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COMMITTEE ON SCHOOL
MATHEMATICS

HIGH SCHOOL MATHEMATICS
TEACHERS' EDITION

HIGH SCHOOL MATHEMATICS

SECOND COURSE

TEACHERS' EDITION

UNIT ONE 1957-58

INTRODUCTION TO
DEDUCTIVE GEOMETRY

UNIVERSITY OF ILLINOIS
COMMITTEE ON
SCHOOL MATHEMATICS

5

Unit 1

INTRODUCTION TO DEDUCTIVE GEOMETRY

Introduction

UICSM
University High School
Urbana, Illinois
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ERRATA

Unit 1, SECOND COURSE   Introduction to Deductive Geometry

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Table of Contents	21	Theorems 2 and 3
	1 bottom	on parallels--4 and 5
Introduction	24	in proving many
	3b	ability in deducing
1-1	6, 7	we drew a part...we emphasized some
1-3	Delete Exercises (7), (8), and (10).	
1-4	Exercise 8	Give examples of a set...least number. Give examples of a set...nor a least. Illustrate your answers with pictures like those in Part A.
1-5	3b	picture (1-1)-correspondences
	2b	page 1-1] which conform with
1-6	10b, 11b	A correspondence which conforms
	12b	with the two boxed statements [above, and on page 1-1] is called <u>a</u> <u>uniform</u> <u>scale</u>.
	5b	Use a picture of a uniform
1-10	13	<u>plane</u>, and each ordered pair of real numbers is called <u>a</u> <u>point</u> (<u>of</u> <u>the</u> <u>number</u> <u>plane</u>).
1-11	1	$\sqrt{(5 - 2)^2 + (7 - 3)^2}$
	2	$\sqrt{(3)^2 + (4)^2}$
1-14	Delete present Exercise *7 and replace with: *7. If d((6, 9), (x, y)) = 4, what is d((3, 5), (x, y))?	
1-17	13 and 1b	Interchange 'a' and 'c', 'b' and 'd', 'm' and 'p', and 'n' and 'q'.
1-19	4b	[<u>The</u> set

Page	Line	
1-22	7	I: (-2, 135411)
1-23	Write 'first component axis' at right end of horizontal line, and 'second component axis' at upper end of vertical line as in diagram on page 1-9.	
1-25	10	3(x) +
	15	the only number which
1-29	Write '(continued on next page)' at bottom.	
1-30	3, 4	enclosed in red boxes, and theorems
	12b	<u>Sample</u>.
1-31	11b	For each of the following equations,
	3b	Write five equations
1-32	1, 2	Find equations for
	19-23	'b' in 'a(x - 8) + b(y - 3) = 0' and transform. Here is how to obtain three linear equations.

$$4(x - 8) + 5(y - 3) = 0 \longrightarrow 4x + 5y - 47 = 0$$

$$\sqrt{2}(x - 8) - 5\pi(y - 3) = 0 \longrightarrow \sqrt{2}x - 5\pi y + (15\pi - 8\sqrt{2}) = 0$$

$$(x - 8) = 0 \longrightarrow x - 8 = 0$$

1-34	11	Hence, one equation
	15	= 0.
	16	A linear equation which has the same locus is:
	19	whose locus contains
1-35	following 4	<u>Two-point form of equations for lines</u>
	9b, 8b	Delete: Just for ... = 0'.
1-42	9	statement about his model
	12	of his two guesses
	14	of Carl's model. And ... false for Carl's

Page	Line	
1-47	9b, 8b	Equivalent equations are equations with the same locus.
	7b	The question facing
	6b, 5b	prove that an equation of each line which contains (3, 4) and
	3b	m is any line
1-48	7b, 6b	(8, 13), an equation for m can be transformed by the multiplication transformation principle into
1-49	Write at bottom:	Since each of these three statements can be deduced from
1-50	Write at top:	postulates III and IV, let us agree to call the first <u>Theorem 2</u>.
	7	to it as <u>Theorem 3</u>.
1-52	9, 10	Two equivalent equations have the same locus, and two equations which
	12, 13	In Exercise 5 on page 1-49, you showed that, for
1-53	2	Theorem 3
1-54	12	$p + 2q - 8 = 0$
	15	$-5p + 2q + 1 = 0$
1-57	2	systems of equations
1-65	2	[Theorem 3,
	7b	Part F on page 1-52
1-69	12b	systems of equations
1-70	3	are given an equation
	4	three linear equations
1-61	6	i.p.t. (y, x) then (y, x) i.p.t.
	7	If (x, y) . . . i.p.t. (x, y).

INTRODUCTION

During most of your mathematics course this year you will study geometry. The study of geometry is a very ancient one; many of the facts of geometry which almost everyone knows were discovered by the Egyptians, the Babylonians, and the Greeks. Euclid and other Greek mathematicians collected the geometric knowledge of their day and attempted to organize it into what is called a postulational system. In a postulational system one shows how certain "facts" can be deduced from a few statements which are called postulates.

Euclid's system of geometry served for hundreds of years as a model for organizing other bodies of knowledge, non-mathematical as well as mathematical. Within the last two hundred years mathematicians have come to understand better just what a postulational system is. They noticed certain flaws in Euclid's system and constructed better postulational systems for geometry and for other branches of mathematics. Some of these systems are rather complicated and you would need much mathematical training in order to understand them.

In this course you will study a postulational system based partly on ideas from Euclid and partly on ideas from mathematicians who worked during more recent times. You will see how the knowledge of points and lines which you acquired in your graphing work is organized into a system. The system will contain postulates based on this knowledge. You will use these postulates in deriving many other statements, some which you already know and some which will be new to you. The ability to derive or deduce statements from others is needed in all branches of mathematics, as well as in all other fields of knowledge. Studying a postulational system of geometry provides a good opportunity for you to acquire this ability. So, to give you practice in deduction you will at times be asked to derive statements which you may already know. At other times you will use this ability in deriving statements which are new to you. In fact, some of these new statements may even be discovered by you.

TEACHERS COMMENTARY

Introduction

In this course we have a development of Euclidean geometry which is as rigorous as, for example, that due to Hilbert, and yet which is, we believe, accessible to students who have mastered FIRST COURSE.* The difficulties in the way of such a development are of two kinds. In the first place, when constructing rigorous proofs, one must pay attention to various "details" which, in conventional treatments of geometry for high school students, either are taken to be "obvious from the picture", or are completely overlooked. In the second place, one must

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*The desirability (and the difficulty) of formulating such a development was pointed out by O. Veblen in his monograph "The Foundations of Geometry" which appeared in the collection Monographs on Topics of Modern Mathematics, edited by J. W. A. Young [Dover reprint, pp. 1-51]:

> A book giving a complete and rigorous treatment of elementary geometry would be a most important influence in improving the teaching of the most ancient and perfect of sciences. Such a book could rarely, if ever, be used in the classroom, but if it were in the hands of the teachers it would serve to keep before them in something like its actual form the structure of which they are trying to give their students a first glimpse. [p. 48, footnote.]

Unlike Veblen, we believe that it is possible to present to high school students "a ... rigorous treatment of elementary geometry". While to this extent disagreeing with one of the most distinguished geometers of our times, we are glad to acknowledge our great debt to him for his work in the foundations of geometry, without which the present development would have been impossible.

make clear to the students the nature of geometry as a pure deductive theory which, in itself, has only logical structure (but no content). Such a theory is actually nothing more than a class of uninterpreted (and, hence, strictly meaningless) sentences.* It obtains a content only when its primitive terms are interpreted according to some arbitrarily chosen convention. For example, you will find that the words 'point' and 'line' are taken as primitive terms. These words occur in the theorems of our geometry [our first postulate, on p. 1-28, is: Each line is a set of points, and contains at least two points.], and these theorems become meaningful only when meanings are assigned to the primitive terms. We might decide that 'point' is to mean person and that 'line' is to mean committee. With this interpretation, postulate I becomes a meaningful statement. [The fact that there may be "a committee of one" shows that postulate I becomes a false statement.] When the choice of meanings for the primitive terms is such that all the theorems become <u>true</u> sentences, we say that we have a <u>model</u> for our deductive theory. A theorem-by-theorem check is of course not necessary in order to assure ourselves that we are in possession of a model. It is sufficient that the suggested interpretation of the

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*<u>A deductive theory</u> is a set of sentences which is such that every sentence which is, logically, a consequence of members of the set also belongs to the set. The sentences which are members of the set are called <u>theorems</u>. In exploring the logical structure of a particular deductive theory, one frequently chooses a subset of its theorems of such a nature that every theorem can be deduced from the members of this subset. The latter are then called <u>postulates</u>. A deductive theory, then, determines, and is determined by, its theorems; on the other hand, there is a considerable degree of arbitrariness as to which of its theorems one chooses to take as postulates to form a basis for developing a given deductive theory.

primitive terms be such that each of some set of postulates for the deductive theory becomes a true sentence. [For this reason, one generally tries to choose postulates which are few in number, and simple.]

One reason for studying a pure deductive theory is that each such theory has many models so that, while one is developing the theory, one is exploring many subjects simultaneously. As an example of this, consider the deductive theory which can be based on our first four postulates:

I. Each line is a set of points, and contains at least two points. [page 1-28.]

II. There are three points which do not belong to the same line. [page 1-28.]

III. For each two points, there is a line which contains them. [page 1-36.]

IV. For each two points, there is at most one line which contains them. [page 1-49.]

Most of the words which occur in these four sentences are, for our present purposes, to be thought of as belonging to logic [for example: 'each', 'is', 'set', 'and', 'contains', and 'at least two']. The two words 'point' and 'line' are primitive terms. We see that if we choose to use 'point' to mean an ordered pair of real numbers and 'line' to mean the solution set of a linear equation (in two pronumerals), then I, II, III, and IV become true statements about linear equations and their solutions. On the other hand, if one imagines a "physical plane", one can decide to interpret 'point' as a position on this plane, and 'line' to mean path (in the plane) of a ray of light. Whether the postulates become true sentences under this interpretation is a matter for the physicist to decide, on the basis of experiments with light rays. That he may never be able to decide this with absolute certainty need not concern us here. The fact remains that such conceptual physical models are of help in the study of the real world and form the basis for our application of the theorems of geometry to the solution of "practical" problems.

From postulate II we can, without reference to any model, deduce

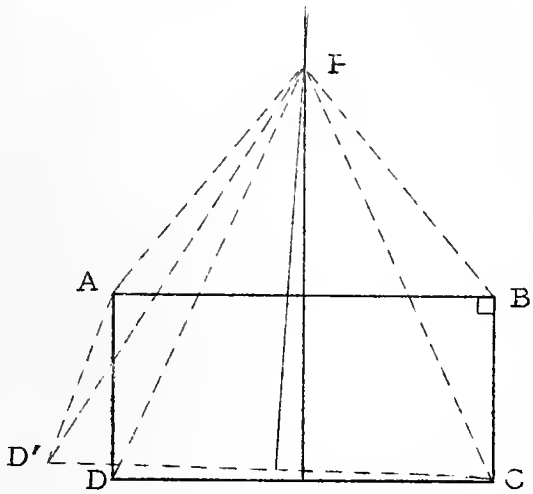
<u>Theorem 1</u>. For each line, there is a point not on the line.

Since this sentence is a consequence of the postulates we know that it will become true if 'point' and 'line' are interpreted, as above, in terms of real numbers and equations. Thus we learn that, for each linear equation there is an ordered pair of real numbers which fails to satisfy the equation. Granted that this is not in itself particularly surprising, still it is of interest to realize that it follows from the fact that there are three ordered pairs of real numbers which are not solutions of any one linear equation. If we believe strongly that the physical interpretation suggested above describes a model for our postulate set, we shall probably feel equally strongly that Theorem 1 assures us that, for each ray of light, there is a position which is not on its path. Again, we may be more struck by the logical connection between this fact and the fact that there are three positions which are not on the path of any single light ray, than we are in the fact expressed by Theorem 1, itself. [We may, of course, find a theorem whose interpretation in the <u>physical</u> "model" can be shown to be a false sentence. This will show us that we were in error in believing that the interpretations of the postulates were true sentences. Such a possibility suggests the process of experimental check of predictions by which physical theories are tested. Since, in the case of the mathematical model, we can derive the interpretations of our postulates from the properties of real numbers, the discovery of a theorem whose interpretation for this model was a false sentence would mean that the logic which we use to deduce theorems from our postulates will result in contradictions when applied in deriving properties of real numbers. This is most unlikely, and if such an event occurred, we should feel very nearly certain that we had blundered in constructing the purported proof for the purported theorem.]

We began by suggesting that the two main obstacles to a rigorous treatment of elementary geometry were, first, that the construction of rigorous proofs requires paying attention to matters which are easily either overlooked or "assumed from the figure", and, second, that appreciation of a rigorous treatment requires some understanding of the nature of a deductive theory. We shall indicate the kind of care which

one must exercise in stating a rigorous proof by presenting a careless "proof" of the statement:

Every right angle is congruent to an obtuse angle.



Let $\angle ABC$ be any right angle, and construct a rectangle ABCD. Choose D′, outside of ▭ ABCD so that the segments AD′ and AD have the same length. The perpendicular bisectors of segments DC and D′C intersect in a point P, as shown in the figure. $\triangle APB$, $\triangle D'PC$, and $\triangle DPC$ are isosceles triangles, so $\triangle AD'P$ and $\triangle BCP$ are congruent. In particular, $\angle PAD'$ and $\angle PBC$ are congruent and, since $\triangle APB$ is isosceles, $\angle PAB$ and $\angle PBA$ are congruent. Since differences of congruent angles are congruent, $\angle BAD'$ and $\angle ABC$ are congruent. But $\angle BAD'$ is an obtuse angle, so $\angle ABC$ is congruent to an obtuse angle.

[Before reading further you may wish to discover the error in this reasoning. A carefully drawn figure will help.]

The error in the supposed proof lies in the tacit assumption that the point B is interior to $\angle PAD'$, just as A <u>is</u> interior to $\angle PBC$. It is on the basis of this assumption that one argues from the congruence of $\angle PAD'$ and $\angle PBC$, and the congruence of $\angle PAB$ and $\angle PBA$, to the conclusion that $\angle BAD'$ and $\angle ABC$ are congruent. A "carefully drawn figure" will show, for example, that A and B are on the same side of the line through P and D′, rather than on opposite sides of this line, as suggested by the figure above. But this should not restore your feeling of satisfaction (if any) with conventional proofs. A proof for a theorem of geometry should show by logically justifiable steps that, the theorem is a consequence

of the postulates. When one introduces into one's reasoning a conclusion drawn only from a picture, whether the picture is the one above, or your "carefully drawn" one, one has departed from this standard of rigor. Without making recourse to the postulates, you have no more justification for introducing into a proof your "correct" conclusion as to the relative position of the points A and B than we had for assuming that B is interior to $\angle BAD'$.

Thus, a set of postulates for a rigorous treatment of geometry must be such that we can deduce from them sentences which assert that, in appropriate circumstances, two points are (or are not) on the same side of a given line. More basically, in the preceding argument we needed to decide whether the intersection of the line through P and D' with the line through A and B was (or was not) between the points A and B. We shall, in fact, take the notion of betweenness, like those of point and line, as primitive (a fourth, and final, such notion will be a weak notion of congruence), and our postulates will contain phrases of the form '... is between ___ and - - -' (abbreviated to '[. . . ___ - - -]'). The first kind of obstacle to developing a treatment of geometry, which is both rigorous and suitable for study by high school students, is illustrated by the difficulty, which we believe we have overcome, of choosing suitable postulates of this sort. The test of whether or not we have succeeded will be the degree of success you have in teaching the course to your students.

We turn now to our method of overcoming the second obstacle, the fact that one must understand the abstract nature of a deductive theory in order to appreciate a rigorous treatment of geometry. It would be futile (or, at best, uneconomical in terms of classroom time) to attempt to give students such an understanding before they have any experience with deductive geometry. Moreover, it is essential that one realizes that, despite the abstractness of deductive theories, the value of a deductive theory derives principally from the importance of its models. It is for these reasons that we begin with a study of the number plane, the set of all ordered pairs of real numbers together with the notion of distance between ordered pairs defined by the usual formula:

$$d\Big((x_0, y_0), (x_1, y_1)\Big) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} .$$

On adopting the usual conventions [p. 1-10] as to how one is to draw pictures of the number plane, one finds that pictures of the solution sets [or: loci] of linear equations are just those pictures which can be drawn by running a pencil along the edge of a ruler, and this, together with the fact that such pencil streaks are customarily called 'straight lines', motivates the decision to define a straight line in the number plane to be the solution set of a linear equation. [Since penciled dots are often called 'points', and, in our pictures of the number plane, represent ordered pairs of real numbers, we decide to define a point in the number plane to be an ordered pair of real numbers.] The fact, easily established by elementary algebra, that the solution set of each linear equation consists of ordered pairs of real numbers and has at least two members can now be stated in terms of 'point and 'line' as postulate I (see above). Other statements concerning points and lines, as these have been defined with reference to the number plane, can also be derived algebraically from properties of real numbers and the definition of distance. Some of these are accepted as postulates, and other such statements are deduced from these, without making use of the number plane interpretations. We also point out [pp. 1-36 through 1-46, and pp. 1-72 through 1-80] that it is possible to give other interpretations of 'point' and 'line', thus obtaining other models of our postulate system. We discuss also [section 1.04] the role of primitive terms and, in illustration of this, introduce the "words" '[... ___ ___]' and '... ≅ ___' to refer to the basic notions of betweenness and congruence.

Our general procedure, then, is to derive algebraically some property of the number plane, state this as a postulate in terms of the notions of point, line, betweenness, and congruence, and then use it, in conjunction with the postulates we have previously obtained in this way, as a basis for proving additional theorems. From time to time we consider alternative interpretations of the undefined terms, thus showing that our deductive theory has numerous models, and emphasizing the fact that proofs must not depend on any properties of a model other than those which have been formulated in the postulates.

In this unit we introduce the first seven of our fifteen postulates, and deduce some of their consequences. The theorems which we (and the students) prove bear little resemblance to those usually met with in

elementary geometry courses but do, for the most part, form a necessary foundation for the proofs of the standard theorems. In succeeding units additional postulates will be introduced and by the end of Unit 2 most of the theorems will have a familiar look.

The derivation of the statements about the number plane, from which the postulates are obtained by abstraction, requires of the students a considerable knowledge of algebra. We consider it one of the great advantages of this course that it furnishes an opportunity for teaching these algebraic techniques in a context which furnishes the student ample opportunities to use them in a non-trivial fashion.

[At this point you may find it helpful to read the SUMMARY, pp. 1-128 through 1-131.]

Three additional comments may help you to orient yourselves in this course. [You will understand them better as you proceed with your study of this and later unit.]

(1) You may have heard Euclidean geometry described as the geometry of ruler and compass. This refers to the fact that Euclid assumed the existence of straight lines and circles. In the light of our development, Euclidean geometry can be described as the geometry of elastic scale and try square. Our postulate VI says, in essence, that one can lay off a scale on a line (but does not prescribe a unit of length; this is where the elasticity comes in), and postulate XIV asserts the existence of right angles (which can be drawn with a try square). [In a later unit, we shall deduce from our postulates the usual theorems concerning circles.]

(2) As in Euclid's geometry (and unlike some conventional treatments of elementary geometry) our deductive theory makes, at first, nothing of the notion of distance. We do make much use of the distance formula in the number plane while establishing those properties which suggest to us our postulates. But these properties themselves involve at most the notion of equality of distances rather than the notion of distance itself; and, although our primitive notion of congruence corresponds

to equality of distances in the number plane we have no primitive notion which corresponds with the concept of distance. However, using our four primitive terms we can define the notion of the ratio of two segments, and, in terms of this we shall eventually define the notion of distance relative to a given unit-segment. Even then, there will be no absolute notion of distance as exists, by definition, in the number plane itself, or as is, in some geometries, taken as a primitive notion.

(3) Our algebraic derivations of the statements from which we abstract our postulates should not be confused with "analytic geometry". Algebraic derivations of statements concerning the number plane are nothing more or less than proofs of theorems of a deductive theory which may be called 'the algebra of real numbers'. A set of postulates for this theory might consist of sentences stating the commutative principle for addition, the distributive principle (for multiplication over addition), etc. [Incidentally, in our development of geometry we assume a prior knowledge of those properties of real numbers which we need to use, just as we assume a knowledge of the rules of logic, including set-theory. (This does not mean that we expect our students to know all this, but merely that we consider this knowledge as prior to, and not a part of deductive geometry.)] Our use of algebraic procedures in obtaining our postulates has as sole purpose the verification of the fact that the number plane furnishes a model for our deductive theory of geometry.

'Analytic geometry', on the other hand, refers to the fact that, from an adequate set of postulates for Euclidean geometry, one can deduce a theorem to the effect that there are (1-1)-correspondences of a certain kind between the set of all points and the set of all ordered pairs of real numbers. The kind of (1-1)-correspondences in question can be described as follows:

If the components of the ordered pair which corresponds with a given point are called the coordinates of the point with respect to the (1-1)-correspondence (or: coordinate system) under discussion, then

(i) the points of each line are just those points whose coordinates satisfy some linear equation,

and

(ii) the relative distance between two points, when the segment whose end-points have the coordinates (0, 0) and (1, 0), respectively, is taken as unit-segment, can be obtained by applying the usual distance formula to the coordinates of the two points.

The existence of such coordinate systems (which, as mentioned above, can be deduced from the postulates of geometry) makes possible the application of algebraic methods in proofs of theorems of geometry. Such proofs are called 'analytic proofs' while those in which coordinate systems are not mentioned are called 'synthetic proofs'. [Thus, the terms 'analytic geometry' and 'synthetic geometry' are misleading. In both cases, the same geometry is under discussion; the distinction is merely in the methods used in constructing proofs.]

In a later unit we shall prove the existence of coordinate systems and then use analytic methods whenever it is advantageous to do so. In fact, one of our most important postulates, VI, asserts the existence of a kind of coordinate system on each <u>line</u> and so allows us some use of analytic methods in the present unit.

Summarizing, in using the number plane to get a <u>model</u> of geometry, we <u>interpret</u> 'point' as an ordered pair of real numbers. The use of analytic methods depends on a <u>theorem</u> of geometry which assures us that the number plane can be "mapped in a nice way" on <u>any</u> model of geometry.*

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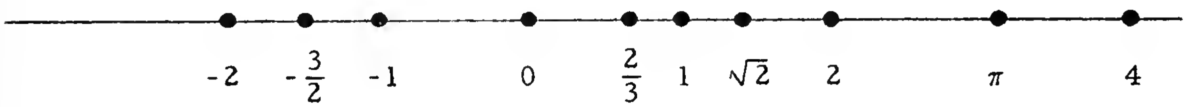
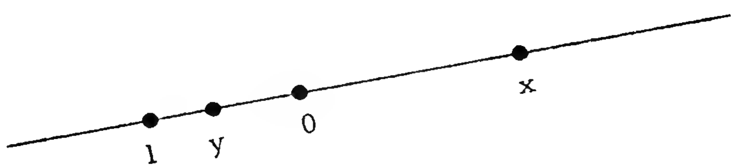
*In classical Greek geometry the words 'analysis' and 'synthesis' were used in discussing proofs, but with meanings entirely different from those given above. [A discussion of these concepts can be found in the Introduction to Heath's translation of "Euclid's Elements" [Dover reprint, vol. 1, page 138 ff.].

1.01 <u>The number line and the number plane</u>.--When we studied directed numbers [or, <u>real numbers</u>, as they are often called], we found it useful to imagine a one-to-one correspondence between the real numbers and the points of a straight line. [That is, we "paired up" all real numbers with all points of a straight line so that each number and each point occurred in one and only one pair.] To aid our imagination we drew a picture of a part of the line. On this we pictured some points of the line by dots, and indicated the imagined (1-1)-correspondence between points and real numbers by writing a numeral for a number below the



dot associated with that number. After choosing the points for 0 and 1, it turned out to be convenient to choose a point for, say, 5 in such a way that the direction from the point for 1 to the point for 5 was the same as the direction from the point for 0 to the point for 1. In general, points were associated with numbers so that the direction from the point for a number to the point for a <u>larger</u> number was the same as the direction from the point for 0 to the point for 1. So, our (1-1)-correspondence between real numbers and points on a line conformed to the following statement.

For each number x and each number y, $x < y$ if and only if the direction from the point for x to the point for y is the same as the direction from the point for 0 to the point for 1.



$x < y$

A. For each of the following pictures fill in the dotted blank with '<' or with '>' according to the statement given in the box above.

(1)

x . . . . . y   y x

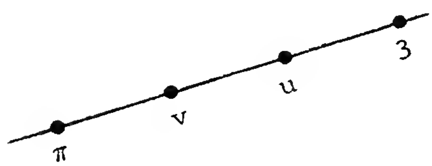
1 . . . . . y   0 x

(2)

x . . . . . y   x 0

1 x   y 1

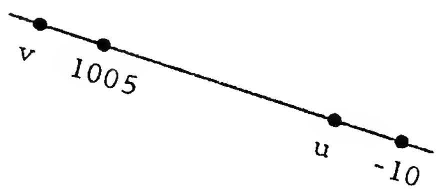
(3)



u v   v 1

0 . . . . . u   1 . . . . . u

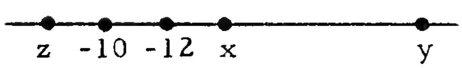
(4)



u v   1005 . . . . -10

v . . . . . -10   u 1005

(5)



z x   y x

-10 . . . . . y   y . . . . . z

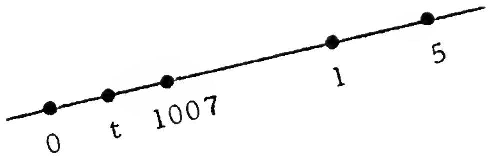
(6)

t . . . . . s   s . . . . . t

1007. . . . . t   1 . . . . . s

(continued on next page)

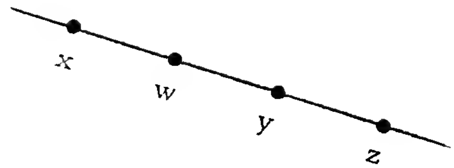
(7)



1 . . . . 5 0 1007

1007 . . . . t 1 1007

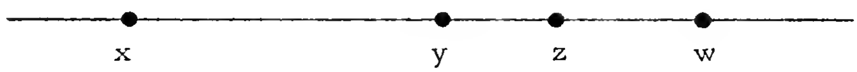
(8)



If x > y then w . . . . z.

If w . . . . y then z . . . . x.

(9)



If $x - y > 0$ then $y - x$ . . . . . 0.

If $|x - y| > 0$ then $|y - x|$ . . . . . 0.

If $x - y > 0$ then $z - w$ . . . . . 0.

If $|x - y| > 0$ then $z - w$ . . . . . 0.

(10) Tell how pictures of lines which conform to the boxed statement on page 1-1 enable us to tell at a glance which of two directed numbers is the larger.

B. When you are working only with whole numbers [or, integers] it makes sense to talk about the next larger integer. The integer next larger than 78 is 79; the integer next larger than -6 is -5. But when you are working with real numbers, it does not make sense to talk about the next real number larger than, say, 78.

1. What integer is "half-way" between the integers 4 and 8? Between the integers 4 and 9? Between the integers -7 and 3?

2. What real number is "half-way" between 4 and 7? Between 4 and $\frac{9}{2}$? Between 4 and 4.1? Between 3.71 and 5.29?

