510.7
Iℓ63h1 ILLINOIS. UNIVERSITY--
1956 COMMITTEE ON SCHOOL
v.3 MATHEMATICS
pt.1

HIGH SCHOOL MATHEMATICS

LIBRARY OF THE
UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

510.7
Iℓ63hl
1956
v.3
pt.1

Digitized by the Internet Archive
in 2011 with funding from
University of Illinois Urbana-Champaign

http://www.archive.org/details/highschoolmathem31univ

[redacted]

510.7
Il65t
1955
v.3 pt.1
uncat.

HIGH SCHOOL MATHEMATICS

THIRD COURSE

TEACHERS' EDITION

UNIT ONE 1956

MATHEMATICAL INDUCTION

UNIVERSITY OF ILLINOIS
COMMITTEE ON
SCHOOL MATHEMATICS

[redacted]

Copyright 1957 by the University of Illinois

This material may be quoted at a length of not over
100 words for each quotation in professional journals
or reviews without further written permission.
Requests for permission to quote under other conditions
should be addressed to the University of Illinois Committee
on School Mathematics, University High School, 1208 West
Springfield, Urbana, Illinois.

3h1
6

Unit 1

MATHEMATICAL INDUCTION

UICSM
University High School
Urbana, Illinois
1956

LIBRARY
UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

UICSM-2-56, Third Course

ERRATA

Unit One THIRD COURSE

Page	Line	
1-5	7	has a given property.
T.C. 11A	7	no algebraic expression of degree $n > 0$ whose...
1-20	16	expressed by (☆)...
T.C. 24B	15	delete the line
1-29	1b	$\sum_{x=1}^{3} 4x = \sum_{x=1}^{2} (4x) + [4(3)]$
1-30	5b	$\sum_{k=p}^{q+1} a_k = \sum_{k=p}^{q} (a_k) + [a_{q+1}]$
1-33	6b	and for every real integer $x > 0$...

urg

When Mr. Jones hears no response to his request "Call out 'absent' if your follower is absent", he knows precisely that the property expressed by (1) is hereditary. From this he knows that if any student is present then so is every student whose name follows this one's name alphabetically. (It may still be the case that Mr. Jones is alone in the assembly hall!) Hence, in order to know that (2) is true it is sufficient that he check on Bill Aaron.

If someone calls out 'absent' then Mr. Jones knows that the property expressed by (1) is not hereditary. He knows then that someone is present whose follower is absent. If there is only one call, he knows that the absentees form a group of one or more "consecutive" students. In fact if there are calls of 'absent' then the number of calls is the number of such groups of absentees.

* * *

Exercises

1.-2. See discussion above.

3. Bill Aaron is the only absentee.

4. (a) The absentees form a block of consecutive students (starting with Bill Aaron, of course). [In an extreme case there may be no students present.]

(b) Yes. In this case the first ten students would necessarily be among the absentees.

(c) Yes. In this case every student would be absent.

urg

Before reading further in the Commentary you should read rapidly through Sections 1.01, 1.02, and 1.03. The illustration on page 1-1 embodies many of the ideas which are introduced later in the unit. You should become familiar with the new terms in those sections as you read the following discussion.

* * *

Mr. Jones wishes to know whether every student at Zabranchburg High School has the property expressed by:

(1) ... is present today

or, in other words, whether the generalization:

(2) Every student is present today.

is true.

He does this by introducing the notion of <u>follower</u> which permits him to ask the pertinent question:

Is the property expressed by (1) hereditary?

That is, is it the case, for each student S who has a follower, that if S is present today then so is his follower? He knows that if the answer to this question is 'yes', <u>and</u> Bill Aaron is present, then (2) is true.

One way he might explain his certainty is by saying that if anyone were absent, there would be an alphabetically first absentee; that if Bill Aaron were present then this first absentee would have to be someone's follower; and that in this case this "someone" would then be present but his follower absent--so the property of being present would not be hereditary. Hence, if Bill Aaron has the property expressed by (1), and this property is hereditary, then every student has the property in question.

[As you have just seen, what is essential for this argument is that every non-empty subset of the set of students have a member who is not the follower of any member of this subset. As long as 'follower' is defined in such a way that this is the case, every property which is hereditary with respect to this notion of follower, and which holds for every student who is no one's follower, holds for every student.]

(continued on T. C. 1B)

1.01 <u>If for one then for the next.</u>--Mr. Jones, principal of Zabranchburg High School, likes to keep track of how many assemblies have perfect attendance. He has an interesting way of finding out whether everyone is present at an assembly. At the beginning of the school year all of the students in school line up in "alphabetical order". Each student in the school (except Dick Zilch who is last in line) memorizes the name of his FOLLOWER (the student directly behind him) and is responsible for reporting the absence of his FOLLOWER throughout the year.

Each time the students are seated in the auditorium, Mr. Jones first checks to see that Bill Aaron (who was first in the original "line-up") is present. Then Mr. Jones instructs the audience:

"Call out 'absent' if your FOLLOWER is absent."

No one calls out 'absent'. What can Mr. Jones conclude?

EXERCISES

<u>A.</u> Answer the following questions about Mr. Jones' way of checking for perfect attendance.

1. Why does Mr. Jones check to see if Bill Aaron is present? Could Bill Aaron be absent and no one call out 'absent'?
2. If after Mr. Jones makes his request, he hears exactly one call, would he be correct in concluding that exactly one student is absent?
3. On one assembly day Bill Aaron is absent but his FOLLOWER is present. Mr. Jones notes this fact and then asks each student to call out if his FOLLOWER is absent. No one calls out. What can he conclude?
4. On another assembly day Bill Aaron is again absent. Mr. Jones notes this fact and then asks each student to call out if his FOLLOWER is absent.
   (a) No one calls out. What can Mr. Jones conclude?
   (b) Could ten people be absent and no one call out?
   (c) Could Dick Zilch be absent and no one call out?

Exercises

1. Mr. Jones' conclusion is correct.

2. In this case Mr. Jones is wrong. There are 299 counter-examples: student 1 is the only absentee; students 1 and 2 are the only absentees; ...; students 1 through 299 are the only absentees. The strongest conclusion that Mr. Jones can draw is that either one of these 299 possibilities is the case or that all students are present. He might, for example, say that if any students are absent then those who are present are precisely those whose numbers follow that of the "last" absentee, and that student 300 is among these. Or he might draw the weaker conclusion that all students with numbers ≥ 300 are present.

3. Correct. In this case, for every student $S \neq 1$, S's follower may be defined to be student $S - 1$, and, if so, the conditions mentioned in the bracket on T. C. 1A are satisfied.

4. Mr. Jones is again in error. The situation now is essentially the same as that in Exercise 2. There are now precisely 627 counter-examples.

5. Similar to Exercise 2.

6. Mr. Jones' instructions are not quite complete, but he obviously means they should apply to all students except student 1 and student 628. With this understanding, students 250 and 251 are not "followers" of any students and, to justify his conclusion, it would be necessary to determine that these two are present. In the actual situation there are 250×378 counter-examples, and the most that Mr. Jones can conclude is that the absentees, <u>if any</u>, form a block of consecutive students to which one or both of students 250 and 251 belong.

B. There are other ways in which Mr. Jones could check for perfect attendance. He could line the students up in any way (without bothering to get them into alphabetical order) and then assign numbers in succession to the students. The first student in line is student 1, the second student is student 2, and so on with, say, student 628 being the last student in line. Each student remembers his number.

Below are described various methods Mr. Jones could use when the students have been numbered as described above. Some of the methods work and some do not. For the methods which do not work, find a counter-example, an example showing that students could be absent when Mr. Jones concludes that they are all present. In such cases, if there is a conclusion which Mr. Jones could make, tell it.

1. Mr. Jones checks that student 1 is present. Then he gives the instructions: "Every student S except 628 check if student S + 1 is present. If he isn't, call out 'absent'." Mr. Jones hears no calls so he concludes that every student is present.
2. Mr. Jones checks that student 300 is present and gives the same instructions as above. There are no calls; he concludes that every student is present.
3. Mr. Jones checks that student 628 is present and then gives the instructions: "Every student S except 1 check if student S - 1 is present." There are no calls and he concludes that everyone is present.
4. Mr. Jones gives the same instructions as in Exercise 3 and checks that student 1 is present. No calls. He concludes that everyone is present.
5. Mr. Jones gives the same instructions as in Exercise 3 and checks that student 475 is present. No calls. He concludes that everyone is present.
6. Mr. Jones gives the instructions: "Every student S, for $S > 250$, check if student S + 1 is present; every student S, for $S \leq 250$, check if student S - 1 is present." No calls. He concludes that every student is present.

(continued on next page)

[1-3]

628

7. Mr. Jones' conclusion follows from the evidence.

* * *

Part C

1. Student 628 has no follower; student 1 is no one's follower.

2. Student 1 has no follower; student 628 is no one's follower.

3. Student 252 is the follower of student 251; student 249 is the follower of student 250; student 1 has no follower; student 1 is the follower of student 2; student 628 is the follower of student 627; student 628 has no follower; students 250 and 251 are not followers.

4. Student 3 is the follower of student 1; student 4 is the follower of student 2; student 627 has no follower; the follower of student 626 is student 628.

5. No one.

* * *

Part D

The three preceding parts have explored the method of establishing universal generalizations which was discussed on T. C. 1A and 1B. As indicated on T. C. 1A, such generalizations assert that some <u>property</u> holds for every member of a certain <u>set</u>, i. e., that every member of this set <u>has</u> the property in question. An <u>instance</u> of a generalization of this type is a sentence which asserts that some named member of the set has the property. For example, 'Bill Aaron is present today' is an instance of 'Every student is present today'. The property in question is that of being present today. We shall also say that the property in question is that which is <u>expressed by</u> the expression '... is present today'. [The circumlocutions 'that of being ...' and 'expressed by '...' 'are necessitated by the fact that most of the properties which can be referred to in English (or in any other "natural" language) have no English names. Perhaps the closest one can come to an English name for the property in our example is 'present-today-ness'. If this is accepted then one can say that the property in question <u>is</u> present-today-ness.]

7. Mr. Jones checks that students 1 and 2 are present. Then he gives the instructions: "Every student S except 627 and 628 check if student S + 2 is present." No calls. Mr. Jones concludes that every student is present.

C. In Exercise 1 of Part B each student is told to check on his FOLLOWER by looking for the student who has been assigned the next higher number. Student 1's FOLLOWER is student 2, student 2's FOLLOWER is student 3, etc.

In Exercise 3 the student makes a different interpretation of 'FOLLOWER'. In that case student 628's FOLLOWER is student 627, student 627's FOLLOWER is student 626, etc.

Notice that no student has more than one FOLLOWER.

1. Which student does not have a FOLLOWER in Exercise 1? Which student is not a FOLLOWER in Exercise 1?
2. Which student does not have a FOLLOWER in Exercise 3? Which student is not a FOLLOWER in Exercise 3?
3. Who is the FOLLOWER of student 251 in Exercise 6? Who is the FOLLOWER of student 250? Does student 1 have a FOLLOWER? Does student 2 have a FOLLOWER? Does student 627 have a FOLLOWER? Does student 628 have a FOLLOWER? Which student is not a FOLLOWER?
4. In Exercise 7 who is the FOLLOWER of student 1? Of student 2? Of student 627? Of student 626?
5. For whom is it the case in Exercise 6 that he FOLLOWS his FOLLOWER?

D. Many things can be said of each of the students at Zabranchburg High School. Take Bob Floogle, for example:

> Bob is 6 feet tall, has blue eyes, is present when Mr. Jones checks for perfect attendance, does well in mathematics, is co-captain of the football team, has a hi-fi set, etc.

(e) This sentence can be analyzed in several ways. If taken as asserting a property of a thing, the more usual interpretation would be that it asserts that John has the property of loving Joan. An equally correct interpretation would be that it asserts that Joan has the property of being loved by John. A reason for preferring the first to the second is that if, as has been done here, an English sentence is analyzed as being of subject-predicate form then it is customary to take as subject the name which first occurs in the sentence. With this point of view, the first analysis would be more natural for (e) while the second would be more natural for (f).

[A third way of analyzing the sentence in question is that it asserts that John and Joan (in this order) have the relation expressed by '... loves ---'. You should not go out of your way to head off this analysis but, since we are interested in properties rather than in relations, it should not be stressed.]

(f) See discussion for (e).

(g) [This is a quotation from a poem by Edward Lear. See LAUGHABLE LYRICS (1888).] The thing is the Pobble who has no toes; the property is that expressed by '... swam across the Bristol Channel'.

(h) (The operation of) addition of integers; commutativity.

(i) Truth; that of being beautiful, or beauty. [An alternative rendering of the same sentiment is: Truth has beauty.]

(j) The last dodo; that expressed by '... died before 1628'.

(k) (3, 4, 5); that of being a Pythagorean triple or, in jest, Pythagorean triplicity. [You may need to review the concept of Pythagorean triples.]

Exercises 1 and 2 are intended to familiarize the student with the notion of a property. In Exercise 3 we return to the concept of "follower" and the question as to whether a given property is hereditary [See page 1-8] with respect to a given definition of 'follower'.

1. One such statement is 'Bob Floogle goes steady with Mary Jones'. The property in question is that of going steady with Mary Jones and is expressed by '... goes steady with Mary Jones'. It has no systematically formed name but it might be worthwhile bringing out the point that it, like anything else, might be assigned a name if it were expedient to do so. For example, we might decide to call it 'Bill'. This would of course require a definition, for example: 'Bill' is a name for the property of going steady with Mary Jones, or: 'Bill' is a name for the property expressed by '... goes steady with Mary Jones'. [But 'Bill' is not an abbreviation for '... goes steady with Mary Jones' since the latter is not a <u>name</u> for the property in question.] So, a student who gives this example might indicate the property in question by saying the following:

(a) The property is that of going steady with Mary Jones.

(b) The property is expressed by:

... goes steady with Mary Jones.

(c) The property is Bill.

[If he gives (c), and this is highly unlikely, then he would have to supply a definition of 'Bill' such as we have given above.]

2.

(a) The thing is George Washington; the property is that of being a president of the United States.

(b) 2; evenness.

(c) Donald Duck; that of quacking, or that expressed by '... quacks'.

(d) 2; primeness, or that of being prime, or that expressed by '... is prime'. [You may need to review here the concept of prime number. We use this concept later in the unit. A prime number is a counting number other than 1 whose only divisors are itself and 1.]

(continued on T. C. 4B)

Each of these facts can be stated in a single sentence. For example:

(1) Bob Floogle has blue eyes.

(2) Bob Floogle is present when Mr. Jones checks for perfect attendance.

(3) Bob Floogle is co-captain of the football team.

Each of these sentences states that Bob Floogle has a certain <u>property.</u> Sentence (1) states that he has the property BLUE-EYEDNESS. Sentence (2) states that he has the property expressed by:

... is present when Mr. Jones checks for perfect attendance.

Sentence (3) states that he has the property of being co-captain of the football team.

1. Make up five other statements which might be true for Bob Floogle and indicate in two or three ways what property of Bob Floogle is referred to in each of your statements.
2. Each of the following sentences states that a certain thing has a certain property. What is the thing? What is the property?

   (a) George Washington was one of the presidents of the U.S.

   (b) 2 is even.

   (c) Donald Duck quacks.

   (d) 2 is prime.

   (e) John loves Joan.

   (f) Joan is loved by John.

   (g) "The Pobble who has no toes swam across the Bristol Channel."

   (h) Addition of integers is commutative.

   (i) Truth is beautiful.

   (j) The last dodo died before 1628.

   (k) (3, 4, 5) is a Pythagorean triple.

(continued on next page)

In Exercise 3 we bring together in a formal way the notions of set, follower, and property, and , without using the word, the notion of hereditariness. The principal purpose of these exercises is to get the student to <u>state</u> the question which he is told "to ask and to answer". [Another purpose of these exercises is to give the student opportunities to review mathematical ideas from earlier courses.] The student's justification for his answer may be informal. In the case of Sample 1 we have given an informal justification. A more formal justification is that, for every counting number n and for every counting number k, if $n = 2k + 1$ then $n + 2 = 2(k + 1) + 1$. Hence, if n is odd then so is $n + 2$.

3. In checking for perfect attendance Mr. Jones found that it was helpful to line the students up and if, say, Norman is directly behind James, then to declare that Norman is James' FOLLOWER. You will learn in this unit that such a procedure of "lining things up" and introducing the notion of "FOLLOWER" is often helpful in proving that each of the things in question has a certain property. (In Mr. Jones' case this was the property of being present at that assembly.)

   In each of the following exercises you are given a set, a definition of 'follower', and some properties. For each of the properties you are <u>to ask and to answer</u> a question of the form:

   > Is it the case that whenever a member of the set has this property, then so does its follower?

<u>Sample 1.</u> <u>Set:</u> All of the counting numbers: 1, 2, 3, ...

<u>Follower:</u> For every counting number n, n's follower is n + 2.

<u>Property:</u> ODDNESS

<u>Solution.</u> First, state the question in terms of the given information:

> Is it the case that whenever a counting number is odd, then the result of adding 2 to it is also odd?

You can answer this question from your knowledge of counting numbers. The answer is 'yes'. In other words, if any member of the set has this property, then so does its follower.

Suppose $\frac{1 + r}{2}$ were rational. Then $2 \times \frac{1 + r}{2}$ would be rational, and then $(2 \times \frac{1 + r}{2}) - 1$ would be rational also. But, this is r. So, r would be rational. But, we said that r was irrational, and we know that an irrational real number can't be a rational real number.

No. Each number r such that $-1 \leq r < 0$ is a counter-example. [After finding a few counter-examples the students should be able to describe the set of all counter-examples.]

(7) "Is it the case that, for every real number r, if r is irrational (i.e., if r is an irrational real number) then $\frac{1 + r}{2}$ is irrational?"

This is a good example of a situation where it is simpler to replace a statement by its contrapositive. The statement in question is:

(i) if r is irrational then $\frac{1 + r}{2}$ is irrational

and this is equivalent to its contrapositive:

(ii) if $\frac{1 + r}{2}$ is rational then r is rational

(since 'not irrational' means rational).

Hence, we can reformulate the question as: "Is it the case that, for every real number r, if $\frac{1 + r}{2}$ is rational then r is rational?"

Now, the answer is obviously 'yes' since, for every real number r, if $\frac{1 + r}{2}$ is rational then so is $2(\frac{1 + r}{2}) - 1$ which is r. (cf. (4).)

[It may be desirable to stress the difference between (ii), the contrapositive of (i), and:

(iii) if $\frac{1 + r}{2}$ is irrational then r is irrational

which is the converse of (i).]

Before dealing formally with the concept of contrapositive, you might encourage the following informal justification:

(continued on T.C. 6E)

6]

(3) "Is it the case that, for every real number r, if $r = 1$ then $\frac{1 + r}{2} = 1$?"

Yes. $\frac{1 + (1)}{2} = 1$.

(4) "Is it the case that for every real number r, if r is a rational real number, then $\frac{1 + r}{2}$ is a rational real number?"

[Here you should review briefly the fact that a subset of the real numbers is isomorphic to the set of rational numbers. This subset of the real numbers is called the <u>set of rational real numbers</u>. For example, the real number $\frac{3}{2}$ corresponds to the rational number $\frac{3}{2}$ and is a rational real number. The sum or product of two rational real numbers is a rational real number.]

Yes. For every rational real number r, $1 + r$ is a rational real number, and $\frac{1 + r}{2}$ is a rational real number.

(5) "Is it the case that, for every real number r, if $r < 1$ then $\frac{1 + r}{2} < 1$?"

Yes. Informal justification:

> The midpoint of the interval $\overline{r, 1}$ belongs to the interval $\overline{r, 1}$.

Formal justification:

> For every real number r,
> if $r < 1$ then $r + 1 < 2$,
> and $\frac{r + 1}{2} < 1$.

(6) "Is it the case that, for every real number r, if $r < 0$, then $\frac{1 + r}{2} < 0$?"

(continued on T.C. 6D)

* * *

(b) Students should discover that for every real number r, the graph (on the number line) of $\frac{1 + r}{2}$ (i.e., the graph of r's follower) is the midpoint of the segment $\overleftrightarrow{1, r}$.

(1) "Is it the case that, for every real number r, if $1 < r < 3$ then $1 < \frac{1 + r}{2} < 3$?"

Yes. An informal justification: If r belongs to the segment $\overleftrightarrow{1, 3}$ then the segment $\overleftrightarrow{1, r}$ is a subset of the segment $\overleftrightarrow{1, 3}$, and the midpoint of $\overleftrightarrow{1, r}$ belongs to $\overleftrightarrow{1, 3}$.

More formally:

For every real number r, if
$1 < r < 3$ then $\frac{1}{2} < \frac{r}{2} < \frac{3}{2}$ and
$1 = \frac{1 + 1}{2} < \frac{1 + r}{2} < \frac{1 + 3}{2} = 2 < 3.$

(2) "Is it the case that, for every real number r, if $|r - 1| < 3 - 1 = 2$, then $\left|\frac{1 + r}{2} - 1\right| = \left|\frac{r - 1}{2}\right| < 2$?"

Yes. An informal justification: If the segment $\overleftrightarrow{1, r}$ is a subset of the interval $\overline{-2, 3}$, then the midpoint of $\overleftrightarrow{1, r}$ belongs to $\overline{-2, 3}$.

A more formal justification:

For every real number r, if
$-2 < r - 1 < 2$ then $\frac{-2}{2} < \frac{r - 1}{2} < \frac{2}{2}$
or $-1 < \frac{r - 1}{2} < 1$. Since $-2 < -1$
and $1 < 2$, it follows that
$-2 < \frac{r - 1}{2} < 2.$

(continued on T.C. 6C)

Note that in Sample 2 the counter-example is a number, the number 2. The demonstration that it is a counter-example amounts to proving that 2's follower (the next larger prime) is not even. [Since 2 is the only even prime, there are no other counter-examples.]

* * *

(a)

(1) "Is it the case that whenever a counting number is even, the result of adding 12 to it is even?"
Yes. See discussion for Sample 1; $2k + 12 = 2(k + 6)$.

(2) Similar to (1).

(3) "Is it the case that whenever a counting number is prime, the result of adding 12 to it is prime?"
No. Counter-examples: 2, 3, 13, and others.

(4) Similar to (1).

(5) "Is it the case that whenever a counting number is greater than 5, the result of adding 12 to it is greater than 5?"
Yes. For every n, $n + 12 > n$; and $>$ is a transitive relation.

(6) "Is it the case that whenever a counting number is a multiple of 5, the result of adding 12 to it is a multiple of 5?"
An alternative formulation of the question is: "Is it the case that for every counting number n, if there exists a counting number k such that $n = 5k$, then there exists a counting number k' such that $n + 12 = 5k'$?"
No. Counter-example: 5.

(continued on T.C. 6B)

Sample 2. Set: All of the prime numbers.

Follower: For every prime number n, n's follower is the next larger prime number.

Property: EVENNESS

Solution. Again, you state the question in terms of the given information:

> Is it the case that whenever a prime number is even, then the next larger prime is also even?

You can answer 'no' to this question if you can find a counter-example. That is, you can answer 'no' if you can find an even prime number such that the next larger prime number is not even. The prime number 2 is one such. (Is it the only one?)

(a) Set: All of the counting numbers.

Follower: For every n, n's follower is n + 12.

Property: (1) EVENNESS

(2) of being divisible by 3

(3) PRIMENESS

(4) ODDNESS

(5) of being greater than 5

(6) of being a multiple of 5

(b) Set: All of the real numbers.

Follower: For every r, r's follower is $\frac{1 + r}{2}$.

Property: (1) of being between 1 and 3

(2) of being closer to 1 than 3 is

(3) of being 1

(4) of being a rational real number (i.e., RATIONALITY)

(5) of being less than 1

(6) of being a negative number

(7) IRRATIONALITY

(10) Another exercise of the same type as Exercises (8) and (9). In this case another explanation of the answer 'yes' can be given along the lines of that given for Exercise (6).

(11) This exercise is essentially different from the preceding three exercises. Instead of referring to two parallel lines it refers to the same line twice. The question may be stated, "Is it the case that, for all real numbers x and y, if $4x - 2y + 1$ then $4(x - 2) + 5(y - 3) = 7$?" The answer is 'no' and an explanation is that any point on the locus of '$4x - 2y = 1$' (and there are such points) is a counter-example.

(12) No. There is only one point (x, y) in the intersection of the loci of '$2y - 3x = 4$' and '$3x - 5y = 1$' and, for all x and y, $(x - 2, y - 3) \neq (x, y)$. In other words, the intersection of the loci is the set consisting of the single point $(-\frac{22}{9}, -\frac{5}{3})$; it is clear that the point $(-\frac{22}{9} - 2, -\frac{5}{3} - 3)$ does not belong to the intersection.

'true' in this sense it will be convenient to use instead the phrase 'materially true'. In mathematics, when one says that a sentence is true one means something quite different. What one means is that the sentence in question is a theorem, i.e., that it is a logical consequence of the postulates and definitions on which mathematics is based. Instead of using the word 'true' in this sense it will be convenient to use instead the phrase 'mathematically true'. For example, the question with which Exercise (8) is concerned: Is it the case that, for ... = 9? can be restated: Is 'For ... = 9' a theorem? or: Is 'For ... = 9' mathematically true? As indicated above, the answer to this question is 'yes'. A more satisfactory explanation for this answer than that given above is: Since we are able to prove 'There do not exist real numbers x and y such that 4x + 5y = 7 and 4x + 5y = 9', it is mathematically true that 'For ... = 9' has no counterexamples. Hence, the latter sentence is a theorem. In general, any sentence of the form:

For every x and y, if ...x...y..., then ________ .

is a theorem if:

There do not exist real numbers
x and y such that ...x...y...

can be proved.

