the equation $3x + 2y = 31$, because $3 \cdot 7 + 2 \cdot 5 = 31$.

45. Root of an Equation. A number that satisfies an equation is called a **root** of the equation.

46. Solving Equations. To **solve** equations is to find their roots.

ORAL EXERCISES

What number satisfies each of the following equations?

1. $3x = 6$.
2. $9x = 18$.
3. $7x = 35$.
4. $4x = 32$.
5. $5x + 2 = 22$.
6. $8x + 12 = 20$.
7. $7y + 5 = 40$.
8. $2y + 1 = 3$.
9. $30 - 6 = 4y$.
10. $4w + 2 = 10$.
11. $4w + 6 = 46$.
12. $\frac{1}{2}z + 3 = 9$.
13. $2n = 90$.
14. $2y + 7 = 13$.
15. $\frac{1}{3}w + 3 = 2$.
16. $2n = 4800$.
17. $2n + 1 = 27$.
18. $2n + 1 = 625$.

Solve the equations:

19. $4x = 20$.
20. $3y + 4 = 25$.
21. $4t + 1 = 27$.
22. $2r + 5 = 13$.
23. $4v + 1 = 9$.
24. $6u = 14 - 2$.
25. $8 + 2 = 5s$.
26. $7x + 7 = 28$.
27. $14 = 6x + 2$.
28. $20 - 4 = 4y$.
29. $9r = 360$.
30. $17u = 3400$.
31. $20p = 50 - 10$.
32. $60 + 15 = 25x$.
33. $18y = 360$.

47. PREPARATORY.

If two weights are in balance, and if the following changes are made in one weight, what change, in each case, must be made in the other to preserve the balance?

1. Two ounces added.
2. Two ounces taken away.
3. The number of ounces in one weight made three times as great.
4. The number of ounces in one weight made $\frac{1}{4}$ as great.

48. Properties used in Solving Equations. The preceding exercises suggest the following properties:

1. *If the same number is added to equal numbers, the results are equal.*
2. *If the same number is subtracted from equal numbers, the results are equal.*
3. *If equal numbers are multiplied by the same number, the results are equal.*
4. *If equal numbers are divided by the same number (not zero), the results are equal.*

NOTE. The reason for excluding zero as a divisor is explained in Chapter IX.

49. The following examples show how these properties are used in solving equations:

EXAMPLES

1. Solve: $3x + 5 = 23$. (*1*)

Subtracting 5 from both members, $3x = 18$. (*2*)

Dividing both members of (*2*) by 3, $x = 6$. (*3*)

TEST. 6 *satisfies* $3x + 5 = 23$, because $3 \cdot 6 + 5 = 23$.

2. Solve: $p + 2 + \frac{1}{3}p = \frac{2}{3}p + 6.$ (*1*)

Subtracting $\frac{2}{3}p$ from both members, $\frac{2}{3}p + 2 = 6.$ (*2*)

Subtracting 2 from both members of (*2*), $\frac{2}{3}p = 4.$ (*3*)

Dividing both members of (*3*) by $\frac{2}{3}$, $p = 6.$ (*4*)

TEST. 6 *satisfies* (*1*), because $6 + 2 + \frac{1}{3} \cdot 6 = \frac{2}{3} \cdot 6 + 6.$
$10 = 10.$

50. Testing. The correctness of the work of solving an equation should be tested by substituting the result in the given equation. If the members become identical, the number substituted is a root of the equation.

WRITTEN EXERCISES

Solve and test:

1. $4x + 1 = 7.$
2. $3x + 1 = 10.$
3. $5x = x + 16.$
4. $2x + 7 = 27.$
5. $16t + 5 = 37.$
6. $14x = 25 + 9x.$
7. $48 = 8y + 16.$
8. $12t + 13 = 49.$
9. $3s + 2 = 19.$
10. $z + 4z = 85.$
11. $15s + 2 = 12.$
12. $\frac{1}{2}x + 2 = \frac{7}{3}.$
13. $6y + 2 = 20.$
14. $8z + 2 = 42.$
15. $72 = 12x.$
16. $6x = 9 + 3x.$
17. $\frac{3}{2}x = 25 + \frac{1}{2}x.$
18. $11y + 1 = 9y + 3.$
19. $3x + 2x = 4x + 16.$
20. $11m + 3 = 2m + 9 + 2m.$
21. $8p + 5 = 2p + 14 + 3p.$
22. $12z + 28 = 7z + 53.$
23. $40 + 3z = 58 + z.$
24. $\frac{2}{3}y + 45 = \frac{1}{3}y + 55.$
25. $\frac{4}{5}y + 31 = \frac{2}{5}y + 41.$
26. $\frac{5}{7}x + 20 = 32 + \frac{3}{7}x.$

51. Use of the Equation in Solving Problems. Equations may be used in solving problems.

EXAMPLES

1. If a certain number is doubled and 16 is added to the product, the result is 46. What is the number?

SOLUTION. 1. Let n be the number.
2. Then $2n + 16$ is double the number plus 16.
3. But 46 is given as double the number plus 16.
4. Therefore, $2n + 16 = 46.$
5. Therefore, $2n = 30$, and $n = 15.$ Why?

TEST. 15 doubled makes 30, and 30 plus 16 is 46.

2. A salesman sold twice as many articles on Friday as on Thursday, and 5 more on Saturday than on Friday; on Saturday he sold 15. How many did he sell on Thursday?

SOLUTION. 1. Let x be the number that he sold on Thursday.

2. What does $2x$ represent? $2x + 5$?

3. State two expressions, each of which is the number sold on Saturday.

4. Since $2x + 5 = 15$, $x = 5$.

TEST. $2 \cdot 5 + 5 = 15$.

Any letter may be used to represent the unknown, as y for the number of years, d for the number of dollars, r for rate, or p for the number of pounds pressure as in physics, but in algebra x is most frequently used.

52. Finding the Equation. In each solution above, step 4 contains the statement of the problem in the form of an equation. *This statement is reached by finding two expressions for the same number and using them as the members of an equation.*

53. Sign of Deduction. Instead of the word "hence," or "therefore," the sign $\therefore$ is often used. It is called the **sign of deduction.**

Thus, $\therefore 2n + 16 = 46$ is read, "Therefore, $2n + 16 = 46$."

WRITTEN EXERCISES

Write the solution of each problem in steps as shown above:

1. A house and lot are worth \$4800, and the house is worth 7 times as much as the lot. Find the value of each.

2. Lucy thought of a number, doubled it, added 16, and obtained 50. Of what number did she think?

3. The continued height of a tower and flagstaff is 120 ft.; the height of the tower is 5 times that of the flagstaff. Find the height of each.

4. $\frac{2}{3}$ of the total height of a bridge pier is out of the water, and 10 ft. of the height is under water. What is the height of the pier?

5. When goods are sold at a gain of $\frac{1}{5}$ of their cost, what is the cost of goods which sell for \$12?

6. A man's salary was increased by $\frac{1}{3}$ of itself; he then received \$1600. What was his salary before the increase?

7. A merchant gained in one year an amount equal to $\frac{1}{4}$ of his capital. He then had \$6250. How many dollars had he at the beginning of the year?

8. The area of Kansas is twice that of Ohio. The sum of their areas is 123,000 sq. mi. Find the area of each.

9. A freight train consisted of 48 cars. The number of closed cars was 6 more than twice the number of open cars. Find how many there were of each.

REVIEW

WRITTEN EXERCISES

1. $\frac{7}{15}$ of the distance from Boston to Cincinnati is 441 mi. Find the distance between these two cities.

2. The product of a certain number and 13 is 221. Write an equation expressing this fact, and find the number.

3. 3 times a certain number increased by 5 equals twice the number increased by 17. Find the number.

4. 3 times a number plus 3 equals $\frac{1}{3}$ of the number plus 27. Find the number.

5. In a recent year France mined twice as much coal as Russia, and together they produced 48 million tons. How many tons did each produce?

6. In a recent year the United States mined 12 times as much coal as Belgium, and together they produced 286 million tons. How many tons did each produce?

7. The length of a garden was 3 times its width, and the distance around it was 72 yd. Find its length and width.

8. If a tennis ball rebounds $\frac{2}{3}$ of the height from which it was dropped, from what height must it be dropped to rebound $3\frac{1}{2}$ ft.?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define an *equation*. Sec. 36.
2. What are the *members* of an equation? Sec. 37.
3. What is an *identical equation?* Sec. 38.
4. What is a *conditional equation?* Sec. 39.
5. What is meant by *substitution?* Sec. 42.
6. What is an *unknown?* Sec. 43.
7. Explain the meaning of *satisfy an equation.* Sec. 44.
8. Define a *root* of an equation. Sec. 45.
9. Define *to solve* an equation. Sec. 46.
10. What are the *properties* used in solving equations? Sec. 48.
11. How is the correctness of the work in solving equations *tested?* Sec. 50.
12. How is the *solution* of a problem tested? Sec. 51.
13. How is a problem *stated* in the form of an equation? Sec. 52.
14. What is the sign of deduction? Sec. 53.

CHAPTER IV

RELATIVE NUMBERS

54. Numbers used in Algebra. The preceding work is much like that of arithmetic; in fact, it might be called *literal arithmetic.* We take up now a class of numbers belonging to algebra proper, and the following examples will illustrate them:

I. *Distances measured in opposite directions.*

1. 40 ft. up an elevator shaft + 20 ft. down the shaft is the same as how many feet up the shaft?

2. 20 ft. up + 40 ft. down is the same as how many feet down?

Similarly, to what is each of the following equivalent?

3. 40 rd. traveled to the right + 20 rd. traveled to the left.

4. 20 rd. traveled to the right + 20 rd. traveled to the left.

5. 20 rd. traveled to the right + 50 rd. traveled to the left.

II. *Rise and fall of temperature.*

1. Calling rise of temperature R and fall of temperature F $15° R + 10° F$ is the same as (?)° R.

Similarly:

2. $10° R + 15° F = ?$

3. $35° F + 45° R = ?$

4. $30° F + 15° R = ?$

5. $40° R + 40° F = ?$

III. *Amounts gained and lost.*

1. $22 gain + $26 loss = ? loss.

Similarly, using G for gain and L for loss:

2. $\$40\,G + \$30\,L = ?$

3. $\$17\,G + \$17\,L = ?$

4. $\$35\,L + \$20\,G = ?$

5. $\$45\,G + \$15\,L = ?$

55. Relative Numbers. In each of the preceding illustrations we have considered quantities which had two opposite directions, or **senses.** Numbers which measure quantities having opposite senses are called **relative numbers.**

56. In algebra we distinguish two opposite senses by calling one **positive** and the other **negative.** Either may be called positive, but the opposite to the positive is always called negative.

For example:

If distance *upward* is called *positive*, distance *downward* is called *negative.* If a *rise* of temperature is called *positive*, a *fall* is called *negative.*

57. Positive and Negative Number. A number that measures a quantity taken in the positive sense is called a **positive number**; one that measures a quantity taken in the negative sense is called a **negative number.**

ORAL EXERCISES

What must be taken as negative when each of the following is taken as positive?

Number of:

1. Feet to the right.
2. Miles southward.
3. Dollars gained.
4. Degrees upward.
5. Points won.
6. Pounds lifted.

58. Notation for Positive and Negative Numbers. Since a number added is offset by the same number subtracted, and relative numbers similarly offset each other, the signs + and − are used to designate positive and negative numbers respectively.

Thus:

+ 3 means 3 positive units, and denotes 3 units to be added.

− 3 means 3 negative units, and denotes 3 units to be subtracted.

59. Signs of Character and Signs of Operation. Thus, in algebra the signs +, −, are used to indicate the *operations* of adding or subtracting numbers, and also to indicate the positive or negative *character* of numbers.

If it is necessary to distinguish a sign of character from a sign of operation, the former is put into a parenthesis with the number it affects.

Thus, + 8 − (− 3) means: positive 8 minus negative 3.

When no sign of character is expressed, the sign plus is understood.

Thus, 5 − 3 means: positive 5 minus positive 3.

Similarly, $8a + 9a$ means: positive $8a$ plus positive $9a$.

60. Signed Numbers. In algebra every number is understood to have either the sign + or the sign −. Consequently the numbers of algebra are often called **signed numbers.**

61. Absolute Value. The value of a signed number apart from its sign is called its **absolute, or numerical, value.**

Thus, 6° is the absolute value of 6° above zero or 6° below zero.
And 6 is the absolute value of either $+6$ or -6.

ORAL EXERCISES

Read the following in full, in accordance with Sec. 59:

1. $7-4$.
2. $-6-8$.
3. $-8+25$.
4. $-2+7$.
5. $9-(-3)$.
6. $14-(-6)$.
7. $-14-(+6)$.
8. $-6-(-8)$.
9. $7-15$.
10. $-7-(+2)$.
11. $6-8$.
12. $6+8$.
13. $-6+(-8)$.
14. $-4+(-9)$.
15. $15-(-9)$.

WRITTEN EXERCISES

Indicate, using the signs +, −:

1. The sum of positive 6 and positive 4.
2. The sum of positive a and negative b.
3. The difference of positive x and positive y.
4. The difference of negative 6 and positive 3.
5. The difference of negative a and positive b.
6. The sum of negative c and negative d.

62. PREPARATORY.

1. 4 points won + 3 points won = —— points won.
2. 4 points lost + 5 points lost = —— points lost.
3. 7° above zero + 9° above zero = —— degrees above zero.

63. Addition of Numbers having Like Signs. When the numbers added are positive, the sum is positive; and when they are negative, the sum is negative.

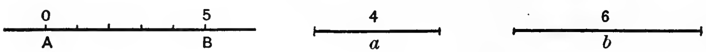
Therefore, to add either positive numbers or negative numbers, find the sum of their absolute values and prefix the corresponding sign.

ORAL EXERCISES

Add:

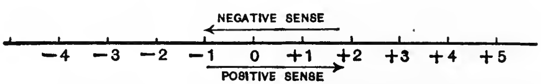
1. $\begin{array}{r}+2\\ \underline{+3}\end{array}$	5. $\begin{array}{r}+6\\ \underline{+5}\end{array}$	9. $\begin{array}{r}+2a\\ \underline{+3a}\\ (\ \ )\,a\end{array}$	13. $\begin{array}{r}-6x\\ \underline{-5x}\end{array}$
2. $\begin{array}{r}-2\\ \underline{-3}\end{array}$	6. $\begin{array}{r}-6\\ \underline{-5}\end{array}$	10. $\begin{array}{r}-2a\\ \underline{-3a}\\ (\ \ )\,a\end{array}$	14. $\begin{array}{r}-5y\\ \underline{-8y}\end{array}$
3. $\begin{array}{r}+4\\ \underline{+3}\end{array}$	7. $\begin{array}{r}-4\\ \underline{-7}\end{array}$	11. $\begin{array}{r}+4b\\ \underline{+3b}\\ (\ \ )\,b\end{array}$	15. $\begin{array}{r}-10x\\ \underline{-\ \ 5x}\end{array}$
4. $\begin{array}{r}-4\\ \underline{-3}\end{array}$	8. $\begin{array}{r}-5\\ \underline{-8}\end{array}$	12. $\begin{array}{r}-4b\\ \underline{-3b}\end{array}$	16. $\begin{array}{r}-12nx\\ \underline{-\ \ 8nx}\end{array}$

64. Number Pictures or Graphs. Numbers are often represented by lines. Such representations are called **graphs.**



Thus, the line AB represents 5, the line a represents 4, and the line b represents 6.

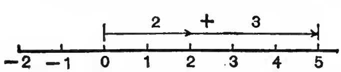
65. Graphical Addition. Positive and negative numbers may be arranged on a straight line as follows:



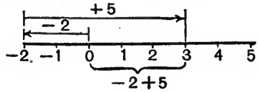
This arrangement is called the **number scale,** and it may be used to perform additions graphically.

EXAMPLES

1. To add $+2$ and $+3$: Let a moving point start at 0 and proceed 2 units in the positive direction (to the right), and from the place where it then is, proceed 3 units farther in the positive direction. The final position of the moving point will be distant $2+3$ units from the starting point. That is, $2+3=5$.

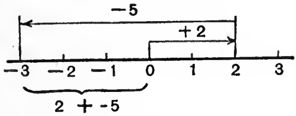


2. To add -2 and $+5$: Let the point proceed 2 units in the negative sense (toward the left), and from there 5 units in the positive sense. The final position is three positive units from the starting point. That is, $-2+5=3$.



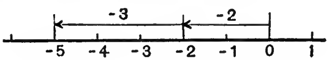
3. To add $+2$ and -5: Proceed 2 units to the right, and from there 5 units to the left. The final position of the moving point is three units to the left of the starting point. That is,

$$2+(-5)=-3.$$



4. Similarly, to add (-2) and (-3) proceed two units to the left, and from there three units to the left. The final position is 5 units to the left.

That is, (-2) and $(-3)=-5$.



WRITTEN EXERCISES

Add by means of the number scale as above:

1. $3+4$.	4. $-2+8$.	7. $4+(-3)$.	10. $-2+(-5)$.
2. $8+3$.	5. $-2+9$.	8. $2+(-6)$.	11. $-3+(-3)$.
3. $-2+5$.	6. $-9+5$.	9. $7+(-3)$.	12. $-4+(-7)$.

66. PREPARATORY.

1. Add 5 and -3.

Regard 5 as made up of $+3$ and $+2$; then

$$\left.\begin{array}{r}+3+2\\ -3\\ \hline 0+2\end{array}\right\}\text{ or }\left\{\begin{array}{r}5\\ -3\\ \hline 2\end{array}\right.$$

That is, the -3 offsets $+3$ of the $+5$, and the sum is $+2$.

2. Add -8 and 6.

Regard -8 as made up of -6 and -2; then

$$\left.\begin{array}{r}-6-2\\ +6\\ \hline 0-2\end{array}\right\}\text{ or }\left\{\begin{array}{r}-8\\ 6\\ \hline -2\end{array}\right.$$

That is, the $+6$ offsets -6 of the -8, and the sum is -2.

Zero, as in arithmetic, means the difference between two equal numbers like $3-3$; hence, in accordance with Sec. 58, $3+(-3)=0$.

Thus the addition of zero has no effect; $6+0=6$, or $-6+0=-6$.

67. Addition of Numbers with Unlike Signs. In adding a positive and a negative number, a positive unit and a negative unit offset each other.

Therefore, *to add a positive and a negative number find the difference of their absolute values and prefix to it the sign of the number having the greater absolute value.*

ORAL EXERCISES

State the sums:

1. 7 negative units + 4 positive units.
2. 7 negative units + 12 positive units.
3. 8 negative units + 7 positive units.
4. 9 positive a's + 9 negative a's.
5. 10 positive x's + 15 negative x's.

WRITTEN EXERCISES

Add, separating the numbers as in Sec. 66:

1. $\begin{array}{r} 5 \\ -2 \\ \hline \end{array}$

2. $\begin{array}{r} 6 \\ -4 \\ \hline \end{array}$

3. $\begin{array}{r} 8 \\ -5 \\ \hline \end{array}$

4. $\begin{array}{r} 10 \\ -7 \\ \hline \end{array}$

5. $\begin{array}{r} -3 \\ 7 \\ \hline \end{array}$

6. $\begin{array}{r} -5 \\ 9 \\ \hline \end{array}$

7. $\begin{array}{r} -6 \\ 12 \\ \hline \end{array}$

8. $\begin{array}{r} -11 \\ 20 \\ \hline \end{array}$

9. $\begin{array}{r} -9 \\ 3 \\ \hline \end{array}$

10. $\begin{array}{r} -6 \\ 2 \\ \hline \end{array}$

11. $\begin{array}{r} -12 \\ 8 \\ \hline \end{array}$

12. $\begin{array}{r} -17 \\ 9 \\ \hline \end{array}$

13. $\begin{array}{r} -40 \\ 30 \\ \hline \end{array}$

14. $\begin{array}{r} -25 \\ 17 \\ \hline \end{array}$

15. $\begin{array}{r} 43 \\ -55 \\ \hline \end{array}$

16. $\begin{array}{r} 39 \\ -44 \\ \hline \end{array}$

17. $\begin{array}{r} -9a \\ +5a \\ \hline \end{array}$

18. $\begin{array}{r} +9b \\ -5b \\ \hline \end{array}$

19. $\begin{array}{r} +9b \\ -9b \\ \hline \end{array}$

20. $\begin{array}{r} -15p \\ +10p \\ \hline \end{array}$

21. $\begin{array}{r} -3x \\ +9x \\ \hline \end{array}$

22. $\begin{array}{r} +8x \\ -13x \\ \hline \end{array}$

23. $\begin{array}{r} -12ab \\ +10ab \\ \hline \end{array}$

24. $\begin{array}{r} -8mn \\ +9mn \\ \hline \end{array}$

25. $\begin{array}{r} +4xy \\ -8xy \\ \hline \end{array}$

68. Preparatory.

1. $5 + ? = 8.$
2. $5 + ? = 3.$
3. $-5 + ? = -7.$
4. $-5 + ? = 3.$

5. Question 1 may be read "8 less 5 are how many?" Read the questions in 2, 3, and 4, and state the difference in each case.

69. Subtraction of Positive and Negative Numbers. The difference is the number which added to the subtrahend produces the minuend.

Thus:

$6 \text{ less } -4 = 10$ because $-4 + 10 = 6.$

$-15 \text{ less } -7 = -8$ because $-7 \text{ plus } -8 = -15.$

$4a \text{ less } -2a = 6a$ because $-2a + 6a = 4a.$

The terms **subtrahend** and **minuend** are used as in arithmetic, the former to mean the number taken away and the latter the number from which the subtrahend is taken.

ORAL EXERCISES

State the numbers to fill the blanks:

1. $\begin{cases} 8 + (\ ) = 15, \\ 15 - 8 = (\ ). \end{cases}$
2. $\begin{cases} 8 - (\ ) = 2, \\ 2 - 8 = (\ ). \end{cases}$
3. $\begin{cases} 12 - (\ ) = -1, \\ -1 - 12 = (\ ). \end{cases}$
4. $\begin{cases} -7 + (\ ) = 3, \\ 3 - (-7) = (\ ). \end{cases}$
5. $\begin{cases} -6 + (\ ) = 8, \\ 8 - 14 = (\ ). \end{cases}$
6. $\begin{cases} -5 + (\ ) = 20, \\ 20 - 25 = (\ ). \end{cases}$
7. $\begin{cases} -6 + (\ ) = -9, \\ -9 - (-3) = (\ ). \end{cases}$
8. $\begin{cases} -11 + (\ ) = -15, \\ -15 - (-4) = (\ ). \end{cases}$
9. $\begin{cases} 14 + (\ ) = -7, \\ -7 - (-21) = (\ ). \end{cases}$
10. $\begin{cases} 15 + (\ ) = -5, \\ -5 - (-20) = (\ ). \end{cases}$
11. $20 - 8 = (\ ).$
12. $8 - 20 = (\ ).$
13. $3 - (-9) = (\ ).$
14. $16 - (-5) = (\ ).$
15. $-17 - 13 = (\ ).$
16. $-15 - (-5) = (\ ).$

70. We have shown in Sec. 58 that a negative and a positive unit offset each other. Hence, to subtract 1 is the same as to add -1, and *vice versa;* and to subtract any number a is to add $-a$, and *vice versa.*

Thus: 7 less $5 = 2$ is the same as 7 plus $-5 = 2$,
or, 7 less $-5 = 12$ is the same as 7 plus $5 = 12$.

By its meaning the subtraction of zero has no effect. (Sec. 66.)

71. *To subtract a number, change its sign and add it.*

ORAL EXERCISES

Find these differences by adding:

1. $\begin{array}{r}6\\ \underline{-3}\end{array}$	5. $\begin{array}{r}12\\ \underline{-9}\end{array}$	9. $\begin{array}{r}7a\\ \underline{4a}\\ (\ )a\end{array}$	13. $\begin{array}{r}-4a\\ \underline{-7a}\\ (\ )a\end{array}$
2. $\begin{array}{r}3\\ \underline{-8}\end{array}$	6. $\begin{array}{r}15\\ \underline{-15}\end{array}$	10. $\begin{array}{r}4a\\ \underline{7a}\\ (\ )a\end{array}$	14. $\begin{array}{r}8x\\ \underline{7x}\\ (\ )x\end{array}$
3. $\begin{array}{r}5\\ \underline{-3}\end{array}$	7. $\begin{array}{r}7\\ \underline{-15}\end{array}$	11. $\begin{array}{r}7a\\ \underline{-4a}\end{array}$	15. $\begin{array}{r}7x\\ \underline{8x}\end{array}$
4. $\begin{array}{r}10\\ \underline{-5}\end{array}$	8. $\begin{array}{r}8\\ \underline{-17}\end{array}$	12. $\begin{array}{r}-4a\\ \underline{7a}\end{array}$	16. $\begin{array}{r}-7x\\ \underline{-8x}\end{array}$

Perform the indicated operations:

17. $13-(-5)+8$.

18. $13-(-6)+(-4)$.

19. $-15+(-8)-(+22)$.

20. $-17-9-(-20)$.

72. Preparatory.

1. A man earned \$3 on Monday and \$3 on Tuesday. How many dollars did he earn in the two days?

2. To multiply 3 by 2 is to take 3 how many times as the addend?

3. A man lost \$3 on Monday and \$3 on Tuesday. How many dollars did he lose in the two days?

4. To multiply -3 by 2 is to take -3 how many times as an addend?

73. Multiplication of Positive and Negative Numbers. *Multiplication by a positive integer means taking the multiplicand as an addend as many times as there are units in the multiplier.*

Correspondingly, *multiplication by a negative integer means taking the multiplicand as a subtrahend as many times as there are units in the multiplier.*

For example:

4 multiplied by $3 = 4 + 4 + 4 = 12.$

-4 multiplied by $3 = -4 + (-4) + (-4) = -12.$

4 multiplied by $-3 = -4 - 4 - 4 = -12.$

-4 multiplied by $-3 = -(-4) - (-4) - (-4) = +4 + 4 + 4 = 12.$

The numerical value of the product is the product of the numerical values of the factors.

74. The law of signs in multiplication, which applies to integral and fractional numbers alike, may be stated thus: If both factors are positive or if both are negative, their product is *positive*. If one is positive and the other negative, their product is *negative*.

In symbols: $+$ times $+ = +$ $\quad$ $-$ times $+ = -$.
$-$ times $- = +$ $\quad$ $+$ times $- = -$.

75. This law is easily remembered in the form:

The product of two factors of like signs is positive, and of two factors of unlike signs is negative.

ORAL EXERCISES

State the product in each of the following:

1. $+5 \cdot +3.$
2. $-5 \cdot +3.$
3. $+5 \cdot -3.$
4. $-5 \cdot -3.$
5. $-7 \cdot +7.$
6. $+8 \cdot -9.$
7. $-7 \cdot -6.$
8. $-8 \cdot -8.$
9. $-7 \cdot -11.$
10. $-12 \cdot +6.$
11. $+5 \cdot -11.$
12. $+6 \cdot -\frac{2}{3}.$
13. $-8 \cdot +\frac{1}{4}.$
14. $+3 \cdot -4 \cdot +5.$
15. $-5 \cdot +2 \cdot -3.$
16. $+14 \cdot +\frac{1}{2}.$
17. $+2 \cdot -3 \cdot -4.$
18. $-4 \cdot -5 \cdot -3.$

4

76. Division of Positive and Negative Numbers. We have seen in arithmetic that

$$\text{Quotient} \times \text{Divisor} = \text{Dividend}.$$

Thus, we know that $24 \div 8$ is 3, because $3 \times 8 = 24$.

Applying this relation to division with positive and negative numbers, we can determine the sign of the quotient. There are four cases, as follows:

$+24 \div +8 = (\ )$; $-24 \div -8 = (\ )$; $+24 \div -8 = (\ )$; $-24 \div +8 = (\ )$.

According to Sec. 74:

$$\begin{array}{lll} +24 \div +8 = +3 & \text{because} & +3 \times +8 = +24. \\ -24 \div -8 = +3 & \text{because} & +3 \times -8 = -24. \\ +24 \div -8 = -3 & \text{because} & -3 \times -8 = +24. \\ -24 \div +8 = -3 & \text{because} & -3 \times +8 = -24. \end{array}$$

In symbols:

$$\frac{+}{+} = +;\quad \frac{-}{-} = +;\quad \frac{+}{-} = -;\quad \frac{-}{+} = -.$$

77. This law is easily remembered in the form:

The quotient of two numbers of like signs is positive, and the quotient of two numbers of unlike signs is negative.

ORAL EXERCISES

State the quotient:

1. $-6 \div +3$.
2. $-12 \div +4$.
3. $+16 \div -4$.
4. $+20 \div -5$.
5. $+15 \div -3$.
6. $-8 \div -8$.
7. $-8 \div +8$.
8. $-15 \div -5$.
9. $-36 \div -6$.
10. $-35 \div +7$.
11. $+10 \div -4$.
12. $+18 \div -12$.
13. $20 \div -4$.
14. $30 \div -10$.
15. $-42 \div +7$.

Perform the indicated operations:

16. $6(-3) \div 9$.
17. $4(-6) \div -12$.
18. $18 \div (-3) \div -2$.
19. $4(-9) - 3 \times 3 - (18 \div -2)$.
20. $6 \times 9 - 4 \times 9 - (-24 \div -4)$.

78. The Greater of Two Numbers. Of two given numbers that one is the greater which can be produced by adding a positive number to the other. The other number is the less.

For example:

11 is greater than 8 because it is necessary to add $+3$ to 8 to make 11.

7 is greater than -2 because it is necessary to add $+9$ to -2 to make 7.

-4 is greater than -9 because it is necessary to add $+5$ to -9 to make -4.

79. The symbol $>$ is read "is greater than," and $<$ is read "is less than."

For example:

$8 > 2$ is read "8 is greater than 2."

$-1 > -5$ is read "-1 is greater than -5."

$-5 < -1$ is read "-5 is less than -1."

The terms "algebraically greater" and "algebraically less" are used when positive and negative numbers are compared. The terms "numerically greater" and "numerically less" apply to absolute values.

For example:

4 is greater than -9, but 4 is numerically less than -9.

-6 is greater than -15, but -15 is numerically greater than -6.

-2 is algebraically greater than -12, but numerically less than -12.

ORAL EXERCISES

Read the following and state why each is correct:

1. $7 > 5$.	3. $-2 > -5$.	5. $3 < 5$.	7. $-1 < 0$.
2. $4 > -8$.	4. $0 > -7$.	6. $-4 < 2$.	8. $-8 < -6$.

From the following list select the numbers that are:

a. Greater than 6.	*d.* Numerically greater than -4.
b. Less than -5.	*e.* Numerically greater than 6.
c. Greater than -4.	*f.* Numerically less than -5.

9. 7.	12. -18.	15. -3.	18. 5.
10. -10.	13. -1.	16. 2.	19. -6.
11. -8.	14. 0.	17. -4.	20. $-\frac{1}{2}$.

25. State the absolute value of each of the numbers in Exercises 9–20.

WRITTEN EXERCISES

Determine which is the greater in each of the following pairs of numbers, and write the relation by use of the sign $>$:

1. 8, 6.
2. 3, 4.
3. -5, 6.
4. -6, 5.
5. -6, -5.
6. 6, -9.
7. 0, 10.
8. 0, -10.
9. -4, -2.

80. Processes with inequalities.

1. Addition:

(1)
$$\begin{array}{r} 4>3 \\ 5>2 \\ \hline \text{Adding, } 4+5>3+2 \end{array}$$

(2)
$$\begin{array}{r} 3<4 \\ -5<-2 \\ \hline \text{Adding, } 3-5<4-2 \end{array}$$

That is, if two inequalities of the same kind are added the result will be an inequality of the same kind, but the sum of two inequalities of different kinds may result in an equality or in an inequality of either kind. Similarly, the subtraction of inequalities is uncertain.

2. Multiplication:

(1) $3<5$ then $2 \cdot 3 < 2 \cdot 5$

(2) $-2>-5$ then $3(-2)>3(-5)$

If the members of an inequality be multiplied by any number not zero or negative, the result will be an inequality of the same kind. If the multiplier is negative, the result will be an inequality of the opposite kind.

ORAL EXERCISES

Add:

1. $\begin{array}{r} 3>2 \\ 4>3 \\ \hline \end{array}$

2. $\begin{array}{r} 3>2 \\ -4>-5 \\ \hline \end{array}$

3. $\begin{array}{r} -3<-2 \\ -7<-5 \\ \hline \end{array}$

4. Multiply: $5>2$ by 3; also by -2; also by -1.
5. Multiply: $a<b$ by 3; also by -2; also by -1.

REVIEW

ORAL EXERCISES

1. The temperature was $-8°$ at 6 o'clock and $+5°$ at 9 o'clock. How many degrees did it rise in this interval?

2. A ship sailed on a meridian from Lat. $+12°$ to Lat. $-2°$. Through how many degrees did it sail?

Read in full:

3. $11+18$.	**6.** $3+(-2)$.	**9.** $xy-(-xy)$.
4. $14-9$.	**7.** $p+(-q)$.	**10.** $ab-(-ab)$.
5. $-2-(+3)$.	**8.** $-3a-(+2b)$.	**11.** $mn+(-2m)$.

Add:

12. $\begin{array}{r}-6\\ \underline{-9}\end{array}$	**14.** $\begin{array}{r}-16\\ \underline{7}\end{array}$	**16.** $\begin{array}{r}-\ \ 8\\ \underline{26}\end{array}$	**18.** $\begin{array}{r}-18\\ \underline{36}\end{array}$
13. $\begin{array}{r}15\\ \underline{-9}\end{array}$	**15.** $\begin{array}{r}23\\ \underline{-9}\end{array}$	**17.** $\begin{array}{r}40\\ \underline{-15}\end{array}$	**19.** $\begin{array}{r}33\\ \underline{-13}\end{array}$

Subtract:

20. $\begin{array}{r}9\\ \underline{-5}\end{array}$	**22.** $\begin{array}{r}-15\\ \underline{-\ \ 6}\end{array}$	**24.** $\begin{array}{r}-14\\ \underline{8}\end{array}$	**26.** $\begin{array}{r}8\\ \underline{-16}\end{array}$
21. $\begin{array}{r}-8\\ \underline{4}\end{array}$	**23.** $\begin{array}{r}-32\\ \underline{-\ \ 3}\end{array}$	**25.** $\begin{array}{r}-\ \ 3\\ \underline{15}\end{array}$	**27.** $\begin{array}{r}-14\\ \underline{-14}\end{array}$

Multiply:

28. $\begin{array}{r}6\\ \underline{-5}\end{array}$	**30.** $\begin{array}{r}-4\\ \underline{-8}\end{array}$	**32.** $\begin{array}{r}-6\\ \underline{10}\end{array}$	**34.** $\begin{array}{r}-12\\ \underline{3}\end{array}$
29. $\begin{array}{r}-8\\ \underline{3}\end{array}$	**31.** $\begin{array}{r}-15\\ \underline{3}\end{array}$	**33.** $\begin{array}{r}-11\\ \underline{-\ \ 3}\end{array}$	**35.** $\begin{array}{r}-10\\ \underline{0}\end{array}$

Divide:

36. -15 by 3.	**38.** 18 by -6.
37. -20 by -4.	**39.** 40 by -10.

WRITTEN EXERCISES

1. Write the sum of positive a and negative b.

Indicate, by using the signs $+$, $-$:

2. \$15 lost plus \$10 gained.
3. The sum of positive x and negative y.
4. The sum of negative m and positive n.
5. The difference of positive m and negative n.
6. The difference of positive x and negative y.

Perform the indicated operations:

7. $-6+5-4(-3)$.
8. $9-3(-3)+6\cdot 0$.
9. $15\div -5+(3\cdot -5)$.
10. $16\div -2-(4\cdot -3)$.
11. $12\cdot -3+24-(6\cdot -3)$.
12. $17(2-8)-(70\div -7)$.
13. $(-5\cdot 15-3)\div(15-4\cdot 6)$.
14. $-6a-2(-4a)-(8a\div -2)$.
15. Multiply $-2x>-5x$ by -3.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What are *relative numbers?* *Negative numbers?* Sec. 56.
2. What signs are used to indicate the *positive and negative character* of numbers respectively? Sec. 59.
3. State the rules that determine when the signs $+$, $-$, indicate *signs of operation.* Sec. 59.
4. Why are the numbers of algebra called *signed* numbers? Sec. 60.
5. Define *absolute,* or *numerical* value. Sec. 61.
6. If the addends have *like* signs, what is the sign of their sum? Sec. 63.
7. In adding two numbers with *unlike* signs, how is the sign of the sum found? Sec. 67.
8. How are *algebraic* numbers subtracted? Secs. 70, 71.
9. State the *law of signs in multiplication.* Sec. 75.

10. State the *law of signs in division.* Sec. 77.

11. Define the *greater* of two numbers; the *less.* Sec. 78.

12. Define *numerically greater; algebraically greater.* Sec. 79.

13. What is the result of adding inequalities of the *same kind?* Of *different kind?* Sec. 80.

14. What is the result of multiplying both members of an inequality by a *positive* number? By a *negative* number?

HISTORICAL NOTE

For the first use of negative number we must turn to the Brahmin schools of India. The Hindoo priests were clever mathematicians, and tradition relates that the great reformer, Buddha, in his youth won the maiden he loved by solving a difficult set of problems. Hindoo scholars did much to develop algebra, and the writings of Bhaskara, who lived about 1150 A.D., is a summary of their work. The poetic tendency of the Hindoos affected all their thinking, and they expressed their mathematics in flowery language and in verse. Bhaskara called one part of his work *Lilavati* ("noble science"), and proposed many problems like the following: "The square root of half the number of bees in a swarm flew to a jasmine bush; $\frac{8}{9}$ of the whole swarm remained behind; one bee, allured by the sweet odor of a lotus flower, became entangled in it while his excited mate lingered about. Tell me the number of bees." (Answer, 72.)

The Hindoos were the first to explain positive and negative numbers by reference to debits and credits, and the modern interpretation of these numbers by opposite directions on a straight line was not unknown to them. They discovered that the solution of certain equations gave negative roots, but, strange as it may seem, they rejected them and negative numbers received no further acceptance until Descartes fully interpreted them (1637). They had so long remained a mystery that they were known as "absurd" or "fictitious" numbers. Even as late as 1545 Cardan called them *numeræ fictæ.* Hence, we may in a measure be forgiven for calling negative number artificial in comparison with positive number. It is unfortunate, however, that the negative number came down to us labeled "artificial" or "unreal" number, because it is just as real as the positive number. Both kinds have to exist, or neither could exist. Both positive and negative numbers are concrete. They are denominate, or named numbers, just as much as 3 feet, or 7 quarts. It is quite as natural to think three positive and three negative units as it is to think three dollars' gain and three dollars' loss. When the unit, *negative one*, has been defined to be an absolute one taken in the opposite sense to positive one, we can as easily count or reckon with negative numbers as with positive numbers.

CHAPTER V

ADDITION

ADDITION OF MONOMIALS

81. Algebraic Sum. The result of adding numbers some or all of which are negative is called their **algebraic sum.**

82. Like Terms. Terms or monomials that have the same literal factors, are called **like terms** or **like monomials.**

For example, the following are pairs of like terms:

ab and ab; $5a$ and $-3a$; $4ab$ and $\frac{7}{8}ab$; $2a^2b$ and $\frac{1}{3}a^2b$.

Like terms in algebra correspond to numbers in arithmetic having a common factor.

For example, 5×7 and 3×7; or 7×15 and 6×15.

ORAL EXERCISES

From the following list select terms like the first; like the second; like the third:

1. ab^3x.
2. cy^3.
3. ab^2.
4. mp.
5. $2b^2x^2$.
6. amp.
7. $-\frac{2}{3}ab^2$.
8. $5ab^3x$.
9. $-4cy^3$.
10. $-6\frac{1}{5}ab^2$.
11. ax.
12. $24cy^2$.

83. Terms are *alike in any letter* if they contain the same power of that letter.

Thus, $2ax^2$ and bx^2 are alike in x.

Which of the terms of Exercises 1–12 are alike in a? In b? In x?

84. *To add two like terms or monomials, prefix to their like factor the sum of their coefficients.*

In arithmetic we add 3×7 and 5×7 by adding 3 and 5 and writing 8×7.

Similarly, in $3\,ax + 5\,ax = 8\,ax$, we add 3 and 5 and write $8\,ax$.

In $-4\,ab^2 + 6\,ab^2$ the like factor is ab^2; hence, adding the coefficients -4 and 6 the whole sum is $+2\,ab^2$.

The addition of dissimilar terms can only be indicated.

ORAL EXERCISES

Add:

1. $-5\,x$ $-2\,x$	3. $8\,y$ $-5\,y$	5. $-15\,y^2$ $-10\,y^2$	7. $-6\,a^2b$ $-3\,a^2b$
2. $-6\,x$ $2\,x$	4. $6\,a^2b$ $2\,a^2b$	6. $-10\,xy$ $5\,xy$	8. $16\,mn$ $-\;9\,mn$

9. $-7\,ab + (-3\,ab) = (\;\;)ab.$ 11. $.7\,x + (-.3\,x) = (\;\;)x.$

10. $3\,abc + (-5\,abc) = (\;\;)abc.$ 12. $8\,a^3 + (-5\,a^3) = (\;\;)a^3.$

85. To add several like terms, *combine them in order, or add the positive and the negative terms separately, and then combine these two sums.*

$$\begin{array}{r} 3\,a \\ -5\,a \\ 12\,a \\ -6\,a \\ \hline 4\,a \end{array}$$

Thus, in the column on the left, adding upward, $-6\,a + 12\,a = +6\,a$, then $6\,a + (-5\,a) = a$, and finally $a + 3\,a = 4\,a$; or the sum of the positive terms is $15\,a$, the sum of the negative terms is $-11\,a$, and the sum of these two is $4\,a$.

WRITTEN EXERCISES

Add:

1. $7\,b$ $-3\,b$ $20\,b$	2. $12\,y$ $-9\,y$ $5\,y$	3. $4\,m^3$ $-m^3$ $25\,m^3$	4. $-2\,x$ $8\,x$ $-9\,x$	5. $8\,xy^2$ $10\,xy^2$ $-20\,xy^2$

6.	7.	8.	9.	10.
$4s$	$-x^2y$	$7b$	t^2	$20w$
$-12s$	$9x^2y$	$8b$	$2t^2$	$30w$
$6s$	$-2x^2y$	$-20b$	$-5t^2$	$-19w$
$-9s$	$4x^2y$	$3b$	$25t^2$	$11w$

11. $3x - 8x + 15x = ?$ 12. $6x + 8x - 3x - 4x = ?$

13. $ax^2 - \frac{1}{2}ax^2 + \frac{3}{2}ax^2 = (\ \ )ax^2.$

14. $\frac{1}{3}ab^2 + \frac{1}{6}ab^2 - \frac{4}{3}ab^2 + \frac{5}{6}ab^2 = (\ \ )ab^2.$

15. $-14x + 23x + 99x = ?$

16. $40a - 75a + 89a = ?$

17. $12ab - 18ab + 75ab = ?$

18. $13xy - 50xy + 113xy = ?$

19. $-x^2 + 4x^2 - 10x^2 = ?$

20. $-xy - 40xy + 9xy = ?$

21. $-mn^2 + 12mn^2 - 15mn^2 = ?$

22. $15xy^2 + 33xy^2 - 48xy^2 = ?$

23. $4z^2 - 52z^2 + 109z^2 - 12z^2 = ?$

24. $29x^2 - 43x^2 + 87x^2 = ?$

25. $-20pq - 100pq + 7pq = ?$

26. $-y^2 + 4y^2 - 117y^2 + 3y^2 = ?$

27. $-12pqr^2 - 17pqr^2 - 53pqr^2 = ?$

Add the terms in x and the other terms separately:

28. $17x - 5x + 3x + 2 - 11x - 4x + 10x - 27.$

29. $22x - 3x - 6 - 4x + 5x - x - 2x - 9.$

30. $2x + 3a + 15x - 12x + 6a - 24a.$

ADDITION OF POLYNOMIALS

86. The addition of polynomials is similar to that of denominate numbers.

For example:

	Denominate Numbers		Polynomials
Just as:	3 bu. 4 qt.	so:	$3b + 4q$
plus	5 bu. 3 qt.	plus	$5b + 3q$
equals	8 bu. 7 qt.	equals	$8b + 7q$

ORAL EXERCISES

1. Add: 4 mi. 3 rd. 2 ft. also, $4m + 3r + 2f$
6 mi. 7 rd. 8 ft. $\underline{6m + 7r + 8f}$

State the numbers to fill the blanks in the following additions:

2.

$$\begin{array}{r} 2a + 4b \\ 3a + 5b \\ \hline (\ )a + (\ )b \end{array}$$

3.

$$\begin{array}{r} 2a + \ b + \ c \\ 5a + 3b + 2c \\ \hline (\ )a + (\ )b + (\ )c \end{array}$$

87. To Add Polynomials: *Arrange the like terms in columns and add as in the case of monomials, using the signs obtained as the signs of the result.*

For example:

NOT ARRANGED	ARRANGED
$a + c + b$	$a + b + c$
$-3b + a + c$	$a - 3b + c$

Here the first column is $+a + a = +2a$; the second column is $+b - 3b = -2b$; the third column is $+c + c = +2c$. The terms thus obtained with their signs, namely, $2a - 2b + 2c$, constitute the sum of the polynomials.

88. Commutative Law of Addition. *The sum of two or more terms of a polynomial is the same in whatever order the terms are taken.*

The rearrangement of the terms of polynomials before adding is an application of this law.

WRITTEN EXERCISES

Add:

1. $\begin{array}{r} 5a - 3b \\ a - b \\ \hline \end{array}$

2. $\begin{array}{r} 2a - 6c \\ 3c + 2a \\ \hline \end{array}$

3. $\begin{array}{r} \frac{1}{2}x - \frac{1}{3}y \\ y + \frac{2}{3}x \\ \hline \end{array}$

4. $\begin{array}{r} 3m - 1.1n \\ 6m - .9n \\ \hline \end{array}$

5. $\begin{array}{r} \frac{2}{3}p + 1\frac{4}{9}q \\ \frac{1}{3}q + \frac{5}{9}p \\ \hline \end{array}$

6. $\begin{array}{r} -7x + 4y \\ 9x - 10y \\ \hline \end{array}$

7. $\begin{array}{r} x^2+2y^2-z^2 \\ \underline{z^2-3y^2+2x^2} \end{array}$

8. $\begin{array}{r} at^2-6at+c \\ \underline{2at-8at^2+2c} \end{array}$

9. $\begin{array}{r} m^2+m+1 \\ \underline{m-2m^2-8} \end{array}$

10. $\begin{array}{r} p^2+p+8 \\ \underline{p^2+9p+6} \end{array}$

11. $\begin{array}{r} x^3+6x^2-9x \\ \underline{2x^3+8x^2-10x} \end{array}$

12. $\begin{array}{r} 17p^2-pq+q^2 \\ \underline{-6p^2+2pq-q^2} \end{array}$

13. $\begin{array}{r} 2a+7b+11c \\ \underline{2b+9c} \end{array}$

14. $\begin{array}{r} 12x+8y+17z \\ \underline{9x+12z+13y} \end{array}$

15. $\begin{array}{r} \frac{1}{8}p+\frac{2}{3}q+\frac{3}{7}r \\ \underline{7\frac{1}{8}p+9\frac{1}{3}q+r} \end{array}$

16. $\begin{array}{r} \frac{1}{3}r-\frac{3}{4}s+4t \\ \underline{\frac{4}{3}r+1\frac{1}{2}s-\frac{1}{2}t} \end{array}$

89. PREPARATORY.

Find the value of each expression when each letter $=1$:

1. $a+2b$.
2. $a+b+c$.
3. $c+4d-2a$.
4. $2a-2b$.
5. $b+2c-a$.
6. $2a+3b+3c$.
7. $3a-c$.
8. $a+d+3c$.
9. $3b+c+a$.

90. Test of Addition. To test the work of addition, substitute unity for the letters. The value of the sum must equal the sum of the values of the expressions.

In practice the work and test are written as follows:

SOLUTION	TEST
$2a-5b$	-3
$4a+4b$	$+8$
$6a-b$	$+5$

The use of unity tests the coefficients including their signs, but does not check mistakes made in writing the literal parts. Such errors, however, rarely occur, and are easily discovered by inspection. Numbers other than unity are apt to make the work of checking too complicated.

WRITTEN EXERCISES

Add and test:

1. $\begin{array}{r} a+3b \\ \underline{11a+10b} \end{array}$

2. $\begin{array}{r} 6c+d \\ \underline{3c+2d+e} \end{array}$

3. $\begin{array}{r} 12t-6t^2 \\ \underline{8t+12t^2} \end{array}$

4.
$$\begin{array}{l} 4x \quad\quad + z \\ 2z + y + x \\ \hline \end{array}$$

5.
$$\begin{array}{r} 40m + \quad n \\ 5m - 39n \\ \hline \end{array}$$

6.
$$\begin{array}{r} 17a + 4b \\ 3a - 16b \\ \hline \end{array}$$

7.
$$\begin{array}{l} 6a + 9b \\ 4a - \quad b \\ \hline \end{array}$$

8.
$$\begin{array}{r} 12a + 5b \\ 6a - 3b \\ \hline \end{array}$$

9.
$$\begin{array}{r} -15x + 12 \\ -\ 8x - 3 \\ \hline \end{array}$$

10.
$$\begin{array}{r} 5x^2 + 3x \\ -3x + 7x^2 \\ \hline \end{array}$$

11.
$$\begin{array}{r} 4xy - 2z^2 \\ 5xy + 10z^2 \\ \hline \end{array}$$

12.
$$\begin{array}{l} 40 - \frac{1}{2}gt^2 \\ 50 + \frac{3}{2}gt^2 \\ \hline \end{array}$$

13.
$$\begin{array}{r} 45m - 2n + q \\ 5m + 3n + q \\ \hline \end{array}$$

14.
$$\begin{array}{l} 3a^2 - 5a + 1 \\ 4a + 8a^2 - 3 \\ \hline \end{array}$$

15.
$$\begin{array}{r} 5x + \quad y - \quad z \\ 3x - 7y + 8z \\ \hline \end{array}$$

16.
$$\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y + .9z \\ \frac{1}{2}x + 1\frac{2}{3}y + 1.1z \\ \hline \end{array}$$

17.
$$\begin{array}{l} 1.1a - 8.9b + \quad c \\ 3.9a + .5b - 5c \\ \hline \end{array}$$

18.
$$\begin{array}{r} x^2 + \quad xy + 3y^2 \\ 6x^2 + 10xy + 5y^2 \\ \hline \end{array}$$

19.
$$\begin{array}{r} x^2 + \quad x^2y + 8y^2 \quad\quad \\ 4x^2y + \quad y^2 + z^2 \\ \hline \end{array}$$

20.
$$\begin{array}{r} 5a - 7b \\ 3a + 10b \\ -6a + 18b \\ -7a + 12b \\ \hline \end{array}$$

21.
$$\begin{array}{r} x^2 - 5x \\ -\ 4x^2 + 3x \\ -12x^2 + \quad x \\ 15x^2 - 12x \\ -10x^2 + 16x \\ \hline \end{array}$$

22.
$$\begin{array}{r} a + 2b + 3c^2 \\ 2a \quad\quad + 66c^2 \\ 9b + \quad c^2 \\ a + \quad b + \quad c^2 \\ 5a - \quad b \quad\quad\quad \\ \hline \end{array}$$

23.
$$\begin{array}{r} p + 3q \quad\quad \\ m + 3p + \quad q + r \\ 5m + 2p \quad\quad\quad + 6r \\ 10q + 5r \\ \hline \end{array}$$

24.
$$\begin{array}{l} a^2 + 4 \quad - \quad a \\ 5 \quad - 3a^2 + 2a \\ 6a^2 + 3a - 5 \\ 4a - 7a^2 - 2 \\ 8a - 12 \quad - 15a^2 \\ \hline \end{array}$$

25.
$$\begin{array}{r} 4g + 3v - 7x \\ 5x + 2y - 4v \\ 2y - 8g - 7v \\ 13v - 11x + 2g \\ 12x - 15g + 4y \\ \hline \end{array}$$

26. A dealer bought at one time 3 kinds of coal, $50a$ tons of the first kind, $10b$ tons of the second kind, and $12c$ tons of the third; at another time he bought $75a$ tons, $15b$ tons, $10c$ tons respectively of the same kinds. How many tons did he buy in all?

REVIEW

ORAL EXERCISES

Add:

1. $\begin{array}{r} -6\,ab \\ -8\,ab \\ \hline \end{array}$

2. $\begin{array}{r} 12\,ac^2 \\ -\ \ 8\,ac^2 \\ \hline \end{array}$

3. $\begin{array}{r} mr^2 \\ -mr^2 \\ \hline \end{array}$

4. $\begin{array}{r} \frac{1}{2}\,mv^2 \\ -\ \ mv^2 \\ \hline \end{array}$

5. $\begin{array}{r} -\ \ xy \\ 10\,xy \\ -15\,xy \\ 11\,xy \\ \hline \end{array}$

6. $\begin{array}{r} 4\,\pi r^2 \\ 2\,\pi r^2 \\ \hline \end{array}$

7. $\begin{array}{r} 12\,xyz \\ 13\,xyz \\ \hline \end{array}$

8. $\begin{array}{r} -\ \ mpq \\ -6\,mpq \\ \hline \end{array}$

9. $\begin{array}{r} -\frac{1}{2}\,mv^2 \\ -2\,mv^2 \\ \hline \end{array}$

10. $\begin{array}{r} 43\,w^3 \\ -23\,w^3 \\ 20\,w^3 \\ -10\,w^3 \\ \hline \end{array}$

11. $\begin{array}{r} -9\,pq^2 \\ -7\,pq^2 \\ \hline \end{array}$

12. $\begin{array}{r} \frac{1}{3}\,\pi r^3 \\ \frac{5}{3}\,\pi r^3 \\ \hline \end{array}$

13. $\begin{array}{r} -6\,a^3b \\ 8\,a^3b \\ \hline \end{array}$

14. $\begin{array}{r} 4\,xy^2 \\ -7\,xy^2 \\ \hline \end{array}$

15. $\begin{array}{r} 16\,uv^2 \\ -\ \ 9\,uv^2 \\ -\ \ 7\,uv^2 \\ 25\,uv^2 \\ \hline \end{array}$

16. $\begin{array}{r} 7\,x^2-2\,x-5 \\ 4\,x^2+4\,x+5 \\ \hline \end{array}$

17. $\begin{array}{rrr} 4.5\,m+3.2\,n+ & & p \\ .5\,m & & +.1\,p \\ \hline \end{array}$

18. $\begin{array}{r} a^3+3\,a^2b+3\,ab^2+b^3 \\ a^3-3\,a^2b+3\,ab^2-b^3 \\ \hline \end{array}$

19. $\begin{array}{r} 3\,x-4\,y+7\,z-9 \\ 8\,y-2\,z+4\,x-3 \\ \hline \end{array}$

WRITTEN EXERCISES

Add and test:

1. $a+b+c,\ \ 2\,a+b+3\,c,\ \ a+b,\ \ 6\,b+5\,c.$
2. $10\,a+9\,b+c,\ \ 9\,a+10\,c,\ \ \frac{1}{3}\,a+\frac{1}{4}\,b,\ \ 2\,b+c.$
3. $.9\,x+.3\,y+z,\ \ .1\,x+.7\,y,\ \ 5\,y+\frac{1}{2}\,z,\ \ 4\,x+z.$
4. $3\,a-2\,b,\ \ 4\,a+7\,b,\ \ -6\,a-b,\ \ 14\,a-21\,b.$
5. $m^2+5\,mp,\ \ 7\,m^2-8\,mp,\ \ -12\,m^2+3\,p^2,\ \ 6\,m^2-4\,mp+7\,p^2.$
6. $5\,ax+b,\ \ 2\,ax-3\,b,\ \ -8\,ax+7\,b,\ \ -20\,ax-18\,b.$
7. $\frac{1}{2}\,x-\frac{1}{3}\,y,\ \ -\frac{3}{5}\,x+\frac{1}{4}\,y,\ \ .7\,x+\frac{2}{3}\,y,\ \ -.3\,x+\frac{5}{6}\,y.$
8. $7+2\,x^2,\ \ 3\,x^2-1,\ \ 4-5\,x^2.$
9. $4\,a+7\,b,\ \ 2\,a-6\,c,\ \ 3\,c+5\,b,\ \ 4\,b-7\,a.$
10. $x^2+7\,x-4,\ \ 3\,x^2-5\,x,\ \ 4\,x^2-11\,x+2.$

11. $5t - 3 + 7t^2,\ \ t - 3t^3,\ \ t^3 + 9t^2 - 15 + 8t.$

12. $3ab + 7ac,\ \ 5ac - 2bc,\ \ 6bc + 9ab,\ \ 8bc - 18ac.$

13. $x + .4,\ \ x^2 - .5,\ \ 3x - .7,\ \ x^2 - .9 + 4x.$

14. $2y^2 - 4y + y^4 - 1,\ \ 8y - y^3 + 3y^2 - 15,\ \ 3y - 7 + 11y^4 - 15y^2,\ \ 4y^3 + 12y^2 - 6 + y^4,\ \ 11 - y^4 + y^3 - 8y^2.$

15. A grocer had $7a$ dollars on hand; his ten salesmen took in $4a, 5c, 2b, 6a, 3a, 7c, 4b, 2c, 5c, 11b$, dollars, respectively. How much had he then?

16. A merchant made the following bank deposits: On Monday $3a$ dollars in gold, $4b$ dollars in silver, and $9c$ dollars in bills; on Tuesday a dollars in gold and $15c$ dollars in bills; on Wednesday b dollars in silver and $12c$ dollars in bills. How much did he deposit all together?

SUMMARY

The following questions summarize the definitions and processes of this chapter:

1. What is meant by *algebraic sum?* — Sec. 81.
2. What are *like terms* or *monomials?* — Sec. 82.
3. How may like terms be added? — Secs. 84, 85.
4. How may polynomials be added? — Sec. 87.
5. State the *Commutative Law of Addition.* — Sec. 88.
6. Explain a *test* for the work of addition. — Sec. 90.

CHAPTER VI

SUBTRACTION

SUBTRACTION OF MONOMIALS

91. Subtraction is the process of finding the difference between two numbers, called the *minuend* and the *subtrahend.*

92. Difference. The **difference** is the number which added to the subtrahend makes the minuend. Sec. 69.

ORAL EXERCISES

State the differences of the following:

1. $\begin{array}{r}20\,a\\ \underline{15\,a}\end{array}$	4. $\begin{array}{r}21\,x^2\\ \underline{14\,x^2}\end{array}$	7. $\begin{array}{r}90\,pq\\ \underline{45\,pq}\end{array}$	10. $\begin{array}{r}40\,mn\\ \underline{39\,mn}\end{array}$
2. $\begin{array}{r}47\,b\\ \underline{27\,b}\end{array}$	5. $\begin{array}{r}17\,bc\\ \underline{9\,bc}\end{array}$	8. $\begin{array}{r}50\,zw\\ \underline{25\,zw}\end{array}$	11. $\begin{array}{r}54\,d\\ \underline{24\,d}\end{array}$
3. $\begin{array}{r}12\,b^2c\\ \underline{7\,b^2c}\end{array}$	6. $\begin{array}{r}39\,c^2\\ \underline{19\,c^2}\end{array}$	9. $\begin{array}{r}30\,xy\\ \underline{17\,xy}\end{array}$	12. $\begin{array}{r}19\,abx\\ \underline{12\,abx}\end{array}$

93. Since subtraction is the reverse of addition, we can subtract a number by adding its opposite.

For example, "to subtract 3" and "to add -3" mean the same thing. Likewise "to subtract $-5\,a$" and "to add $5\,a$" mean the same thing.

To subtract one number from another, change the sign of the subtrahend and add the result to the minuend.

Thus, to subtract $\begin{array}{r}30\,a\\ \underline{-10\,a}\end{array}$ change to $\begin{array}{r}30\,a\\ \underline{10\,a}\end{array}$ and add.

The pupil should learn to make the change of sign mentally.

ORAL EXERCISES

Find the differences:

1. 18 -99	8. $6\,mv^2$ $19\,mv^2$	15. $-8\,x$ $-2\,x$	22. $2\,m^2$ $2\,m^2$
2. $6\,ab$ $6\,ab$	9. $-1.5\,s$ $-3.5\,s$	16. $-14\,y$ $-6\,y$	23. $23\,a$ $-6\,a$
3. $5\,x$ $-15\,x$	10. $-4\frac{1}{2}\,w$ $-9\frac{1}{2}\,w$	17. $16\,b$ $11\,b$	24. $23\,a$ $6\,a$
4. $5\,xy$ $-10\,xy$	11. $5\,a$ $8\,a$	18. $-14\,d$ $-7\,d$	25. $-23\,a$ $6\,a$
5. $-3\,abc$ $3\,abc$	12. $4\,x$ $7\,x$	19. $8\,g$ $-3\,g$	26. $-23\,a$ $-6\,a$
6. $-12\,m$ $8\,m$	13. $12\,y$ $3\,y$	20. $-4\,p$ $-7\,p$	27. $23\,x^3$ $-6\,x^3$
7. $12\,pq$ $-2\,pq$	14. $-8\,x$ $2\,x$	21. $2\,m^2$ $-2\,m^2$	28. $-t$ $-3\,t$

29. $46 - 52$.
30. $4\,a - 7\,a$.
31. $8\,x^2 - 10\,x^2$.
32. $8\,ab - 15\,ab$.
33. $40\,m - 46\,m$.
34. $13\,pq - 15\,pq$.
35. $\frac{1}{2}\,mv^2 - mv^2$.
36. $\pi r^2 - 4\,\pi r^2$.
37. $\frac{4}{3}\,\pi r^3 - 2\,\pi r^3$.
38. $\frac{1}{2}\,gt^2 - gt^2$.
39. $10\,fs - 15\,fs$.
40. $45\,rt - 50\,rt$.

SUBTRACTION OF POLYNOMIALS

94. The subtraction of polynomials is similar to the subtraction of denominate numbers.

For example:

Denominate Numbers	Polynomials
Just as: 5 lb. 4 oz.	so: $5\,l + 4\,z$
minus 3 lb. 3 oz.	minus $3\,l + 3\,z$
equals 2 lb. 1 oz.	equals $2\,l + 1\,z$

ORAL EXERCISES

Subtract:

1. 12 bu. 3 pk. 7 qt.
9 bu. 2 pk. 4 qt.

2. $12\,b + 3\,p + 7\,q$
$9\,b + 2\,p + 4\,q$

State the numbers to fill the blanks in the following:

3. $$\begin{array}{r} 6\,xy + 8\,y^2 \\ 3\,xy + 2\,y^2 \\ \hline (\ )xy + (\ )y^2 \end{array}$$

4. $$\begin{array}{r} 5\,a + 3\,b^2 + 8\,c \\ a + \ \ b^2 + 5\,c \\ \hline (\ ) + (\ ) + (\ ) \end{array}$$

Subtract:

5. $$\begin{array}{r} 6\,a + 5b \\ a + 3\,b \\ \hline \end{array}$$

6. $$\begin{array}{r} 16\,m + 2\,n \\ 12\,m + \ \ n \\ \hline \end{array}$$

7. $$\begin{array}{r} 17\,c + 9\,d \\ 17\,c + \ \ d \\ \hline \end{array}$$

8. $$\begin{array}{l} 20\,x + 5\,y + z \\ \ \ 9\,x + 5\,y \\ \hline \end{array}$$

9. $$\begin{array}{r} 45\,a + 3\,b + 12\,c \\ 5\,a + \ \ b + \ \ 2\,c \\ \hline \end{array}$$

10. $$\begin{array}{r} 10\,x^2 + 6\,y^2 + 5\,z^2 \\ 5\,x^2 + \ \ y^2 + 5\,z^2 \\ \hline \end{array}$$

95. To subtract a polynomial *arrange its terms under the like terms of the minuend, subtract each term from the one above it, and use the signs obtained as the signs of the result.*

$$\begin{array}{r} 2\,a - 2\,b + \ \ c \\ a + \ \ 0 - 2\,c \\ \hline a - 2\,b + 3\,c \end{array}$$

In the example, the first column is $2\,a - a = a$; the second column is $-2\,b - 0 = -2\,b$; the third column is $+c - (-2\,c) = +3\,c$. The terms thus obtained with their signs constitute $a - 2\,b + 3\,c$, the difference of the polynomials. When either polynomial lacks a term to correspond to a term of the other polynomial, supply zero in its place.

96. Test of Subtraction. Use arbitrary values to test sub traction. The sum of the values of the difference and the subtrahend must equal the value of the minuend.

SOLUTION

$$\begin{array}{r} 3\,x - 4\,y + \ \ c \\ x + \ \ y - 3\,c \\ \hline 2\,x - 5\,y + 4\,c \end{array}$$

TEST: Let each letter = 1

$$\begin{array}{r} 3 - 4 + 1 = \ \ 0 \\ 1 + 1 - 3 = -1 \\ \hline 2 - 5 + 4 = +1 \end{array}$$

In practice it is sufficient to write the following:

SOLUTION	TEST
$3\,x - 4\,y + \ \ c$	0
$x + \ \ y - 3\,c$	-1
$2\,x - 5\,y + 4\,c$	1

WRITTEN EXERCISES

Subtract and test:

1. $\begin{array}{r} 3a+b \\ a-b \\ \hline \end{array}$

2. $\begin{array}{r} 6a-3b \\ 5a+7b \\ \hline \end{array}$

3. $\begin{array}{r} \frac{1}{2}a-\frac{1}{3}b \\ a+\frac{2}{3}b \\ \hline \end{array}$

4. $\begin{array}{r} 3m-.1n \\ 6m-.9n \\ \hline \end{array}$

5. $\begin{array}{r} \frac{2}{3}p+\frac{1}{9}q \\ \frac{2}{3}p-\frac{8}{9}q \\ \hline \end{array}$

6. $\begin{array}{r} -7x+4y \\ 10x-9y \\ \hline \end{array}$

Arrange the like terms in columns and subtract:

7. $\begin{array}{l} 8x+12q+6a \\ 4a+\ \ 5x+3q \\ \hline \end{array}$

8. $\begin{array}{l} p+3n+45m \\ 5m+2n+p \\ \hline \end{array}$

9. $\begin{array}{l} 5c+6b+10a \\ b+5a+\ \ 5c \\ \hline \end{array}$

10. $\begin{array}{l} 10ax+\ \ mp+5pq \\ 2pq+\frac{1}{2}mp+5ax \\ \hline \end{array}$

11. $\begin{array}{r} 3xy-\ \ \ \ z^2 \\ 5xy+10z^2 \\ \hline \end{array}$

12. $\begin{array}{r} 12t-\ \ 6t^2 \\ 9t-12t^2 \\ \hline \end{array}$

13. $\begin{array}{r} 40-\frac{1}{2}gt^2 \\ 50-\frac{1}{2}gt^2 \\ \hline \end{array}$

14. $\begin{array}{r} 5x+7y-8z \\ 3x+\ \ y-4z \\ \hline \end{array}$

15. $\begin{array}{r} 2a+c-2b \\ a+b-2c \\ \hline \end{array}$

16. $\begin{array}{r} 7x^2-2x+4 \\ 2x^2+3x-1 \\ \hline \end{array}$

17. $\begin{array}{r} 2z+4x+y \\ 2x+\ \ y-z \\ \hline \end{array}$

18. $\begin{array}{l} 6c+3d+e \\ 3c+2d \\ \hline \end{array}$

19. $\begin{array}{r} 3x^2+2x^2y^2+z^2 \\ x^2+2x^2y^2\ \ \ \ \ \ \\ \hline \end{array}$

20. $\begin{array}{r} 4x-3y+8 \\ 2y+5z-1 \\ \hline \end{array}$

21. $\begin{array}{r} 4a+2b-9 \\ 8c+4a-6d \\ \hline \end{array}$

22. $\begin{array}{r} 2x^4+5x\ \ -1 \\ 3x^3-7x^2+8x \\ \hline \end{array}$

23. A broker had $7a+5b$ dollars in a bank and withdrew $a+4b$ dollars. How much did he still have in the bank?

24. A coal dealer bought c carloads of coal containing 40 tons each, and d carloads of 50 tons each; he sold $5c$ tons to one customer, $8d$ tons to another, and $12c$ tons to another. How many tons had he left?

97. Removal of Parentheses.

1. If a parenthesis is preceded by the sign $+$, the terms within the parenthesis are to be added to what precedes, hence the parenthesis may be removed without altering the value of the expression.

For example:

$$a+(b+c)=a+b+c.$$
$$a+(b-c)=a+b-c.$$
$$a+(-b-c)=a-b-c.$$

2. If a parenthesis is preceded by the sign $-$, the terms within the parenthesis are to be subtracted from what precedes; hence the parenthesis may be removed provided the sign of each term within the parenthesis is changed, each sign $+$ to the sign $-$, and each sign $-$ to the sign $+$. Sec. 93.

For example:

$$a-(b+c)=a-b-c.$$
$$a-(b-c)=a-b+c.$$
$$a-(-b-c)=a+b+c.$$

WRITTEN EXERCISES

Remove parentheses and unite terms when possible:

1. $a+(b-3a)$.
2. $5-(t+2)$.
3. $3a-(2a+b)$.
4. $7b-(4a-6b)$.
5. $a-(2b-5a)-4b$.
6. $11t+(-3t-1)$.
7. $4x+7y-(3x+2y)$.
8. $d+3d^2-(2d-d^2)$.
9. $5-3p+(-18+2p)$.
10. $a-bx-(2a+bx)$.
11. $7q+5-(-11-3q)$.
12. $-(-3x-2y+11z)$.
13. $-(4a+3b-6c)$.
14. $x-3x^2+7-(2x^2+5-3x)$.
15. $9t+3-(2t-1)$.
16. $5x-12y-(3x+2y)$.
17. $4a+7b-(2a+3b)$.
18. $7-6m-(1+3m)$.
19. $11p+1-(-p+3)$.
20. $15x^2+7x-(18x^2-3x)$.
21. $4m^3-7m^2+3m-(-2m^3-9m^2)$.
22. $a^2-5ab+7ac+b^2-(4ab+7a^2-6b^2)$.
23. $x^2+3xy-4xz+7y^2-(z^2-3xz-4x^2-y^2)$.
24. Calculate the value of $A-(2B-C)$ when

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$A=$	24	1	-4	x	$4d$	$a+x$	$p+3$
$B=$	3	-5	3	$2x$	$-6d$	$3a-x$	$q+5$
$C=$	8	9	-5	$3x$	$3d$	$4a+7x$	$2p+3q$

98. The methods of Sec. 97 may be applied when there is a parenthesis within a parenthesis. In this case the signs, $\{\ \}$, $[\ ]$, $\overline{\qquad}$, are commonly used to distinguish the different parentheses. They may be removed one at a time, usually the inner one first, although it is likewise possible to begin with the outer one.

For example:

Beginning with the inner parenthesis,

$$\begin{aligned} 5a-\{6a+3b-(2a-5b)\} &= 5a-\{6a+3b-2a+5b\} \\ &= 5a-\{4a+8b\} \\ &= 5a-4a-8b \\ &= a-8b. \end{aligned}$$

Beginning with the outer parenthesis,

$$\begin{aligned} 5a-\{6a+3b-(2a-5b)\} &= 5a-6a-3b+(2a-5b) \\ &= -a-3b+2a-5b \\ &= a-8b. \end{aligned}$$

When the first parenthesis is removed the signs within the second one are not changed because the expression is taken as a single term.

WRITTEN EXERCISES

Remove parentheses and unite terms as much as possible:

1. $4x-\{3x-(2+x)\}$.
2. $7+\{4-(5x+2)+3x\}$.
3. $2+(5a-\overline{3a+4a})+6a$.
4. $x^2-\{3x^2-(2x^2+1)\}$.
5. $-\{4x^2+(3x-\overline{5x^2-9x})\}$.
6. $a-3b-\{b-3a+(3b-a)\}$.
7. $p+(3p-q)-(3q-p)+q$.
8. $a^2-(b^2-c^2+a^2)-(c^2-b^2)$.
9. $3xy+2y^2-(x^2-xy+2y^2)$.
10. $3a+b-\{c-[d-\overline{3a+b}-c]+d\}$.
11. $m-\{3m-[4m-(5m-\overline{6m-7m})-8m]+9m\}+10m$.
12. $5a-[2x-\{4c+(7c-x)-3c\}+4a]$.
13. $a^4-[a^2-(1-a)]-[1+\{a^2-(1-a)+a^4\}]$.
14. $6m-\{p+2q-(m+q)+3-(5p-3q-4m)\}$.
15. $ab-ac-\{2ab-(3ac+bc)-2bc+2ac-(ab-2ac)\}$.
16. $b-[x-\{3+3b-(5x+2)\}-(4x-3b-1)]$.

17. $m+p-\{-[2m+2p-(q+r)]-[4p$
$-(r-q+m)-p]+3m\}.$

18. $2a+b+(12a-4b)-\{3a+3b-(5a-\overline{3a-3b})\}.$

19. $\frac{1}{3}a-\frac{1}{2}x-(\frac{3}{4}a-\frac{1}{6}x)-(2n-\overline{\frac{5}{4}x-\frac{1}{4}a})+\frac{5}{6}a.$

20. $3-\{[4y-(3-\overline{a-2})]-[a+(5y-\overline{a+5})]\}.$

21. $xy-[3+a-(x+z-xy+a)]+\{a-(y-z-5)\}.$

22. $-[3ab-\{5bc-3d\}+d-(4bc+\{2d+3ab\}+d)].$

23. $x-(y-z)-\{x-[y-z-(y+z-x)+(x-y)]\}.$

24. $1-y-\{1-y-[1-y-(1-y)-(y-1)]\}.$

25. $.5a-\{a+\overline{.3ab-.3a}-[.25a+\overline{c+1.5a}]\}.$

99. Introduction of Parentheses.

The value of a polynomial is not changed:

1. If any number of terms with their *signs unchanged* are grouped in a parenthesis preceded by the *sign* $+$.

2. If any number of terms with their *signs changed* are grouped in a parenthesis preceded by the *sign* $-$. Sec. 97.

For example:

$$2a+5b-6c=2a+(5b-6c).$$
$$2a+5b-6c=2a-(-5b+6c).$$
$$4a-7m+2x=4a-(7m-2x).$$

100. The fact that the terms of a polynomial may be grouped as stated in Sec. 99, without changing the value of the polynomial, is called the **Associative Law of Addition.**

WRITTEN EXERCISES

Write as a plus a parenthesis:

1. $a+6b-4c.$ 3. $a-4b-c+5d.$ 5. $a+9b-7c+3d.$

2. $a-3b+7c.$ 4. $a-8b+c-1.$ 6. $a+2b+7c+4.$

7–12. Write each of the expressions in Exercises 1–6 as a minus a parenthesis.

13–18. In each of the expressions in Exercises 1–6 place the terms involving b and c in a parenthesis preceded by the sign $-$.

Group the like terms in a parenthesis preceded by a minus sign:

19. $bc - a^2 + d + 3a^2 + c - 5a^2$.

20. $x^2 - 2xy + y^2 - 3xy + z$.

21. $ab + bc - 2ab - cd + 5ab$.

22. Collect in a parenthesis the like terms in x in $y^2 - 2x + x^2 - px + z^2 - pqx$.

23. Collect the like terms in x^2, also those in y, each in a parenthesis preceded by a minus sign:

$$x^2 - y^2 - px^2 + 2y + 5x^2 - xy + py.$$

24. Taking the terms of this expression in order, group each pair in a parenthesis:

$$3a^2 - b - c^2 - d^2 + bc + ac - x^2 - xyz.$$

101. The sum or the difference of terms alike with respect to certain letters may be indicated by the use of parentheses.

EXAMPLES

1. Add ax and bx.

Addend	ax
Addend	bx
Sum	$(a+b)x$

2. Subtract $(c+n)xy$ from $(b+2c)xy$.

$$\begin{array}{r} (b+2c)xy \\ (c+n)xy \\ \hline (b+c-n)xy \end{array}$$

WRITTEN EXERCISES

Add:

1. $\begin{array}{r}(a+b)x \\ bx \\ \hline\end{array}$

3. $\begin{array}{r}(a+c)xy \\ (a-c)xy \\ \hline\end{array}$

5. $\begin{array}{r}(a+c)(m^2+n) \\ c(m^2+n) \\ \hline\end{array}$

2. $\begin{array}{r}(1+2b)x^2 \\ -bx^2 \\ \hline\end{array}$

4. $\begin{array}{r}(a+b)(p+q) \\ (c+d)(p+q) \\ \hline\end{array}$

6. $\begin{array}{r}(m-14+p)x^2y \\ (-m+2n-p)x^2y \\ \hline\end{array}$

7. Collect the terms in x and put the coefficients in a parenthesis with the proper sign:

$$a - x + bx - abx.$$

8. Similarly, collect the terms in x, also those in y:

$$ax - by - bx - ay.$$

9. Similarly, collect the like terms of each kind:

$$3a - 2b - 6a - c - 5b + 10c.$$

10. Similarly, collect the like terms in x^2, and also in xy:

$$2x^2 - xy + ax^2 - 3xy - bx + 2axy.$$

Group in parentheses the same powers of x in each:

11. $ax^2 - bx^3 + cx - ax^3 + bx^2 - dx.$

12. $x^2 - 3x^3 + ax^2 - 2x + cx^3 + bx.$

13. $mx^3 + 2px + qx^2 - qx - px^3 - px^2.$

14. $ax^3 - x^4 + bx^2 - ax^4 + bx^3 - px.$

Collect in alphabetical order the coefficients of x, also of y:

15. $2ax - by + 3bx - 4ay - cx + 2cy.$

16. $mx + py - qx + ny - 5px + 4nx.$

Subtract:

17. $\begin{array}{r}(a+2b)x^p\\ \underline{bx^p}\end{array}$

19. $\begin{array}{r}(a+c)(m^2+n)\\ \underline{(2a+c)(m^2+n)}\end{array}$

21. $\begin{array}{r}(a+c)(x+y).\\ \underline{2c(x+y)}\end{array}$

18. $\begin{array}{r}(2m-4)x^2y\\ \underline{(37-n)x^2y}\end{array}$

20. $\begin{array}{r}(a+c)x^p\\ \underline{(a+2c)x^p}\end{array}$

22. $\begin{array}{r}(1+b+c)p^2q^2\\ \underline{(a-2b-2)p^2q^2}\end{array}$

First collect like terms, then subtract:

23. $ax + bx$ from $cx + dx$.

24. $ax - bx$ from $cx - dx$.

25. $ax - 3bx$ from $ax - 4bx$.

26. $ax + 4bx$ from $2ax - bx + cx$.

27. $ax + 2y - 3x$ from $ax - y + 2x$.

28. From $(a+b)x^2$ plus $(c-a)x^2$ subtract $(b-c)x^2$.

29. From $(a-2b)xy$ plus $(b-2a)xy$ subtract $(a+b+c)xy$.

30. From $(a+b)x + (c+d)y$ plus $2bx - 2cy$ take $2ax - (2d-c)y$.

REVIEW

ORAL EXERCISES

Subtract:

1. $\begin{array}{r} 5n \\ 16n \\ \hline \end{array}$

2. $\begin{array}{r} 7a \\ -9a \\ \hline \end{array}$

3. $\begin{array}{r} 7a-5b \\ a-2b \\ \hline \end{array}$

4. $\begin{array}{r} x+y \\ x-x \\ \hline \end{array}$

5. $\begin{array}{r} a-7 \\ 2a-3 \\ \hline \end{array}$

6. $\begin{array}{r} x^2+y \\ 2x^2-y \\ \hline \end{array}$

7. $\begin{array}{r} 11a+10b \\ a+3b \\ \hline \end{array}$

8. $\begin{array}{r} 3a-b \\ 3a+b \\ \hline \end{array}$

9. $\begin{array}{r} cx-dy \\ x-y \\ \hline \end{array}$

10. $\begin{array}{r} a-x \\ a-1 \\ \hline \end{array}$

11. $\begin{array}{r} p^2-q^2 \\ p^2+q^2 \\ \hline \end{array}$

12. $\begin{array}{r} ax-b \\ bx-c \\ \hline \end{array}$

13. $\begin{array}{r} (a-2b)xy \\ (b-a)xy \\ \hline \end{array}$

14. $\begin{array}{r} (2a+b-c)x^2 \\ (a-2b-c)x^2 \\ \hline \end{array}$

15. $\begin{array}{r} (a+2+2c)y^{28} \\ (1+a)y^{28} \\ \hline \end{array}$

WRITTEN EXERCISES

Remove the parentheses:

1. $(a+b-c)+(a-b+c)$.
2. $(a+b-c)-(a-b+c)$.
3. $7a-3b-(5a+3b)$.
4. $3x-7-(9x-11)$.
5. $2y-(\frac{1}{2}tx+\frac{1}{3}y)$.
6. $(\frac{1}{3}tx-\frac{1}{2}y)-(\frac{1}{6}y-\frac{1}{6}tx)$.
7. $m-[(a-b)-(c-m)]$.
8. $m+[(a-b)+(b+d)]$.
9. $6x+5y-32-(5x-3y+22)$.
10. $6a^2-(3ab+2ac)-(2ac+3ab)$.

Subtract and test:

11. $\begin{array}{r} 3x^2+2x^2y^2+z^2 \\ x^2-2x^2y^2 \\ \hline \end{array}$

12. $\begin{array}{r} m-3n+p-7 \\ m-4n-p+8 \\ \hline \end{array}$

13. $\begin{array}{r} 6c+3d+a-3z \\ 3c-2d-2a \\ \hline \end{array}$

14. $\begin{array}{r} 4x+3y-4z+8 \\ -7x-3y-2z+17 \\ \hline \end{array}$

15. $\begin{array}{l} 8x^3-9x^2+x^4-x+16+x^5 \\ 2x-7+5x^4-x^2-x^3+6x^5 \\ \hline \end{array}$

16. $\begin{array}{r} 5.65a+7\frac{3}{5}b-27\frac{3}{4}c+.76x-1\frac{1}{6}y \\ 4\frac{1}{4}a-9.38b+2.65c-13\frac{1}{2}x-0.375y \\ \hline \end{array}$

17. Group the last four terms in a bracket and the last three in a parenthesis:

$$mp - 3\,mn + 2\,pq + m^2 - n^2.$$

18. Group as the difference of two trinomials without changing the order of the terms:

$$x^2 + 2\,xy + y^2 - x^2 + 2\,xy - y^2.$$

Test by removing the parentheses and point out any errors:

19. $a^3 - a^2 + a - 1 = (a^3 - 1) - (a^2 + a).$

20. $a^2 - b^2 + 2\,bc - c^2 = a^2 - (b^2 + 2\,bc - c^2).$

21. From $(2\,b + c)\,x$ take $(b - c)\,x$.

22. From $(a + 2\,b - c)\,xy$ take $(2\,a - b + 2\,c)\,xy$.

23. From the sum of $(2\,a + b)\,x^2$ and $(c - b)\,x^2$ take $(a + b - c)x^2$.

24. Collect into the first member the terms in x and unite:

$$15 - (2\,a + b)\,x + 3\,cx = 20 + (c - 2\,b)\,x - 4\,ax.$$

SUMMARY

The following questions summarize the definitions and processes in this chapter:

1. Define *subtraction;* also *difference.* Secs. 91, 92.

2. Explain how to *subtract monomials.* Sec. 93.

3. Explain how to *subtract polynomials.* Sec. 95.

4. Explain the test of subtraction by arbitrary values. Sec. 96.

5. State the sign rules for removing parentheses. Sec. 97.

6. State the sign rules for introducing parentheses. Sec. 99.

CHAPTER VII

EQUATIONS

102. Practice in writing expressions containing positive and negative numbers is a necessary preparation for solving equations involving these numbers.

EXAMPLE

By how much does $b+3$ exceed $b-3$?

Indicating the difference and simplifying:

$$b+3-(b-3)=b+3-b+3=b-b+3+3=6.$$

WRITTEN EXERCISES

By how much does

1. $2x-5$ exceed $x-5$?

2. $2a-x$ exceed $2a+x$?

3. $2x-5$ exceed $x-6$?

4. $4a-1$ exceed $2a-20$?

5. B's age is $3x-1$ years. What was it 5 years ago? What will it be $x+2$ years hence?

If A is $n+2$ years old and B is $2n-1$ years old, express:

6. A's age 4 years ago.

7. B's age 4 years ago.

8. A's age 5 years hence.

9. B's age 5 years hence.

10. The sum of their ages now.

11. The sum of their ages 4 years ago.

12. The sum of their ages 5 years hence.

13. How much older is B than A?

If A has $d+25$ dollars and B has $2d-6$ dollars, write the equation expressing that:

14. A and B together have $100.

15. A has $20 more than B.

16. B has $20 less than A.

17. B's money increased by \$15 equals A's money decreased by \$10.

18. If A loses \$10 and B gains \$10, they will have the same amounts.

19. Express the next larger integer after n. Also the second one. Also the third.

20. Express the next integer smaller than n; the second integer smaller than n; the third.

Express:

21. The sum of two consecutive integers is 27.

22. The sum of three consecutive integers is 15.

23. The next larger integer after $n-5$; after $n-1$.

24. The next integer smaller than $n-5$; than $n-1$.

25. The sum of $n-4$ and the two next smaller integers.

103. Certain classes of numbers are represented by algebraic formulas.

EXAMPLES

1. $2n$ represents all even numbers, if n be given all positive integral values.

2. $2n-1$ represents all odd numbers, if n be given all positive integral values.

3. $n+\frac{m}{10}+\frac{p}{100}$ represents any number with two decimal places, where n is the whole number and m and p are the decimal figures.

4. $\frac{n}{m}$ represents any fraction. If n and m are integers and $n < m$, it is a proper fraction.

5. n^2 represents square numbers. It represents the squares of all integers if n has all positive or all negative integral values.

WRITTEN EXERCISES

1. In $2n$ substitute for n the numbers that will produce the even numbers between 99 and 111.

2. In $2n$ substitute for n the numbers that will produce the multiples of 4 between 15 and 25.

3. If the negative integers be substituted for n in $2n$, what kind of numbers are produced?

4. When n is 1, what is the value of $2n+1$? When n is 2? When n is 3? 4? 0? What class of numbers does $2n+1$ represent when n may be zero or any positive integer?

5. Write a formula to represent all positive cube numbers. Can the values of the letter be negative? If they are taken as negative, what class of numbers is produced?

6. Can you substitute in n^2 any numbers that will make the series, -1, -4, -9, -16, and so on?

7. What number does $n+\frac{m}{10}+\frac{p}{100}$ become when $n=1$, $m=2$, $p=3$? Also when $n=63$, $m=0$, $p=8$? Also when $n=0$, $m=2$, $p=5$?

8. Write a formula to represent any number with three decimal places.

9. If q is 10 in $\frac{p}{q}$, what is the greatest integral value p can have and make $\frac{p}{q}$ a proper fraction?

10. What kind of a fraction is $\frac{n}{n+1}$ for all positive integral values of n?

104. The processes with negative numbers may be applied in substitution.

EXAMPLES

1. What is the value of d in $d=vt$, when $v=-3$ and $t=8$?

Given	$d = vt.$	(*1*)
When $v=-3$ and $t=8$,	$d=-3\cdot 8.$	(*2*)
Therefore,	$d=-24.$ (Sec. 75.)	(*3*)

2. What is the value of t in $d=vt$, when $d=-16$, $v=-2$?

Given	$d = vt.$	(*1*)
Dividing by v,	$t=\frac{d}{v}.$	(*2*)
When $d=-16$ and $v=-2$,	$t=\frac{-16}{-2}.$	(*3*)
Therefore,	$t=8.$ (Sec. 77.)	(*4*)

3. Find a in $a = \pi r^2$, when $r = -2$.

When $r = -2$, $a = \pi(-2)^2$. (1)

Therefore, $a = 3.1416 \times (-2)(-2)$. (Sec. 75.) (2)

$= 12.5664$.

ORAL EXERCISES

1. In $w = fs$, find w when $f = 6$ and $s = -3$. Also find w when $f = 12$ and $s = -\frac{1}{3}$.

2. In $w = fs$, find f when $w = 20$ and $s = -5$.

3. In $d = vt$, find d when $v = -12$ and $t = \frac{1}{4}$. Also find t when $d = -2$ and $v = -\frac{1}{8}$.

4. In $s = \frac{1}{2}ab$, find s when $a = 6$ and $b = -5$. Also find s when $a = -8$ and $b = -3$.

5. In $x = 2ay$, find x when $y = -15$. Also find y when $x = -10$.

6. In $3x = az$, find x when $z = -9$. Also find z when $x = 4a$.

WRITTEN EXERCISES

1. In $v = \pi r^2 a$, find v when $\pi = 3.1416$ and $r = -25$.

2. In $v = \pi r^2 a$, find a when $v = 9.4248$ and $r = -\frac{1}{2}$.

3. In $k = \frac{1}{2}mv^2$, find k when $m = -12$ and $v = 3$. Also find k when $m = -15$ and $v = -2$. Also find k when $m = 20$ and $v = -5$.

4. In $a = \pi r^2$, find a when $r = -4$; also when $r = -\frac{1}{3}$.

5. In $c = 2\pi r$, find c when $r = 20$ ft.; also when $r = -15$ ft.

6. In $s = \frac{1}{2}ab$, find s when $a = -17$ and $b = -12$; also when $a = 15$ and $b = -9$.

7. In $s = \frac{1}{2}ab$, find a when $s = -42$ and $b = -7$.

8. In $x = 3y^2$, find x when $y = 14$; also when $y = -14$.

9. In $3x + 1 = 4y - 4$, find x when $y = -5$.

10. In $ax = by + bc + a$, find x when $y = -c$.

11. In $ay - b = bx + c$ find x when $y = \frac{c}{a}$. Also find y when $x = -1$.

105. The processes with negative numbers may be applied in solving equations.

EXAMPLES

1. Solve for x: $3x - 4 = 11$ (1)

Add 4 to both members, $4 = 4$

Then, $3x = 11 + 4 = 15$, (Sec. 67.) (2)

and $x = 5$. (3)

TEST. $3 \cdot 5 - 4 = 11$.

2. Solve for x: $5x + 21 = 6$ (1)

Subtract 21 from both members, $21 = 21$

Then, $5x = 6 - 21 = -15$, (Sec. 71.) (2)

and $x = -3$. (Sec. 77.) (3)

TEST. $5(-3) + 21 = 6$.

The effect of adding or subtracting as above is equivalent to taking a number from one side of the equation and placing it on the other with the sign changed, and the term **transpose** is often used to name this process; compare steps (1) and (2) in each example. But it is much better at first in explaining the work to say that a certain number has been added to, or subtracted from, both members of the equation.

3. Solve for y: $2y - 6 = 3y + 1$. (1)

Subtracting $3y$ from both members, $-y - 6 = 1$. (Sec. 71.) (2)

Adding 6 to both members, $-y = 7$. (Sec. 67.) (3)

Multiplying both members by -1, $y = -7$. (Sec. 75.) (4)

TEST. $2(-7) - 6 = 3(-7) + 1$.

The change from step (3) to step (4) follows directly from the meaning of relative numbers. That is, if negative y equals positive 7, then positive y must equal negative 7. But since the sign of a number is changed by multiplying or dividing it by -1, it is customary to refer such a change of signs to one or the other of these processes.

ORAL EXERCISES

Solve:

1. $x + 1 = -8$.

2. $x - 2 = -5$.

3. $2x - 2 = 6$.

4. $2y - 1 = 11$.

5. $3y - 5 = 13$.

6. $7x + 5 = -9$.

7. $2 - 3x = -4$.

8. $5 - 4x = 21$.

9. $\frac{1}{2}y - 4 = 8$.

10. $5y + 5 = -20$.

11. $x - a = b - c$.

12. $ax - 2b = c - 3b$.

13. $4c - 2bx = 8 - 4bx$.

14. $y - a = -b$.

15. $ay + b = -c$.

WRITTEN EXERCISES

Solve for x:

1. $2x-1=3x-7.$
2. $5x+3=2x-15.$
3. $4x-5=10-x.$
4. $\frac{1}{2}x+6=x-18.$
5. $\frac{3}{2}x-5=20-x.$
6. $x+7=2-\frac{1}{4}x.$
7. $\frac{2}{3}-3x=4x-\frac{1}{3}.$
8. $\frac{5}{6}x-8=-17-\frac{1}{6}x.$
9. $ax-b=\frac{1}{2}ax+c.$
10. $ax+m=bc-(mx+m).$

Solve for the letter in each case:

11. $x-3+4x+21-2x=0.$
12. $y+8+6y-4-3y=0.$
13. $2y-5-y+20=-15.$
14. $m-7+3m+42=-4m.$
15. $5t+6-3t-10=t.$
16. $12-h+8-4h=-10.$
17. $8-2p+6-5p=21.$
18. $7s-16=3-3s-29.$
19. $35-4y=6y+5+5y.$
20. $2+x-5=8x+11.$
21. $10+z-4=-5z-12.$
22. $8-8w=28-28w.$

106. The process of removing parentheses applies to equations.

EXAMPLE

Solve: $x-(2x+6)=3-(4x-6).$ (*1*)

Removing the parentheses, $x-2x-6=3-4x+6.$ (Sec. 97.) (*2*)

Then, $3x=15,$

and $x=5.$

TEST. $5-(2\cdot 5+6)=-11=3-(20-6).$

WRITTEN EXERCISES

Solve and test:

1. $2x-(2+x)=6.$
2. $\frac{1}{2}x+(2-x)=6.$
3. $4x-(2x+1)=5.$
4. $5-(4x+2)=4.$
5. $7-(4x+3)=0.$
6. $6y+(4y-10)=10.$
7. $12z-(2+6z)=16.$
8. $2\frac{1}{2}+(2\frac{1}{2}-5t)=0.$
9. $2y-(y+6)=2y-7.$
10. $3x-(2x+8)=2-4x.$
11. $7+t=3-(2t-17).$
12. $5-3s=6-(s+15).$
13. $z-(6-5z-18)=30.$
14. $4p-(3p+1)=2p+1.$
15. $5w-(w+6)=2-(w+1).$
16. $z-6=z-(3z+17).$

17. $-(x+5)=8+(x-5)$.
18. $15-(z+12)=4z-20$.
19. $2y-(y+5)=4-(y-3)$.
20. $16-(2x+8)=3x+(4x-1)$.

107. In solving problems by means of equations, the parenthesis is often used.

EXAMPLE

The sum of two numbers is 100, and 3 times one of them is 7 times the other; what are the numbers?

SOLUTION.
1. Let x be the smaller number.
2. Then, $100 - x$ is the larger number.
3. Then, $7x$ is 7 times the smaller one and $3(100 - x)$ is 3 times the larger one.
4. $\therefore 7x = 3(100 - x)$, according to the problem.
5. $\therefore 7x = 300 - 3x$. (Sec. 24.)
6. $\therefore 10x = 300$.
7. $\therefore x = 30$, and the numbers are 30, 70.

TEST. $30 + 70 = 100$, and $7 \times 30 = 3 \times 70$. Therefore the numbers found fulfill the conditions of the problem.

The use of the parenthesis is seen in the third and fourth steps.

WRITTEN EXERCISES

1. Write an equation stating that if the cost (c) of a lot be diminished by \$200 and the remainder multiplied by 5, the result will be the value (v) of the house.

2. Write the equation which states that c times the sum of x and a equals d.

3. A father is twice as old as his son; 11 years ago he was three times as old as his son. Find the age of each.

4. If a certain number is diminished by 9 and the remainder multiplied by 9, the result is the same as if the number were diminished by 6, and the remainder multiplied by 6. Find the number.

5. Divide 48 into two such parts that one part shall exceed the other by 6.

6. A man is four times as old as his son; in 18 years he will be only twice as old. Find the age of each.

7. Two men enter a partnership and together furnish a capital of \$5000; twice what one furnishes is 3 times what the other furnishes. How much does each furnish?

8. The sum of two numbers is 40; the smaller number, x, is $\frac{1}{3}$ of the larger number. Write the equation needed to find x. Find the numbers.

9. A rectangular lot is 20 ft. longer than it is wide. Using x to represent the width, state what represents the length. Write an equation stating that 4 times the width equals 2 times the length. Find the dimensions of the lot.

10. In a certain post office there are three rates of pay: \$50, \$100, and \$150 per month. There are 5 more men receiving \$100 than \$150, and 2 more receiving \$50 than \$100; the monthly pay roll is \$1150. Letting x represent the number receiving \$150, write the equation needed to find x.

108. The solution of problems must be proved by substituting the results in the conditions of the problems.

EXAMPLE

1. A commission merchant remitted \$475 as the proceeds of a sale of 500 bu. of potatoes after deducting his commission of 5%. For how much did he sell the potatoes?

SOLUTION. 1. Let $x =$ the number of dollars received for the potatoes.
2. Then, $x - .05x =$ the amount remitted.
3. $\therefore x - .05x = 475$, according to the problem.
4. $\therefore .95x = 475$.
5. Dividing by .95, $x = 500$.
6. $\therefore$ the potatoes sold for \$500.

TEST. $500 - .05 \cdot 500 = 475$.

This merely tests the correctness of the work after step 3; it does not test the correctness of the equation in step 3; to do this the result, 500, must be tested *in the conditions of the problem itself.*

Thus, (1) 5% of \$500 = \$25, (2) \$500 − \$25 = \$475, the proceeds. If equation 3 had been incorrectly written, the result found in step 6 might have been a correct solution of equation 3 without giving the correct proceeds.

WRITTEN EXERCISES

Solve and test:

1. If A has x dollars and B has twice as much, express what they both have. If this amount is \$120, write and solve an equation, thus finding what each has.

2. The sum of a number and .05 of itself is 210, what is the number?

3. Separate the number 72 into two such parts that one part shall be $\frac{1}{3}$ of the other.

4. Divide 100 into two parts such that one is 24 less than the other.

5. Three times a number less twice the difference between the number and 5 is 30. Find the number.

6. Find a number such that when 9 is added to three times the number the sum is 42.

7. A tree 120 ft. high was broken so that the length of the part broken off was four times the length of the part left standing. Find the length of each part.

8. Three men together have \$1800. The first has twice as much as the second, and the third has as much as the first and second. How much has each?

9. A man was hired for 50 da. Each day he worked he was to receive \$2 and each day he was idle he was to forfeit 50¢ instead of getting his wages. At the end of the 50 da. his balance was \$80. How many days did he work?

10. A farmer bought 16 sheep. If he had bought 4 sheep more for the same money, each sheep would have cost him one dollar less. How much did he pay for a sheep?

11. A, B, and C bought a summer cottage for \$3000. B pays twice as much as A, and C pays as much as A and B together. How much does each pay?

12. A tree 90 ft. high was broken so that the length of the part broken off was 5 times the length of the part left standing. Find the length of each part.

HISTORICAL NOTE

The earliest attempts to solve equations mark the beginning of algebra. About 1700 B.C. there lived an Egyptian priest named Ahmes. He was probably a teacher of mathematics, and was the first writer on the subject whose work is still in existence, for there has been recovered from the ancient pyramids a manual written by him which contains, among other mathematical subjects, a simple treatment of equations. Ahmes' work, called *Directions for obtaining the Knowledge of all Dark Things*, is written in hieroglyphics on papyrus, and is now, thirty-six centuries after its production, the property of the British Museum. It was deciphered by the German scholar, Eisenlohr, in 1877.

The equations of Ahmes were expressed in words, not in symbols; for example, one of his problems, when translated, reads: "Hau (literally *heap*, meaning unknown quantity) its seventh, its whole, makes nineteen," which in present-day algebraic notation is $x + \frac{x}{7} = 19$. There are few rules in Ahmes' handbook to show how his results were obtained, but it contains certain tables of numbers that indicate the processes. For example, he could write only unit-fractions, like $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{1}{7}$, other fractions being expressed by the sum of unit-fractions. Accordingly, he gives the solution of the above equation as follows:

(1) $8 \quad \frac{1}{7}x = 19$ (2) $\frac{1}{7}x = 2 \quad \frac{1}{4} \quad \frac{1}{8}$ (3) $x = 16 \quad \frac{1}{2} \quad \frac{1}{8}$

(In the first step the adjacent numbers are supposed to be multiplied, and in the other steps they are to be added.)

Ahmes was greatly hindered by lack of suitable notation and other limitations, and knowing nothing of negative number, he could not solve so simple an equation as $x + 3 = 1$.

Little progress was made in solving equations during the twenty centuries from Ahmes to Diophantos. During the Middle Ages the Arabs and Hindoos simplified the solutions of Diophantos, but the perfection of modern symbols and the explanation of positive and negative roots remained for the mathematicians of the sixteenth century.

CHAPTER VIII

MULTIPLICATION

MULTIPLICATION OF MONOMIALS

109. PREPARATORY.

1. $5 \cdot 3¢ = (\ )¢$.
2. $6 \cdot 3d = (\ )d$.
3. $5 \cdot 3$ ft. $= (\ )$ ft.
4. $7 \cdot 5x = (\ )x$.
5. $5 \cdot 3f = (\ )f$.
6. $6 \cdot 6abc = (\ )abc$.
7. $3a \cdot 4b = 3 \cdot 4ab = ?$
8. $\frac{1}{2}a \cdot 8b = \frac{1}{2}$ of $8ab = ?$
9. $4x \cdot 6y \cdot \frac{2}{3}z = 4 \cdot 6 \cdot \frac{2}{3}xyz = (\ )xyz$.

110. To find the product of two monomials, *take the product of their literal parts for the literal part of the product, and the product of their coefficients for the coefficient of the product.*

Prefix to the result the sign + or − according to Sec. 75.

111. Repeated Factors. If the same letter occurs more than once as a factor in the product, it should be written only once, with the proper exponent (Sec. 14).

For example:

$$-3a \cdot 2ab = -6aab = -6a^2b.$$
$$a^2b^3 \cdot ab^2 = aabbb \cdot abb = aaa \cdot bbbbb = a^3b^5.$$

ORAL EXERCISES

State the products:

1. $t^2 \cdot t^4$.
2. $y^5 \cdot y^2$.
3. $x^2y \cdot y^3$.
4. $8c \cdot 8c$.
5. $7a \cdot 6ab$.
6. $a^3 \cdot -a^2$.
7. $4a \cdot 2ax$.
8. $4x^3 \cdot -2x^2$.
9. $a^2x \cdot -a^3x^4$.
10. $12ab \cdot 2ab^2$.
11. $3xy^2 \cdot -5x^2y$.
12. $2a^4 \cdot a^5$.
13. $(4p^2)(8p^4)$.
14. $(am^3)(-am^3)$.
15. $2a^3r \cdot 6a^2r^5$.
16. $7a^2 \cdot -a^2$.
17. $15v^3 \cdot v^5$.
18. $13x^7 \cdot x^3$.

WRITTEN EXERCISES

Multiply as indicated:

1. $48ab^3 \cdot -10bc^2$.
2. $24a^2bc \cdot 12c$.
3. $214bc \cdot 6b^2c^3$.
4. $16a^2 \cdot 15a^8c$.
5. $36x^2 \cdot 3xz^2$.
6. $12m^5 \cdot -16mr^3$.
7. $30x^2 \cdot -4ax^3$.
8. $65a^2 \cdot -4a^9b$.
9. $27a^6 \cdot -63a^9b^2$.

112. Law of Exponents in Multiplication. *The exponent of any letter in a product is the sum of the exponents of that letter in all of the factors.*

In symbols this law is expressed:

$$a^m \cdot a^r = a^{m+r}.$$

EXAMPLES

1. $a^5 \cdot a^7 = a^{5+7} = a^{12}$.
2. $x^n \cdot x^{n+2} = x^{n+n+2} = x^{2n+2}$.
3. $a^n \cdot -a^7x \cdot ax^{p-3} = -a^n \cdot a^7 \cdot a^1 \cdot x^1 \cdot x^{p-3}$
$= -a^{n+7+1} \cdot x^{1+p-3}$
$= -a^{n+8}x^{p-2}$.

113. To find the product of several monomials:

1. *Find the sign of the product.*

The sign is plus when the number of negative factors is even, and minus when the number of negative factors is odd.

2. *Find the numerical coefficient of the product.*
3. *Find the literal part of the product.*

EXAMPLE

Find the product of $-2a$, $+3b$, $-2ab$, $-4bc$.

1. The sign is $-$.
2. $2 \cdot 3 \cdot 2 \cdot 4 = 48$.
3. $a \cdot b \cdot ab \cdot bc = a^2b^3c$.
4. $\therefore -2a \cdot 3b \cdot -2ab \cdot -4bc = -48a^2b^3c$.

114. *If any factor is zero, the product is zero.*

For example:

$$3 \cdot 0 = 0; \text{ similarly, } -x \cdot 0 = 0, \text{ and } x \cdot y \cdot z \cdot 0 = 0.$$

This follows from the meaning of zero Sec. 66. For if $3 - 3 = 0$, then $(3-3)4$ represents $0 \cdot 4$; but $(3-3)4 = 12 - 12 = 0$.

Hence, $0 \cdot 4$ should be taken to mean 0.

ORAL EXERCISES

Multiply:

1. $6, -3, 5.$
2. $-7, 3, -1.$
3. $a, 0, -c.$
4. $-6, -0, -4.$
5. $2x, \ -3y, \ -2z.$
6. $ax, -bx, -cx.$
7. $-a, -a, 0.$
8. $(-q), (-q), (-q).$
9. $(-6), (-6).$
10. $(-4x), (-4x).$
11. $(-2)^3, (-a)^3.$
12. $x, (-x)^2, (-x)^3.$
13. $x^3, (-x)^3.$
14. $-x^2, (-x)^2.$
15. $2gx, 4gx, -gx^6.$
16. $m^2, -n^2, (-p)^2$

WRITTEN EXERCISES

Multiply:

1. $(-2)^2 \cdot (-3)^3 \cdot (-1)^2.$
2. $(-3ab)^3 \cdot (-3ab)^2 \cdot 5a^2b^3.$
3. $abxy \cdot -21a^3b^5x^my^n.$
4. $31a^rx^s \cdot -15a^{r-1}x^{s-1}.$
5. $-4a^2x^{n-5} \cdot -64a^{n-2}x^5.$
6. $20xy^pz^q \cdot -71x^py^qz^r.$
7. $15a^{m+1} \cdot -12a^{3m}b \cdot 3a^5b^{m-1}.$
8. $17a^mx^{p-2} \cdot -5a^{p-m}x^4.$
9. $-24x^7y^{p-4} \cdot -8x^{r-1}y^5.$
10. $12m^2n \cdot -15mn^8 \cdot -m^pn^s.$
11. $-\frac{1}{8}x^py^qz^s \cdot \frac{1}{7}x^{1-p}y^{1-q}z^{5-s} \cdot -\frac{7}{15}x^5y^6z.$
12. $121s^xt^yy^z \cdot -17s^3t^5z^4 \cdot 0.$

13. Suppose $12a^2b^3c$ to be taken as the product of $4ab^2$ and $3a^2c$. Test this product by letting a, b, and c each equal 1.

14. Also test it by letting $a = 2$, $b = 3$, and $c = 5$. Why did not the test applied in Exercise 13 reveal the error?

115. Signs of Factors. It follows from Sec. 75 that, if the product of two factors is positive, the factors must have like signs; but if the product is negative, the factors must have unlike signs.

For example:

$$8ax = 2a \cdot 4x, \text{ or } (-2a)(-4x).$$
$$-14b^2c = (7b)(-2bc), \text{ or } (-7b)(2bc).$$

WRITTEN EXERCISES

Write a set of two factors for each of the following:

1. $b^2h.$
2. $\frac{1}{2}gt^2.$
3. $\pi r^2h.$
4. $\frac{4}{3}\pi r^3.$
5. $15ax^2y^2.$
6. $-7mn^2p.$
7. $14y^2.$
8. $-48x.$

9. mx^2.	12. $\frac{1}{2}mv^2$.	15. $16\,ab^2c$.	18. $21\,a^3x^3$.
10. $-\pi r^2$.	13. $\frac{1}{6}\pi d^3$.	16. $-8\,a^3b$.	19. $-md^3$.
11. $4\,\pi r^2$.	14. $6\,ax^2$.	17. $14\,br^2$.	20. $2\,\pi rh$.

MULTIPLICATION OF POLYNOMIALS

116. To multiply a polynomial by a monomial *multiply each term of the multiplicand by the monomial and use the signs obtained as the signs of the product.*

For example:

$$\begin{array}{r} 2\,a - 3\,b + 5\,c \\ \underline{6\,a \qquad\qquad\quad} \\ 12\,a^2 - 18\,ab + 30\,ac \end{array}$$

117. Distributive Law. The fact that a polynomial is multiplied by multiplying each of its terms separately and taking the algebraic sum of the partial products thus found is called the **Distributive Law of Multiplication.**

The formula, $a(b+c)=ab+ac$, expresses this law in symbols.

ORAL EXERCISES

Read and supply the numbers for the blanks:

1. $\begin{array}{l} 3\,a + 4\,b \\ \underline{3 \qquad\qquad} \\ (\;)\,a + (\;)\,b \end{array}$

2. $\begin{array}{l} 5\,a^2 + 2\,b \\ \underline{10 \qquad\qquad} \\ (\;)\,a^2 + (\;)\,b \end{array}$

3. $\begin{array}{l} 10\,x + y \\ \underline{5 \qquad\qquad} \\ (\;)\,x + (\;)\,y \end{array}$

4. $\begin{array}{l} 3\,a + 4\,b \\ \underline{3\,a \qquad\qquad} \\ (\;)\,a^2 + (\;)\,ab \end{array}$

5. $\begin{array}{l} 10\,m + 2\,n \\ \underline{5\,mn \qquad\qquad} \\ (\;)\,m^2n + (\;)\,mn^2 \end{array}$

6. $\begin{array}{l} 4\,x^2y + xy^2 \\ \underline{3\,xy \qquad\qquad} \\ (\;)\,x^3y^2 + (\;)\,x^2y^3 \end{array}$

7. $\begin{array}{l} 6\,a^2 + 2\,b \\ \underline{3\,b \qquad\qquad} \\ 18\,(\;) + 6\,(\;) \end{array}$

8. $\begin{array}{l} 5\,m + 3\,p^2 \\ \underline{3\,mp \qquad\qquad} \\ (\;)\,m^2p + (\;)\,mp^3 \end{array}$

9. $\begin{array}{l} 9\,x^2 + y^2 \\ \underline{5\,xy \qquad\qquad} \\ 45\,(\;) + 5\,(\;) \end{array}$

118. To test the work of multiplication, use arbitrary values. The product of the values of the multiplicand and the multiplier must equal the value of the product.

Unity is the easiest number to substitute for the letters; it tests the coefficients and signs in the work of multiplication. It does not, however, test the exponents (see Exercises 13 and 14, p. 77); but this does not impair the test materially, since errors in exponents alone seldom occur.

WRITTEN EXERCISES

Multiply and test:

1. $ax+4$ 3	4. $7ax+3a$ $5x$	7. a^2+2bx^3 $3b^2x$
2. $a+b$ c	5. $8a^2+2b^2$ ab	8. $3m^2+r^2$ $4a$
3. $2a+3b$ $4c$	6. $7ax+3bx^2$ cx	9. mt^2+v vt

10. $4x^2(a-5x)$.	13. $2x(3x+5y)$.	16. $6q^2(5q+18q^3)$.
11. $a(2x^3+1)$.	14. $(7x-5)2a$.	17. $\frac{1}{2}r(2r^2+\frac{1}{4}r)$.
12. $a^2b(a^2c-b^2d)$.	15. $(4x+5t)2tx$.	18. $6t(t^2+\frac{1}{2}at)$.

119. Special Tests. Certain polynomials have properties which aid in testing the work of multiplication.

Thus, in $(x+y)(x^2-xy+y^2)=x^3+y^3$, each term of the first factor is of the first degree, and each term of the second is of the second degree; hence, if each term of the product were not of the third degree, the product would be incorrect.

120. Expressions all of whose terms are of the same degree are called **homogeneous**; when factors are homogeneous, their product is homogeneous, and its degree is the sum of the degrees of its factors.

ORAL EXERCISES

1. Without multiplying, state the degree of each term in the product of $a+b$ and a^2+b^2.

2. Similarly for x^2+xy and $x+y$. Also for $mn+n^2$ and m^2+n^2.

3. Without multiplying, determine whether or not x^3+xy+y^3 is the product of x^2+xy+y^2 and $x+y$.

WRITTEN EXERCISES

Multiply and test first by seeing whether the product is homogeneous. If it is homogeneous, test further by substituting arbitrary values:

1. $(a^3+2b^3)(a^3-3b^3)$.
2. $(x+y)(x^2-3xy+5y^2)$.

3. $(x^2 - y^2)(x^2 + 6xy - y^2)$.
4. $(a + x)^2(a^2 - x^2)$.
5. $(a^2 - 5ax + x^2)(x^3 + 2a^3)$.
6. $(x^2 - 2xy + y^2)(x^2 + y^2)$.

121. Removal of Parentheses. *If a parenthesis used to indicate multiplication is removed, the multiplication must be performed.*

Thus: $7a - 5(9a - 4b) = 7a - 45a + 20b = -38a + 20b$.

WRITTEN EXERCISES

Remove parentheses and unite terms as much as possible:

1. $3 + 5(6 - 4)$.
2. $5x + 3(11x - 5)$.
3. $7(4a - 2b) + 10b$.
4. $6y - 7(4y + 3t)$.
5. $11 - 3(7 - 2x)$.
6. $4a - 12(7 - 6a)$.
7. $9(a - x) - a(5 + x)$.
8. $-5(2x - 1) + 3(4x - 8)$.
9. $a(b + c) - c(a + b) + b(a - c)$.
10. $2m - \{4m + 7(6m - 1)\}$.
11. $p\{4r - 3r(1 - a) + 5ar\}$.
12. $x^2\{x^2 - 2a(2x - 3a)\} - a^3(4x - a)$.
13. $-10\{x - 6[x - (y - z)]\} + 60\{y - (z + x)\}$.

122. The multiplication of literal numbers is similar to the multiplication of numbers expressed by figures.

For example:

Multiplication with Figures	Multiplication with Letters	Test
32	$3a + 2b$	5
14	$a + 4b$	5
$128 = 4 \times 32$	$3a^2 + 2ab = a(3a + 2b)$	
$320 = 10 \times 32$	$12ab + 8b^2 = 4b(3a + 2b)$	
$448 = 14 \times 32$	$3a^2 + 14ab + 8b^2 = (a + 4b)(3a + 2b)$	25

123. To multiply by a polynomial, *multiply by each term of the polynomial, add like terms, and use the signs obtained as the signs of the result.*

Thus:

$$\begin{array}{r} 2a^2 + 5a - 2 \\ a^2 - 3a + 1 \\ \hline 2a^4 + 5a^3 - 2a^2 \qquad\qquad \\ -6a^3 - 15a^2 + 6a \qquad \\ 2a^2 + 5a - 2 \\ \hline 2a^4 - a^3 - 15a^2 + 11a - 2 \end{array}$$

WRITTEN EXERCISES

Multiply and test:

1. $\begin{array}{l} a + b \\ a + b \\ \hline \end{array}$

2. $\begin{array}{l} a + b \\ a - b \\ \hline \end{array}$

3. $\begin{array}{l} a - b \\ a - b \\ \hline \end{array}$

4. $\begin{array}{l} c + 1 \\ c - 1 \\ \hline \end{array}$

5. $\begin{array}{l} x + 2 \\ x + 2 \\ \hline \end{array}$

6. $\begin{array}{l} z^2 + 5 \\ z^2 + 5 \\ \hline \end{array}$

7. $\begin{array}{l} 2a - 1 \\ 2a - 1 \\ \hline \end{array}$

8. $\begin{array}{l} 12 + x \\ 12 - x \\ \hline \end{array}$

9. $\begin{array}{l} 3y - 5 \\ 2y + 4 \\ \hline \end{array}$

10. $\begin{array}{l} x + a \\ x + b \\ \hline \end{array}$

11. $\begin{array}{r} 3a + x \\ a + b \\ \hline \end{array}$

12. $\begin{array}{r} 4a + 5 \\ x - a \\ \hline \end{array}$

13. $\begin{array}{r} m + 3 \\ 3m + 2 \\ \hline \end{array}$

14. $\begin{array}{l} m^2 - n^2 \\ m^2 + n^3 \\ \hline \end{array}$

15. $\begin{array}{l} x^2 + 1 \\ x^2 - 1 \\ \hline \end{array}$

16. $\begin{array}{l} 2a + b \\ a + 2b \\ \hline \end{array}$

17. $\begin{array}{r} 2a - b \\ c - 3a \\ \hline \end{array}$

18. $\begin{array}{l} 3x + 2y \\ 2x + 3y \\ \hline \end{array}$

19. $\begin{array}{l} 3ab + 4b^2 \\ 2ab - 3b^2 \\ \hline \end{array}$

20. $\begin{array}{l} x^2 + 3x - 1 \\ x + 3 \\ \hline \end{array}$

21. $\begin{array}{l} x^2 - 4x + 3 \\ x - 2 \\ \hline \end{array}$

22. $\begin{array}{l} x^2 - ax + b \\ x - c \\ \hline \end{array}$

23. $\begin{array}{l} x^2 - ax + b \\ 3x + a \\ \hline \end{array}$

24. $\begin{array}{l} t^2 + tu + u^2 \\ t - u \\ \hline \end{array}$

25. $\begin{array}{l} a + b - c \\ a - b + c \\ \hline \end{array}$

26. $\begin{array}{l} x^2 + y^2 - z^2 \\ x + y - z \\ \hline \end{array}$

27. $\begin{array}{r} xy + yz + xz \\ x - y + z \\ \hline \end{array}$

124. *To find the product of expressions involving literal coefficients and exponents, find the product of the coefficients and add the exponents as in numerical cases.*

EXAMPLES

1. Multiply ax and $(a - b)x$.

$$\begin{array}{r} (a - b)x \\ ax \\ \hline a(a - b)x^2 \end{array}$$

2. Multiply $x^n + 1$ by $x^n - 2$.

$$\begin{array}{l} x^n + 1 \\ x^n - 2 \\ \hline x^{2n} - x^n - 2 \end{array}$$

WRITTEN EXERCISES

Multiply:

1. $(5a + x)(5a - x)$.
2. $(x - y)(x^m - y^m)$.
3. $(x^m + y^n)(x + y)$.
4. $(4x^m + 3y^n)(4x^m - 3y^n)$.
5. $(x^m - y^n)(x^{2m} - y^{2n})$.
6. $(a^{2n} + c^{2n})(a^{mn} + c^{mn})$.
7. $z^n(z^r + z^s)$.
8. $(xc - 1)cx$.
9. $(a - b)y \cdot aby$.
10. $(c + d)x \cdot (c - d)x$.
11. $(a^m + c)(a^m - c)$.
12. $(a - 3ab^3)(a + 3ab^3)$.
13. $(a + b)x^n \cdot (a - 2b)xy^p$.
14. $(m - n - 2p)x^r \cdot -3m^2npx^s$.
15. $(m^2 - n^2)x^4y^5 \cdot (m^2 + n^2)x^2y^2$.
16. $(p^2 - p + 1)x^{2a}y^{2b} \cdot (p + 1)x^by^a$.