3. For each real number a and each real number b, what number is "half-way" between a and b?

4. Think of all the real numbers less than 2. What is the largest such number? What is the smallest such number?

5. Suppose someone claims he knows the largest real number less than 2. How would you prove to him that there is a real number larger than his and which is also less than 2?

6. Suppose someone claims that he knows the smallest real number less than 2. How would you prove to him that there is a real number smaller than his?

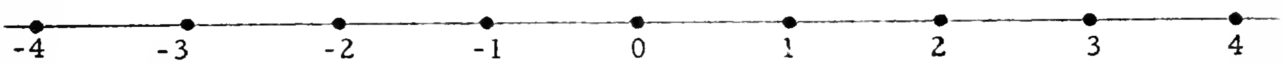
7. Think of the set of numbers consisting of $4\frac{1}{2}$ and all the real numbers larger than $4\frac{1}{2}$. What is the smallest number in that set?

8. Describe a set of numbers which has a largest number and a least number. Describe a set which has a largest and no least. A set with a least and no largest. A set with neither a largest nor a least.

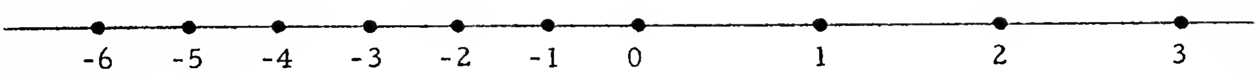
DISTANCE AND PAPER-DISTANCE

Here are two pictures which satisfy the boxed statement on page 1-1.

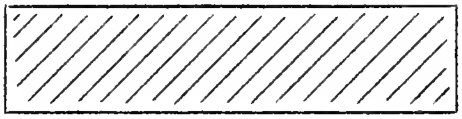
(I)



(II)



If you take a strip of paper as long as:



and lay it along picture (I), the upper corners give you two numbers. Some of these pairs of numbers are

-3 and 0,	-1 and 2,	4 and 1,
$3\frac{1}{2}$ and $\frac{1}{2}$,	$-\frac{3}{4}$ and $-3\frac{3}{4}$,	$-1\frac{1}{2}$ and $1\frac{1}{2}$.

Notice that, for each of these pairs of numbers, if you subtract one number from the other, you get either 3 or -3. That is, the absolute value of each of these differences is 3.

If you put the strip on picture (II), you obtain such pairs as

$\frac{1}{2}$ and 3,	$2\frac{2}{3}$ and $\frac{1}{6}$,	-1 and 2,
-3 and 1,	-5 and 0,	-1 and -6.

The absolute values of the differences for these pairs are, respectively,

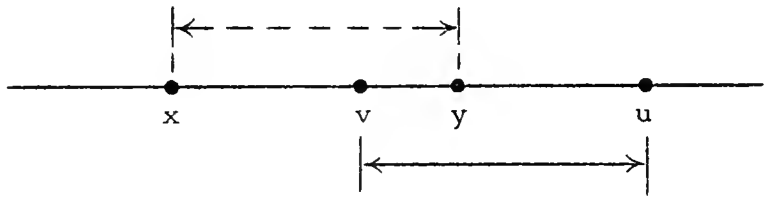
$2\frac{1}{2}$,	$2\frac{1}{2}$,	3,
4,	5,	5.

Notice that, for picture (II), the <u>paper-distance</u> between two dots does not tell you the absolute value of the difference between the corresponding numbers. Picture (I) is certainly more convenient than picture (II) for visualizing the absolute values of differences between numbers.

So, we usually choose to picture a (1-1)-correspondence between real numbers and points of a line [see page 1-1] in conformity with the following statement.

For each pair of numbers x and y, and each pair of numbers u and v, the paper-distance between the dots for x and y is equal to the paper-distance between the dots for u and v if and only if

$$|x - y| = |u - v|.$$



$$|x - y| = |u - v|$$

A straight line whose points correspond to the real numbers in a way which conforms to the statement in the box above [and, therefore, to the boxed statement on page 1-1] is called *a uniform scale*.

Because certain pictures of lines enable us to visualize properties of real numbers, we often call the set of real numbers *the number line*, and speak of a real number as a *point* of the number line.

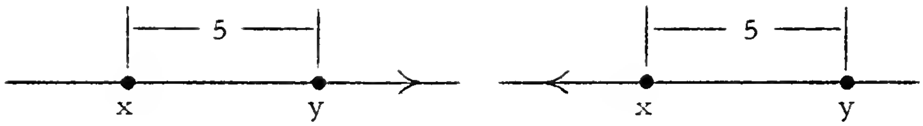
EXERCISES

Use a uniform scale in answering the following questions.

Sample. For all numbers x, y, and z, if $|x - y| = 5$ and $|x - z| = 7$, what is $|y - z|$?

Solution. There are two cases: $x < y$ and $y < x$.

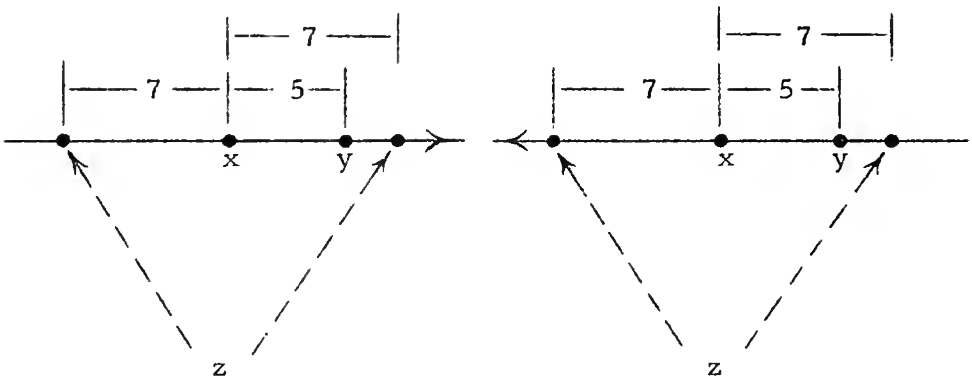
(continued on next page)



$x < y$ $\qquad\qquad$ $y < x$

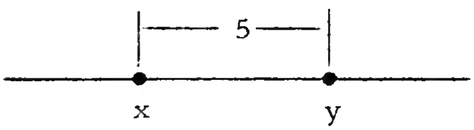
[The arrowhead in each picture gives the direction from 0 to 1.]

Since $|x - z| = 7$, z must be either 7 greater than x or 7 less than x, as illustrated below for each of the two cases.



In two of these four cases, $|y - z| = 2$; in the other two cases $|y - z| = 12$.

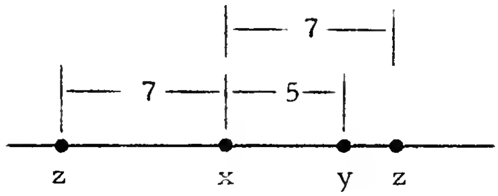
[Notice that the two pictures above which illustrate the four cases differ only in the location of the arrowhead.



$\left\{ \begin{array}{l} \text{If } x < y, \longrightarrow; \\ \text{if } y < x, \longleftarrow. \end{array} \right\}$

(continued on next page)

Consequently, we could have solved the exercise by using just one picture:



.]

1. For all numbers x, y, and z, if $|x - y| = 8$ and $|y - z| = 9$, what is $|x - z|$?

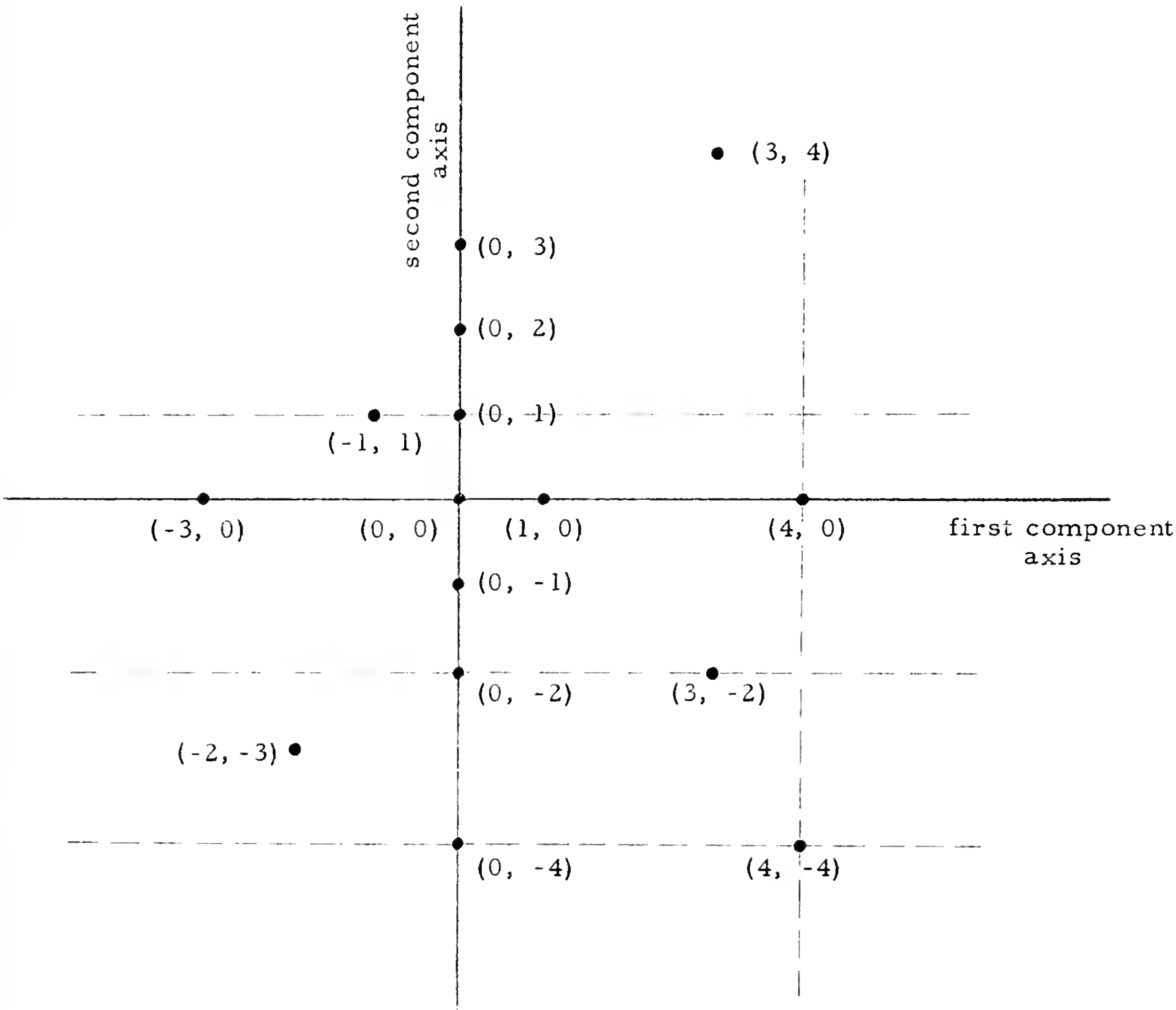
2. $|x - y| = 5$, $|x - z| = 5$, $|y - z| = ?$

3. $|x - y| = 8$, $|y - z| = 0$, $|z - x| = ?$

4. $|y - x| = 5$, $|y - u| = 3$, $|u - v| = 4$, $(|x - v|, |y - v|) = (?, ?)$

THE NUMBER PLANE

In working with <u>ordered pairs</u> of real numbers we can use a (1-1)-correspondence between the set of such ordered pairs and the set of points of a plane. You can picture this correspondence by drawing a part of a plane and writing a name for an ordered pair close to the dot which pictures the point associated with that ordered pair. Here is a picture of the (1-1)-correspondence you used in your earlier work with ordered pairs of numbers.



There are several important things to notice about the picture of this correspondence:

(1) All ordered pairs with the same first component [or, with the same second component] correspond to the points of a straight line.

(2) The same uniform scale is used for both component axes.

(3) The paper-distance between the dots for (0, 1) and (1, 0) is the same as the paper-distance between the dots for (0, -1) and (1, 0).

Pictures of the system of ordered pairs of real numbers which conform to conditions (1), (2), and (3) are called <u>pictures of the number plane</u>. The set of all ordered pairs of real numbers is often called <u>the number plane</u>.

EXERCISES

<u>A</u>. Use UICSM Coordinate Plane Paper B as a picture of the number plane. Locate and label the dots corresponding to

R:	(2, 3)	and	S:	(5, 7),		P:	(11, 9)	and	Q:	(15, 6),
U:	(1, 7)	and	V:	(-2, 11),		M:	(-7, -3)	and	N:	(-3, -6),
D:	(-2, 0)	and	E:	(3, 0),	and	G:	(0, 13)	and	H:	(0, 8).

1. Use a ruler to measure the paper-distance between the dots R and S, P and Q, U and V, M and N, D and E, and G and H.

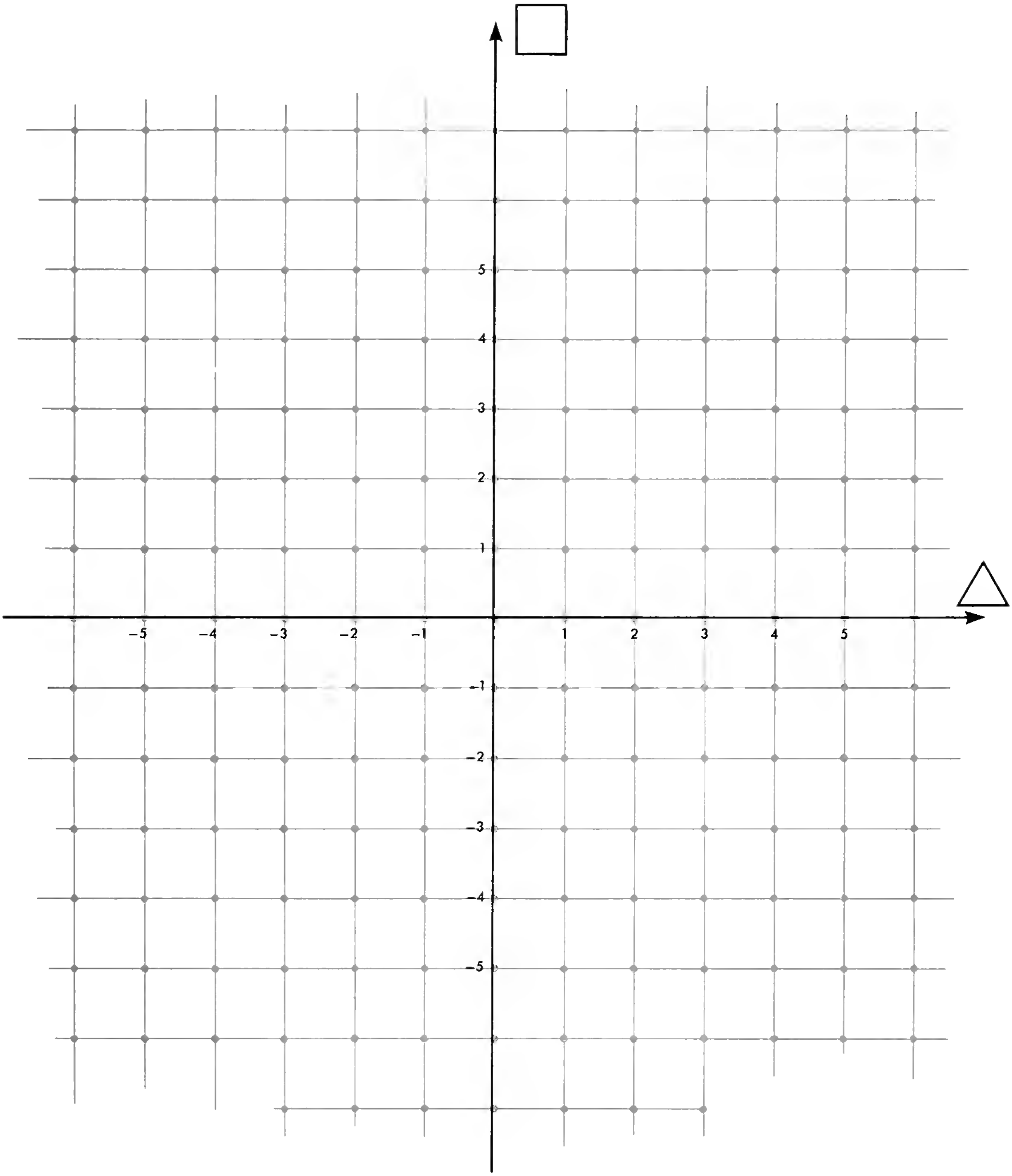
2. Use the "distance formula" you learned in an earlier unit to <u>compute</u> the distance between the ordered pairs corresponding to the pairs of dots mentioned in 1.

<u>Sample</u>. The dots R and S.

<u>Solution</u>. For the dots R and S, the distance between the corresponding ordered pairs (2, 3) and (5, 7), is

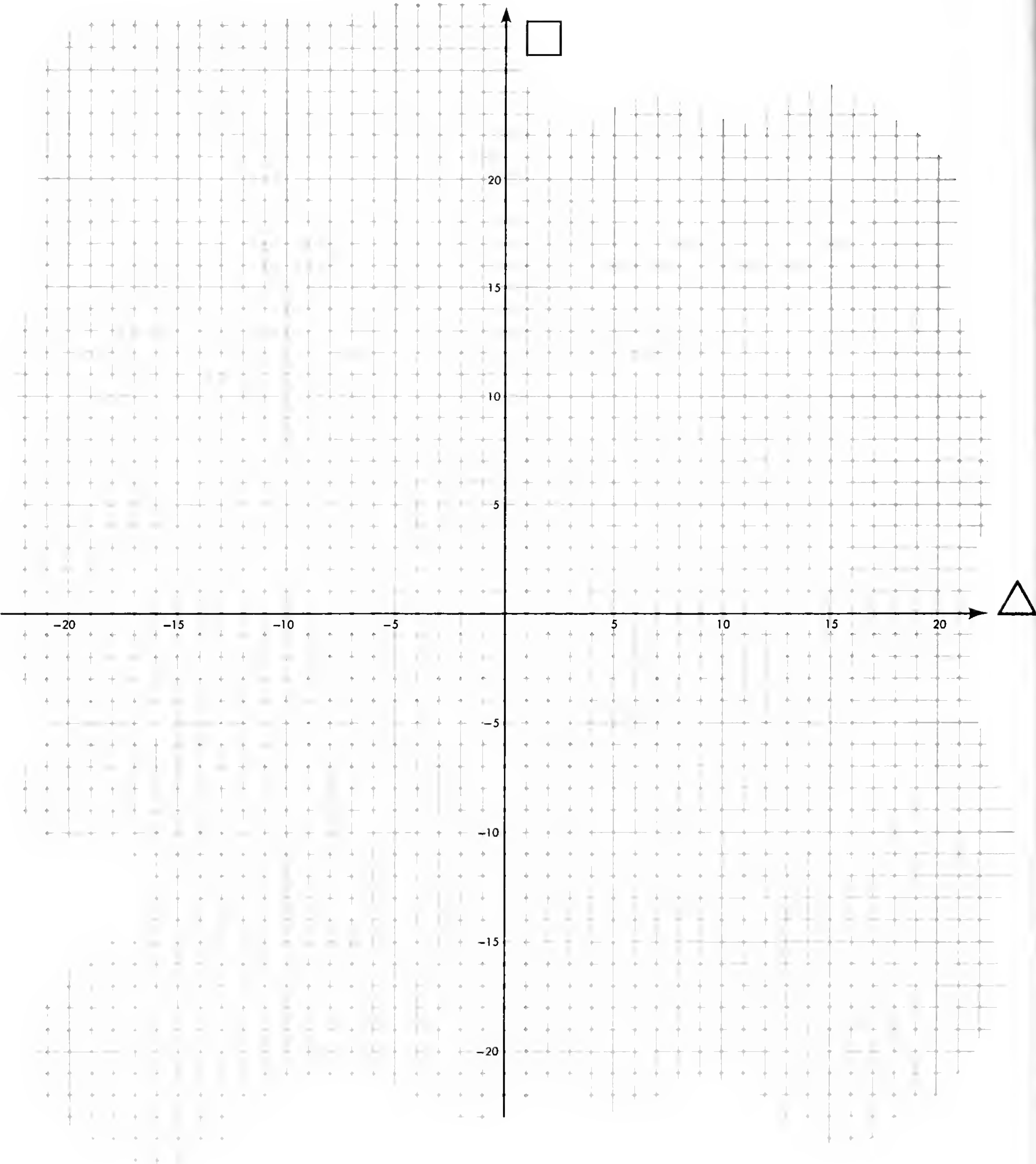
(continued on next page)

NAME ______________________



COORDINATE PLANE PAPER A

NAME ______________________________



COORDINATE PLANE PAPER B

$$\sqrt{(2 - 5)^2 + (3 - 7)^2}$$

$$= \sqrt{(-3)^2 + (-4)^2}$$

$$= \sqrt{9 + 16}$$

$$= 5.$$

3. Use UICSM Coordinate Plane Paper A as a picture of the number plane. Pick a pair of dots A and B, measure the paper-distance between them, and then pick another pair of dots C and D such that the paper-distance between C and D is the same as the paper-distance between A and B. Then compute the distance between the ordered pairs for A and B and the distance between the ordered pairs for C and D.

4. Use the results of 1, 2, and 3 to complete the following statement.

> For any picture of the number plane, the paper-distance between the dots corresponding to the ordered pairs (a, b) and (c, d) is equal to the paper-distance between the dots corresponding to the ordered pairs (m, n) and (p, q) if and only if
>
> $$\sqrt{\qquad\qquad} = \sqrt{\qquad\qquad}.$$

5. Simplify the equation in 4 for the case of four dots corresponding to (a, b), (c, d), (m, n), and (p, q), in which

(a) the four dots belong to the first component axis,

(b) the four dots belong to the second component axis,

(c) $b = d = n = q = 3$,

(d) $a = c = m = p = 97.38$.

(continued on next page)

6. Check your simplified equation in part (c) of 5 for a = 7, c = 9, m = 1, and p = -1. If your equation does not check, refer to the boxed statement on page 1-6.

7. Complete the following statements with expressions which do not contain square root signs.

(a) For every x and every y,

$$\sqrt{(x - y)^2} = \sqrt{(y - x)^2} = \underline{\hspace{3cm}}.$$

(b) For every k,

$$\sqrt{k^2} = \sqrt{(-k)^2} = \underline{\hspace{3cm}}.$$

(c) For every m and every p,

$$\sqrt{m^2 - 2mp + p^2} = \sqrt{(m - p)^2} = \underline{\hspace{3cm}}.$$

(d) For every n and every q,

$$\sqrt{q^2 - 2nq + n^2} = \underline{\hspace{3cm}}.$$

(e) For every r and every s, the distance between the ordered pairs (r, 2) and (s, 2) is ______________.

(f) For every t and every v, the distance between the ordered pairs (-3, t) and (-3, -v) is ______________.

(g) For every a and every b,

$$\sqrt{a^2 + 2ab + b^2} = \underline{\hspace{3cm}}.$$

(h) For every a and every b,

$$\sqrt{(a + 2b)^2 + 2(a + 2b)(a - 3b) + (a - 3b)^2} = \underline{\hspace{3cm}}.$$

8. How many pairs of ordered pairs (a, b) and (c, d) are there if $|a| = 3$, $|b| = 4$, $|c| = 9$, and $|d| = 12$? [One such pair of ordered pairs is (3, -4) and (-9, -12).] List all of these pairs in an orderly way, and then compute, for each pair, the distance between its ordered pairs. Do as little work as possible!

* * *

In Part A you used a formula to compute the <u>distance between two ordered pairs of numbers</u>. This notion of distance between ordered pairs of numbers is quite different from the notion of paper-distance between two dots. In fact, it may even sound strange to talk about distance between ordered pairs of numbers. So, we should state precisely what we mean by 'the distance between ordered pairs of real numbers'. Here is such a statement.

> For all ordered pairs (x_0, y_0) and (x_1, y_1), the distance between (x_0, y_0) and (x_1, y_1) is the number
>
> $$d\Big((x_0, y_0), (x_1, y_1)\Big),$$
>
> where
>
> $$d\Big((x_0, y_0), (x_1, y_1)\Big) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

The equation in the box is often called <u>the distance formula</u>. Note that it tells you that the distance between ordered pairs is a number, and that it tells you how to compute this number.

* * *

<u>B</u>. Use the distance formula given above in answering the following questions.

1. If $(x_0, y_0) = (7, 3)$ and $(x_1, y_1) = (-2, 5)$, what is $d\Big((x_0, y_0), (x_1, y_1)\Big)$?

2. If $(u_2, v_2) = (9, -3)$ and $(a, b) = (3, -4)$, what is $d\Big((a, b), (u_2, v_2)\Big)$?

3. If A = (3, 7) and B = (-2, 5), what is d(A, B)? d(B, A)? d(A, A)?

4. Check <u>three</u> instances of the following statement.

   For all points A, B, and C, $d(A, B) + d(B, C) \geq d(A, C)$.

(continued on next page)

[An instance of this generalization is obtained by selecting values for 'A', 'B', and 'C', and replacing the occurrences of 'A', 'B', 'C' in:

'd(A, B) + d(B, C) $\geq$ d(A, C)'

by names for the values selected. Of course, values of 'A', 'B', and 'C' are ordered pairs of real numbers.]

5. (a) Plot the points A, B, and C where A = (2, 4), B = (-1, 2), and C = (5, 6), and note that they appear to be in a straight line. Compute d(A, B), d(B, C), and d(A, C).

   (b) Repeat (a) for the points A, B, and C where A = (3, -2), B = (1, 0), and C = (-3, 4).

   (c) What do the results of (a) and (b) suggest to you?

6. Solve the equation:

$$d\Big((4, 5), (1, 1)\Big) = d\Big((4, 5), (7, x)\Big).$$

*7. If $(x_0, y_0) = (3, 5)$, $(x_1, y_1) = (6, 9)$, and $d\Big((x_1, y_1), (x_2, y_2)\Big) = 4$, what is $d\Big((x_0, y_0), (x_2, y_2)\Big)$?