(9) This exercise is essentially the same as Exercise (8). The two lines referred to are parallel, so their intersection is the empty set.

(continued on T.C. 7G)

not assert that there <u>are</u> any counter-examples. This should suggest to them that they return to Exercise (7) and complete the explanation given there (if they have not already done so) by inserting ' , and there <u>are</u> points which belong to the union of the two lines ' before the period. Since there are no points which belong to the intersection of the loci of '$4x + 5y = 7$' and '$4x + 5y = 9$' students may strongly suspect that there are no counter-examples. This is, of course, the case since a counter-example to:

> For all real numbers x and y, if $4x + 5y = 7$ and $4x + 5y = 9$ then $4(x - 2) + 5(y - 3) = 7$ and $4(x - 2) + 5(y - 3) = 9$.

would be a pair (x, y) of real numbers such that

1) $4x + 5y = 7$ and $4x + 5y = 9$, and
2) either $4(x - 2) + 5(y - 3) \neq 7$ or $4(x - 2) + 5(y - 3) \neq 9$.

Since there is no pair of real numbers which satisfy condition 1), there is no pair of real numbers which satisfy conditions 1) and 2), i.e., there is no counter-example. <u>A generalization which has no counter examples is true.</u> Hence, the correct answer and explanation for Exercise (8) is: Yes. There are no counter-examples because there do not exist real numbers x and y such that $4x + 5y = 7$ and $4x + 5y = 9$.

A further point which should be understood has to do with the use of the word 'true'. In most situations of every day life, when one says that a sentence is true, one means that what it asserts is a fact. Instead of using the word

(continued on T.C. 7F)

Since we obtain the same conclusion:

$$2(y - 3) - 3(x - 2) = 7$$

$$\text{or} \quad 3(x - 2) + 2(y - 3) = 9$$

Since we obtain the same conclusion:

$$2(y - 3) - 3(x - 2) = 7$$

<u>or</u> $3(x - 2) - 2(y - 3) = 9$

under either hypothesis:

$$2y - 3x = 7$$

or:

$$3x - 2y = 9$$

this conclusion is a consequence of the alternation (disjunction) of the hypotheses. [Note that since we are considering the union of the loci, we are concerned with the alteration ("or") of the hypotheses, rather than with the conjunction ("and") of the hypotheses.]

(7) "Is it the case that, for all real numbers x and y, if $4x + 5y = 7$ or $4x + 5y = 9$ then $4(x - 2) + 5(y - 3) = 7$ or $4(x - 2) + 5(y - 3) = 9$?"
No. Any point which belongs to the union of the given lines is a counter-example.

(8) "Is it the case that, for all real numbers x and y, if $4x + 5y = 7$ <u>and</u> $4x + 5y = 9$ then $4(x -2) + 5(y - 3) = 7$ <u>and</u> $4(x - 2) + 5(y - 3) = 9$?"
A student's natural inclination, after answering Exercise (7) may be to give the same answer (and a similar explanation) for the present exercise (of course, replacing 'union' by 'intersection'). We hope that some students will at first answer in this way, and <u>then</u> realize that there are no points in the intersection of the given lines. Some will, of course, become aware of this immediately on reading the problem. At any rate, they should all see that the true sentence 'Any point which belongs to the intersection of the given lines is a counter-example' does

(continued on T.C. 7E)

No. The set of all counter-examples is the set of all points (x, y) in the first quadrant with $x \leq 2$ or $y \leq 3$.

(2) No. The set of all counter-examples is the set of all points (x, y) in the second quadrant with $y \leq 3$.

(3) Yes.

(4) No. The set of all counter-examples is the set of all points (x, y) in the fourth quadrant with $x \leq 2$.

(5) Yes. For all real numbers x and y, if $2y - 3x = 7$ then $2(y - 3) - 3(x - 2) = 7$. [Here is a place to stress again the fact that a point is in the locus of an equation if and only if its coordinates satisfy the equation.]

(6) "Is it the case that, for all real numbers x and y, if $2y - 3x = 7$ <u>or</u> $3x - 2y = 9$, then $2(y - 3) - 3(x - 2) = 7$ <u>or</u> $3(x - 2) - 2(y - 3) = 9$?"

Yes. Informal justification:

The union of the loci in question is a pair of parallel lines each with slope $\frac{3}{2}$. If the point (x, y) belongs to one of these lines, then the point $(x - 2, y - 3)$ also belongs to that line since the slope of the segment $\overline{(x, y)(x - 2, y - 3)}$ is $\frac{3}{2}$.

More formal justification:

For every x and y, if $2y - 3x = 7$ then $2(y - 3) - 3(x - 2) = 7$; so, certainly,

$2(y - 3) - 3(x - 2) = 7$

<u>or</u> $3(x - 2) - 2(y - 3) = 9$;

if $3x - 2y = 9$ then $3(x - 2) - 2(y - 3) = 9$; so certainly,

$2(y - 3) - 3(x - 2) = 7$

<u>or</u> $3(x - 2) - 2(y - 3) = 9$

(continued to T.C. 7D)

* * *

It would be instructive if at sometime during the discussion of Exercise 3 (c) the students discovered the point of view that (c) deals with a transformation, $(x, y) \rightarrow (x, y+2)$, of the coordinate plane onto itself, and that each part of (c) is concerned with the question of whether a certain subset of the coordiante plane is <u>mapped</u> ("moved") onto itself (or onto a part of itself) by this transformation. For example, the transformation in question clearly maps:

the first quadrant onto part of itself;
the second quadrant onto part of itself;
the third quadrant onto the union of three sets: the third quadrant itself, the "negative half" of the first coordinate axis, and part of the second quadrant;
the locus of '$x = 7$' onto (the whole of) itself;
the curve whose equation is '$y = x^2$' onto a second curve which does not intersect the first;
the set of all points with real integral coordinates onto itself.

A similar point of view can be adopted in the case of Exercise 3(d).

* * *

(d)

(1) "Is it the case that, for all real numbers x and y, if $x > 0$, and $y > 0$. then $x - 2 > 0$ and $y - 3 > 0$?"

(continued on T.C. 7C)

(c) [Coordinate plane paper should be used for (c) and (d).]

(1) "Is it the case that, for all real numbers x and y, if $x > 0$ and $y > 0$ then $x > 0$ and $y + 2 > 0$?"

Yes. For every real number y, if $y > 0$ then $y + 2 > 0$. [See (a) (5).] [Compare these problems with the coordinate planes "games" in FIRST COURSE.]

(2) Yes. Similar to (1).

(3) No. Counter-example: (-3, -1). [Students should be able to describe the set of all counter-examples for this Exercise and for (4).]

(4) No. Counter-example: (5, -1).

(5) "Is it the case that, for all real numbers x and y, if (x, y) belongs to the locus of '$x = 7$' then (x, y + 2) belongs to the locus of '$x = 7$'?"

Yes. For every real number x, if $x = 7$ then $x = 7$.

(6) No. Counter-example: (3, 9). (3, 9) satisfies '$y = x^2$' but (3, 11) does not. In fact, for all real numbers x and y, if $y = x^2$ then (x, y) is a counter-example.

(7) Yes. For every real number y, if y is a real integer then y + 2 is a real integer.

[As in the case of (b) (4), you should review the fact that that subset of the real numbers which is isomorphic to the set of integers is called the set of real integers.]

(continued on T. C. 7B)

(c) Set: All the points in the coordinate plane.

Follower: For every x and y, the follower of the point (x, y) is the point $(x, y + 2)$.

Property:

(1) of being in the first quadrant

(2) of being in the second quadrant

(3) of being in the third quadrant

(4) of being in the fourth quadrant

(5) of being in the locus of '$x = 7$'

(6) of being in the locus of '$y = x^2$'

(7) of being a point with real integral coordinates

(d) Set: All of the points in the coordinate plane.

Follower: For every x and y, (x, y)'s follower is $(x - 2, y - 3)$.

Property:

(1) of being in the first quadrant

(2) of being in the second quadrant

(3) of being in the third quadrant

(4) of being in the fourth quadrant

(5) of being in the locus of '$2y - 3x = 7$'

(6) of being in the union of the locus of '$2y - 3x = 7$' and the locus of '$3x - 2y = 9$'

(7) of being in the union of the loci of '$4x + 5y = 7$' and of '$4x + 5y = 9$'

(8) of being in the intersection of the loci of '$4x + 5y = 7$' and of '$4x + 5y = 9$'

(9) of being in the intersection of the loci of '$5(x - 4) + 7(3 - y) = 8x$' and of '$6x + 14y = 13$'

(10) of being in the intersection of the loci of '$3x - 2y = 9$' and of '$2y - 3x = 7$'

(11) of being in the intersection of the loci of '$8x - 4y = 2$' and '$2y - 4x + 1 = 0$'

(12) of being in the intersection of the loci of '$2y - 3x = 4$' and of '$3x - 5y = 1$'

Line 21:

The phrase 'a property which is meaningful for the members of the set' should be interpreted as follows: If 'P' is a name of a property, then P is meaningful for the members of the set if the result of replacing ' ... ' in:

... has P

by a name for a member of the set is a sentence which "makes sense" i. e., is well-formed. For example, the property beauty is not meaningful for the counting numbers but is meaningful for physical objects. On the other hand, primeness is meaningful for the counting numbers but is not meaningful for people.

***

Exercises

Although the exercises here are similar to those on page 1-1, the student now has a framework of ideas on which to base his answers.

1. No, but the absentees, if any, must be consecutive and include Bill Aaron.

PROPERTIES WHICH ARE INHERITED

Let us return to Mr. Jones and the Zabranchburg High assembly as described in Exercise 3 on page 1-5.

> The set Mr. Jones deals with is the set of all students enrolled at the school.
>
> 'follower' is defined by lining up the students and declaring that the follower of each student (except the last in line) is the student directly behind him.
>
> The property he is interested in is that of being present.

When no student calls 'absent', Mr. Jones knows that, in the case of each student, if that student is present then so is his follower. For short, we say in this case that the property of being present is hereditary.

[You can guess the meaning of the word 'hereditary' if you know the meaning of the word 'inherit'. If a property is hereditary, and if Ezra is Milton's follower and Milton has the property in question, then Ezra "inherits" the property from Milton. If you answered the questions in Exercise 3 correctly, then each yes-answer indicated an hereditary property.]

In general, given a set and a definition of 'follower', then a property which is meaningful for the members of the set is hereditary over the set if it is the case that whenever a member has the property in question, then so does its follower (if it has a follower).

EXERCISES

Recall the situation referred to in the first paragraph of Exercise 3 on page 1-5. Two bits of additional information are that Bill Aaron is first in line and Dick Zilch is last in line. Now answer the following questions assuming that Mr. Jones has just given his familiar instructions: "Call out 'absent' if your follower is absent."

1. If Mr. Jones hears no response, can he conclude that every student is present?

(continued on next page)

Sometimes a convention is made that the "domain of the index" is the desired set and, in this case it is not necessary to refer to the set in the quantifier. [Compare Exercises 1 and 2 with 3 on page 1-10. With respect to Exercise 3 it is to be understood that the domain of 'n' is the set of all counting numbers.]

***

A generalization such as the boxed one has many other instances besides those we have called 'first', 'second', etc. For example, 'the (5 + 2)<u>nd</u> odd number is $2 \cdot (5 + 2) - 1$' is an instance of the boxed statement and is a different sentence from any of those listed. [For one thing, it contains a ' + ' while this symbol does not occur in any of the enumerated instances.] However, the instance in question is <u>equivalent</u> to what we would call the seventh instance of the boxed statement.

2. Yes.

3. All are present.

4. The absentees, if any, form a consecutive group of students, beginning with Bill Aaron, and ending before James Monk.

5. See solution for Exercise 4.

6. Some student is present whose follower is absent; the property of being present is not hereditary.

***

We are using the word 'generalization' as short for 'universal generalization'. Logicians call statements like:

> There is a counting number n such that the nth odd number is 2n - 1.

'existential generalizations'. Note the "tie" between 'existential' and 'There is'.

This is the first formal use of the word 'generalization' which the student has encountered in the UICSM program. We intend to use the word in a technical sense. A generalization is a statement. A <u>universal</u> generalization such as we are concerned with here asserts that <u>every</u> member of some set has a given property. A generalization consists of a <u>quantifier</u> [for example, 'for every counting number',] a phrase which expresses the property in question [for example, 'the ....<u>th</u> odd number is 2·... -1'], and an "index", [for example, 'n'] which "links" the quantifier with the blanks in the phrase. The quantifier usually contains reference to a set. [The assertion made by the generalization is that the property in question is "generalized over the set".]

(continued on T. C. 9B)

2. If Mr. Jones hears no response, can he conclude that the property of being present is hereditary?

3. If Mr. Jones hears no response <u>and</u> knows that Bill Aaron has the property, what can he conclude?

4. If Mr. Jones hears no response and knows that James Monk has the property, what can he conclude?

5. If Mr. Jones hears no response and knows that Dick Zilch has the property, what can he conclude?

6. If Mr. Jones hears a response, what can he conclude? Can he conclude that the property is hereditary?

1.02 <u>Generalizations about counting numbers.</u>--Given a property which is meaningful for each counting number, a problem which often arises is that of proving that each counting number has this property. A statement which asserts that every counting number has a certain property is a <u>generalization</u> about counting numbers. For example:

For every counting number n, the n<u>th</u> odd number is 2n - 1 .

To begin with, let us make sure we know what this <u>generalization</u> states. What it states is most easily understood by considering some of its <u>instances</u>. An instance of it is obtained by filling the two blanks in:

the ... th odd number is $2 \cdot \ldots - 1$

with a name for a counting number. For convenience, we shall say that its <u>first</u> instance is :

the 1st odd number is $2 \cdot 1 - 1$

that its <u>second</u> instance is:

the 2nd odd number is $2 \cdot 2 - 1$

that its <u>third</u> instance is:

the 3rd odd number is $2 \cdot 3 - 1$

etc.

4. True

5. False

6. False

7. True

8. True. This generalization is itself the first instance of the generalization:

$$\text{For every } m \text{ (and) for every } n,$$
$$(n + m)^2 = n^2 + 2nm + m^2.$$

9. True. This generalization is itself the 112<u>th</u> instance of the generalization which states that addition of counting numbers is commutative:

$$\text{For every } m \text{ (and) for every } n,$$
$$m + n = n + m.$$

10. True.

11. True. [The first instance is 'the (1 + 4)<u>th</u> even number is $2 \cdot 1 + 8$' which is a true sentence.]

12. True.

Students should become convinced that whether a sentence is a consequence of another (i. e., can be inferred from it) is independent of the truth or falsity of the latter. Of course, if a sentence is true then so is each of its consequences, and a sentence from which a false sentence can be inferred is itself false. But whether a sentence is a consequence of another is a question of grammar alone (or, as a logician would be more likely to say, of syntax). The answer to it depends only on the forms of the two sentences.

Point out to students that when they were dealing with the True-False exercises in the geometry unit, they established the falsity of some generalizations by discovering counter-examples. A counter-example to a generalization shows that some instance of the generalization is false. Since any instance is a consequence of the generalization, the falsity of any instance shows that the generalization is false. On the other hand, we hope that they recognize that they can not establish any generalization by establishing any number of its instances.

***

Exercises.

A.

1. Examples of instances: 1 is even, 2 is even, 3 is even, etc. These are alternately false and true. Since the generalization has at least one (in fact, infinitely many) false instances, it, itself, is false.

2. $2 \cdot 1$ is even, $2 \cdot 2$ is even, $2 \cdot 3$ is even, $2 \cdot 5285$ is even, etc. Each of these instances is true and in fact from the definition of 'even number' we know that every instance of the generalization is true. So the generalization is true, by definition of 'even number'.

3. The generalization is true.

(continued on T. C. 10B)

Each instance of the generalization can be <u>inferred</u> from the generalization. In particular, if we accept the generalization, we must, logically, accept each of its instances; moreover, if the generalization is true, then each of its instances is true. Of course, there are generalizations some or all of whose instances are false. [Examples: (1) Every horse is white. (That is, for every horse h, h is white.) (2) Every dog has seven legs. (That is, for every dog d, d has seven legs.)] Such generalizations are false. Even so, each instance of a false generalization can be inferred from it.

EXERCISES

<u>A</u>. Give three instances of the following generalizations. State for each instance whether the instance is true or false. Make a <u>guess</u> about the truth of each generalization.

1. For every counting number n, n is even.

2. For every counting number n, 2n is even.

<u>Note</u>: Since it is understood that we are talking about no other numbers than counting numbers in these exercises, we shall abbreviate 'For every counting number n' to 'For every n'.

3. For every n, $2n + 1$ is odd.

4. For every n, the <u>nth</u> even number is 2n.

5. For every n, $n > 2$.

6. For every n, $n > 7$.

7. For every n, $n \geq 1$.

8. For every n, $(n + 1)^2 = n^2 + 2n + 1$.

9. For every n, $112 + n = n + 112$.

10. For every n, $(n + 1)^2 - n^2 = 2n + 1$.

11. For every n, the $(n + 4)$<u>th</u> even number is $2n + 8$.

12. For every n, the $(n + 1)$<u>th</u> odd number is $2n + 1$.

(continued on next page)

13. True.

14. True.

15. This is a classic problem frequently used to illustrate the fact that even a large number of true instances of a generalization do not constitute evidence that the generalization is a theorem. Each of the first 40 instances of this generalization is true but the 41st is false. [As a matter of fact there is no algebraic expression whose value for each counting number is a prime number. (But don't bother trying to prove this!)]

***

B.

1. For every n, $2n + 3$ is an odd number.
2. For every n, $3(n + 2)$ is the nth multiple of 3.

   There are other possible answers:

   For every n, $3(n + 4)$ is the $(n + 2)$th multiple of 3.

   For every n, $3(n + 5)$ is the $(n + 3)$th multiple of 3.

[Notice that although (a) and (c) are instances of:

For every n, $3(n + 8)$ is the $(n + 6)$th multiple of 3.

(b) is not. Note also that the three correct answers are not equivalent generalizations since they do not have the same instances.]

3. For every n, the $(n + 1)$th square number is $n^2 + 2n + 1$.
4. For every n, n is even.

13. For every n, the n<u>th</u> multiple of 5 is 5n.
14. For every n, the n<u>th</u> multiple of 6 is 6n.
15. For every n, $n^2 - n + 41$ is a prime number.

<u>B</u>. For each of the sets of statements given below state a generalization of which each statement in the set is an instance.

1. a) $2 \cdot 9 + 3$ is an odd number
   b) $2 \cdot 1 + 3$ is an odd number
   c) $2 \cdot 8 + 3$ is an odd number
2. a) $3 \cdot 9$ is the 7<u>th</u> multiple of 3
   b) $3 \cdot 7$ is the 5<u>th</u> multiple of 3
   c) $3 \cdot 100$ is the 98<u>th</u> multiple of 3
3. a) the 3<u>rd</u> square number is 4 + 4 + 1
   b) the 7<u>th</u> square number is 36 + 12 + 1
   c) the 2<u>nd</u> square number is 1 + 2 + 1
   d) the 100<u>th</u> square number is 9801 + 198 + 1
4. a) 28 is even
   b) 13 is even
   c) 97 is even
   d) 104 is even

THE PROBLEM OF PROVING A GENERALIZATION

In order to prove a generalization about counting numbers such as the boxed statement of page 1-9

For every counting number n, the n<u>th</u> odd number is 2n - 1.

it is sufficient to show for each n how one can prove its n<u>th</u> instance. Since, for example, the 4<u>th</u> instance of this generalization:

the 4<u>th</u> odd number is $2 \cdot 4 - 1$

asserts that 4 has the property expressed by:

the ... <u>th</u> odd number is $2 \cdot \ldots -1$,

we need to show for each n how to prove that n has this property.

For any particular counting number, say 4, we could give such a proof as follows:

Line 6:

"....we have no way..." This is not quite exact. As in some of the preceding exercises, a generalization may be a rather immediate consequence of a definition. [For example, Exercise 2 on page 1-10. Each instance is a consequence of the definition of 'even number' and is, therefore, a theorem.]

***

Lines 18 and 19:

Every counting number has an immediate successor. 1 is the only counting number which is not the immediate successor of any counting number.

The first 4 odd numbers are 1, 3, 5, 7. So, 7 is the <u>4th</u> odd number. Since $7 = 2 \cdot 4 - 1$, it is the case that the <u>4th</u> odd number is $2 \cdot 4 - 1$.

Of course, the same procedure could be used to test any instance of the generalization, but up to now we have no way of being certain that each such test would result in showing that the instance under test was true.

Our earlier work suggests that it might be of use to see whether the property in question is hereditary. Before investigating this we must, of course, decide upon a definition of 'follower'. The most commonly useful definition when the set in question is the set of all counting numbers is, as we shall see:

> For every counting number n, n's follower is n + 1.

[The follower, according to this definition, of any counting number is often called its <u>immediate successor</u>. Thus 7 is the immediate successor of 6. Is there any counting number which does not have an immediate successor?]

With this definition of 'follower' we are led to the definition:

> A property is <u>hereditary over the set of all counting numbers</u> if it is meaningful for each counting number and if, for every counting number k, if k has the property in question then so does k + 1.

[Instead of saying that a property is hereditary over the set of all counting numbers we shall usually say that it is an <u>hereditary property of counting numbers</u>.]

Now it is easy to prove that the property expressed by:

the ... <u>th</u> odd number is 2 · ... -1

is an hereditary property of counting numbers. To do so we must show that for every counting number k,

if the <u>kth</u> odd number is $2k - 1$ then the <u>(k + 1)th</u> odd number is $2(k + 1) - 1$.

Notice that in order to prove that a property is hereditary we have to prove a generalization, in this case: For every k, if the k_th_ odd number is 2k - 1 then the (k + 1)_th_ odd number is (2k + 1) - 1. We do so by deriving the needed generalization from other generalizations. Ideally, these other generalizations should be previously proved theorems (or postulates) but in practice we are likely to use as premises generalizations which we may not actually have proved in this course but are sure that we could prove. In the derivation on page 1-13 we have referred (in lines 1 and 2) to the generalization:

> For every n and k, if n is the k_th_ odd number then n + 2 is the (k + 1) _th_ odd number.

We have not previously proved this theorem but it is a consequence of theorems proved or assumed in earlier courses. You can take the theorem as an assumption at this point but you should make this fact explicit to your students. [Incidentally, the theorem is a simple consequence of the definitions of 'odd number' and 'even number', and elementary properties of the relation < .]

***

Exercises.

A.

Sample 1.

Here the generalization which asserts that the property in question is hereditary is derived from the sentence '> is transitive'. [This sentence is actually an abbreviation of the generalization: For every m, n, and p, if m > n and n > p then m > p.]

We know, however, that by adding 2 to the k<u>th</u> odd number we shall obtain the (k + 1)<u>th</u> odd number. For every counting number k, if the k<u>th</u> odd number is 2k - 1 then for this k, the (k + 1)<u>th</u> odd number is (2k - 1) + 2. But,

$$\begin{aligned}(2k - 1) + 2 &= (2k + 2) - 1 \\ &= 2(k + 1) - 1.\end{aligned}$$

Hence, it <u>is</u> the case that for every counting number k, if the k<u>th</u> odd number is 2k - 1, then the (k + 1)<u>th</u> odd number is 2(k + 1) - 1.

We have shown that the property in question is an hereditary property of counting numbers. After the following exercises we shall show how this fact is used in proving the generalization which states that every counting number has this property.

EXERCISES

<u>A</u>. Consider each of the properties expressed below. If it is an hereditary property of counting numbers, prove that it is; if it is not, give a counter-example.

<u>Sample 1</u>. ... > 5

<u>Solution</u>. If we are to prove that this property is an hereditary property of counting numbers, we must prove:

For every counting number k,
if k > 5 then k + 1 > 5.

Now, for every counting number k, if k > 5 then, since k + 1 > k and since > is transitive, it follows that k + 1 > 5.

[<u>Note</u>: It is clear that not all counting numbers have this property. Nonetheless, as we have proved, the property is hereditary. Compare this situation with that described in Exercise 2 on page 1-2.]

<u>Sample 2</u>. ... < 5

<u>Solution</u>. In order to decide whether this is an hereditary property of counting numbers, we must consider the statement:

Note that the property in question, though hereditary, does not hold for any counting number. [Compare this situation with one in which Mr. Jones hears no calls of 'absent' and correctly concludes that the property (expressed by '... is present today') is hereditary <u>even though no student is present</u>.]
Contrast the <u>theorem</u> just established in Exercise 9:

> The property expressed by:
>
> the ... <u>th</u> odd number is $2 \cdot \ldots + 5$
>
> is hereditary over the set of all counting numbers.

with the generalization:

> For every k, the k<u>th</u> odd number is $2k + 5$.

which is <u>not</u> a theorem.

10. Hereditary. Proof like that for Exercise 4.

[In doing the exercises of Part A students should state the generalization they are trying to prove. Their work in formulating questions for exercises on pages 1-5 through 1-7 should have prepared them for stating the appropriate generalizations.]