REVIEW

ORAL EXERCISES

State the products:

1. $\begin{array}{r} 4x \\ \underline{9x} \end{array}$
2. $\begin{array}{r} 8y \\ \underline{-3y} \end{array}$
3. $\begin{array}{r} 7a \\ \underline{4a} \end{array}$
4. $\begin{array}{r} -6t \\ \underline{-2t} \end{array}$
5. $\begin{array}{r} -9m \\ \underline{3m} \end{array}$
6. $\begin{array}{r} 5ab \\ \underline{7ac} \end{array}$
7. $\begin{array}{r} -2ax \\ \underline{+5x^2} \end{array}$
8. $\begin{array}{r} -6ay \\ \underline{-4ay} \end{array}$
9. $\begin{array}{r} 7x^2y \\ \underline{-\ \ xy} \end{array}$
10. $\begin{array}{r} -9am \\ \underline{\frac{1}{3}ar} \end{array}$

WRITTEN EXERCISES

Multiply and test:

1. $\begin{array}{l} ax + 3 \\ \underline{ax + 5} \end{array}$
2. $\begin{array}{l} 4ab + c \\ \underline{2ab + 3c} \end{array}$
3. $\begin{array}{l} y + \ \ x + b \\ \underline{y + 5x - b} \end{array}$
4. $\begin{array}{l} a - b - c \\ \underline{a - b - c} \end{array}$
5. $\begin{array}{l} 1 - x + x^2 \\ \underline{1 + x - x^2} \end{array}$
6. $\begin{array}{l} a + 3x - 1 \\ \underline{x + 2a + 1} \end{array}$
7. $\begin{array}{l} x^3 - 7x^2 + 5x - 3 \\ \underline{2x - 4 \qquad\qquad\quad} \end{array}$
8. $\begin{array}{l} a^3 - 3a^2y + 3ay^2 - y^3 \\ \underline{a - y \qquad\qquad\qquad\quad} \end{array}$
9. $\begin{array}{r} x^2 + 3px - 4p^2 \\ \underline{2x^2 - 7px - p^2} \end{array}$
10. $\begin{array}{l} x^3 + 3x^2y + 3xy^2 + y^3 \\ \underline{x + y \qquad\qquad\qquad\quad} \end{array}$

11. $\begin{array}{l} x-5 \\ \underline{x+6} \end{array}$

12. $\begin{array}{r} 2x+3 \\ \underline{x-1} \end{array}$

13. $\begin{array}{l} 3x+5 \\ \underline{2x-4} \end{array}$

14. $\begin{array}{r} 6t-3u \\ \underline{t+2u} \end{array}$

15. $\begin{array}{r} 3x^2+5 \\ \underline{x-9} \end{array}$

16. $\begin{array}{l} x^2+x+1 \\ \underline{x \quad +1} \end{array}$

17. $\begin{array}{r} x^2-2x+1 \\ \underline{x-1} \end{array}$

18. $\begin{array}{l} x^2+3x+2 \\ \underline{x^2+\quad x+2} \end{array}$

19. $\begin{array}{l} 2x^2-x+1 \\ \underline{2x \quad -5} \end{array}$

20. $(2a^n+3b^n)(2a^n+3b^n)$.

21. $(r^{p-1}-s^{p-1})(r-s)$.

22. $(4z^{3n}-2^{2n}+z^n-1)(3z^n+1)$.

23. $(3x^m-y^r)(3x^m+y^r)$.

24. $(x-1)(x-2)(x-3)$.

25. $(x+6)(x-5)(x-3)$.

26. $(a^3-a^2-1)(a+1)$.

27. $(m^2-m+1)(m+1)$.

28. $(x^4-x^2+1)(x^2+6)$.

29. $x^2y^s(x^ny^r-x^qy^4)$.

30. $(a-b)x\cdot(a+b)x$.

31. $ab\cdot(c-1)b$.

32. $ax^2\cdot(a-b)x^2\cdot(b+c)x^2$.

33. $(x^2y-xy^2)(x+y)$.

34. $(m^5+m^3+1)(m^2+1)$.

35. $(y^4+y+2)(y^3-1)$.

36. $(y^3-3y+5)(y^2+10)$.

37. $(rs-r^2s^2)(rs+r^2s^2)$.

SUMMARY

The following questions summarize the definitions and processes in this chapter:

1. State how to find the product of two monomials. Sec. 110.

2. State the *Law of Exponents in Multiplication.* Sec. 112.

3. State how to find the product of several monomials. Sec. 113.

4. State how to multiply a polynomial by a monomial Sec. 116

5. State the *Distributive Law of Multiplication.* Illustrate this law in symbols. Sec. 117.

6. State how to test the work in multiplication. Secs. 118, 119.

7. State how to multiply one polynomial by another. Sec. 123.

CHAPTER IX

DIVISION

DIVISION OF MONOMIALS

125. Preparatory.

1. 4 lb. $\div 2 = (\ )$ lb. 4 yd. $\div 2 = (\ )$ yd. $4\,y \div 2 = (\ )\,y$.

2. 6 oz. $\div 3 = (\ )$ oz. 6 ft. $\div 3 = (\ )$ ft. $6\,f \div 2 = (\ )\,f$.

3. $3\,a \cdot (\ ) = 6\,a^2b$, then $6\,a^2b \div 3\,a = ?$

4. $8\,b^2 \cdot (\ ) = 16\,ab^3$, then $16\,ab^3 \div 8\,b^2 = ?$

5. $a^2b \cdot (\ ) = a^2bcd$, then $a^2bcd \div a^2b = ?$

126. Division. Division is the process of finding one of two factors when the product and the other factor are given. Division is thus the inverse of multiplication.

The problem $18\,a^3b^7c \div 3\,a^2b^4c$ means to find the number by which $3\,a^2b^4c$ must be multiplied to produce $18\,a^3b^7c$.

To do this:

3 must be multiplied by 6 to produce 18.

a^2 must be multiplied by a to produce a^3.

b^4 must be multiplied by b^3 to produce b^7.

c multiplied by 1 produces c.

Hence, the quotient of $18\,a^3b^7c \div 3\,a^2b^4c$ is $6 \cdot a \cdot b^3 \cdot 1$, or $6\,ab^3$.

Test. The product of the divisor and the quotient must equal the dividend.

127. Zero cannot be used as a divisor.

This may be seen by reference to Sec. 114.

For, $0 \cdot 3 = 0$ and $0 \cdot 5 = 0$.

Therefore, $0 \cdot 3 = 0 \cdot 5$.

Now if we could divide both numbers by 0, the result would be $3 = 5$, which is not true.

WRITTEN EXERCISES

Divide and test:

1. $65\,a^2b \div 13\,ab$.
2. $48\,x^2y \div 12\,xy$.
3. $63\,m^2n^2 \div 21\,mn$.
4. $96\,a^3b^2 \div 12\,ab$.
5. $45\,p^2q^3 \div 15\,p^2q^2$.
6. $120\,abc \div 20\,a$.
7. $\frac{1}{2}\,mv^2 \div mv$.
8. $\frac{1}{2}\,gt^2 \div \frac{1}{2}\,t$.
9. $\frac{4}{3}\,\pi r^3 \div \pi r^2$.
10. $\frac{1}{6}\,\pi d^3 \div \frac{1}{6}\,\pi d$.

11. Find the numbers to fill the blanks:

	(1)	(2)	(3)	(4)	(5)
Dividend:	$12\,x^2$	$27\,ab$	$33\,ay$	$42\,p$	$36\,mn$
Divisor:	$3\,x$	——	——	$7\,p$	$9\,m$
Quotient:	——	$3\,b$	$11\,a$	——	——

128. The equation:

$$\text{Dividend} = \text{Divisor} \times \text{Quotient}$$

remains true if the two numbers, dividend and divisor, are multiplied (or divided) by the same number.

In other words:

The quotient is not altered, if both dividend and divisor are multiplied or divided by the same number.

Canceling may be used as in arithmetic:

Thus, $$\frac{8\,ab^6}{4\,ab^2} = \frac{\overset{2\,b^4}{\cancel{8}\,\cancel{a}\cancel{b}^6}}{\cancel{4}\,\cancel{a}\cancel{b}^2} = 2\,b^4.$$

Exponents must not be canceled. E.g. if we should cancel the exponent 2 from the exponent 6, the result would be b^3 instead of b^4.

Taking out factors results in subtracting not dividing exponents.

129. Law of Exponents. Since $a^5 = a^3 \cdot a^2$, it follows by dividing both members by a^2 that $\frac{a^5}{a^2} = a^3$.

Likewise, from $a^{m+r} = a^m \cdot a^r$, it follows by dividing both members by a^r that $\frac{a^{m+r}}{a^r} = a^m$.

Accordingly, the exponent of any letter in the quotient is the difference between the exponent of that letter in the dividend and its exponent in the divisor.

For example:

$$\frac{a^{12}}{a^7} = a^5, \text{ also } \frac{a^{3n}}{a^n} = a^{2n},$$
$$\frac{a^{n+2}}{a^2} = a^n, \text{ also } \frac{(ab)^5}{(ab)^2} = (ab)^3.$$

ORAL EXERCISES

Divide:

1. $a^5 \div a^3$.
2. $a^7 \div a^5$.
3. $b^{10} \div b^3$.
4. $b^7 \div b^4$.
5. $a^3b \div a^2$.
6. $a^4b^2 \div a^2b$.
7. $m^7 \div m^2$.
8. $\frac{14\,a^3b}{2\,a}$.
9. $\frac{12\,a^{4s}}{10\,a^{3s}}$.
10. $\frac{a^{4m}}{a^{2m}}$.
11. $\frac{3\,a^2b^q}{3\,ab^q}$.
12. $\frac{b^{5n+3}}{b^3}$.
13. $\frac{25\,m^{2x}n}{5\,m^x}$.
14. $\frac{x^3y^{3n}}{xy^n}$.
15. $\frac{p^2qr}{pqr}$.
16. $\frac{65\,x^3y^5}{13\,xy^3}$.
17. $\frac{18\,abx^3y^{2p}}{9\,abxy^p}$.
18. $\frac{6\,a^2b^{x+2}}{3\,ab^2}$.
19. $\frac{20\,m^2n^2q}{6\,mn^2q}$.

130. To find the quotient of two monomials:

1. *Find the sign of the quotient,* by the rule of signs. (Sec. 77.)

2. *Find the numerical coefficient,* by dividing the given numerical coefficients.

3. *Find the literal part,* by dividing the given literal parts.

Thus, in $18\,a^3b \div -3\,ab$, the sign is $-$, the numerical coefficient is $18 \div 3$ or 6, the literal part is $a^3b \div ab$ or a^2; $\therefore$ the quotient is $-6\,a^2$.

WRITTEN EXERCISES

Divide:

1. $\frac{2\,a^2cx^2y}{-2\,axy}$.
2. $\frac{-abcx^2y}{-cx}$.
3. $\frac{-5\,abc^2}{-5\,bc}$.
4. $\frac{10\,a^3x^2y^3}{-5\,ax^2}$.
5. $\frac{10\,m^3n^3}{-\frac{1}{6}\,mn^2}$.
6. $\frac{1.4\,a^5b}{.7a^3b}$.
7. $\frac{a^{7m}b^{4c}}{a^{3m}b^c}$.
8. $\frac{18\,a^3b^3}{3\,a^2b}$.
9. $\frac{24\,abc}{-3\,b}$.

10. $\frac{1.5\,x^3yz}{-5\,x^2y}$.

11. $\frac{m^2n^2p^2}{-n\cdot -p}$.

12. $\frac{-a^4b^3c^3}{-a^2\cdot -bc}$.

13. $\frac{-35\,b^6c^6}{7\,b^3c^3}$.

14. $\frac{10^{6n}x^{3n+8}}{10^{2n}x^{3n+4}}$.

15. $\frac{-2\,a^3x^3}{\frac{1}{2}\,ax^2}$.

16. $\frac{-2\,p^3q^4r^4t^4}{-\frac{1}{5}\,pqrt^3}$.

17. $\frac{a^4b^2c^2x}{-a^2bcx}$.

18. $\frac{-16\,a^2x^2y^3}{4\,y}$.

19. $\frac{27\,xy^2z^4}{-3\,z^3}$.

20. $\frac{16\,b^6cx^6}{-4\,b^3cx^3}$.

21. $\frac{54\,m^2p^3x^5}{-9\,mp^2x^3}$.

22. $\frac{a^{n+7}b^{3n+6}c^{2n+9}}{-a^4b^3c^7}$.

23. $3\,a^4b^4c^3 \div -abc$.

24. $2\cdot 5\,a^2x^3y \div -5\,ax^2y$.

25. $55\,a^2x^2 \div -11\,a^2x$.

26. $\frac{-18\,x^{a+5}y^{2a+7}}{3\,x^4y^{a+3}}$.

27. $\frac{4}{3}\,\pi r^3 \div \frac{1}{3}\,\pi r$.

28. $\frac{1}{6}\,\pi d^3 \div \frac{1}{2}\,\pi d^2$.

29. $-27\,x^2y^2z^2 \div 3\,xyz^2$.

DIVISION OF POLYNOMIALS

131. PREPARATORY.

1. 2) 4 bu. 2 pk. → ( ) bu. ( ) pk. 2) 4 lb. 2 oz. → ( ) lb. ( ) oz. $2)\,4\,l+2\,z$ → $(\ )l+(\ )z$

2. 3) 9 ft. 6 in. → ( ) ft. ( ) in. 3) 9 mi. 6 rd. → ( ) mi. ( ) rd. $3)\,9\,m+6\,r$ → $(\ )m+(\ )r$

MULTIPLICATION	DIVISION
Since (1)	then, (2)
$a-bd+5\,c$	
m	$m)\,ma-mbd+5\,mc$
$ma-mbd+5\,mc$	$a-bd+5\,c$

132. *Accordingly, to divide a polynomial by a monomial, divide each term of the polynomial by the monomial, and use the signs obtained as the signs of the quotient.*

WRITTEN EXERCISES

Divide and test:

1. $(6\,a^2+3\,a)\div 3$.
2. $(12\,ab+4\,b)\div 4$.
3. $(6\,a^2+3\,a)\div a$.
4. $(12\,ab+5\,b)\div b$.
5. $(6\,a^2+3\,a)\div 3\,a$.
6. $(12\,ab+4\,b)\div 4\,b$.

7. $5a^2 - 4ab + 4a$ by a.

8. $a^6 - 5a^5 + 3a^4$ by a^2.

9. $(x^2y + xy^2) \div xy$.

10. $(3x^2y - 6xy) \div 3xy$.

11. $25a^2 + 10ab + 5b^2$ by 5.

12. $27a^2b - 9ab^2 + 9a^2b^2$ by $9ab$.

13. $(12mn + 27mn^2p) \div 3mn$.

14. $(6x^2y - 4x^2z + 6xyz) \div 2x$.

15. $4x^4y^4 - 8x^3y^3 + 6xy^3$ by $-2xy$.

16. $-3a^2 + \frac{9}{2}ab - 3ac$ by $-\frac{3}{2}a$.

17. $x^3y - 3x^2y + 9xy^2$ by $3xy$.

18. $(x^3y - 3x^2y^3 + 9xy^2) \div 3xy$.

19. $(a^2c^2 - 2abc^2 + 3ac^3) \div ac^2$.

20. $a^2c^2 - 2abc^2 + 3ac^3$ by $-ac^2$.

21. $(5a^3b^3 - 35a^2b^2c^2 + 2ab^2c^4) \div 5ab$.

22. $(2m^2n^2 - 3mn^3 + 4m^2n - n^4) \div 3n$.

23. $(a^3x^3y - 3a^2bx^2y + 3ab^2xy^2 - a^2b^3xy^3) \div axy$.

24. $x^{7n} + 4x^{9n}y^2 + 3x^{8n}y^{2n}$ by x^{3n}.

25. $a^{7c+5} + a^{4c+8} + a^{11c+12}$ by a^4.

26. $10^{4n+12} + 10^{8n+9} + 10^{6n+6}$ by 10^{4n+3}.

133. To Divide by a Polynomial. This process is seen best from examples.

1. Divide $x^2 + 3xy + 2y^2$ by $y + x$.

$$
\begin{array}{rl}
 & \text{QUOTIENT} \\
 & x + 2y \\
\text{DIVISOR } x + y) & x^2 + 3xy + 2y^2 \\
 & x^2 + xy \\
 & \quad 2xy + 2y^2 \\
 & \quad 2xy + 2y^2
\end{array}
$$

Arrange the terms of the divisor and the dividend according to the powers of the same letter (x in this example).

Divide the first term of the dividend by the first term of the divisor.

The result (in this example, x) is the first term of the quotient.

Multiply the entire divisor by this term and subtract.

Divide the first term of the remainder by the first term of the divisor. The result (in this example, $2y$) is the second term of the quotient.

Multiply the entire divisor by this term and subtract.

Continue the process until a remainder zero is reached.

Cases in which such a remainder cannot be reached are treated later.

Test. Multiply the quotient by the divisor. If the work has been correctly done (including the multiplication), the result will equal the dividend. Or, substitute arbitrary values and proceed as in previous cases.

2. Divide $6a^3 - 17a^2b + 16b^3$ by $3a - 4b$.

SOLUTION

$$\begin{array}{r|l} & 2a^2 - 3ab - 4b^2 \\ 3a - 4b) & 6a^3 - 17a^2b + 16b^3 \\ & 6a^3 - 8a^2b \\ & -9a^2b \\ & -9a^2b + 12ab^2 \\ & -12ab^2 + 16b^3 \\ & -12ab^2 + 16b^3 \end{array}$$

TEST (Substitute $a = b = 1$)

Divisor × Quotient = Dividend

$$(3-4)(2-3-4) = 6-17+16$$
$$-1 \cdot (-5) = 5$$
$$5 = 5$$

WRITTEN EXERCISES

Divide and test:

1. $a^2 + 2ab + b^2$ by $a + b$.
2. $a^2 + 3a + 2$ by $a + 1$.
3. $x^2 - 11x + 30$ by $x - 5$.
4. $x^2 + 4xy + 4y^2$ by $x + 2y$.
5. $3c^2 + 7cd + 2d^2$ by $d + 3c$.
6. $6a^2 - 7a - 3$ by $2a - 3$.
7. $2x^2 - xy - 3y^2$ by $x + y$.
8. $3a^2 + ab - 2b^2$ by $3a - 2b$.
9. $6m^2 + mn - 2n^2$ by $3m + 2n$.
10. $12y^2 + 19y - 21$ by $3y + 7$.
11. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
12. $96a^2 - 4ab - 15b^2$ by $12a - 5b$.
13. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$.
14. $a^4 + a^2b^2 + b^4$ by $a^2 - ab + b^2$.
15. $x^3 - 2x^2 - 2x + 1$ by $x + 1$.
16. $3p^2 - 115 - 8p$ by $p + 5$.
17. $3a^4 - 2a - 1865$ by $a - 5$.
18. $14y^4 - 13y - 43y^2 + 32y^3 + 3$ by $7y^2 - 5y - 3$.
19. $21a^3 - 4a^2 - 16 - 46a$ by $4a + 2 - 3a^2$.
20. $\dfrac{1 + a^6}{1 - a^2 + a^4}$.
21. $\dfrac{a^8 - b^8}{a^2 + b^2}$.

22. $\dfrac{x^{2p}+8x^p+16}{x^p+4}$.

23. $\dfrac{9m^2+12mn+4n^2}{3m+2n}$.

24. $\dfrac{a^4+4b^4}{a^2+2ab+2b^2}$.

25. $\dfrac{6a^2b^2-ab^3-12b^4}{3ab+4b^2}$.

26. $\dfrac{4x^4y^4+1}{2x^2y^2-2xy+1}$.

27. $\dfrac{a^3+b^3}{a^2-ab+b^2}$.

28. $\dfrac{a^3-b^3}{a^2+ab+b^2}$.

29. $\dfrac{x^6-4x^4y^2+4x^2y^4-y^6}{x^2-y^2}$.

30. $\dfrac{a^5-a^4-11a^3+16a^2-2a-3}{a^2-4a+3}$.

31. $\dfrac{6x^4+17x^3-19x-4}{2x^3+3x^2-4x-1}$.

32. $\dfrac{x^{2n}+3x^ny^{2n}-x^ny^n-3y^{3n}}{x^n-y^n}$.

134. Remainders. In algebra, as in arithmetic, the division may not be exact. That is, no remainder may be zero, however far the division is carried.

Thus, in the example at the right, there is a remainder, 2. The division might be continued, the next term of the quotient being $\frac{2}{x}$; but it is customary to stop as soon as a remainder is reached that is of lower degree than the divisor. The integral part of the quotient has now been found; it is called the integral quotient.

$$\begin{array}{r|l} & x^3-x^2+x-1 \\ x+1) & x^4+1 \\ & x^4+x^3 \\ & \quad -x^3+1 \\ & \quad -x^3-x^2 \\ & \qquad x^2+1 \\ & \qquad x^2+x \\ & \qquad\quad -x+1 \\ & \qquad\quad -x-1 \\ & \qquad\qquad 2 \end{array}$$

By using the fractional form to indicate the division of the remainder, the result of the above division may be expressed thus:

$$\frac{x^4+1}{x+1}=x^3-x^2+x-1+\frac{2}{x+1}.$$

The right member of the equation is called the complete quotient.

Test. Dividend = divisor × integral quotient + the remainder.

Substituting $x=1$.

$$\text{Dividend} = \text{Divisor} \times \text{Integral Quotient} + \text{Remainder}.$$
$$1+1=(1+1)(1-1+1-1)+2.$$
$$2=2\cdot 0+2.$$

WRITTEN EXERCISES

Find the quotient and remainder:

1. $(x^3-1)\div(x+1)$.
2. $(a^2+x^2)\div(a+x)$.
3. $\dfrac{x^5+x+1}{x^2-1}$.
4. $\dfrac{x^4-2x^2+1}{x^2+1}$.
5. $(3x^2-5x+2)\div(x-4)$.
6. $(8y^2+7y-1)\div(2y+3)$.
7. $\dfrac{6ax-9ay-4bx+8by}{3a-2b}$.
8. $\dfrac{2a^3-2a^2-6a+4}{2a-3}$.

REVIEW

ORAL EXERCISES

Divide and test:

1. $\dfrac{-32ax^4}{4x^3}$.
2. $\dfrac{-42m^6y^3}{6m^2y}$.
3. $\dfrac{144a^5x^4}{-16ax^3}$.
4. $\dfrac{-14641p^2q}{-11p}$.
5. $\dfrac{t^2-14t+49}{t-7}$.
6. $\dfrac{1-16y+64y^2}{1-8y}$.
7. $\dfrac{ax^2+bx^2}{x^2}$.
8. $\dfrac{x^2-y^2}{x-y}$.
9. $\dfrac{x^2-y^2}{x+y}$.
10. $\dfrac{a^2-x^2}{a+x}$.
11. $\dfrac{m^2p^2-n^2q^2}{mp+nq}$.
12. $\dfrac{x^4-y^4}{x^2-y^2}$.
13. $\dfrac{m^4-n^4}{m^2+n^2}$.
14. $\dfrac{x^2y^2-m^2n^2}{xy-mn}$.
15. $\dfrac{x^{2m}-y^{2r}}{x^m-y^r}$.
16. $(14ax^2+6ax)\div 2a$.
17. $(25t^3-40t)\div -5$.
18. $(18x^2z^3-24x^3z^2)\div -6xz^2$.
19. $(a^2bc-ab^2c+2abc^2)\div abc$.
20. $(-27m^{2x}y^{2a}-9m^{5x}y^{3a})\div 9m^xy^a$.
21. $(-25p^{2a}q^{3r}+10p^{3a}q^{2r})\div 5p^aq^r$.

WRITTEN EXERCISES

Find the quotient and remainder, if any:

1. p^4-16 by $p+2$.
2. a^4-10a^2+25 by a^2-5.
3. $6y^2+2y-20$ by $2y+4$.

4. $2a^2 + 5ab + 2b^2$ by $a + 2b$.

5. $6a^2b^2 - ab^3 - 12b^4$ by $2ab - 3b^2$.

6. $x^2 + ax + bx + ab$ by $x + a$.

7. $m^4 - m^2p^2 + m^2p^3 - p^5$ by $m^2 - p^2$.

8. $x^3 + 6x^2 + 8x - 3$ by $x^2 + 3x - 1$.

9. $2ac - bc - 6a^2 + 3ab$ by $c - 3a$.

10. $1 - x^2 + 2x^3 - x^4$ by $x^2 - x + 1$.

11. $\dfrac{x^5 - 1}{x^2 - 1}$.

12. $\dfrac{x^3 - 7x^2 + 3x - 1}{x - 3}$.

13. $\dfrac{a^4 - 5a^2 + 4}{a + 2}$.

14. $\dfrac{x^3 - 2x^2 + x - 4}{x - 3}$.

15. $\dfrac{x^4 - 2x^2 + 3x}{x^2 - 1}$.

16. $\dfrac{8 - 12a + 6a^2 - a^3}{2 - a}$.

17. $\dfrac{15a^2b + 6ab^2 + 8a^3 + 3b^3}{8a - b}$.

18. $\dfrac{-7r^2 + 3r^3 + 5 + r^4}{r^2 - 5r + 2}$.

19. $\dfrac{18x^4 - 24x^3 + 38x^2 - 68x + 32}{3x - 2}$.

Divide:

20. $z^4 - 1$ by $z - 1$.

21. $x^6 - x^3y^3$ by $x - y$.

22. $a^4 - 2a^2 + 1$ by $a - 1$.

23. $a^2 + 2ab + b^2 - c^2$ by $a + b + c$.

24. $m + \dfrac{1}{p}$ by $\dfrac{1}{m} + p$ to four terms.

25. $x^4 + x^2y^2 + y^4$ by $x^2 + y^2 - xy$.

26. $\dfrac{1}{a^2} + 1 + a^2$ by $\dfrac{1}{a^2} - \dfrac{1}{a} + 1$.

27. $x^2 - y^2 - 2y - 1$ by $x + y + 1$.

28. $6x^2 - 2xy - 3x + y$ by $3x - y$.

29. $x^{3m} + y^{3n}$ by $x^m + y^n$.

30. $6r^6 - 18r^5 + 18r^4 - 3r^3 - 9r^2 + 9r - 3$ by $2r^3 + 1$.

31. $a^{4x} + 4a^{3x}b + 6a^{2x}b^2 + 4a^xb^3 + b^4$ by $a^{2x} + 2a^xb + b^2$.

SUMMARY

These questions summarize the definitions and processes in this chapter:

1. State in the form of an equation the relation between the dividend, divisor, and quotient. Sec. 128.

2. State the *Law of Exponents* in division. Sec. 129.

3. State how to find the quotient of two monomials. Sec. 130.

4. Explain how to divide a polynomial by a monomial. Sec. 132.

5. State how to divide by a polynomial. Sec. 133.

6. State how to test the work of division. Secs. 126, 133.

7. When is the remainder reached in the process of division? Sec. 134.

HISTORICAL NOTE

After solving many exercises in the four processes with polynomials, algebra may seem to consist mostly of such manipulations; but we shall see as we proceed that the solution of equations becomes more and more prominent and that the facility we have gained in the use of symbols is a help in handling equations. In fact, the word "algebra" itself is connected in its origin with the equation. There lived in Bagdad, Arabia, about 825 A.D., a famous mathematician known from his birthplace, Kharezm, as Al-khowarazmi, and in the title of his work signifying "The science of transposing and combining in solving equations," there appeared the Arabic word *Al-Jabr* to denote these processes. When this manuscript was translated into Latin in the thirteenth century, the word *Al-Jabr* was transferred as *algebræ*, from which we have our word "algebra." Thus, *algebra*, instead of meaning originally "science of symbols," as we might imagine, means "the science of processes in equations."

Although Diophantos (300 A.D.) could find the product of two binomials, the knowledge of how to multiply or divide one polynomial by another is comparatively recent. Ordinary algebraic division was developed in the seventeenth century, and Newton, in his *Arithmetica Universalis* (1707), showed how arranging the terms in both dividend and divisor according to the descending powers of the same letter facilitated the work.

CHAPTER X

EQUATIONS

135. The parenthesis is often used in equations to indicate multiplication.

EXAMPLES

1. Solve: $(x+1)(x-5)=x(x-1)$. (1)

Removing the parentheses, $x^2-4x-5=x^2-x$. (2)

Then, $-3x=5$,

and $x=-\frac{5}{3}$. (3)

TEST. $(-\frac{5}{3}+1)(-\frac{5}{3}-5)=\frac{40}{9}=-\frac{5}{3}(-\frac{5}{3}-1)$.

2. Solve: $(ax-1)x+2-b=ax^2-c$. (1)

Removing the parenthesis, $ax^2-x+2-b=ax^2-c$. (2)

Uniting terms, $-x+2-b=-c$. (3)

Then, $-x=-c+b-2$,

and $x=c-b+2$. (4)

Test by substitution.

WRITTEN EXERCISES

Solve:

1. $x(x-3)+1-x(x-5)=0$.
2. $x^2+3-x(x+4)=15$.
3. $x^2(x-1)-x^3+x^2-2x=12$.
4. $5(x-4)=6(x+1)$.
5. $-3(x+7)=2(1-3x)$.
6. $a(x-b)=3ab$.
7. $(x-4)(x+4)=x^2-8x$.
8. $(x-1)^2=(x-3)^2$.
9. $(x-5)(x+8)=(x-7)^2$.
10. $y(9y-5)=(3y+1)^2$.
11. $t^2-1=(t+4)^2$.
12. $15p=-29-(4-4p)$.
13. $1-(16+7w)=8w$.
14. $2x-(7x-18)=4x$.
15. $(x+5)(2x-1)=x(2x+4)$.
16. $(2x+1)(2x-1)=x(4x-2)$.
17. $3x(6x+5)=18x^2-(x+32)$.

18. $4y(y-1)=(2y-1)(2y+1)$.

19. $4x+8=2-(9x+20)$.

20. $4s=7-(s-3)$.

21. $(x+2)(x-5)=x(x-1)$.

22. $(cx+2)x=cx^2-a$.

23. $ax(x-5)=ax^2-1$.

24. $(a+b)x=cx+5$.

25. $(x-1)(ax+b)=ax^2+b$.

26. $(m+1)x-px=q$.

27. $p(x-5)+4p=1$.

28. $(x-1)(x+2)=(x-p)(x+q)$.

29. $(ax+b)(c+d)=f$.

30. $(ax-1)(bx-1)=(ax+b)(bx-c)$.

136. Problems often result in equations in which parentheses may be used.

EXAMPLES

1. Express by an equation: 2% of $(100-x)$ dollars equals $1.40.

2. Express by an equation: $\frac{4}{5}$ of the quantity $75-2x$ equals 44.

3. The difference between twice a number and the number less 10 is 22. Express this fact by an equation.

4. A man has $500 of which x dollars is in the bank drawing interest at 4%, and the remainder is lent at 6%. Express the annual interest received on the $500.

5. The amount of $100 for one year at 5% simple interest is $(1+.05)$ 100 dollars. Express the amount of x dollars at r% for 1 yr.; for 6 yr.

WRITTEN EXERCISES

Solve and test:

1. A man had $500, of which he invested x dollars at 4% per annum, and lent the remainder at 6% per annum. His annual interest was $28. Find how many dollars were invested at 4% and how many were lent at 6%.

2. Find the principal that will amount to $32.70 at the rate of $4\frac{1}{2}$% per annum for two years.

3. The amount of a certain principal at 4 % simple interest for $2\frac{1}{2}$ years is $220. What is the principal?

4. The amount of a certain principal at $5\frac{1}{2}$ % simple interest for $2\frac{1}{2}$ years is $284.375. What is the principal?

5. $100 is divided into two parts, of which one is x dollars. What is the other? If the first part exceeds the second by $10, find each part.

6. $600 is divided into two parts, one of which is $2x$. Write an expression for the other. The first part equals $\frac{1}{2}$ the second. Find each part.

7. The total amount of insurance in force in New York City and Buffalo in a recent year was $2,700,000,000; Buffalo had $\frac{2}{25}$ as much as New York. How much had each?

Solve Exercises 8–12 by equations requiring the use of the parenthesis: Then solve each by an equation not requiring a parenthesis:

8. Japan recently gave American manufacturers an order for 2000 cars and locomotives; it consisted of 19 times as many cars as locomotives. How many of each were ordered?

9. The amount of condensed milk produced by New York and Illinois is $\frac{3}{4}$ of that produced by the rest of the country; the total amount produced in the country annually is about 154 million pounds. How many pounds are produced by the two states together?

10. The United States produces 3 times as much cotton as the rest of the world; the total cotton production in a recent year was 14 million bales. What was the number of bales produced by the United States?

11. Mississippi and Texas together produced 4 million bales; Texas produced $1\frac{2}{3}$ times as much as Mississippi. How many bales did each produce?

12. The United States consumes $\frac{2}{5}$ as much cotton as does the rest of the world. How many bales is this, when the whole world consumes 14 million bales annually?

137. The use of the parenthesis often makes it possible to reduce the number of unknowns.

EXAMPLE

The sum of two numbers is 20, and 3 times one of them less 4 times the other is 4. Find the numbers.

This problem suggests at once two unknowns, but it can readily be solved by the use of one unknown, and practice in this work is good training in mathematics.

1. Let x be one of the required numbers.
2. Then, $20 - x$ is the other.
3. $3x - 4(20 - x) = 4$, by the conditions of the problem.
4. Therefore, $7x - 80 = 4$, and $x = 12$.
5. Then, $20 - x = 20 - 12$, or 8, and the numbers are 12 and 8.

The use of the parenthesis in step (3) takes the place of a second unknown.

WRITTEN EXERCISES

Solve and test:

1. The sum of two numbers is 25, and twice one of them plus 3 times the other is 60. What are the numbers?

2. The sum of two numbers is 30. $\frac{1}{3}$ of one of them less $\frac{1}{4}$ of the other is 3. What are the numbers?

3. The sum of two numbers is 38. One of them less $\frac{2}{3}$ of the other is 13. What are the numbers?

4. The sum of 2 numbers is 20. 2 times the larger number less 3 times the smaller is 5. Find the numbers.

5. The sum of two numbers is 42. When the larger number is diminished by 5, $\frac{1}{2}$ the result is the smaller number less 7. Find the numbers.

6. The product of two consecutive whole numbers diminished by the square of the smaller is 29. Find the numbers.

7. The product of two consecutive whole numbers less the square of the smaller is 49. Find the numbers.

8. The difference between two numbers is 2, and their product diminished by the square of the larger is -16. Find the numbers.

138. Certain problems of measurement are best solved by equations requiring the use of the parenthesis.

EXAMPLES

1. The inside measurements of a rectangular garden are 20 ft. by 30 ft. The outer margin is $1\frac{1}{2}$ ft. wide. What are the outside measurements of the garden? Express its area. If the margin were x ft. wide, express the area of the garden.

2. Express the outside measurements of a rectangular garden consisting of a walk $2a$ ft. wide around an inner plot 50 ft. by 70 ft. Express its area.

3. Express the difference between the area of a square x ft. on a side and the area of a square $(x+1)$ ft. on a side.

4. Express the difference between the area of a square $(x+2)$ ft. on a side, and the area of a square formed by diminishing this length by 5 ft.

WRITTEN EXERCISES

Solve and test:

1. The difference between the area of a square x ft. on a side and the area of a square $(x+1)$ ft. on a side is 13 sq. ft. Find the side of the first square.

2. The difference between the area of a square $(x+2)$ ft. on a side, and the area of a square formed by diminishing this length by 5 ft. is 75 sq. ft. Find the side of each square.

3. The inside measurements of a picture frame are 10 in. by 14 in. The width of the frame is x in. What are its outside measurements? If the area of the frame less $4x^2$ is 120 sq. in., what is its width?

4. The area of a square of side x equals the area of a rectangle, one of whose sides is $x-6$, and the other $x+12$. Find x.

5. The area of a square equals the area of a rectangle, one of whose sides exceeds the side of the square by 10 in., and the other is less than the side of the square by 6 in. Find the dimensions of each figure.

Problems of Motion:

1. If an electric car moves at the rate of $1\frac{1}{2}$ blocks per minute, how far will it move in 4 min.? If it moves at the rate of x miles per hour, how far will it move in $(m+1)$ hr.?

2. Let d equal the distance traveled in t hr. at the rate of r mi. per hour. Express d in terms of r and t.

3. Solve the equation $d = rt$ for t; solve it for r.

4. In $d = rt$, find d when $t = a + 1$ and $r = b - 1$. Find r when $d = x^2 - y^2$ and $t = x - y$.

5. $d = rt$ is the equation for distance when a body so moves that its rate may be taken as uniform. If sound travels 1100 ft. per second, how far away is a gun when the report of firing is heard $3\frac{1}{2}$ sec. after it occurred?

6. A fort is 10 mi. away. According to Exercise 5, how long after firing will it be before the report is heard?

7. If A travels x mi. an hour and B follows A at the rate of y mi. an hour, going faster than A, express the number of miles that B gains on A per hour. Express how long it will take B to gain d mi.

8. If A travels x mi. an hour and has c hr. the start of B, how far ahead is he when B starts? According to Exercise 7, in how many hours will B overtake A?

9. An automobile leaves city A at 7 A.M., going 20 mi. an hour. A motor cycle follows 2 hr. later, going 25 mi. an hour. How far ahead is the automobile at 9 A.M.? How far from A will the motor cycle overtake the automobile?

10. A steamer leaves its dock and travels 16 mi. per hour. It is followed 30 min. later by a motor boat traveling 20 mi. per hour. How far will they be from the dock when the motor boat overtakes the steamer?

11. Two cyclists A and B start at the same time from M and P respectively, 100 mi. apart, and travel toward each other, A at the rate of 15 mi. per hour and B at the rate of 20 mi. per hour. How many hours after starting do they meet, and how far from M?

REVIEW

WRITTEN EXERCISES

1. If C has x dollars and B has half as much, express what they both have. If this amount is \$75, write and solve an equation that will find what each has.

2. Express the area of a rectangle whose length is n ft. and whose width is 4 ft. If the area is 48 sq. ft., find n.

3. The width of a rectangle is x ft. and its length is twice the width. Express its perimeter. If the perimeter is 24 ft., find the length and width of the rectangle.

4. Express the bank discount on m dollars for 4 mo. at 6%. If the discount is \$3, find m.

5. Express the interest on s dollars at 6% for 3 yr. If this interest is \$36, find s.

6. Express the area of a triangle whose base is 6 in. and whose altitude is a in. If its area is 24 sq. in., find a.

7. A merchant marked an article d dollars and sold it at a 10% discount. Express the selling price. If the article brought \$7.20, find d.

8. In how many years will \$300 yield \$108 at 6% interest?

9. The perimeter of a rectangle is 30 ft. If the length of the base is twice the altitude, find the area of the rectangle.

10. A certain number plus twice the same number is 51. Find the number.

11. What number added to 3 times itself equals 64?

12. Divide the number 21 into three such parts that the first is twice the second and the second is twice the third.

13. If a certain number is multiplied by 12, the product is 168. Find the number.

14. A man sold a quantity of wood for \$49, half of it at \$3 a cord and the other half at \$4 a cord. How many cords of wood did he sell?

15. The sum of the ages of a father and son is 42 yr. and the father is five times as old as the son. What is the age of each?

16. Two men start from the same place and travel in opposite directions, one 35 mi. a day and the other 25 mi. a day. In how many days will they be 360 mi. apart?

17. Two men start from the same place and travel in the same direction, one 35 mi. a day and the other 25 mi. a day. In how many days will they be 360 mi. apart?

18. A, whose horse travels at the rate of 10 mi. an hour, starts 2 hr. after B from the same place. If B's horse travels at the rate of 8 mi. an hour, how many miles must A drive in the same direction to overtake B?

SUGGESTION. Let x equal the number of hours traveled by A before he overtakes B.

19. A flag pole 105 ft. high was broken so that the length of the part broken off was six times the length of the part left standing. Find the length of each part.

CHAPTER XI

TYPE PRODUCTS

139. Certain products are specially important because they serve as types or models for other multiplications. They apply to positive and negative numbers alike.

140. Type I: $x(y + z) = xy + xz.$

Type II: $x(y - z) = xy - xz.$

For example:

$a(b + c) = ab + ac.$
$5\,x(3 - y) = 15\,x - 5\,xy.$
$2\,a^2(a - 5\,b) = 2\,a^3 - 10\,a^2b.$
$-3\,ab(c^2 - 4\,ad + b) = -3\,abc^2 + 12\,a^2bd - 3\,ab^2.$

WRITTEN EXERCISES

Multiply:

1. $-x(x + y).$
2. $c(a - b).$
3. $a(t + t^2).$
4. $cx(w + z).$
5. $-y(x - y).$
6. $t(u + \frac{1}{2}\,at).$
7. $4\,ab(a + 2\,b).$
8. $5\,xy(x^2 - y^2).$
9. $pq(m - n).$
10. $-2\,x(3\,x^2 - 2\,xy).$
11. $-3\,a^xb^yx(a^2 - b^2).$
12. $\frac{2}{3}\,xy(\frac{1}{2}\,x^2y^2 - 1).$
13. $-4\,x^2(3\,x - 2\,y).$
14. $2\,m^2(m - n^2).$
15. $3\,y(4\,x - y).$
16. $(5\,a^2 - 4\,b^2)(-a^2b^2).$
17. $(5\,x - acy)(-acxy).$
18. $(3\,a^2x - 8\,ax^3)(-3\,a^3x).$
19. $(6\,am^p + 2\,bn^q)(-6\,m^2n)$
20. $(9\,a^3b^2 - 3\,cd^2)(-abcd).$
21. $x(y + z + w).$
22. $-3\,ab(a^2 - b^2 + c^2).$
23. $\pi(r_1^2 + r_2^2 + r_1r_2).$

141. Type III: $(x+y)^2 = x^2 + 2\,xy + y^2.$

In words:

The square of the sum of two numbers is the square of the first, plus twice the product of the first and second, plus the square of the second.

142. Type IV: $(x-y)^2 = x^2 - 2\,xy + y^2.$

In words:

The square of the difference of two numbers is the square of the first, minus twice the product of the first and the second, plus the square of the second.

For example:

$$(a+b^2)^2 = a^2 + 2\,ab^2 + b^4.$$

$$\begin{aligned}(2\,a - b)^2 &= (2\,a)^2 - 2(2\,a)b + b^2 \\ &= 4\,a^2 - 4\,ab + b^2.\end{aligned}$$

$$\begin{aligned}23^2 &= (20+3)^2 = 20^2 + 2 \cdot 20 \cdot 3 + 3^2 \\ &= 400 + 120 + 9 = 529.\end{aligned}$$

Evidently the above types include expressions either of the form $(ax+b)^2$ or $(ax-b)^2$.

WRITTEN EXERCISES

Square as indicated:

1. $(n+u)^2.$
2. $(x^2+y^2)^2.$
3. $(a+3\,b)^2.$
4. $(m+2\,n)^2.$
5. $(3\,x+2\,y)^2.$
6. $(a+2)^2.$
7. $(a^2+1)^2.$
8. $(a-b)^2.$
9. $(x-1)^2.$
10. $(2\,x-1)^2.$
11. $(x-\frac{1}{2})^2.$
12. $(t+w)^2.$
13. $(x+1)^2.$
14. $(2\,x+1)^2.$
15. $(2\,x^2+1)^2.$
16. $(2\,x^2+3\,y^2)^2.$
17. 25^2 or $(20+5)^2.$
18. $41^2.$
19. $82^2.$
20. 76^2 or $(80-4)^2.$
21. $(a^2-b^2)^2.$
22. $(3\,a-2\,b)^2.$
23. $(t-u)^2.$
24. $(abc-1)^2.$
25. $(mn+u^2)^2.$
26. $(2\,ab+bc)^2.$
27. $(abc+1)^2.$
28. $33^2.$
29. $52^2.$
30. $91^2.$
31. $17^2.$
32. $97^2.$
33. $46^2.$
34. $89^2.$
35. $11^2.$
36. $36^2.$

143. Trinomials may also be squared by Types III and IV.

For example:

$$(a+b+c)^2 = (\overline{a+b}+c)^2 = (a+b)^2 + 2(a+b)c + c^2$$
$$= a^2 + 2\,ab + b^2 + 2\,ca + 2\,bc + c^2$$
$$= a^2 + b^2 + c^2 + 2\,ab + 2\,bc + 2\,ca.$$
$$(2\,x - y + z)^2 = (2\,x - y)^2 + 2(2\,x - y)z + z^2$$
$$= 4\,x^2 - 4\,xy + y^2 + 4\,xz - 2\,yz + z^2.$$

In words:

The square of a polynomial is the sum of the squares of each of its terms and twice the product of every two.

WRITTEN EXERCISES

Square the trinomials as indicated:

1. $(\overline{a+b}-c)^2$.
2. $(2\,a-\overline{b+c})^2$.
3. $(a-\overline{3\,b-c})^2$.
4. $(x-y+z)^2$.
5. $(x+w-2\,z^3)^2$.
6. $(\frac{1}{2}\,a-\frac{1}{3}\,b+\frac{1}{5}\,c)^2$.
7. $(mn+pq+rs)^2$.
8. $(1-6\,y+y^2)^2$.
9. $(m^2+mp-q^2)^2$.

144. Type V: $\qquad (x+y)(x-y) = x^2 - y^2.$

In words:

The product of the sum and the difference of two numbers is the difference of their squares.

For example:

$$(a+b)(a-b) = a^2 - b^2.$$
$$(2\,a+b)(2\,a-b) = (2\,a)^2 - b^2 = 4\,a^2 - b^2.$$
$$(m^2+n^2)(m^2-n^2) = (m^2)^2 - (n^2)^2 = m^4 - n^4.$$

Evidently the above type includes expressions of the form $(ax+b)(ax-b)$.

ORAL EXERCISES

Multiply:

1. $(m-n)(m+n)$.
2. $(a-x)(a+x)$.
3. $(p-q)(p+q)$.
4. $(t+u)t(-u)$.
5. $(x-1)(x+1)$.
6. $(x-2)(x+2)$.
7. $(x+b)(x-b)$.
8. $(2\,x-1)(2\,x+1)$.
9. $(2\,x-y)(2\,x+y)$.
10. $(1+x^3)(1-x^3)$.
11. $(ax+by)(ax-by)$.
12. $(2\,x-3\,y)(2\,x+3\,y)$.
13. $(2\,a^2+3)(2\,a^2-3)$.
14. $(a^2+b)(a^2-b)$.
15. $(a^2-3\,ax)(a^2+3\,ax)$.

16. $(ax - x^2)(ax + x^2)$. 17. $(\frac{1}{2}x - \frac{1}{3}y)(\frac{1}{2}x + \frac{1}{3}y)$.

18. $(a - x)(a + x)(a^2 + x^2)$.

19. $(a - x)(a + x)(a^2 + x^2)(a^4 + x^4)$.

20. $(1 - r)(1 + r)(1 + r^2)(1 + r^4)(1 + r^8)$.

21. $(1 - r)(1 + r)(1 + r^2)(1 + r^4)(1 + r^8)(1 + r^{16})$.

145. Two numbers, one greater than a multiple of 10, and the other less than this multiple by the same amount, may be multiplied according to Type V.

Thus, $93 \cdot 87 = (90 + 3)(90 - 3) = 90^2 - 3^2 = 8100 - 9 = 8091$.

WRITTEN EXERCISES

1. $31 \cdot 29 = (30 + 1)(30 - 1) = ?$
2. $42 \cdot 38 = (40 + 2)(40 - 2) = ?$
3. $35 \cdot 45 = ?$
4. $57 \cdot 63 = ?$
5. $21 \cdot 19 = ?$
6. $32 \cdot 28 = ?$
7. $29 \cdot 31 = ?$
8. $66 \cdot 54 = ?$
9. $44 \cdot 36 = ?$
10. $91 \cdot 89 = ?$
11. $53 \cdot 47 = ?$
12. $16 \cdot 24 = ?$
13. $99 \cdot 101 = ?$
14. $98 \cdot 102 = ?$
15. $90 \cdot 110 = ?$
16. $127 \cdot 113 = ?$
17. What is the cost of 21 doz. eggs at 19 ¢ a dozen?
18. What is the cost of 28 lb. of butter at 32 ¢ a pound?
19. How many oranges in 146 crates of 154 oranges each?
20. What is the area of a rectangle whose dimensions are 62 ft. and 58 ft.?
21. How far does a train travel in 37 hr. at the rate of 43 mi. per hour?

146. Type VI: $(x + a)(x + b) = x^2 + (a + b)x + ab$.

For example:

$(x + 5)(x + 3) = x^2 + 8x + 15.$

$(x - 3)(x + 7) = x^2 + 4x - 21.$

$(3x + c)(3x - d) = (3x)^2 + (c - d)3x - cd = 9x^2 + 3(c - d)x - cd.$

$$91 \cdot 87 = (100 - 9)(100 - 13) = 100^2 - 22 \cdot 100 + 9 \cdot 13$$
$$= 10000 - 2200 + 117 = 7917.$$

Evidently this type includes expressions of the form

$$(ax + b)(ax + c).$$

ORAL EXERCISES

State the products:

1. $(x+2)(x+5)$.
2. $(a+3)(a+6)$.
3. $(b-5)(b+2)$.
4. $(4x+7)(4x-5)$.
5. $(5+m)(5+r)$.
6. $(4g-5)(4g-9)$.
7. $(p+q)(p+2q)$.
8. $(7-x)(7-y)$.
9. $(ab+c)(ab+d)$.
10. $(2a-7b)(2a+8b)$.

WRITTEN EXERCISES

Find the products:

1. $93 \cdot 95$.
2. $197 \cdot 191$.
3. $(x+14)(x-19)$.
4. $(2z-3a)(2z+5a)$.
5. $993 \cdot 985$.
6. $(-3x+11)(-3x-6)$.
7. $(15x+23)(15x-25)$.
8. $(4w+a)(4w-2a)$.

147. Type VII. Expressions of the form $(ax+b)(cx+d)$ have the product $acx^2+(bc+ad)x+bd$.

EXAMPLES

$$(3a+5)(4a+7) = 12a^2+(20+21)a+35$$
$$= 12a^2+41a+35.$$

$$(ax-b)(cx-d) = acx^2-(bc+ad)x+bd.$$

The coefficient of the middle term is the sum of the products indicated by the curved lines above the given expression.

Actual multiplication is quite as simple, but practice in forming products as above indicated is a good preparation for factoring expressions of this type.

ORAL EXERCISES

Find the products:

1. $(2a+5)(3a-1)$.
2. $(a-5)(4a+2)$.
3. $(3a+1)(a-3)$.
4. $(x+4)(3x-5)$.
5. $(2x-a)(x-b)$.
6. $(3xy-1)(5xy+2)$.
7. $(x^n-2)(3x^n-4)$.
8. $(2x^n-a)(x^n+b)$.

WRITTEN EXERCISES

Multiply:

1. $(ay+3)(by+5)$.
2. $(2\,ay+4)(3\,ay-5)$.
3. $(3\,ax+b)(ax-c)$.
4. $(15\,b+2\,c)(5\,b-10\,c)$.
5. $(40\,b^2c-1)(5\,b^2c+d)$.
6. $(2\,x^p-ab)(5\,x^p+bc)$.
7. $(mx^n-y^p)(px^n+2\,y^p)$.
8. $(aby^r-c)(ab^2y^r-d)$.
9. $(cdx^p-y^2)(acx^p-dy^2)$.
10. $(xy^n-mn)(zy^n+pq)$.

148. Type VIII. $(x+y)^3=x^3+3\,x^2y+3\,xy^2+y^3$.

Since by actual multiplication

$$(x+y)^3=x^3+3\,x^2y+3\,xy^2+y^3,$$

therefore, $(x+a)^3=(\ \ )^3+3(\ \ )^2a+3(\ \ )a^2+(\ \ )^3$.

Also, $(a+b^2)^3=(\ \ )^3+3(\ \ )b^2+3(\ \ )(b^2)^2+(b^2)^3$
$=(\ \ )+3(\ \ )+3(\ \ )+(\ \ )$.

Since by actual multiplication

$$(x-y)^3=x^3-3\,x^2y+3\,xy^2-y^3,$$

therefore, $(m-n)^3=(\ \ )^3-3(\ \ )+3(\ \ )-(\ \ )^3$.

Similarly, $(ab-c)^3=(ab)^3-3(ab)^2c+3\,abc^2-c^3$
$=(\ \ )-3(\ \ )+3\,abc^2-c^3$.

Similarly, $(\overline{a-b}+c)^3=(a-b)^3+3(a-b)^2c+3(a-b)c^2+c^3$
$=(\ \ )+3(\ \ )+3(\ \ )+(\ \ )$.

WRITTEN EXERCISES

Expand by Type VIII:

1. $(m+n)^3$.
2. $(p-q)^3$.
3. $(a-x)^3$.
4. $(2\,a+x)^3$.
5. $(a^2+b^2)^3$.
6. $(m^2-n^2)^3$.
7. $(a+2\,b)^3$.
8. $(a-3\,c)^3$.
9. $(2\,a-3\,b)^3$.
10. $a(a+b)^3$.
11. $ax(x-y)^3$.
12. $(ab+cd)^3$.
13. $(x+1)^3$.
14. $(3\,x-1)^3$.
15. $(y^2-1)^3$.
16. $(a^m+b^m)^3$.
17. $(x^n-y^n)^3$.
18. $(\frac{1}{2}\,x-\frac{1}{3}\,y)^3$.
19. $(x^n-y^m)^3$.
20. $(a^2b+ab^2)^3$.
21. $(\overline{m+n}-p)^3$.
22. $(m+\overline{n-p})^3$.
23. $(a+b+c)^3$.
24. $(2\,a+b+c)^3$.

REVIEW

ORAL EXERCISES

State the products of:

1. $x(x+3)$.
2. $x(a+b)$.
3. $4r(r-2)$.
4. $3a(3b-4c)$.
5. $b(a-c)$.
6. $p(m-p)$.
7. $(3+t)(3-t)$.
8. $(a-y)(a+y)$.
9. $(7-x)(7+x)$.

State the square as indicated:

10. $(a+x)^2$.
11. $(1-x)^2$.
12. $(7-y)^2$.

WRITTEN EXERCISES

Write the products:

1. $(2a-1)^3$.
2. $(10+3)^3$.
3. $(5x-y)(5x+y)$.
4. $(ax+8)(ax-8)$.
5. $(ax^m-1)(ax^m-3)$.
6. $(20+1)(20-1)$.
7. $(3a+x)(3a-x)$.
8. $(3a-5)(4a-6)$.
9. $(6a-12)(7a+15)$.
10. $(2a-3b)(7a+5b)$.

Remove the parentheses and unite terms where possible:

11. $(3x-1)^2+2(4x+3)^2$.
12. $5(7y-4)-(4y+3)^2$.
13. $(79)^2+(92)^2$.
14. $(ab-c)^2+(ab+c)^2$.
15. $(x-3y)(x+3y)+(x-5y)^2$.
16. $3x(7y-4)-(2x+3y)^2$.
17. $133 \cdot 127$ or $(130+3)(130-3)$.
18. $4a(b-1)+2(3a-b)^2$.

Find the product:

19. $(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})$.

20. Show by multiplying that

$$s(s-a)(b+c)+a(s-b)(s-c)-2bcs$$

is identical with

$$s(s-b)(a+c)-b(a-s)(s-c)-2acs.$$

CHAPTER XII

FACTORING

149. The factoring of algebraic expressions depends largely upon a knowledge of products, and for this reason the special products most used in factoring were brought together and emphasized in the previous chapter.

I. ROOTS

150. PREPARATORY.

1. $(+2)(+2) = ?$ $\qquad$ $(-2)(-2) = ?$

2. State a number which taken twice as a factor produces 4. State another number which taken twice as a factor produces 4.

3. According to Exercise 2, how many square roots has 4? What are they?

4. Similarly, name the square roots of 9; 16; 25; 36.

151. Signs of Square Roots. Every number has two square roots which differ only in their signs.

Thus, $\sqrt{4} = +2$ or -2; because $(+2)(+2) = 4$, and $(-2)(-2) = 4$.

$\sqrt{a^2} = +a$ or $-a$; because $(+a)(+a) = a^2$, and $(-a)(-a) = a^2$.

It should be noticed that, although either square root taken twice as a factor produces the given number, the product of the two square roots is not equal to the given number.

152. The sign $\pm$ is used to denote that a number may be taken either positively or negatively.

Thus, $\sqrt{4} = +2$ or -2 is written $\sqrt{4} = \pm 2$.
Also, $\sqrt{a^2} = +a$ or $-a$ is written $\sqrt{a^2} = \pm a$.

The positive square root of a number is called the **principal square root.**

ORAL EXERCISES

State the two square roots of each number:

1. 25.
2. 49.
3. 121.
4. 196.
5. 225.
6. 81.
7. 625.
8. 169.
9. 144.
10. 36.
11. a^2b^2.
12. $36\,a^2b^2$.
13. $49\,a^2x^4$.
14. $25\,p^2q^6$.
15. $\frac{1}{4}g^2t^2$.
16. $\frac{9}{16}m^2n^8$.

153. The square roots of a monomial may often be found by factoring.

EXAMPLE

Find the square root of $576\,m^2n^4$.

By trial, $576 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 2^3 \cdot 3 \cdot 3$,

and $m^2n^4 = mn^2 \cdot mn^2$.

$$\therefore \sqrt{576\,m^2n^4} = \pm\, 2^3 \cdot 3 \cdot m \cdot n^2$$
$$= \pm\, 24\,mn^2.$$

WRITTEN EXERCISES

Find by factoring:

1. $\sqrt{625\,m^2n^2}$.
2. $\sqrt{11025\,x^6}$.
3. $\sqrt{256\,a^4b^4}$.
4. $\sqrt{225\,x^6y^6}$.
5. $\sqrt{3025\,x^2y^2}$.
6. $\sqrt{961\,a^6b^8}$.
7. $\sqrt{3136\,m^4n^8}$.
8. $\sqrt{169\,(a+b)^2}$.
9. $\sqrt{9216\,(x-y)^4}$.

154. Cube Root. A **cube root** of a given number is a number whose third power (or cube) equals the given number.

For example:

> 4 is a cube root of 64 because $4 \cdot 4 \cdot 4 = 64$.
> bc is a cube root of b^3c^3 because $bc \cdot bc \cdot bc = b^3c^3$.
> $-3\,ab$ is a cube root of $-27\,a^3b^3$.
> $2(a+b)$ is a cube root of $8(a+b)^3$.
> $-a(b+c)$ is a cube root of $-a^3(b+c)^3$.

The sign of the cube root is the same as the sign of the power.

155. Evolution. The finding of roots of numbers is called **evolution**.

ORAL EXERCISES

Name a cube root of:

1. a^6.
2. $27\,a^3$.
3. $-8\,a^3b^3$.
4. $8\,x^3y^3z^3$.
5. $27\,x^3y^3m^6$.
6. $8\,(x+y)^3$.
7. $-a^3(b+c)^3$.
8. $8\,m^3(m+n)^3$.
9. $125\,(p-q)^3$.
10. $x^6y^6(b-c)^3$.
11. $(a+b)^3(x+y)^3$.
12. $(x-y)^6(x+y)^3$.

II. MONOMIAL FACTORS

156. When every term of a polynomial contains a common factor, that factor may usually be found by inspection. (Sec. 140.)

For example:

$3\,ab$ is a factor of $3\,abx - 6\,aby - 9\,abz$, for $3\,abx - 6\,aby - 9\,abz = 3\,ab \cdot x - 3\,ab \cdot 2\,y - 3\,ab \cdot 3\,z$.

ORAL EXERCISES

State the monomial factor of each expression:

1. $ab + ac + ad$.
2. $ab + bc + b$.
3. $2\,ax + 2\,ay + 2\,az$.
4. $m^2 + m^2y + m^2z$.
5. $3\,mx - 6\,my - 9\,mz$.
6. $5\,ab + 10\,a^2b^2 - 5\,abc$.
7. $2\,a(x-y) + 2\,axy$.
8. $a^2b^2 - 3\,ab^3 + 5\,a^3b$.
9. $10^{n+3} + 10^{n+5} + 10^n$.
10. $7\,acx^2 - 3\,bcx^3 - 2\,c^2x^4$.
11. $12\,am^2 + 4\,ar^2 - 6\,a^2rm$.
12. $6\,x^2y + 10\,xy^2 - 18\,x^2y^2$.
13. $2\,ax^3 + 12\,a^2x^2 - 24\,abx^2$.
14. $\frac{m^2p^2}{5\,a} + \frac{3\,mp^2}{2\,a^2} - \frac{7\,m^2p}{10\,a^3}$.
15. $6\,a^2b^2c^2 - 2\,a^3bc - 2\,abc$.
16. $3\,ay^2 + 6\,a^2y - 9\,a^3y^3$.
17. $x^3 - 6\,x^2 + 12\,x$.
18. $x^2y^2 + xy^2 + xyz$.
19. $3\,x^3y - 6\,x^2y^2 + 9\,x^2y^3$.
20. $4\,a^2x - 8\,a^2y - 6\,a^2b^2$.
21. $3\,a^4b^3 - 3\,a^5 - 15\,a^3b^2$.
22. $15\,a^2x - 10\,a^2y + 5\,a^2z$.
23. $a^{2n+4}b^3 - a^{3n+4}b^7$.
24. $4\,px^2 - 20\,p^2x^3 - 10\,pxy$.
25. $36\,q^3 + 108\,q^5 - 18\,pq^4$.
26. $5^{a+1} - 5^{a+5} + 3 \cdot 5^a$.
27. $3x(2y-1) - 6x^2y - 9xyz$.
28. $\frac{3\,xy^2}{2\,p^2} - \frac{6\,x^2y}{p^3} + \frac{12\,xyz}{5\,p^5}$.

WRITTEN EXERCISES

1–28. Write the other factor in Exercises 1–28 above.

III. POLYNOMIAL FACTORS

157. An expression may have a binomial or other polynomial factor that can readily be found by inspection.

For example:

1. $a+b$ is a factor of $(a+b)x+(a+b)y$.
2. And $x+y-z$ is a factor of $(x+y-z)ab-3(x+y-z)cd$.
3. $3x^3-5x^2+3x-5=x^2(3x-5)+1(3x-5)=(3x-5)(x^2+1)$.

Note that $3x-5$ divides each term of the middle, or grouped, expression giving the quotients x^2 and $+1$.

ORAL EXERCISES

State a factor of:

1. $(a+1)x-(a+1)y$.
2. $(a+x)x-(a+x)y$.
3. $a(b+c)x^2-a(b+c)y^2$.
4. $(m+n+p)ab+(m+n+p)cd$.
5. $(x+y)^2-(x+y)$.
6. $(a+1)^3+(a+1)^2+(a+1)$.

Supply the blanks:

7. $ax+ay+bx+by=(\quad)(x+y)+(\quad)(x+y)$
$=[(\quad)+(\quad)](x+y)$.

8. $ax+bx-ay-by=(\quad)(a+b)-(\quad)(a+b)$
$=[(\quad)-(\quad)](a+b)$.

9. $2ax^2-4ax+3x-6=(\quad)(x-2)+(\quad)(x-2)$
$=[(\quad)+(\quad)](x-2)$.

10. $6a+3b+9c+2ax+bx+3cx=(\quad)(2a+b+3c)$
$+(\quad)(2a+b+3c)=[(\quad)+(\quad)](2a+b+3c)$.

WRITTEN EXERCISES

Factor:

1. x^3+x^2+x+1.
2. x^3-2y-x^2y+2x.
3. $ax-ay+bx-by$.
4. $ax+3a+bx+3b$.
5. $a^2+ab-ac-bc$.
6. $ax+x-ay-y$.
7. $x-a+(x-a)^2$.
8. $5h^3-4h^2+10h-8$.

9. $6m^3 + 4m^2 - 9m - 6.$

10. $xy - by - b + x.$

11. $x(z-a)^2 - y(a-z).$

12. $a(x+1)^2 + 3x + 3.$

13. $4ax + bx - 4ay - by.$

14. $4a^3 + a^2 - 4a - 1.$

15. $x^3 - 4x^2 + 2x - 8.$

16. $x^2 - x - 2x + 2.$

17. $(a+b+c)^2 + ax + bx + cx.$

18. $(m+r)^2x^2 + my + ry - m - r.$

19. $x(p-y) + ap - ay.$

20. $ax + ay - x(x+y)$

IV. SQUARES OF BINOMIALS

158. Since $(x \pm y)^2 = x^2 \pm 2xy + y^2$, a trinomial is the square of a binomial, if one term is twice the product of the square roots of the other two, but not otherwise. (Secs. 141, 142.)

For example:

$$a^2 + 14a + 49 = a^2 + 2 \cdot 7a + 7^2 = (a+7)^2.$$

Here $14a$ is twice the product of $\sqrt{a^2}$ and $\sqrt{49}$.

$$25m^2 - 30m + 9 = (5m)^2 - 2 \cdot 3 \cdot 5m + 3^2 = (5m-3)^2.$$

$$16a^6 - 8a^3 + 1 = (4a^3)^2 - 2 \cdot 4a^3 \cdot 1 + (1)^2 = (4a^3 - 1)^2.$$

Test by squaring.

WRITTEN EXERCISES

Factor:

1. $x^2 + 2ax + a^2.$

2. $x^2 - 2mx + m^2.$

3. $4x^2 - 4x + 1.$

4. $9x^2 - 12x + 4.$

5. $x^2y^2 + 2xy + 1.$

6. $a^{2x}b^{2y} + 2a^xb^yc + c^2.$

7. $a^2b^2 + 2abmn + m^2n^2.$

8. $a^2x^2 - 8ax + 16.$

9. $p^2 - 4pq + 4q^2.$

10. $4p^2x^2 - 4px + 1.$

11. $25a^2b^2 - 10abc + c^2.$

12. $9x^{2a} - 12x^a + 4.$

13. $x^2y^4 - 2xy^2 + 1.$

14. $\frac{4}{9}a^2 - \frac{2}{3}a + \frac{1}{4}.$

15. $\frac{1}{4}x^2 + \frac{1}{2}xy + \frac{1}{4}y^2.$

16. $p^2q^2 - \frac{2}{3}pq + \frac{1}{9}.$

17. $1 - \frac{2}{x^a} + \frac{1}{x^{2a}}.$

18. $(1-p)^2 - 6(1-p) + 9.$

19. $x^2 - 2(a-b) + (a-b)^2.$

20. $4(a^2+1)^2 + 4(a^2+1) + 1.$

21. $\frac{4}{x^2} + 20 + 25x^2.$

22. $y^{2a} - 4y^a + 4$.

23. $a^4x^2 - a^2x + \frac{1}{4}$.

24. $(x+y)^2 + 2(x+y) + 1$.

25. $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2}$.

26. $1 + \frac{2}{x} + \frac{1}{x^2}$.

27. $(m-n)^2 - 2(m-n)x + x^2$.

28. $(a+b)^2 - 2(a+b)y + y^2$.

29. $9(p+1)^2 - 6(p+1) + 1$.

30. $x^{2m} + 4x^m + 4 + 2(x^m + 2) + 1$.

31. $p^2q^2x^2y^2 - 6pqxy + 9$.

32. $a^{2p} + 144 - 24a^p$.

33. $p^{6n} + 49 - 14p^{3n}$.

34. $49x^2 + 81y^2 - 126xy$.

35. $\frac{1}{m^4} + \frac{2}{m^2} + 1$.

36. $\frac{x^2}{y^2} + 2 + \frac{y^2}{x^2}$.

37. $x^2 + 2xy + y^2 + 2(x+y)z + z^2$.

38. $a^2 + 2a(p-q) + p^2 - 2pq + q^2$.

REVIEW EXERCISES

Factor:

1. $ax + ab + ay^2$.

2. $1 - 2y + y^2$.

3. $3a^2x - 6ay + 9abxy$.

4. $(x+y)^2 + a(x+y)$.

5. $16m^2 - 24mn + 9n^2$.

6. $px + py + qx + qy$.

7. $\frac{4}{x^2} + \frac{4}{x} + 1$.

8. $\frac{p^2}{q^2} - 2 + \frac{q^2}{p^2}$.

9. $\frac{1}{y^4} - \frac{2}{y^2} + 1$.

10. $ax - bx - a + b$.

11. $(m-2)^2 - 3m + 6$.

12. $16a^4 - 8a^2 + 1$.

13. $ap + bp - aq - bq$.

14. $4a^{2p}b^{4p} - 4a^pb^{2p} + 1$.

15. $\frac{6x^2}{y} - \frac{9x}{y^2} + \frac{12x^2}{5y^3}$.

16. $81x^2 + 121y^2 - 198xy$.

17. $x^{n+1} + x^{n+2} - x^n$.

18. $5a^{2n+1}b^p - 15a^{3n-1}b^{p+1}$.

19. $mxy + nxy + pxy + mcd + ncd + pcd$.

20. $4x + 6y + 8z + 2ax + 3ay + 4az$.

21. $p^2 - 2pq + q^2 + 2(p-q)r + r^2$.

22. $y^{2p} - 4y^p + 4 - 2(y^p - 2) + 1$.

23. $abx + aby - abz - 3cdx - 3cdy + 3cdz$.

V. THE DIFFERENCE OF TWO SQUARES

159. The factors of the difference of two squares are the sum and the difference of the numbers whose squares are given.

For example:

The factors of $a^2b^2 - 1$ are $ab + 1$, $ab - 1$, because
$$(ab + 1)(ab - 1) = a^2b^2 - 1.$$
The factors of $a^4 - 4c^2d^2$ are $a^2 + 2cd$, $a^2 - 2cd$, because
$$(a^2 + 2cd)(a^2 - 2cd) = a^4 - 4c^2d^2.$$

ORAL EXERCISES

Read and supply the blanks:

1. $t^2 - v^2 = (t - v)(\quad)$.
2. $a^{2x} - 4b^{2y} = (a^x - 2b^y)(\quad)$.
3. $4x^2 - y^2 = (2x + y)(\quad)$.
4. $9x^2 - 4y^2 = (3x - 2y)(\quad)$.
5. $a^2x^{2m} - y^{2n} = [ax^m - (\quad)][ax^m + (\quad)]$.
6. $25s^2 - 49t^2 = [5s + (\quad)][5s - (\quad)]$.

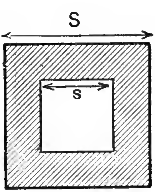
WRITTEN EXERCISES

Write the factors of:

1. $1 - y^2$.
2. $81u^2 - 64v^2$.
3. $121t^2 - 4$.
4. $1 - 144q^2$.
5. $x^2y^2 - 25$.
6. $144a^2b^2 - 49c^2$.
7. $4x^4 - y^4$.
8. $a^4 - 9b^4$.
9. $x^{10} - y^8$.
10. $y^4 - \frac{1}{4}$.
11. $92^4 - 1$.
12. $36x^4 - 49y^4$.
13. $26^2 - 24^2$.
14. $41^2 - 31^2$.
15. $32^2 - 28^2$.
16. $a^2 - 36b^4$.
17. $763^2 - 663^2$.
18. $36x^2y^2 - 169t^2u^2$.

19. Calculate the area of the shaded portion of this square, if

SIDE	(1)	(2)	(3)	(4)	(5)	(6)
$S =$	49	290	597	73	$6a$	$24q$
$s =$	45	280	497	27	$4a$	$14q$



Factor:

20. $a^2 - b^2x^2$.
21. $x^4 - y^2$.
22. $4a^2c - b^2c$.
23. $9x^2 - 16y^2$.
24. $25a^2b^2 - 1$.
25. $25a^2t^3 - 4b^2t^3$.
26. $49x^{2p+6} - 1$.
27. $49a^2 - 4b^2$.
28. $1 - a^2b^2c^2$.
29. $1 - 121x^2y^2$.
30. $1 - x^4$.
31. $7a^2b^2 - 7c^2d^2$.
32. $16x^2y^2 - 4m^2n^2$.
33. $ax^4 - ay^4$.
34. $x^6 - 4y^6$.
35. $c^5x^8 - 4c^5y^8$.
36. $81x^2y^2 - 9$.
37. $225a^4b^4 - 1$.
38. $25x^2y^4 - 36x^4y^2$.
39. $100a^2b^6 - 25a^6b^2$.
40. $\frac{a^2}{b^2} - 1$.
41. $\frac{x^{2m}}{y^{2m}} - 9$.
42. $16 - \frac{p^{2n+4}}{q^{2n+4}}$.
43. $\frac{4x^2}{9y^2} - \frac{252^2}{49}$.
44. $\frac{16}{625x^2} - 1$.

160. The terms of the given square may be polynomials, but the method of factoring is the same.

For example:

$$(a-b)^2 - (b+c)^2 = [(a-b)+(b+c)][(a-b)-(b+c)]$$
$$= (a+c)(a-2b-c).$$
$$(a^2+b)^2 - (x^2+y)^2 = (a^2+b+x^2+y)(a^2+b-x^2-y).$$

WRITTEN EXERCISES

Factor:

1. $(x+y)^2 - (x-y)^2$.
2. $(a+b)^2 - (a-b)^2$.
3. $(p+q)^2 - (m+n)^2$.
4. $(a+b+c)^2 - z^2$.
5. $(a-2b)^2 - (3b+c)^2$.
6. $(a^2-1)^2 - (b^2-1)^2$.
7. $a^2 + 2ab + b^2 - c^2$.
8. $x^2 - 2xy + y^2 - z^2$.
9. $4a^2 - 4a + 1 - 9b^2$.
10. $16a^2b^2 - a^2c^2 - 6ac - 9$.
11. $p^2t^2 - 10pt + 25 - p^2 + 10pt - 25t^2$.
12. $a^2 - 4ab + 4b^2 - 9y^2$.
13. $4a^2 + 8ab + 4b^2 - 1$.
14. $4 - x^2 - 2xy - y^2$.
15. $4x^2 - 4a^2 - 4ab - b^2$.
16. $x^2 - 2xy + y^2 - 16c^2$.
17. $(3x-2y)^2 - (2x+3y)^2$.
18. $(x^n - y^n)^2 - (x^n + y^n)^2$.
19. $x^{2p} + 2x^p + 1 - a^2b^2$.
20. $(m-2n)^2 - 36y^2$.
21. $(2x+y)^2 - (3x-4)^2$.
22. $(3a-4b)^2 - (5a+6b)^2$.
23. $(2x+y-z)^2 - (5x+6y)^2$.
24. $9a^{2x} - 6a^xb + b^2 - 9c^2$.
25. $(5z^m + 2p - 1)^2 - x^{2a}y^{4p}$.

VI. FORMING THE DIFFERENCE OF TWO SQUARES

161. It is often possible to factor an expression by first making it the difference of two squares.

If the given expression can be made the square of a binomial by adding a square, it can be made the difference of two squares by adding and subtracting the same square.

For example:

$$a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = (a^2+b^2)^2 - a^2b^2$$
$$= (a^2 + b^2 + ab)(a^2 + b^2 - ab).$$
$$x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2+2)^2 - (2x)^2$$
$$= (x^2 + 2 - 2x)(x^2 + 2 + 2x).$$
$$16x^4 - x^2 + 1 = 16x^4 + 8x^2 + 1 - 9x^2 = (4x^2+1)^2 - 9x^2$$
$$= (4x^2+1)^2 - (3x)^2 = (4x^2 + 1 + 3x)(4x^2 + 1 - 3x)$$

TEST. $16 - 1 + 1 = 16 = (4 + 1 + 3)(4 + 1 - 3).$

WRITTEN EXERCISES

Express as a difference of two squares and factor:

1. $a^4 + 4.$
2. $x^4y^4 + 4.$
3. $a^8 + 64.$
4. $64 + b^4.$
5. $x^4 + 4 \cdot 10^{8n}.$
6. $x^{4n} + 2x^{2n} + 9.$
7. $x^4 - 6x^2y^2 + y^4.$
8. $x^4 + 3x^2y^2 + 4y^4.$
9. $16a^4 + 4a^2 + 1.$
10. $4x^4y^4 + 3x^2y^2 + 1.$
11. $x^4 + 4y^4.$
12. $81p^4 + 9p^2 + 1.$
13. $x^4 + 25x^2y^2 + 625y^4.$
14. $16a^4b^4 + 4a^2b^2 + 1.$
15. $81a^4 + 225a^2b^2 + 625b^4.$
16. $625x^4 + 400x^2y^2 + 256y^4.$
17. $a^4 + 2a^2b^2 + 9b^4.$
18. $x^4 - 8x^2y^2 + 4y^4.$
19. $4a^4 - 16a^2b^2 + 9b^4.$
20. $x^2 + 2xy - 15y^2.$

REVIEW EXERCISES

Factor:

1. $3x^3 - 9x^2 + x - 3.$
2. $m^4 - m^2 - 5m - 5.$
3. $16x^4 + 4x^2 + 1.$
4. $4p^4 - 8p^2q^2 + q^4.$
5. $25a^2 - 40ab + 16b^2 - 9c^2.$
6. $81x^2 - y^2 - 4yz - 4z^2.$
7. $b(y-z)^2 + 5y - 5z.$
8. $(p+q-r)^2 + ap + aq - ar.$

9. $64 + y^4$.

10. $(m - n)^2 - 4x^4y^2$.

11. $p^4 + 4z^{4n}$.

12. $(x^a - b)^2 + 2x^a - 2b + 1$.

13. $(x + y)^2 + 2(x + y)z + z^2$.

14. $bx^2 + x^2 - by^2 - y^2$.

15. $(2x - y)^{n+1} - (2x - y)^n$.

16. $64r^4 - \frac{1}{16}$.

17. $36x^{2m+6} - 1$.

18. $\frac{x^2y^2}{p^2} - \frac{p^2}{m^2q^2}$.

19. $x^{2a} + 81 - 18x^a$.

20. $x^{2a}y^{2b} + 2x^ay^b + 1$.

21. $36m^2 + 49p^2 - 84mp$.

22. $15a^2xy + 25a^2x^2y - 10a^2xy^2$.

23. $\frac{3xy}{5} - \frac{x^2}{15} + \frac{4x^2z}{75}$.

24. $4x + 3y - z - 12ax - 9ay + 3az$.

VII. TYPE $x^2 + px + q$

162. $(x + a)(x + b) = x^2 + (a + b)x + ab$. Hence if a trinomial is the product of two factors like $x + a$ and $x + b$, the sum of a and b is the coefficient of the middle term, and their product, ab, is the third term.

The factors of such a trinomial are seen at once if a and b can be found by inspection.

For example:

In $x^2 + 5x + 6$, $5 = 2 + 3$, and $6 = 2 \cdot 3$.
$\therefore x^2 + 5x + 6 = x^2 + (2 + 3)x + 2 \cdot 3 = (x + 2)(x + 3)$.

In $x^2 - 7x + 6$, $6 = (-6)(-1)$, and $-7 = -6 + (-1)$.
$\therefore x^2 - 7x + 6 = x^2 + (-6 - 1), x + (-6)(-1) = (x - 6)(x - 1)$.

In $x^2 - 5x - 6$, $-6 = 6(-1)$, and $5 = 6 + (-1)$.
$\therefore x^2 + 5x - 6 = x^2 + (6 - 1)x + 6(-1) = (x + 6)(x - 1)$.

ORAL EXERCISES

Factor:

1. $a^2 + 3a + 2$.
2. $d^2 - 3d + 2$.
3. $a^2 - 5a + 4$.
4. $z^2 - 8z + 15$.
5. $m^2 - m - 12$.
6. $x^2 + 5x + 4$.
7. $y^2 - 6y + 5$.
8. $m^2 + 6m + 5$.
9. $w^2 + 4w - 21$.
10. $z^2 - 2z - 15$.
11. $x^2 + 8x + 7$.
12. $x^2 - 8x + 7$.
13. $a^2 - a - 2$.
14. $m^2 - 7m + 12$.
15. $w^2 - 4w - 21$.

163. Although the inspection method of Section 162 should be generally used, it may be helpful in some cases to write the various pairs of factors of the third term and then compare their sum with the coefficient of the middle term.

For example:

In $x^2 - 17x + 72$, the pairs of factors of $+72$ are

72	36	24	18	12	9
1	2	3	4	6	8

and the same pairs taken negatively.

Since the sum of the factors is negative, only the negative pairs need be examined, and by trial the pair -8, -9 is found to have the sum -17.

$$\therefore x^2 - 17x + 72 = (x - 8)(x - 9).$$

Likewise, in $x^2 - x - 56$, the factors of -56 are

−28	−14	−8
2	4	7

or the same numbers with the signs changed; but since the coefficient of x is negative, only those pairs need be examined in which the negative number is the larger.

By inspection, -8, 7 are found to have the sum -1.

$$\therefore x^2 - x - 56 = (x - 8)(x + 7).$$

TEST by multiplication.

WRITTEN EXERCISES

Factor (use Section 163 for 7, 10, 12, 16):

1. $x^2 - x - 30$.
2. $x^2 + x - 30$.
3. $x^2 - x - 20$.
4. $x^2 + x - 20$.
5. $x^2 - 3x - 18$.
6. $x^2 + 3x - 18$.
7. $a^2 + a - 42$.
8. $a^2 - a - 12$.
9. $m^2 - \frac{5}{6}m + \frac{1}{6}$.
10. $x^2 - 7x - 18$.
11. $x^2 + 7x - 18$.
12. $x^2 + 17x + 60$.
13. $m^2 + 11m + 28$.
14. $m^2 - 3m - 28$.
15. $m^2 + 3m - 28$.
16. $y^2 + 6y - 40$.
17. $y^2 - 6y - 40$.
18. $t^2 - \frac{3}{2}t + \frac{1}{2}$.
19. $x^2 + 6x + 5$.
20. $x^2 + 9x + 20$.
21. $x^2 - 8x + 15$.
22. $x^2 + 8x + 7$.
23. $x^2 - 10x + 9$.
24. $x^2 + 7x + 12$.
25. $x^2 - 5x - 14$.
26. $x^2 + 2x - 15$.
27. $s^2 - \frac{3}{4}s - \frac{1}{4}$.
28. $(a + b)^2 - 10(a + b) + 9$.
29. $(m + n)^2 + 2(m + n) - 15$.

VIII. TYPE $mx^2 + px + q$

164. The type $mx^2 + px + q$ may be factored by a comparison of factors similar to that in Section 163. This trial method is commonly called *the method of cross products.*

EXAMPLE

Factor: $12x^2 - 7x - 10.$

From the factors of the first and third terms without taking into account the middle term, the possible factors are

$12x - 10$	$2x - 5$	$3x + 2$
$x + 1$	$6x + 2$	$4x - 5$

in which the two coefficients in any column may be interchanged.

To find which are the actual factors, it is only necessary to multiply the coefficients and observe what combination produces the coefficient of the second term of the original trinomial, in this case -7.

Thus,			
	$12 - 10$	$2 - 5$	$3 + 2$
	$\times$	$\times$	$\times$
	$1 + 1$	$6 + 2$	$4 - 5$
	-2 (not -7)	-26 (not -7)	-7 (correct)

Therefore, the factors are $3x + 2$ and $4x - 5$. Test as usual.

It is seldom necessary to try all the sets of factors in their different combinations. Simple conditions will eliminate them at once. For example, $12x - 10$ could not be a factor, because, if it were, the factor 2 which it contains would be a factor of the given trinomial, and it is not. For the same reason, $6x + 2$, in the second set, could not be a factor.

WRITTEN EXERCISES

Factor:

1. $6x^2 - 25x + 24.$
2. $4x^2 - 27x - 7.$
3. $12x^2 + 11x - 5.$
4. $3p^2 + 16p - 35.$
5. $88p^2 - 31p - 15.$
6. $24x^2 + 73x + 24.$
7. $7y^2 + 48y - 7.$
8. $10y^2 + 13y - 3.$
9. $21x^2 + 46x - 7.$
10. $6x^2 + 47x + 35.$
11. $8p^2 - 30p + 7.$
12. $20y^2 - 49y + 30.$
13. $18x^2 - 19x + 5.$
14. $24z^2 - 103z + 55.$

15. $3m^2 + 20m + 32$.

16. $21y^2 - 61y + 28$.

17. $26p^2x^2 - 10px - 36$.

18. $15m^2 + 8m - 23$.

19. $15z^2 - 17z - 42$.

20. $22x^2 - 19x - 21$.

21. $26p^2x^2 + 62px + 36$.

22. $3p^2 - 20p + 32$.

23. $15a^2 + 17a - 42$.

24. $25x^2 - 5x - 56$.

25. $15m^2 - 38m + 23$.

26. $21b^2 - 37b - 28$.

27. $25c^2 + 75c + 56$.

28. $22x^2 + 19x - 21$.

29. $25z^2 + 5z - 56$.

30. $21y^2 + 37y - 28$.

31. $3m^2 - 4m - 32$.

32. $22x^2 + 47x + 21$.

165. The methods for factoring given in Sections 162, 163, and 164 are of limited value, for they determine the factors only in favorable instances. Thus, $x^2 - 17x + 72$ was readily factored in Section 163, but $x^2 - 16x + 72$ could not be so factored. A general method will be given in Chapter XXX by means of which all such trinomials can readily be factored.

REVIEW EXERCISES

Factor:

1. $x^8 - 49y^4$.

2. $200 - 2x^4$.

3. $a^{12}b^9 - bc^{16}$.

4. $243 - 3x^{12}$.

5. $121a^{2p} - y^2$.

6. $a^8 - 256b^8$.

7. $1 - 25(a + 2b)^2$.

8. $1 - 81(x^2 + x - 1)^2$.

9. $2a^2 - 4a - 2a^4 + 2$.

10. $14x^7 - 21x^5y$.

11. $3a^nb^3 - 6a^{2n}b^4 + 15a^{3n}b^5$.

12. $x^5 - 2x^4 + x^3$.

13. $b^3z - 2b^2z^2 + bz^3$.

14. $4a^3 - 48a^2b^2 + 144ab^4$.

15. $x^my^5z^{2p} + 7x^my^3z^{2p-1}$.

16. $100a^4 - 61a^2 + 9$.

17. $a^2 - (b + 1)^2$.

18. $z^5 + z - 2z^3$.

19. $25(a - b)^2 + 20c(a - b) + 4c^2$.

20. $a^2 + b^2 + c^2 - 2ab - 2bc + 2ac$.

21. $a^2 + b^2 - x^2 - y^2 + 2ab - 2xy$.

22. $2ab - b^2 + x^2 - a^2$.

23. $16(2x - y)^2 - (x - 3y)^2$.

24. $4a^2 + 9b^2 + 25c^2 - 12ab + 20ac - 30bc$.

25. $a^4 - 169\,b^{10}$.

26. $36\,c^4 + 47\,c^2d^2 + 16\,d^4$.

27. $a^5 - 20\,a + a^3$.

28. $x^{2p} - x^p - 72$.

29. $ay^2 - ay - 20\,a$.

30. $2\,x^3 - 10\,x^2 + 12\,x$.

31. $x^2y^2 - 3\,xy - 88$.

32. $(x^2-1)^2 - 2(x^2-1) - 63$.

33. $8\,a^2 + 2\,a - 15$.

34. $25\,x^4 - 41\,x^2y^2 + 16\,y^4$.

35. $9\,z^4 - 148\,z^2 + 64$.

36. $14\,x^2 - 39\,x + 10$.

37. $7\,x^2 - 3\,x - 4$.

38. $10 - 5\,a - 15\,a^2$.

39. $3\,t^2 + 7\,t - 20$.

40. $10\,s^2 - 34\,s + 28$.

41. $6\,c^2 - 5\,cd - 21\,d^2$.

42. $m^2 + 26\,m + 133$.

43. $(a-b)^2 - 15(a-b) + 50$.

44. $21\,r^2 - 18\,rs - 48\,s^2$.

45. $x^4y^4 + 2\,x^2y^2 - 63$.

46. $9\,x^{2k} - x^ky - 10\,y^2$.

47. $4\,x^4 - 9\,x^2 + 6\,x - 1$.

48. $(2\,a+b)^2 - 16(3\,a-b)^2$.

49. $(3+b)^2 - 2(3+b)(x-1) + (x-1)^2$.

50. $4\,x^{4n+2}y^{3n+4} - 8\,x^{6n+4}y^{3n+3} + 12\,x^{5n+3}y^{5n+6}$.

IX. TYPE x^3+y^3 AND x^3-y^3.

166. We know by multiplying that

$$(x+y)(x^2-xy+y^2) = x^3+y^3,$$

and that

$$(x-y)(x^2+xy+y^2) = x^3-y^3.$$

Hence, one factor of the sum of two cubes is the sum of the numbers, and the other is the sum of the squares of the numbers minus their product.

Also one factor of the difference of two cubes is the difference between the numbers, and the other is the sum of their squares plus their product.

EXAMPLES

1. Factor: $27\,x^3 + 8\,y^3$.

$$27\,x^3 = (3\,x)^3.$$
$$8\,y^3 = (2\,y)^3.$$
$$\therefore 27\,x^3 + 8\,y^3 = (3\,x+2\,y)[(3\,x)^2 - 3\,x\cdot 2\,y + (2\,y)^2]$$
$$= (3\,x+2\,y)(9\,x^2 - 6\,xy + 4\,y^2).$$

TEST by multiplication.

2. Factor: $8a^3b^3 - 125c^3$.

$$8a^3b^3 = (2ab)^3.$$
$$125c^3 = (5c)^3.$$
$$\therefore 8a^3b^3 - 125c^3 = (2ab - 5c)[(2ab)^2 + 2ab \cdot 5c + (5c)^2]$$
$$= (2ab - 5c)(4a^2b^2 + 10abc + 25c^2).$$

Test by letting $a = b = c = 1$: $8 - 125 = -3 \cdot 39$.

3. Factor: $8x^6 - a^6$.

$$8x^6 - a^6 = (2x^2)^3 - (a^2)^3$$
$$= (2x^2 - a^2)[(2x^2)^2 + (2x^2)(a^2) + (a^2)^2]$$
$$= (2x^2 - a^2)(4x^4 + 2a^2x^2 + a^4).$$

167. Types $x^3 - y^3$ and $x^3 + y^3$ may be used in calculation.

EXAMPLE

Calculate: $14^3 - 13^3$.

$$14^3 - 13^3 = (14 - 13)(14^2 + 14 \cdot 13 + 13^2)$$
$$= 14^2 + 14 \cdot 13 + 13^2$$
$$= 14(14 + 13) + 13^2$$
$$= 14 \cdot 27 + 13^2$$
$$= 378 + 169$$
$$= 547.$$

WRITTEN EXERCISES

Factor:

1. $a^3 - b^3$.
2. $a^3 - 8b^3$.
3. $8a^3 + b^3$.
4. $27x^3 - y^3$.
5. $y^3 + 27z^3$.
6. $a^{3n} + 1$.
7. $\frac{1}{a^3} - 1$.
8. $a^3 - 1$.
9. $x^6 - y^6$.
10. $a^6 + b^6$.
11. $64a^6 - 1$.
12. $8x^3 + 1$.
13. $27y^{6p} - 1$.
14. $a^3 - \frac{1}{27}$.
15. $xy^4 - x^4y$.
16. $125a^6 - b^3$.
17. $(a + b)^3 - 1$.
18. $27x^3y^3z^3 + 8$.
19. $a^3b^3 + (a + b)^3$.
20. $8m^3 - 27p^3q^3$.
21. $10^3 - h^{3n}$.
22. $(a + b)^3 - (b - c)^3$.
23. $(3a + b)^3 - (2a - b)^3$.
24. $(x + y)^3 - (x - y)^3$.
25. $(c + d)^3 + (2c - d)^3$.
26. $(x^3 + 1)^3 - (y^3 + 1)^3$.
27. $8a^3 + 125(b + c)^3$.

Calculate:

28. $16^3 - 15^3$. **29.** $19^3 - 18^3$. **30.** $23^3 - 21^3$. **31.** $57^3 - 56^3$.

32. It is known that the volume of a sphere is $\frac{4}{3}\pi r^3$, r being the length of the radius. Using $\frac{22}{7}$ as an approximate value of π, calculate the number of cubic inches in a spherical shell whose outer radius is 14 in., and inner radius 13 in.

Solution. The volume of the outer sphere is $\frac{4}{3} \cdot \frac{22}{7} \cdot 14^3$, and that of the inner sphere is $\frac{4}{3} \cdot \frac{22}{7} \cdot 13^3$.

Hence the volume of the shell in cubic inches is

$$\frac{4}{3} \cdot \frac{22}{7}(14^3 - 13^3) = \frac{4}{3} \cdot \frac{22}{7} \cdot 547 = 2292.+$$

33. Find similarly the volumes of spherical shells if:

	(1)	(2)	(3)	(4)
Outer radius =	16	19	25	36
Inner radius =	15	18	23	32

168. We have seen how to factor the difference of two squares and the difference of two cubes; it will be sufficient for present purposes to note that:

The difference of any two even powers is always divisible by the difference of their square roots, and the difference of two odd powers is divisible by the difference of the numbers.

Furthermore, the sum of any two odd powers is divisible by the sum of the numbers.

EXAMPLES

1. Factor: $x^{10} - y^{10}$.

$x^{10} - y^{10} = (x^5)^2 - (y^5)^2 = (x^5 - y^5)(x^5 + y^5)$. Then $x^5 - y^5$ is divisible by $x - y$, and $x^5 + y^5$ is divisible by $x + y$.

2. Factor: $x^5 - y^5$.

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$$

3. Factor: $a^5 + 32\,b^5$.

$$a^5 + 32\,b^5 = (a + 2\,b)(a^4 - 2\,a^3b + 4\,a^2b^2 - 8\,ab^3 + 16\,b^4)$$

WRITTEN EXERCISES

Factor into two binomials:

1. $m^2n^2 - p^2q^2$.
2. $4m^2 - 9n^2$.
3. $16 - y^2$.
4. $4a^4b^2 - 1$.
5. $m^4 - p^4$.
6. $x^4y^4 - 64$

Factor into three binomials:

7. $a^4 - b^4$.
8. $16 - p^4$.
9. $x^4y^4 - 1$.
10. $81m^4 - p^4$.

State one factor of each of these expressions and prove by division that it is a factor:

11. $m^5n^5 - 1$.
12. $x^7 - a^7$.
13. $x^{3a} + y^{3a}$.
14. $x^5 - 32$.
15. $8a^3b^3 + 27$.
16. $x^{5a} + 1$.

REVIEW

169. In attempting to factor an expression not an indicated root:

First. Remove any monomial factor, as in Type II.

Second. If the resulting expression is a binomial, apply Type V, or VI, or IX.

Third. If the resulting expression is a trinomial, apply Type IV, or VII, or VIII.

Fourth. If the expression is not one of the above types, attempt to group it, as in Type III.

WRITTEN EXERCISES

1. $3a^2 - 15ab$.
2. $a^2 - x^2$.
3. $p^2 - 9q^2$.
4. $3a^4 - 6a^2b$.
5. $xy^3z - xyz^2$.
6. $16a^4 - 9c^4$.
7. $a^3 - x^3$.
8. $x^3 + z^3$.
9. $a^2x^4 - 2ax^2 + 1$.
10. $m^2 + 6m + 5$.
11. $x^6 + 1$.
12. $x^2 - 2ax - 3a^2$.
13. $y^{3b} + 1$.
14. $ac - ad + bc - bd$.

15 $2x^3 - 54$.

16. $-x^4 + x^2 + 12$.

17. $2x^2 + 5bx - 12b^2$.

18. $3a^2 + 12ab - 2a - 8b$.

19. $x^4 - y^4$.

20. $4a^2 - b^2 + 6a - 3b$.

21. $a^2 - 4b^2 - ac + 2bc$.

22. $25x^4 - 10x^2y + y^2 - 9z^2$.

23. $x^4 - 6x^2y^2 + y^4$.

24. $16a^4 + 24a^2b + 9b^2$.

25. $4x^2 - 4xy + y^2 - x^2y^2$.

26. $25a^6 + 20a^3 + 4$.

27. $a^9 + b^9$.

28. $3x^2 - 7xy + 2y^2$.

29. $3x^2 + 6xy - 24y^2$.

30. $x^8 + 3x^4 - 4$.

31. $3x^3 - 6x^2 - 9x$.

32. $3a^2 + 6ab - 24b^2$.

33. $x^{2m} - 2x^m - 3$.

34. $a^2 - 2ab - ac + 2bc$.

35. $m^4 - 11m^2n^2 + n^4$.

36. $a^5 + b^5$.

37. $x^4 - 3x^2y^2 + y^4$.

38. $8 + z^6$.

39. $4x^4 + 1$.

40. $x^8 - y^8$.

41. $a^2 - b^2 - (a - b)^2$.

42. $27x^2 + 3x - 2$.

43. $x^{2n} - y^{2b}$.

44. $a(a + b) - c(c + b)$.

45. $ab - 3a - 2b + 6$.

46. $3x^3 - 6x^2 - 9x$.

47. $2a^3 - 3a^2 - 2a + 3$

48. $4x^4 + y^4 - 5x^2y^2$.

49. $a^4 - 3a^2 + 1$.

50. $4x^2 + 11x - 20$.

51. $x^2 - 4ax - 4b^2 + 8ab$.

52. $a^3 + a^2 - a - 1$.

53. $x^{2a} + 2x^a + 1$.

54. $(a + b)^2 - (c - d)^2$.

55. $b^2 - a^2 + 2ac - c^2$.

56. $a^2 - b^2 - a - b$.

57. $\frac{c^2}{d^2} + \frac{2c}{d} - 3$.

58. $16x^2 + 10xy - 9y^2$.

59. $x^8 + x^4 + \frac{1}{4}$.

60. $x^6 - x^4 - x^2 + 1$.

61. $x^3 - 216$.

62. $a^4 + a^2b^2 + b^4$.

63. $x^3 + 8$.

64. $2a^2 + 13a - 24$.

65. $a^{2n} + 6a^n + 9$.

66. $x^4 - 81$.

67. $6x^2 - 11xy - 2y^2$.

68. $6x^2 - 13xy + 6y^2$.

69. $xy^7 - xy$.

70. $x^{2p} - 2x^py + y^2$.

71. $x^4 + x^2y^4 + y^8$.

72. $x^4 - 2x^2 + 1$.

73. $a^2b - a^2 - ab + a$.

74. $a^4 - 6a^2 + 1$.

75. $x^8 + x^4y^2 + y^4$.

76. $x^{2a} - 4x^a + 4$.

77. $\frac{a^2}{4} - ab + b^2$.

78. $1 + a - b - ab$.

79. $12x^2 - 27y^2$.

80. $64 - a^6$.

81. $3a^2 - 15a + 18$.

82. $a^2 + a - 30$.

83. $x^2 + 3x + 2$.

84. $16 + 4a^2 + a^4$.

85. $x^{3p} + y^{3q}$.

86. $16x^2 - 48x + 35$.

87. $a^4 + a^2 + 1$.

88. $2a^2 + 3ab - 2b^2$.

89. $2x^2 + xy - 3y^2$.

90. $6a^2 + 10ab - 4b^2$.

91. $a^{x+1} - a^{x-2}b^3$.

92. $p^3 + 3p^2 + 3p + 1$.

93. $x^4 + 2x^3 + 2x^2 + 2x + 1$.

94. $\frac{1}{x^2} + 1 + x^2$.

95. $2x^3 + 3x^2 - 2x - 3$.

96. $4b^3 - 9b$.

97. $55x^2 - x - 2$.

98. $6y^3 - y^2 + 6y - 1$.

99. $\frac{1}{9}x^2 - \frac{4}{25}y^2$.

100. $a^4 - 3a^2 + 1$.

101. $x^3 - x^2 - x + 1$.

102. $a^2 + \frac{5}{6}a + \frac{1}{6}$.

103. $1 - \frac{1}{a^2}$.

104. $2a^2 + ab - 6b^2$.

105. $12a^2 - 5ab - 3b^2$.

106. $x^{2n} - 2x^n + 1$.

107. $abx^2 + (a^2 + b^2)xy + aby^2$.

108. $a^3x - a^2c + a^2by - ab^2x - b^3y + cb^2$.

109. $(a + b)(c^2 - d^2) - (a^2 - b^2)(c - d)$.

110. $a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$.

111. $a^2 + b^2 + 1 + 2b + 2a + 2ab - d^2$.

Factor by reference to:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

112. $m^2 + q^2 + r^2 + 2mq + 2mr + 2qr$.

113. $a^2 + b^2 + x^2 + 2ab - 2ax - 2bx$.

114. $x^2 + y^2 + 25 + 2xy + 10x + 10y$.

115. $9 + a^2 + m^2 + 2am - 6a - 6m$.

116. $t^2 + 2tv + v^2 - 6t + 9 - 6v$.

117. $25x^2 + 9y^2 + 4z^2 + 30xy - 20xz - 12yz$.

118. $x^4 - 2x^2a^2 - 2x^2b^2 + a^4 + b^4 + 2a^2b^2$.

CHAPTER XIII

EQUATIONS

170. Factoring is an important process in the solution of equations.

171. PREPARATORY.

1. Find the values of the trinomial x^2-x-2 when $x=1$; 2; 0; -1; -2; 5; for which values of x does it become 0?

2. Find the value of the binomial x^2-2x when $x=1$; -1; 2; -2; 0; 3; for which values of x does it become 0?

3. According to Exercise 1, what are the roots of the equation $x^2-x-2=0$?

4. According to Exercise 2, what are the roots of the equation $x^2-2x=0$?

5. What roots are common to the two equations?

172. Equivalent Equations. If two equations have the same roots, the equations are said to be **equivalent**.

Thus, $$4x=12$$
and $$5x-15=0$$
are equivalent, each having the root 3, and no others.

Also, $$x^2-25=0$$
and $$4x^2-100=0$$
are equivalent, each having the root 5, -5, and no others.

But, $$x^2-25=0$$
and $$x^2-8x+15=0$$
are not equivalent, for the first has the roots 5, -5, while the second has the roots 5 and 3.

WRITTEN EXERCISES

Write the roots of these equations and find which exercises contain equivalent equations:

1. $3x - 6 = 0$, $x - 3 = 4$.
2. $x - 3 = 5$, $2x = 16$.
3. $x^2 = 4$, $x + 2 = 0$.
4. $x^2 = 2$, $2x^2 = 4$.
5. $x - 4 = 8$, $x + 2 = 6$.
6. $x^2 = 9$, $x + 3 = 0$.

173. If two equations have between them all the roots of a third equation and no others, the two together are said to be equivalent to the third.

EXAMPLES

1. One of the equations $x - 5 = 0$, and $x - 3 = 0$, has the root 5, the other the root 3. Between them, they have the roots 3 and 5, which are the roots of $x^2 - 8x + 15 = 0$. The equations $x - 3 = 0$ and $x - 5 = 0$ are together equivalent to $x^2 - 8x + 15 = 0$.

2. The equation $(x - 1)(x - 2) = 0$ asks: *For what values of x does the product $(x - 1)(x - 2)$ have the value zero?*

The product is zero, if either factor is zero, and not otherwise (Sec. 114). $\therefore (x - 1)(x - 2) = 0$, if $x - 1 = 0$, or if $x - 2 = 0$, and not otherwise.

Thus, the solution of the equation $(x - 1)(x - 2) = 0$ depends upon the solution of $x - 1 = 0$ and $x - 2 = 0$. The roots of these being 1 and 2, the roots of $(x - 1)(x - 2) = 0$, are likewise 1 and 2.

The pair of equations $x - 1 = 0$, $x - 2 = 0$ is equivalent to the equation $(x - 1)(x - 2) = 0$.

ORAL EXERCISES

State the equations of the first degree that are equivalent of each of the following:

1. $(x - 3)(x - 2) = 0$.
2. $(x - 5)(x - 3) = 0$.
3. $(x - 3)(x + 2) = 0$.
4. $(x + 5)(x + 3) = 0$.
5. $(x - 3)(x + 10) = 0$.
6. $x(x - 5) = 0$.
7. $(x + 7)(x + 1) = 0$.
8. $x(x + 3) = 0$.
9. $x(x - a) = 0$.
10. $(x + 8)(x - 11) = 0$.

Factor the first member of each of the following equations and state the equations equivalent to each :

11. $x^2 - 3x + 2 = 0.$

12. $x^2 - 5x + 4 = 0.$

13. $x^2 - 6x + 5 = 0.$

14. $x^2 + 6x + 5 = 0.$

15. $x^2 + 8x + 7 = 0.$

16. $x^2 - 4x - 5 = 0.$

17. $x^2 - 9 = 0.$

18. $x^2 - 2x = 0.$

19. $3x^2 - 2x = 0.$

20. $x^2 - 14x + 33 = 0.$

WRITTEN EXERCISES

1–10. Solve the equation in each exercise above from 11–20 by solving the equivalent equations.

Find the roots of each equation by factoring the left member and solving the equivalent equations :

11. $x^2 - x - 20 = 0.$

12. $x^2 + x - 30 = 0.$

13. $x^2 - 3x - 18 = 0.$

14. $x^2 - x - 30 = 0.$

15. $x^2 + 3x - 18 = 0.$

16. $x^2 - x - 42 = 0.$

17. $x^2 - 7x - 18 = 0.$

18. $x^2 - 17x + 72 = 0.$

19. $x^2 - x - 56 = 0.$

20. $x^2 + 7x - 18 = 0.$

21. $x^2 + 11x + 28 = 0.$

22. $x^2 - 3x - 28 = 0.$

23. $x^2 + 6x - 40 = 0.$

24. $x^2 + 11x + 24 = 0.$

25. $x^2 - 3x - 28 = 0.$

26. $x^2 - 6x - 40 = 0.$

27. $x^2 + 18x + 81 = 0.$

28. $x^2 - 20x + 100 = 0.$

29. $x^2 - 8x + 15 = 0.$

30. $x^2 + 9x + 20 = 0.$

31. $x^2 + 8x + 7 = 0.$

32. $x^2 + 2x - 15 = 0.$

33. $x^2 - x - 6 = 0.$

34. $x^2 - 5x - 14 = 0.$

35. $x^2 + x - 110 = 0.$

36. $x^2 - 5x - 24 = 0.$

37. $-x^2 + x + 12 = 0.$

38. $3x^2 - 7x + 2 = 0.$

39. $4z^2 + 12z + 9 = 0.$

40. $25x^2 + 20x + 4 = 0.$

41. $x^2 - x - 2 = 0.$

42. $x^2 - 64 = 0.$

43. $1 - x^2 = 0.$

44. $2y^2 + 12y + 10 = 0.$

174. Quadratic Equations. Equations of the second degree are called **quadratic equations.**

For example, $x^2 = 16$, $x^2 - 3x = 0$, and $x^2 - 5x + 6 = 0$ are quadratic equations with one unknown, x.

175. The way in which quadratic equations occur in problems is illustrated in the following exercises.

WRITTEN EXERCISES

Translate each statement into an equation:

1. The product of a certain number and the number increased by 3 is 70.

2. The product of two consecutive integers is 132.

3. The area of a rectangle whose length is three times its height is 75 sq. in.

4. In the case of a body falling from rest, the distance d fallen in the time t is one half the product of a fixed number g (the constant of gravity) and the square of the time.

176. General Form. The *general form* of the quadratic equation is

$$ax^2 + bx + c = 0,$$

in which a, b, c, are any known numbers, except that a may not be zero.

For example:

	a	b	c
$3x^2 - x + 5 = 0,$	3	-1	5
$x^2 + 7x = 0,$	1	7	0
$4x^2 - 12 = 0,$	4	0	-12
$x^2 = 0,$	1	0	0

177. It is often necessary to simplify equations apparently involving x^2 to see whether or not a is zero; that is, whether or not the equations are really quadratics.

For example:

$\frac{x^2 - 3}{5x} = \frac{x + 2}{7}$ can more readily be seen to be a quadratic equation when reduced to $2x^2 - 10x - 21 = 0$.

In this form it is apparent that $a = 2$, $b = -10$, $c = -21$.

WRITTEN EXERCISES

Reduce each equation to the type form and write the value of a; of b; of c:

1. $\frac{x^2-1}{2}=1.$

2. $3x^2-5=\frac{x-1}{2}.$

3. $\frac{x^2-1}{2}=\frac{x^2-4}{5}.$

4. $\frac{2x^2+3}{5}=\frac{x^2}{7}.$

5. $x+\frac{1}{x}=3.$

6. $12-x=\frac{x^2}{8}.$

7. $\frac{x+6}{7}=\frac{x^2}{2}.$

8. $\frac{2x-9}{3}=\frac{2x^2}{5}.$

178. Incomplete Quadratic Equations. A quadratic equation which lacks either its absolute term or its term in x, that is, in which either c or b is zero, is an **incomplete quadratic equation.** The two forms of the incomplete quadratic equation are

$$x^2+\frac{c}{a}=0.$$

$$ax^2+bx=0.$$

It is unnecessary to consider the case where b and c are both 0, because x would always be 0 whatever the value of a.

179. Solution of Incomplete Quadratic Equations.

I. *The incomplete quadratic equation* $x^2+\frac{c}{a}=0$ *is solved by transposing and extracting the square root of both members.*

EXAMPLES

1. $x^2=4. \quad \therefore \; x=\sqrt{4}=\pm 2.$

2. $3x^2=75. \quad \therefore \; x^2=25$ and $x=\pm 5.$

3. $ax^2=b. \quad \therefore \; x^2=\frac{b}{a}$ and $x=\pm\sqrt{\frac{b}{a}}.$

In Example 3 the value of x is the square root of the fraction $\frac{b}{a}$, and it is sufficient to indicate this by the use of the radical sign.

II. *The incomplete quadratic equation* $ax^2 + bx = 0$ *is solved by factoring* (Sec. 173).

Thus, $x(ax + b) = 0$, in which $x = 0$ and $ax + b = 0$ from which $x = -\frac{b}{a}$ One value of x in this case is always zero (Sec. 114), and the other is the root of the linear equation $ax + b = 0$.

EXAMPLES

1. $x^2 - x = 0$. $\therefore x(x - 1) = 0$ and $x = 0$, $x = 1$.
2. $3x^2 - 10x = 0$. $\therefore x(3x - 10) = 0$ and $x = 0$, $x = \frac{10}{3}$.

WRITTEN EXERCISES

Solve the following equations:

1. $x^2 = 169$.
2. $x^2 - 121 = 0$.
3. $x^2 - 144 = 0$.
4. $x^2 - 81 = 0$.
5. $x^2 - 49 = 0$.
6. $x^2 - 625 = 0$.
7. $3x^2 - 75 = 0$.
8. $4x^2 - 100 = 0$.
9. $5x^2 - 500 = 0$.
10. $12x^2 - 1728 = 0$.
11. $s^2 - \frac{1}{4} = 0$.
12. $t^2 - \frac{9}{16} = 0$.
13. $r^2 - 25 = 0$.
14. $121x^2 = 1089$.
15. $7x^2 - 448 = 0$.
16. $w^2 - \frac{16}{25} = 0$.
17. $2s^2 - \frac{8}{9} = 0$.
18. $3x^2 - \frac{12}{25} = 0$.
19. $3x^2 + 8 = 5x^2 + 8$.
20. $17x^2 + x = 0$.
21. $40x^2 - 25x = 0$.
22. $9x^2 - 17 = 4x^2 - 12$.
23. $29x^2 - 30 = 10x^2 + 46$.
24. $40x^2 - 43 = 7 - 10x^2$.
25. $7x^2 - 5 = 4x^2 + 7$.
26. $3(x^2 - x) = 2x^2 - 3x + 1$.

27. In the equation $16t^2 = 256$, t is the number of seconds required for a body to fall 256 ft. How many seconds is this?

28. The area inclosed by a circle is πr^2. Using 3.1416 as the approximate value of π, find the radius of a circle whose area is 12.5664 sq. in.

29. Find the radius of a circle whose area is 3.1416 sq. ft.

30. Find the diameter of a circle whose area is 28.2744 sq. yd.

31. If a stands for the area of a circle of radius r and π be taken as $\frac{22}{7}$, $a = \frac{22}{7}r^2$. The area of a certain circle is $\frac{22}{7}$ sq. ft. Find the length of the radius in feet.

32. Find the radius of a circle whose area is $\frac{22}{63}$ sq. in.; $\frac{22}{175}$ sq. in.; $\frac{22}{112}$ sq. yd.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What are *equivalent equations?* Sec. 172.

2. What is a *quadratic equation?* Sec. 174.

3. State the *general form* of quadratic equations. Sec. 176.

4. What is an *incomplete quadratic equation?* Illustrate its two forms. Sec. 178.

5. How may each kind of incomplete quadratic equations be solved? Sec. 179.

HISTORICAL NOTE

We have seen that the solution of equations and the factoring of algebraic expressions are closely related, and we have found it advantageous to employ factoring in solving equations. If Diophantos had known this method, he would have found two roots of the quadratic equation instead of stopping with one. For example, he showed in his work *Arithmetica* that $(x - 1)(x - 2) = x^2 - 3x + 2$, and could find one root of equations like $x^2 - 3x + 2 = 0$, but he did not discover that this equation is the same as $(x - 1)(x - 2) = 0$, which evidently has two roots, 1 and 2.

It is strange that the application of factoring to solving equations was so long overlooked, for it remained unknown long after other more obscure processes were discovered. To Thomas Harriot, an English algebraist of the seventeenth century, belongs the credit of first reducing an equation by factoring. The English were proud to boast of Harriot's knowledge of mathematics, and Sir Walter Raleigh sent him to Virginia to survey the new colonial territory. He afterward returned to England and made other improvements in algebra, among them the use of small letters in place of capital letters to represent numbers; but Harriot's ignorance of negative number prevented him from applying the factoring process to any but particular cases of equations.

CHAPTER XIV

FACTORS AND MULTIPLES

180. Integral Expressions. Algebraic expressions which do not contain fractions are called integral expressions, but this classification is generally understood to refer to the letters involved.

Thus, a, $2\,ab$, a^2b, $a + x$, $(2\,x - y)^2$, are integral expressions.

And, $\frac{1}{2}\,a$, $\frac{1}{a}$, $\frac{b}{3\,a}$, $\frac{2\,a - b}{c}$, $\frac{c - 3\,b}{4}$, are fractional expressions, although $\frac{1}{2}\,a$ and $\frac{c - 3\,b}{4}$ are integral with respect to the letters involved.

Equations are called "integral" whenever the fractions involved are confined to the coefficients.

ORAL EXERCISES

Select the fractional expressions from the following:

1. $a + \frac{1}{2}\,b.$
2. $x + \frac{1}{b}.$
3. $3\,x + \frac{2}{c}.$
4. $\frac{a + b + c}{5}.$
5. $\frac{1}{a}(b + c).$
6. $\frac{x - y + z}{2\,a}.$
7. $\frac{2\,y}{3} + q.$
8. $\frac{4\,x + 8\,y}{4}.$
9. $\frac{2\,a}{2\,c} - \frac{3\,b}{5\,d}.$
10. $a - \frac{1}{3}\,x + \frac{4}{3}\,y.$

11. In Exercises 1–10, select the expressions that are fractional with respect to the coefficients only.

12. From the same Exercises, select the expressions that are fractional with respect to the letters only.

13. Select all the fractional expressions after all possible reductions have been made.

181. Common Factor. An expression that is a factor of each of two or more expressions is called a common factor of the expressions.

182. This chapter is concerned with integral factors only.

183. Algebraic expressions which have no common literal factors are said to be algebraically prime to each other.

184. PREPARATORY.

1. What is the degree of $2a^2$? Of $3a^2b$? Of $(a+x)^2$? (See Sec. 33, 34.)

2. What is the degree of $3a^2b$ with respect to a? With respect to b? With respect to ab?

185. Highest Common Factor. The highest common factor (h. c. f.) of two or more algebraic expressions is the algebraic expression of highest degree that is an exact divisor of each expression, including both the numerical and literal parts.

For example, to find the h. c. f. of $3a^2b$, $-6ab^2$, and $9abc$:

The literal common factor of highest degree is ab. The greatest common divisor (g. c. d.) of 3, -6, and 9 is 3. Hence the h. c. f. of $3a^2b$, $-6ab^2$, and $9abc$ is $3ab$.

Although -3 is also a common divisor of 3, -6, and 9, it is customary to take the g. c. d. with the positive sign.

If the given expressions are factored so as to have the h. c. f. as one factor, the set of second factors will have no further common factor, other than unity.

186. *In the case of monomials*, the h. c. f. is seen by inspection. Its coefficient is the g. c. d. of the given numerical coefficients, and its literal part is the product of all the different letters, each with the lowest exponent that it has in any of the monomials.

187. *If expressions not monomials* are given, they must first be factored if possible, after which the h. c. f. can usually be seen.

1. Find the h. c. f. of ab^2+abc, and b^2c+bc^2:

Factoring,
$$ab^2+abc=ab(b+c).$$
$$b^2c+bc^2=bc(b+c).$$

The h. c. f. is the product of the common factors b and $b+c$, or $b(b+c)$.