C. Use the distance formula to find simple expressions for the distance between the given ordered pairs.

Sample. (2a + 3, 4a - 5) and (a - 7, a + 6)

Solution. The distance is given by the expression:

$$\sqrt{[(2a + 3) - (a - 7)]^2 + [(4a - 5) - (a + 6)]^2},$$

which can be simplified as follows:

$$= \sqrt{(2a + 3 - a + 7)^2 + (4a - 5 - a - 6)^2}$$

$$= \sqrt{(a + 10)^2 + (3a - 11)^2}$$

$$= \sqrt{a^2 + 20a + 100 + 9a^2 - 66a + 121}$$

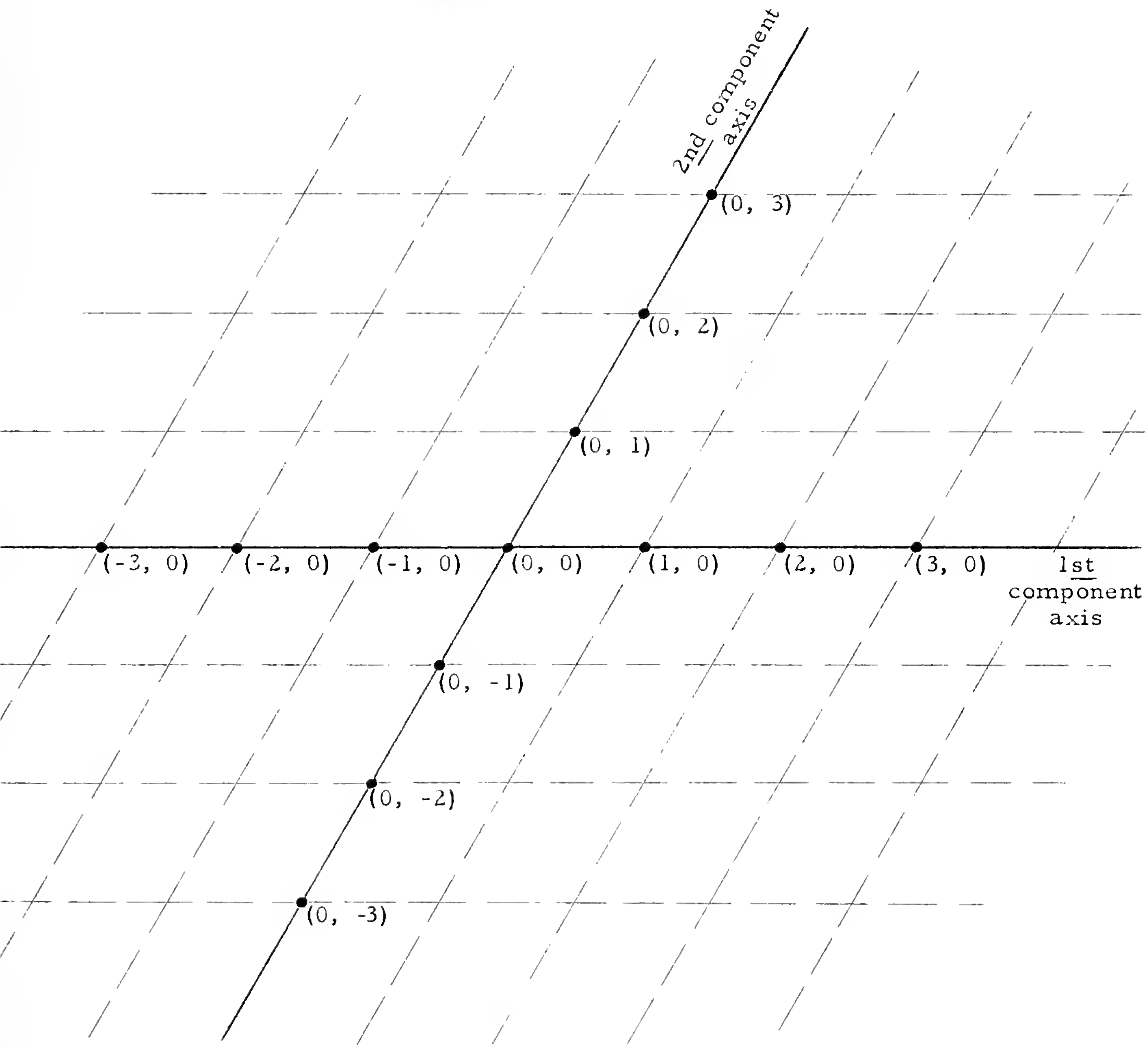
$$= \sqrt{10a^2 - 46a + 221}.$$

1. $(4x + 3, 2x - 1)$ and $(x - 2, x + 5)$
2. $(a + 4, a - 5)$ and $(a + 7, a - 2)$
3. $(5y - 3, 2y + 4)$ and $(3y + 5, 4y - 2)$
4. $(b + a, c + d)$ and $(2b - 3a, 5c - 2d)$
5. $(\frac{1}{2}x + 3, \frac{2}{3}x - 2)$ and $(\frac{1}{3}x - 5, \frac{1}{4}x + 5)$
6. $(3 + 7y, 2 - 6y)$ and $(3 + 7y, 1 + 4y)$
7. $(2x - 3y, 3x + 4y)$ and $(x + 5y, 2x - 7y)$
8. $(x - y, m)$ and $(x + y, -m)$
9. $(5x - 3, x + 2)$ and $(x - 4, -4x + 1)$
10. $(\frac{4}{x}, \frac{x}{2})$ and $(\frac{3}{x}, \frac{5}{x})$
11. $\left(\frac{3x}{4}, \sqrt{\frac{3x^2}{64}}\right)$ and $\left(\frac{5x}{8}, 0\right)$

D. Compute the perimeters of <u>and</u> the areas enclosed by the figures formed by joining A_1 to A_2, A_2 to A_3, A_3 to A_4, ..., A_{n-1} to A_n, and A_n to A_1.

1. $A_1 = (2, 1)$, $A_2 = (5, 1)$, $A_3 = (5, 5)$.
2. $A_1 = (-1, -1)$, $A_2 = (4, -1)$, $A_3 = (1, 4)$.
3. $A_1 = (-1, -3)$, $A_2 = (3, -3)$, $A_3 = (3, 2)$, $A_4 = (-1, 2)$.
4. $A_1 = (-3, -4)$, $A_2 = (5, -4)$, $A_3 = (3, 1)$, $A_4 = (-1, 1)$.
5. $A_1 = (-2, -5)$, $A_2 = (3, -2)$, $A_3 = (2, 5)$.
6. $A_1 = (4, 6)$, $A_2 = (-4, 3)$, $A_3 = (-5, -5)$, $A_4 = (3, -2)$.
7. $A_1 = (0, 0)$, $A_2 = (a, 0)$, $A_3 = (a + b, c)$, $A_4 = (b, c)$.

*E. Here is a picture of a (1-1)-correspondence between the set of all ordered pairs of real numbers and the set of all points in a plane.



This picture satisfies conditions (1) and (2) given on page 1-10. But it does not satisfy condition (3), a fact which you can tell by using a ruler, or even by comparing paper-distances by eye.

1. Prove that this picture does not conform to the description indicated in Exercise 4 on page 1-11. That is, prove that it is <u>not</u> the case that

   for every (a, b) and (c, d), and for every (m, n) and (p, q), the paper-distance between the dots corresponding to the ordered pairs (a, b) and (c, d) is equal to the paper-distance between the dots corresponding to the ordered pairs (m, n) and (p, q) if and only if

$$\sqrt{(a - c)^2 + (b - d)^2} = \sqrt{(m - p)^2 + (n - q)^2} .$$

   [<u>Hint</u>: You can do this in either of two ways.

   (1) Find a pair of dots and measure the paper-distance between them. Find another pair with the same paper-distance between them. Then show that the equation above is not satisfied by the components of the ordered pairs corresponding to the dots.

   (2) Find two pairs of ordered pairs which satisfy the equation. Plot the points corresponding to these ordered pairs. Then measure the paper-distances for each pair of dots, and show that the paper-distances are not equal.]

2. Check several instances of the following statement.

   For every (a, b) and (c, d), and for every (m, n) and (p, q), the paper-distance [on <u>this</u> picture] between the dots for (a, b) and (c, d) is equal to the paper-distance between the dots for (m, n) and (p, q) if and only if

$$\sqrt{(a - c)^2 + (a - c)(b - d) + (b - d)^2} = \sqrt{(m - p)^2 + (m - p)(n - q) + (n - q)^2}.$$

SETS OF POINTS

In your work in geometry in this course you will use the notion of a set of points. In an earlier unit, you learned about intersections of sets and unions of sets. Now, you will learn more about sets and learn to use several new words and symbols.

EXERCISES

Consider the three sets r, b, and g. We can tell what points are elements of these sets by listing their points between curly braces. So,

$r = \{(-5, 5), (-2, 3), (5, 5), (0, 0), (2, 3)\}$,

$b = \{(2, 3), (8, 7), (5, 5), (2, -2), (-1, 1)\}$, and

$g = \{(-3, 4), (0, 4), (2, -4), (2, -2), (7, 4), (3, 4)\}$.

Do you see that the set r consists of exactly five elements? One of these elements is the point (-2, 3). You can state that (-2, 3) is an element of r by writing:

$$(-2, 3) \in r.$$

You can state that (9, 12) is not an element of g by writing:

$$(9, 12) \notin g.$$

A. Plot, on a picture of the number plane, the points in each of the sets r, b, and g. [Make your dots so that you can tell the three sets apart.]

B. 1. What set is the union of r and b? [Recall that the union of a set m and a set n is a set which consists of just those elements which belong to set m or to set n. An abbreviation for 'the union of set m and set n' is:

$$m \cup n,$$

which is read as 'em union en'.

The question asked in the exercise is to be answered by listing between curly braces the elements in $r \cup b$.]

(continued on next page)

2. What set is $g \cup b$?

3. What is $b \cup g$?

> For each set m, each set n, and each point P,
>
> $$P \in m \cup n$$
>
> if and only if
>
> $$P \in m \text{ or } P \in n.$$

* * *

C. 1. What set is the intersection of r and b? [Recall that the intersection of a set m and a set n (symbolized by: $m \cap n$, and read as 'em intersection en') is a set which consists of just those elements which belong both to m and to n.

The question asked in this exercise is to be answered by listing between curly braces the elements in $r \cap b$.]

2. What set is $b \cap g$?

3. What is $g \cap b$?

> For each set m, each set n, and each point P,
>
> $$P \in m \cap n$$
>
> if and only if
>
> $$P \in m \text{ and } P \in n.$$

4. What is $r \cap g$? [The set which consists of no elements is called 'the empty set'. Another name for the empty set is '$\emptyset$'.]

* * *

D. 1. What is $r \cap r$? What is $r \cup r$?

2. What is $b \cap b$? What is $b \cup b$?

E. True or false?

1. $(5, 5) \in r$ 2. $(0, 4) \notin b$ 3. $(7, 4) \in b$

4. $(2, 3) \in r$ 5. $(5, 5) \in r \cap b$ 6. $(0, 4) \notin b \cup g$

7. $(1, 1) \in r$ 8. $(2, -4) \in b \cap g$ 9. $(2, 3) \in \emptyset$

10. $(8, 7) \in (r \cup b) \cup g$ 11. $(8, 7) \in r \cup (b \cup g)$

12. $(8, 7) \in (r \cap b) \cap g$ 13. $(8, 7) \in r \cap (b \cap g)$

14. $(2, -4) \in (r \cap b) \cup g$ 15. $(2, -4) \in r \cap (b \cup g)$

16. $(5, 5) \in r$ and $(7, 4) \in g$

17. $(5, 5) \in r$ or $(7, 4) \in g$

18. $(2, -2) \in g$ and $(0, 0) \in b$

19. $(2, -2) \in g$ or $(0, 0) \in b$

* * *

20. $(-5, 5) \in r$ and $(2, 3) \in r$

21. $(-3, 4) \in g$ and $(2, -4) \in g$ and $(7, 4) \in g$

22. $(2, 3) \in b$ and $(8, 7) \in b$ and $(-1, 1) \in b$

23. $(0, 4) \in g$ and $(-1, 1) \in g$ and $(3, 4) \in g$

[Another way of stating what is given in Exercise 20 is to write:

$\{(-5, 5), (2, 3)\}$ is a <u>subset</u> of r,

or, more simply:

$$\{(-5, 5), (2, 3)\} \subseteq r.$$

Short ways of stating what is given in Exercises 21, 22, and 23 are:

$$\{(-3, 4), (2, -4), (7, 4)\} \subseteq g,$$
$$\{(2, 3), (8, 7), (-1, 1)\} \subseteq b,$$

and

$$\{(0, 4), (-1, 1), (3, 4)\} \subseteq g,$$

(continued on next page)

respectively. Since what is stated in Exercise 23 is false, the following statement is true:

$$\{(0, 4), (-1, 1), (3, 4)\} \not\subseteq g. \;]$$

24. $\{(0, 0), (2, 3)\} \subseteq r$
25. $\{(-3, 4)\} \subseteq g$
26. $\{(8, 7), (7, 4)\} \subseteq b$
27. $\{(-2, 3), (0, 4)\} \not\subseteq r$
28. $\emptyset \subseteq r$
29. $\emptyset \subseteq b$
30. $\emptyset \subseteq g$
31. $\{(2, 3), (-3, 4)\} \subseteq b \cup g$
32. $\{(2, 3), (-3, 4)\} \subseteq b \cap g$
33. $r \cap b \subseteq r \cup b$
34. $b \cap g \subseteq b \cup g$
35. $r \subseteq r$
36. $b \subseteq r$
37. $b \subseteq x \cup b$

* * *

For each set m, and for each n,

$$m \subseteq n$$

if and only if every element of m is an element of n; that is, if and only if, for every point P, if $P \in m$ then $P \in n$.

* * *

38. For each set m and each set n,

if $m \subseteq n$ and $n \subseteq m$, then $m = n$.

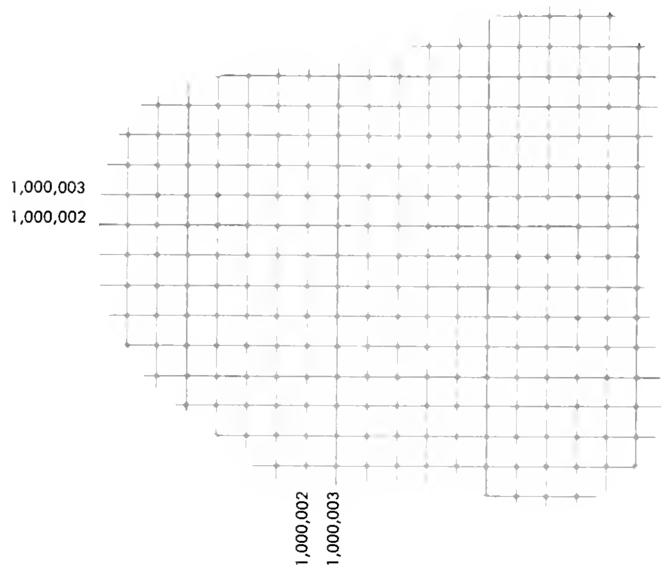
* * *

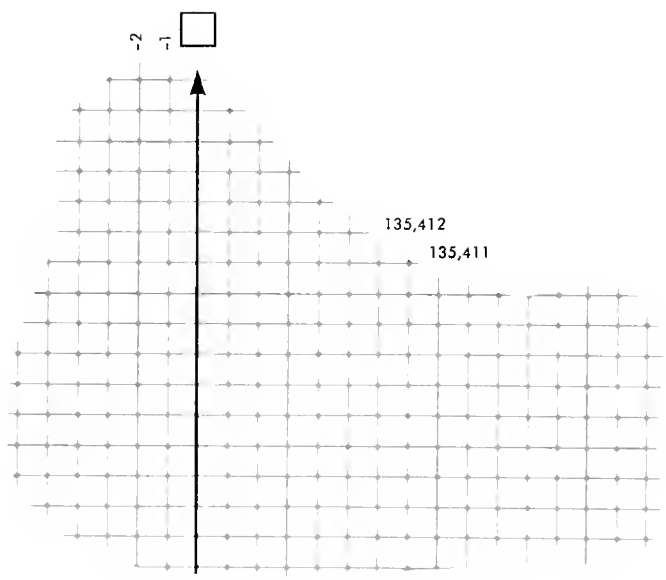
F. Use UICSM Coordinate Plane Paper C_1 and C_2 for the following exercises. Notice that page C_1 shows two subsets of the number plane (call them 'C_1-up' and 'C_1-down'), and that page C_2 shows two other subsets of the number plane (call them 'C_2-up' and 'C_2-down'). [We shall abbreviate by calling these subsets:

$$C_{1u}, \quad C_{1d}, \quad C_{2u}, \quad C_{2d}. \;]$$

(continued on next page)

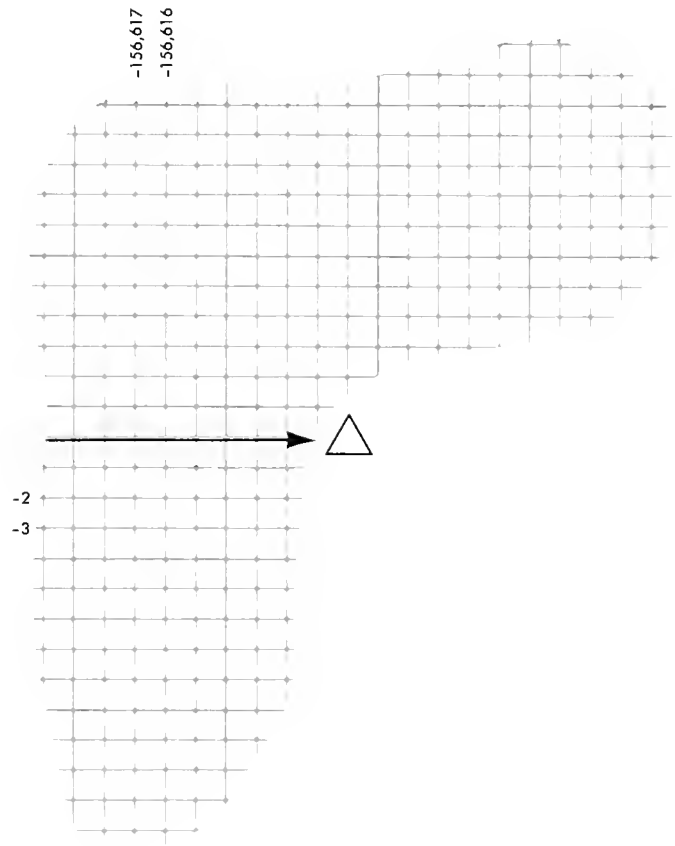
NAME ______________________

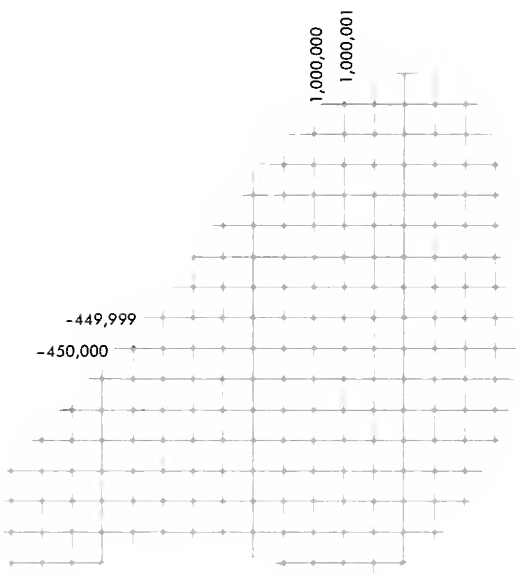




COORDINATE PLANE PAPER C_1

NAME ________________





COORDINATE PLANE PAPER C_2

1. Plot each of the given points and tell which subset it belongs to. [The Δ-axis is the first component axis.]

A: (1 000 003, 1 000 003)	B: (-156 616, -2)
C: (5, 135 410)	D: (999 999, -450 003)
E: (1 000 008, 999 998)	F: (-156 620, 0)
G: (0, 135 406)	H: (-156 602, 4)
I : (135 411, -2)	J : (135 408, -5)

2. Make a rough sketch of the number plane which shows the point (0, 0) and the four subsets.

3. Is there a straight line in the number plane which intersects each of the sets C_{1u} and C_{2d} in a non-empty set?

4. Suppose you were in C_{1u} and met (0, 0). Indicate [by drawing an arrow] which way you would point to show him the way to go home. [Repeat for the other three regions.]

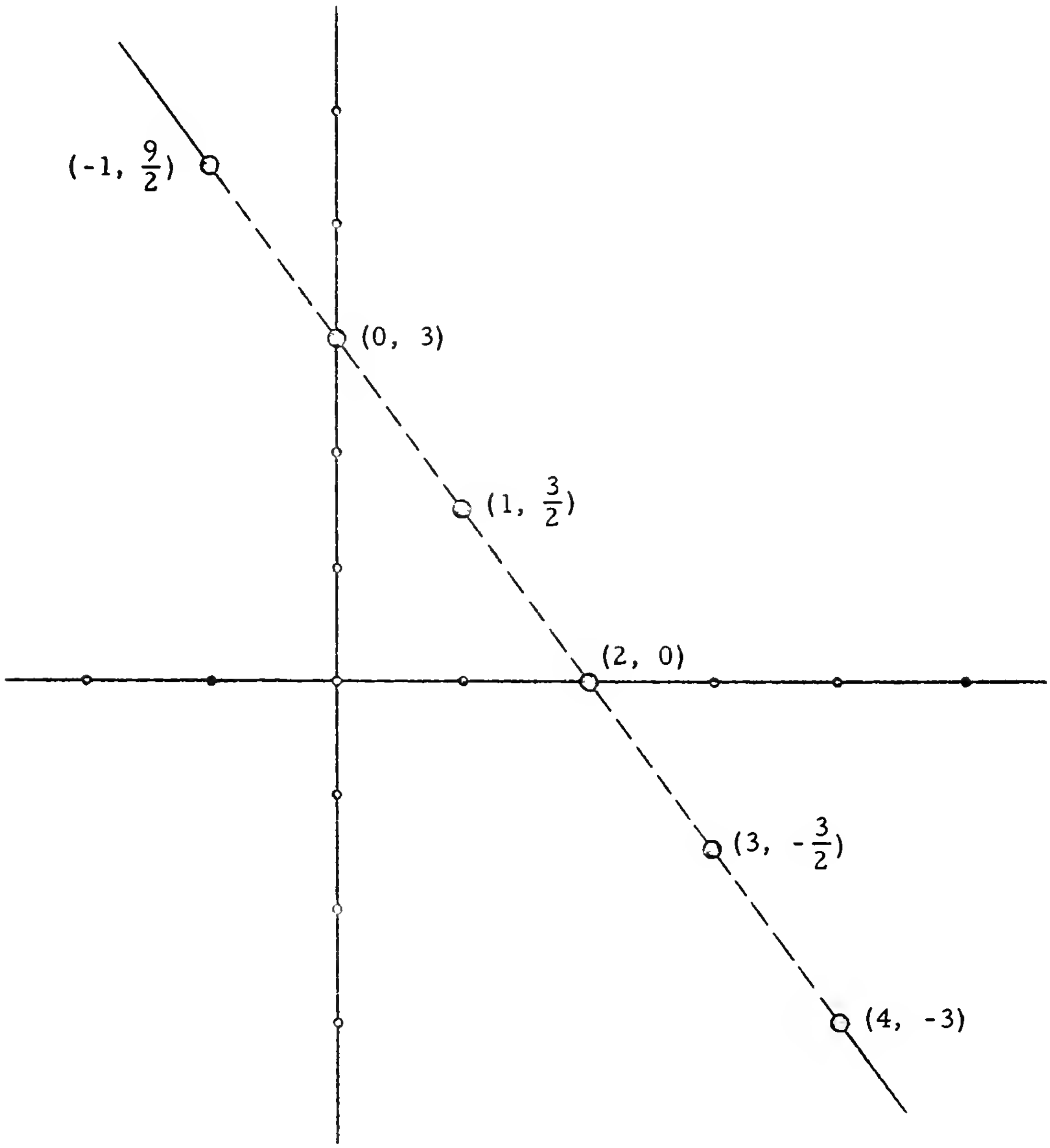
5. Suppose you were in C_{1u} and you wanted to travel along a straight line to reach the closest point in

   the set of all points (x, y) such that y = x.

   Indicate the direction in which you would walk by drawing an arrow. [Repeat for the other three regions.]

1.02 Straight lines in the number plane. --You may recall drawing pictures of straight lines in the number plane by finding ordered pairs of numbers which satisfy a certain kind of equation. For example, consider the equation '$3x + 2y = 6$'. If, on a picture of the number plane, you make dots corresponding to several ordered pairs (x, y) which satisfy this equation, it is easy to convince yourself that no matter how many such

dots you make, you will be able to draw a straight line which passes through all of them.



We say that the set of all ordered pairs (x, y) which satisfy [explain this word] the equation '3x + 2y = 6' is a straight line in the number plane. This set of ordered pairs is also called the solution set of (x, y) [or: the locus in (x, y)] of '3x + 2y = 6'. [For short, we sometimes omit 'of (x, y)' or 'in (x, y)' and refer simply, to the solution set of '3x + 2y = 6', or to the locus of '3x + 2y = 6'.]

EXERCISES

A. 1. Show that (-10, 18) is an element of the solution set of (x, y) of '3x + 2y = 6'.

2. Show that (4, 7) does not belong to the locus in (x, y) of '3y - 2x + 5 = 0'.

3. Does (5, 7) belong to the locus of 'x - y = 2'?

4. Does (5, 7) belong to the locus in (y, x) of 'x - y = 2'?

5. Is (3, 12) a member of the solution set of 'b - 2a = 6'?

6. Show that (3, 12) belongs to the solution set of (a, b) of 'b - 2a = 6'.

* * *

Note: A short expression which means the same as:

the locus in (a, b) of '7a - 3b = 5'

and:

the solution set of (a, b) of '7a - 3b = 5'

is:

$\{(a, b): 7a - 3b = 5\}$

which is read as

'the set of all ordered pairs (a, b) such that 7a - 3b = 5'.

The curly braces in '$\{(a, b): 7a - 3b = 5\}$' tell you that we are talking about a set; '(a, b)' tells you that the members of the set are ordered pairs of real numbers; '7a - 3b = 5' tells you that the members of the set are just those ordered pairs (a, b) which satisfy '7a - 3b = 5'.

* * *

7. Give an ordered pair which belongs to $\{(x, y): 3x + 2y = 6\}$ and which is not labeled in the diagram on page 1-23.

8. Show that $(3, 2) \not\in \{(a, t): 2t + 3a = 12\}$.

9. Show that $(3, 2) \in \{(t, a): 2t + 3a = 12\}$.

10. Show that $\{(3, -1), (-3, 7)\} \subseteq \{(p, t): 4p + 3t - 9 = 0\}$.

(continued on next page)

11. Show that $\{(4, 5), (4, 8), (4, -3)\} \subseteq \{(x, y): 3x = 12\}$.
12. Show that $\{(0, -8), (\frac{1}{2}, -8), (-306, -8), (\pi, -8)\} \subseteq \{(a, b): 5b + 40 = 0\}$.
13. Describe each set.
    (a) $\{(x, y): 0 = 0\}$ (b) $\{(x, y): 1 = 0\}$

B. Find all values of 'x' which satisfy each of the following.

Sample. $(x, 4) \in \{(a, b): 3a + 2b - 7 = 0\}$

Solution. For every x,

$$(x, 4) \in \{(a, b): 3a + 2b - 7 = 0\}$$

if and only if x satisfies the equation:

$$3(x,) + 2(4) - 7 = 0,$$

which is the equation we get from '$3a + 2b - 7 = 0$' by replacing 'a' by 'x' and 'b' by '4'.