1. Not hereditary. 1 is a counter-example, and so is every odd counting number.

2. Hereditary. For every counting number k, if the k<u>th</u> even number is 2k then the (k + 1)<u>th</u> even number is 2k + 2, i.e. 2(k + 1). [See discussion on T. C. 13A.]

3. Not hereditary. Any prime number other than 2 is a counter-example.

4. Hereditary. Proof like that on pages 1-12 and 1-13.

5. Hereditary. For every k, if k is a counting number, then k + 1 is a counting number. [Justification: The operation of addition of counting numbers is <u>closed.</u> See Units 1 and 2 of SECOND COURSE, 1955-56.]

6. Not hereditary. 7 is the sole counter-example.

7. Hereditary. Compare with Sample 1.

8. Not hereditary. Compare with Sample 2.

9. Hereditary. Generalization to be proved:

   For every k, if k<u>th</u> odd number is 2k + 5
   then the (k + 1)<u>th</u> odd number is 2(k + 1) + 5.

   Proof:

   For every k, if the k<u>th</u> odd number is 2k + 5 then, for this k, the (k + 1)<u>th</u> odd number is (2k + 5) + 2, i. e., 2(k + 1) + 5.

(continued on T. C. 14B)

For every k,

if $k < 5$ then $k + 1 < 5$.

This statement is certainly false since its 4th instance:

if $4 < 5$ then $4 + 1 < 5$

if false. Therefore, in view of the counter-example 4, we know that the property in question is not an hereditary property of counting numbers.

1. ... is odd
2. the ... th even number is $2 \cdot \ldots$
3. ... is a prime number
4. the $(\ldots + 1)$th odd number is $2 \cdot \ldots + 1$
5. ... is a counting number
6. $\ldots = 7$
7. $\ldots > 1000$
8. $\ldots < 1000000$
9. the ... th odd number is $2 \cdot \ldots + 5$
10. the $(\ldots + 10)$th even number is $2 \cdot \ldots + 20$

B. Figurate numbers

One of the mathematical pastimes of the ancient Greeks was the discovery of interesting generalizations concerning counting numbers. For example, consider the following diagram:

If we count the dots in each of these "triangles" and in those we would get by continuing the obvious construction, we obtain a sequence of numbers which begins:

1, 3, 6, 10, 15, 21, 28, ...

nbers.

Other examples of recursive definitions are:

> For every real number a
>
> $$a^1 = a$$
>
> and, for every counting number k
>
> $$a^{k+1} = a^k \cdot a.$$

and:

> $$1\,! = 1$$
>
> and, for every counting number k,
>
> $$(k + 1)! = k! \cdot (k + 1).$$

Recursive definition is a way of avoiding the use of '. . .' and 'etc.'. For example, compare the above recursive definitions with the statements:

$$a^k = \underbrace{a \cdot a \cdot \ldots \cdot a}_{k \text{ factors}}$$

and:

$$k! = 1 \cdot 2 \cdot \ldots \cdot k,$$

both of which leave a good deal to the imagination.

Recursive definitions form a natural basis for proof that properties are hereditary (See below).

The Greeks called the numbers in this sequence <u>triangular numbers</u>. Having obtained these numbers in sequence it is clear what we shall mean by 'the first triangular number', 'the second triangular number', etc. As you can see from the picture the second triangular number is 2 more than the first, the third triangular number is 3 more than the second, the fourth is 4 more than the third, etc. In general, we see that

For every counting number k,
the (k + 1)th triangular number=
the kth triangular number + (k + 1).

[If, for every n, T_n is the nth triangular number, then the above can be written more simply as:

For every counting number k,
$T_{k+1} = T_k + (k + 1)$.]

Notice that the boxed statement:

$$\boxed{\begin{array}{l} \qquad T_1 = 1 \\ \text{and, for every } k, \\ \quad T_{k+1} = T_k + (k+1). \end{array}}$$

provides us with a way of computing successive triangular numbers. For example,

$$T_1 = 1,$$
$$T_2 = T_1 + (2) = 1 + (2) = 3,$$
$$T_3 = T_2 + (3) = 3 + (3) = 6,$$
$$T_4 = T_3 + (4) = 6 + (4) = 10,$$

etc.

[Such a statement as the boxed one which specifies the first term of a sequence and tells you how, for every k, to compute the (k + 1)th term once you know the kth term is called a <u>recursive definition</u>. We shall make considerable use of recursive definitions in this and the following unit.]

Lines 5 and 6:

Note how the generalization is used to provide an expression for the property in question. Then note the fact that although we can prove that this property is an hereditary property of counting numbers, we still do not have the machinery necessary to prove the generalization itself.

***

Note how the recursive definition is used in proving that the property is an hereditary property of counting numbers.

***

A recursive definition defines a sequence, i.e. a function whose domain is the counting numbers. An explicit definition of the sequence of triangular numbers is:

> 'T' is a name for the **set** whose members are the ordered pairs $(n, \frac{n(n+1)}{2})$, for every counting number n.

Rather than spend space in this unit on the function concept, we limit ourselves to giving explicit definitions for the separate terms of the sequence of triangular numbers.

The Greeks discovered the following generalization concerning triangular numbers:

$$\boxed{\begin{array}{l}\text{For every counting number } k,\\ \qquad T_k = \dfrac{k(k+1)}{2}\ .\end{array}}$$

That is, every counting number has the property expressed by:

$$T_{\ldots} = \frac{\ldots(\ldots + 1)}{2}$$

We shall prove this generalization <u>later</u>. In doing so we shall want to make use of the fact that the property is an hereditary property of counting numbers. This latter fact we can prove <u>now</u>:

For every k, if $T_k = \frac{k(k+1)}{2}$ then by the second equation in the recursive definition,

$$\begin{aligned} T_{k+1} &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= (k+1)\left(\frac{k+2}{2}\right) \\ &= \frac{(k+1)([k+1]+1)}{2}. \end{aligned}$$

Hence, for every k, if k has the property in question then so does k + 1. We have proved that the property is an hereditary property of counting numbers.

It is important to note that the boxed generalization stated (but not proved) above suggests the following sequence of <u>explicit definitions</u>:

$$\boxed{\begin{array}{l} \text{`}T_1\text{' is a name for } \dfrac{1(1+1)}{2} \\[2ex] \text{`}T_2\text{' is a name for } \dfrac{2(2+1)}{2} \\[2ex] \text{`}T_3\text{' is a name for } \dfrac{3(3+1)}{2} \\[2ex] \qquad\qquad \text{etc.} \end{array}}$$

[1-17]

1. $O_1 = 1$ and, for every n, $O_{n+1} = O_n + 2$;

$E_1 = 2$ and, for every n, $E_{n+1} = E_n + 2$.

2. For every n, $O_n = 2n - 1$; for every n, $E_n = 2n$.

Explicit definitions are often easier to use (but harder to "discover") than recursive definitions. For example, to compute T_{1000} is an easy job using the explicit definition:

$$T_{1000} = \frac{1000(1001)}{2} = 500500$$

but it would require pages of computations using the recursive definition:

$$\begin{aligned} T_{1000} &= T_{999} + 1000 \\ &= (T_{998} + 999) + 1000 \\ &= \text{etc.} \end{aligned}$$

1. Use 'O_1', 'O_2', etc. to name the successive odd numbers, and 'E_1', 'E_2', etc. to name the successive even numbers. Using this notation give a recursive definition of the sequence of odd numbers, and a recursive definition of the sequence of even numbers.
2. What are the generalizations which, like the boxed one atop page 1-16, suggest explicit definitions of 'O_1', 'O_2', etc., and 'E_1', 'E_2', etc.?
3. Another kind of figurate number is the square number. The sequence of square numbers can be guessed from the following diagram.

It begins:

1, 4, 9, 16, 25, ...

3. For every k, if $Sq_k = k^2$ then, by the recursive definition, $Sq_{k+1} = k^2 + (2k + 1) = (k + 1)^2$.

4. (b) For every k, if $P_k = \frac{k(3k - 1)}{2}$ then, by the recursive definition, $P_{k+1} = \frac{k(3k - 1)}{2} + (3k + 1)$

$$= \frac{(k + 1)(3[k + 1] - 1)}{2}.$$

You can see that the following is a recursive definition of the sequence:

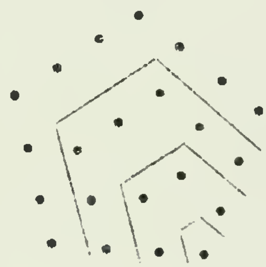
$$\boxed{\begin{array}{c} Sq_1 = 1 \\ \text{and, for every } k, \\ Sq_{k+1} = Sq_k + (2k+1). \end{array}}$$

Prove that the property expressed by:

$$Sq_{\ldots} = (\ldots)^2$$

is an hereditary property of counting numbers.

4. Still another kind of figurate number is the pentagonal number. The fourth pentagonal number is the number of dots in the following figure:



The recursive definition of the sequence of pentagonal numbers is:

$$\boxed{\begin{array}{c} P_1 = 1 \\ \text{and, for every } k, \\ P_{k+1} = P_k + (3k+1). \end{array}}$$

(a) Make diagrams like the above corresponding to P_1, P_2, P_3, P_5, and P_6.

(b) Prove that the property expressed by:

$$P_{\ldots} = \frac{\ldots(3 \cdot \ldots - 1)}{2}$$

is an hereditary property of counting numbers.

The logical status of the principle of mathematical induction depends on how the counting numbers were arrived at. In a postulational treatment the principle might be taken as a postulate. This was done by Peano [See Stabler's INTRODUCTION TO MATHEMATICAL THOUGHT (Cambridge, Mass.: Addison-Wesley Publishing Company, 1953).] On the other hand, if one defines cardinal numbers as equivalence classes of equinumerous sets as did Frege and Russell [See SECOND COURSE, 1955-1956.], the principle of mathematical induction for counting numbers follows from the definition:

> 'counting number' is an abbreviation
> for 'cardinal number which has all
> the hereditary properties of 1'.

Thus, in Peano's system there are two postulates according to which 1 is a counting number and the follower of each counting number is a counting number, and the role of the principle of mathematical induction (a third postulate) is to assert that there are no counting numbers other than those whose existence is ensured by the first two postulates. On the other hand, in the Frege-Russell development the principle of mathematical induction serves to single out the counting numbers from the set of all cardinals.

In the Commentary for page 1-22, we describe a postulational basis for the counting numbers, different from Peano's, with respect to which the principle of mathematical induction is a theorem.

'For every n, the $n\underline{th}$ odd number is $2n - 1$', and the set of all its instances, for example, the set whose members are 'the $1\underline{st}$ odd number is $2 \cdot 1 - 1$', 'the $2\underline{nd}$ odd number is $2 \cdot 2 - 1$', etc. The nature of this gap can perhaps be seen more clearly in a simpler situation in which the generalization covers only a finite number of instances. Suppose, whether you know it or not, that there are only three boys, say John, Charles, and Henry, in a room. If you notice that John has red hair, Charles has red hair, and Henry has red hair <u>and</u> if you also observe that <u>there are no other boys in the room</u>, then you are entitled to assert that every boy in the room has red hair. If you don't make the last observation, then all that you are justified in inferring is that John, Charles, and Henry have red hair. Similarly, if you could somehow prove separately each instance of 'For every n, the $n\underline{th}$ odd number is $2n - 1$', you would still only be justified in claiming to have proved 'the $n\underline{th}$ odd number is $2n - 1$ when $n = 1, 2, 3, \ldots$'. Clearly, it is impossible to write the infinitely long "sentence" suggested by the dots. But even if it could be written, it would still be the case that before you could assert 'For every n, the $n\underline{th}$ odd number is $2n - 1$', you would also have to prove 'Every counting number is either 1 or 2 or 3 or ...'. The principle of mathematical induction for counting numbers, by implying that every counting number > 1 can be obtained from 1 by successive additions of 1, has the effect of ensuring that every counting number is either 1 or $1 + 1$ or $1 + 1 + 1$ or .... As in the case of the red-headed boys in the room, aside from establishing each instance of the generalization, you must also establish that these <u>are</u> all the instances.

* * *

(continued on T. C. 19C)

Stress the last paragraph on page 1-19. A proof by mathematical induction is <u>not</u> an infinite sequence of syllogisms:

$$\begin{array}{ll} & 1 \text{ has the property.} \\ & \underline{\text{If } 1 \text{ has the property then } 2 \text{ has the property}}. \\ \therefore & 2 \text{ has the property.} \\ & \underline{\text{If } 2 \text{ has the property then } 3 \text{ has the property}}. \\ \therefore & 3 \text{ has the property.} \\ & \text{etc.} \end{array}$$

A proof must be of finite length. However, even if we allowed "proofs" of infinite length and so admitted as a proof such an infinite sequence of syllogisms, we would still need a principle to the effect that each instance of the generalization:

For every x, x has the property

is equivalent to the conclusion of one of these syllogisms. The principle of mathematical induction for counting numbers is, essentially, such a principle. In fact, the principle of mathematical induction for counting numbers allows us to infer the generalization from the premises

(i) the property is hereditary, and

(ii) 1 has the property

and, hence, obviates the necessity for infinite proofs.

The principle of mathematical induction is not merely a handy gadget which allows one to by-pass the impossible task of stating a proof for each of infinitely many cases, or to abbreviate the practically impossible task of, say, proving step-by-step that the 1000th odd number is $2 \cdot 1000 - 1$. Rather, its importance is in bridging the gap between a generalization about counting numbers, for example,

(continued on T. C. 19B)

1.03 Mathematical Induction. -- Mr. Jones (see Exercises on page 1-1) knew that if the property of being present was hereditary (with respect to the notion of 'follower' he had defined) and if Bill Aaron was present, then every student had the property of being present. Perhaps we, having proved:

(i) the property expressed by:

(☆) the ... th odd number is 2 · ... - 1

is an hereditary property of counting numbers

can, if we also prove:

(ii) 1 has the property expressed by (☆)

conclude that every counting number has this property. That is, perhaps we can, by using these facts, establish the boxed statement on page 1-9:

For every counting number n, the nth odd number is 2n - 1.

[Before continuing you should check that 1 does have the property expressed by (☆).]

Now, it is easy to see that, because of (i) and (ii) alone, 2, 3, 4, etc. must have the property expressed by (☆). For, by (i), if 1 has this property, then so does its follower 1 + 1. But by (ii), 1 does have this property, so 1 + 1 also has it. Since 1 + 1 = 2, 2 has the property. Again by (i), if 2 has the property then so does 2 + 1. But we have just shown that 2 has the property, so 2 + 1 also has it. Since 2 + 1 = 3, 3 has the property. Again by (i), if 3 has the property then so does 3 + 1. But we have just shown that 3 has the property, so 3 + 1 also has it. Since 3 + 1 = 4, 4 has the property. Etc.

Evidently (i) and (ii) make it possible for us to show that every number which we can obtain from 1 by a finite number of steps, each of which consists in adding 1 to the number obtained in the preceding step, has the property in question. Now, it is a fundamental property of the set of all counting numbers that each of its members can be obtained by such a step-by-step process. Consequently the following principle is a theorem:

4. The property in question is that expressed by:

the $(\ldots + 10)$th even number is $2 \cdot \ldots + 20$.

(i) The property is hereditary.

For every k, if the $(k + 10)$th even number is $2k + 20$ then the $[(k + 1) + 10]$th, or the $(k + 11)$th even number is $(2k + 20) + 2$, or $2(k + 1) + 20$.

(ii) 1 has the property.

The $(1 + 10)$th, or the 11th even number is $2 \cdot 1 + 20$, or 22.

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

-20]

(i) The property is hereditary.

For every k, if the $(k + 1)$th odd number is $2k + 1$ then the $[(k + 1) + 1]$th or $(k + 2)$th odd number is $(2k + 1) + 2$, or $2(k + 1) + 1$.

(ii) 1 has the property.

The $(1 + 1)$th, or the 2nd odd number is $2 \cdot 1 + 1$, or 3.

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

3. The property in question is that expressed by:

... is a counting number.

(i) The property is hereditary.

For every k, if k is a counting number then, since addition over the counting numbers is closed, $k + 1$ is a counting number.

(ii) 1 has the property.

1 is a counting number.

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number, that is, every counting number is a counting number! [This generalization is, of course, trivial and could be proved without the principle of mathematical induction. It is included for comic relief.]

(continued on T. C. 20C)

Note, in the résumé, that we conclude the proof by citing the principle of mathematical induction. Although it is not customary in proofs to cite the justification of <u>each</u> step, we believe that this should be done for any step whose justification is the principle of mathematical induction. Our reason for this insistence is the widespread misconception of inductive proofs as consisting of infinitely many syllogisms [See T. C. 19A ff.]. We hope that by doing this consistently students will be reminded of the distinction.

* * *

<u>A</u>. The property in question in each of these exercises was shown to be hereditary in an exercise on page 1-14. We repeat the proofs here in order to exhibit the style of proofs by mathematical induction.

1. The property in question is that expressed by:

    the ...<u>th</u> even number is $2 \cdot \ldots$.

    (i) <u>The property is hereditary</u>.

    For every k, if the k<u>th</u> even number is 2k then the (k + 1)<u>th</u> even number is $2k + 2 = 2(k + 1)$.

    (ii) <u>1 has the property</u>.

    The first even number is $2 \cdot 1$, or 2.

    Therefore, in view of (i) and (ii), by [the definition of 'even number', and the distributive principle, and] the principle of mathematical induction for counting numbers, the property holds for every counting number.

2. The property in question is that expressed by:

    the (... + 1)<u>th</u> odd number is $2 \cdot \ldots + 1$.

(continued on T. C. 20B)

Every hereditary property of counting numbers which holds for 1 holds for every counting number.

This principle is called <u>the principle of mathematical induction for counting numbers</u>. It is important!

By virtue of this principle we can conclude from (i) and (ii) that every counting number has the property expressed by (☆), that is, that

For every counting number n,
the nth odd number is 2n - 1.

We give now a resume of the proof of this generalization:

(i) The property expressed by:

(☆) the ... th odd number is 2 · ... - 1

is hereditary. For, for every k, if the kth odd number is 2k - 1 then the (k + 1)th odd number is (2k - 1) + 2 = 2(k + 1) - 1.

(ii) The number 1 has the property expressed by (1). For, the first odd number is 1 and 2 · 1 - 1 = 1.

Hence, by the principle of mathematical induction for counting numbers, every counting number has the property expressed by (☆), that is the generalization:

For every counting number n,
the nth odd number is 2n - 1.

is a theorem.

EXERCISES

A. Use the principle of mathematical induction for counting numbers to prove each of the following generalizations. Whenever possible, make use of what you have already done in answer to the exercises on page 1-14.

1. For every n, the nth even number is 2n.
2. For every n, the (n + 1)th odd number is 2n + 1.
3. For every n, n is a counting number.
4. For every n, the (n + 10)th even number is 2n + 20.

10. Property is that expressed by:

$$P_{\ldots + 1} = Sq_{\ldots + 1} + T_{\ldots}\ .$$

(i) <u>The property is hereditary.</u>

Suppose that, for a given k,

$$P_{k+1} = Sq_{k+1} + T_k\ .$$

Then, for this k,

$$\begin{aligned} P_{(k+1)+1} &= P_{k+1} + [3(k+1)+1] \\ &= \left[Sq_{k+1} + T_k\right] + [3(k+1)+1] \\ &= \left\{Sq_{k+1} + [2(k+1)+1]\right\} + \left\{T_k + (k+1)\right\} \\ &= Sq_{(k+1)+1} + T_{k+1}\ . \end{aligned}$$

Hence, for every k, if $P_{k+1} = Sq_{k+1} + T_k$

then $P_{(k+1)+1} = Sq_{(k+1)+1} + T_{k+1}$.

(ii) <u>1 has the property.</u>

$$P_{1+1} = P_1 + (3 \cdot 1 + 1) = 1 + 4 = 5.$$

$$Sq_{1+1} = Sq_1 + (2 \cdot 1 + 1) = 1 + 3 = 4.$$

$$T_1 = 1$$

$$5 = 4 + 1\ .$$

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

The property is that expressed by:

$$Sq_{\ldots +1} = T_{\ldots +1} + T_{\ldots} \ .$$

(i) <u>The property is hereditary.</u>

Suppose that, for a given k,

$$Sq_{k+1} = T_{k+1} + T_k \ .$$

Then, for this k, by the recursive definitions on page 1-18 and on page 1-15,

$$\begin{aligned} Sq_{(k+1)+1} &= Sq_{k+1} + [2(k+1)+1] \\ &= \left[T_{k+1} + T_k\right] + [2(k+1)+1] \quad \text{[hypothesis]} \\ &= \left\{T_{k+1} + [(k+1)+1]\right\} + \left\{T_k + (k+1)\right\} \\ &= T_{(k+1)+1} + T_{k+1} \ . \end{aligned}$$

Hence, for every k, if $Sq_{k+1} = T_{k+1} + T_k$ then $Sq_{(k+1)+1} = T_{(k+1)+1} + T_{k+1}$.

(ii) <u>1 has the property.</u>

$$Sq_{1+1} = Sq_1 + (2 \cdot 1 + 1) = 1 + 3 = 4$$

$$T_{1+1} = T_1 + (1+1) = 1 + 2 = 3$$

$$T_1 = 1$$

$$4 = 3 + 1$$

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

(continued on T. C. 21I)

(i) <u>The property is hereditary.</u>

Suppose that, for a given k, the kth pentagonal number is $\frac{k(3k - 1)}{2}$. Then, for this k, by the recursive definition on page 1-18, the (k + 1)th pentagonal number is

$$\frac{k(3k - 1)}{2} + (3k + 1)$$

$$= \frac{3k^2 - k + 6k + 2}{2}$$

$$= \frac{(k + 1)(3k + 2)}{2}$$

$$= \frac{(k + 1)[3(k + 1) - 1]}{2}.$$

Hence, for every k, if the kth pentagonal number is $\frac{k(3k - 1)}{2}$ then the (k + 1)th pentagonal number is $\frac{(k + 1)[3(k + 1) - 1]}{2}$.

(ii) <u>1 has the property.</u>

By the recursive definition, the first pentagonal number is 1, and

$$\frac{1(3 \cdot 1 - 1)}{2} = 1.$$

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

9. Before attempting to prove this generalization (and the one in Exercise 10) students should verify a few of its instances by making dot-diagrams in order to achieve an intuitive feeling for what is to be proved.

(continued on T. C. 21H)

Remind students that the generalization in Exercise 6 is the one mentioned on page 1-16.

* * *

Note that the properties in Exercises 7 and 8 were shown to be hereditary in Exercises 3 and 4 on pages 1-17, 18.

* * *

7. Property is that expressed by:

   the ...th square number is $(\ldots)^2$.

   (i) The property is hereditary.

   Suppose that, for a given k, the kth square number is k^2. Then, for this k, by the recursive definition on page 1-18, the (k + 1)th square number is

$$\begin{aligned} & k^2 + (2k + 1) \\ = \ & (k + 1)^2 . \end{aligned}$$

   Hence, for every k, if the kth square number is k^2 then the (k + 1)th square number is $(k + 1)^2$.

   (ii) 1 has the property.

   By the recursive definition, the first square number is 1, and

$$1^2 = 1.$$

   Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

8. Property is that expressed by:

   the ...th pentagonal number is $\dfrac{\ldots(3 \cdot \ldots - 1)}{2}$.

(continued on T. C. 21G)

6. Property is that expressed by:

the ...th triangular number is $\frac{\ldots(\ldots + 1)}{2}$.

(i) The property is hereditary.

Suppose that, for a given k, the kth triangular number is $\frac{k(k + 1)}{2}$. Then, for this k, by the recursive definition, the $(k + 1)$th triangular number is

$$\frac{k(k + 1)}{2} + (k + 1)$$

$$= \frac{k(k + 1) + 2(k + 1)}{2}$$

$$= \frac{(k + 1)(k + 2)}{2}$$

$$= \frac{(k + 1)[(k + 1) + 1]}{2} .$$

Hence, for every k, if the kth triangular number is $\frac{k(k + 1)}{2}$ then the $(k + 1)$th triangular number is $\frac{(k + 1)[(k + 1) + 1]}{2}$.

(ii) 1 has the property.

By the recursive definition, the first triangular number is 1, and

$$\frac{1(1 + 1)}{2} = 1.$$

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

(continued on T. C. 21F)

So, we have proved that, for every k,

$$\text{if } k \geq 1 \text{ then } k + 1 \geq 1.$$

(ii) <u>1 has the property</u>.

$$1 \geq 1 \quad \text{because} \quad 1 = 1.$$

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

5. Property is that expressed by:

$$(\ldots)^2 + \ldots \text{ is even.}$$

(i) <u>The property is hereditary</u>.

Suppose that, for a given k, $k^2 + k$ is even.

Then, for this k,

$$(k + 1)^2 + (k + 1) = k^2 + 2k + 1 + k + 1$$
$$= (k^2 + k) + 2(k + 1).$$

Since, by hypothesis, $k^2 + k$ is even, and since, by definition, $2(k + 1)$ is even, and since the sum of two even numbers is even, we know that $(k^2 + k) + 2(k + 1)$, or $(k + 1)^2 + (k + 1)$ is even.

Hence, for every k, if $k^2 + k$ is even then $(k + 1)^2 + (k + 1)$ is even.