2. Find the h. c. f. of $(1-x)^2$, x^2-1, x^2-2x+1.

Factoring,

$$(1-x)^2=(1-x)(1-x).$$
$$x^2-1=(x-1)(x+1).$$
$$x^2-2x+1=(x-1)(x-1).$$

The h. c. f. is $1-x$ or $x-1$, because these factors differ only in sign and the h. c. f. may have either sign.

188. The h. c. f. of expressions not readily factored can be found by the long division process, but no problems will be given in this chapter requiring the general method.

189. To find the h. c. f.: *Factor each expression into its prime factors. Then find the product of all the common prime factors, using each the least number of times it occurs in any of the given expressions.*

WRITTEN EXERCISES

Find the h. c. f. of:

1. $3xy^3$, x^2y.
2. x^4y^3, x^3y, xy^3.
3. $10a^2$, $15a^3$, $5a$.
4. x^2+xy, $(x+y)^3$.
5. $3a^2+3ab$, a^2-b^2.
6. ax^4-ay^4, x^2-y^2.
7. $3x^4$, $2x^3$, $4x^5$, x^3.
8. $3a^2bc^3$, $15ab^3c^2$, $10a^2b^2$.
9. $4a^3b^2c$, $8ab^5c^2$, $12axybc^2$.
10. $10a^2b^4c$, $15ab^3c^2$, $20a^2c^3$.
11. a^4+4, a^2-2a+2.
12. a^3-1, a^2+a+1.
13. $3ax^2$, $-2a^2x$, a^2x^2, $-3abx$.
14. $3a^3+2a^2b-5ab^2$, $2a^2b+2ab^2$.
15. x^2-a^2, $x^2-2ax+a^2$.
16. $a^2x^2-8ax+16$, $5ax-20$.
17. $ax+ay+bx+by$, $x^2+2xy+y^2$.
18. $a^2-b^2x^2$, $a^2-2abx+b^2x^2$.
19. x^2+2x+1, x^3+x^2+x+1.
20. $6x^2+19x+10$, $3x^2-13x-10$.
21. $(a+b)^3-1$, $a+(b-1)$.
22. $8x^3-1$, $4x^2-4x+1$.
23. $x^2-8x+15$, x^2-5x.

24. $x^2+8x+15,\ x^2+3x.$

25. $3x^2+x-2,\ 3x^2-2x.$

26. $a^3-x^3,\ a^3-3a^2x+3ax^2-x^3.$

27. $x^2+5x+6,\ x^2-4.$

28. $x^2-2xy-3y^2,\ 2x-6y.$

29. $a^2-a-6,\ a^3-9a^2+27a-27.$

30. $a^2-16,\ 16-8a+a^2.$

31. $m^2-2mr+r^2,\ m^3-r^3.$

32. $a^2+2ap+p^2,\ ab+bp.$

33. $d^2-9,\ d^2+6d+9.$

34. $(a-b)^3,\ a^2-b^2.$ 35. $a^3+x^3,\ (a+x)^2.$ 36. $a^2-ay,\ a^4-y^4.$

190. Multiples. A product is called a **multiple** of any of its factors.

Thus, abc is a multiple of a, of b, of c, of ab, of bc, and of ac.
Also, x^4 is a multiple of x, x^2, x^3, and x^4.

191. Common Multiple. An expression that is a multiple of two or more expressions is called a **common multiple** of these expressions.

Thus, $12x^3y^2$ is a common multiple of $3xy$ and $6x^3$.

192. Lowest Common Multiple. The lowest common multiple (l. c. m.) of two or more algebraic expressions is the algebraic expression of lowest degree that is divisible, without a remainder, by each of the given expressions, including both the numerical and literal parts.

For example, find the l. c. m. of $8a^2b$, $6abc$, and $-4ac^3$:

The literal common multiple of lowest degree is a^2bc^3. The least common multiple of 8, 6, and -4 is 24.

Thus, the lowest common multiple of $8a^2b$, $6abc$, and $-4ac^3$ is $24a^2bc^3$.

Although -24 is a common multiple of 8, 6, and -4, it is customary to take the lowest common multiple with the positive sign.

If the least common multiple is divided by each of the given expressions, the quotients will have no common divisor, other than unity.

193. *In the case of monomials,* the lowest common multiple is seen by inspection. Its numerical coefficient is the least common multiple of the given coefficients, and its literal part is the product of all the different letters, each with the highest exponent that it has in any of the given expressions.

194. *If expressions not monomials* are given, they must first be factored if possible, after which the factors of the lowest common multiple may be seen.

1. Find the l. c. m. of ax, $ac + ab$, and $cx^2 + bx^2$.

1. $ax = ax$.
2. $ac + ab = a(b + c)$.
3. $cx^2 + bx^2 = x^2(b + c)$.
4. $\therefore$ the l. c. m. is the product of a, x^2, and $b + c$, or $ax^2(b + c)$.

195. To find the l. c. m.: *Factor each expression into its prime factors. Then find the product of all the different prime factors, using each the greatest number of times it occurs in any of the given expressions.*

WRITTEN EXERCISES

Find the l. c. m. of:

1. $4\,ab$, $6\,ac$.
2. $5\,a^2$, $10\,ax$.
3. $6\,pr$, $9\,pq$.
4. $7\,x^2$, $3\,xy$.
5. xyz, yzw.
6. abc^2, a^2b^2c.
7. $x + y$, $ax + ay$.
8. $a^2 + ac$, $ab + bc$.
9. $bcx + bcy$, abc.
10. $13\,a^2 - 13\,b^2$, $39\,ab$.
11. $ax + xy$, $abc + bcy$.
12. pq, $apq - bpq$.
13. $a(b - c)$, $xb - xc$.
14. $17\,x^2$, $51\,y^2$, $17\,a^2$.
15. $(a + b)n$, $(a + b)r$.
16. $3\,a^2bc$, $5\,a^3b^2$, $15\,a^2b^3c$.
17. $(t - u)x$, $(t - u)xyz$.
18. $8\,xyz^2$, $24\,x^2y^2z$, $6\,xy^2z^2$.
19. $ax + x^2$, $ax - x^2$.
20. $a^3 + b^3$, $a^2 - b^2$.
21. $a - b$, $a + b$, $a^2 - b^2$.
22. $3\,r + 2$, $9\,r + 6$.
23. $x^2 - y^2$, $(x + y)^2$.
24. $(a - b)(b - c)$, $a^2 - b^2$.
25. $2(a - b)$, $2(a + b)$, $a^2 + b^2$.
26. $x^2 + y^2$, $x^2 - y^2$.
27. $1 - x$, $1 - x^2$.
28. $x^2 - 3\,x + 2$, $x - 2$.

29. $c+d,\ 2a-3b$.

30. $x-1,\ x+1,\ x^2-1$.

31. $a^3-b^3,\ a^2+ab+b^2$.

32. $x^3+y^3,\ x^2-xy+y^2$.

33. $(x-y)^3,\ x^2+2xy+y^2$.

34. $c(a-b),\ a(b-a)$.

35. $(b-c)(c-a),\ (c-a)(a-b),\ (a-b)(b-c)$.

REVIEW

WRITTEN EXERCISES

Cancel the h. c. f. from the numerator and denominator of each fraction:

1. $\dfrac{(a^2-b^2)x^2-2ax+1}{ax-bx-1}$.

2. $\dfrac{a^3-2a^2-a+2}{a^2-1}$.

3. $\dfrac{x^4-10x^2+9}{(x-1)(x-3)}$.

4. $\dfrac{y^4-5y^2+4}{y^2+y-2}$.

5. By factoring the expressions find the highest common factor of $x^4+x^2y^2+y^4$ and x^3+y^3.

6. By factoring the expressions find the lowest common multiple of x^2-3x+2, x^2-1, and x^2+2x+1.

7. By factoring the expressions find the highest common factor of $(2x-1)(x^3-1)$ and $(x^3+x^2+x)(x-1)(x^2-1)$.

8. By factoring the expressions find the highest common factor and the lowest common multiple of the two expressions:

$$(x^2-1)(x^2+5x+6),\ (x^2+3x)(x^2-x-6).$$

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define *an integral expression.* Sec. 180.

2. Define *common factor;* also *highest common factor.* Secs. 181, 185.

3. What is meant by *algebraically prime?* Sec. 183.

4. Define *multiple;* also *common multiple;* also *lowest common multiple.* Secs. 190–192.

5. State how to find the *h. c. f.* Sec. 189.

6. State how to find the *l. c. m.* Sec. 195.

CHAPTER XV

FRACTIONS

DEFINITIONS AND LAWS

196. Meaning of Fraction. In arithmetic the fraction $\frac{4}{12}$ is taken to mean either 4 of the 12 equal parts of a unit, the quotient of $4 \div 12$, or the ratio of 4 to 12.

All three questions:

What part of 12 *is* 4?
What is the quotient of $4 \div 12$?
What is the ratio of 4 *to* 12?

are answered by one fraction, $\frac{4}{12}$.

197. Fractions in Algebra. In algebra, similarly, the symbol $\frac{a}{b}$ stands for a of the b equal parts of a unit, or for the quotient of $a \div b$, or for the ratio of a to b; but it is usually regarded as an indicated division.

Symbols like $\frac{a}{b}$ and $\frac{a+x}{4q+p^2}$ are therefore usually read "a divided by b," and "$a+x$ divided by $4q+p^2$"; but for brevity, they may be read "a over b," and "$a+x$ over $4q+p^2$."

198. The dividend and the divisor of the indicated division are called the **numerator** and the **denominator** of the fraction; together they are called the **terms** of the fraction.

199. A fraction is said to be in its **lowest terms** when its numerator and denominator have no common factor.

200. The Sign of a Fraction. Every fraction, taken as a whole, has a sign before it, expressed or understood, in addition to the signs that the numerator and the denominator may contain.

201. The Law of Signs in Fractions. The sign of the quotient is changed if the sign of either the divisor or the dividend is changed (Sec. 77); hence,

To change the sign of either the numerator or the denominator is equivalent to changing the sign of the fraction.

Thus, if $\frac{2}{3}$ is changed to $\frac{-2}{3}$, the latter is the same as $-\frac{2}{3}$.

Similarly, if $\frac{2}{3}$ is changed to $\frac{2}{-3}$, the latter is the same as $-\frac{2}{3}$.

If the signs of both numerator and denominator are changed, the value of the fraction is unchanged.

Thus, $\frac{-a}{-b}=\frac{a}{b}$, also $\frac{b(-a)}{d(-c)}=\frac{-b(-a)}{-d(-c)}=\frac{a\cdot b}{c\cdot d}$.

ORAL EXERCISES

State expressions equal to these, and having no negative signs in the numerator or denominator of the fractions:

1. $\frac{-3}{5}$. 2. $\frac{7}{-9}$. 3. $\frac{-4}{-7}$. 4. $\frac{-a}{x}$.

5. $\frac{4}{-d}$. 9. $\frac{(-2)(3x)}{(-a)(-b)}$. 13. $\frac{3a(-2b)}{5t}$.

6. $\frac{-gt}{3}$. 10. $\frac{-4}{(-3)5}$. 14. $\frac{4h}{(-3)(-5x)}$.

7. $\frac{1}{-am}$. 11. $\frac{-m}{p(-q)}$. 15. $\frac{(-2)(-3x)}{(-a)(-b)}$.

8. $\frac{(-a)(-b)}{c}$. 12. $\frac{2(-3x)}{a(-b)}$. 16. $\frac{-8x}{5(-3y}$.

Express with denominator $x-y$:

17. $\frac{-7}{y-x}$. 19. $\frac{-a}{y-x}$. 21. $\frac{a-b}{y-x}$.

18. $\frac{(-3)(-m)}{y-x}$. 20. $\frac{(-2a)(-5b)}{y-x}$. 22. $\frac{36(a-b)}{y-x}$.

Express with numerator $m-r$:

23. $\frac{r-m}{2}$. 24. $\frac{r-m}{2x+3y}$. 25. $\frac{r-m}{-3q}$. 26. $\frac{r-m}{(-5x)(3y)}$.

27. How are the fractions $\frac{a+b}{a-b}$, $\frac{a+b}{b-a}$ related?

Compare similarly:

28. $\frac{b-a}{-2}$ and $\frac{a-b}{2}$; $\frac{b-a}{-c-d}$ and $\frac{a-b}{c+d}$.

29. $\frac{a(-b)(-c)}{ef}$ and $\frac{abc}{ef}$; $\frac{(-a)(-b)(-c)}{ef}$ and $\frac{abc}{ef}$.

WRITTEN EXERCISES

For each of the following write an equal fraction preceded by the sign + and having the same denominator as the original fraction:

1. $-\frac{-x}{y}$.
2. $-\frac{x}{-y}$.
3. $-\frac{5x}{3q}$.
4. $-\frac{-3m}{8}$.
5. $-\frac{1}{a}$.
6. $-\frac{-1}{b}$.
7. $-\frac{a-b}{2m}$.
8. $-\frac{a-b}{a+b}$.
9. $-\frac{3a+2b}{4x+5}$.
10. $-\frac{5m-1}{3+4x}$.
11. $-\frac{2-x^2}{x^2-3}$.
12. $-\frac{t^3+5}{1-t}$.

13. For each of the fractions in Exercises 1–12 write an equal fraction preceded by the sign + and having the same numerator as the given fraction.

202. If the numerator of a fraction is of the same degree as the denominator, or of higher degree, the fraction may be reduced to an integral expression or to an integral expression and a fraction (mixed expression).

To do this, *divide the numerator of the given fraction by the denominator.*

For example:

$$\frac{x^2+1}{x^2+x+1}=1-\frac{x}{x^2+x+1}.$$

$$\frac{20x^2-5x+3}{10x}=\frac{20x^2}{10x}-\frac{5x}{10x}+\frac{3}{10x}=2x-\frac{1}{2}+\frac{3}{10x}.$$

WRITTEN EXERCISES

Reduce to integral or mixed forms:

1. $\frac{at + at^2}{a}$.
2. $\frac{w - w^2}{w}$.
3. $\frac{2kt - k}{k}$.
4. $p\left(\frac{100 + a}{100}\right)$.
5. $\frac{(t - tr)^2}{t^2}$.
6. $\frac{s(p+u)^2}{pu}$.
7. $\frac{25a^2b^2}{5ab}$.
8. $\frac{375x^3y^2}{25x^2y}$.
9. $\frac{a^2 + b^2}{a + b}$.
10. $\frac{x^2 + a^2}{x - a}$.
11. $\frac{a^2 - 3a}{a - 2}$.
12. $\frac{36xy + 5}{9x}$.
13. $\frac{2a^3 + 3a^2 + 1}{a}$.
14. $\frac{x^2 + y + 3}{x^2}$.
15. $\frac{4x^4 + 10x^2 + 5}{2x^2}$.

203. Mixed expressions are changed to the fractional form by reversing the process of Section 202. That is,

Multiply the integral part by the denominator of the fraction and add the product to the numerator of the fraction. The sum is the numerator of the result.

For example:

$$x^2 + x + \frac{2}{x} = \frac{x(x^2 + x) + 2}{x} = \frac{x^3 + x^2 + 2}{x}.$$

WRITTEN EXERCISES

Change each expression to a fraction:

1. $a + 2b + \frac{c}{d}$.
2. $x + \frac{2y}{3z}$.
3. $a^2 + \frac{3b}{5a^2}$.
4. $x^2 + 2xy + \frac{3}{y}$.
5. $x - 1 + \frac{5}{x+1}$.
6. $17x^4 + 3x^3 + \frac{1}{x^2}$.
7. $a^2 + 2ab + b^2 + \frac{c}{b}$.
8. $x^3 + x^2 + \frac{1}{x+1}$.
9. $x^2 + 2 - \frac{1}{x^2 - 2}$.
10. $ay^2 - 3 - \frac{y - 4}{y^2}$.
11. $p^2 - \frac{su}{pv}$.
12. $1 - \frac{w^2}{x}$.
13. $p + \frac{pr}{100}$.
14. $vt + \frac{1}{v}$.
15. $3x - \frac{4x^2 - 2x - 5}{x - 1}$.

204. Principle of Reduction. Applying Section 128, *the value of a fraction is unchanged if both numerator and denominator are multiplied or divided by the same number.*

For example:

$$\frac{6}{2} = 3 \text{ and } \frac{2 \cdot 6}{2 \cdot 2} = \frac{12}{4} = 3.$$

$$\frac{24}{6} = 4 \text{ and } \frac{24 \div 3}{6 \div 3} = \frac{8}{2} = 4.$$

$$\frac{b}{c} = \frac{ab}{ac}; \text{ also } \frac{ax}{ay} = \frac{x}{y}.$$

205. To Reduce Fractions to Lowest Terms, *divide both numerator and denominator by all factors common to them, or by the h.c.f. of the numerator and denominator.*

The division may be indicated by canceling.

For example, $\frac{x(\cancel{a+x})}{2y(\cancel{a+x})}$ is reduced to $\frac{x}{2y}$ by dividing both numerator and denominator by $a + x$, their only common factor.

206. Law of Exponents. The law of exponents in division (Sec. 129) applies to fractions.

For example:

$$\frac{a^2}{a^5} = \frac{a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{a^2}{a^2 \cdot a^3} = \frac{1}{a^3}; \text{ also } \frac{a^4b}{a^7c} = \frac{a^4b}{a^4a^3c} = \frac{b}{a^3c}.$$

$$\frac{x^4}{x^3} = \frac{x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \frac{x^3 \cdot x}{x^3} = x; \text{ also } \frac{14\,a^3b^2c}{7\,a^2bc} = \frac{2 \cdot 7\,a^2bcab}{7\,a^2bc} = 2\,ab.$$

$\frac{a^m}{a^r}$ means m factors, each a, in the numerator, and r factors, each a, in the denominator. They can be canceled from both numerator and denominator, one by one, until they are exhausted in either the numerator or the denominator.

ORAL EXERCISES

Reduce to lowest terms, and express without negative signs in the numerator or the denominator of the fraction itself:

1. $\frac{a^2}{a^5}$.
2. $\frac{3\,b^5}{4\,b^9}$.
3. $\frac{-x^7}{x^{13}}$.
4. $\frac{m^5}{m^2}$.
5. $\frac{-y^4}{2\,y^3}$.
6. $\frac{-t^5}{-t^3}$.
7. $\frac{-12\,p^3}{16\,p^5}$.
8. $\frac{20\,r^3}{-15\,r}$.

9. $-\dfrac{3c^3}{-2c^4}$.

10. $-\dfrac{-a}{-a^4}$.

11. $-\dfrac{-g^7}{4g^3}$.

12. $-\dfrac{3a^2}{15a^8}$.

13. $\dfrac{-x^{7n}}{x^{10n}}$.

14. $\dfrac{a^{4n}}{a^{10n}}$.

15. $\dfrac{10^8}{10^{11}}$.

16. $\dfrac{2^{4a}}{-2^{7a}}$.

WRITTEN EXERCISES

Reduce to lowest terms and express without using negative signs in the numerator or denominator of the fraction:

1. $\dfrac{-a^2bx}{-ab^2x^5}$.

2. $\dfrac{-a^2bc^n}{a^2c^2}$.

3. $-\dfrac{3ab^2c^2}{-6a^3bc^3}$.

4. $\dfrac{2^7a^{3p}}{2^4a^{2p}}$.

5. $\dfrac{m^{p+1}qv^2}{2qm}$.

6. $\dfrac{\frac{1}{2}mv^2}{m}$.

7. $\dfrac{mas}{am}$.

8. $\dfrac{-a^{3n}b^{7q}}{a^{6n}b^{5q}}$.

9. $\dfrac{m^pat}{m^n}$.

10. $\dfrac{-12x^2y^2z^{10}}{24x^4y^3z^5}$.

11. $\dfrac{a(-b^2)(-c^3)}{3a^2b(-c^2)}$.

12. $\dfrac{x^{n-1}y^{2n+2}}{x^{n-1}y^{n+2}}$.

207. Sometimes the common factors are made more apparent by factoring either the numerator or the denominator, or both.

For example:

$$\frac{ax-ay}{cx-cy}=\frac{a(x-y)}{c(x-y)}=\frac{a}{c}.$$

$$\frac{a^2-m^2}{4(a-m)}=\frac{(a+m)(a-m)}{4(a-m)}=\frac{a+m}{4}.$$

$$\frac{-5x-10a}{x^2+4ax+4a^2}=\frac{-5(x+2a)}{(x+2a)^2}=\frac{-5}{x+2a}=-\frac{5}{x+2a}.$$

WRITTEN EXERCISES

Reduce to lowest terms:

1. $\dfrac{a^2-b^2}{b-a}$.

2. $\dfrac{3x-3y}{7x-7y}$.

3. $\dfrac{ab-ac}{3b-3c}$.

4. $\dfrac{a^2-x^2}{4a+4x}$.

5. $\dfrac{1-y^2}{a+ay}$.

6. $\dfrac{a-b}{b-a}$.

7. $\frac{a-t}{a^2-t^2}$.

8. $\frac{a^2+ax}{ab+bx}$.

9. $\frac{x^2-2x+1}{x^2-x}$.

10. $\frac{a^2+2ap+p^2}{ab+bp}$.

11. $\frac{x^2-y^2}{y-x}$.

12. $\frac{d^2-9}{d^2+6d+9}$.

13. $\frac{(a^2-x^2)z}{(a-x)2y}$.

14. $\frac{a^2+b^2}{a^4-b^4}$.

15. $\frac{m^2-2mr+r^2}{m^3-r^3}$.

208. When several fractions have the same denominator, that denominator is called their **common denominator.** The common denominator must evidently be a multiple of the given denominators.

209. When the common denominator of several fractions is the l. c. m. of their denominators it is called the **lowest common denominator** (l. c. d.) of the given fractions.

210. *To reduce fractions to their lowest common denominator:*

Find the l. c. m. of their denominators for the l. c. d.

Divide the l. c. d. by the denominator of each fraction and multiply both terms of each fraction by the corresponding quotient.

WRITTEN EXERCISES

Change to fractions having the l. c. d.:

1. $\frac{2}{3x}, \frac{3}{4x^2}, \frac{1}{6x^2}$.

2. $\frac{a}{b}, \frac{b}{3c}, \frac{c}{2d}$.

3. $\frac{a}{2bx}, \frac{c}{abxy}, \frac{b}{3acx}$.

4. $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$.

5. $\frac{x^2}{2ab}, \frac{y^2}{3ac}, \frac{z^2}{4bc}$.

6. $\frac{a}{1-x}, \frac{1}{1-x^2}$.

7. $\frac{c}{a-c}, \frac{a}{a+c}$.

8. $\frac{4}{ax+x^2}, \frac{2}{ax-x^2}$.

9. $\frac{a+x}{a-x}, \frac{a-x}{a+x}$.

10. $\frac{4x^2}{3(a+b)}, \frac{xy}{6(a^2-b^2)}$.

11. $\frac{x^2}{a^2+b^2}, \frac{y^2}{a^2-b^2}$.

12. $\frac{1}{a-b}, \frac{1}{a+b}, \frac{abc}{a^2-b^2}$.

13. $\frac{3}{8(1-x)}, \frac{1}{8(1+x)}, \frac{x-1}{4(1+x^2)}$.

14. $\frac{1}{4a^3(a+b)}, \frac{1}{4a^3(a-b)}, \frac{1}{2a^2(a^2-b^2)}$.

15. $\frac{1}{2(x-y)}, \frac{1}{4(x+y)}, \frac{1}{6(x-y)^2}$.

16. $\frac{x}{(x-y)(y-z)}, \frac{y}{(x-y)(x-z)}, \frac{z}{(y-z)(x-z)}$.

ADDITION AND SUBTRACTION OF FRACTIONS

211. PREPARATORY.

1. What is the sum of $\frac{2}{3}$ and $\frac{1}{6}$? Of $\frac{1}{3}$ and $\frac{4}{5}$?

2. How must fractions be expressed before being added or subtracted?

3. What is the sum of $\frac{a}{b}$ and $\frac{c}{b}$? Of $\frac{x}{abc}$ and $\frac{y^2}{abc}$?

4. Subtract $\frac{ax}{bc}$ from $\frac{ay}{bc}$. Also, $\frac{m^2}{pq}$ from $\frac{n^2}{pq}$.

212. To Add or Subtract Fractions. 1. *Find the l.c.d. of the given fractions. This is the denominator of the result.*

2. *Reduce the given fractions to fractions having the l.c.d.*

3. *Find the algebraic sum of the numerators of the fractions resulting from step 2. This is the numerator of the result.*

4. *Reduce the results to lowest terms.*

EXAMPLES

1. Add $\frac{a}{a-x}$ and $\frac{3a}{a+x}$.

1. The l. c. d. is $(a-x)(a+x)=a^2-x^2$.

2. $\frac{a}{a-x}=\frac{a(a+x)}{a^2-x^2}=\frac{a^2+ax}{a^2-x^2}$.

3. $\frac{3a}{a+x}=\frac{3a(a-x)}{a^2-x^2}=\frac{3a^2-3ax}{a^2-x^2}$.

4. $\therefore \frac{a}{a-x}+\frac{3a}{a+x}=\frac{a^2+ax}{a^2-x^2}+\frac{3a^2-3ax}{a^2-x^2}=\frac{4a^2-2ax}{a^2-x^2}$.

2. Add $\dfrac{a}{b(c-a)(b-c)}$, $-\dfrac{b}{c(a-b)(a-c)}$, $-\dfrac{c}{a(a-b)(c-b)}$.

1. The l. c. m. is $abc\,(a-b)(b-c)(c-a)$.

Before dividing the l. c. m. by the given denominators we notice that $a-c$ in the second denominator is the same as $c-a$ in the l. c. m. with the signs changed. Also $c-b$ in the third denominator is the same as $b-c$ in the l. c. m. with the signs changed. Hence, the signs of each of these factors may be changed and by changing the signs before the second and third fractions, the denominators of the three fractions are now, $b(c-a)(b-c)$, $c(a-b)(c-a)$, and $a(a-b)(b-c)$, and the given fractions are all positive.

2. Dividing the l. c. m. by each denominator in turn we have $ac(a-b)$, $ab(b-c)$, and $bc(c-a)$.

3. Multiplying the results in step 2 by the numerators of the corresponding fractions and placing the sum over the l. c. m. we have the final sum:

$$\frac{a^2c\,(a-b)+ab^2\,(b-c)+bc^2\,(c-a)}{abc\,(a-b)(b-c)(c-a)}.$$

If the numerator and denominator of the sum have a common factor, it should be canceled so that the result will be in its lowest terms.

TEST: Let $a=1$, $b=2$, $c=3$. Then the given fractions become

$$\frac{1}{2\cdot 2(-1)}-\frac{2}{3(-1)(-2)}-\frac{3}{1(-1)(+1)}=-\frac{1}{4}-\frac{1}{3}+3=\frac{29}{12}.$$

The result becomes $\dfrac{3(-1)+4(-1)+18(+2)}{6(-1)(-1)(+2)}=\dfrac{29}{12}$.

These being equal, we are reasonably sure that the work is correct.

If in checking the work in fractions with arbitrary values, the numbers chosen make any denominator zero, other values must be used.

WRITTEN EXERCISES

Add:

1. $\dfrac{a}{bx}, \dfrac{b}{x}$.
2. $\dfrac{a}{bc}, \dfrac{c}{xy}$.
3. $\dfrac{m}{n}, \dfrac{p}{q}$.
4. $\dfrac{3p}{4q}, \dfrac{p}{2q}$.
5. $\dfrac{1}{6z}, \dfrac{5}{12z}$.
6. $\dfrac{a}{x}, \dfrac{b}{y}$.
7. $\dfrac{1}{x^2}, \dfrac{5}{xy}$.
8. $\dfrac{b}{a}, \dfrac{w}{p}$.
9. $\dfrac{4}{1-x}, \dfrac{5}{1+x}$.
10. $\dfrac{2t-4}{(t+3)^2}, \dfrac{1}{t+3}$.
11. $\dfrac{c}{a}, \dfrac{c+x}{2a}$.
12. $\dfrac{a+x}{a-x}, \dfrac{3x+5a}{2(a-x)}$.

13. $\frac{1}{mn}, \frac{1}{pq}$.
14. $\frac{3}{a}, \frac{5}{a+1}$.
15. $\frac{1-p}{3v+2}, \frac{4p+5}{9v+6}$.

Subtract the second fraction from the first:

16. $\frac{3}{b}, \frac{2}{b}$.
17. $\frac{3a}{b}, \frac{2b}{a}$.
18. $\frac{1}{x}, \frac{1}{3x}$.
19. $\frac{5}{x}, \frac{a^2}{x^3}$.
20. $\frac{7}{a}, \frac{4}{ab}$.
21. $\frac{a}{b}, \frac{c}{d}$.
22. $\frac{4x}{a+5}, \frac{2x}{3(a+5)}$.
23. $\frac{7}{1-x}, \frac{5}{(1-x)^2}$.
24. $\frac{8}{m+1}, \frac{2}{m-1}$.

213. Signs before Fractions. Since the sign before the fraction relates to the fraction as a whole, the *whole* numerator is added or subtracted, as the case may be, the bar of the fraction having upon the numerator the effect of a parenthesis.

For example:

$$\frac{a}{c}-\frac{d-e}{c}=\frac{a-(d-e)}{c}=\frac{a-d+e}{c}.$$

$$\frac{a}{c}-\frac{d-e+f}{c}=\frac{a-(d-e+f)}{c}=\frac{a-d+e-f}{c}.$$

$$\frac{a}{x+1}+\frac{2a}{x-1}-\frac{5a-7ax}{x^2-1}=\frac{ax-a}{x^2-1}+\frac{2ax+2a}{x^2-1}-\frac{5a-7ax}{x^2-1}$$

$$=\frac{ax-a+(2ax+2a)-(5a-7ax)}{x^2-1}$$

$$=\frac{10ax-4a}{x^2-1}.$$

WRITTEN EXERCISES

Perform the operations indicated:

1. $\frac{5}{x}-\frac{4}{7x}-5x$.
2. $\frac{x-xy}{4}-\frac{x-xy}{2}$.
3. $\frac{x+y}{z}+\frac{x-2y}{2z}$.
4. $\frac{1}{x}-\frac{3-x}{x^2}$.
5. $\frac{x-2y}{5}+\frac{2x-y}{3}$.
6. $\frac{z}{2m^2q}+\frac{w}{2mq^2}$.
7. $\frac{3a^2b}{5x^2y}-\frac{5ab^2}{3xy}$.
8. $\frac{x}{ab}+\frac{y}{bc}-4xbc$.

9. $\frac{a}{c}-\frac{c}{d}-3ad+\frac{1}{2}c^2.$

10. $\frac{3a}{5x}-\frac{6b}{10y}+15xy.$

11. $\frac{a^2}{2xy}-\frac{4b^2+3a^2}{x^2}.$

12. $\frac{1}{a-b}+\frac{1}{a+b}.$

13. $\frac{a}{3}+\frac{a}{5}-\frac{a}{15}-5a.$

14. $\frac{a}{b^3c^2}+\frac{3b}{a^2c^3}+\frac{1}{ab^2c^2}.$

15. $\frac{a}{b}-\frac{b}{a}+\frac{b^2}{ab}+a^2b.$

16. $\frac{x^6}{x^4-a^4}-\frac{a^4x^2}{a^4+x^4}.$

17. $\frac{m}{q}+n-\frac{2n}{3q}.$

18. $\frac{x}{y}-\frac{w}{xy}-\frac{z}{wy}.$

19. $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}.$

20. $a+\frac{c}{6abxy}+\frac{b}{3acx}.$

21. $\frac{x-y}{x^2y}-\frac{x-y}{xy^2}.$

22. $\frac{a}{2b}-\frac{a-b}{2(a+b)}.$

23. $\frac{x}{2y}+\frac{x+y}{3(x-y)}.$

24. $\frac{a^2+b^2}{ab}-\frac{b^2+c^2}{bc}.$

25. $\frac{a+b}{a}+\frac{b+c}{b}+\frac{c+a}{c}.$

26. $\frac{a-b}{ab}-\frac{c-a}{ac}-\frac{b-c}{bc}.$

27. $\frac{1}{a-x}-3+\frac{2a}{(a+x)^2}.$

28. $\frac{1}{(x^2+1)^2}-\frac{x+1}{x^2+1}.$

29. $\frac{1}{2(x-1)}-\frac{1}{3(x+1)}-\frac{1}{x^2}.$

30. $\frac{2x^2-y^2}{x^2}-\frac{y^2-z^2}{y^2}-\frac{z^2-x^2}{z^2}.$

31. $\frac{1}{2(a-b)}+\frac{1}{2(a+b)}+\frac{a}{a^2+b^2}.$

32. $2-\frac{x^2-y^2}{x^2+y^2}+\frac{x^2+y^2}{x^2-y^2}.$

33. $\frac{1}{(a-b)(b-c)}+\frac{1}{a^2-b^2}.$

34. $\frac{x}{x^2-y^2}-\frac{y}{(x+y)^2}.$

35. $\frac{a}{c(a-b)}-\frac{c}{a(b-a)}.$

36. $\frac{1}{(a-b)(c-a)}-\frac{1}{(b-a)(c-b)}.$

37. $\frac{a}{(b-c)(b-a)}+\frac{b}{(c-a)(c-b)}+\frac{c}{(a-b)(a-c)}.$

38. $\frac{a}{(b-c)(c-a)}+\frac{b}{(c-a)(a-b)}+\frac{c}{(a-b)(b-c)}.$

39. $\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}.$

40. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}$.

41. $\dfrac{a(b+c)}{(a-b)(c-a)}+\dfrac{b(c+a)}{(b-c)(a-b)}+\dfrac{c(a+b)}{(c-a)(b-c)}$.

42. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$.

43. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-z)(y-x)}+\dfrac{1}{z(z-x)(z-y)}$.

214. Preparatory.

Read as the sum or difference of two fractions:

1. $\dfrac{x+y}{3}$. 2. $\dfrac{a+5}{b}$. 3. $\dfrac{x+1}{x-1}$. 4. $\dfrac{4a+x}{a+9}$.

215. Separating Fractions into Parts. It is sometimes desirable to separate fractions into two or more addends, by reversing the process used in the addition of fractions.

For example:

1. $\dfrac{a+b}{c}=\dfrac{a}{c}+\dfrac{b}{c}$.

2. $\dfrac{ax+5y}{10ay}=\dfrac{ax}{10ay}+\dfrac{5y}{10ay}=\dfrac{x}{10y}+\dfrac{1}{2a}$.

3. $\dfrac{x^3+x^2+1}{x+1}=x^2+\dfrac{1}{x+1}$.

4. $\dfrac{(a+x)^2-3(a-x)}{a^2-x^2}=\dfrac{(a+x)^2}{a^2-x^2}-\dfrac{3(a-x)}{a^2-x^2}=\dfrac{a+x}{a-x}-\dfrac{3}{a+x}$

$$=\frac{a}{a-x}+\frac{x}{a-x}-\frac{3}{a+x}.$$

WRITTEN EXERCISES

Write each fraction as the algebraic sum of two or more fractions, reduce the fractions to lowest terms, and unite those with the same denominator:

1. $\dfrac{5x+8y}{10a}$.

2. $\dfrac{6a^2-3b^2}{14ab}$.

3. $\dfrac{a^2x+4b^2y}{12ay}$.

4. $\dfrac{a+x}{y}+\dfrac{a^2b-5dx}{aby}$.

5. $\dfrac{a+7b+c}{21a^2x}$.

6. $\dfrac{3g+5h}{30(g+1)}$.

7. $\dfrac{a^2-b^2+5}{a^2-2ab+b^2}$.

8. $\dfrac{(a-1)^2-11(a+1)}{a^2-1}$.

9. $\dfrac{9y-14x}{6xy}+\dfrac{15z-2ax}{6xz}-xyz$.

10. $\dfrac{7y-3x}{3y}-\dfrac{4ay-9ax+15y}{3ay}+6a^2y^2$.

11. $\dfrac{ab^2+8b+8a}{8ab}-\dfrac{6bc-18ac-6ab}{6abc}$.

12. $\dfrac{4x^2+4x+1-(1-3x-10x^2)}{10x+5}$.

13. $\dfrac{a^3-b^3+(2a^2-7ab+5b^2)}{3(a-b)}$.

14. $\dfrac{a^2-b^2-3(a-b)}{a+b}-\dfrac{7a^2+7a-(a^2-1)}{5(a+1)}$.

15. $\dfrac{4(1+r)+3r(1-r)}{1-r^2}-\dfrac{7a(1+r)-6r(1-r)}{a-ar^2}$.

MULTIPLICATION OF FRACTIONS

216. Multiplying Fractions. *The product of two or more fractions is the product of their numerators divided by the product of their denominators.*

For example: $\dfrac{2}{3}\cdot\dfrac{5}{7}=\dfrac{2\cdot 5}{3\cdot 7}=\dfrac{10}{21}$.

$$\frac{m}{n}\cdot\frac{p}{q}=\frac{mp}{nq}, \text{ and } \frac{a}{b}\cdot\frac{c}{d}\cdot\frac{3e}{4}=\frac{3ace}{4bd}.$$

$$\frac{a}{c}\cdot\frac{-b}{d}=\frac{a\cdot -b}{c\cdot d}=\frac{-ab}{cd}=-\frac{ab}{cd}.$$

Since every integer or integral expression can be regarded as a fraction with denominator 1, the above definition of product also includes the case where one of the factors is an integral expression.

217. In simplifying the results of multiplication, canceling may be helpful.

For example:

$$\frac{4a}{7b}\cdot\frac{14b}{a^2}=\frac{4\not{a}}{\not{7}\not{b}}\cdot\frac{\overset{2}{\not{14}}\not{b}}{\not{a}a}=\frac{8}{a}.$$

If necessary, factor before canceling.

For example:

$$\frac{x^2}{34}\cdot\frac{51}{ax+x^2}=\frac{\not{x}^2}{2\cdot\not{17}}\cdot\frac{3\cdot\not{17}}{\not{x}(a+x)}=\frac{3x}{2(a+x)}.$$

$$\frac{x^2-4}{3+a}\cdot\frac{9+6a+a^2}{5x+10}=\frac{(x-2)\not{(x+2)}\not{(3+a)}^2}{5\not{(3+a)}\not{(x+2)}}$$
$$=\frac{(x-2)(3+a)}{5}.$$

Only common factors of the whole numerator and denominator may be canceled. *E.g.* in $\frac{5x-2}{5x+2}$ neither the 5 nor the x nor the 2 may be canceled; but in $\frac{5(x-2)}{5(x+2)}$ the 5 may be canceled. What is wrong with this indicated work, $\frac{\not{3}\not{a}+5b}{\not{a}-\not{3}c}$?

ORAL EXERCISES

Multiply:

1. $\frac{4b}{3x}\cdot\frac{1}{5}$.
2. $\frac{x^2}{5}\cdot\frac{x^3}{4}$.
3. $\frac{1}{a^3}\cdot\frac{1}{a^4}$.
4. $\frac{x^n}{3}\cdot\frac{x^n}{7}$.
5. $\frac{a}{4b}\cdot\frac{c}{3d}$.
6. $6a\cdot\frac{4b}{5c}$.
7. $p^3\cdot\frac{p^7}{4a}$.
8. $\frac{a^n}{b^{2n}}\cdot\frac{a^5}{b^8}$.

Multiply each of the following in turn by 12, and reduce to lowest terms:

9. $\frac{8}{3ax}$.
10. $\frac{4}{2by}$.
11. $\frac{5x}{ac}$.
12. $\frac{4y}{3x}$.
13. $\frac{2m}{6aby}$.
14. $\frac{7ax}{3b}$.
15. $\frac{8d}{12c}$.
16. $\frac{5t}{24s}$.

Multiply Exercises 9–16 above in turn by:

17. $3a$. 18. $2b$. 19. xy. 20. $4ax$. 21. $3cy$.

WRITTEN EXERCISES

Multiply:

1. $\frac{x^5}{a^4} \cdot \frac{x^7}{a^7}$.
2. $\frac{a^2b}{cd} \cdot \frac{3a^4}{c^3d}$.
3. $\frac{4x}{5a} \cdot \frac{2y}{3ab}$.
4. $\frac{a^{5n+1}}{10^{3n+2}} \cdot \frac{a^6}{10^{2n}}$.
5. $\frac{a^2b^{n+2}}{7c^3} \cdot \frac{a^{3n}b^5}{4d^{3n}}$.
6. $\frac{2^8}{y^5} \cdot \frac{x^{3n}}{y^7z^{3n}}$.

Multiply and reduce to lowest terms:

7. $\frac{7}{13a} \cdot \frac{39a^2}{49}$.
8. $\frac{6x^2}{a} \cdot \frac{a^2}{15x}$.
9. $\frac{a}{b} \cdot \frac{cb^2}{ad}$.
10. $\frac{p}{q} \cdot \frac{q^2}{3p^3}$.
11. $\frac{a}{3c} \cdot \frac{9c}{ad}$.
12. $\frac{x}{y^2} \cdot \frac{z^2}{w}$.
13. $z^2 \cdot \frac{xy^2z}{zw}$.
14. $\frac{x}{y} \cdot \frac{x+y}{5xy}$.
15. $\frac{p^2}{q^2} \cdot \frac{aq}{bp}$.
16. $\frac{m+n}{a} \cdot ab$.
17. $\frac{1}{c^2} \cdot \frac{bc}{(x+y)^2}$.
18. $\frac{x}{y} \cdot \frac{y^2}{z^2} \cdot \frac{z^3}{w^3}$.
19. $p^4q^4 \cdot \frac{ab}{p^3q^3}$.
20. $\frac{d^2c^2}{a^2b^2} \cdot \frac{ab}{cd}$.
21. $\frac{4b^2}{xy^2} \cdot xy$.
22. $\frac{mn}{3x} \cdot 3mx$.
23. $\frac{3c}{abx} \cdot ab$.
24. $\frac{a}{bx} \cdot \frac{cx}{d}$.
25. $\frac{2x}{a} \cdot \frac{3ab}{c} \cdot \frac{3ac}{2b}$.
26. $a \cdot \frac{1}{bc} \cdot \frac{c}{a}$.
27. $3bc \cdot \frac{b+c}{2bc}$.
28. $-x^2 \cdot \frac{-x}{ab}$.
29. $\frac{1}{x} \cdot \frac{x^2-xy}{6bc}$.
30. $\frac{-2a^2}{c^2} \cdot \frac{-b}{d}$.
31. $\frac{1}{a^3b^4xy^2} \cdot \frac{1}{abx^2y}$.
32. $\frac{x^3b^2z}{ab^2c} \cdot \frac{a^3b^2c}{xy^2z^4}$.
33. $\frac{3m^4n^3s}{4xy^2z^3} \cdot \frac{6x^4y^3z^2}{5m^2ns}$.
34. $\frac{ab^2}{cd^2} \cdot \frac{c^2d}{e^2f} \cdot \frac{ef}{ab}$.
35. $(x+3) \cdot \frac{x-2}{x-3}$.
36. $\frac{4x}{a+5} \cdot \frac{2(a+5)}{3x^3}$.
37. $\frac{am-bm}{c+1} \cdot \frac{2}{ar-br}$.
38. $(a^2-1) \cdot \frac{ax}{a+1}$.

39. $\frac{1}{2}xy^2\left(\frac{1}{x}+\frac{1}{y}\right)$.

40. $-\frac{3}{4}a^2b^2\left(\frac{a^2}{b^2}-\frac{a}{b}\right)$.

41. $abc^3\left(\frac{1}{a}-\frac{1}{b}\right)$.

42. $x^2y^2\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$.

43. $\frac{1}{2}xy\left(\frac{x}{y}-\frac{y}{x}\right)$.

44. $\left(x+\frac{1}{y}\right)\left(x-\frac{1}{y}\right)$.

45. $\left(\frac{m}{n}+n\right)\left(\frac{m}{n}-n\right)$.

46. $\frac{x^2-2x+1}{q^3}\cdot\frac{3q^2}{x-1}$.

47. $\frac{1+6a+9a^2}{21}\cdot\frac{7t}{2+6a}$.

48. $\frac{4d^3}{x^2-10x+25}\cdot\frac{x^2-25}{8d}$.

49. $\frac{9-a^2}{a-4}\cdot\frac{16-8a+a^2}{a^2+6a+9}$.

50. $\frac{7x+14}{1-5a}\cdot\frac{2a-3}{(x+2)^2}\cdot\frac{x+2}{6a-9}$.

51. $\frac{a}{x-1}\cdot\frac{1-x^2}{2a+ay}\cdot\frac{2+y}{1+x}$.

52. $\frac{3m^2+2m}{6m^2-11m+3}\cdot\frac{3m^2+2m-1}{3m+2}$.

Simplify, but do not multiply the factors in the terms of the final result:

53. $\frac{x^2+2y^2-2xy}{a^4-b^4}\cdot\frac{a^2-2ab+b^2}{x^4+4y^4}$.

54. $\frac{c^2-d^2}{m^3-n^3}\cdot\frac{m^2+mn+n^2}{2ad-2ac+ce-ed}$.

55. $\frac{3x^2-xy}{x^2-x-12}\cdot\frac{2x^2-7x-4}{12x^2+11xy-5y^2}$.

56. $\frac{ax+an-am}{\frac{1}{x^2}+2+\frac{1}{y^2}}\cdot\frac{\frac{1}{y}+\frac{1}{x}}{x^2-2(m-n)+(m-n)^2}$.

57. $\frac{10bx+5by-20bz}{a+b+x}\cdot\frac{a^2-x^2+b^2+2ab}{8ax+4ay-16az}$.

58. $\frac{m^4-1}{(a+b+c)^2+ax+bx+cx}\cdot\frac{a^2+b^2+c^2+2ab+2ac+2bc}{p(m+1)^2+3m+3}$.

59. $\frac{a^4+4b^4}{c+2c^2-3c^3-cd}\cdot\frac{ab+2abc-3abc^2-abd}{a^2+2b^2-2ab}$.

218. Powers of Fractions. *Any power of a fraction equals that power of the numerator divided by the same power of the denominator.*

For example:

$$\left(-\frac{2}{3}\right)^2=\frac{(-2)^2}{(3)^2}=\frac{4}{9}.$$

$$\left(\frac{a}{b}\right)^3=\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}=\frac{a^3}{b^3}.$$

$$\left(\frac{a}{b}\right)^n=\frac{a}{b}\cdot\frac{a}{b}\cdots \text{to } n \text{ factors.}$$

$$=\frac{a\cdot a\cdots \text{to } n \text{ factors}}{b\cdot b\cdots \text{to } n \text{ factors}}=\frac{a^n}{b^n}.$$

$$\left(\frac{3a^3x^2}{-5bc}\right)^3=\frac{(3a^3x^2)^3}{(-5bc)^3}=\frac{27a^9x^6}{-125b^3c^3}=-\frac{27a^9x^6}{125b^3c^3}.$$

$$\left(\frac{m+r}{m-r}\right)^2=\frac{(m+r)^2}{(m-r)^2}=\frac{m^2+2mr+r^2}{m^2-2mr+r^2}.$$

Reversing the order,

$$\frac{a^2+6a+9}{x^2-10x+25}=\frac{(a+3)^2}{(x-5)^2}=\left(\frac{a+3}{x-5}\right)^2.$$

WRITTEN EXERCISES

Write without parentheses:

1. $\left(\frac{a^3}{b^n}\right)^2$.
2. $\left(\frac{a^{4n}}{x^5}\right)^3$.
3. $\left(\frac{-2a3b^2}{5xy}\right)^3$.
4. $\left(1+\frac{x}{y}\right)\left(1+\frac{x}{y}\right)$.
5. $\left(\frac{a}{b}\right)^{10}$.
6. $\left(\frac{a}{b}\right)^n$.
7. $\left(\frac{-3abc^2}{4m^3n}\right)^4$.
8. $\left(-\frac{2x-3}{12}\right)^3$.
9. $\left(\frac{a^2}{x^3}\right)^{3p}$.
10. $\left(\frac{1}{x^2}\right)^{q+1}$.
11. $\left(\frac{2ax}{3b^2y}\right)^5$.
12. $\left(\frac{2a^2b^3}{3x^3y}\right)^4$.
13. $\left(\frac{a}{x}-\frac{b}{y}\right)^2$.
14. $\left(\frac{-abx}{c^2dy}\right)^7$.
15. $\left(1+\frac{ab}{a+b}\right)\left(1+\frac{ab}{a+b}\right)$.
16. $\left(\frac{p}{q}+\frac{q}{p}\right)\left(\frac{p}{q}+\frac{q}{p}\right)$.
17. $\left(\frac{1}{a}+\frac{1}{b}\right)^2$.
18. $\left(\frac{a+bc}{a+b}\right)^2$.

19. $\left(\frac{1}{x}+\frac{1}{y}\right)^2$.
23. $\left(\frac{1}{x}-1\right)^2$.
27. $\left(\frac{1}{x}-x\right)^2$.
20. $\left(\frac{1}{m^2}+1\right)^2$.
24. $\left(\frac{1}{x}-\frac{1}{y}\right)^2$.
28. $\left(\frac{a}{b}+1\right)^2$.
21. $\left(\frac{x}{y}+\frac{y}{x}\right)^2$.
25. $\left(\frac{x}{y}-\frac{y}{x}\right)^2$.
29. $\left(\frac{c}{2}+a\right)^2$.
22. $\left(1+\frac{1}{x}\right)^2$.
26. $\left(\frac{1}{m^2}-1\right)^2$.
30. $\left(\frac{a}{b}-c\right)^2$.

Write each of the following as a power of a fraction:

31. $\frac{x^2-2x+1}{a^2+2a+1}$.
32. $\frac{a^2+2ab+b^2}{x^4-2x^3y+x^2y^2}$.
33. $\frac{x^2+4x+4}{4x^2+4x+1}$.
34. $\frac{a^2-4ab+4b^2}{9a^2-12ab+4b^2}$.
35. $\frac{a^6b^3}{x^3+3x^2+3x+1}$.
36. $\frac{100p^4-20p^2+1}{q^4-20q^2+100}$.
37. $\frac{a^3-9a^2+27a-27}{27x^3-27x^2+9x-1}$.
38. $\frac{x^2-2x+1}{x^4+2x^3y+x^2y^2}$.

DIVISION OF FRACTIONS

219. Reciprocal. If the product of two numbers is 1, each is called the **reciprocal** of the other.

Thus,

5 and $\frac{1}{5}$ are reciprocals of each other, because $5 \cdot \frac{1}{5} = 1$.

$\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals of each other, because $\frac{a}{b} \cdot \frac{b}{a} = 1$.

220. Division of Fractions. *To divide by a fraction multiply by the reciprocal of the fraction, that is, by the fraction inverted.*

Thus, to divide $\frac{4}{5}$ by $\frac{2}{3}$ multiply $\frac{4}{5}$ by $\frac{3}{2}$,

and to divide $\frac{a}{b}$ by $\frac{x}{3y}$ multiply $\frac{a}{b}$ by $\frac{3y}{x}$.

Test. If the quotient is correct, the product of the quotient and the divisor equals the dividend.

WRITTEN EXERCISES

Divide:

1. $\frac{1}{a} \div \frac{1}{3}$.
2. $\frac{1}{a} \div \frac{3}{a}$.
3. $\frac{1}{a} \div \frac{1}{b}$.
4. $\frac{3}{a} \div \frac{5}{b}$.
5. $\frac{a}{b} \div \frac{1}{b}$.
6. $\frac{m}{n} \div \frac{a}{b}$.
7. $\frac{a}{2b} \div \frac{c}{3d}$.
8. $\frac{m}{5n} \div \frac{5m}{21n}$.
9. $\frac{p}{7q} \div \frac{3p}{14q}$.
10. $\frac{n}{m} \div \frac{p}{q}$.
11. $\frac{x}{y} \div \frac{y}{x}$.
12. $\frac{1}{3ab} \div \frac{3}{ab}$.
13. $\frac{x^2}{y} \div \frac{x}{y^2}$.
14. $\frac{mn}{pq} \div \frac{m}{p}$.
15. $\frac{mx}{ny} \div \frac{m}{n}$.
16. $\frac{a^2}{b^2} \div \frac{a}{b}$.
17. $\frac{3a^2}{2b^2} \div \frac{2a}{3b}$.
18. $\frac{5x^2}{y^2} \div \frac{x^2}{y}$.
19. $\frac{15m^2}{n^2} \div \frac{5}{n^2}$.
20. $\frac{ab}{c} \div \frac{b}{c^2}$.
21. $\frac{6a}{b} \div \frac{a}{x}$.
22. $\frac{7a^2}{5b} \div \frac{14}{5ab^2}$.
23. $\frac{y^2z}{xw} \div \frac{yz}{x}$.
24. $\frac{m^2}{xyz} \div \frac{10m}{3x}$.
25. $\frac{a}{bx} \div \frac{d}{cx} = ?$
26. $\frac{5ax}{3cy} \div \frac{5x}{cy} = ?$
27. $\frac{b}{x-y} \div \frac{a}{x+y} = ?$
28. $\frac{3bx}{2az^q} \div \frac{ax}{2z^{b+q}} = ?$
29. $\frac{ac}{bd} \div \frac{ab}{cd} = ?$
30. $\frac{4a}{x^{m+1}} \div \frac{2b}{x} = ?$
31. $\frac{3ab}{5a^2c} \div \frac{6a^2b}{5a^3c^2} = ?$
32. $\frac{4a^2b^2}{15c^2d^2} \div \frac{4ab}{3c^2d} = ?$
33. $\frac{2x}{3y} \div \frac{4x}{3y} = ?$
34. $\frac{2x^2}{yz} \div \frac{3xyc}{z^2d} = ?$
35. $\frac{3abx}{5c^2} \div \frac{6a^2x^2}{10cb^2} = ?$
36. $\frac{2y}{3x} \div \frac{a}{b} \cdot \frac{9x^2}{4y^2} = ?$
37. $\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{e} \cdot \frac{e}{f} \div \frac{a}{f} = ?$
38. $\frac{a^{2n}b^{2n}c^{2n}}{xyz} \div \frac{a^nb^nc^n}{x^2y^2z^2} = ?$
39. $\frac{x^{2p}}{y} \cdot \frac{y^2}{x^p} \div \frac{x^3}{y^3} = ?$
40. $\frac{m^2n}{p^2q} \cdot \frac{n^2p}{q^2r} \div \frac{mnp}{qr} = ?$

221. Since every integral expression can be regarded as a fraction with denominator 1, the process of division includes also the case where one of the fractions is an integral expression.

For example:

$$-\tfrac{3}{5} \div 6 = -\tfrac{3}{5} \div \tfrac{6}{1} = -\tfrac{3}{5} \cdot \tfrac{1}{6} = -\tfrac{1}{10}.$$

$$\frac{a}{-b} \div c = \frac{a}{-b} \div \frac{c}{1} = \frac{a}{-b} \cdot \frac{1}{c} = \frac{a}{-bc} \text{ or } -\frac{a}{bc}.$$ Sec. 201.

$$\frac{3a^2x}{by} \div (-3a) = \frac{3a^2x}{by} \cdot \frac{1}{-3a} = -\frac{ax}{by}.$$ Sec. 201.

$$x^2y \div \frac{-x}{z} = \frac{x^2y}{1} \cdot \frac{-z}{x} = -xyz.$$

WRITTEN EXERCISES

Perform the divisions indicated and reduce the results to lowest terms:

1. $\dfrac{3a^2x}{by} \div 2c.$
2. $\dfrac{2a}{b^2c} \div \dfrac{3a^2b}{c}.$
3. $\dfrac{-3a^2}{bx^2} \div a^2.$
4. $\dfrac{a^3+b^3}{a^2-b^2} \div \dfrac{a+b}{a-b}.$
5. $\dfrac{2ab}{3c^2} \div \dfrac{2b}{-a^3c}.$
6. $\dfrac{9x^2y}{12mn^2} \div \dfrac{3xy^2}{4m^2n}.$
7. $\dfrac{-ba^2}{7cb^2} \div \dfrac{3a}{21c^2b}.$
8. $\dfrac{-7x^3y}{12^2} \div \dfrac{7xy}{-4zw}.$
9. $\dfrac{5ab^2}{-21cd} \div \dfrac{-5a^2b}{7c^2d^2}.$
10. $\dfrac{-15mn^3}{25a^2b} \div \dfrac{3m^2n^2}{5ab}.$
11. $\dfrac{8a^2}{a^2b^2} \div \dfrac{4a}{a-b}.$
12. $\dfrac{a-x}{y} \div \dfrac{a^2-x^2}{-xy}.$
13. $\dfrac{x^3+y^3}{x(x-y)} \div \dfrac{x^2+2xy+y^2}{x^2-y^2}.$
14. $\dfrac{2a+3b}{c+d} \div \dfrac{c-d}{2a-3b}.$
15. $\dfrac{a}{a+x} \div \dfrac{a^2-a}{a+x}.$
16. $\dfrac{4x^2-9y^2}{a^2-b^2} \div \dfrac{2x+3y}{a-b}.$
17. $\dfrac{4x^2}{3(a+b)} \div \dfrac{xy}{6(a^2-b^2)}.$
18. $\dfrac{a^2}{a^2-b^2} \div \dfrac{2ab}{(a-b)^2}.$

19. $\frac{a^3-x^3}{a+x} \div \frac{a-x}{-(a+x)^2}.$

20. $\frac{x^2-b^2}{-bc} \div \frac{b+c}{-b(x+b)^2}.$

21. $\frac{2ax-x^2}{a(x+a)} \div \frac{x^2-a^2}{x+a}.$

22. $\frac{x^2-1}{x^2-3x+2} \div \frac{x-1}{x-2}.$

23. $\frac{x^2-2xy-3y^2}{x^2+2xy+y^2} \div \frac{x-3y}{x+y}.$

24. $\frac{a^2-ay}{m-n} \div \frac{a^4-y^4}{-(a-y)^2}.$

25. $\frac{3x^2-8x+4}{3x^2+2x-5} \div \frac{15x^2+11x-14}{12x^2+11x-15}.$

COMPLEX FRACTIONS

222. We have so far considered *simple* fractions, those in which neither the numerator nor the denominator is in fractional form.

Since a fraction indicates division, the division of two fractions may be indicated in fractional form.

Thus, $\frac{2}{3} \div \frac{7}{5}$ may be written $\frac{\frac{2}{3}}{\frac{7}{5}}$,

and $\frac{a}{b} \div \frac{c}{d}$ may be written $\frac{\frac{a}{b}}{\frac{c}{d}}$.

223. Complex Fractions. A fraction whose numerator or denominator, or both, are fractional expressions is called a **complex fraction.**

For example:

$$\frac{\frac{a+b}{c}}{d},\quad \frac{2}{\frac{7}{3}},\quad \frac{\frac{31}{2+a}}{\frac{x+2}{3}},\quad \frac{\frac{3x}{4y}}{\frac{7x}{12y}},$$ are complex fractions.

224. A complex fraction may be reduced to a simple fraction either by performing the indicated divisions, or by multiplying both numerator and denominator by the l. c. m. of their respective denominators. In each case any operations indicated in the numerator or the denominator should be performed first, as far as possible.

EXAMPLES

1. Simplify: $\dfrac{\dfrac{x^2-y^2}{a+b}}{\dfrac{x+y}{a^2-b^2}}$.

$$\frac{\dfrac{x^2-y^2}{a+b}}{\dfrac{x+y}{a^2-b^2}}=\frac{x^2-y^2}{a+b}\div\frac{x+y}{a^2-b^2}=\frac{x^2-y^2}{a+b}\times\frac{a^2-b^2}{x+y}=(x-y)(a-b).$$

2. Simplify: $\dfrac{\dfrac{a}{ax}+\dfrac{b}{ax}}{\dfrac{b+c}{by}}$.

$$\frac{\dfrac{a}{ax}+\dfrac{b}{ax}}{\dfrac{b+c}{by}}=\frac{\dfrac{a+b}{ax}}{\dfrac{b+c}{ay}}=\frac{\dfrac{axy(a+b)}{ax}}{\dfrac{axy(b+c)}{ay}}=\frac{y(a+b)}{x(b+c)}.$$

WRITTEN EXERCISES

Reduce to simple fractions:

1. $\dfrac{\dfrac{24x}{y}}{6x}$.

2. $\dfrac{\dfrac{4}{8}}{a^2}$.

3. $\dfrac{\dfrac{7a}{5c}}{21b}$.

4. $\dfrac{\dfrac{m^2-9}{m+3}}{a}$.

5. $\dfrac{\dfrac{1}{a-b}}{a+b}$.

6. $\dfrac{\dfrac{12ab}{35cd}}{\dfrac{25d}{16ac}}$.

7. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{x-y}$.

8. $\dfrac{\dfrac{1-49t^2}{7t+1}}{3ab}$.

9. $\dfrac{\dfrac{x^2-1}{5a}}{\dfrac{x-1}{15a^2b}}$.

10. $\dfrac{2+ax+\dfrac{1}{ax}}{a^2x^2-1}$.

11. $\dfrac{\dfrac{1}{x-y}-\dfrac{1}{x+y}}{\dfrac{y}{x-y}}$.

12. $\dfrac{\dfrac{2ax}{a+x}-a}{\dfrac{1}{x}+\dfrac{1}{a-2x}}$.

13. $\dfrac{\dfrac{a-3}{3b}-\dfrac{3b}{a-3}}{\dfrac{1}{3b}-\dfrac{1}{a-3}}.$

15. $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}.$

14. $\dfrac{\dfrac{a}{b^2}+\dfrac{b}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ab}+\dfrac{1}{b^2}}.$

16. $\dfrac{\dfrac{a^2-3ab}{x^2-1}}{\dfrac{9b^2-6ab+a^2}{1+x}}.$

17. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b}{a}-1}-\dfrac{1+\dfrac{b^2}{a^2}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b^2}{a^2}}.$

18. $\dfrac{\dfrac{ab}{c}+\dfrac{bc}{a}+\dfrac{ca}{b}}{\dfrac{a}{bc}+\dfrac{b}{ca}+\dfrac{c}{ab}}\cdot\left[\dfrac{(a+b+c)^2}{ab+bc+ca}-2\right].$

19. $\dfrac{x^4+x^3-x-1}{1-y^2}\cdot\dfrac{y^2-1}{x^2-x}\cdot\left(1-\dfrac{1}{1-\dfrac{1}{x}}\right).$

REVIEW

WRITTEN EXERCISES

Reduce to an integral or mixed expression:

1. $\dfrac{x^2+b}{x^2+c}.$ 2. $\dfrac{6x^2+5}{x+5}.$ 3. $\dfrac{x^3-a^3}{x-a}.$

Reduce each expression to a fraction:

4. $a+\dfrac{b}{c}.$ 5. $x+1-\dfrac{1}{x+1}.$ 6. $c+\dfrac{b^2}{a+b}.$

Reduce to lowest terms:

7. $\dfrac{x^2+2xy+y^2}{x^2-y^2}.$ 8. $\dfrac{9x^2-12x+4}{9x^2-4}.$

9. $\frac{a^2-4}{a^2-4a+4}$.

10. $\frac{8ax+4ay-16az}{10bx+5by-20bz}$.

11. $\frac{x^3-1}{x^2-1}$.

12. $\frac{(a+b)^2-c^2}{ax+bx+cx}$.

13. $\frac{a^2-x^2+b^2+2ab}{a+b+x}$.

14. $\frac{x^4+3x^3+x+3}{2x+6}$.

Add:

15. $\frac{a}{x}-\frac{1}{2x}$.

16. $\frac{x}{12a}-\frac{y}{4}$.

17. $\frac{a-2b}{c}+\frac{3b}{2c}$.

18. $\frac{x-2y}{8y}-\frac{2x-y}{12y}$.

19. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

20. $\frac{1}{1+x}+\frac{1}{1-x}-\frac{2}{1-x^2}$.

21. $\frac{1}{x+a}+\frac{2}{x+b}-\frac{3}{x+c}$.

22. $\frac{5x+4}{x-2}-\frac{3x-2}{x-3}-\frac{x^2-2x-17}{x^2-5x+6}$.

23. $\frac{a-b}{a^2-ab+b^2}+\frac{1}{a+b}+\frac{ab}{a^3+b^3}$.

24. $\frac{(x+y)(x^2+y^2-1)}{xy}+\frac{(y+1)(y^2-x^2+1)}{y}+\frac{(x+1)(x^2-y^2+1)}{x}$.

25. $\frac{4}{x^2-y^2}+\frac{5}{x^3-y^3}$.

26. $\frac{x}{x^4-1}-\frac{x}{x^2+1}$.

27. $\frac{a}{a^4+a^2+1}+\frac{1}{a^2+a+1}$.

28. $\frac{x-3}{x^2-7x-18}+\frac{2x-1}{x^2-8x-9}$.

Multiply:

29. $\frac{x-1}{x}\cdot\frac{x+1}{-y}$.

30. $\frac{5ax}{3cy}\cdot\frac{xy-y^2}{x^2-xy}$.

31. $\frac{ax}{(a-x)^2}\cdot\frac{a^2-x^2}{-ab}$.

32. $\frac{x^2-2xy+x^2}{x+y}\cdot\frac{1}{x^2-y^2}$.

33. $\frac{x+2}{x-1}\cdot\frac{x^2-1}{x^2-4}$.

34. $\frac{a+b}{-b}\cdot\frac{ac}{a^2-b^2}$.

35. $\dfrac{x^2-9}{x^2+4x}\cdot\dfrac{x^2-16}{x^2-3x}$.

37. $\dfrac{-4ab}{6c}\cdot\dfrac{9c^2d}{-7a^2b}\cdot\dfrac{-21c}{12abc}$.

36. $\dfrac{4x^2}{3(a+b)}\cdot\dfrac{6(a^2-b^2)}{-xy}$.

38. $\dfrac{a-b}{a^2+2ab}\cdot\dfrac{a^2-4b^2}{a^2-ab}$.

39. $\dfrac{x^3-8y^3}{x^2-y^2}\cdot\dfrac{x^2-xy-2y^2}{x^2-4xy+4y^2}$.

40. $\dfrac{4m+r}{25a^2-9}\cdot\dfrac{10am-6m}{16m^2+8mr+r^2}$.

41. $\dfrac{x^2+6x+5}{a^3+10a^2b+25ab^2}\cdot\dfrac{a^2-25b^2}{x^2+2x+1}$.

42. $\dfrac{x^2+6ax+5a^2}{x^2+2ax+a^2}\cdot\dfrac{ab+bx}{5ac+cx}$.

Divide:

43. $\dfrac{a^2}{b}\div\dfrac{a}{b^2}$.

48. $\left(\dfrac{ax}{by}\right)^2\div\left(\dfrac{cy}{bx}\right)^3$.

44. $\dfrac{ac}{bd}\div\dfrac{ad}{bc}\div\dfrac{x^2}{y}\div\dfrac{x}{y^2}$.

49. $\left(\dfrac{a-b}{x-y}\right)^3\div\left(\dfrac{a^2-b^2}{x^2-y^2}\right)^2$.

45. $\dfrac{a+b}{a-b}\div\dfrac{(a+b)^2}{a-b}$.

50. $\left(\dfrac{m-n}{m-p}\right)^4\div\left(\dfrac{n-m}{n-p}\right)^4$.

46. $\dfrac{x^2+5x+6}{x^2-1}\div\dfrac{x+5}{x+1}$.

51. $\dfrac{x}{1+x}\div\dfrac{x+x^2}{(1+x)^2}$.

47. $\dfrac{a^2bc}{x^3-1}$ by $\dfrac{a^4bc^3}{x-1}$.

52. $\dfrac{a^2-16}{p+t}$ by $\dfrac{a^2-4a}{pt+t^2}$.

53. $\dfrac{a^2-3a-28}{x^4+4y^4}$ by $\dfrac{a^2-7a}{x^2+2xy+2y^2}$.

54. $\dfrac{a^4+10a^2x^2+25x^4}{5x^2}$ by $\dfrac{a^4+7a^2x^2+10x^4}{10x^2-35x^3}$.

Simplify:

55. $1-\cfrac{1}{a-\cfrac{1}{b-\cfrac{1}{c}}}$.

56. $a-\cfrac{b}{c-\cfrac{d}{e-\cfrac{f}{g}}}$.

57. $\dfrac{\dfrac{x^2 - y^2}{a^3 + b^3}}{\dfrac{x^2 - 2\,xy + y^2}{a^2 + 2\,ab + b^2}}$.

58. $\dfrac{\dfrac{15(x - 2\,y)}{m^2 + n^2}}{\dfrac{5\,x^2 - 10\,xy}{m^4 - n^4}}$.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. State three meanings of a fraction. Sec. 196.

2. Which meaning of a fraction is commonly used in algebra? Sec. 197.

3. When is a fraction in its *lowest terms?* Sec. 199.

4. What is meant by the sign of a fraction? Sec. 200.

5. State the *Law of Signs* of fractions. Sec. 201.

6. When may the numerator be divided by the denominator? Sec. 202.

7. How may a mixed expression be changed to a fraction? Sec. 203.

8. State the principle that applies to the reduction of fractions. Sec. 204.

9. How is a fraction *reduced to its lowest terms?* Sec. 205.

10. What law of exponents applies to fractions? Sec. 206.

11. Define *common denominator;* also *lowest common denominator.* Secs. 208, 209.

12. How may fractions be changed to equivalent fractions having the lowest common denominator? Sec. 210.

13. How are fractions added? Multiplied? Divided? Secs. 212, 216, 220.

14. How is a power of a fraction obtained? Sec. 218.

15. What are *complex fractions* and how are they simplified? Secs. 223, 224.

CHAPTER XVI

EQUATIONS

PROPERTIES

225. Degree of an Equation. The degree of an equation is stated with respect to its unknowns. It is the highest degree to which the unknowns occur in any term in the equation. Unless otherwise stated, all the unknowns are considered.

For example:

1. $3x + 1 = 0$, and $4x + 7y - 3z = 6$, are equations of the first degree.
2. $4y^2 - y + 3 = 0$, and $x^2 + y^2 - 4 = 0$, are equations of the second degree.
3. $x^3 - 1 = 0$, $x^3 + 2y^2 - 4x + 3 = 0$, $5xyz + x^2 = 2y$, and $x^3 - 3y^3 + z = 5x$ are equations of the third degree.
4. $v = \frac{4\pi R^3}{3}$ is of the first degree in v and of the third degree in R.

226. An equation of the first degree is called a **linear equation.**

227. An equation of the second degree is called a **quadratic equation.**

228. An equation of the third or higher degree is called a **higher equation.**

It is unnecessary to define here the degree of expressions containing radicals.

229. In order to state the degree of an equation its terms must be united as much as possible.

Thus, $x^2 + 2x + 1 = x^2$ appears to be a quadratic equation.
But $2x + 1 = 0$, to which it reduces, is a linear equation.

230. Terms not involving the unknowns are called **absolute terms.**

Thus, in $x^2 + 5 = 3x - 2a$, 5 and $-2a$ are absolute terms.

231. Fractional Equations. An equation in which the unknown quantity occurs in any denominator is called a *fractional equation.*

Thus, $\frac{2}{x} - 3x + 1 = 0$ is a fractional equation; also $\frac{1}{x^2} - \frac{a}{x+1} = b$.

We shall not define here the degree of fractional equations. It will be shown later that the process of clearing of fractions may change the degree of an equation.

ORAL EXERCISES

State the degree of each of the following equations with respect to each unknown:

1. $x + 2 = 0$.
2. $ax + 2 = 0$.
3. $z^2 - mz = 4$.
4. $x^2 + xy = 5$.
5. $\frac{x^2 - 4}{2} = 0$.
6. $\frac{1}{2}mv^2 = 6$.
7. $gt^2 = 32$.
8. $at + \frac{1}{2}gt^2 = 0$.
9. $ax^2 + b = 0$.
10. $ax^3 - by^2 = 1$.
11. $a^2x^2 + b^2y^2 = a^2b^2$.
12. $\frac{4}{3}\pi r^3 = 100$.
13. $\frac{w^2 - 1}{4} = 6$.
14. $2\pi r^2 h = 50$.
15. $mv^2 - 16 = 0$.
16. $xy^2 - x^2y = 6$.

Select the linear, the quadratic, and the higher equations from the following:

17. $x + 3 = 0$.
18. $x^2 = 9$.
19. $x^3 = 27$.
20. $2x - 6 = 0$.
21. $ax = b + c$.
22. $mx^2 + px = q$.
23. $x^3 - x^2 - x = 3$.
24. $ax^2 + ax = c$.
25. $2x - 8x = 0$.
26. $3x^2 - x^2 + 3x = x^2 + 1$.
27. $2x^3 = 2x^2 + x + 2x^3$.
28. $x^4 + 5 = 3$.
29. $ay^3 - by^2 + 1 = 0$.
30. $5y^2 + 2y = 6 + 5y^2$.

SOLUTION OF LINEAR EQUATIONS

232. To Solve a Linear Equation with One Unknown. In general:

1. *Clear it of fractions, if there are any.*
2. *Remove the parentheses, if there are any.*
3. *Transpose all the terms containing the unknown quantity to one member, preferably the left, and all the other terms to the other member.*
4. *Unite the terms in each member as much as possible.*
5. *Divide both members by the coefficient of the unknown.*

EXAMPLE

Solve: $$\frac{2x-5}{6}+\frac{6x+3}{4}=5x-\frac{35}{2}. \quad (1)$$

Multiplying each term by 12, the l. c. m. of the denominators, $$2(2x-5)+3(6x+3)=60x-6\cdot 35. \quad (2)$$

Removing the parentheses, $$4x-10+18x+9=60x-210. \quad (3)$$

Transposing -10, 9, and $60x$, $$4x+18x-60x=10-9-210. \quad (4)$$

Uniting terms, and multiplying both members by -1, $$38x=209. \quad (5)$$

Dividing both members by 38, $$x=\tfrac{11}{2}. \quad (6)$$

TEST. $$\frac{2(\frac{11}{2})-5}{6}+\frac{6(\frac{11}{2})+3}{4}=5(\tfrac{11}{2})-\frac{35}{2}.$$

WRITTEN EXERCISES

Solve and test:

1. $2(x+3)-3(x+2)=6(x+5).$
2. $x(x-1)-(2x-1)=x(x+6).$
3. $2x-\dfrac{11+x}{2}=\dfrac{19+x}{3}.$
4. $(x+3)^2=-7+(5-x)^2.$
5. $\dfrac{z-1}{2}+\dfrac{z-2}{3}-6=\dfrac{z-3}{4}.$
6. $\dfrac{x}{4}+\dfrac{4x-12}{5}-\dfrac{9x+2}{20}=0.$

7. $\frac{y}{12}-\frac{8-y}{8}+2\frac{3}{4}=\frac{5+y}{4}.$

8. $5(x-4)+4(x-3)-x(x-1)=x(3-x).$

9. $\frac{1}{2}(x-5)-\frac{1}{3}(x-4)=\frac{1}{2}(x-3)-(x-2).$

10. $\frac{y}{2}+\frac{y}{4}-\frac{y}{6}=7.$

11. $\frac{y}{12}-\frac{3-y}{8}=\frac{5+y}{4}-2\frac{3}{4}.$

12. $\frac{x}{3}-\frac{x}{4}=\frac{1}{2}+\frac{x}{5}-\frac{x}{6}.$

13. $\frac{x-1}{2}+\frac{x-2}{3}=6+\frac{x-3}{4}.$

14. $\frac{y}{3}-\frac{y}{4}-\frac{1}{2}=\frac{3y}{4}+1.$

15. $\frac{2x}{3}+\frac{5x}{9}=\frac{x}{6}+\frac{x}{2}+10.$

16. $\frac{x}{5}-\frac{x-8}{4}=\frac{x}{20}.$

17. $\frac{7}{10}x+\frac{2}{5}(20-x)=10.$

18. $\frac{1}{3}(x-3)-\frac{1}{5}(x-5)=\frac{1}{15}(x-15)+5.$

233. A fraction that is not given in its lowest terms should first be reduced.

EXAMPLE

Solve: $$\frac{x(1+6x)}{x^2-2x}+\frac{1}{x}=6. \quad (1)$$

Simplifying the first fraction, $$\frac{1+6x}{x-2}+\frac{1}{x}=6. \quad (2)$$

Multiplying both members of (2) by the l. c. d.,

$$\frac{x(x-2)(1+6x)}{x-2}-\frac{x(2-x)}{x}=6x(x-2). \quad (3)$$

Canceling common factors in (3) from numerators and denominators, $$x(1+6x)+x-2=6x(x-2). \quad (4)$$

Removing parentheses, $$x+6x^2+x-2=6x^2-12x. \quad (5)$$

Simplifying, $$14x=2. \quad (6)$$

$$\therefore x=\tfrac{1}{7}. \quad (7)$$

TEST. $$\frac{\frac{1}{7}(1+\frac{6}{7})}{(\frac{1}{7})^2-2\cdot\frac{1}{7}}+\frac{1}{\frac{1}{7}}=\frac{\frac{1}{7}(\frac{13}{7})}{\frac{1}{7}(\frac{1}{7}-2)}+7 \;=6 \text{ as in step } (1).$$

If the given equation was cleared of fractions before reducing the first fraction an equation of higher degree would result which would not be *equivalent* (Sec. 172) to the given equation. There would be a factor x in every term, giving the value 0, which does not satisfy the given equation.

WRITTEN EXERCISES

Solve and test:

1. $\frac{x-2}{x^2-4}=\frac{2}{x-2}$.

2. $\frac{x^2-3}{x-1}=x+2$.

3. $\frac{1}{3}=x-\frac{x^2+1}{x-1}$.

4. $\frac{y}{2}-\frac{y^2-1}{y+1}=y$.

5. $\frac{5}{x^2}-\frac{1}{x}=-\frac{2x-1}{x^2}$.

6. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{4}{x^2-1}$.

7. $\frac{2}{w-2}-\frac{4}{w+2}=\frac{7}{w^2-4}$.

8. $\frac{3}{x}-\frac{5}{x+6}=\frac{1}{x^2+6x}$.

9. $\frac{x+1}{x-1}=\frac{x}{(1-x^2)x}+\frac{x}{x-1}$.

10. $\frac{3x}{x(x+2)}=\frac{5}{x-5}$.

11. $\frac{(6x-2)x}{5x^2-8x}=4$.

12. $x-\frac{8x}{3x^2}=\frac{2x^2-x}{2x}$.

13. $\frac{1}{x-2}-\frac{x}{x+2}=\frac{4-x}{x+2}$.

14. $\frac{2}{y}+\frac{y}{y^2-y}=\frac{3}{y(y-1)}$.

15. $\frac{2y+1}{y-3}+5=\frac{y^2-7y}{(y-3)y}$.

16. $\frac{8}{2x-8}-\frac{5x}{3x-12}+5=0$.

17. $\frac{3x}{x+5}-\frac{6}{2x+10}=\frac{7x+9}{3x+15}$.

18. $\frac{3}{4}-\frac{2x-5}{3x}+\frac{(x-3)x}{4x^2}=0$.

19. $x+2=\frac{1}{x+1}+\frac{x^2-25}{x+5}$.

20. $\frac{1}{y-3}+\frac{3}{y+3}=\frac{6y+18}{y^2-9}$.

21. $\frac{w+1}{w-4}+\frac{w+2}{w+7}=\frac{2w^2-3w+5}{w^2+3w-28}$.

22. $\frac{x-2}{x+3}-\frac{x+2}{x+5}=\frac{4x-1}{x^2+8x+15}$.

234. Decimal Coefficients. Equations in fractional form, like those in Secs. 231–233 should be distinguished from equations having decimal coefficients. The former present peculiar difficulties, as explained above, but the latter yield to the same processes as equations with integral coefficients.

If equations fractional in themselves contain decimals, the processes of Secs. 232, 233 are to be applied first.

235. Preparatory.

We have already used decimal coefficients in evaluating, in adding, in subtracting, and in solving equations containing per cents.

Example: A number less 5 % of itself is 4.75. What is the number?

Solution. 1. $x - .05\,x = 4.75.$

2. $.95\,x = 4.75.$

3. $x = \frac{4.75}{.95} = \frac{475}{95} = 5$, the number.

This equation could be written $x - \frac{5}{100}\,x = 4\frac{75}{100}$, but this is not so simple.

236. *In simplifying equations with decimal coefficients, it is generally better to work with the decimals or multiply both members by a power of 10 so as to make all the decimals whole numbers, instead of substituting for them their common fractional equivalents.*

EXAMPLE

A part of \$100 was lent at 5 % annually, and the rest at 7 % annually; the total interest for one year was \$5.50. What was each sum?

Solution. 1. $.05\,x + .07(100 - x) = 5.50.$

2. $5\,x + 7(100 - x) = 550.$ Multiplying (1) by 100.

3. $\therefore 2\,x = 150$, and $x = 75.$

4. The parts were \$75 and \$25.

WRITTEN EXERCISES

1. $.5\,x + 8 = 20.5.$

2. $.05\,x - 6.25 = -3.75.$

3. $.15\,x + .03\,x = .25\,x - 2.1.$

4. $3.5\,x - 1.6\,x = 3\,x - .75.$

5. $1.01\,x + 2.005 - .003\,x = 45.306.$

6. $4.2(3\,x - 1.06) - .04(3.5\,x - 5) = 20.668.$

7. $\frac{.1\,x + .05}{2} - \frac{.01\,x + .005}{3} = .0009\frac{1}{3}.$

8. $\frac{.03(5-.4x)}{.2}+\frac{.7(2x-\frac{1}{2})}{.5}=9.914.$.

9. A bank lent \$500 in two parts, one at 6% and the other at 4%. The annual interest on the \$500 was \$25.90. What was each part?

10. In a certain solution of camphor and alcohol, the amount of camphor was 25% of the amount of alcohol. How much of each in 4.5 oz. of the mixture?

11. In a certain grade of concrete composed of pure cement and gravel the weight of the cement is $66\frac{2}{3}$% of the weight of the gravel. How many tons of pure cement are there in a block of this concrete weighing 6.6 T.?

237. The Linear Form $ax + b$. Every polynomial of the first degree can be put into the form $ax + b$. That is, by rearranging the terms suitably, it can be written as *the product of* x *by a number not involving* x, *plus an absolute term.* Hence, the form $ax + b$ is called a *general form* for all polynomials of the first degree in x.

Thus, $4x + 3(2 - 5x)$ can be written $-11x + 6$ which is in the form of $ax + b$, because -11 takes the place of a and $+6$ takes the place of b.

238. General Equations. Just as every linear polynomial is of the general form $ax + b$, so every equation of the first degree is equivalent to an equation of the form

$$ax + b = 0;$$

consequently the latter is called *a general equation of the first degree with one unknown.*

239. General Solution. From the equation $ax + b = 0$, (*1*)

we have $$ax = -b, \quad (2)$$

and hence, $$x = \frac{-b}{a}. \quad (3)$$

240. $\frac{-b}{a}$ is the *general form of the root* of the equation of the first degree. There is always *one root,* and only one.

241. To solve an equation of the first degree in an unknown:

Put the equation into the form $ax + b = 0$. *Then divide the absolute term,* b, *with its sign changed by the coefficient of* x.

WRITTEN EXERCISES

Simplify and solve according to Sec. 241:

1. $1 = 4x(3 + 7)$.
2. $6 + 8(1 - x) = 2$.
3. $5x + 3(x - 2) = 4x$.
4. $a = bx + c$.
5. $\frac{1}{x} = \frac{3}{4}$.
6. $\frac{x+1}{x+2} = 7$.
7. $\frac{x+1}{x} = \frac{4}{7}$.
8. $\frac{x}{1+x} = 2$.
9. $a = \frac{4y+5}{7-y}$.
10. $3px + 4bd = 7ax - b + d$
11. $w(1 - a) + a(w - 1) = 0$.
12. $5z + 15 + \frac{3z+16}{5} = \frac{6z+91}{5}$.
13. $3(x + 1) + 7(x + 2) = 4(x + 3)$.
14. $(x - a)(x - b) = x^2 - a$.
15. $3bx - 7(x + b) + ac - cx = 0$.
16. $4 - \frac{5-x}{2} = \frac{3a+x}{4}$.
17. $2[3(x + 1) + 1] + 1 = 0$.
18. $\frac{1}{x+1} - \frac{a}{x+a} = 0$.
19. $(3 - x)(7 - x) = 10 + x^2$.
20. $x(1 + a) + 2x(1 + b) = 3x(1 + c)$.
21. $(1 + 3x)^2 + (3 + 4x)^2 = (1 - 5x)^2$.
22. $(1 - x)(4 + x) + (2 + x)(6 + x) = 0$.
23. $(a - x)(b + x) = c^2 - x^2$.
24. $(x + b)(x + d) = (x + 1)^2$.
25. $\frac{1}{a+b} + \frac{a+b}{x} = \frac{1}{a-b} + \frac{a-b}{x}$.

Solve for a:

26. $ab + ac = 4.$ 27. $ag + 3 = 2ah - 5.$

28. $(a - 1)(a + 2) = (a - b)(a + b).$

29. Solve this equation for e; also for b:

$$\frac{ab}{e} = bc + d + \frac{1}{e}.$$

Solve for g:

30. $v = at + \frac{gt^2}{2}.$ 31. $gh + gh^2 = r^3.$

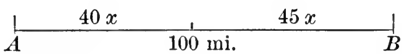
32. How much alcohol must be added to 2 qt. of a solution 90% pure alcohol to make a solution 95% pure?

SUGGESTION. Let x equal the number of quarts of alcohol to be added. Then, the new solution will contain $(2 + x)$ qt. The water in the new solution, or 5% of $(2 + x)$ qt., will be the same as that in the original solution, or 10% of 2 qt.

33. 18% by weight of wheat is lost (as bran, etc.) in grinding it into flour. How many 60-lb. bushels of wheat are used in making 738 lb. of flour?

34. Divide the number n into two parts A, B, such that A shall be $\frac{3}{2}$ of B.

35. A train leaves A for B 100 mi. distant, traveling 40 mi. per hour. At the same time a train leaves B for A, traveling 45 mi. per hour. How far from A will they meet, and how long after starting?



SOLUTION. 1. Let x be the number of hours from the time the trains start until they meet.

2. Then, the train from A travels $40x$ mi., and the train from B, $45x$ mi.
3. Therefore, $40x + 45x = 100$, as in the figure.
4. Therefore, $x = \frac{100}{85} = 1\frac{3}{17}$, the number of hours.
5. And $40x = 47\frac{1}{17}$, the number of miles from A.

CHECK. Find the distance from B and add to $47\frac{1}{17}$ mi.

36. A train leaves A for a station B, d miles distant; its rate is r miles per hour; h hours later another train leaves B for A, running R miles per hour. How far will each train have traveled when they meet?

SOLUTION. 1. Let $x =$ the number of hours traveled by the first train before they meet.

2. Then, $x - h =$ the number of hours traveled by the second train.
3. $rx =$ number of miles traveled by first train.
4. $R(x - h) =$ number of miles traveled by second train.
5. $\therefore\ rx + R(x - h) = d$, by the conditions of the problem.
6. $\therefore\ (r + R)x = d + Rh$, performing the operations.
7. $\therefore\ x = \dfrac{d + Rh}{r + R}$, solving (6).
8. $\therefore\ rx = \dfrac{r(d + Rh)}{r + R}$, the distance traveled by the first train.
9. And $R(x - h) = \dfrac{R(d - rh)}{r + R}$, the distance traveled by the second train.

37. Read the above problem, taking $d = 450$, $r = 50$, and $R = 40$, and $h = 4\frac{1}{2}$. Solve the problem by substituting these numbers in the expressions of steps 8 and 9.

242. Formula. An expression which shows how a desired number is to be obtained from given numbers is called a formula.

Thus, the expressions in steps 8 and 9 of Problem 36 are formulas.

A formula contains the solution of all problems which may be made by simply changing the numbers in the given problem.

WRITTEN EXERCISES

1. The sum of three numbers is 40. The second is 6 more than the first, and the third is the sum of the other two. Find the numbers.

2. The sum of three numbers is s; the second is a greater than the first, and the third is the sum of the other two. Find each number.

3. The sum of three numbers is s. The second is a less than the first, and the last is b less than the second. Find the numbers.

4. Apply Exercise 3 to the problem: The sum of three numbers is 35. The second is 3 less than the first, and the third is 4 less than the second. Find the numbers.

243. Special Problems. The solution of many problems is made easier by a proper selection of the unknown quantities.

WRITTEN EXERCISES

1. A certain number consists of two digits whose difference is 3; and, if the digits are interchanged, the number so formed is $\frac{4}{7}$ of the original number. Find the latter.

SUGGESTION. 1. Let x be the smaller digit.
2. Then $3 + x$ is the greater.
3. $\therefore 10(x + 3) + x$ is the given number.
4. $\therefore 10x + (x + 3)$ is the number with the digits interchanged.
5. According to the problem, $10x + (x + 3) = \frac{4}{7}[10(x + 3) + x]$.
6. Solve for x.

2. The tens' digit of a certain number exceeds the units' digit by 4, and when the number is divided by the sum of its digits, the quotient is 7. Find the number.

3. A boatman who rows $3\frac{1}{2}$ mi. in still water finds that it takes him 12 hr. to row upstream a distance that he can row down in 2 hr. What is the rate of the current?

SUGGESTION. Let x be the rate of the current. Then, the boatman rows upstream at the rate of $3\frac{1}{2} - x$ miles per hour, and downstream at the rate of $3\frac{1}{2} + x$ miles per hour.

4. The sum of three numbers, a, b, c, is 3036; a is the same multiple of 7 that b is of 4, and also the same multiple of 5 that c is of 2. What are the numbers?

SUGGESTION. Show that a, b, and c may be denoted by $7x$, $4x$, $\frac{7x}{5}\cdot 2$.

5. A number is composed of three digits each greater by one than that on its right; the difference between the number and $\frac{1}{4}$ of the number formed by reversing the order of the digits is 36 times the sum of the digits. Find the number.

SUGGESTION. A convenient selection of the digits is $x + 1$, x, $x - 1$.

REVIEW

WRITTEN EXERCISES

Solve for x:

1. $\frac{x+1}{x+2}+\frac{x+3}{x+4}=2.$

2. $(a-x)(b+x)=b-x^2.$

3. $(a+x)(b+x)=ab+x^2.$

4. $(a-x)(b-x)=b+x^2.$

5. $(a+bx)(a-bx)=(a^2+b^2x)(a-x).$

6. $(a+bx)(b-ax)=(a-bx)(ax-b).$

7. $(1-x)(2-x)-(3-x)(4-x)=5-x.$

8. $(a+x)(b-x)=a-x^2.$

9. $ax(bx+1)=bx(ax+1)+a+b.$

10. $(1+x)(2+x)+(5+6x)(1+8x)=(4+7x)^2$

11. $(a+x)(d+x)-(c-x)(a-x)=0.$

12. How much water must be added to 80 lb. of a 5% solution of salt to obtain a 4% solution?

13. Two men are 25 mi. apart and walk toward each other at the rates of $3\frac{1}{2}$ mi. and 4 mi. an hour respectively. After how long do they meet?

14. A man travels 50 mi. in an automobile in $3\frac{1}{4}$ hr. If his car runs at the rate of 20 mi. an hour in the country, and at the rate of 8 mi. an hour when within city limits, find how many miles of his journey are made in the country.

15. A train running 30 mi. an hour requires 21 min. longer to go a certain distance than does a train running 36 mi. per hour. What is this distance?

16. A and B set out on an automobile trip, A having $\frac{2}{3}$ as much money with him as B; after A had paid out \$1 less than $\frac{2}{3}$ of his money, and B had paid \$1 more than $\frac{7}{8}$ of his, it was found that B had left only half as much as A. How much had each at the outset?

17. In forming a regiment into a solid square 60 men were left over; but, when formed into a rectangle with 5 men more in front than before and 3 less in depth, there was one man wanting to complete it. Find the total number of soldiers in the regiment.

18. Two automobiles start from the same place; one goes east at the rate of 18 mi. an hour and the other west at 15 mi. an hour. In how many hours are they 330 mi. apart?

19. A train going from New York to Chicago at the average rate of 40 mi. an hour takes $4\frac{4}{5}$ hr. longer than one going 50 mi. an hour. Find the distance between these places.

20. Given $a =$ the amount, $r =$ the rate, and $p =$ the principal in a problem of simple interest, what is the time?

21. In Exercise 20 let $a = \$560$, $r = 4\%$, and $t = 3$ yr. Find the principal.

22. Divide the number a into two parts such that m times the greater may exceed n times the less by b.

23. A certain number consists of two digits whose difference is 5; and, if the digits are interchanged, the number so formed is $\frac{3}{8}$ of the original number. Find the latter number.

24. A certain number consists of two digits of which the tens' digit is 3 times the units' digit; and, if these digits are interchanged a number is formed which is less than the original number by 36. Find the original number.

25. Separate 275 into two parts such that $\frac{1}{5}$ of the smaller equals $\frac{1}{6}$ of the larger.

26. The sum of three consecutive even numbers is 306. Find the numbers.

27. One-fourth of a man's age now equals $\frac{1}{2}$ of what it was 20 yr. ago. Find his age now.

28. In a certain number of two digits the tens' digit exceeds the units' digit by 2; if the number is diminished by $\frac{3}{2}$ of the sum of its digits, they are interchanged. Find the number.

29. Separate 150 into two numbers such that if one be divided by 23 and the other by 27, the sum of the quotients is 6.

30. The current of a certain river is 6 mi. per hour. A steamer can go upstream in 9 hr., a distance that takes only 3 hr. to come down using the same power per hour. Find the rate of the steamer in still water.

31. A boatman who could row 5 mi. per hour in still water rowed a certain distance up a stream and back again. The current was 3 mi. per hour, and it took the boatman 10 hr. to make the round trip. How far up the stream did he go?

32. A can do a piece of work in 3 da. But A and B working together can do the same work in 2 da. How many days would it take B to do it alone?

Suggestion. Let $x =$ the number of days required by B. Then, A can do $\frac{1}{3}$ in 1 da. and B can do $\frac{1}{x}$ in 1 da. The sum of these multiplied by 2 will be the whole, or 1.

33. To a mass of metal composed of 4 parts of silver to 1 part of tin, enough silver is added to make a mass containing 6 parts of silver to 1 part of tin. How many ounces of silver are added per ounce of the original metal?

34. Twenty coins composed of dimes and quarters amount to $3.20. How many coins of each kind are there?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. How is the degree of an equation determined? Sec. 225.

2. Define and illustrate *linear equation;* also *quadratic equation;* also *higher equation.* Secs. 226–228.

3. What is the *absolute term?* Sec. 230.

4. What is the first step in the solution of a fractional equation? Secs. 231–233.

5. How are decimal coefficients best removed? Sec. 236.

6. What is the *general form* of an equation of the first degree? What is the form of the root? Secs. 238–240.

7. What is a *formula?* Sec. 242.

CHAPTER XVII

RATIO, PROPORTION, AND VARIATION

RATIO

244. Ratio. The quotient of two numbers of the same kind is often called their **ratio.**

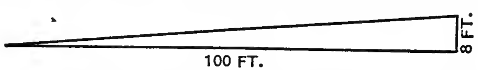
The following examples illustrate the use of the word "ratio":

A solution consists of sulphuric acid and water in the ratio of 2 to 3. This means that $\frac{2}{5}$ of the whole is sulphuric acid and $\frac{3}{5}$ is water.

Sterling silver requires a little other metal (alloy) to harden the silver in it; the ratio by weight of the amount of pure silver to the entire mass is usually 925 to 1000. This means that $\frac{925}{1000}$ of the whole is silver.

The specific gravity of a solid is the ratio of its weight to the weight of an equal volume of water under standard conditions.

The birth rate per annum in a city is said to be 23 when the ratio of the total number of births in a certain year to the total number of inhabitants at the beginning of that year is that of 23 to 1000.



A road bed is said to have an 8 % grade when the ratio of the vertical rise to the horizontal distance is that of 8 to 100.

245. Ratio is commonly expressed by a fraction.

Thus, the ratio of 2 to 3 is written $\frac{2}{3}$; the old form 2:3 is less convenient in calculation.

Before ratios were written in fractional form it was convenient to have names for the terms. Thus, the first number was called the *antecedent*, and the second number the *consequent*.

In division the divisor may be abstract or concrete. If it is abstract, the quotient is of the same character as the dividend. If it is concrete, the dividend must be concrete and expressed in the same unit.

Consequently, we cannot speak directly of the ratio of 12 gal. to 3 qt.; we must first express both numbers in the same unit, as 48 qt. to 3 qt.

Similarly, when we speak of the ratio of the distance to the time, we mean the ratio of the corresponding abstract numbers, as in Sec. 244.

WRITTEN EXERCISES

1. How much pure silver is there in 200 oz. of sterling silver (Sec. 244)?

SUGGESTION. $\frac{x}{200}=\frac{925}{1000}$.

2. A silversmith buys $46\frac{1}{4}$ oz. of pure silver; how much sterling silver can be made from it?

SUGGESTION. The equation is $\frac{46\frac{1}{4}}{x}=\frac{925}{1000}$.

3. What is the rate per second of a train which travels uniformly 635 ft. in 5 sec.?

4. The weight of a piece of gold is 94.5 oz., and the weight of an equal volume of water is 5 oz. What is the specific gravity (Sec. 244) of the gold?

5. There are 4053 births in a certain city, making its birth rate 21 (Sec. 244). What is the population of the city?

6. If the population of a city is p and the birth rate is b, indicate the number of births.

7. The top of a mountain pass is 1200 ft. vertically above the level of the base; the top is reached by a zigzag road 5 mi. long. What is the average grade of the road?

8. A road m mi. long, measured horizontally, ascends to a height of f ft. above the level of its starting point. Indicate the average grade of the road.

9. Two men, A and B, divide $963 of profits so that A's part is to B's in the ratio of 2 to 1. How many dollars has each?

SUGGESTION. 1. Let x be the amount A receives.

2. Then, $963 - x$ is the amount B receives.

3. $\therefore \frac{x}{963-x}=\frac{2}{1}$, the ratio of the shares, as given.

4. Clear the equation of fractions and solve for x.

10. Two partners, A and B, divide $575 in the ratio of 2 to 3. How many dollars does each receive?

246. Special Notations. When a letter stands for different values of the same variable quantity, these values are commonly designated either by:

1. *Small letters and capital letters.*

For example, if we have to represent the rates of two moving bodies in the same problem, we may use r for the rate of one and R for the rate of the other.

2. *Primes and seconds.*

For example, if two weights occur in the same problem, we may express one by w' and the other by w'', read: "w prime" and "w second."

3. *Subscripts.*

If a problem has three quantities indicating time, we may express these by t_1, t_2, t_3, read: "t sub-one," "t sub-two," "t sub-three," or simply "t-one," "t-two," and "t-three."

Primes and subscripts should be carefully distinguished from exponents or coefficients, because they have no significance other than to distinguish numbers of the same kind one from another.

WRITTEN EXERCISES

1. Solve for r the equation $\frac{r}{3}=\frac{5}{R}$.

2. Solve the equation in Exercise 1 for R.

3. Solve for r' the equation $\frac{1}{r'}=\frac{r''}{5}$.

4. Solve the equation in Exercise 3 for r''.

5. Solve $\frac{t_1}{.5}=\frac{.1}{t_2}$ for t_1; also for t_2.

6. Solve $3\,t_1=\frac{5}{6}tt_2$ for t_2. Solve the equation for t.

7. It is known that in triangles whose corresponding angles are equal, the corresponding sides have the same ratio.

In the figure, what is the ratio of a to a'? Of 20 to x? Find x. What is the ratio of y to 180? Find y.

Supply the numbers to fill the blanks in the table:

	Substance	w = weight of substance	w' = weight of an equal volume of water	Specific gravity $= \frac{w}{w'}$
8.	Lead	56.5 oz.	5 oz.	——
9.	Oak	12.75 oz.	15 oz.	——
10.	Tin	153.3 lb.	21 lb.	——
11.	Coal	18 T.	10 T.	——
12.	Alcohol	13.6 oz.	17 oz.	——

247. A property of ratio:

The numerical value of a ratio whose numerator is less than its denominator is increased by adding the same positive number to both terms.

The numerical value of a ratio whose numerator is greater than its denominator is decreased by adding the same positive number to both terms.

1. For, let $\frac{a}{b}$ be the given ratio and x the given positive number.

2. To determine whether $\frac{a}{b}$ or $\frac{a+x}{b+x}$ is the greater, subtract the second ratio from the first.

3. Then, $\frac{a}{b} - \frac{a+x}{b+x} = \frac{ab+ax-ab-bx}{b(b+x)} = \frac{(a-b)x}{b(b+x)}$.

(1) If $a < b$, the last ratio is a negative number, therefore the ratio of less inequality $\frac{a}{b}$ would be increased by adding x to each of its terms.

(2) If $a > b$, the last ratio is a positive number, therefore the ratio of greater inequality $\frac{a}{b}$ would be diminished by adding x to each of its terms.

A ratio whose numerator is less than its denominator was formerly called a **ratio of less inequality.**

A ratio whose numerator is greater than its denominator was formerly called a **ratio of greater inequality.**

(3) If $a = b$, the last ratio is zero, therefore the ratio $\frac{a}{b}$, if equal to unity, would be unchanged by adding x to each of its terms.

A knowledge of this property will remove the temptation to say that adding or subtracting the same number from both terms of a fraction does not alter its value.

ORAL EXERCISES

1. What is the effect on $\frac{2}{3}$ of adding 4 to each term?
2. What is the effect on $\frac{3}{2}$ of adding 5 to each term?
3. What is the effect on $\frac{4}{5}$ of adding a to each term?
4. What is the effect on $\frac{5}{4}$ of adding b to each term?
5. What is the effect on $\frac{6}{6}$ of adding c to each term?
6. Which is the greater, $\frac{2}{3}$ or $\frac{4}{5}$? Also $\frac{4}{7}$ or $\frac{2}{5}$? Why?
7. Which is the greater, $\frac{a}{b}$ or $\frac{a+1}{b+1}$? Why?
8. Which is the less, $\frac{a}{b}$ or $\frac{a-2}{b-2}$? Why?
9. Which is the less, $\frac{a}{b}$ or $\frac{2a}{a+b}$? Why?
10. Compare $\frac{n}{n+1}$ and $\frac{n+1}{n+2}$; also, $\frac{n-2}{n+1}$ and $\frac{n}{n+3}$.

PROPORTION

248. Proportion. An equation between two ratios is called a **proportion.**

Thus, $\frac{2}{3} = \frac{6}{9}$, $\frac{3a}{7a} = \frac{6b}{14b}$, $\frac{25}{5} = \frac{5}{1}$, $\frac{5}{6} = \frac{3}{x}$ are proportions.

A proportion is usually read in one of two ways: For example, $\frac{3}{5} = \frac{a}{x}$ is read "3 is to 5 as a is to x," or "the ratio three-fifths equals the ratio a over x."

249. The numbers forming one of the ratios are said to be "proportional to" the numbers forming the other.

Thus, in the proportion $\frac{12}{9} = \frac{20}{15}$ the numbers 12 and 9 are proportional to 20 and 15.

250. The terms "proportional" and "proportionally" are used with the meaning "in the same ratio."

For example:

The express rate from Chicago to New York is \$2.50 per 100 lb., the excess above 100 lb. being charged proportionally. This means the charge for the excess has the same ratio to the excess as 2.50 has to 100.

Of two men in business one furnished $\frac{2}{3}$ of the capital, and the other $\frac{1}{3}$ of it. They divided their gain of \$9000 in proportion to their capitals. This means that they divided the \$9000 into two parts having the ratio of 2 to 1.

251. The expression *pro rata* is often used with the same meaning as "proportionally" or "in the same ratio."

For example:

A and B hire an automobile for a trip and agree to pay $\frac{3}{5}$ and $\frac{2}{5}$ of the rental respectively, and other expenses occurring on the trip *pro rata*. This means that the other expenses are to be divided between A and B in the same ratio as the rental of the automobile.

WRITTEN EXERCISES

1. Two families of 3 members and 5 members respectively camp out together at an expense of \$160; they divide this amount in proportion to the size of the families. How much did each family pay?

2. A man hires a piano at \$5 per month of 30 days, and is to pay *pro rata* for any part of a month. What does he pay, if he keeps the piano 51 days? d days?

3. Three farmers share in the purchase of a steam thresher. A pays \$400, B pays \$600, and C pays \$1000. In the course of the year the thresher is rented to other farmers 27 days at \$10 per day, and the earnings divided among the owners *pro rata*. What does each receive?

4. If the thresher of Exercise 3 is rented d days at r dollars per day and the earnings divided *pro rata*, what does each owner receive?

5. It cost R dollars to repair the thresher, and the cost is divided *pro rata* among the owners. What does each pay?

WRITTEN EXERCISES

1. Find x in the proportion $\frac{x}{15} = \frac{160}{25}$. (*1*)

SOLUTION:

Multiplying both members by 15, $x = \frac{15 \times 160}{25}$. (*2*)

Simplifying the fraction in (*2*), $x = 96$. (*3*)

2. Find x in the proportion $\frac{7}{x} = \frac{.14}{42}$. (*1*)

SOLUTION:

Clearing of fractions, $7 \times 42 = .14\,x$. (*2*)

Simplifying (*2*), $x = \frac{7 \times 42}{.14} = 2100$. (*3*)

3. Solve the proportion $\frac{h}{l} = \frac{h'}{l'}$ for h. For h'. For l'.

4. Solve the proportion $\frac{p}{P} = \frac{W}{w}$ for p. For P. For W. For w.

5. Solve the equation $\frac{p}{p'} = \frac{b}{b'}$ for p. For b. For b'. For p'.

6. In Exercise 5 what is the value of p', if $p = 15$, $b = 28$, and $b' = 30.5$?

7. Two partners, A and B, in business divide \$9000 between them in the ratio of 2 to 1. How much does each receive?

8. At \$2.80 for 8 hours' work, overtime paid proportionally, how much does a workman receive for $2\frac{1}{2}$ hr. overtime?

9. After rents rose $\frac{1}{3}$ and other things in proportion, a family's expenses for one month were \$132. What were their expenses before the rise?

10. 1 cu. ft. of lime and 2 cu. ft. of sand are used in making 2.4 cu. ft. of mortar. How much of each is needed to make 72 cu. ft. of mortar?

11. In making glass, 1, 5, and 15 portions by weight of chalk, potash, and sand respectively are used. How many pounds of each are required to produce 1176 lb. of glass?

252. Problems of the Lever:

1. The point of support of a lever is called the fulcrum and in the figure is denoted by F.

If P denotes the power and W the weight (expressed in the same unit), and if p denotes the length of the arm FP and w the length of the arm FW (both expressed in the same unit), then it is known that

$$\frac{w}{p} = \frac{P}{W}$$

In words: *The ratio of the lengths of the arms equals the reciprocal of the ratio of the corresponding forces.*

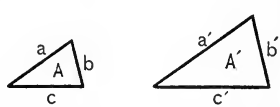
Express w in terms of P, W, and p. Express W in terms of P, p, and w.

2. Find p if $P = 10$ lb., W $= 60$ lb., $w = 2$ ft.

Supply the numbers to fill the blanks in the following table concerning levers:

	p	w	P	W
3.	( )	20 in.	1.60 lb.	90 lb.
4.	1.5 ft.	( )	2.25 lb.	1125 lb.
5.	12 ft.	$\frac{1}{2}$ ft.	( )	960 lb.
6.	.3 ft.	12.9 ft.	2.5 T.	( )

253. Problems of Similar Triangles:



1. Areas of similar triangles (those of the same shape) are proportional to the squares of the lengths of their corresponding sides. Let A and A' be the areas of two similar triangles, and a and a' be a pair of corresponding sides. Then $\frac{A}{A'} = \frac{a^2}{a'^2}$.

Solve the proportion $\frac{A}{A'} = \frac{a^2}{a'^2}$ for A. For A'. For a. For a'.

2. Express the ratio of a to a' in Exercise 1.

3. The areas of two similar triangles are 64 sq. in and 625 sq. in. What is the ratio of any pair of corresponding sides?

4. If the side a of the smaller triangle in Exercise 3 is 8 in., what is the corresponding side a', in the larger triangle? If the side b of the smaller triangle is 5 in., what is b'?

5. The lengths of a pair of corresponding sides in two similar triangles are 3 ft. and 8 ft.; the area of the larger one is 640 sq. ft. What is the area of the smaller one?

VARIATION

254. If related numbers change so as always to remain in the same ratio, one is said to *vary as* the other.

For example:

At 5 ¢ per pound, the amount paid varies as the number of pounds purchased.

If a body moves at the same rate, the distance varies as the time of motion.

If $100 is placed at simple interest, the amount of interest varies as the time.

255. "Varies as" is thus seen to be merely another expression for "is proportional to" or "varies proportionally with."

ORAL EXERCISES

1. How much more can a man lift with a lever 8 ft. long than with a lever 4 ft. long? What is the weight lifted proportional to? What does the weight vary as?

2. How much more does $10 earn at 6 % simple interest in one year than $5? When the rate and time are the same what does the interest vary as?

3. When the principal and the rate are the same, what does the simple interest vary as?

4. An automobile is running at a uniform rate. The distance traveled varies as what number?

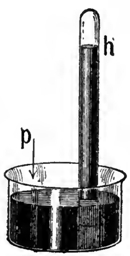
5. If the machine in Exercise 4 traveled 90 mi. in 3 hr., how far would it travel in 5 hr.?

WRITTEN EXERCISES

1. When the grade of a roadbed is $8\frac{1}{2}\%$, what is the rise in a horizontal distance of 175 ft.?

2. What weight can a man weighing 175 lb. raise with a lever 5 ft. long, if the fulcrum is 6 in. from the end acting on the weight?

3. What weight can he raise, if he weighs 180 lb., the lever being $4a$ ft. long, and the support being a ft. from the end acting on the weight?



4. The height to which the mercury rises in a barometer varies as the pressure of the air on the mercury. Let the heights at two readings be h and h', and the corresponding pressure p and p'; then $\frac{p}{p'} = \frac{h}{h'}$. Express p in terms of h, h', and p'.

5. In Exercise 4, let two readings of the barometer be $h = 28.5$ and $h' = 29.5$, and the corresponding pressures be p and $p' = 15$ lb. What is the value of p?

Supply the blanks in the following table of readings:

	p	p'	h	h'
(6)	( ) lb.	14.5 lb.	30 in.	29 in.
(7)	14.75 lb.	( )	29.5 in.	30 in.
(8)	15 lb.	15.25 lb.	( )	30.5 in.
(9)	14.2 lb.	14.8 lb.	28.75 in.	( )

PROPERTIES OF PROPORTION

256. Means and Extremes. The middle numbers in a proportion are called the **means** and the end numbers the **extremes.**

Thus, in the older form of writing a proportion, $a : b :: c : d$, b and c are called the *means* and a and d the *extremes*. When so written, the proportion is read "a is to b as c is to d." At present, it is more customary to use the form $\frac{a}{b} = \frac{c}{d}$ and to read it "a over b equals c over d."

257. Fourth Proportional. In the proportion $\frac{a}{b}=\frac{c}{d}$, the fourth number, d, is called the **fourth proportional.**

Numbers that are said to form a proportion must be placed in the proportion in the order in which they are given. Thus, if 3, 5, 10, and x are said to form a proportion, then $3:5=10:x$.

258. Third Proportional. In the proportion $\frac{a}{b}=\frac{b}{c}$, the third number, c, although in the fourth place, is called the **third proportional** to a and b.

259. Mean Proportional. In the proportion $\frac{a}{b}=\frac{b}{c}$, b is called the **mean proportional** between a and c.

260. Relation to the Equation. A proportion is an equation expressing the equality of two ratios.

EXAMPLES

1. Find the fourth proportional to the numbers 6, 8, 30.

Let x be the fourth proportional, then, by Sec. 257, $\frac{6}{8}=\frac{30}{x}$. (1)

Multiplying by $8x$, $6x=8\cdot 30$. (2)

Dividing by 6, $x=40$. (3)

Therefore 40 is the fourth proportional to 6, 8, 30.

2. Find the third proportional to 5 and 17.

Let x be the third proportional, then, by Sec. 258, $\frac{5}{17}=\frac{17}{x}$. (1)

Multiplying both members by $17x$, $5x=17^2$. (2)

Solving (2), $x=\frac{289}{5}=57\frac{4}{5}$. (3)

Therefore $57\frac{4}{5}$ is the third proportional to 5 and 17.

WRITTEN EXERCISES

1. Write m, n, and p so that p shall be the third proportional to m and n.

2. Write m, n, and p so that n shall be the mean proportional between m and p.

3. Solve: $-\frac{5}{x}=\frac{1}{4}$.

4. Solve: $\frac{1.21}{x}=\frac{x}{.09}$.

Solve:

5. $\frac{x}{11} = \frac{7}{1331}$.

6. $x \div 16 = \frac{1}{4} \div x$.

7. $-24 \div x = 2x \div -12$.

8. $\frac{1}{6} = \frac{42}{x}$.

9. $\frac{75}{-x} = \frac{-x}{3}$.

10. Find the third proportional to 8 and 5.

11. Find the third proportional to -6 and -4.

12. Find the mean proportionals between the following numbers: 9 and 16; -25 and -4.

13. If a sum of money earns $48 interest in 5 yr., how much will it earn in 16 yr. at the same rate per cent?

14. A city whose population was 40,000 had 2500 school children; the total population increased to 48,000, and the number of children of school age increased proportionally. How many children of school age were there then?

15. What number must be added to each of the four numbers, 5, 29, 10, 44, to make the results proportional?

261. Relation of Means to Extremes. *In any proportion the product of the means equals the product of the extremes,*

For, $\frac{a}{b} = \frac{c}{d}$, and multiplying both members by bd, $ad = bc$.

262. Conversely, *If the product of two numbers equals the product of two other numbers, the four numbers can be arranged in a proportion, the two factors of one product being the means, and the two factors of the other product the extremes.*

For, if $ad = bc$, divide both members by bd, and $\frac{a}{b} = \frac{c}{d}$.

Also, if
$$a^2 - b^2 = xy,$$
$$\frac{a+b}{x} = \frac{y}{a-b}.$$

263. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

For, if two numbers are equal, their reciprocals are equal. Sec. 219.

The older form of statement still sometimes used is: If four numbers form a proportion, they are in proportion by **inversion.**

264. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

For, multiplying both members by $\frac{b}{c}$, $\frac{b}{c}\cdot\frac{a}{b}=\frac{b}{c}\cdot\frac{c}{d}$,

and canceling b and c, $\frac{a}{c}=\frac{b}{d}$.

The older form of statement is: If four numbers form a proportion, they are in proportion by **alternation.**

265. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

Also $\frac{a-b}{b}=\frac{c-d}{d}$.

For, adding $+1$ to both members, $\frac{a}{b}+1=\frac{c}{d}+1$.

Therefore, performing the processes, $\frac{a+b}{b}=\frac{c+d}{d}$.

Similarly, by adding -1 to both members of the given proportion the second form results.

The older form of statement is: If four numbers form a proportion, they are in proportion by **composition** or by **division.**

266. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

For, from Sec. 265, $\frac{a+b}{b}=\frac{c+d}{d}$,

and $\frac{a-b}{b}=\frac{c-d}{d}$.

Dividing these equations, member for member,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

The older form of statement for this is: If four numbers form a proportion, they are in proportion by **composition and division.**

267. The **mean proportional** between two numbers is the square root of their product.

For, from $\frac{a}{b}=\frac{b}{c}$, Sec. 259, we find $b=\sqrt{ac}$.

268. Series of Equal Ratios. The following theorem concerning ratios is sometimes used:

In a series of equal ratios (a continued proportion), the sum of the numerators divided by the sum of the denominators is equal to any one of the ratios.

Proof: Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots$ be the equal ratios and r be their common value. (1)

Then $\frac{a}{b}=r,\ \frac{c}{d}=r,\ \frac{e}{f}=r,$ and so on. (2)

$\therefore\ a=br,\ c=dr,\ e=fr,$ and so on. (3)

$\therefore$ adding the equations in (3),

$$(a+c+e\cdots)=(b+d+f\cdots)r. \qquad (4)$$

$\therefore$ solving (4) for r, $\frac{a+c+e\cdots}{b+d+f\cdots}=r.$ (5)

$\therefore\ \frac{a+c+e\cdots}{b+d+f\cdots}=\frac{\text{the sum of the numerators}}{\text{the sum of the denominators}}=\frac{a}{b},$ or $\frac{c}{d}$, and so on, since r is any of these ratios. (6)

269. These properties may be applied to certain equations.

EXAMPLES

1. Given $\frac{a}{b}=\frac{c}{d}$; show that $\frac{3a-5b}{5b}=\frac{3c-5d}{5d}$. (1)

Multiplying both members of the given equation by 3, $\frac{3a}{b}=\frac{3c}{d}$. (2)

Dividing (2) by -5, $\frac{3a}{-5b}=\frac{3c}{-5d}$. (3)

Applying Sec. 265 to (3), $\frac{3a-5b}{5b}=\frac{3c-5d}{5d}$. (4)

2. Three numbers are in the ratio of 3 to 4 to 7 and their sum is 42. Find the numbers.

Solution:

1. Let $3x$ be the first number, then $4x$ and $7x$ are the others.
2. $\therefore\ 3x+4x+7x=42$, by the conditions of the problem.
3. $\therefore\ 14x=42$ and $x=3$.
4. The numbers are 9, 12, and 21 by steps (1) and (3).

3. Solve: $$\frac{2x-2}{4-9x}=\frac{2}{9x}. \qquad (1)$$

Applying Sec. 268, $$\frac{2x}{4}=\frac{2}{9x}. \qquad (2)$$

Simplifying (2), $$\frac{x}{2}=\frac{2}{9x}. \qquad (3)$$

Solving (3), $$9x^2=4, \qquad (4)$$

and $$x=\pm\tfrac{2}{3}.$$

TEST by substitution.

WRITTEN EXERCISES

1. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a-4b}{4b}=\frac{c-4d}{4d}$.

2. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{b}{b-a}=\frac{d}{d-c}$.

3. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{2a}{a-b}=\frac{2c}{c-d}$.

4. Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a+c-e}{b+d-f}=\frac{c}{d}$.

5. Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a^2+c^2+e^2}{b^2+d^2+f^2}=\frac{e^2}{f^2}$.

6. Given $\frac{m}{n}=\frac{p}{q}=\frac{r}{s}$, show that $\frac{2m-3p+5r}{2n-3q+5s}=\frac{m}{n}$.

7. Given $\frac{x}{a}=\frac{b}{c}=\frac{e}{d}$, show that $x=\frac{a(b+e)}{c+d}$.

8. Write in the form of a proportion (Sec. 262):

(1) $a^2+2ab+b^2=cd$.
(2) $a^2-b^2=a(b+c)$.
(3) $x^2-5x+6=7x$.

9. Find x in $\frac{x-7}{x+2}=\frac{5x+7}{3x-2}$.

10. There are two numbers with the ratio of 3 to 5 and one is 24 less than the other. Find the numbers.

REVIEW

ORAL EXERCISES

1. What is the fourth proportional in $\frac{3}{3}=\frac{5}{d}$?

2. What is the third proportional in $\frac{3}{5}=\frac{5}{x}$?

3. What is the mean proportional between 4 and 9?

4. State the principle according to which it follows from

$$\frac{a}{b}=\frac{c}{d} \text{ that } \frac{a+b}{b}=\frac{c+d}{d}.$$

5. What is the value of x in $\frac{x}{a}=\frac{b}{x}$?

WRITTEN EXERCISES

Find the fourth proportional to:

1. x, y, ab.
2. p^2, q, m^2q.
3. x^2, y^2, z^2.

Find the mean proportional between:

4. a^2, a^2b^2.
5. $16, 4\,a^2b^2$.
6. $27\,x^2y^3, 3\,y$.

When $\frac{a}{b}=\frac{c}{d}$ show that:

7. $\frac{ac}{bd}=\frac{a^2}{b^2}$.
8. $\frac{a}{c}=\frac{a-b}{c-d}$.
9. $\frac{a+2\,b}{2\,b}=\frac{c+2\,d}{2\,d}$.
10. $\frac{2\,a-3\,b}{3\,b}=\frac{2\,c-3\,d}{3\,d}$.
11. $\frac{4\,a-5\,b}{5\,b}=\frac{4\,c-5\,d}{5\,d}$.
12. $\frac{a-b}{c-d}=\frac{b}{d}$.

13. The ratio of two numbers is 3 to 5 and their sum is 48. Find the numbers.

14. The lengths of the sides of a triangle are in the ratio of 3 to 4 to 5 and their sum is 120 ft. What is the length of each side?

15. Denoting by b the arm of a lever on which the power p is applied and by a the arm on which the weight w is applied:

What force is required to raise 500 lb., if $a = 2$ and $b = 10$? Also, if $a = 8$ and $b = 40$? Also, if $a = 9$ and $b = 36$?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

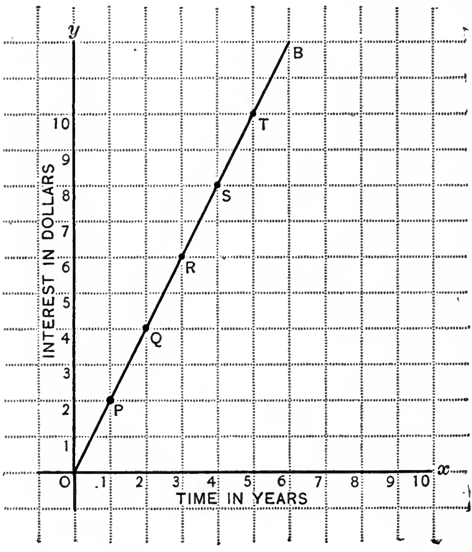
1. Define *ratio*; also *proportion*. Secs. 244, 248.
2. What is the effect of adding the same positive number to both terms of a fraction? Sec. 247.
3. When are two numbers proportional to two other numbers? Sec. 249.
4. Explain "*proportional*," "*pro rata*," and "*in the same ratio*." Secs. 250, 251.
5. What is meant by "*one quantity varies as another*"? Sec. 254.
6. What is the relation of variation to proportion? Sec. 255.
7. Define the *extremes* and *means* of a proportion. Sec. 256.
8. What is the *fourth proportional* to three numbers? Sec. 257.
9. What is the *third proportional* to two numbers? Sec. 258.
10. What is the *mean proportional* between two numbers? Sec. 259.
11. When can four numbers form a proportion? Sec. 262.
12. Illustrate the result of taking a proportion by inversion; also by alternation; by composition; by division; by composition and division. Secs. 263–266.
13. State a property of a series of equal ratios. Sec. 268.

CHAPTER XVIII

GRAPHS OF LINEAR EQUATIONS

270. PREPARATORY:

1. If $100 is lent at 2 % simple interest, show graphically how the amount of interest (I) varies with the number of years (t).



In the diagram the spaces on the horizontal scale represent the number of years, and those on the vertical scale the number of dollars. The positions of the points on the line OB show the amounts of interest for the periods of time indicated on the horizontal scale. Thus, point P shows that the interest is $2 when the time is 1 yr.; point Q shows that the interest is $4 when the time is 2 yr.

2. What does point R show? Point O? Point S? Point T?

3. What does the point halfway between Q and R show?

The number of dollars interest is always twice the number of years. This is expressed by the formula, $I = 2t$.

From the graph we can read either the amount of interest for a given time, or the time required to earn a given interest.

ORAL EXERCISES

From the graph read:

1. The amount of interest on \$100 at 2% for 2 yr. For 3 yr. For $1\frac{1}{2}$ yr.

2. The time in which \$100 at 2% will earn \$4. \$5. \$3.

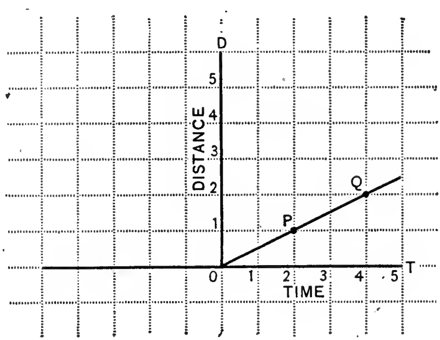
WRITTEN EXERCISES

1. Make a graph like that on page 198, taking the rate of interest to be 3%.

2. From the graph answer questions like 1 and 2 above. State the relation between interest and time in this case. Write an equation expressing this relation.

3. Treat similarly each of the rates: 4%; $2\frac{1}{2}$%; 5%; 6%.

4. In the figure values of t are represented on the horizontal line OT, and corresponding values of d on the line OD. What is the value of t for point P? Of d for point P? What is the value of t for point Q? Of d for point Q? Every value of d is what part of the corresponding value of t? Express this relation by an equation.



5. An elevator goes up at the rate of 4 ft. per second. What distance does it ascend in 2 sec.? In 3 sec.? In 5 sec.? In 1 min.?

6. Letting $d =$ the number of feet passed over in any number of seconds (t), and using horizontal spaces to represent the number of seconds, and vertical spaces to represent the number of feet, make a graph to represent the relation $d = 4t$.

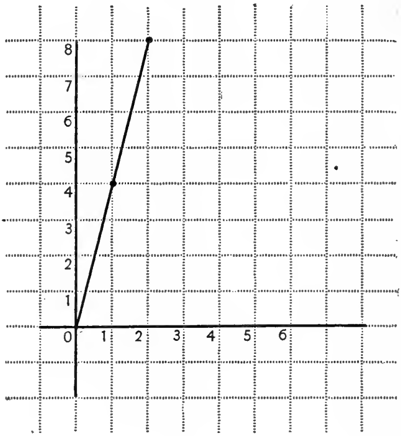
271. Before constructing a graph, the corresponding values of the letters may be conveniently arranged in a table as in the following example:

Construct the graph of $y = 4x$.

TABLE

x	y
0	0
1	4
$1\frac{1}{2}$	6
2	8
$2\frac{1}{2}$	10
3	12
4	16
5	20

GRAPH



WRITTEN EXERCISES

Construct the graphs of:

1. $2y = 3x$.
2. $4y = x$.
3. $y = \frac{1}{3}x$.
4. $y = \frac{1}{2}x$.
5. $y = \frac{4}{5}x$.
6. $\frac{1}{2}y = x$.
7. $y = \frac{3}{2}x$.
8. $y = \frac{2}{3}x$.
9. $d = \frac{1}{4}t$.
10. $d = \frac{3}{4}t$.
11. $d = 5t$.
12. $\frac{1}{3}d = t$.
13. $d = 3\frac{1}{2}t$.
14. $2\frac{1}{2}y = x$.
15. $1\frac{1}{4}x = y$.

272. Graphs may be constructed for negative values as well as for positive values of the numbers involved.

EXAMPLES

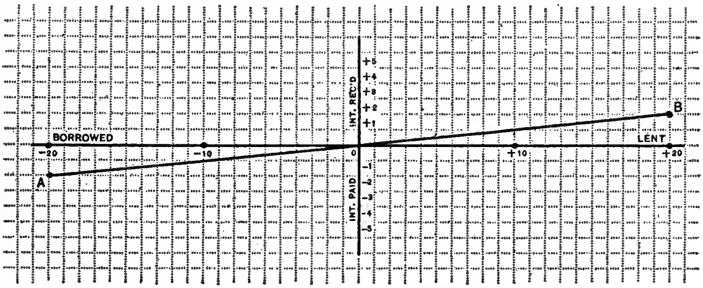
1. If we regard money borrowed (which is the opposite to money lent) as negative, and interest paid (which is the opposite to interest received) as negative, we may express by a single line the changes in interest and principal, both for money borrowed and for money lent.

The table shows the change in interest at 5% for 2 yr. corresponding to the change in the principal from + $ 20 to − $ 20, the positive values denoting money lent and interest received, the negative ones denoting money borrowed and interest paid.

TABLE

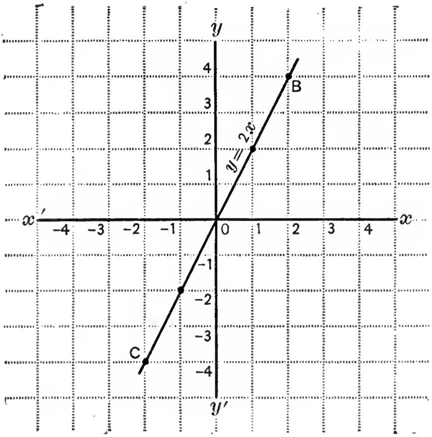
$p =$	20	10	0	− 10	− 20
$i =$	2	1	0	− 1	− 2

The line AOB is the graph representing these changes.



2. The line CB in the figure below is the graph of the equation $y = 2x$,

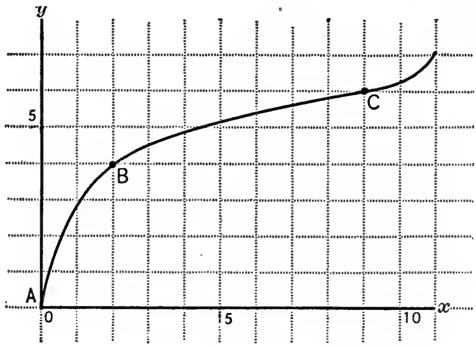
y	x
−4	−2
−2	−1
−1	$-\frac{1}{2}$
0	0
2	1
3	$1\frac{1}{2}$
4	2



for both positive and negative values. The line BO is the graph for all positive values of x and y and the line OC, the extension of OB, is the graph for negative values. The negative values of x are marked off to the left of O and the negative values of y downward from O.

273. Any change that takes place throughout at a constant rate is said to take place "uniformly."

For example, a body moving continually at the same speed is said to move uniformly. That *the graph of uniform change* will be a straight line appears from the fact that the straight line is the only path along which a point moves upward or downward uniformly. Along any curved



line it moves upward or downward more rapidly at some times than at others. Thus, in the figure the increase from A to B is much more rapid than from B to C.

If x and y vary, subject to the equation $y = 4x + 3$, or to $y = mx + b$, the change will be uniform. For, in the first case, y changes by 4 times the change in x, and, in the second case, by m times the change in x; that is, y varies uniformly with x. Since we know that only straight lines represent uniform change, we know that the graphs of the above equations are straight lines.

ORAL EXERCISES

1. How many points are necessary to fix the position of a straight line?

2. How many points must be fixed in order to draw a graph which is known to be a straight line?

3. When several points are constructed corresponding to the values of the unknowns in an equation and a straight line is drawn through any two of them, where will the others lie?

274. Constructing the graph of an equation is called **plotting** the equation.

WRITTEN EXERCISES

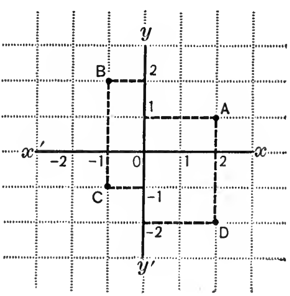
Plot the following equations:

1. $3y = x$.	7. $2x - y = 0$.	13. $y = x + 4$.
2. $y = 3x$.	8. $5p - w = 0$.	14. $y = 5x$.
3. $c = \frac{1}{2}p$.	9. $x - 6y = 0$.	15. $y = -3x$.
4. $y = \frac{1}{3}x$.	10. $y = 2x + 1$.	16. $y = x$.
5. $p = 4t$.	11. $y = 1 - 2x$.	17. $y = \frac{1}{2}x + 6$.
6. $s = \frac{1}{4}t$.	12. $y = x - 4$.	18. $y = .3x - 2$.

275. Preparatory.

1. In the diagram how many spaces is point A to the right of line yy'? How many spaces is point A above the line xx'?

The position of point A is fixed by the distances 2, 1. (It is customary to name the distance along the line xx' first.)



2. How many spaces is point B to the left of line yy'? How many spaces above xx'?

The position of point B is fixed by the distances -1, 2.

3. How far is point C to the left of yy'? How far below xx'?

The position of point C is fixed by the distances -1, -1.

4. How far is point D from yy'? From xx'? What distances fix the position of this point?

The position of point D is fixed by the distances $+2$, -2.

276. Axes. The lines of reference designated by xx' and yy' are called the **axes**.

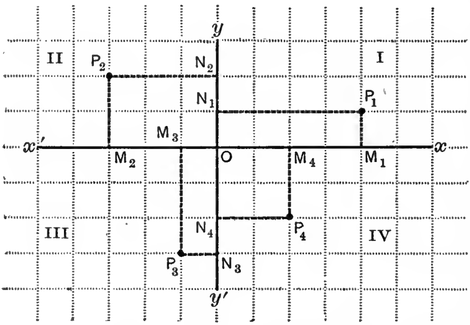
277. Quadrants. The axes divide the diagram into four quarters called *quadrants.* These are numbered I, II, III, and IV as shown in the figure in Section 280.

278. The position of the point P_1 is fixed by its perpendicular distances, P_1M_1, P_1N_1, from the axes. These two perpendicular distances are called the **coördinates** of point P_1. Similarly, the coördinates of P_2 are P_2M_2, P_2N_2.

The position of any point is fixed by its two coördinates.

279. The point of intersection of the axes is called the **origin of coördinates.**

280. Abscissas and Ordinates. The distance of a point from the axis yy', as P_1M_1, is called the **abscissa** of the point, and the distance of the point from the axis xx' is called the **ordinate** of the point.



The ordinates of all points in quadrants I and II are positive, and of those in III and IV are negative.

The abscissas of all points in quadrants I and IV are positive, and those of all points in II and III are negative.

281. Variables. A number symbol that may assume different values is called a **variable.**

Thus, the number of degrees of temperature from time to time, the number of hours from sunrise to sunset, the price of wheat, the number of inhabitants of the United States, are variable numbers. They measure physical or other quantities that are in themselves variable.

282. Constants. A number that has a fixed value is called a **constant.**

Thus, 3, $\sqrt{5}$, -4, are constants.

283. When numbers are indicated by letters, the conditions of the problem must specify which are constant and which are variable.

It is customary to use the earlier letters of the alphabet to denote constants, and the later ones to denote variables, but it is not necessary to do so.

284. Function. When the value of one variable depends upon that of another the first variable is said to be a **function** of the second.

The function is called the **dependent** variable, and the other the **independent variable.**

For example:

1. The cost of a railroad ticket depends upon the number of miles to be traveled. That is, the cost is a function of the distance. The distance is the independent variable, and the cost is the dependent variable. Likewise, the distance one can ride depends upon the cost of the ticket, that is, the distance is a function of the cost. In the latter way of looking at it, the cost is the independent variable and the distance the dependent variable.

2. If a train moves uniformly, the distance traversed is a function of the time.

3. An iron bar expands when heated. Its length is a function of the temperature.

4. $3x - 2$ is a function of x because its value depends upon that of x.

285. "Function of x" is often briefly expressed by $f(x)$.

Thus, if $\quad f(x) = 5x - 4,$

then, $\quad f(2) = 5 \cdot 2 - 4 = 6,$

$\quad f(0) = 5 \cdot 0 - 4 = -4,$

and $\quad f(r) = 5r - 4.$

286. Graphs of Functions. The **graph of a function** is a diagram representing the variation of the function due to the variation in the value of its variable.

Thus, the first graph in Sec. 271, p. 200, shows that the value of $f(x) = 4x$ varies from 0 to 8 as x varies from 0 to 2.

287. Since the graph of the function $ax + b$ is a straight line (Sec. 273), this function is called a **linear function,** and the corresponding equation, $y = ax + b$, a linear equation.

288. Graphs of Incomplete Equations. Either x or y may be lacking in an equation to be plotted.

Thus, the equation $y = 3$ means $y = 0x + 3$, in which $y = 3$ for all values of x; consequently the graph of $y = 3$ is a line parallel to the x-axis and 3 units above it. Similarly, the graph of $x = 2$ is a line parallel to the y-axis, and 2 units to the right of it.

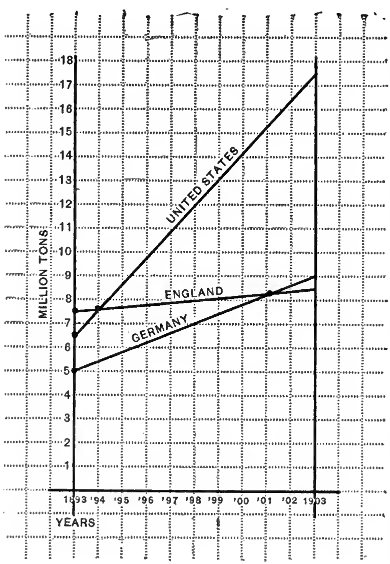
WRITTEN EXERCISES

Construct the graphs of:

1. $x + y = 1.$
2. $x + y = 0.$
3. $2y - 4x = 2.$
4. $x - y = 2.$
5. $x = 4y - 1.$
6. $3x + 2y = 6.$
7. $0 = x + 3.$
8. $2x - 2y = 4.$
9. $3x + 3y = 9.$

10. What line must the graph cross when $x = 0$? When $y = 0$? Construct the graph of $y = 2x - 3$ by using the points for which $x = 0$, and $y = 0$, respectively.

GRAPHS OF EQUATIONS WITH TWO UNKNOWNS



289. PREPARATORY.

1. The diagram shows the general trend in the increase of the pig-iron product in the United States, Germany, and England during a period of ten years.

(*a*) In which country has the increase been the most rapid? The least rapid?

(*b*) In what year was the amount produced in England and the United States the same?

(*c*) Answer the same question for England and Germany.

2. A factory has a fixed charge of \$20 daily, and makes an average gross profit of \$2 daily per workman employed. What remains after the fixed charge is paid is net profit. Make a graphic representation of how the net profit (p) varies while the number of workmen (w) varies from 0 to 50.

The graph exhibits the change in p due to a change in w subject to the above relation; or it is the graph of the equation $p = 2w - 20$.

3. A second factory has a fixed charge of \$60, and makes an average gross profit of \$3 daily per workman employed. Construct a graph to represent the net profits of the second factory in the diagram made for Exercise 2.

What is the relation between p and w, of which this is the graph?

4. From the diagram read:

(*a*) The number of workmen for which each factory makes the same net profit.

(*b*) The amount of the profit mentioned in (*a*).

(*c*) The number of workmen beyond which the second factory makes the larger net profit.

(*d*) The net profit of each factory where 25 workmen are employed. 40 workmen. 15 workmen. 10 workmen. 20 workmen.

(*e*) The number of workmen each factory must employ to make \$30 net profit. Also \$60 net profit. No net profit.

290. Graphs may be employed to represent the solution of any set of two simultaneous linear equations with two unknowns.

For example:

$$\begin{cases} x - y = 1, & (1) \\ x + 2y = 4. & (2) \end{cases}$$

We have already seen how to represent graphically all of the solutions of a single equation of the first degree in two unknowns.

If we represent the equation $x - y = 1$, and the equation $x + 2y = 4$, in the same diagram it is possible at once to read the values of x and y that satisfy both equations.

TABLE OF VALUES FOR EQUATION (1)

x	y
0	-1
1	0
2	1

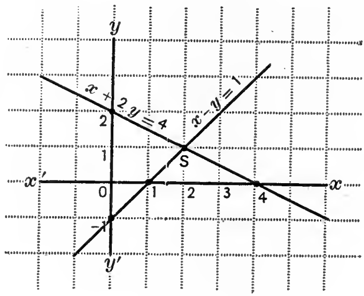


TABLE OF VALUES FOR EQUATION (2)

x	y
0	2
1	$1\frac{1}{2}$
2	1

ORAL EXERCISES

1. Read from the graph of the first equation the value of y when $x = 0$. When $x = 1$. When $x = 2$. When $x = 3$.

2. How many points are needed to fix the position of a straight line?

3. Read from the graph of the second equation the value of y when $x = 0$. When $x = 2$. When $x = 4$.

4. What point in the diagram is common to the two graphs? What are the values of x and y for point s?

5. What point in the diagram represents the solution of the system of equations?

291. *The solution of two simultaneous equations of the first degree in two unknowns is represented graphically by the point of intersection of the graphs of the equations.*

NOTE. Since the accuracy of the graphical solution depends upon the precision of the diagram, the results so found must be regarded as approximate. Their accuracy must be tested in the usual way. The graphical method of solution is of practical use chiefly in applied mathematics where an approximate result is commonly sufficient, but in theoretic mathematics it is important as exhibiting clearly to the eye the relations between the variables involved.

WRITTEN EXERCISES

Solve graphically:

1. $x+y=5$, $3x-2y=0$.
2. $3x-y=1$, $x+2y=12$.
3. $x+y=0$, $y+3=4$.
4. $x+2y=6$, $2x-y=2$.
5. $x+y=7$, $2x-y=5$.
6. $3x+y=12$, $x-y=0$.
7. $x-y=1$, $2x-3y=1$.
8. $x+3y=8$, $x-2y=-2$.
9. $2x-y=1$, $x-2y=-4$.
10. $4x+y=6$, $2x-y=0$.
11. $x+3y=12$, $3x-y=6$.
12. $x-\frac{1}{2}y=2$, $\frac{1}{3}x+y=3$.

SUMMARY

1. In graphical work what are the *axes?* The *quadrants?* The *coördinates?* The *origin?* The *abscissas?* The *ordinates?* Secs. 276–280.

2. What is a *variable?* A *constant?* Secs. 281, 282.

3. What is a *function?* What is meant by *dependent* and *independent variables?* Sec. 284.

4. What does the graph of a function show? Sec. 286.

5. When two linear equations are plotted on the same axes, what does their intersection represent? Sec. 291.

HISTORICAL NOTE

We have stated in an earlier note that the chief symbols of algebra were perfected during the sixteenth century, and that scholars had then discovered many properties of algebraic expressions as well as methods of solving equations. But at the beginning of the seventeenth century a new channel for algebraic study was opened by the French philosopher and mathematican, René Descartes, who laid the foundation for what we now call "graphical algebra."

René Descartes, was born at La Haye in 1596. At the age of twenty-one he enlisted as a soldier under Prince Maurice of Orange, pursuing the study of mathematics in his leisure hours. In 1829 he went to Holland, Dutch culture and learning then being at its height, and there devoted twenty years to the preparation of his works in mathematics and philosophy. His belief in the certainty and accuracy of the reasoning used

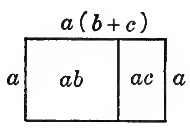
RENE DESCARTES

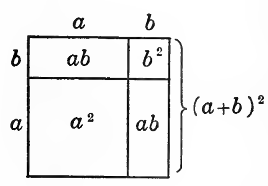
in arithmetic and algebra led him to apply the same method to physics and other sciences. Descartes adopted the plan of locating points in a plane by means of their distances from two fixed lines or axes, and was the first to plot lines representing algebraic expressions, as explained in Chapter XVIII. The words "ordinate" and "coördinates" were introduced by him, probably from the Latin phrase *lineæ ordinatæ*, used by the Roman surveyors to express parallel lines. The word *abscissa* was first used by an Italian writer (1659).

Graphical algebra, in a broader sense, was used many centuries before Descartes. For example, the products:

$$a(b+c) = ab + ac \text{ and } (a+b)^2 = a^2 + 2ab + b^2,$$

which were not considered direct results of multiplication before Diophantos, were represented graphically by Euclid (300 B.C.) in the following way:





The first step toward constructing graphs of equations was taken by the Hindoos in Bhaskara's time, when they represented positive and negative numbers by opposite segments on a straight line; from this conception it was only one step more to the expression of the relation of one variable to another by measurements along two lines, or by coördinates. This was done by Nicholas Horem, a French teacher in Paris in the fourteenth century. Naturally he began with positive numbers and constructed graphs in the first quadrant only, so it remained for Descartes to show how the changes in a function of x can be shown graphically in the four quadrants of a plane for all real values of x.

CHAPTER XIX

SYSTEMS OF LINEAR EQUATIONS

EQUATIONS WITH TWO UNKNOWNS

292. Problems involving more than one unknown often require more than one equation in their solution.

EXAMPLES

1. A street car of a certain make has 48 seats, some single and some double; the seating capacity of the car is 56. How many single seats are there? How many double seats?

SOLUTION. Let $x =$ the number of single seats,
and $y =$ the number of double ones.

Then, $\begin{cases} x + y = 48, & (1) \\ x + 2y = 56. & (2) \end{cases}$

Subtracting equation (1) from equation (2), $y = 8.$ (3)

Substituting this value of y in equation (1), $x + 8 = 48.$ (4)

$\therefore x = 40.$ (5)

Therefore there were 40 single seats and 8 double ones. (6)

TEST. $\begin{cases} 40 + 8 = 48. \\ 40 + 2 \cdot 8 = 56. \end{cases}$

2. Solve: $\begin{cases} 10x + 5y = 45, & (1) \\ 6x + 8y = 47. & (2) \end{cases}$

Multiplying (1) by 3, $30x + 15y = 135.$ (3)
Multiplying (2) by 5, $30x + 40y = 235.$ (4)
Subtracting (3) from (4), $25y = 100$, and $y = 4.$ (5)
Substituting $y = 4$ in (1), $x = 2\frac{1}{2}.$ (6)

Test the values $x = 2\frac{1}{2}$ and $y = 4$ as in equation (2).

Where in the solution was one unknown made to disappear, and how?

WRITTEN EXERCISES

Solve for x and y:

1. $x + y = 4,$
$3x + 5y = 18.$

2. $x + y = 7,$
$3x + 10y = 42.$

3. $x + y = 5,$
$2x + 5y = 22.$

4. $x + y = 0,$
$5y + 2x = -3.$

5. $2x + y = 4,$
$x + 2y = 5.$

6. $3x + 3y = 0,$
$x + 4y = 3.$

7. $x - y = 1,$
$2x + y = 5.$

8. $x - y = 1,$
$3x + 2y = 13.$

9. $x - 2y = -4,$
$2x - y = 1.$

10. $3x - 3y = 0,$
$x + y = 6.$

11. $4x - 2y = 0,$
$x + 3y = 7.$

12. $5x + 3y = 23,$
$3x + 5y = 17.$

13. $5x - 3y = 11,$
$3x - y = 5.$

14. $4x + y = -7,$
$y - 4x = 9.$

15. $2x - y = 0,$
$y + 2x = 4.$

16. $10y + x = 11,$
$y + 10x = 11.$

17. $5y - x = 3,$
$y - 5x = -9.$

18. $4y - 4x = 8,$
$y + x = 8.$

19. $2x + y = 9,$
$x + 2y = 12.$

20. $7x + y = 14,$
$x + 9y = 64.$

21. $2x + 3y = 22,$
$5x + 4y = 48.$

22. $2x + 6y = 34,$
$6x + 8y = 62.$

23. $2x + 4y = 38,$
$3x + y = 27.$

24. $x + 10y = 11,$
$\frac{1}{2}x + \frac{1}{5}y = \frac{7}{10}.$

25. $\frac{1}{2}x - \frac{1}{2}y = 6,$
$x - 2y = 0.$

26. $5x - 3y = 1,$
$2x + y = 7.$

27. $2x + 3y = 2,$
$8x - 6y = 2.$

28. $2x - 3y = 4,$
$7x - 6y = 3.$

293. In the work of solving the preceding problems the process has always been to combine the given equations so as to obtain an equation involving only one of the unknowns.

The other unknown is said to have been **eliminated**.

294. Independent Equations. Equations which express different relations between the same unknowns are called **independent equations.**

Thus, $x - y = 1$ and $x + y = 7$ are independent equations; for one expresses the *difference* between two numbers, x and y, while the other expresses the *sum* of the same two numbers.

$x - y = 1$ and $2x - 2y = 2$ are not independent, for the second when divided by 2 is the same as the first.

295. While a linear equation in one unknown has but one root, a linear equation in two unknowns is satisfied by any number of sets of values of the unknowns. In the former the unknown is a *constant*, while in the latter the unknowns are *variables.*

296. Simultaneous Equations. Two or more equations are said to be **simultaneous** when all of them are satisfied by the same values of the unknowns.

297. Systems of Equations. Two or more equations considered together are called a **system** of equations.

Thus, each of the exercises of p. 212 contains a system of equations.

298. A system of independent simultanous linear equations in which the number of equations is the same as the number of unknowns is satisfied by only one set of values of the unknowns.

299. Method of Addition and Subtraction. All systems of two independent simultaneous equations of the first degree in two unknown quantities can be solved by the **method of addition and subtraction.**

The method of addition and subtraction consists in multiplying one or both of the given equations by such numbers that the coefficients of one of the unknowns become equal. Then by subtraction this unknown is eliminated, and the solution is reduced to that of a single equation.

If the coefficients of one unknown are made numerically equal, but have opposite signs, the equations should be added.

EXAMPLE

Solve: $\begin{cases} 4x - 3y = 5, & (1) \\ 6x + 2y = 14. & (2) \end{cases}$

Multiplying (1) by 2, $8x - 6y = 10,$ (3)

and (2) by 3, $18x + 6y = 42.$ (4)

Adding (3) and (4), $26x = 52,$ (5)

and $x = 2.$ (6)

Substituting 2 for x in (2), $y = 1.$ (7)

Test. Substitute $x = 2$, $y = 1$ in equations (1), (2).

WRITTEN EXERCISES

Solve and test:

1. $2x + y = 5,$
   $x - y = 1.$
2. $3x + 2y = 7,$
   $x - 2y = 3.$
3. $2x + y = 1\frac{1}{2},$
   $x - y = 0.$
4. $3x + 2y = 2,$
   $x + y = \frac{5}{6}.$
5. $x + 5y = 35,$
   $5x + y = 31.$
6. $4x + 3y = 18,$
   $2x + 2y = 10.$
7. $x + 2y = 0,$
   $4x - 3y = 4.$
8. $2x + 3y = 4,$
   $3x + 2y = 1.$
9. $\frac{1}{2}x + \frac{1}{3}y = 4,$
   $\frac{3}{4}x + \frac{2}{3}y = 5.$
10. $.3x + .2y = .1,$
    $.2x + .3y = .4.$
11. $4x + y = 34,$
    $4y + x = 16.$
12. $4x - y = 7,$
    $3x + 4y = 29.$
13. $2x + 3y = 4,$
    $3x - 2y = -7.$
14. $2x + 3y - 8 = 0,$
    $7x - y - 5 = 0.$

15. Find two numbers whose sum is 13 and the difference between twice the first and three times the second is 3.

16. The cost of a house and lot was \$6500; the house cost \$3500 more than the lot. What was the cost of each?

17. In a recent year the value of the hay crop in the United States exceeded that of the cotton crop by \$30 millions; the two amounted to \$1180 millions. Find the value of each.

18. The value of the cotton crop in a certain year exceeded that of the wheat crop by \$50 millions; the two amounted to \$1100 millions. What was the value of each?

19. If the value of the sugar produced, in a given year, had been increased by $\frac{4}{25}$ of itself, it would have equaled the value of the barley crop; the two amounted to $108,000,000. What was the value of each?

300. Method of Substitution. The method of addition and subtraction can be used to solve all systems of two simultaneous equations with two unknowns, but occasionally problems occur in which other methods are shorter. The most useful of these is the **method of substitution.**

EXAMPLES

1. Solve: $\begin{cases} 3x+2y=6, & (1) \\ 2x-y=4. & (2) \end{cases}$

From equation (2), $y = 2x - 4.$ (3)

Substituting in (1), $3x + 4x - 8 = 6.$ (4)

$\therefore\ 7x = 14,$ (5)

and $x = 2.$ (6)

From (6) and (3), $y = 0.$ Verify. (7)

2. Solve: $\begin{cases} 4x-3y=9, & (1) \\ 5x=15. & (2) \end{cases}$

From equation (2), $x = 3.$ (3)

Substituting in (1), $12 - 3y = 9.$ (4)

$\therefore\ y = 1.$ Verify. (5)

301. *The method of substitution consists in expressing one unknown in terms of the other by means of one equation and substituting this value in the other equation, thus eliminating one of the unknowns.*

This may be the shorter method when an unknown in either equation has the coefficient 0, +1, or −1.

WRITTEN EXERCISES

Solve by substitution:

1. $x+y=75,$ $3x-3y=15.$

2. $5x+2y=31,$ $x=12y.$

3. $3x+2y=12,$ $x+y=5.$

4. $x+2y=7,$ $x=\frac{3}{2}y.$

5. $x+3y=11,$ $3x+y=9.$

6. $x-4y=8,$ $x+2y=14.$

7. $y + 15x = 53,$
$x + 3y = 27.$

8. $5x - 7y = -35\frac{1}{2},$
$2x - y = \frac{5}{7}.$

9. $3x - 5y = 31,$
$4y = -10x.$

10. $x + y = \frac{7}{6},$
$x + 7y = \frac{31}{6}.$

11. $x + y = 12,$
$3x + y = 24.$

12. $x + 2y = 10,$
$5x - 2y = 2.$

Solve and test:

13. $x + y = 29,$
$2x + 5y = 103.$

14. $x + y = 480,$
$12x + 20y = 7520.$

15. $x = 4y + 76,$
$x - y = 430.$

Show that each of the following exercises can be solved by using one equation with one unknown, or by using two equations with two unknowns:

16. The continued height of a tower and flagstaff is 100 ft.; the height of the tower is 60 ft. more than the length of the staff; find the height of each.

17. A house and lot are worth \$3500; the house is worth \$2500 more than the lot; find the value of each.

18. The area of the United States and the British Isles together is 3,146,600 sq. mi.; the area of the United States diminished by 600 sq. mi. is 25 times that of the British Isles. What is the area of each?

19. Japan's exports in a recent year plus its exports 10 yr. ago were approximately \$180,000,000; this was a gain of 400% on their value 10 yr. before. What was their value at that time?

20. It is estimated that the part of the population of the United States living on farms is $\frac{2}{3}$ of the rest of the population. Taking the total to be 93 millions, how many live on farms?

21. The average creamery of the Eastern States produces only $\frac{3}{5}$ as much butter as the average creamery of the Western States; two average eastern creameries and three average western creameries together produce 42,000 lb. annually. What is the annual output of each?

302. General Form of Linear Equations with Two Unknowns. A *general form* for an equation of the first degree with two unknowns is

$$ax + by = e.$$

A *general form* for two such equations is

$$\begin{cases} ax + by = e & (1) \\ cx + dy = f. & (2) \end{cases}$$

303. General Solution. From the general equations (*1*) and (*2*) it is possible (without knowing the values of a, b, c, d, e, f) to find a *general form* for the solution.

Solve:
$$\begin{cases} ax + by = e, & (1) \\ cx + dy = f. & (2) \end{cases}$$

Multiplying (*1*) by c and (*2*) by a,	$cax + cby = ce,$	(*3*)
	$acx + ady = af.$	(*4*)
Subtracting (*3*) from (*4*),	$ady - cby = af - ce.$	(*5*)
Thus,	$(ad - bc)y = af - ce.$	(*6*)
Dividing by the coefficient of y,	$y = \dfrac{af - ce}{ad - bc}.$	(*7*)
Eliminating by multiplying (*1*) by d, and (*2*) by b, and subtracting,	$x = \dfrac{de - bf}{ad - bc}.$	(*8*)

From these values of x and y the values in any particular case can be found by substituting for a, b, c, d, e, f, their particular values. But there are other forms of these values better adapted to substitution, and these are given in Chapter XXV.

304. Literal Equations. The general solution of Section 303 shows that the processes of elimination previously explained serve fully to solve systems of simultaneous equations with *literal coefficients.*

WRITTEN EXERCISES

Solve:

1. $12x + 6y = -18,$
   $48x - 9y = 60.$
2. $x + 17y = 300,$
   $11x - y = 104.$
3. $2x - 3y = 5a - b,$
   $3x - 2y = b + 5a.$
4. $26x + 42y = 33,$
   $39x + 28y = 44.$

5. $3x - 5y = -\frac{1}{6}$,
$6x + 2y = 3\frac{2}{3}$.

6. $12x + 6y = 13$,
$48x - 9y = 30$.

7. $5x + 45y = 145$,
$15x - 9y = 3$.

8. $\frac{x}{a} + \frac{y}{b} = 1$,
$\frac{x}{2a} - \frac{y}{3b} = 4$.

9. Solve for a and l:
$l = a + (n-1)d$.
$s = \frac{n}{2}(a + l)$.

10. $3p + q = 3$,
$5p - q = 13$.

11. $4h - 2b = 1$,
$3h + b = 5$.

12. $4v - 5w = 6$,
$-3v + 4w = 1$.

13. $14R + 3r = 2$,
$9R + 2r = 6$.

14. $(a + c)x - (a - c)y = 2ab$,
$(a + b)x - (a - b)y = 2ac$.

15. $(a+1)x - (b+1)y = c$,
$(a-1)x + (b-1)y = d$.

16. $(m - n)x + (m + n)y = mn$,
$(m - p)x + (m + p)y = mp$.

17. An investor purchases two kinds of securities; one kind pays 2 % and the other 4 % annually; his annual income from both sources is \$900; if he had invested as much in the 2 % securities as he did in the 4 per cents, and *vice versa*, his income would have been \$600. Find his investment in each.

SOLUTION. 1. Let x and y be the number of dollars invested at 2 % and 4 %, respectively.

2. Then, by the conditions, $.02x + .04y = 900$,
3. and $.04x + .02y = 600$.
4. Multiplying (2) by 2, $.04x + .08y = 1800$.
5. Subtracting (3) from (4), $.06y = 1200$.
6. Solving (5), $y = 20,000$.
7. Substituting $y = 20,000$ in (1), $.02x + 800 = 900$.
8. Solving (7), $x = 5000$.

∴ he invested \$5000 at 2 % and \$20,000 at 4 %.

TEST. 2 % of \$5000 + 4 % of \$20,000 = \$900;
2 % of \$20,000 + 4 % of \$500 = \$600.

18. An investor purchased Pennsylvania Railroad stock paying 6 % annually, and municipal bonds paying 4 %; his annual income from both was \$2100; if the stock had paid 1 % less and the bonds 1 % more, his total income would have been \$2000. How much did he invest in each?

19. An investment of $20,000 in one stock, and one of $10,000 in another, together yield $1300 annually; an investment of half as much in the first and twice as much in the second would together yield $1100 annually. What is the annual rate of dividend in each stock?

20. A standard daily ration for an adult laborer requires 4 oz. of protein and 4 oz. of fat.

The following table shows the approximate amounts of protein and fat in various foods:

Food	Per Cent of Fat	Per Cent of Protein
Mutton	37	14
Pork (fresh)	26	13
Eggs	9	13
Bread (white)	1	9
Beans (dried)	2	22
Corn (green)	1	3
Rice	$\frac{1}{2}$	8

How many ounces of mutton and bread are needed to make a standard ration for one day?

Solution. 1. Let x and y be the numbers of ounces required of mutton and bread, respectively.

2. Then $0.14\,x + 0.09\,y$ is the amount of protein in these foods according to the table.

3. Also $0.37\,x + 0.01\,y$ is the amount of fat in these foods.

4. Thus, $0.14\,x + 0.09\,y = 4,$

5. and $0.37\,x + 0.01\,y = 4.$

6. Multiplying (4) by 100, $14\,x + 9\,y = 400.$

7. Multiplying (5) by 100, $37\,x + y = 400.$

8. Subtracting (7) from (6), $23\,x - 8\,y = 0. \quad \therefore\ x = \frac{8}{23}\,y.$

9. Using (8) in (7), $\dfrac{37 \cdot 8\,y}{23} + y = 400.$

10. $\therefore\ y = 28.8$ (to one decimal place) and $x = 10.0+$

11. $\therefore$ the ration is 10 oz. of mutton and 28.8 oz. of bread.

A negative result for either unknown quantity would show that it is impossible to make up the standard ration out of the foods named.

Find which of the following combinations of foods can make a standard ration, and the number of ounces of each food required:

21. Mutton and beans.

22. Mutton and rice.

23. Bread and eggs.

24. Pork and beans.

25. Eggs and corn.

26. Mutton and corn.

27. Bread and pork.

28. Bread and rice.

29. Pork and rice.

30. Eggs and rice.

31. A man has a certain sum of money invested at 5 %; he reinvests the whole sum, placing three times as much of it at 8 % as he does at 4 %. His income is increased $200 a year by the change. How much money has he, and what is his income?

305. Solving Fractional Equations. I. *In case of fractional equations, when the unknown quantities occur only in monomial denominators, it is best not to clear of fractions.*

EXAMPLES

1. Solve:
$$\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=2. & (1) \\ \dfrac{1}{2x}-\dfrac{1}{3y}=\dfrac{5}{36}. & (2) \end{cases}$$

Multiplying (2) by 4, $$\frac{4}{2x}-\frac{4}{3y}=\frac{20}{36}. \quad (3)$$

Simplifying (3), $$\frac{2}{x}-\frac{4}{3y}=\frac{5}{9}. \quad (4)$$

Subtracting (4) from (1), $$\frac{13}{3y}=\frac{13}{9}. \quad (5)$$

Dividing (5) by 13, $$\frac{1}{3y}=\frac{1}{9}. \quad (6)$$

Clearing (6) of fractions, $$3y=9. \quad (7)$$

$$\therefore y=3. \quad (8)$$

Substituting 3 for y in (1), $$x=2. \quad (9)$$

TEST. $$\frac{2}{2}+\frac{3}{3}=2 \text{ and } \frac{1}{2\cdot 2}-\frac{1}{3\cdot 3}=\frac{5}{36}.$$

It is evident that the above equations, if cleared of fractions, would contain terms in xy which would complicate the solution.

2. In such equations the unknown quantities may be thought of as $\frac{1}{x}$ and $\frac{1}{y}$.

Thus, to solve $\frac{10}{x}+\frac{9}{y}=5$, and $\frac{35}{x}-\frac{6}{y}=5$, is to solve

$$10\left(\frac{1}{x}\right)+9\left(\frac{1}{y}\right)=5, \text{ and } 35\left(\frac{1}{x}\right)-6\left(\frac{1}{y}\right)=5.$$

The coefficients are now manipulated as in the case of integral equations. It is often convenient to introduce new unknowns in such problems; in this case, for example, by putting

$\frac{1}{x}=x'$ and $\frac{1}{y}=y'$, the equations become

$$10\,x'+9\,y'=5,$$
$$35\,x'-6\,y'=5.$$

306. The same principles apply in exactly the same way to equations with *literal coefficients.*

EXAMPLE

1. Solve:
$$\begin{cases}\frac{a}{x}+\frac{b}{y}=m, & (1)\\ \frac{c}{x}+\frac{d}{y}=n. & (2)\end{cases}$$

Regard $\frac{1}{x}, \frac{1}{y}$ as the unknowns, multiply (1) by d and (2) by b and subtract.

Then, $$\frac{ad}{x}-\frac{bc}{x}=md-nb,$$

or, $$(ad-bc)\frac{1}{x}=md-nb.$$

Hence, $$\frac{1}{x}=\frac{md-nb}{ad-bc}. \qquad (3)$$

Similarly, multiply (1) by c and (2) by a and subtract.

Then, $$(ad-bc)\frac{1}{y}=an-mc,$$

and $$\frac{1}{y}=\frac{an-mc}{ad-bc}. \qquad (4)$$

From (3), $$x=\frac{ad-bc}{md-bn}. \qquad (5)$$

From (4), $$y=\frac{ad-bc}{an-mc}. \qquad (6)$$

Test. In literal equations it is generally more convenient to test by reworking in a different way, when possible, than by substitution.

WRITTEN EXERCISES

Solve for x and y:

1. $\frac{x}{5}+y=3,$
$x+\frac{y}{2}=11.$

2. $\frac{x}{3}+\frac{3y}{2}=4,$
$\frac{5x}{6}-2y=-13.$

3. $\frac{6}{x}+5y=18,$
$\frac{8}{x}-y=1.$

4. $\frac{x}{3}+\frac{3y}{4}=-4,$
$\frac{x}{y}=\frac{3}{4}.$

5. $x+y=5,$
$\frac{x}{y}=\frac{1}{4}.$

6. $x+\frac{3}{4}-\frac{x}{3}-\frac{5y}{6}=\frac{5x}{9}+\frac{17}{36},$
$\frac{1}{7}(y-3x)+\frac{1}{3}(9x-3y)=12.$

7. $\frac{20}{x}+\frac{4}{3y}=0,$
$\frac{x}{2}-\frac{y}{5}=4.$

8. $\frac{x+1}{y}=1,$
$\frac{y+1}{x}-1=\frac{3}{2}.$

9. $\frac{7x}{5}-\frac{5y}{6}=2,$
$\frac{3x}{10}-\frac{5y}{12}=4.$

10. $\frac{1}{x}+\frac{1}{y}=\frac{3}{4},$
$\frac{1}{x}-\frac{1}{y}=\frac{1}{4}.$

11. $\frac{6}{x}+\frac{8}{y}=3,$
$\frac{12}{x}-\frac{20}{y}=-3.$

12. $\frac{1}{3x}+\frac{1}{2y}=-\frac{1}{6},$
$\frac{1}{x}+\frac{2}{y}=-1.$

13. $\frac{3}{x}-\frac{2a}{y}=b,$
$\frac{5}{2x}+\frac{3b}{y}=c.$

14. $2x+\frac{1}{y}=14,$
$4x-\frac{5}{y}=14.$

SUGGESTION. Regard x and $\frac{1}{y}$ as the unknowns.

15. $\frac{6}{x}-2y=6,$
$\frac{9}{x}+y=1.$

16. $\frac{5}{x}+3y=17,$
$\frac{2}{x}-2y=-6.$

17. $\frac{1}{x}+\frac{a}{y}=2,$
$\frac{b}{x}-\frac{1}{y}=a.$

18. $\frac{2a}{x}-\frac{3b}{y}=4,$
$\frac{3a}{x}-\frac{4b}{y}=2c.$

19. $\frac{3}{x}+\frac{2}{y}=-12,$
$\frac{4}{x}-\frac{3}{y}=1.$

20. $\frac{1}{x}+\frac{a}{y}=1,$
$\frac{a}{x}+\frac{1}{y}=1.$

21. The sum of the reciprocals of two numbers is $\frac{5}{4}$, and the difference of the reciprocals is $\frac{1}{4}$. Find the numbers.

22. Find two numbers such that 3 times the reciprocal of the first added to 5 times the reciprocal of the second makes 2, and 24 times the reciprocal of the first diminished by 10 times the reciprocal of the second makes 1.

307. II. *It is usually best to clear of fractions when the unknowns occur in polynomial denominators.*

EXAMPLES

1. Solve for x and y:
$$\begin{cases} \frac{x+1}{y+1}=\frac{1}{2}, & (1) \\ \frac{x-1}{y-1}=\frac{1}{4}. & (2) \end{cases}$$

Clearing (1) of fractions, $2x+2=y+1$, or $2x-y=-1$. (3)
Clearing (2) of fractions, $4x-4=y-1$, or $4x-y=3$. (4)
Solving (3) and (4), $x=2$ and $y=5$. (5)

2. Solve:
$$\begin{cases} \frac{2x+1}{x-4}=\frac{2y+18}{y+6}, & (1) \\ \frac{x+1}{y}=-2. & (2) \end{cases}$$

Clearing (1) of fractions, $2xy+y+12x+6=2xy+18x-8y-72.$ (3)
Collecting terms in (3), $-6x+9y=-78.$ (4)
From (2), $x+2y=-1.$ (5)
Solving (4) and (5), $x=7,\ y=-4.$ (6)

3. Solve:

$$\begin{cases} \dfrac{2x+6}{x} = y + \dfrac{10x+6}{x}, & (1) \\[2ex] \dfrac{x+4}{y+3} = -2. & (2) \end{cases}$$

Separating the fractions of (1), $\dfrac{2x}{x} + \dfrac{6}{x} = y + \dfrac{10x}{x} + \dfrac{6}{x}$. (3)

Simplifying the terms of (3), $2 + \dfrac{6}{x} = y + 10 + \dfrac{6}{x}$. (4)

Hence, $2 = y + 10$. (5)

Or, $y = -8$. (6)

Substituting $y = -8$ in (2), $x = 6$. (7)

WRITTEN EXERCISES

Solve:

1. $\dfrac{2x+3y}{1+4y} = 1,\quad \dfrac{4x-2y}{x-y} = -2.$

2. $\dfrac{x+1}{y} = \dfrac{x+5}{y+2},\quad \dfrac{2x-3y}{4x-2y} = \dfrac{1}{2}.$

3. $\dfrac{x+1}{y} = 4,\quad \dfrac{y+1}{x} = 3.$

4. $\dfrac{x}{y} = \dfrac{5}{4},\quad \dfrac{x+1}{y+1} = \dfrac{16}{13}.$

5. $\dfrac{x+5}{y-3} = \dfrac{x+1}{y-2},\quad \dfrac{x+2}{y-3} = 3.$

6. $\dfrac{x+1}{5y} = -1,\quad \dfrac{3x+2y}{y+6} = 2.$

7. $y + 1 + \dfrac{1}{3x} = 2y - \dfrac{27x-3}{9x},\quad \dfrac{1}{3} + \dfrac{y}{2x} = \dfrac{x}{3} - \dfrac{30x-5y}{10x}.$

8. $\dfrac{c-2y}{a+b-2x} \cdot \dfrac{c}{b-a} = -1,\quad \dfrac{c-2y}{b-2x} \cdot \dfrac{c}{b} = -1.$

9. $\dfrac{y}{x+a} \cdot \dfrac{c}{a} = 1,\quad \dfrac{y}{x-a} = -\dfrac{c}{a}.$

10. $\dfrac{2}{x} + \dfrac{1}{y} = 2,\quad \dfrac{3}{x} - \dfrac{2}{y} = 5.$

11. $\dfrac{2x}{3} - \dfrac{5y}{12} - \left(\dfrac{3x}{2} - \dfrac{4y}{3}\right) = -\dfrac{2}{3},\quad \dfrac{x-y}{x+y} = \dfrac{1}{5}.$

12. $3x - 5ay = 8ab,\quad \dfrac{x}{7a} - 7y + 3b = 0.$

13. $\frac{2x}{3} - 4 + \frac{y}{2} + x = 8 - \frac{3y}{4} + \frac{1}{12}, \quad \frac{y}{6} - \frac{x}{2} + 2 = \frac{1}{6} - 2x + 9\frac{5}{6}.$

14. $\frac{x+y}{x-y} = 4,$ $\quad 3x + 5y = \frac{3}{4}.$

15. $\frac{3x + 12y}{1+x} = 3,$ $\quad \frac{x+4y}{1+4y} = -\frac{1}{4}.$

16. $\frac{4(3x-2y)}{5(x+2y)} = \frac{8}{10},$ $\quad \frac{1}{3x} + \frac{3}{5} = 0.$

TYPES OF LINEAR SYSTEMS

308. Certain systems of equations have more than one solution.

EXAMPLES

1. Solve: $3x + y = 5.$

We may express x in terms of y, or express y in terms of x, but neither is eliminated. To obtain sets of numerical values for both of them we may give arbitrary values to one, and find the corresponding values for the other. In the above equation, if we give x the values,

$-2, -1, 0, 1, 2$; and so on.

y will have the values, $11, 8, 5, 2, -1$; and so on.

2. Solve:

$$\begin{cases} x + y = 3, \\ 2x + z = 5. \end{cases}$$

This is a system of two equations with three unknowns. We can dispose of only one of them by the methods of elimination, hence the result is an equation in two unknowns which may have any number of solutions as in the case of Exercise 1. Thus, the given system has an unlimited number of solutions.

309. Indeterminate Systems. Systems of equations like those in Section 308, admitting an unlimited number of solutions, are called *indeterminate systems.*

310. The equations of a system may be dependent.

EXAMPLE

Solve:

$$\begin{cases} x - 2y = 1, \\ 3x - 6y = 3. \end{cases}$$

If the first equation is multiplied by 3, the result is the second equation. Hence, nothing is gained by attempting to eliminate one of the unknowns.

311. Dependent Equations. If, in a system of equations, two or more equations express the same relation between the unknowns, the equations are said to be *dependent*.

In such a case, if the number of unknowns is the same as the number of equations, the system is indeterminate.

312. The equations of a system may be inconsistent.

EXAMPLE

Solve: $\begin{cases} x + 2y = 3, \\ 2y + x = 4. \end{cases}$

The first equation asserts that $x + 2y = 3$, but the second equation asserts that $x + 2y = 4$. Evidently both of these relations cannot be true for the same values of x and y. If equation (1) is subtracted from equation (2), the result, $0 = 1$, shows that they are incompatible.

313. Inconsistent Equations. Equations which express contradictory relations between the unknowns are called *inconsistent*, or *incompatible, equations*.

314. Number of Solutions. Two linear equations in two unknowns may be:

1. Determinate and have one solution.
2. Dependent, and have an unlimited number of solutions.
3. Contradictory and have no solution.

ORAL EXERCISES

Classify the following systems according to Secs. 308–313:

1. $\begin{cases} x + y = 2, \\ x + y - z = 3. \end{cases}$
3. $\begin{cases} 3x - y = 2, \\ 9x - 3y = 6. \end{cases}$
5. $\begin{cases} ax - by = c, \\ ax - by = bc. \end{cases}$
2. $\begin{cases} 2x - y = 4, \\ 4x - 2y = 12. \end{cases}$
4. $\begin{cases} 2x - y = 7, \\ x + y = 5. \end{cases}$
6. $\begin{cases} 5x - y = 4, \\ 10x - 8 = 2y. \end{cases}$

INTERPRETATION

315. Problems may lead to equations whose solutions are inconsistent with the given conditions. Hence, any solutions that do not admit of interpretation should be rejected.

EXAMPLES

1. A shelf contained 20 books; some were histories and the rest biographies. If 4 times the number of biographies, less 2 times the number of histories, was 35, how many were there of each kind?

SOLUTION.

$$x + y = 20, \qquad (1)$$
$$4y - 2x = 35. \qquad (2)$$

Multiplying (1) by 2 and adding, $6y = 75$.

Therefore, $y = 12\frac{1}{2}$, and $x = 7\frac{1}{2}$.

DISCUSSION. The equations are correctly written and solved, but the values show that the conditions of the problem are impossible, because we cannot have a fractional number of books.

2. If the sum of the length and width of a rectangular garden is 5 ft. and the difference between the dimensions is 19 ft., find the length and width of the garden.

SOLUTION.

$$x + y = 5, \qquad (1)$$
$$x - y = 19. \qquad (2)$$

Adding (1) and (2), $2x = 24$, and $x = 12$. (3)

Substituting x in (2), $y = -7$. (4)

DISCUSSION. The equations are correctly written and solved; hence, the statement of the problem is at fault, because an actual garden could not have a side measuring -7 ft.

WRITTEN EXERCISES

1. If the sum of two consecutive even numbers is 26, and one of them is $\frac{5}{6}$ of the other, find the numbers.

2. If a positive fraction is equal to $\frac{1}{2}$, and the numerator exceeds the denominator by 5, find the fraction.

3. If the sum of two integers is 24, and one of them is $\frac{3}{4}$ of the other, find the numbers.

4. The difference between two numbers is 2, and one of them is 10. Find the other.

5. If there are two numbers such that the first plus 3 times the second equals 8, and twice the first plus 6 times the second equals 15, find the numbers.

EQUATIONS WITH THREE OR MORE UNKNOWNS

316. The definitions and methods given for the solution of two equations with two unknowns may be applied equally well to a greater number of equations and unknowns.

To solve three linear equations with three unknowns, eliminate one unknown from any pair of the equations and the same unknown from any other pair; two equations are thus formed which involve only two unknowns and which may be solved by methods previously given.

Four or more equations with four or more unknowns may be solved similarly.

EXAMPLE

Solve:
$$\begin{cases} x - 2y + 3z = 2, & (1) \\ 2x - 3y + z = 1, & (2) \\ 3x - y + 2z = 9. & (3) \end{cases}$$

To eliminate x from (*1*) and (*2*):

Multiplying (*1*) by 2, $2x - 4y + 6z = 4.$ (*4*)

Subtracting (*2*) from (*4*), $-y + 5z = 3.$ (*5*)

To eliminate x from (*1*) and (*3*):

Multiplying (*1*) by 3, $3x - 6y + 9z = 6.$ (*6*)

Subtracting (*3*) from (*6*), $-5y + 7z = -3.$ (*7*)

To eliminate y from (*5*) and (*7*):

Multiplying (*5*) by 5, $-5y + 25z = 15.$ (*8*)

Subtracting (*7*) from (*8*), $18z = 18.$ (*9*)

$\therefore z = 1.$ (*10*)

To eliminate z from (*8*):

Substituting $z = 1$ in (*8*), $-5y + 25 = 15.$ (*11*)

Solving (*11*), $y = 2.$ (*12*)

Substituting $y = 2$, $z = 1$ in (*1*), $x - 4 + 3 = 2.$ (*13*)

Solving (*13*), $x = 3.$ (*14*)

TEST. (*2*) $2 \cdot 3 - 3 \cdot 2 + 1 \cdot 1 = 1.$
(*3*) $3 \cdot 3 - 1 \cdot 2 + 2 \cdot 1 = 9.$

Since x was found in step (*13*) by substituting $y = 2$, $z = 1$ in equation (*1*), it is not necessary to substitute again in this equation when testing the results.

317. Literal equations are solved in the same way; when fractions are involved use the method of Sections 305 and 307.

WRITTEN EXERCISES

Solve:

1. $2x+3y+4z=20,$
$3x+4y+5z=26,$
$3x+5y+6z=31.$

2. $x+y+z=5,$
$x+y-z=7,$
$x-y-z=3.$

3. $x+2y=7,$
$y+2z=2,$
$3x+2y=z-1.$

4. $5x+3y=65,$
$2y-z=11,$
$3x+4z=57.$

5. $y+z=-a,$
$x+z=-b,$
$x+y=-c.$

6. $x+y-z=1,$
$8x+3y-6z=1,$
$4x+y-3z=1.$

7. $x+y+2z=2(b+c),$
$x+2y+z=2(a+c),$
$2x+y+z=2(a+b).$

8. $\frac{1}{2}x+\frac{1}{3}y=12-\frac{1}{6}z,$
$\frac{1}{2}y+\frac{1}{3}z=8+\frac{1}{6}x,$
$\frac{1}{2}x+\frac{1}{3}z=10.$

9. $x+ay=b,$
$ax+z=c,$
$z+cy=a.$

10. $x+\frac{1}{2}y=100,$
$y+\frac{1}{3}z=100,$
$z+\frac{1}{4}x=100.$

11. $7x+13y=205,$
$14x+5z=300,$
$12y+20z=140.$

12. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=31,$
$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=23.5,$
$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=19.$

13. $x+z=1-cy,$
$y+z=-cx,$
$x+y=-1-cz.$

14. $x=6+\frac{y}{3},$
$y=4+\frac{z}{5},$
$z=8+\frac{x}{4}.$

15. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1,$
$\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=-3,$
$\frac{3}{x}-\frac{4}{y}-\frac{5}{z}=14.$

Suggestion. First solve for $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}.$

16. $ax+by=1,$
$cy-az=1,$
$bz-cx=1.$

Suggestion. Multiply the equations by c, b, a, respectively, and add.

17. A man has three sums invested, the first at 4 %, the second at 5 %, and the third at 6 % per annum. The annual yield of the first and second sums together is $220, that of the first and third together is $420, and that of the second and third is $400. How many dollars are there in each sum?

18. A tourist spent $520 on a trip. If he had cut down his transportation expenses $\frac{1}{3}$, his hotel bill $\frac{1}{4}$, and his miscellaneous expenses $\frac{1}{2}$, his trip would have cost him $350. If he had cut down his transportation expenses $\frac{1}{2}$, increased his hotel bills by $\frac{1}{3}$, and his miscellaneous expenses by $\frac{1}{4}$, the trip would have cost him $535. Find the amount he actually spent for each of the three items.

19. Find three numbers such that the difference between the reciprocals of the first and second is $\frac{1}{6}$, between the reciprocals of the first and third is $\frac{1}{4}$, and the sum of the reciprocals of the second and third is $\frac{7}{12}$.

20. The sum of the reciprocals of three numbers is $\frac{13}{6}$; the difference between the reciprocals of the first and second equals that between the reciprocals of the second and third. The third number is twice the first. Find the numbers.

21. Three brothers, A, B, C, at a family reunion were discussing their ages. C said to A, "Thirty years ago my age was double yours." Then B said to A, "Twenty-three years ago *my* age was double yours." If C's present age exceeds A's by four years, and B's exceeds A's by eleven years, find the age of each.

318. Equations with More than Three Unknowns. The solution of a particular example will serve to indicate the method to be used when there are more than three unknowns.

EXAMPLE

Solve:

$$\begin{cases} w + x + y + z = 8, & (1) \\ 2w - \frac{1}{2}x + \frac{1}{3}y - z = 1, & (2) \\ 3w - 2x - y + 2z = 17, & (3) \\ 5w - x - 2y - 3z = -9. & (4) \end{cases}$$

To eliminate z:

Adding (1) and (2),	$3w + \frac{1}{2}x + \frac{4}{3}y = 9.$	(5)
Subtracting twice (1) from (3),	$w - 4x - 3y = 1.$	(6)
Adding three times (1) and (4),	$8w + 2x + y = 15.$	(7)

To eliminate x:

Subtracting (7) from 4 times (5),	$4w + \frac{13}{3}y = 21.$	(8)
Adding twice (7) to (6),	$17w - y = 31.$	(9)

To solve for w:

Substituting y from (9) into (8),	$4w + \frac{13}{3}(17w - 31) = 21.$	(10)
Solving (10),	$233w = 466$, and $w = 2.$	(11)

To solve for the remaining unknowns:

Substituting $w = 2$ in (9),	$y = 3.$	(12)
Substituting $y = 3$, $w = 2$ in (7),	$16 + 2x + 3 = 15.$	(13)
Solving (13),	$x = -2.$	(14)
Substituting $x = -2$, $y = 3$, $w = 2$ in (1),	$2 - 2 + 3 + z = 8.$	(15)
Solving (15),	$z = 5.$	(16)

Test. (1) Used in (15),
(2) $2 - \frac{1}{2}\cdot(-2) + \frac{1}{3}\cdot 3 - 5 = 1.$
(3) $3\cdot 2 - 2\cdot(-2) - 3 + 2\cdot 5 = 17.$
(4) $5\cdot 2 - (-2) - 2\cdot 3 - 3\cdot 5 = -9.$

In elementary algebra little emphasis should be placed on problems with more than three unknowns; in later mathematics better and more general methods are given for dealing with them, together with proofs of the assumptions which are tacitly made at this stage.

WRITTEN EXERCISES

Solve:

1. $v + w = 6,$
$2 - w = 8,$
$3z - 2v = 10,$
$2x - y = 12,$
$y + z = 14.$

2. $x - 2y + 32 = 32,$
$2w - 3v + 4x = 13,$
$4y - 2 + 3w = -4,$
$3z - 7w + 5v = 0,$
$5v - x + 3y = -14.$

3. $\frac{1}{3}x + 3y = 23,$
$x + \frac{1}{4}z = 8,$
$y + 3z = 31,$
$x + w + y = 22.$

4. $w + x + y = 9,$
$x + y + z = 9,$
$y + z + w = 9,$
$z + w + x = 9.$

Suggestion. Add the equations, divide by 3, and subtract each from the result.

5. $3w - x + y - 2z = -3,$
$w - 3x - 2y + z = 1\frac{1}{2},$
$5w + 2x - 3y + 2z = 19,$
$2w - 2x + 4y - 3z = -15\frac{1}{2}.$

6. $w - x + y = 6,$
$x - y + z = -4,$
$y - z + w = 5,$
$z - w + x = -6.$

7. $w + y + z = 2x,$
$w + x + 2 = 3y,$
$w + x + y = 42,$
$w - x - y = w.$

8. $7x - 3y = 1,$
$8z - 7x = 1,$
$4z - 7y = 1,$
$14x - 3w = 1.$

9. $\frac{1}{x} - \frac{1}{y} = \frac{1}{6},$
$\frac{1}{y} - \frac{1}{z} = \frac{1}{12},$
$\frac{1}{x} + \frac{1}{z} = \frac{3}{4}.$

10. $\frac{a}{x} + \frac{b}{y} = m,$
$\frac{b}{y} + \frac{c}{z} = n,$
$\frac{a}{x} + \frac{c}{z} = p.$

11. $x + y + z = 1,$
$5x + y - .22 = .5,$
$2x + 3y + 3z = 1.$

REVIEW

WRITTEN EXERCISES

Solve and test:

1. $x + y = 27,$
$x - y = 17.$

2. $3x + 5y = 19,$
$7x - 4y = 13.$

3. $\frac{1}{3}(7 + x) = \frac{1}{5}(9 + y),$
$\frac{1}{7}(11 + x + y) = \frac{1}{5}(9 + y).$

4. $bx + ay = b,$
$ax - by = a.$

5. $x + y + z = 4,$
$2x + 3y - z = 1,$
$3x - y + 2z = 1.$

6. $\frac{1}{8}x + \frac{1}{4}y = 5,$
$\frac{5}{6}x - \frac{3}{8}y = 8\frac{5}{6}.$

7. $2(3x + 1) - 3(4y - 26) = 2,$
$3(x - 5) + 2(y - 14) = 6.$

8. $2p + 3w = 1,$
$5p + 7w = 6.$

9. $4h + 7v = 9,$
$2h + 5v = 6.$

10. $2x + 6y + 7z = 4,$
$3x + 8y + 9z = -2,$
$4x + 9y + 10z = 1.$

11. $4x + 11y + 2z = 33,$
$15x + 39y + 7z = 115,$
$23x + 56y + 10z = 162.$

12. $2x - 9y = -1,$
$5x - 24y = 2.$

13. $\frac{x}{a}+\frac{y}{b}=7,$

$\frac{x}{2}+\frac{y}{3}=2a+b.$

14. $3x+5y=1,$

$4x+6y=3.$

15. $\frac{x}{a}+\frac{y}{b}=1,$

$\frac{x}{b}-\frac{y}{a}=1.$

16. $6x+2y+8z=1,$

$9x+8y+11z=-1,$

$15x+12y-3z=-1.$

17. $x-2y+3z-u=5,$

$y-2z+3u-x=0,$

$z-2u+3x-y=0,$

$u-2x+3y-z=0.$

18. Solve for a and b:

$$ac+bq=d,$$
$$ad+5b=q.$$

Solve the equations of Exercise 18 for:

19. a and q.

20. c and d.

21. c and q.

22. b and d.

23. A certain number is written with two digits; twice the tens' digit plus the units' digit makes 9; when the digits are interchanged, the number formed is 27 greater than the given number. Find the original number.

Suggestion. 1. Let $x=$ the tens' digit, and $y=$ the units' digit.

2. Then the number is $10x+y$, and $10y+x$ is the number with digits interchanged.

3. Then, by the conditions of the problem,

$$2x+y=9; \text{ and } 10y+x-27=10x+y.$$

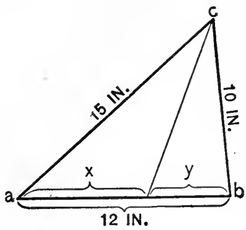
4. Solve these equations for x and y.

24. In a certain number of two digits the sum of three times the tens' digit and twice the units' digit is 36; if the digits are interchanged, the number formed is 27 greater than the original number. What is the original number?

25. The sum of two numbers is 1.6 and their difference is .2. What are the numbers?

26. The difference between two numbers is 30, and the less is $\frac{2}{5}$ of the greater. What are the numbers?

27. Two numbers are such that if the first is increased by 14, the result is twice the second; if the second is diminished by 12, the result is $\frac{1}{3}$ the first. What are the numbers?



28. The angle c in the figure is bisected; and it is known that the bisector of an angle divides the opposite side of the triangle into parts proportional to the adjacent sides; what is the ratio of x to y in the figure? What is the sum of x and y? Find the length of each segment.

29. A person leaves an estate worth $13,000; some of it is willed to a college, and 12 times as much to an eldest son, whose share is $1\frac{1}{2}$ times as much as that of each of his 2 brothers, and $1\frac{2}{3}$ times that of each of 5 sisters. Find the amount left the college.

30. The sum of two numbers is a^2 and their difference is b^2; what are the numbers?

31. A and B play at a game with counters. In the first game A loses as many counters as B has; in the second game B loses as many counters as A then has; at the end of the second game A has 16 counters and B has 4. How many had each at first?

32. A certain number is twice another; their difference divided by their sum equals the smaller. Find the numbers.

33. What fraction becomes equal to $\frac{2}{3}$ if the numerator and the denominator are each increased by 1, and equal to $\frac{1}{2}$ if they are each diminished by 1?

SUGGESTION. Let $\frac{x}{y}$ be the fraction.

34. A fraction whose value is $\frac{5}{6}$ assumes the value $\frac{7}{8}$ if the numerator and the denominator are each increased by 8. Find the fraction.

35. A fraction becomes equal to $\frac{2}{3}$, if the numerator and the denominator are each increased by 3; and equal to $\frac{4}{7}$, if the numerator and the denominator are each diminished by 3. Find the fraction.

36. The number 128 is the sum of two numbers such that $\frac{1}{7}$ of one equals $\frac{1}{25}$ of the other. What are these numbers?

37. A certain capital is invested in two kinds of securities, one paying 4%, the other $4\frac{1}{2}$%; $\frac{2}{3}$ of the capital is invested in the first kind and the rest in the second; the total income is $75. What is the capital?

38. A certain kind of woolen cloth 1 yd. wide shrinks $\frac{1}{20}$ of its length and $\frac{1}{16}$ of its width in washing. How many yards must be bought in order to have 38 sq. yd. after shrinking?

39. A band of smugglers found a cave, which would exactly hold the cargo of their boat, namely, 13 bales of silk and 33 casks of rum. While they were unloading a revenue cutter was sighted, and they sailed away, leaving 9 casks and 5 bales; these filled only one third of the cave. How many bales alone would the cave hold? How many casks?

SUGGESTION. Let x equal the number of bales alone required to fill the cave and y the number of casks. Then, 1 bale will fill $\frac{1}{x}$ of the cave and 1 cask will fill $\frac{1}{y}$ of it; also according to the problem $\frac{13}{x}+\frac{33}{y}=$ the whole capacity of the cave, or 1.

Similarly, form the equation corresponding to $\frac{1}{3}$ of the capacity.

40. A man had two sums invested, one at 4 %, the other at 5 %, simple interest, and thus received $500 annually. If the rates of interest had been 5 % and 6 %, respectively, he would have received $110 more per annum. Find the sums invested.

41. A man can walk $2\frac{1}{2}$ miles an hour up hill and $3\frac{1}{2}$ miles an hour down hill. He walks 56 miles in 20 hours on a road no part of which is level. How much of it is up hill?

42. Three thousand dollars was given annually to a college to provide annual scholarships of grades a, b, c. When two of grade a, six of grade b, and one of grade c were granted, the gift was just sufficient; similarly, when four of grade a, two of grade b, and two of grade c were granted, and also when one of grade a, five of grade b, and five of grade c were granted, the gift was sufficient. What was the value of each scholarship?

43. The number of adults and the number of children likely to attend a certain entertainment was estimated in advance; the sum to be raised by the entertainment was \$550; if the admission for adults was fixed at 40 cents and that for children at 30 cents, the estimated receipts would lack \$90 of the required amount; but if the admission was fixed at 50 cents for adults and 25 cents for children, the exact sum would be raised. How many of each were expected to attend?

44. A tank contains 20 gal. of water, and water flows in at the rate of 5 gal. per minute. At the same time a second tank contains 50 gal. of water, and water flows in at the rate of 2 gal. per minute. Construct a graph to represent the amount of water in the first tank for each minute from 0 to 15. In the same figure draw a graph to represent the amount of water in the second tank for each minute. From the graph read at what time the two tanks will contain equal amounts of water and what the amount is. Verify by solving algebraically.

45. A merchant pays \$10 rent weekly. His profits on his sales average 20%. Represent graphically his net profits corresponding to weekly sales ranging from 0 to \$200. A second merchant sublets part of his store for \$5 per week more than his own rental, but he makes only 10% average profits on his sales. In the same figure represent his total profits for sales ranging from 0 to \$200. From the graph read the amount of sales for which both merchants make the same net profit. Verify by solving algebraically.

46. If tin and lead lose, respectively, $\frac{5}{37}$ and $\frac{2}{23}$ of their weights when weighed in water, and a 60-lb. mass of lead and tin loses 7 lb., find the weight of the tin in this mass.

47. What are the sides of a rectangle such that: (a) the area is not changed if the base is diminished by 2 and the altitude increased by 2; (b) the area is increased by 10, if both base and altitude are increased by 1?

48. The income from a certain investment is devoted to scholarships of two grades; the higher grade receives $200 more than the lower per scholarship. When there are 7 students holding the lower grade and 7 holding the higher, the income exceeds the expense by $100; but the income is exactly sufficient to provide 8 scholarships of the higher grade and 5 of the lower grade. Find the amount of the income, and the amount of each grade of scholarships.

49. The sum of the reciprocals of the first and third of three numbers is twice the reciprocal of the second; the reciprocal of the third is 4 times that of the first; the sum of the reciprocals of the first and second is 7. Find the numbers.

50. A man has three debtors, of whom A and B together owe him 60 pounds, A and C 80 pounds, and B and C 92 pounds. How much did each one owe? (Saunderson's *Algebra*, 1740.)

51. A vessel filled with water has three orifices, A, B, C. If all three are opened, it is emptied in 6 hr.; through B alone it is emptied in $\frac{3}{4}$ of the time that it would take through A alone; and the time through C is twice as great as through B. In what time is the vessel emptied through each orifice alone? (Bossut's *Algebra*, 1773.)

52. The price of a house is 100 dollars. A could pay for it if he had half of B's money in addition to his own; B could pay for it if he had one third of C's; and C could pay for it if he had one fourth of A's money. How much had each? (Euler's *Algebra*, 1770.)

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What are *independent equations?* Sec. 294.

2. What are *simultaneous equations?* Sec. 296.

3. State and explain a method of solving simultaneous equations. Another method. Secs. 299, 301.

4. When should the method of adding be preferred? Sec. 299.

5. When should the method of substitution be preferred? Sec. 301.

6. What is a general form of a system of two simultaneous linear equations? Sec. 302.

7. State the formulas for the values of the unknowns in a system of two simultaneous linear equations. Sec. 303.

8. In solving fractional simultaneous equations, when is it best *not to clear of fractions?* When may the equations be cleared of fractions? Secs. 305, 307.

9. When are equations *dependent?* Sec. 311.

10. When are two equations *inconsistent* or *contradictory?* Sec. 313.

11. What is meant by *interpretation of results?* Sec. 315.

12. State a method for solving a system of three or more simultaneous linear equations of the first degree. Sec. 316.

CHAPTER XX

INVOLUTION AND EVOLUTION

INVOLUTION

319. The operation of raising an expression to a given power is called **involution.**

An important case is the involution of a binomial.

320. PREPARATORY.

1. We have already found that

$$(a+b)^2 = a^2 + 2\,ab + b^2$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

$(a+b)^4$ may be found by multiplying the last result by $a+b$. Thus,

$$\begin{array}{rl} & a^3 + 3\,a^2b + 3\,ab^2 + b^3 \\ & a \ + b \\ \hline & a^4 + 3\,a^3b + 3\,a^2b^2 + \quad ab^3 \\ & \qquad\quad a^3b + 3\,a^2b^2 + 3\,ab^3 + b^4 \\ \hline \therefore\ (a+b)^4 = & a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4 \end{array}$$

2. From this find similarly $(a+b)^5$.
3. From $(a+b)^5$ find similarly $(a+b)^6$.

321. The result of multiplying out a power of a binomial is called a **binomial expansion.**

322. The coefficients of the successive powers of a binomial may be arranged in a triangular table:

EXPANSIONS	COEFFICIENTS
$(a+b)^0 = 1$	1
$(a+b)^1 = a + b$	1 1
$(a+b)^2 = a^2 + 2\,ab + b^2$	1 2 1
$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$	1 3 3 1
$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$	1 4 6 4 1

Supply the next two lines, using the results of Exercises 2 and 3, Section 320.

323. Binomial Coefficients. The table on the right in Section 322 is called **Pascal's triangle,** and the numbers the **binomial coefficients.**

Each number of Pascal's triangle is the sum of the number directly above it and the number to the left of that.

That this must be so follows from the process of multiplying by $a + b$, and is readily seen when the method of detached coefficients is used (Chapter XXIV). The table enables us easily to write expansions of successive powers of $a + b$.

324. Any expression that can be put into the form of a binomial expansion may be written as a power of a binomial by inspection.

For example:

1. $16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$
$= (2x)^4 - 4 \cdot 8x^3y + 6 \cdot 4x^2y^2 - 4 \cdot 2xy^3 + y^4$
$= (2x)^4 + 4 \cdot (2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$
$= (2x - y)^4.$

2. $a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$
$= (a+b)^3 + 3(a+b)^2(-c) + 3(a+b)(-c)^2 + (-c)^3$
$= (\overline{a+b} - c)^3.$

ORAL EXERCISES

Express as a power of a binomial:

1. $-a^3 + 3a^2 - 3a + 1.$
2. $a^{12} + 3a^8 + 3a^4 + 1.$
3. $1 - 3x^4 + 3x^8 - x^{12}.$
4. $x^6 - 3x^4y^2 + 3x^2y^4 - y^6.$
5. $8x^3 - 12a^2b + 6ab^2 - b^3.$
6. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.$
7. $x^4 - 4x^3 + 6x^2 - 4x + 1.$
8. $(2x)^4 - 4(2x)^3 + 6(2x)^2 - 4(2x) + 1.$
9. $16x^4 + 32x^3 + 24x^2 + 8x + 1.$
10. $16x^4 - 32x^3 + 24x^2 - 8x + 1.$
11. $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1.$
12. $a^2 + 2ab + b^2 - 2ac - 2bc + c^2.$

13. $(2a)^4 + 4(2a)^3 + 6(2a)^2 + 4(2a) + 1$.

14. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

15. $(a-b)^4 - 4(a-b)^3c + 6(a-b)^2c^2 - 4(a-b)c^3 + c^4$.

16. $x^6 - 6x^5(2y) + 15x^4(2y)^2 - 20x^3(2y)^3 + 15x^2(2y)^4 - 6x(2y)^5 + (2y)^6$.

17. State in order the coefficients in the expansion of a binomial of the fourth degree. Of the third degree. Of the fifth degree.

325. The Binomial Formula. It can be proved that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots.$$

This is known as the **binomial formula.**

The factor 1 is understood in each denominator; and by denoting the product $1\cdot 2\cdot 3$ by $3!$ (read "three factorial"), and generally $1\cdot 2\cdot 3 \cdots k$ by $k!$, the above formula can be written:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \cdots.$$

326. The expansion of $(a+b)^n$ may be written by observing that:

1. *The first term* a^n, *the last is* b^n, *and the number of terms is* $n+1$.

2. *The exponent of* a *is one less in each succeeding term.* b *occurs in the second term, and its exponent increases by one in each succeeding term.*

3. *The coefficient of any term is the coefficient of the next preceding term multiplied by the exponent of* a *in that term, and divided by one more than the exponent of* b.

4. *All of the signs are plus if the sign of* b *is plus, or alternately plus and minus if the sign of* b *is minus.*

WRITTEN EXERCISES

1. Test the binomial formula just given for $n = 3$.

It should reduce to the known expression for $(a+b)^3$. The formula comes to an end because the factor $n-3$ occurs in all the terms after a certain one, and when $n = 3$, this factor is zero.

2. Test similarly for $n = 4, 5, 6, 2, 1$.

Expand:

3. $(x+y)^4$.
4. $(x-y)^4$.
5. $(x-y)^5$.
6. $(x+y)^5$.
7. $(x-1)^5$.
8. $(x-2y)^6$.
9. $(ab+1)^4$.
10. $(bc-1)^4$.
11. $(ab-cd)^4$.
12. $(t-u)^6$.
13. $(3a+b)^7$.
14. $(x+5)^5$.

15. Write the next two terms of the binomial formula as given in Sec. 325.

16. Observe that if the successive terms were written according to the same law, the *tenth* term of the binomial formula would be $\frac{n(n-1)(n-2)\cdots(n-8)}{9!}a^{n-9}b^9$. Write the fifteenth term.

17. By reference to the binomial formula, state the number of the last term written in the following expression:

$$(x+2y)^8 = x^8 + 8x^7(2y) + \frac{8\cdot 7}{2!}x^6(2y)^2 + \cdots$$
$$+ \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{6!}x^2(2y)^6 + \cdots.$$

Write the first three terms and the seventh term in the expansion of each of the following:

18. $(x+y)^{12}$.
19. $(1+5x)^{14}$.
20. $(ab-y)^{11}$.
21. $\left(\frac{x}{2}+\frac{y}{3}\right)^{12}$.
22. $(2x-1)^9$.
23. $(4x+3)^{20}$.

24. Reduce the terms written for Exercise 21 to their simplest form.

25. If the expansions in Exercises 18–23 were written out in full, how many terms would each have?

26. In each of Exercises 18–23, determine whether or not there is a middle term in the expansion. If there is a middle term:

(*a*) Determine its number.
(*b*) Write it out, without simplifying.
(*c*) Simplify the result.

Expand:

27. $\left(\frac{1}{a^2} - \frac{a}{3}\right)^5$.

28. $(a^2b - \frac{2}{3})^7$.

29. $\left(\frac{1}{x} - \frac{2}{y}\right)^5$.

30. $(x^p - 5\,y^q)^5$.

31. Write the first five terms of $\left(y^2 - \frac{b}{2\,y}\right)^8$.

32. Write the first five terms of $\left(2\,x - \frac{y^2}{3}\right)^6$.

33. Write the first four terms of $(2\,x^2 - y^3)^8$.

Write the last four terms of:

34. $(x^2 - \frac{1}{3})^9$.

35. $\left(x^2 - \frac{y}{2}\right)^6$.

EVOLUTION

327. The process of extracting an indicated root is called **evolution.** The most important case of evolution is the extraction of square root.

All the numbers that we have hitherto considered, whether positive or negative, have positive squares; none of them has a negative square, consequently the square root of a negative number (as, for example, $\sqrt{-5}$) has no meaning at this stage of our work, but will be explained in Chapter XXVIII.

328. Square Root by Inspection. The formula

$$[\pm(a + b)]^2 = a^2 + 2\,ab + b^2.$$

shows that when one of the terms of the trinomial is twice the product of the square roots of the other two, the trinomial is the square of the sum of these square roots. By aid of this relation the square roots of certain trinomials can be found readily by inspection.

For example:

$\sqrt{4\,a^2 - 12\,ab + 9\,b^2} = \pm\,(2\,a - 3\,b)$, since $-12\,ab$ is twice the product of the square roots of $4\,a^2$ and $9\,b^2$. These roots must be taken with opposite signs in this case because twice their product is to be negative.

ORAL EXERCISES

Find the square root of:

1. $x^2 + 4\,x + 4$.
2. $4\,x^2 + 8\,x + 4$.
3. $a^2b^2 + 2\,ab + 1$.
4. $a^2 - 4\,ab + 4\,b^2$.
5. $4\,m^2 - 4\,mn + n^2$.
6. $x^4 - 4\,x^2 + 4$.
7. $16\,x^8 + 8\,x^4 + 1$.
8. $4\,a^2b^2 - 4\,ab + 1$.
9. $a^6 + 2 + \frac{1}{a^6}$.
10. $x^{4a} + 2\,x^{2a}y^b + y^{2b}$.
11. $25\,a^4 - 10\,a^2 + 1$.
12. $25\,a^2 - 30\,ab + 9\,b^2$.
13. $49\,a^2b^2 - 14\,a^3b + a^4$.
14. $25\,a^4b^2c^2 + 10\,a^2bc^5 + c^8$.

329. Square Roots of Arithmetical Numbers. The square root of arithmetical numbers can be found approximately by inspection.

EXAMPLES

1. Find approximately $\sqrt{19}$.

19 lies between 16 and 25.

Therefore $\sqrt{19}$ lies between $\sqrt{16}$ and $\sqrt{25}$, or between 4 and 5.

That is, $\sqrt{19}$ is 4 plus a decimal.

2. Find approximately $\sqrt{643}$.

643 lies between 400 and 900.

Therefore, $\sqrt{643}$ lies between $\sqrt{400}$ and $\sqrt{900}$, or between 20 and 30.

That is, it is 20 plus a number less than 10.

The numbers to be added in any case will not change the first figure of the root found. That is, *by inspection we can find exactly the first figure of the square root.*

330. Pointing off into Periods. Since $10^2 = 100$, we know that the square root of any number greater than 1 but less than 100 is less than 10. Its integral part consists of one figure.

Since $100^2 = 10{,}000$, we know that the square root of any number greater than 100 but less than 10,000 is greater than 10 but less than 100.

That is, if the given number has 3 or 4 digits in its integral part, its square root will have 2 digits in its integral part. If larger numbers are given, the above reasoning can be repeated for 1000^2, etc., showing that in all cases if the number be pointed off into periods of 2 digits each (or possibly fewer in the left period), then each period will correspond to a digit of the root.

Thus in 67′62′31, there are three periods, hence there are three places in the integral part of the root. Since 67 lies between 64 and 81, the square root of 67 is approximately 8, and of 676,231 approximately 800.

ORAL EXERCISES

By the method above state an approximate square root of:

1. 12′36.
2. 30′95.
3. 1′25′00.
4. 8′23′00.
5. 43′21′00.
6. 58′61′23.

331. When the first digit of the square root has been found by inspection, the process may be continued thus:

EXAMPLE

Find $\sqrt{2209}$:

1. The approximate value, as above, is 40.
2. Let $\sqrt{2209} = 40 + r$, where r is the rest of the root.
3. $\therefore 2209 = (40 + r)^2 = 40^2 + 2 \cdot 40 \cdot r + r^2$.
4. $\therefore 2209 - 40^2 = 2 \cdot 40 \cdot r + r^2$, or $609 = 80r + r^2$.
5. Then 609 is greater than $80r$, or $\frac{609}{80} = 7 +$ dec. is greater than r.

$\therefore 7 +$ decimal is greater than r, and it is possible that $7 = r$.

Trying, we find that $80 \cdot 7 + 7^2 = 609$, hence, $r = 7$.

Therefore, $\sqrt{2209} = 40 + 7 = 47$.

332. Thus, when once an approximate value, a, has been found for the root, an approximate value for the remainder, r, of the root can be found by means of the formula: $(a+r)^2 = a^2 + 2ar + r^2$.

1. Let n denote the number whose square root is sought, a denote the approximate root at any stage, and r the remainder of the root.
2. Then, $n = (a + r)^2 = a^2 + 2ar + r^2$.
3. $\therefore n - a^2 = 2ar + r^2$.
4. Or, $n - a^2$ is greater than $2ar$, or, $\frac{n - a^2}{2a}$ is greater than r.
5. Hence, $\frac{n - a^2}{2a}$ may be tried as an approximate value of r.

333. What precedes may be formulated into a process or working rule, thus:

1. *Point off the number into periods of two figures each, beginning at units' place (at the decimal point).*

2. *By inspection find the largest integer whose square is not greater than the left period.* (In Example A it is 9.)

3. *Use this integer as the first digit of the root. Subtract its square from the left period.* (In Example A this square is 81.)

4. *Bring down the next period.* (In Example A this makes 364.)

5. *Multiply the part of the root already found by 2. This number is called the* **trial divisor.** (18 in Example A.)

(A)

Root		9 2
Number		84'64
		81
	18	364
	182	364

6. *Divide the remainder (omitting the right digit) by the trial divisor and use the digit found as the next digit of the root.* (In Example A, $36 \div 18 = 2$.)

7. *Annex this digit to the trial divisor. This forms the* **complete divisor.** (182 in A.)

8. *Multiply the complete divisor by the digit of the root just found and subtract.*

NOTE. It may happen that the product to be subtracted is larger than the number from which it is to be subtracted. This indicates that the trial divisor produced too large a digit. Try the next smaller digit for the figure of the root last found.

(B)

Root		3 0. 6 9
Number		9'41'.87'61
		9
	6	41
	60	4187
	606	3636
	612	55161
	6129	55161

9. *Repeat the steps 4 to 8 until all of the periods have been brought down.*

If the last remainder is zero, as in Example B, the process is ended, the given number is a perfect square, and its root has been found exactly. If the last remainder is not zero, as in C, the process may be continued as far as desired by supplying zeros.

TEST. The square of the root, if complete, equals the given number.

(C)

Root		1 4. 1 4+
Number		2'00.
		1
	2	100
	24	96
	28	400
	281	281
	282	11900
	2824	11296
		604

334. When a number contains a decimal the decimal point of its root is placed between the figures furnished by the integral periods and those furnished by the decimal periods (as in C, Sec. 333).

WRITTEN EXERCISES

Find the square roots:

1. 361.	3. 625.	5. 2025.	7. 177,241.	9. 4,334,724.
2. 784.	4. 841.	6. 1936.	8. 120,409.	10. 4,888,521.

Find the square roots to two decimal places:

11. 2.25.	14. 19.36.	17. 2000.	20. 3.
12. 7.84.	15. 90.25.	18. 0.03.	21. 111.
13. 6.25.	16. 1.21.	19. 5.	22. $0.00\dot{1}11$.

335. To find the square roots of fractional numbers, *either first reduce the fraction to a decimal or extract the square root of both numerator and denominator.*

WRITTEN EXERCISES

Find the square roots:

1. $\frac{81}{144}$.	3. $\frac{256}{441}$.	5. $\frac{625}{3025}$.	7. $\frac{622521}{724201}$.
2. $\frac{169}{225}$.	4. $\frac{361}{841}$.	6. $\frac{289}{961}$.	8. $\frac{7695076}{7371225}$.

9. It is known that if the sides of a rectangular prism are a, b, and c, the diagonal, d, is $\sqrt{a^2+b^2+c^2}$. If the sides of such a solid are 20 yd., 30 yd., and 50 yd., find its diagonal to three decimal places.

10. Compute $\sqrt{s(s-a)(s-b)(s-c)}$ to two decimal places, when $s=18.5$ in., $a=10$ in., $b=15$ in., $c=12$ in.

Solve and compute the roots to three decimal places:

11. $9x^2=21$. 12. $16.1x^2=21$.

336. Square Roots of Polynomials. The square root of every polynomial that is a square may be extracted according to the process given in Sec. 333. If the polynomial is not a square, the square root may be approximated to any number of terms.

17

EXAMPLE

Extract the square root of $a^2 - 2ab + b^2 - 2ac + 2bc + c^2$.

$$\begin{array}{lrl} \text{Root} & & \pm(a-b-c) \\ \hline \text{Power} & & a^2 - 2ab - 2ac + b^2 + 2bc + c^2 \\ & & a^2 - 2ab \qquad\quad + b^2 \\ \hline & & \qquad\quad -2ac \qquad + 2bc + c^2 \\ & & \qquad\quad -2ac \qquad + 2bc + c^2 \\ \hline \end{array}$$

1. As far as possible, arrange the terms according to the descending powers of some letter, as a in this case.

2. The square root of the first term is the first term of the root. (Corresponds to steps 2, 3, Sec. 333.)

3. Divide the second term of the power by twice the first term of the root, $2a$ in this case. The result is the second term of the root. (Steps 5, 6.)

4. Subtract from the power the square of the binomial found.

5. If there is a remainder (as $-2ac + 2bc + c^2$ in this case) it shows that the power contains the square of a trinomial and that there is at least another term in the root.

This term (c) is found by dividing the remainder by twice the part of the root found [$2(a - b)$ in this case], for the same reason as in the square root of numbers. (Step 6.)

6. The square of the entire root so far found $(a - b - c)^2$ must now be subtracted. We have already subtracted the square of the first part of the new binomial [$(a - b)^2$ in this case]. Therefore, subtract the rest of the square [$-2(a - b)c + c^2$, or $-2ac + 2bc + c^2$].

Briefly, the trial divisor, $2(a - b)$, is augmented by the next term, $-c$, resulting in $2(a - b) - c$, and this is multiplied by $-c$. This gives the part $-2(a - b)c + c^2$, still to be subtracted. This is analogous to what is done in extracting the square roots of numbers. (Steps 7, 8.)

If there is a new remainder, divide it by twice the entire part of the root found and proceed as before. Test by squaring the root.

WRITTEN EXERCISES

Extract the square root of:

1. $49a^2b^2 - 14a^3b + a^4$.
2. $16x^2y^2 + 40xy^2z + 25y^2z^2$.
3. $4x^4 + 4x^3 + 5x^2 + 2x + 1$.
4. $x^4 - 6x^3 + 11x^2 - 6x + 1$.
5. $1 + 4x + 10x^2 + 12x^3 + 9x^4$.
6. $9x^4 + 12x^3 + 22x^2 + 12x + 9$.
7. $9a^2 + 12ab + 4b^2 + 6ac + 4bc + c^2$.
8. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.
9. $x^6 - 4x^5 + 6x^4 + 2x^3 - 11x^2 + 6x + 9$.

10. $\frac{4}{x^2} - \frac{9}{y^2} + \frac{25}{z^2} - \frac{12}{xy} - \frac{30}{yz} + \frac{20}{xz}$.

11. $4\,ac - \frac{4\,c}{b} + a^2 - \frac{2\,a}{b} + 4\,c^2 + \frac{1}{b^2}$.

12. $\frac{1}{4}x^4 - 2\,x^2 - x + 4 + \frac{4}{x} + \frac{1}{x^2}$.

13. $2\,xw - 4\,xz^3 + x^2 + w^2 + 4\,z^6 - 4\,wz^3$.

14. $a^2 + 9\,b^2 - 6\,ab - 6\,bc + 2\,ca + c^2$.

15. $\frac{1}{4}a^2 - \frac{1}{3}ab + \frac{1}{9}b^2 + \frac{1}{5}ac - \frac{2}{15}bc + \frac{1}{25}c^2$.

16. $1 - 12\,y + 38\,y^2 - 12\,y^3 + y^4$.

17. $2\,x^2z^2 - 2\,x^2y - 2\,yz^2 + x^4 + y^2 + z^4$.

18. $a^4 + 6\,a^3 + \frac{29}{3}a^2 + 2\,a + \frac{1}{9}$.

19. $x^4y^4 - 2\,x^2y^2z^2 + 4\,x^3y^3z + 9\,z^4 - 12\,xyz^3$.

20. $9\,a^2 + 25\,b^2 + 9\,c^2 - 30\,ab + 18\,ac - 30\,bc$.

21. $m^2n^2 + p^2q^2 + r^2s^2 + 2\,mnpq + 2\,mnrs + 2\,pqrs$.

22. $m^4 + q^4 + m^2p^2 + 2\,m^3p - 2\,m^2q^2 - 2\,mpq^2$.

23. $4 + 29\,a^2 - 12\,a - 30\,a^3 + 25\,a^4$.

24. $\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{1}{3}x^2y + 4\,x^2 - 2\,xy + \frac{1}{4}y^2$.

Extract the square root to 4 terms:

25. $a^2 - 1$. 27. $y^2 + 1$. 29. $4 - z$. 31. $16\,a^2 + 12\,ab$.

26. $1 - x$. 28. $x^2 + 5$. 30. $x^2 + 4\,y$. 32. $9\,m^2 + 9\,mn$.

REVIEW

WRITTEN EXERCISES

Extract the square root of:

1. 14,641. 2. 1.5625. 3. $a^4 + 2\,a^2x + x^2$.

4. $9\,a^4 + 12\,a^3 - 20\,a^2 - 16\,a + 16$.

5. $a^2 + b^2 + 4\,c^2 - 2\,ab + 4\,ac - 4\,bc$.

6. $m^4 + 4\,am^3 + 6\,a^2m^2 + 4\,a^3m + a^4$.

7. $\frac{y^4}{16} - \frac{y^3}{4\,z} + \frac{3\,y^2}{20\,z^2} + \frac{y}{5\,z^3} + \frac{1}{25\,z^4}$.

8. $x^6 - 6\,x^5 + 15\,x^4 - 20\,x^3 + 15\,x^2 - 6\,x + 1$.

Expand by the binomial formula:

9. $(a+\frac{1}{2})^7$.
10. $(ax+1)^5$.
11. $(mx^2-1)^6$.
12. $(a^3+5)^4$.
13. $(ax^2-2y)^7$.
14. $(3x^2-\frac{1}{2})^5$.
15. $\left(\frac{a}{b}-\frac{b}{a}\right)^4$.
16. $\left(\frac{1}{2x^2}-\frac{1}{3y^2}\right)^5$.

17. Find the middle term of the expansion of $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$.

18. Raise 98 to the 5th power by the binomial theorem.

SUGGESTION. Use 100 − 2 for 98.

19. Find the ratio between the 6th term in the expansion of $\left(\frac{1+3x}{2}\right)^{10}$ and the 5th term in the expansion of $\left(\frac{1+3x}{2}\right)^{9}$.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What is *involution?* Sec. 319.
2. What is a *binomial expansion?* Sec. 321.
3. How else can an expression in the form of a binomial expansion be stated? Sec. 324.
4. State the rule for writing a binomial expansion. Sec. 326.
5. Define *evolution.* Sec. 327.
6. On what formula is the general process of finding *square root* based? Sec. 332.
7. How may the square root of a fraction be found? Sec. 335.

HISTORICAL NOTE

The most important process in involution is the binomial expansion. It is the basis for finding the powers of all algebraic polynomials, and likewise the basis of determining roots. The general principle for writing the terms of this series, known as the Binomial Theorem, was discovered by Sir Isaac Newton, the greatest English mathematician of the seventeenth century. Special cases of this formula, like $(a+b)^2$ and $(a+b)^3$, were known to the Hindoos and Arabs, who used them to find square and cube roots of numbers, Vieta in the sixteenth century knew the expansion of $(a + b)^4$, and Pascal constructed a table for mechanically computing the numerical coefficients; but the full significance of the Binomial Theorem was first discovered by Newton while investigating an expression for the value of π, the ratio of the circumference to the diameter of a circle.

Newton was born in a small town, Lincolnshire, in 1642, and his health was so delicate in his childhood that it interfered with his education. It is said that his backwardness provoked the ridicule of his companions, under the sting of which he soon surpassed all of them in learning. He entered Cambridge University in 1660 and readily mastered the works of Descartes, Vieta, and Wallis. While developing methods for finding areas bounded by various curved lines, he discovered, not only the Binomial Theorem, but also the method now known as Calculus, for which Newton will ever be famous as a mathematician. His discovery of the law of falling bodies and the general principle of gravitation has placed his name for all time among the foremost men of science. The English writer Pope has paid a graceful tribute to Newton's greatness in the following couplet:

SIR ISAAC NEWTON

"Nature and nature's laws lay hid in night,
God said, 'let Newton be' and all was light."

But we should not conclude from this poetic conception that Newton's great achievements were mere flashes of thought. They were the product of a mind trained by labor and informed by painstaking study. Genius pointed out the way to greater things, but not till Newton had climbed the height of other men's knowledge.

CHAPTER XXI

RADICALS AND EXPONENTS

DEFINITIONS AND PROPERTIES

337. Rational Numbers. Integers and other numbers expressible as the quotient of two integers are called **rational numbers.**

Thus, 5 and $\frac{2}{3}$ are rational numbers.
Also, .2, which is expressible as $\frac{2}{10}$, is a rational number.

338. Irrational Numbers. Any number not rational is called an **irrational number.**

Thus, $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{10}$, $\frac{1}{\sqrt{5}}$, $1+\sqrt{3}$, $\sqrt{2}-\sqrt{3}$, are irrational numbers.

Numbers like $\pi = 3.14159^+$, the ratio of the circumference to the diameter of a circle, are also called irrationals.

A special class of irrational numbers, namely, even roots of negative numbers are called imaginaries. We have seen that $(+2)(+2) = 4$, and that $(-2)(-2) = 4$; hence, $\sqrt{4} = \pm 2$, similarly $\sqrt{3} = \pm 1.732^+$; but we have not yet found two equal numbers whose product is -4, or any other negative number. Hence, the use of numbers like $\sqrt{-4}$ and $\sqrt{-a}$, will not appear in the processes or in the roots of equations until Chapter XXVIII has been studied.

339. An indicated root of any number is called a **radical.**

Thus, $\sqrt{5}$, $\sqrt[3]{8}$, $\sqrt{\frac{a}{3b}}$, $\sqrt{a+x^2}$, are radicals.

In the present chapter all roots that cannot be exactly extracted by inspection are indicated.

340. Although all indicated even roots may be taken either + or −, it is customary in the treatment of radicals to omit these signs, regarding the radical as positive.

341. Surd. A monomial containing an indicated root of a rational number is sometimes called a **surd.** The part under the radical is called the **radicand.**

342. The number denoting the root to be taken is called the **index** of the root.

Thus, $\sqrt{2}$, $\sqrt[3]{5}$, $3\sqrt[5]{7}$, $5\sqrt{14-a}$, are surds the radicands in order are 2, 5, 7, $14-a$, and the indices of the roots are 2, 3, 5, 2.

343. A radical expression with no rational factor is called an **entire surd**; otherwise a **mixed surd.**

344. The index of the root is called the **order** of a surd.

For example, $\sqrt{2}$ is of the second order, $\sqrt[3]{2}$ is of the third order, $\sqrt[5]{3}$ is of the fifth order.

345. A surd of the second order is also called a **quadratic** surd, of the third order a **cubic** surd, and one of the fourth order a **biquadratic** surd.

346. An expression involving one or more radicals is called a **radical expression.**

Thus, $5+2\sqrt{3}$, $\dfrac{4}{\sqrt{x}}-1$, $\dfrac{8+\sqrt{4a}}{2-\sqrt{3b}}$ are radical expressions.

347. Some Properties of Radicals. A few important properties of radicals are given here. The fuller treatment is contained in Chapter XXVI on Exponents.

348. I. $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$.

For example, $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$.
That this is true may be seen by squaring both members.

Thus, $(\sqrt{2}\cdot\sqrt{3})(\sqrt{2}\cdot\sqrt{3})=\sqrt{6}\cdot\sqrt{6}$,

or, $\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{3}=\sqrt{6}\cdot\sqrt{6}$,

or, $2\cdot 3=6$, which is known to be true.

In the same way, it may be seen that for every a and b, $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$.

In words:

The product of two square roots is the square root of the product of the numbers.

WRITTEN EXERCISES

Show by squaring that:

1. $\sqrt{3} \cdot \sqrt{5} = \sqrt{15}$.
2. $\sqrt{4} \cdot \sqrt{7} = \sqrt{28}$
3. $\sqrt{3} \cdot \sqrt{7} = \sqrt{21}$.
4. $\sqrt{5} \cdot \sqrt{11} = \sqrt{55}$.
5. $\sqrt{2a} \cdot \sqrt{3b} = \sqrt{6ab}$.
6. $\sqrt{x^3} \cdot \sqrt{5y} = \sqrt{5x^3y}$.
7. $\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{5} = \sqrt{30}$.
8. $\sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c} = \sqrt{abc}$.

349. II. $\sqrt{a^2b} = \sqrt{a^2}\ \sqrt{b} = a\sqrt{b}$.

In words:

Factors which are perfect squares may be taken from under the radical sign.

Thus, $\sqrt{18} = \sqrt{9} \cdot \sqrt{2} = \sqrt{3^2} \cdot \sqrt{2} = 3\sqrt{2}$.

WRITTEN EXERCISES

Take all factors which are perfect squares from under the radical sign:

1. $\sqrt{20}$.
2. $\sqrt{27}$.
3. $\sqrt{50}$.
4. $\sqrt{48}$.
5. $\sqrt{45}$.
6. $\sqrt{75}$.
7. $\sqrt{24}$.
8. $\sqrt{32}$.
9. $\sqrt{12}$.
10. $\sqrt{40}$.
11. $\sqrt{500}$.
12. $\sqrt{128}$.
13. $\sqrt{8a^2}$.
14. $\sqrt{x^3}$.
15. $\sqrt{48x^3y^2}$.
16. $\sqrt{45a^7y^4}$.

350. III. $a\sqrt{b} = \sqrt{a^2} \cdot \sqrt{b} = \sqrt{a^2b}$.

In words:

Any factor outside the radical sign may be placed under the radical sign provided the factor is squared.

Thus, $3\sqrt{2} = \sqrt{9} \cdot \sqrt{2} = \sqrt{18}$.

WRITTEN EXERCISES

Place under one radical sign:

1. $6\sqrt{2}$.
2. $5\sqrt{3}$.
3. $2 \cdot \sqrt{3}$.
4. $5 \cdot \sqrt{7}$.
5. $3 \cdot \sqrt{7} \cdot 2$.
6. $5 \cdot \sqrt{3} \cdot \sqrt{2}$.
7. $2 \cdot \sqrt{3} \cdot \sqrt{11}$.
8. $5 \cdot \sqrt{3} \cdot \sqrt{7}$.
9. $4\sqrt{2} \cdot 3$.
10. $b\sqrt{2}$.
11. $2x\sqrt{3x}$.
12. $ab\sqrt{bc}$.
13. $t\sqrt{g}$.
14. $r\sqrt{\pi r}$.
15. $\frac{x}{3}\sqrt{18xy}$.
16. $a\sqrt{b-a}$.

351. Fractional Exponents. A more convenient notation to indicate roots is found in **fractional exponents.** In order to find the proper meaning of such exponents, we shall assume that the laws of integral exponents apply also to fractional exponents.

Assume the law $a^m \cdot a^m = a^{m+n}$ to hold when $m = \frac{1}{2}$ and $n = \frac{1}{2}$.

Then, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

Therefore, $a^{\frac{1}{2}}$ is one of the two equal factors of a, or the *square root of* a.

(1) Hence we may write $\sqrt{a} = a^{\frac{1}{2}}$.

Similarly, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$.

Hence, $a^{\frac{1}{3}}$ is the *cube root of* a.

(2) and $\sqrt[3]{a} = a^{\frac{1}{3}}$.

352. *In general* $\sqrt[n]{a} = a^{\frac{1}{n}}$.

EXAMPLES

1. $9^{\frac{1}{2}} = \sqrt{9} = 3$.
2. $(4\,a^2)^{\frac{1}{2}} = \sqrt{4\,a^2} = 2\,a$.
3. $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$.
4. $(8\,a^3b^6)^{\frac{1}{3}} = \sqrt[3]{8\,a^3b^6} = 2\,ab^2$.

WRITTEN EXERCISES

Write with radical sign and simplify:

1. $4^{\frac{1}{2}}$.
2. $9^{\frac{1}{2}}$.
3. $(9\,x^2)^{\frac{1}{2}}$.
4. $(x^2y^2)^{\frac{1}{2}}$.
5. $(4\,y^2)^{\frac{1}{2}}$.
6. $(25\,x^2)^{\frac{1}{2}}$.
7. $8^{\frac{1}{3}}$.
8. $(27\,a^3)^{\frac{1}{3}}$.
9. $(x^3y^3)^{\frac{1}{3}}$.
10. $(27\,b^6)^{\frac{1}{3}}$.

353. Applying the law of multiplication to $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}}$.

we have $a^{\frac{1}{3}+\frac{1}{3}} = a^{\frac{2}{3}}$.

But, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = (a^{\frac{1}{3}})^2$.

Therefore, $a^{\frac{2}{3}} = (a^{\frac{1}{3}})^2 = (\sqrt[3]{a})^2$.

Similarly, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = (a^{\frac{1}{2}})^3 = (\sqrt{a})^3$.

354. *In general the numerator of the fractional exponent denotes the power to be taken, and the denominator denotes the root.*

In symbols, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.

EXAMPLES

1. $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4.$
2. $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 2^3 = 8.$
3. $9^{\frac{3}{2}} = (\sqrt{9})^3 = 27.$
4. $(4\,a^2b^4)^{\frac{5}{2}} = (2\,ab^2)^5 = 32\,a^5b^{10}.$

WRITTEN EXERCISES

Write with radical sign:

1. $a^{\frac{2}{3}}.$
2. $b^{\frac{4}{3}}.$
3. $c^{\frac{3}{4}}.$
4. $x^{\frac{3}{2}}.$
5. $(2\,a)^{\frac{1}{3}}.$
6. $(xy)^{\frac{3}{2}}.$
7. $3\,b^{\frac{1}{3}}.$
8. $5\,mx^{\frac{2}{3}}.$
9. $a^{\frac{2}{3}}b^{\frac{2}{3}}.$
10. $5\,x^{\frac{1}{3}}(y+z)^{\frac{2}{3}}.$
11. $2\,r^{\frac{5}{4}}s^{\frac{3}{4}}.$
12. $5\,a^{\frac{1}{x}}b^{\frac{m}{x}}.$

Find the rational values of:

13. $16^{\frac{1}{2}}.$
14. $27^{\frac{1}{3}}.$
15. $16^{\frac{1}{4}}.$
16. $27^{\frac{2}{3}}.$
17. $9^{\frac{3}{2}}.$
18. $8^{\frac{4}{3}}.$
19. $64^{\frac{2}{3}}.$
20. $32^{\frac{1}{5}}.$
21. $(-27)^{\frac{2}{3}}.$
22. $(-125)^{\frac{1}{3}}.$
23. $(\frac{1}{4})^{\frac{1}{2}}.$
24. $(\frac{1}{8})^{\frac{1}{3}}.$
25. $(\frac{4}{9})^{\frac{1}{2}}.$
26. $(-\frac{8}{27})^{\frac{1}{3}}.$
27. $(\frac{4}{9})^{\frac{3}{2}}.$
28. $9^{\frac{1}{2}}(\frac{1}{9})^{\frac{1}{2}}.$
29. $16^{\frac{1}{2}}(\frac{1}{4})^{\frac{1}{2}}.$
30. $27^{\frac{2}{3}}(\frac{4}{9})^{\frac{1}{2}}.$
31. $\sqrt[3]{64}\cdot(\frac{1}{16})^{\frac{3}{2}}.$
32. $(-125)^{\frac{2}{3}}\sqrt{\frac{1}{25}}.$
33. $(-343)^{\frac{2}{3}}\sqrt{\frac{1}{49}}.$

Write with fractional exponents and simplify when possible:

34. $\sqrt{x}.$
35. $\sqrt{2\,x}.$
36. $3\sqrt{x}.$
37. $\sqrt{a^3}.$
38. $\sqrt{4\,a}.$
39. $5\sqrt{25\,b^2}.$
40. $3\sqrt[3]{4\,m^2}.$
41. $5\sqrt[3]{8\,z^4}.$
42. $2\sqrt[3]{27\,xy^3}.$
43. $3\sqrt[4]{a^2b^3}.$
44. $6\sqrt[5]{ab^6}.$
45. $-2\,a\sqrt[5]{32\,a^3}.$
46. $5\,a^2\sqrt[3]{3\,a^3}.$
47. $x\sqrt{(yz)^3}.$
48. $ab\sqrt{(a+b)^3}.$
49. $2\sqrt{p^3}\cdot\sqrt[3]{q^2}.$
50. $\sqrt[n]{a^r}\cdot\sqrt[n]{b^{2n}}.$
51. $\sqrt[r]{a^x}\cdot\sqrt[r]{a^{5x}}.$

PROCESSES

355. Reduction. The reduction of radicals means the changing from one form to another equivalent form, not necessarily to a simpler one.

1. To reduce a mixed surd to an entire surd.

EXAMPLES

1. $3\sqrt{5} = \sqrt{3^2 \cdot 5} = \sqrt{45}.$ Sec. 354.
2. $2\sqrt[3]{3a} = \sqrt[3]{2^3 \cdot 3a} = \sqrt[3]{24a}.$

To reduce a mixed surd to an entire surd raise the rational factor to a power equal to the index of the root and place the result under the radical sign as a factor.

WRITTEN EXERCISES

Reduce to an entire surd:

1. $\frac{1}{2}\sqrt{2}.$
2. $\frac{1}{a}\sqrt{a}.$
3. $\frac{2}{3}\sqrt{x}.$
4. $2\sqrt[3]{2}.$
5. $3\sqrt[3]{5}.$
6. $a\sqrt[3]{a}.$
7. $a\sqrt{ab}.$
8. $\frac{2}{3}\sqrt{\frac{15}{7}x^2}.$
9. $\frac{4}{5}\sqrt{\frac{7}{a}}.$
10. $x^2\sqrt[3]{y}.$
11. $2\sqrt[4]{3}$
12. $\frac{1}{3}\sqrt[4]{3\frac{3}{4}}.$
13. $\frac{1}{2}\sqrt[3]{2\frac{4}{5}}.$
14. $\frac{a}{2}\sqrt{bc}.$
15. $ab\sqrt[3]{\frac{1}{abc}}.$
16. $(a+b)\sqrt{ab}.$
17. $2(a+b)\sqrt{\frac{a}{a+b}}.$
18. $\frac{1}{bc}\sqrt{a-b}.$
19. $\frac{a+b}{a-b}\sqrt{\frac{a-b}{a+b}}.$
20. $\frac{x+2}{x-2}\sqrt{1-\frac{4}{x+2}}.$

2. To change the order of a surd.

EXAMPLES

1. $\sqrt{3} = \sqrt[4]{3^2} = \sqrt[4]{9}.$
2. $3\sqrt[3]{2a^2} = 3\sqrt[9]{2^3a^6} = 3\sqrt[9]{8a^6}.$

To multiply or divide the order of a surd by a positive integer multiply or divide the exponent of each factor under the radical sign by the same number.

This process is made easier by using fractional exponents. Thus, to change $\sqrt[3]{2a^2}$ to the ninth order,

$$\sqrt[3]{2a^2}=(2a^2)^{\frac{1}{3}}=(2a^2)^{3\cdot\frac{1}{9}}=(8a^6)^{\frac{1}{9}}=\sqrt[9]{8a^6}.$$

WRITTEN EXERCISES

1. Reduce the radicals to the fourth order:

$$\sqrt{2},\quad \sqrt{a},\quad 2\sqrt{ab},\quad \sqrt{\tfrac{1}{2}},\quad \frac{1}{2}\sqrt{\frac{b}{2a}},\quad a\sqrt{a+b}.$$

2. Reduce the above radicals to the sixth order.

3. Reduce the radicals to the ninth order:

$$\sqrt[3]{3},\quad \sqrt[3]{a},\quad 3\sqrt[3]{ab},\quad \sqrt[3]{\tfrac{1}{3}},\quad a\sqrt[3]{\frac{b}{2c}},\quad ab\sqrt[3]{x-y}.$$

4. Reduce the radicals to the second order:

$$\sqrt[4]{4},\quad \tfrac{1}{2}\sqrt[10]{a^5},\quad \sqrt[6]{\frac{1}{(3x)^3}},\quad a+b\sqrt[4]{(a-b)^2},\quad xy\sqrt[8]{(a-2b)^4}.$$

3. To reduce radicals to the same order.

Apply the processes in case 2 above to reduce each radical to the given order.

WRITTEN EXERCISES

1. Reduce to radicals of the eighth order: $\sqrt{3}$, $\sqrt[4]{2}$.

2. Reduce to radicals of the sixth order: $\sqrt{5}$, $\sqrt[3]{4}$.

Reduce to entire surds of the order named:

3. $\sqrt{2}$, $\sqrt[4]{3a}$, 8th order.

4. $\sqrt[6]{5}$ $2\sqrt[3]{a}$, 12th order.

5. $3\sqrt[5]{a^3x^4}$, $\sqrt{mn}$, 10th order.

6. $\sqrt{\frac{1}{2}}$, $\sqrt[4]{\frac{1}{2}x^3y}$, 8th order.

7. $p\sqrt[4]{p^3r}$, $\sqrt{\frac{1}{2}ax}$, 12th order.

8. $6\sqrt{5}$, $.3\sqrt[3]{.5}$, 6th order.