This equation is equivalent to:

$$3x + 1 = 0.$$

Hence, $-\frac{1}{3}$ is the only value which satisfies the expression:

$$(x, 4) \in \{(a, b): 3a + 2b - 7 = 0\}.$$

1. $(5, x) \in \{(a, b): 2a - 3b = 7\}$
2. $(3, \frac{2}{x}) \in \{(a, b): a = 3b - 1\}$
3. $(2x + 1, 5) \in \{(a, b): 5a - 4a = 6\}$
4. $(x, 2x) \in \{(b, a): 5a + b - 5 = 0\}$
5. $(x, x^2) \in \{(a, b): 4(a + 1) = -b\}$
6. $(x, x^2) \in \{(a, b): (a + 1)^2 - a^2 = a + b + 1\}$
7. $(\frac{1}{x}, \frac{1}{x}) \in \{(c, d): 3c^2 + d = 0\}$
8. $(x, 3) \in \{(x, y): 3x - y = 5\}$
9. $(7x - 2, 5x + 7) \in \{(a, b): 5a - 2b + c = 8 + a + \frac{1}{2}(1 + 2c)\}$
10. $(-2, x) \in \{(x, y): x + |y| \leq x\}$
11. $(x, 3x + 7) \in \{(a, b): \frac{b}{3} = a + 3\}$
12. $(x, 2x) \in \{(r, s): |s - r| + 2r > s\}$

C. Graph each of the sets given below.

1. $\{(x, y): x = 2y\}$
2. $\{(x, y): 4y - 3x = 12\}$
3. $\{(x, y): y + 2x - 6 = 0\}$
4. $\{(a, b): b + 2a - 6 = 0\}$
5. $\{(r, s): r \geq s\}$
6. $\{(x, y): y \leq x\}$
7. $\{(x, y): x^2 = y^2\}$
8. $\{(y, x): 3x + 2y - 9 = 0\}$
9. $\{(s, t): |s| + |t| = 10\}$
10. $\{(y, x): y = |x| + 1\}$
11. $\{(y, x): y = 3x^2\}$
12. $\{(k, m): k^2 + m^2 = 25\}$
13. $\{(u, v): 3u = 12\}$
14. $\{(c, d): 8 = 4d - 3\}$
15. $\{(x, y): 0x + 0y = 1\}$
16. $\{(x, y): 0x + 0y = 0\}$

D. Separate the equations below into two groups. Put in one group those equations whose solution sets [sets of points (x, y)] are straight lines and in the other group those equations whose loci are not straight lines.

$x + 5y = 12$	$x\sqrt{3} - 7y = 15$	$4x + 9y - 8 = 0$
$3x + 4 = 0$	$9y = x^2$	$2x = \sqrt{y}$
$\frac{2}{3}x - \frac{3}{5}y + 7 = 0$	$8 - 2y = 0$	$3 + x + y^2 = 0$
$x^2 + y^2 = 17$	$x^2 + y^2 = 0$	$\|x\| + y = 0$
$0x + 0y - 1 = 0$	$2x + 9y - 71 = 0$	$5x^2 + 3y^2 = 9$
$2y - 3x^3 = 0$	$7x - 2y + 8 = 0$	$4x - 7y + 1 = 0$

* * *

You may have guessed from your work in Parts C and D the kind of equation whose locus is a line ['line', from now on, means straight line]. We describe below what we mean by 'line in the number plane'.

> A line in the number plane is a set ℓ for which there exist numbers a, b, and c [a and b not both 0] such that
>
> $$\ell = \{(x, y): ax + by + c = 0\}.$$

This description tells you that each time you replace 'a', 'b', and 'c' in the equation '$ax + by + c = 0$' by numerals [at least one of the first two must not name 0], you get an equation whose locus in (x, y) is a line. Each such equation is called <u>a linear equation in x and y</u>.

Can you find a linear equation whose solution set is empty? Can you find one whose solution set consists of just one point? Can you <u>prove</u> that the solution set of each linear equation consists of at least two points? [In other words, can you prove that there is no linear equation whose solution set consists of no points or one point?] Try it before reading the following proof.

Each linear equation in x and y can be obtained from:

$$ax + by + c = 0$$

by assigning values to 'a', 'b', and 'c' [but you may not assign the value 0 to both 'a' and 'b'].

(i) Suppose $a \neq 0$. Then $(-\frac{c}{a}, 0)$ is one point in the solution set. [Check by substituting.] A second point is $(\frac{-c - b}{a}, 1)$. [Why doesn't this argument hold when $a = 0$?]

(ii) Suppose $a = 0$. Then $b \neq 0$ [Why?], and two points in the solution set are $(0, -\frac{c}{b})$ and $(1, -\frac{c}{b})$.

Since either $a \neq 0$ or $a = 0$, we have shown that the solution set of each linear equation consists of at least two points. [You can see that such solution sets consist of many points. However, we wished to prove merely that there were at least two points.]

In the boxed description on page 1-27 we agreed to use the word 'line' in referring to the solution set of a linear equation. Therefore, from this description and from the proof we just gave, it follows that

> I. Each line is a set of points, and contains at least two points.

Note that this statement was derived from the technical description of a line given on page 1-27 and from your knowledge of the properties of ordered pairs of real numbers.

[Statement I is enclosed in a red box. It is one of the <u>postulates</u> we referred to in the Introduction. Other postulates will be enclosed in red boxes and will be assigned Roman numerals for easy reference.]

* * *

<u>E</u>. 1. Prove that each line consists of at least 3 points.

2. How would you go about proving that each line consists of at least 1492 points?

3. Refer to the boxed description of lines on page 1-27, and show that if we omitted 'a and b not both 0' from the description, it would not be possible to prove statement I. [<u>Hint</u>: Find a counter-example.]

* * *

From our description of lines we can also prove that

> II. There are three points that do not belong to the same line.

In order to prove this, all we have to do is find three points which do not belong to the same line. Consider the three points

$$(0, 0), \quad (1, 0), \text{ and } (0, 1).$$

If these three points belong to a line then there are numbers a, b, and c [$a \neq 0$ or $b \neq 0$] such that

$$\{(0, 0), (1, 0), (0, 1)\} \subseteq \{(x, y): ax + by + c = 0\}.$$

That is,

$$\begin{aligned} & a \cdot 0 + b \cdot 0 + c = 0, \\ \text{and} \quad & a \cdot 1 + b \cdot 0 + c = 0, \\ \text{and} \quad & a \cdot 0 + b \cdot 1 + c = 0, \end{aligned}$$

or, more simply,

$$\begin{aligned} & c = 0, \\ \text{and} \quad & a + c = 0, \\ \text{and} \quad & b + c = 0. \end{aligned}$$

If $c = 0$ and $a + c = 0$ then $a = 0$.

If $c = 0$ and $b + c = 0$ then $b = 0$.

So, if we assume that

$$\{(0, 0), (1, 0), (0, 1)\} \subseteq \{(x, y): ax + by + c = 0\},$$

it follows that $a = 0$ and $b = 0$, or, in other words, that $\{(x, y): ax + by + c = 0\}$ is not a line.

* * *

F. 1. Show that the three points (0, 0), (0, -1), and (5, 7) do not belong to the same line.

2. The postulates (enclosed in the red box) can be used as a basis for proving statements.

Such statements are called <u>theorems</u>. In proving theorems you may use statements enclosed in red boxes, definitions, and theorems which you have already proved. <u>But you must not use any other properties of the number plane</u>.

<u>Theorem 1</u>.

For each line, there is a point not on the line.

Use Postulate II to prove this theorem.

[<u>Hint</u>: If you know that there are three points which do not belong to the same line then, given a line, how many of <u>these</u> three points can belong to it?]

<u>G</u>. In each of the following exercises you are given a point. Find linear equations for at least 3 lines which pass through that point.

<u>Sample 1</u>. (2, 5)

<u>Solution</u>. There are many lines which pass through (2, 5). Every such line is a set $\{(x, y): ax + by + c = 0\}$ where a and b are not both 0, and

$$(*) \quad a(2) + b(5) + c = 0.$$

It is easy to find values of 'a' and 'b' [not both 0] and of 'c' which satisfy (*). One choice of values is 4, 5, and -33 for 'a', 'b', and 'c', respectively. Hence, the locus of the equation:

$$(1) \quad 4x + 5y - 33 = 0$$

passes through (2, 5).

(continued on next page)

We can find other loci containing (2, 5) by choosing other values of 'a' and 'b' and determining the appropriate value of 'c' from (*). For example, if a = 9 and b = 4 then

$$9(2) + 4(5) + c = 0,$$

that is, c = -18 - 20 = -38. So, the locus of:

$$(2) \quad 9x + 4y - 38 = 0$$

contains the point (2, 5).

If a = 7 and b = 12 then by (*), c = -7(2) - 12(5) = -74. So, a third set which contains (2, 5) is the locus of:

$$(3) \quad 7x + 12y - 74 = 0.$$

[Suppose someone claimed that a <u>fourth</u> locus is that of the equation:

$$14x + 24y - 148 = 0.$$

Prove to him that the locus of this equation is the same as the locus of (3). Also, prove that the sets which are the loci of (1), (2), and (3) are, indeed, <u>three</u> sets.]

1. (-8, 3) 2. (4, -7) 3. $(\frac{1}{3}, -1)$

4. (r, s) 5. (-2, s) 6. (x_0, y_0)

<u>H</u>. For each of the following linear equations, give one ordered pair which satisfies it.

1. $3x + 5y - 8 = 0$
2. $18x - 17y - 19 = 0$
3. $3(x - 1) + 5(y - 1) = 0$
4. $18(x - 2) - 17(y - 1) = 0$
5. $87(x - 5) + 93(y - 6) = 0$
6. $7.03(x - 2) + 6.54(y + 9) = 0$
7. $63(x + 4) - 76(y - 7) = 0$
8. $631(x - 17.5) + 12.2(y - 60.3) = 0$
9. $98(x + 4) - 95(y - 7) = 0$
10. $932(x - 17.5) - 61.8(y - 60.3) = 0$

* * *

11. Write five linear equations each of which is satisfied by (17.5, 60.3).

12. Repeat Exercise 11 for the ordered pair (m, -6); for (7, t); for (x_1, y_1).

I. In each of the following exercises you are given a point. Find linear equations for at least 3 lines which contain that point.

Sample. (8, 3)

Solution. For all numbers a, b, and c [$a \neq 0$ or $b \neq 0$],

$$\{(x, y): ax + by + c = 0\}$$

is a line. [And, each line in the number plane is one of these sets.] Some of these lines contain the point (8, 3). Those which do are just those for which

$$8a + 3b + c = 0,$$

that is, for which

$$c = -8a - 3b.$$

Hence, each of those lines which contains (8, 3) is a set

$$(1) \qquad \{(x, y): ax + by + (-8a - 3b) = 0\}$$

for which $a \neq 0$ or $b \neq 0$. Show that the set

$$(2) \qquad \{(x, y): a(x - 8) + b(y - 3) = 0\}$$

for which $a \neq 0$ or $b \neq 0$ is the same set as (1).

So, we have an easy way of finding each linear equation whose locus contains the point (8, 3). Just replace 'a' and 'b' in '$a(x - 8) + b(y - 3) = 0$' by the appropriate numerals. Here are three linear equations obtained in this way:

$$4(x - 8) + 5(y - 3) = 0$$

$$\sqrt{2}(x - 8) - 5\pi(y - 3) = 0$$

$$(x - 8) = 0.$$

[Prove that these equations really have three loci.]

1. (2, 7)
2. (-7, -3)
3. (6.2, -8.3)
4. (0, 0)
5. (0, 2)
6. (-s, t)
7. (x_0, y_0)

* * *

Your work in Exercise 7 should help you to fill in the blank spaces in the following.

For every point (x_0, y_0) and for each a and b, not both 0, the set $\{(x, y): a(\quad\quad) + b(\quad\quad) = 0\}$ is a line which contains the point (x_0, y_0).

* * *

J. Suppose you are given two points, say, (2, 3) and (7, 5), and you want to find a linear equation whose locus contains both of these points. You know from the box above that, for every a and b [not both 0], the set

$$\{(x, y): \ a(x - 2) + b(y - 3) = 0\}$$

is a line which contains (2, 3). Our problem is to determine a and b so that the corresponding line also contains (7, 5). That is, find values of 'a' and 'b' which satisfy the equation:

$$a(7 - 2) + b(5 - 3) = 0,$$

that is:

$$a(5) \quad + \quad b(2) \quad = 0.$$

There is an easy way to find these values. Just use the numbers already available. If we assign 'a' the value 2:

$$2(5) \quad + \quad b(2) \quad = 0,$$

then the equation is satisfied by what value of 'b'?

(continued on next page)

So, for the equation:

$$a(7 - 2) + b(5 - 3) = 0,$$

if we assign to 'a' the value $5 - 3$:

$$(5 - 3)(7 - 2) + b(5 - 3) = 0$$

then the corresponding value of 'b' is $-(7 - 2)$:

$$(5 - 3)(7 - 2) - (7 - 2)(5 - 3) = 0.$$

With these values of 'a' and 'b' we obtain from:

$$\{(x, y): a(x - 2) + b(y - 3) = 0\}$$

the expression:

$$\{(x, y): (5 - 3)(x - 2) - (7 - 2)(y - 3) = 0\}.$$

Hence, one linear equation whose locus contains (2, 3) and (7, 5) is:

$$(5 - 3)(x - 2) - (7 - 2)(y - 3) = 0,$$

or, equivalently:

$$2(x - 2) - 5(y - 3) = 0,$$

or:

$$2x - 5y + 11 = 0.$$

In each of the following exercises you are given two points. Find a linear equation whose locus is a line which contains both of these points.

1. (7, -4) and (-2, -5)
2. (3, -2) and (-8, -2)
3. (-2, -7) and (2, 7)
4. (0, 5) and (6, 0)
5. (4, 4) and (4, 9)
6. (-2, 9) and (7, -12)
7. (11, 10) and (10, 11)
8. (1, 5) and (1, 5)
9. (0, 0) and (2, 7)
10. (r, s) and (5, 2)
11. $(\frac{8}{3}, -7)$ and $(4, -\frac{2}{3})$
12. (6, 2) and (8, -7)

(continued on next page)

13. (m, n) and (p, q)

14. (x_0, y_0) and (x_1, y_1)

* * *

Your work in Exercise 14 should help you to fill in the blank space in the following.

> For every two points (x_0, y_0) and (x_1, y_1),
>
> $\{(x, y) : \qquad\qquad\qquad = 0\}$
>
> is a line which contains (x_0, y_0) and (x_1, y_1).

* * *

Use the boxed formula to obtain a linear equation in x and y whose locus passes through the given points. Just for practice, simplify the equation to the form of '$ax + by + c = 0$'.

15. $(3, 2)$ and $(4, 8)$

16. $(9, 7)$ and $(-6, -5)$

17. $(12, -10)$ and $(-7, 9)$

18. $(a, 0)$ and $(0, b)$

19. (k, m) and (k, n)

20. (a, b) and $(a + 1, b + m)$

21. $(r, s - t)$ and $(-r, t - s)$

22. $\left(\frac{a}{b + c}, d\right)$ and $\left(\frac{a}{b - c}, b\right)$

23. $(a^2 + b^2, 4c)$ and $(2ab, c^2 + 4c)$

24. $(m + 2n, m - 2n)$ and $(m - 2n, m + 2n)$

* * *

You have seen that one can find an equation of a line which passes through two given points, whatever these may be. In other words, you have proved that for every two points P and Q, there is a line ℓ such that ℓ contains P and Q, that is, $\{P, Q\} \subseteq \ell$. We restate this as postulate

> III. For every two points, there is a line which contains them.

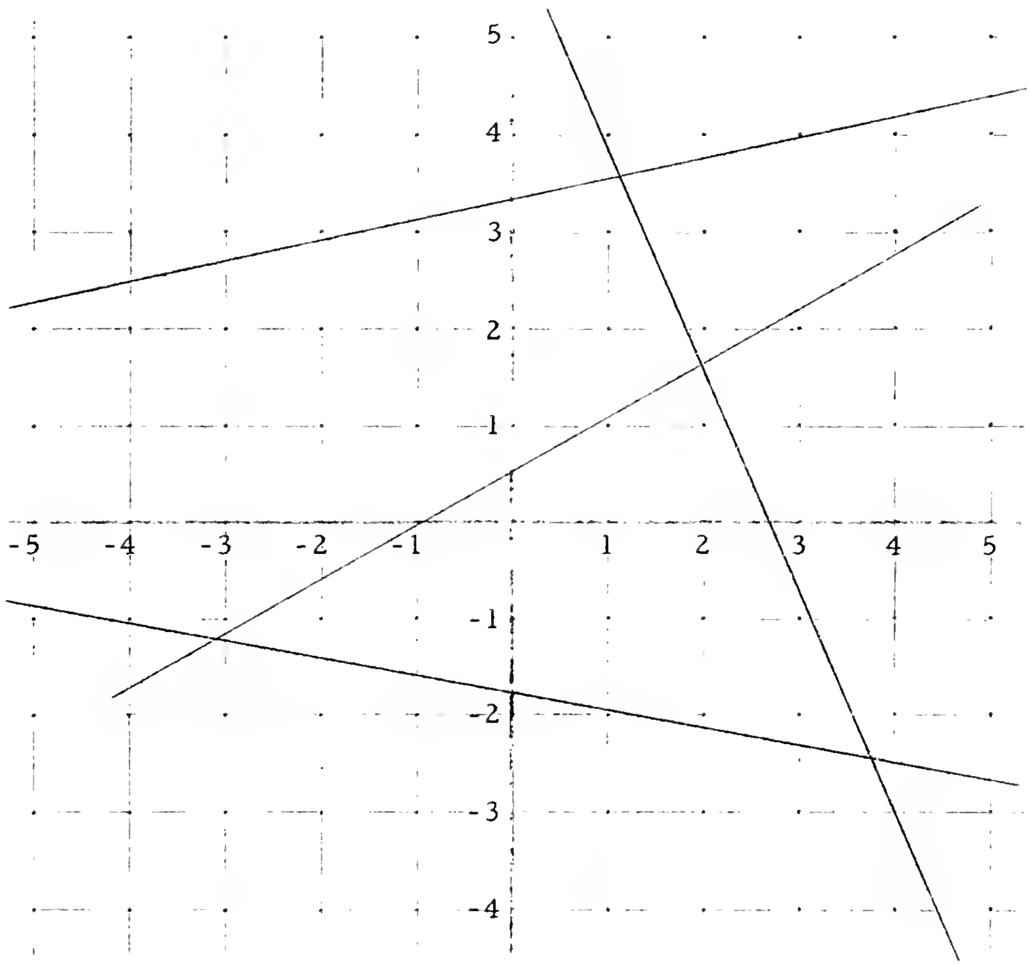
MODELS FOR POSTULATE SYSTEMS

Postulates I, II, and III are true statements about the number plane if we interpret the word 'line' as meaning the solution set of a linear equation, and the word 'point' as meaning an ordered pair of real numbers. [Theorem 1 is also a true statement about the number plane under this interpretation of 'line' and 'point'.] The number plane, with these definitions of 'point' and 'line', is a <u>model</u> for our system of three postulates.

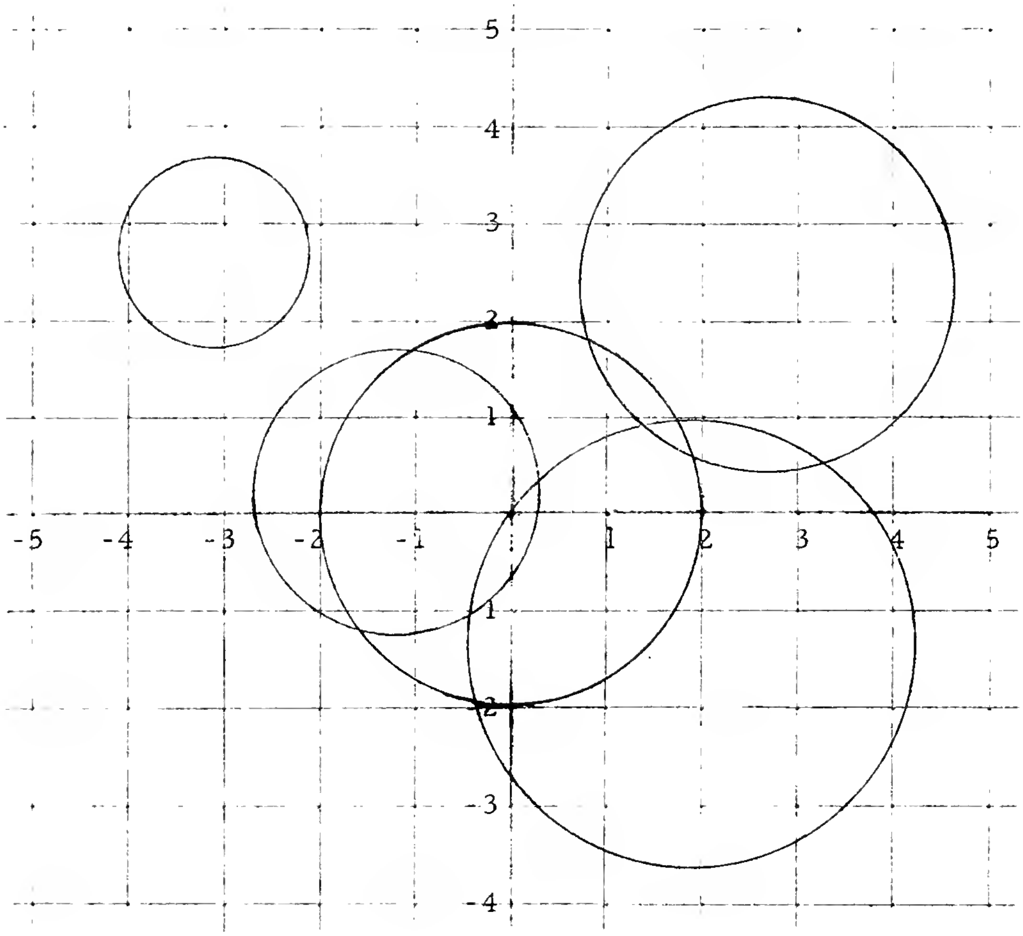
Any set of objects some of which one decides to call <u>points</u> and some of which one decides to call <u>lines</u> can serve as a model for our three-postulate system provided that the postulates are true statements about this set of objects.

Stan Straight and Carl Circle each have a set of objects. Stan calls some of his objects, <u>points</u>, and some of his objects, <u>lines</u>. Carl also calls some of his objects, <u>lines</u>, and some, <u>points</u>. Stan's points are ordered pairs of real numbers, and so are Carl's. But Stan's lines are solution sets of linear equations, whereas Carl's lines are solution sets of equations of a different kind. When Stan Straight draws

pictures of his lines, the pictures look like these:



When Carl Circle draws pictures of his lines, the pictures look like these:



Suppose that Straight and Circle want to know if their points and lines are models for our three-postulate system. This will be the case for Stan if each of the postulates is a true statement about his points and lines, and for Carl, if each postulate is a true statement about <u>his</u>

points and lines. So they check each postulate.

> I. Each line is a set of points, and contains at least two points.

Since Stan's points and lines are what <u>we</u> have been considering as points and lines, we are sure that postulate I is true for Stan. In fact, since we derived all three postulates from properties of the number plane and used the same interpretation of 'point' and 'line' that Stan does, we know that the three postulates must be true statements about Stan's points and lines. Carl's lines are solution sets of equations of a certain type. For instance, one of his lines [pictured in the diagram] is $\{(x, y): x^2 + y^2 = 4\}$. This line contains many points; for example: $(2, 0)$, $(0, 2)$, $(1, \sqrt{3})$, and $\left(\frac{3}{2}, \frac{\sqrt{7}}{2}\right)$. Similarly, each of his other lines is a set of points, and contains at least two points. So, postulate I is true for Carl.

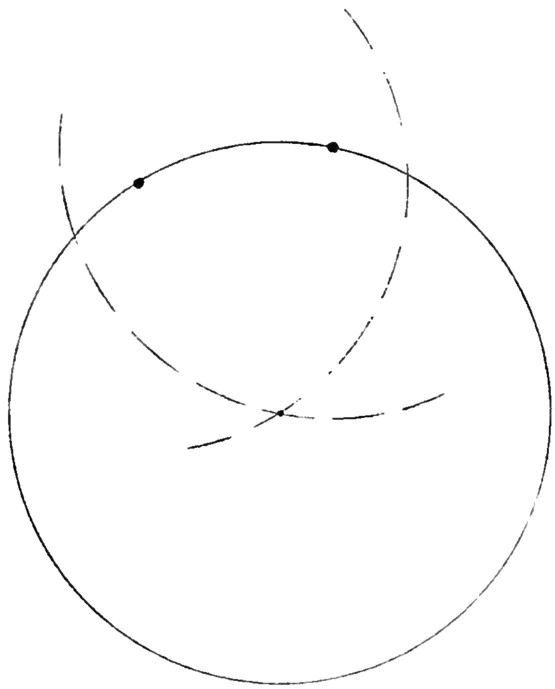
> II. There are three points which do not belong to the same line.

As we mentioned above, postulate II is true for Stan since we proved [page 1-29] that the points $(0, 0)$, $(0, 1)$, and $(1, 0)$ do not belong to the same line. Carl can prove (and does) that the points $(0, 0)$, $(1, 0)$, and $(2, 0)$ do not belong to any one of his lines.

> III. For every two points, there is a line which contains them.

Carl proves that this is true for his points and lines. [Stan draws pictures of his line by laying a ruler down on his paper with the edge of the ruler just touching both of the two dots. Carl draws a

picture of one of his lines by using compasses.

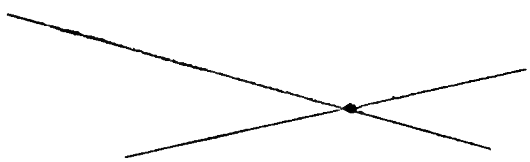


Explain how Carl can start with two dots and use compasses to draw one of his lines through the dots.]

So, with the interpretation each boy has for 'point' and 'line', the three postulates are true for both. Thus, Stan has a model for the three-postulate system, and Carl has a model for the same three-postulate system.

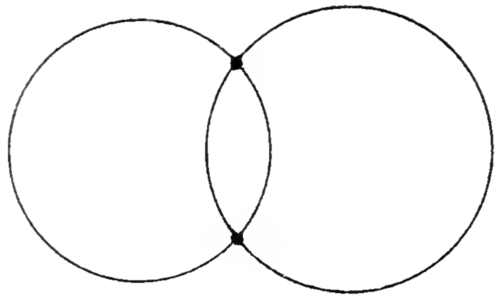
Now, suppose that neither boy knows that the other has different meanings for 'point' and 'line'. If they were telephoning each other about postulates I, II, and III, they would agree that the three postulates were true statements. They might even conclude that their models were the same. Suppose Stan says during this telephone conversation,

"You know, it looks as if two lines intersect in exactly one point."



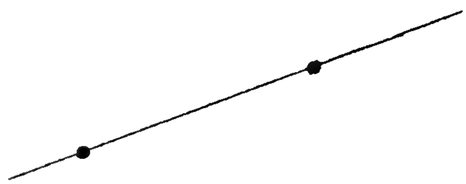
Carl would certainly reply,

"That's wrong. I know lots of cases in which two lines intersect in two points."



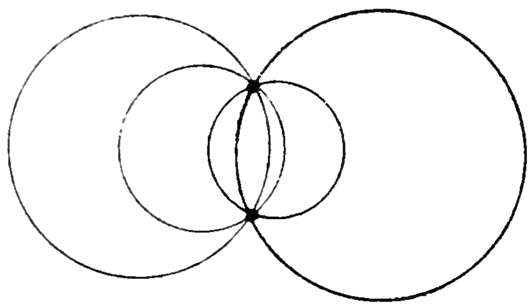
Stan might answer this by saying,

"But I think I'm right, because it seems to be the case that for each two points, there is exactly one line which contains them."



Carl replies,

"Well, I guess that's just where your trouble is. Anybody can see that there are lots and lots of lines which contain two given points."



Stan Straight made two guesses about his model and both of these were false statements for Carl Circle. Do you think Stan could prove either of his guesses from the three postulates alone?