(ii) <u>1 has the property</u>.

$$1^2 + 1 = 2$$

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

(continued on T. C. 21E)

For example, in order to prove:

$$\text{if } 1 + 2 = 2 + 1 \text{ then } 1 + (2 + 1) = (2 + 1) + 1$$

one might proceed as follows:

From the associative principle

$$1 + (2 + 1) = (1 + 2) + 1.$$

If $1 + 2 = 2 + 1$ then $(1 + 2) + 1 = (2 + 1) + 1$.

Hence, if $1 + 2 = 2 + 1$ then

$$1 + (2 + 1) = (2 + 1) + 1.$$

The foregoing proof would be equally valid if, throughout it, the numeral '2' were replaced by any other numeral, for example, '7' or '27'. The variation essentially consists in exhibiting the form of all proofs so obtained. In the proof the symbol 'k' serves as a parameter; if one replaces 'k' by the appropriate numeral [and omits such explanatory phrases as 'for this k'] one obtains a proof of any instance of the generalization.

* * *

3. The proof is similar to that given for Exercise 1.

4. Property is that expressed by:

$$\ldots \geq 1.$$

(i) <u>The property is hereditary.</u>

Suppose that, for a given k, $k \geq 1$.

Then, for this k,

$$k + 1 \geq k. \qquad \left\{ \begin{array}{r} k = k \\ 1 \geq 0 \\ \hline k + 1 \geq k \end{array} \right.$$

Hence, since $\geq$ is transitive:

$$\text{if } k \geq 1 \text{ then } k + 1 \geq 1.$$

(continued on T. C. 21D)

[1-21]

Hence, for every k,

if $1 + k = k + 1$

then $1 + (k + 1) = (k + 1) + 1$.

(ii) <u>1 has the property</u>.

$1 + 1 = 1 + 1$ (= is reflexive)

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

* * *

The foregoing proof of (i) is a variation of the kind of proof we have illustrated in connection with Exercise 1. Let students compare this variation with the following:

For every k,

if $1 + k = k + 1$

then since, by the associative principle,

$1 + (k + 1) = (1 + k) + 1$,

$1 + (k + 1) = (k + 1) + 1$.

In connection with the variation it is important to note that the inductive hypothesis does not assert that, for every k, $1 + k = k + 1$. This is what we are trying to prove! The phrase 'Suppose that, for a given k', gives notice that we shall not justify a step in the proof by inferring from the inductive hypothesis one of its <u>substitution instances</u> [e.g., we shall not infer '$1 + (k + 1) = (k + 1) + 1$' from the inductive hypothesis <u>by substituting</u> '<u>k + 1</u>' <u>for</u> '<u>k</u>']. To get students to understand this variation, let them consider the problem of <u>proving</u> an instance of the generalization:

For every k,

if $1 + k = k + 1$ then $1 + (k + 1) = (k + 1) + 1$.

(continued on T. C. 21C)

B.

1. Property is that expressed by:

   the ...th multiple of 5 is 5 · ... .

   (i) The property is hereditary.

   For every k, if the kth multiple of 5 is 5k then the (k + 1)th multiple of 5 is, by the recursive definition, (the kth multiple of 5) + 5, or 5k + 5, or 5(k + 1).

   (ii) 1 has the property.

   By the recursive definition, the first multiple of 5 is 5. Hence, the first multiple of 5 is 5 · 1.

   Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers, the property holds for every counting number.

2. Property is that expressed by:

   1 + ... = ... + 1.

   (i) The property is hereditary.

   Suppose that, for a given k,

   $1 + k = k + 1.$

   [The foregoing supposition '1 + k = k + 1' is often called the inductive hypothesis. Students should be taught this term.]

   Then, for this k,

   $$\begin{aligned} 1 + (k + 1) &= (1 + k) + 1 && \text{[associative principle]} \\ &= (k + 1) + 1 && \text{[inductive hypothesis]} \end{aligned}$$

(continued on T. C. 21B)

B. Use the principle of mathematical induction for counting numbers to prove the following generalizations.

1. For every n, the nth multiple of 5 is 5n.

   [Hint: Use the recursive definition:

   The 1st multiple of 5 is 5 and,
   for every k, the (k + 1)th multiple
   of 5 is (the kth multiple of 5) + 5.]

2. For every n, $1 + n = n + 1$.

   [Of course, you are expected not to use the commutative principle for addition of counting numbers in your proof.]

3. For every n, the nth multiple of 6 is 6n.

4. For every n, $n \geq 1$.

5. For every n, $n^2 + n$ is even.

6. For every n, the nth triangular number is $\frac{n(n + 1)}{2}$.

   [See the recursive definition on page 1-15.]

7. For every n, the nth square number is n^2.

8. For every n, the nth pentagonal number is $\frac{n(3n - 1)}{2}$.

☆9. For every n, $Sq_{n+1} = T_{n+1} + T_n$.

☆10. For every n, $P_{n+1} = Sq_{n+1} + T_n$.

THE PRINCIPLE OF MATHEMATICAL INDUCTION FOR OTHER SETS OF NUMBERS

You have seen that the reason that the principle of mathematical induction for counting numbers holds is that each counting number can be reached by starting at the first counting number (1) and proceeding through a finite number of steps each of which consists in passing from the number reached by the preceding step to its follower (its immediate successor). There are many other sets of numbers for which one can define 'first member' and 'follower' in such a way that each member can be reached from the first member in the set through such a step-by-step process. In fact, one can do this for each set which can be "ordered (by some relation) in the same way" in which the set of counting numbers is ordered by the relation <.

If $A \subseteq K$, and

(a) $\ell_K \in A$, and

(b) for every $x \in K$, if $x \neq g_K$ and $x \in A$ then $x^+ \in A$,

then $A = K$.

This principle of finite induction holds for every simply ordered finite set. It can be proved, using (I^1), (II^1), (III^1), the denial of (IV^1), and (V^1) in just the same way as the principle of mathematical induction is proved from (I^1)-(V^1). It is sufficient to remark, at the appropriate point of the proof, that $(\ell_{\tilde{A}})^- \neq g_K$ and so, as required in the proof, has an immediate successor.

Pictorially, with 'R' corresponding to 'is to the left of':

1, 3, 5, . . . ; 2, 4, 6, ...

It is clear that the relation in question satisfies postulates (I^1)-(IV^1) but does not satisfy postulate (V^1), since 2 has no immediate predecessor. Also, the usual form of the principle of mathematical induction is not valid here, but that

if $A \subseteq K$ and

(a) $1 \epsilon A$ and $2 \epsilon A$, and

(b) for every $x \epsilon C$, if $x \epsilon A$ then $x^+ \epsilon A$,

then $A = C$.

[Here, of course, for every $x \epsilon C$, $x^+ = x + 2$.]

In the preceding, we have merely touched on the theory of simply ordered sets. A convenient source for further material is Huntington's THE CONTINUUM (Cambridge, Mass.: Harvard University Press, 1942).

* * *

The examples used at the beginning of this unit to introduce the notion of mathematical induction dealt with <u>finite</u> ordered sets, and a few words concerning these are in order. If postulate (IV^1) is not satisfied [but (I^1)-(III^1) and (V^1) are] then there is an $x \epsilon K$ such that, for every $y \epsilon K$, $y R x$ or $y = x$. [The proof of this, in particular the proof that if it is not the case that $x R y$ then either $y R x$ or $y = x$, depends on the connectedness of R which we have said follows from (I^1) and (V^1).] Thus K has a "greatest number" and, using (I^1) it is easily shown that there can be only one such. If we denote the greatest member of K by 'g_K' then it is easy to prove, under the specified assumptions concerning K and R, that

(continued on T. C. 22P)

is necessarily asymmetric. Such a relation is called an irreflexive order relation. [A good example of an irreflexive order relation is that expressed by '... is supported by ...' and defined over the parts (trunk, boughs, branches, twigs) of a tree.] A relation R defined over K is said to be connected if, for every x and y in K, if $x \neq y$ then either x R y or y R x. [The example just referred to is not connected (except in the case of an extremely one-sided tree).] A connected irreflexive order relation is called a simple (or sometimes a linear) order relation.

It is not difficult to show that a relation R which satisfies postulates (I^1) and (II^1) is necessarily a simple order relation. Consequently, every progression is a simply ordered set. Postulate (II^1), however, says much more. It is equivalent, for a simply ordered set, to the statement that no subset of K is ordered by R in the same way that the set of negative integers is ordered by <. [Other ways of saying this are (1) that no subset of K is ordered by R as a regression, or (2) no subset of K is ordered by the converse of R as a progression.] Thus, for example, the set of integers (negative, zero, and positive) is simply ordered by <, but is not so ordered as a progression. Another example of a simply ordered set which is not a progression is given by the peculiar ordering of the counting numbers suggested in Part B on page 1-25.

A relation which satisfies postulates (I^1) and (II^1) is called a well-ordering relation. An example of a well-ordering of C which is not a progression is given by the relation R such that:

x R y if (1) x and y are both odd or both even, and $x < y$, or
(2) x is odd and y is even.

(continued on T. C. 22O)

These serve as a recursive definition of the relation $<$. [The word 'must' above should be qualified, but a discussion of how and why would take us too far into logical theory.]

Neither of the above systems of postulates is adequate for arithmetic. [This also should be qualified but, as in the case of 'must', we cannot do it here.] Either of them becomes so if we adjoin postulates which define recursively the operations of addition and multiplication. In the case of the Peano postulates these would be:

(VII^2_a) For every x, if $x \in K$ then $x + 1 = x'$.

(VII^2_b) For every x and y, if $x \in K$ and $y \in K$ then $x + y' = (x + y)'$.

($VIII^2_a$) For every x, if $x \in K$ then $x \cdot 1 = x$.

($VIII^2_b$) For every x and y, if $x \in K$ and $y \in K$ then $x \cdot y' = x \cdot y + x$.

These recursive definitions form bases for inductive proofs of the usual theorems of arithmetic.

* * *

A better understanding of the nature of a progression can be obtained from a more detailed investigation of possible kinds of relations. A relation R defined over a set K is said to be <u>transitive</u> if, for every x, y and z in K, if x R y and y R z then x R z. A relation which satisfies postulate (I^1) is said to be <u>asymmetric</u>. A relation which is asymmetric has the further property of being <u>irreflexive</u>: for every x in K, it is not the case that x R x. It is readily seen that a relation which is both irreflexive and transitive

(continued on T. C. 22N)

for every counting number n, the set of all counting numbers $\leq$ n: $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, ... This set is, when ordered by $\subset$, a progression.

For comparison we append a version of the Peano postulates. These refer to a set K, an object 1 and an operation $'$.

(I^2) $1 \in K$.

(II^2) For every x, if $x \in K$ then $x' \in K$.

(III^2) For every x, if $x \in K$ then $x' \neq 1$.

(IV^2) For every x and y, if $x \in K$, $y \in K$, and $x' = y'$ then $x = y$.

(V^2) For every $A \subseteq K$, if

(a) $1 \in A$, and

(b) for every $x \in K$, if $x \in A$ then $x' \in A$,

then $A = K$.

['$(\ )'$' is to be read as 'the immediate successor of ', and, for ease of comparison, we have replaced the usual form of the principle of mathematical induction by its set-theoretic variant, (V^2).]

If, on the basis of (I^1)-(V^1), we define '1' to be an abbreviation for 'ℓ_K' and '$(\ )'$' to be an abbreviation for '$(\ )^+$', (I^2)-(V^2) become theorems. If we wish to go the other way, and derive theorems corresponding to (I^1)-(V^1) from (I^2)-(V^2), one must adjoin to the latter two additional postulates:

(VI^2_a) For every x, if $x \in K$ then it is not the case that $x < 1$,

and:

(VI^2_b) For every x and y, if $x \in K$ and $y \in K$ then $x < y'$ if and only if $x < y$ or $x = y$.

(continued on T. C. 22M)

Then [if 'ℓ_K' and '$(\ )^+$' are defined as in the preceding discussion]:

For every $A \subseteq K$, if
(a) $\ell_K \in A$, and
(b) for every $x \in K$, if $x \in A$ then $x^+ \in A$,
then $A = K$.

The boxed statement is an expansion of the last sentence on page 1-21. The situation described in the first sentence of the boxed statement can be more briefly characterized by saying that K, as ordered by the relation R, is a progression. The boxed statement asserts that for each progression there is a corresponding principle of mathematical induction.

In (I^1)-(V^1) we have an example of a system of postulates which has many different interpretations. Nine such interpretations are suggested on page 1-22. It happens that in each of these the value of 'K' is a set of numbers and the value of 'R' is the appropriate one of the meanings of either '<' or '>'. ['<' and '>' are, of course, ambiguous since they are used in various contexts to refer to a relation defined for counting numbers, another defined for real numbers, etc., and these are, strictly, different relations.] Non-numerical interpretations are a little difficult to find since, for example, it is doubtful that there are infinitely many physical objects. Availing oneself of some degree of poetic license, one can, however, imagine an infinite row of people for which the relation expressed by '... is to the right of ...' satisfies the above postulates. Another interesting mathematical interpretation consists of the set whose members are,

(continued on T. C. 22L)

them is that the former does so by speaking of properties of counting numbers, while the latter refers to sets of counting numbers.

The preceding development shows that the validity of the principle of mathematical induction for counting numbers is a consequence solely of the manner in which the relation $<$ orders these numbers. Since we have used only those properties of the counting numbers which are expressed by (I), (II), (III), (IV), and (V) it is evident that the following is true:

> Let K be any set for which there is a relation R such that
>
> (I^1) For every x and y, if $x \in K$ and $y \in K$ then it is not the case that both $x\ R\ y$ and $y\ R\ x$,
>
> (II^1) If $A \subseteq K$ and $A \neq \emptyset$ (i.e. A is non-empty) then there is an x such that $x \in A$ and, for every z, if $z \in A$ then either $x\ R\ z$ or $x = z$,
>
> (III^1) $K \neq \emptyset$,
>
> (IV^1) For every x, if $x \in K$ then there is a y such that $y \in K$ and $x\ R\ y$,
>
> (V^1) For every x, if $x \in K$ and there exists a $z \in K$ such that $z\ R\ x$ (i.e. $x \neq \ell_K$), then there exists a $y \in K$ such that $y\ R\ x$ and, for every $z \in K$, if $z\ R\ x$ then either $z\ R\ y$ or $z = y$.
>
> (continued on T. C. 22K)

(1) $\ell_{\tilde{A}} \in \tilde{A}$, and that

(2) for every $z \in C$, if $z \in \tilde{A}$ then $\ell_{\tilde{A}} \leq z$.

From (1) and (a) it follows that $\ell_{\tilde{A}} \neq \ell_C$. Hence, by (V), $\ell_{\tilde{A}}$ has an immediate predecessor, $(\ell_{\tilde{A}}^{-})$. We know that

(3) $(\ell_{\tilde{A}})^{-} < \ell_{\tilde{A}}$, and that

(4) for every z, if $z < \ell_{\tilde{A}}$ then $z \leq (\ell_{\tilde{A}})^{-}$.

Evidently $(\ell_{\tilde{A}})^{-} \in A$ for, if not, it would follow from (2) that $\ell_{\tilde{A}} \leq (\ell_{\tilde{A}})^{-}$ and this, by (I), contradicts (3). Hence, by (b), $((\ell_{\tilde{A}})^{-})^{+} \in A$. But, by the previous theorem, $((\ell_{\tilde{A}})^{-})^{+} = \ell_{\tilde{A}}$. Consequently, $\ell_{\tilde{A}} \in A$, in contradiction to (1).

Since our assumption that $\tilde{A}$ is non-empty has led to a contradiction, $A = C$.

In order to relate the theorem we have just proved to the principle of mathematical induction, we notice that to say that a property holds for every counting number is to say that the set A of those numbers for which the property holds is C. And to say that the property holds for 1 is to say that $1 \in A$ (or, alternatively, that $\ell_C \in A$); while to say that the property is hereditary is to say that, for every $x \in C$, if $x \in A$ then so does its follower (or, equivalently, then $x^{+} \in A$). Consequently, the principle of mathematical induction and the theorem which we have just proved express the same characteristic of the set of all counting numbers, as it is ordered by <. The only difference between

(continued on T. C. 22J)

(1) $k^- < k$, and

(2) for every $z \epsilon C$, if $z < k$ then $z \leq k^-$.

From the similar theorem concerning immediate successors, whose statement precedes that of postulate (V), we know that

(3) $k^- < (k^-)^+$, and

(4) for every $z \epsilon C$, if $k^- < z$ then $(k^-)^+ \leq z$.

Now from (1) and (4) it follows that $(k^-)^+ \leq k$; that is, that either $(k^-)^+ = k$ or $(k^-)^+ < k$. But, if $(k^-)^+ < k$ then, by (2), $(k^-)^+ \leq k^-$. However, by (3), $k^- < (k^-)^+$. But, by (I), it is impossible that $(k^-)^+ \leq k^-$ and $k^- < (k^-)^+$. Hence it is impossible that $(k^-)^+ < k$, and we are left with the alternative that $(k^-)^+ = k$.

We shall now prove a theorem which is merely a restatement of the principle of mathematical induction for counting numbers:

> If $A \subseteq C$ and
> (a) $\ell_C \epsilon A$, and
> (b) for every $x \epsilon C$, if $x \epsilon A$ then $x^+ \epsilon A$,
> then $A = C$.

<u>Proof</u>.

Suppose that there is a counting number which does not belong to A. Then $\tilde{A}$, the <u>complement</u> of A, is non-empty and, by (II), has a least member, $\ell_{\tilde{A}}$. We know that

(continued on T. C. 22I)

Our last postulate is:

(V) Every counting number except ℓ_C has an immediate predecessor.

[In other words: For every $x \epsilon C$, if $x \neq \ell_C$ then there is a $y \epsilon C$ such that $y < x$ and, for every $z \epsilon C$, if $z < x$ then $z \leq y$.]

From (I) it follows that each number has at most one immediate predecessor. For suppose that some number x had two immediate predecessors, say y_1 and y_2. Then, for every $z < x$, $z \leq y_1$ and $z \leq y_2$. Since $y_2 < x$, it follows that $y_2 \leq y_1$; since $y_1 < x$, that $y_1 \leq y_2$. Hence, by (I), $y_1 = y_2$. Every number, then, has at most one immediate predecessor. Hence, from (V) it follows that every number other than ℓ_C has a unique immediate predecessor. Thus we are justified in speaking of <u>the</u> immediate predecessor of any number other than ℓ_C. Using a self explanatory abbreviation for 'immediate predecessor of ' we have the theorem:

For every $x \epsilon C$, if $x \neq \ell_C$ then $x^- < x$ and, for every $z \epsilon C$, if $z < x$ then $z \leq x^-$.

In deriving the principle of mathematical induction from our five postulate's we shall make use of the following theorem:

For every $x \epsilon C$, if $x \neq \ell_C$ then $(x^-)^+ = x$.

In order to prove this let us suppose that $k \epsilon C$ such that $k \neq \ell_C$. Then, from the theorem noted at the end of the preceding paragraph, it follows that

(continued on T. C. 22H)

for the number 1. However, before we can introduce the symbol 'ℓ_C' we need an additional postulate:

(III) The set C of all counting numbers is non-empty (i.e. there are counting numbers).

As a consequence of the definition of 'ℓ_C' we know that $\ell_C \epsilon C$ and, for every $z \epsilon C$, $\ell_C \leq z$.

(IV) There is no greatest counting number.

[This means that, for every $x \epsilon C$, there exists a $y \epsilon C$ such that $x < y$.]

From (IV) it follows that, for every $x \epsilon C$, the set G(x) of all numbers y such that $y > x$ is non-empty. Hence $\ell_{G(x)} \epsilon G(x)$, and, for every z, if $z \epsilon G(x)$ then $\ell_{G(x)} \leq z$. In other words: <u>for every $x \epsilon C$, $x < \ell_{G(x)}$ and, for every $z \epsilon C$, if $x < z$ then $\ell_{G(x)} \leq z$.</u> It is convenient to abbreviate '$\ell_{G(\ )}$' to '$(\ )^+$', to omit the parentheses when doing so will not lead to confusion, and to read the resulting symbol as 'the immediate successor of '. For example, '$\ell_C{}^+$' denotes the immediate successor of the least member of C, which is, of course, the number two. We can now restate the theorem underlined above:

For every $x \epsilon C$, $x < x^+$ and, for every $z \epsilon C$, if $x < z$ then $x^+ \leq z$.

[As a matter of fact, for every $x \epsilon C$, $x^+ = x + 1$, and we shall want to recall this later. But, at present, since we wish to use only properties of the relation <, it is better not to think of the operation of addition.]

(continued on T. C. 22G)

immediately that for every x and y, if $x \leq y$ and $y \leq x$ then $x = y$; and that for no x and y do we have both $x \leq y$ and $y < x$.]

(II) Every non-empty set of counting numbers has a least member.

[This means that if A is any non-empty set of counting numbers then there is a number x such that $x \epsilon A$ (i.e. x is a member of A) and such that, for every number z, if $z \epsilon A$ then $x \leq z$. If, for example, A is the set of even numbers, then $2 \epsilon A$ and, for every even number z, $2 \leq z$. 2 is the least even number.]

From (I) it follows that there is at most one least member of any given set A. For suppose that there were two least members, x_1 and x_2. Then, for every $z \epsilon A$, $x_1 \leq z$ and $x_2 \leq z$. Since $x_2 \epsilon A$, it follows that $x_1 \leq x_2$. Since $x_1 \epsilon A$ it follows that $x_2 \leq x_1$. Hence, by (I), $x_1 = x_2$. So, every set of counting numbers has at most one least member. But, by (II), every non-empty set of counting numbers has at least one least number. Therefore, by (I) and (II), every non-empty set of counting numbers has a unique least member. This last assertion justifies our use of the definite article in speaking of the least number of a given non-empty set of counting numbers.

It will be convenient to name the least member of a non-empty set by a symbol obtained by attaching a name for the set as a subscript to the letter 'ℓ'. Thus, if we use 'C' as a name for the set of all counting numbers, then 'ℓ_C' is a name

(continued on T. C. 22F)

With this definition the principle of mathematical induction is again a theorem.]

[B.] We shall now take up a postulational basis for the arithmetic of the counting numbers which is quite different from Peano's. The latter emphasizes the operation of obtaining the immediate successor of a counting number; the one which we are about to take up emphasizes, instead, the order relation $<$. What we shall do is to show how, from five postulates concerning $<$, one can establish the existence and uniqueness of an immediate successor of each counting number, and derive the principle of mathematical induction. The procedure to be followed is implicit in the last two paragraphs of T. C. 1A (q. v.). Our postulates will be chosen in such a way as to justify the following argument:

> If there exists a counting number which does not have a given property P then there is a least such number. If 1 has P then this least number which does not have P is not 1, and so has an immediate predecessor. This latter number must then have P and so, if P is hereditary, its immediate successor has P. But its immediate successor is the least number which does not have P. Consequently, if P is an hereditary property of 1 then there is no counting number which does not have P; i.e., every counting number has P.

Our first two postulates are:

(I) For no counting numbers x and y do we ever have both $x < y$ and $y < x$.

[From (I) and the obvious definition of '$\leq$' it follows

(continued on T. C. 22E)

We shall now consider two alternative postulational treatments of the principle of mathematical induction.

[A.] In explaining the principle of mathematical induction for counting numbers we have said that it is valid because <u>each counting number other than 1 can be reached by starting at 1 and making a finite sequence of steps each of which consists in passing to the follower</u> (or, as we shall now say: the immediate successor) <u>of the number reached in the preceding step</u> (or, in the first step, to the immediate successor of 1). It might seem natural to take this as a postulate, in place of the principle of mathematical induction, and it is perfectly feasible to do this provided that one has first defined the notion of finite sequence. One cannot use the most natural definition according to which a finite sequence is a function whose domain is the set of all counting numbers $\leq$ some counting number, for then the underlined statement above would assert no more than that, for every counting number n, there is a 1-1 correspondence between the counting numbers $\leq$ n and the counting numbers $\leq$ n. But it is possible to define 'finite sequence' without referring to counting numbers, and on this basis to carry out the program suggested above. However, this procedure is somewhat complicated and while the underlined statement may in itself seem more intuitively simple than does the principle of mathematical induction, it becomes much less so when backed up by an appropriate definition of 'finite sequence'.

[A similar modification can be made in the Frege-Russell procedure: first define 'finite set' without reference to counting numbers and then define 'counting number' as an abbreviation for 'cardinal number whose members are finite sets'.

(continued on T. C. 22D)

'1 + 2 = 3'

and

' for all counting numbers a, b and c,

$$a(b + c) = ab + ac',$$

are true

if 'counting number' means even counting number,

if '1' stands for the number two,

if, for every even counting number a, a's follower is the next even counting number,

and

if, for all even counting numbers a and b, a + b is the ordinary sum of a and b, and a · b is half the ordinary product of a and b.

With this interpretation of the symbolism,'1 + 2' denotes the sum of two and four, or six, whose name, in the present interpretation, is '3', and '3 · 5' denotes half the product of six and ten, or thirty, whose name is now '15'. Hence '1 + 2 = 3' now states that the sum of two and four is six, while '3 · 5 = 15' states that half the product of six and ten is thirty.

On the other hand, the Frege-Russell definition, also referred to on T. C. 19C, not only gives a basis from which to develop the arithmetic of the counting numbers, but also tells us, in terms of the notion of set, what the counting numbers are.]