9. Reduce $\sqrt[3]{\frac{1}{2}a^2b}$, $7\sqrt[5]{\dfrac{a-b}{2}}$, to the 15th order.

10. Reduce $\sqrt{2}$, $\sqrt[4]{3}$, $\sqrt[5]{4}$, $\sqrt[10]{5}$, to the 20th order.

11. Reduce to radicals of the sixth order:

$$\sqrt{2},\quad \sqrt[3]{ab},\quad \sqrt{a+b}.$$

12. Reduce to radicals of the tenth order:

$$\sqrt{2},\quad \sqrt{ax},\quad 5\sqrt{2\,xy},\quad \sqrt[5]{a+b},\quad (a-b)\sqrt[5]{x-2\,y}.$$

Reduce to radicals of the same order:

13. $(x+y)\sqrt{x-y},\ \sqrt[3]{x-y}.$

14. $\sqrt{a},\ \sqrt[4]{a^2-b^2},\ \sqrt{a-b}.$

15. $\sqrt{m},\ \sqrt[5]{m-1},\ \sqrt[10]{m-2}.$

16. $\sqrt[2n]{a},\ \sqrt{ab},\ \sqrt[n]{a^3b^3}.$

17. $\sqrt[4p]{x},\ \sqrt{xy},\ \sqrt[p]{x-y}.$

18. $\sqrt[n]{a-b},\ \sqrt[5n]{a-b+c}.$

Which is the greater

19. $\sqrt[3]{2}$ or $\sqrt[6]{5}$? **20.** $\sqrt{2}$ or $\sqrt[3]{3}$? **21.** $\sqrt{2\frac{1}{2}}$ or $\sqrt[3]{4}$?

4. To reduce a radical to its simplest form.

EXAMPLES

1. $\sqrt{8\,a^3}=\sqrt{4\,a^2\cdot 2\,a}=2\,a\sqrt{2\,a}.$ Sec. 349.
2. $\sqrt[3]{2\,a^6x^5}=\sqrt[3]{(a^2x)^3\cdot 2\,x^2}=a^2x\sqrt[3]{2\,x^2}.$
3. $\sqrt{\frac{1}{2}}=\sqrt{\frac{2}{4}}=\sqrt{\frac{1}{4}\cdot 2}=\frac{1}{2}\sqrt{2}.$
4. $\sqrt[8]{a^2x^4}=\sqrt[4\times 2]{(ax^2)^2}=\sqrt[4]{ax^2}.$

Note that different exponents like the 2 and 3 in $\sqrt[6]{a^2b^3}$ must not be canceled from the common index.

356. The reduction of radicals is facilitated by the use of fractional exponents.

Consider Example 2 above:

$$\sqrt[3]{2\,a^6x^5}=(2\,a^6x^5)^{\frac{1}{3}}=2^{\frac{1}{3}}(a^6)^{\frac{1}{3}}\cdot(x^5)^{\frac{1}{3}}=2^{\frac{1}{3}}\,a^2\cdot(x^3\cdot x^2)^{\frac{1}{3}}$$
$$=2^{\frac{1}{3}}\,a^2x\,(x^2)^{\frac{1}{3}}=a^2x(2\,x^2)^{\frac{1}{3}}=a^2x\sqrt[3]{2\,x^2}.$$

Similarly, for Example 3:

$$\sqrt{\tfrac{1}{2}}=(\tfrac{1}{2})^{\frac{1}{2}}=(\tfrac{1}{4}\cdot 2)^{\frac{1}{2}}=\tfrac{1}{2}(2)^{\frac{1}{2}}=\tfrac{1}{2}\sqrt{2}.$$

Also for example 4:

$$\sqrt[8]{a^2x^4}=(a^2x^4)^{\frac{1}{8}}=(a^2)^{\frac{1}{8}}\cdot(x^4)^{\frac{1}{8}}=(a)^{\frac{1}{4}}\cdot(x^2)^{\frac{1}{4}}=(ax^2)^{\frac{1}{4}}=\sqrt[4]{ax^2}.$$

357. A radical is understood to be in its simplest form when:

(1) *No factor can be taken from under the radical sign* (*Examples 1 and 2*, Sec. 355, 4);

(2) *The radicand is integral* (*Example 3*).

(3) *The radicand is not a power whose exponent has a common factor with the order of the radical* (*Example 4*).

WRITTEN EXERCISES

Reduce to simplest form:

1. $\sqrt{18}$.
2. $\sqrt{27}$.
3. $\sqrt{20}$.
4. $\sqrt{32}$.
5. $2\sqrt{8}$.
6. $3\sqrt{12}$.
7. $5\sqrt{50}$.
8. $\sqrt[3]{16}$.
9. $3\sqrt[3]{32}$.
10. $3\sqrt{75}$.
11. $\sqrt[4]{320}$.
12. $\sqrt[3]{3000}$.
13. $\sqrt{\frac{1}{3}}$.
14. $3\sqrt{\frac{1}{2}}$.
15. $3\sqrt{\frac{1}{27}}$.
16. $2\sqrt[3]{\frac{1}{4}}$.
17. $5\sqrt[6]{\frac{1}{16}}$.
18. $\sqrt{a^2x^2y}$.
19. $\sqrt{\frac{ax^2y}{b}}$.
20. $\sqrt{\frac{1}{a}}$.
21. $\sqrt{\frac{b^2}{2x^3}}$.
22. $a\sqrt[3]{\frac{b}{2a}}$.
23. $4\sqrt[4]{15\frac{1}{2}}$.
24. $2\sqrt{\frac{a^4b^3}{2x^3}}$.
25. $\sqrt[2x]{a^{3x}b^{2x}}$.
26. $\frac{a}{b^n}\sqrt{\frac{b^{2n}}{a^5}}$.
27. $\sqrt[4]{\frac{81}{1296}a^{20}b^8c^{12}}$.
28. $\sqrt{1-(\frac{1}{3})^2}$.
29. $\sqrt{1+(\frac{1}{2})^2}$.
30. $\sqrt{3-(\frac{3}{4})^2}$.
31. $\sqrt{1-(\frac{2}{3})^2}$.
32. $3\sqrt{\frac{x^3}{27}}$.
33. $\sqrt{a^2-\left(\frac{a}{2}\right)^2}$.
34. $\sqrt{18+9\sqrt{2}}$.
35. $\sqrt{\frac{r^2-r^2\sqrt{3}}{2}}$.
36. $ab\sqrt{2+\frac{a^2+b^2}{ab}}$.
37. $\sqrt[3]{a^6-3a^3}$.
38. $\sqrt{7x^2-14xy+7y^2}$.
39. $(5x^2-10ax+5a^2)^{\frac{1}{2}}$.
40. $\sqrt{m^2-6}\,(m^4-6m^2)^{\frac{1}{2}}$.
41. $\sqrt[3]{a \div b} \cdot \sqrt[3]{a^2c \div b^2}$.
42. $(a-b)\sqrt{a^2-2ab+b^2}$.

43. $\sqrt{x^2 + \left(\frac{x}{4}\right)^2}$.

44. $\sqrt[3]{p^3 - \left(\frac{p}{2}\right)^3}$.

45. $m\sqrt[3]{\frac{10}{3m^2}}$.

46. $\sqrt[3]{27 - 81\sqrt{3}}$.

47. $\sqrt{(a^4 - 2a^3b + a^2b^2) \div b^2}$.

48. $ab\sqrt{3x^2 + 6xy + 3y^2}$.

49. $(x+y)\sqrt{a^2b - a^3}$.

50. $\sqrt{p + \left(\frac{p}{c}\right)^2}$.

51. $\sqrt{\left(x + \frac{x}{2}\right)\left(x - \frac{x}{2}\right)}$.

358. Addition and Subtraction of Radical Expressions. Radicals can be united by addition or subtraction only when the same root is indicated and the expressions under the radical sign are the same in each.

359. When the expression cannot be put into this form the sum or the difference can only be indicated.

360. *To add or subtract radical expressions having the same radical part, add or subtract the coefficients of their radical parts.*

For example:

1. $2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$. 2. $2\sqrt{12} + \sqrt{300} = 4\sqrt{3} + 10\sqrt{3} = 14\sqrt{3}$.

3. Add $\sqrt{2}$, $-\sqrt{8}$, $\sqrt[3]{16}$, $\sqrt[3]{-54}$:

$$-\sqrt{8} = -2\sqrt{2};\ \sqrt[3]{16} = 2\sqrt[3]{2};\ \sqrt[3]{-54} = -3\sqrt[3]{2}.$$

$$\therefore \sqrt{2} - \sqrt{8} + \sqrt[3]{16} + \sqrt[3]{-54} = \sqrt{2} - 2\sqrt{2} + 2\sqrt[3]{2} - 3\sqrt[3]{2} = -\sqrt{2} - \sqrt[3]{2}.$$

Since only like radical parts may be added, it will be clear that $\sqrt{a} + \sqrt{b}$ does not mean $\sqrt{a+b}$. Show this by letting $a = 9$, $b = 16$.

WRITTEN EXERCISES

Find the sum:

1. $\sqrt{2}$, $\sqrt{8}$, $\sqrt{18}$.
2. $\sqrt{75}$, $-\sqrt{12}$, $-\sqrt{3}$.
3. $\sqrt{8}$, $\sqrt{5}$, $-\sqrt{18}$.
4. $\sqrt{128}$, $-2\sqrt{50}$, $\sqrt{72}$.
5. $\sqrt[3]{40}$, $-\sqrt[3]{320}$, $\sqrt[3]{135}$.
6. $8\sqrt{48}$, $-\frac{1}{2}\sqrt{12}$, $4\sqrt{27}$.
7. $\sqrt[3]{72}$, $-3\sqrt[3]{9}$, $6\sqrt[3]{243}$.
8. $\sqrt{6}$, $\sqrt{24}$, $\sqrt{63}$.
9. $\sqrt{108}$, $-\sqrt{12}$, $\sqrt{48}$.
10. $\sqrt{75}$, $\sqrt{48}$, $-\sqrt{27}$.
11. $\sqrt{80}$, $\sqrt{20}$, $-\sqrt{45}$.
12. $\sqrt{44}$, $-\sqrt{99}$, $\sqrt{121}$.
13. $5\sqrt{24}$, $-\sqrt{54}$, $3\sqrt{96}$.
14 $\sqrt[3]{27x^4}$, $-\sqrt[3]{64x^4}$, $\sqrt[3]{16x^4}$.

361. Multiplication of Radical Expressions containing Square Roots. In multiplying expressions containing indicated square roots, make use of the relation $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

EXAMPLES

1.
$$\begin{array}{r} 3 - 4\sqrt{5} \\ 6 + 2\sqrt{3} \\ \hline 18 - 24\sqrt{5} \qquad\qquad\qquad\qquad \\ 6\sqrt{3} - 8\sqrt{15} \\ \hline 18 - 24\sqrt{5} + 6\sqrt{3} - 8\sqrt{15}. \end{array}$$

2.
$$\begin{array}{r} 2 - \sqrt{3} \\ 5 + 2\sqrt{3} \\ \hline 10 - 5\sqrt{3} \qquad\qquad \\ 4\sqrt{3} - 6 \\ \hline 10 - \sqrt{3} - 6 = 4 - \sqrt{3}. \end{array}$$

WRITTEN EXERCISES

Multiply:

1. $2 + \sqrt{5}$ by $2 - \sqrt{5}$.
2. $1 + \sqrt{3}$ by $2 + \sqrt{5}$.
3. $2 + \sqrt{3}$ by $2 + \sqrt{3}$.
4. $\sqrt{2} + \sqrt{3}$ by $1 - \sqrt{3}$.
5. $\sqrt{3} - \sqrt{5}$ by $\sqrt{3} + \sqrt{5}$.
6. $\sqrt{5} - \sqrt{6}$ by $\sqrt{5} - \sqrt{6}$.
7. $4 + \sqrt{5}$ by $\sqrt{10}$.
8. $3 - \sqrt{15}$ by $2 + \sqrt{5}$.
9. $1 + \sqrt{2}$ by $1 - \sqrt{8}$.
10. $2\sqrt{3} - 3\sqrt{5}$ by $\sqrt{3} - \sqrt{5}$.
11. $\sqrt{14} + \sqrt{7}$ by $\sqrt{8} - \sqrt{21}$.
12. $\sqrt{5} - \sqrt{48}$ by $\sqrt{5} + \sqrt{12}$.

362. Division of Square Roots. The quotient of the square roots of two numbers is the square root of the quotient of the numbers. In symbols, $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.

Thus, $\dfrac{\sqrt{5}}{\sqrt{6}} = \sqrt{\dfrac{5}{6}}$, because, multiplying each member by itself,

$$\frac{\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{5}}{\sqrt{6}} = \sqrt{\frac{5}{6}} \cdot \sqrt{\frac{5}{6}},$$

or $$\frac{\sqrt{5} \cdot \sqrt{5}}{\sqrt{6} \cdot \sqrt{6}} = \sqrt{\frac{5}{6}} \cdot \sqrt{\frac{5}{6}},$$

or $$\frac{\sqrt{5^2}}{\sqrt{6^2}} = \sqrt{\left(\frac{5}{6}\right)^2},$$

or $$\frac{5}{6} = \frac{5}{6},$$ which is evidently true.

ORAL EXERCISES

Read each of the following as a fraction under one radical sign:

1. $\dfrac{\sqrt{3}}{\sqrt{5}}$. 3. $\dfrac{\sqrt{5}}{\sqrt{7}}$. 5. $\dfrac{1}{\sqrt{5}}$. 7. $\dfrac{\sqrt{5}}{\sqrt{15}}$.

2. $\dfrac{\sqrt{3}}{\sqrt{7}}$. 4. $\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{1}}{\sqrt{2}}$. 6. $\dfrac{\sqrt{2}}{\sqrt{8}}$. 8. $\dfrac{\sqrt{10}}{\sqrt{20}}$.

363. In the four processes with radicals explained in Secs. 359–362 we have admitted only the exact values of the radical expressions, but if we accept approximate values of the radicals, we may further simplify the results obtained, and this is often done for practical purposes.

EXAMPLE

Each of the two legs of a right triangle is 1 in., and the hypotenuse is $\sqrt{2}$ in. Find the perimeter.

The exact result is $(1 + 1 + \sqrt{2})$ in. $= (2 + \sqrt{2})$ in.

If we accept an approximate result, taking three decimal places as the degree of accuracy, we have

$(2 + 1.414)$ in. $= 3.414$ in. Secs. 333, 334.

WRITTEN EXERCISES

Find to two decimal places the value of:

1. $2+\sqrt{3}$. 4. $4-\sqrt{2}$. 7. $\frac{1}{2}\sqrt{2}-1$.

2. $3-\sqrt{3}$. 5. $\sqrt{2}+\sqrt{3}$. 8. $\frac{1}{2}\sqrt{3}+2$.

3. $3-\sqrt{2}$. 6. $\sqrt{3}-\sqrt{2}$. 9. $\sqrt{5}-2$.

Find to three decimal places the value of:

10. $\frac{1}{3}\sqrt{3}$. 12. $\dfrac{3-\sqrt{2}}{2}$. 14. $\sqrt{5}-\sqrt{2}$.

11. $\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{2}$. 13. $\sqrt{2}-\sqrt{3}$. 15. $\sqrt{6}-\sqrt{3}$

16. Find x to two decimal places in the equation $x^2 - 3 = 0$.

17. The altitude of an equilateral triangle with sides 1 in. is $\frac{1}{2}\sqrt{3}$ in. Find the altitude to two decimal places.

18. Find the perimeter of a right triangle to two decimal places whose sides are 1 in., 2 in., and $\sqrt{5}$ in.

19. Find the perimeter of a right triangle to two decimal places whose sides are 1 ft., $\frac{1}{2}$ ft., and $\frac{1}{2}\sqrt{3}$ ft.

364. Rationalizing the Denominator. Multiplying both numerator and denominator of a fraction by an expression that will make the denominator rational is called **rationalizing the denominator.**

Thus, multiplying both numerator and denominator of $\frac{\sqrt{3}}{\sqrt{2}}$ by $\sqrt{2}$, we obtain

$$\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{6}}{2}.$$

WRITTEN EXERCISES

Rationalize the denominator of:

1. $\frac{1}{\sqrt{2}}$.
2. $\frac{1}{\sqrt{3}}$.
3. $\frac{1}{\sqrt{5}}$.
4. $\frac{2}{\sqrt{3}}$.
5. $\frac{3}{\sqrt{7}}$.
6. $\frac{\sqrt{2}}{\sqrt{3}}$.
7. $\frac{\sqrt{5}}{\sqrt{7}}$.
8. $\frac{5}{\sqrt{3}}$.
9. $\frac{\sqrt{5}}{\sqrt{3}}$.
10. $\frac{10}{\sqrt{5}}$.
11. $\frac{6}{\sqrt{3}}$.
12. $\frac{8}{\sqrt{2}}$.

365. Rationalizing Factors. When the denominator is of the form $\sqrt{a}+\sqrt{b}$ or $a+\sqrt{b}$, the rationalizing factor is the same binomial with the connecting sign changed, often called the **conjugate binomial.**

It is not necessary in elementary algebra to take up the rationalizing of more complicated denominators.

EXAMPLES

1. Rationalize the denominator in $\frac{3}{2-\sqrt{5}}$.

The conjugate of $2-\sqrt{5}$ is $2+\sqrt{5}$.

Then, $\frac{3}{2-\sqrt{5}} = \frac{3(2+\sqrt{5})}{(2-\sqrt{5})(2+\sqrt{5})} = \frac{6+3\sqrt{5}}{4-5} = -(6+3\sqrt{5})$.

2. Rationalize the denominator in $\dfrac{2+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$.

The conjugate of $\sqrt{3}+\sqrt{5}$ is $\sqrt{3}-\sqrt{5}$.

Then, $\dfrac{2+\sqrt{3}}{\sqrt{3}+\sqrt{5}}=\dfrac{(2+\sqrt{3})(\sqrt{3}-\sqrt{5})}{(\sqrt{3}+\sqrt{5})(\sqrt{3}-\sqrt{5})}=\dfrac{2\sqrt{3}+3-2\sqrt{5}-\sqrt{15}}{3-5}$

$=-\dfrac{3+2\sqrt{3}-2\sqrt{5}-\sqrt{15}}{2}.$

3. Rationalize the denominator of $\dfrac{3-\sqrt{5}}{\sqrt{2}-\sqrt{5}+\sqrt{3}}$.

First multiply the terms by the expression $\sqrt{2}+(\sqrt{5}-\sqrt{3})$.

Then, we have $\dfrac{(3-\sqrt{5})(\sqrt{2}+\sqrt{5}-\sqrt{3})}{2-(\sqrt{5}-\sqrt{3})^2}=\dfrac{(3-\sqrt{5})(\sqrt{2}+\sqrt{5}-\sqrt{3})}{-6+2\sqrt{15}}.$

Then, multiply by the conjugate $-6-2\sqrt{15}$ and simplify as usual.

WRITTEN EXERCISES

Rationalize the denominators:

1. $\dfrac{2+\sqrt{3}}{3+\sqrt{3}}$.

2. $\dfrac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

3. $\dfrac{1}{1-\sqrt{2}}$.

4. $\dfrac{3}{2-\sqrt{5}}$.

5. $\dfrac{5}{\sqrt{3}+\sqrt{7}}$.

6. $\dfrac{2-\sqrt{5}}{3-\sqrt{5}}$.

7. $\dfrac{3}{\sqrt{5}+\sqrt{2}}$.

8. $\dfrac{\sqrt{5}+2\sqrt{2}}{4-2\sqrt{2}}$.

9. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.

10. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.

11. $\dfrac{2+4\sqrt{7}}{2\sqrt{7}-1}$.

12. $\dfrac{2\sqrt{15}-6}{\sqrt{5}+2\sqrt{2}}$.

13. $\dfrac{1}{1-\sqrt{2}+\sqrt{3}}$.

14. $\dfrac{1+\sqrt{2}}{\sqrt{3}-\sqrt{5}+4}$.

15. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}+\sqrt{5}}$.

16. $\dfrac{3-\sqrt{5}+\sqrt{7}}{3+\sqrt{5}-\sqrt{7}}$.

17. $\dfrac{3}{4-(\sqrt{6}+\sqrt{5})}$.

18. $\dfrac{c}{2-\sqrt{a-b}}$.

19. $\dfrac{1}{a-\sqrt{2b+c}}$.

20. $\dfrac{a+\sqrt{b}}{2\sqrt{a}-\sqrt{b}+\sqrt{c}}$.

21. $\dfrac{\sqrt{a^2-1}+\sqrt{a^2+1}}{\sqrt{a^2+1}-\sqrt{a^2-1}}$.

22. $\dfrac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

23. $\dfrac{\sqrt{x^2+x+1}-1}{\sqrt{x^2+x+1}+1}$.

24. $\dfrac{\sqrt{p+q}-\sqrt{p-q}}{\sqrt{p+q}+\sqrt{p-q}}$.

366. Chapter XII dealt only with rational factors, but the properties of radicals make it possible to find irrational factors.

EXAMPLES

1. Factor: $2x^2-1$.

This is the difference of two squares, if we admit radicals, because 2 is the square of $\sqrt{2}$.

Then, $2x^2-1=(\sqrt{2}\,x-1)(\sqrt{2}\,x+1)$.

2. Factor: x^2-5.

Using radicals, $x^2-5=x^2-(\sqrt{5})^2$.

Then, $x^2-5=(x-\sqrt{5})(x+\sqrt{5})$.

3. Similarly, $x-y=(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$.

4. $x-y=(\sqrt[3]{x}-\sqrt[3]{y})(\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2})$.

367. Irrational factors may be used to solve equations.

Thus, In $2x^2-1=0$, $(\sqrt{2}\,x-1)(\sqrt{2}\,x+1)=0$.

Therefore, $\sqrt{2}\,x-1=0$, and $\sqrt{2}\,x+1=0$.

Solving, $x=\dfrac{1}{\sqrt{2}}=\dfrac{1}{2}\sqrt{2}$, also $x=-\dfrac{1}{\sqrt{2}}=-\dfrac{1}{2}\sqrt{2}$.

WRITTEN EXERCISES

Factor:

1. x^2-3.
2. $4x^2-5$.
3. $8x^2-1$.
4. $5x^2-4$.
5. $6x^2-9$.
6. $2a^2-b$.
7. z^2-7.
8. $a-b^2$.
9. What factor taken with $\sqrt[3]{a}-\sqrt[3]{b}$ makes $a-b$?
10. What factor taken with $x-\sqrt[3]{2}$ makes x^3-2?

Solve.

11. $y^2-2=0$.
12. $z^2-3=0$.
13. $2p^2-1=0$.
14. $ax^2-b=0$.
15. $2m^2-5=0$.
16. $2x^2-12=0$.

REVIEW

WRITTEN EXERCISES

Simplify:

1. $\dfrac{\sqrt{5}}{\sqrt{60}}$.
2. $\sqrt{\frac{27}{8}}-\sqrt{\frac{3}{8}}$.
3. $\sqrt{6}\cdot\sqrt{125}$.
4. $\dfrac{\sqrt{10}}{\sqrt{40}}$.
5. $\sqrt{50}+\sqrt{128}$.
6. $\sqrt{3}\div\sqrt{5}$.
7. $\dfrac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
8. $\sqrt{6}\div\sqrt{2}$.
9. $(2+\sqrt{3})^2$.
10. $(5+\sqrt{7})(5-\sqrt{7})$.
11. $(2\sqrt{3}+3\sqrt{5})\div\sqrt{15}$.
12. $(\sqrt{6}+\sqrt{15})(\sqrt{8}-\sqrt{20})$.

Express with rational denominators, and with at most one radical sign in the dividend:

13. $\sqrt{12}\div\sqrt{3}$.
14. $\sqrt{7}\div\sqrt{11}$.
15. $2\sqrt{24}\div 2\sqrt{6}$.
16. $2\div 3\sqrt{5}$.
17. $\dfrac{4}{\sqrt{5}-1}$.
18. $1\div(\sqrt{2}-10)$.
19. $\sqrt{2}\div(\sqrt{2}-\sqrt{3})$.
20. $(2\sqrt{6}+5\sqrt{12})\div\sqrt{6}$.
21. $(5\sqrt{18}-8\sqrt{50})\div 2\sqrt{2}$.
22. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.
23. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.
24. $\dfrac{\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.
25. $\dfrac{2+4\sqrt{7}}{2\sqrt{7}-1}$.
26. $\dfrac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}$.

Solve by factoring:

27. $3x^2-1=0$.
28. $4x^2-2=0$.
29. $5x^2-a=0$.
30. $3x^3-9x=0$.

31. Computing the square root to two decimal places, find r in the equation $\pi r^2=8$. (Use $\pi=3.1416$.)

32. Find t to three decimal places, using $g=32$, in the equation $48=\frac{1}{2}gt^2$.

33. Find the value of $\frac{1}{3}h(B+b+\sqrt{Bb})$, taking $h=5$, $b=3$, $B=8$, and computing the radical to two decimal places.

34. Find the value of the expression in Exercise 33 when $h=9$, $b=8$, $B=17$, computing the radical to 3 decimal places.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define *rational numbers;* also *irrational numbers.* Secs. 337, 338.

2. Define a *radical;* also a *surd;* also a *radical expression.* Secs. 339, 341, 346.

3. What sign is chosen for radicals of even order? Sec. 340.

4. What is the *radicand?* Illustrate. Sec. 341.

5. What is an *entire surd?* A *mixed surd?* Sec. 343.

6. What is meant by the *order* of a surd or radical? State a *quadratic* surd; a *cubic* surd; a *biquadratic* surd. Sec. 344, 345.

7. What is another expression for the product of two square roots? Sec. 348.

8. How may factors be placed under the radical sign? Sec. 350.

9. How is a mixed radical reduced to an entire radical? Sec. 355.

10. Illustrate how to change the order of a radical. Sec. 355.

11. State how to *simplify* a radical. Sec. 355.

12. Illustrate how the use of fractional exponents simplifies the process of reduction of radicals. Sec. 356.

13. What radicals may be added? How are they added or subtracted? Secs. 358–360.

14. What is another expression for the quotient of two square roots? Sec. 362.

15. What is meant by *rationalizing the denominator* of a fraction containing radicals? Sec. 364.

CHAPTER XXII

QUADRATIC EQUATIONS

RATIONAL

368. The general form of a quadratic equation is

(1) $$ax^2 + bx + c = 0.$$

By dividing this equation by a, the coefficient of x^2, the equation becomes $x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$

Putting $p = \frac{b}{a}$, $q = \frac{c}{a}$ to replace the fractional forms, the equation becomes

(2) $$x^2 + px + q = 0$$

which is also a general form for quadratic equations.

369. Kinds of Quadratic Equations. The equation $ax^2 + bx + c = 0$ is said to be a **complete quadratic equation** when neither b nor c is zero; that is, when there are three terms, one containing x^2, another x, and a term without x (absolute term). In any other case it is called an **incomplete quadratic equation.**

370. A quadratic in which the first power of the unknown does not occur is called a **pure** quadratic, and one in which it occurs is called an **affected** quadratic.

Thus, $x^2 + 5x - 2 = 0$ is a complete quadratic equation, while $2x^2 - 5 = 0$, and $x^2 + 7x = 0$ are incomplete quadratics. The second equation is a pure quadratic and the others are affected quadratics.

371. Solution by Completing the Square. The following example shows a general method for solving quadratic equations with one unknown:

EXAMPLE

Solve: $x^2 + 8x + 7 = 0.$ (1)

Transposing the absolute term, $x^2 + 8x = -7.$ (2)

Making the left member a square by adding 16 to *both* members, $x^2 + 8x + 16 = 9.$ (3)

$\therefore (x + 4)^2 = 9.$ (4)

Extracting the square root of both members, $x + 4 = \pm 3.$ (5)

$\therefore x = -1 \text{ or } -7.$ (6)

The process consists of two essential parts:

(1) *Making the left member a square while the right member does not contain the unknown.*

This is called completing the square.

It is based upon the relation $(x + a)^2 = x^2 + 2ax + a^2$ (Sec. 141, p. 103) in which it appears that the last term, a^2, is the square of one half of the coefficient of x.

(2) *Extracting the square roots of both members and solving the resulting linear equations.*

WRITTEN EXERCISES

Solve completely:

1. $x^2 + 8x = 9.$
2. $x^2 + 4x = 12.$
3. $x^2 + 12x = -11.$
4. $x^2 - 8x = 9.$
5. $x^2 + 10x = 11.$
6. $x^2 + 5x = \frac{11}{4}.$
7. $x^2 - 20x = -75.$
8. $x^2 - 6x = 7.$
9. $z^2 - 40z = 41.$
10. $w^2 - w = \frac{3}{4}.$
11. $t^2 - 2t = 8.$
12. $u^2 - u = -\frac{1}{4}.$
13. $v^2 + 3v = \dfrac{-5}{4}.$
14. $s^2 - 18s = 19.$

372. Approximate Roots Expressed Decimally. Square roots which cannot be found exactly may be indicated and should be so used for checking, but the roots of quadratic equations expressing the solution of practical problems are often irrational and require approximating. The nature of the problem must determine to how many decimal places the result should be expressed, but for practice we shall use two decimal places.

EXAMPLE

Solve: $x^2 - 6x + 3 = 0.$ (1)

Rearranging and completing the square, $x^2 - 6x + 9 = 6.$ (2)

$\therefore x - 3 = \pm\sqrt{6}.$ (3)

$\therefore x = 3 + \sqrt{6}$ and $3 - \sqrt{6}.$ (4)

To test this work we must substitute $3 + \sqrt{6}$ and $3 - \sqrt{6}$ in equation (1). But, by taking $\sqrt{6}$ to the nearest hundredth, or as 2.45 these roots are $3 + \sqrt{6} = 5.45$ and $3 - \sqrt{6} = 0.55$. These roots will not check since they are only approximations, but they are definite rational numbers and sufficiently accurate for many practical purposes.

WRITTEN EXERCISES

Solve, computing all irrational roots to two decimal places:

1. $y^2 - y = 1.$
2. $x^2 + x = 5.$
3. $x^2 - 6x = -1.$
4. $x^2 - 6x = -3.$
5. $x^2 + 5x = 6.$
6. $x^2 - 4x = 20.$
7. $x^2 + \frac{1}{2}x = \frac{3}{2}.$
8. $x^2 - \frac{1}{4}x = \frac{1}{2}.$
9. $z^2 - 16z = -15.$
10. $t^2 - 2t = 6.$
11. $u^2 - u = 1.$
12. $x^2 + 13x = -30.$
13. $x^2 + 6x = -4.$
14. $x^2 - 8x = -8.$
15. $x^2 + 13x = \frac{3}{4}.$
16. $x^2 + 15x = -25.$
17. $x^2 - 8x = 3.$
18. $m^2 + 8m = 4.$
19. $x^2 + 10x = 1.$
20. $x^2 - 5x + 6 = 0.$
21. $x^2 - 8x = 7.$
22. $x^2 + 10x = 5.$
23. $p^2 - 6p = -1.$
24. $x^2 - 6x = 13.$
25. $x^2 + 5x = 7.$
26. $x^2 - 10x = 25.$
27. $x^2 - 9x = \frac{3}{2}.$
28. $x^2 - \frac{3}{4}x = \frac{1}{2}.$
29. $z^2 - 16z = 9.$
30. $t^2 - 8t = 6.$
31. $u^2 - u = 5.$
32. $x^2 + 3x = 10.$
33. $m^2 - 4m = 1.$
34. $w^2 + 5w + 6 = 0.$
35. $t^2 + 9t + 20 = 0.$
36. $v^2 - v - 20 = 0.$
37. $x^2 - x - 42 = 0.$
38. $x^2 - 5x - 84 = 0.$
39. $u^2 + 19u + 84 = 0.$
40. $z^2 - 9z + 14 = 0.$

373. Solution by Formula. When the coefficient of x^2 is not unity, the equation must be divided by that coefficient before using the method above to complete the square.

If the square of the general equation $x^2 + px + q = 0$ be completed, we have

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} - q. \qquad \text{Sec. 371.}$$

Extracting the square root of both members,

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q}, \qquad \text{Sec. 151.}$$

or,

$$x = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q}. \qquad \text{Sec. 364.}$$

If the general equation $a^2 + bx + c = 0$ be divided by a and solved as above the roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These results are general formulas for finding the values of x in any quadratic equation.

EXAMPLE

Solve: $5x^2 + 7x - 2 = 0.$

Dividing by 5, $x^2 + \frac{7}{5}x - \frac{2}{5} = 0.$

Here $p = +\frac{7}{5}$ and $q = -\frac{2}{5}.$

Hence, $\frac{p}{2} = \frac{7}{10}$ and $\sqrt{p^2 - 4q} = \sqrt{\frac{49}{25} - 4\left(-\frac{2}{5}\right)},$

and $x = -\frac{7}{10} \pm \frac{1}{2}\sqrt{\frac{49}{25} + \frac{8}{5}}$

$= -\frac{7}{10} \pm \frac{1}{10}\sqrt{89}.$

WRITTEN EXERCISES

Solve:

1. $4x^2 + 6x - 4 = 0.$
2. $9x^2 + 15x + 6 = 0$
3. $4x^2 - 2x - 2 = 0.$
4. $9x^2 + 3x - 6 = 0.$
5. $25z^2 - 20z + 4 = 0.$
6. $3x^2 - 7x - 20 = 0.$
7. $4x^2 + 12x - 55 = 0.$
8. $9w^2 + 6w - 35 = 0.$
9. $9v^2 - 39v + 22 = 0$
10. $4y^2 - 12y = 91.$
11. $16t^2 - 8t = 15.$
12. $4z^2 + 20z = -21.$

13. $6x^2 + x = 12.$

14. $6x^2 = -5x + 4.$

15. $12x^2 = 5x + 2.$

16. $9x^2 = 18x - 5.$

Clear each of the following of fractions and solve the resulting equation:

17. $\frac{120}{x+3} = \frac{120}{x} - 2.$

18. $\frac{5x}{x+4} - \frac{3x-2}{2x-3} = 2.$

19. $x - \frac{x^3-8}{x^2+5} = 2.$

20. $\frac{1}{3} + \frac{1}{3+x} + \frac{1}{3+2x} = 0.$

21. $\frac{x+22}{3} - \frac{4}{x} = \frac{9x-6}{2}.$

22. $\frac{x+2}{x-1} - \frac{4-x}{2x} = 2\frac{1}{3}.$

23. $\frac{48}{x+3} = \frac{165}{x+10} - 5.$

24. $\frac{x}{x+1} + \frac{x+1}{x} = \frac{13}{6}.$

25. $\frac{2x}{x-4} + \frac{2x-5}{x-3} = \frac{25}{3}.$

26. $\frac{3x-7}{x} + \frac{4x-10}{x+5} = \frac{7}{2}.$

27. $\frac{4x+7}{19} + \frac{5-x}{3+x} = \frac{4x}{9}.$

28. $\frac{3}{5-x} + \frac{1}{4-x} = \frac{6}{x+2}.$

29. The perimeter of a rectangular field is 100 yd. and its area is 600 sq. yd. Find its length and breadth.

SOLUTION. 1. Let x be the length of the field.

2. Then $\frac{600}{x}$ is its width, and

3. $2x + \frac{2 \cdot 600}{x}$ is its perimeter, being twice the sum of its sides.

4. $\therefore 2x + \frac{2 \cdot 600}{x} = 100$, by the conditions of the problem.

5. $\therefore x^2 - 50x + 600 = 0$, simplifying (4).

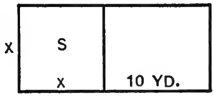
6. $\therefore (x-20)(x-30) = 0$, factoring (5).

7. $\therefore x = 20$, $x = 30$, solving (6).

8. If 20 yd. be taken as the length, the width is 30 yd., by step (2).

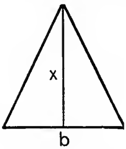
9. If 30 yd. be taken as the length, the width is 20 yd., by step (2).

30. The area of the whole plot shown in the diagram is 96 sq. yd. What is the length of a side of the square (s)?



31. The sum of two unequal sides of a rectangular court is 19 yd.; the sum of the areas of the squares

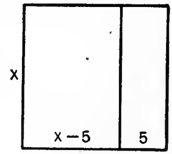
on these two sides is 181 sq. yd. What are the dimensions of the court?



32. A triangle whose area is 200 sq. yd. has its altitude equal to its base. Find the base of the triangle, using the formula, area $= \frac{1}{2}$ base $\times$ altitude.

33. The breadth of a room is 4 ft. more than its height and 20 ft. less than its length. The cost of cleaning and painting the side walls was \$20.70, at $2\frac{1}{2}$¢ per square foot. Find the dimensions of the room.

34. A partition is built 5 yd. from one side of a square room as shown in the diagram; the area of the floor remaining is 24 sq. yd. What are the dimensions of the floor?



35. One side of a rectangle is $\frac{3}{4}$ as long as the other; if 10 ft. be added to the shorter side and 10 ft. be subtracted from the longer side, the area will not be changed. Find the dimensions of the original rectangle.

36. If there are s subscribers in a telephone exchange, the total number of different connections of any subscriber with any other is $\frac{s(s-1)}{2}$. If in an exchange the total number of different connections is 3240, find the number of subscribers.

37. A builder used steel bars weighing 120 lb. each for a certain purpose. By changing the mode of support, he found that he could get the same service from bars weighing 2 lb. less per running foot, but 2 ft. longer than the original bars. The new bars also weighed 120 lb. Find the length of the original bars.

374. Quadratic Forms. Certain equations of higher degree have the form of the quadratic equation and may be solved like a quadratic. These are said to be in **quadratic form.**

Only a few of these will be given here because their roots are usually complex in form and equations of this kind are given in Chapter XXX.

EXAMPLES

1. Solve: $x^4 - 13x^2 + 36 = 0.$ (1)

Let $y = x^2$, then $y^2 - 13y + 36 = 0.$ (2)

Factoring, $(y-9)(y-4) = 0.$ (3)

Hence, $y = 9$ and $y = 4.$ (4)

Since $y = x^2$ $x^2 = 9$ and $x^2 = 4.$ (5)

$\therefore x = +3, -3$ and $x = +2, -2.$ (6)

All of these roots check, making four values of x for the given equation of the fourth degree.

2. Solve: $x^4 - 9x^2 + 8 = 0.$ (1)

Let $y = x^2$; then the given equation becomes, $y^2 - 9y + 8 = 0.$ (2)

Solving for y, $y = 8$ or $1.$ (3)

Therefore $x^2 = 8.$ (4)

Or, $x^2 = 1.$ (5)

Solving (4), (5), $x = \pm\sqrt{8}, \pm 1.$ (6)

Test these four values of x by substitution in the given equation.

WRITTEN EXERCISES

Solve and test:

1. $x^4 - 5x^2 + 4 = 0.$
2. $x^4 - 17x^2 + 16 = 0.$
3. $36x^4 - 13x^2 + 1 = 0.$
4. $4y^4 - 17y^2 + 4 = 0.$
5. $R^4 - 18R^2 + 81 = 0.$
6. $(x-1)^4 - 5(x-1)^2 + 4 = 0.$
7. $(y-5)^4 - 17(y-5)^2 + 16 = 0.$
8. $(t-3)^4 - 11(t-3)^2 - 42 = 0.$

RADICAL EQUATIONS

375. *To solve equations in which only a single square root occurs, transpose so that the square root constitutes one member. Square both members and solve the resulting equation.*

EXAMPLE

Solve: $2x - 3 = \sqrt{x^2 + 6x - 6}.$ (1)

Squaring both members, $4x^2 - 12x + 9 = x^2 + 6x - 6.$ (2)

Collecting terms, $3x^2 - 18x + 15 = 0.$ (3)

Solving (3), $x = 5$ or $1.$ (4)

TEST. By trial, it appears that 5 satisfies the given equation, taking the radical as positive, while 1 satisfies the equation $2x - 3 = -\sqrt{x^2 + 6x - 6}.$

1. It must be remembered that the equation resulting from squaring will usually not be *equivalent* to the given equation (Sec. 172, p. 128). It may have additional roots, and substitution must determine which of the values found satisfy the given equation.

2. In order that the given problem may be definite, the radical must be taken with a given sign. If every possible square root is meant, two different equations are really given for solution. Thus, unless restricted, $2x = \sqrt{4-6x}$ is a compact way of uniting the two different equations, $2x = +\sqrt{4-6x}$, and $2x = -\sqrt{4-6x}$. If solved as indicated above, it appears that the first is satisfied when $x = \frac{1}{2}$, the second when $x = -2$.

3. In the exercises of the following set the radical sign is to be understood to mean the *positive* square root.

376. Extraneous Roots. The ambiguity of signs is removed by squaring; hence, the resulting equation appears to have more roots than the original. These roots of the final equation which do not satisfy the original are called **extraneous roots,** and should be omitted.

WRITTEN EXERCISES

Solve, and test only rational roots:

1. $x = \sqrt{10x+7}$.
2. $x = \sqrt{b + x - bx}$.
3. $3x - 7\sqrt{x} = -2$.
4. $\sqrt{x+4} - x = 4$.
5. $\dfrac{\sqrt{x}-1}{3} = \dfrac{x}{16}$.
6. $x + 5\sqrt{37-x} = 43$.
7. $\sqrt{x+5} = x - 7$.
8. $\sqrt{2x+7} = \dfrac{x}{3}$.
9. $x + \sqrt{x+5} = 2x - 1$.
10. $30 = x - 29\sqrt{x}$.
11. $x = 2 + \sqrt{3 - 11x}$.
12. $x - \sqrt{x-2} = 2$.
13. $x + 1 = \sqrt{x+3}$.
14. $\sqrt{100 - x^2} = 10 - x$.
15. $x + \sqrt{2x - x^2} = 2$.
16. $\dfrac{x+2}{2} - \dfrac{x+1}{3} - \dfrac{\sqrt{2x+1}}{2} = 0$.

377. If the equation involves but one radical, the method of Sec. 375 can be used; but when more than one radical occurs, successive squaring may be necessary.

EXAMPLE

Solve: $\sqrt{2x+6}+\sqrt{3x+1}=8.$ (1)

Rearranging,	$\sqrt{2x+6}=8-\sqrt{3x+1}.$	(2)
Squaring,	$2x+6=64-16\sqrt{3x+1}+3x+1.$	(3)
Collecting,	$16\sqrt{3x+1}=x+59.$	(4)
Squaring again and collecting,	$x^2-650x+3225=0.$	(5)
Solving,	$x=5$, or 645.	(6)

TEST. Trial shows that the first of these values satisfies the given equation, and it is obvious by inspection that the second cannot satisfy the equation.

378. Sometimes it is best first to transform the given expression.

EXAMPLE

Solve: $$2x+\sqrt{4x^2+9}=\frac{2x+1+\sqrt{5x+6}}{2x-\sqrt{4x^2+9}}.\quad (1)$$

Clearing of fractions,	$9=2x+1+\sqrt{5x+6}.$	(2)
Rearranging,	$8-2x=\sqrt{5x+6}.$	(3)
Squaring and collecting,	$4x^2-37x+58=0.$	(4)
Hence,	$x=2$, or $7\frac{1}{4}$.	(5)

TEST. By trial, 2 is seen to satisfy the given equation. To avoid the complete work of substituting $7\frac{1}{4}$, we note that every root of (1) must satisfy (3). If $x=7\frac{1}{4}$, the left member of (3) is negative and the right member positive. Hence, $7\frac{1}{4}$ is an extraneous root (Sec. 376) and should be disregarded. It would satisfy (3) if the radical had the negative sign.

WRITTEN EXERCISES

Solve:

1. $x-\sqrt{3x+10}=6.$
2. $\sqrt{x}-\sqrt{x-15}=1.$
3. $\sqrt{x^2-5}+\dfrac{6}{\sqrt{x^2-5}}=5.$
4. $x+2+(x+2)^{\frac{1}{2}}=20.$
5. $\sqrt{x+7}+\sqrt{3x-2}=\dfrac{4x+9}{\sqrt{3x-2}}.$

6. $4\sqrt{x^2+5x+6}=\sqrt{5}\cdot\sqrt{x^2+5x+6}$.

7. $\sqrt{x^2-a^2x^2}=\sqrt{x^4+b^4x^2}$.

8. $\sqrt{x}+\sqrt{a-x}=\sqrt{a+b}$.

9. $\sqrt{6z+6}-\sqrt{3z+1}=\sqrt{5z-21}$.

10. $\sqrt{4+z}+\dfrac{10}{\sqrt{5z}}=\sqrt{5z}$.

11. $\sqrt{\dfrac{2x^2-2x+1}{2x^2+2x+1}}-2=0$.

379. An equation whose radicals have been removed by squaring may be linear.

WRITTEN EXERCISES

Solve and test:

1. $\sqrt{y}-\sqrt{y-8}=\dfrac{2}{\sqrt{y-8}}$.

2. $\sqrt{x-1}-\sqrt{x}=\dfrac{2}{\sqrt{x}}$.

3. $\sqrt{x-1}-\sqrt{a-1}=0$.

4. $\dfrac{1}{1-x}+\dfrac{1}{1+\sqrt{x}}=\dfrac{1}{1-\sqrt{x}}$.

REVIEW

WRITTEN EXERCISES

Solve, approximating any irrational roots to two decimal places:

1. $x^2+7x=8$.
2. $3x^2=48$.
3. $x^2+25x=-100$.
4. $(3x+4)^2=96$.
5. $x^2-25x+144=0$.
6. $x^2+3x-28=0$.
7. $x^2-13x=68$.
8. $x^2-12x+27=0$.
9. $x^2+111x=3400$.
10. $5x^2+13x=370$.
11. $3z^2-2z=1$.
12. $y^2+4ay-2=0$.
13. $t^2+3t-6=0$.
14. $13x^2-39x=0$.
15. $a^2x^2-abx=0$.
16. $(a+b)x^2+cx=0$.
17. $m^2x^2-(m+r)x=0$.
18. $ax^2-bx=cx^2+dx$.
19. $\frac{1}{2}x^2=14-3x^2$.
20. $x^2+5=\frac{10}{3}x^2-16$.

21. $\frac{2x^2+4}{9}=x+1.$

22. $x-2+\sqrt{2-x}=0.$

23. $x^2+\frac{10}{3}x=19.$

24. $5-3x+\frac{1}{4}x^2=0.$

25. $x^2-7x=0.$

26. $y^2-ay=c.$

27. $x^2-12=30+x.$

28. $12x^2-3x=7x^2+2x.$

29. $10x^2+5x=15x^2+9x.$

30. $(a+b)^2x^2+(a+b)x=0.$

31. $\frac{6}{x}+4=2x.$

32. $x+\sqrt{9-x^2}=4.$

33. $x^4-10x^2+9=0.$

34. $(z-3)^4-5(z-3)=-4.$

35. $(w-7)^4-(w-7)^2=2.$

36. The perimeter of a rectangular field is 200 ft., and its area is 2400 sq. ft. Find its length and breadth.

37. The area of a rectangular field is 2000 sq. ft., and its length is 10 ft. more than its breadth. Find its dimensions.

38. After a man had lived in his house 4 months longer than the number of dollars monthly rental that he paid for his house, he had paid altogether $320 rent. How much was the monthly rental?

39. If d is the diagonal of a square of side s, then $d^2=2s^2$. Solve this equation for d. For s.

40. In Exercise 39, when $s=4$, find d, taking 1.414 as the square root of 2.

41. The volume (v) of a cylinder is the product of the area of the base (πr^2) and the height (h). That is, $v=\pi r^2h$. Solve this equation for r. For h.

42. Using $\frac{22}{7}$ for π in $v=\pi r^2h$, find the radius of the base of a cylinder, if its volume is $\frac{22}{112}$ sq. ft. and its altitude 4 ft.

43. Given $m=\frac{gv^2}{c^2}$. Express v in terms of the other letters.

44. In the preceding exercise, express c in terms of the other letters.

45. A room is 1 yd. longer than it is wide; at 75¢ per square yard, a covering for the floor of the room costs $31.50. Find the dimensions of the floor.

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46. The length of a room is twice its height, and the breadth is 6 ft. more than the height. At 10¢ per square foot it costs \$72 to decorate the side walls of the room, no allowance being made for openings. Find the dimensions of the room.

47. The cost of decorating a certain square ceiling is \$45. If a second square ceiling of side 5 yd. longer were decorated at the same rate, the cost would be \$80. Find the dimensions of the first ceiling.

48. A hall is lighted by a certain number of incandescent electric lights and 5 fewer of gas mantles. The candle power of each of the former is 70 greater than that of each mantle. The total candle power of the gas mantles is 500, and that of the electric lights 1800. How many lamps of each sort?

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define and illustrate a *complete* quadratic equation; also an *incomplete* quadratic equation. Sec. 369.

2. What is a *pure quadratic?* An *affected quadratic?* Sec. 370.

3. Explain the process of completing the square. Sec. 371.

4. Why approximate roots? Why not use such roots for testing? Sec. 372.

5. Explain how to proceed when the coefficient of x^2 is not unity. Sec. 373.

6. State the *formula* for solving quadratic equations. Sec. 373.

7. What kind of higher equations may be solved by quadratic methods? Sec. 374.

8. How are *radical equations* solved? Secs. 375, 377, 378.

9. Define *extraneous* roots. Sec. 376.

HISTORICAL NOTE

The solution of quadratic equations dates from Diophantos, but he did not leave any description of a general method. The Hindoos could solve special cases by completing the square when the number to be added for this purpose was obvious, and Cridharra is said to have formulated a rule for this. They discussed the existence of two roots, although they felt that positive roots only were significant, and Bhaskara called a negative root "inadequate, because people don't approve of negative roots." The Arab, Mohammed ben Musa, or Al-Khowarazmi (p. 93), helped to simplify the solution of equations by giving directions "first to transpose terms, then combine them," but added nothing else of importance.

The European scholars of the Middle Ages contented themselves with translating the Greek and the Arab manuscripts so that further development in solving equations awaited the mathematicians of the fifteenth century. By that time there were current more than twenty special rules for solving quadratics, which Michael Stifel (about 1550) reduced to three.

Stifel, the greatest German algebraist of the sixteenth century, was born in Esslingen and was educated by the monks for the ministry; but his interest in the mystic numbers found in the prophetic books of the Bible led him to study mathematics. The new science of algebra was called by the Germans "Coss," and Stifel is now known as the greatest "cossist." He made many improvements in algebra, but his limited idea of the negative number hindered him, particularly in the solution of equations. Thus, he succeeded only in decreasing the number of rules for solving the quadratic equation, whereas Stevin, in the next century, with full knowledge of the negative number, reduced all of these rules to our present method of completing the square.

CHAPTER XXIII

SYSTEMS OF QUADRATIC EQUATIONS

SIMULTANEOUS QUADRATIC EQUATIONS

380. Two simultaneous quadratic equations with two unknowns cannot in general be solved by the methods used in solving quadratic equations, because an equation of higher degree usually results from eliminating one of the unknowns. But many systems containing quadratic equations can be solved by quadratic methods, among them the following:

381. Class I. In which the result of substitution is a quadratic form.

1. *A system of equations composed of a linear equation and a quadratic equation can always be solved by substitution.*

EXAMPLE

Solve:
$$\begin{cases} 3x + 4y = 24. & (1) \\ x^2 + y^2 = 25. & (2) \end{cases}$$

From (1), $$3x = 24 - 4y. \qquad (3)$$

From (3), $$x = 8 - \frac{4y}{3}. \qquad (4)$$

Substituting (4) in (2), $$\left(8 - \frac{4y}{3}\right)^2 + y^2 = 25. \qquad (5)$$

Simplifying (5), $$25y^2 - 192y + 351 = 0. \qquad (6)$$

Factoring (6), $$(y - 3)(25y - 117) = 0. \qquad (7)$$

Solving (7), $$y = 3 \text{ and } y = \frac{117}{25}. \qquad (8)$$

From (4), $$x = 4 \text{ and } x = \frac{44}{25}. \qquad (9)$$

TEST. $$\left.\begin{matrix} 3 \cdot 4 + 4 \cdot 3 = 24 \\ 3^2 + 4^2 = 25 \end{matrix}\right\} \text{ and } \begin{cases} \dfrac{3 \cdot 44}{25} + \dfrac{4 \cdot 117}{25} = 24. \\ \left(\dfrac{44}{25}\right)^2 + \left(\dfrac{117}{25}\right)^2 = 25. \end{cases}$$

2. *When both equations are quadratic, substitution is applicable if the result of substitution is an equation having the quadratic form.*

EXAMPLE

Solve: $\begin{cases} x^2 + y^2 = 25, & (1) \\ xy = 12. & (2) \end{cases}$

From (2), $$x = \frac{12}{y}. \quad (3)$$

Substituting from (3) in (1), $$\left(\frac{12}{y}\right)^2 + y^2 = 25. \quad (4)$$

Simplifying (4), we obtain an equation in quadratic form, $$y^4 - 25\,y^2 + 144 = 0. \quad (5)$$

Factoring (5), $$(y^2 + 16)(y^2 - 9) = 0. \quad (6)$$

Solving (6), $$y = \pm 4, \text{ and } \pm 3. \quad (7)$$

Substituting (7) in (3), $$x = \pm 3, \text{ and } \pm 4. \quad (8)$$

TEST. Taking both values to be positive, or both to be negative,

$$(\pm 3)^2 + (\pm 4)^2 = 25.$$
$$(\pm 3) \cdot \pm 4 = 12.$$

There are frequently various methods of solving the same problem. Thus, in the last example, multiply (2) by 2, add it to or subtract it from (1); the resulting equations $(x + y)^2 = 49$, and $(x - y)^2 = 1$, can be solved by extracting the square roots of both members and adding and subtracting the resulting equations.

382. Corresponding Values. The sets of values of x and y which form solutions may be determined by noticing which value of one unknown furnishes a given value of the other in the process of solution.

Thus, in Example 1, p. 282, $y = 3$ produces $x = 4$.

WRITTEN EXERCISES

Solve and test:

1. $x^2 + y^2 = 2\,xy,$
 $x + y = 8.$

2. $x^2 + y^2 = 25,$
 $x + y = 1.$

3. $x^2 + y^2 = 13,$
 $2\,x + 3\,y = 13.$

4. $x - y = 1,$
 $x^2 - y^2 = 16.$

5. $m^2 - n^2 = 16,$
 $m - n = 2.$

6. $2\,R^2 - R_1{}^2 = 14,$
 $3\,R + R_1 = 11.$

7. $x + y = a,$
 $x^2 - y^2 = b.$

8. $x - y = b,$
 $xy = a^2.$

9. $2\,k = k^2 - k'^2,$
 $2\,k = 4\,kk'.$

383. Class II. In which one equation is homogeneous.

A system in which one equation has only terms of the second degree in x *and* y *can be solved by finding* x *in terms of* y, *or* vice versa, *from this equation and substituting in the other.*

EXAMPLE

Solve:
$$\begin{cases} x^2 - 5xy + 6y^2 = 0, & (1) \\ x^2 - y^2 = 27. & (2) \end{cases}$$

Dividing (1) by y^2, $\frac{x^2}{y^2} - \frac{5xy}{y^2} + \frac{6y^2}{y^2} = 0.$ (3)

Simplifying (3), $\left(\frac{x}{y}\right)^2 - 5\left(\frac{x}{y}\right) + 6 = 0.$ (4)

Factoring (4), $\left(\frac{x}{y} - 2\right)\left(\frac{x}{y} - 3\right) = 0.$ (5)

Solving (5), $\frac{x}{y} = 2,\ \frac{x}{y} = 3.$ (6)

From (6), $x = 2y,\ x = 3y.$ (7)

Substituting $x = 2y$ in (2), $(2y)^2 - y^2 = 27.$ (8)

Solving (8), $y = \pm 3.$ (9)

Substituting (9) in $x = 2y$, $x = \pm 6.$ (10)

Substituting $x = 3y$ in (2), $(3y)^2 - y^2 = 27.$ (11)

Solving (11), $y = \frac{\pm 9}{2\sqrt{6}} = \pm \frac{3}{4}\sqrt{6}.$ (12)

Substituting (12) in $x = 3y$, $x = \frac{\pm 27}{2\sqrt{6}} = \pm \frac{9}{4}\sqrt{6}.$ (13)

TEST.

Taking both values to be positive or both negative,
$$\begin{cases} (\pm 6)^2 - 5(\pm 3)(\pm 6) + 6(\pm 3)^2 = 0. \\ (\pm 6)^2 - (\pm 3)^2 = 27. \end{cases}$$

Taking the signs as before,
$$\begin{cases} \left(\frac{\pm 27}{2\sqrt{6}}\right)^2 - 5\left(\frac{\pm 27}{2\sqrt{6}}\right)\left(\frac{\pm 9}{2\sqrt{6}}\right) + 6\left(\frac{\pm 9}{2\sqrt{6}}\right)^2 = 0. \\ \left(\frac{\pm 27}{2\sqrt{6}}\right)^2 - \left(\frac{\pm 9}{2\sqrt{6}}\right)^2 = 27. \end{cases}$$

NOTES. 1. When the equation in $\frac{x}{y}$, as in step (4), cannot be factored by inspection, the formula for solving the quadratic equation is used.

2. When an equation has the right member zero, it is unnecessary to divide by x^2 or y^2, if the left member can be factored by inspection. Thus, in (1) above, $(x - 3y)(x - 2y) = 0$, hence $x = 3y$ and $x = 2y$, as in (7).

WRITTEN EXERCISES

Solve:

1. $x^2 - 2xy - 3y^2 = 0,$
   $x^2 + 2y^2 = 12.$
2. $x^2 + xy - 2y^2 = 0,$
   $x^2 + y^2 = 50.$
3. $x^2 - y^2 = 0,$
   $x^2 - 3xy + 2y = \frac{1}{2}.$
4. $2x^2 - 5xy + 2y^2 = 0,$
   $4x^2 - 4y^2 + 12x = 3.$
5. $m(m + n) = 0,$
   $m^2 - mn + n^2 = 27.$
6. $6x^2 + 5xy + y^2 = 0,$
   $y^2 - x - y = 32.$
7. $3t^2 - 2tu - u^2 = 0,$
   $t + u + u^2 = 32.$
8. $x^2 + a^2y^2 = 0,$
   $x^2 + ay - y^2 = a^2.$
9. $x^2 + 3x - 5y + xy = 400,$
   $x^2 + 7xy + 10y^2 = 0.$
10. $4x^2 + 4xy + y^2 = 0,$
    $x^2 + 3y^2 - 2x = 195.$

384. Class III. In which the unknowns are symmetrically involved.

A system in which each equation is unaltered when x *and* y *are interchanged can be solved by letting* $x = u + v$ *and* $y = u - v$.

EXAMPLE

Solve:
$$\begin{cases} x^2 + y^2 = 5, & (1) \\ xy + x + y = 5. & (2) \end{cases}$$

Let $x = u + v$, $y = u - v$,

then (1) becomes $$(u + v)^2 + (u - v)^2 = 5, \quad (3)$$

and (2) becomes
$$(u + v)(u - v) + (u + v) + (u - v) = 5. \quad (4)$$

Simplifying (3), $$2u^2 + 2v^2 = 5. \quad (5)$$

Simplifying (4), $$u^2 - v^2 + 2u = 5. \quad (6)$$

Dividing (5) by 2, and subtracting it from (6), $$2u^2 + 2u = \tfrac{15}{2}. \quad (7)$$

Simplifying, $$u^2 + u - \tfrac{15}{4} = 0. \quad (8)$$

Solving (8), $$u = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 + 15}, \quad (9)$$
$$= -\tfrac{1}{2} \pm 2 = 1\tfrac{1}{2}, -2\tfrac{1}{2}.$$

Substituting in (5), $$2 \cdot \tfrac{9}{4} + 2v^2 = 5. \quad (10)$$

Solving (10), $$v^2 = \tfrac{1}{4}, \text{ and } v = -\tfrac{1}{2}, +\tfrac{1}{2}. \quad (11)$$

Finally, $$x = u + v = 1\tfrac{1}{2} \mp \tfrac{1}{2} = 1 \text{ or } 2, \quad (12)$$

and, $$y = u - v = 1\tfrac{1}{2} \pm \tfrac{1}{2} = 2 \text{ or } 1. \quad (13)$$

If the other value, $u = -2\frac{1}{2}$, step (9), be substituted in (5), then $v = \pm \frac{1}{2}\sqrt{-15}$. But square roots of negative numbers, called imaginaries, will not be admitted until they have been studied in Chapter XXVIII.

Special systems of symmetric equations may be solved by dividing one equation by the other.

For example,

$$\begin{cases} x^3 + y^3 = 18, \\ x + y = 6, \end{cases} \quad \text{or} \quad \begin{cases} x^4 + x^2y^2 + y^4 = 8, \\ x^2 - xy + y^2 = 4. \end{cases}$$

WRITTEN EXERCISES

Solve by Sec. 384 or by some of the previous methods:

1. $x^2 - xy + y^2 - 7 = 0,$
   $x - y + 1 = 0.$
2. $x^2 - xy + y^2 = 12,$
   $x^2 + xy + y^2 = 4.$
3. $xy - (x + y) - 1 = 0,$
   $xy = -3.$
4. $x^2 + y^2 = 39,$
   $y - x = 3.$
5. $2x^2 + 2y^2 - (x - y) = 9,$
   $y^2 = 1.$
6. $x^2 - xy + y^2 = 19,$
   $xy = 15.$
7. $xy = 3(x + y),$
   $x^2 + y^2 = 160.$
8. $\frac{x}{y} + \frac{y}{x} = \frac{5}{2},$
   $\frac{1}{x} + \frac{1}{y} = 4.$
9. $x^2 + y^2 + x + y = 188,$
   $xy = 77.$
10. $x^3 - y^3 = 9,$
    $x - y = 3.$
11. $x^4 + x^2y^2 + y^4 = 8,$
    $x^2 + xy + y^2 = 2.$

385. The foregoing classes of simultaneous quadratic equations are applied in the following problems.

WRITTEN EXERCISES

1. Two square floors are paved with stones 1 ft. square; the length of the side of one floor is 12 ft. more than that of the other, and the number of stones in the two floors is 2120. Find the length of the side of each floor.

SUGGESTION. Let x be the length in feet of a side of the smaller floor and y be the length of the other, then

$$x = y - 12, \quad (1)$$

and by the given conditions,
$$x^2 + y^2 = 2120. \quad (2)$$

Substituting (1) in (2),
$$(y - 12)^2 + y^2 = 2120. \quad (3)$$

Simplifying and factoring (3),
$$(y - 38)(y + 26) = 0 \quad (4)$$

The negative values not being admissible, the squares are 26 ft. and 38 ft. on a side.

2. The sum of the sides of two squares is 7 and the sum of their areas is 25. Find the side of each square.

3. The hypotenuse of a certain right triangle is 50, and the length of one of its sides is $\frac{3}{4}$ that of the other. Find the sides.

4. The difference between the hypotenuse of a right triangle and the other two sides is 3 and 6 respectively. Find the sides.

5. A number consists of two digits; the sum of their squares is 41. If each digit is multiplied by 5, the sum of these products is equal to the number. Find the number.

6. The difference between two numbers is 5; their product exceeds their sum by 13. Find the numbers.

7. The diagonal of a rectangle is 13 in.; the difference between its sides is 7 in. Find the sides.

8. The diagonal of a rectangle is 29 yd., and the sum of its sides is 41 yd. Find the sides.

9. The sum of the perimeters of two squares is 104 ft.; the sum of their areas is 346 sq. ft. Find their sides.

10. The difference between the areas of two squares is 231 sq. in.; the difference between their perimeters is 28 in. Find their sides.

11. Two trains leave New York simultaneously for St. Louis, which is 1170 mi. distant; the one goes 10 mi. per hour faster than the other and arrives $9\frac{3}{4}$ hr. sooner. Find the rate of each train.

12. In going 120 yd. the front wheel of a wagon makes 6 revolutions more than the rear wheel; but if the circumference of each wheel were increased 3 ft., the front wheel would make only 4 revolutions more than the rear wheel in going the same distance. Find the circumference of each wheel.

13. The sum of the volumes of two cubes is 35 cu. in. and the sum of the lengths of their edges is 5 in. Find the length of the edge of each.

14. The difference between the volumes of two cubes is 37 cu. ft. and the difference between their edges is 1 ft. Find their edges.

386. Higher Equations. Certain systems containing higher equations can be solved by the methods of this chapter.

EXAMPLES

1. Solve:
$$\begin{cases} x^3+y^3=18xy, & (1) \\ x+y=12. & (2) \end{cases}$$

Put $x=u+v$, and $y=u-v$,

Then, from (1) $(u+v)^3+(u-v)^3=18(u+v)(u-v)$, (3)

and from (2), $(u+v)+(u-v)=2u=12$. (4)

$\therefore u=6$.

Combining (4) and (3), $216+18v^2=9(36-v^2)$. (5)

Solving (5), $v=\pm 2$. (6)

From (4) and (6), $x=u+v=8$ and 4, (7)

and $y=u-v=4$ and 8. (8)

TEST as usual.

2. Solve:
$$\begin{cases} x^4+y^4=706, & (1) \\ x-y=2. & (2) \end{cases}$$

Put $x=u+v$, $y=u-v$,

Then from (1), $(u+v)^4+(u-v)^4=706$, (3)

and from (2), $(u+v)-(u-v)=2$. (4)

Simplifying (4), $v=1$. (5)

From (5) and (3), $(u+1)^4+(u-1)^4=706$. (6)

Simplifying (6), $u^4+6u^2-352=0$. (7)

Solving (7), $u^2=16$, (8)

and $u^2=-22$. (9)

From (8), $u=\pm 4$. (10)

From (5) and (10), $x=\pm 4+1=5,\ -3$. (11)

and $y=\pm 4-1=-5,\ +3$. (12)

The imaginary values of u in (9), namely $\pm\sqrt{-22}$, would give other values of x and y, but such values will be omitted here and considered in Chapter XXVIII.

TEST as usual.

WRITTEN EXERCISES

Solve:

1. $x^3+y^3=189$,
   $x+y=9$.
2. $x^3+y^3=72$,
   $x+y=6$.
3. $x^3+y^3=189$,
   $x^2y+xy^2=180$.
4. $x^4+y^4=2$,
   $x+y=0$.
5. $x^5+y^5=2$,
   $x+y=2$.
6. $x^4+y^4=17$,
   $x-y=1$.

REVIEW

WRITTEN EXERCISES

Solve:

1. $x + y = 7.5,$
   $xy = 14.$

2. $3x - 2y = 0,$
   $xy = 13.5.$

3. $x + y = 7,$
   $x^2 - y^2 = 21.$

4. $x - y = 5,$
   $x^2 + y^2 = 37.$

5. $x - y = 1,$
   $3x^2 + y^2 = 31.$

6. $x - y = 5,$
   $x^2 + 2xy + y^2 = 75.$

7. $x + y = 7(x - y),$
   $x^2 + y^2 = 225.$

8. $5(x^2 - y^2) = 4(x^2 + y^2),$
   $x + y = 8.$

9. $\frac{x^2y}{x + 5y} = 4,$
   $xy = 20.$

10. $x^2 + xy + y^2 = 19,$
    $xy = 6.$

11. $(x + 2)(x - 3) = 0,$
    $x^2 + 3xy + y^2 = 5.$

12. $x + y = 5,$
    $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}.$

13. $x - y = -3,$
    $x^3 - y^3 = -9.$

14. $x^2 - y^2 = 144,$
    $x - y = 8.$

15. $\frac{x}{2} - \frac{y}{3} = 0,$
    $x^2 + y^2 = 5(x + y) + 2.$

16. $x + y = 9,$
    $\frac{xy}{\sqrt{xy}} = \frac{10}{\sqrt{5}}.$

17. $x^3 - y^3 = 665,$
    $x - y = 5.$

18. $x + y = \frac{x}{y} = x^2 - y^2.$

19. $x + y = a,$
    $x^3 + y^3 = b^3.$

20. $x - y = a,$
    $x^3 - y^3 = b^3.$

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Name three classes of *simultaneous quadratic equations* that can be solved by quadratic methods. Secs. 381, 383, 384.

2. State how the first class may be solved; also the second class; the third class. Secs. 381, 383, 384.

3. How is it possible to determine what values of one unknown are to be associated with those of the other? Sec. 382.

4. State a set of higher equations that can be solved by quadratic methods. Sec. 386.

HISTORICAL NOTE

Simultaneous quadratic equations in determinate systems, such as we study in elementary algebra, received little attention until recent times. Mathematicians of the past favored the study of indeterminate systems, that is, systems in which there are more unknown quantities than equations. These problems offered a wide range for ingenuity, and Diophantos invented many special devices for their solution. The Hindoos and Arabs were likewise attracted to this class of problems. Thus, no real progress was made by Eastern scholars in finding general solutions of simultaneous quadratic equations.

To illustrate how little was known about this subject in the middle ages, we may refer to Gerbert (990) who, while teaching at Rheims, gained much fame as a mathematician by solving the problem : *To find the sides of a right-angled triangle given its area and hypotenuse.* It seems strange, from our present day point of view, that the solution of so simple a pair of equations as:

$$(1)\ x^2 + y^2 = h^2 \qquad (2)\ \tfrac{1}{2}xy = a$$

should have brought great distinction to any one. Gerbert is often referred to as Sylvester, because he later became Pope under the name of Sylvester II.

CHAPTER XXIV

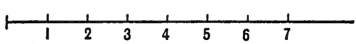
REVIEW AND EXTENSION OF PROCESSES

POSITIVE AND NEGATIVE NUMBERS

387. Algebra is concerned with the study of numbers. The number of objects in any set (for example, the number of books on a shelf) is found by counting. Such numbers are called **whole numbers** or **integers**; also **primitive** or **absolute numbers.**

In arithmetic, numbers are usually represented by means of the numerals, 0, 1, 2, 3 . . . 9, according to a system known as the **decimal notation.** In algebra, numbers are represented by numerals and also by letters, either singly or in combinations.

388. Graphical Representation. The natural integers may be represented by equidistant points of a straight line, thus:



389. Addition. If two sets of objects are united into a single set (for example, the books on two shelves placed on a single shelf), the number of objects in the single set is called the sum of the numbers of objects in the two original sets. The process of finding the sum is called **addition.**

The sign, +, between two number symbols indicates that the numbers are to be added. In the simplest instances the sum is found by counting.

Thus, to find $5 + 7$, we first count 5, and then count 7 more of the number words next following (six, seven, eight, etc.). The number word with which we end (twelve) names the sum.

390. Graphical Representation. The sum of two integers may be represented graphically thus:

Theoretically, the sum of two integers can in every instance be found by counting. But it is not necessary or desirable to do so when either (or both) of the numbers is larger than nine. In this case, the properties of the decimal notation, as learned in arithmetic, enable us to abridge the process of counting.

391. Commutative Law of Addition. If two sets of objects are to be united into a single set, the number of objects in the result is obviously the same whether the objects of the second set are united with those of the first, or those of the first united with those of the second.

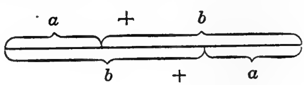
For example, the number of books is the same whether those on the first shelf be placed on the second, or those on the second be placed on the first.

In symbols: $$a + b = b + a.$$

This fact is called the **commutative law of addition.**

The letters a and b are here used to stand for integers, but the law will apply when they stand for any algebraic numbers.

392. Graphical Representation. The commutative law may be represented graphically thus:



393. Addition of Two or More Whole Numbers. If more than two sets of objects are united into a single set, the number of objects in the resulting set is called the **sum** of the number of objects in the original sets, and the process of finding the sum is called **addition.** As in the case of two numbers, the sum of three or more numbers may be found by counting in the simplest instances, and for larger numbers, the process may be abridged by use of the properties of the decimal notation.

394. The commutative law likewise applies to the sum of three or more integers. That is:

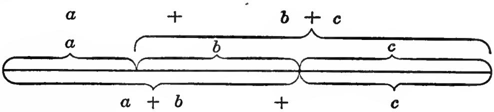
The sum is the same for every order of adding the numbers.

395. Associative Law of Addition. If we have three rows of books, the number of books is the same whether those in the second row are first placed with the first row, and then those in the third row placed with these, or those in the third row placed with the second, and then all of these with the first row.

In symbols: $(a+b)+c=a+(b+c)$.

This fact is called the **associative law of addition.**

396. Graphical Representation. The associative law may be represented graphically thus:



The properties stated above are often used to abridge calculations. Thus, $7+4+3+6$, are more easily added thus: $(7+3)+(4+6)$.

ORAL EXERCISES

Add in the easiest way:

1. $8+3+2+7$.
2. $91+43+9$.
3. $87+26+13$.
4. $13a+5a+17a+5a$.
5. $7x+12x+3x+18x$.
6. $8y+10y+7y+5y$.
7. $23a+6a+2a+4a$.
8. $48x+73x+2x+7x$.
9. $19y+54y+6y+y$.
10. $73b+186b+14b$.
11. $279t+347t+21t$.
12. $624p+45p+6p+5p$.
13. $93t+9t+7t+t$.
14. $144m+7m+6m+3m$.

WRITTEN EXERCISES

Show graphically that:

1. $11+4+6=11+(4+6)$.
2. $3+(4+1)=1+3+4$.
3. $8+5=5+8$.
4. $4a+5b=5b+4a$.

397. Subtraction. It often happens that we wish to know how many objects are left when some of a set are taken away, or to know how much greater one number is than another. The process of finding this number is called **subtraction.** The number taken away is called the **subtrahend**, that from which it is taken, the **minuend**, and the result, the **difference** or the **remainder.**

398. The sign of subtraction is $-$.

399. Subtraction is the reverse of addition, and from every sum one or more differences can at once be read.

Thus, from $5 + 7 = 12$ we read at once $12 - 5 = 7$,
$12 - 7 = 5$.
And from $5 + 5 = 10$, we read $10 - 5 = 5$.
And from $a + b = c$, we read $c - a = b$,
$c - b = a$.
Likewise, from $a + b + c = d$ we read $d - a = b + c$,
$d - (a + b) = c$, etc.

400. There is no commutative law of subtraction. For $7 - 4$ is not the same as $4 - 7$. In fact, the latter indicated difference has no meaning in arithmetic. We cannot take a larger number of objects from a smaller number.

401. In algebra, where numbers are often represented by letters, we may not know whether the minuend is larger than the subtrahend or not. For example, in $a - b$, we do not know whether a is larger than b or not. But it is desirable that such expressions should have a meaning in all cases, and this is accomplished by the definition and use of **relative numbers.**

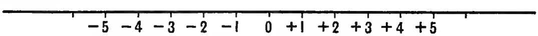
402. The First Extension of the Number System. Relative Numbers. Whenever quantities may be measured in one of two opposite senses such that a unit in one sense offsets a unit in the other sense, it is customary to call one of the senses the positive sense, and the other the negative sense, and numbers measuring changes in these senses are called **positive** and **negative** numbers respectively. (For examples, see Chapter IV.)

403. A number to be added is offset by an equal number to be subtracted; hence such numbers satisfy the above definition, and numbers to be added are called positive, and those to be subtracted are called negative. Consequently, positive and negative numbers are denoted by the signs + and − respectively.

Thus, + 5 means positive five, and denotes five units to be added or to be taken in the positive sense.

− 5 means negative five, and denotes five units to be subtracted, or to be taken in the negative sense.

404. Graphical Representation. Relative integers may be represented graphically thus:



It appears that the positive integers are represented by just the same set of points as the natural or absolute integers. For this and other reasons the absolute numbers are usually identified with positive numbers. Although it is usually convenient to do this, we have in fact the three classes of numbers: the absolute, the positive, and the negative. Thus, we may consider $ 5 without reference to its relation to an account, or we can consider it as $ 5 of assets, or we may consider it as $ 5 of debts.

405. According to Sec. 403, the signs +, − denote the operations of addition or subtraction, or the positive or negative character of the numbers which these signs precede.

If it is necessary to distinguish a sign of character from a sign of operation, the former is put into a parenthesis with the number it affects.

Thus, $+8-(-3)$, means: positive 8 minus negative 3.

When no sign of character is expressed, the sign plus is understood.

Thus, $5-3$ means: positive 5 minus positive 3.
Similarly, $8a+9a$ means: positive $8a$ plus positive $9a$.

406. Absolute Value. The value of a relative number apart from its sign is called its **absolute value.**

ORAL EXERCISES

Read the following in full, according to Sec. 405:

1. $6-4$.
2. $-5-8$.
3. $-8+20$.
4. $2a+3a$.
5. $2b-3b$.
6. $-2a-3a$.
7. $2y+(+3y)$.
8. $3p-(-2p)$.
9. $12-(-5)$.
10. $-12-(+5)$.
11. $-7-(-9)$.
12. $2a-(+3a)$.
13. $c+d$.
14. $c-d$.
15. $m+(-n)$.
16. $4x+(-2x)$.
17. $7-9$.
18. $7+9$.
19. $-7+(-9)$.
20. $2y-(-3y)$.
21. $-2x-(-3x)$.
22. $-2x+(-3x)$.
23. $-2b-(-5c)$.
24. $3a-(+5y)$.

WRITTEN EXERCISES

Indicate, using the signs $+$, $-$:

1. The sum of positive 5 and positive 3.
2. The sum of positive a and negative b.
3. The difference of positive p and positive q.
4. The difference of negative 5 and positive 3.
5. The difference of negative x and positive y.
6. The sum of positive a and positive b.
7. The sum of negative ab and negative ab.
8. The sum of positive y and negative x.
9. The difference of positive xy and negative xy.
10. The difference of negative pq and positive mn.

407. Addition of Relative Numbers.

Just as	3 pounds	+ 5 pounds	= 8 pounds,
so	3 positive units	+ 5 positive units	= 8 positive units,
and	3 negative units	+ 5 negative units	= 8 negative units.

To add units of opposite character, use is made of the defining property of relative numbers, that a unit in one sense offsets a unit in the other sense. Thus, to add 3 positive units and 7 negative units we notice that the 3 positive units offset

3 of the negative units and the result of adding the two will be 4 negative units.

That is, $(+3)+(-7)=(+3)+(-3)+(-4)=-4.$

In general:

I. *If two relative numbers have the same sign, the absolute value of the sum is the sum of the absolute values of the addends, and the sign of the sum is the common sign of the addends.*

II. *If two relative numbers have opposite signs, the absolute value of the sum is the difference of the absolute values of the addends, and the sign of the sum is the sign of the addend having the larger absolute value.*

408. More than two numbers are added by repetition of the process just described. This may be done either:

(1) *by adding the second number to the first; then the third number to the result, and so on;* or

(2) *by adding separately all the positive numbers and all the negative numbers, and then adding these two results.*

409. It may be verified that the Commutative and the Associative Laws of Addition hold also for relative integers.

410. **Subtraction of Relative Numbers.** Since n units of one sense are offset by adding n units of the opposite sense, we may subtract n units of one sense by adding n units of the opposite sense.

Thus, $7-(+3)=7+(-3).$

And, $7-(-3)=7+(+3).$

And, $4-(+7)=4+(-7).$

411. Accordingly, subtraction may be regarded as the inverse of addition: *To subtract a monomial, we add its opposite.*

To subtract an algebraic expression consisting of more than one term, we subtract the terms one after another.

In general, *to subtract any algebraic expression we may change the sign of each of its terms and add the result to the minuend.*

ORAL EXERCISES

State the sums:

1. $5+(-3)$.
2. $-6a+(-7a)$.
3. $-11y+3y$.
4. $-12z+(-18z)$.
5. $11x+(-2x)+(-5x)$.
6. $-3q+7q+(-6q)$.

7. How may the correctness of a result in subtraction be tested? State the differences:

8. $11-6$.
9. $-11-6$.
10. $-11a-(-6a)$.
11. $31x-(+5x)$.
12. $-31y-(-3y)$.
13. $17p-(-17p)$.

14. How may a parenthesis preceded by the sign + be removed without changing the value of the expression? One preceded by the sign −?

15. How may terms be introduced in a parenthesis preceded by the sign + without changing the value of the expression? In a parenthesis preceded by the sign −?

WRITTEN EXERCISES

Add:

1. $\begin{array}{r} 2a+5 \\ a+4 \\ \hline \end{array}$

2. $\begin{array}{r} 3a+8 \\ a-4 \\ \hline \end{array}$

3. $\begin{array}{r} 6b+c \\ 3b-2c \\ \hline \end{array}$

4. $\begin{array}{r} -3a+b \\ 2a-3b \\ \hline \end{array}$

5. $\begin{array}{r} 4a-5 \\ 3a+7 \\ \hline \end{array}$

6. $\begin{array}{r} c+d-5 \\ c-d+5 \\ \hline \end{array}$

7. $\begin{array}{r} x+y+2z \\ x-y+4z \\ \hline \end{array}$

8. $\begin{array}{r} p+q-m \\ p-q+2m \\ \hline \end{array}$

9. $\begin{array}{r} 2x-y+z \\ 2x+2y-4z \\ \hline \end{array}$

10. $\begin{array}{r} ax+by+c \\ ax+y-c \\ \hline \end{array}$

11. $\begin{array}{r} 4x-2z+y \\ 2x-y+z \\ \hline \end{array}$

12. $\begin{array}{r} 1+m^3+p^2 \\ 1-m^3-p^2 \\ \hline \end{array}$

13. $\begin{array}{r} ax+by+cz^2 \\ bx+ay-z^2 \\ \hline \end{array}$

14. $\begin{array}{r} 1.5x+3.5y+z \\ .5x+6.5y-.1z \\ \hline \end{array}$

15. $\begin{array}{r} \frac{1}{2}x+\frac{4}{5}y-\frac{1}{3}z \\ \frac{1}{4}x-\frac{3}{5}y+\frac{2}{3}z \\ \hline \end{array}$

Subtract:

16. $\begin{array}{r} 4a+6 \\ 2a-9 \\ \hline \end{array}$

17. $\begin{array}{r} 8x+3 \\ -3x+2 \\ \hline \end{array}$

18. $\begin{array}{r} 11x-4y \\ 19x \\ \hline \end{array}$

19. $\begin{array}{r} 12t \\ -6t+3 \\ \hline \end{array}$

20. $\begin{array}{r} 7x+3y \\ 2y-4x \\ \hline \end{array}$

21. $\begin{array}{r} 4a+3b \\ 2c-5a \\ \hline \end{array}$

22. $\begin{array}{r} 16y+-z \\ 8y-10z \\ \hline \end{array}$

23. $\begin{array}{r} 10x^2-16y^2 \\ 20x^2+\ \ 4y^2 \\ \hline \end{array}$

24. $\begin{array}{r} 41x^2-16y^2 \\ 15x^2-20y^2 \\ \hline \end{array}$

25. $\begin{array}{r} -11m+40p \\ -40m-12p \\ \hline \end{array}$

26. $\begin{array}{r} x-7y+5z \\ -x+4y-6z \\ \hline \end{array}$

27. $\begin{array}{r} p+\ \ q-\ \ m \\ 6p-2q+4m \\ \hline \end{array}$

28. $\begin{array}{r} ax^2+by^2+c \\ ax^2-by^2-c \\ \hline \end{array}$

29. $\begin{array}{r} m^2+2p^2-6q^2 \\ -m^2-4p^2+5q^2 \\ \hline \end{array}$

30. $\begin{array}{r} 40x^2-10y^2+\ \ z^2 \\ 50x^2+40y^2-7z^2 \\ \hline \end{array}$

31. $\begin{array}{r} 2.5a+6.3b-.1c \\ 1.5a-3.5b-.9c \\ \hline \end{array}$

32. $\begin{array}{r} \frac{1}{2}x-\frac{2}{3}y+\frac{1}{5}z \\ \frac{3}{4}x-\frac{1}{3}y+\frac{4}{5}z \\ \hline \end{array}$

33. $\begin{array}{r} a^2x+b^2y+c^2z^2 \\ a^2x+by^2+cz^2 \\ \hline \end{array}$

Remove parentheses and unite terms as much as possible:

34. $3a-2b+(3b-7a)$.

35. $4m+5-(63-3p)$.

36. $(11x+5y)-(-3x+2z)$.

37. $7-[2-(3-5)]$.

38. $(4a+2a)-[(7a-5a)+(-6a-17a)]$.

39. $3+\{5x-2-(7x-1-2+3x)\}$.

40. $4x+\{2x-[3-(7x+5)-1+x]\}$.

41. $ab-\{2ab+c-[3c-(6-ab)]\}$.

42. $12xy-\{2xy-6z+[4xy-(2z+xy)]\}$.

43. $-(5pq-3xy+1)-\{-2pq+3xy-(xy+2)\}$.

44. $-\{4abc-(2ac+bc)\}+\{6abc+(2ac-bc)\}$.

Write the expressions of Exercises 45–56, as x minus a parenthesis. Also group the terms involving x and y in each expression in a parenthesis preceded by the sign $-$.

45. $3a+x+2y$.

46. $-5y+7c-8+x$.

47. $6t^2+4y+x-3$.

48. $2m+p+x-y$.

49. $a+x-b+cy$.

50. $p+x-y+by$.

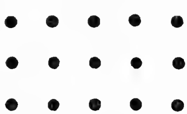
51. $x - 3b + 2y - c$.
52. $3b + x + ay + by$.
53. $by + m - n + x$.
54. $2p - q + x - y$.
55. $ay + x - z + zy$.
56. $c + d + x - (a + b)y$.

412. Multiplication. To multiply two absolute integers means to use the one (called the **multiplicand**) as addend as many times as there are units in the other (called the **multiplier**). The result is called the **product**.

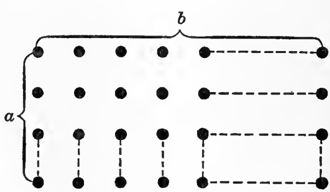
Thus, 3 times 4 means $4 + 4 + 4$.

The simpler products are obtained by actually making the additions that are implied. For large numbers, as we have seen in arithmetic, the process may be much abridged by use of properties of the decimal notation.

413. Commutative Law of Multiplication. The expression 3 times 5 means $5 + 5 + 5$, and may be indicated as follows:



That is, since each horizonal line (or *row*) contains 5 dots, there are all together 3 times 5 dots. But each vertical line (or *column*) contains 3 dots and there are 5 columns. Hence there are 5 times 3 dots all together. But the number of dots is the same whether we count them by rows or by columns, hence *5 times 3 equals 3 times 5*. Quite similarly, if we have a rows of dots with b dots in each row, it follows that a times b equals b times a.



The result may be stated in symbols thus:

$$ab = ba.$$

This fact, called the **Commutative Law of Multiplication,** means that the product is not altered if multiplier and multiplicand are interchanged. Consequently, these names are frequently replaced by the name **factor** applied to each of the numbers multiplied.

414. Let each dot of the block of dots of (A) above have the value of 6. Since there are 3×5 dots, the value of the block would be $(3 \times 5) \times 6$.

A second expression for the value of the block is obtained by finding the value of the first row (viz. 5×6) and multiplying it by the number of rows, or 3. The expression resulting must be equal to that already found, or:

$$3 \times (5 \times 6) = (3 \times 5) \times 6.$$

Similarly, if each dot of the block (B) above has the value c, it follows that

$$a(bc) = (ab)c$$

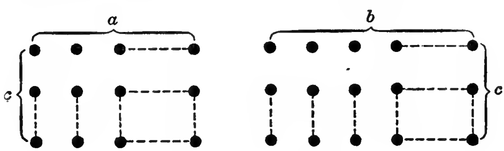
That is: *The product of three absolute integers is not altered if they be grouped for multiplication in any way possible without changing the order. This is a case of what is known as the* **Associative Law of Multiplication** *for absolute integers.*

415. By similar methods and use of these results it can be proved that both the Commutative Law and the Associative Law apply to all products of absolute integers. That is:

Commutative Law. *The product of any number of given factors is not changed, if the order of the factors be changed in any way.*

Associative Law. *The product of any number of given factors in a given order is not changed if the factors be grouped in any way.*

The Distributive Law. From the block of dots we see that



$$c(a + b) = ca + cb.$$

This is called the **Distributive Law** of Multiplication, and the above proof covers the case in which a, b, and c are absolute integers.

416. Multiplication of Relative Integers. To multiply by a positive integer means to take the multiplicand as addend as many times as there are units in the multiplier, and to multiply by a negative integer means to take the multiplicand as subtrahend as many times as there are units in the multiplier.

Consequently,

$$4 \cdot 3 = 3 + 3 + 3 + 3 = 12.$$
$$4(-3) = -3 + (-3) + (-3) + (-3) = -12.$$
$$(-4)3 = -3 - 3 - 3 - 3 = -12.$$
$$(-4)(-3) = -(-3) - (-3) - (-3) - (-3) = 12.$$

So, generally,

$$(+a)(+b) = +ab,$$
$$(+a)(-b) = -ab,$$
$$(-a)(+b) = -ab,$$
$$(-a)(-b) = +ab.$$

417. In words: *The product of two (integral) factors of like signs is positive, and of two factors of unlike signs is negative; in each case the absolute value of the product is the product of the absolute values of the factor.*

418. We observe that the Commutative Law holds in this case also.

For example: $(-b)a = a(-b)$. Since $ba = ab$, by the Commutative Law for absolute integers, and by Sec. 416, $(-b)a = -ba = -ab = a(-b)$.

It may be shown that the Associative and the Distributive Laws also hold.

ORAL EXERCISES

State the laws that are applied in the various steps of the following calculations:

1. $2 \cdot 8 \cdot 3 \cdot 5 = 2 \cdot 5 \cdot 8 \cdot 3 = 10 \cdot 24 = 240.$

2. $5(17 \cdot 2 - 6c) = 5(17 \cdot 2) - 5(6c) = (5 \cdot 2)17 - (5 \cdot 6)c$
$= 10 \cdot 17 - 30c = 170 - 30c.$

3. $7b(5x + ab) = (7b)(5x) + (7b)(ab) = 35bx + 7ab^2.$

4. $(a+b)(c+d)=a(c+d)+b(c+d)=ac+ad+bc+bd.$

State the products:

5. $\begin{array}{r} ax \\ 5\,b \\ \hline \end{array}$

6. $\begin{array}{r} -2\,a \\ -3\,a^2 \\ \hline \end{array}$

7. $\begin{array}{r} 8\,x \\ -4\,xy \\ \hline \end{array}$

8. $\begin{array}{r} a^5b \\ a^2c \\ \hline \end{array}$

9. $\begin{array}{r} 5\,x^2 \\ 2\,x^2 \\ \hline \end{array}$

10. $\begin{array}{r} -x^2y \\ -xy^2 \\ \hline \end{array}$

11. $\begin{array}{r} 6\,xyz \\ -2\,xz^3 \\ \hline \end{array}$

12. $\begin{array}{r} -4\,a^2t \\ -3\,b^2t \\ \hline \end{array}$

13. $\begin{array}{r} -5\,x^3y^2 \\ 5\,x^2y^3 \\ \hline \end{array}$

14. $(a+b)^2$.
15. $(a-c)^2$.
16. $(x+y)^2$.
17. $(x-a)^2$.
18. $(x-2\,y)^2$.
19. $(x+y)(x-y)$.
20. $(2\,x-1)^2$.
21. $(2\,m+n)^2$.
22. $(x-1)(x+1)$.
23. $(1-y)(1+y)$.
24. $(m+2\,x)^2$.
25. $(3-2\,d)(3+2\,d)$.
26. $(4-x)^2$.
27. $-3\,x(x+2)$.
28. $(1-2\,d)(1+2\,d)$.
29. $(\frac{1}{2}\,x-\frac{1}{3}\,y)(\frac{1}{2}\,x+\frac{1}{3}\,y)$.

WRITTEN EXERCISES

Multiply and test:

1. $\begin{array}{r} 2\,a+3 \\ 2\,a+4 \\ \hline \end{array}$

2. $\begin{array}{r} 3\,a-2\,b \\ 2\,a-3\,b \\ \hline \end{array}$

3. $\begin{array}{r} x+2 \\ x-3 \\ \hline \end{array}$

4. $\begin{array}{r} x^2-3\,x+1 \\ x-2 \\ \hline \end{array}$

5. $\begin{array}{r} y^2-3\,y+4 \\ y-2 \\ \hline \end{array}$

6. $\begin{array}{r} p-3\,t+t^3 \\ p-2\,t \\ \hline \end{array}$

7. $(3\,x-4\,y)^2$.
8. $(5\,y-7\,t)(8\,y+2\,t)$.
9. $(6\,a+13\,q)^2$.
10. $(1+x)(2+x)(1-x)$.
11. $(1-2\,y)^2$.
12. $(b-3)(b+7)(b-3)$.
13. $(x^n+y^n)(x^n-y^n)$.
14. $(a^p-b^m-1)^2$.
15. $(4\,x-3\,xy)(2\,x+2\,y^2)$.
16. $(ax+by)(cx+dy)$.
17. $(ax^2-by)(cx-dy^2)$.
18. $(m+n+p)(m+n-p)$.
19. $(a+b+c-d)^2$.
20. $(a-2\,b)(a-2\,b)(a-2\,b)$.

419. Detached Coefficients. — When two polynomials can be arranged in descending powers of the same letter, the work of multiplication may often be shortened by *using the coefficients only.* This is called **multiplication by detached coefficients.**

Thus, to multiply $3x^2-2x+1$ by $x-1$:

Work in Full	Work with Coefficients Only	Test
$3x^2-2x+1$	$3-2+1$	$x=1$ | 2
$\underline{x-1}$	$\underline{1-1}$	$\underline{0}$
$3x^3-2x^2+\quad x$	$3-2+1$	
$\underline{\quad -3x^2+2x-1}$	$\underline{\quad -3+2-1}$	
$3x^3-5x^2+3x-1$	$3-5+3-1$	0

The processes in the two cases are identical with the exception of the omission of the letters in the second case. The product of $3x^2$ and x, the first terms of the factors, show that the first term of the product is $3x^3$, the next term must contain x^2 and the next x, because the terms in the result must be in order of degree. Thus we may perform the operation with coefficients, and then supply the proper letters and exponents.

When some powers of the letters are missing, zeros must be supplied as coefficients of the missing powers in order to keep a record of the places in which the powers are missing.

Thus, to multiply $2x^3-7x+3$ by $2x-5$, write $2x^3-7x+3$ as $2x^3+0x^2-7x+3$, and perform the multiplication as shown at the right.

	Test
$2+\ 0-\ 7+3$	-2
$\underline{2-\ 5}$	$\underline{-3}$
$4+\ 0-14+6$	
$\underline{\quad -10-\ 0+35-15}$	
$4-10-14+41-15$	$+6$

$\therefore\ (2x-5)(2x^3-7x+3)=4x^4-10x^3-14x^2+41x-15.$

WRITTEN EXERCISES

Multiply by detached coefficients and test:

1. $m+n$	3. a^2+2a+1	5. y^3-y^2+5
$\underline{2m-3n}$	$\underline{a+1}$	$\underline{2y+3}$
2. x^3-5x+3	4. x^4-3x^3+5	6. x^4+2x^2+7x
$\underline{x-1}$	$\underline{x+2}$	$\underline{2x-1}$

7. x^2-x-6 by $2x^2-4x-16$.
8. a^2-a-12 by $3a^2-15a+18$.

9. a^4-2a^2+1 by a^4+2a^2+1.

10. m^3-m^2-m+1 by $2m-1$.

11. $1-x-x^2$ by $2+x^3-x^2+2x$.

12. x^2+xy+y^2 by x^2+y^2-xy.

13. $x^{2a}-4x^a+4$ by $1-x^a+x^{3a}-2x^{2a}$.

420. Division. Division is the process of finding a number called the **quotient**, such that when multiplied by a given number, called the **divisor**, the product is a given number, called the **dividend.**

421. The fundamental relation of division is:

Divisor times Quotient equals Dividend.

From this it follows at once that if dividend and divisor have the same sign, the quotient is positive, and if they have unlike signs, the quotient is negative.

422. In dividing polynomials, it is best to arrange both dividend and divisor in the order of the powers of the same letter.

Thus, $(x^3-3x^2+3x-1)\div(x-1)$.
instead of $(3x-3x^2+x^3-1)\div(x-1)$.

WRITTEN EXERCISES

Divide:

1. x^3+x^2-x-1 by $x+1$.

2. x^4-1 by $x-1$.

3. y^3-z^3 by y^2+yz+z^2.

4. $x^{3a}-y^{3a}$ by x^a-y^a.

5. x^6-1 by x^2-1.

6. p^5+q^5 by $p+q$.

7. m^3+m^2+m+1 by $m+1$.

8. $x^2-px-qx+pq$ by $x-p$.

9. $4x^4+1$ by $2x^2-2x+1$.

10. $x^2-2xy-3y^2$ by $x-3y$.

11. $6a^3-10a^2+13a-6$ by $3a-2$.

12. $6m^4-2m^3-8m^2+4m-8$ by $3m^2-m+2$.

13. $x^{2a}+x^a-6$ by x^a-2.

14. $x^{4a}+4x^{3a}y+6x^{2a}y^2+4x^ay^3+y^4$ by $x^{2a}+2x^ay+y^2$.

423. Detached Coefficients. When both divisor and dividend involve a regular series of powers of the same letters, it is easier to divide with *coefficients only.*

Thus, to divide $x^3 - 4x^2 + 4x - 1$ by $x - 1$.

Work in Full

$$\begin{array}{r|l} & x^2 - 3x + 1 \\ \hline x - 1 & x^3 - 4x^2 + 4x - 1 \\ & \underline{x^3 - x^2} \\ & -3x^2 \\ & \underline{-3x^2 + 3x} \\ & \quad\quad\quad x - 1 \\ & \quad\quad\quad \underline{x - 1} \end{array}$$

Work with Coefficients Only

$$\begin{array}{r|l} & 1 - 3 + 1 \text{ or } x^2 - 3x + 1 \\ \hline 1 - 1 & 1 - 4 + 4 - 1 \\ & \underline{1 - 1} \\ & \;\; -3 \\ & \;\; \underline{-3 + 3} \\ & \quad\quad\quad 1 - 1 \\ & \quad\quad\quad \underline{1 - 1} \end{array}$$

Test. Letting $x = 2$,

$$(2 - 1)(4 - 6 + 1) = 8 - 16 + 8 - 1.$$
$$\therefore -1 = -1.$$

424. If the series of powers is not complete, zero coefficients must be used.

Thus, $x^3 + 3x^2 + 1$ must be regarded as $x^3 + 3x^2 + 0x + 1$, and the coefficients $1 + 3 + 0 + 1$ must be used in the division.

WRITTEN EXERCISES

Divide by detached coefficients and test:

1. $\dfrac{a^4 - 1}{a - 1}$.
2. $\dfrac{a^3 - 1}{a^2 + a + 1}$.
3. $\dfrac{8x^3 + 1}{2x + 1}$.
4. $(x^2 + 5x + 6) \div (x + 2)$.
5. $(x^2 - 2x - 3) \div (x - 3)$.
6. $(10 + x^2 - 7x) \div (x - 2)$.
7. $(x^4 - 1) \div (x^2 - 1)$.
8. $(x^3 + 3x^2 + 3x + 1) \div (x + 1)$.
9. $(y^3 - 3y^2 + 3y - 1) \div (y - 1)$.
10. $(6x + 1 + 9x^2) \div (3x + 1)$.
11. $(a^4 + a^2 + 1) \div (a^2 - a + 1)$.
12. $\dfrac{m^4 + 4m^3 + 6m^2 + 4m + 1}{m^2 + 2m + 1}$.

425. Special Quotients. The general form of quotients represented by the type $(x^n \pm y^n) \div (x \pm y)$ may readily be seen by division.

Thus, considering $(x^n + y^n) \div (x + y)$, we have

$$
\begin{array}{r|l}
 & x^{n-1} - x^{n-2}y + x^{n-3}y^2 \cdots \\
\hline
x + y & x^n \qquad\qquad + y^n \\
 & \underline{x^n + x^{n-1}y} \\
 & \quad - x^{n-1}y \\
 & \quad \underline{- x^{n-1}y - x^{n-2}y^2} \\
 & \qquad\qquad + x^{n-2}y^2 + \cdots
\end{array}
$$

By inspection, what do you think the fourth term of the quotient will be? The fifth? Verify your opinion by continuing the above division.

WRITTEN EXERCISES

Find the quotient and remainder if any:

1. $(x^5 - 1) \div (x - 1)$.
2. $(x^6 + 1) \div (x - 1)$.
3. $(x^5 + 1) \div (x + 1)$.
4. $(x^6 - 1) \div (x + 1)$.
5. $(x^4 - y^4) \div (x + y)$.
6. $(x^4 - y^4) \div (x - y)$.
7. $(x^5 + y^5) \div (x + y)$.
8. $(x^7 + y^7) \div (x + y)$.
9. $\dfrac{x^7 - y^7}{x - y}$.
10. $\dfrac{x^7 + y^7}{x - y}$.
11. $\dfrac{x^6 - y^6}{x - y}$.
12. $\dfrac{x^6 + y^6}{x - y}$.
13. $\dfrac{x^6 - y^6}{x + y}$.
14. $\dfrac{x^6 + y^6}{x + y}$.

15. Substitute 5 for n in the first five terms of the above division, and compare with the result of Exercise 7.

16. Substitute 6 for n similarly in the first six terms, and compare with the result of Exercise 14.

17. Use $n = 7$ in the first 7 terms; compare with Exercise 8.

18. Take $n = 5$ and $y = 1$, and compare with Exercise 3.

19. Divide $x^n + y^n$ by $x - y$ to seven terms. Let $n = 6$ and compare with the result in Exercise 12. Let $n = 7$ and compare with the result in Exercise 10.

20. Divide $x^n - y^n$ by $x + y$ to six terms. Let $n = 6$ and compare with the result of Exercise 13.

21. Divide $x^n - y^n$ by $x - y$ to six terms. Let $n = 6$ and compare with the result of Exercise 11.

ZERO AND ITS RELATION TO THE PROCESSES

426. Definition of Zero. Zero may be defined as the result of subtracting a number from itself.

$$a - a = 0.$$

427. Addition. By definition of zero, $a + 0 = a + b - b = a$, since to add b and immediately take it away leaves the original number a.

$$a + 0 = a.$$

428. Subtraction. Similarly, $a - 0 = a - (b - b) = a - b + b = a$, since to take away b and then replace it leaves the original number a.

$$a - 0 = a.$$

To add or subtract zero does not alter the original number.

429. Multiplication. By definition of zero,

$$0 \cdot a = (b - b)a = ba - ba = 0.$$

Or, $$0 \cdot a = 0.$$

That is, if one factor is zero, the product is zero.

Multiplication by zero simply causes the multiplicand to vanish.

It follows directly that $\frac{0}{a} = 0.$

430. Division. According to the definition of division $a \div 0$, or $\frac{a}{0}$, asks: By what must zero be multiplied to produce a?

Let x denote the desired number. Then $0 \cdot x = a$.

But we know that zero times any number is zero. If a is not zero, there is no number x that satisfies the above equation.

That is, $\frac{a}{0} =$ **no number.**

If a is zero, every number x satisfies the equation.

That is, $\frac{0}{0} =$ **any number,** since 0 times any number = 0.

Division by zero is therefore either entirely indefinite or impossible. In either case it is not admissible.

If we divide one literal expression by another, there is no guarantee that the result is correct for those values of the letters that make the divisor zero.

EXAMPLE

Let	$a = b.$	(1)
Multiplying both members by a,	$a^2 = ab.$	(2)
Subtracting b^2 from both members,	$a^2 - b^2 = ab - b^2$	(3)
Factoring,	$(a + b)(a - b) = b(a - b).$	(4)
Dividing both members by $a-b$,	$a + b = b.$	(5)
Substituting the value of a from (1),	$b + b = b.$	(6)
Or,	$2b = b.$	(7)
Dividing by b,	$2 = 1.$	(8)

The work is quite correct to equation (4). But by dividing equation (4) by an expression that, according to the conditions of the problem, is zero, we find as the result an incorrect equation.

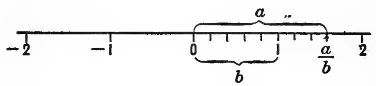
FRACTIONS

431. The Second Extension of the Number System; Fractions. We found the primitive or absolute integers by counting. We defined the operation of addition for these integers and saw that it was always possible. Next we defined the operation of subtraction for these integers, and found that it was not always possible. This led us to define relative numbers (positive and negative). In the system of numbers as enlarged by this first extension, we saw that both addition and subtraction are always possible. Then we defined multiplication for all integers and saw that it was always possible. We now examine the operation of division as just defined.

The operation $12 \div 4$ is possible in the system of integers because there exists an integer, 3, whose product with 4 is 12. But the operation $12 \div 5$ is impossible in the system of integers, since there exists no integer whose product with 5 is 12. This leads us to define another kind of number, the fraction. This is done by dividing the unit into b equal parts, and taking a of these parts. A symbol for the new number is $\frac{a}{b}$, in which a and b may be any integers. This constitutes our **second extension** of the number system.

432. The new number, $\frac{a}{b}$, is called a **fraction**; a is called the **numerator** and b the **denominator** of the fraction; a and b together are called the **terms** of the fraction.

433. Graphical Representation of Fractions. Fractions may be represented by points of the number scale that we have already had.



Thus, if the distance from *zero* to 1 is divided into b equal parts, and then a of these parts laid off from *zero*, the end point represents the fraction $\frac{a}{b}$. $\left(\text{In the figure the fraction is } \frac{8}{5}.\right)$ Fractions may also be taken in the negative sense. Two fractions are said to be equal, or to have the same value, when they are represented by the same point of the number scale.

434. If each of the b equal parts is halved, making $2b$ parts, and each of the a parts taken is halved, making $2a$ parts, the end point remains the same.

That is:

$$\frac{a}{b} = \frac{2a}{2b}.$$

Similarly:

$$\frac{a}{b} = \frac{na}{nb}.$$

(A) In words: *The value of a fraction is not altered if both numerator and denominator is multiplied by the same integer.*

(B) Reading the above equation from right to left: *The value of a fraction is not altered if both numerator and denominator be divided by a common integral factor.*

Every integer may be regarded as a fraction.

Thus, $3 = \frac{3}{1}$ or $\frac{6}{2}$ or $\frac{15}{5}$ etc. $a + b = \frac{a+b}{1}$.

435. Addition and Subtraction of Fractions. The properties (A) and (B), Sec. 434, enable us to add and subtract fractions.

If fractions have not the same denominator, they are first reduced to the same denominator, and then the results added.

WRITTEN EXERCISES

Add and express the results in lowest terms:

1. $\frac{3}{5}+\frac{1}{10a}$.
2. $\frac{4a}{5}-\frac{5a}{4}$.
3. $\frac{3}{b-1}-\frac{2}{b+1}$.
4. $\frac{x-1}{x+1}-\frac{x+4}{2(x+1)}$.
5. $\frac{x^2}{2y}+\frac{3y}{4x}$.
6. $\frac{5}{a-b}+\frac{2}{b}$.
7. $\frac{2}{a+p}-\frac{3p}{(a+p)^2}$.
8. $\frac{5}{x^4}+\frac{3}{2x^2}-1$.
9. $\frac{2q}{1-q^2}+\frac{5}{1+q}$.
10. $\frac{x}{(1-x)(1-y)}+\frac{1}{(y-x)(y-1)}+\frac{y}{(x-1)(1-y)}$.
11. $\frac{1}{a^2-(b+c)a+bc}+\frac{1}{a^2-(b+d)a+bd}+\frac{1}{a^2-(c+d)a+cd}$.
12. $\frac{2xy+y^2}{(x-y)^2}-\frac{y}{x-y}-\frac{x^2+5xy}{(x+y)^2}$.

436. The addition and reduction of fractions is facilitated by the processes for finding the l. c. m. and the h. c. f. The methods for finding these by factoring given in Chapter XIV are sufficient for elementary algebra.

WRITTEN EXERCISES

Find the l. c. m. of:

1. 48, 60, 120.
2. $15a^2$, $21a^2x$, $105abxy$.
3. $5abx$, $25a^2x$, $40b^2x^2$.
4. x^2-x, x^2-2x+1.
5. x^2-5x-6, x^2+6x+5.
6. $a(x-1)$, $b(1-x^2)$, $c(x-x^2)$.
7. $(1-3x)^2$, $1-9x^2$, $1-4x+3x^2$.
8. m^2+m+1, m^2-m+1, m^4+m^2+1.
9. $5x^2-18x+9$, $4x^2-11x-3$.
10. $x-1$, $(x-1)^3$, x^2-2x+1.
11. $(p-q)(p+r)$, $(q-r)(q-p)$, $(r-q)(r+p)$.
12. $(a-c)(a-b)$, $(c-a)(b-c)$, $(b-a)(c-b)$.
13. $x^3-6x^2+11x-6$, x^2-4x+3, x^2-3x+2.

Find the h. c. f. of:

14. 28, 35, 105.

15. $21x^2$, $42ax^2$, $63abx^3y$.

16. $1-a$, a^2-2a+1.

17. $1-x^3$, $1-x^2$, x^2-2x+1.

18. $a^2+2ab+b^2$, a^5+b^5.

19. $x+y$, x^3-xy^2, $x^2-2xy-3y^2$.

20. $x^4+x^2y^2+y^4$, x^2+y^2-xy.

21. a^3+5a^2-6a, $2a^2-2$.

22. $3z^3-7z^2+4$, $5z^3-17z^2+16z-4$.

23. $a^2+b^2+2c^2+2ab+3ac+3bc$, $ac+bc+c^2$.

437. Fractions may sometimes be added more easily by adding them in an order different from that in which they were given.

EXAMPLES

1. In adding $\frac{1}{a-1}+\frac{1}{a-3}+\frac{-1}{a+1}+\frac{1}{a+3}$ it is especially easy to add the first and third fractions; then the second and fourth; and, finally, the results thus obtained.

Thus,

$$\frac{1}{a-1}-\frac{1}{a+1}=\frac{a+1-a+1}{a^2-1}=\frac{2}{a^2-1}.$$

$$\frac{1}{a-3}+\frac{1}{a+3}=\frac{a+3+a-3}{a^2-9}=\frac{2a}{a^2-9}.$$

$$\frac{2}{a^2-1}+\frac{2a}{a^2-9}=\frac{2a^2-18+2a^3-2a}{(a^2-1)(a^2-9)}=\frac{2(a^3+a^2-a-9)}{(a^2-1)(a^2-9)}.$$

2. In adding $\frac{3}{a-b}+\frac{3}{a+b}+\frac{6a}{a^2+b^2}+\frac{12a^3}{a^4+b^4}$ the first and second are easily combined, then that result is easily added to the third, and finally that result to the fourth.

(1) $$\frac{3}{a-b}+\frac{3}{a+b}=\frac{6a}{a^2-b^2}.$$

(2) $$\frac{6a}{a^2-b^2}+\frac{6a}{a^2+b^2}=\frac{12a^3}{a^4-b^4}.$$

(3) $$\frac{12a^3}{a^4-b^4}+\frac{12a^3}{a^4+b^4}=\frac{24a^7}{a^8-b^8}.$$

WRITTEN EXERCISES

1. Add the fractions in the examples above in the ordinary way; that is, find the l. c. d. of the several fractions and add at once.

Add by successive combinations, as above:

2. $\frac{1}{a-x}+\frac{1}{a+x}-\frac{2a}{a^2+x^2}$.

3. $\frac{1}{a-1}+\frac{3}{a-2}-\frac{3}{a+2}-\frac{1}{a+1}$.

4. $\frac{2}{a-b}+\frac{2}{a+b}+\frac{4a}{a^2+b^2}$.

5. $\frac{1}{a-b}+\frac{a-b}{c}-\frac{1}{a+b}-\frac{a+b}{c}$.

6. $\frac{1}{p+1}-\frac{2}{p-2}+\frac{2}{p+2}-\frac{1}{p+1}$.

7. Add several of the preceding problems in the ordinary way. Which method is shorter?

438. To multiply a fraction by an integer we extend the definition of multiplication to this case, and we have:

$$3(\tfrac{4}{5})=\tfrac{4}{5}+\tfrac{4}{5}+\tfrac{4}{5}=\tfrac{12}{5}.$$

$$c\left(\frac{a}{b}\right)=\frac{a}{b}+\frac{a}{b}+\cdots\text{ to }(c\text{ terms})=\frac{a+a+a\cdots\text{ to }(c\,a\text{'s})}{b}=\frac{ca}{b}.$$

Similarly, $(-c)\left(\frac{a}{b}\right)=-\frac{a}{b}-\frac{a}{b}\cdots\text{ to }(c\text{ terms})=-\frac{ca}{b}.$

439. That is, *to multiply a fraction by an integer we multiply the numerator by that integer.*

The fraction $\frac{a}{b}$ is the quotient called for in the indicated division $a \div b$, because the product of the divisor b and the asserted quotient $\frac{a}{b}$ is the dividend a, as seen in

$$b\cdot\frac{a}{b}=\frac{ba}{b}=a\quad\text{(by property } B\text{, Sec. 434).}$$

In our present number system the fraction $\frac{a}{b}$ may therefore always be regarded as indicating the division of a by b (provided b is not zero).

440. The operation of division is thus seen to be possible in the new number system for any integral dividend and any integral divisor (except zero).

We shall see, under Division of Fractions, that if either dividend or divisor or both are fractions, the operation is still possible without enlarging the number system further.

441. Multiplication of Fractions. *The product of two or more fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

442. Since an integer or an integral expression may be regarded as a fraction with denominator 1, this definition applies also when one or more of the factors are integral.

443. The Associative, Commutative, and Distributive Laws of Multiplication apply to fractions as well as to integers.

WRITTEN EXERCISES

Multiply:

1. $\frac{5}{7a} \cdot \frac{14}{15b}$.

2. $\frac{x^3}{y^2} \cdot \frac{3xy}{4z^5}$.

3. $\frac{t^2-4}{(1+q)^2} \cdot \frac{q^2-1}{(2+t)^2}$.

4. $\frac{5x}{3y} \cdot \frac{9t^2}{16x^3y} \cdot \frac{4yz}{15t^3x}$.

5. $bc \cdot \frac{b}{c^2} \cdot \frac{ac^4}{2b^3}$.

6. $\frac{-3a}{x^2-9} \cdot \frac{4b}{9x}$.

7. $\left(\frac{3}{a} - \frac{2}{b}\right)^2$.

8. $x^3\left(1 - \frac{1}{x}\right)^2$

9. $pqr\left(\frac{p}{q} - \frac{q}{r}\right)^2$.

10. $\frac{a(1-a)}{1+2a+a^2} \cdot \frac{1+a}{1-2a+a^2}$.

11. $\frac{1-a^2}{1+2b+b^2} \cdot \frac{1-b^2}{b^2-2ab+a^2}\left(\frac{a}{1-a} - \frac{b}{1-b}\right)$.

12. $\left(\frac{t+1}{t-1} - \frac{t-1}{t+1} - \frac{4t^2}{1-t^2}\right)\frac{t^2+2t+1}{4t}$.

444. Reciprocals. If the product of two numbers is 1, each is called the **reciprocal** of the other.

Thus, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, since $\frac{a}{b}\cdot\frac{b}{a}=\frac{ab}{ab}=1$.

445. Division of Fractions. Since *divisor* × *quotient* = *dividend*, to divide $\frac{a}{b}$ by $\frac{c}{d}$ means to find a number q such that $\frac{c}{d}\cdot q=\frac{a}{b}$.

Solving the above equation, we have,

$$q=\frac{a}{b}\cdot\frac{d}{c}.$$

That is: *To obtain the quotient, multiply the dividend by the reciprocal of the divisor.*

We have thus seen that in the number system as now extended the four fundamental operations are possible when any or all of the numbers involved are fractions. (Division by zero is always excepted.)

446. Complex Fractions. If one or both of the terms of a fraction are themselves fractions, the given fraction is called a **complex fraction.** Since a fraction indicates division, a complex fraction may be simplified by performing the indicated division of its numerator by its denominator.

WRITTEN EXERCISES

Divide, and express results in lowest terms:

1. $\frac{x}{2}\div\frac{y}{6}.$
2. $\frac{ab}{18c}\div\frac{4b^2}{3ac}.$
3. $\frac{ax^2}{2y^3}\div\frac{bx^3}{ay^2}.$
4. $\frac{9a^2}{4b}\div 2c^2.$
5. $\frac{x^2-1}{7h}\div\frac{x+1}{2m}.$
6. $\frac{-3x}{4y+1}\div\frac{1}{2x-1}.$
7. $\frac{gt^2}{2}\div\frac{ag^3}{8}.$
8. $pv^3\div\frac{6p^4}{v^5}.$
9. $\frac{-6}{x^3}\div\frac{-18}{x^7}.$
10. $\dfrac{\frac{5x}{3y^2}}{\frac{2x}{9m^3}}.$
11. $\dfrac{\frac{1}{x}}{\frac{3}{x}}.$
12. $\dfrac{\frac{5a}{3a^2}}{2b}.$
13. $\dfrac{\frac{4x}{7y^2}}{9x^3}.$
14. $\dfrac{\frac{1}{t^2}}{t^2}.$
15. $\dfrac{t^2}{\frac{1}{t^2}}.$

16. $\left(\frac{a^2+b^2}{b}-a\right)\div\left(\frac{1}{b}-\frac{1}{a}\right)$.

17. $\frac{a}{b}+\frac{c}{d}\div\left(1-\frac{ac}{bd}\right)$.

18. $\left(1-\frac{1}{a^3}\right)\div\left(\frac{1}{a^2}-\frac{1}{a^3}\right)$.

Perform the operations indicated:

19. $x-\frac{1}{1+\frac{1}{x}}$. 20. $t+\frac{1}{\frac{1}{t}+1}$. 21. $\left(\frac{p-q}{p+q}+1\right)\div\left(1-\frac{2q}{p+q}\right)$.

FACTORING

447. Factoring. The process of finding two or more factors whose product is a given algebraic expression is called **factoring** the given expression.

In division one factor (the divisor) is given and another factor (the quotient) is to be found such that the product of the two factors is the given expression (the dividend). Division is thus really a variety of factoring, although the name "factoring" is not usually applied to it. As a rule, when an expression is to be factored, none of the factors is specified in advance, and any set of factors is acceptable on the sole condition that their product is the given expression.

448. The type products of special importance were applied in Chapter XII and are collected here for reference.

I. $xy+xz=x(y+z)$.

II. $x^2\pm 2xy+y^2=(x\pm y)^2$.

III. $x^2-y^2=(x+y)(x-y)$.

IV. $x^2+(a+b)x+ab=(x+a)(x+b)$.

V. $acx^2+(ad+bc)x+bd=(ax+b)(cx+d)$.

VI. $x^3\pm 3x^2y+3xy^2\pm y^3=(x\pm y)^3$.

VII. $x^3+y^3=(x+y)(x^2-xy+y^2)$.

VIII. $x^3-y^3=(x-y)(x^2+xy+y^2)$.

These formulas apply when the letters are either positive or negative numbers, and a detailed treatment of them was given in Chapter XI. The following miscellaneous exercises cover all the types.