Since Stan and Carl both agree that the postulates are true, anything that could be proved from the postulates alone would be true for both Stan and Carl. As an example of this, consider Theorem 1 which says that for each line, there is a point not on the line. This theorem was deduced from the postulates. If Stan and Carl wanted to, each could prove this statement in his model, just as we proved postulates I, II, and III in Stan's model. But it isn't necessary to do this because Theorem 1 has been deduced from the postulates. So, we know that Stan could not deduce either of his guesses from the three postulates alone; if he could, it would be a theorem and would have to be true in Carl's model. And we know it is false in Carl's model.

If Stan Straight feels sure about his guesses, he should try to derive at least one of them from the properties of his model. He could then add this proved statement to the set of three postulates, making it into a set of four postulates. Then, he might be able to deduce his other guess from the four postulates. If Stan did add one of these guesses to the postulate set, would Carl's set of objects which he called 'lines' and 'points' be a model for the four-postulate set?

EXERCISES

A. There are many models for the three-postulate system. Some of them are sets of just a few objects which when called 'points' and 'lines' satisfy the three postulates.

Sample. Three businessmen, A, B, and C, belong to just the three partnerships $\{A, B\}$, $\{B, C\}$, and $\{A, C\}$. If each businessman is called 'a point' and each partnership is called 'a line', are these points and lines a model for the three postulates?

(continued on next page)

Solution. We check each postulate.

I. Each line is a set of points, and contains at least two points.

Each partnership is a set of two businessmen, so postulate I is satisfied.

II. There are three points which do not belong to the same line.

None of the partnerships contains all three businessmen.

III. For every two points, there is a line which contains them.

Each two businessmen belong to a partnership.

Thus we see that the three businessmen (points) and partnerships (lines) are a model for the three postulates.

In each of the following exercises a set of objects is described. Some of the objects are called 'points' and some are called 'lines'. Determine for each exercise whether the objects are a model for the set of postulates I, II, and III.

1. Four class presidents in a school are appointed to three committees.

Class presidents	Committees
A, B, C, D	{A, C, D}, {A, B, D}, {B, C, D}

The class presidents are points and the committees are lines. [Hint: An easy way to work with these objects is to visualize them in a sketch.

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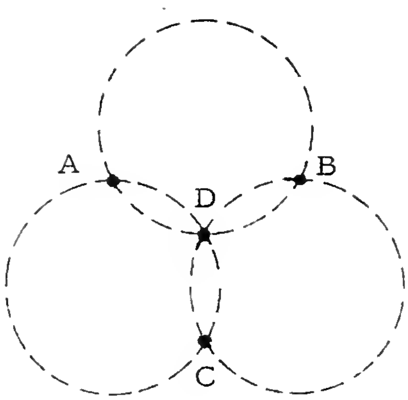
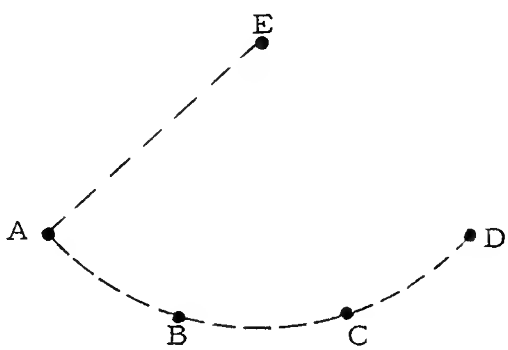


The loops help you to remember which presidents belong to each committee.]

2. Five businessmen form two partnerships.

Businessmen	Partnerships
A, B, C, D, E	{A, E}, {A, B, C, D}

The businessmen are points and the partnerships are lines.



3. The baseball players in the American League form eight teams. The players are points and the teams are lines.

4. The debating squad in Zabranchburg High School has seven members. The coach forms teams of three students each to enter competitions.

Squad members	Teams
A, B, C, D, E, F, G	{A,B,D}, {A,G,E}, {A,C,F}, {B,G,F}, {C,G,D}, {D,E,F}, {B,E,C}

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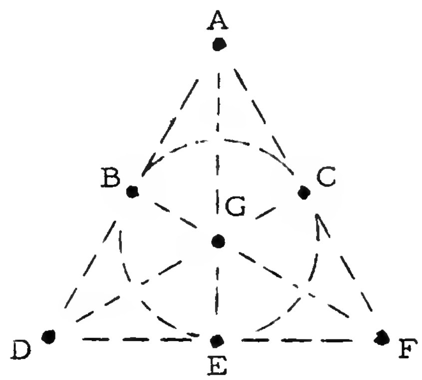
The squad members are points and the teams are lines. Use this sketch to help you check.



5. The three members of one of the teams mentioned in Exercise 4 drop off the squad. The coach then forms from the remaining four members as many teams of two each as he can. Again, the squad members are points and the teams are lines.

*B. Stan Straight has a precise interpretation of 'line' in his set of objects [that is, he can tell you exactly which objects he calls 'lines']. This description was enclosed in the box on page 1-27. Although we did not mention this in our discussion, Carl Circle also knows exactly which objects to call 'lines'. Here is his description.

> A line is a set ℓ for which there exist real numbers a, b, and c [$c > 0$] such that
> $\ell = \{(x, y): (x - a)^2 + (y - b)^2 = c^2\}$.

1. Prove that the three points (0, 0), (1, 0), (2, 0) do not belong to any of Carl's lines.

2. Find values of 'a', 'b', and 'c' which give an equation whose solution set contains the points (6, 0) and (0, 6) and is one of Carl's lines.

3. Repeat Exercise 2 for the points (3, 4) and (4, 3).

(continued on next page)

4. In Carl's model the line

$$\{(x, y): (x - a)^2 + (y - b)^2 = c^2\}$$

has the point (a, b) as "center" and c as "radius". Find equations of <u>two</u> of Carl's lines which intersect in the two points (0, 4) and (0, -4).

* * *

<u>Note</u>: Stan Straight's set of points is the number plane, and so is Carl Circle's. Their interpretations of the word 'line' differ, however. So, their models are different. And, there exist many other models in which the set of points is the number plane. It would be natural to call any such model '<u>a</u> number plane model'. Since we are going to make much use of Stan Straight's model, it will be convenient to single it out and refer to it as <u>the number plane model</u>. In other words, whenever we speak about the number plane model, we mean Stan Straight's model.

1.03 <u>Intersections of lines</u>.--If we look at pictures of lines in the number plane model, it certainly seems to be the case that for each two lines, their intersection is either the empty set or a set consisting of just one point. We know that we cannot deduce this from our three postulates [Why?]. But we might be able to derive it from the properties of the number plane model. Instead of doing this, however, we shall derive a different statement from the number plane model, add this statement to the set of postulates, and then deduce from the set of four postulates the statement concerning the number of points of intersection of two lines.

TWO POINTS DETERMINE A LINE

Postulate III tells us that there is a line through every two given points. [Carl Circle would say '<u>at least</u> <u>one</u> line'.] The postulate leaves open the question: Is there more than one line which contains the two points?

Consider this question for the points (3, 4) and (8, 13). From the boxed statement on page 1-35, we know that there is <u>one</u> line which contains these two points. Let this line be ℓ, where

$$\ell = \{(x, y): (13 - 4)(x - 3) - (8 - 3)(y - 4) = 0\}.$$

[There are countless other equations which can be derived from '$(13 - 4)(x - 3) - (8 - 3)(y - 4) = 0$' by the equation transformation principles. For example:

$$2(13 - 4)(x - 3) - 2(8 - 3)(y - 4) = 0,$$

$$\frac{1}{3}(13 - 4)(x - 3) - \frac{1}{3}(8 - 3)(y - 4) = 0,$$

$$(13 - 4)(x - 3) - (8 - 3)(y - 4) + 17 = 17.$$

Since these equations are equivalent to each other [that is, they are satisfied by the same set of ordered pairs], the expressions:

$$\{(x, y): 2(13 - 4)(x - 3) - 2(8 - 3)(y - 4) = 0\},$$

$$\{(x, y): \frac{1}{3}(13 - 4)(x - 3) - \frac{1}{3}(8 - 3)(y - 4) = 0\},$$

and $$\{(x, y): (13 - 4)(x - 3) - (8 - 3)(y - 4) + 17 = 17\}$$

are just different names for the same line, ℓ. Equivalent linear equations are equations of the same line.

The questions facing us, 'Is there more than one line through (3, 4) and (8, 13)?', will be answered if we prove that each linear equation which is satisfied by (3, 4) and (8, 13) is an equation equivalent to '$(13 - 4)(x - 3) - (8 - 3)(y - 4) = 0$'.]

Suppose that m is another line which contains (3, 4) and (8, 13). Since $(3, 4) \in m$, there are numbers a and b [not both 0] such that

$$(1) \quad m = \{(x, y): a(x - 3) + b(y - 4) = 0\}.$$

Since $(8, 13) \in m$, these numbers a and b are such that

$$(2) \qquad a(8 - 3) + b(13 - 4) = 0.$$

Since $8 - 3 \neq 0$, we can transform, by the multiplication transformation principle, the equation within the braces in (1) into the equivalent equation:

$$(8 - 3)[a(x - 3)] + (8 - 3)[b(y - 4)] = (8 - 3)[0],$$

or, simply:

$$(3) \qquad [(8 - 3)a](x - 3) + (8 - 3)b(y - 4) = 0.$$

If $a(8 - 3) + b(13 - 4) = 0$ then $a(8 - 3) = -b(13 - 4)$. So, by equation (2), equation (3) is equivalent to:

$$[-b(13 - 4)](x - 3) + (8 - 3)b(y - 4) = 0,$$

that is, to:

$$(4) \qquad -b[(13 - 4)(x - 3) - (8 - 3)(y - 4)] = 0.$$

In other words, if m is a line through (3, 4) and (8, 13), there is a number b such that

$$(5) \quad m = \{(x, y): \ -b[(13 - 4)(x - 3) - (8 - 3)(y - 4)] = 0\}.$$

Since m is a line, $b \neq 0$. Hence, by the multiplication transformation principle, the equation in the braces in (5) is equivalent to:

$$\frac{-b[(13 - 4)(x - 3) - (8 - 3)(y - 4)]}{-b} = \frac{0}{-b},$$

or, simply:

$$(13 - 4)(x - 3) - (8 - 3)(y - 4) = 0.$$

So, if m is a line through (3, 4) and (8, 13), each linear equation of m can be transformed by the equation transformation principles into the equivalent equation '$(13 - 4)(x - 3) - (8 - 3)(y - 4) = 0$' which is the equation given for the line ℓ. Thus, for every m through (3, 4) and (8, 13), $m = \ell$. We have proved that in the number plane there can be no more than one line through the two points (3, 4) and (8, 13).

EXERCISES

A. Use the procedure demonstrated above to prove that there can be no more than one line through the two given points.

1. (9, 17) and (2, -11)
2. (3, -5) and (-8, 2)
3. (4, -7) and (4, 3)
4. (2, 9) and (3, 9)
5. (x_0, y_0) and (x_1, y_1)

* * *

In Exercise 5 above, you proved that given a line ℓ which contains the two points (x_0, y_0) and (x_1, y_1), there is no other line m through these points. In other words, you proved that there is at most one line through these points. This is a result that cannot be deduced from postulates I, II, and III, but that is true for the number plane model. Hence, we add it to our postulates.

IV.	For each two points, there is at most one line which contains them.

Postulate III assures you that, for each two points, there is at least one line through the points. Postulate IV assures that if there is one line through two points then this is the only one. With postulate III alone, you can talk about a line through two points. With postulate IV alone, you can be sure that there are less than two lines [perhaps, no lines] through a pair of points. Together, III and IV enable you to speak about the line through two given points. Other ways of referring to the results of III and IV are·

There is one and only one line through two points.
There is a unique line through two points.
Two points determine a line.

B. As we indicated earlier, adding IV to our postulate set will enable us to deduce more statements than we could with the smaller number of postulates. One such statement deals with an important property of the intersection of two lines:

> The intersection of two lines
> consists of at most one point.

Prove this statement. We shall refer to it as Theorem 2.

C. Some of the following statements can be deduced from the postulates and theorems. Others cannot because they are false for at least one model for the four postulates.

Prove those statements which are theorems, and show that the others are not theorems by finding, for each of them, a model for which it is false.

Sample. For each line, there are at least two points not on the line.

Solution. We try to deduce this from the postulate set and are not successful. It is not false in the number plane model, but perhaps we can find another model for the four-postulate set for which the statement is false. The easiest kind of model to work with in this case is one which contains just a few objects.

The set of objects described in the Sample on page 1-42 in which there were three businessmen (points) and three partnerships (lines) each containing just two of the businessmen is a model of postulates I, II, and III. It is easy to see

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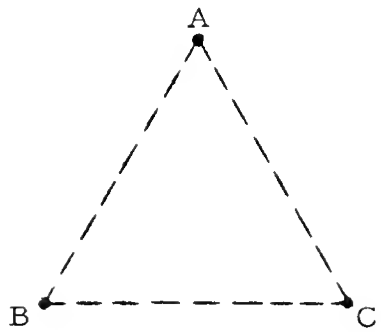


that it also satisfies postulate IV. So, it is a model for the four postulates. The statement we are considering is false, since for each partnership, there is only one businessman not contained in the partnership.

1. There are three lines which do not all contain a same point.

2. For each line, there is at least one point not on the line.

3. For each point, there are at least two lines which contain it.

4. For each point, there are at least three lines which contain it.

5. For each two lines, there is at least one point which belongs to both lines.

D. 1. Tell which of the sets of objects described in Exercises 1-5 on pages 1-43 through 1-45 are models of the four-postulate set.

*2. Describe a model for the four-postulate set which consists of 13 things called 'points' and 13 things called 'lines'.

E. Transform each of the following equations into the form of:

$$ax + by + c = 0,$$

and tell which of them have the same locus [set of points (x, y)] as:

$$2x + 6y - 24 = 0.$$

1. $x + 3y - 12 = 0$	2. $8x + 24y - 96 = 0$
3. $3x + 9y - 36 = 0$	4. $-2x - 6y + 24 = 0$
5. $3x - 9y - 36 = 0$	6. $-x - 3y = 12$

(continued on next page)

7. $6y + 2x = 24$

8. $-y - \frac{1}{3}x + 4 = 0$

9. $(x - 3) + 3(y - 7) = 0$

10. $(3 - x) - 3(y - 7) = 0$

11. $2x + 6(y - 7) = 6$

12. $x + 3y = 10$

13. $2x + 6y = 0$

14. $0(x - 3) = 0(y - 7)$

15. $2k(x - 3) + 6k(y - 3) = 0$

16. $2k(x - 3) + 3k(y - 7) = 0$

17. $(x + 3)^2 + 7(3 - y) = (4 - x)(3 - x) - 2(5y - 6x - 15)$

18. $5(2x) + 5(3y) = 120$

19. $4x + 12y - 72 = 0$

20. $\frac{2}{3}x + 2y - 8 = 0$

21. $r(x - 3) + s(y - 7) = 0$

F. Two equivalent linear equations have the same locus, and two linear equations which have the same locus are equivalent. This last statement should not be surprising for it simply expresses what is meant by 'equivalent equations'. In Exercise 14 on page 1-35, you showed that for each a and b, not both 0, if

$$\{(x, y): a(x - x_0) + b(y - y_0) = 0\}$$

is the same as

$$\{(x, y): (y_1 - y_0)(x - x_0) - (x_1 - x_0)(y - y_0) = 0\}$$

then there is a number $r \neq 0$ such that $ra = y_1 - y_0$ and $rb = -(x_1 - x_0)$ [Why '$r \neq 0$'?].

Use this result to show that, for each a and b, not both 0, and each a' and b', not both 0, if

$$\{(x, y): a(x - x_0) + b(y - y_0) = 0\}$$

is the same as

$$\{(x, y): a'(x - x_0) + b'(y - y_0) = 0\}$$

then there is a number $k \neq 0$ such that $a' = ka$ and $b' = kb$. [Hint: There is a point $(x_1, y_1) \neq (x_0, y_0)$ which belongs to both sets [Why?].]

FINDING THE POINT OF INTERSECTION OF TWO LINES

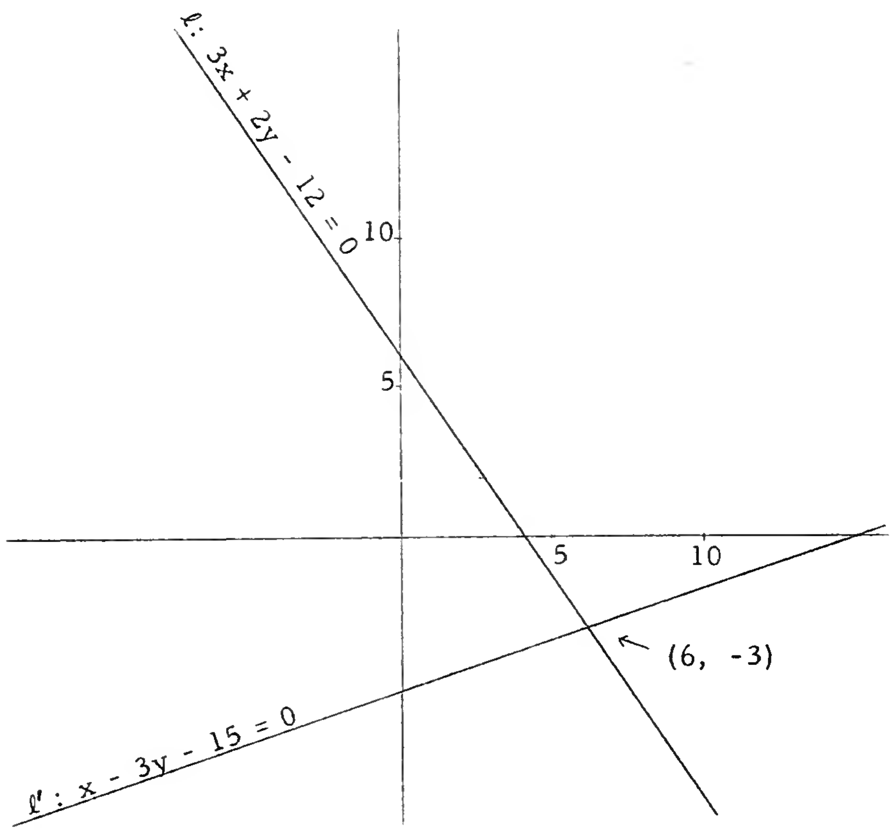
Theorem 2 on page 1-50 tells you that the intersection of two lines contains either no points or one point. Suppose you are given two lines in the number plane,

$$\ell = \{(x, y): ax + by + c = 0\},$$
$$\text{and} \quad \ell' = \{(x, y): a'x + b'y + c' = 0\}.$$

If you are told that their intersection is not empty [that is, $\ell \cap \ell' \neq \emptyset$], then you know that there is exactly one ordered pair which belongs to ℓ and to ℓ' [Why?]. How can you find this ordered pair?

One way is to draw on a picture of the number plane the graphs of ℓ and ℓ'. The dot where the lines cross tells you the ordered pair. Here is an illustration of this procedure for

$$\ell = \{(x, y): 3x + 2y - 12 = 0\},$$
$$\text{and} \quad \ell' = \{(x, y): x - 3y - 15 = 0\}.$$



The picture indicates that (6, -3) is the ordered pair of real numbers which belongs to each of the lines ℓ and ℓ'. To check our work, we need to check that (6, -3) satisfies both of the equations.

$$
\begin{array}{rl|rl}
 & 3x + 2y - 12 = 0 & & x - 3y - 15 = 0 \\
 & 3(6) + 2(-3) - 12 & & 6 - 3(-3) - 15 \\
= & 18 - 6 - 12 & = & 6 + 9 - 15 \\
= & 0 & = & 0
\end{array}
$$

Practice this method of finding the point of intersection of two lines in the following exercises.

(1) $\ell = \{(x, y) : 3x - 5y + 10 = 0\}$
$\ell' = \{(x, y) : 2x - 3y + 3 = 0\}$

(2) $\ell = \{(p, q) : p + 2q - 7 = 0\}$
$\ell' = \{(p, q) : 3p - q - 14 = 0\}$

(3) $\ell = \{(r, s) : 2r + s - 4 = 0\}$
$\ell' = \{(p, q) : -5p + 2q - 5 = 0\}$

* * *

The problem of finding the point of intersection of two lines is often referred to as the problem of solving a system of two linear equations. In the "graphical procedure", you draw the graphs of the two lines and guess at the components of their common point. Sometimes this is easy to do [Exercise (1)], sometimes harder [Exercise (2)], and sometimes practically impossible [suppose the point of intersection were (93.7183, -81.0006)]. There are other procedures which do not require guesswork. Here is an illustration of such a procedure [an "algebraic procedure"].

Consider the system of equations:

$$
\begin{cases}
(1) \quad 3x + 5y - 13 = 0 \\
(2) \quad 7x - 2y - 3 = 0.
\end{cases}
$$

If there is an ordered pair (x, y) which satisfies both (1) and (2) then it satisfies both:

$$7[3x + 5y - 13] = 0$$

and:

$$3[7x - 2y - 3] = 0,$$

or, more simply, both:

$$(1') \qquad 21x + 35y - 91 = 0$$

and:

$$(2') \qquad 21x - 6y - 9 = 0.$$

If this ordered pair satisfies (1′) and (2′) then it also satisfies:

$$[21x + 35y - 91] - [21x - 6y - 9] = 0,$$

or, more simply:

$$(3) \qquad 41y - 82 = 0.$$

Now, every ordered pair (x, y) which satisfies (3) must have 2 as its second component. So, if there is an ordered pair which satisfies both (1) and (2), its second component is 2. We can find the first component of such an ordered pair by replacing 'y' in either (1) or (2) by '2'. Replace 'y' in (1):

$$3x + 5(2) - 13 = 0.$$

Since the root of this last equation is 1, then if there is an ordered pair which satisfies both equations (1) and (2), it must be the ordered pair (1, 2). We know that this ordered pair satisfies equation (1). Check to see that it also satisfies equation (2).

Let us analyze the steps we took in solving the system of equations. If there is an ordered pair (x, y) which satisfies (1), then, for these values of 'x' and 'y', the value of '$3x + 5y - 13$' is 0. Since, for every number z, $z \times 0 = 0$, if (x, y) satisfies (1), then $7[3x + 5y - 13] = 0$. Similarly, if (x, y) satisfies (2), then $3[7x - 2y - 3] = 0$. Since $0 - 0 = 0$, if (x, y) satisfies both (1) and (2), then

$$[21x + 35y - 91] - [21x - 6y - 9] = 0,$$

that is, $41y - 82 = 0$. Do you see why 7 and 3 were chosen as multipliers in going from the system [(1) and (2)] to the equation '$41y - 82 = 0$'?

This process of going from the system of two equations in two pronumerals to the single equation in one pronumeral is often called '<u>eliminating</u> one of the pronumerals from the system'.

In practice, this algebraic procedure for solving a system of equations is carried out in a very brief form. We illustrate.

$$\begin{cases} 3x - 8y - 15 = 0 \\ 23x + 5y + 16 = 0 \end{cases}$$

$$\begin{aligned} 15x - 40y - 75 &= 0 \\ 184x + 40y + 128 &= 0 \\ \hline 199x \qquad + 53 &= 0 \\ x &= -\frac{53}{199} \end{aligned}$$

$$\begin{aligned} 3\left(-\frac{53}{199}\right) - 8y - 15 &= 0 \\ -159 - 1592y - 2985 &= 0 \\ -1592y &= 3144 \\ y &= -\frac{393}{199} \end{aligned}$$

<u>Check</u>

$$\begin{aligned} & 23\left(-\frac{53}{199}\right) + 5\left(-\frac{393}{199}\right) + 16 \\ = \ & -\frac{1219}{199} - \frac{1965}{199} + \frac{3184}{199} \\ = \ & \frac{-3184 + 3184}{199} \\ = \ & 0 \end{aligned}$$

So, the ordered pair which satisfies the system is $\left(-\frac{53}{199}, -\frac{393}{199}\right)$.

EXERCISES

A. Solve each of these systems of linear equations by the graphical procedure <u>and</u> by the algebraic procedure illustrated above.

1. $\begin{cases} 5a - 7b - 17 = 0 \\ 3a + 5b - 1 = 0 \end{cases}$

2. $\begin{cases} 10t + 5u - 14 = 0 \\ 2t - 5u - 40 = 0 \end{cases}$

3. $\begin{cases} 7y + 4 = 6x \\ -3x + 17 = 4y \end{cases}$

4. $\begin{cases} x + 1 = 2y \\ 3y + 1 = x \end{cases}$

B. Solve each of these systems.

1. $\begin{cases} 3x + 7y + 6 = 0 \\ 8x + 4y - 28 = 0 \end{cases}$

2. $\begin{cases} 2x - 3y - 5 = 0 \\ 7x + 8y + 34 = 0 \end{cases}$

3. $\begin{cases} 5x - 7y + 4 = 0 \\ -2x + 8y - 3 = 0 \end{cases}$

4. $\begin{cases} 2x - 12y - 7 = 0 \\ 13x + 2y + 4 = 0 \end{cases}$

5. $\begin{cases} 3x - 7y = 2 \\ 14y = 12x \end{cases}$

6. $\begin{cases} 5 + 3x = 13y \\ 3y - 7 = 9x \end{cases}$

C. Solve each of these systems by the graphical procedure and by the algebraic procedure.

1. $\begin{cases} x + 2y = 4 \\ 3x + 6y = 18 \end{cases}$

2. $\begin{cases} x + 2y = 4 \\ 3x + 6y = 12 \end{cases}$

PROPORTIONALITY

You have probably discovered an easy way to tell when two equations have the same locus, and when the two equations have loci which intersect in the empty set. Explain why each pair in the left column has the same locus, and why each pair in the right column has an empty intersection.

Same locus for each pair	Empty intersection for each pair
$3x + 7y + 4 = 0$ $6x + 14y + 8 = 0$	$3x + 7y + 4 = 0$ $6x + 14y + 9 = 0$
$3x + 2y + 12 = 0$ $9x + 6y + 36 = 0$	$3x + 2y + 12 = 0$ $9x + 6y + 5 = 0$
$8x + 12y - 20 = 0$ $2x + 3y - 5 = 0$	$8x + 12y - 20 = 0$ $2x + 3y - 20 = 0$
$13x - 5y + 17 = 0$ $-39x + 15y - 51 = 0$	$13x - 5y + 17 = 0$ $-39x + 15y + 5 = 0$
$10x + 21y - 34 = 0$ $2x + \frac{21}{5}y - \frac{34}{5} = 0$	$10x + 21y - 34 = 0$ $2x + \frac{21}{5}y - 1 = 0$

For those pairs of equations which have the same locus, the three coefficients [that is, the values of 'a', 'b', and 'c'] of one equation can be obtained by multiplying the three coefficients of the other by some number other than 0. For those pairs of equations whose loci intersect in the empty set, only the coefficients of 'x' and 'y' in one equation can be obtained in this way.

Let us examine more closely the coefficients of the first pair of equations in the left column.

3	7	4
6	14	8

Each number in the second group is obtained by multiplying the corresponding number in the first group by 2. In order to avoid confusion in telling which numbers correspond, it is convenient to treat these groups of numbers as the ordered triples (3, 7, 4) and (6, 14, 8). Then, the first components correspond, the second components correspond, and the third components correspond. We say that the ordered triple (3, 7, 4) is proportional to the ordered triple (6, 14, 8) because

$$6 = \underline{2} \times 3, \qquad 14 = \underline{2} \times 7, \qquad \text{and} \qquad 8 = \underline{2} \times 4.$$

The ordered pair (7, 9) is not proportional to the ordered pair (21, 18) because

$$21 = \underline{3} \times 7 \qquad \text{but} \qquad 18 \neq \underline{3} \times 9.$$

Study each of the following statements and tell whether it is true or false.