(continued on T. C. 22C)

(3) The first member is 2 and, for every counting number n, n's follower is n + 2.

(4) 2; for every prime number n, n's follower is the least prime number > n.

(9) 1; for every unit fraction n, n's follower is $\frac{n}{1 + n}$.

* * *

The following comments have relevance for the text material on pages 1-21 through 1-23 as well as for Part B of the Exercises on page 1-25.

* * *

As was pointed out on T. C. 19C, the principle of mathematical induction may be taken as one of a set of postulates chosen to characterize the arithmetic of the counting numbers.

[We say '<u>the</u> <u>arithmetic</u> of the counting numbers' because a postulational treatment such as Peano's cannot characterize the counting numbers themselves (i.e., cannot tell <u>what</u> they are), but only their behavior, with respect to operations such as addition and multiplication. For example, all the theorems of arithmetic, such as

(continued on T. C. 22B)

Here are some examples:

(1)	1,	2,	3,	4,	5,	6,	7,	8,	...
(2)	4,	5,	6,	7,	8,	9,	10,	11,	...
(3)	2,	4,	6,	8,	10,	12,	14,	16,	...
(4)	2,	3,	5,	7,	11,	13,	17,	19,	...
(5)	0,	1,	2,	3,	4,	5,	6,	7,	...
(6)	+1,	+2,	+3,	+4,	+5,	+6,	+7,	+8,	...
(7)	-5,	-4,	-3,	-2,	-1,	0,	+1,	+2,	...
(8)	-1,	-2,	-3,	-4,	-5,	-6,	-7,	-8,	...
(9)	1,	$\frac{1}{2}$,	$\frac{1}{3}$,	$\frac{1}{4}$,	$\frac{1}{5}$,	$\frac{1}{6}$,	$\frac{1}{7}$,	$\frac{1}{8}$,	...

Set (2) is the set of all counting numbers ≥ 4. The first member in that set is 4, and for every x, x's follower is x + 1. Set (8) is the set of all negative real integers. The first member in that set is -1, and for every x, x's follower is x - 1. The student should define in a similar way 'first member' and 'follower' for (3), the set of even counting numbers; (4), the set of prime counting numbers; (5), the set of finite cardinal numbers; (6), the set of positive real integers; (7), the set of real integers $\geq$ -5; (9), the set of unit fractional numbers. [Note: Actually 'first member' and 'follower' could be defined for each of the above sets in many different ways. We have indicated the definitions which we shall find most useful in practice.]

Once 'follower' has been defined with respect to the members of a set, one can introduce a notion of "hereditariness". For example, in case (3) above we would make the following definition:

> A property is hereditary over the set of all even counting numbers if it is meaningful for each even counting number and if, for every even counting number k, if k has the property in question then so has k + 2.

The student should make corresponding definitions for each of the sets (2), (4), (5), (6), (7), (8), and (9).

Since for each of the given sets each of its members can be reached from its first member through a finite number of "steps", we have in each case a "principle of mathematical induction". For example, in case (8), the principle of mathematical induction for negative real integers is:

> Every hereditary property of negative real integers which holds for -1 holds for every negative real integer.

The student should state principles of mathematical induction for the other sets.

We give now an example of the use of one such principle.

Let us attempt to prove the following generalization:

> For every real integer $x < 0$, $1 - 2x > 0$.

Note that this is a generalization about the negative real integers, set (8). As remarked before, the first member in that set is -1 and for every x, x's follower is x - 1. A property is hereditary over the set of all negative real integers if it is meaningful for each negative real integer, and if, for every negative integer y, if y has the property in question then so does y - 1. The principle of mathematical induction for negative real integers has been stated above. The property in which we are interested is that expressed by:

$$1 - 2 \cdot \ldots > 0$$

The proof of the generalization runs as follows:

(i) <u>The property is hereditary</u>.

Suppose that, for a given real integer $y < 3$,

the $(3 - y)$th even integer is $6 - 2y$.

Then, for this integer y, the $[3 - (y - 1)]$th, or $[3 - y + 1]$th even integer is $6 - 2y + 2$, or $6 - 2(y - 1)$.

(ii) <u>2 has the property</u>.

The $(3 - 2)$th or first even integer is $6 - 2 \cdot 2$, or 2.

Therefore, in view of (i) and (ii), by (d), the property holds for every real integer < 3.

4. (a) All counting numbers which are multiples of 5: $\{5, 10, 15, \ldots\}$.

(b) 5; k's follower is $k + 5$.

(c) A property is hereditary over the set of all counting numbers which are multiples of 5 if it is meaningful for each multiple of 5 and if, for every k which is a multiple of 5, if k has the property in question then so has $k + 5$.

(d) Every hereditary property of counting numbers which are multiples of 5 which holds for 5 holds for every counting number which is a multiple of 5.

(e) The property in question is that expressed by:

$$\ldots + 1 \text{ is even.}$$

The generalization is false; 10 is a counter-example.

(ii) <u>6 has the property</u>.

$$T_{6-5} = \frac{(6-4)(6-5)}{2} = 1; \; T_{6-5} = T_1 = 1.$$

Therefore, in view of (i) and (ii), by (d), the property holds for every counting number $\geq$ 6.

2. (a) The set of all counting numbers $\geq$ 3.
   (b) First number is 3; k's follower is k + 1.
   (c) A property is hereditary over the set of all counting numbers $\geq$ 3 if it is meaningful for each counting number $\geq$ 3 and if, for every counting number k $\geq$ 3, if k has the property in question then so has k + 1.
   (d) Every hereditary property of counting numbers $\geq$ 3 which holds for 3 holds for every counting number $\geq$ 3.
   (e) The property in question is that expressed by:

   $(\ldots)^2 - 1$ is a multiple of 8.

   The generalization is false; 4 is a counter-example.

3. (a) All real integers < 3.
   (b) 2; y's follower is y - 1.
   (c) A property is hereditary over the set of all real integers < 3 if it is meaningful for each real integer < 3 and if, for every real integer y < 3, if y has the property in question then so has y - 1.
   (d) Every hereditary property of real integers < 3 which holds for 2 holds for every real integer < 3.
   (e) The property in question is that expressed by:

   the (3 - ...)th even integer is 6 - 2 · ... .

(continued on T. C. 24C)

A.

1. (a) The set of all counting numbers ≥ 6.

   (b) First number is 6; k's follower is k + 1.

   (c) A property is hereditary over the set of all counting numbers ≥ 6 if it is meaningful for each counting number ≥ 6 and if, for every counting number $k \geq 6$, if k has the property in question, then so has k + 1.

   (d) Every hereditary property of counting numbers ≥ 6 which holds for 6 holds for every counting number ≥ 6.

   (e) The property in question is that expressed by:

$$T_{\ldots - 5} = \frac{(\ldots - 4)(\ldots - 5)}{2} .$$

(i) <u>The property is hereditary</u>.

Suppose that, for a given $k \geq 6$,

$$T_{k - 5} = \frac{(k - 4)(k - 5)}{2} .$$

Then, for this k,

$$\begin{aligned} T_{(k + 1) - 5} &= T_{k - 5} + [(k - 5) + 1] \\ &= \frac{(k - 4)(k - 5)}{2} + [(k - 5) + 1] \\ &= \frac{(k - 4)[(k - 5) + 2]}{2} \\ &= \frac{(k - 3)(k - 4)}{2} \\ &= \frac{[(k + 1) - 4][(k + 1) - 5]}{2} . \end{aligned}$$

(continued on T. C. 24B)

(i) <u>The property in question is hereditary</u>:

For every y, if $1 - 2y > 0$ then
$1 - 2(y - 1) = (1 - 2y) + 2 > 0 + 2 > 0.$

(ii) <u>-1 has the property in question</u>:

$$1 - 2(-1) = 3 > 0.$$

Having established (i) and (ii) the principle of mathematical induction for negative real integers now tells us that every negative real integer has the property in question. That is, for every real integer $x < 0$, $1 - 2x > 0$.

EXERCISES

A. For each of the following generalizations

(a) identify the set with which the generalization is concerned,

(b) decide on appropriate definitions of 'first member' and 'follower',

(c) define 'hereditary property' appropriately,

(d) state the appropriate principle of mathematical induction, and

(e) either use the answer to (d) to prove the generalization, or give a counter-example.

1. For every counting number $n \geq 6$, $T_{n - 5} = \frac{(n - 4)(n - 5)}{2}$.

2. For every odd counting number $n \geq 3$, $n^2 - 1$ is a multiple of 8.

3. For every real integer $x < 3$, the $(3 - x)$<u>th</u> even integer is $6 - 2x$.

4. For every counting number m which is a multiple of 5, $m + 1$ is even.

B.

1. 3, 11, 10, 98.
2. Yes, 2.
3. Yes, 1.
4. 1, 3, 5, ..., ..., 6, 4, 2.
5. [See Commentary to page 1-22.] With this definition of 'follower' for the counting numbers it is not the case that each counting number which is a follower can be reached by a finite number of steps each of which consists in passing from the number reached in the preceding step to its follower. In fact, no even counting number can be so reached. Hence, (although there is just one counting number which is not the follower of any other) there is no principle of mathematical induction corresponding to this notion of follower.

   The given notion of follower does correspond to an ordering relation (see answer to Exercise 4, above) but, with this ordering there are non-empty sets of counting numbers which have no least members. In fact, the set of all even counting numbers is of this kind.

B. Suppose you make the following definition of 'follower' for counting numbers:

For every odd counting number n,
n's follower is $n + 2$;

for every even counting number $n \neq 2$,
n's follower is $n - 2$.

1. What is the follower of 1? Of 9? Of 12? Of 100?
2. Is there a counting number which does not have a follower?
3. Is there a "first member"?
4. Indicate how to list all of the counting numbers in a horizontal line such that the first member is "first in line" and the follower of each number except 2 is listed immediately to the right of it.
5. We are going to "prove" now that <u>every counting number is odd</u>.

   (i) <u>The property is hereditary</u>:

   For every counting number k, if k is odd, then it has a follower, $k + 2$, and its follower is odd.

   (ii) <u>1 has the property</u>:

   1 is odd.

   Therefore, every counting number is odd. Criticize the above "proof".

C. We are going to "prove" that for every counting number n, every n boys have the same weight.

   (i) <u>The property is hereditary</u>:

   Suppose that every k boys have the same weight.

(continued on next page)

The fallacy is in the proof of (i), and illustrates the dangers of using ' ... ' (as in line 2). The error can be discovered by attempting to specialize the given proof of (i) to a proof that if all the boys in every set of 1 boys have the same weight then so do all the boys in every set of 1 + 1 boys.

Consider a group of k + 1 boys, say,

$$B_1, B_2, B_3, \ldots, B_k, B_{k+1}.$$

The boys

$$B_1, B_2, B_3, \ldots, B_k$$

form a set of k boys. Therefore, they have the same weight. Also, the group

$$B_2, B_3, \ldots, B_k, B_{k+1}$$

form a set of k boys; so they too have the same weight. But, since B_2 belongs to both sets, each boy weighs the same as B_2. Hence, for every counting number k, if all the boys in every set of k boys have the same weight then so do all the boys in every set of k + 1 boys.

(ii) <u>1 has the property</u>:

Clearly, all the boys in every set which consists of just 1 boy have the same weight.

Therefore, by the principle of mathematical induction for counting numbers, for every counting number n, all the boys in every set of n boys have the same weight. Note that the second instance of this generalization states that any 2 boys have the same weight!

Criticize the above "proof".

1.04 <u>Sums of progressions</u>.--Sequences of numbers such as those given on page 1-22 are often called <u>progressions</u>. We are sometimes interested in computing the sum of the members of a set of successive terms in a progression. For example, you have learned earlier in the unit that, for every counting number n, the <u>nth</u> triangular number is the <u>sum</u> of the first n terms of the progression which begins:

$$1, 2, 3, 4, 5, \ldots$$

[1-27]

in a 1-1 way on the set $\{0, 1, 2, 3, 4\}$, and $\sum_{k=1}^{5}(k^2 + k) = \sum_{j=0}^{4}[(j+1)^2 + j]$. Also, $\sum_{k=1}^{5}(k^2 + k) = \sum_{m=-5}^{-1}[(-m)^2 + (-m)]$; here the transformation is expressed by '$m = -k$'. In general, allowable transformations of $\sum$-expressions will be of the form '$j = ak + b$' with 'a' replaced by '1' and 'b' by a counting numeral, if the interval of summation is a set of counting numbers; or 'a' replaced by '1' or by '-1' and 'b' by and integer numeral, if the interval of summation is a set of integers. The restriction on replacements for 'a' is necessitated by the requirement that the transformation be 1-1. Other examples of such transformations will be found on T.C. 29A.

in the range of the variables. For example, if 'u' is a variable whose range is the set of real numbers then '$\sqrt{u(1 - u)}$' has a value for each value of 'u' between 0 and 1, inclusive. Similarly, if the range of 'p' is the set of integers, '$\sum_{r=p}^{5} r^2$' has a value for each value of 'p' $\leq$ 5, i.e. for ..., -3, -2, ..., 4, 5, and these values are the same as the "corresponding" values of the expression '$\sum_{x=p}^{5} x^2$'. $\sum$-expressions like '$\sum_{k=1}^{5} k$' in which no variables occur are themselves numerals. [If one replaces 'c', in the example above in which it occurs, by a name for a real number then the resulting expression names a set of real numbers. The original expression, then, has sets as values.]

Returning to the analogy between $\sum$-expressions and $\int$-expressions, you will recall that in the case of the latter one can "transform variables" as illustrated by: $\int_1^x \sqrt{t}\, dt = \int_1^{\sqrt{x}} y \cdot 2y \cdot dy$.

Here the transformation is that expressed by '$y = \sqrt{t}$', and for every real number $x \geq 0$, transforms the segment $\overline{1, x}$ (or $\overline{x, 1}$) in a 1-1 manner on the segment $\overline{1, \sqrt{x}}$ (or $\overline{\sqrt{x}, 1}$). In a similar way, a $\sum$-expression, say '$\sum_{k=1}^{5} (k^2 + k)$' can be transformed by using any transformation which maps the "interval of summation" in a 1-1 way on a similar set of numbers. For example, the transformation expressed by '$j = k - 1$' maps the set $\{1, 2, 3, 4, 5\}$

(continued on T. C. 27C)

There is an analogy between the notations '$\sum_{k=p}^{q} f(k)$' and '$\int_a^b f(x)\,dx$'. [The concepts expressed by the notations are, of course, quite different.] The symbols 'k' and 'x' are apparent variables (or "dummy" variables) and can be replaced by any appropriate symbols without changing the meaning of the expressions: for every p, q and f, $\sum_{k=p}^{q} f(k) = \sum_{*=p}^{q} f(*)$; for every a, b and f, $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$. Other examples of apparent variables are the symbols 't' and 'y' in 'for every counting number t, t + 1 = 1 + t' and 'the set of all real numbers y such that y + 2 < 3c'.

The symbols 'p', 'q', 'a', 'b', and 'c' in these examples are variables (i.e. pronumerals)(and 'f' is also a variable, a "profunctional"), 'p' and 'q' are called 'limits of summation' and 'a' and 'b' are called 'limits of integration'. [The same terminology is also applied to any expressions which occupy the same positions with respect to a '$\sum$' as do 'p' and 'q', or with respect to an '$\int$' as do 'a' and 'b'. For example, in '$\sum_{n=1}^{x+5}$', '1' and 'x + 5' are the limits of summation.]

In interpreting a $\sum$-expression such as '$\sum_{r=p}^{5} r^2$' the variables which occur in the limits of summation (in this case, the variable 'p') must have specified ranges, for example, the set of counting numbers, or the set of non-negative integers, or the set of integers, etc. The expression, then, like any other algebraic expression has a value corresponding to each of a set of values

(continued on T. C. 27B)

It is convenient to have a simple notation to use when referring to such sums. Such is the so-called "Σ-notation". As an example of the use of this notation consider the 5th triangular number, T_5, which, you remember, is 1 + 2 + 3 + 4 + 5. Using the Σ-notation we would write:

$$\sum_{k=1}^{5} k$$

as a name for the 5th triangular number. Similarly, we could use as a name for the 4th square number:

$$\sum_{k=1}^{4} (2k - 1)$$

[This symbol is read "sigma from k = 1 to 4 of 2k - 1". 'Σ' is the Greek letter corresponding to our 'S' and is meant to remind you of 'sum'.]

The expression:

$$\sum_{k=1}^{4} (2k - 1)$$

stands for the sum of the four values of '2k - 1' which corresponds to the values 1, 2, 3, and 4 of 'k'. In other words:

$$\begin{aligned}\sum_{k=1}^{4} (2k - 1) &= [2(1) - 1] + [2(2) - 1] + [2(3) - 1] + [2(4) - 1] \\ &= [\ 1\ ] + [\ 3\ ] + [\ 5\ ] + [\ 7\ ] \\ &= 16\end{aligned}$$

EXERCISES

A. Study each of the first six statements until you understand them, and then complete the last seven in the same manner.

1. $\sum_{x=1}^{3}(2 - 3x) = [2 - 3(1)] + [2 - 3(2)] + [2 - 3(3)] = [-1] + [-4] + [-7]$

2. $\sum_{y=1}^{3}(2 - 3y) = [2 - 3(1)] + [2 - 3(2)] + [2 - 3(3)] = [-1] + [-4] + [-7]$

(continued on next page)

7. $\frac{-3}{2} + \frac{5}{2} + \frac{13}{2} + \frac{21}{2} + \frac{29}{2} = \frac{65}{2}$

8. $9 + 4 + 1 + 0 + 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = 399$

9. $26 + 33 + 40 + 47 = 146$

10. $-10 - 5 + 0 + 5 + 10 + 15 + 20 = 35$

11. 22

12. $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} = \frac{7}{8}$

Remind students of the following procedure for finding the lowest common multiple for the denominator numbers:

$$\left.\begin{aligned} 56 &= 2^3 \cdot 7 \\ 42 &= 2 \cdot 3 \cdot 7 \\ 30 &= 2 \cdot 3 \cdot 5 \\ 20 &= 2^2 \cdot 5 \\ 12 &= 2^2 \cdot 3 \\ 6 &= 2 \cdot 3 \\ 2 &= 2 \end{aligned}\right\} \quad \text{L.C.M.} = 2^3 \cdot 3 \cdot 5 \cdot 7$$

13. $\frac{-4}{10} + \frac{-3}{22} + 0 + \frac{5}{52} = -\frac{1259}{2860}$

3. $\sum_{x=10}^{13} 4x = 40 + 44 + 48 + 52$

4. $\sum_{k=-2}^{3} (3k + 7) = 1 + 4 + 7 + 10 + 13 + 16$

5. $\sum_{n=0}^{3} (a + nd) = [a] + [a + d] + [a + 2d] + [a + 3d]$

6. $\sum_{k=1}^{5} \frac{k(3k - 1)}{2} = 1 + 5 + 12 + 22 + 35$

7. $\sum_{t=0}^{4} \frac{8t - 3}{2} =$

8. $\sum_{s=-3}^{10} s^2 =$

9. $\sum_{i=3}^{6} (7i + 5) =$

10. $\sum_{x=1}^{7} 5(x - 3) =$

11. $\sum_{p=3}^{3} (5p + 7) =$

12. $\sum_{n=1}^{7} \frac{1}{n(n + 1)} =$

13. $\sum_{n=2}^{5} \frac{n(n - 4)}{n^2 + 7n - 8} =$

6. $\sum_{k=1}^{4} 4$ [Correction: insert '+' between the first two terms in Exercise 6.]

7. By the recursive definition on page 1-18,

$$\begin{aligned} P_5 &= P_4 + (3 \cdot 4 + 1) \\ &= P_3 + (3 \cdot 3 + 1) + (3 \cdot 4 + 1) \\ &= P_2 + (3 \cdot 2 + 1) + (3 \cdot 3 + 1) + (3 \cdot 4 + 1) \\ &= P_1 + (3 \cdot 1 + 1) + (3 \cdot 2 + 1) + (3 \cdot 3 + 1) + (3 \cdot 4 + 1) \\ &= [3 \cdot 0 + 1] + (3 \cdot 1 + 1) + (3 \cdot 2 + 1) + (3 \cdot 3 + 1) + \\ &\qquad (3 \cdot 4 + 1). \end{aligned}$$

Therefore,

$$P_5 = \sum_{k=0}^{4} (3k + 1), \text{ or } \sum_{k=1}^{5} (3k - 2).$$

8. (a) $T_n = \sum_{k=0}^{n-1} (k + 1)$, or $\sum_{k=1}^{n} k$.

(b) $Sq_n = \sum_{k=0}^{n-1} (2k + 1)$, or $\sum_{k=1}^{n} (2k - 1)$.

(c) $P_n = \sum_{k=0}^{n-1} (3k + 1)$, or $\sum_{k=1}^{n} (3k - 2)$.

* * *

The exercises in Part C are extremely easy. Note that they lead to the recursive definition given on page 1-30.

B.

Here is an illustration of a procedure for obtaining an unlimited number of answers to problem posed in the Sample [See T. C. 27A ff.]:

$$\sum_{k=0}^{3}(3k+4) = \sum_{k-1=0}^{k-1=3}[3(k-1)+4] = \sum_{k=1}^{4}(3k+1);$$

$$\sum_{k=0}^{3}(3k+4) = \sum_{k-2=0}^{k-2=3}[3(k-2)+4] = \sum_{k=2}^{5}(3k-2);$$

$$\sum_{k=0}^{3}(3k+4) = \sum_{k+5=0}^{k+5=3}[3(k+5)+4] = \sum_{k=-5}^{-2}(3k+19).$$

A second procedure:

$$\sum_{k=0}^{k=3}(3k+4) = \sum_{-k=0}^{-k=3}[3(-k)+4] = \sum_{k=-3}^{k=0}(4-3k);$$

$$\sum_{k=5}^{k=8}(3k-11) = \sum_{-k=5}^{-k=8}[3(-k)-11] = \sum_{k=-8}^{k=-5}(-3k-11).$$

1. $\sum_{k=2}^{6}(5k+3)$

2. $\sum_{k=-3}^{0}(k^2-1)$

3. $\sum_{k=5}^{11}10k$

4. $\sum_{k=3}^{9}5k, \quad \sum_{k=1}^{7}(10+5k)$

5. $\sum_{k=2}^{5}(0\cdot k+5)$, or $\sum_{k=2}^{5}5$

(continued on T. C. 29B)

B. Express each of the following in Σ-notation.

Sample. $[3(0) + 4] + [3(1) + 4] + [3(2) + 4] + [3(3) + 4]$

Solution. Here is an answer: $\sum_{k=0}^{3} (3k + 4)$.

Other possible answers are:

$$\sum_{k=5}^{8} (3k - 11) \quad \text{and:} \quad \sum_{k=-3}^{0} (4 - 3k) .$$

1. $[5(2) + 3] + [5(3) + 3] + [5(4) + 3] + [5(5) + 3] + [5(6) + 3]$

2. $[(-3)^2 - 1] + [(-2)^2 - 1] + [(-1)^2 - 1] + [(0)^2 - 1]$

3. $10(5) + 10(6) + 10(7) + 10(8) + 10(9) + 10(10) + 10(11)$

4. $15 + 20 + 25 + 30 + 35 + 40 + 45$

5. $[0(2) + 5] + [0(3) + 5] + [0(4) + 5] + [0(5) + 5]$

6. $[\quad 4 \quad] \quad [\quad 4 \quad] + [\quad 4 \quad] + [\quad 4 \quad]$

7. the 5th pentagonal number

☆8. (a) the nth triangular number

(b) the nth square number

(c) the nth pentagonal number

C. Verify each of the following.

1. $\sum_{k=0}^{0} (9 + 3k) = 9$

2. $\sum_{k=5}^{5} (3 - 2k) = -7$

3. $\sum_{x=1}^{3} 4x = \sum_{x=1}^{2} 4x + [4(3)]$

(continued on next page)

The recursive definition in the box on page 1-30 is ambiguous to the extent that we have not specified the range of the variables 'p' and 'q'. Their range may be specified as <u>any</u> progression but in practice will be either the counting numbers, or the real integers.

4. $\sum_{y=-2}^{0}(3y + 5) = \sum_{y=-2}^{-1}(3y + 5) + [5]$

5. $\sum_{x=6}^{1000}(5x^2 + 2x - 3) = \sum_{x=6}^{999}(5x^2 + 2x - 3) + [5(1000)^2 + 2(1000) - 3]$

6. $\sum_{k=3}^{9} a_k = \sum_{k=3}^{8} a_k + [a_9]$

[Note: Think of 'a_k' as an abbreviation of an expression which is meaningful when 'k' is replaced by '3', '4', '5', '6', '7', '8', or '9'. For example, the expression '12 + 7k', in which case, 'a_9' is an abbreviation for '12 + 7(9)'.]