ORAL EXERCISES

Factor:

1. $ax + ay$.
2. $ax + a$.
3. $ax + a^2$.
4. $abx - ay$.
5. $x^2 + ax$.
6. $x^3 + ax^2$.
7. $a^2x^2 - ax$.
8. $mx + my + m$.
9. $px^2 + p^2x + pq$.
10. $x^3 - x^2 + x$.
11. $x^2y^2 + xy + y$.
12. $(a + b)x - (a + b)y$.
13. $s^2t + st^2 + s^2t^2$.
14. $\frac{1}{2}gt^2 + gt$.
15. $abc - acd + bcd$.
16. $a^2x^2 - b^2y^2$.
17. $4x^2 - 9z^2$.
18. $a^2b^2x^2 - c^2$.
19. $1 - p^2q^2y^2$.
20. $m^2p^2 - s^2t^2$.
21. $(a + b)^2 - (c + d)^2$.
22. $1 - (m + p)^2$.
23. $a^2 + 2abc + b^3c^2$.
24. $a^4x^4 + 2a^2x^2 + 1$.
25. $a^2 + 2a(b + c) + (b + c)^2$.
26. $1 - 2(x + y) + (x + y)^2$.
27. $49 + 14x + x^2$.
28. $9 - 12y^2 + 4y^4$.
29. $16 - 40z + 25z^2$.
30. $a^2b^2 + 2abcd + c^2d^2$.
31. $\frac{1}{4} + t + t^2$.
32. $y^4 + .4y^2 + .04$.
33. $m^2x^2 + 4mx + 4$.
34. $\frac{1}{16}z^4 - \frac{1}{2}z^2 + 1$.
35. $x^2 - 5x + 6$.
36. $x^2 + 7x + 12$.
37. $x^2 - x - 12$.
38. $x^2 + x - 6$.
39. $t^2 - 12x + 35$.
40. $s^2 - 2x - 35$.
41. $12x^2 + 7x + 1$.
42. $15y^2 - 2y - 1$.
43. $6x^2 - 12x + 6$.
44. $3y^2 + 8y + 5$.
45. $a^3 - b^3$.
46. $a^3 + b^3$.
47. $8 - a^3b^3$.
48. $64 - x^3y^3$.
49. $8x^3 - 12x^2 + 6x - 1$.
50. $27y^3 + 27y^2 + 9y + 1$.

WRITTEN EXERCISES

Factor:

1. $x^2 - 49$.
2. $t^2 - 7t + 6$.
3. $5y^3 - 3y^2 + y$.
4. $4v^2 - 9u^2$.

5. $1 + 10a + 25a^2$.

6. $a^3 + 3a^2b + 3ab^2 + b^3$.

7. $(x+1)^2 + 2(x+1) + 1$.

8. $1 - \frac{2}{a} + \frac{1}{a^2}$.

9. $x^2 - 2x - 15$.

10. $a^2b^2 - 1$.

11. $x^4 + 6x^2y + 9y^2$.

12. $v^8 - t^4$.

13. $1 - x^3y^3$.

14. $a^4 - 4a^2b^2$.

15. $(x+y)^4 - 1$.

16. $a^3b^3 - c^3d^3$.

17. $x^4 - 2x^2y^2 + y^4$.

18. $(m+p)^3 - 1$.

19. $(x+y)^2 - (p+q)^2$.

20. $(a+b)^3 + x^3$.

21. $x^3 + \frac{3}{2}x^2 + \frac{3}{4}x + \frac{1}{8}$.

22. $a^3 - 1\frac{1}{2}a^2 + \frac{3}{4}a - \frac{1}{8}$.

23. $x^2 + (a+b)x + ab$.

24. $y^2 + (ac - bd)y - abcd$.

25. $acx^2 + (cb + ad)x + bd$.

26. $.125x^3 + .75x^2 + 1.5x + 1$.

27. $\frac{x^3}{8} - \frac{3x^2}{4} + \frac{3x}{2} - 1$.

28. $8a^3 + 12a^2t + 6at^2 + t^3$.

29. $g^4 - 16$.

30. $a^2y^2 - 11ay + 24$.

31. $4h^2 - 12ht + 9t^2$.

32. $1 - m^3$.

33. $(2a+b)^2 - (3a - 2b)^2$.

34. $8a^3 - 32a^5$.

35. $10m^2x - 60mx + 90x$.

36. $m^4p^4 + 4s^4$.

37. $x^4 + 3x^2 - 28$.

38. $s^3 - 64$.

39. $56x^2 - 68x + 20$.

40. $24x^2 + x - 10$.

41. $15x^2 - 34x - 16$.

42. $a^2z^2 - 7az + 12$.

43. $y^4 - 13y^2 + 42$.

44. $(a+1)^2 - (a-1)^2$.

45. $x^2 - \frac{1}{2}x - \frac{1}{2}$.

Calculate:

46. $27^2 - 25^2$.

47. $387^2 - 377^2$.

48. $26^3 - 25^3$.

General Methods

449. The General Trinomial. The factors of the general trinomial $mx^2 + px + q$ have the form $(ax+b)(cx+d)$, and they may be found by reducing the general form to the type $x^2 + bx + c$. To do this, multiply and divide the expression by m (the coefficient of x^2) and put $mx = y$. The method will be made sufficiently clear by an example.

EXAMPLE

Factor: $6x^2 + 19x + 10.$

Multiplying and dividing the given trinomial by 6, we obtain

$$6x^2 + 19x + 10 = \frac{1}{6}(36x^2 + 6 \cdot 19x + 60).$$

Put $6x = y.$ (1)

Then, $36x^2 + 6 \cdot 19x + 60 = y^2 + 19y + 60$ (2)

$= (y + 4)(y + 15)$ Sec. 163. (3)

Replacing y by $6x$, $= (6x + 4)(6x + 15)$ (4)

$= 2(3x + 2)3(2x + 5)$

$= 6(3x + 2)(2x + 5).$ (5)

And finally, $6x^2 + 19x + 10 = \frac{1}{6} \cdot 6(3x + 2)(2x + 5)$

$= (3x + 2)(2x + 5).$ (6)

Test by multiplication.

WRITTEN EXERCISES

Factor:

1. $3x^2 + 7x + 2.$
2. $3x^2 - 5x - 2.$
3. $4x^2 + 13x - 12.$
4. $6x^2 - 22x + 20.$
5. $3a^2 + a - 10.$
6. $4m^2 + 9m - 9.$
7. $4p^2 + 8p + 3.$
8. $9x^2 - 3x - 2.$
9. $5x^2 - 9x - 2.$
10. $6a^2 - 8a - 8.$
11. $25y^2 - 50y - 24.$
12. $10x^2 + 9xy - 9y^2.$
13. $15p^2 - 18p - 24.$
14. $12s^2 - 10s - 12.$
15. $6y^2 + 35y - 6.$
16. $28z^2 - 44z + 16.$
17. $7p^2 + 22p + 3.$
18. $15z^2 + z - 6.$
19. $3a^2 + 14ab - 5b^2.$
20. $15x^2 + 2xy - 8y^2.$
21. $14m^2 - mp - 3p^2.$
22. $10s^2 + 9st - 9t^2.$
23. $12x^4 - 47x^2 - 17.$
24. $40x^{2a} - 2x^a - 2.$
25. $3a^2 - 8a - 115.$
26. $6a^2b^2 - ab^3 - 12b^4.$
27. $25x^2 + 20x + 4.$
28. $4z^2 + 4az - 15a^2.$
29. $3a^2 + ab - 2b^2.$
30. $12y^2 + 19y - 21.$
31. $6m^2 + mn - 2n^2.$
32. $96a^2 - 4ab - 15b^2.$

450. Expressions that can be made the Difference of Two Squares.

1. Expressions of the form $a^{4n} + 4\,b^{4n}$ may be factored by adding and subtracting $4\,a^{2n}b^{2n}$. Thus,

$$a^{4n} + 4\,a^{2n}b^{2n} + 4\,b^{4n} - 4\,a^{2n}b^{2n} = (a^{2n} + 2\,b^{2n})^2 - (2\,a^n b^n)^2$$
$$= (a^{2n} + 2\,b^{2n} - 2\,a^n b^n)(a^{2n} + b^{2n} + 2\,a^n b^n).$$

EXAMPLE

$$x^8 + 4\,y^8 = x^8 + 4\,x^4y^4 + 4\,y^8 - 4\,x^4y^4$$
$$= (x^4 + 2\,y^4 - 2\,x^2y^2)(x^4 + 2\,y^4 + 2\,x^2y^2).$$

2. Expressions of the form $a^{4n} \pm pa^{2n}b^{2n} + b^{4n}$ may be factored if a number can be found such that when added to p the result is 2.

EXAMPLE

Factor $a^4 - 6\,a^2b^2 + b^4$.

Adding and subtracting $4\,a^2b^2$,

$$a^4 - 6\,a^2b^2 + b^4 = a^4 - 2\,a^2b^2 + b^4 - 4\,a^2b^2.$$
$$= (a^2 - b^2)^2 - 4\,a^2b^2.$$
$$= (a^2 - b^2 - 2\,ab)(a^2 - b^2 + 2\,ab).$$

3. Some other expressions may be made the difference of two squares.

EXAMPLE

Factor $16\,m^4 + 36\,m^2p^2 + 25\,p^4$.

Since twice the product of the square roots of the end terms is $40\,m^2p^2$, the expression can be made the difference of two squares by adding and subtracting $4\,m^2p^2$. Thus,

$$16\,m^4 + 36\,m^2p^2 + 25\,p^4 = 16\,m^4 + 40\,m^2p^2 + 25\,p^4 - 4\,m^2p^2$$
$$= (4\,m^2 + 5\,p^2)^2 - (2\,mp)^2$$
$$= (4\,m^2 + 5\,p^2 - 2\,mp)(4\,m^2 + 5\,p^2 + 2\,mp).$$

WRITTEN EXERCISES

Factor:

1. $p^4 + 4\,q^4$.
2. $m^4 + 4$.
3. $4\,x^4 + 1$.
4. $4\,y^8 + 1$.
5. $y^{12} + 4$.
6. $4\,m^8 + y^8$.
7. $a^4 + a^2 + 1$.
8. $a^4 + 4\,a^2 + 16$.
9. $a^4 + a^2b^2 + b^4$.

10. $64p^4+1$. 11. m^8+64. 12. $4p^{8x}+q^{4x}$.
13. $9x^4-10x^2+1$. 15. $4c^4+3c^2d^2+9d^4$.
14. $16y^4+4y^2+1$. 16. $36x^4-84x^2y^2+16y^4$.

451. The General Binomial. The factors of the general binomial $x^n \pm y^n$ depend upon the nature of n. The following table shows for what values of n the division is exact in $(x^n \pm y^n) \div (x \pm y)$:

a. $(x^n+y^n)\div(x+y)$, if n is odd.
b. $(x^n+y^n)\div(x-y)$, not for any value of n.
c. $(x^n-y^n)\div(x+y)$, if n is even.
d. $(x^n-y^n)\div(x-y)$, for every integral value of n.

ORAL EXERCISES

Name a factor of each of the following:

1. x^5-y^5. 3. $1-x^6$. 5. $x^{10}-1$. 7. $a^{14}-b^{14}$.
2. x^7+a^7. 4. x^9+a^9. 6. $x^{13}-y^{13}$. 8. $32x^5-1$.

452. The factor of the first degree in $x^n \pm y^n$ having been found by inspection, the other is a regular series of powers that can be written directly (Sec. 425); it has alternate signs in the cases (*a*) and (*c*), and plus signs in case (*d*).

For example:

$$x^7+a^7=(x+a)(x^6-ax^5+a^2x^4-a^3x^3+a^4x^2-a^5x+a^6).$$

$$\begin{aligned}32a^5-1&=(2a)^5-1\\&=(2a-1)[(2a)^4+(2a)^3\cdot 1+(2a)^2\cdot 1^2+2a\cdot 1^3+1^4]\\&=(2a-1)(16a^4+8a^3+4a^2+2a+1).\end{aligned}$$

$x^{10}-r^{10}$ has both $x-r$ and $x+r$ as factors, but it is better first to factor as a difference of two squares; then apply (*d*) and (*a*).

WRITTEN EXERCISES

Factor:

1. $32a^5-b^5$. 5. a^7-b^7. 9. $(x-3y)^5+1$.
2. $128a^7+1$. 6. $1-m^{10}$. 10. x^8-y^8 (four factors).
3. $32-243t^5$. 7. x^6-a^6. 11. $(a+b)^5-(a-b)^5$.
4. $(c-d)^7+1$. 8. $1-(a+b)^4$. 12. $(x-y)^4-(x+y)^4$.

453. Factor Theorem. *If any polynomial in* x *assumes the value zero when* a *is substituted for* x, *then* x — a *is a factor of the polynomial.*

For example, substitute 2 for x in the polynomial:

$$x^4 - 3x^3 + 7x^2 - 2x - 16.$$

Then, $2^4 - 3 \cdot 2^3 + 7 \cdot 2^2 - 2 \cdot 2 - 16 = 16 - 24 + 28 - 4 - 16 = 0.$

Suppose the polynomial to be divided by $x - 2$, and denote the quotient by Q and the remainder by R, the latter being numerical.

Then, $x^4 - 3x^3 + 7x^2 - 2x - 16 = (x - 2)Q + R.$

In this equation substitute 2 for x; the left number becomes zero as just seen; $2 - 2$ is also 0, and 0 times Q is 0; hence the result is:

$$0 = 0 + R, \text{ or } R = 0.$$

Consequently, $x^4 - 3x^3 + 7x^2 - 2x - 16 = (x - 2)Q$, or $x - 2$ is a factor of $x^4 - 3x^3 + 7x^2 - 2x - 16$. Thus we may know *without actual division* that $x - 2$ is a factor of the polynomial.

In general: Let $P(x)$ denote the given polynomial.

Suppose $P(x)$ to be divided by $x - a$. There will be a certain quotient, call it $Q(x)$, and a remainder, R. This remainder will not involve x, otherwise the division could be continued.

We have then: $P(x) = (x - a)Q(x) + R.$ (1)

Put a for x, and denote the resulting value of $P(x)$ by $P(a)$, and of $Q(x)$ by $Q(a)$:

Then, $P(a) = (a - a)(Q)(a) + R.$ (2)
But, $(a - a)(Q)a = 0,$
hence, if $P(a) = 0,$
then, $R = 0,$
and, $P(x) = (x - a)Q(x).$

That is, $x - a$ is a factor of $P(x)$, if the expression is 0 when $x = a$.

WRITTEN EXERCISES

By use of the Factor Theorem test each expression and find if the binomial at the right is a factor of it:

Expression	Binomial
1. $x^2 - 4x + 4.$	$x - 2.$
2. $x^2 - 5x + 6.$	$x - 3.$

3. $x^2 - x - 6.$	$x - 3.$
4. $3x^2 - x - 2.$	$x - 1.$
5. $3y^2 - 20y + 25.$	$y - 5.$
6. $x^3 - 3x^2 + 3x - 1.$	$x - 1.$
7. $p^3 - 4p^2 - 4p + 16.$	$p - 4.$
8. $m^3 - 2m^2 + m - 2.$	$m - 5.$

454. *If the polynomial becomes* 0 *when* $-$ a *is put for* x, *then* $x - (-a)$, or $x + a$ *is a factor.*

Example: $x + 2$ is a factor of $x^3 + x^2 - 2x$ because

$$(-2)^3 + (-2)^2 - 2(-2) = -8 + 4 + 4 = 0.$$

WRITTEN EXERCISES

By use of the Factor Theorem, prove that each polynomial has the factor named:

Polynomial	Factor
1. $x^3 + 12x^2 + 31x - 20.$	$x + 5.$
2. $(x - a)^2 + (x - b)^2 - (a - b)^2.$	$x - b.$
3. $x^3 + 2x^2 + 3x + 2.$	$x + 1.$
4. $x^3 - 2x^2 + \frac{3x}{m} - \left(m^3 - 2m^2 + \frac{3m}{x}\right).$	$x - m.$
5. $x^{3n} + 2x^{2n} + 3x^n + 2.$	$x^n + 1.$
6. $x^3 + ax^2 - a^2x^2 - a^5.$	$x - a^2.$
7. $9x^5 + (3a^2 - 12a)x^4 - 4a^3x^3 + 3a^2x + a^4.$	$x + \frac{a^2}{3}.$

In each polynomial, substitute the values given for x, and use the Factor Theorem to find the possible factors:

Polynomial	Values for x
8. $x^4 + 3x^3 + 4x^2 - 12x - 32.$	$1, 2, -2.$
9. $x^4 - 7x^3 + 14x^2 + x - 21.$	$1, -1, 2, 3.$
10. $2x^4 + 5x^3 - 41x^2 - 64x + 80.$	$4, -4, 5, -5.$

11. Apply the Factor Theorem to $(a^n \pm b^n) \div (a \pm b)$ and prove the results of Secs. 451 and 452.

REVIEW

WRITTEN EXERCISES

Perform the indicated operations, expressing fractional results in lowest terms:

1. $2a - 3b + 4a + 11c - 2d + 6 - 8a - 9b + 3c - 4 - 5d + 2a - b$.

2. $x^2y + 3xy^2 + 4x^2 - 2xy - 3x^2y + 2x^2y^2 - 4xy + 8xy^2 - 7y^2 - 3x^2y^2$.

3. $5x - 7y - (3x + 4y - 2) + 3 + (8x - 7) - (2x - 8y + 13) + 8(2x - 1)$.

4. $4x - \{2c - (3x + 2y + 5c)\}$.

5. $5ac - \{4b + 2c[3a - 5b - (6a - 2b - \overline{7c + 3})]\}$.

If $L = 4x + 2y - 3$, $M = x - 9y + 1$, $R = y - 5x$, find:

6. $L + M + R$.
7. $3L - 2M + R$.
8. $LM - 3R$.
9. $L^2 - M^2$.
10. $(L + M)^2$.
11. $3L^2 - 5MR$.

If $X = 7a + 2b$, $Y = 2a - 9b + 3$, $Z = 4b - 7a - 3$, find:

12. $X - (3Y - 2Z)$.
13. $2Y - [Z + 3(4X - 8Y)]$.
14. $X^2 - 16Y^2$.
15. $Y^2 + 4YZ$.
16. XYZ.

17. $\dfrac{a+5}{a-5} - \dfrac{a^2+75}{a^2-25} + \dfrac{a-5}{a+5}$.

18. $\dfrac{6m+4p}{15m} - \dfrac{3m^2+4p^2}{15mp} + \dfrac{m-2p}{5p}$.

19. $1 - \dfrac{9+a}{1+a^2} - \dfrac{a-3}{a+1}$.

20. $\left(\dfrac{1}{x} - \dfrac{1}{y}\right)(x + y)$.

21. $\dfrac{2}{t^2+9t+20} + \dfrac{1}{t^2+7t+12} - \dfrac{3}{t^2+11t+28}$.

22. $c^4\left(\dfrac{x^2}{c^2} - \dfrac{3x}{c^2} + \dfrac{14}{c}\right)$.

23. $\dfrac{1}{3a^2}\left(\dfrac{6a^3+9a^2}{4a+2}\right)$.

24. $(b + 3)^3 - (b - 2)^3$.

25. $(x^n + 3x^{n-1} + 5x^{n-2})(x^2 + 4x)$.

26. $\left(\dfrac{10x^3}{3m^2} - \dfrac{5x^3}{9m^2} - \dfrac{5x^2}{m^4}\right) \div \dfrac{15x}{m^2}$.

27. $\left(\frac{2x^2}{y^2} - \frac{5y^2}{x^2} + 3\right) \div \left(\frac{x}{y} - \frac{y}{x}\right)$.

28. $\frac{35 + 12g + g^2}{15 + 8g + g^2} \div \frac{56 + 15g + g^2}{24 + 11g + g^2}$.

29. $\dfrac{\frac{1}{a} - \frac{3}{a^2}}{\frac{2}{a^3} - \frac{3}{a}}$.

30. $\dfrac{\frac{q+1}{q-1} + \frac{q-1}{q+1}}{\frac{q+1}{q-1} - \frac{q-1}{q+1}}$.

31. $\left(\frac{3a}{b} - \frac{11b}{3a}\right)\left(\frac{3a}{b} - \frac{b}{3a}\right) - \left(\frac{3a}{b} - \frac{2b}{3a}\right)^2$.

32. $\frac{2}{x^2 + 4x + 3} + \frac{1}{x^2 + 3x + 2} - \frac{3}{x^2 + 5x + 6}$.

33. $\frac{x^2y}{a^3 + 1} \div \frac{2x^3}{a + 1}$.

34. $\frac{4a + 7}{a^2 - 16x + 64} \div \frac{16a^2 + 56a + 49}{a^2 - 64}$.

Divide by detached coefficients:

35. $16 - 32x + 24x^2 - 8x^3 + x^4 \div (2 - x)$.

36. $16m^4 - 32m^3 + 24m^2 - 8m + 1 \div (2m - 1)$.

Factor by the Factor Theorem:

37. $x^3 + x^2 - x - 1$.

38. $4x^4 + x^3 - 5$.

39. $a^3 - ab^2 + a^2 - ab$. (Substitute b for a.)

SUMMARY

1. State the laws that govern the processes of addition for algebraic numbers. Secs. 389, 395.

2. State the laws that govern the processes of multiplication for algebraic numbers. Secs. 412, 418.

3. State and illustrate two extensions of the number system of algebra. Secs. 402, 431.

4. Name eight type expressions used in factoring. Sec. 448.

5. State the Factor Theorem. Sec. 453.

HISTORICAL NOTE

We have seen that algebra embraces the enlargement of the number field of arithmetic to include negative number and the extension of the basic processes to those numbers. Thus, the equation $x + 3 = 1$, which Ahmes could not solve and which was regarded as negligible until the sixteenth century, is no more exceptional than $x + 1 = 3$, since algebra defines $1 - 3$ to be -2. In a similar way the field of number was enlarged to include positive and negative fractions, for the equation $3x = -2$ could not be solved until $\frac{-2}{3}$ was understood. In arithmetic the processes $b - a$ and $\frac{b}{a}$ are limited to positive numbers, but in algebra each of these has a meaning for all values of b and a, positive or negative. This process of generalizing was first explained by the English algebraist, George Peacock (1830), and called by him the *principle of permanence.* That is, in order that any number or symbol may be made a part of algebra, it must conform to the basic laws governing the processes, namely, The Associative Law, The Commutative Law, and The Distributive Law.

Sir William Rowan Hamilton

The famous scientist, Sir William Rowan Hamilton (1840), regarded these laws as distinguishing algebraic number from other numbers. In doing so, he discovered numbers which do not obey the Commutative Law of Multiplication, and to these numbers he gave the name Quaternions. Their study has since become a new branch of mathematics.

Hamilton was of Scotch parentage, but Ireland shares his fame, because he was born and educated at Dublin. Like Tartaglia, he received instruction at home when a boy, and showed exceptional ability at an early age. When only thirteen he could read a dozen languages, at eighteen he had mastered Newton's *Principia,* and shortly became professor of Astronomy in Trinity College, Dublin. Hamilton did much for mechanics and astronomy, but his greatest achievement in mathematics was the discovery of Quaternions.

CHAPTER XXV

EQUATIONS

EQUATIONS WITH ONE UNKNOWN

455. Two algebraic expressions are **equal** when they represent the same number.

456. If two numbers are equal, the numbers are equal which result from:

1. *Adding the same number to each.*
2. *Multiplying each by the same number.*

Subtraction and division are here included as varieties of addition and multiplication.

457. The equality of two expressions is indicated by the symbol, =, called "the sign of equality."

458. Two equal expressions connected by the sign of equality form an **equation.**

459. Such values of the letters as make two expressions equal are said to **satisfy** the equation between these expressions.

460. Equations that are satisfied by any set of values whatsoever for the letters involved are called **identities.**

461. Equations that are satisfied by particular values only are called **conditional** equations, or, when there is no danger of confusion, simply equations.

462. The numbers that satisfy an equation are called the **roots** of the equation.

463. **To solve** an equation is to find its roots.

464. The letters whose values are regarded as unknown are called the **unknowns.**

465. The **degree** of an equation is stated with respect to its unknowns. It is the highest degree to which the unknowns occur in any term in the equation. Unless otherwise stated, all the unknowns are considered.

466. An equation of the first degree is called a **linear** equation.

467. An equation of the second degree is called a **quadratic** equation.

468. An equation of the third or higher degree is called a **higher** equation.

469. In order to state the degree of an equation its terms must be united as much as possible.

470. Terms not involving the unknowns are called **absolute** terms.

471. Equivalent Equations. If two equations have the same roots, the equations are said to be equivalent. If two equations have together the same roots as a third equation, the two equations together are said to be equivalent to the third.

472. The Linear Form, $ax + b$. Every polynomial of the first degree can be put into the form $ax + b$. That is, by rearranging the terms suitably, it can be written as the product of x by a number not involving x, plus an absolute term. Hence, the form $ax + b$ is called a general form for all polynomials of the first degree in x.

473. Every equation of the first degree in one unknown can be put into the form:

$$ax + b = 0.$$

Consequently this is called *a general equation of the first degree in one unknown.*

474. General Solution. From the equation $ax + b = 0$, (1)

we have $$ax = -b, \quad (2)$$

and hence, $$x = \frac{-b}{a}. \quad (3)$$

475. $-\frac{b}{a}$ is the general form of the root of the equation of the first degree. There is always one root, and only one.

The advantage of a general solution like this is that it leads to a formula which is applicable to all equations of the given form.

In words: *When the equation has been put into the form* $ax + b = 0$ *the root is the negative of the absolute term divided by the coefficient of* x.

Test. The correctness of a root is tested by substituting it in the *original equation.*

If substituted in any later equation, the work leading to that equation is not covered by the test.

Results for problems expressed in words should be tested by substitution in the *conditions of the problem.*

If tested by substitution in the equation only, the correctness of the solution is tested, but the setting up of the equation is not tested. Negative results that may occur in such problems are always correct as solutions of the equations, but they are admissible as results in the concrete problem only when the unknown quantity is such that a unit of the unknown quantity is offset by a unit of its opposite.

For example, if the unknown measures distance forward, a negative result means that a corresponding distance backward satisfies the conditions of the problem. But, if the unknown is a number of men, a negative result is inadmissible, since no opposite interpretation is possible.

ORAL EXERCISES

Solve for x:

1. $3x = 15$.
2. $2x = 11$.
3. $4\frac{1}{2}x = 9$.
4. $x - 6 = 10$.
5. $x + 6 = 12$.
6. $2x + 1 = 13$.

Solve for t:

7. $6t = 36$.
8. $t - 5 = 20$.
9. $2t + 5 = 25$.
10. $3t - 8 = 22$.
11. $at = ab$.
12. $at + b = c$.

Solve for y:

13. $3y - 1 = 2$.
14. $2y - 1 = 7$.
15. $5y + 5 = 35$.
16. $by = bc$.
17. $by = b + c$.
18. $ay - b = c$.

Solve for a:

19. $6a = 18$.
20. $2\frac{1}{2}a = 10$.
21. $a + 1 = 13$.
22. $3a + 1 = 13$.
23. $2a - 5 = 15$.
24. $5a - b = c$.

WRITTEN EXERCISES

Solve for x:

1. $x - 75 = 136$.
2. $2x - 15 = 45$.
3. $3x + 6 = 48$.
4. $5x - 10 = 55$.
5. $1.5x - 5 = 70$.
6. $3.5x - 5 = 100$.
7. $2.1x - 41 = 400$.
8. $1.3x + .1 = 1.79$.
9. $.25x + .50 = 3.25$.
10. $.11x + .11 = 1.32$.
11. $325x + 60 = 400$.
12. $125x + 5 = 10$.
13. $8x + 625 = 105$.
14. $ax + bx = c$.
15. $abx + ax = ab$.
16. $cx + dx = c + d$.
17. $mx + px = p + q$.
18. $lx + mx = l - m$.
19. $ax + bx = 2(a + b)$.
20. $2cx + dx = 1$.

Solve and test:

21. $2(x-1) + 3(2x+5) = 0$.

22. $\frac{4(x+2)}{5} + 3 = 2x + 1$.

23. $\frac{x-1}{2} + \frac{x-1}{3} + \frac{x-1}{4} = x$.

24. $\frac{3y+1}{4y-2} = \frac{1-3y}{4-4y}$.

25. $\frac{1+3p}{p} = \frac{17-5p}{p}$.

Solve for c:

26. $5c + 3a = ac.$

27. $\dfrac{4c-1}{3m} = \dfrac{2c+5}{6m}.$

28. $v = ct + \dfrac{gt^2}{2}.$

29. $(a+c)(a-c) = -(c+a)^2.$

30. $\dfrac{3c}{c-3} - \dfrac{c-2}{c-4} = \dfrac{(c-2)(2c-7)}{c^2-7c+12}.$

31. $\dfrac{L}{1+ct} = l.$

32. An inheritance of $2000 is to be divided between two heirs, A and B, so that B receives $100 less than twice what A receives. How much does each receive?

33. Twenty steps, each of a given height, are required to build a certain staircase. If each step is made 2 inches higher, 16 steps are required. Find the vertical height of the staircase.

34. In a certain hotel the large dining room seats three times as many persons as the small dining room. When the large dining room is $\frac{2}{3}$ full and the small dining room $\frac{1}{2}$ full, there are 100 persons in both together. How many does each room seat?

35. Tickets of admission to a certain lecture are sold at two prices, one 25 cents more than the other. When 100 tickets at the lower price and 60 at the higher price are sold, the total receipts are $95. Find the two prices.

36. Originally, $\frac{10}{11}$ of the area of Alabama was forest land. $\frac{1}{3}$ of this land has been cleared, and now 20 million acres are forest land. Find the area of Alabama in million acres.

37. In a recent year the railroads of the United States owned 70,000 cattle cars. Some of these were single-decked, and others double-decked. There were 44,000 more of the former than of the latter. Find how many cars there were of each kind.

38. The average number of sheep carried per deck is 45 larger than the average number of calves. If a double-decked car has the average number of calves on the lower deck and of sheep on the upper deck, it contains 195 animals. Find the number of sheep and of calves.

39. The average number of inhabitants per square mile for Indiana is $\frac{7}{4}$ of that for Iowa, and that for Ohio is 32 greater than that for Indiana, and 62 greater than that for Iowa. Find the number for each state.

40. Lead weighs $\frac{63}{15}$ times as much as an equal volume of aluminium. A certain statuette of aluminium stands on a base of lead. The volume of the base is twice that of the statuette, and the whole weighs 282 oz. Find the weight of the statuette and of the base.

41. A man inherits $10,000. He invests some of it in bonds bearing $3\frac{1}{2}$ % interest, the rest in mortgages bearing $5\frac{1}{2}$ % interest per annum. His entire annual income from these investments is $ 510. Find the amount of each investment.

42. A pile of boards consists of inch boards and half-inch boards. There are 80 boards and the pile is 58 in. high. How many boards of each thickness are there?

43. A hardware dealer sold a furnace for $180 at a gain of 20 %. What did the furnace cost him?

44. A merchant sold a damaged carpet for $ 42.50 at a loss of 15 %. What did the carpet cost him?

45. A collector remitted $475 after deducting from the amount collected a fee of 5 %. How many dollars did he collect?

46. The amount of a certain principal at 4 % simple interest for 1 yr. was $ 416. What was the principal?

47. Pythagoras being asked the time of day, replied: "There remains of the day (from 6 A.M. to 6 P.M.) twice the number of hours already passed." What time was it?

48. The three Graces, carrying 4 apples each, met the 9 Muses; they gave each Muse the same number of apples; then the Muses and Graces had equal shares. How many had each?

49. A fruit vender gave a boy 4 dozen oranges to sell and agreed to pay him $\frac{3}{4}$¢ for each orange sold, but demanded a payment of 2 ¢ for each orange eaten; the boy disposed of all the oranges and received 25 ¢. How many oranges did he eat?

50. A robber in escaping from a castle met a guard whom he bribed with $\frac{1}{5}$ of his plunder; at the next gate he bribed another guard with $\frac{1}{4}$ of the plunder remaining; at the third gate he bribed another guard with $\frac{1}{3}$ of the plunder remaining; the robber then escaped with 2000 ducats. How many ducats did he steal?

51. A servant agreed to work for £ 10 a year and his livery. At the end of 7 months his lord discharged him, giving him the livery only. How many pounds was the livery worth?

SPECIAL PROBLEMS

476. The solution of many problems is made easier by a special plan of work.

EXAMPLE

How much water must be added to a 20 % solution of ammonia to make a 10 % solution?

Plan. 1. Consider an arbitrary quantity of the given mixture; for example, 1 gallon.

2. Let x be the number of gallons added; then the two quantities are:

1st Quantity	2d Quantity
1 gal.	$(1 + x)$ gal.

3. Decide which substance (the ammonia in this problem) has not changed in quantity; state the amount in each solution. Thus,

Ammonia in 1st Solution	Ammonia in 2d Solution
20 % of 1 gal.	10 % of $(1 + x)$ gal.

4. ∴ the equation is 20 % of 1 = 10 % of $(1 + x)$, or $.2 = .1\,(1 + x)$.

5. Therefore, $x = 1$, and 1 gallon of water must be added for each gallon of the original solution.

WRITTEN EXERCISES

Solve the following, using the tabular plan given above:

1. How much water must be added to a 95 % solution of alcohol to make an 80 % solution?

2. A certain paint consists of equal parts of oil and pigment. How much oil must be added to a gallon of this paint to make a paint $\frac{2}{3}$ of which is oil?

3. How much potash must be added to a 10 % solution to make a 20 % solution?

4. Spirits of camphor is camphor gum dissolved in alcohol. How many ounces of camphor gum must be added per ounce to a 5 % solution in order to make an 8 % solution?

477. Problems involving the rates of two moving bodies have received much attention in mathematics.

For example, a courier, or messenger, leaves the rear of an army, 5 miles long, to deliver a dispatch to the officer at the front, and rides at the rate of 10 miles per hour. 10 minutes later he is followed by another messenger riding at the rate of 15 miles per hour. If the army is not moving, where will the second messenger overtake the first?

The following are further examples commonly known as "clock" problems and "planet" problems.

EXAMPLE

At what time between 3 and 4 o'clock are the hands of a clock together?

SOLUTION. 1. The minute hand moves 1 minute space per minute.

2. The hour hand moves $\frac{1}{12}$ of a minute space per minute.

3. Let $x =$ the number of minutes *after* 3 *o'clock* when the hands are together. Then, x is the number of spaces moved by the minute hand and $\frac{x}{12}$ is the number of spaces moved by the hour hand.

4. When they are together the hour hand is $15 + \frac{x}{12}$ minute spaces from XII, and the minute hand is x spaces.

5. Hence, $x = 15 + \frac{x}{12}$, and $x = 16\frac{4}{11}$, the number of minutes past 3 o'clock.

WRITTEN EXERCISES

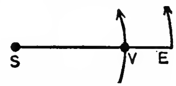
1. How many minute spaces must the minute hand gain on the hour hand from the time they meet until they lie opposite to each other in the same straight line? At what time are the hands of a clock opposite to each other for the first time after 12 o'clock?

2. At what time between 7 and 8 o'clock are the hands of a clock opposite each other?

3. At how many different times, and when, are the hands of a clock at right angles between 4 and 5 o'clock?

4. A and B enter a race together; at the end of 5 minutes A is 900 yd. from the starting line and 75 yd. ahead of B; at this point he falls, and though he renews the race, his rate is 20 yd. a minute less for the rest of the course; he crosses the line $\frac{1}{2}$ minute after B. How long did the race last?

5. In astronomy it is important to know when planets are in line between the earth and the sun. This is called *conjunction.* Taking the earth's time of revolution about the sun as 365 days and that of Venus as 225 days, how long after one conjunction of Venus until the next one occurs?



SUGGESTION. The problem is quite analogous to that of the hands of a watch. For the purposes of this problem we suppose all the planets to move in the same plane and in circular paths (orbits) in the same direction of revolution about the sun as a center.

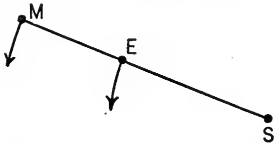
1. Let $x =$ the number of days.
2. Venus will have made $\frac{x}{225}$ revolutions.
3. The earth will have made $\frac{x}{365}$ revolutions.
4. But to be in conjunction, Venus (which goes faster) must have made one more revolution than the earth.

$$\text{Hence, } \frac{x}{225} = \frac{x}{365} + 1,$$

$$\text{and } \quad x = 586\tfrac{17}{28}.$$

6. Taking 88 days as Mercury's time of revolution about the sun, how long from one conjunction of Mercury to the next?

7. When the earth is between a planet and the sun, in the same line with them, the planet is said to be in opposition to the sun. Taking 687 days as Mars' time of revolution about the sun, how long is it from one opposition of Mars to the next?



8. Taking 4307 days as Jupiter's time of revolution about the sun, how long is it from one opposition of Jupiter to the next?

9. Answer the same question for Saturn, whose time of revolution is 28.5 yr.

10. Also for Neptune, whose time of revolution is 165.5 yr.

NOTE. Those who have studied geometry may take up here some of the problems based upon geometric properties found in Chapter XXXIII.

EQUATIONS WITH TWO UNKNOWNS

478. Systems of Equations. Two or more equations considered together are called a **system of equations.**

479. Simultaneous Equations. Two or more equations are said to be **simultaneous** when all of them are satisfied by the same values of the unknowns.

480. All systems of two independent simultaneous equations of the first degree in two unknowns can be solved by the **method of addition and subtraction,** which consists in multiplying one or both of the given equations by such numbers that the coefficients of one of the unknowns become numerically equal. Then by addition or subtraction this unknown is eliminated, and the solution is reduced to that of a single equation.

481. Occasionally the **method of substitution** is useful. This consists in expressing one unknown in terms of the other by means of one equation and substituting this value in the other equation, thus eliminating one of the unknowns.

This may be the shorter method when an unknown in either equation has the coefficient 0, $+1$, or -1.

General Solution. A general form for two equations of the first degree is

$$ax + by = e, \tag{1}$$
$$cx + dy = f. \tag{2}$$

From these it is possible (without knowing the values of a, b, c, d, e, f) to find a *general form* for the solution, namely:

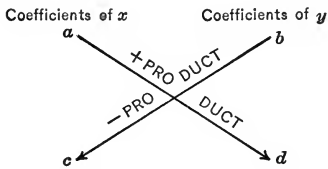
$$x = \frac{de - bf}{ad - bc}, \qquad y = \frac{af - ce}{ad - bc}. \tag{3}$$

These are the *formulas* for the roots of any two independent linear simultaneous equations with two unknowns.

482. Relation of the Roots to the Constants in the Equations.

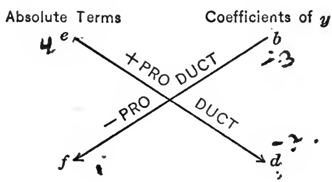
1. The denominator is the same in each result and is made up from the coefficients as follows:

$$\begin{cases} ax + by = e. \\ cx + dy = f. \end{cases}$$



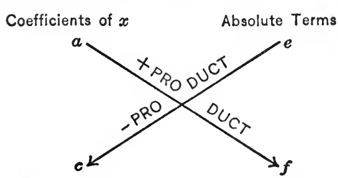
2. The numerator of the value of x is made up thus:

$$\begin{cases} ax + by = e. \\ cx + dy = f. \end{cases}$$



3. The numerator of the value of y is made up thus:

$$\begin{cases} ax + by = e. \\ cx + dy = f. \end{cases}$$



The following examples will illustrate the use of these formulas in solving equations:

1. Solve: $\begin{cases} 2x - 3y = 4, & (1) \\ 4x + 2y = 1. & (2) \end{cases}$

$$x = \frac{\begin{array}{cc} 4 & -3 \\ 1 & +2 \end{array}}{\begin{array}{cc} 2 & -3 \\ 4 & 2 \end{array}} = \frac{4 \cdot 2 - 1(-3)}{2 \cdot 2 - (-3) \cdot 4} = \frac{8+3}{4+12} = \frac{11}{16}. \qquad (3)$$

$$y = \frac{\begin{array}{cc} 2 & 4 \\ 4 & 1 \end{array}}{\begin{array}{cc} 2 & -3 \\ 4 & 2 \end{array}} = \frac{2 \cdot 1 - 4 \cdot 4}{2 \cdot 2 - 4(-3)} = \frac{2-16}{16} = \frac{-14}{16} = \frac{-7}{8}. \qquad (4)$$

TEST. $\frac{2 \cdot 11}{16} - \frac{3 \cdot -7}{8} = \frac{64}{16} = 4$; $\frac{4 \cdot 11}{16} + \frac{2 \cdot -7}{8} = \frac{16}{16} = 1.$

483. The above form of expressing cross products is derived from the **Determinant Notation**, and while it is not necessary to know **Determinants** in order to solve such simultaneous equations by inspection, it is well to know the basis of the method.

The symbol $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ is defined to mean $ad - bc$, and is called a Determinant of the *second order.*

EXAMPLES

$$\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 2 \cdot 5 - 3 \cdot 4 = -2, \text{ and } \begin{vmatrix} 2 & -1 \\ -3 & 5 \end{vmatrix} = 2 \cdot 5 - (-3 \cdot -1) = 13.$$

ORAL EXERCISES

Find the value of:

1. $\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}$.

2. $\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix}$.

3. $\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix}$.

4. $\begin{vmatrix} \frac{1}{2} & 3 \\ 6 & 4 \end{vmatrix}$.

5. $\begin{vmatrix} \frac{1}{2} & \frac{1}{4} \\ 8 & 6 \end{vmatrix}$.

6. $\begin{vmatrix} -\frac{1}{2} & 9 \\ \frac{1}{3} & 4 \end{vmatrix}$.

7. $\begin{vmatrix} \frac{1}{2} & -\frac{1}{8} \\ 2 & \frac{1}{3} \end{vmatrix}$.

8. $\begin{vmatrix} 2 & a \\ 3 & b \end{vmatrix}$.

9. $\begin{vmatrix} a & b \\ x & y \end{vmatrix}$.

484. The values of x and y in Section 481 may be expressed in Determinant Notation.

Thus, $$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

EXAMPLE

Solve: $$\begin{cases} 4x + 7y = -27, & (1) \\ x - 2y = 12. & (2) \end{cases}$$

$$x = \frac{\begin{vmatrix} -27 & 7 \\ 12 & -2 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 1 & -2 \end{vmatrix}} = \frac{-27\cdot(-2) - 7\cdot 12}{4\cdot(-2) - 7} = \frac{-30}{-15} = 2. \qquad (3)$$

$$y = \frac{\begin{vmatrix} 4 & -27 \\ 1 & 12 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 1 & -2 \end{vmatrix}} = \frac{4\cdot 12 - 1(-27)}{-15} = \frac{75}{-15} = -5. \qquad (4)$$

Test. $4\cdot 2 + 7(-5) = -27$; $2 - 2(-5) = 12$.

485. This method usually gives the values by inspection, for the products and differences can be read direct from the equations themselves.

For example:

$4x + 2y = 1,$ $3x - 5y = 4.$	$4x + 2y = 1,$ $3x - 5y = 4.$	$4x + 2y = 1,$ $3x - 5y = 4.$
Denominators $= 4(-5) - 2\cdot 3$ $= -26.$	Numerator of y $= 4\cdot 4 - 1\cdot 3 = 13.$ $\therefore y = \frac{13}{-26} = -\frac{1}{2}.$	Numerator of x $= 1(-5) - 2\cdot 4 = -13.$ $\therefore x = \frac{-13}{-26} = \frac{1}{2}.$

WRITTEN EXERCISES

Solve, using determinant forms:

1. $x + y = 5,$ $x - y = 3.$
2. $x + y = 5,$ $x - y = 1.$
3. $2x + y = 3,$ $x + y = 2.$
4. $2x + 3y = 7,$ $x - y = 1.$
5. $2x + 2y = 8,$ $2x - y = 2.$
6. $3x - y = -5,$ $2x - y = -3.$

7. $4x-3y=7,$
$3x-4y=7.$

8. $5x+y=9,$
$3x+y=5.$

9. $4x+5y=22,$
$3x+2y=13.$

10. $x-5y=-22,$
$5x-y=10.$

11. $4x-3y=3,$
$3x-4y=-3.$

12. $4x+2y=1,$
$3x-2y=\frac{5}{2}.$

13. $12x-11y=87,$
$4x+2y=46.$

14. $7x-2y=3,$
$7x-4y=-1.$

15. $9x-3y=-6,$
$8x-2y=-6.$

16. $ax+y=1,$
$bx+y=2.$

17. $ax+by=c,$
$px+qy=d.$

18. $x-my=a,$
$x+py=b.$

19 $ax-by=e,$
$cx-dy=f.$

20. $ax-y=b,$
$cx+y=d.$

21. $2x+7y=11,$
$5x-9y=1.$

22. $3x+7y=-1,$
$2x-3y=7.$

23. $4x-7y=5,$
$8x+15y=39.$

24. $2x-y=6,$
$10x-3y=7.$

25. $7x+9y=20,$
$8x-4y=-20.$

26. $2x-6y=6,$
$4x+2y=16.$

27. $2x+y=0,$
$x+2y=-3.$

28. $11x+22y=33,$
$4x+18y=22.$

29. $1.4x+2.1y=1,$
$2.8x+3.3y=2.$

30. $30x+25y=40,$
$13x+16y=22\frac{1}{2}.$

31. $9x-12y=-51,$
$21x-35y=-133.$

32. $3x+5y=37,$
$10x-5y=15.$

33. $5x=3y,$
$2x+8y=4.$

34. $ax-by=c,$
$cx+ay=b.$

35. $4a^2x+5ay=3,$
$6ax+7y=2.$

36. $ax+by=c,$
$a^2x+b^2y=c^2.$

37. $x+by=1,$
$\frac{x}{a}+y=3.$

38. $\frac{2}{3}x+\frac{5}{6}y=\frac{1}{2},$
$\frac{3}{8}x+\frac{7}{16}y=\frac{1}{3}.$

NOTE. See Chapter XXXIII for problems relating to geometry.

SPECIAL SYSTEMS

486. The study of graphs of systems of equations helps to interpret special cases.

487. We have solved (Sec. 481) the equations

$$ax + by = e,$$
$$cx + dy = f,$$

and found that $$x = \frac{de - bf}{ad - bc};\ y = \frac{af - ec}{ad - bc}.$$

I. Let us give to the letters a, b, c, d, such values that $ad - bc = 0$; for example, $a = 2$, $b = 1$, $c = 4$, $d = 2$. And let us give to e and f such values that $de - bf$ is not 0; for example, $e = 5$, $f = 4$.

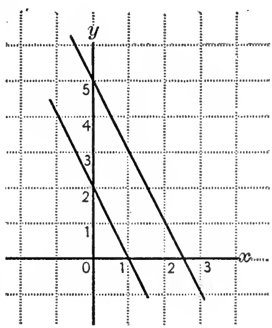
Then the above results become:

$$x = \frac{6}{0};\ y = -\frac{12}{0}.$$

The indicated division by zero means that the solution is impossible, Sec. 430. There is no pair of values that satisfies both equations. This appears readily also by substituting the values 2, 1, 4, 2, 5, 4, for a, b, c, d, e, f, in the given equations, which then become:

$$\left.\begin{matrix} 2x + y = 5, \\ 4x + 2y = 4, \end{matrix}\right\}\ \text{or}\ \left\{\begin{matrix} 2x + y = 5, \\ 2x + y = 2. \end{matrix}\right.$$

It is obvious that no set of values of x and y can make $2x + y$ equal to 5 and also equal to 2.



This condition can be illustrated graphically:

For drawing the graphs of

$$2x + y = 5 \text{ and } 2x + y = 2,$$

the two lines are parallel. That two parallel straight lines do not intersect is the geometric condition corresponding to the fact that a system of two incompatible equations has no solution.

The two equations are called **incompatible** or **contradictory.**

II. Retaining the values of a, b, c, d, above, let us give e and f such values that the numerators of the result both become zero; for example, $e=5, f=10$.

The result assumes the form:

$$x=\frac{0}{0};\ y=\frac{0}{0}.$$

Under these conditions we have seen, Section 430, that x and y may have any values. But this may also be seen by reference to the equations.

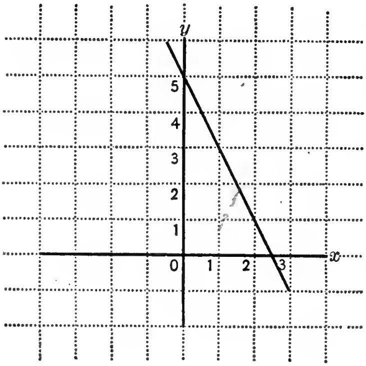
Substituting the values 2, 1, 4, 2, 5, 10, for a, b, c, d, e, f, in the given equations, they become:

$$2x+y=5.$$
$$4x+2y=10.$$

It appears that the second equation is twice the first, and hence equivalent to it. Any values of x and y that satisfy the first, will also satisfy the second.

We can choose arbitrarily any value for x and then determine a value of y to go with it by means of the first equation. For example, choosing $x=3$, then $2\cdot 3+y=5$, which gives $y=-1$. These values of x and y satisfy both equations. Similarly, any value can be chosen for y, and a value of x found such that the pair of values satisfies the given system.

The two equations are **dependent**. Every solution of one is a solution of the other.



If we undertake to make the graphs of the two equations as given, we find that they lead to the same straight line. The two graphs are coincident; every point of the straight line is a common point of the two graphs. Any abscissa x is the abscissa of a common point of the graphs; any ordinate y is the ordinate of a common point of the graphs.

NOTE. The study of expressions which may assume the exceptional forms mentioned above, especially those which may assume the form $\frac{0}{0}$, is very important, both from the point of view of later mathematics and the physical sciences.

488. Number of Solutions. We have thus seen that systems of two linear equations in two unknowns may be classified as follows:

1. Independent (the ordinary case, admitting one solution).
2. Contradictory (admitting no solution).
3. Dependent (admitting a boundless number of solutions).

WRITTEN EXERCISES

Construct the graphs of each of the following systems and classify them according to Section 488:

1. $3x + y = 2,$
   $x + y = 0.$
2. $2x - y = 1,$
   $4x - 2y = 2.$
3. $s - t = 6,$
   $s + t = 6.$
4. $x + 2z = 10,$
   $x + 3z = 11.$
5. $x = 25,$
   $y = 10.$
6. $10x + 5y = 25,$
   $2x + y = 5$
7. $7x + 14y = 7,$
   $x + 2y = 2.$
8. $12x - 3y = 8,$
   $3y - x = 4.$
9. $2x - 3y = -5,$
   $x + 2y = 8.$

EQUATIONS WITH THREE OR MORE UNKNOWNS

489. The definitions and methods for the solution of two equations with two unknowns may be applied equally well to a greater number of equations and unknowns.

To solve three linear equations with three unknowns, eliminate one unknown from any pair of the equations and the same unknown from any other pair; two equations are thus formed which involve only two unknowns and which may be solved by methods previously given.

Four or more equations with four or more unknowns may be solved similarly.

490. Determinants of the third and higher orders have been defined, and can be used to solve linear equations with three or more unknowns, but the method is too complicated to be of practical value here.

WRITTEN EXERCISES

Solve:

1. $4x - 2y + z = 3,$
$x + 3y + 2z = 13,$
$-8x + 12y + z = 21.$

2. $5x + 4y + 2z = 17,$
$3x - 2y + 5z = 2,$
$2x - y + 3z = 2.$

3. $x - y - z = a,$
$3y - x - z = 2a,$
$7z - y - x = 4a.$

4. $3x - 2z + 5 = 0,$
$2x + 3y - 21 = 0,$
$4y + 7z - 69 = 0.$

5. $x + y = \frac{1}{a},$
$y + z = \frac{1}{b},$
$x + z = \frac{1}{c}.$

6. $$\frac{x}{a+b} + \frac{y}{b-c} + \frac{z}{a+c} = 2c,$$
$$\frac{x}{a-b} - \frac{y}{b-c} - \frac{z}{a-c} = 2a,$$
$$\frac{x}{a-b} + \frac{y}{c-b} - \frac{z}{a+c} = 2a - 2c.$$

7. $$x - y - z - 2w = -12,$$
$$3x - y - 2z + 8w = 40,$$
$$4x - 4y + 7z - 5w = 52,$$
$$3x - y + 2z + w = 44.$$

8. A and B can do a piece of work in 6 days; A and C can do it in 9 days, and A, B, and C can do 8 times the whole work in 45 days. In how many days can each do it alone?

9. A sum of money is divided into 3 parts such that the first part exceeds the second part by $100. The annual income from the second and third parts, if invested at 6% per annum, is $42. The sum of the first and second parts equals the sum of the second and third parts. Find the number of dollars in each part.

10. A number consists of 3 digits. If 99 is added to it the sum is a number having the same digits, but in reverse order. The sum of the hundreds' and tens' digits equals the units' digit, and the units' digit exceeds the tens' digit by 1.

11. The sum of the digits in a certain number of three figures is 13, the hundreds' digit exceeds the tens' digit by 1, and the units' digit exceeds the hundreds' digit by 2. Find the number.

12. Three casks together contain 79 gallons; the second contains 3 gallons more than $\frac{1}{2}$ as much as the first, and the third contains 7 gallons less than the second. How many gallons are there in each? (From a fourteenth century manuscript.)

NOTE. Those who have studied geometry may take up here some of the problems based upon geometric properties found in Chapter XXXIII.

QUADRATIC EQUATIONS

491. Quadratic Equations. Equations of the second degree are called **quadratic equations.**

A general form for quadratic equations in one unknown is

$$ax^2 + bx + c = 0,$$

in which a, b, c represent any known numbers, except that a may not be zero.

492. Solution of Quadratic Equations.

(1) The incomplete quadratic equation $x^2 = a$ is solved by extracting the square root of both members. The roots are: $x = \pm\sqrt{a}$.

(2) The incomplete quadratic equation $ax^2 + bx = 0$ is solved by factoring. The roots are $x = 0$ and $x = -\frac{b}{a}$.

(3) Complete quadratic equations are solved by completing the square.

The process consists of two main parts:

(a) *Making the left member a square while the right member does not contain the unknown.*

This is called completing the square.

It is based upon the relation $(x + a)^2 = x^2 + 2\,ax + a^2$, in which it appears that the last term, a^2, is the square of one half of the coefficient of x.

(b) *Extracting the square roots of both members and solving the resulting linear equations.*

Square roots which cannot be found exactly should be indicated.

EXAMPLE

Solve: $x^2 - 8x + 9 = 0.$ (*1*)

Transposing,	$x^2 - 8x = -9.$	(*2*)
Completing the square,	$x^2 - 8x + 16 = -9 + 16.$	(*3*)
Rearranging,	$(x-4)^2 = 7.$	(*4*)
Extracting the square root,	$x - 4 = \pm\sqrt{7}.$	(*5*)
Solving (*5*) for x,	$x = 4 \pm \sqrt{7}.$	(*6*)

(4) If any quadratic equation has zero for the right member, and if the polynomial constituting the left member can be factored, the quadratic is equivalent to two linear equations whose roots can readily be found. (See Chapter XIII.)

WRITTEN EXERCISES

Solve:

1. $3x^2 = 18.$
2. $x^2 - 5x = 0.$
3. $x^2 - 5x + 6 = 0.$
4. $x^2 + 4x - 3 = 0.$
5. $t^2 - 2t - 6 = 0.$
6. $8p^2 = 5p.$
7. $x^2 + 11x + 24 = 0.$
8. $5x^2 - 13x + 5 = 0.$
9. $15y^2 + 134y + 288 = 0.$
10. $\frac{x}{4} = \frac{9}{x-8} + \frac{2x}{5} + 1.35.$
11. $\frac{1}{y-3} + \frac{1}{1-y} - \frac{1}{y-2} = 0.$
12. $\frac{6-2w}{w-2} + \frac{5+w}{3+w} = \frac{w-5}{2-w}.$
13. $7(7-z)(z-6) + 3(5-z)(2-z) - 40 = 0.$
14. Find two numbers whose sum is 10, and the sum of whose squares is 68.

Suggestion. Let x represent one number, and $10 - x$ the other.

15. A room is 3 yd. longer than it is wide; at \$1.75 per square yard, carpet for the room costs \$49. Find the dimensions of the room.

16. A man bought for \$300 a certain number of oriental rugs, each at the same price. If he had bought rugs each costing \$40 more, he would have obtained 2 fewer rugs. How many rugs did he buy?

17. A dealer bought a number of similar tables for $153. He sold all but 7 of them at an advance of $1 each on their cost, thus receiving $100. How many tables did he buy?

18. A man invested $6000 at a certain rate of simple interest during 4 years. At the end of that time he reinvested the capital and the interest received during the 4 years at a rate of interest 1 % lower than at first. His annual income from the second investment was $372. What was the original rate of interest?

19. A rectangle whose area is 84 sq. in. is 5 in. longer than it is wide. Find its dimensions.

20. A certain number of men hire an automobile for $156. Before they start, two others join them, sharing equally in the expense. The amount to be paid by each of the original renters is thus reduced by $13. How many men were there at first?

21. A man rows down a stream a distance of 21 mi. and then rows back. The stream flows at 3 mi. per hour and the man makes the round trip in $13\frac{1}{8}$ hours. What is his rate of rowing in still water?

22. The product of a number and the same number increased by 40 is 11,700; what is the number?

23. If each side of a certain square is increased by 5 the area becomes 64; what is the length of a side?

24. Find two numbers whose sum is 16 and the difference of whose squares is 32.

25. A number multiplied by 5 less than itself is 750. Find the number.

26. The product of two consecutive even numbers is 728. Find the numbers.

INTERPRETATION OF RESULTS

493. Interpretation of Results. After the conditions of a problem have been expressed by equations, and the equations solved, the result must be examined to see whether it is admissible under the conditions of the problem.

EXAMPLES

1. Find three consecutive integers such that their sum shall be equal to 3 times the second.

SOLUTION.
1. Let $x =$ the first.
2. Then, $x + 1 =$ the second,
3. and $x + 2 =$ the third.
4. $\therefore x + (x + 1) + (x + 2) = 3(x + 1)$.
5. $\therefore (3 - 3)(x + 1) = 0$, or $0(x + 1) = 0$.

INTERPRETATION OF THE RESULT. The equation determines no particular value of x; it exists for every value of x. Consequently, every three consecutive integers must satisfy the given conditions.

2. Find three consecutive integers whose sum is 57, and the sum of the first and third is 40.

SOLUTION.
1. Let $x =$ the first.
2. Then $x + 1 =$ the second,
3. and $x + 2 =$ the third.
4. Then, $x + (x + 1) + (x + 2) = 57$,
5. and $x + (x + 2) = 40$, by the given conditions.
6. From (4), $x = 18$.

INTERPRETATION OF THE RESULT. $x = 18$ will not satisfy equation (5); therefore no three consecutive integers satisfy the problem.

3. The town B is d mi. from A; two trains leave A and B simultaneously, going in the same direction (that from A towards B), A at the rate of m mi. per hour and B q mi. per hour. How far from B will the trains be together?

Solving this problem by the usual method, we find as the result $\frac{qd}{m - q}$.

INTERPRETATION OF THE RESULT. If d is not equal to 0, and if $m = q$, the result assumes the form $\frac{qd}{0}$. This means that the problem is impossible under these conditions. This is evident also from the meaning of m and q in the problem. If the two trains go in the same direction at the same rate, the one will always remain d miles behind the other.

If, however, $d = 0$, and $m = q$, the result assumes the form $\frac{0}{0}$, which equals any number whatever. This also agrees with the conditions of the problem. If d is zero, B and A are coincident, and the two trains are together at starting. If $m = q$, they both run at the same rate, and always remain together. They are therefore together at every distance from B.

WRITTEN EXERCISES

Solve and interpret the results:

1. Fifteen clerks receive together \$150 per week; suppose that some receive \$8 and others \$12 per week. How many would there be receiving each salary?

2. A train starts from New York for Richmond via Philadelphia and Baltimore at the rate of 30 miles an hour, and two hours later another train starts from Philadelphia for Richmond at the rate of 20 miles an hour. How far beyond Baltimore will the first train overtake the second, given that the distance from New York to Philadelphia is 90 miles and from Philadelphia to Baltimore 96 miles?

3. If the freight on a certain class of goods is 2 cents per ton per mile, together with a fixed charge of 5 cents per ton for loading, how far can 2000 tons be sent for \$80?

4. Find three consecutive integers whose sum equals the product of the first and the last.

5. The hot-water faucet of a bath tub will fill it in 14 minutes, the cold-water faucet in 10 minutes, and the waste pipe will empty it in 4 minutes. How long will it take to fill the tub when both faucets and the waste pipe are opened?

REVIEW

WRITTEN EXERCISES

Solve:

1. $\frac{x+3}{x-1}=\frac{x-1}{x+3}$.
2. $x^2-14x+33=0$.
3. $a+\frac{b}{x}=c$.
4. $2x+3(4x-1)=5(2x+7)$.
5. $14x-5(2x+4)=3(x+1)$.
6. $\frac{3x}{2}-\frac{4x}{3}=50$.
7. $\frac{x+150}{x+50}=\frac{6}{5}$.
8. $4x+19=5x-1$.
9. $4-\frac{3}{2x}=6$.
10. $3x-40=100-\frac{x}{2}$.
11. $\frac{x}{2}-2-\frac{x}{14}+\frac{2}{7}=24$.

12. $\frac{2x}{3}+4=\frac{4x}{5}-2.$

13. $2x-\frac{30-x}{2}=35.$

14. $\frac{5x}{16}+875=4000.$

15. $7x+4-15x=6x+2.$

16. $5x+4=9x+1.$

17. $\frac{x}{x+60}=\frac{7}{3x-5}.$

18. $\frac{2(2a-bx)}{3b^2}=\frac{a}{b}.$

19. $\frac{180}{2x+6}=\frac{x}{2}.$

20. $100x-x^2=-2400.$

21. $x(x+4)=45.$

22. $\frac{120}{x}-\frac{120}{x+3}=2.$

23. $x(7-x)=12.$

24. $x^2-8x=9.$

25. $\frac{x+7}{x}+\frac{x}{x+7}=\frac{5}{2}.$

26. $y^2+3y=70.$

27. $z^2-3z=70.$

28. $11m^2-10m=469.$

29. $\frac{90}{r}-\frac{27}{r+2}=\frac{90}{r+1}.$

30. $\frac{12}{5-x}+\frac{8}{4-x}=\frac{32}{x+2}.$

31. $2y-3y+120$
$=4y-6y+132.$

32. $7x^2-10x=120+4x^2-19x.$

33. $(5x-2)(6x+1)-(10x+3)(3x+10)=0.$

34. $(2x-3)(x+1)-(3x-7)(x-4)=36.$

35. $\frac{1}{x+2}+\frac{2}{x-2}=\frac{7}{2x-3}.$

36. $2x+5y=6,$
$4x+11y=3.$

37. $5x-3y=1,$
$13x-8y=9.$

38. $\frac{x}{y}=7,$
$2x-10y=3y+2.$

39. $\frac{7y-6}{3x+4}=\frac{1}{2},$
$\frac{4y-5}{2x+1}=\frac{1}{3}.$

40. $\frac{3x+2y+5}{2}+\frac{x-3y-13}{5}+\frac{y-3x+3}{6}=3,$
$\frac{2x-4y+6}{y-5x+11}=-2.$

41. $6z=43-5y,$
$3z=37-4x,$
$4y=55-5x.$

42. $3x+4y+2z=-4,$
$2x-5y-z=9,$
$-4x+2y+3z=-23.$

43. $\frac{y-z}{x+y}=\frac{1}{5},\ \frac{x-z}{y+z}=\frac{2}{3},\ \frac{x-y}{x+z}=\frac{1}{4}.$

44. $(z-2)(x+3)=(x-1)(z-1),$
$(z+8)(y-2)-(y+2)(z+2)=0,$
$y(3-2x)+(2y-3)(1+x)=0.$

45. $\frac{3}{2x+10}-\frac{3x^2+14}{7(4-x)}+\frac{1+3x}{5+x}-\frac{3x}{7}=0.$

46. $2(2x+3y)-\frac{5(x+3)}{8}-\frac{3y}{4}=9,\ x+y=1.$

47. A rectangle whose length is greater than its breadth by 1 yd. has an area of 6 sq. yd. Find its dimensions.

48. A square $(2x-3)$ ft. on a side has taken from it a square x ft. on a side. The remaining area is 24 sq. ft.; find the side of each square.

49. The product of two consecutive numbers is 380; find the numbers.

50. The product of two consecutive even numbers is 840. Represent the smaller by $2n$ and find both numbers.

51. In a certain election 36,785 votes were cast for the three candidates A, B, C. B received 812 votes more than twice as many as A; and C had a majority of one vote over A and B together. How many votes did each receive?

52. In a certain election there were two candidates, A and B. A received 10 votes more than half of all the votes cast. B received 4 votes more than one third of the number received by A. How many votes did each receive?

53. A group of friends went to dine at a certain restaurant. The head waiter found that if he were to place five persons at each table available, four would have no seats, but by placing six at each table, only three persons remained for the last table. How many guests were there, and how many tables?

54. A flower bed of uniform width is to be laid out around a rectangular house 20 ft. wide and 36 ft. long. What must be the width of the bed in order that its area may be one third of that of the ground on which the house stands?

55. Wood's metal, which melts in boiling water, is made up of one half (by weight) of bismuth, a certain amount of lead, half that much zinc, and half as much cadmium as zinc. How much of each is there in 100 lb. of Wood's metal?

56. If in the preceding exercise $\frac{3}{4}$ as much cadmium as zinc is used, a different metal is formed. How many pounds of each constituent metal in 100 lb. of this metal?

57. A cask contains 10 gal. of alcohol. A certain number of quarts are drawn out; the cask is then filled up with water and the contents thoroughly mixed. Later, twice as many quarts are drawn out as the previous time and the cask filled up with water. There now remain only 4.8 gal. of alcohol in the mixture. How many gallons were drawn out at first?

SUGGESTION. Let $x =$ the number of gallons first withdrawn. Then $10 - x =$ the number of gallons left.

When the cask is filled again with water any part of the mixture is $\frac{10-x}{10}$ alcohol. Then, $\frac{10-x}{10}$ of the $2x$ gallons of mixture withdrawn the second time is alcohol. Hence, $10 - x - \left(\frac{10-x}{10}\right)2x$ is the number of gallons of alcohol left in the cask.

58. How much water must be added to 30 oz. of a 6 % solution of borax to make a 4 % solution?

59. How much acid must be added to 10 quarts of a 2 % solution to make a 5 % solution?

60. Alcides was asked, "How many are there of your numerous herd?" He replied: "If I had 6 less than twice as many more, the number would be 306. Find the number."

61. A courier went from Paris to Grenoble, 120 leagues, in 4 days, each day's journey being 2 leagues shorter than that of the preceding day. How many leagues did he travel each day? (Ozanam's *Algebra*, 1702.)

62. Two messengers, A and B, set out towards each other from two places 59 mi. apart, B starting 1 hr. after A. A goes 7 mi. in 2 hr., and B 8 mi. in 3 hr. How far will A have gone when he meets B? (Newton's *Arithmetica Universalis*, 1707.)

63. A merchant bought a certain number of platters for \$ 366. Three were broken during shipment. He sold $\frac{1}{6}$ of the remainder for \$ 75 at a profit of 25 %. Find the number of platters bought and the price per platter.

64. A certain hall contains both gas jets and electric lights. When 60 gas jets and 80 electric lights are used, the cost for an evening is \$ 4. If 90 gas jets and 60 electric lights are used, the cost is \$ 4.05. Find the cost per gas jet and electric light.

65. A tailor paid \$ 12 for 4 yd. of cloth and 8 yd. of lining. At another time he paid \$ 21 for 6 yd. of the cloth and 16 yd. of the lining. Find the price of each per yard.

66. Two wheelmen are 328 ft. apart and ride toward each other. If A starts 3 seconds before B, they meet in 14 seconds after A starts; or if B starts 2 seconds before A, they meet in 14 seconds after B starts. Find the rate of each.

67. A man had a portion of his capital invested in stocks paying 6 % dividends, the remainder in mortgages paying 5 %. His annual income was \$ 700. The next year the dividend on the stock was reduced to 5 %, but by reinvestment he replaced his old mortgages by new ones paying $5\frac{1}{2}$ %. His income for this year was \$ 690. How much had he invested in stocks? Also in mortgages?

68. A company at a tavern, when they came to pay, found that if the same bill were divided among three persons more, the amount would be one shilling less per person; and, if it were divided among two persons fewer, it would be one shilling more per person. Find the number of persons in the original company, and the amount of the bill. (Saunderson's *Algebra*, 1740.)

69. One person says to another, "If you give me three of your coins, I shall have as many as you." The second person replies, "If you give me three of yours, I shall have twice as many as you have." Find the numbers that each has. (Ozanam's *Algebra*, 1702.)

70. A coach set out from Cambridge to London with four more passengers outside than within. Seven outside passengers could travel at 2 shillings less expense than 4 inside passengers. The fares of all the passengers amounted to 180 shillings. At the end of half the journey the coach took up 1 more inside and 3 more outside passengers; these paid $\frac{2}{15}$ as much as the others. Required the number of passengers and the fare of each. (Bland's *Algebraical Problems*, 1816.)

71. Seven years ago a man was 4 times as old as his son; 7 years hence he will be only double his age. Find the age of each. (Simpson's *Algebra*, 1767.)

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. State the general form of a linear equation with one unknown, and the formula for its root. Secs. 472–475.

2. When is a system of equations simultaneous? Sec. 479.

3. State the general form of a system of two linear equations with two unknowns. Sec. 481.

4. State two methods of solving simultaneous equations. Secs. 480, 481.

5. What is the general form of the values of the unknowns in simultaneous equations with two unknowns? Sec. 481.

6. What use may be made of determinants in solving simultaneous equations? Secs. 483, 484.

7. What is the general form of quadratic equations with one unknown? Sec. 491.

8. Name two kinds of quadratic equations, and the method of solving each. Sec. 492.

CHAPTER XXVI

EXPONENTS AND ROOTS

LAWS OF EXPONENTS

494. Preparatory

1. What is the meaning of a^2? Of a^4? Of a^n?
2. What is the meaning of $\sqrt{a^2}$? Of $\sqrt[3]{a^3}$? Of $\sqrt[n]{a^n}$?
3. $a^2 \cdot a^3 = ?$ $a^3 \cdot a^2 = ?$ $a^5 \cdot a^3 = ?$ $a^8 \cdot a^5 = ?$
4. $a^3 \div a^2 = ?$ $a^4 \div a^2 = ?$ $a^5 \div a^2 = ?$ $b^{10} \div b^4 = ?$
5. $(a^2)^2 = ?$ $(a^2)^3 = ?$ $(c^5)^2 = ?$ $(x^{10})^3 = ?$

495. We shall soon define negative and fractional exponents, but until this is done literal exponents are to be understood to represent positive integers.

496. Law of Exponents in Multiplication.

I. $$\boldsymbol{a^m \cdot a^r = a^{m+r}}.$$

For $\quad a^m = a \cdot a \cdot a \cdots$ to m factors,

and $\quad a^r = a \cdot a \cdot a \cdots$ to r factors.

$$\therefore a^m \cdot a^r = (a \cdot a \cdot a \cdots \text{ to } m \text{ factors})(a \cdot a \cdot a \cdots \text{ to } r \text{ factors})$$
$$= a \cdot a \cdot a \cdot a \cdots \text{ to } m + r \text{ factors}$$
$$= a^{m+r}, \text{ by the definition of exponent.}$$

Similarly, $\quad a^m \cdot a^r \cdot a^p \cdots = a^{m+r+p} \cdots$.

ORAL EXERCISES

Multiply:

1. $a^2 \cdot a^4$.
2. $a^3 \cdot a^3$.
3. $a^5 \cdot a^7$.
4. $m^1 \cdot m^5$.
5. $x \cdot x$.
6. $2^3 \cdot 2^4$.
7. $(-1)^3 \cdot (-1)^5$.
8. $6^2 \cdot 6^2$.
9. $5 \cdot 5 \cdot 5^2$.
10. $2^3 \cdot 2^3 \cdot 2^2$.
11. $7 \cdot 7^2 \cdot 7^3$.
12. $3 \cdot 3^5 \cdot 3^2$.
13. $(-1)^2 \cdot (-1)^3 \cdot (-1)^5$.
14. $(-a)^2 \cdot (-a)^4 \cdot (-a)$.

497. Law of Exponents in Division.

II. $$\frac{a^m}{a^r} = a^{m-r}, \text{ if } m > r.$$

For $a^m = a \cdot a \cdot a \cdots m$ factors,

and $a^r = a \cdot a \cdot a \cdots r$ factors.

$\therefore \frac{a^m}{a^r} = \frac{a \cdot a \cdots m \text{ factors}}{a \cdot a \cdots r \text{ factors}}$

$= a \cdot a \cdot a \cdots m - r$ factors, canceling the r factors from both terms

$= a^{m-r}$, by definition of exponent.

ORAL EXERCISES

Divide:

1. $\frac{a^4}{a^2}$.	5. $\frac{a^8}{a^5}$.	9. $\frac{6^3}{6^2}$.	13. $\frac{x^2y}{x}$.
2. $\frac{a^5}{a^3}$.	6. $\frac{(-a)^5}{(-a)}$.	10. $\frac{5^4}{5^3}$.	14. $\frac{mv^2}{v}$.
3. $\frac{a^7}{a^5}$.	7. $\frac{(-1)^5}{(-1)^3}$.	11. $\frac{4\pi r^2}{4r}$.	15. $\frac{\frac{1}{2}gt^2}{t}$.
4. $\frac{2^4}{2^2}$.	8. $\frac{(ab)^2}{(ab)}$.	12. $\frac{x^5}{x}$.	16. $\frac{\pi r^3}{r}$.

498. Laws of Exponents for Powers.

III. $$(a^m)^r = a^{mr}.$$

For $(a^m)^r = a^m \cdot a^m \cdot a^m \cdots$ to r factors

$= (a \cdot a \cdots$ to m factors$)(a \cdot a \cdots$ to m factors$)$ to r such parentheses

$= a \cdot a \cdot a \cdots$ to mr factors

$= a^{mr}$, by definition of exponent.

ORAL EXERCISES

Apply this law to:

1. $(4^3)^2$.	4. $(a^3)^2$.	7. $(x^2)^5$.	10. $[(b)^4]^5$.
2. $(3^2)^5$.	5. $(a^2)^3$.	8. $(x^3)^3$.	11. $[(-a)^5]^2$.
3. $(2^5)^4$.	6. $(a^5)^2$.	9. $(y^4)^4$.	12. $[(-8)^2]^3$.

IV. $$(ab)^n = a^n b^n.$$

For $(ab)^n = (ab) \cdot (ab) \cdots$ to n factors

$= (a \cdot a \cdots$ to n factors$)(b \cdot b \cdots$ to n factors$)$

$= a^n b^n$, by definition of exponent.

Similarly, $(abc \cdots)^n = a^n b^n c^n \cdots$.

ORAL EXERCISES

Apply this law to:

1. $(8 \cdot 3)^2$.
2. $(4 \cdot 5)^2$.
3. $(2 \cdot 5)^3$.
4. $(ab)^5$.
5. $(cd)^3$.
6. $(abc)^4$.
7. $(mn)^p$.
8. $(xy)^a$.
9. $(ab)^{2r}$.
10. $(x^2y)^3$.
11. $(x^a y^2)^5$.
12. $(ax^3)^b$.

V. $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

For $\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdots$ to n factors

$= \dfrac{a \cdot a \cdots \text{ to } n \text{ factors}}{b \cdot b \cdots \text{ to } n \text{ factors}}$

$= \dfrac{a^n}{b^n}$, by definition of exponent.

ORAL EXERCISES

Apply this law to:

1. $(\frac{2}{3})^2$.
2. $(\frac{1}{5})^3$.
3. $(\frac{2}{5})^5$.
4. $(\frac{3}{5})^4$.
5. $(-\frac{6}{7})^2$.
6. $(-\frac{3}{4})^3$.
7. $\left(\frac{a}{b}\right)^2$.
8. $\left(\frac{a}{b}\right)^5$.
9. $\left(\frac{c}{d}\right)^3$.
10. $\left(\frac{x}{y}\right)^u$.
11. $\left(\frac{m}{n}\right)^a$
12. $\left(\frac{p}{q}\right)^x$.
13. $\left(\frac{2a}{b}\right)^5$.
14. $\left(\frac{c}{5d}\right)^7$.
15. $\left(\frac{-xy}{z}\right)^5$.

499. Collected Laws of Exponents.

I.	$a^m \cdot a^r = a^{m+r}$.		Sec. 496.
II.	$a^m \div a^r = a^{m-r}$.	$(m > r.)$	Sec. 497.
III.	$(a^m)^r = a^{mr}$.		Sec. 498.
IV.	$(ab)^n = a^n b^n$.		Sec. 498.
V.	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.		Sec. 498.

FRACTIONAL EXPONENTS

500. Hitherto we have spoken only of positive integers as exponents, the exponent meaning the number of times the base is used as a factor. This meaning does not apply to fractional and negative exponents, because it does not mean anything to speak of using a as a factor $\frac{2}{3}$ of a time, or -6 times. But it is possible to find meanings for fractional and negative exponents such that they will conform to the laws of integral exponents.

501. Preparatory.

Find the meaning of $a^{\frac{1}{2}}$.

Assuming that Law I applies, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a,$

or $$(a^{\frac{1}{2}})^2 = a.$$

That is, $a^{\frac{1}{2}}$ is one of the two equal factors of a,

or, $$a^{\frac{1}{2}} = \sqrt{a}.$$

Thus, the fractional exponent $\frac{1}{2}$ means *square root.*

Similarly, $$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a,$$

or, $$(a^{\frac{1}{3}})^3 = a.$$

That is, $a^{\frac{1}{3}}$ is one of the three equal factors of a,

or, $$a^{\frac{1}{3}} = \sqrt[3]{a}.$$

Thus, the fractional exponent $\frac{1}{3}$ means *cube root.*

WRITTEN EXERCISES

Find similarly the meaning of:

1. $a^{\frac{1}{4}}$.
2. $a^{\frac{1}{5}}$.
3. $x^{\frac{1}{8}}$.
4. $b^{\frac{1}{10}}$.
5. $n^{\frac{1}{8}}$.
6. $c^{\frac{1}{7}}$.
7. $m^{\frac{1}{6}}$.
8. $a^{\frac{1}{50}}$.

502. The meaning of $a^{\frac{1}{n}}$ is found as follows:

Assuming that Law I applies,

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots \text{to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\cdots \text{to } n \text{ terms}}$$
$$= a^{n \cdot \frac{1}{n}}$$
$$= a^1 = a.$$

That is, $a^{\frac{1}{n}}$ is one of n equal factors of a, or $a^{\frac{1}{n}} = \sqrt[n]{a}$.

503. Preparatory.

Find the meaning of $a^{\frac{2}{3}}$.

Assuming that Law I applies, $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2$, or $(a^{\frac{2}{3}})^3 = a^2$.

That is, $a^{\frac{2}{3}}$ is one of the three equal factors of a^2, or $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

504. The meaning of $a^{\frac{p}{q}}$ is found as follows:

Assuming that Law I applies,

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots \text{to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q} \cdots \text{to } q \text{ terms}}$$
$$= a^{q \cdot \frac{p}{q}} = a^p.$$

That is, $a^{\frac{p}{q}}$ is one of the q equal factors of a^p,

or,
$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Similarly,
$$a^{\frac{p}{q}} = a^{\frac{1}{q} \cdot p} = (\sqrt[q]{a})^p.$$

In words:

a *with the exponent* $\frac{\text{p}}{\text{q}}$ *denotes the* q*th root of the* p*th power of* a, *or the* p*th power of the* q*th root of* a.

This definition applies when p and q are positive integers. The meaning of negative fractional exponents is found in Section 513, p. 366.

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ORAL EXERCISES

State the meaning of:

1. $5^{\frac{2}{3}}$.
2. $a^{\frac{3}{4}}$.
3. $6^{\frac{1}{2}}$.
4. $a^{\frac{2}{5}}$.
5. $4^{\frac{5}{7}}$.
6. $b^{\frac{5}{6}}$.
7. $8^{\frac{3}{4}}$.
8. $c^{\frac{4}{3}}$.

Find the value of:

9. $8^{\frac{4}{3}}$.
10. $16^{\frac{3}{4}}$.
11. $25^{\frac{3}{2}}$.
12. $32^{\frac{3}{5}}$.

WRITTEN EXERCISES

Express with fractional exponents:

1. $\sqrt[4]{a^3}$.
2. $\sqrt[5]{a^2}$.
3. $\sqrt[3]{mn}$.
4. $\sqrt{a}\sqrt[3]{b}$.
5. $\sqrt[3]{ab}$.
6. $\sqrt[3]{ab^2xy}$.
7. $\sqrt[3]{\frac{x^4y^2}{4}}$.
8. $\sqrt{\frac{9\,a^2b}{x}}$.
9. $\sqrt[8]{16\,x^4y^8}$.
10. $\sqrt{m}\sqrt[3]{n^2}\sqrt[5]{p}$.
11. $\sqrt[5]{\frac{32\,a^5b^8}{c^{15}}}$.
12. $\sqrt[3]{\frac{16\,a^2}{9\,b^2}}$.
13. $\sqrt[3]{\frac{-8\,x^3y^6}{5}}$.
14. $\sqrt[6]{\frac{64\,x^{12}}{y}}$.
15. $\sqrt{a+b}$.
16. $\sqrt{a^3+b^3}$.
17. $\sqrt[3]{b^2}\cdot\sqrt{a}$.
18. $\sqrt[5]{-a}\cdot\sqrt[5]{-b}$.
19. $\sqrt[3]{(a-b)^2}$.
20. $\sqrt[n]{a}$.
21. $\sqrt[n]{a}\cdot\sqrt[m]{b}$.
22. $\sqrt[n]{a^m}$.
23. $\sqrt[m]{a^n}$.
24. $\sqrt[pn]{b^{2n}}$.
25. $\sqrt[2m]{a^{mn}}$.
26. $\sqrt[2mn]{a^{2mn}}$.
27. $\sqrt[mn]{a^nb^m}$.
28. $\sqrt[n]{a}\cdot\sqrt[p]{q}$.
29. $\sqrt[2n]{abc}$.
30. $\sqrt[p]{a^q}\cdot\sqrt[q]{a^p}$.

505. The definition of positive fractional exponents has been found as a consequence of the assumption that Law I applies to them. It can be shown that the other laws of Section 499, p. 358, also apply to this class of exponents, as thus defined, and we shall so apply them, although the proof is omitted here.

506. According to Law I (Sec. 499), *when the bases are the same, the exponent of the product is found by adding the exponents.*

For example: $a^{\frac{1}{2}} \cdot a^{\frac{3}{4}} = a^{\frac{1}{2}+\frac{3}{4}} = a^{\frac{5}{4}}$

A general formula for this statement is,

$$a^{\frac{m}{n}} \cdot a^{\frac{p}{q}} = a^{\frac{mq}{nq}+\frac{np}{nq}} = a^{\frac{mq+np}{nq}}.$$

The number a, or the base, must be the same in all factors. When it is not, as in $a^{\frac{1}{2}} \cdot b^{\frac{3}{4}}$, the product cannot be found by adding the exponents.

WRITTEN EXERCISES

Find the products:

1. $a^{\frac{1}{2}} \cdot a^{\frac{2}{3}}$.
2. $4^2 \cdot 4^{\frac{3}{5}}$.
3. $7^{\frac{1}{2}} \cdot 7^{\frac{3}{5}}$.
4. $a^{\frac{3}{2}} \cdot a^{\frac{2}{3}}$.
5. $b^{\frac{1}{2}} \cdot b^{\frac{1}{8}}$.
6. $a^{\frac{2}{3}} \cdot a^{\frac{3}{4}}$.
7. $m^{\frac{3}{4}} \cdot m^{\frac{3}{5}}$.
8. $a^{\frac{5}{3}} \cdot a^{\frac{1}{2}}$.
9. $r^{\frac{1}{n}} \cdot r^{\frac{1}{n}}$.
10. $a^{\frac{1}{n}} \cdot a^{\frac{1}{m}}$.
11. $x^{\frac{1}{a}} \cdot x^{\frac{1}{b}}$.
12. $x^{\frac{m}{n}} \cdot x^{\frac{p}{n}}$.
13. $p^{\frac{2}{3}} \cdot p^{\frac{1}{2}} \cdot p$.
14. $a^{\frac{4}{5}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{1}{2}}$.

507. According to Law II (Sec. 499), *when the bases are the same, the exponent of the quotient is found by taking the difference between the exponents.*

For example: $a^{\frac{2}{3}} \div a^{\frac{1}{2}} = a^{\frac{2}{3}-\frac{1}{2}} = a^{\frac{1}{6}}$.

$a \div a^{\frac{3}{4}} = a^{1-\frac{3}{4}} = a^{\frac{1}{4}}$.

A general formula for this statement is,

$$a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}} = a^{\frac{mq-np}{nq}}.$$

$\frac{m}{n}$ is here supposed to be greater than $\frac{p}{q}$, but this restriction will be removed later.

WRITTEN EXERCISES

Find the quotients:

1. $a^{\frac{3}{4}} \div a^{\frac{1}{2}}$.
2. $a \div a^{\frac{7}{12}}$.
3. $a \div a^{\frac{1}{n}}$.
4. $a \div a^{\frac{p}{q}}$.
5. $a^{\frac{4}{5}} \div a^{\frac{3}{4}}$.
6. $x^{\frac{7}{8}} \div x^{\frac{2}{3}}$.
7. $a^{\frac{3}{4}} \div a^{\frac{2}{3}}$.
8. $x^{\frac{2}{5}} \div x^{\frac{1}{7}}$.
9. $ab \div (ab)^{\frac{1}{7}}$.
10. $\left(\frac{1}{a}\right)^{\frac{5}{6}} \div \left(\frac{1}{a}\right)^{\frac{1}{2}}$.
11. $5^{\frac{2}{3}} \div 5^{\frac{1}{2}}$.
12. $x^{\frac{m}{n}} \div x^{\frac{p}{n}}$.
13. $m^{\frac{1}{a}} \div m^{\frac{1}{b}}$.
14. $6^{\frac{2}{3}} \div 6^{\frac{1}{6}}$.
15. $m^{\frac{3}{8}} \div m^{\frac{1}{8}}$.
16. $p^{\frac{4}{5}} \div p^{\frac{3}{4}}$.

508. According to Law III (Sec. 499), *when an exponent is applied to a base having an exponent, the product of the exponents is the exponent of the result.*

For example:

$$(a^{\frac{1}{2}})^2 = a^{2 \cdot \frac{1}{2}} = a^1 = a.$$

$$(a^2)^{\frac{1}{4}} = a^{2 \cdot \frac{1}{4}} = a^{\frac{2}{4}} = a^{\frac{1}{2}}.$$

$$(a^{\frac{1}{3}})^{\frac{2}{5}} = a^{\frac{1}{3} \cdot \frac{2}{5}} = a^{\frac{2}{15}}.$$

A general formula for this statement is,

$$(a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{m}{n} \cdot \frac{p}{q}} = a^{\frac{mp}{nq}}.$$

ORAL EXERCISES

Simplify by applying Law III:

1. $(2^{\frac{1}{2}})^{\frac{1}{3}}$.
2. $(3^{\frac{1}{3}})^2$.
3. $(3^{\frac{2}{3}})^3$.
4. $(5^{\frac{3}{4}})^2$.
5. $(b^{\frac{1}{3}})^{\frac{1}{2}}$.
6. $(a^{\frac{1}{2}})^{\frac{1}{3}}$.
7. $(a^{\frac{2}{3}})^{\frac{5}{6}}$.
8. $(a^{\frac{3}{4}})^{\frac{2}{3}}$.
9. $(x^{\frac{5}{6}})^{\frac{7}{8}}$.
10. $(y^9)^{\frac{1}{3}}$.
11. $(5^{\frac{1}{2}})^{\frac{2}{3}}$.
12. $(3^{\frac{2}{3}})^{\frac{1}{3}}$.
13. $(a^{\frac{1}{5}})^{\frac{1}{2}}$.
14. $(a^{\frac{3}{4}})^4$.
15. $(10^{\frac{1}{6}})^{\frac{1}{3}}$.
16. $[(a+b)^{\frac{1}{2}}]^{\frac{1}{2}}$.
17. $[(a-b)^{\frac{1}{3}}]^{\frac{3}{4}}$.
18. $[(a^2-b^2)^{\frac{1}{4}}]^{\frac{1}{5}}$.
19. $[(x^n-y^n)^{\frac{1}{p}}]^{\frac{p}{q}}$.

509. According to Law IV (Sec. 499), *an exponent affecting a product is applied to each factor*, and according to Law V (Sec. 499), *an exponent affecting a fraction is applied to both numerator and denominator.*

For example:

$$(8\,x^6y^2z)^{\frac{1}{3}} = 8^{\frac{1}{3}}x^{\frac{6}{3}}y^{\frac{2}{3}}z^{\frac{1}{3}} = 2\,x^2y^{\frac{2}{3}}z^{\frac{1}{3}}.$$

$$(a^m b^{\frac{m}{n}} c)^{\frac{p}{q}} = a^{\frac{mp}{q}} \cdot b^{\frac{mp}{nq}} \cdot c^{\frac{p}{q}}.$$

$$\left(\frac{x^2}{y^8}\right)^{\frac{1}{4}} = \frac{(x^2)^{\frac{1}{4}}}{(y^8)^{\frac{1}{4}}} = \frac{x^{\frac{2}{4}}}{y^{\frac{8}{4}}} = \frac{x^{\frac{1}{2}}}{y^2}.$$

A general formula for an exponent affecting a fraction is,

$$\left(\frac{a^{\frac{m}{n}}}{b^{\frac{p}{q}}}\right)^{\frac{r}{s}} = \frac{a^{\frac{mr}{ns}}}{b^{\frac{pr}{qs}}}.$$

WRITTEN EXERCISES

Simplify by applying Law IV:

1. $(a^3b^2)^{\frac{1}{2}}$.
2. $(a^2b^{\frac{1}{3}})^3$.
3. $(a^mb^n)^{\frac{1}{p}}$.
4. $(x^{\frac{1}{2}}y^{\frac{1}{4}})^3$.
5. $(16\,x^4y^8)^{\frac{1}{4}}$.
6. $(a^6bn^{9m})^{\frac{1}{3}}$.
7. $(27\,a^{12}b^9c^6)^{\frac{1}{3}}$.
8. $(m^{\frac{1}{2}}n^{\frac{1}{4}}p^{\frac{1}{5}})^{20}$.
9. $(86\,x^6y)^{\frac{1}{3}}$.
10. $\left(\frac{32\,a^5b^{10}}{c^{15}}\right)^{\frac{1}{5}}$.
11. $\left(\frac{64\,x^{12}}{y}\right)^{\frac{1}{6}}$.

510. When the bases are different and the fractional exponents are different, the exponents must be made to have a common denominator, before any simplification is possible.

For example: $a^{\frac{1}{2}}b^{\frac{1}{3}} = a^{\frac{3}{6}}b^{\frac{2}{6}} = (a^3b^2)^{\frac{1}{6}}.$

A general formula for this statement is,

$$a^{\frac{m}{n}}b^{\frac{p}{q}} = a^{\frac{mq}{nq}} \cdot b^{\frac{np}{nq}} = (a^{mq} \cdot b^{np})^{\frac{1}{nq}}.$$

This is *simplifying by reducing exponents to the same order.*

WRITTEN EXERCISES

Simplify by reducing the exponents to the same order:

1. $a^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$.
2. $a^{\frac{2}{3}} \cdot b^{\frac{3}{4}}$.
3. $v^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$.
4. $a^{\frac{1}{5}} \cdot b^{\frac{1}{4}}$.
5. $a^{\frac{1}{3}} \cdot b^{\frac{3}{4}}$.
6. $b^{\frac{1}{2}} \cdot 6^{\frac{1}{6}}$.
7. $5^{\frac{1}{5}} \cdot b^{\frac{1}{4}}$.
8. $a^{\frac{1}{2}} \div b^{\frac{1}{5}}$.
9. $m^{\frac{1}{3}} \cdot n^{\frac{1}{2}} \cdot p^{\frac{1}{4}}$.
10. $m^{\frac{2}{3}} \cdot n^{\frac{3}{5}} \div p^{\frac{1}{2}}$.
11. $p^{\frac{1}{n}} \cdot q^{\frac{1}{m}} \cdot r^{\frac{1}{3}}$.
12. $x^{\frac{p}{q}} \cdot y^{\frac{m}{n}} \cdot z^{\frac{1}{n}}$.

511. It is usually preferable to indicate roots by fractional exponents instead of by radical signs, since operations are thus more easily seen.

COMPARISON

	By Radicals	By Exponents
1.	$\sqrt{a}\sqrt[3]{a} = \sqrt[6]{a^3}\sqrt[6]{a^2} = \sqrt[6]{a^3a^2} = \sqrt[6]{a^5}$.	$a^{\frac{1}{2}}a^{\frac{1}{3}} = a^{\frac{3}{6}}a^{\frac{2}{6}} = a^{\frac{3}{6}+\frac{2}{6}} = a^{\frac{5}{6}}$.
2.	$\sqrt{b} \div \sqrt[3]{b} = \sqrt[6]{b^3} \div \sqrt[6]{b^2} = \sqrt[6]{b^3 \div b^2} = \sqrt[6]{b}$.	$b^{\frac{1}{2}} \div b^{\frac{1}{3}} = b^{\frac{3}{6}} \div b^{\frac{2}{6}} = b^{\frac{3}{6}-\frac{2}{6}} = b^{\frac{1}{6}}$.

WRITTEN EXERCISES

Simplify by use of fractional exponents as in the examples above:

1. $2^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}$.
2. $\sqrt{\frac{49}{121}}$.
3. $\sqrt{8} \cdot 3\sqrt{2}$.
4. $2\sqrt{3} \cdot 3 \cdot \sqrt{10}$.
5. $2\sqrt[3]{3} \cdot 3\sqrt[3]{2}$.
6. $5^{\frac{1}{2}} \cdot 3 \cdot 5^{\frac{1}{3}}$.
7. $\sqrt{7} \cdot 11^{\frac{1}{2}}$.
8. $-8\sqrt{2} \cdot 12\sqrt{3}$.
9. $\sqrt{81a^{10}}$.
10. $\sqrt{5} \cdot \sqrt{75}$.
11. $(16x^4p)^{\frac{1}{4}}$.
12. $\sqrt[5]{32a^3b^{10}}$.
13. $\sqrt{36m^4n^2}$.
14. $\sqrt{64m^3n^6}$.
15. $(49a^4b^6)^{\frac{1}{2}}$.
16. $\sqrt[3]{2a^2b^2}$.
17. $3^{\frac{1}{2}} \cdot 2 \cdot 7^{\frac{1}{2}}$.
18. $3 \cdot 5^{\frac{1}{2}} \cdot 2\sqrt{3}$.
19. $\left(\frac{49a^4}{64b^6}\right)^{\frac{1}{2}}$.
20. $\sqrt[3]{a^3\sqrt{25b}}$.
21. $\sqrt{x^2y\sqrt[4]{25yz}}$.
22. $\sqrt[6]{a\sqrt[5]{ab^3}}$.
23. $\sqrt{\sqrt[3]{4x^2}}$.

24. $-\sqrt{12} \cdot 2\sqrt{3}$.

25. $a\sqrt{b} \cdot b\sqrt{a}$.

26. $(\sqrt{2}-\sqrt{3})2\sqrt{3}$.

27. $3^{\frac{1}{2}}(6^{\frac{1}{2}} - 2 \cdot 5^{\frac{1}{2}})$.

28. $(a\sqrt[3]{x})^2$.

29. $\sqrt{a^6b^8}$.

30. $\sqrt[3]{8\,a^5b^6}$.

31. $(27\,xy^2)^{\frac{1}{3}}$.

32. $\sqrt{32} \cdot \sqrt{2}$.

33. $(a^2b\sqrt[4]{xy^2})^3$.

34. $(m^3\sqrt[5]{p^4})^4$.

35. $\sqrt{6\,ab} \cdot \sqrt{2\,a}$.

36. $[2\,a(4\,a^2)^{\frac{1}{3}}]^2$.

37. $11^{\frac{1}{2}} \cdot 11^{\frac{1}{3}} \cdot 11^{\frac{1}{7}}$.

38. $\sqrt[8]{m^2\sqrt[3]{m^2}}$.

39. $\sqrt{\sqrt[4]{8\,x^3}}$.

40. $\sqrt{\sqrt[5]{81\,a^4}}$.

41. $\sqrt[3]{x^2\sqrt[3]{9\,a^2}}$.

42. $\sqrt[3]{x^3y^6\sqrt[4]{27\,a^3}}$.

43. $a(a^2b)^{\frac{1}{3}} \cdot b(ab^2)^{\frac{1}{3}}$.

44. $3\sqrt{9\,a} \cdot 3^{\frac{1}{2}}$.

45. $5\sqrt[4]{a^3x^3} \cdot 2\sqrt[3]{ax}$.

46. $3\sqrt{8} \cdot 2\sqrt[3]{6} \cdot 3\sqrt[4]{54}$.

MEANING OF ZERO AND NEGATIVE EXPONENTS

512. The meaning of a^0 may be found as follows:

Assuming that Law I holds for a^0, $a^5 \cdot a^0 = a^{5+0}$

$= a^5$.

Dividing by a^5, $a^0 = \frac{a^5}{a^5} = 1$.

That is:

Any number (not zero) with the exponent zero equals 1.

Thus, $5^0 = 1$, $10^0 = 1$, $(\frac{10}{3})^0 = 1$, $\left(\frac{a}{b}\right)^0 = 1$.

ORAL EXERCISES

State the value of:

1. $(ab)^0$.

2. $.9^0$.

3. 100^0.

4. $\left(\frac{a}{bc}\right)^0$.

5. $(-3)^0$.

6. $(\frac{1}{2})^0$.

7. $(-\frac{1}{27})^0$.

8. $3^2 \cdot 3^0$.

9. $3^2 \cdot 5^0$.

10. $\frac{1}{2} \cdot a^0$.

11. a^5a^0.

12. $a^m \cdot b^0$.

13. $a^5 \div a^0$.

14. $4^3 \div 4^0$.

15. $(\frac{1}{2})^0$.

16. $.9 \div 100^0$.

17. $9\frac{1}{2} \cdot 5^0$.

18. $3^0 \cdot 27^0$.

19. $(2^3 \cdot 3^2)^0$.

513. The meaning of the negative exponent may be found as follows:

Assuming that Law I holds for negative exponents,

$$5^{-3} \cdot 5^{+3} = 5^{-3+3} = 5^0 = 1.$$

That is, 5^{-3} is a multiplier such that its product with 5^{+3} is 1. But if the product of two numbers is 1, one is the reciprocal of the other.

Therefore, 5^{-3} is the reciprocal of 5^3 which is $\frac{1}{5^3}$.

Expressed in general terms:

$$a^{-n} \cdot a^n = a^{-n+n}$$
$$= a^0 = 1.$$
$$a^{-n} = \frac{1}{a^n}.$$

In words:

a^{-n} *means* $\frac{1}{a^n}$, *for all values of* n, *positive or negative, integral or fractional.*

ORAL EXERCISES

Find similarly the meaning of:

1. 4^{-3}.
2. 2^{-4}.
3. $(\frac{3}{4})^{-2}$.
4. $(-5)^{-6}$.
5. a^{-3}.
6. $a^{-\frac{3}{4}}$.
7. $(\frac{1}{2})^{-\frac{2}{3}}$.
8. $(-\frac{2}{3})^{-\frac{4}{5}}$.

State the value of:

9. $4^{-\frac{1}{2}}$.
10. $8^{-\frac{1}{3}}$.
11. $8^{-\frac{2}{3}}$.
12. $(.2^0)^{\frac{1}{2}}$.
13. $16^{-\frac{1}{4}}$.
14. $16^{-\frac{3}{4}}$.
15. $.125^{-\frac{1}{3}}$.
16. $.125^{-\frac{2}{3}}$.
17. $32^{-\frac{3}{5}}$.
18. $27^{-\frac{2}{3}}$.
19. $.36^{-\frac{3}{2}}$.
20. $64^{-\frac{5}{6}}$.
21. a^{-3}.
22. $(a^0)^{-2}$.
23. $a^{-\frac{3}{5}}$.
24. $a^{-\frac{1}{3}}$.

WRITTEN EXERCISES

Perform the operations indicated, admitting negative exponents to the results:

1. $(4^2 \cdot 5^4)^{-\frac{1}{2}}$.
2. $2^{14} \cdot 2^{-6}$.
3. $2^{\frac{2}{3}} \cdot 2^{-\frac{5}{6}}$.
4. $2^{-\frac{2}{3}} \cdot 2^{-\frac{5}{6}}$.
5. $\frac{5^{-\frac{1}{2}} \cdot 5^{\frac{3}{2}}}{5^0}$.
6. $10^{-5} \cdot 10^2 \cdot 10^0$.

7. $2^{-\frac{2}{3}} \cdot 2^{\frac{5}{6}}$.
8. $27^{-\frac{3}{4}} \cdot 9^{\frac{3}{2}}$.
9. $3^{\frac{7}{8}} \cdot 3^{-\frac{1}{2}} \cdot 4^0$.
10. $(x^{-4})^2$.
11. $10^7 \cdot 10^{-7}$.
12. $a^0 \cdot a^{\frac{1}{3}}$.
13. $\sqrt[4]{18^{-4}} \cdot \sqrt[3]{18^{-3}}$.
14. $\sqrt[3]{10^{-4}} \cdot \sqrt[5]{10^0}$.
15. $(10^{-4} \cdot 10^{-3})^{\frac{2}{7}}$.
16. $(10^{-\frac{2}{3}} \cdot 10^{\frac{1}{2}}$.
17. $10^{\frac{1}{2}} \cdot 10^{\frac{5}{6}} \cdot 10^{\frac{1}{3}} \cdot (\frac{4}{7})^0$.
18. $\dfrac{a^{-\frac{1}{2}}}{a^{-\frac{1}{3}}}$ (or $a^{-\frac{1}{2}-(-\frac{1}{3})}$).
19. $a^{\frac{1}{2}} \cdot a^{-\frac{1}{2}}$(or $a^{\frac{1}{2}-\frac{1}{2}}$).
20. $\dfrac{a^{\frac{1}{2}}}{a^{-\frac{1}{2}}}$ (or $a^{\frac{1}{2}-(-\frac{1}{2})}$).
21. $(a^{-\frac{1}{2}})^{-\frac{1}{3}}\,(x^{-4})^5$.

USE OF ZERO, NEGATIVE, AND FRACTIONAL EXPONENTS

514. We have defined zero and negative exponents so that Law I holds for them. It can be shown that the other four laws hold for these exponents as defined, but the laws will be applied here without proof.

515. The relation $a^{-n} \cdot a^n = a^0 = 1$ can be used to change the form of expressions.

I. **To free an expression from a negative exponent,** *multiply both numerator and denominator by a factor that will so combine with the factor having the negative exponent as to produce unity. If more than one negative exponent is involved, apply the process for each.*

For example:

$$7^{-3} \cdot 2^4 = \frac{7^3 \cdot 7^{-3} \cdot 2^4}{7^3} = \frac{2^4}{7^3}.$$

$$\frac{a^3}{b^{-5}} = \frac{b^5 \cdot a^3}{b^5 \cdot b^{-5}} = \frac{b^5a^3}{1} = b^5a^3$$

$$\frac{x^{-2}}{t^{-5}} = \frac{t^5x^2 \cdot x^{-2}}{t^5x^2 \cdot t^{-5}} = \frac{t^5}{x^2}.$$

WRITTEN EXERCISES

Write equivalent expressions without negative exponents:

1. $\dfrac{3^{-2}}{2^{-3}}$.
2. $\dfrac{a^{-1}b^{-2}}{x^{-4}}$.
3. $2\,a^{-\frac{1}{4}}$.
4. $\dfrac{ax^{-4}}{b^{-3}x^2}$.

5. $\dfrac{a^{-5}}{b^{-3}}$.

6. $\dfrac{3^{-2} \cdot 4^{-1}}{5^{-3}}$.

7. $\dfrac{x^{-a}}{a^{-x}}$.

8. $\dfrac{a^{-3}}{b^{3}}$.

9. $\dfrac{ab}{c^{-5}}$.

10. $a^{-1}b^{-2}$.

11. $\dfrac{5\,y^3}{5^{-1}y^{-4}}$.

12. $\dfrac{1}{a^{-3}x^{-3}}$.

II. **To free an expression from a fractional form,** *multiply both numerator and denominator by a factor that, in combination with the denominator, will produce unity. If more than one such form is involved, apply the process for each.*

For example:

$$\frac{a^3}{b^2} = \frac{b^{-2}a^3}{b^{-2}b^2} = b^{-2}a^3;$$

also,

$$\frac{2}{4^{-5}} + \frac{a^3}{x^2y^{-3}} = \frac{4^5 \cdot 2}{4^5 \cdot 4^{-5}} + \frac{x^{-2}y^3a^3}{x^{-2}y^3x^2y^{-3}}$$

$$= 4^5 \cdot 2 + \frac{x^{-2}y^3a^3}{(x^{-2}x^2)\,(y^3y^{-3})}$$

$$= 2 \cdot 4^5 + a^3x^{-2}y^3.$$

WRITTEN EXERCISES

Free from fractional forms:

1. $\dfrac{3}{5^2}$.

2. $\dfrac{1}{a^2b^3}$.

3. $\dfrac{a}{bx^3}$.

4. $\dfrac{a^{-2}}{t^{-3}}$.

5. $\dfrac{1}{5^2} + \dfrac{3}{2^3}$.

6. $\dfrac{x^2}{y^3} + \dfrac{y^2}{x^3}$.

7. $\dfrac{x}{y^{-1}} + \dfrac{a^2}{x^{-3}}$.

8. $\dfrac{a}{b} + \dfrac{a^{-1}}{b^{-1}}$.

III. **To transfer any specified factor from the numerator into the denominator, or vice versa,** *multiply the numerator and denominator by a factor that, in combination with the factor to be transferred, will produce unity.*

EXAMPLES

1. Transferring the factors of the denominator to the numerator:

$$\frac{x^7}{x^4y^{-3}} = \frac{x^{-4}y^3x^7}{x^{-4}y^3x^4y^{-3}} = x^{-4}y^3x^7 = x^3y^3.$$

2. Transferring the literal factors of the numerator to the denominator:

$$\frac{5\,a^3b^2}{4\,a^4b^{-3}} = \frac{5\,a^{-3}b^{-2}a^3b^2}{4\,a^{-3}b^{-2}a^4b^{-3}} = \frac{5}{4\,ab^{-5}}.$$

WRITTEN EXERCISES

In the following expressions:

(*a*) Transfer all literal factors to the numerator.

(*b*) Transfer all literal factors to the denominator.

1. $\dfrac{a^2x^3}{a^4x^7}$.
2. $\dfrac{a^{\frac{3}{4}}}{b^{\frac{7}{8}}}$.
3. $\dfrac{6\,ay^2z^{-5}}{11\,a^3x^{-3}y^4}$.
4. $\dfrac{4\,a^2b^3c^5}{17\,a^3b^5c^{-3}}$.
5. $\dfrac{5\,a^{-\frac{1}{3}}b^{\frac{2}{5}}}{8\,a^{\frac{5}{6}}b^{-\frac{3}{5}}}$.
6. $\dfrac{a^{-\frac{2}{3}}b^{-5}c^{\frac{3}{4}}}{a^{\frac{1}{3}}b^{-5}c^{-\frac{3}{4}}}$.
7. $\dfrac{5\,a^{-\frac{1}{3}}}{abx}$.
8. $\dfrac{6\,x^{-\frac{4}{5}}y^2}{5\,pq}$.

516. The laws of exponents enable us to perform operations with polynomials containing fractional and negative exponents.

Thus:
$$(a^{\frac{2}{3}}+b^{\frac{1}{2}})^2=(a^{\frac{2}{3}})^2+2\,a^{\frac{2}{3}}b^{\frac{1}{2}}+(b^{\frac{1}{2}})^2$$
$$=a^{\frac{4}{3}}+2\,a^{\frac{2}{3}}b^{\frac{1}{2}}+b.$$

WRITTEN EXERCISES

Perform the indicated operations, admitting negative exponents to the results:

1. $(a^{\frac{3}{2}}-b^{\frac{5}{4}})^2$.
2. $(4\,a^{\frac{2}{3}}x^{\frac{1}{2}}+2^3)^2$.
3. $(a^{-\frac{1}{2}}+b^{\frac{3}{2}})^2$.
4. $(a^n-3\,b^r)^2$.
5. $(x^{-\frac{3}{4}}-y^{-\frac{2}{3}})(x^{-\frac{3}{4}}+y^{-\frac{2}{3}})$.
6. $(x^{-n}+1)(x^{-n}-1)$.
7. $(x^{\frac{6}{5}}+3)(x^{\frac{6}{5}}+5)$.
8. $\dfrac{a^5-x^4}{a^{\frac{5}{2}}-x^2}$.
9. $(x^n-y^n)^2$.
10. $(a^{2n}b^{3r}-1)^2$.
11. $\dfrac{a^p\cdot b^q}{a^{p+1}\cdot b^{q-1}}$.
12. $(a^n+t^n)^2$.
13. $(a^{\frac{1}{3}}b^{\frac{3}{2}}+x^{-\frac{5}{2}})(a^{\frac{1}{3}}b^{\frac{3}{2}}-x^{-\frac{5}{2}})$.

Express as a product of two factors:

14. x^6-m^{-4}.
15. $a^8-2\,a^4x^{\frac{1}{3}}+x^{\frac{2}{3}}$.
16. $a^{2n}+2\,a^nb^n+b^{2n}$.
17. $x^{4n}-4\,x^{2n}y^{2n}+4\,y^{4n}$.
18. $y^{-7}-x^{-10}$.
19. x^r-4.
20. $1+8\,x^{-\frac{5}{2}}+16\,x^{-5}$.
21. $x^{12}+6\,x^6y^{-\frac{7}{2}}+9\,y^{-7}$.

EVOLUTION

Square Root of Binomials of the Form $a+\sqrt{b}$

517. Binomials of the form $a+\sqrt{b}$ can often be put into the form $x+y+2\sqrt{xy}$, or $(\sqrt{x}+\sqrt{y})^2$, and hence the square root, $\sqrt{x}+\sqrt{y}$, of the binomial can be written at once.

EXAMPLES

1. Find the square root of $4+2\sqrt{3}$.

$$4+2\sqrt{3}=3+1+2\sqrt{3\cdot 1}.$$

Hence, $x+y=3+1$ and $xy=3\cdot 1$, from which $x=3$ and $y=1$.

$$\therefore \sqrt{4+2\sqrt{3}}=\pm(\sqrt{3}+\sqrt{1})=\pm(\sqrt{3}+1).$$

The coefficient of the radical must be made 2 in order to apply the formula $x+y+2\sqrt{xy}$.

2. Find the square root of $3-\sqrt{8}$.

$$3-\sqrt{8}=3-\sqrt{4\cdot 2}=3-2\sqrt{2}.$$

$$\therefore x+y=3, \text{ and } xy=2. \quad \therefore x=2,\ y=1 \text{ by inspection.}$$

$$\therefore \sqrt{3-\sqrt{8}}=\pm(\sqrt{2}-\sqrt{1})=\pm(\sqrt{2}-1).$$

3. Find the square root of $7+4\sqrt{3}$.

$$7+4\sqrt{3}=7+2\sqrt{4\cdot 3};\ x+y=7,\ xy=12;\ \therefore x=4,\ y=3.$$

$$\therefore \sqrt{7+4\sqrt{3}}=\pm(\sqrt{4}+\sqrt{3})=\pm(2+\sqrt{3}).$$

The square root as a whole may be taken positively or negatively, as in the case of rational roots.

The solution of these problems depends upon finding two numbers whose sum and product are given. This can sometimes be done by inspection, but the general problem is one of simultaneous equations.

WRITTEN EXERCISES

Find the square root of:

1. $11+6\sqrt{2}$.
2. $8-2\sqrt{15}$.
3. $49-12\sqrt{10}$.
4. $41-24\sqrt{2}$.
5. $2\frac{1}{4}-\sqrt{5}$.
6. $2\frac{1}{3}-\frac{4}{3}\sqrt{3}$.
7. $17+12\sqrt{2}$.
8. $\frac{3}{2}\sqrt{5}+3\frac{1}{2}$.
9. $56-24\sqrt{5}$.

Cube Root of Arithmetical Numbers

518. Pointing off into Periods. Since $10^3 = 1000$, we know that the cube root of any number greater than 1 but less than 1000 is less than 10. Its integral part consists of one figure.

Since $\overline{100}^3 = 1{,}000{,}000$, we know that the cube root of any number greater than 1000 but less than 1,000,000 is greater than 10 but less than 100. That is, if the given number has from 4 to 6 digits in its integral part, its cube root will have 2 digits in its integral part. If larger numbers are given, the above reasoning can be repeated for 1000^3, etc., showing that in all cases if the number be pointed off into periods of 3 digits each (or possibly fewer in the left period), then each period will correspond to a digit of the root.

519. The cube root of the left period can be found approximately by inspection, and the number so found, with a zero annexed for each other period, will be an approximate value for the root.

EXAMPLE

Find $\sqrt[3]{481890304}$.

Pointing off as above:

481′890′304.

By trial we find $7^3 = 343$, and $8^3 = 512$. Hence 700 is an approximate value for the root. It may be verified that $\overline{700}^3$ is less than the given number, and that $\overline{800}^3$ is more than the given number. That is, the hundreds' figure of the root is 7.

WRITTEN EXERCISES

Find, as above, the first figure of:

1. $\sqrt[3]{493039}$.
2. $\sqrt[3]{2924207}$.
3. $\sqrt[3]{57066625}$.
4. $\sqrt[3]{254840104}$.
5. $\sqrt[3]{1345572864}$.
6. $\sqrt[3]{62287505344}$.

520. When once an approximate value a has been found for the root, an approximate value for the remainder, r, of the root can be found by means of the formula:

$$(a + r)^3 = a^3 + 3\,a^2r + 3\,ar^2 + r^3.$$

For example:

In $\sqrt[3]{238328}$ we find as above that $a = 60$.

$$
\begin{array}{lrl}
\text{Then,} & a^3 + 3a^2r + 3ar^2 + r^3 = & 238'328, \text{ the whole cube} \\
\text{Subtracting} & a^3 = & 216000, \text{ the cube of the part found,} \\
\hline
& 3a^2r + 3ar^2 + r^3 = & 22'328, \text{ the first remainder.}
\end{array}
$$

Since something must be added to $3a^2r$ to make it equal to 22,328,

$$3a^2r \text{ is less than } 22'328,$$

or

$$r \text{ is less than } \frac{22'328}{3a^2}.$$

Consequently, the first figure of this quotient will either be the first figure of r or greater than it. In this instance $3a^2$ is 10,800, hence the first figure of the quotient is 2.

Trying 2 as r, we have to calculate $3a^2r + 3ar^2 + r^3$. This is most conveniently done by using the form $(3a^2 + 3ar + r^2)r$.

$$
\begin{array}{lrr}
\text{We have already} & 3a^2 = & 10800. \\
\text{We find:} & 3ar = & 360 \\
& r^2 = & 4 \\
\hline
\text{Adding:} & 3a^2 + 3ar + r^2 = & 11164 \\
\text{Then,} & r(3a^2 + 3ar + r^2) = & 22328
\end{array}
$$

The calculation should be arranged thus:

$$
\begin{array}{lrr|l}
& & & \;\;6\;\;\;2 \\
& & & 238'328 \\
& a^3 = & & 216\,000 \\
\text{Trial divisor:} & 3a^2 = & 10800 & 22328 \\
& 3ar = 3 \cdot 2 \cdot 60 = & 360 & \\
& r^2 = 2^2 = & 4 & \\
\hline
\text{Complete divisor:} & & 11164 & 22328 \text{ or } 2 \times 11164
\end{array}
$$

If the root consists of more than two figures, the above work is repeated, using the part of the root already found as a.

If it should happen that the product of r and the complete divisor is larger than the remainder of the number, try the next smaller digit for r.

When necessary, periods are pointed off to the right of the decimal point.

WRITTEN EXERCISES

Find:

1. $\sqrt[3]{74088}$.
2. $\sqrt[3]{148877}$.
3. $\sqrt[3]{2803221}$.
4. $\sqrt[3]{16387064}$.
5. $\sqrt[3]{55306341}$.
6. $\sqrt[3]{143055667}$

Find these roots to one decimal place:

7. $\sqrt[3]{637}$. 8. $\sqrt[3]{3485}$. 9. $\sqrt[3]{263488}$.

Find these roots to two decimal places:

10. $\sqrt[3]{5}$. 11. $\sqrt[3]{17}$ 12. $\sqrt[3]{269}$. 13. $\sqrt[3]{4.763}$.

Cube Root of Polynomials

521. The *cube roots* of polynomials may be extracted in a similar manner:

EXAMPLE

Extract the cube root of $27x^3 - 27x^2y + 9xy^2 - y^3$.

		Root	$3x - y$
$a^3 + 3a^2r + 3ar^2 + r^3 =$		Power	$27x^3 - 27x^2y + 9xy^2 - y^3$
$a = 3x$	$a^3 = (3x)^3 =$		$27x^3$
Trial divisor $= 27x^2$	$\therefore r = -y$		$-27x^2y + 9xy^2 - y^3$
Complete divisor $= [3(3x)^2 - 3(3x)y + y^2]$			$-27x^2y + 9xy^2 - y^3$

EXPLANATION.