$(2, 9, 12)$	is proportional to	$(1, 4\frac{1}{2}, 6)$.	....
$(-3, 8, -\frac{1}{4})$	is proportional to	$(-120, 320, -10)$.	....
$(8, 2, -3)$	is proportional to	$(-8, -2, -3)$.	....
$(2, 0, 5)$	is proportional to	$(14, 0, 35)$.	....
$(1, 3, -9)$	is proportional to	$(1, 3, -9)$.	....
$(4, 3, -2)$	is proportional to	$(8, -4, 6)$.	....
$(8, 4)$	is proportional to	$(\frac{1}{2}, \frac{1}{4})$.	....
$(9, 3)$	is proportional to	$(\frac{1}{9}, \frac{1}{3})$.	....
$(9, -2, 3, -5)$	is proportional to	$(-3, \frac{2}{3}, -1, \frac{5}{3})$.	....

In general,

> $(a, b, c, \ldots)$ is proportional to $(a', b', c', \ldots)$ if and only if there is a number $k \neq 0$ such that
>
> $$a = ka', \quad b = kb', \quad c = kc', \quad \ldots .$$

EXERCISES

A. 1. Give five ordered triples each proportional to (2, 5, 7).

2. Give five ordered pairs each proportional to $(3, \frac{1}{2})$.

3. Give five ordered pairs each proportional to (0, 0).

4. For each of the following, find all values of the pronumerals which satisfy the statement.

(a) $(3, x)$ is proportional to $(2, 7)$

(b) $(9, x, 5)$ is proportional to $(1, 4, \frac{5}{9})$

(c) $(x, 8)$ i.p.t. $(2, 7)$

(d) $(5, x)$ i.p.t. $(x, 5)$

(e) $(0, x)$ i.p.t. $(0, 9)$

(f) $(0, 0)$ i.p.t. $(3, x)$

(g) $(x, y, 3)$ i.p.t. $(2, -1, 6)$

(h) $(3, -1, 5)$ i.p.t. $(3a, -a, 5a)$

(i) $(a - b, 3, 0)$ i.p.t. $(b - a, x, y)$

(j) $(x^2, x, -1)$ i.p.t. (x, x^2, x^2)

(k) $(x, 2x, -x)$ i.p.t. $(x^2, 2x^2, x^2)$

*(l) $(a^2 - 4a + 4, x, 4 - a^2)$ i.p.t. $(a - 2, 1, y)$

(continued on next page)

5. (a) Solve.

(1) (4, 7) i.p.t. (x, 7) (2) (3, 5) i.p.t. (3, x)

(3) (x, -1) i.p.t. (13.4, -1) (4) $(\frac{7}{9}, x)$ i.p.t. $(\frac{7}{9}, \frac{15}{17})$

(b) Show that, for each x and y, (x, y) is proportional to (x, y).

6. (a) Solve.

(1) If (4, 7) i.p.t. (8, x) then (8, x) i.p.t. (4, 7).

(2) If (x, 3) i.p.t. (45, 15) then (45, 15) i.p.t. (x, 3).

(3) If (3x, 7x) i.p.t. (3, 7) then (3, 7) i.p.t. (3x, 7x).

(b) Show that, for every x, y, x′, and y′, if (x, y) is proportional to (x′, y′) then (x′, y′) is proportional to (x, y).

7. (a) If $(\frac{27}{71}, x)$ is proportional to (3, 4), then $(\frac{27}{71}, x)$ is proportional to which of the ordered pairs listed below?

(7, 15)	(31, 58)	(27, 43)
(9, 71)	(-18, -37)	$(-1, -\frac{4}{3})$
(10, 29)	(30, 40)	(-33, -44)

(b) Prove that, for every x, y, x′, y′, x″, y″, if (x, y) is proportional to (x′, y′), and if (x′, y′) is proportional to (x″, y″), then (x, y) is proportional to (x″, y″).

* * *

The principle stated in Exercise 5(b) tells you that proportionality is a <u>reflexive</u> relation; 6(b) tells you that it is a <u>symmetric</u> relation; 7(b) tells you that it is a <u>transitive</u> relation. [Look up the three underlined words in an unabridged dictionary and show the connection between these meanings of the words and their dictionary meanings.]

* * *

(continued on next page)

8. Use the symmetric and transitive properties of the relation of proportionality to prove the following.

For every x, y, x', y', x'', and y'',
if (x', y') is proportional to (x, y),
and if (x'', y'') is proportional to (x, y),
then (x', y') is proportional to (x'', y'').

9. Prove that if two ordered pairs are proportional, the line which passes through them also passes through $(0, 0)$. [Hint: Suppose (p, q) and (r, s) are the proportional pairs. Then, there exists a number $k \neq 0$ such that $r = kp$ and $s = kq$.]

10. Prove that if three ordered pairs are proportional, the line which passes through two of them also passes through the third.

11. List all positive integral solutions for each of the following.

(a) $(r, 2)$ i.p.t. $(15, s)$ (b) $(a, 3)$ i.p.t. $(8, b)$

(c) $(5, x)$ i.p.t. $(y, 4)$ (d) $(7, c)$ i.p.t. $(d, 4)$

(e) (a, b) i.p.t. $(\frac{12}{b}, c)$ (f) (r, s) i.p.t. $(\frac{15}{s}, \frac{15}{r})$

* * *

Find a counter-example for each of the following generalizations.

(g) For every r, and every s, and every t,

(r, s) i.p.t. $(\frac{t}{s}, \frac{t}{r})$.

(h) For every m, and every n, and every p,

(m, n) i.p.t. $(p, \frac{pn}{m})$.

* * *

(continued on next page)

Complete each of the following.

(i) For every x and every y,

$(x, 3)$ i.p.t. $(10, y)$ if and only if $xy =$ _______.

(j) For every x and every y,

$(2, x)$ i.p.t. $(y, 10)$ if and only if $xy =$ _______.

(k) For every x and every y,

$(1, x)$ i.p.t. $(y, 17)$ if and only if $xy =$ _______.

(l) For every x and every y,

$(x, 0)$ i.p.t. $(0, y)$ if and only if $xy =$ _______.

(m) For every x and every y,

$(0, x)$ i.p.t. $(y, 17)$ if and only if $xy =$ _______.

(n) For every m and every p,

$(m, 0)$ i.p.t. $(22, p)$ if and only if $p =$ _______.

* * *

12. Exercise 11 illustrated the following principle. Prove it.

> For every $(a, b) \neq (0, 0)$ and
> for every $(a', b') \neq (0, 0)$,
> (a, b) is proportional to (a', b')
> if and only if
> $$ab' = a'b.$$

[Do you see that '$(a, b) \neq (0, 0)$' is equivalent to 'a and b not both 0'?]

(continued on next page)

[Hint: First, prove the "if part":

if $ab' = a'b$ then (a, b) is proportional to (a', b').

To do this, consider the case $a' \neq 0$ and then the case $a' = 0$.

If $a' \neq 0$ and $ab' = a'b$ then [solving for 'b'] we see that

$$(1) \qquad b = \frac{ab'}{a'} = \left(\frac{a}{a'}\right)b'.$$

Also,

$$(2) \qquad a = a\left(\frac{a'}{a'}\right) = \left(\frac{a}{a'}\right)a'.$$

Now, $a \neq 0$ because if $a = 0$ then, from (1), $b = 0$. So, if $k = \frac{a}{a'}$ then $k \neq 0$ and, from (1) and (2),

$$b = kb' \quad \text{and} \quad a = ka'.$$

Therefore,

(a, b) is proportional to (a', b').

Next, consider the case $a' = 0$. Settle this, and finally, prove the "only if part":

if (a, b) is proportional to (a', b') then $ab' = a'b$.]

13. True or false? [Use the boxed principle of Exercise 12.]

(a) $(2, 3)$ i.p.t. $(6, 9)$ (b) $\left(\frac{2}{9}, \frac{3}{5}\right)$ i.p.t. $\left(\frac{2}{21}, \frac{9}{35}\right)$

(c) $\left(\frac{21}{49}, -\frac{5}{8}\right)$ i.p.t. $\left(\frac{6}{7}, -\frac{30}{24}\right)$

(d) $\left(8, \frac{2}{3}\right)$ i.p.t. $\left(\frac{2}{3}, 8\right)$ (e) $(0, 5)$ i.p.t. $\left(0, -\frac{2}{3}\right)$

(f) $(x, 0)$ i.p.t. $(3.5, 0)$ (g) $(a - b, d - c)$ i.p.t. $(b - a, c - d)$

MORE ON SOLVING SYSTEMS OF EQUATIONS

You have proved [Theorem 2, page 1-50] that the intersection of two lines is either empty or consists of just one point. When you are given the equations for two intersecting lines in the number plane, you know that you can find the components of their common point by eliminating one of the pronumerals from the given equations.

Let us apply this procedure in the case of two lines ℓ and ℓ', whose equations are:

$$(1) \qquad 3x + 6y + 1 = 0$$

and: $$(2) \qquad 2x + 4y - 5 = 0.$$

If there is a point (x, y) in $\ell \cap \ell'$ then

$$2[3x + 6y + 1] = 0 \quad \text{and} \quad 3[2x + 4y - 5] = 0,$$

so

$$[6x + 12y + 2] - [6x + 12y - 15] = 0,$$

and

$$17 = 0!$$

Since $17 \neq 0$, there is no point (x, y) in $\ell \cap \ell'$.

Note that in our attempt to eliminate one of the pronumerals from (1) and (2), we have succeeded in eliminating both of them. This happened because the coefficients of the pronumerals in equation (1) are proportional to those in equation (2); that is, (3, 6) is proportional to (2, 4). Is it always the case that when (a, b) is proportional to (a', b'), the intersection of the lines $\{(x, y): ax + by + c = 0\}$ and $\{(x, y): a'x + b'y + c' = 0\}$ is the empty set?

Your work in Part E on page 1-51 shows that if

$$\ell = \{(x, y): ax + by + c = 0\}$$

and: $$\ell' = \{(x, y): a'x + b'y + c' = 0\},$$

then $\ell = \ell'$ if and only if (a, b, c) is proportional to (a', b', c'). [Did you consider such a case when answering the question at the end of the last paragraph?] You now see that if (a, b) is proportional to (a', b') but (a, b, c) is <u>not</u> proportional to (a', b', c') then $\ell \cap \ell' = \emptyset$.

What do you think will be the case for ℓ and ℓ' if (a, b) is not proportional to (a′, b′)? [Your work in finding points of intersection may have suggested that the answer to this question is that $\ell \cap \ell'$ consists of just one point. In one of the exercises which follow, you will prove that this is the case.]

Summarizing: If (a, b, c) is not proportional to (a′, b′, c′) then

(i) if (a, b) is proportional to (a′, b′) then $\ell \cap \ell' = \emptyset$, and

(ii) if (a, b) is not proportional to (a′, b′) then $\ell \cap \ell' \neq \emptyset$.

* * *

In carrying out proofs of statements (such as those given above, and others) it is convenient to be aware of some facts of logic.

Consider three statements:

(1) if it is raining then it is cloudy,

(2) if it is not cloudy then it is not raining,

(3) if it is cloudy then it is raining.

You will probably agree that statements (1) and (2) "say the same thing". That they do say the same thing is just a result of the way we ordinarily use the words 'if . . . then ---' and 'not . . .'. Statement (2) is called <u>the contrapositive of statement (1)</u>. Statement (1) is of the form:

if . . . then ---

[that is, it is a <u>conditional</u> statement], and statement (2) [also a conditional statement] is of the form:

if not --- then not . . . .

Like (1) and (2), a conditional statement and its contrapositive are equivalent. Here are several examples of pairs of conditional statements in which the second in each pair is the contrapositive of the first.

if $5 > 4$ then $5 > 3$

if $5 \not> 3$ then $5 \not> 4$

---

if Bangor is in Maine then Bangor is in the U. S.

if Bangor is not in the U. S. then Bangor is not in Maine

---

if St. Louis is north of Chicago then Chicago is south of New Orleans

if Chicago is not south of New Orleans then St. Louis is not north of Chicago

---

if peaches squiffle then hefferdunks are aquatic

if no hefferdunk is aquatic then there isn't a squiffling peach

---

Now, consider statements (1) and (3):

(1) if it is raining then it is cloudy,

(3) if it is cloudy then it is raining.

You will probably agree that statements (1) and (3) do not make the same assertion. In fact, it is certainly not a result of the way we use the words 'if ... then ---' that either of these statements is a logical consequence of the other. Statements (1) and (3) are conditional statements, and each is called <u>the converse of the other</u>.

In proving statements it will often be helpful to use the fact that each conditional statement implies its contrapositive, and <u>vice versa</u>. You will often make use of this fact by proving the contrapositive of the conditional statement you wish to prove. On the other hand, although it <u>may</u> happen that a statement is implied by its converse, this is not usually the case. Study these pairs of converses.

if $5 > 4$ then $5 > 3$

if $5 > 3$ then $5 > 4$

(continued on next page)

if Bangor is in Maine then Bangor is in the U. S.
if Bangor is in the U. S. then Bangor is in Maine

---

if St. Louis is north of Chicago then Chicago is south of New Orleans
if Chicago is south of New Orleans then St. Louis is north of Chicago

---

if peaches squiffle then hefferdunks are aquatic
if hefferdunks are aquatic then peaches squiffle

The conjunction of statement (1) and its converse, (3), is:

if it is raining then it is cloudy
and
if it is cloudy then it is raining.

This can be simplified to:

(4) it is raining if and only if it is cloudy.

The first 'if' in (4) corresponds to statement (3) [(3) might be written:

it is raining if it is cloudy].

The 'only if' in (4) corresponds to statement (1) [(1) might be written:

it is raining only if it is cloudy].

Statement (4) follows from the two statements (1) and (3) together, and each of (1) and (3) follows from (4). This is a result of the way in which we use the words '... if and only if ---'.

A statement like (4) is called a biconditional.

* * *

Returning to the summary on page 1-66, you will see that the contrapositive of (i) is:

(i′) if $\ell \cap \ell' \neq \emptyset$ then (a, b) is not proportional to (a′, b′).

Since we know that if (a, b, c) is not proportional to (a′, b′, c′) then

(i) if (a, b) is proportional to (a′, b′) then $\ell \cap \ell' = \emptyset$,

we know that if (a, b, c) is not proportional to (a′, b′, c′) then

(i′) if $\ell \cap \ell' \neq \emptyset$ then (a, b) is not proportional to (a′, b′).

Consequently, we can restate our summary as follows .

If (a, b, c) is not proportional to (a′, b′, c′) then

(i′) if $\ell \cap \ell' \neq \emptyset$ then (a, b) is not proportional to (a′, b′) and

(ii) if (a, b) is not proportional to (a′, b′) then $\ell \cap \ell' \neq \emptyset$.

Since (i′) and (ii) are converses, our summary is equivalent to:

If (a, b, c) is not proportional to (a′, b′, c′) then $\ell \cap \ell' \neq \emptyset$ if and only if (a, b) is not proportional to (a′, b′).

EXERCISES

A. From the summary on page 1-66, derive the following statement:

If (a, b, c) is not proportional to (a′, b′, c′) then (a, b) is proportional to (a′, b′) if and only if $\ell \cap \ell' = \emptyset$.

[Hint: You can do this in just the way we derived the statement immediately preceding these exercises.]

B. For each of the following systems of linear equations whose loci are ℓ and m, tell whether $\ell \cap m = \emptyset$ or $\ell \cap m \neq \emptyset$.

1. $\begin{cases} 5x + 2y = 8 \\ 15x + 6y = 9 \end{cases}$

2. $\begin{cases} 5x + 2y = 8 \\ 15x + 6y = 24 \end{cases}$

3. $\begin{cases} 5x + 2y = 8 \\ 15x + 7y = 2 \end{cases}$

4. $\begin{cases} 5x + 2y = 8 \\ 15x + 7y = 24 \end{cases}$

5. $\begin{cases} x + 8y = 15 \\ 2x + 16y = 30 \end{cases}$

6. $\begin{cases} 3x - 5y = 7 \\ -3x + 5y = 7 \end{cases}$

7. $\begin{cases} 8x - 5y = 12 \\ 9x + 4y = 3 \end{cases}$

8. $\begin{cases} x = 5 \\ y = 7 \end{cases}$

9. $\begin{cases} 3(x + 2) + 6(y - 1) = 5 - y \\ x = \dfrac{7(1 - y)}{3} \end{cases}$

<u>C</u>. For those exercises in Part B where you decided that $\ell \cap m \neq \emptyset$, tell whether $\ell = m$ or $\ell \neq m$.

<u>D</u>. In each of the following exercises you are given a linear equation with locus ℓ. Write three other linear equations with loci m, n, and p, respectively, such that

$$\ell = m, \quad \ell \cap n = \emptyset, \quad \text{and} \quad \ell \cap p \neq \emptyset \text{ but } \ell \neq p .$$

1. $7x + 5y = 17$
2. $x - 4y + 3 = 0$
3. $5y - 2x - 1 = 0$
4. $7 + 2x - 3y = 0$
5. $8 - 3y = 9x$
6. $1 + \frac{1}{2}x - \frac{1}{3}y = 0$

<u>E</u>. You have not yet proved (ii) of the summary on page 1-66. You will have done so when you have completed the next two exercises.

1. Use the method of eliminating a pronumeral to show that if $(x, y) \in \ell \cap \ell'$, where

$$\ell = \{(x, y):\ ax + by + c = 0\},$$

and $$\ell' = \{(x, y):\ a'x + b'y + c' = 0\}, \text{ then}$$

(1) $[ab' - a'b]x + [cb' - c'b] = 0$

and (2) $[ab' - a'b]y + [ac' - a'c] = 0.$

2. Show that if (a, b) is not proportional to (a′, b′) then

$$\left(-\frac{cb' - c'b}{ab' - a'b},\ -\frac{ac' - a'c}{ab' - a'b}\right) \in \ell \cap \ell' .$$

[Remember the need for checking.]

PARALLEL LINES

We know that two lines intersect in either a set consisting of just one point or in the empty set. In the latter case, we say that the one line is parallel to the other. We shall use the word 'parallel' to abbreviate many of our postulates and theorems. Therefore, we should be careful to say precisely what we mean by this word. Statements which assign meanings to new words in terms of familiar words are called definitions. [Each definition used to abbreviate postulates and theorems will be enclosed in a red box. Definitions for terms used in connection with models for the postulate system, such as the definition of 'line' on page 1-27, are not enclosed in red boxes.]

> Definition
>
> A line is parallel to a line if and only if their intersection is the empty set.
>
> $m \parallel n$ if and only if $m \cap n = \emptyset$.

[Note the abbreviation '$\parallel$' for 'is parallel to'.]

EXERCISES

Prove each of the following theorems. [Remember that in proving theorems we deduce them from the postulate system and do not use a model.]

1. For each line ℓ, $\ell \nparallel \ell$.

2. For each line m and each line n,
   if $m \parallel n$ then $n \parallel m$.
   [This theorem tells you that parallelism is a symmetric relation. Hence, we are justified in saying
   'm and n are parallel'

(continued on next page)

when we know that $m \parallel n$ or $n \parallel m$. Is parallelism a reflexive relation?]

3. For all lines m and n,
   if m and n are parallel then $m \neq n$.

MODELS AND POSTULATES

You will recall from our earlier discussion of models that in Stan Straight's model for the system of postulates I, II, and III, 'point' means an ordered pair of real numbers, and 'line' means $\{(x, y): ax + by + c = 0\}$, a and b not both 0. In Carl Circle's model for the three-postulate system, 'point' means an ordered pair of real numbers, and 'line' means $\{(x, y): (x - a)^2 + (y - b)^2 = c^2\}$, $c > 0$. Stan's guess that for each two points there is at most one line which contains them was false for Carl's model. Therefore, Stan knew that he could not deduce this statement from postulates I, II, and III, alone. [Naturally, if Stan could deduce the statement from the three postulates, so could Carl, because deducing statements from postulates has nothing to do with models for the postulates.] However, Stan's guess was true for his model. You know it was true because you proved this [Exercise 5 on page 1-49] by deriving it from the properties of Stan's model. We formulated Stan's guess as postulate IV. Carl's interpretation of the three-postulate system did not give him a model for the four-postulate system although Stan's interpretation did.

The way in which Stan draws pictures of his model [See conditions (1), (2), and (3) at the top of page 1-10.] is just the way we picture things we call 'lines' and 'points' in the everyday world. Statements deduced from postulates which are satisfied by Stan's model often give us useful information about the world. In fact, they permit us to make predictions about it. For this reason, the only postulates we shall add to those already set down are statements which are true of the number plane model.

Now, there are many statements about points and lines which are true of the number plane model. Which of these shall we add to the

postulate set? What we want is a set of postulates from which we can deduce all the "geometrical" statements about points and lines which are true of the number plane. We could take all such statements as postulates, but one of our purposes is to show that all these statements can be deduced from a few of them. Consequently, we shall try to choose as postulates those statements which we think will be most fruitful as a basis for deducing others. [Of course, you can't be sure in advance about which statements will be fruitful postulates, any more than you can be sure of success in tackling any difficult problem. But, the more experience you have had, the more likely you are to be successful.] Since we wish to end up with only a few postulates, we shall try to add to our set of postulates only statements which we think cannot be deduced from the postulates already chosen. A way to tell that a statement <u>cannot</u> be deduced from the postulates is to find a model for these postulates for which the statement is false. [This is just what we did with Stan's guess. Carl's model was one for which Stan's guess was false.]

Before adding a statement to the postulate set, we make sure that it is true of the number plane model by deriving it from properties of ordered pairs of real numbers and the definitions of 'point' and 'line' which describe this model. [Sometimes it takes a lot of complicated algebra to derive the statement. In fact, it may take more algebra than we want to take time to teach you. In such cases (there will be very few of these), we shall ask you to take our word that the statements <u>can</u> be derived.]

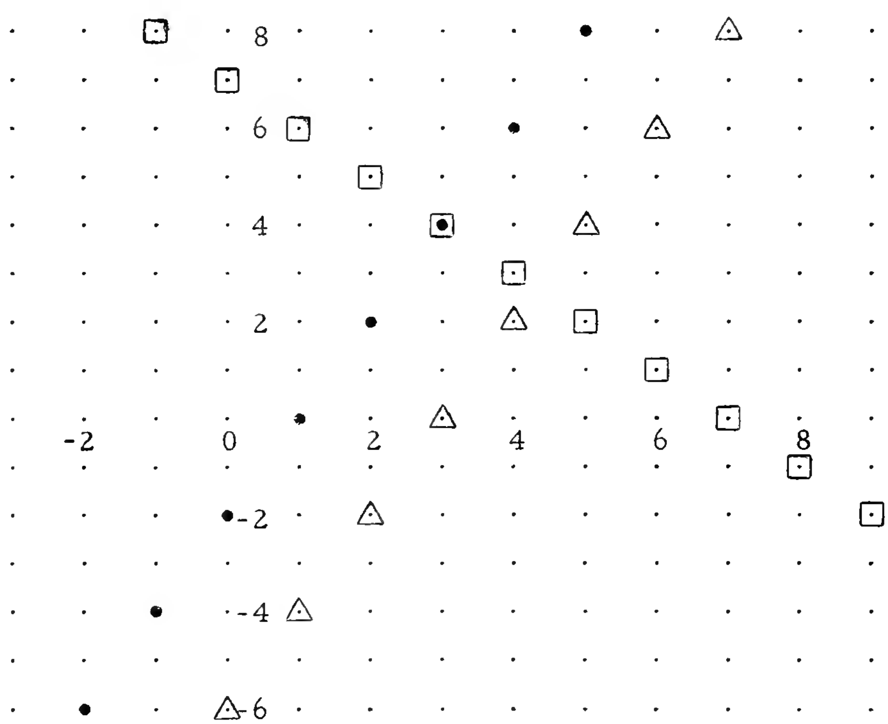
* * *

Here is an example of a statement which Stan Straight thinks is true of his model.

> For each three lines ℓ, m, and n,
> if ℓ is parallel to m, and
> m is parallel to n,
> then ℓ is parallel to n.

[It is natural for Stan to make this guess because he has been thinking a great deal about parallel lines, and has been studying them by means of pictures of his model.] He tries to establish this statement by trying to deduce it from postulates I, II, III, and IV, the definition of 'parallel lines', and the theorems. He is unsuccessful, and so would you be, no matter how hard you tried. Here is why.

Stan Straight, after long hours of trying to deduce the statement, telephones Larry Lattice and tells Larry about his guess. Now, Larry doesn't interpret 'point' and 'line' in the same way Stan does. Larry thinks a point is an ordered pair of <u>integers</u>, and that a line is a non-empty set of all ordered pairs of integers (x, y) which satisfy '$ax + by + c = 0$' [a, b, and c are integers, a and b are not both 0]. Larry's interpretation does give him <u>a</u> number plane model for the four-postulate system. Pictures of points and lines in his model are like pictures you drew when you studied plane lattices in an earlier unit. When Larry hears Stan's statement, he makes a quick sketch like this:



and says, "No wonder you couldn't deduce it from the four postulates. I've got an example which shows that the statement you made is false."

Larry's $\bullet$ -line is parallel to his $\triangle$ -line, and his $\triangle$ -line is parallel to his $\square$ -line, because they have no points in common. Yet, the $\bullet$ -line is not parallel to the $\square$ -line since each line contains (3, 4).

Do you see why it is impossible for Stan to deduce his statement from the four-postulate system? [If he could, it would be a theorem in the system, and would then be true in every model for the system, including Larry's.]

EXERCISES

A. Stan realizes that he cannot deduce his statement from the four-postulate system. Also, he has not been able to find a counter-example in his own model. So, he tries to derive the statement from properties of his model. Here is how he starts his proof; <u>your job is to finish it.</u>

For each three lines ℓ, m, and n, if $\ell \parallel m$, there are linear equations of ℓ and m:

$$(1) \quad ax + by + c = 0$$

and: $$(2) \quad a'x + b'y + c' = 0,$$

respectively, such that (a, b) is proportional to (a′, b′). [(a, b, c) is not proportional to (a′, b′, c′) because $\ell \neq m$.] If $m \parallel n$, there is a linear equation of n:

$$(3) \quad a''x + b''y + c'' = 0$$

such that (a′, b′) is proportional to (a″, b″). [Since $m \neq n$, (a′, b′, c′) is not proportional to (a″ , b″, c″), and since $\ell \neq n$, (a, b, c) is not proportional to (a″, b″, c″).]

Now, ...

(more space on next page)

B. In Part A you derived Stan's statement from properties of his model. You also know that it cannot be deduced from the four-postulate system. We shall soon conjecture other statements about parallel lines. If we add Stan's statement to the postulate set, it may prove useful to us in trying to deduce other propositions from the postulates. So, we add to our set the postulate

(continued on next page)

V. For each three lines ℓ, m, and n, if $\ell \parallel m$, and $m \parallel n$, then $\ell \parallel n$.

In the following exercises you are given several statements about parallel lines. Some can be deduced from the five-postulate system, some can be derived by using the properties of Stan's model but cannot be deduced from the postulate system, and some are false in Stan's model. Those which are deducible from the five-postulate system are theorems. Those which can be derived from properties of Stan's model, but cannot be deduced from the five-postulate system, could be added as postulates to the five-postulate set.