MORE ABOUT Σ-NOTATION

The exercises in Part C suggest the following recursive definition:

$$\sum_{k=p}^{p} a_k = a_p$$

and for every $q \geq p$,

$$\sum_{k=p}^{q+1} a_k = \sum_{k=p}^{q} a_k + [a_{q+1}]$$

As in the case of any recursive definition [See page 1-15.] this provides us with a way of computing sums of successive values of 'a_k'. For example, if we replace 'a_k' by '5 - 2k' and 'p' by '3', we get the following recursive definition:

$$\sum_{k=3}^{3}(5 - 2k) = 5 - 2(3)$$

and for every $q \geq 3$,

$$\sum_{k=3}^{q+1}(5 - 2k) = \sum_{k=3}^{q}(5 - 2k) + [5 - 2(q + 1)]$$

Hence,

$$\sum_{k=3}^{3}(5 - 2k) = -1$$

$$\sum_{k=3}^{4}(5 - 2k) = \sum_{k=3}^{3}(5 - 2k) + [5 - 2(4)]$$

$$= -1 + [-3] = -4$$

$$\sum_{k=3}^{5}(5 - 2k) = \sum_{k=3}^{4}(5 - 2k) + [5 - 2(5)]$$

$$= -4 + [-5] = -9$$

etc.

Another important use of recursive definitions is to prove generalizations like the following:

For every counting number n, the sum of the first n even numbers is $n(n + 1)$.

Introducing Σ-notation this can be written:

For every counting number n,

$$\boxed{\sum_{k=1}^{n} 2k = n(n + 1).}$$

rs,

Therefore, in view of (a) and (b), by the principle of mathematical induction for counting numbers, the property in question holds for every counting number.

Be sure to call attention to the similarity between this proof and the proof for Exercise 6 on page 1-21.

rs,

The proof that $\frac{1}{2}n(n + 1)$ is the sum of the first n counting numbers makes use of the principle of mathematical induction for counting numbers.

* * *

Property is that expressed by:

$$\sum_{k=1}^{\cdots} k = \frac{\ldots(\ldots + 1)}{2} .$$

(a) 1 has the property.

By the recursive definition on page 1-30,

$$\sum_{k=1}^{1} k = 1;$$

and

$$\frac{1(1 + 1)}{2} = 1 .$$

(b) The property is hereditary.

For every q,

$$\text{if } \sum_{k=1}^{q} k = \frac{q(q + 1)}{2}$$

$$\text{then } \sum_{k=1}^{q+1} k = \sum_{k=1}^{q} k + (q + 1) \qquad \text{[recursive definition]}$$

$$= \frac{q(q + 1)}{2} + (q + 1) \qquad \text{[inductive hypothesis]}$$

$$= \frac{(q + 1)(q + 2)}{2}$$

$$= \frac{(q + 1)[(q + 1) + 1]}{2} .$$

(continued on T. C. 32C)

rs,

Be sure that students understand the inductive proof given on page 1-32. The answer to the first '[Why?]' is:

the recursive definition

and the answer to the second '[Why?]' is:

the inductive hypothesis.

* * *

1. Point out to students that in many college algebra texts (in which $\sum$-notation is not discussed) exercises of this type are often stated:

   Prove:

$$1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}.$$

   The use of $\sum$-notation avoid the vagueness of the three dots.

   Students should be able to state what the generalization in this exercise asserts, viz. that the sum of the first n counting numbers is $\frac{n(n + 1)}{2}$. It is natural for students to wonder about the source of the form '$\frac{n(n + 1)}{2}$'. They may guess at the derivation of this form if you lead them through the following:

$$\begin{aligned} \text{Sum} &= 1 + 2 + 3 + \ldots + n \\ \text{Sum} &= \underline{n + (n - 1) + (n - 2) + \ldots + 1} \\ 2 \times \text{Sum} &= (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1) \\ &= n(n + 1) \\ \text{Sum} &= \frac{1}{2}n(n + 1). \end{aligned}$$

(continued on T. C. 32B)

According to the principle of mathematical induction for counting numbers, it is sufficient to show that

(a) $\sum_{k=1}^{1} 2k = 1(1 + 1)$

and that

(b) for every counting number q,

$$\text{if } \sum_{k=1}^{q} 2k = q(q + 1) \text{ then } \sum_{k=1}^{q+1} 2k = (q + 1)[(q + 1) + 1].$$

Proof of (a):

$$\sum_{k=1}^{1} 2k = 2(1) = 1(2) = 1(1 + 1).$$

Proof of (b):

For every q,

$$\text{if } \sum_{k=1}^{q} 2k = q(q + 1)$$

$$\text{then } \sum_{k=1}^{q+1} 2k = \sum_{k=1}^{q} 2k + [2(q + 1)] \qquad \text{[Why?]}$$

$$= q(q + 1) + [2(q + 1)] \qquad \text{[Why?]}$$

$$= (q + 1)(q + 2)$$

$$= (q + 1)[(q + 1) + 1].$$

Hence, the boxed statement follows by the principle of mathematical induction for counting numbers.

EXERCISES

Prove each of the following. Be sure to state the principle of mathematical induction you use for each proof.

1. For every counting number n, $\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}$.

(continued on next page)

Hence, for every real number y different from 0 and -1,

$$3 - \frac{1}{y} + \frac{1}{(y+1)^2} < 3 - \frac{1}{y+1}.$$

So, for every real integer $y \geq 1$,

$$\text{if} \quad \sum_{k=1}^{y} \frac{1}{k^2} < 3 - \frac{1}{y}$$

$$\text{then} \quad \sum_{k=1}^{y+1} \frac{1}{k^2} < 3 - \frac{1}{y+1} .$$

Therefore, in view of (a) and (b), by the principle of mathematical induction for positive real integers, the property in question holds for every positive real integer.

(b) The property is hereditary.

Suppose that, for a given $y > 0$,

$$\sum_{k=1}^{y} \frac{1}{k^2} < 3 - \frac{1}{y} .$$

Then, for this y,

$$\sum_{k=1}^{y+1} \frac{1}{k^2} = \sum_{k=1}^{y} \frac{1}{k^2} + \frac{1}{(y+1)^2} .$$

Since, by the inductive hypothesis,

$$\sum_{k=1}^{y} \frac{1}{k^2} < 3 - \frac{1}{y} ,$$

$$\sum_{k=1}^{y+1} \frac{1}{k^2} < 3 - \frac{1}{y} + \frac{1}{(y+1)^2} .$$

Now,

$$\begin{aligned} 3 - \frac{1}{y} + \frac{1}{(y+1)^2} &= 3 \ \frac{(y+1)^2 - y}{y(y+1)^2} \\ &= 3 - \frac{y^2 + y + 1}{y(y+1)^2} \\ &< 3 - \frac{y^2 + y}{y(y+1)^2} \qquad \left[\text{"add } \frac{1}{y(y+1)^2} \text{"}\right] \\ &= 3 - \frac{1}{y+1} . \end{aligned}$$

(continued on T. C. 33N)

Then, for this z,

$$\sum_{k=y}^{z+1} c = \sum_{k=y}^{z} c + c$$

$$= c(z - y + 1) + c$$

$$= c[(z + 1) - y + 1].$$

Hence, for every real integer $z \geq y$,

if $$\sum_{k=y}^{z} c = c(z - y + 1)$$

then $$\sum_{k=y}^{z+1} c = c[(z + 1) - y + 1].$$

Therefore, for every real integer y, by the principle of mathematical induction for real integers $\geq$ y, it follows in view of (i) and (ii), that the property in question holds for every real integer $\geq$ y.

11. Property is that expressed by:

$$\sum_{k=1}^{\cdots} \frac{1}{k^2} < 3 - \frac{1}{\cdots} .$$

(a) <u>1 has the property</u>.

$$\sum_{k=1}^{1} \frac{1}{k^2} = \frac{1}{1^2} = 1; \text{ and}$$

$$1 < 3 - \frac{1}{1} = 2 .$$

(continued on T. C. 33M)

Therefore, in view of (i) and (ii), by the principle of mathematical induction for positive real integers, the property in question holds for every positive real integer.

Note that the generalization in this exercise asserts that, for every real number c and for every real integer $x > 0$,

$$\sum_{k=1}^{x} c = \underbrace{c + c + c + \ldots + c}_{x \text{ terms}}.$$

Compare with Exercises 5 and 6 on page 1-29.

10. (b) Property in question is that expressed by:

for every real number c and for every real integer y,

$$\sum_{k=y}^{\ldots} c = c(\ldots - y + 1).$$

(i) <u>For every real integer y, y has the property.</u>

$$\sum_{k=y}^{y} c = c;$$

and $c(y - y + 1) = c$.

(ii) <u>For every real integer y, the property is hereditary over the set of real integers $\geq$ y,</u>

Suppose that, for a given $z \geq y$,

$$\sum_{k=y}^{z} c = c(z - y + 1).$$

(continued on T. C. 33L)

10. (a) Property is that expressed by:

for every real number c $\sum_{k=1}^{\cdots} c = c \cdot \ldots$.

(i) 1 has the property.

For every real number c,

$$\sum_{k=1}^{1} c = c; \qquad [c = c + 0 \cdot k]$$

and $$c \cdot 1 = c.$$

(ii) The property is hereditary.

Suppose that, for a given y,

$$\sum_{k=1}^{y} c = cy \quad \text{(for every real number c).}$$

Then, for this y,

$$\sum_{k=1}^{y+1} c = \sum_{k=1}^{y} c + c$$

$$= cy + c$$

$$= c(y + 1) .$$

Hence, for every y,

if, for every real number c, $\sum_{k=1}^{y} c = cy$

then, for every real number c,

$$\sum_{k=1}^{y+1} c = c(y + 1) .$$

(continued on T. C. 33K)

Then, for this q,

$$\sum_{k=1}^{q+1} \frac{1}{k(k+1)} = \sum_{k=1}^{q} \frac{1}{k(k+1)} + \frac{1}{(q+1)(q+2)}$$

$$= \frac{q}{q+1} + \frac{1}{(q+1)(q+2)}$$

$$= \frac{1}{q+1}\left[q + \frac{1}{q+2}\right]$$

$$= \frac{1}{q+1}\left[\frac{q^2+2q+1}{q+2}\right]$$

$$= \frac{q+1}{(q+1)+1} .$$

Hence, for every q,

$$\text{if} \quad \sum_{k=1}^{q} \frac{1}{k(k+1)} = \frac{q}{q+1}$$

$$\text{then} \quad \sum_{k=1}^{q+1} \frac{1}{k(k+1)} = \frac{q+1}{(q+1)+1} .$$

Therefore, in view of (a) and (b), by the principle of mathematical induction for positive real integers, the property in question holds for every positive real integer.

***

Correction: Insert 'every' between 'for' and 'real' in the sixth line from the bottom.

***

$$= (q+1)(q+2)(\frac{q}{3}+1)$$

$$= \frac{(q+1)[(q+1)+1][(q+1)+2]}{3}.$$

Hence, for every q,

$$\text{if } \sum_{k=1}^{q} k(k+1) = \frac{q(q+1)(q+2)}{3}$$

$$\text{then } \sum_{k=1}^{q+1} k(k+1) = \frac{(q+1)[(q+1)+1][(q+1)+2]}{3}.$$

Therefore, in view of (a) and (b), by the principle of mathematical induction for counting numbers, the property in question holds for every counting number.

9. Property is that expressed by:

$$\sum_{k=1}^{\cdots} \frac{1}{k(k+1)} = \frac{\cdots}{\cdots + 1}.$$

(a) _1 has the property._

$$\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{2};$$

$$\text{and } \frac{1}{1+1} = \frac{1}{2}.$$

(b) _The property is hereditary._

Suppose that, for a given real integer $q > 0$,

$$\sum_{k=1}^{q} \frac{1}{k(k+1)} = \frac{q}{q+1}.$$

(continued on T. C. 33I)

$$\text{then } \sum_{z=-4}^{x+1} (5z + 10) = \frac{[5(x+1)][(x+1)+5]}{2}.$$

Therefore, in view of (a) and (b), by the principle of mathematical induction for the set of real integers $\geq$-4, the property in question holds for every real integer $\geq$ -4.

7. Similar to foregoing proof except that here we use the principle of mathematical induction for real integers $\geq$ -2.

8. Property is that expressed by:

$$\sum_{k=1}^{\cdots} k(k+1) = \frac{\ldots(\ldots+1)(\ldots+2)}{3}.$$

(a) <u>1 has the property.</u>

$$\sum_{k=1}^{1} k(k+1) = 1(1+1) = 2;$$

$$\frac{1(1+1)(1+2)}{3} = 2.$$

(b) <u>The property is hereditary.</u>

Suppose that, for a given q,

$$\sum_{k=1}^{q} k(k+1) = \frac{q(q+1)(q+2)}{3}.$$

Then, for this q,

$$\sum_{k=1}^{q+1} k(k+1) = \sum_{k=1}^{q} k(k+1) + [(q+1)(q+2)].$$

$$= \frac{q(q+1)(q+2)}{3} + (q+1)(q+2)$$

(continued on T. C. 33H)

(a) $\underline{-4}$ has the property.

By the recursive definition,

$$\sum_{z=-4}^{-4} (5z + 10) = 5(-4) + 10 = -10;$$

and $$\frac{5(-4)(-4+5)}{2} = -10.$$

(b) The property is hereditary.

Suppose that, for a given $x \geq -4$,

$$\sum_{z=-4}^{x} (5z + 10) = \frac{5x(x+5)}{2}.$$

Then, for this x,

$$\begin{aligned}\sum_{z=-4}^{x+1} (5z + 10) &= \sum_{z=-4}^{x}(5z + 10) + [5(x+1) + 10] \\ &= \frac{5x(x+5)}{2} + [5(x+1) + 10] \\ &= \frac{5x^2 + 25x + 10x + 30}{2} \\ &= \frac{5x^2 + 35x + 30}{2} \\ &= \frac{5(x+1)(x+6)}{2} \\ &= \frac{[5(x+1)][(x+1)+5]}{2}.\end{aligned}$$

Hence, for every $x \geq -4$,

if $\sum_{z=-4}^{x} (5z + 10) = \frac{5x(x+5)}{2}$

(continued on T. C. 33G)

Hence, equation (2) becomes:

$$(3) \qquad (x+1)^3 - 1 = 3\sum_{k=1}^{x} k^2 + 3\left[\frac{x(x+1)}{2}\right] + x .$$

Solving for '$\sum_{k=1}^{x} k^2$' we obtain:

$$(4) \qquad \sum_{k=1}^{x} k^2 = \frac{1}{3}\left[(x+1)^3 - 1 - \frac{3x(x+1)}{2} - x\right]$$

$$= \frac{1}{3}\left[(x+1)\left\{(x+1)^2 - \frac{3x}{2} - 1\right\}\right]$$

$$= \frac{1}{6}[(x+1)(2x^2 + 4x + 2 - 3x - 2)]$$

$$= \frac{1}{6}[(x+1)(2x^2 + x)]$$

$$= \frac{x(x+1)(2x+1)}{6} .$$

In a similar manner you can derive a form for $\sum_{k=1}^{x} k^3$ by considering the generalization:

for every k,

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1.$$

***

6. Property is that expressed by:

$$\sum_{z=-4}^{\cdots} (5z + 10) = \frac{5 \cdot \ldots(\ldots +5)}{2} .$$

(continued on T. C. 33F)

[1-33]

Therefore, in view of (a) and (b), by the principle of mathematical induction for positive real integers, the property in question holds for every positive real integer.

***

Students may inquire about the derivation of the form $\frac{x(x + 1)(2x + 1)}{6}$, in Exercise 5. This form can be derived with the help of the theorems stated in the box on page 1-34 and of the theorem in Exercise 10(a) on page 1-33.

Since, for every k,

$$(k + 1)^3 = k^3 + 3k^2 + 3k + 1$$

then

$$(k + 1)^3 - k^3 = 3k^2 + 3k + 1.$$

Now,

(1) $$\sum_{k=1}^{x} [(k + 1)^3 - k^3] = \sum_{k=1}^{x} (3k^2 + 3k + 1)$$

or

(2) $$\sum_{k=1}^{x} (k+1)^3 - \sum_{k=1}^{x} k^3 = 3\sum_{k=1}^{x} k^2 + 3\sum_{k=1}^{x} k + \sum_{k=1}^{x} 1 .$$

Since $$\sum_{k=1}^{x} (k+1)^3 = 2^3 + 3^3 + 4^3 + \ldots + x^3 + (x + 1)^3$$

and $$\sum_{k=1}^{x} k^3 = 1^3 + 2^3 + 3^3 + \ldots + x^3,$$

the left member of (2) simplifies to:

$$(x + 1)^3 - 1^3 .$$

(continued on T. C. 33E)

[1-33]

and $$\frac{1(1+1)(2\cdot 1+1)}{6} = 1.$$

(b) The property is hereditary.

Suppose that, for a given real integer $y \geq 1$,

$$\sum_{r=1}^{y} r^2 = \frac{y(y+1)(2y+1)}{6}.$$

Then, for this y,

$$\begin{aligned}
\sum_{r=1}^{y+1} r^2 &= \sum_{r=1}^{y} r^2 + (y+1)^2 \\
&= \frac{y(y+1)(2y+1)}{6} + (y+1)^2 \\
&= \frac{y(y+1)(2y+1) + 6(y+1)^2}{6} \\
&= \frac{(y+1)[y(2y+1) + 6(y+1)]}{6} \\
&= \frac{(y+1)(2y^2 + y + 6y + 6)}{6} \\
&= \frac{(y+1)(2y^2 + 7y + 6)}{6} \\
&= \frac{(y+1)(y+2)(2y+3)}{6} \\
&= \frac{(y+1)[(y+1)+1][2(y+1)+1]}{6}.
\end{aligned}$$

Hence, for every y,

$$\text{if } \sum_{r=1}^{y} r^2 = \frac{y(y+1)(2y+1)}{6}$$

$$\text{then } \sum_{r=1}^{y+1} r^2 = \frac{(y+1)[(y+1)+1][2(y+1)+1]}{6}.$$

Then, for this q,

$$\sum_{k=1}^{q+1} (5 + 4k) = \sum_{k=1}^{q} (5 + 4k) + [5 + 4(q + 1)]$$

$$= q(2q + 7) + [4q + 9]$$

$$= 2q^2 + 11q + 9$$

$$= (q + 1)(2q + 9)$$

$$= (q + 1)[2(q + 1) + 7].$$

Therefore, in view of (a) and (b), by the principle of mathematical induction for counting numbers, the property in question holds for every number.

3. Similar to foregoing proof.

4. Proof is similar to that for Exercise 1 except that here we use the principle of mathematical induction for non-negative real integers.

5. The generalization to be proved asserts that the sum of the squares of the first x real positive integers is

$$\frac{x(x + 1)(2x + 1)}{6}.$$

Property is that expressed by:

$$\sum_{r=1}^{\cdots} r^2 = \frac{\ldots(\ldots + 1)(2 \cdot \ \ldots + 1)}{6}.$$

(a) <u>1 has the property</u>.

By the recursive definition,

$$\sum_{r=1}^{1} r^2 = 1^2, \text{ or } 1;$$

(continued on T. C. 33C)

2. Here the student is to find the sum of the first n terms of the progression

$$9, 13, 17, \ldots .$$

The kth term of this progression is $5 + 4k$. In traditional terminology, the student is asked to prove:

$$9 + 13 + 17 + \ldots + (5 + 4n) = n(2n + 7).$$

As in Exercise 1, the form '$n(2n + 7)$' is suggested by:

$$\begin{array}{rcl} \text{Sum} & = & 9 + 13 + 17 + \ldots + (5 + 4n) \\ \text{Sum} & = & (5 + 4n) + (1 + 4n) + (-3 + 4n) + \ldots + 9 \\ \hline 2 \times \text{Sum} & = & (14 + 4n) + (14 + 4n) + (14 + 4n) + \ldots + (14 + 4n) \\ & = & n(14 + 4n) \\ \text{Sum} & = & \frac{1}{2}n(14 + 4n) = n(7 + 2n). \end{array}$$

Property is that expressed by:

$$\sum_{k=1}^{\ldots} (5 + 4k) = \ldots (2 \cdot \ldots + 7).$$

(a) <u>1 has the property</u>.

By the recursive definition,

$$\sum_{k=1}^{1} (5 + 4k) = 5 + 4(1) = 9;$$

and $1(7 + 2 \cdot 1) = 9$.

(b) <u>The property is hereditary</u>.

Suppose that, for a given q,

$$\sum_{k=1}^{q} (5 + 4k) = q(2q + 7).$$

(continued on T.C. 33B)

2. For every counting number n, $\sum_{k=1}^{n} (5 + 4k) = n(2n + 7)$.

3. For every counting number n, $\sum_{k=1}^{n} (3k - 2) = \frac{n(3n - 1)}{2}$.

4. For every real integer $x > 0$, $\sum_{k=1}^{x} k = \frac{x(x + 1)}{2}$.

5. For every real integer $x > 0$, $\sum_{r=1}^{x} r^2 = \frac{x(x + 1)(2x + 1)}{6}$

6. For every real integer $y \geq -4$, $\sum_{z=-4}^{y} (5z + 10) = \frac{5y(y + 5)}{2}$.

7. For every real integer $w \geq -2$, $\sum_{b=-2}^{w} (b + 7) = \frac{(w + 3)(w + 12)}{2}$.

8. For every counting number n, $\sum_{k=1}^{n} k(k + 1) = \frac{n(n + 1)(n + 2)}{3}$.

9. For every real integer $x > 0$, $\sum_{k=1}^{x} \frac{1}{k(k + 1)} = \frac{x}{x + 1}$.

10. (a) For every real number c and for real integer $x > 0$,

$$\sum_{k=1}^{x} c = cx.$$

(b) For every real number c and for all real integers x and y such that $x \geq y$,

$$\sum_{k=y}^{x} c = c(x - y + 1).$$

☆ 11. For every real integer $x > 0$, $\sum_{k=1}^{x} \frac{1}{k^2} < 3 - \frac{1}{x}$.

STILL MORE ABOUT Σ-NOTATION

The recursive defintion of a Σ-expression makes it possible to use mathematical induction to prove theorems about sums of any number of terms of progression. We shall state, illustrate, and prove two such theorems:

$$\text{(I)} \quad \sum_{k=p}^{q} a_k + \sum_{k=p}^{q} b_k = \sum_{k=p}^{q} (a_k + b_k).$$

$$\text{(II)} \quad \text{For every real number } c, \quad \sum_{k=p}^{q} c(a_k) = c\sum_{k=p}^{q} a_k .$$

Illustrations:

$$\text{(I):} \quad \sum_{k=2}^{5} (3k+1) + \sum_{k=2}^{5} (k-2)$$

$$= [7 + 10 + 13 + 16 + 19] + [0 + 1 + 2 + 3 + 4]$$

$$= (7+0) + (10+1) + (13+2) + (16+3) + (19+4)$$

$$= \sum_{k=2}^{5} ([3k+1] + [k-2])$$

$$\text{(II):} \quad \sum_{m=-3}^{2} 2m^2 = 2(-3)^2 + 2(-1)^2 + 2(0)^2 + 2(1)^2 + 2(2)^2$$

$$= 2[(-3)^2 + (-2)^2 + (-1)^2 + (0)^2 + (1)^2 + (2)^2]$$

$$= 2\sum_{m=-3}^{2} m^2$$

Proofs:

Theorem (I)

$$\text{(a)} \quad \sum_{k=p}^{p} a_k + \sum_{k=p}^{p} b_k = a_p + b_p \qquad \text{[Why?]}$$

$$= \sum_{k=p}^{p} (a_k + b_k). \qquad \text{[Why?]}$$

(b) For every $q \geq p$,

$$\text{if } \sum_{k=p}^{q} a_k + \sum_{k=p}^{q} b_k = \sum_{k=p}^{q} (a_k + b_k)$$

$$\text{then } \sum_{k=p}^{q+1} a_k + \sum_{k=p}^{q+1} b_k = [\sum_{k=p}^{q} a_k + a_{q+1}] + [\sum_{k=p}^{q} b_k + b_{q+1}]$$

$$= [\sum_{k=p}^{q} a_k + \sum_{k=p}^{q} b_k] + [a_{q+1} + b_{q+1}]$$

$$= [\sum_{k=p}^{q} (a_k + b_k)] + a_{q+1} + b_{q+1}$$

$$= \sum_{k=p}^{q+1} (a_k + b_k).$$

Hence, theorem (I) follows either by the principle of mathematical induction for the set of counting numbers $\geq p$ (or by the principle of mathematical induction for the set of real integers $\geq p$, according to how the theorem is interpreted).

Theorem II

(a) For every real number c, $\sum_{k=p}^{p} c(a_k) = c(a_p)$

$$= c \sum_{k=p}^{p} a_k .$$

(b) For every real number c and for every $q \geq p$,

$$\text{if } \sum_{k=p}^{q} c(a_k) = c \sum_{k=p}^{q} a_k$$

$$\text{then } \sum_{k=p}^{q+1} c(a_k) = \sum_{k=p}^{q} c(a_k) + c(a_{q+1})$$

(continued on next page)

1.
$$\sum_{k=p}^{q} 2k = \sum_{k=p}^{q} (k + k)$$

$$= \sum_{k=p}^{q} k + \sum_{k=p}^{q} k .$$

2.
$$\sum_{k=p}^{q} (k + 3)^2 = \sum_{k=p}^{q} (k^2 + 6k + 9)$$

$$= \sum_{k=p}^{q} k^2 + \sum_{k=p}^{q} 6k + \sum_{k=p}^{q} 9$$

$$= \sum_{k=p}^{q} k^2 + 6 \sum_{k=p}^{q} k + \sum_{k=p}^{q} 9 .$$

3.
$$\sum_{k=1}^{q} (k + 3)^2 = \sum_{k=1}^{q} k^2 + 6 \sum_{k=1}^{q} k + \sum_{k=1}^{q} 9$$

$$= \frac{q(q + 1)(2q + 1)}{6} + 6 \left[\frac{q(q + 1)}{2} \right] + 9q.$$

$$= c\sum_{k=p}^{q} a_k + c(a_{q+1})$$

$$= c[\sum_{k=p}^{q} a_k + a_{q+1}]$$

$$= c\sum_{k=p}^{q+1} a_k .$$

Hence, theorem (II) follows either by the principle of mathematical induction for the set of counting numbers $\geq p$ (or by the principle of mathematical induction for the set of real integers $\geq p$).