1. Arrange the expression in the order of the powers of some letter, as x.

2. Take the cube root of the first term of the power for the first term of the root, as $3x$.

3. Divide the second term of the power by 3 times the square of the first term of the root. The result is the second term of the root, as $-y$.

4. If a denote the approximate value of the root already found ($3x$ in the above instance), and r the value to be used as the next term ($-y$ in the above instance), form the complete divisor $3a^2 + 3ar + r^2$, multiply it by r, and subtract.

5. If there is a further remainder, proceed as before, using the entire part of the root already found as a.

WRITTEN EXERCISES

Extract the cube root of:

1. $8x^3 + 12x^2 + 6x + 1$.
2. $27 - 27x + 9x^2 - x^3$.
3. $8a^3b^3 - 12a^2b^2 + 6ab - 1$.
4. $x^3 - 3x^2y + 3xy^2 - y^3$.
5. $8x^6 - 12x^4 + 6x^2 - 1$.
6. $x^{12} - 3x^8m + 3x^4m^2 - m^3$.

7. $-a^6-6a^5+40a^3-96a+64.$

8. $54x^2y^2-36x^4y+8x^6-27y^3.$

9. $64x^6-144x^5+156x^4-99x^3+39x^2-9x+1.$

10. $-64+96x-40x^3+6x^5+x^6.$

11. $27y^3-27y^2+63y-37+\frac{42}{y}-\frac{12}{y^2}+\frac{8}{y^3}.$

12. $8x^3-12x+\frac{6}{x}-\frac{1}{x^3}.$

13. $a^3-b^3+c^3+3ab^2+3ac^2-3bc^2+3a^2c+3b^2c$
$-3a^2b-6abc.$

14. $27x^3-27x+\frac{9}{x}-\frac{1}{x^3}.$

15. $64(ab)^{3p}-48(ab)^{2p}+12(ab)^p-1.$

16. $30a^{-1}+8a^{-3}+8a^3+30a-12a^2-25-12a^{-2}.$

17. $a^6-20a^3-6a+15a^4-6a^5+15a^2+1.$

18. $x^3-12x^2+54x-112+\frac{108}{x}-\frac{48}{x^2}+\frac{8}{x^3}.$

19. $x^{3p}-y^{3q}-z^{3r}-3x^{2p}y^q+3x^py^{2q}-3x^{2p}z^r+3x^pz^{2r}-3y^{2q}z^r$
$-3y^qz^{2r}+6x^py^qz^r.$

20. $27x^3+y^3-\frac{1}{z^3}+9xy^2+\frac{9x}{z^2}+27x^2y-27\frac{x^2}{z}-18\frac{xy}{z}$
$+\frac{3y}{z^2}-\frac{3y^2}{z}.$

REVIEW

WRITTEN EXERCISES

Express with positive exponents:

1. $m^{-\frac{3}{4}}n^8.$ 2. $4x^{-\frac{1}{2}}y^{-1}z.$ 3. $3a^{-5}b^5.$ 4. $17x^{-\frac{5}{2}}y^{-7}z^{-\frac{7}{8}}.$

Transfer all literal factors from the denominator to the numerator:

5. $\frac{x^{\frac{3}{4}}}{x^{-\frac{2}{3}}}.$ 6. $\frac{ab}{a^{-\frac{1}{7}}b^{-3}}.$ 7. $\frac{1}{6x^{-2}y^{\frac{3}{2}}}.$ 8. $\frac{5}{ay^{-4}}.$

Multiply:

9. $(2+\sqrt{x+1})^2$.
10. $\sqrt{5}\cdot\sqrt[3]{6}$.
11. $5\sqrt[4]{m^{-3}}\cdot 2\,m^{-1}$.
12. $p\cdot p^{-\frac{3}{4}}$.
13. $(a^{-1}-b^{-1})(a^{-\frac{1}{2}}-b^{-\frac{1}{2}})$.
14. $(x^2-1)(x^{\frac{1}{2}}+1)$.

Remove the parentheses:

15. $(a^{-\frac{3}{4}})^2$.
16. $(\sqrt[3]{a^{-1}})^3$.
17. $\left(\frac{1}{\sqrt[4]{p^3}}\right)^{\frac{4}{5}}$.
18. $\left(\frac{1}{n^{-2}}\right)^5$.
19. $\left(a^{\frac{5n}{6}}\right)^{\frac{3}{10n}}$.
20. $\left(x^{\frac{m^2}{n^2}-1}\right)^{\frac{n}{m-n}}$.
21. $\left(\frac{1}{x^{-\frac{p}{q}}}\right)^{\frac{q}{p}}$.
22. $\left(m^{\frac{pq}{xy}}\right)^{\frac{x^2y}{pq}}$.
23. $\left(s^{\frac{1}{n-1}}\right)^{n^2-1}$.

Simplify as far as possible, admitting negative exponents to the results:

24. $(8\,a^3b^{\frac{1}{2}}c^{-\frac{3}{4}})^{\frac{2}{3}}$.
25. $(16\,a^{\frac{2}{3}}bc^{-4})^{\frac{3}{4}}$.
26. $(5\,ax^3y^2)^{\frac{1}{2}}$.
27. $(\sqrt[3]{4\,x^2})^{\frac{1}{4}}$.
28. $(64\,x^{-3}y^{\frac{1}{2}})^{-\frac{1}{6}}$.
29. $(-\,250\,x^3y^{-\frac{3}{5}}z^{-3})^{\frac{2}{3}}$.
30. $(-\,243\,a^{-\frac{1}{2}}y^{-1}z^{\frac{1}{4}})^{-\frac{4}{5}}$.
31. $\sqrt[p-q]{x^{p^2-q^2}}$.
32. $\sqrt[p]{x^{-2p}y^{-p^2}z^{-p}}$.
33. $\sqrt{a^{-m}b^{4m}}$.
34. $3\,a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-1}\cdot 2\,a^{\frac{1}{4}}b^{\frac{1}{2}}c$.
35. $10\,ab^{-\frac{1}{2}}c^{\frac{3}{4}}\div 2\,a^{-1}b^2z^{\frac{1}{2}}$.
36. $a^mb^{-2n}c^{-2}\div ab^nz^{-3}$.
37. $(3\,a^{-2}\div b^{-2})^{-5}$.
38. $(a^{p-q})^{p+q}\cdot a^{p^2}a^{q^2}$.
39. $\frac{\sqrt{a}}{\sqrt[5]{a^2b^3}}\cdot\frac{\sqrt[4]{a^3b^2}}{\sqrt[10]{a^7b^9}}$.
40. $x^{-3}\cdot x^{\frac{3}{4}}\cdot x^2\cdot\sqrt[5]{x}$.
41. $\frac{\sqrt[3]{x}}{\sqrt{x}}\cdot\frac{x^3}{x}\cdot\frac{\sqrt[3]{x^2}}{x^2}$.
42. $[\{(a^2-b^{-2})^{-1}\}^{-2}]^5$.
43. $[(a^{-\frac{1}{2}})^{\frac{2}{3}}]^{-12}$.
44. $\left\{\sqrt{ab^{-2}\sqrt{ab}}\right\}^4$.
45. $\{(a^{-3}b^2)^{\frac{1}{2}}\}^{-\frac{2}{3}}$.
46. $\sqrt[3]{a^2\sqrt{a^{-1}}}$.
47. $[(x^a)^{-b}]^{-\frac{1}{a}}\div[(x^{-b})^c]^{-\frac{1}{c}}$.
48. $-a^3b^{-4}c^{-3}d^5\cdot -a^{-2}b^5c^4d^{-5}$.

49. $a^2x - 3\,y^6 \cdot a^3x^6y^9.$

50. $3\,x^{-\frac{1}{2}}5\,x^{\frac{4}{5}} \cdot 10\,x^{-\frac{1}{2}}.$

51. $\sqrt{a^{-1}\sqrt{a^3\sqrt{a-4}}}.$

52. $\{x^{-\frac{3}{2}}y(xy^{-2})^{-\frac{1}{2}}(x^{-1}y)^{\frac{2}{3}}\}^{\frac{3}{2}}.$

53. $$\frac{\dfrac{x-y}{2y+x}+\dfrac{1}{2}}{3\frac{1}{2}-\left(\dfrac{x+2\,y}{5\,x+7\,y}\right)^{-1}}.$$

54. $\left(x^{\frac{n+1}{n-1}} \div x^{\frac{n-1}{n+1}}\right)^{\frac{n-1}{2n}}.$

55. $\left[(a^{p+q})^{p-q}(a^{q^3})^{\frac{1}{q}}\right]^{\frac{1}{p^2}}.$

56. $$\frac{\sqrt{(2^{\frac{1}{3}}+2^{-\frac{1}{3}})^2-4}}{4\,(2^{\frac{1}{3}}-2^{-\frac{1}{3}})} \cdot \frac{5+\sqrt{21}}{5-\sqrt{21}}.$$

57. Add: $\frac{1}{3}\sqrt{45}+4\sqrt{\frac{5}{4}}-\sqrt{125}.$

58. Express the product $\sqrt[3]{a^2}\sqrt{a^3}$ as a single surd.

59. Divide $2\,x^{\frac{9}{2}}y^{-3} - 5\,x^{\frac{7}{2}}y^{-2} + 7\,x^{\frac{5}{2}}y^{-1} - 5\,x^{\frac{3}{2}} + 2\,x^{\frac{1}{2}}y$ by $x^3y^{-3} - x^2y^{-2} + xy^{-1}.$

Find equivalent expressions with rational divisors:

60. $3\sqrt{2\,a} \div 2\sqrt{3\,b}.$

61. $b\sqrt{a^2} \div \sqrt{ab}.$

62. $\sqrt{40\,x^3y} \div x\sqrt{5\,y}.$

63. $x\sqrt[3]{y} \div y\sqrt[3]{x}.$

64. $2\sqrt[3]{2\,a^2} \div \sqrt[3]{4\,a}.$

65. $-\frac{3}{4}\sqrt{\frac{3}{5}} \div \frac{3}{10}\sqrt{3}.$

66. $6\sqrt[3]{54\,x^2} \div 2\sqrt[3]{2\,x^2}.$

67 $a^2\sqrt[4]{48\,ab^3} \div 2\,ab\sqrt[4]{3\,ab^2}.$

68. $4\,ax \div \sqrt[4]{ax}.$

69. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}.$

70. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}.$

71. $\dfrac{\sqrt{x}-\sqrt{x+y}}{\sqrt{x}+\sqrt{x+y}}.$

72. $\dfrac{\sqrt{a^2+b^2}+b}{\sqrt{a^2+b^2}-b}.$

73. $\dfrac{3\sqrt{x-3}+\sqrt{x+3}}{3\sqrt{x-3}-\sqrt{x+3}}.$

74. $\dfrac{1}{a-\sqrt{a^2-x^2}}.$

75. $\dfrac{p-\sqrt{q}}{p+\sqrt{q}}.$

76. $\dfrac{1}{2+\sqrt{5}-\sqrt{2}}.$

77. $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}.$

78. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}.$

Solve:

79. $3x+\sqrt{x^2-2x+5}=1.$ 80. $\frac{\sqrt{x}+a}{b}=\frac{x}{a^2-2ab+b^2}.$

81. $\frac{x}{a+b}+\sqrt{(a+b)^2+ab-x}=a+b+\frac{2ab}{a+b}.$

82. $\sqrt{x}\,(2a-b+\sqrt{x})=3a^2-ab.$

Extract the square root of:

83. $7-2\sqrt{10}.$ 84. $a+2b+\sqrt{8ab}.$

Extract the cube root of:

85. $8x^3-12x^2y^{\frac{1}{3}}+6xy^{\frac{2}{3}}-y.$

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Explain the meaning of the exponent in $a^{\frac{p}{q}}$. Sec. 504.

2. Explain the meaning of the exponent zero. Sec. 512.

3. Explain the meaning of the exponent in a^{-n}. Sec. 513.

4. State the *Law of Exponents* for multiplication; for division; for involution. Secs. 496–499.

5. How are expressions multiplied whose bases are different and whose exponents are different? Secs. 510, 514.

6. How may an expression with negative exponents be changed to an equal one with positive exponents? Sec. 515, I.

7. How free an expression from the fractional form by the use of exponents? Sec. 515, II.

8. How may a factor be transferred from one term of a fraction to the other by use of an exponent? Sec. 515, III.

9. How may the square root of a binomial surd be found by inspection? Sec. 517.

10. What formula may be used to extract the cube root of a number or of an algebraic expression? Sec. 520.

HISTORICAL NOTE

The use of exponents to denote powers and roots seem so essential to our present-day work in algebra that it is difficult to imagine how algebra could exist without this notation; but their general use is comparatively recent. Vieta (1591), the founder of modern algebra, had no knowledge of them. In place of our notation he used a system much like that of Diophantos. Thus, for the expression

$$x^3 - 2x^2 + 5x - 3 = 7,$$

Vieta wrote $1\,C - 2\,Q + 5\,N - 3$ aequ. 7,

in which C, Q, and N are the first letters of the Latin words meaning "cube," "square," and "number." By number is meant the unknown quantity, and by C and Q are meant the square and cube of this number. In the sixteenth century Stifel used integral exponents, and Stevin invented fractional exponents, if we pass over the beginnings made by Oresme (1382), but it remained for Wallis and Newton, in the seventeenth century, to popularize these improvements.

John Wallis, born in 1616, was second only to Newton among English mathematicians of the seventeenth century, and became a professor of geometry at Oxford College at the age of 33. He was a charter member of the famous Royal Society of Great Britain, founded in 1663.

JOHN WALLIS

Exponents, like the binomial theorem and many other principles of algebra, were standardized and made popular through their application to concrete problems, for Wallis, in seeking the areas inclosed by various curves, used the series of powers, x^3, x^2, x^1, x^0, x^{-1}, x^{-2}, x^{-3} in its present meaning, thus placing upon these symbols his stamp of authority. In this way our present definitions of negative exponents and of the zero exponent were established. The fact that these interpretations satisfy the fundamental laws of algebra was shown by Peacock and Hamilton, as noted (p. 326).

CHAPTER XXVII

LOGARITHMS

MEANING AND USE OF LOGARITHMS

522. Use of Exponents in Computation. By applying the laws of exponents certain mathematical operations may be performed by means of simpler ones. The following table of powers of 2 may be used in illustrating some of these simplifications:

$1 = 2^0$	$32 = 2^5$	$1024 = 2^{10}$	$32768 = 2^{15}$
$2 = 2^1$	$64 = 2^6$	$2048 = 2^{11}$	$65536 = 2^{16}$
$4 = 2^2$	$128 = 2^7$	$4096 = 2^{12}$	$131072 = 2^{17}$
$8 = 2^3$	$256 = 2^8$	$8192 = 2^{13}$	$262144 = 2^{18}$
$16 = 2^4$	$512 = 2^9$	$16384 = 2^{14}$	$524288 = 2^{19}$

523. Application of Law I, Section 496, p. 355.

EXAMPLES

1. Find: $8 \cdot 32$.

From the table, $8 = 2^3$, *(1)*

and $32 = 2^5$. *(2)*

Then, $8 \cdot 32 = 2^3 \cdot 2^5 = 2^8$, *(3)*

and, according to the table, $2^8 = 256$. *(4)*

2. Find: $2048 \cdot 64$.

From the table, $2048 = 2^{11}$, *(1)*

and $64 = 2^6$. *(2)*

Then, $2048 \cdot 64 = 2^{11} \cdot 2^6 = 2^{17}$, *(3)*

and, according to the table, $2^{17} = 131072$. *(4)*

Thus, the process is simply one of inspection. In the above example we merely added 11 and 6 and looked in the table for the number opposite to 2^{17}.

ORAL EXERCISES

State the following products by reference to the table:

1. $16 \cdot 256$.
2. $32 \cdot 128$.
3. $64 \cdot 512$.
4. $8 \cdot 2048$.
5. $32 \cdot 32$.
6. $64 \cdot 64$.
7. $32 \cdot 2048$.
8. $16 \cdot 4096$.
9. $128 \cdot 512$.
10. $128 \cdot 1024$.
11. $8 \cdot 16384$.
12. $32 \cdot 4096$.

524. Application of Law II, Section 497, p. 356.

EXAMPLES

1. Find: $\frac{256}{32}$.

From the table, $256 = 2^8$, (1)

and $32 = 2^5$. (2)

Hence, $\frac{256}{32} = \frac{2^8}{2^5} = 2^{8-5} = 2^3$, (3)

and, according to the table, $2^3 = 8$. (4)

2. Find: $\frac{65536}{2048}$.

As above, $\frac{65536}{2048} = \frac{2^{16}}{2^{11}} = 2^5 = 32$.

ORAL EXERCISES

By use of the table determine the value of the following:

1. $\frac{1024}{128}$.
2. $\frac{8192}{64}$.
3. $\frac{32768}{1024}$.
4. $\frac{64 \cdot 512}{16 \cdot 128}$.
5. $\frac{32 \cdot 2048}{512}$.
6. $\frac{128 \cdot 131072}{64 \cdot 8 \cdot 8192}$.

525. Application of Law III, Section 498, p. 356.

EXAMPLES

1. Find: 16^3.

By the table, $16 = 2^4$. (1)

Hence, $16^3 = (2^4)^3 = 2^{12}$. (2)

and, according to the table, $2^{12} = 4096$. (3)

2. Find: $\sqrt{1024}$.

As above, $1024 = (1024)^{\frac{1}{2}} = (2^{10})^{\frac{1}{2}} = 2^5 = 32$, according to the table.

3. Find: $\sqrt[5]{32768}$.

As above, $\sqrt[5]{32768} = (2^{15})^{\frac{1}{5}} = 2^3 = 8$.

ORAL EXERCISES

By use of the table find the value of:

1. 32^3.	4. 64^2.	7. $\sqrt{8192}$.	10. $512^{\frac{2}{3}}$.
2. 3^5.	5. 256^2.	8. $\sqrt[3]{4096}$.	11. $\sqrt[3]{32768}$.
3. 32^5.	6. 16^4.	9. $\sqrt[4]{65536}$.	12. $\sqrt[5]{1024}$.

526. The examples and exercises above show that the laws of exponents furnish a powerful and remarkably easy way of making certain computations.

In the above illustrations we have used a table based on the number 2, and have limited the table to integral exponents; but for practical purposes a table based on 10 is used and is made to include fractional exponents.

For example:

1. It is known that approximately,

$$2 = 10^{\frac{3}{10}} \text{ or } 10^{.3} \text{ (more accurately } 10^{.301}\text{)}.$$

From this we can express 20 as a power of 10, for

$$20 = 10 \cdot 2 = 10^1 \cdot 10^{.301} = 10^{1.301}.$$

Similarly, $$200 = 10 \cdot 20 = 10^1 \cdot 10^{1.301} = 10^{2.301},$$

and $$2000 = 10 \cdot 200 = 10^1 \cdot 10^{2.301} = 10^{3.301}.$$

2. It is known that approximately $763 = 10^{2.88}$.

Then $$7630 = 10 \cdot 763 = 10^1 \cdot 10^{2.88} = 10^{3.88},$$

and $$76300 = 100 \cdot 763 = 10^2 \cdot 10^{2.88} = 10^{4.88}.$$

Similarly, $$76.3 = \frac{763}{10} = \frac{10^{2.88}}{10^1} = 10^{2.88-1} = 10^{1.88},$$

and $$7.63 = \frac{763}{100} = \frac{10^{2.88}}{10^2} = 10^{2.88-2} = 10^{0.88}.$$

WRITTEN EXERCISES

Given $48 = 10^{1.68}$; express as a power of 10:

1. 480.
2. 4800.
3. 48,000.
4. 4.8.

Given $649 = 10^{2.81}$; express as a power of 10:

5. 6490.
6. 64,900.
7. 649,000.
8. 64.9.
9. 6.49.
10. 649,000,000.

Given $300 = 10^{2.47}$; express as a power of 10:

11. 3.
12. 30.
13. 3000.
14. 30,000.
15. 300,000.
16. 3,000,000.

527. The use of the base 10 has several advantages.

I. The exponents of numbers not in the table can readily be found by means of the table.

To make this clear, let us suppose that a certain table expresses all integers from 100 to 999 as powers of 10; then 30, although not in this table, can be expressed as a power of 10 by reference to the table.

For, $30 = \frac{300}{10}$, and since 300 is in the supposed table we may find by reference to the table that $300 = 10^{2.47}$, and hence, $30 = \frac{10^{2.47}}{10^1} = 10^{1.47}$.

Similarly, 3.76 is not in the supposed table, but 376 is and $3.76 = \frac{376}{100} = \frac{376}{10^2}$. Therefore it is necessary only to subtract 2 from the power of 10 found for 376 in order to find the power of 10 equal to 3.76.

Similarly, 4680 is not in the table, but 468 is, and $4680 = 468 \cdot 10^1$. Therefore it is necessary only to add 1 to the power of 10 found for 468 in order to find the power of 10 equal to 4680.

Such a table would not enable us to express in powers of 10 numbers like 4683, 46.83, and 356,900, but only numbers of 3 or fewer digits, which may be followed by any number of zeros.

Similar conditions would apply to a table of powers for numbers from 1000 to 9999, from 10,000 to 99,999, and so on.

II. The integral part of the exponent can be written without reference to a table.

For example:

1. 879 is greater than 100, which is the second power of 10, and less than 1000, or the third power of 10. That is, 879 is greater than 10^2 but less than 10^3. Therefore the exponent of the power of 10 which equals 879 is 2. + a decimal.

2. Similarly, 87.9 lies between 10 and 100, or between 10^1 and 10^2, hence the exponent of the power of 10 that is equal to 87.9 is 1. + a decimal.

ORAL EXERCISES

State the integral part of the exponent of the power of 10 equal to each of the following:

1. 35.	4. 25.	7. 25.5.
2. 350.	5. 2500.	8. 365.5.
3. 36.5.	6. 36,500.	9. 17.65.

III. If two numbers have the same sequence of digits but differ in the position of the decimal point, the exponents of the powers of 10 which they equal have the same decimal part.

For example:

Given that $274.3 = 10^{2.43}$,

we have $$27.43 = \frac{274.3}{10} = \frac{10^{2.43}}{10^1} = 10^{1.43},$$

also $$2743 = 10 \cdot 274.3 = 10^1 \cdot 10^{2.43} = 10^{3.43},$$

also $$274,300 = 1000 \cdot 274.3 = 10^3 \cdot 10^{2.43} = 10^{5.43}.$$

In each instance the decimal part of the exponent is the same. It is evident that this will be the case in all similar instances, for shifting the decimal point is equivalent to multiplying or dividing repeatedly by 10, which is equivalent to changing the integral part of the exponent by adding or subtracting an integer.

ORAL EXERCISES

Given $647 = 10^{2.81}$, state the decimal part of the exponent of the power of 10 that equals:

1. 64.7.	3. 6470.	5. 647,000.
2. 6.47.	4. 64,700.	6. 6,470,000.

Given $568.1 = 10^{2.75}$, state the decimal part of the exponent of the power of 10 that equals:

7. 56.81.
8. 5.681.
9. 5681.
10. 56,810.
11. 568,100.
12. 56,810,000.

528. Logarithms. Exponents when used in this way for computation are called **logarithms**, abbreviated **log**.

529. The number to which the exponents are applied is called the **base**.

For the purposes of computation the base used is 10.

According to the above definition the equation $30 = 10^{1.48}$ may be written $\log 30 = 1.48$, which is read "The logarithm of 30 is 1.48." These equations mean the same thing; namely, that 1.48 is (approximately) the power of 10 that equals 30.

WRITTEN EXERCISES

Write the following in the notation of logarithms:

1. $700 = 10^{2.84}$.
2. $75 = 10^{1.87}$.
3. $6 = 10^{0.77}$.
4. $50 = 10^{1.69}$.
5. $361 = 10^{2.55}$.
6. $45 = 10^{1.65}$.

Write the following as powers of 10:

7. $\log 20 = 1.3$.
8. $\log 500 = 2.70$.
9. $\log 3 = 0.47$.
10. $\log 7 = 0.84$.
11. $\log 111 = 2.04$.
12. $\log 21 = 1.32$.

530. Characteristic and Mantissa. The integral part of a logarithm is called its **characteristic**, and the decimal part its **mantissa**.

The characteristic of the logarithms of a number greater than unity is one less than the number of digits at the left of the decimal point.

ORAL EXERCISES

1–12. State the characteristic and the mantissa in each of the logarithms in Exercises 1–12 above.

531. Since the characteristics of logarithms can be determined by inspection, tables of logarithms furnish only the mantissas.

EXPLANATION OF THE TABLES

532. The use of the tables, pp. 389 and 390, is best seen from an example.

Find the logarithm of 365.

The first column in the table, p. 389, contains the first two figures of the numbers whose mantissas are given in the table, and the top row contains the third figure.

Hence, find 36 in the left-hand column, p. 389, and 5 at the top.

In the column under 5 and opposite to 36 we find 5623, the required mantissa.

Since 365 is greater than 100 (or 10^2) but less than 1000 (or 10^3), the characteristic of the logarithm is 2.

Therefore, log 365 is 2.5623.

WRITTEN EXERCISES

By use of the table find the logarithms of:

1. 25.	**5.** 99.	**9.** 9.9.	**13.** 1000.
2. 36.	**6.** 86.	**10.** 8.6.	**14.** 5000.
3. 50.	**7.** 999.	**11.** 33,000.	**15.** 505.
4. 75.	**8.** 800.	**12.** 99,900.	**16.** 5.05.

533. Negative Characteristics. An example will serve to show how negative characteristics arise:

From log 346 = 2.5391, we find,

$$\log 34.6 = \log \frac{346}{10} = \log 346 - \log 10 = 2.5391 - 1 = 1.5391.$$

$$\log 3.46 = \log \frac{34.6}{10} = 1.5391 - 1 = 0.5391.$$

$$\log .346 = \log \frac{3.46}{10} = 0.5391 - 1.$$

In the last line we have a positive decimal less 1, and the result is a negative decimal; viz. $-.4609$. But to avoid this change of mantissa, it is customary not to carry out the subtraction, but simply to indicate it. It might be written $-1 + .5391$, but it is customarily abridged into $\bar{1}.5391$. The mantissa is kept positive in all logarithms. The logarithm $\bar{1}.5391$ says that the corresponding number is greater than 10^{-1} (or $\frac{1}{10}$) but less than 10^0 or 1.

We now write $\log\ .346 = \bar{1}.5391.$

Similarly, $\log .0346 = \log \frac{.346}{10} = \bar{1}.5391 - 1 = \bar{2}.5391.$

Thus we see that the mantissa remains the same, no matter how the position of the decimal point is changed. The mantissa is determined solely by the sequence of digits constituting the number.

The characteristic is determined solely by the position of the decimal point. The characteristics of the logarithm of a number smaller than unity is negative and equals the number of the place occupied by the first significant figure of the decimal.

EXAMPLES

1. What is the characteristic of log .243?

.243 is more than .1 or 10^{-1} but less than 1 or 10^0. Hence,

$$.243 = 10^{-1+\text{a decimal}}.$$

The characteristic is $\bar{1}$.

2. Similarly, since .00093 is greater than .0001 or 10^{-4} but less than .001 or 10^{-3},

$$\log .00093 = \bar{4} + \text{a decimal.}$$

534. The characteristic having been determined, the mantissa is found from the table in the usual way.

For example:

$$\log\ .243 = \bar{1}.3856,$$
$$\log .00093 = \bar{4}.9685.$$

WRITTEN EXERCISES

Find the logarithms of:

1. .35.	3. .105.	5. .0023.	7. .00342.
2. .634.	4. .027.	6. .0123.	8. .0004.

535. In finding the number corresponding to a logarithm with negative characteristic, the same method is followed as when the characteristic is positive. The mantissa determines the sequence of digits constituting the number; the characteristic fixes the position of the decimal point.

For example:

If $\log n = \bar{2}.5955$, the digits of n are 394. The characteristic $\bar{2}$ says that n is greater than 10^{-2} (or .01) but less than 10^{-1} (or .1). Hence, the decimal point must be so placed that n has no tenths but some hundredths. Therefore $n = .0394$.

WRITTEN EXERCISES

By use of the tables find the numbers whose logarithms are:

1. $\bar{1}.6232$.	7. $\bar{2}.0792$.	13. $\bar{3}.0969$.
2. $\bar{1}.4914$.	8. $\bar{2}.7076$.	14. $\bar{1}.6972$.
3. 2.4281.	9. 4.9196.	15. 3.9284.
4. 1.9196.	10. 3.2201.	16. 2.9284.
5. 0.9196.	11. 0.2201.	17. 1.9284.
6. 3.4281.	12. 1.2201.	18. 5.7832.

Find n if:

19. $\log n = 1.9289$.

20. $\log n = 0.9289$.

21. $\log (n-1) = 3.9294$.

22. $\log (\frac{1}{2} n) = 1.6128$.

USE OF THE TABLES FOR COMPUTATION

536. For use in computation by logarithms the laws of exponents may be expressed thus:

I. $10^m \cdot 10^r = 10^{m+r}$. *The logarithm of a product is the sum of the logarithms of the factors.*

II. $\dfrac{10^m}{10^r} = 10^{m-r}$. *The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.*

III. $(10^m)^r = 10^{mr}$. *The logarithm of a number with an exponent is the product of the exponent and the logarithm of the number.*

Since r may be positive or negative, integral or fractional, Law III provides not only for raising to integral powers, but also for finding reciprocals of such powers, and for extracting roots.

EXAMPLES

1. Multiply 21 by 37.

    1. $\log 21 = 1.3222$, table, p. 390.
    2. $\log 37 = 1.5682$, table, p. 390.
    3. Adding, $\log 21 + \log 37$, or $\log(21 \times 37) = 2.8904$ (Sec. 536).
    4. $\therefore 21 \times 37 = 777$ from table, p. 391.

2. Divide 814 by 37.

    1. $\log 814 = 2.9106$, table, p. 391.
    2. $\log 37 = 1.5682$, table, p. 390.
    3. $\therefore \log 814 - \log 37 = 2.9106 - 1.5682 = 1.3424$.
    4. $\therefore 814 \div 37 = 22$, table, p. 390.

3. Extract the cube root of 729.

    1. $\sqrt[3]{729} = (729)^{\frac{1}{3}}$.
    2. $\log 729^{\frac{1}{3}} = \frac{1}{3} \log 729$ (Sec. 536).
    3. $\log 729 = 2.8627$, table, p. 391.
    4. $\frac{1}{3}$ of $2.8627 = 0.9542$.
    5. $\therefore (729)^{\frac{1}{3}}$ or $\sqrt[3]{729} = 9$, table, p. 391.

WRITTEN EXERCISES

Compute by use of logarithms:

1. 8×15.
2. 41×23.
3. 37×17.
4. 12×17.
5. $893 \div 19$.
6. 5^4.
7. 31^2.
8. $\sqrt{484}$.
9. $\sqrt[3]{343}$.
10. $940 \div 47$.
11. 19^2.
12. 7^3.
13. 4^4.
14. $\sqrt{196}$.
15. $\sqrt[3]{216}$.

Since the logarithm is approximate, the result in general is approximate. Thus, $\log \sqrt[4]{(256)} = 0.60205$, which is not the logarithm of 4, but is sufficiently near to be recognized in the table.

537. How to calculate the logarithm of a number not found in the table is best seen from an example.

EXAMPLE

Find the logarithm of 257.3 (see 16th line of table, p. 390):

	0	1	2	3	4	5	6	7	8	9
25	3979	3997	4014	4031	4048	4065	4082	**4099**	**4116**	4133

The numbers nearest to 257.3, whose logarithms are given in the table, are 257 and 258. We have

$$\begin{array}{rl} \log 258 = & 2.\mathbf{4116} \\ \log 257 = & 2.\mathbf{4099} \\ \hline \text{Difference,} & .0017 \end{array}$$

That is, an increase of 1 in the number causes an increase of .0017 in the logarithm. Assuming that an increase of .3 in the number would cause an increase of .3 of .0017, or .0005, in the logarithm, then

$$\log 257.3 = 2.4099 + .0005 = 2.4104.$$

NOTES: 1. The product of .3 and .0017 is .00051, but we take only four places, because the mantissas as given in the table are expressed to four places only. If the digit in the fifth decimal place of the correction is more than 5, we replace it by a unit in the fourth place.

2. The difference between two succeeding mantissas of the table (called the **tabular difference**) can be seen by inspection.

3. What is written in finding the logarithm of 257.3 should be at most the following:

$$\begin{array}{rr} \log 257 = & 2.4099 \\ \text{correction for } .3 = & 5 \\ \hline \log 257.3 = & 2.4104 \end{array} \qquad \begin{array}{rr} \text{tabular difference} & 17 \\ & .3 \\ \hline & 5.1 \end{array}$$

4. The corrections are made on the assumption that the change in the logarithm is proportional to the change in the number. This is sufficiently accurate when used within the narrow limits here prescribed.

WRITTEN EXERCISES

Find the logarithm of:

1. 1235.	5. 1425.	9. 3.142.	13. .4071.	17. .3002.
2. 23.5.	6. 1837.	10. 1.414.	14. 85.51.	18. 9009.
3. 2.36.	7. 6720.	11. 1.732.	15. .0125.	19. 12.02.
4. .0237.	8. 67.25.	12. .6226.	16. .4267.	20. 5.008.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

538. To calculate the number whose logarithm is given apply the table as follows:

(1) If the given logarithm is in the table, the number can be seen at once.

(2) If the given logarithm is not in the table, the number corresponding to the nearest logarithm of the table may be taken.

A somewhat closer approximation may be found by using the method of the following example:

EXAMPLE

Find the number whose logarithm is 1.4271. The mantissas nearest to this are found in the 17th line of table, p. 389.

	0	1	2	3	4	5	6	7	8	9
26	4150	4166	4183	4200	4216	4232	4249	**4265**	**4281**	4298

The process is the reverse of that of finding the logarithm.

The next smaller mantissa in the table is 4265, corresponding to the number 267. The difference between this mantissa and the given mantissa is 6. The tabular difference between 4265 and the next larger mantissa is 16. An increase of 16 in the logarithm corresponds to an increase of .1 in the number. Hence, an increase of 6 in the logarithm corresponds to an increase of $\frac{6}{16}$ of 1, or .4, in the number. This means .4 of one unit in the number 267. What its place value is in the final result depends upon the characteristic. The digits of the result are 2674.

The characteristic 1 shows that the desired number is greater than the first power of 10, but less than the second power of 10 or 100. Hence, the decimal point must be placed between 6 and 7, and the final result is 26.74.

NOTES: 1. For reasons similar to those of Note 1, p. 391, the correction should be carried to one place only.

2. At most the following should be written:

$$\begin{array}{rl} \log n = 1.4271 & \text{tab. diff. } 16 \\ \text{mantissa for } 267 = \underline{\ \ .4265} & \frac{6}{16} = .4 \\ \text{diff.} \quad\quad 6 & \end{array}$$

Therefore, $n = 26.74.$

WRITTEN EXERCISES

Find the number whose logarithm is:

1. 0.7305.	4. 2.9023.	7. $\bar{1}.1962$.	10. $\bar{3}.9485$.
2. 0.5029.	5. 3.1467.	8. $\bar{2}.0342$.	11. 4.6987.
3. 1.4682.	6. 3.6020.	9. $\bar{3}.3920$.	12. $\bar{2}.6376$.

539. Exponential Equations. When the unknown quantity occurs in an equation as an exponent, the equation is called an **exponential equation.**

EXAMPLE

1. Solve: $$3^x = 729. \quad (1)$$

Taking the log of both members, $$\log 3^x = \log 729. \quad (2)$$

Or, $$x \cdot \log 3 = \log 729. \quad \text{Sec. 536, III.} \quad (3)$$

Dividing by log 3, $$x = \frac{\log 729}{\log 3} = \frac{2.8627}{.4771} = 6. \quad (4)$$

TEST: $$3^6 = 729.$$

The test is especially important because the division in step (*4*) is not exact. The approximation is so close, however, that the correct number will always be suggested, and the test will finally verify it.

It should also be noted that step (*4*) contains the quotient of two logarithms, which is not the same as the logarithm of the quotient; hence, Sec. 536, II, is not applicable.

2. Solve: $$8^{7-2y} = 512. \quad (1)$$

$$(7 - 2y) \log 8 = \log 512. \quad (2)$$

$$\therefore 7 - 2y = \frac{\log 512}{\log 8} = \frac{2.7093}{.9031} = 3. \quad (3)$$

$$\therefore -2y = -4, \text{ or } y = 2. \quad (4)$$

TEST: $$8^{7-2\cdot 2} = 8^3 = 512.$$

WRITTEN EXERCISES

Solve:

1. $2^x = 512$.
2. $3^y = 243$.
3. $2^{2x} = 256$.
4. $8^w = 4096$.
5. $10^{z-2} = 1000$.
6. $10^y = 500$.

7. $12^x = 20{,}736$.

8. $9^{6-x} = 243$.

9. $7^{2x} = 343$.

10. $15^{a-x} = 3375$.

11. $a^x = b^{x-1}$.

12. $c^{x+1} = e^{-x}$.

13. Solve for x and y, computing the values to 2 decimal places:

$$\begin{cases} 3^x = 2^y. \\ 3^{x-2} = 8^y. \end{cases}$$

REVIEW

WRITTEN EXERCISES

Find exactly or approximately by use of logarithms the value of:

1. $\sqrt{2}$.

2. $\sqrt{5}$.

3. $\sqrt[3]{9}$.

4. $\sqrt[5]{7}$.

5. $\sqrt{926}$.

6. $\sqrt{656}$.

7. $\sqrt[3]{756}$.

8. $\sqrt[5]{812}$.

9. $(1.5)^4$.

10. $(1.03)^7$.

11. $(1.04)^{10}$.

12. $(1.06)^9$.

13. $\dfrac{(164)(798)}{779}$.

14. $\dfrac{(732)(774)}{(731)(671)}$.

15. $\sqrt{(624)(598)(178)}$.

16. $\sqrt{\dfrac{(651)(654)(558)}{763}}$.

17. It is known that the volume of a sphere is $\frac{4}{3}\pi r^3$, r being the length of the radius. Using 3.14 as the approximate value of π, find by logarithms the volume of a sphere of radius 7.3 in.

18. Find, as above, the volume of a sphere whose radius is 36.4 ft.

Calculate by logarithms:

19. $\dfrac{(132)(1837)}{167}$.

20. $\dfrac{(2076)(379)}{173}$.

21. $\dfrac{(3059)(349)}{(19)(23)(2443)}$.

22. $\sqrt{\dfrac{(294)(1842)}{307}}$.

23. $\sqrt{\dfrac{(2373)(675)}{113}}$.

24. $\sqrt[3]{.0578}$.

25. $\dfrac{(15.61)^2}{(700)^{\frac{1}{3}}}$.

26. $\sqrt{\dfrac{(7688)(7719)}{(248)(249)}}$.

27. $(217.6)(.00681)$.

28. $\dfrac{[\sqrt{278.2}(2.578)]^2}{\sqrt[3]{.00231}\cdot\sqrt{76.19}}$.

29. Given $a = 0.4916$, $c = 0.7544$, and $b = c^2 - a^2$. Find b.

SUGGESTION. $b = (c - a)(c + a)$.

30. It is known that in steam engines, the piston head's average velocity (c) per second is approximately given by the formula:

$$c = 1.7\sqrt[3]{s\sqrt{\frac{p}{15}}},$$

where s denotes the distance over which the piston moves (expressed in the same unit as c), and p the number of pounds pressure in the cylinder.

(1) Find c, if $s = 32.5$ in., $p = 110$ lb.

(2) Find p, if $c = 15$ ft., $s = 2.6$ ft.

31. Solve for x and y, computing decimal results to 2 decimal places:

$$2^x - 6^y = 0,$$
$$2^{x+1} - 12^y = 0.$$

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What is a *logarithm?* Sec. 528.

2. What is the meaning of *base* in logarithms? Sec. 529.

3. What is the integral part of a logarithm called? What is the decimal part called? Sec. 530.

4. State how to find the logarithm of a product. Sec. 536, I.

5. State how to find the logarithm of a quotient. Sec. 536, II.

6. How is the logarithm of a power found? Sec. 536, III.

7. State each of the processes named in statements 4, 5, and 6 in the form of laws of exponents. Sec. 536.

HISTORICAL NOTE

It is remarkable that the discovery of Logarithms occurred so timely, for this great machine of computation was invented by Napier at the beginning of the seventeenth century, just in time to aid the new work in astronomy and navigation; Galileo had devised the telescope and Kepler was ready to calculate the orbits of the planets. It is also remarkable that Napier worked out the principle of Logarithms without the use of exponents, and this peculiar method, which is too complex to be explained here, produced tables quite different from those now in use. As soon as Napier's great work on Logarithms was published, Henry Briggs, a teacher in Gresham College, London, hastened to visit Napier, and suggested the advantages of the base 10, and thus laid the foundation of our tables of common Logarithms.

John Napier, also known as the Baron of Merchiston, was a Scotchman, born in 1550, and published his work on Logarithms in 1614, only three years before his death. The modesty and simplicity of the great Scottish philosopher is shown by his attitude toward Briggs, for when informed that the latter had been obliged to postpone his promised visit, Napier regretfully replied, "Ah, Mr. Briggs will not come."

JOHN NAPIER

Napier's discovery, whose importance is not exaggerated by the claim that "it doubled the life of the astronomer by shortening his labor," followed immediately upon the general acceptance of the Hindoo notation and the introduction by Stevin of decimal fractions. Thus, the seventeenth century saw the perfection of the three greatest instruments of modern calculation, the Hindoo notation, decimal fractions, and Logarithms.

CHAPTER XXVIII

IMAGINARY AND COMPLEX NUMBERS

540. Imaginary Numbers. The numbers defined in what precedes have all had positive squares. Consequently, among them the equation $x^2 = -3$, which asks, "What is the number whose square is -3?" has no solution.

A solution is provided by defining a new number, $\sqrt{-3}$, as a number whose square is -3. Similarly we define $\sqrt{-a}$, a being a positive number, as a number whose square is $-a$.

The square roots of negative numbers are called **imaginary numbers.**

541. If a is positive, $\sqrt{-a}$ may be expressed $\sqrt{a}\sqrt{-1}$.

Similarly, $\sqrt{-49} = \sqrt{49(-1)} = 7\sqrt{-1}$.

542. Real Numbers. In distinction from imaginary numbers, the numbers hitherto studied are called **real numbers.**

WRITTEN EXERCISES

Express as in Section 541:

1. $\sqrt{-9}$.
2. $\sqrt{-16}$.
3. $\sqrt{-25}$.
4. $\sqrt{-100}$.
5. $-\sqrt{-64}$.
6. $\sqrt{-8}$.
7. $\sqrt{-18}$.
8. $-\sqrt{-32}$.
9. $-\sqrt{-7}$.
10. $\sqrt{-12}$.
11. $\sqrt{-50}$.
12. $-\sqrt{-75}$.

543. The positive square root of -1 is frequently denoted by the symbol i; that is, $\sqrt{-1} = i$.

Using this we write:

$$\sqrt{-5} = \sqrt{5} \cdot i; \quad \sqrt{-49} = \pm 7i; \text{ also } \sqrt{-75a^2b} = 5a\sqrt{3b} \cdot i.$$

NOTE. Throughout this chapter the radical sign is taken as positive.

WRITTEN EXERCISES

Rewrite the following, using the symbol i as in Section 543:

1. $2+\sqrt{-4}$.
2. $3-\sqrt{-9}$.
3. $4+\sqrt{-4}$.
4. $5-\sqrt{-16}$.
5. $25-\sqrt{-25}$.
6. $5-\sqrt{-3}$.
7. $3+\sqrt{-6}$.
8. $7+\sqrt{-12}$.
9. $12-\sqrt{-9}$.
10. $2\sqrt{-100}$.
11. $4\sqrt{-(a+b)}$.
12. $\sqrt{a}+\sqrt{-b^2c^2}$.
13. $-\sqrt{-b^2c}$.
14. $a+\sqrt{-(a^2+x^2)}$.
15. $x+y-\sqrt{-xy^2}$.
16. $p^2+\sqrt{-(p+q)^3}$.
17. $\sqrt{x}-\sqrt{-(a+b)^2}$.
18. $\sqrt{x+y}+\sqrt{-(x+y)^4}$.

544. Complex Numbers. A binomial one of whose terms is real and the other imaginary is called a **complex number.**

The general form of a complex number is $a+bi$, where a and b may be any real numbers.

NOTE. Complex numbers are also simply called **imaginary,** any expression which involves i being called imaginary. Single terms in which i is a factor (those which we have called imaginary above) are often called **pure imaginaries,** while the others are called **complex imaginaries.** Thus, $\sqrt{-2}$, $3\sqrt{-a}$, $5i$ are pure imaginaries and $1-\sqrt{-3}$, $a-\sqrt{-b}$ are complex imaginaries.

545. Processes with Imaginary and Complex Numbers. After introducing the symbol i for the imaginary unit $\sqrt{-1}$, the operations with imaginary and complex numbers are performed like the operations with real numbers.

I. *Addition and Subtraction.*

EXAMPLE

Add $\sqrt{-9}$, $-\sqrt{-25}$, $\sqrt{-3}$.

$$\sqrt{-9}=3i.$$
$$-\sqrt{-25}=-5i.$$
$$\sqrt{-3}=\sqrt{3}\cdot i.$$

$\therefore$ the sum is $(3-5+\sqrt{3})\,i=-(2-\sqrt{3})\,i$.

WRITTEN EXERCISES

Add:

1. $2i, 3i, -i$.
2. $\sqrt{16}\,i, -2i$.
3. $\sqrt{-16}, -2\sqrt{-1}$.
4. $\sqrt{-4}, \sqrt{-9}, \sqrt{-1}$.
5. $6-\sqrt{-5}, 2\sqrt{-4}, \sqrt{-25}$.
6. $3+4i, 2-3i, 5+5i$.
7. $\sqrt{-9x^2}, -\sqrt{-8x^2}$.
8. $\sqrt{-(a+b)^2}, -\sqrt{(b+c)^2}$.
9. $2\sqrt{-32a^3}, 3\sqrt{-8a^3}, 6\sqrt{2}\,i$.
10. $\sqrt{3}\,i-1, \sqrt{2}\,i+2, i-2\sqrt{2}$.

II. *Multiplication.*

To multiply complex numbers we apply the relation that $\sqrt{-1}\cdot\sqrt{-1}=-1$, or $i^2=-1$, since the square of the square root of a number is the number itself.

EXAMPLES

Multiply:

1. $\sqrt{-16}$ by $\sqrt{-9}$.

$$\sqrt{-16}=4\sqrt{-1}=4i.$$
$$\sqrt{-9}\ =3\sqrt{-1}=3i.$$

$\therefore$ the product is $12(\sqrt{-1})^2=(12)(-1)=-12$.
This may be written $(4i)(3i)=12i^2=-12$.

2. $3-\sqrt{-3}$ by $2-\sqrt{-5}$.

$$\begin{array}{l} 3-\sqrt{3}\,i \\ 2-\sqrt{5}\,i \\ \hline 6-2\sqrt{3}\,i \\ \quad -3\sqrt{5}\,i+\sqrt{15}\,i^2 \\ \hline 6-(2\sqrt{3}+3\sqrt{5})i-\sqrt{15}. \end{array}$$

3. $a+bi$ by $a-bi$.

$$\begin{array}{ll} a+bi & \\ a-bi & \\ \hline a^2+abi & \\ \quad -abi & -b^2i^2 \\ \hline a^2 & +b^2. \end{array}$$

546. $a+bi$ and $a-bi$ are called **conjugate** complex numbers.

WRITTEN EXERCISES

Multiply:

1. $5-3i$ by $5+3i$.
2. $3+\sqrt{-3}$ by $2+\sqrt{-5}$.
3. $5-2\sqrt{-1}$ by $3+2\sqrt{-1}$.
4. $5+\sqrt{-3}$ by $5-\sqrt{-3}$.
5. $3-\sqrt{-2}$ by $3+2\sqrt{-2}$.
6. $1-\sqrt{-7}$ by $2+3\sqrt{-7}$.

7. $4+i$ by $5-i$.

8. $a+xi$ by $a-xi$.

9. a^2+b^2i by a^2-b^2i.

10. $\sqrt{r}+3i$ by $\sqrt{r}-3i$.

11. $\sqrt{-25}$ by $\sqrt{-9}$ by $\sqrt{-5}$.

12. $\sqrt{-a}$ by $\sqrt{-b}$ by $-ci$.

III. *Division.*

Fractions (that is, indicated quotients) may be simplified by rationalizing the denominator (Sec. 364, p. 264).

For example:

1. $$\frac{\sqrt{-7}}{\sqrt{-5}}=\frac{\sqrt{-7}\sqrt{-5}}{\sqrt{-5}\sqrt{-5}}=\frac{\sqrt{7}\cdot\sqrt{5}(-1)}{-5}=\frac{\sqrt{35}}{5}.$$

2. $$\frac{2+\sqrt{-3}}{3-\sqrt{-5}}=\frac{(2+\sqrt{-3})(3+\sqrt{-5})}{(3-\sqrt{-5})(3+\sqrt{-5})}=\frac{6+3\sqrt{-3}+2\sqrt{-5}-\sqrt{15}}{9-(-5)}$$
$$=\frac{1}{14}(6+3\sqrt{-3}+2\sqrt{-5}-\sqrt{15}).$$

3. $$\frac{x+yi}{x-yi}=\frac{(x+yi)^2}{(x-yi)(x+yi)}=\frac{x^2+2xyi-y^2}{x^2+y^2}.$$

WRITTEN EXERCISES

Write in fractional form and rationalize the denominators:

1. $\sqrt{-6}\div\sqrt{2}$.

2. $1\div(a+xi)$.

3. $\sqrt{-3}\div\sqrt{-5}$.

4. $\sqrt{ax}\div\sqrt{-a}$.

5. $1\div(2-\sqrt{-3})$.

6. $4\sqrt{-1}\div-2\sqrt{-4}$.

7. $a\div(a-bi)$.

8. $(a+bi)\div(a-bi)$.

9. $(3+6i)\div(5+4i)$.

10. $(\sqrt{3}-9i)\div(\sqrt{2}-9i)$.

11. $(x-\sqrt{-7})\div(x+\sqrt{-7})$.

12. $\left(a-\frac{\sqrt{-5}}{2}\right)\div\left(a+\frac{\sqrt{-5}}{2}\right)$.

13. $(\sqrt{-2}+\sqrt{-5})\div(\sqrt{-5}-\sqrt{-2})$.

547. Powers of the Imaginary Unit. Beginning with $i^2=-1$ and multiplying successively by i we find:

$i^2=-1$.

$i^3=i^2\cdot i=-i$.

$i^4=i^2\cdot i^2=-1(-1)=+1$.

$i^5=i^4\cdot i=i$.

$i^6=i^4\cdot i^2=i^2=-1$.

$i^7=i^6\cdot i=-1\cdot i=-i$.

$i^8=i^4\cdot i^4=(+1)^2=+1$.

$i^9=i^8\cdot i=i$.

548. By means of the values of i^2, i^3, i^4, any power of i can be shown to be either $\pm i$ or ± 1.

For example: $i^{63} = i^{60} \cdot i^3 = (i^4)^{15} \cdot i^3 = 1^{15} \cdot i^3 = i^3 = -i$.

WRITTEN EXERCISES

Simplify similarly:

1. i^9.
2. i^{10}.
3. i^{12}.
4. i^{16}.
5. i^{21}.
6. i^{27}.
7. i^{54}.
8. i^{56}.
9. i^{198}.
10. i^{143}.
11. i^{400}.
12. i^{3001}.

Perform the operations indicated:

13. $(1+i)^2$.
14. $(1-i)^3$.
15. $(1-i)^3 \cdot i^4$.
16. $\left(\frac{-1+3i}{2}\right)^3$.
17. $(1+i) \cdot i^6$.
18. $\left(\frac{-1-3i}{2}\right)^3$.
19. $(1+i) \cdot (1-i)^2$.
20. $(1+i)^2 \div (1-i)^2$.

IMAGINARIES AS ROOTS OF EQUATIONS

549. Complex numbers often occur as roots of quadratic equations.

EXAMPLE

Solve: $$x^2 + x + 1 = 0. \quad (1)$$

$$x^2 + x = -1. \quad (2)$$

Completing the square, $$x^2 + x + \tfrac{1}{4} = \tfrac{1}{4} - 1. \quad (3)$$

$$\therefore x + \tfrac{1}{2} = \pm\sqrt{-\tfrac{3}{4}}. \quad (4)$$

$$\therefore x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i. \quad (5)$$

TEST: $(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i)^2 + (-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i) + 1 = 0$.

WRITTEN EXERCISES

Solve and test, expressing the imaginary roots in the form $a + bi$:

1. $x^2 + 5 = 0$.
2. $x^2 + 2x + 2 = 0$.
3. $x^2 + 2x + 37 = 0$.
4. $x^2 - 8x + 25 = 0$.
5. $x^2 - 6x + 10 = 0$.
6. $m^2 + 4m + 85 = 0$.
7. $x^2 + 10x + 41 = 0$.
8. $x^2 + 30x + 234 = 0$.

9. $y^2 - 4y + 53 = 0.$
10. $z^2 - 6z + 90 = 0.$
11. $p^2 + 20p + 104 = 0.$
12. $2x^2 + 4x + 3 = 0.$
13. $3x^2 + 2x + 1 = 0.$
14. $12t^2 + 24 = 0.$
15. $6w^2 + 30 = 0.$
16. $x^2 - x + 1 = 0.$
17. $4x^2 + 4x + 3 = 0.$
18. $x^2 + x + 5 = 0.$
19. $6x^2 + 3x + 1 = 0.$
20. $10x^2 - 2x + 3 = 0.$
21. $7t^2 - t + 1 = 0.$
22. $12x^2 + x + 1 = 0.$
23. $8t^2 + t + 6 = 0.$
24. $7x^2 + x + 5 = 0.$
25. $15z^2 + 5z - 1 = 0.$
26. $8v^2 + 3v + 6 = 0.$

550. The occurrence of imaginary roots in solving equations derived from problems often indicates the impossibility of the given conditions.

EXAMPLE

A rectangular room is twice as long as it is wide; if its length is increased by 20 ft. and its width diminished by 2 ft., its area is doubled. Find its dimensions.

SOLUTION. 1. Let $x =$ the width of the room, and $2x$ its length.
2. Then $(2x + 20)(x - 2) = 2 \cdot 2x \cdot x$, or $x^2 - 8x + 20 = 0.$
3. Solving (2), $x = 4 \pm 2i.$

The fact that the results are complex numbers shows that no actual room can satisfy the conditions of the problem.

WRITTEN EXERCISES

Solve and determine if the problems are possible:

1. In remodeling a house a room 16 ft. square is lengthened on one side a certain number of feet and decreased on the other by twice that number. If the area of the room as changed is 296 sq. ft., what are the original dimensions?

2. A triangle has an altitude 2 in. greater than its base, if it has an area of 32 sq. ft., find the length of its base.

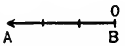
3. If a train moving x mi. per hour travels 90 mi. in $15 - x$ hours, what is its rate per hour?

GRAPHICAL REPRESENTATION

551. We have seen that positive integers and fractions can be represented by lines.

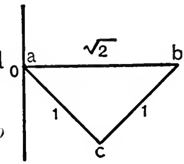
Thus, the line AB represents 3, and the line BC represents $3\frac{1}{2}$.

Similarly, we have seen that negative integers and fractions, which for a long time were considered to be meaningless, can be represented by lines.

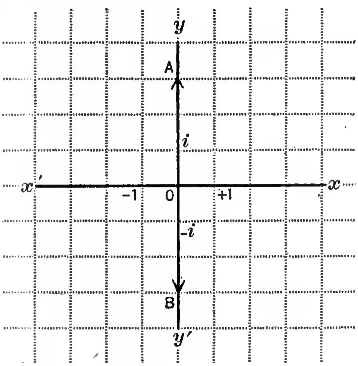


Thus, the line BA represents -3, and the line CB represents $-3\frac{1}{2}$.

Irrational numbers can also be represented by lines.



Thus, in the right-angled triangle abc, the line ab represents the $\sqrt{2}$.



Like the negative number the imaginary number remained uninterpreted several centuries. But this number also can be represented graphically.

Thus, if a unit length on the y-axis be chosen to represent $\sqrt{-1}$ or i, the negative unit $-\sqrt{-1}$ or $-i$ should evidently be laid off in the opposite direction. $3\sqrt{-1}$ or $3i$ would then be represented by OA and $-3i$ by OB, as in the figure, and others similarly.

The reason for placing $\sqrt{-1}$ or i on a line at right angles to the line on which real numbers are plotted may be seen in the fact that multiplying 1 by $\sqrt{-1}$ twice changes $+1$ into -1. On the graph $+1$ can be changed into -1 by turning it through 180°. If multiplying 1 by $\sqrt{-1}$ twice turns the line 1 through 180°, multiplying 1 by $\sqrt{-1}$ once should turn $+1$ through 90°.

For example:

1. Represent graphically $\sqrt{-4}$:

$\sqrt{-4} = \sqrt{4}\,i = 2\,i$; this is represented by a line 2 spaces long drawn upward on the y-axis.

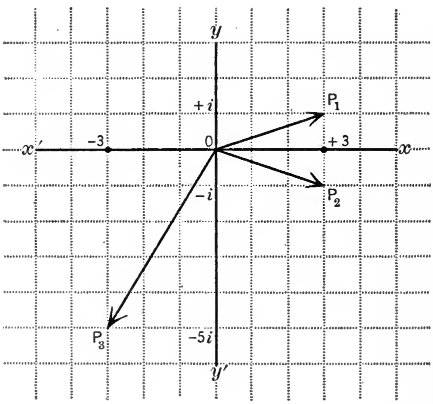
2. Represent graphically $-\sqrt{-3}$:

$-\sqrt{-3} = -\sqrt{3}\,i = -1.7\,i$ (approximately); this is represented by a line 1.7 spaces long drawn downward on the y-axis.

WRITTEN EXERCISES

Represent graphically:

1. $3\,i$.
2. $-2\,i$.
3. $\sqrt{-9}$.
4. $\sqrt{-16}$.
5. $-5\,i$.
6. $5\,i$.
7. $\sqrt{-3}$.
8. $-\sqrt{-12}$.
9. $-5\sqrt{-4}$.
10. $-3\,i$.
11. $+2\sqrt{-3}$.
12. $5\sqrt{-9}$.



Complex numbers may be represented graphically by a modification of the plan used in representing imaginary numbers.

EXAMPLES

1. Represent graphically $3 + i$.

To do this 3 is laid off on the axis of real numbers (xx'), and i upward on the axis of imaginaries (yy'). As in other graphical work this locates the point P_1 which is taken to represent the complex number, $3 + i$.

The number $\sqrt{a^2 + b^2}$ is called the *modulus* of the complex number $a + bi$. As appears from the figure $OP_1 = \sqrt{3^2 + 1^2}$, and hence OP_1 represents the modulus of $3 + i$.

2. Represent graphically $3 - i$.

The point P_2 is the graph of the complex number $3 - i$, and OP_2 represents its modulus.

3. Represent graphically $-3 - 5i$.

The point P_3 is the graph of the complex number $-3 - 5i$, and OP_3 represents its modulus.

We have thus *interpreted by means of diagrams* positive and negative integers, positive and negative fractions, positive and negative irrational numbers, and positive and negative complex numbers; in fact, all of the numbers used in elementary algebra.

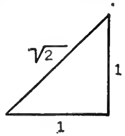
SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. What is an *imaginary number?* Sec. 540.
2. What term is used to designate numbers not imaginary? Sec. 542.
3. Define and illustrate a *complex number.* Sec. 544.
4. What is the meaning of the symbol i in complex numbers? Sec. 543.
5. How may any integral power of i be expressed? Secs. 547, 548.
6. How do imaginary numbers often occur in practice? What do imaginary roots indicate in solving problems? Secs. 549, 550.

HISTORICAL NOTE

Negative numbers were so long a stumbling block to mathematicians that their square roots were naturally regarded as impossible until recent times. The Greeks understood irrationals and could express many of these numbers concretely; for example, the $\sqrt{2}$ was shown by them to be the length of the hypotenuse of a right triangle whose sides are unity, as in the figure, but the $\sqrt{-2}$ had no meaning to them. The Hindoo, Bhaskara, said:



"The square of a positive, as also of a negative number, is positive, but there is no square root of a negative number, for it is not a square." Even the great scholars of the sixteenth and seventeenth centuries did little more than to accept imaginaries as numbers, and it remained for Caspar Wessel (1797) to make the first concrete representation of complex numbers; but his discovery made little impression until Gauss emphasized its importance.

Karl Friedrich Gauss was born at Brunswick, Germany, in 1777. His father was a mason and took little interest in his son's education, but the boy's wonderful genius for numbers attracted the attention of his teachers, who induced the Duke of Brunswick to send young Gauss to a preparatory school. He entered the University of Göttingen in 1795 and soon made discoveries in the properties of numbers that won for him high rank among mathematicians. On the appearance of his great work, *Disquisitiones Arithmeticæ*, published in 1801 when Gauss was only twenty-four years old, his contemporary, Laplace, declared Gauss to be the greatest mathematician of all Europe. Gauss died in 1855 after a life devoted to mathematics. He enriched all its branches, including its applications in astronomy and physics, with the lasting products of his wonderful genius. He has been worthily called: *Princeps Mathematicorum*, the Prince of Mathematicians.

Karl Friedrich Gauss

CHAPTER XXIX

GRAPHS OF QUADRATIC EQUATIONS

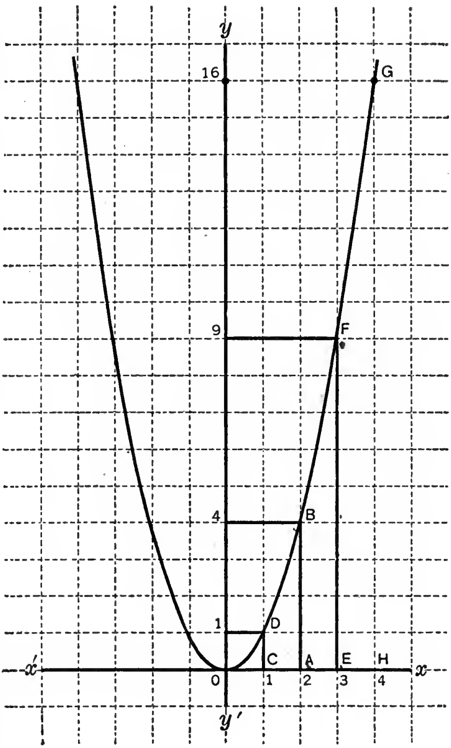
552. PREPARATORY.

1. By counting the spaces read the length of *EF* in the figure.

2. Is it the square of the length of *OE*?

3. Answer similar questions for *GH* and *OH*.

Every point of the curve is so located that *the length of its ordinate is the square of its abscissa.*



553. Quadratic expressions may be represented graphically.

For example:

The curve in the figure is the graph of $y = x^2$. That is, the length of *CD* is the square of that of *OC*; the length of *AB* is the square of that of *OA*; etc.

ORAL EXERCISES

1. In $y = x^2$ what is y when $x = 2$? Locate in the figure the point having these values of x and y.

2. Answer the same question when $x = 1$; also 3; 0; 4.

WRITTEN EXERCISES

1. Construct on a large sheet of squared paper the points corresponding to the table of squares given below.

2. Then sketch a smooth curve through the points beginning with $-5, 25$.

The work should be carefully done, and the result preserved for later use. As there are no negative values of x^2, the x-axis should be taken near the lower edge of the paper. The unit chosen should be quite large; for example, 10 spaces. Then the table might include squares of numbers increasing by tenths: 1, 1.1, 1.3, etc. The curve will be a graphical table of squares and square roots.

NUMBER	SQUARE
-5	25
-4	16
-3	9
-2	4
-1	1
-0	0
1	1
2	4
3	9
4	16
5	25

3. Read to one decimal place from the graph $\sqrt{2}$; $\sqrt{3}$; $\sqrt{5}$; $\sqrt{6}$; $\sqrt{7}$; $\sqrt{8}$.

Every y-distance is the square of the corresponding x-distance; and every x-distance is the square root of the corresponding y-distance. We see that for every y-distance there are two corresponding x-distances, one plus and the other minus, corresponding to the two square roots. Thus the points of the curve for which $y = 4$ are those whose values of x are 2 and -2, respectively, *i.e.* $\sqrt{4} = \pm 2$.

554. Graphical Solution of Quadratic Equations. Any value of x which satisfies the system

$$\begin{cases} y = x^2, \\ y = -px - q \end{cases}$$

makes x^2 equal to $-px - q$, or $x^2 + px + q = 0$.

The values of x satisfying the system may be read from the graph of $y = x^2$.

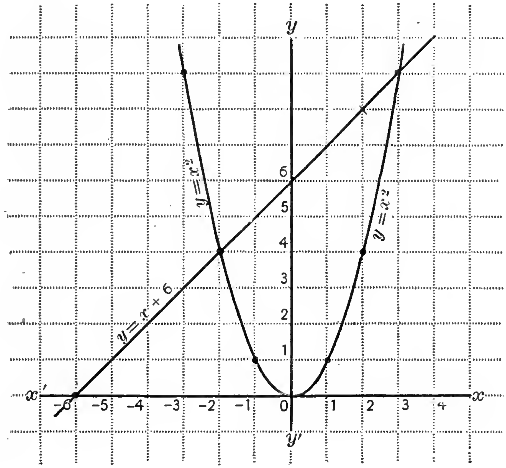
EXAMPLE

Solve graphically $x^2 - x - 6 = 0$.

1. Construct the graph of $y = x^2$. (As in Sec. 553.)
2. Construct the graph of $y = x + 6$.
3. They intersect at points for which $x = -2$ and $+3$.

$\therefore$ the roots of $x^2 - x - 6 = 0$ are $-2, 3$.

NOTES: 1. Step 2 may be done by simply noting two points of the graph of $y = x + 6$ and laying a ruler connecting them. The roots can



be read while the ruler is in position, and thus the same graph for $y = x^2$ can be used for several solutions.

2. The equation must first be put in the form $x^2 + px + q = 0$, if not so given.

WRITTEN EXERCISES

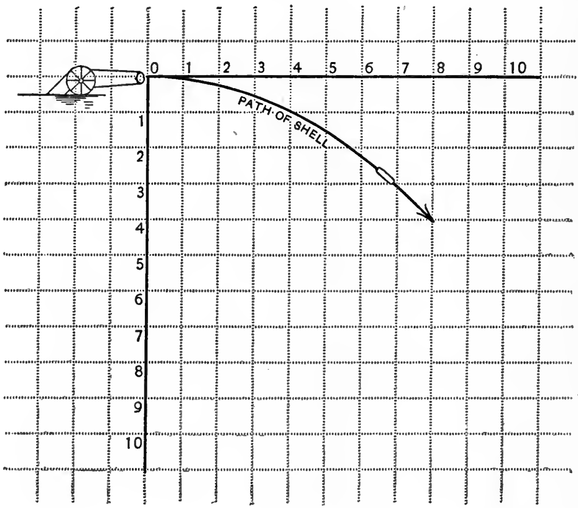
Solve graphically:

1. $x^2 - 5x + 6 = 0$.
2. $x^2 + 3x + 2 = 0$.
3. $x^2 - 2x - 3 = 0$.
4. $x^2 + 2x - 3 = 0$.
5. $x^2 - 3x - 40 = 0$.
6. $x^2 + 4x + 4 = 0$.
7. $2x^2 - x - 1 = 0$.
8. $3x^2 - 2x - 1 = 0$.
9. $x^2 + x + \frac{1}{4} = 0$.
10. $x^2 + x - 2 = 0$.
11. $4x^2 + 4x + 1 = 0$.
12. $x^2 - 9 = 0$.
13. $x^2 - 4x + 4 = 0$.
14. $x^2 + \frac{2}{3}x + \frac{1}{9} = 0$.
15. $x^2 - 4x + 3 = 0$.
16. $x^2 - 4x - 5 = 0$.
17. $x^2 - 7x + 12 = 0$.
18. $x^2 - 3x - 10 = 0$.

19. The path of a projectile fired horizontally with a given velocity from an elevation, as at 0 in the figure, may be represented by the graph of the equation $y = \frac{gx^2}{2\,v^2}$, where $g = 32$ and v is the initial velocity of the projectile in feet per second. Let $v = 16$ ft. per second and compute the numbers to complete the table of values of x and y.

TABLE

x	y
0	0
1	( )
4	( )
8	( )
9	( )
16	( )
24	( )
32	( )
48	( )



Read from the graph of this table the horizontal distance traveled by the projectile when it is 4 ft. below the starting point.

20. Construct similarly the path of a projectile whose initial velocity is 32 ft. per second.

21. A cannon of a fort on a hill is 300 ft. above the plane of its base. The cannon can be charged so as to give the projectile an initial velocity of 100 ft. per second. How far does a projectile travel horizontally before it strikes the plane?

22. The enemy is observed at a point $2\frac{1}{2}$ mi. from the foot of the vertical line in which the cannon stands. With what initial velocity must the ball start to strike the enemy?

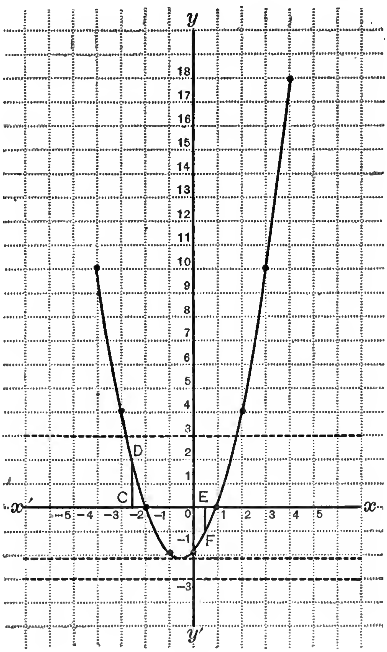
555. Graphs of Quadratic Functions. The solutions of $ax^2 + bx + c = 0$ may also be represented graphically by plotting the curve corresponding to $y = ax^2 + bx + c$. The real roots of the equation will be represented by the points where the curve cuts the x-axis.

EXAMPLE

Represent the solutions of $x^2 + x - 2 = 0$.

To draw the curve $y = x^2 + x - 2$, we make a table of a sufficient number of pairs of values of x and y, plot the corresponding points, and sketch a smooth curve through them.

x	y
-4	10
-3	4
-2	0
-1	-2
$-\frac{3}{4}$	$-2\frac{3}{16}$
$-\frac{1}{2}$	$-2\frac{1}{4}$
$-\frac{1}{4}$	$-2\frac{3}{16}$
0	-2
1	0
2	4
3	10
4	18



NOTE. When the integral values do not satisfactorily outline the shape of the curve, fractional values of x must also be used as above.

This graph not only represents the solutions of $x^2 + x - 2 = 0$; but, what is far more important, it represents the values and variation of the polynomial $x^2 + x - 2$ for all values of x (within the limits of the figure). Thus EF represents the value of the trinomial for $x = \frac{1}{2}$, and CD that for $x = -2\frac{1}{2}$.

The figure also tells us that $x^2 + x - 2$ is positive when x is negative and numerically greater than -2 (algebraically less than -2). That when x is algebraically less than -2 and increasing, the trinomial is positive and decreasing, reaching the value zero when $x = -2$. As x increases, the trinomial continues to decrease, becoming negative, reaching its least value for $x = -\frac{1}{2}$. It then begins to increase, through negative values, reaching zero when $x = 1$. As x increases further, the trinomial continues to increase, becoming positive and continuing to increase as long as x increases. Owing to the fact that a quadratic equation has two and only two roots, the graph of a quadratic function must consist of a curve having a single bend.

The graph enables us to read approximately not only the roots of $x^2 + x - 2 = 0$, but also those of any equation of the form $x^2 + x - 2 = a$, if they are real.

For example:

1. The roots of $x^2 + x - 2 = 3$ are indicated by the points where the line drawn parallel to the x-axis through the point 3 on the y-axis cuts the graph.

2. The line parallel to the x-axis through the point -3 does not cut the graph at all. This tells us that there are no real values of x which make the trinomial $x^2 + x - 2$ equal to -3. The equation $x^2 + x - 2 = -3$ has imaginary roots.

The graph shows (1) that the equation $x^2 + x - 2 = a$ has two real and distinct roots whenever a is positive; also when a ranges from zero to a little beyond -2; (2) that it has imaginary roots when a is less than a certain value between -2 and -3; (3) at a certain point the line parallel to the x-axis just touches the graph. This corresponds to the case of equal roots of the equation. The value of a in this case may be read from the graph as $-2\frac{1}{4}$. This means that the equation $x^2 + x - 2 = -2\frac{1}{4}$, or $x^2 + x + \frac{1}{4} = 0$, has equal roots. This may be verified by solving the equation.

Note. Since even the best drawing is not mathematically accurate, results read from a graph are usually only approximately correct. The closeness of the approximation depends on the degree of accuracy in the drawing.

WRITTEN EXERCISES

Treat each trinomial of Nos. 1 to 9 below as follows:

(1) Draw the graph of the trinomial.

(2) From the graph discuss the variations of the trinomial as in Section 555.

(3) From the graph read approximately the roots of the equation resulting from equating the trinomial to zero.

(4) If the trinomial is equated to a, read from the graph the range of values of a, for which the roots of the equation are, (i) real and distinct, (ii) real and equal, (iii) imaginary.

(5) Last of all, verify those of the preceding results which relate to roots of the equations by solving the equations.

1. $x^2 + 5x - 6.$	**4.** $x^2 + 4x - 5.$	**7.** $x^2 + 2x + 3.$
2. $x^2 + 3x + 2.$	**5.** $3x^2 - 2x.$	**8.** $x^2 - 2x - 1.$
3. $2x^2 - 5x + 2.$	**6.** $x^2 - 4.$	**9.** $8 + 2x - x^2.$

10. Draw the graph of the function $2x^2 + 5x - 3$, and state how it varies as x varies from a negative value, numerically large at will, through zero to a large positive value. For what values of x is the function positive? For what values negative?

11. Draw the graph of the function $6x^2 - x - 5$, and state how it varies as x varies from a negative value, numerically large at will, through zero to a large positive value. For what values of x is the function positive? For what values negative?

12. What is the graphic condition that $ax^2 + bx + c$ shall have the same sign for all values of x? What must therefore be the character of the roots of $ax^2 + bx + c = 0$, if the trinomial $ax^2 + bx + c$ has the same sign for all values of x?

Determine m so that each of the following trinomials shall be positive for all values of x:

13. $x^2 + mx + 5.$ **14.** $3x^2 - 5x + m.$ **15.** $mx^2 + 6x + 8.$

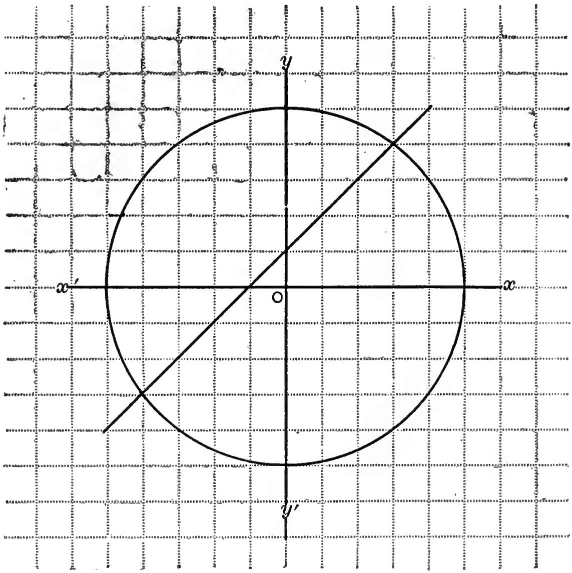
GRAPHS OF SIMULTANEOUS QUADRATIC EQUATIONS

556. Preparatory.

1. In the same diagram construct graphs to represent the equations:

$$\begin{cases} x^2 + y^2 = 25, \\ x - y = -1. \end{cases}$$

Compare the result with this figure.



2. Solve the system of equations in Exercise 1.

Compare the values of x and y with the coördinates of the intersections of the graphs.

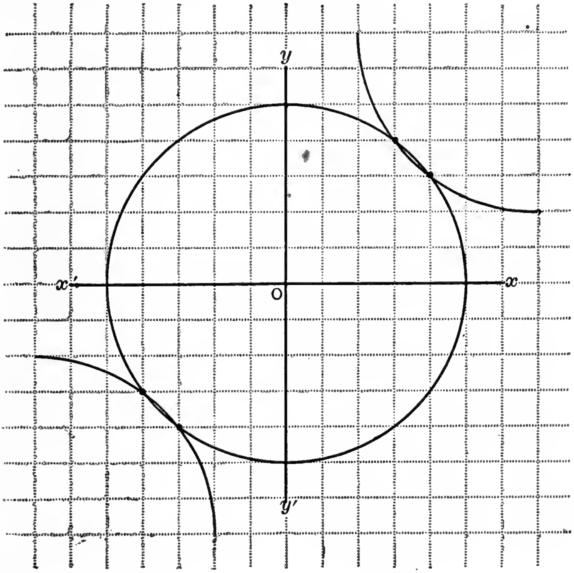
In how many points does the straight line intersect a circle?

How many solutions has the given system of equations?

3. Construct in one diagram the graphs of the equation:

$$\begin{cases} x^2 + y^2 = 25, \\ xy = 12. \end{cases}$$

Compare the result with this figure.



4. Solve the system of equations in Exercise 3.

Compare the values of x and y with the coördinates of the intersections of the graphs.

557. Graphical Solution of Two Simultaneous Equations. Every point of the graph of one of the equations has coördinates, x and y, that satisfy that equation; the points of intersection are points of both graphs and therefore have coördinates, x and y, that satisfy both equations. Hence, to solve two simultaneous equations graphically, draw their graphs and read the coördinates of their intersections. The coördinates of each point of intersection correspond to a solution. If the graphs do not intersect, the system of equations has no real roots, all of them being imaginary numbers.

WRITTEN EXERCISES

Solve graphically, and test by computing x and y:

1. $2x^2 + y = 1$,
   $x - y = 2$.
2. $x^2 - y^2 = 25$,
   $x + y = 1$.
3. $x^2 + y^2 = 10$,
   $xy = 3$.
4. $x^2 + y^2 = 13$,
   $xy = 6$.
5. $x^2 + y^2 = 13$,
   $x + 2y = 1$.
6. $x^2 - y^2 = 5$,
   $3x - y = 7$.
7. $x^2 + y^2 = 25$,
   $x^2 - y^2 = 7$.
8. $x^2 - 3xy + 2y^2 = 0$,
   $x^2 + 3y^2 = 16$.
9. $4x^2 + 9y^2 = 36$,
   $x^2 + y^2 = 25$.
10. $4x^2 + 9y^2 = 36$,
    $2x - 3y = 5$.
11. $4x^2 - 9y^2 = 36$,
    $x^2 + y^2 = 16$.
12. $4x^2 - 9y^2 = 36$,
    $xy = 18$.

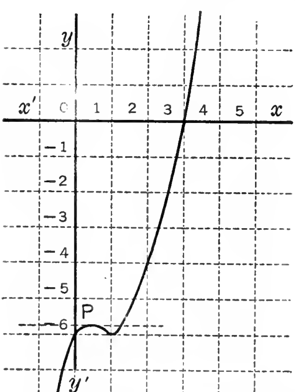
GRAPHS OF HIGHER EQUATIONS

558. The graphs of higher equations serve to show the character of the roots.

EXAMPLE

The diagram is the graph of

$$y, \text{ or } f(x) = x^3 - 3x^2 + 2x - 6.$$



x	$f(x)$
−2	−30
−1	−12
0	−6
1	−6
2	−4
3	0
4	66

The roots of the equation

$$x^3 - 3x^2 + 2x - 6 = 0$$

are $3, -\sqrt{-2}, +\sqrt{-2}$.

The curve shows that 3 is the only real root. The downward bend at P

indicates that the other two roots are imaginary, for the curve does not rise high enough to cut the axis again, so as to make two more real intersections. The imaginary roots are $-\sqrt{-2}$ and $+\sqrt{-2}$.

Thus we have an illustration of the fact that imaginary roots always occur in pairs. Consequently, a cubic equation can only have two imaginary roots and must have either one or three real roots. An equation of the fourth degree may have either one pair or two pairs of imaginary roots; hence, it may have either none, two, or four real roots.

If the curve just touches the x-axis and then turns away from it, the equation has an even number of *equal* real roots corresponding to this point of contact.

ORAL EXERCISES

1. How many roots has a cubic equation? If the roots of the cubic equation are all real and unequal, in how many points does the graph cut the x-axis?

2. Can a cubic equation have two and only two real roots?

3. If the roots of a cubic equation are 2, 2 and -5, in how many places does its graph cross the x-axis? What will indicate the equal roots?

4. When might the graph of a fourth degree equation not cut the x-axis at all?

5. What is the greatest number of intersections with the x-axis possible for the graph of a fifth degree equation?

559. Graphs of Radical Equations. The graphs of radical equations picture the changes which account for extraneous roots.

EXAMPLES

1. Given $x - \sqrt{2x^2 + 17} = -3.$ *(1)*

Transposing and squaring, $x^2 - 6x + 8 = 0.$ *(2)*

Then, $(x-2)(x-4) = 0.$ *(3)*

Therefore the roots are 2 and 4, both of which satisfy *(1)*.

To find the relation between the graphs of equation (1) and equation (2), they are plotted on the same axes.

In equation (1) y, or $f(x) = x - \sqrt{2x^2 + 17} + 3$.
In equation (2) y, or $f(x) = x^2 - 6x + 8$.

VALUES FOR PLOTTING

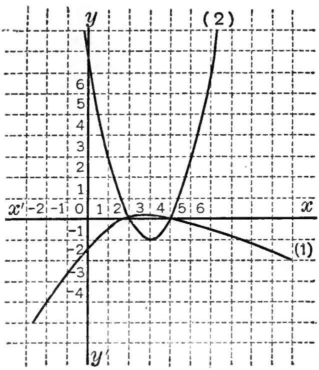
EQUATION (1)

x	$f(x)$
-2	-4
-1	-2.3
0	-1.1
1	-0.3
2	0
3	0.1
4	0
6	-0.4

EQUATION (2)

x	$f(x)$
-2	24
-1	15
0	8
1	3
2	0
3	-1
4	0
5	3

In finding values of y always take the positive sign of the radical.



Although the graphs of equations (1) and (2) are entirely different, each cuts the x-axis in points 2 and 4. This shows why both roots of equation (2) also satisfy equation (1). Hence, no *extraneous* roots are introduced in this case by squaring.

The equations not being of higher degree than the second can not have other roots.

2. Given $\quad \sqrt{x-1} - \sqrt{x-5} = \sqrt{2x-6}$. *(1)*

Squaring, $\quad \sqrt{(x-1)(x-5)} = 0$. *(2)*

Squaring, $\quad x^2 - 6x + 5 = 0$. *(3)*

Solving (3), $x = 1, 5$. But 1 is not a root of (1). 5 satisfies equations (1), (2), (3), and 1 satisfies (2) and (3).

To find the relation between the graphs of equations (1), (2), and (3), they are plotted on the same axes.

In equation (1) y, or $f(x) = \sqrt{x-1} - \sqrt{x-5} - \sqrt{2x-6}$.
In equation (2) y, or $f(x) = \sqrt{(x-1)(x-5)}$.
In equation (3) y, or $f(x) = x^2 - 6x + 5$.

VALUES FOR PLOTTING

EQUATION (1)

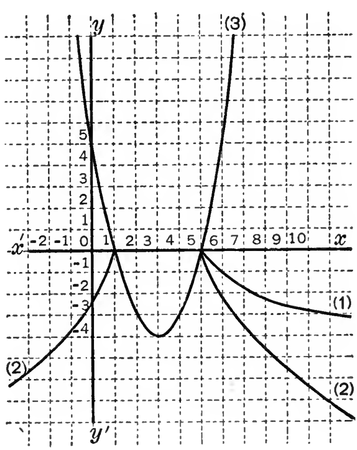
x	$f(x)$
-1	Imag.
0	"
1	"
5	0
6	-1.2
8	-2.2
10	-2.9

EQUATION (2)

x	$f(x)$
-2	-4.6
-1	-3.5
0	-2.2
1	0
2, 3, 4	Imag.
5	0
6	-2.2
7	-3.5
8	-4.6

EQUATION (3)

x	$f(x)$
-2	21
-1	12
0	5
1	0
2	-3
3	-4
4	-3
5	0
6	5



The graphs show that equations (2) and (3) have a root, 1, which equation (1) does not have. Hence, squaring in this case has introduced the extraneous root 1.

The diagram also shows that the real part of the graph of equation (1) ends at 5; also that equation (2) has two real branches, one extending downward from $x = 5$ and the other downward from $x = 1$. The imaginary values of the radicals account for the interruptions in the graphs at points 1 and 5.

560. Since squaring may, or may not, introduce extraneous roots, as shown in the above examples, *all roots must be tested, and only those preserved which satisfy the given equation.*

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Write the equation whose graph expresses the square roots of positive numbers. Sec. 553.

2. What is the general shape of the graph of a quadratic function? In how many places can it cut the x-axis? Sec. 555.

3. What do the intersections of a graph with the x-axis represent? Sec. 555.

4. If the roots of a quadratic equation are imaginary, how will the graph show it? Sec. 558.

5. How are the solutions of simultaneous quadratic equations represented graphically? Sec. 557.

6. How do graphs show that the result of squaring produces, in general, a different equation? Sec. 559.

7. What must determine which are extraneous roots resulting from transforming a given equation? Sec. 560.

CHAPTER XXX

QUADRATIC EQUATIONS

THEORY

561. The **general form** for a quadratic polynomial with one unknown quantity is $ax^2 + bx + c$, where a, b, and c denote any algebraic expressions not involving x, and where a is not zero. If a is zero the polynomial is linear.

For example: **1.** $5x^2 - 7x + 8.$

Here $a = 5$, $b = -7$, $c = 8$.

2. $\dfrac{7m}{2n+1}x^2 + 3x - \dfrac{5m}{2n-1}.$

Here $a = \dfrac{7m}{2n+1}$, $b = 3$, $c = \dfrac{-5m}{2n-1}$.

WRITTEN EXERCISES

Put the following expressions into the form $ax^2 + bx + c$:

1. $3x + 5x(x-2) + 4(x^2-5)$.
2. $7(4x-1) + (x+3)(x-2)$.
3. $a(bx+c)(2dx+3e)$.
4. $(x+q) - q(x^2-11)$.
5. $(ax+b)(cx+d)$.
6. $(x^2-a) + (x^2-b)$.
7. $(x+q)(x+p) - (x-q)(2x-p)$.
8. $\left(\dfrac{x}{2}+\dfrac{1}{3}\right)^2 - \dfrac{2}{3}\left(\dfrac{9x}{8}-\dfrac{16}{3}\right)^2.$
9. $x^2 + ab - ax - b(a+x+x^2)$.
10. $x(x-2)(x-4) - x^2(x-5)$.
11. $(2x+1)^2 - (3x+1)^2 + (4x+1)^2$.
12. $(x-1)(x-2)(x-3) - (x+1)(x+2)(x+3)$.

562. Similarly, every quadratic equation can be put into the form $ax^2 + bx + c = 0$ by transposing all terms to the left member and then putting the polynomial which constitutes the left member into the form $ax^2 + bx + c$.

For example:

Given $(mx + 3a)^2 = mx^2 - 5(amx - 2)$,

then $m^2x^2 + 6amx + 9a^2 = mx^2 - 5amx + 10$,

transposing, $(m^2 - m)x^2 + 11amx + 9a^2 - 10 = 0$.

Hence, $a = m^2 - m$, $b = 11am$, $c = 9a^2 - 10$.

METHODS OF SOLUTION

563. General Solution. By solving the general quadratic equation $ax^2 + bx + c = 0$, general formulas for the roots are obtained.

Solve: $$ax^2 + bx + c = 0. \quad (1)$$

Dividing by a, which is not 0, $$x^2 + \frac{bx}{a} + \frac{c}{a} = 0. \quad (2)$$

Adding $\frac{b^2}{4a^2}$ to complete the square and subtracting the same, $$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0. \quad (3)$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = 0. \quad (4)$$

Writing the second term as the square of its square root, $$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0. \quad (5)$$

Factoring (5),

$$\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0. \quad (6)$$

$$\therefore x = -\frac{b}{2a} + \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right) \text{ and } x = -\frac{b}{2a} - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right). \quad (7)$$

Denoting these roots by r_1 and r_2:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

By substituting in these formulas the values, including the signs, that a, b, c have in any particular equation, the roots of that equation are obtained. This is called **solution by formula.**

EXAMPLE

Solve: $3x^2 - 9x + 5 = 0.$ (1)

Here, $a = 3,\ b = -9,\ c = 5,$ (2)

and $x = \dfrac{-(-9) \pm \sqrt{(-9)^2 - 4 \cdot 3 \cdot 5}}{2 \cdot 3}.$ (3)

$= \dfrac{9 \pm \sqrt{81 - 60}}{6} = \dfrac{9 \pm \sqrt{21}}{6}.$ (4)

WRITTEN EXERCISES

Solve by formula:

1. $x^2 - 4x + 4 = 0.$
2. $x^2 - 5x + 6 = 0.$
3. $x^2 - 3x + 2 = 0.$
4. $x^2 - x - 1 = 0.$
5. $x^2 + 3x + 1 = 0.$
6. $x^2 + 2x - 1 = 0.$
7. $x^2 - 13x + 9 = 0.$
8. $2x^2 - 7x - 3 = 0.$
9. $x^2 - x + 6 = 0.$
10. $2x^2 - x + 2 = 0.$
11. $3x^2 - 2x + 1 = 0.$
12. $7x^2 + 6x - 4 = 0.$
13. $4x^2 - 12x + 9 = 0.$
14. $3x^2 + 5x - 2 = 0.$
15. $5x^2 - 4x + 6 = 0.$
16. $7x^2 + 5x - 8 = 0.$

564. Literal Quadratic Equations. When any of the coefficients of a quadratic equation involve letters, the equation is called a **literal quadratic equation.**

Such equations are solved in the usual way.

EXAMPLES

1. Solve: $x^2 + 6mx + 8 = 0.$ (1)

$x^2 + 6mx = -8.$ (2)

Completing the square, $x^2 + 6mx + 9m^2 = 9m^2 - 8.$ (3)

$\therefore (x + 3m)^2 = 9m^2 - 8.$ (4)

$\therefore x + 3m = \pm\sqrt{9m^2 - 8}.$ (5)

$\therefore x = -3m \pm \sqrt{9m^2 - 8}.$ (6)

2. Solve: $t^2 + gt + h = 0.$ (1)

Here, $a = 1,\ b = g,\ c = h.$ (2)

Hence, by Sec. 563, $t = \dfrac{-g \pm \sqrt{g^2 - 4h}}{2}.$ (3)

3. Solve: $$gt^2 + 2\,vt = 2\,s. \qquad (1)$$

$$gt^2 + 2\,vt - 2\,s = 0. \qquad (2)$$

Here, $$a = g,\ \ b = 2\,v,\ \ c = -\,2\,s. \qquad (3)$$

Hence, by Sec. 563, $$t = -\frac{2\,v}{2\,g} \pm \frac{1}{2\,g}\sqrt{(2\,v)^2 - 4\,g(-\,2\,s)} \qquad (4)$$

$$= -\frac{v}{g} \pm \frac{1}{g}\sqrt{v^2 + 2\,gs}. \qquad (5)$$

$$= \frac{1}{g}(-\,v \pm \sqrt{v^2 + 2\,gs}). \qquad (6)$$

WRITTEN EXERCISES

Solve:

1. $t^2 - 8\,t + 24\,d = 9\,d^2.$
2. $ay^2 - (a - b)y - b = 0.$
3. $w^2 + 4\,aw + a^2 = 0.$
4. $v^2 - 4\,amv = (a^2 - m^2)^2.$
5. $w^2 - a^2 = 2\,b(a - w).$
6. $t^2 + at = k.$
7. $u^2 + ku + 1 = 0.$
8. $v^2 + mv = 1.$
9. $ax^2 + bx + c = 0.$
10. $x^2 + ax + b = 0.$
11. $m^2x^2 + 2\,mx = -\,1.$
12. $x^2 + 2\,px - 1 = 0.$
13. $4\,x^2 - 4\,ax + 16 = 0.$
14. $a^2x^2 + 2\,ax + 5 = 0.$
15. $m^2x^2 + 4\,mx - 6 = 0.$
16. $x^2 - 4\,ax = 9.$
17. $5\,ax^2 + 3\,bx + 2\,b^3 = 0.$
18. $b^2x^2 - 2\,bx = ac - 1.$
19. $x^2 - 3\,ax + 10\,a^2 = 0.$
20. $2\,x^2 - 3\,x = a(3 - 4\,x).$

565. Collected Methods. We have used three methods of solving quadratic equations:

1. *Factoring.*

Equation	Factors	Roots
$x^2 - 3\,x + 2 = 0.$	$(x - 2)(x - 1).$	$x = 2,\ x = 1.$
$x^2 - (a + b)x + ab = 0.$	$(x - a)(x - b).$	$x = a,\ x = b.$

2. *Completing the square.*

Equation	Solution	Roots
$x^2 + x + 2 = 0.$	See Sec. 371.	$x = \dfrac{-1 \pm \sqrt{-7}}{2}.$

$ax^2 + bx + c = 0.$	See Sec. 371.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

3. *Formula.*

Equation	Solution	Roots
$3x^2 + 2x - 7 = 0.$	See Sec. 563.	$x = \frac{-1 \pm \sqrt{22}}{3}.$
$ax^2 + bx + c = 0.$	See Sec. 563.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

WRITTEN EXERCISES

Solve by factoring:

1. $x^2 - x - 6 = 0.$
2. $x^2 - x - 2 = 0.$
3. $x^2 + x - 2 = 0.$
4. $x^2 + x - 6 = 0.$
5. $x^2 + 3x + 2 = 0.$
6. $x^2 - x - 30 = 0.$
7. $x^2 + x - 12 = 0.$
8. $x^2 - 3x + 2 = 0.$
9. $x^2 + 11x + 30 = 0.$
10. $x^2 - 7x + 12 = 0.$

Solve by completing the square:

11. $x^2 + x + 1 = 0.$
12. $x^2 + 3x + 1 = 0.$
13. $x^2 - \frac{1}{2}x + 1 = 0.$
14. $x^2 - 1.5x + .5 = 0.$
15. $x^2 - 5x + 10 = 0.$
16. $x^2 - 16x + 60 = 0.$
17. $x^2 + \frac{3}{4}x + \frac{1}{4} = 0.$
18. $x^2 + 7.5x - 3.5 = 0.$

Solve by formula:

19. $3x^2 + x + 5 = 0.$
20. $2x^2 - 5x - 3 = 0.$
21. $4x^2 + 3x - 1 = 0.$
22. $5x^2 + 2x + 6 = 0.$
23. $x^2 + x + 1 = 0.$
24. $x^2 + 1 = 0.$
25. $x^2 + 15x + 56 = 0.$
26. $x^2 + 8x + 33 = 0.$
27. $x^2 - 10x + 34 = 0.$
28. $2x^2 + 3x - 27 = 0.$

Solve and test, using whichever of the methods in Sec. 563 seems most convenient:

29. $9y^2 - 4 = 0.$
30. $6x^2 - 13x + 6 = 0.$
31. $5x^2 - 4x + 4 = 0.$
32. $t^2 + 11t + 30 = 0.$

33. $6s^2 - 5s - 6 = 0.$

34. $6r^2 - 2r - 4 = 0.$

35. $w^2 + 4w - 3 = 0.$

36. $\dfrac{2x-1}{2x+1} + \dfrac{2x+1}{2x-1} = 3.$

37. $x^2 - 2x + 3 = 0.$

38. $x^2 - 0.5x + 0.06 = 0.$

39. $x^2 - .7x + .12 = 0.$

40. $11x^2 + 1 = 4(2 - x)^2.$

41. $x^2 + (a + b)x + ab = 0.$

42. $x^2 - (b + c)x + bc = 0.$

43. $2ax + (a - 2)x - 1 = 0.$

44. $\dfrac{a^2}{b+x} - \dfrac{a^2}{b-x} = c.$

45. The product of two consecutive positive integers is 306. Find the integers.

SOLUTION.

1. Let x be the smaller integer.
2. Then $x + 1$ is the larger.
3. $\therefore x(x + 1)$ is their product.
4. $\therefore x(x + 1) = 306$, by the given conditions.
5. $\therefore x^2 + x - 306 = 0$, from (4).
6. $\therefore x = \dfrac{-1 \pm \sqrt{1 + 1224}}{2} = \dfrac{-1 \pm 35}{2} = 17$, or -18, solving (5).

Since the integers are to be positive, the value -18 is not admissible. $x = 17$, $\therefore x + 1 = 18$, and the integers are 17 and 18.

TEST. $17 \cdot 18 = 306.$

46. There is also a pair of consecutive negative integers whose product is 306. What are they?

47. If the square of a certain number is diminished by the number, the result is 72. Find the number.

48. A certain number plus its reciprocal is -2. What is the number?

49. A certain positive number minus its reciprocal is $\frac{5}{6}$. What is the number? What negative number has the same property?

50. The perimeter of the rectangle shown in the figure is 62 in. Find the sides.

51. One perpendicular side of a certain right triangle is 31 units longer than the other; the square of their sum exceeds the square of the hypotenuse by 720. Find the sides.

52. In a right triangle of area 60 sq. ft.; the difference between the perpendicular sides is 7. Find the three sides.

NOTE. Those who have studied geometry may take up some of the problems based upon geometric properties found in Chapter XXXIII.

RELATIONS BETWEEN ROOTS AND COEFFICIENTS

566. Relation of Roots to Coefficients. By adding the values found for the roots (Sec. 563), we obtain

$$r_1 + r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}.$$

Multiplying the values, we find

$$r_1 r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \times \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{c}{a}.$$

Applying these results to the equation $x^2 + px + q = 0$, we have:

$$r_1 + r_2 = -p,$$
$$r_1 r_2 = q.$$

In words:

In the equation $x^2 + px + q = 0$, *the coefficient of* x *with its sign changed is the sum of the roots, and the absolute term is their product.*

Every quadratic equation can be put into the form $x^2 + px + q = 0$ by dividing both members by the coefficient of x^2.

567. Symmetric Functions of the Roots. It is apparent that the equations in Section 566 remain unchanged if r_1 and r_2 are interchanged. On this account the expressions $r_1 + r_2$ and $r_1 r_2$ are called symmetric functions of the roots of the quadratic equation. There are other such functions, but these only will be treated here. The two following sections show some of their uses.

568. By means of Section 566 a quadratic equation may be written whose roots are any two given numbers.

EXAMPLES

1. Write an equation whose roots are 2, -3.

$$-p = r_1 + r_2 = 2 + (-3) = -1. \quad \therefore p = 1.$$
$$q = r_1 r_2 = 2(-3) = -6.$$
$$\therefore x^2 + x - 6 = 0 \text{ is the equation sought.}$$

2. Write an equation whose roots are $\frac{1}{2} + \sqrt{-3}$, $\frac{1}{2} - \sqrt{-3}$.

$$-p = r_1 + r_2 = (\tfrac{1}{2} + \sqrt{-3}) + (\tfrac{1}{2} - \sqrt{-3}) = 1. \quad \therefore p = -1.$$
$$q = r_1 r_2 = (\tfrac{1}{2} + \sqrt{-3})(\tfrac{1}{2} - \sqrt{-3}) = \tfrac{1}{4} - (-3) = \tfrac{13}{4}.$$
$$\therefore x^2 - x + \tfrac{13}{4} = 0 \text{ is the equation sought.}$$

WRITTEN EXERCISES

Write the equations whose roots are:

1. 4, 5.
2. $\frac{3}{4}$, $\frac{4}{3}$.
3. 7, $-1\frac{3}{7}$.
4. -4, $+4$.
5. $\frac{2}{3} \pm \sqrt{-5}$.
6. 24, 30.
7. $8\frac{3}{5}$, 10.
8. -5, -20.
9. $-\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$.
10. $\frac{5}{6} \pm \frac{1}{6}\sqrt{-47}$.
11. a, $-b$.
12. 8, -40.
13. $a - bi$, $a + bi$.
14. $1 + 2i$, $1 - 2i$.
15. $\frac{1}{2} - \sqrt{2}$, $\frac{1}{2} + \sqrt{2}$.

569. Testing Results. The ultimate test of the correctness of a solution is that of substitution; but this is not always convenient, especially when the roots are irrational. In such cases, the relations between the roots and coefficients are of use.

For example: Solving $2x^2 - 5x + 6 = 0$,

or $x^2 - \frac{5}{2}x + 3 = 0$,

the roots are $r_1 = \frac{5}{4} + \frac{1}{4}\sqrt{-23}$ and $r_2 = \frac{5}{4} - \frac{1}{4}\sqrt{-23}$.

Adding, $-(r_1 + r_2) = -\frac{10}{4} = -\frac{5}{2}$, the coefficient of x.

Multiplying, $r_1 r_2 = (\frac{5}{4})^2 - (\frac{1}{4}\sqrt{-23})^2 = \frac{25}{16} + \frac{23}{16} = 3$, the absolute term.

Therefore, the roots are correct. (Sec. 566.)

570. In what follows, the coefficients a, b, c are restricted to rational numbers.

571. Character of the Roots. By examining the formula for the roots, $-\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$, it appears that the character of the roots as real or imaginary, rational or irrational, equal or unequal, depends upon the value of the expression $b^2 - 4ac$.

1. If $b^2 - 4ac$ is positive, the roots are real.

Thus, in $x^2 + 4x - 3 = 0$, $b^2 - 4ac = 16 + 12$, or 28, $\therefore$ the roots are real and unequal.

2. If $b^2 - 4ac$ is a perfect square, the indicated square root can be extracted, and the roots are rational.

Thus, in $x^2 - 4x + 3 = 0$, $b^2 - 4ac = 16 - 12$, or 4, $\therefore$ the roots are rational and unequal.

3. If $b^2 - 4ac$ is not a perfect square, the indicated root cannot be extracted and the roots are irrational.

Thus, in $x^2 + 5x + 1 = 0$, $b^2 - 4ac = 25 - 4 = 21$, $\therefore$ the roots are irrational.

4. If $b^2 - 4ac = 0$, the radical is zero, and the two roots are equal.

Thus, in $x^2 - 10x + 25 = 0$, $b^2 - 4ac = 100 - 4 \cdot 25 = 0$, $\therefore$ the roots are equal.

5. If $b^2 - 4ac$ is negative, the roots are imaginary.

Thus, in $2x^2 - x + 1 = 0$, $b^2 - 4ac = 1 - 8$, or -7, $\therefore$ the roots are complex numbers.

Consequently, it is merely necessary to calculate $b^2 - 4ac$ to know in advance the nature of the roots of a quadratic equation.

572. Discriminant. Because its value determines the character of the roots, the expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

ORAL EXERCISES

Without solving the equations, find the nature of the roots of:

1. $x^2 + x - 20 = 0$.
2. $x^2 + x - 3 = 0$.
3. $2x^2 - x + 2 = 0$.
4. $3x^2 - x + 3 = 0$.
5. $2x^2 + 2x - 4 = 0$.
6. $5x^2 - 3x + 6 = 0$.
7. $3x^2 - 4x + 5 = 0$.
8. $6x^2 + x - 1 = 0$.
9. $7x^2 + 3x - 4 = 0$.
10. $-4x + 8x^2 + 1 = 0$.
11. $5 + 4x^2 - 3x = 0$.
12. $7x + 6 + x^2 = 0$.
13. $-6x + 9x^2 + 3 = 0$.
14. $x^2 - 6x + 4 = 0$.

573. The relation $x^2 + px + q = x^2 - (r_1 + r_2)x + r_1 r_2$ may be written:

(1) $$x^2 + px + q = (x - r_1)(x - r_2).$$

And since $x^2 + px + q = \frac{ax^2 + bx + c}{a}$ in which $p = \frac{b}{a}$, $q = \frac{c}{a}$, we have

(2) $$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

574. The solution of a quadratic equation, therefore, enables us to factor every polynomial of either form (1) or (2).

Since r_1 and r_2 involve radicals:

1. *The factors will generally be irrational.*

2. *The factors will be rational when* r_1 *and* r_2 *are so; that is when* $b^2 - 4ac$ *is a perfect square.*

3. *The two factors involving* x *will be the same when the roots are equal; that is, when* $b^2 - 4ac = 0$.

In the last case the expressions are squares and

(1) becomes $(x - r_1)^2$, and
(2) becomes $[\sqrt{a}(x - r_1)]^2$.

EXAMPLES

Trinomial	$b^2 - 4ac$	Nature of Factors
1. $3x^2 - 7x + 2$	$49 - 4 \cdot 3 \cdot 2 = 25$	rational of 1st degree.
2. $3x^2 - 7x + 3$	$49 - 4 \cdot 3 \cdot 3 = 13$	irrational.
3. $2x^2 - 8x + 8$	$64 - 4 \cdot 2 \cdot 8 = 0$	equal.

ORAL EXERCISES

By means of the above test, select the squares; also the trinomials with rational factors of the first degree:

1. $8x^2 - 8x + 2$.
2. $\frac{y^2}{3} + 4y + 12$.
3. $3x^2 + 3x + 1$.
4. $3z^2 + 2z + 12$.
5. $x^2 + 3x - 2$.
6. $a^2x^2 + 2ax + 1$.
7. $4x^2 + 4x + 1$.
8. $x^2 - 8x + 15$.
9. $6x^2 + 5x - 4$.
10. $6x^2 - 5x + 9$.
11. $4x^2 - 4x - 3$.
12. $8x^2 - 9x + 3$.

575. General Method of Factoring Quadratic Trinomials. The actual factors of any quadratic trinomial of the form $ax^2 + bx + c$ can be found by solving the quadratic equation:

$$ax^2 + bx + c = 0,$$

and substituting the roots in the relation:

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

EXAMPLES

1. Factor: $\quad 6x^2 + 5x - 4. \quad (1)$

Solving $6x^2 + 5x - 4 = 0$, $\quad x = -\frac{4}{3}, \quad (2)$

$x = \frac{1}{2}.$

Substituting $-\frac{4}{3}$ for r_1, $\frac{1}{2}$ for r_2, and 6 for a. $\quad a(x - r_1)(x - r_2) = 6(x + \frac{4}{3})(x - \frac{1}{2}). \quad (3)$

Therefore, $\quad 6x^2 + 5x - 4 = 6(x + \frac{4}{3})(x - \frac{1}{2}). \quad (4)$

Using the factor 6 as $3 \cdot 2$, the result may be written $(3x + 4)(2x - 1)$.

2. Factor: $\quad x^2 - x + 1. \quad (1)$

Solving $x^2 - x + 1 = 0$, $\quad x = \frac{1}{2} + \frac{1}{2}\sqrt{-3}. \quad (2)$

$x = \frac{1}{2} - \frac{1}{2}\sqrt{-3}.$

Substituting these values of x for r_1 and r_2,

$$a(x - r_1)(x - r_2) = \left(x - \frac{1 + \sqrt{-3}}{2}\right)\left(x - \frac{1 - \sqrt{-3}}{2}\right). \quad (3)$$

Thus, we have a general method of factoring which does not depend upon the trial methods of Chapter XII.

WRITTEN EXERCISES

Factor:

1. $3x^2 - 2x - 5.$
2. $9x^2 - 3x - 6.$
3. $6y^2 + y - 1.$
4. $15y^2 - 4y - 35.$
5. $10w^2 - 12w + 2.$
6. $9v^2 - 17v - 2.$
7. $6x^2 + 25x + 14.$
8. $2z^2 + 5z + 2.$
9. $6x^2 - 7x + 3.$
10. $5x^2 - 40x + 6.$
11. $a^{2m} - 2a^m - 3.$
12. $c^4 - 13c^2 + 36.$

RADICAL EQUATIONS

576. The simpler forms of radical equations can be solved by squaring both members, but other forms require special methods.

EXAMPLE

Solve: $5x^2 - 3x + \sqrt{5x^2 - 3x + 2} = 18.$ (1)

Putting $5x^2 - 3x = y$, the given equation becomes $y + \sqrt{y+2} = 18.$ (2)

Subtracting y, $\sqrt{y+2} = 18 - y.$ (3)

Squaring, $y + 2 = 324 - 36y + y^2.$ (4)

Rearranging, $y^2 - 37y + 322 = 0.$ (5)

Solving, $y = \frac{37 \pm \sqrt{37^2 - 4 \cdot 322}}{2}$ (6)

$= 23 \text{ or } 14.$ (7)

Hence the values of x are determined from $5x^2 - 3x = 23,$ (8)

and $5x^2 - 3x = 14.$ (9)

Solving these, $x = \frac{3 \pm \sqrt{469}}{10},$ (10)

and $x = 2, \text{ or } -1.4.$ (11)

Test. Substituting, it appears that 2 and -1.4 satisfy the equation.

The values $\frac{3 \pm \sqrt{469}}{10}$ satisfy

$$5x^2 - 3x - \sqrt{5x^2 - 3x + 2} = 18.$$

577. Radical equations in quadratic form are more easily handled in the notation of exponents.

EXAMPLE

Solve: $x^{-\frac{1}{2}} + x^{-1} - 6 = 0.$ (1)

Rearranging, $x^{-1} + x^{-\frac{1}{2}} - 6 = 0.$ (2)

Factoring, $(x^{-\frac{1}{2}} + 3)(x^{-\frac{1}{2}} - 2) = 0.$ (3)

Therefore, $x^{-\frac{1}{2}} = -3, \text{ and } x^{-\frac{1}{2}} = 2.$ (4)

Or, $x = \frac{1}{9}, \text{ and } x = \frac{1}{4}.$ (5)

Substituting in step (1), $\frac{1}{4}$ satisfies the equation, but $\frac{1}{9}$ is an extraneous root.

578. If the factors of a quadratic form are not readily apparent, the quadratic formula may be applied.

579. If the equation contains denominators irrational in the unknowns, it is generally best to clear of fractions.

WRITTEN EXERCISES

Solve:

1. $6x - x^{\frac{1}{2}} - 12 = 0.$
2. $x^{\frac{2}{5}} - x^{\frac{1}{5}} - 6 = 0.$
3. $x^2 + x + \sqrt{x^2 + x + 1} = -1.$
4. $x^2 + a + \sqrt{x^2 + 2a} = -a.$
5. $x^2 + 6x - 5 = \sqrt{x^2 + 6x + 7}.$
6. $\sqrt{x^2 + 6x - 16} + (x + 3)^2 = 25.$
7. $2x - 5x^{\frac{1}{2}} + 2 = 0.$
8. $x^{-\frac{2}{5}} - x^{-\frac{1}{5}} - 6 = 0.$
9. $3x^{-\frac{3}{2}} - 4x^{-\frac{3}{4}} = 7.$
10. $x^2 + 3x - 3 + \sqrt{x^2 + 3x + 17} = 0.$
11. $x + \sqrt{x^2 - 8} = \dfrac{3x - \sqrt{x^2 - 8}}{x - \sqrt{x^2 - 8}}.$
12. $\sqrt{x(a + x)} + \sqrt{x(a - x)} = 2\sqrt{ax}.$
13. $\sqrt{(\sqrt{1 + x^2} + x) \div (\sqrt{1 + x^2} - x)} = 4.$
14. $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}.$
15. $\sqrt{2x + 11} - \sqrt{x - 3} - \sqrt{x + 2} = 0.$
16. $\sqrt{y + 1} + \sqrt{y - 2} = \sqrt{2y + 3}.$
17. $\sqrt{\dfrac{x}{c} + \dfrac{a}{b}} + \sqrt{\dfrac{x}{c} - \dfrac{a}{b}} = \sqrt{\dfrac{4x}{c} - \dfrac{2a}{b}}.$
18. $\sqrt{2y - 2} + \sqrt{y} = \sqrt{6y - 5}.$
19. $2a = \sqrt{2ax + 5a^2} - \sqrt{2ax - 3a^2}.$
20. $\dfrac{\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2}}{\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2}} - 1 = 0.$

CERTAIN HIGHER EQUATIONS SOLVED BY THE AID OF QUADRATIC EQUATIONS

580. We have found the general solution of linear and quadratic equations with one unknown. Equations of the third and the fourth degree can also be solved generally by algebra, and certain types of equations of still higher degree as well; but these solutions do not belong to an elementary course. We shall take up only certain equations of higher degree whose solution is readily reduced to that of quadratic equations.

EXAMPLES

1\. Solve: $x^6 - 3x^3 - 4 = 0.$ (*1*)

Let $y = x^3$, then $y^2 - 3y - 4 = 0.$ (*2*)

Solving (*2*), $y = 4,$

and $y = -1.$ (*3*)

$\therefore$ by the substitution in (*2*), $x^3 = 4$, or $x^3 - 4 = 0,$

and $x^3 = -1$, or $x^3 + 1 = 0.$ (*4*)

Factoring (*4*), $x^3 - 4 = (x - \sqrt[3]{4})(x^2 + \sqrt[3]{4} \cdot x + \sqrt[3]{4^2}) = 0,$ (*5*)

and $x^3 + 1 = (x + 1)(x^2 - x + 1) = 0.$ (*6*)

Solving (*5*), $x = \sqrt[3]{4}$, or $\sqrt[3]{4}(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})$, or $\sqrt[3]{4}(-\frac{1}{2} - \frac{1}{2}\sqrt{-3}).$ (*7*)

Solving (*6*), $x = -1$, or $\frac{1}{2} - \frac{1}{2}\sqrt{-3}$, or $\frac{1}{2} + \frac{1}{2}\sqrt{-3}).$ (*8*)

2\. Solve: $(x^2 - 3x + 1)(x^2 - 3x + 2) = 12.$ (*1*)

This may be written $(x^2 - 3x + 1)[(x^2 - 3x + 1) + 1] = 12.$ (*2*)

If y is put for $x^2 - 3x + 1$, the equation becomes $y(y + 1) = 12,$ (*3*)

or $y^2 + y - 12 = 0.$ (*4*)

Solving, $y = 3$ or $-4.$ (*5*)

Then from (*3*), $x^2 - 3x + 1 = 3,$ (*6*)

and $x^2 - 3x + 1 = -4.$ (*7*)

Solving (*6*), $x = \dfrac{3 \pm \sqrt{17}}{2}.$ (*8*)

Solving (*7*), $x = \dfrac{3 \pm \sqrt{-11}}{2}.$ (*9*)

These are the four roots of the given equation of the fourth degree.

WRITTEN EXERCISES

Solve as above:

1. $x^4 - 3x^2 + 1 = 0.$
2. $12 - x^4 = 11x^2.$
3. $x^4 + ax^2 - 8a^2 = 0.$
4. $x^2 + 5 = \frac{5}{x^2 + 3}.$
5. $x^6 - 7x^3 + 6 = 0.$
6. $x^8 - 3x^4 + 2 = 0.$
7. $x^{10} - 5x^5 + 6 = 0.$
8. $x^4 + 13x^2 + 36 = 0.$
9. $ax^{2n} - bx^n + c = 0.$
10. $\frac{1}{x^2 + 1} + \frac{1}{x^2 + 2} = \frac{1}{x^2 + 3}.$
11. $x^{2n} - 4x^n - 5 = 0.$
12. $2x^6 + 5x^3 + 2 = 0.$
13. $(x^2 + 4)^2 - 4(x^2 + 4) + 4 = 0.$
14. $x^2 + 3x = 1 - \frac{1}{x^2 + 3x + 1}.$
15. $(x^2 - 3x + 1)(x^2 - 3x + 2) = 12.$
16. $(x^2 + 5x - 1)(x^2 + 5x + 1) = -1.$

581. Binomial Equations. Equations of the form $x^n \pm a = 0$ are called **binomial equations.** The simpler cases admit of being solved by elementary processes.

EXAMPLES

1. Solve: $x^3 - 1 = 0$, or $x^3 = 1.$ *(1)*

Factoring $x^3 - 1$, $(x - 1)(x^2 + x + 1) = 0.$ *(2)*

Finding equations equivalent to (2), $x = 1,\ x^2 + x + 1 = 0.$ *(3)*

Solving (3) $x = 1,\quad x = \frac{-1 \pm \sqrt{-3}}{2}.$ *(4)*

Thus we have found the three numbers such that the cube of each is 1, or *the three cube roots of unity.*

Verify this statement by cubing each number in step *(4)*.

Note that $\left(-\frac{1}{2} + \frac{1}{2}\sqrt{-3}\right)^2 = \frac{-1 - \sqrt{-3}}{2} = \frac{-1 - i\sqrt{3}}{2},$

also that $\left(-\frac{1}{2} - \frac{1}{2}\sqrt{-3}\right)^2 = \frac{-1 + \sqrt{-3}}{2} = \frac{-1 + i\sqrt{3}}{2}.$

Hence, if ω denotes one of the complex cube roots of 1, ω^2 is the other.

Every number has three cube roots; for example, the cube roots of 8 are 2, and 2ω, and $2\omega^2$.

2. Solve: $x^4 + 1 = 0$, or $x^4 = -1$. (1)

Factoring x^4+1, $(x^2 - i)(x^2 + i) = 0$. (2)

Solving (2), $x^2 = i$, $x^2 = -1$. (3)

Solving (3), $x = \pm\sqrt{i}$, $x = \pm\sqrt{-i}$. (4)

These are the four numbers, each of which raised to the fourth power equals -1 or *the four fourth roots of* -1.