Deduce from the five-postulate system as many of these statements as you can. For those which you cannot deduce, it may be the case that

(1) you have not been clever enough, or
(2) it is not a theorem in the five-postulate system.

Show that it is the second of these cases by exhibiting a counter-example in some model.

1. For each line m and each point P, there is at most one line n such that $P \in n$ and $n \parallel m$.

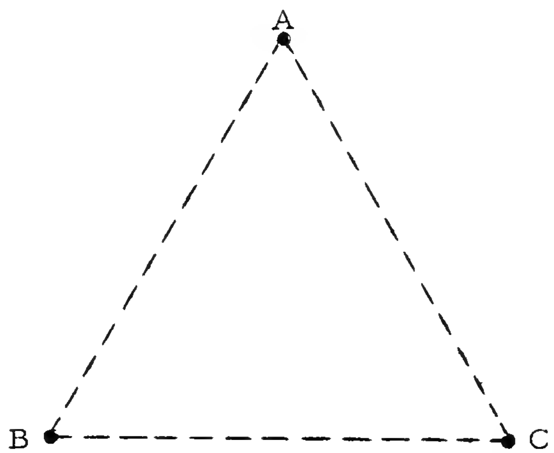
2. For each three lines, if two of these lines are parallel and one of them is intersected by the third, the other is intersected by the third.

3. For each three lines p, q, and r, if $p \not\parallel q$, and $q \not\parallel r$, then $p \not\parallel r$.

4. For each line m and each point $P \notin m$, there is at least one line n such that $P \in n$ and $n \parallel m$.

C. Tell which of the sets of objects described on pages 1-42 through 1-45 are models for the postulates I - V.

Sample. We showed on page 1-50 that the set of objects described in the Sample on page 1-42 was a model for postulates I - IV. Recall that the points are the three businessmen A, B, and C, and that the lines are the three partnerships $\{A, B\}$, $\{B, C\}$, and $\{A, C\}$.



Solution. We need to check postulate V. Let us restate this postulate in terms of 'businessmen' and 'partnerships'.

> For each three partnerships ℓ, m, and n, if ℓ and m have no common partners, and m and n have no common partners, then ℓ and n have no common partner.

Now, for this postulate to fail to check, we would have to find a counter-example in this model. This means that we would have to find two things:

(1) A pair of partnerships, ℓ and m, with no common partners,

and

(2) another pair of partnerships, m and n, with no common partners,

(continued on next page)

such that the pair of partnerships, ℓ and n, do have a common partner. It is clear that we cannot find (1) and (2). So, this model for postulates I - IV does not have a counter-example for postulate V. Therefore, the set of objects is a model for postulates I - V. That is, postulate V is true of this model. [Note carefully the two statements

'there are no parallel lines in the model'

and

'<u>if</u> $\ell \parallel m$, and $m \parallel n$, <u>then</u> $\ell \parallel n$'.

Both of these are true of the model in question.]

*<u>D</u>. Which of the sets given below are lines in Larry Lattice's model?

1. $\{(x, y):$ x and y are integers, and $x + 4y - 7 = 0\}$
2. $\{(x, y):$ x and y are integers, and $3x + 3y - 4 = 0\}$
3. $\{(x, y):$ x and y are integers, and $\frac{2}{3}x - \frac{5}{7}y + 4 = 0\}$
4. $\{(x, y):$ x and y are integers, and $\sqrt{2}x + y = 0\}$

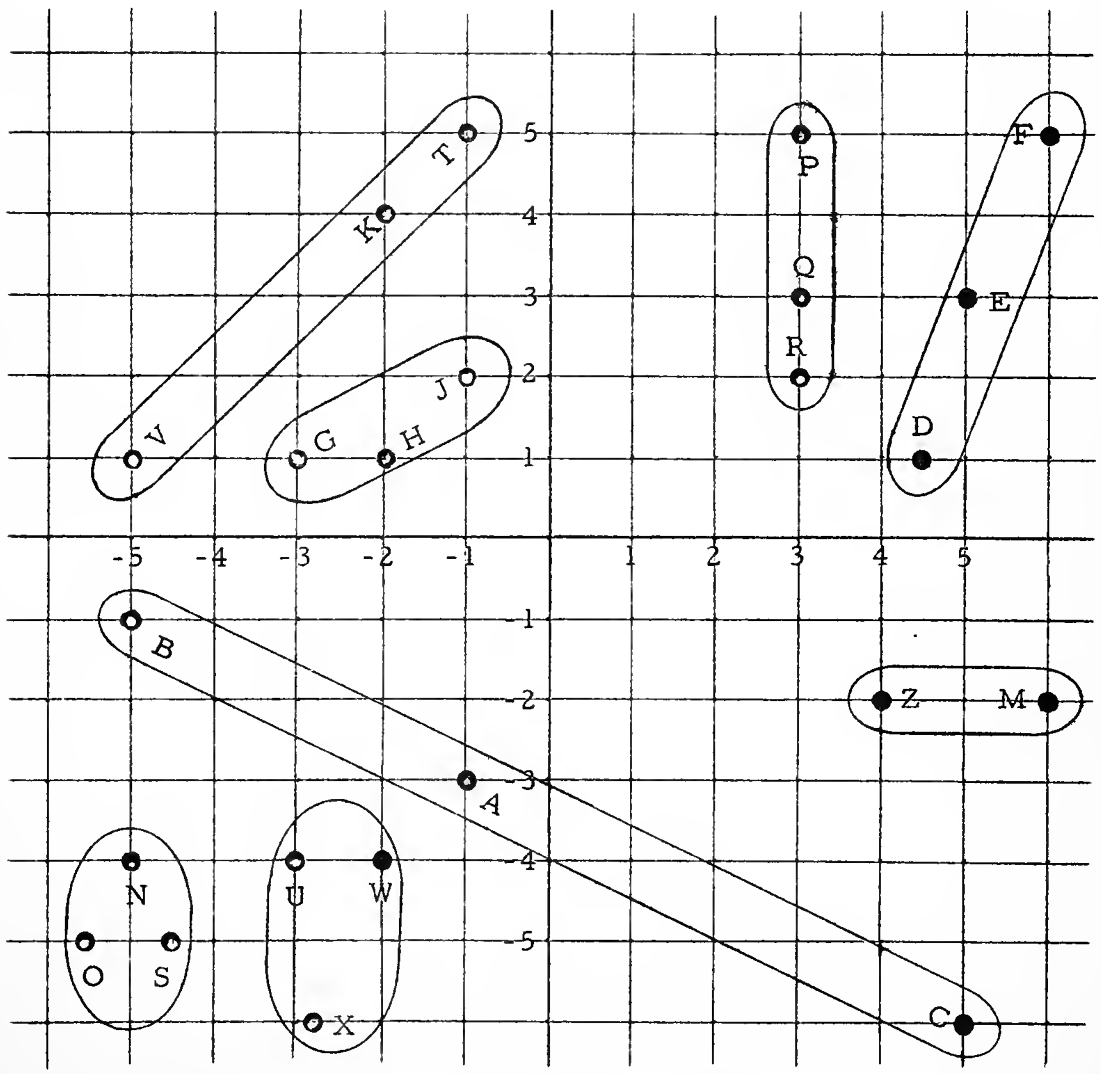
1.04 <u>Primitive terms</u>.--A model for a postulate system is obtained by assigning meanings to certain of the words which occur in the postulates. These words are called <u>primitive terms</u>. Up to this point in the development of our postulate system, we have used two primitive terms, 'point' and 'line'. [We have assumed that everyone has the same interpretation for all the other words (for example, 'each', 'three', 'set', 'ϵ', and 'there is') used in the postulates, theorems, and definitions. Also, we have assumed that everyone who is willing to work with the postulates uses the same kind of logic in his reasoning.] A model for a postulate set in which 'point' and 'line' are primitive terms is obtained by describing what are to be called 'points' and 'lines'. Of course, you must make certain that the postulates are satisfied by these points and lines.

In the development of our full postulate set, we shall introduce just two more primitive terms. Anyone who wants to construct a model for a postulate set in which the four primitive terms are used must give an interpretation not only for 'point' and 'line' but also for the other two primitive terms. Since we want to continue with our number plane model, we shall have to assign number plane meanings to these new primitive terms.

From another point of view, what we need to do is to study the number plane model, and, in this way, discover new properties which involve other concepts than those of point and line. In stating these properties we shall have to use new words for these concepts, and these words will be our new primitive terms.

BETWEENNESS

Here is a picture of several sets of points in Stan's model.



Stan notices that {K, T, V} differs from {U, W, X} in a way that seems interesting to him. He expresses this by saying that K is between T and V, but none of the points U, W, and X is between the other two. He also says that Q is between P and R. Look at the pictures of the other sets, and guess what Stan said about the points in each of them.

Do you agree with Stan that none of the points D, E, and F is between any two of them? Stan decided that he would not say that a point is between two points unless these points all belong to the same line [that is, unless the points are <u>collinear</u>]. His decision agrees more or less with the way people ordinarily use the word 'between' in talking about the world. When they say one object is between two others, they usually imply that the objects are "lined up".

Since we are going to use the words '__ is between . . . and - - -' frequently, it will be convenient to abbreviate them by '[. . . __ - - -]'. For example, the statement:

K is between V and T

will be abbreviated as:

[VKT].

[Remember that '[VKT]' is a complete statement, and it is read as 'K is between V and T'.] Practice using this new notation by reading aloud each of the following statements which refer to the sets pictured on page 1-80.

(1) [RQP].

(2) [PQR].

(3) It is not the case that [QPR].

(4) [BAC].

(5) It is not the case that [SON].

(6) It is not the case that [LZM].

(7) [(-5, 2)(-4, 3)(-3, 4)].

(8) It is not the case that [(1, 0)(0, 0)(0, 1)].

After reading the eight statements above [all of which are correct] you see that it is not difficult to tell, in most cases, whether or not one of three pictured points is between the others. But, since we want to refer to betweenness in later postulates and theorems, we need a <u>precise</u> interpretation of '[. . . —— - - -]' so that for any three ordered pairs (x_0, y_0), (x_1, y_1), and (x_2, y_2), we can tell whether or not $[(x_0, y_0), (x_1, y_1), (x_2, y_2)]$. The following exercises should help you arrive at a precise interpretation.

EXERCISES

<u>A</u>. In each of the following exercises you are given a set of collinear points. Your job is to state which of the points is between the other two. Try to do this without making a picture.

<u>Sample</u> <u>1</u>. $\{(-2,0), (5.5,0), (3,0)\}$ . . . . . $[(-2,0)(3,0)(5.5,0)]$

<u>Sample</u> <u>2</u>. $\{(4,0), (6, -1\frac{1}{2}), (2, 1\frac{1}{2})\}$ . . . . . $[(2, 1\frac{1}{2})(4,0)(6, -1\frac{1}{2})]$

1. $\{(7, 0), (2, 0), (4, 0)\}$
2. $\{(-8, 0), (0, 0), (-10, 0)\}$
3. $\{(0, 0), (0, 5), (0, 5.1)\}$
4. $\{(0, -7), (0, -9), (0, -1)\}$
5. $\{(3, 4), (-2, 4), (7, 4)\}$
6. $\{(-2, 7), (-2, -3), (-2, 10)\}$
7. $\{(3, 3), (5, 5), (4, 4)\}$
8. $\{(7, 10), (9, 13), (5, 7)\}$
9. $\{(3, 8), (5, 6), (-1, 12)\}$
10. $\{(5, -1), (2, 2), (11, -7)\}$
11. $\{(6, y_0), (3, y_1), (12, y_2)\}$
12. $\{(-5, y_0), (0, y_1), (7, y_2)\}$
13. $\{(x_0, 2), (x_1, -5), (x_2, 1)\}$

(continued on next page)

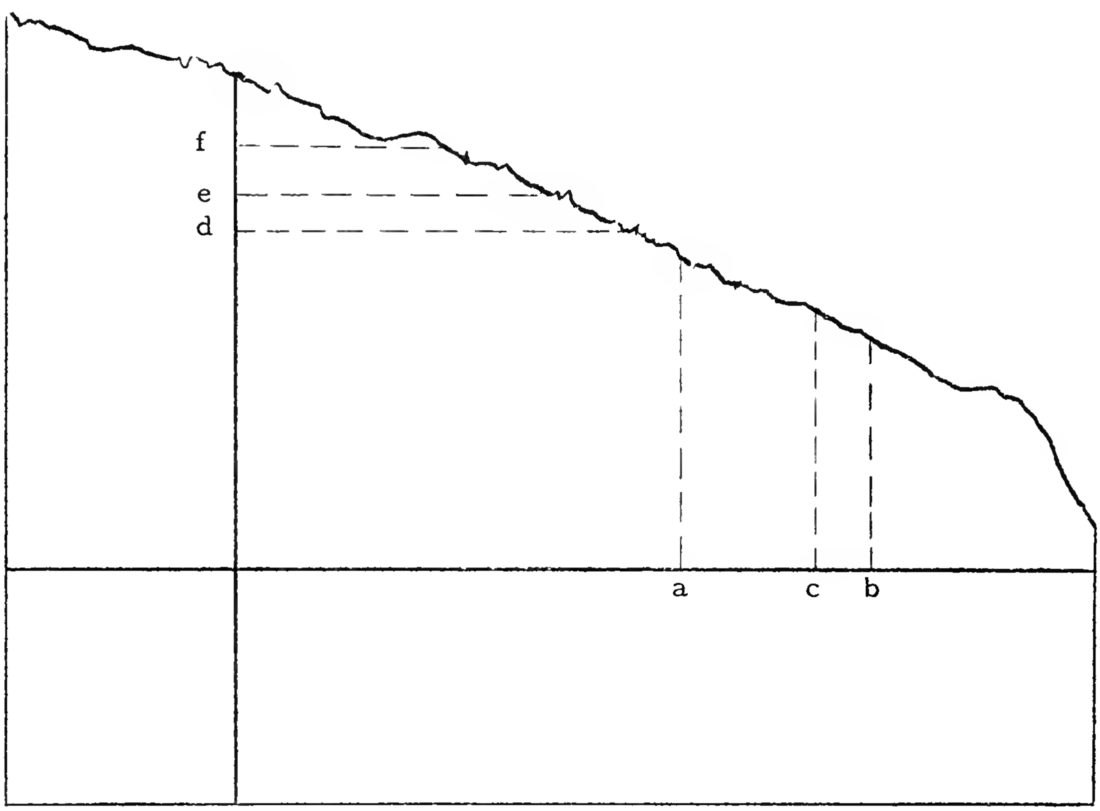
14. $\{(x_0, -6), (x_1, -12), (x_2, -3)\}$

15. $\{(3, 7), (8, 5), (3, 7)\}$

16. $\{(x_0, 6), (x_1, 7), (x_0, 6)\}$

17. $\{(3, y_0), (4, y_0), (4, y_0)\}$

B. The picture below is one of a drawing of the number plane model in which part of the drawing has been torn away. The part torn away contained a picture of part of a line with three dots marked on it. The dotted lines point to the projections of the three dots on the component axes. Thus, a, b, and c are the first components of the three points, and d, e, and f are the second components.



1. Sketch in the missing line showing on it the dots corresponding to the ordered pairs (a, d), (b, f), and (c, e).
2. Repeat for the ordered pairs (a, f), (b, d), and (c, e).
3. Repeat for the ordered pairs (a, e), (b, f), and (c, d).
4. Repeat for the ordered pairs (a, d), (b, e), and (c, f).

* * *

Here is a precise interpretation of '[. . . _____]' for the number plane model.

For all ordered pairs (x_0, y_0), (x_1, y_1), and (x_2, y_2),

$$[(x_0, y_0)(x_2, y_2)(x_1, y_1)]$$

if and only if

(1) there is a linear equation whose solution set contains $\{(x_0, y_0), (x_1, y_1), (x_2, y_2)\}$,

and (2) $x_0 < x_2 < x_1$ or $x_0 > x_2 > x_1$ or $y_0 < y_2 < y_1$ or $y_0 > y_2 > y_1$.

* * *

C. True or false?

Sample 1. $[(-2, 2)(8, 12)(3, 7)]$.

Solution. We see immediately that it is not the case that either $-2 < 8 < 3$, or $-2 > 8 > 3$, or $2 < 12 < 7$, or $2 > 12 > 7$. So, by (2) above, we know that the statement is false.

Sample 2. $[(-2, 2)(3, 7)(8, 12)]$.

Solution. We see immediately that at least one of the four conditions given in (2) is satisfied. [Which of the four?] So, now we check to see that the points are collinear. On page 1-35, we have a formula for a linear equation which is satisfied by two

(continued on next page)

points, (x_0, y_0) and (x_1, y_1).

$$(y_1 - y_0)(x - x_0) - (x_1 - x_0)(y - y_0) = 0$$

Using the given points (-2, 2) and (3, 7), we obtain:

$$(7 - 2)(x + 2) - (3 + 2)(y - 2) = 0.$$

Check the third point, (8, 12):

$$\begin{aligned} &(7 - 2)(8 + 2) - (3 + 2)(12 - 2) \\ = &\quad 50 \quad - \quad 50 \\ = &\ 0. \end{aligned}$$

So, the points are collinear and the given statement is true.

1. [(4, 16)(12, 8)(8, 12)]
2. [(157, 84)(162, 86)(217, 108)]
3. [(5, -1)(7, 4)(11, 8)]
4. [(367, -187)(303, -110)(297, -103)]
5. [(-15, -20)(-4, -16)(-7, 11)]
6. [(365, 4)(366, 5)(367, 6)]
7. [(-4, 8)(0, 5)(2, 0)]
8. [(-14, -10)(-13.8, 0)(8, 8)]

D. Given two ordered pairs with first components equal to k. Show that every ordered pair which is on the line containing the given two ordered pairs must have first component k.

Repeat for two ordered pairs with second components equal to k.

E. 1. You know that, for every a and b, a < b if and only if b - a is a positive number. Use this fact to show that

$$x_0 < x_2 < x_1 \quad \text{or} \quad x_0 > x_2 > x_1$$

if and only if

$$(x_1 - x_2)(x_2 - x_0) > 0.$$

2. Use the result of Exercise 1 to replace the four double inequalities in (2) of the dashed box by two inequalities.

F. Show that if the points (x_0, y_0), (x_1, y_1), and (x_2, y_2) are collinear then

$$(y_0 - y_2)(x_1 - x_2) - (x_0 - x_2)(y_1 - y_2) = 0.$$

G. 1. Show that, for every three ordered pairs (x_0, y_0), (x_1, y_1), and (x_2, y_2), if (x_0, y_0), (x_1, y_1), and (x_2, y_2) are collinear, and if $x_0 < x_2 < x_1$, then either $y_0 < y_2 < y_1$ or $y_0 > y_2 > y_1$ or $y_0 = y_2 = y_1$.

[Hint: Use the result of Part F and the fact that if $x_0 < x_2 < x_1$ then $x_1 - x_2$ is positive and $x_0 - x_2$ is negative.]

2. Use Exercise 1 to show that the points referred to in Exercise 5 of Part C are not collinear.

H. Use the results of Exercises 1 of Parts E and G to show that for the collinear ordered pairs (x_a, y_a), (x_b, y_b), and (x_c, y_c),

if $(y_c - y_b)(y_b - y_a) > 0$

then either

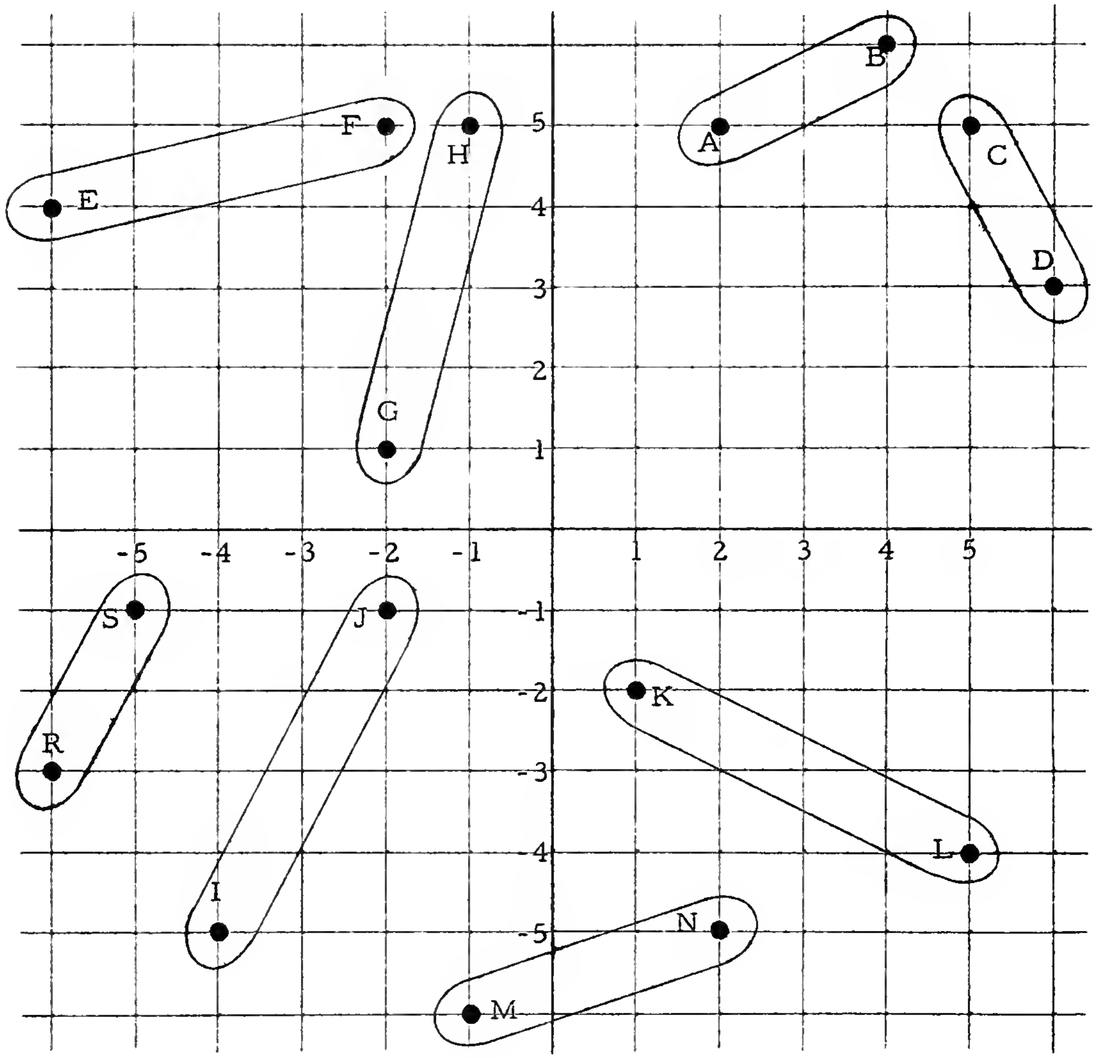
$$(x_c - x_b)(x_b - x_a) > 0 \text{ or } x_a = x_b = x_c.$$

I. Show that if some two points of a line have different first components, then each two points of the line have different first components. [Compare with Part D.]

CONGRUENCE

We now have three primitive terms, 'point', 'line', and '[. . . ___ - - -]' for which we have interpretations in the number plane model. We have one more to deal with.

Here is a picture of several sets of points in Stan's model.



Stan notices that the distance between A and B is the same as the distance between C and D. He also sees that he can make similar statements about the members of other sets he has pictured. Since he believes that he will often want to make statements of this kind, he decides on a briefer way of making them. Instead of saying:

the distance between A and B
is the same as
the distance between C and D

he says:

$\{A, B\}$ is congruent to $\{C, D\}$.

[The word congruent is frequently used in everyday speech to refer to objects which have the same "size and shape". So Stan's choice of abbreviation is a pretty good one.]

Look at the pictures of the other sets and check these statements.

(1) $\{H, G\}$ is congruent to $\{E, F\}$

(2) $\{C, D\}$ is congruent to $\{R, S\}$

(3) $\{A, B\}$ is congruent to $\{R, S\}$

(4) $\{E, F\}$ is not congruent to $\{B, A\}$

[We abbreviate '... is congruent to ---' by '... ≅ ---'.]

(5) $\{J, I\} \cong \{K, L\}$

(6) $\{J, I\} \not\cong \{M, N\}$

(7) $\{M, N\} \cong \{M, N\}$

(8) $\{(0, 0), (1, 1)\} \cong \{(0, 1), (1, 0)\}$

(9) $\{(-6, -6), (-1, 6)\} \cong \{(6, -2), (-6, 3)\}$

Statement (9) is not as easy to check as the others. However, as you may already have done in checking the other statements, you can use the distance formula [page 1-13].

$$\sqrt{(-1 + 6)^2 + (6 + 6)^2} = \sqrt{(-6 - 6)^2 + (3 + 2)^2} \quad .$$

So, our fourth (and, last) primitive term, '... $\cong$ ---', and its interpretation for the number plane model is given in the following statement.

> For every (x_1, y_1), (x_2, y_2), (x_1', y_1'), and (x_2', y_2'),
>
> $$\{(x_1, y_1), (x_2, y_2)\} \cong \{(x_1', y_1'), (x_2', y_2')\}$$
>
> if and only if
>
> $$d\Big((x_1, y_1), (x_2, y_2)\Big) = d\Big((x_1', y_1'), (x_2', y_2')\Big).$$

EXERCISES

A. Use the above interpretation of '... $\cong$ ---' to tell which of the following statements are true of the number plane model and which are false for it.

1. $\{(4, 3), (9, 15)\} \cong \{(6, -2), (18, -7)\}$
2. $\{(5, 5), (0, 0)\} \cong \{(6, 6), (1, 1)\}$
3. $\{(9, 4), (10, 0)\} \cong \{(10, 0), (14, -2)\}$
4. $\{(7, 2), (-2, 11)\} \cong \{(19, -13), 7, -8)\}$
5. $\{(19, -50), (28, 41)\} \cong \{(7, 2), (-2, 11)\}$
6. $\{(3, 5), (9, 7)\} \cong \{(9, 7), (3, 5)\}$
7. $\{(8, -2), (3, 12)\} \cong \{(8, -2), (3, 12)\}$

B. Show that, for every (x_1, y_1) and (x_2, y_2),

$$\{(x_1, y_1), (x_2, y_2)\} \cong \{(x_1, y_1), (x_2, y_2)\}.$$

In other words, show that in the case of the number plane model, congruence is a reflexive relation.

C. Show that, for every (x_1, y_1), (x_2, y_2), (x_1', y_1'), and (x_2', y_2'),

$$\text{if } \{(x_1, y_1), (x_2, y_2)\} \cong \{(x_1', y_1'), (x_2', y_2')\}$$
$$\text{then } \{(x_1', y_1'), (x_2', y_2')\} \cong \{(x_1, y_1), (x_2, y_2)\}.$$

In other words, show that congruence, in the case of the number plane model, is a symmetric relation.

D. 1. If $\{(87t + 43, 16t - 13), (84t + 42, 12t - 14)\} \cong \{(0, 0), (1, 1)\}$, then $\{(87t + 43, 16t - 13), (84t + 42, 12t - 14)\}$ is congruent to which point couples in the following list?