EXERCISES

Prove each of the following.

Sample. $\sum_{k=p}^{q} (1 + 3k + k^2) = \sum_{k=p}^{q} 1 + 3\sum_{k=p}^{q} k + \sum_{k=p}^{q} k^2$

Solution. $\sum_{k=p}^{q} (1 + 3k + k^2)$

$= \sum_{k=p}^{q} 1 + \sum_{k=p}^{q}(3k + k^2)$ [Associative principle for addition and Theorem (I)]

$= \sum_{k=p}^{q} 1 + \sum_{k=p}^{q} 3k + \sum_{k=p}^{q} k^2$ [Theorem (I)]

$= \sum_{k=p}^{q} 1 + 3\sum_{k=p}^{q} k + \sum_{k=p}^{q} k^2$ [Theorem (II)]

1. $\sum_{k=p}^{q} 2k = \sum_{k=p}^{q} k + \sum_{k=p}^{q} k$

2. $\sum_{k=p}^{q} (k + 3)^2 = \sum_{k=p}^{q} k^2 + 6\sum_{k=p}^{q} k + \sum_{k=p}^{q} 9$

3. Use the result in Exercise 2 together with the results in Exercises 4, 5, and 10(a) on page 1-33 to show that

$$\sum_{k=1}^{q} (k + 3)^2 = \frac{q(q + 1)(2q + 1)}{6} + 6[\frac{q(q + 1)}{2}] + 9q .$$

(continued on next page)

-37]

$$= \frac{s}{2}\left[\, 6s^2 + 9s + 3 + 24s + 24 + 32 \,\right]$$

$$= \frac{s}{2}\left[\, 6s^2 + 33s + 59 \,\right].$$

5. $$\sum_{k=2}^{12} (k - 12)(k - 2) = \sum_{k=2}^{12} (k^2 - 14k + 24)$$

$$= \sum_{k=2}^{12} k^2 - 14\sum_{k=2}^{12} k + \sum_{k=2}^{12} 24$$

$$= \sum_{k=2}^{12} k^2 - 14 \left[\, \sum_{k=0}^{12} k - (0 + 1) \,\right] + 24(12 - 2 + 1)$$

$$= \sum_{k=2}^{12} k^2 - 14 \sum_{k=0}^{12} k + 14 + 264$$

$$= \sum_{k=2}^{12} k^2 - 14 \sum_{k=0}^{12} k + 278 .$$

-37]

$$= \frac{6(k^3 + 3k^2 + 3k + 1) + 33(k^2 + 2k + 1) + 59(k + 1)}{2}$$

$$= \frac{k + 1}{2}\left[6(k + 1)^2 + 33(k + 1) + 59 \right].$$

Hence, for every k,

$$\text{if } \sum_{t=1}^{k} (3t + 4)^2 = \frac{k}{2}(6k^2 + 33k + 59)$$

$$\text{then } \sum_{t=1}^{k+1} (3t + 4)^2 = \frac{k + 1}{2}\left[6(k + 1)^2 + 33(k + 1) + 59\right].$$

Therefore, in view of (i) and (ii), by the principle of mathematical induction for counting numbers (or for real integers ≥ 1), the property holds for every counting number (or for every real integer ≥ 1).

(b) $$\sum_{t=1}^{s} (3t + 4)^2 = \sum_{t=1}^{s} (9t^2 + 24t + 16)$$

$$= 9\sum_{t=1}^{s} t^2 + 24 \sum_{t=1}^{s} t + \sum_{t=1}^{s} 16$$

$$= 9\left[\frac{s(s + 1)(2s + 1)}{6}\right] + 24\left[\frac{s(s + 1)}{2}\right] + 16s$$

$$= \frac{3s(s + 1)(2s + 1) + 24s(s + 1) + 32s}{2}$$

$$= \frac{s}{2}\left[3(s + 1)(2s + 1) + 24(s + 1) + 32\right]$$

(continued on T. C. 37C)

4. (a) Property is that expressed by:

$$\sum_{t=1}^{\cdots} (3t + 4)^2 = \frac{\cdots}{2} [6(\ldots)^2 + 33 \cdot \ldots + 59].$$

(i) <u>1 has the property.</u>

$$\sum_{t=1}^{1} (3t + 4)^2 = (3 + 4)^2 = 49; \text{ and}$$

$$\frac{1}{2}(6 + 33 + 59) = 49.$$

(ii) <u>The property is hereditary.</u>

Suppose that, for a given k,

$$\sum_{t=1}^{k} (3t + 4)^2 = \frac{k}{2}(6k^2 + 33k + 59).$$

Then, for this k,

$$\sum_{t=1}^{k+1} (3t + 4)^2 = \sum_{t=1}^{k} (3t + 4)^2 + [3(k + 1) + 4]^2$$

$$= \frac{k}{2}(6k^2 + 33k + 59) + (9k^2 + 42k + 49)$$

$$= \frac{6k^3 + 51k^2 + 143k + 98}{2}$$

(continued on T. C. 37B)

4. Prove that, for every $s \geq 1$,

$$\sum_{t=1}^{s} (3t + 4)^2 = \frac{s}{2}(6s^2 + 33s + 59)$$

(a) by mathematical induction, and (b) by the method suggested by Exercise 3 on page 1-36.

5. $$\sum_{k=2}^{12} (k - 12)(k - 2) = \sum_{k=2}^{12} k^2 - 14 \sum_{k=0}^{12} k + 278$$

ARITHMETIC PROGRESSIONS

Progressions such as

(1) 5, 7, 9, 11, 13, 15, 17, 19, ...

(2) -3, 1, 5, 9, 13, 17, 21, 25, ...

(3) $\frac{1}{2}$, 2, $\frac{7}{2}$, 5, $\frac{13}{2}$, 8, $\frac{19}{2}$, 11, ...

(4) 8, 1, -6, -13, -20, -27, -34, -41, ...

(5) π, 2π, 3π, 4π, 5π, 6π, 7π, 8π, 9π, ...

are called arithmetic progressions. Notice that a characteristic property of arithmetic progressions is this:

The difference of any term of an arithmetic progression from its follower is the same as the difference of any other term from its follower.

We frequently abbreviate the boxed statement by saying:

> The difference between successive terms of an A. P. is constant. This difference is the common difference of the A. P. .

-38]

A. Let students discover their own methods for the exercises in this part. Formal methods are developed in subsequent parts.

* * *

Exercise 11 does not have a unique solution.

For example, for the A. P. (1) the common difference is 2, and for the A. P. (4) the common difference is -7.

For all real numbers a and d, the successive terms of the A. P. whose first term is a and whose common difference is d are the values of:

$$a + (x - 1)d$$

for the values 1, 2, 3, etc. of 'x'.

For example, in A. P. (1) above, the first term is 5 and the common difference is 2. Identify the first term and the common difference in (2), (3), (4), and (5).

EXERCISES

A. Fill in the blanks in each of the following so that the result gives an arithmetic progression.

1. -2, -1, ___, 1, 2, ___, ___. ...
2. 3, ___, ___, 66, ___, ___, ...
3. 9, 10, 11, ___, ___, ___, ___, ...
4. -8, ___, ___, ___, ___, ___, 17, ...
5. ___, ___, ___, 16, ___, 24, ...
6. ___, ___, ___, ___, ___, -30, -35, ...
7. 6, ___, ___, ___, ___, ___, 7, ...
8. $\frac{1}{3}$, $\frac{5}{6}$, ___, ___, ___, ___, ...
9. 3, ___, ___, ___, 9, ...
10. -3, ___, $-3 + 4\sqrt{2}$, ___, ___, ___, ...
11. ___, ___, ___, ___, 7, ___, ___, ...

B. In filling the blanks between '3' and '66' in Exercise 2 of Part A you "inserted two <u>arithmetic</u> <u>means</u> between 3 and 66". In Exercise 4 you inserted five arithmetic means between -8 and 17.

[1-39]

C.

4. x, $x + d$, and $x + 2d$ are consecutive terms in an A.P., and $x + d$ is the arithmetic mean of x and $x + 2d$. Since $x + 2d = y$, we have $d = \frac{y - x}{2}$ and

$$x + d = x + \frac{y - x}{2} = \frac{x + y}{2}.$$

In each of the following exercises insert arithmetic means as indicated.

1. two arithmetic means between 4 and 10
2. three between 1 and 7
3. five between 8 and 3
4. ten between -6 and -14
5. two between 1 and -3

C. If three numbers are consecutive terms of an A. P., then the second is called the arithmetic mean of the first and third.

1. Find the arithmetic mean of 6 and 10.
2. Find the arithmetic mean of 10 and 6.
3. Find the arithmetic mean of -3 and 3
4. Prove the following generalization:

For every x and y, the arithmetic mean of x and y is $\frac{x + y}{2}$.

D. We can develop a procedure for inserting any number of arithmetic means between any two real numbers.

Sample. Develop a procedure for inserting three arithmetic means between any two real numbers.

Solution. For all real numbers x and y, there exists three arithmetic means between x and y if and only if there exists a real number d

such that

$$x, x + d, x + 2d, x + 3d, y$$

3. Suppose you want to insert p arithmetic means between x and y. This would give us $p + 2$ consecutive terms of an A. P. in which x is the first term and y is the $(p + 2)$th term. Hence,

$$y = x + (p + 1) d,$$

so

$$d = \frac{y - x}{p + 1} .$$

Then the $p + 2$ terms of the A. P. are:

$$x, \frac{px + y}{p + 1}, \frac{(p - 1) x + 2y}{p + 1}, \frac{(p - 2) x + 3y}{p + 1}, \ldots, \frac{x + py}{p + 1}, y.$$

D.

1. If x and y are two numbers such that x is the first term of an A. P. and y is the seventh term then

$$y = x + 6d$$

so

$$d = \frac{y - x}{6}.$$

For every x and y, the five arithmetic means between x and y are the values of:

$$x + k\left(\frac{y - x}{6}\right)$$

for the values 1, 2, 3, 4, and 5 of 'k'.

Hence, the following are successive terms of an A. P.:

$$x, \quad \frac{5x + y}{6}, \quad \frac{2x + y}{3}, \quad \frac{x + y}{2}, \quad \frac{x + 2y}{3}, \quad \frac{x + 5y}{6}, \quad y.$$

2. If x is the first term of an A. P. and y is the 102nd term then

$$y = x + 101d,$$

so

$$d = \frac{y - x}{101}.$$

The 102 terms of the A. P. are:

$$x, \quad \frac{101x + y}{101}, \quad \frac{99x + 2y}{101}, \quad \frac{98x + 3y}{101}, \quad \ldots, \quad \frac{2x + 99y}{101}, \quad \frac{x + 100y}{101}, \quad y.$$

are successive terms in an A. P. This is the case if and only if the difference between successive terms is constant. Since each of the first three differences is d, all four differences will be the same if and only if

$$y - (x + 3d) = d,$$

that is,

$$d = \frac{y - x}{4}.$$

Hence, for all x and y, there exist three arithmetic means between x and y and, in fact, these are the values of the expression:

$$x + k(\frac{y - x}{4})$$

for the values 1, 2, and 3 of 'k'.

[You could have applied this procedure in Exercise 2 of Part B. In that case the required arithmetic means are the values of:

$$1 + k(\frac{7 - 1}{4})$$

for the values 1, 2, and 3 of 'k':

$$1 + 1(\frac{7 - 1}{4}) = \frac{5}{2},$$

$$1 + 2(\frac{7 - 1}{4}) = 4,$$

$$1 + 3(\frac{7 - 1}{4}) = \frac{11}{2}.]$$

1. Develop a procedure for inserting five arithmetic means between any two numbers.

2. Tell how to insert one hundred arithmetic means between any two numbers.

3. Tell how to insert any number of arithmetic means between any two numbers.

SUMS OF ARITHMETIC PROGRESSIONS

Consider each of the sums

(I) $1 + 2 + 3 + 4 + 5$

(II) $7 + 10 + 13 + 16 + 19$

(III) $(+4) + (-1) + (-6) + (-11) + (-16)$

Notice that each sum can be computed by adding the first 5 terms of an arithmetic progression. In (I) the terms of the progression are the values of '$1 + (x - 1)(1)$' for the values $+1$, $+2$, $+3$, ... of 'x'. The first five terms are obtained by using the first five values: $+1$, $+2$, $+3$, $+4$, $+5$. Therefore, (I) can be abbreviated as:

$$\sum_{x=+1}^{+5} [+1 + (x - 1)(1)]$$

Similarly,

$$(\mathrm{II}) = \sum_{x=+1}^{+5} [7 + (x - 1)(3)]$$

and

$$(\mathrm{III}) = \sum_{x=+1}^{+5} [4 + (x - 1)(-5)]$$

In general,

> For every counting number n, the sum of the first n terms of the arithmetic progression whose first term is a and whose common difference is d is
>
> $$\sum_{x=1}^{n} [a + (x - 1)d] .$$

Let us use the boxed theorems on page 1-34 and compute (III).

$$\sum_{x=1}^{5}[4 + (x - 1)(-5)] = \sum_{x=1}^{5}[9 - 5x]$$

$$= \sum_{x=1}^{5} 9 + \sum_{x=1}^{5}(-5x)$$

$$= \sum_{x=1}^{5} 9 - 5\sum_{x=1}^{5} x$$

By Exercises 4 and 10a on page 1-33,

$$\sum_{x=1}^{5} 9 - 5\sum_{x=1}^{5} x = 9(5) - 5\left[\frac{(5)(5+1)}{2}\right] = -30$$

In this particular example the result, -30, could have been obtained without using Σ-notation. However, in cases involving a large number of terms or when proving generalizations, the use of Σ-notation results in a great saving of labor. Using Σ-notation we can prove a generalization covering all cases. Suppose we wish to find the sum of the first n terms of the A. P. whose first term is a and whose common difference is d. We have seen that this sum is

$$\sum_{x=1}^{n}[a + (x - 1)d].$$

Proceeding as in the example above

$$\sum_{x=1}^{n}[a + (x - 1)d] = \sum_{x=1}^{n}[(a - d) + dx]$$

$$= \sum_{x=1}^{n}(a - d) + d\sum_{x=1}^{n} x$$

$$= (a - d)(n) + d\left[\frac{n(n+1)}{2}\right]$$

$$= (n)\left[(a - d) + \frac{d(n+1)}{2}\right]$$

$$= \frac{n}{2}[2a - 2d + d(n + 1)]$$

$$= \frac{n}{2}[2a + (n - 1)d].$$

Students should be able to supply the proof of the theorem stated in the box on page 1-43. They can give the proof we give on page 1-42 or they can use mathematical induction directly. It is instructive to go through an inductive proof of this theorem.

***

A.

1. $\frac{20}{2}[2(-2) + 19 \cdot 7] = 10(-4 + 133) = 1290$

2. $\frac{12}{2}[2(-100) + 11 \cdot 100] = 6(-200 + 1100) = 5400$

3. $\frac{7}{2}[2(1) + 6(-0.5)] = \frac{7}{2}(2 - 3) = -3.5$

We have just proved the following theorem:

> For every counting number n, and for all real numbers a and d, the sum of the first n terms of the A. P. whose first term is a and whose common difference is d is
>
> $$\frac{n}{2}[\ 2a + (n - 1)d\ ],$$

The use of this theorem is illustrated by the following

Example. Find the sum of the first 53 terms of the A. P. which begins:

$$8,\ 17,\ 26,\ \ldots$$

Solution. The first term is 8 and the common difference is 9. Therefore, the sum of the first 53 terms is

$$\begin{aligned} &\frac{53}{2}[\ 2(8) + (53 - 1)(9)\ ] \\ = &\frac{53}{2}[\ 16 + 468\ ] \\ = &\ 12826 \end{aligned}$$

The labor-saving value of the theorem is obvious.

EXERCISES

A. Find the sum as indicated.

1. First 20 terms of $-2, 5, 12, \ldots$
2. First 12 terms of $-100, 0, 100, \ldots$
3. First 7 terms of $1, 0.5, 0, \ldots$

(continued on next page)

Upon simplification, (1) and (2) become:

$$(1') \qquad 3a + 21d = 20$$

$$(2') \qquad 2a + 19d = 13$$

or:

$$(1'') \qquad 6a + 42d = 40$$

$$(2'') \qquad \underline{6a + 57d = 39}$$

$$-15d = 1$$

$$d = -\frac{1}{15}, \quad a = \frac{107}{15}.$$

6. $630 = \frac{n}{2}[12 + (n - 1)3]$

$$1260 = 12n + 3n(n - 1)$$

$$1260 = 12n + 3n^2 - 3n$$

$$3n^2 + 9n - 1260 = 0$$

$$n^2 + 3n - 420 = 0$$

$$n = \frac{-3 \pm \sqrt{1689}}{2}$$

Since the given equation does not have a positive integral root, there is no A. P. which meets the stated conditions. Modify the problem so that students are asked to find the number of terms of an A. P. (first term 6, difference 3) for which the sum is 627.

2. $d = \dfrac{2S_n - 2na}{n(n-1)}$, $[n(n-1) \neq 0]$

3. Since $S_{10} = 98$, $a = 10$, $n = 10$, and

$$d = \frac{2S_n - 2na}{n(n-1)}$$

then

$$d = \frac{2(98) - 2(10)(10)}{10(10-1)}$$

$$= \frac{196 - 200}{90}$$

$$= -\frac{2}{45}.$$

4. $a = \dfrac{2S_n - n(n-1)d}{2n}$

$$= \frac{2(\frac{1}{2}) - 30(29)(\frac{1}{2})}{2(30)}$$

$$= \frac{1 - 435}{60}$$

$$= -\frac{217}{30}.$$

5. (1) $100 = \frac{15}{2}[2a + (15-1)d]$

(2) $130 = \frac{20}{2}[2a + (20-1)d]$

(continued on T. C. 44C)

A. (continued)

4. $\frac{50}{2}[2(-3) + 49(0.5)] = 25(-6 + 24.5) = 462.5$

5. $\frac{12}{2}[2(9) + 11(-1)] = 6(18 - 11) = 42$

6. $\frac{19}{2}[2(2) + 18(2)] = 19(2 + 18) = 380$

7. $\frac{1000}{2}[2(1) + 999(2)] = 1000(1000) = 1,000,000$

8. Correction: Replace '$6, \sqrt{2}$' by '$6 + \sqrt{2}$'.

$$\frac{28}{2}[2\sqrt{2} + 27(3)] = 1134 + 28\sqrt{2}$$

9. $\frac{1001}{2}[2(1) + 1000(-5)] = 1001(1 - 2500) = -2,501,499$

10. $\frac{1001}{2}[2(1) + 1000(-\frac{2}{1001})] = 1001[1 - \frac{1000}{1001}] = 1$

11. In this case we let the first term of the A. P. be $7 + 4(4)$.

$$\frac{20}{2}[2(7 + 16) + 19(4)] = 10(46 + 76) = 1220$$

12. $\frac{48}{2}\left[2[64 + 47(-6)] + 47(-6)\right] = 24[128 + 47(-18)] = -17,232$

B.

1. $2S_n = 2na + n(n - 1)d$

$$a = \frac{2S_n - n(n - 1)d}{2n}, \quad [n \neq 0]$$

4. First 50 terms of $-3, -2.5, -2, \ldots$

5. First 12 terms of $9, 8, 7, \ldots$

6. First 19 terms of $2, 4, 6, \ldots$

7. First 1000 terms of $1, 3, 5, \ldots$

8. First 28 terms of $\sqrt{2},\ 3 + \sqrt{2},\ 6, \sqrt{2}, \ldots$

9. First 1001 terms of $1, -4, -9, \ldots$

10. First 1001 terms of $1, \frac{999}{1001}, \frac{997}{1001}, \ldots$

11. Twenty successive terms starting with the fifth term of

$$7, 11, 15, \ldots$$

12. Forty-eight successive terms starting with the forty-eighth term of $64, 58, 52, \ldots$

B. The boxed theorem on page 1-43 is often abbreviated to:

$$S_n = \frac{n}{2}[\ 2a + (n - 1)d\ ]$$

1. Solve this equation for 'a'.

2. Solve the equation for 'd'.

3. Find the common difference of an arithmetic progression whose first term is 10 and the sum of whose first 10 terms is 98.

4. Find the first term of an arithmetic progression whose common difference is $\frac{1}{2}$ and the sum of whose first 30 terms is $\frac{1}{2}$.

5. Find the first term and the common difference of an arithmetic progression the sum of whose first 15 terms is 100 and the sum of whose first 20 terms is 130.

6. Given the arithmetic progression with first term 6 and common difference 3. How many terms of this arithmetic progression must be added starting with the first to get the sum 630?

5]

Note that each of the sample problems can be "translated" into one involving arithmetic progressions. It is instructive to do this once or twice in order to increase the student's familiarity with Σ-notation and with arithmetic progressions. For example, in Sample 1. the problem can be viewed as finding the sum of the first 14 terms of the A. P. whose first term is 7 and whose common difference is 4.

$$
\begin{aligned}
S_{14} &= \frac{14}{2}[14 + 13 \cdot 4] \\
&= 7(66) \\
&= 462
\end{aligned}
$$

However, it is probably more important that a student develop dexterity with Σ-notation by solving the exercises on 1-46 using the Σ-notation theorems.

***

The bracketed note at the bottom of page 1-45 exemplifies the "change of apparent variable" procedure illustrated on T. C. 29A. Each occurrence of 'x' is replaced by 'x + 8' in the expression on the left of ' = '. Of course, the limits of summation are changed automatically.

$$\sum_{x=9}^{x=20} x = \sum_{x+8=9}^{x+8=20} (x+8) = \sum_{x=1}^{12} (x+8)$$

C. Compute each of the given sums.

Sample 1. $\sum_{x=1}^{14} (4x + 3)$

Solution.
$$\sum_{x=1}^{14} (4x + 3) = 4 \sum_{x=1}^{14} x + \sum_{x=1}^{14} 3$$
$$= 4\left[\frac{14(14 + 1)}{2} \right] + 3(14)$$
$$= 28(15) + 42$$
$$= 462$$

Sample 2. $\sum_{x=9}^{20} x$

Solution. Note that in this case you are asked to compute a sum of 12 successive terms starting with the ninth term. We can compute the sum of these 12 terms by subtracting the sum of the first 8 terms from the sum of the first 20 terms:

$$\sum_{x=9}^{20} x = \sum_{x=1}^{20} x - \sum_{x=1}^{8} x$$
$$= \frac{20(20 + 1)}{2} - \frac{8(8 + 1)}{2}$$
$$= 210 - 36 = 174$$

[Note: Do you see that $\sum_{x=9}^{20} x = \sum_{x=1}^{12} (x + 8)$?]

(b) The property is hereditary.

Suppose that, for a given $k > 1$,

$$Sq_k > T_k .$$

Then, for this k,

$$\begin{aligned} Sq_{k+1} &= Sq_k + (2k + 1) \\ &> T_k + (2k + 1) && \text{[hypothesis]} \\ &= [T_k + (k + 1)] + k \\ &= T_{k+1} + k \\ &> T_{k+1}. && [k > 0] \end{aligned}$$

Hence, for every $k > 1$,

$$\text{if } Sq_k > T_k$$

$$\text{then } Sq_{k+1} > T_{k+1} .$$

Therefore, in view of (a) and (b), by the principle of mathematical induction for counting numbers > 1, the property in question holds for every counting number > 1.

[Ask students why the set in question is not the set of all counting numbers.]

2. Property is that expressed by:

$$Sq \ldots - T \ldots \geq \ldots$$

2 is a counter-example.

$$Sq_2 - T_2 = 4 - 3 = 1 < 2.$$

[Note: The property in Exercise 2 is hereditary. Ask students to prove this and then to modify the Exercise so that it becomes a theorem. (Modification: replace '1' by '2'.) The proof of hereditariness is apparent if one analyzes the proof of (b) in Exercise 1.]

10. Since for every positive integer x,

$$\sum_{k=1}^{x} (2k + 1) = 2\left[\frac{x(x + 1)}{2}\right] + x,$$

the equation in question is equivalent to:

$x(x + 1) + x = 15$ and x is a positive integer.

$x^2 + 2x - 15 = 0$

$(x + 5)(x - 3) = 0$

$x = 3$; 3 is the root.

***

Review Exercises

A.

1. Property is that expressed by:

$Sq \ldots > T \ldots$

(a) 2 has the property.

$Sq_2 = Sq_1 + (2 \cdot 1 + 1) = 1 + 3 = 4$

$T_2 = T_1 + (1 + 1) = 1 + 2 = 3$; and

$4 > 3$.

(continued on T. C. 46F)

Alternative method for Exercise 7:

$$\sum_{x=-3}^{18} (x + 5) = \sum_{x=1}^{22}(x - 4 + 5) = \sum_{x=1}^{22} (x + 1)$$

$$= \frac{22(23)}{2} + 22$$

$$= 253 + 22$$

$$= 275$$

8. $$\sum_{k=-5}^{12} (11 - 2x) = (11 - 2x)(12 + 5 + 1) \quad \text{[Ex. 10(b), page 1-33]}$$

$$= 18(11 - 2x)$$

9. The assertion to be proved states that the sum of the first 100 odd numbers and the first 100 even numbers is the sum of the first 200 counting numbers. Students who recognize this need do no more!