WRITTEN EXERCISES

1. Find the 3 cube roots of -1 by solving $x^3 + 1 = 0$.

2. Find the 4 fourth roots of 1 by solving $x^4 - 1 = 0$.

3. Find the 6 sixth roots of 1 by solving

$$(x^6 - 1) = (x^3 - 1)(x^3 + 1) = 0.$$

4. Find the 4 fourth roots of 16 by solving $x^4 - 16 = 0$.

5. Find the 3 cube roots of 8 by solving $x^3 - 8 = 0$.

6. Show that the square of either irrational cube root of -1 is the negative of the other irrational cube root.

7. Show that the sum of the three cube roots of unity is zero; also that the sum of the 6 sixth roots of unity is zero.

SIMULTANEOUS QUADRATIC EQUATIONS WITH TWO UNKNOWNS

582. The simpler cases in which one equation is linear or homogeneous of the second degree, or in which the equations are symmetric, have been solved in Chapter XXIII. The following Sections 583, 584 treat of special cases, showing how these may be reduced to the simpler systems.

A system of two simultaneous quadratic equations whose terms are of the second degree in x *and* y *with the exception of the absolute terms can be solved by reducing to a system in which one equation is entirely homogeneous.* (Sec. 120.)

This can be done in two ways:

1. *Make their absolute terms alike, and subtract. The resulting equation has every term of the second degree in* x *and* y.

EXAMPLE

Solve: $\begin{cases} x^2 + xy = 66, & (1) \\ x^2 - y^2 = 11. & (2) \end{cases}$

Multiplying (2) by 6, $6x^2 - 6y^2 = 66.$ (3)

Subtracting (1) from (3), $5x^2 - xy - 6y^2 = 0.$ (4)

Factoring (4), $(5x - 6y)(x + y) = 0.$ (5)

Expressing x in terms of y, $x = \frac{6}{5}y$ and $x = -y.$ (6)

Substituting $x = \frac{6}{5}y$ in (2), $\frac{36}{25}y^2 - y^2 = 11.$ (7)

Solving (7), $y = \pm 5.$ (8)

Substituting $y = \pm 5$ in $x = \frac{6}{5}y$, $x = \pm 6.$ (9)

Similarly, substituting $x = -y$ in (2), $y^2 - y^2 = 11.$ (10)

But this leads to $0 = 11.$ (11)

(11) being impossible, the solution is

$$x = \pm 6, \; y = \pm 5. \quad (12)$$

TEST. Taking the values to be both positive or both negative, $\begin{cases} (\pm 6)^2 + (\pm 6)(\pm 5) = 66. \\ (\pm 6)^2 - (\pm 5)^2 = 11. \end{cases}$

2. *Substitute* vx *for* y *throughout the equations and solve for* v.

EXAMPLE

Solve: $\begin{cases} 2x^2 - 3xy + y^2 = 4, & (1) \\ x^2 - 2xy + 3y^2 = 9. & (2) \end{cases}$

Putting $y = vx$ in (1) and (2), $2x^2 - 3vx^2 + v^2x^2 = 4,$ (3)

and, $x^2 - 2vx^2 + 3v^2x^2 = 9.$ (4)

Factoring (3) and (4), $x^2(2 - 3v + v^2) = 4.$ (5)

$x^2(1 - 2v + 3v^2) = 9.$ (6)

Equating the values of x^2 in (5) and (6), $\dfrac{4}{2 - 3v + v^2} = \dfrac{9}{1 - 2v + 3v^2}.$ (7)

Clearing (7) of fractions, $3v^2 + 19v - 14 = 0.$ (8)

Solving (8), $v = -7, \frac{2}{3}.$ (9)

Since $y = vx$, $y = -7x,$ (10)

and, $y = \dfrac{2x}{3}.$ (11)

From (1), $2x^2 + 21x^2 + 49x^2 = 72x^2 = 4.$ (12)

Solving (12), $x = \pm \dfrac{1}{3\sqrt{2}}.$ (13)

From (10), $y = \mp \dfrac{7}{3\sqrt{2}}.$ (14)

Similarly, from (11) and (1), $x = \pm 3,$ (15)

From (15) and (11), $y = \pm 2.$ (16)

TEST as usual.

WRITTEN EXERCISES

Solve:

1. $x^2 + y^2 = 13,$
   $xy = 6.$

2. $x^2 - xy = 54,$
   $xy - y^2 = 18.$

3. $x^2 + xy = 12,$
   $y^2 + xy = 24.$

4. $4x^2 + 3y^2 = 43,$
   $3x^2 - y^2 = 3.$

5. $\frac{x+y}{x-y} + \frac{x-y}{x+y} = \frac{5}{4},$
   $x^2 + y^2 = 20.$

6. $x^2 + y^2 = 41,$
   $xy = 20.$

7. $x^2 + xy + y^2 = 7,$
   $3x^2 - xy = 1.$

8. $x^2 + xy - 2y^2 = 4,$
   $x^2 - 3xy + 2 = 0.$

9. $x^2 + 6xy + 2y^2 = 133,$
   $x^2 - y^2 = 16.$

10. $2x^2 + 3xy + y^2 = 80,$
    $xy + x^2 = 6.$

11. $x^2 - 4y^2 = 20,$
    $xy = 12.$

12. $4x^2 + 3y^2 = 43,$
    $3x^2 - y^2 - 3 = 0.$

583. The substitution of a single variable for a function is the most successful means of simplifying quadratic systems that do not yield to the methods already given.

EXAMPLE

$$\begin{cases} x - y - \sqrt{x-y} = 2, & (1) \\ x^3 - y^3 = 2044. & (2) \end{cases}$$

Put $\sqrt{x-y} = z.$ (3)

From (1), $z^2 - z - 2 = 0.$

$(z-2)(z+1) = 0.$ (4)

$z = 2,$ or $-1.$ (5)

$\therefore$ from (3), $x - y = 4,$ or $1.$ (6)

From (2),

$(x-y)(x^2 + xy + y^2) = 2044.$ (7)

From (7) and (6),

$x^2 + xy + y^2 = 2044,$ or $511.$ (8)

From (6), $y = x - 1.$ Put this in (8).

Then, $x^2 + x(x-1) + (x-1)^2 = 2044.$ (9)

$x^2 + x^2 - x + x^2 - 2x + 1 = 2044.$ (10)

$x^2 - x - 681 = 0.$ (11)

Solving (11), $x = \frac{1 \pm 5\sqrt{109}}{2}$ and $y = \frac{-5 \pm \sqrt{109}}{2}.$

From (*6*), $y = x - 4$. Put this in (*8*).

$$x^2 + x(x-4) + (x-4)^2 = 511. \quad (13)$$

$$3x^2 - 12x - 495 = 0. \quad (14)$$

$$x^2 - 4x - 165 = 0. \quad (15)$$

$$x = \frac{4 \pm \sqrt{16 + 4.165}}{2} \quad (16)$$

$$= \frac{4 \pm 2\sqrt{4 + 165}}{2} \quad (17)$$

$$= 2 \pm 13 \quad (18)$$

$$= 15, \text{ or } -11. \quad (19)$$

$$\therefore y = 11, \text{ or } -15. \quad (20)$$

Test by substitution.

WRITTEN EXERCISES

Solve:

1. $x^2 - y^2 = 3,$
   $x^2 + y^2 - xy = 3.$

2. $\frac{x+y}{x-y} + \frac{x-y}{x+y} = \frac{5}{2},$
   $x^2 + y^2 = 90.$

3. $x^2 + xy + y^2 = 7,$
   $x^4 + x^2y^2 + x^4 = 21.$

4. $x^2 + y^2 - 1 = 2xy,$
   $xy(xy + 1) = 8190.$

5. $x^2 + xy = \frac{5}{12},$
   $xy + y^2 = \frac{5}{18}.$

6. $x^2 + 2xy = 55,$
   $2x^2 - xy = 35.$

7. $2x^2 + xy = 24,$
   $x^2 - y^2 = 5.$

8. $x^2 - y^2 = 8\frac{3}{4},$
   $xy + y^2 = \frac{7}{4}.$

9. $x^2 - y^2 = 45,$
   $xy + y^2 = 18.$

10. $x - 2y = 2,$
    $x^2 - 6y^2 = 14 - xy.$

11. $x^2 + y^2 = 25,$
    $x + y = 1.$

12. $x - y = 2,$
    $x^2y - xy^2 = 30.$

13. $x^2 + y^2 + 2(x + y) = 12,$
    $xy - (x + y) = 2.$

14. $x^2 + 4y^2 = 13,$
    $x + 2y = 5.$

15. $2x + y = 5,$
    $3x^2 - 7y^2 = 5.$

16. $x - y = 5,$
    $\sqrt{x} - \sqrt{y} = 1.$

17. $x^2 + xy + y^2 = 21,$
    $x - \sqrt{xy} + y = 3.$

    Suggestion. In Ex. 17 divide (1) by (2), obtaining

    $$x + y + \sqrt{xy} = 7.$$

    Add this equation to (2) and find x in terms of y.

18. $x^2 + tx + t^2 = 133,$
    $t + x - \sqrt{tx} = 7.$

19. $x^2 + 2xy + 7y^2 = 24,$
$2x^2 - xy - y^2 = 8.$

20. $x(x - y) = 0,$
$x^2 + 2xy + y^2 = 9.$

21. $\sqrt{x} - \sqrt{y} = 2,$
$(\sqrt{x} - \sqrt{y})\sqrt{xy} = 30.$

22. $(2x - 3)(3y - 2) = 0,$
$4x^2 + 12xy - 3y^2 = 0.$

SUGGESTION. In Ex. 22, from equation (1), $x = \frac{3}{2}$. The corresponding value of y is found by substituting this value of x in equation (2).

23. Two men, A and B, dig a trench in 20 days. It would take A alone 9 days longer to dig it than it would B. How long would it take A and B each working alone?

NOTE. Those who have studied geometry and are ready for problems based upon geometric properties will find them in Chapter XXXIII.

REVIEW

WRITTEN EXERCISES

Solve:

1. $x^2 = 6x - 5.$
2. $w^2 - w - 1 = 0.$
3. $x^2 - 6x - 7 = 0.$
4. $v^2 + 2v + 6 = 0.$
5. $x^2 - 5x + 1 = 0.$
6. $2x^2 - x + 3 = 0.$
7. $3x^2 - x + 7 = 0.$
8. $x^2 - 5x + 11 = 0.$
9. $x^2 - 14x + 5 = 0.$
10. $3x^2 - 9x - \frac{5}{2} = 0.$
11. $x^2 - 4x - 45 = 0.$
12. $x^2 - 4x + 45 = 0.$
13. $x^2 + 4x - 45 = 0.$
14. $x^2 + 4x + 45 = 0.$
15. $x^2 + 30x + 221 = 0.$
16. $x^2 - 30x - 221 = 0.$
17. $x^2 = 6x + 16.$
18. $24 - 10x = x^2.$
19. $(x - 1)^2 = x + 2.$
20. $5x + x^2 + 6 = 0.$
21. $x^2 - 9x + 14 = 0.$
22. $x^2 + 3x - 70 = 0.$
23. $4x^2 - 4x - 3 = 0.$
24. $3x^2 - 7x + 2 = 0.$
25. $x^2 - 10x + 21 = 0.$
26. $x^2 - 10x + 24 = 0.$
27. $x^2 + 10x + 24 = 0.$
28. $x^2 = 9x^2 - (x + 1)^2.$
29. $9x^2 + 4x - 93 = 0.$
30. $4x^2 + 3x - 22 = 0.$
31. $6x^2 - 13x + 6 = 0.$
32. $(x + 1)(x + 2) = x + 3.$

33. $x+\frac{a^2}{x}=\frac{a^2}{b}+b.$

34. $cx^2+bx+a=0.$

35. $(x-2)(x+3)=16.$

36. $\frac{a^2}{b+x}+\frac{a^2}{b-x}=c.$

37. $\frac{x+1\frac{1}{3}}{x+2}-\frac{x+12}{2(x+19)}=\frac{1}{2}.$

38. $5x-\frac{3(x-1)}{x-3}=2x+\frac{3(x-2)}{2}.$

39. $(x-1)^2(x+3)=x(x+5)(x-2).$

40. $x^2-6acx+a^2(9c^2-4b^2)=0.$

41. $\frac{1}{a}+\frac{1}{b}+\frac{1}{x}-\frac{1}{a+b+x}=0.$

42. $\frac{1}{a}+\frac{1}{a+x}+\frac{1}{a^2-x^2}=0.$

43. $(2a-5-x)^2+(3a-3x)^2=(a+5-2x)^2.$

44. $(3x-4a+3b)^2+(2x+a)^2=(x-4a+b)^2 + (2x-3a+4b)^2.$

45. $(7a+3b+x)^2+(4a-b-8x)^2-(4a+3b+4x)^2 =(7a+b-5x)^2.$

46. $(x+a)(5x-3a-4b)=(x+a-2b)^2.$

47. $(5x+4a+3b)(10x-6a+8b)=(5x+a+7b)^2.$

48. $(26x+a+22b)(14x+13a-2b)=(16x+11a+8b)^2.$

49. $(8c+10+4x)(18c+160+24x)=(12c+40+11x)^2.$

50. $\frac{14y^2+16}{21}-\frac{2y^2+8}{8y^2-11}=\frac{2y^2}{3}.$

51. $\frac{a^2+x^2}{a^2}=\frac{(b+c)^2}{(b-c)^2}.$

52. $\frac{3x^2-27}{x^2+3}+\frac{90+4x^2}{x^2+9}=7.$

53. $\frac{x^2+x-2}{x^2+2x-3}-\frac{x+1}{x-3}=0.$

54. $\frac{1}{a+b-x}=\frac{1}{a}+\frac{1}{b}-\frac{1}{x}.$

55. $(a-x)\left(1-\frac{3a+3x}{c-\frac{1}{2}}\right)-2=\frac{1-2a}{1-2c}(c-3)-\frac{1+a}{1-\frac{1}{2c}}.$

56. $x^2 + xy + 4y^2 = 6,$
$3x^2 + 8y^2 = 14.$

57. $x^3 + y^3 = a,$
$x + y = b.$

58. $2x^2 + xy - 6y^2 = 3\frac{3}{4},$
$x + 2y = 2\frac{1}{2}.$

59. $x^2 + xy = 36,$
$x^2 - y^2 = -9.$

60. $x^2 + 3xy = 22,$
$x + y = 5.$

61. $x^2 + 4y^2 = 85,$
$x - y = 2.$

62. $x^2 - y^2 = 112,$
$x + y = 14.$

63. $x + y = 7,$
$x^3 + y^3 = 133.$

64. $x^3 + y^3 = 61,$
$x^2 - xy + y^2 = 61.$

65. $x^3 - y^3 = 665,$
$x - y = 5.$

66. $x^3 + y^3 = 28,$
$x^2y + xy^2 = 12.$

67. $x^2 - xy + y^2 = 37,$
$x + y = 10.$

68. $\dfrac{x}{x^2 + x + 5} + \dfrac{5}{\sqrt{x^2 + x + 5}} = \dfrac{24}{4x}.$

SUGGESTION. Multiply both members by x and let

$$y = \frac{x}{\sqrt{x^2 + x + 5}}.$$

69. $x^3 - 6x^{\frac{3}{2}} - 16 = 0.$

State what can be known by means of the discriminant and without solving, concerning the factors of:

70. $3x^2 - 2x + 1.$

71. $4x^2 + 11x - 1.$

72. $5y^2 + 20y + 20.$

73. $2x^2 - x + 3.$

Similarly, what can be known about the roots of:

74. $t^2 - 9 = 0?$

75. $5x^2 + x + 2 = 0?$

76. $z^2 - z - 1 = 0?$

77. $5x^2 = 9x?$

How must a be chosen in order that:

78. The roots of $x^2 + ax + 5 = 0$ shall be imaginary?

79. The roots of $ax^2 + 6x + 1 = 0$ shall be real?

80. The roots of $x^2 + 4x + 2a = 0$ shall be real and of opposite signs?

81. The roots of $(a + 1)x^2 + 3x - 2 = 0$ shall be imaginary?

82. The roots of $4x^2 - ax + 2 = 0$ shall be real and both positive?

Find the values of m for which the roots of the following equations are equal to each other. What are the corresponding values of x?

83. $x^2 - 12x + 3m = 0$.

84. $mx^2 + 8x + m = 0$.

85. $4x^2 + mx + x + 1 = 0$.

86. $mx^2 + 3mx - 5 = 0$.

87. A number increased by 30 is 12 less than its square. Find the number.

88. The product of two consecutive odd numbers is 99. What are the numbers? Is there more than one set?

89. Find two consecutive even numbers the sum of whose squares is 164.

90. Find a positive fraction such that its square added to the fraction itself makes $\frac{4}{9}$.

91. If a denotes the area of a rectangle and p its perimeter, show that the lengths of the sides are the roots of the equation

$$x^2 - \frac{p}{2}x + a = 0.$$

92. The diagonal and the longer side of a rectangle are together equal to 5 times the shorter side, and the longer side exceeds the shorter by 35 meters. Find the area of the rectangle.

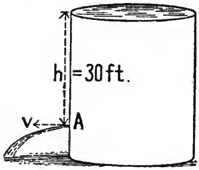
93. A company of soldiers attempts to form in a solid square, and 56 are left over. They attempt to form in a square with 3 more on each side, and there are 25 too few. How many soldiers are there?

94. It took a number of men as many days to dig a ditch as there were men. If there had been 6 more men, the work would have been done in 8 days. How many men were there?

95. Solve: $2x^2 + 6x + c = 0$.
What value must c have to make the two values of x equal?

96. A library spends \$180 monthly for books. In June the average cost per book was 15¢ less than in May, and 60 books more were bought. How many were bought in May?

97. When water flows from an orifice in a tank the square of the velocity (v) equals $2g$ times the height (h) of the surface above the orifice. Write the equation that denotes this fact. g is the "constant of gravity" and may be taken as 32.



98. What does the square of the velocity, (v^2), at A in the figure equal? Find this velocity.

99. What would be the velocity of the water if an opening were made halfway up from A shown in the figure?

100. Find the velocity with which water rushes through an opening at the base of a dam against which the water stands 25 ft. high.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. State the general forms of the roots of the general quadratic equation. Sec. 563.

2. State the relation between the *roots* and *coefficients*. Sec. 566.

3. Name two *symmetric* functions of the roots of the quadratic equation. Sec. 567.

4. What is the *discriminant* of the quadratic? Sec. 572.

5. What is the nature of the discriminant $b^2 - 4ac$ when the roots are:

1. Real and unequal?
2. Real and equal?
3. Complex numbers?
4. Irrational numbers? Sec. 571.

6. What methods are used to solve radical equations? Secs. 576–579.

7. What kind of higher equations may be solved like quadratics? Sec. 580.

8. What kind of equations does $x^n \pm a = 0$ represent? Sec. 581.

9. State the three cube roots of unity in terms of 1 and ω. Sec. 581.

10. What method of solving simultaneous quadratics has the broadest application? Sec. 583.

HISTORICAL NOTE

Among the many improvements in algebra made in the sixteenth century was that masterly achievement, the general solution of the cubic equation. We have explained how for centuries the solution of the quadratic equation $x^2 + px = q$ baffled the skill of the keenest mathematicians, but the solution of the cubic $x^3 + px^2 = q$ had been regarded quite unattainable, so much so that the great Italian Paciuolo (1494) closed his famous work *Summa* with the remark that the solution of this equation was as impossible as the squaring of the circle. But to assert that the solution of a problem is impossible is dangerous ground to take, as was proved in this case. For within fifty years after this remark of Paciuolo, his countryman Niccolo solved the cubic equation.

Niccolo was born at Brescia just at the beginning of the sixteenth century, and came to be known as Tartaglia, meaning, "The stammerer." This defect in his speech was caused by a saber cut received at the hands of a French soldier when Niccolo was six years old. Although poverty deprived him of the advantages of school instruction, his industry and persistence enabled him to master the classics and mathematics, and to add (1541) to the science of algebra the most significant discovery of the sixteenth century. It was Tartaglia's ambition to write a great work on algebra and to use this as a means of giving to the world his method of solving the cubic equation. But before he could accomplish this, he was betrayed by his friend Cardan, to whom, after much pleading, he had revealed the nature of his discovery. Cardan published his now famous work, the *Ars Magna*, in 1545,

NICCOLO, OR TARTAGLIA

and included, as the crown to his treatment of equations, the solution by Tartaglia, claiming for himself full credit for the discovery. This deception and robbery was a bitter disappointment to Tartaglia, who never recovered sufficiently to complete his projected writings, and, although the claim of Cardan is now conceded by all to be fraudulent, the solution of the cubic equation is still commonly called Cardan's method.

The solution of the equation of the fourth degree immediately followed that of the cubic equation. This was first effected, in a similar way, by Ferrari, a pupil of Cardan, and was also published in the *Ars Magna.* It was naturally supposed that these methods could be extended to equations of degrees higher than the fourth, and prodigious labor was expended in the effort to do this, until Abel, a Norwegian mathematician, proved in 1824 that the methods of elementary algebra are not sufficient to solve general equations of degree higher than the fourth.

CHAPTER XXXI

PROPORTION, VARIATION, AND LIMITS

PROPORTION

584. Proportion. If four numbers, a, b, c, d, are such that $\frac{a}{b}=\frac{c}{d}$, they are said to form the proportion, ***a* is to *b* as *c* is to *d*.**

585. Fourth Proportional. If four numbers a, b, c, d are in proportion, d is called the *fourth proportional* to a, b, and c.

586. Third Proportional. If three numbers, a, b, c, are such that a, b, b, c are in proportion, c is called the *third proportional* to a and b.

587. Mean Proportional. If a, b, b, c are in proportion, b is called the *mean proportional* to a and c.

588. Means and Extremes. If a, b, c, d are in proportion, b and c are called the *means*, and a and d the *extremes*.

589. Relation of Means to Extremes. *In any proportion the product of the means equals the product of the extremes, and conversely, if the product of two numbers equals the product of two other numbers, the four numbers can be arranged in a proportion.*

For, the two factors of one product may be made the means, and the two factors of the other the extremes, or *vice versa*.

590. Inversion. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$.

For, the members of the second equation are the reciprocals of the members of the first.

591. Alternation. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

For, the second equation results if the first is multiplied by $\frac{b}{c}$.

592. Composition. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

For, the second equation results from the first by adding 1 to both members.

593. Division. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$.

For, the second equation results from the first if 1 be subtracted from both members.

594. Composition and Division. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

For, the result of Sec. 592 divided by the result of Sec. 593 gives the second equation.

595. Continued Proportion. If several ratios are equal, as in $\frac{a}{b}=\frac{c}{d}=\frac{e}{f},\ldots$ the numbers, a, b, c, d, e, $f,\ldots$, are said to be in continued proportion.

In the continued proportion $\frac{a}{b}=\frac{c}{d}=\frac{e}{f},\ldots$, any one of the ratios equals $\frac{a+c+e\ldots}{b+d+f\ldots}$.

596. These properties may be applied to certain problems.

EXAMPLES

1. If p and w are the power and weight applied to a lever whose power and weight arms are, respectively, a and b, then $pa=bw$. Write a proportion between these four numbers.

SOLUTION. By Sec. 589, $a:b=w:p$.

2. Find the third proportional to $a-b$ and a^2-b^2.

SOLUTION. 1. By Sec. 589, $\frac{a-b}{a^2-b^2}=\frac{a^2-b^2}{x}$.

2. $\therefore (a-b)x=(a^2-b^2)(a^2-b^2)$.

3. $\therefore$ the third proportional, $x=(a+b)(a^2-b^2)$.

3. Form a proportion from the equation, $x^2-y^2=z^2$.

SOLUTION. 1. Factoring, $(x-y)(x+y)=zz$.

2. $\therefore \frac{x-y}{z}=\frac{z}{x+y}$. Sec. 589.

4. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{c+d}=\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$. (1)

By Sec. 592 $$\frac{a+b}{b}=\frac{c+d}{d}. \qquad (2)$$

∴ by Sec. 591, $$\frac{a+b}{c+d}=\frac{b}{d}. \qquad (3)$$

Squaring both members of the given equation, $$\frac{a^2}{b^2}=\frac{c^2}{d^2}. \qquad (4)$$

Applying Sec. 592 to (4), $$\frac{a^2+b^2}{b^2}=\frac{c^2+d^2}{d^2}. \qquad (5)$$

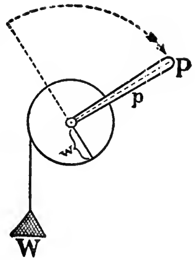
Applying Sec. 591 to (5), $$\frac{a^2+b^2}{c^2+d^2}=\frac{b^2}{d^2}. \qquad (6)$$

From (6), $$\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}=\frac{b}{d}. \qquad (7)$$

Equating values of $\frac{b}{d}$ in (3) and (6), $$\frac{a+b}{c+d}=\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}. \qquad (8)$$

WRITTEN EXERCISES

1. A lever need not be straight, although it must be rigid. Thus, the crank and the wheel and axle are varieties of the lever, and the law of the lever (p. 188) applies to them. Thus, in the figure,



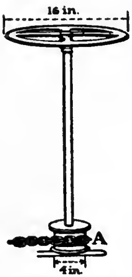
$$\frac{P}{W}=\frac{w}{p}.$$

Find W, if $P=14$, $p=16$, and $w=4$.

2. Find the unknown number:

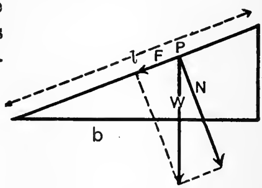
	(1)	(2)	(3)
$P=$	$3a$	—	$a-b$
$W=$	$6b$	$5p$	$(a+b)^2$
$p=$	$2c$	$8p$	$a+b$
$w=$	—	$2p$	—

3. If an axle is 6 in. in diameter, what must be the diameter of the wheel in order that a boy exerting a force of 50 lb. may be able to raise 800 lb. weight?



4. A brakeman pulls with a force of 150 lb. on a brake wheel 16 in. in diameter. The force is communicated to the brake by means of an axle, A, 4 in. in diameter. What is the pull on the brake chain?

5. In the figure below the weight W acts at P on an inclined plane, whose rate of slope is a vertical units to b horizontal units. It is known that the weight W acts in two ways: a force N pressing directly against the surface and tending to produce friction, and a force F parallel to the plane and tending to cause the weight to slide down the plane. It is known that these various quantities are related to each other thus:



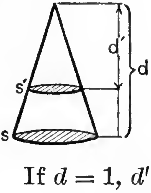
$$\frac{F}{W} = \frac{a}{l}. \qquad \frac{N}{W} = \frac{b}{l}.$$

Find F and N, if $W = 15$ lb., $a = 20$ in., $b = 21$ in., $l = 29$ in.

6. Find l and N, if $F = 16$ lb., $W = 34$ lb., $a = 4$ ft., $b = 7\frac{1}{2}$ ft.

7. Find b and l, if $F = 66$ lb., $N = 112$ lb., $W = 130$ lb., $a = 33$ in.

8. Find W and a, if $F = 200$ lb., $N = 45$ lb., $b = 9$, $l = 41$.



9. It is known that in any cone the area of any section parallel to the base varies as the square of its distance from the vertex. Thus, in the figure, $\frac{s'}{s} = \frac{d'^2}{d^2}$.

If $d = 1$, $d' = \frac{1}{2}$, and $s = 40$ sq. in., what is the area of s'?

10. The area of a section $\frac{1}{3}$ of the way from the vertex to the base and parallel to it is what part of the base?

11. Given $\frac{a}{b}=\frac{b}{c}$, show that $\frac{a^2+b^2}{a+c}=\frac{a^2-b^2}{a-c}$.

12. Given $\frac{a}{b}=\frac{b}{c}$, show that $\frac{a^2-b^2}{a}=\frac{b^2-c^2}{c}$.

13. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a}{d(a+b)^2}=\frac{c}{b(d+c)^2}$.

14. If $a:b=p:q$, prove that

$$a^2+b^2:\frac{a^3}{a+b}=p^2+q^2:\frac{p^3}{p+q}.$$

SUGGESTION. The given proportion may be written :

$$\frac{a}{p}=\frac{b}{q}.$$

Let $\frac{a}{p}=r$, then $a=pr$, $b=qr$.

The proportion to be proved may be written :

$$\frac{(a+b)(a^2+b^2)}{a^3}=\frac{(p+q)(p^2+q^2)}{p^3}.$$

Substituting the values of a and b above, the left member readily reduces to the right.

NOTE. A good method for proving such identities is to begin with the required relation and transform it into the given relation, or to transform both the given and the required relation until they reduce to the same thing.

15. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{4a^2-5b^2}{5b^2}=\frac{4c^2-5d^2}{5d^2}$.

16. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a^2-b^2}{c^2-d^2}=\frac{b^2}{d^2}$.

17. Write $x^2-4y^2=x^2-xy$ in the form of a proportion.

18. Given $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a}{c}=\frac{\sqrt{a^2-b^2}}{\sqrt{c^2-d^2}}$.

VARIATION

597. Direct Variation. When two variable quantities vary so as always to remain in the same ratio, each is said to **vary directly** as the other. Each increases or decreases at the same rate that the other increases or decreases.

Consequently, if x and y are two corresponding values of the variables and k the fixed ratio, then,

$$\frac{x}{y} = k \text{ and } x = ky.$$

For example, at a fixed price (k) per article, the total cost (x) of a number of articles of the same sort varies directly with the number of articles (y). That is $\frac{x}{y} = k$.

Likewise in the case of motion at a uniform rate (r), the distance traversed (d) varies as the time of motion (t). That is, $\frac{d}{t} = r$.

598. A symbol still occasionally used for "varies as" is $\propto$.

Thus, "x varies as y" is written $x \propto y$,
and "d varies as t" is written $d \propto t$.

599. Relation of Variation to Proportion. When one variable varies directly as another, any pair of values of the variables forms a proportion with any other pair.

For, $\frac{x'}{y'} = r$, and $\frac{x''}{y''} = r$; $\therefore \frac{x'}{y'} = \frac{x''}{y''}$, which is a proportion.

600. Expressions for Direct Variation. We have thus seen that the relation *x varies directly as y* may be expressed in any one of three ways:

(*a*) $x = ky$, by use of the equation.

(*b*) $x \propto y$, by use of the symbol of variation.

(*c*) $\frac{x'}{y'} = \frac{x''}{y''}$ by use of the proportion; x', y' and x'', y'' being any two pairs of corresponding values of the variables.

WRITTEN EXERCISES

1. Write the statement "v varies as w" in the form of an equation, also in the form of a proportion.

2. Write the statement $x = ky$ by use of the symbol $\propto$; also in the form of a proportion.

3. Write $\frac{t'}{v'} = \frac{t''}{v''}$ by use of the symbol $\propto$.

4. The weight (w) of a substance varies as the volume (v) when other conditions are unchanged. Express this law by use of the equation. By use of the symbol $\propto$. Also in the form of a proportion.

5. In the equation $w = kv$, if $w = 4$ and $v = 2$, what is the value of k? Using this value of k, what is the value of w when $v = 25$?

6. When 1728 cu. in. of a substance weigh 1000 ounces, what is the ratio of the weight (w) to the volume (v)? What volume of this substance will weigh 5250 ounces?

7. The cost (c) of a grade of silk varies as the number of yards (n). Find the ratio (r) of c to n when c is $7.00 and $n = 4$.

8. In Exercise 7 if $\frac{c}{n} = r$, what does $\frac{c'}{n'}$ equal? Given that 40 yd. of silk cost $60, find the cost of 95 yd. by means of the proportion $\frac{c'}{n'} = \frac{c}{n}$. Also by means of the equation $c = nr$.

601. Inverse Variation. A variable x is said to **vary inversely** as a variable y, if it varies directly as $\frac{1}{y}$.

Inverse variation means that when one variable is doubled the other is halved; when one is trebled the other becomes $\frac{1}{3}$ of its original value, and so on.

602. Expressions for Inverse Variation. The relation "x varies inversely as y" may be expressed:

(a) By the equation, $x = k\left(\frac{1}{y}\right)$, or $x = \frac{k}{y}$. $\therefore xy = k$.

(b) With the symbol of variation $y \propto \frac{1}{x}$.

(c) As a proportion $\frac{x'}{x''} = \frac{y''}{y'}$.

WRITTEN EXERCISES

1. Write the statement "v varies inversely as w" in the form of an equation; also in the form of a proportion.

2. Write $t=\frac{k}{p}$ by use of the symbol $\propto$; also in the form of a proportion.

3. In a bicycle pump the volume (v) of air confined varies inversely as the pressure (p) on the piston. Write the relation between v and p in three ways.

4. In Exercise 3, if $v=18$ (cu. in.) and $p=15$ (lb.), what is k in $v=\frac{k}{p}$? What is the pressure (p) when $v=1$ (cu. in.)?

5. In an auditorium whose volume (v) is 25,000 cu. ft. there are 2000 persons (p). What is the number (n) of cubic feet of air space to the person? What will be the number when 1000 more persons come in?

6. The area of a triangle varies as the base times the altitude. If the area is 12 when the base is 8 and the altitude 3, what is the area of a triangle whose base is 40 and altitude 20?

7. The area of a circle varies as the square of its radius. The area of a circle of radius 2 is 12.5664; what is the area of a circle whose radius is 5.5?

8. The volume of a sphere varies as the cube of its radius. If the volume of a sphere whose radius is 3 is 113.0976, what is the volume of a sphere whose radius is 5?

9. If $x \propto y$ and $x=6$ when $y=2$, find x when $y=8$.

SUGGESTION. $x=ky$. $\therefore 6=k\cdot 2$ and $k=3$. Substitute $y=8$ in the equation $x=3y$.

10. Determine k in $x \propto y$, if $x=10$ when $y=20$. Also if $x=1$ when $y=5$. If $x=100$ when $y=10$.

11. If $x \propto w$ and $y \propto w$, prove that $x+y \propto w$.

12. If $x \propto w$ and $w \propto y$, prove that $xy \propto w^2$.

13. If $x \propto y$ and $w \propto z$, prove that $\frac{x}{w} \propto \frac{y}{z}$.

14. If $x \propto y^2z$ and $x = 1$ when $y = 2$ and $z = 3$, find the constant k.

15. Given $y = z + w$, $z \propto x$ and $w \propto x$; and that $x = 1$ when $w = 6$, and that $x = 2$ when $z = 20$. Express y in terms of x.

SUGGESTION. $y = kx + k'x$; determine k and k'.

16. Solve Exercise 15 under the conditions that $x = 2$ when $w = 12$, and that $x = 1$ when $z = 10$.

17. Given $z \propto x + y$ and $y \propto x^2$, and that $x = \frac{1}{2}$ when $y = \frac{1}{3}$ and $z = \frac{1}{4}$. Express z in terms of x.

603. A large number of problems in science may be solved by the following plan:

EXAMPLES

1. The "law of gravitation" states that the weight of a given body varies inversely as the square of its distance from the center of the earth. What is the weight of a body 5 mi. above the surface of the earth, which weighs 10 lb. at the surface (4000 mi. from the center)?

METHOD. There are two variables in the problem, the weight (w) and the distance (d). There are also two parts or cases in the statement, one in which the value of one variable is unknown, and one in which the values of both variables are given.

Arrange the data as follows:

	w	d
1st case	x	4005
2d case	10	4000

To this table apply the law of variation expressed in the physical law. Since the law is: *w varies inversely as the square of d*, the values of d must be squared, and the ratio $x : 10$ equals the inverse ratio of 4005^2 to 4000^2; that is,

$$\frac{x}{10} = \frac{4000^2}{4005^2}.$$

$$4000 \div 4005 = .99874 + \text{ and } .9987^2 = 9.96\ldots$$

$$\therefore x = 9.96, \text{ and the weight is } 9.96 \text{ lb.}$$

2. The squares of the times of revolution of the planets about the sun vary directly as the cubes of their distances from the sun. The earth is 93,000,000 mi. from the sun, and makes a revolution in approximately 365 da. How far is Venus from the sun, its time of revolution being 226 da.?

Solution.

	t = time of revol.	d = distance from sun
1st case	365	93,000,000
2d case	226	x

According to the astronomical law the times must be squared and the distances cubed; then, since the law is that of direct variation:

$$\frac{365^2}{226^2} = \frac{93{,}000{,}000^3}{x^3}.$$

$\therefore x = 93{,}000{,}000\sqrt[3]{\left(\frac{226}{365}\right)^2} = 68{,}900{,}000$, and the distance of Venus from the sun is approximately 69,000,000 miles.

WRITTEN EXERCISES

1. The intensity of light from a given source varies inversely as the square of the distance from the source. If the intensity (candle power) of an electric light is 4 at a distance of 150 yd., what is its intensity at a distance of 25 yd.?

2. According to the first sentence of Exercise 1, how much farther from an electric light must a surface be moved to receive only $\frac{1}{4}$ as much light as formerly?

3. The time of oscillation of a pendulum varies directly as the square root of its length. What is the length of a pendulum which makes an oscillation in 5 sec., a 2-second pendulum being 156.8 in. long?

4. According to Exercise 3, what is the time of oscillation of a pendulum 784 in. long?

5. The distance through which a body falls from rest varies as the square of the time of falling. A body falls from rest 576 ft. in 6 sec.; how far does it fall in 10 sec.?

6. Volumes of similar solids vary as the cubes of their linear dimensions. The volume of a sphere of radius 1 in. is 4.1888 cu. in.; what is the volume of a sphere whose radius is 5 in.?

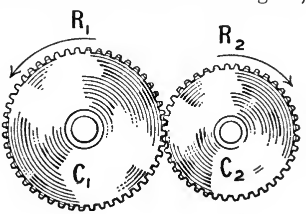
7. According to Exercise 6, what is the radius of a sphere whose volume is 33.5104 cu. ft.?

8. According to Exercise 6, if a flask holds $\frac{1}{2}$ pt., what is the capacity of a flask of the same shape 4 times as high?

9. In compressing a gas into a closed receptacle, as in pumping air into an automobile tire, the pressure varies inversely as the volume. If the pressure is 25 lb. when the volume is 125 cu. in., what is the pressure when the volume is 115 cu. in.?

10. According to Exercise 9, if the pressure is 50 lb. when the volume is 250 cu. in., what is the volume when the pressure is 10 lb.?

11. It is known that if one gear wheel turns another as in the figure, the number of revolutions of the two are to each other inversely as their number of teeth. That is, if the first has C_1 teeth and makes R_1 revolutions, and the second has C_2 teeth and makes in the same time R_2 revolutions,



then $$\frac{R_1}{R_2} = \frac{C_2}{C_1}.$$

Find R_2 if $C_1 = 25$, $C_2 = 15$, and $R_1 = 6$.

Find the numbers to fill the blanks:

	(1)	(2)	(3)	(4)
$C_1 =$	42	60	$5n$	—
$C_2 =$	—	50	$3n$	40
$R_1 =$	12	—	21	$8n$
$R_2 =$	9	15	—	$6n$

12. A diamond worth \$2000 was broken into two parts, together worth only \$1600. If the value of a diamond is proportional to the square of its weight, into what fractions was the original diamond broken? (Find result to nearest hundredth.)

LIMITS

604. In the construction of graphs we have studied the changes in functions corresponding to given values of their variables, but we shall now consider an important particular case in which the successive values of a variable approach nearer and nearer to a fixed number.

For example, if we take the values of the decimal .444 . . . corresponding to the successive decimal places, we have .4, .44, .444, .4444, . . . , a series of increasing numbers so limited that no one of them, however far we go, can be as large as .45. In fact, no one of them can equal $\frac{4}{9}$, but by using more and more decimal places we may come to terms differing as little from $\frac{4}{9}$ as we choose.

605. Limit. If a variable x has a boundless number of successive values which approach nearer and nearer to a fixed number l, so that the difference $l - x$ may become and remain numerically as small as we choose, *x is said to approach l as a limit.*

606. This relation is expressed in symbols by $x \doteq l$, read, "x approaches l as a limit."

For example: In Sec. 604, if the variable x be taken to represent the different numbers in the series .4, .44, .444, . . . , then $x \doteq \frac{4}{9}$.

Or, if x is successively .02, .002, .0002, and so on as far as we choose, then $x \doteq 0$.

607. Meaning of $\frac{0}{0}$. We have shown (Sec. 487, II) how the indicated operation, $\frac{0}{0}$, may arise from solving systems of equations, and that its value is indeterminate. We will further illustrate it here by use of expressions having limiting values.

For example, the fraction $\frac{x-1}{x^2-1}$ becomes $\frac{0}{0}$, if $x = 1$, but it can be shown to have the limiting value $\frac{1}{2}$. If we let $x = .9, .99, .999$, then $\frac{x-1}{x^2-1}$ becomes $\frac{10}{19}, \frac{100}{199}, \frac{1000}{1999}$, a series of fractions whose values decrease toward $\frac{1}{2}$ when $x \doteq 1$. Furthermore, $\frac{x-1}{x^2-1} = \frac{1}{x+1}$ by reducing to lowest terms, and $\frac{1}{x+1} \doteq \frac{1}{2}$, when $x \doteq 1$. Thus, $\frac{x-1}{x^2-1}$ has the limiting value $\frac{1}{2}$ when $x \doteq 1$.

Similarly, the fraction $\frac{x}{x^2-x}$ has the limiting value -1, when $x \doteq 0$.

Thus the expression $\frac{0}{0}$ may represent different numbers, and as a symbol taken by itself, it must be regarded as indeterminate.

608. Meaning of ∞. Opposed to quantities which tend toward zero, are quantities which grow large without bound. Such variables are said to have the property of becoming infinitely great.

For example, if x assumes in turn all integral values, 1, 2, 3, 4, and so on, without end, x is said to have the property of becoming infinite.

The symbol, ∞, called *infinity*, is commonly used to express the fact that a variable has the above property, but it must not be regarded as a particular *number* nor as a *limit;* it is merely a sign of the infinitely great.

Thus, $n = \infty$ means "when n becomes infinitely great."

609. Meaning of $\frac{a}{\infty}$. This means a fixed number divided by a variable number which grows large without bound. Under these conditions, $\frac{a}{\infty} \doteq 0$.

For example, let a be represented by a line 1 ft. long, and suppose it to be bisected. The result is $\frac{12 \text{ in.}}{2} = 6$ in. Then, suppose each division to be bisected again. The result is $\frac{12 \text{ in.}}{4} = 3$ in. Bisect each division again, and suppose the process to be continued indefinitely. The denominator becomes 32,768 when the division has been made 15 times, and $\frac{12 \text{ in.}}{32,768}$ is less than $\frac{1}{2730}$ in. By taking more and more divisions, the fraction expressing the length of a division approaches zero as a limit.

610. Meaning of $\frac{\infty}{\infty}$. This means the quotient of two variable numbers each of which grows large without bound, and the expression can have no fixed value. But the quotient may tend to a limit as both numerator and denominator increase.

For example, when $x = \infty$, $\frac{x}{1+x} = \frac{\infty}{\infty}$, but it has the limiting value 1, as may be seen by dividing both terms by x; thus, $\frac{x}{1+x} = \frac{1}{1+\frac{1}{x}} \doteq \frac{1}{1+0}$ or 1, when $x = \infty$.

WRITTEN EXERCISES

What is the limiting value of each of the following when $n = \infty$:

1. $\frac{1}{x}$.
2. $\frac{3}{x}$.
3. $\frac{5}{y}$.
4. $\frac{y}{y-2}$.
5. $\frac{z+1}{z}$
6. $\frac{z-5}{z}$.
7. $\frac{x(x+2)}{x^2}$. (Reduce and separate into a whole number and a fraction before substituting $x = \infty$.)
8. Apply the suggestion in Ex. 7 to $\frac{x(x-1)(x-2)}{x^3}$.

What value does each expression approach as $v \doteq 0$:

9. $\frac{1}{v}$?
10. $\frac{\frac{3}{2}}{v}$?
11. $3v^2$?
12. $\frac{v}{v+1}$?

Find the limiting value of:

13. $\frac{z-2}{z^2-4}$ as $z \doteq 2$.
14. $\frac{z^2-9}{z-3}$ as $z \doteq 3$.
15. $\frac{x^2+2x-3}{x^2-1}$ as $x \doteq 1$.
16. $\frac{x^2-4}{x^2+x-2}$ as $x \doteq -2$.

GRAPHICAL WORK

611. The relation "x varies as y" has been expressed by the linear equation $x = ky$, and is represented by a *straight line* (Sec. 287, p. 205).

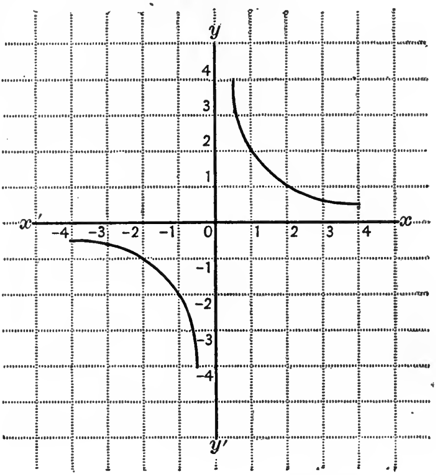
The relation "x varies inversely as y" has been expressed by the equation $x = \frac{k}{y}$. The graph of this equation is a *curve*.

Suppose, for illustration, that $k = 2$. Then $x = \frac{2}{y}$. For this equation:

The table is

x	y
5	$\frac{2}{5}$
4	$\frac{1}{2}$
3	$\frac{2}{3}$
2	1
1	2
$\frac{1}{2}$	4
-1	-2
-2	-1
-3	$-\frac{2}{3}$
-4	$-\frac{1}{2}$
-5	$-\frac{2}{5}$

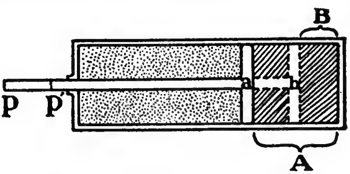
The graph is



If large numerical values are given to y, x will become correspondingly small so that for $y = \infty$, $x \doteq 0$. Thus, the curves approach nearer and nearer to the y-axis. Similarly, when $x = \infty$, then $y \doteq 0$, and the curves approach nearer and nearer to the x-axis.

WRITTEN EXERCISES

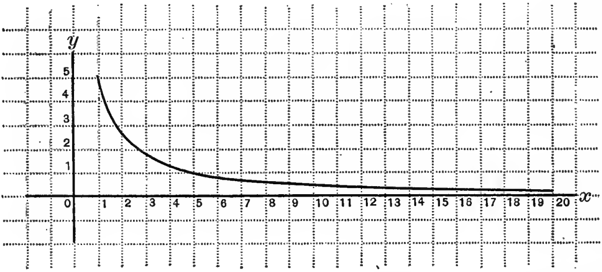
1. Under standard conditions the volume (v) of a confined gas varies inversely as the pressure (p). That is, $v = \frac{k}{p}$. This is known as Boyle's Law. Suppose when the piston is at a in the figure, the volume of the gas in part A is 1 cu. ft., and the pressure at P is 5 lb. When the pressure is 10 lb. the piston moves to b, and the volume of the gas B becomes $\frac{1}{2}$ cu. ft. What will be the volume of the gas when $P = 20$ lb.?



Taking $k = 5$ in the equation $v = \frac{k}{p}$, the table of values is

$p =$	1	2	3	4	5	6	7	8	9	10	15	20
$v =$	5	$2\frac{1}{2}$	$1\frac{2}{3}$	$1\frac{1}{4}$	1	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\frac{5}{9}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

The graph of the table is ·



Read from the graph the pressure on the piston necessary to hold this volume of gas at $\frac{1}{2}$ cu. ft.; at $\frac{1}{4}$ cu ft.; at 5 cu. ft.; at 2 cu. ft.

What is the limiting value of v when $p = \infty$?

Represent graphically:

2. $x = \frac{1}{y}$, or $xy = 1$.

3. $x = \frac{-2}{y}$, or $xy = -2$.

4. $x = \frac{4}{y}$, or $xy = 4$.

5. $3x = \frac{-1}{y}$, or $3xy = -1$.

6. The attraction or "pull" of the earth on bodies in its neighborhood is the cause of their weight. The law of gravitation states that the weight of a given body varies inversely as the square of its distance from the center of the earth. This law may be expressed by the equation $w = \frac{k}{d^2}$.

To construct the graph which shows the nature of the relation between w and d as they vary, k may be taken to be 1.

Fill the blanks in the table:

d	± 1	± 5	$\pm \frac{1}{3}$	$\pm .25$	$\pm .2$
w	1	( )	( )	( )	( )

7. Plot the graph for the table in Exercise 6.

REVIEW

WRITTEN EXERCISES

1. Solve, using the principle of composition and division,

$$(a-\sqrt{2bx+x^2}):(a-b)=(a+\sqrt{2bx+x^2}):(a+b).$$

2. If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\cdots=\frac{a_n}{b_n}$, prove that

$$\frac{a_1+a_2+a_3+\cdots+a_n}{b_1+b_2+b_3+\cdots+b_n}=\frac{a_1}{b_1}.$$

3. Given $x+\sqrt{x}:x-\sqrt{x}::3\sqrt{x}+6:2\sqrt{x}$, to find x.

4. If the volume of a sphere varies as the cube of its radius, find the radius of a sphere whose volume equals that of the sum of two spheres whose radii are, respectively, 6 ft. and 3.5 ft.

5. The number of vibrations (swings) made by two pendulums in the same time are to each other inversely as the square roots of their lengths. If a pendulum of length 39 in. makes 1 vibration per second (called a seconds pendulum), about how many vibrations will a pendulum 10 in. long make? How long must the pendulum be to make 10 vibrations per second?

6. Two towns join in building a bridge which both will use, and agree to share its cost, \$5000, in direct proportion to their populations and in inverse proportion to their distances from the bridge. One town has a population of 5000 and is 2 mi. from the bridge; the other has a population of 9000 and is 6 mi. from the bridge. What must each pay?

7. In $x=\frac{3}{y}$ what is the limiting value of x when $y=\infty$?

8. In $x=\frac{a}{\frac{1}{y}}$ what does x become when $y=\infty$?

9. Find the limiting value of $\frac{x^2-25}{x^2+2x-15}$ as $x \doteq -5$.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. Define a *proportion;* define *means;* also *extremes.* Secs. 584, 588.

2. What is a *fourth proportional?* A *third proportional?* A *mean proportional?* Secs. 585–587.

3. Define *alternation;* also *inversion; composition; division.* Secs. 590–594.

4. What is a *continued proportion?* Sec. 595.

5. Define and illustrate *direct variation.* Sec. 597.

6. State the relation of variation to proportion. Sec. 599.

7. Define and illustrate *inverse variation.* Sec. 601.

8. State an equation expressing the law of direct variation; also one expressing inverse variation. Secs. 600, 602.

9. Define and illustrate *limit.* Sec. 605.

10. Explain the meaning of the symbol $\frac{0}{0}$; also ∞; also $\frac{a}{\infty}$; also $\frac{\infty}{\infty}$. Secs. 607–610.

11. What kind of a line is the graph of the equation expressing direct variation? Of the equation expressing inverse variation? Sec. 611.

CHAPTER XXXII

SERIES

612. Series. If a sequence of numbers is determined by a given law, the sequence of numbers is called a **series**.

613. Terms. The numbers constituting the series are called its **terms**, and are named from the left, 1st term, 2d term, etc.

The following are examples of series:

1. $1, 2, 3, 4, 5, \cdots.$
2. $1, 3, 5, 7, 9, \cdots.$
3. $1, 5, 9, 13, 17, \cdots.$
4. $3, 6, 9, 12, 15, \cdots.$
5. $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \cdots.$
6. $2, 4, 8, 16, 32, \cdots.$
7. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots.$
8. $1, 3, 9, 27, 81, 243, \cdots.$
9. $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \cdots.$
10. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots.$
11. $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \cdots.$
12. $100, 99, 98, 97, 96, 95, \cdots.$

ORAL EXERCISES

1–12. State the next five terms of each series above.

614. When the *law* of a series is known, any term may be found directly.

EXAMPLES

1. The law of the second series in Sec. 613 is that each term is two more than the preceding. To get the tenth term we start from 1 and add 2 for each of 9 terms.

That is, the tenth term is $1 + 9 \cdot 2 = 19.$

Similarly, the 12th term is $1 + 11 \cdot 2 = 23,$
the 15th term is $1 + 14 \cdot 2 = 29,$
the 47th term is $1 + 46 \cdot 2 = 93,$
the nth term is $1 + (n - 1)2 = 2n - 1.$

2. The law of the eighth series is that each term after the first is 3 times the preceding term. To get the ninth term we start at 1 and multiply by 3 eight times, or by 3^8. That is, the ninth term is $1 \cdot 3^8 = 6561$.

Similarly, the 12th term is $1 \cdot 3^{11} = 177{,}147$,
the nth term is $1 \cdot 3^{n-1} = 3^{n-1}$.

WRITTEN EXERCISES

1–4. Select four of the series in Sec. 613 that can be treated like the first example and write the 10th, 12th, 15th, and 47th terms of each.

5–7. Select three of the series in Sec. 613 that can be treated like the second example and write the 8th, 10th, and nth terms of each.

8. Write similarly the 7th, 11th, 20th, 47th, and nth term for any 6 of the above series.

615. We shall give only two types of series, the arithmetical and the geometric, the laws of which are comparatively simple.

ARITHMETICAL SERIES

616. Arithmetical Series. A series in which each term after the first is formed by adding a fixed number to the preceding term is called an **arithmetical series** or **arithmetical progression.**

617. Common Difference. The fixed number is called the **common difference**, and may, of course, be negative.

For example:

1. 7, 15, 23, 31, 39, ⋯ is an arithmetical series having the common difference 8.

2. 16, $14\frac{1}{2}$, 13, $11\frac{1}{2}$, 10, ⋯ is an arithmetical series having the common difference $-\frac{3}{2}$.

WRITTEN EXERCISES

1. Select the arithmetical series in the list of Sec. 613. Write the nth term in each.

2. Beginning with 2 find the 100th even number.

3. Beginning with 1 find the 100th odd number.

4. Beginning with 3 find the 200th multiple of 3.

5. A city with a population of 15,000 increased 600 persons per year for 10 yr. What was the population at the end of 10 yr.?

618. A general form for an arithmetical series is:

$$a,\ a+d,\ a+2\,d,\ a+3\,d,\ \cdots,\ a+(n-1)d,\ \cdots,$$

where a denotes the first term,

d denotes the common difference, and

n denotes the number of the term.

619. Last Term. If the last term considered is numbered n and denoted by l, we have for the last of n terms the formula:

$$l = a + (n-1)d.$$

620. The Sum of an Arithmetical Series. The sum of n terms of an arithmetical series can be found readily.

EXAMPLE

Find the sum of the first 6 even numbers.

1. Let $s = 2 + 4 + 6 + 8 + 10 + 12.$
2. We may also write $s = 12 + 10 + 8 + 6 + 4 + 2.$
3. Adding (1) and (2),

$$2\,s = (2+12)+(4+10)+(6+8)+(8+6)+(10+4)+(12+2)$$
$$= 6(2+12);$$ for each parenthesis is the same as $2 + 12$.

4. $\therefore s = \dfrac{6(2+12)}{2} = 42.$

WRITTEN EXERCISES

Find similarly the sum of:

1. The first 6 odd numbers.
2. The first 6 multiples of 3.
3. The first 5 multiples of 7.
4. The first 4 multiples of 8.

621. General Formula for the Sum. The general form of the series may be treated in the same way.

If l denotes the last of n terms, the term before it is denoted by $l-d$, the next preceding by $l-2d$, and so on. Hence, the sum of n terms may be written:

$$s=a+(a+d)+(a+2d)+\cdots+(l-2d)+(l-d)+l.$$

And also, $s=l+(l-d)+(l-2d)+\cdots+(a+2d)+(a+d)+a.$

Whence, adding, $2s=(a+l)+(a+l)+\cdots+(a+l)=n(a+l).$

Therefore, $$s=\frac{n(a+l)}{2}.$$

Or, in words,

The sum of any number of terms of an arithmetical series is one half the sum of the first and the last terms times the number of terms.

By using the value of l in Sec. 619, $s=\dfrac{n[2a+(n-1)d]}{2}.$

This permits the calculation of s without working out separately the value of l.

WRITTEN EXERCISES

For each series in the following list find: First, the sum of 10 terms. Second, the sum of n terms.

1. 1, 2, 3, 4, 5, ⋯.
2. 1, 3, 5, 7, 9, ⋯.
3. 1, 5, 9, 13, 17, ⋯.
4. 3, 6, 9, 12, 15, ⋯.
5. 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, ⋯.
6. 100, 99, 98, 97, 96, 95, ⋯.

7. A man invests \$100 of his earnings at the beginning of each year for 10 yr. at 6 %, simple interest. How much has he at the end of 10 yr.?

Solution.

1. The last investment bears interest 1 yr. and amounts to \$106; the next to the last bears interest 2 yr. and amounts to \$112, etc.; the first bears interest 10 yr. and amounts to \$160.
2. Hence, $a=\$106$, $d=\$6$, and $n=10$.
3. Therefore, $l=106+9\cdot 6=160$.
4. Therefore, $s=\frac{10}{2}(106+160)=1330$.
5. The man has \$1330.

8. If \$50 is invested at the beginning of each year for 20 yr. at 5 % simple interest, what is the amount at the end of 20 yr.?

9. If a body falls approximately 16 ft. the first second and 32 ft. farther in each succeeding second, how far does it fall in 5 sec.?

622. Collected Results. The three chief formulas of arithmetical series are:

1. $l = a + (n-1)d$.
2. $s = \frac{n(a+l)}{2}$.
3. $s = \frac{n[2a + (n-1)d]}{2}$.

GEOMETRIC SERIES

623. Geometric Series. A series in which each term after the first is formed by multiplying the preceding term by a fixed number is called a **geometric series,** or a **geometric progression.**

624. Common Ratio. The fixed multiplier is called the **common ratio,** and may be negative.

For example:

1. 2 is the common ratio in the geometric series 2, 4, 8, 16, ….
2. $-\frac{1}{3}$ is the common ratio in the geometric series 27, -9, 3, -1, $\frac{1}{3}$, ….

ORAL EXERCISES

1. Name all the geometric series in the list, Sec. 613.
2. State the common ratio in each of these series.
3. State the 6th term of each of these series.

WRITTEN EXERCISES

1. The series 1, 3, 9, 27, 81, …, whose ratio is 3, is the same as 1, 3^1, 3^2, 3^3, 3^4, …. Write by use of exponents the 6th term of this series; the 8th term; the 10th; the 15th; the 100th.

2. The series 3, $-\frac{3}{2}$, $\frac{3}{4}$, $-\frac{3}{8}$, …, whose ratio is $-\frac{1}{2}$, is the same as 3, $3(-\frac{1}{2})$, $3(-\frac{1}{2})^2$, $3(-\frac{1}{2})^3$, …. Write by use of exponents the 5th term of this series; the 8th term; the 10th; the 25th; the 50th.

625. A general form for the geometric series is:

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1}\ \cdots,$$

where a denotes the first term,
r denotes the common ratio, and
n denotes the number of the terms.

626. Last Term. If the last term is numbered n, and denoted by l, then we have for the last of n terms the formula,

$$l = ar^{n-1}.$$

627. The Sum of a Geometric Series. The sum of n terms of a geometric series can readily be found.

EXAMPLE

Find the sum of 5 terms of the series 2, 6, 18, 54, 162.

SOLUTION.

Let	$s = 2 + 6 + 18 + 54 + 162.$	(1)
Multiplying by 3, the common ratio,	$3s = 6 + 18 + 54 + 162 + 486.$	(2)
Subtracting (1) from (2),	$3s - s = 486 - 2,$	(3)
or,	$2s = 484.$	(4)
Dividing by 2,	$s = 242.$	(5)

WRITTEN EXERCISES

Find similarly the sum of 5 terms of each of these series:

1. $6, 30, 150, \cdots.$
2. $7, -14, 28, \cdots.$
3. $\frac{1}{2}, \frac{1}{10}, \frac{1}{50}, \cdots.$
4. $\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, \cdots.$

628. General Formula for the Sum. The general form of the series may be treated in the same way. If l denote the last of n terms, the term before it is denoted by $\frac{l}{r}$, the next preceding by $\frac{l}{r^2}$, and so on. Hence, the sum of n terms may be written:

$$1.\quad s = a + ar + ar^2 + \cdots \frac{l}{r^2} + \frac{l}{r} + l.$$

2. Then $rs = ar + ar^2 + \cdots \frac{l}{r^2} + \frac{l}{r} + l + lr.$

3. Subtracting, $s - rs = a - lr.$

4. Or, $(1 - r)s = a - lr.$

$$\therefore s = \frac{a - lr}{1 - r} = \frac{lr - a}{r - 1}.$$

In words,

The sum of any number of terms of a geometric series is the ratio times the last term diminished by the first term and divided by the ratio less 1.

By using the value of l (Sec. 626),

$$s = \frac{ar^{n-1} \cdot r - a}{r - 1} = \frac{ar^n - a}{r - 1}.$$

Thus, s may be found without first computing l.

WRITTEN EXERCISES

1. Find the sum of 6 terms of the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.

2. Find the sum of 10 terms of the series $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \cdots$.

3. Find the sum of 8 terms of the series 1, .25, .0625, ⋯.

4. Find the sum of 12 terms of the series 27, −9, 3, −1, ⋯.

5. An air pump exhausted the air from a cylinder containing 1 cu. ft. at the rate of $\frac{1}{10}$ of the remaining contents per stroke. What part of a cubic foot of air remained in the cylinder after 25 strokes?

6. The population of a town increased from 10,000 to 14,641 in 5 yr. If the population by years was in geometric series, what was the rate of increase per year?

629 Collected Results. The three chief formulas of geometric series are:

1. $l = ar^{n-1}.$

2. $s = \dfrac{lr - a}{r - 1}.$

3. $s = \dfrac{ar^n - a}{r - 1}.$

WRITTEN EXERCISES

1. \$100 is placed on interest at 5%, compounded annually.

(1) What is the amount at the end of the first year?

(2) What is the principal for the second year?

(3) What is the amount at the end of the second year?

1st	\$100 (1.05)
2d	\$100 $(1.05)^2$
3d	\$100 $(1.05)^3$
4th	\$100 $(1.05)^4$

Notice that the amounts appear in the right-hand column of the table.

(4) Indicate similarly the amount of \$100 at the end of 5 yr.; 10 yr.; n yr. Which formula of geometric series expresses the amount for n yr.?

2. Indicate the amount of \$100 at 6%, compounded annually, at the end of 1 yr.; 2 yr.; 5 yr.; 10 yr.; n yr.

3. Many savings banks pay interest at the rate of 3%, compounded semi-annually.

Indicate the amount of \$100 under the above conditions at the end of 6 mo.; 1 yr.; 18 mo.; 2 yr.; 10 yr.; n yr.

NOTE. The numerical value of these expressions can be computed readily by logarithms.

Solve by the use of logarithms:

4. What is the amount of \$1 at 4% compound interest for 8 yr.?

5. A man deposits \$100 in a bank paying 4% interest, compounded annually, on the first day of each year for 5 yr. How much will he have on deposit at the end of 5 yr.?

6. Determine similarly the amounts of:

	DEPOSIT AT BEGINNING OF EACH YEAR	RATE OF INTEREST COMPOUNDED ANNUALLY	NUMBER OF YEARS
(1)	\$25	5	8
(2)	10	6	15
(3)	43.20	4	20
(4)	39.87	3	20

MEANS

630. Means. Terms standing between two given terms of a series are called **means**.

631. Arithmetical Mean. If three numbers are in arithmetical progression, the middle one is called the **arithmetical mean** between the other two.

The arithmetical mean between a and b is found thus:

1. Let A be the mean and d the common difference.
   Then the terms may be written $A - d$ and $A + d$.
2. Whence, $A - d = a$ and $A + d = b$.
3. Adding, $2A = a + b$ and $A = \frac{a+b}{2}$.

632. *The arithmetical mean between two numbers is one half their sum.*

633. Geometric Mean. If three numbers are in geometric progression, the middle one is called the **geometric mean** between the other two.

The geometric mean between a and b is found thus:

1. Let g be the geometric mean.
2. Then $\frac{g}{a} = \frac{b}{g}$.
3. $\therefore g^2 = ab$ and $g = \sqrt{ab}$.

634. *The geometric mean between two numbers is the square root of their product. There are really two geometric means, one negative and one positive.*

The geometric mean between two numbers is the same as their mean proportional.

ORAL EXERCISES

State the arithmetical mean between:

1. 8, 12. 2. 6, 3. 3. 4, -10. 4. $5a$, $13a$.

State the geometric mean, including signs, between:

5. 8, 6. 6. 3, 12. 7. a, a^5. 8. $2x^3$, $32x^7$.

635. Any number of means may be found by use of formulas already given.

EXAMPLES

1. Insert 5 arithmetical means between 4 and 12.

   1. In this case $a = 4$, $l = 12$, and $n = 7$.
   2. $\therefore\ l = a + (n - 1)d$ becomes $12 = 4 + (7 - 1)d$.
   3. Solving for d, $d = \frac{12 - 4}{6} = 1\frac{1}{3}$.
   4. Adding $1\frac{1}{3}$ to 4, and $1\frac{1}{3}$ to that result, and so on, the means are found to be $5\frac{1}{3}$, $6\frac{2}{3}$, 8, $9\frac{1}{3}$, and $10\frac{2}{3}$.

2. Insert 4 geometric means between -27 and $\frac{1}{9}$.

   1. In this case $a = -27$, $l = \frac{1}{9}$, and $n = 6$.
   2. $\therefore\ l = ar^{n-1}$ becomes $\frac{1}{9} = -27\,r^5$.
   3. Solving for r, $r^5 = -\frac{1}{9 \cdot 27} = -\frac{1}{3^5}$.

   Therefore, $r = -\frac{1}{3}$.

   4. Multiplying -27 by $-\frac{1}{3}$, and multiplying this result by $-\frac{1}{3}$, and so on, the means are found to be 9, -3, 1, and $-\frac{1}{3}$.

WRITTEN EXERCISES

1. Insert 3 arithmetical means between 6 and 26.
2. Insert 10 arithmetical means between -7 and 144.
3. Insert 3 geometric means between 2 and 32.
4. Insert 4 geometric means between $-\frac{1}{10}$ and $3\frac{1}{5}$.

OTHER FORMULAS

636. Arithmetical Series. By means of the formulas of Sec. 622, any two of the five numbers a, n, l, d, s, can be found when the other three are given.

EXAMPLES

1. Given $n = 6$, $s = 18$, $l = 8$, find a, d.

   1. For these values, formulas (1) and (2) become:

$$8 = a + (6 - 1)d,$$
$$18 = \frac{6(a + 8)}{2}.$$

2. We have thus two equations to determine and two numbers, a, d.

From the second equation, $a = -2$.

3. Using this value in the first equation, $d = 2$.

2. Given $a = 4$, $l = 12$, $s = 56$, find n and d.

1. For these values formulas (1) and (2) become:

$$12 = 4 + (n - 1)d.$$

$$56 = \frac{n(4 + 12)}{2}.$$

2. $\therefore$ from the second equation, $n = 7$.
3. Substituting in the first, $12 = 4 + 6d$,
4. therefore $d = \frac{4}{3}$.

3. Given $n = 12$, $s = 30$, $l = 10$, find a, d.

1. Formulas (1) and (3) become:

$$10 = a + 11d.$$

$$30 = \frac{12(2a + 11d)}{2} = 12a + 66d.$$

2. Solving these equations for a and d:

$$a = -5, \text{ and } d = \tfrac{15}{11}.$$

The same results would be found by using formulas (1) and (2), since (3) is only another form of (2).

637. The same problems can also be solved generally; that is, without specifying numerical values.

EXAMPLE

Regarding n, s, d as known, find a, l.

1. From (1), Sec. 622, $l - a = (n - 1)d$.
2. From (2), Sec. 622, $a + l = \frac{2s}{n}$.
3. Adding (1) and (2), $2l = (n - 1)d + \frac{2s}{n}$.

Or, $l = \frac{(n-1)d}{2} + \frac{s}{n}$.

4. Substituting in (2) above, $a = \frac{2s}{n} - \left[\frac{(n-1)d}{2} + \frac{s}{n}\right]$

$$= \frac{s}{n} - \frac{(n-1)d}{2}.$$

WRITTEN EXERCISES

By use of the formulas in Sec. 622, find the following:

	Find	In Terms of	Result
1.	l	$a\ d\ n$	$l = a + (n-1)d$
2.	l	$a\ d\ s$	$l = \frac{1}{2}[-d \pm \sqrt{8ds + (2a-d)^2}]$
3.	l	$a\ n\ s$	$l = \frac{2s}{n} - a$
4.	l	$d\ n\ s$	$l = \frac{s}{n} + \frac{(n-1)d}{2}$
5.	s	$a\ d\ n$	$s = \frac{1}{2}n[2a + (n-1)d]$
6.	s	$a\ d\ l$	$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}$
7.	s	$a\ n\ l$	$s = \frac{n}{2}(a + l)$
8.	s	$d\ n\ l$	$s = \frac{1}{2}n[2l - (n-1)d]$
9.	a	$d\ n\ l$	$a = l - (n-1)d$
10.	a	$d\ n\ s$	$a = \frac{s}{n} - \frac{(n-1)d}{2}$
11.	a	$d\ l\ s$	$a = \frac{1}{2}[d \pm \sqrt{(2l+d)^2 - 8ds}]$
12.	a	$n\ l\ s$	$a = \frac{2s}{n} - l$
13.	d	$a\ n\ l$	$d = \frac{l-a}{n-1}$
14.	d	$a\ n\ s$	$d = \frac{2(s-an)}{n(n-1)}$
15.	d	$a\ l\ s$	$d = \frac{l^2 - a^2}{2s - l - a}$
16.	d	$n\ l\ s$	$d = \frac{2(nl - s)}{n(n-1)}$
17.	n	$a\ d\ l$	$n = \frac{l-a}{d} + 1$
18.	n	$a\ d\ s$	$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}$
19.	n	$a\ l\ s$	$n = \frac{2s}{l+a}$
20.	n	$d\ l\ s$	$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}$

NOTE. a, l, d, s may have any values, but n must be a positive integer. Hence, not all solutions of equations 18, 20, represent possible series.

638. Geometric Series. By means of the formulas of Sec. 629, any two of the five numbers a, n, l, r, s, can be found when the other three are given.

EXAMPLES

1. Given $s = 1024$, $r = 2$, $a = 2$, find l.

1. For these values formula (2) becomes:

$$1024 = \frac{2l - 2}{2 - 1} = 2(l - 1).$$

2. $\therefore l = 513$.

2. Given $r = 5$, $n = 5$, $s = 363$, find a.

1. For these values formulas (1) and (2) become:

$$l = a \cdot 3^4.$$

$$363 = \frac{3l - a}{2}.$$

2. Eliminating l, $363 = \dfrac{a(3^5 - 1)}{2}.$

3. Therefore, $363 = \frac{1}{2} a \cdot 242$, and $a = 3$.

3. Given $s = 363$, $a = 3$, $r = 3$, find n.

1. For these values formula (3) becomes:

$$363 = \frac{3 \cdot 3^n - 3}{2}.$$

2. Therefore, $3^n - 1 = 242$, and $3^n = 243$.

3. By factoring 243, n is seen to be 5.

NOTE. In finding n it may not be possible to factor as in the case of 243 above. In this case logarithms may be applied.

639. The same problems can be solved generally, that is, without specifying numerical values.

EXAMPLE

Express l in terms of a, n, and s.

1. From formula (2) $r = \dfrac{s - a}{s - l}$, or $r^{n-1} = \dfrac{(s - a)^{n-1}}{(s - l)^{n-1}}.$

2. $\therefore$ substituting in (1) $l = \dfrac{a(s - a)^{n-1}}{(s - l)^{n-1}}.$

3. $\therefore l(s - l)^{n-1} - a(s - a)^{n-1} = 0.$

This equation is of a degree higher than 2 in l when $n > 3$. But for n equal to or less than 3 it can be solved by methods already explained.

WRITTEN EXERCISES

By use of the formulas in Sec. 629, find the results given in the table:

NOTE. In Exercises 3, 12, and 16, only the equation connecting the unknown numbers with the given ones can be found:

	FIND	IN TERMS OF	RESULT
1.	l	$a\ r\ n$	$l = ar^{n-1}$
2.	l	$a\ r\ s$	$l = \frac{a + (r-1)s}{r}$
3.	l	$a\ n\ s$	$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$
4.	l	$r\ n\ s$	$l = \frac{(r-1)sr^{n-1}}{r^n - 1}$
5.	s	$a\ r\ n$	$s = \frac{a(r^n - 1)}{r - 1}$
6.	s	$a\ r\ l$	$s = \frac{rl - a}{r - 1}$
7.	s	$a\ l\ n$	$s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$
8.	s	$r\ n\ l$	$s = \frac{lr^n - l}{r^n - r^{n-1}}$
9.	a	$r\ n\ l$	$a = \frac{l}{r^{n-1}}$
10.	a	$r\ n\ s$	$a = \frac{(r-1)s}{r^n - 1}$
11.	a	$r\ l\ s$	$a = rl - (r-1)s$
12.	a	$n\ l\ s$	$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$
13.	r	$a\ n\ l$	$r = \sqrt[n-1]{\frac{l}{a}}$
14.	r	$a\ n\ s$	$r^n - \frac{s}{a}r + \frac{s-a}{a} = 0$
15.	r	$a\ l\ s$	$r = \frac{s-a}{s-l}$
16.	r	$n\ l\ s$	$r^n - \frac{s}{s-l}r^{n-1} + \frac{l}{s-l} = 0$

SPECIAL SERIES

640. Binomial Expansion. The Binomial Expansion, Chapter XX, is a series in which each term of $(a+b)^n$ is produced from the next preceding one by multiplying by $\frac{b}{a}$ and inserting in the numerator and the denominator the next factor in each sequence.

For example, the third term in the binomial expansion of $(a+b)^n$ is $\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$, and the fourth term is

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3.$$

To form the fourth term from the third, note that:

1. $a^{n-2}b^2 \times \frac{b}{a} = a^{n-3}b^3$.

2. The numerator, $n(n-1)(n-2)$, of the coefficient of the fourth term has one more factor in the sequence of factors, which begins with n and decreases by 1 each time.

3. The denominator, $1\cdot 2\cdot 3$, has one more factor in the sequence of factors which begins with 1 and increases by 1 each time.

4. In the expansion of $(a-b)^n$, the even terms are negative.

Thus, from any given term of the binomial series, all the subsequent terms can be written.

WRITTEN EXERCISES

1. The fifth term of a binomial expansion is

$$\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4}y^4.$$

Write the sixth term; also the seventh term.

2. The sixth term of a binomial expansion is

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^{n-5}b^5.$$

Find the eighth term.

3. By Sec. 325, $1\cdot 2$ may be written $\underline{|2}$, called *factorial two*, $1\cdot 2\cdot 3$ may be written $\underline{|3}$, called *factorial three*, and so on.

Write the denominators of the coefficients in Exercise 1 as factorials. Also in Exercise 2.

4. Suppose n is 8 in Exercise 1. What do the sixth and seventh terms become? Suppose n is 10 in Exercise 2, what does the eighth term become?

641. Finding the *r*th Term. Any term of the binomial series may be written at once by observing the general form of the terms, as explained in Sec. 640.

Observe that:

1. *The last factor in the numerator of each coefficient is* n *minus a number two less than the number of the term.*

E.g., in the third term the last factor is $n-1$; in the 4th it is $n-2$; in the 5th, $n-3$, and so on.

2. *The denominator is the factorial of the number one less than the number of the term.*

E.g., $\lfloor 2$ in the third, $\lfloor 3$ in the 4th, and so on

3. *The exponent of the first term of the binomial, or* a *in* $(a \pm b)^n$, *is* n *minus a number one less than the number of the term;*

E.g., $n-1$ in the 2d term, $n-2$ in the 3d, and so on.

4. *The exponent of* b *is one less than the number of the term;*

E.g., 1 in the 2d term, 2 in the 3d, and so on.

5. *The signs of all terms are positive in* $(a+b)^n$. *In* $(a-b)^n$ *the odd terms are positive, and the even terms negative.*

EXAMPLES

1. Find the 9th term of $(x-y)^p$.

By (1) the numerator of the coefficient is $p(p-1)(p-2)\cdots(p-7)$.
By (2) the denominator of the coefficient is $\lfloor 8$.
By (3) the exponent of x is $p-8$.
By (4) the exponent of y is 8.
By (5) the sign is $+$.
Therefore, the 9th term of $(x-y)^p$ is

$$+\frac{p(p-1)(p-2)\cdots(p-7)}{\lfloor 8}x^{p-8}y^8.$$

2. Find the rth term of $(x - 2a)^n$.

By (1) the numerator of the coefficient is $n(n-1) \cdots (n - \overline{r-2})$.
By (2) the denominator of the coefficient is $|\underline{r-1}$.
By (3) the exponent of x is $n - \overline{r-1}$.
By (4) the exponent of $(2a)$ is $r - 1$.
By (5) the sign is $\pm$, according as r is odd or even.
Therefore, the rth term of $(x - 2a)^n$ is

$$\pm \frac{n(n-1) \cdots (n-r+2)}{|\underline{r-1}} x^{n-r+1}(2a)^{r-1}$$

WRITTEN EXERCISES

Expand:

1. $(x - a)^5$.
2. $(a + x)^6$.
3. $(2a - x)^7$.

Write without expanding the series:

4. The 5th term of $(a - 2b)^7$.
5. The 7th term of $(a - x)^{10}$.
6. The middle term of $(a + x)^{12}$.
7. The two middle terms of $(2x - y^2)^{13}$.
8. The 10th term of $(a + 3x^3)^{21}$.
9. The qth term in $(\frac{1}{2}x - 2y)^n$.
10. The rth term in $(x - y)^{\frac{1}{2}}$.
11. The $(r + 1)$st term in $(a + b)^n$.
12. The $(r - 5)$th term in $(a - b)^{n+1}$.

642. Finite Series. So far we have treated only series with a fixed number of terms. A series which comes to an end is called a **finite series.**

643. Infinite Series. A series whose law is such that every term has a term following it is called an **infinite series.**

For example:

2, 5, 8, 11, ⋯, 239 as here written ends with 239. But the law of the series would permit additional terms to be specified. In the above example, the next following terms would be 242, 245, etc. It is obvious that however many terms may have been specified, still more can be made by adding 3. The series is thus unending.

Similarly, all of the series so far considered might have been continued by applying their corresponding laws.

The term "infinite" comes from the Latin *infinitus*, and is here used with the meaning *unending*.

If the coefficients of the binomial expansion be regarded as a series,

$$1,\ n,\ \frac{(n-1)}{2\,!},\ \frac{n(n-1)(n-2)}{3\,!},\ \frac{n(n-1)(n-2)(n-3)}{4\,!},\ \cdots$$

they furnish, when n is a positive integer, instances of series that come to an end according to the law of the series. If $n = 3$, the series has 4 terms, and if $n = 10$, the series has 11 terms; for the positive integer n, it has $n + 1$ terms. This is true because the factors n, $n - 1$, $n - 2$, and so on, will finally in the $(n + 2)$d term contain $n - n$ or zero. Therefore the series has $n + 1$ terms.

But if n is a negative integer or a fraction, none of the factors, n, $n - 1$, $n - 2$, and so on, becomes zero, and the series can always be extended farther. That is, if n is a negative integer or any fraction, the series is unending or infinite.

644. Infinite Geometric Series. The subject of infinite series is of great importance, but is too difficult to be taken up here. We shall mention simply a few properties of infinite geometric series *whose ratio is numerically less than* 1.

The following are examples of such series:

1. $4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \cdots.$
2. $3,\ \frac{3}{5},\ \frac{3}{25},\ \frac{3}{125},\ \cdots.$
3. $.5,\ .05,\ .005,\ .0005,\ \cdots.$
4. $1,\ -\frac{1}{2},\ \frac{1}{4},\ -\frac{1}{8},\ \frac{1}{16},\ -\frac{1}{32},\ \cdots.$

State the ratio and the next three terms of each series.

I. *The terms become numerically smaller and smaller.* Each term is numerically smaller than the one preceding it, for it is a fractional part of it.

II. *The terms become numerically small at will.* That is, however small a number may be selected, there are terms in the series smaller than it, and when r is numerically less than 1, the term ar^{n-1} may be made numerically small at will, by taking n sufficiently large.

This seems obvious from the consideration of the series given above as examples. It is not difficult to prolong these series until their terms are less than $\frac{1}{100}$ say, or $\frac{1}{1000}$, and from this it seems plausible to think that the terms would become less than one millionth, or one billionth, or any other number, if a sufficient number of terms are taken. As a matter of fact this is true, but the proof is too difficult to be given here.

III. We have proved that, if s_n denote the sum of the first n terms of a geometric series,

$$s_n = \frac{a - ar^n}{1 - r}.$$

This may be written: $s_n = \dfrac{a}{1-r} - ar^{n-1}\left(\dfrac{r}{1-r}\right).$

By taking n sufficiently large, the product of ar^{n-1} and the fixed number $\dfrac{r}{1-r}$ can be made as small as desired. As more and more terms of the series are added, the sum differs less and less from $\dfrac{a}{1-r}$; and if sufficient terms are taken, the sum comes and remains as close as we please to $\dfrac{a}{1-r}$.

The number $\dfrac{a}{1-r}$ is called the **limit** of the sum of n terms, as n is increased without bound. Denoting this limit by s, we have:

$$s = \frac{a}{1-r}.$$

The number s is not the sum of *all* the terms of the series, for since the terms of the series never come to an end, the operation of adding them cannot be completed.

According to Sec. 644,

$$s \doteq \frac{a}{1-r} \text{ when } n = \infty.$$

For example:

When $a = 4$, and $r = \dfrac{1}{2}$, then the limit of $s = \dfrac{4}{1-\frac{1}{2}} = 8.$

To test this, we form successive values of s_n.

$$s_1 = 4.$$
$$s_2 = 6.$$
$$s_3 = 7.$$
$$s_4 = 7\tfrac{1}{2}.$$
$$s_5 = 7\tfrac{3}{4}.$$
$$s_6 = 7\tfrac{7}{8}.$$

It appears that the values of s_n approximate more and more closely to 8 as n is increased.

WRITTEN EXERCISES

Find the limit of the sum of the series:

1. $1 + \frac{1}{2} + \frac{1}{4} + \cdots.$

2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots.$

3. $5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} \cdots.$

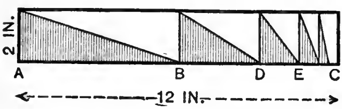
4. $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \cdots.$

5. Test the results of the preceding exercises by finding successive values of s_n.

6. In an infinite geometric series $s = 2$ and $r = \frac{1}{2}$; find a.

7. Find the fraction which is the limit of $.\dot{3}33333 \cdots$, or $.3 + .03 + .003 + \cdots$.

8. Find the limit of $.\dot{2}\dot{3}232323 \cdots$ or $.23 + .0023 + .000023 + \cdots$.



9. Triangles are drawn in a rectangle of dimensions indicated, B being the midpoint of AC, D that of BC, E that of DC, and so on. What limit does the sum of the areas of the triangles approach as more and more triangles are taken?

10. Find the sum of 16 terms of the series,

$$27,\ 22\tfrac{1}{2},\ 18,\ 13\tfrac{1}{2},\ \cdots.$$

11. Find the sum of 18 terms of the series,

$$36,\ 12,\ 4,\ \tfrac{4}{3},\ \cdots.$$

12. The difference between two numbers is 48. Their arithmetical mean exceeds their geometric mean by 18. Find the numbers.

13. Express as a geometric series the decimal fraction.

$$.0373737 \cdots.$$

What is its limiting value?

14. Find the limiting value of each of these series:

(a) $.\dot{3}\dot{5}3535 \cdots.$

(b) $.125\dot{6}66 \cdots.$

(c) $.03\dot{2}\dot{4}24 \cdots.$

(d) $.\dot{1}2\dot{5}125 \cdots.$

(e) $3.\dot{6}0\dot{5}605 \cdots.$

(f) $5.00\dot{8}88 \cdots.$

(g) $9.\dot{1}\dot{0}1010 \cdots.$

(h) $6.04\dot{3}\dot{8}38 \cdots.$

15. If $\frac{1}{b-a}, \frac{1}{2b}, \frac{1}{b-c}$, are in arithmetical progression, show that a, b, c are in geometric progression.

SUGGESTION. The supposition means that

$$\frac{1}{b-a}-\frac{1}{2b}=\frac{1}{2b}-\frac{1}{b-c}.$$

This reduces to $b^2 = ac$.

16. Find the amount in n years of P dollars at r per cent per annum, interest being compounded annually.

17. During a truce, a certain army A loses by sickness 14 men the first day, 15 the second, 16 the third, and so on; while the opposing army B loses 12 men every day. At the end of fifty days the armies are found to be of equal size. Find the difference between the two armies at the beginning of the truce.

18. A strip of carpet one half inch thick and $29\frac{6}{7}$ feet long is rolled on a roller four inches in diameter. Find how many turns there will be, remembering that each turn increases the diameter by one inch, and taking as the length of a circumference $\frac{22}{7}$ times the diameter.

19. Insert between 1 and 21 the arithmetic means such that the sum of the last three terms of the series is 48.

20. If $\frac{a}{b}=\frac{c}{d}$, prove that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

21. The sum of the first ten terms of a geometric series is 244 times the sum of the first five terms; and the sum of the fourth and sixth terms is 135. Find the first term and the common ratio.

REVIEW

WRITTEN EXERCISES

1. Find the 47th multiple of 7.
2. Find the sum of the first 12 multiples of 4.

Find the 20th term, and the sum of 12 terms of each series:

3. $6, 9, \frac{27}{2}, \frac{81}{4}, \cdots$.

4. $8, 11, 14, 17, \cdots$.

5. $2^9, 2^6, 2^3, \cdots$.

6. $a+b, a-b, a-3b, a-5b$.

Find the eighth term, and the sum of 8 terms:

7. $1, 4, 16, \cdots$.

8. $3, 6, 12, \cdots$.

9. $2, -4, 8, -16, \cdots$.

10. $1, -2, 2^2, -2^3, \cdots$.

11. $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \cdots$.

12. $100, -40, 16, \cdots$.

Find the twelfth term, and the sum of 12 terms:

13. $2, 4, 6, \cdots$.

14. $-5, -3, -1, \cdots$.

15. $1, \frac{6}{7}, \frac{5}{7}, \cdots$.

16. $\frac{2}{3}, \frac{7}{15}, \frac{4}{15}, \cdots$.

17. $4, -3, -10, \cdots$.

18. $\frac{1}{2}, -\frac{2}{3}, -\frac{11}{6}, \cdots$.

19. Find three numbers whose common difference is 1 and such that the product of the second and third exceeds that of the first and second by $\frac{1}{2}$.

20. The first term of an arithmetical series is $n^2 - n - 1$, the common difference is 2. Find the sum of n terms.

21. In some countries of Europe the hours of the day are numbered on the clockface from 1 to 24. How many strokes would a clock make per day in striking these hours?

22. How many strokes does a common clock striking the hours make in a day?

23. A man leases a business block for 20 years under the condition that, owing to estimated increase in the value of the property, the rental is to be increased \$50 each year. He pays altogether \$19,500. What was the rental of the first year? The last?

24. A railroad car starting from rest began to run down an inclined plane. It is known that in such motion the distances passed over in successive seconds are in arithmetical progression. It was observed that at the end of 10 sec. the car had passed over 570 ft. and at the end of 20 sec. 2340 ft. from the

starting point. How far did it run the first second? How far from the starting point was it at the end of 15 sec.?

25. It is known that if a body falls freely, the spaces passed over in successive seconds are in arithmetical progression, and that it falls approximately 16 ft. in the first second and 48 ft. in the next second. To determine the height of a tower, a ball was dropped from the top and observed to strike the ground in 4 sec. Find the height of the tower.

26. An employee receives a certain annual salary, and in each succeeding year he receives $72 more than the year before. At the end of the tenth year he had received all together $10,440. What was his salary the first year? The last?

27. The 14th term of an arithmetical series is 72, the fifth term is 27. Find the common difference and the first term.

28. A man is credited $100 annually on the books of a building society as follows: At the beginning of the first year he pays in $100 cash. At the beginning of the second year he is credited with $6 interest on the amount already to his credit; and he is required to pay $94 in cash, making his total credit $200. At the beginning of the third year he is credited with $12 interest, and pays $88 in cash, and so on. How much is his payment at the beginning of the tenth year? What is his credit then? How much cash has he paid in all?

29. At each stroke an air pump exhausts $\frac{2}{3}$ of the air in the receiver. What part of the original air remains in the receiver after the 8th stroke?

30. At the close of each business year, a certain manufacturer deducts 10 % from the amount at which his machinery was valued at the beginning of the year. If his machinery cost $10,000, at what did he value it at the end of the fourth year?

31. In Exercise 30, by use of logarithms, find its valuation at the end of the 20th year.

32. Show that the terms of a geometric progression form a continued proportion, by applying Sec. 595 to the series a, ar, ar^2, ar^3, ar^4, $\cdots$.

33. We have shown in Sec. 644 that $\frac{a}{1-r}$ is the limit of the sum of the terms of a geometric progression whose first term is a and whose common ratio is r. Find the terms of this series by dividing the numerator of the above fraction by its denominator.

34. What kind of a series do the reciprocals of the numbers 2, 3, and 6 form?

35. Find the arithmetical mean between $\frac{1}{a}$ and $\frac{1}{b}$.

36. Find the sum of each of the following infinite series:

(1) $3 + \frac{1}{2} + \frac{1}{12} + \cdots$.
(2) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$.
(3) $1 + \frac{4}{5} + \frac{16}{25} + \cdots$.

37. Find the sum of 10 terms of each series in Exercise 36.

38. Write the *rth* term of the binomial expansion; also the $(r+1)$*st* term. Find the ratio of the *rth* term to the $(r+1)$*st* term, and simplify the result.

39. Write the term which contains x^{10} in the expansion of $(a-x)^{12}$.

SUMMARY

The following questions summarize the definitions and processes treated in this chapter:

1. When is a group of numbers called a *series?* Sec. 612.
2. What is meant by the *terms* of a series? Sec. 613.
3. Define and illustrate an *arithmetical series,* or *progression.* Sec. 616.
4. Define *common difference.* Sec. 617.
5. Define and illustrate a *geometric progression.* Sec. 623.
6. Define *common ratio.* Sec. 624.
7. Define *arithmetical mean; geometric mean.* Secs. 631–633.

8. State the formula for the *last term* or general term of an arithmetical progression; also of a geometric progression. Secs. 619, 626.

9. State the formula for the sum of n terms of an arithmetical progression; also of a geometric progression. Secs. 620, 627.

10. State a formula to find n in terms of a, l, and s in an arithmetical progression. Sec. 637.

11. State a formula to find d in terms of a, n, and l in an arithmetical progression. Sec. 637.

12. State a formula to find a in terms of n, l, and s in an arithmetic progression. Sec. 637.

13. State a formula to determine a in terms of r, n, and l in a geometric progression. Sec. 639.

14. State a formula to determine r in terms of a, l, and s in a geometrical progression. Sec. 639.

15. State the general form of the rth term of the binomial expansion $(a \pm b)^n$. Sec. 641.

16. Define and illustrate an *infinite series.* Sec. 643.

17. State the expression for the limit of s in an infinite geometric series. Sec. 644, III.

HISTORICAL NOTE

The existence of sets of successive numbers, each term of which depends, in a definite way, upon its predecessors for its value, called a progression, or series of numbers, was discovered by the early mathematicians. One of the two Babylonian tablets still in existence gives in cuneiform symbols the squares of the integers from 1 to 60, namely, 1, 4, 9, 16, 25, and so on. The other tablet gives the following numbers: 5, 10, 20, 40, 80, 96, 112, and so on, the first five of which form a geometric progression and the rest an arithmetical series. These numbers were used to represent the illuminated portions of the moon's disk from day to day, from new moon to full moon. Thus, taking the whole as 240 parts, the visible portion of the moon for the first day would be $\frac{5}{240}$, or $\frac{1}{48}$ of the whole disk.

The terms of a geometric series with an integral ratio increase at a very rapid rate, and the earlier mathematicians seem to have taken much interest in the framing of problems intended to exemplify this. Sixteenth-century writers of arithmetic collected many of these problems, of which the following are typical:

A peasant agreed with a blacksmith to pay him for shoeing his horse at the rate of one cent for the first nail, two cents for the second, four cents for the third, and so on, in geometric progression. There being 8 nails in each of the four shoes, how much was the peasant to pay the blacksmith?

Required the number of kernels of wheat needed in order to place one kernel on the first square of the chessboard, two on the second, four on the third, and so on, for the sixty-four squares. This later problem was given by Masudi, in *Meadows of Gold* (950 A.D.).

The process of summing arithmetical and geometric series (Secs. 621, 628), and the methods for finding any required term, were known to the Hindoos and appear in the works of Bhaskara. But these were trifling beginnings compared with the part played by the almost unlimited variety of series used in modern mathematics. The proof of the binomial expansion by Newton, the ability to express logarithms by series, accomplished by Mercator (1668), and the study of other systems of infinite series has opened vast new fields of mathematics.

CHAPTER XXXIII

GEOMETRIC PROBLEMS FOR ALGEBRAIC SOLUTION

The problems in the following list may be used as supplementary work for pupils that have studied plane geometry. In the body of the Algebra numerous problems have been given applying geometric facts which the pupil has learned in the study of mensuration in arithmetic. In the following list the problems contain the application of other relations and theorems of geometry, and typical solutions have been inserted to suggest to the pupil the method of attack.

LINEAR EQUATIONS. ONE UNKNOWN

1. In a given triangle one angle is twice another, and the third angle is 24°. Find the unknown angles.

SOLUTION. Let x be one of the unknown angles,

then $2x$ is the other. (1)

Because the sum of the angles of a triangle = 180°,

$$x + 2x + 24° = 180°. \quad (2)$$

Solving equation (2)

$$x = 52° \text{ and } 2x = 104°. \quad (3)$$

The angles of the triangle are 52°, 104°, 24°.

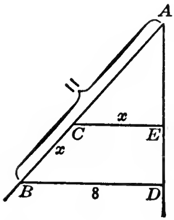
2. In a certain triangle one angle is three times another, and the third angle is 36°. Find the unknown angles.

3. In a given right-angled triangle one acute angle is $\frac{2}{3}$ the other. Find the angles.

4. In a certain isosceles triangle the angle opposite to the base is 18°. Find the angles at the base.

5. The three angles A, B, and C of a given triangle are in the ratio of 2, 3, and 5. Find the angles.

6. Given an angle A such that a point B situated on one side 11 in. from the vertex is 8 in. distant from the other side. Find a point C on the same side as B, and equidistant from B and the other side of the angle.



Solution. Let A be the given angle, then the figure represents the conditions of the problem.

From the similar triangles ACE and ABD, we have

$$\frac{AC}{CE} = \frac{AB}{BD},$$

or,
$$\frac{11 - x}{x} = \frac{11}{8}. \quad (1)$$

From (1), $$88 - 8x = 11x. \quad (2)$$

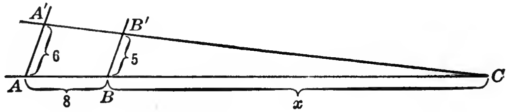
Then, $$88 = 19x \text{ and } x = \tfrac{88}{19}. \quad (3)$$

The distance CB is $\frac{88}{19}$ in., or $4\frac{12}{19}$.

7. Solve problem 6, if the point B is 9 in. from A and 6 in. from side AD.

8. Solve problem 6, if the point B is a in. from the vertex of the angle and b in. from the other side of the angle.

9. Two points A and B are 8 in. apart. Parallels are drawn through A and B; on these parallels the points A' and B' are located on the same side of the straight line through AB and at distances 6 in. and 5 in. from A and B, respectively. Determine the point where the line $A'B'$ cuts the line AB.



Solution. Let C be the desired point and let $BC = x$. Then, by similar triangles,

$$\frac{BC}{BB'} = \frac{AC}{AA'}, \quad (1)$$

or,
$$\frac{x}{5} = \frac{x + 8}{6}. \quad (2)$$

Hence, $6x = 5x + 40.$ (3)

and $x = 40.$ (4)

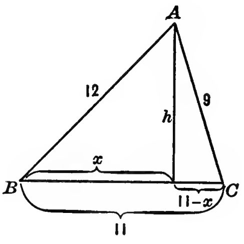
The point is 40 in. from B.

10. Solve the same problem if A' and B' lie on opposite sides of AB.

11. Solve the same problem if the distance AB is d, and the points A', B' lie on the same side of AB, and at distances a and b from A and B, respectively, with $a > b$.

12. Solve the preceding problem if the points A' and B' lie on opposite sides of AB.

13. The three sides of a triangle are 11, 9, 12. A perpendicular is dropped on the side of length 11 from the opposite vertex. Find the lengths of the segments into which the foot of the perpendicular divides that side.



SOLUTION. Using the notations of the figure,

$$h^2 = 12^2 - x^2, \quad (1)$$

and

$$h^2 = 9^2 - (11 - x)^2. \quad (2)$$

From (1) and (2), $12^2 - x^2 = 9^2 - (11 - x)^2.$ (3)

Rearranging (3), $12^2 - 9^2 + 11^2 = 22x.$ (4)

Solving (4), $x = \frac{92}{11}.$ (5)

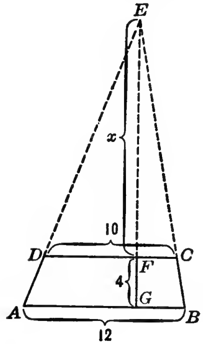
The other segment is $11 - x$, or $\frac{29}{11}$. (6)

The segments are $\frac{92}{11}$ and $\frac{29}{11}$.

14. Solve the preceding problem if the sides are 4, 7, 9, and the perpendicular is dropped on the side of length 7.

15. Solve the same problem if the sides of the triangle are a, b, c, and the perpendicular is dropped on the side of length a.

16. The lower base of a trapezoid is 12, the upper base is 10, and the altitude is 4. Determine the altitude of the triangle formed by the upper base and the prolongation of the two non-parallel sides until they meet.



SOLUTION. Using the notations of the figure,

$$\frac{EF}{EG}=\frac{DC}{AB}, \qquad (1)$$

or

$$\frac{x}{x+4}=\frac{10}{12}. \qquad (2)$$

Hence,
$$12x = 10x + 40, \qquad (3)$$

or
$$x = 20. \qquad (4)$$

The altitude is 20.

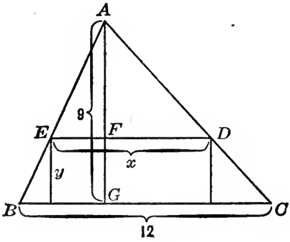
17. Solve the same problem if the lower base is a, the upper base b, and the altitude h.

LINEAR EQUATIONS. TWO UNKNOWNS

18. A rectangle 5 in. longer than it is wide is inscribed in a triangle of base 12 in. and altitude 9 in., the longer side resting on the base of the triangle. Find the dimensions of the rectangle.

SOLUTION. Let x denote the longer side and y the shorter

Then,
$$x - y = 5. \qquad (1)$$



In the similar triangles ABC and AED,

$$\frac{AF}{AG}=\frac{ED}{BC}, \qquad (2)$$

or

$$\frac{9-y}{9}=\frac{x}{12}. \qquad (3)$$

From (3), $108 - 12y = 9x.$ (4)

From (1) and (4), $108 - 12y = 9(y + 5),$ (5)

or $21y = 63.$ (6)

$y = 3.$ (7)

From (7) and (1), $x = 8.$ (8)

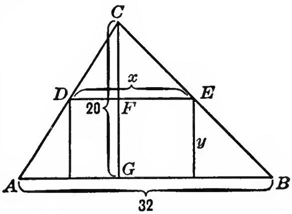
The dimensions of the rectangle are 3 in. and 8 in.

19. Solve the preceding problem if the shorter side rests on the base of length 12 in.

20. Solve Problem 18 if the difference of the sides is d, the length of the base of the triangle is a, the altitude is h, and the longer side of the rectangle rests on the base of the triangle.

21. Solve the preceding problem if the shorter side of the rectangle rests on the given base of the triangle.

22. A rectangle similar to a rectangle whose sides are 5 and 8 is inscribed in a triangle of base 32 and altitude 20. The longer side of the rectangle rests on the given base of the triangle. Find the dimensions of the rectangle.



SOLUTION. Let x and y denote the sides of the inscribed rectangle. Then from the similar triangles EDC and ABC,

$$\frac{DE}{AB} = \frac{CF}{CG}, \quad (1)$$

or $$\frac{x}{32} = \frac{20 - y}{20}. \quad (2)$$

From the similarity of the rectangles,

$$\frac{x}{y} = \frac{8}{5}. \quad (3)$$

From (3), $x = \frac{8}{5}y.$ (4)

From (2), $20x = 640 - 32y.$ (5)

From (4) and (5), $32y = 640 - 32y,$ (6)

or $64y = 640.$ (7)

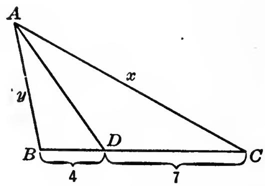
$y = 10.$ (8)

From (8) and (4), $x = 16.$ (9)

23. Solve the preceding problem if the shorter side of the rectangle rests on the given side of the triangle.

24. Solve the same problem if the given side of the triangle is of length a, and the altitude on it is of length h, and the given rectangle has dimensions l and m, provided the inscribed rectangle has its side corresponding to the side l of the given rectangle resting on the given base of the triangle.

25. The bisector of an angle of a given triangle divides the side opposite to the angle into two segments of lengths 4 in. and 7 in. The difference between the other two sides of the triangle is 5 in. Find the perimeter of the triangle.



Solution. Let ABC be the given triangle, AD the bisector of angle A, and x and y the required sides.

Then, $x - y = 5$, given in the problem, (1)

and $\frac{x}{y} = \frac{7}{4}$, by geometry the bisector divides the opposite side into segments proportional to the adjacent sides. (2)

$4x = 7y$, from (2). (3)

$7y - 4y = 20$, from (3) and 4 times (1). (4)

Then, $y = 6\frac{2}{3}$, solving (4). (5)

Then, $x = 11\frac{2}{3}$, from (5) and (1). (6)

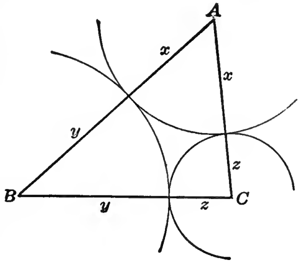
The perimeter is 11 in. + $6\frac{2}{3}$ in. + $11\frac{2}{3}$ in. = $29\frac{1}{3}$ in.

26. Solve the preceding problem, if the segments of the base are l and m and the difference between the sides is d. Also, if the segments of the base are p and q and the difference between the sides is r.

27. The sides of a triangle are 8 ft., 12 ft., and 15 ft., and the angle between the sides 8 and 12 is bisected by a line cutting the side 15. What is the length of each segment of the line 15?

LINEAR EQUATIONS. THREE UNKNOWNS

28. The points A, B, and C are situated so that $AB = 8$ in., $BC = 6$ in., $AC = 5$ in. Find the radii of three circles having the three points as centers and each tangent to the other two externally.



SOLUTION. Let x, y, z, be the radii of the three circles as indicated in the figure.

Then, $x + y = 8$, (1)

$x + z = 5$, (2)

$y + z = 6$. (3)

Adding (1), (2), and (3), and dividing the result by 2,

$x + y + z = \frac{19}{2}$. (4)

From (3) and (4), $x = \frac{7}{2}$. (5)

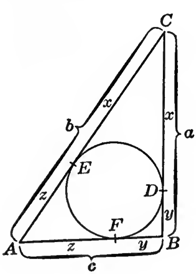
From (2) and (4), $y = \frac{9}{2}$. (6)

From (1) and (4), $z = \frac{3}{2}$. (7)

The radii are $3\frac{1}{2}$ in., $4\frac{1}{2}$ in., and $1\frac{1}{2}$ in.

29. Solve the same problem if $AB = 12$, $BC = 16$, and $AC = 20$.

30. Solve Problem 28, if $BC = a$, $AC = b$, $AB = c$.



31. A triangle ABC is circumscribed about a circle and its sides are tangent at points E, D, F. The sides of the triangle are $a = 8$ in., $b = 15$ in., and $c = 12$ in. Find the segments into which points E, D, F divides the sides.

SOLUTION. The tangents from an external point are equal, hence in the figure,

$x + y = 8$, (1)

$x + z = 15$, (2)

$y + z = 12$. (3)

Subtracting (3) from (2), $x - y = 3$. (4)

Adding (1) and (4) $2x = 11$ (5)

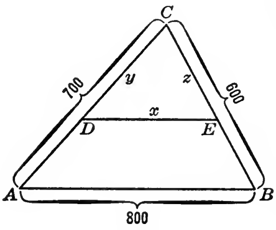
$\therefore \quad x = 5\frac{1}{2}$.

From (1) and (2) $y = 2\frac{1}{2}$ and $z = 9\frac{1}{2}$.

32. Solve the same problem, if $a = 24$ ft., $b = 10$ ft., and $c = 24$ ft.

Also, if $a = 30$ yd., $b = 40$ yd., $c = 50$ yd.

33. A man owned a triangular, unfenced field of sides 600 yd., 700 yd., and 800 yd. He sold a triangular piece cut off by a straight line parallel to the side of length 800 yd. and found that 1800 running yards of fence would be required to inclose what remained. Find the lengths of the sides of the portion sold.



SOLUTION. Using the notations of the figure,

$$AB + BC + AC + 2\,DE = AB + BE + EC + CD + DA + 2\,DE$$
$$= (AB + BE + ED + DA) + (EC + CD + DE). \quad (1)$$

Hence, $$2100 + 2x = 1800 + (x + y + z). \quad (2)$$

But $$\frac{CD}{CA} = \frac{DE}{AB}, \quad (3)$$

and $$\frac{CE}{CB} = \frac{DE}{AB}, \quad (4)$$

or $$\frac{y}{700} = \frac{x}{800}, \quad (5)$$

and $$\frac{z}{600} = \frac{x}{800}. \quad (6)$$

Hence, $$y = \frac{7x}{8}, \quad (7)$$

$$z = \frac{3x}{4}. \quad (8)$$

From (7) and (8), $$x + y + z = \frac{21x}{8}. \quad (9)$$

From (9) and (2), $$2100 + 2x = 1800 + \frac{21x}{8}. \quad (10)$$

Hence, $$\frac{5x}{8} = 300. \quad (11)$$

$$x = 480. \quad (12)$$

From (12) and (7), $$y = 420. \quad (13)$$

From (12) and (8), $$z = 360. \quad (14)$$

The lengths of the sides are: $DE = 480$ yd., $EC = 360$ yd., and $CD = 420$ yd.

34. Solve Problem 33 if the division line runs parallel to the side of length 700 yd.

35. Solve Problem 33 if the division line runs parallel to the side of length 600 yd.

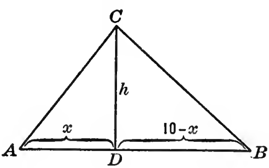
36. Solve Problem 33 if the sides are a, b, c, with the division line parallel to a, and the required length of fence, $2p$.

37. It is known that the sum of the three angles of any triangle is 180°. In a certain triangle the difference between the first angle and the second is 10°, and between the second angle and the third is 25°. Find the number of degrees in each angle.

38. The sum of the dimensions of a rectangular box is $13\frac{1}{2}$ ft.; the height equals one half the sum of the length and breadth and also equals twice their difference. Find the dimensions.

QUADRATIC EQUATIONS

39. The sides of a triangle are $AC = 7$, $BC = 9$, and $AB = 10$. Calculate the length of the altitude on the side 10, and of the two segments into which the altitude divides that side.



Solution. Using the notations of the figure,

$$h^2 = 7^2 - x^2, \quad (1)$$

and $$h^2 = 9^2 - (10 - x)^2. \quad (2)$$

Then, $$7^2 - x^2 = 9^2 - (10 - x)^2, \quad (3)$$

or $$7^2 = 9^2 - 100 + 20x. \quad (4)$$

Then, $$68 = 20x, \quad (5)$$

and $$x = \tfrac{17}{5}; \quad (6)$$

also, $$10 - x = \tfrac{33}{5}. \quad (7)$$

By (1), $$h^2 = 7^2 - (\tfrac{17}{5})^2. \quad (8)$$

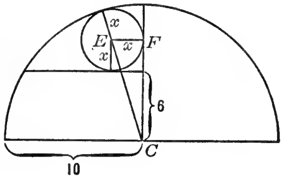
Then, $$h = \frac{\sqrt{7^2 \cdot 5^2 - 17^2}}{5} = \frac{6\sqrt{26}}{5}. \quad (9)$$

The segments are $\frac{17}{5}$ and $\frac{33}{5}$ and the altitude is $\frac{6\sqrt{26}}{5}$.

40. Calculate similarly the length of the altitude on the side of length 7, and of the segments into which the altitude divides that side.

41. Calculate similarly the length of the altitude on the side of length 9, and of the segments into which the altitude divides that side.

42. If the lengths of the sides of a triangle are a, b, c, calculate the lengths of the segments into which each side is divided by the altitude on that side.



43. In a circle of radius 10, a chord is drawn at distance 6 from the center. Find the radius of a circle that is tangent to the circle, to the chord, and to a diameter perpendicular to it.

SOLUTION. Using the notations of the figure:

$$\overline{EC}^2 = \overline{EF}^2 + \overline{FC}^2. \quad (1)$$

$$(10 - x)^2 = x^2 + (6 + x)^2. \quad (2)$$

$$100 - 20x + x^2 = x^2 + 36 + 12x + x^2. \quad (3)$$

$$x^2 + 32x - 64 = 0. \quad (4)$$

$$x = -\frac{32 \pm \sqrt{32^2 + 4 \cdot 64}}{2} \quad (5)$$

$$= -\frac{32 \pm 16\sqrt{4 + 1}}{2} \quad (6)$$

$$= -16 \pm 8\sqrt{5}. \quad (7)$$

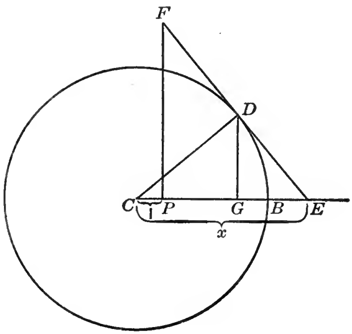
The negative value of x being inadmissible under the geometric conditions, we have:

$$x = -16 + 8\sqrt{5}. \quad (8)$$

44. Solve the same problem, if the radius of the given circle is 12 ft. and the chord is 4 ft. from the center.

45. Solve Problem 43, if the radius of the given circle is r and the chord is at d distance from the center.

46. A point P is selected on a diameter of a circle of radius 6, at the distance 1 from the center. At P a perpendicular is erected to the diameter in question, and a tangent is drawn to the circle such that the point of contact of the tangent bisects the segment of the tangent lying between the perpendicular and the diameter produced. Find the distance from the center to the point where the tangent cuts the diameter produced.



SOLUTION. Let $CE = x.$ (1)

Then in the right triangle CDE,

$$\overline{DE}^2 = x^2 - 6^2. \tag{2}$$

From the similar triangle DCE and DGE,

$$\frac{DE}{CE} = \frac{GE}{DE}, \tag{3}$$

or

$$DE^2 = CE \cdot GE \tag{4}$$

$$= x \cdot GE. \tag{5}$$

Since D bisects FE,

$$GE = \frac{PE}{2} \tag{6}$$

$$= \frac{x-1}{2}. \tag{7}$$

From (2), (5), and (7),

$$\frac{x(x-1)}{2} = x^2 - 6^2, \tag{8}$$

or

$$x^2 - x = 2\,x^2 - 72. \tag{9}$$

$$\therefore\ x^2 + x - 72 = 0. \tag{10}$$

$$x = -\frac{1 \pm \sqrt{1 + 288}}{2} \tag{11}$$

$$= -9,\ 8. \tag{12}$$

The negative root also indicates a solution. It means that a second tangent satisfying the required conditions cuts the diameter produced on the opposite side from P, at the distance 9.

47. Solve the same problem if the radius is 12 and the point lies at the distance 2 from the center.

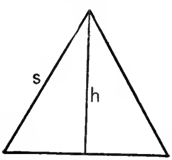
48. Solve the same problem if the radius is 15, and the distance of P from the center is 35.

49. Solve the same problem if the radius is r and the distance from the center is d.

50. The tangent to a circle is a mean proportional between the segments of the secant from the same point. Find the length of the tangent, if the segments of the secant are 4 ft. and 9 ft.

51. Two chords AB and CD intersect at O within the circle. The product of OA and OB equals the product of OC and OD. Given $OA = 4$, $OB = 8$, and $CD = 12$, find OC and OD.

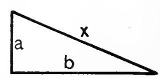
52. 4 times the square of the altitude (h) of an equilateral triangle equals 3 times the square of a side s. Express this by an equation. Solve the equation for s. Also for h in terms of s.



53. Find the altitude of an equilateral triangle whose side is 20 in., using 1.7321 as $\sqrt{3}$.

54. Find the side of an equilateral triangle, if $h = 9\sqrt{3}$ in.

55. The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. Express this relation in the form of an equation, using the letters in the figure.



56. Find the length of the hypotenuse of a right triangle the other two sides of which are 3 ft. and 4 ft. Also of one whose other two sides are 15 ft. and 20 ft.

57. Solve the equation $c^2 = a^2 + b^2$ for a; for b.

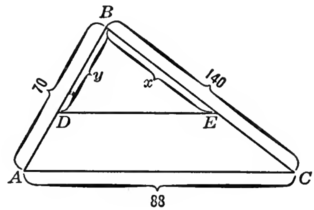
58. Find b in Exercise 57, if $c = 25$ and $a = 15$.

Similarly, determine the numbers to fill the blanks. To simplify calculation use the relation $c^2 - a^2 = (c - a)(c + a)$:

	59.	60.	61.	62.	63.	64.	65.
a.	15	7	40	45	—	208	44
b.	—	24	—	—	171	—	117
c.	17	—	41	53	221	233	—

SIMULTANEOUS QUADRATICS

66. The owner of a triangular lot whose sides are 70, 88, and 140 rd. in length wishes to divide it by a straight fence into two parts that shall be equal in area and also have the same perimeter. If the fence connects the sides of length 70 and 140 rd., how must it be placed?



Solution. Let DE be the desired position of the fence.

Then, $$\triangle ABC = 2\,\triangle DBE. \qquad (1)$$

Since the triangles have one angle in common,

$$\frac{\triangle ABC}{\triangle DBE} = \frac{70 \cdot 140}{xy}, \qquad (2)$$

or, by (1), $$2 = \frac{70 \cdot 140}{xy}, \qquad (3)$$

or $$xy = 35 \cdot 140. \qquad (4)$$

By the conditions of the problem,

$$BD + BE + DE = DA + AC + CE + ED. \qquad (5)$$

Subtracting DE from both members and replacing the other lines by their values,

$$x + y = 70 - y + 140 - x + 88, \qquad (6)$$
$$2(x + y) = 298, \qquad (7)$$
$$x + y = 149. \qquad (8)$$

From (4), $$4\,xy = 4 \cdot 35 \cdot 140 = \overline{140}^2. \qquad (9)$$

Squaring (8) and subtracting (9) from the result,

$$(x - y)^2 = \overline{149}^2 - \overline{140}^2 \qquad (10)$$
$$= (149 + 140)(149 - 140) \qquad (11)$$
$$= 289 \cdot 9. \qquad (12)$$

Then, $$x - y = \pm\, 17 \cdot 3 = \pm\, 51. \qquad (13)$$

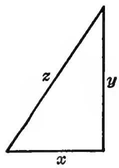
From (8) and (13), $$x = 100,\ 49. \qquad (14)$$
$$y = 49,\ 100. \qquad (15)$$

These values satisfy the algebraic equations, but in the concrete problem $y = 100$ is inadmissible, since y lies on the side of length 70. Hence, in the concrete problem, the result is $x = 100$, $y = 49$.

67. Solve the same problem if the fence connects the sides of length 70 and 88.

68. Solve the same problem if the fence connects the sides of length 88 and 140.

69. Solve the same problem if the sides are of length a, b, c, and the fence connects the sides of lengths a and c.



70. Find the sides of a right-angled triangle, given its area 25, and its perimeter 30.

SOLUTION. Let x, y, z denote the sides of the triangle, z being the hypotenuse.

Then,
$$x^2 + y^2 = z^2, \quad (1)$$
$$x + y + z = 30, \quad (2)$$
and
$$\frac{xy}{2} = 25. \quad (3)$$

Multiplying both members of (3) by 4,
$$2xy = 100. \quad (4)$$

Adding (4) and (1), $$x^2 + 2xy + y^2 = z^2 + 100, \quad (5)$$
or
$$(x + y)^2 = z^2 + 100. \quad (6)$$
From (2),
$$x + y = 30 - z, \quad (7)$$
or
$$(x + y)^2 = (30 - z)^2. \quad (8)$$
From (8) and (6),
$$(30 - z)^2 = z^2 + 100. \quad (9)$$
$$900 - 60z + z^2 = z^2 + 100. \quad (10)$$
$$60z = 800. \quad (11)$$
$$z = \tfrac{40}{3}. \quad (12)$$
From (7),
$$x + y = \tfrac{50}{3}, \quad (13)$$
or
$$x^2 + 2xy + y^2 = \tfrac{2500}{9}. \quad (14)$$

Multiplying (4) by 2 and subtracting the result from (14),
$$x^2 - 2xy + y^2 = \tfrac{700}{9}, \quad (15)$$
or
$$x - y = \pm \frac{10\sqrt{7}}{3}. \quad (16)$$

Adding (13) and (16) and dividing the result by 2,
$$x = \frac{25 \pm 5\sqrt{7}}{3}. \quad (17)$$

Subtracting (16) from (13) and dividing the result by 2,
$$y = \frac{25 \mp 5\sqrt{7}}{3}. \quad (18)$$

The sides are $\frac{25 + 5\sqrt{7}}{3}$, $\frac{25 - 5\sqrt{7}}{3}$, and $\frac{40}{3}$.

71. Solve the same problem if the area of the triangle is 64, and the perimeter 48.

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