Mechanical procedure:

$$\sum_{w=1}^{100} (2w - 1) + \sum_{w=1}^{100} 2w = \sum_{w=1}^{100}(4w - 1)$$

$$= 4\,\frac{100(101)}{2} - 100$$

$$= 100(202 - 1)$$

$$= 20100;$$

$$\sum_{w=1}^{200} = \frac{200(201)}{2}$$

$$= 20100.$$

Alternative method for Exercise 4:

$$\sum_{x=4}^{20} \left(\frac{1}{2} x - 5\right) = \sum_{x=1}^{17} \left[\frac{1}{2}(x + 3) - 5\right] = \sum_{x=1}^{17} \left(\frac{1}{2} x - \frac{7}{2}\right)$$

$$= \frac{1}{2}\left[\frac{17(18)}{2} - 7(17)\right]$$

$$= \frac{17}{2}\left[9 - 7\right]$$

$$= 17$$

5. $\sum_{z=1}^{50} (4 - 3z) + \sum_{z=1}^{50} (3z - 4) = \sum_{z=1}^{50} 0 = 0(50) = 0$

6. Change the apparent variable 't' to 'u' in the second term and proceed as in Exercise 5.

7. $\sum_{x=-3}^{18} (x + 5) = \sum_{x=-3}^{18} x + \sum_{x=-3}^{18} 5$

$$= \left[\sum_{x=-3}^{0} x + \sum_{x=1}^{18} x\right] + 5(18 + 3 + 1)$$

$$= \left[-3 + (-2) + (-1) + \frac{18(19)}{2}\right] + 110$$

$$= 165 + 110$$

$$= 275$$

***

Alternative method for Exercise 3:

$$\sum_{x=8}^{19} (2 + 4x) = \sum_{x+7=8}^{x+7=19} [2 + 4(x+7)]$$

$$= \sum_{x=1}^{12} (30 + 4x)$$

$$= \sum_{x=1}^{12} 30 + 4\sum_{x=1}^{12} x$$

$$= 30(12) + 2(12)(13)$$

$$= 360 + 312$$

$$= 672$$

4. Note first that

$$\sum_{x=4}^{15} (\tfrac{1}{2}x - 5) + \sum_{x=16}^{20} (\tfrac{1}{2}x - 5) = \sum_{x=4}^{20} (\tfrac{1}{2}x - 5).$$

Then,

$$\sum_{x=4}^{20} (\tfrac{1}{2}x - 5) = \frac{1}{2}\sum_{x=4}^{20} x - \sum_{x=4}^{20} 5$$

$$= \frac{1}{2}\left[\sum_{x=1}^{20} x - \sum_{x=1}^{3} x\right] - 5(20 - 4 + 1)$$

$$= \frac{1}{2}\left[\frac{20(21)}{2} - \frac{3(4)}{2}\right] - 85$$

$$= 102 - 85$$

$$= 17.$$

C.

1. $$\sum_{x=1}^{17}(3-2x) = \sum_{x=1}^{17} 3 - 2\sum_{x=1}^{17}$$

$$= 3(17) - 2\,\frac{17(17+1)}{2}$$

$$= 51 - 306$$

$$= -255$$

2. $$\sum_{y=1}^{100}(2y+7) = 2\sum_{y=1}^{100} y + \sum_{y=1}^{100} 7$$

$$= 2\,\frac{100(101)}{2} + 7(100)$$

$$= 10100 + 700$$

$$= 10800$$

3. $$\sum_{x=8}^{19}(2+4x) = \sum_{x=1}^{19}(2+4x) - \sum_{x=1}^{7}(2+4x)$$

$$= \sum_{x=1}^{19} 2 + 4\sum_{x=1}^{19} x - \sum_{x=1}^{7} 2 - 4\sum_{x=1}^{7} x$$

$$= 2(19) + 2(19)(20) - 2(7) - 2(7)(8)$$

$$= 2(19)(21) - 2(7)(9)$$

$$= 2(399 - 63)$$

$$= 672$$

***

1. $\sum_{x=1}^{17} (3 - 2x)$

2. $\sum_{y=1}^{100} (2y + 7)$

3. $\sum_{x=8}^{19} (2 + 4x)$

4. $\sum_{x=4}^{15} (\frac{1}{2}x - 5) + \sum_{x=16}^{20} (\frac{1}{2}x - 5)$

5. $\sum_{z=1}^{50} (4 - 3z) + \sum_{z=1}^{50} (3z - 4)$

6. $\sum_{u=1}^{98} (157 - 63u) + \sum_{t=1}^{98} (63t - 157)$

7. $\sum_{x=-3}^{18} (x + 5)$

8. $\sum_{k=-5}^{12} (11 - 2x)$

***

9. Prove: $\sum_{w=1}^{100} (2w - 1) + \sum_{w=1}^{100} 2w = \sum_{w=1}^{200} w$

10. Solve the equation: $\sum_{k=1}^{x} (2k + 1) = 15$

REVIEW EXERCISES

A. Use mathematical induction to prove each of the following generalizations if it is true; give a counter-example if it is false.

1. For every counting number $n > 1$, the nth square number (Sq_n) is greater than the nth triangular number (T_n).

2. For every counting number $n > 1$, $Sq_n - T_n \geq n$.

(continued on next page)

[1-47]

$$S_{s+t} = \frac{s+t}{2}\left[2a + (s+t-1)d\right]$$

$$= \frac{s+t}{2}\left[(1-t-s)d + (s+t-1)d\right]$$

$$= \frac{s+t}{2}[\,0\,]$$

$$= 0$$

[Students may be puzzled over this theorem in that it is difficult to imagine arithmetic progressions which behave according to the conditions of the theorem. However, the equation '$a = \left[\frac{1-t-s}{2}\right]d$' provides an unlimited number of illustrations.]

6. $S_{10} = 5(2a + 9d)$

$S_6 = 3(2a + 5d)$

Since $S_{10} = S_6$,

$$10a + 45d = 6a + 15d$$
$$4a = -30d$$
$$a = -\frac{15d}{2}.$$

$$S_{16} = 8(2a + 15d)$$
$$= 8\left[2\left(\frac{-15d}{2}\right) + 15d\right]$$
$$= 8[-15d + 15d]$$
$$= 0$$

7. Exercise 6 is a special case of the theorem in Exercise 7.

$$S_s = \frac{s}{2}[2a + (s - 1)d]$$
$$S_t = \frac{t}{2}[2a + (t - 1)d]$$

Since $S_s = S_t$,

$$sa + \frac{s(s - 1)}{2} d = ta + \frac{t(t - 1)}{2} d.$$

$$(s - t)a = \frac{t(t - 1) - s(s - 1)}{2} d$$
$$= \frac{t^2 - t - s^2 + s}{2} d$$
$$= \frac{(t - s)(t + s) - (t - s)}{2} d$$
$$= \frac{(t - s)(t + s - 1)}{2} d;$$
$$a = \frac{(1 - t - s)d}{2} \qquad [s - t \neq 0].$$

[1-47]

Therefore, in view of (a) and (b), by the principle of mathematical induction for real integers ≥ -3, the property in question holds for every real integer ≥ -3.

***

B.

1. $\frac{16}{2}[14 + 15(-8)] = 8[14 - 120] = -848$

2. $8[8\sqrt{2} + 15(-\sqrt{2})] = -56\sqrt{2}$

3. $$\begin{array}{r} a + 4d = 9 \\ \underline{a + 7d = 10} \\ 3d = 1 \end{array}$$

$$d = \frac{1}{3}, \quad a = \frac{23}{3}$$

$$S_{16} = 8\left[\frac{46}{3} + 15\left(\frac{1}{3}\right)\right] = \frac{488}{3}$$

4. $$\frac{10}{2}\left[2a + 9d\right] = 5$$

$$\frac{12}{2}\left[2a + 11d\right] = 13$$

$$-2d = 1 - \frac{13}{6}$$

$$d = \frac{7}{12}, \quad a = -\frac{17}{8}$$

$$S_{16} = 8\left[-\frac{17}{4} + \frac{35}{4}\right] = 36$$

5. $$\begin{aligned} 50[2a + 99(5)] &= 100 \\ 2a + 495 &= 2 \\ 2a &= -493 \end{aligned}$$

$$S_{16} = 8\left[-493 + 15(5)\right] = 8(-418) = -3344$$

(a) -3 has the property.

$$\sum_{k=-3}^{-3} 2k = 2(-3) = -6; \text{ and } (-3+4)(-3-3) = -6.$$

(b) The property is hereditary.

Suppose that, for a given $y \geq -3$,

$$\sum_{k=-3}^{y} 2k = (y+4)(y-3).$$

Then, for this y,

$$\sum_{k=-3}^{y+1} 2k = \sum_{k=-3}^{y} 2y + [2(y+1)]$$

$$= (y+4)(y-3) + 2(y+1)$$

$$= y^2 + y - 12 + 2y + 2$$

$$= y^2 + 3y - 10$$

$$= (y+5)(y-2)$$

$$= [(y+1)+4][(y+1)-3].$$

Hence, for every $y \geq -3$,

if $\sum_{k=-3}^{y} 2k = (y+4)(y-3)$

then $\sum_{k=-3}^{y+1} 2k = [(y+1)+4][(y+1)-3].$

(continued on T. C. 47E)

$$= \frac{1}{3}n[2(n+1)(2n+1) - 6(n+1) + 3]$$

$$= \frac{1}{3}n[4n^2 + 6n + 2 - 6n - 6 + 3]$$

$$= \frac{1}{3}n(4n^2 - 1).$$

***

4. Property is that expressed by:

$$\sum_{k=1}^{\cdots} k^3 = \frac{1}{4}(\ldots)^2(\ldots + 2)^2.$$

1 is a counter-example because

$$\sum_{k=1}^{1} k^3 = 1^3 = 1 \text{ and } \frac{1}{4}(1)^2(1+2)^2 = \frac{9}{4}.$$

***

The generalization that for every real integer $x > 0$,

$$\sum_{k=1}^{x} k^2 = \frac{1}{4}x^2(x+1)^2$$

can be proved by mathematical induction.

***

5. Property is that expressed by:

$$\sum_{k=-3}^{\cdots} 2k = (\ldots + 4)(\ldots - 3).$$

(continued on T. C. 47D)

$$= \frac{(2q + 1)(2q + 3)(q + 1)}{3}$$

$$= \frac{1}{3}(q + 1) [\{2(q + 1) - 1\} \{2(q + 1) + 1\}]$$

$$= \frac{1}{3}(q + 1)[4 (q + 1)^2 - 1] .$$

Hence, for every q,

$$\text{if } \sum_{k=1}^{q} (2k - 1)^2 = \frac{1}{3}q(4q^2 - 1)$$

$$\text{then } \sum_{k=1}^{q+1} (2k - 1)^2 = \frac{1}{3}(q + 1)[4(q + 1)^2 - 1].$$

Therefore, in view of (a) and (b), by the principle of mathematical induction for counting numbers, the property in question holds for every counting number.

***

The generalization in Exercise 3 can be proved, without using mathematical induction directly, by using the results of earlier exercises and theorems. It is instructive to have students go through an alternative proof such as the following:

$$\sum_{k=1}^{n} (2k - 1)^2 = \sum_{k=1}^{n} (4k^2 - 4k + 1)$$

$$= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= \frac{2}{3}n(n + 1)(2n + 1) - 2n(n + 1) + n$$

(continued on T. C. 47C)

3. Property is that expressed by:

$$\sum_{k=1}^{\cdots} (2k - 1)^2 = \frac{1}{3}(\ldots)[4(\ldots)^2 - 1].$$

(a) <u>1 has the property.</u>

$$\sum_{k=1}^{1}(2k - 1)^2 = (2 \cdot 1 - 1)^2 = 1; \text{ and}$$

$$\frac{1}{3}(1)(4 \cdot 1^2 - 1) = 1.$$

(b) <u>The property is hereditary.</u>

Suppose that, for a given q,

$$\sum_{k=1}^{q} (2k - 1)^2 = \frac{1}{3}q(4q^2 - 1).$$

Then, for this q,

$$\begin{aligned}\sum_{k=1}^{q+1} (2k - 1)^2 &= \sum_{k=1}^{q} (2k - 1)^2 + [2(q + 1) - 1]^2 \\ &= \frac{1}{3}q(4q^2 - 1) + (2q + 1)^2 \\ &= \frac{1}{3}q(2q + 1)(2q - 1) + (2q + 1)^2 \\ &= (2q + 1)\left[\frac{q}{3}(2q - 1) + (2q + 1)\right] \\ &= \frac{2q + 1}{3}(2q^2 - q + 6q + 3) \\ &= \frac{2q + 1}{3}(2q^2 + 5q + 3)\end{aligned}$$

(continued on T. C. 47B)

3. For every counting number n,

$$\sum_{k=1}^{n}(2k - 1)^2 = \frac{1}{3}n(4n^2 - 1).$$

4. For every real integer $x > 0$,

$$\sum_{k=1}^{x}k^3 = \frac{1}{4}x^2(x + 2)^2.$$

5. For every real integer $x \geq -3$,

$$\sum_{k=-3}^{x} 2k = (x + 4)(x - 3).$$

B. Find the sum of the first 16 terms of the arithmetic progression

1. whose first term is 7 and whose common difference is -8.
2. whose first term is $4\sqrt{2}$ and whose common difference is $-\sqrt{2}$.
3. whose fifth term is 9 and whose eighth term is 10.
4. the sum of whose first 10 terms is 5 and the sum of whose first 12 terms is 13.
5. whose common difference is 5 and the sum of whose first 100 terms is 100.
6. the sum of whose first 10 terms is equal to the sum of whose first 6 terms.

***

7. Prove: For all counting numbers s and t, if $s \neq t$ and if the sum of the first s terms of an arithmetic progression is equal to the sum of the first t terms of that progression then the sum of the first $s + t$ terms is 0.

·48]

In this section of the review, our aim is to get students to formulate for themselves the multiplication and addition rules for counting number exponents. These rules do not need to be stated; students will encounter no difficulty in discovering them.

REVIEW OF WORK WITH COUNTING NUMBER EXPONENTS

In earlier courses you learned such facts as:

(1) '4^3' is an abbreviation for '$4 \times 4 \times 4$'

(2) $3^5 \times 3^2 = 3^7$

(3) $\frac{5^4}{5^2} = 5^2$

(4) For every x, $x^2 x^4 = x^6$.

(5) For every x and y, $(x^2 y^3)^4 = x^8 y^{12}$.

(6) For every x and y, if $y \neq 0$ then $\frac{x^2 y^5}{y^2} = x^2 y^3$.

Statement (1) indicates that you consider the '3' in '4^3' as a name for a <u>counting number</u> because it tells the number of factors in '$4 \times 4 \times 4$'. However, the '4' in '4^3' may be a name for a real number, or a name for a rational number, etc. In the exercises which follow we shall continue to regard exponent symbols as names for counting numbers and regard the symbols to which they are attached as names for real numbers or, as the case may be, as pronumerals whose domain is the set of all real numbers. In the next unit we shall deal with real number exponents.

Consider the expression:

$$4^3$$

We know that '4^3' is a name for 64 because '4^3' is an abbreviation for '$4 \times 4 \times 4$'. They symbol '4^3' is called an <u>exponential</u>; the number 4^3 is called the <u>third power of 4</u>. When 64 is named by '4^3', we say that

with respect to the <u>base</u> 4,
the <u>exponent</u> of 64 is 3.

[With respect to the base 2, the exponent of 64 is 6.]
Note that exponentials are symbols and that powers, bases, and exponents are numbers.

Study each of the following statements until you are sure you understand them.

(a) 8 is the third power of 2

(b) '2' is the exponent symbol in the exponential '3^2'

(c) 4^2 is the second power of 4

(d) 16 is the fourth power of 2

(e) 4^2 is the fourth power of 2

(f) 81 is the second power of 9

(g) 81 is the fourth power of 3

(h) 2×50 is the second power of 10

(i) $(3^2)^5$ is the second power of 3^5

A. Write the simplest name which is not an exponential for the power given in each of the following exercises.

Sample. $(3.5)^3$

Solution. $(3.5)^3 = 3.5 \times 3.5 \times 3.5$
$= 42.875$

1. 2^2	2. 3^2	3. $(2.5)^2$
4. 2^3	5. 3^4	6. 10^3
7. 15^2	8. 2.3^3	9. 1.1^4
10. $(-2)^2$	11. $(-2)^3$	12. $(-1)^3$
13. $(-1)^4$	14. $(-1)^{15}$	15. $(-2)^7$
16. 3^5	17. 3^6	18. 5^5
19. 3^7	20. 7^3	21. 0^{17}
22. 0^{1000}	23. 1.414^2	24. $(\sqrt{2})^2$

B. Simplify these exponentials. Leave answers in exponential form.

Sample 1. $4^3 \times 4^5$

Solution.
$$\begin{aligned} 4^3 \times 4^5 &= (4 \times 4 \times 4) \times (4 \times 4 \times 4 \times 4 \times 4) \\ &= 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \\ &= 4^8. \end{aligned}$$

Sample 2. $x^2y^3x^3y^5$

Solution. For every x and y,
$$\begin{aligned} x^2y^3x^3y^5 &= x^2x^3y^3y^5 \\ &= xx \times xxx \times yyy \times yyyyy \\ &= xxxxx \times yyyyyyyy \\ &= x^5y^8. \end{aligned}$$

Sample 3. $5.7(5.7r)(5.7r)^2r^3$

Solution. For every r,
$$\begin{aligned} &5.7(5.7r)(5.7r)^2r^3 \\ = \; &5.7(5.7r)(5.7r)(5.7r)rrr \\ = \; &(5.7 \times 5.7 \times 5.7 \times 5.7)(rrrrrr) \\ = \; &5.7^4r^6. \end{aligned}$$

1. $2^3 2$	2. $2^3 2^2$	3. x^3x^2
4. $4^2 \cdot 5 \cdot 4^3$	5. x^2yx^3	6. $6^5 6^9$
7. $3^4 3^9$	8. d^4d^9	9. $19^5 19^{10}$
10. y^5y^9	11. $3^7 2^3 3^2 2^6$	12. $x^7z^3x^2z^5$
13. $3^{17} 3^{17}$	14. $3^{17} 3^{17} 3^{17}$	15. $5^2 9^5 5^9 9$
16. $r^2 9^3 r^9 9$	17. $3^{106} 3^{207}$	18. $x^{674}y^{65}x^{186}$
19. $5^2 100^3 6^3 100^2$	20. $a^2 100^4 b^6 100^3$	21. $0^5 0^{10} 0^6 0^8 0^{179}$
22. $(5a)^2(5ab)^3(ab)^2$	23. $8^5 6 3^9 0^{13} 8^{14}$	

(continued on next page)

24. $(3.14)^2(2.28)^3(3.14)^7$

25. $x^2y^2x^4y^3z^2x^7$

26. $ab^2c^3a^3b^2c$

27. $(2.9)t(2.9t)^2$

28. $(4.6p)(4.6p)(4.6p)^3$

29. $(6.7mn)(6.7mn)^2(6.7mn)(6.7mn)^3$

30. $(2a)(4c)(3a)^2(2c)(4a)^3(2c^2)^5$

C. Partition the set of expressions which follow by using the relation STANDS FOR THE SAME NUMBER AS.

5^3 fourth power of 5 $(4^2)^3$ $4^3 \times 4$ 4^4

5×5^3 4^4 5×5^3 $50^3 \div 10^3$ $(2^2)^3$ $4 \times 4^2 \times 4$

5 cubed $(3^2)^3$ fifth power of 3 $4 \times 4 \times 4 \times 4 \times 4$

$5^8 \div 5^5$ $5^2 \times 5^2$ 5×5^2 the cube of 5 8^2

$4^2 \times 4^2$ $(5^2)^2$ $(5^2 \times 5^3) \div 5$ 4^3 5^4 $4^5 \div 4$

$5^8 \div 5^4$ third power of 5 5^2 3^5 $(3^3)^2$

$(5 \times 5) \times 5$ fifth power of 4 $(2^4)^2$ 4 to the fourth power

$4^8 \div 4^5$ 125 $2 \times 2 \times 2^4$ 2^8 $8^3 \div 4^3$

fourth power of 4 $3^2[(3 \times 3) \times 3]$

D. Partition the set of expressions which follow by using the relation IS EQUIVALENT TO. [Recall that two expressions which contain pronumerals are equivalent if they give a pair of expressions for the same number upon each replacement of the pronumerals by numerals.]

UNIVERSITY OF
ILLINOIS LIBRARY
AT URBANA-CHAMPAIGN

52]

E. We introduce the student to a recursive definition of exponentials. This type of definition will be met repeatedly in the next unit. Note that we do not define an exponential with exponent symbol '1' at this point. The student is asked to do so in Exercise 7 on page 1-53.

$(xxx)^3$	$(x^2)^4$	$(y^4x)^2x$	x^3y^6	x^9	$x \cdot x^5$
$(x^4)^2$	x^6	x^3y^5y	$x^6 \times x^3$	third power of x^2	
$(x^2)^3$	x^2y^5	$x^3(y^2)^4$	$x \cdot x^8$	$x(x^2)^3x$	x^4x^2
$(x^2)^3$	$(x^5)^2$	$x^3x^3x^3$	third power of xy^2		
$xxxyyy(y)^5$	$x^3y^3y^2$	$(x^2x^2)^2$	$x(x^2)^4$	sixth power of x	
x^3x^8	ninth power of x	$(x^3y^2)^4$	$(y^2)^3x^3y^2$		
$x^5(x^2)^4$	$x^2 \times x^3$	$(x^6y^4)^2$	x to the ninth power		
$(x^3)^4y^8$	$(xy^2)^3$	$(xy)^3y^3$	x^2x^7	$(xy)^3y^5$	
$x^2x^3x^4$	$x(xy)^2(y^3)^2$	x^3x^3	$x^2y^4xy^4$		

E. We could define 'a^n' as an abbreviation for 'the product of n factors a'. Instead of this we shall state the following recursive definition.

> For every real number a,
>
> $$a^2 = a \cdot a$$
>
> and, for every counting number $n \geq 2$,
>
> $$a^{n+1} = a^n \cdot a$$

Sample. Prove: $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

Solution. $3^5 = 3^4 \cdot 3$ [Why?]

$$= (3^3 \cdot 3) \cdot 3$$

$$= [(3^2 \cdot 3) \cdot 3] \cdot 3$$

$$= [(3 \cdot 3) \cdot 3] \cdot 3 \cdot 3$$

$$= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3.$$

Hence, for every k,

$$\text{if } (-1)^{2k} = 1$$

$$\text{then } (-1)^{2(k+1)} = 1.$$

Therefore, in view of (a) and (b), by the principle of mathematical induction for counting numbers, the property in question holds for every counting number.

6. For every $n > 1$, by the recursive definition,

$$(-1)^{2n} = (-1)^{2n-1}(-1).$$

But, by the theorem in Exercise 5, for every n,

$$(-1)^{2n} = 1.$$

Hence, for every $n > 1$,

$$1 = (-1)^{2n-1}(-1)$$

or $$\frac{1}{-1} = (-1)^{2n-1}$$

or $$-1 = (-1)^{2n-1}.$$

[The theorem in Exercise 6 can also be proven using mathematical induction.]

7. (a) A reasonable definition:

'$(-1)^1$' means -1.

(b) For every real number a,

$$a^1 = a$$

and, for every counting number n,

$$a^{n+1} = a^n \cdot a.$$

Urge students to use the recursive definition in proving the statements 1 - 4 rather than use their conjectured addition rule.

***

5. Property is that expressed by:

$$(-1)^{2 \cdot \cdots} = 1$$

(a) 1 has the property.

By the recursive definition,

$$(-1)^2 = (-1)(-1) = 1.$$

(b) The property is hereditary.

Suppose that, for a given k,

$$(-1)^{2k} = 1.$$

Then, for this k,

$$\begin{aligned}
(-1)^{2(k+1)} &= (-1)^{(2k+1)+1} \\
&= (-1)^{2k+1} \cdot (-1) \\
&= [(-1)^{2k} (-1)] [-1] \qquad \text{[recursive definition]} \\
&= (-1)^{2k} [(-1)(-1)] \\
&= (-1)^{2k} \\
&= 1. \qquad \text{[inductive hypothesis]}
\end{aligned}$$

As exercises in the use of this recursive definition prove each of the following:

1. $(-2)^5 = -32$
2. $3^2 \cdot 3^3 = 3^5$
3. $(-1)^6 = 1$
4. $(\sqrt{2})^5 = 4\sqrt{2}$

* * *

5. Prove that for every counting number n, $(-1)^{2n} = 1$.
   [Hint: Use mathematical induction.]

6. Prove that for every counting number n > 1, $(-1)^{2n-1} = -1$.

7. (a) Suggest a definition for '$(-1)^1$' so that the theorem in Exercise 6 can be extended to every counting number.
   (b) In light of your answer to (a) frame a reasonable recursive definition of exponentiation which allows every counting number to be an exponent (and every real number to be a base).

JUN 27 1973

UNIVERSITY OF ILLINOIS-URBANA
3 0112 084224622