$\{(7, 5), (9, 8)\}$	$\{(3, 5), (4, 2)\}$	$\{(6, 8), (7, 9)\}$
$\{(-5, 6), (4, 3)\}$	$\{(5, 5), (6, 6)\}$	$\{(k, m), (k + 1, m + 1)\}$

2. Show that in the case of the number plane model, congruence is a transitive relation. That is, show that, for every (x_1, y_1), (x_2, y_2), (x_1', y_1'), (x_2', y_2'), (x_1'', y_1''), and (x_2'', y_2''),

$$\text{if } \{(x_1, y_1), (x_2, y_2)\} \cong \{(x_1', y_1'), (x_2', y_2')$$
$$\text{and } \{(x_1', y_1'), (x_2', y_2')\} \cong \{(x_1'', y_1''+, (x_2'', y_2'')\},$$
$$\text{then } \{(x_1, y_1), (x_2, y_2)\} \cong \{(x_1'', y_1''), (x_2'', y_2'')\}.$$

E. Use the results of Parts C and D to derive the following statement.

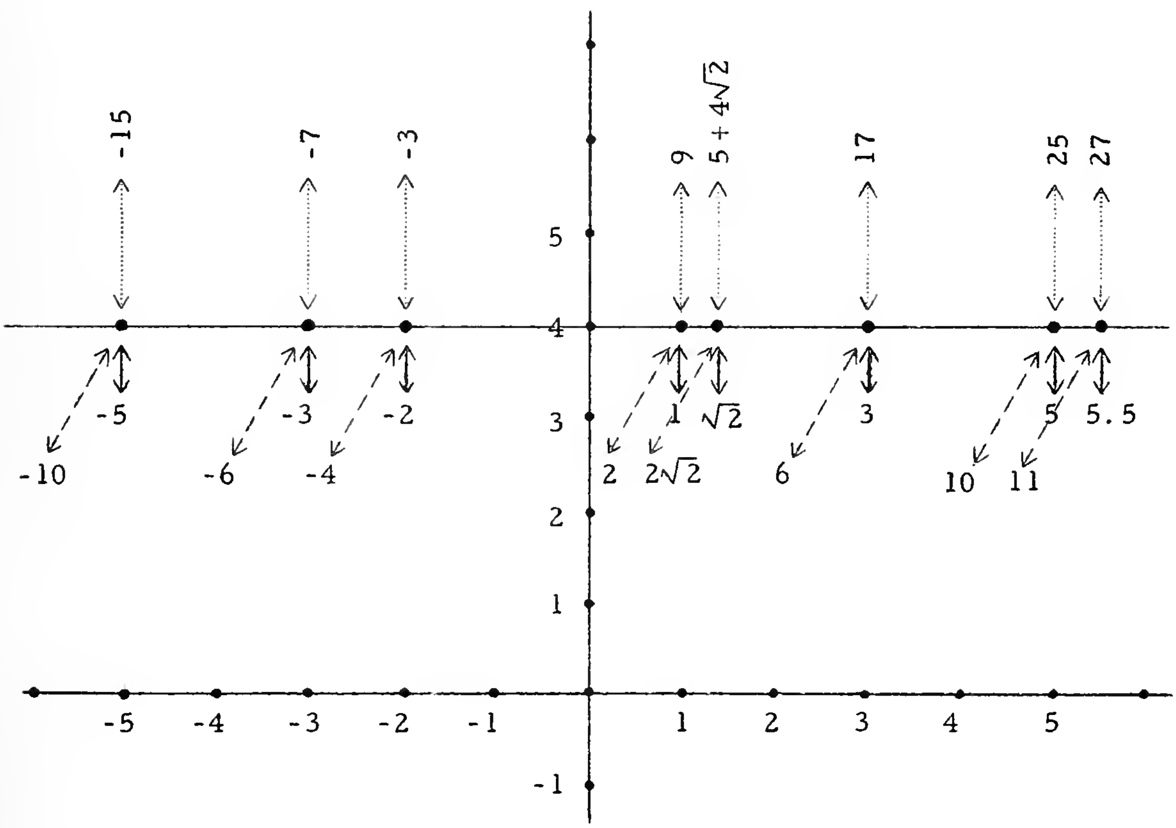
For all points A, B, A′, B′, A″, and B″,

if $\{A', B'\} \cong \{A, B\}$, and $\{A'', B''\} \cong \{A, B\}$,
then $\{A', B'\} \cong \{A'', B''\}$.

* * *

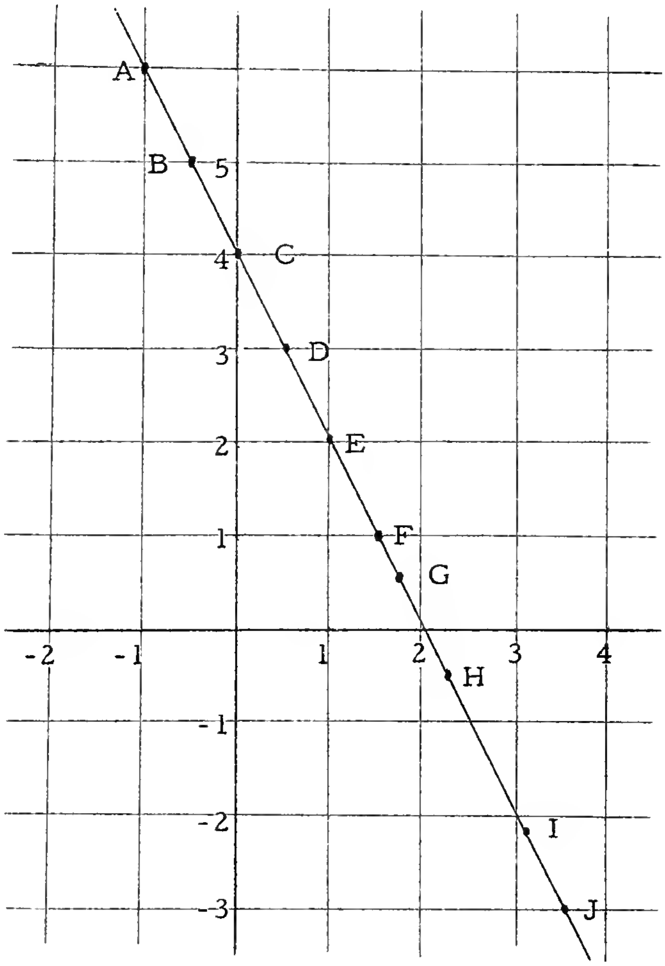
The next statement which we shall add to our postulate set will refer to the concepts of betweenness and congruence. In the next section you will learn some of the algebra which you would need to show that this postulate is true of the number plane model.

1.05 <u>(1-1)-correspondences between points on a line and real numbers</u>.-- Consider one of the lines in the number plane which is parallel to the first component axis, for example, $\{(x, y) : y = 4\}$. There are many (1-1)-correspondences between the points on that line and the real numbers. For example, each point on the line can be paired with its first component. Or, each point can be paired with twice its first component. Or, each point can be paired with 5 more than 4 times its first component. Some of the pairings in each of these three (1-1)-correspondences are shown in the picture. Clearly, there is no end to the number of such (1-1)-correspondences you can construct between the points of this line and the real numbers. Similarly, you can construct many (1-1)-correspondences



between the points of a line parallel to the second component axis, and the real numbers. In fact, you can construct a countless number of (1-1)-correspondences between the points of <u>any</u> line and the real

numbers. In the diagram below there is a picture of part of the line $\{(x, y): 2x + y - 4 = 0\}$ on which are marked several points. A (1-1)-correspondence between the points of this line and the real numbers is



obtained by pairing each point with its first component. If you think of this (1-1)-correspondence as a set of ordered pairs

(point, real number)

and call this set of pairs 'f', then a few of the elements in the set f are

$$(A, -1), \quad (B, -\tfrac{1}{2}), \quad (C, 0), \quad (D, \tfrac{1}{2}), \quad (E, 1),$$

$$(F, \tfrac{3}{2}), \quad (G, \tfrac{7}{4}), \quad (H, \tfrac{9}{4}), \quad (I, \pi), \quad (J, \tfrac{7}{2}).$$

Another (1-1)-correspondence between the points of this line and the real numbers is obtained by pairing each point with its second component. Call this (1-1)-correspondence 's'. A few of the elements in s are

$$(A, 6), \quad (B, 5), \quad (C, 4), \quad (D, 3), \quad (E, 2),$$

$$(F, 1), (G, \frac{1}{2}), (H, -\frac{1}{2}), (I, 4 - 2\pi), (J, -3).$$

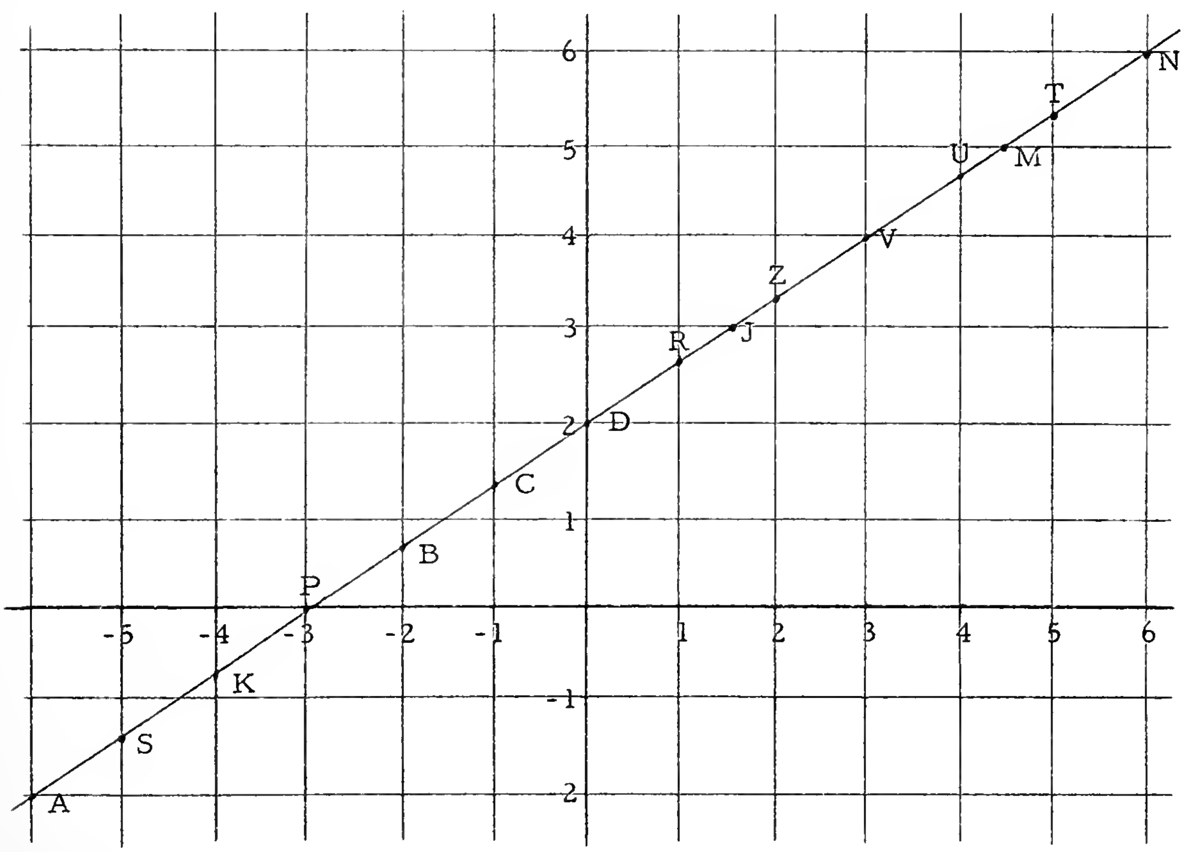
Still another (1-1)-correspondence is obtained by pairing each point with the real number which is 3 more than 4 times its first component. If c is this (1-1)-correspondence then a few of the elements in c are

$$(A, -1), \quad (B, 1), \quad (C, 3), \quad (D, 5), \quad (E, 7),$$

$$(F, 9), (G, 10), (H, 12), (I, 4\pi - 3), (J, 17).$$

EXERCISES

A. Here is a picture of the line $\{(x, y): \ 2x - 3y + 6 = 0\}$.



1. List six of the elements in the set g, where g is the (1-1)-correspondence which pairs each point of this line with the real number which is its first component.

2. List six of the elements in the set h, where h is the (1-1)-correspondence which pairs each point of this line with the real number which is twice its second component.

3. List six of the elements in the set k, where k is the (1-1)-correspondence which pairs each point of this line with the real number which is 3 less than 5 times its first component.

4. Suppose c is the (1-1)-correspondence which pairs each point (x, y) in this line with the real number which is 7 more than 3 times its second component. Another way of describing c is:

$$\text{For each } (x, y) \text{ in } \{(x, y): 2x - 3y + 6 = 0\},$$
$$c((x, y)) = 3y + 7.$$

[Read 'c(---)' as 'see of ---'.]
So, $c(A) = 3(-2) + 7 = 1$, and $c((3, 4)) = c(V) = 3(4) + 7 = 19$.
The symbol 'c((3, 4))' is a name for a real number, and this real number is computed by using the formula '$c((x, y)) = 3y + 7$'.

Compute each of the following. [Refer to Exercises 1, 2, and 3 to obtain descriptions of g, h, and k.]

(a) c(S)	(b) c(B)	(c) c(J)	(d) c(N)
(e) c(D)	(f) c(C)	(g) c(M)	(h) c(K)
(i) g(R)	(j) g(U)	(k) g(N)	(l) g(V)
(m) h(P)	(n) h(R)	(o) h(S)	(p) h(M)
(q) k(S)	(r) k(Z)	(s) k(C)	(t) k(J)
(u) c(A)	(v) g(A)	(w) h(A)	(x) k(A)

B. In Exercise 4 of Part A, we said that the (1-1)-correspondence c is the set of ordered pairs of the form of:

$$\Big((x, y), c((x, y))\Big),$$

where $(x, y) \in \{(x, y): 2x - 3y + 6 = 0\}$ and $c((x, y)) = 3y + 7$. Each element in c is an ordered pair in which the first component is a point, (x, y), of the given line, and the second component is a real number, c((x, y)). There is another set closely associated with c, in which each element is an ordered pair of the form of:

$$\Big(c((x, y)), (x, y)\Big).$$

In these ordered pairs, the first component is a real number, c((x, y)), and the second component is a point, (x, y), of the given line. You can get all the elements in this second set simply by interchanging the components of the elements in c. The second set is called <u>the inverse of c</u> and is labelled 'c^{-1}'. c^{-1} is a (1-1)-correspondence which pairs the real numbers with the points of the given line. Then, just as 'c((3, 4))' is regarded as a name for the real number 19, '$c^{-1}(19)$' [read as 'see inverse of 19'] can be regarded as a name for the point (3, 4). Study each of the following lists. They contain a few of the elements in c and a few of the elements in c^{-1}.

c	c^{-1}
$\big((x, y), c((x, y))\big)$	$\big(c((x, y)), (x, y)\big)$
$\vdots$	$\vdots$
(A, 1)	(1, A)
(S, 3)	(3, S)
(P, 7)	(7, P)
((3, 4), 19)	(19, (3, 4))
$((8, \frac{22}{3}), 29)$	$(29, (8, \frac{22}{3}))$
$\vdots$	$\vdots$
$\big((100, \frac{206}{3}), 213\big)$	$\vdots$
$\vdots$	$\big(373, (180, 122)\big)$
	$\vdots$

How do we know that the point $c^{-1}(373)$ is (180, 122)? $c^{-1}(373)$ is that point in $\{(x, y): 2x - 3y + 6 = 0\}$ which corresponds under c with the real number 373. Since $c((x, y)) = 3y + 7$, the second component, y, of this point is the root of the equation:

$$3y + 7 = 373,$$

that is, the second component is 122. But, if the second component is 122, the first component, x, is the root of the equation:

$$2x - 3(122) + 6 = 0,$$

or, 180. So, $c^{-1}(373) = (180, 122)$.

1. List six of the elements of c^{-1} other than those given in the list above.

2. Compute.

   (a) $c^{-1}(7)$ (b) $c^{-1}(247)$ (c) $c^{-1}(-14)$ (d) $c^{-1}(\pi)$

(continued on next page)

3. List six of the elements of g^{-1}. Of h^{-1}. Of k^{-1}.

4. Compute.

(a) $g^{-1}(4)$ (b) $g^{-1}(12)$ (c) $h^{-1}(6)$ (d) $h^{-1}(17)$

(e) $k^{-1}(12)$ (f) $k^{-1}(32)$ (g) $c^{-1}(0)$ (h) $k^{-1}(0)$

(i) $c^{-1}\left(c((6,\ 6))\right)$ (j) $k^{-1}\left(k((-6,\ -2))\right)$

(k) $g^{-1}(t)$ (l) $h^{-1}(t)$ (m) $k^{-1}(t)$ (n) $c^{-1}(t)$

C. Consider the line ℓ, where $\ell = \{(x, y):\ 4x + 3y - 12 = 0\}$. Let c be the (1-1)-correspondence between the points of this line and the real numbers, such that

$$c((x,\ y)) = 2x - 1.$$

1. Compute.

(a) $c((3, 0))$ (b) $c((-6,\ 12))$ (c) $c^{-1}(17)$ (d) $c^{-1}(0)$

2. Compute $c^{-1}(t)$. That is, find a formula for the point which corresponds, under the (1-1)-correspondence c, with the real number t.

PARAMETRIC EQUATIONS FOR A LINE

In Exercise 2 of Part C above, you were asked to compute $c^{-1}(t)$ where c^{-1} is the inverse of a (1-1)-correspondence c which pairs each point (x, y) of ℓ [$\ell = \{(x, y): 4x + 3y - 12 = 0\}$] with the real number t [$t = 2x - 1$].

For each real number t and its corresponding point (x, y),

$$t = 2x - 1,$$

or,

$$x = \frac{1 + t}{2}.$$

But, if x is the first component of a point on ℓ, then the second component y can be obtained by solving for 'y' the equation:

$$4\left(\frac{1 + t}{2}\right) + 3y - 12 = 0.$$

$$2 + 2t + 3y - 12 = 0$$
$$3y = 10 - 2t$$
$$y = \frac{10 - 2t}{3} .$$

So, for each real number t, the point $\left(\frac{1 + t}{2}, \frac{10 - 2t}{3}\right)$ is on ℓ . In fact, since c^{-1} is a (1-1)-correspondence between all real numbers and all points on ℓ, we can compute the first and second components of each point on ℓ by using the corresponding value of 't' in the equations:

$$\begin{cases} x = \dfrac{1 + t}{2} \\ y = \dfrac{10 - 2t}{3} . \end{cases}$$

For example, for t = 17,

$$x = \frac{1 + 17}{2} = 9$$

and

$$y = \frac{10 - 2(17)}{3} = -8.$$

So, (9, -8) is the point on ℓ which corresponds with the real number 17. [Check this by replacing 'x' by '9' and 'y' by '-8' in the equation of ℓ, '4x + 3y - 12 = 0'.]

You can use the equations:

$$(1) \qquad \begin{cases} x = \dfrac{1 + t}{2} \\ y = \dfrac{10 - 2t}{3} \end{cases}$$

to find points on the line ℓ, even more easily than you can use the equation:

$$(2) \qquad 4x + 3y - 12 = 0.$$

If you want to find the point on ℓ which has first component -7, you can find its second component by using equation (2):

$$4(-7) + 3y - 12 = 0$$
$$3y = 40$$
$$y = \frac{40}{3} .$$

Another way of finding its second component is to use equations (1):

$$\begin{aligned} -7 &= \frac{1 + t}{2}, \\ t &= -15, \\ y &= \frac{10 - 2(-15)}{3}, \\ &= \frac{40}{3}. \end{aligned}$$

The equations:

$$\begin{cases} x = \dfrac{1 + t}{2} \\ \\ y = \dfrac{10 - 2t}{2} \end{cases}$$

are called *parametric equations for the line ℓ*. ['t' is called *a parameter*. In finding values of '(x, y)' you first replace 't' by a name of one of its values.]

EXERCISES

A. Given a linear equation for a line ℓ, how can we obtain a pair of parametric equations for this line?

Suppose $\ell = \{(x, y) : 7x - 2y + 4 = 0\}$. Construct some (1-1)-correspondence c between points of ℓ and the real numbers; for example, let

$$c((x, y)) = 3y + 7.$$

Then, compute $c^{-1}(t)$.

$$\begin{aligned} 3y + 7 &= t \\ y &= \frac{-7 + t}{3} && \longleftarrow \text{one of a pair of parametric equations} \\ 7x - 2\left(\frac{-7 + t}{3}\right) + 4 &= 0 && \longleftarrow \text{'}7x - 2y + 4 = 0\text{' is an equation for } \ell \\ 21x - 2(-7 + t) + 12 &= 0 \\ 21x + 14 - 2t + 12 &= 0 \\ 21x &= -26 + 2t \\ x &= \frac{-26 + 2t}{21} && \longleftarrow \text{the other parametric equation} \end{aligned}$$

So, a pair of parametric equations for ℓ is:

$$\begin{cases} x = \dfrac{-26 + 2t}{21} \\[2ex] y = \dfrac{-7 + t}{3} \end{cases}.$$

Since, for every t, the point $\left(\dfrac{-26 + 2t}{21}, \dfrac{-7 + t}{3}\right)$ is on ℓ, [an equation for which is '$7x - 2y + 4 = 0$'], a check on the computations made in deriving the parametric equations for ℓ is obtained by noticing that, for every t,

$$\begin{aligned} &7\left(\frac{-26 + 2t}{21}\right) - 2\left(\frac{-7 + t}{3}\right) + 4 \\ &= \frac{-26 + 2t}{3} - \frac{-14 + 2t}{3} + \frac{12}{3} \\ &= \frac{-26 + 2t + 14 - 2t + 12}{3} \\ &= 0. \end{aligned}$$

Note that these parametric equations were obtained after choosing a particular (1-1)-correspondence, c. If you picked another (1-1)-correspondence, would you get a different pair of parametric equations? Try it.

Find a pair of parametric equations for the line which is the locus of the given equation. Check your computations.

1. $7x - 2y + 10 = 0$
2. $-\frac{2}{3}x + 9y - 6 = 0$
3. $4x - 5y = 0$
4. $x + 17 = 0$
5. Find a pair of parametric equations for the line which contains the points (3, 5) and (7, 8).

B. In the next section you will learn that each pair of equations of the form of:

$$\begin{cases} x = m_1 + n_1 t \\ y = m_2 + n_2 t, \end{cases}$$

where $(n_1, n_2) \neq (0, 0)$, are parametric equations in 't' for some line. [Parametric equations of other forms may also have lines as loci, but we shall not have occasion to consider them.]

Consider the parametric equations:

$$\begin{cases} x = 3 + 5t \\ y = 4 + 2t\ . \end{cases}$$

How can we obtain a linear equation for their locus ℓ ? From the preceding paragraph, we know that ℓ is a line, and we know, given the components of two points in ℓ, how to find a linear equation for ℓ. We get two points by using the values 0 and 1 for 't' in the parametric equations:

$$\begin{array}{ll|l} x_0 = 3 + 5(0) = 3 & & x_1 = 3 + 5(1) = 8 \\ y_0 = 4 + 2(0) = 4 & & y_1 = 4 + 2(1) = 6 \end{array}$$

So, (3, 4) and (8, 6) are points in ℓ . Then, a linear equation for ℓ is obtained from:

$$(6 - 4)(x - 3) - (8 - 3)(y - 4) = 0$$ [see page 1-35].

$$2(x - 3) - 5(y - 4) = 0,$$

$$2x - 5y + 14 = 0.$$

Another way to obtain a linear equation for the line which is the locus of the given parametric equations is to <u>eliminate</u> the parameter from the pair of equations.

(continued on next page)

$$\begin{cases} x = 3 + 5t \\ y = 4 + 2t \end{cases}$$

$$\begin{aligned} 2x &= 6 + 10t \\ 5y &= 20 + 10t \end{aligned}$$

$$\begin{aligned} 2x - 5y &= -14 \\ 2x - 5y + 14 &= 0 \end{aligned}$$

In each of the following exercises you are given a pair of parametric equations for a line. In each case, find a linear equation whose locus is the given line.

1. $\begin{cases} x = 11 + 6t \\ y = 1 - t \end{cases}$

2. $\begin{cases} x = 10 + t \\ y = 5 + 4t \end{cases}$

3. $\begin{cases} x = 2 + t \\ y = 3 - 2t \end{cases}$

4. $\begin{cases} x = t \\ y = -3 + 7t \end{cases}$

5. $\begin{cases} x = a + bt \\ y = c + dt \end{cases}$

6. $\begin{cases} x = -2z + (1 + 2z)t \\ y = z + (-2z - 1)t \end{cases}$

7. $\begin{cases} x = -b + (2b - \frac{a}{c})t \\ y = c - \frac{a}{b} + (\frac{a}{b} - 2c)t \end{cases}$

8. $\begin{cases} x = x_0 + (x_1 - x_0)t \\ y = y_0 + (y_1 - y_0)t \end{cases}$

TWO-POINT FORM FOR PARAMETRIC EQUATIONS

Consider now the general problem of finding a pair of parametric equations for a line which contains two given points, (x_0, y_0) and (x_1, y_1). From your work in Exercise 8 of Part B, you might guess that the equations given there are such a pair of parametric equations. Your guess would be correct, but the job is not finished because we have not shown that the parametric equations in Exercise 8 are equations of a line. So, let us derive the equations in Exercise 8 from a linear equation for a line which contains (x_0, y_0) and (x_1, y_1).

We are given $(x_0, y_0) \neq (x_1, y_1)$. [That is, we are given <u>two</u> points, so $x_0 \neq x_1$, or $y_0 \neq y_1$.] Then:

$$(y_1 - y_0)(x - x_0) - (x_1 - x_0)(y - y_0) = 0,$$

or:

$$(1) \qquad (x - x_0)(y_1 - y_0) = (x_1 - x_0)(y - y_0)$$

is a linear equation of ℓ. Now, on page 1-63, we established the principle that, for $(a, b) \neq (0, 0)$ and $(a', b') \neq (0, 0)$,

$$ab' = a'b$$

if and only if

(a, b) is proportional to (a', b').

This principle can be applied to (1). Since either $x_0 \neq x_1$ or $y_0 \neq y_1$, we know that $(x_1 - x_0, y_1 - y_0) \neq (0, 0)$. If $(x - x_0, y - y_0) \neq (0, 0)$ then because

$$(1) \qquad (\underset{\substack{\uparrow \\ a}}{x - x_0})(\underset{\substack{\uparrow \\ b'}}{y_1 - y_0}) = (\underset{\substack{\uparrow \\ a'}}{x_1 - x_0})(\underset{\substack{\uparrow \\ b}}{y - y_0}),$$

we have:

$$(x - x_0, y - y_0) \text{ is proportional to } (x_1 - x_0, y_1 - y_0).$$

In other words, for every $(x, y) \neq (x_0, y_0)$, (x, y) satisfies equation (1) if and only if there is a real number $t \neq 0$ such that

$$x - x_0 = t(x_1 - x_0) \quad \text{and} \quad y - y_0 = t(y_1 - y_0).$$

So, we have shown that, for every point (x, y) of ℓ except (x_0, y_0) there is a real number $t \neq 0$ such that

$$(2) \qquad \begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0). \end{cases}$$

Also, because the principle on page 1-63 was of the "if and only if" type, we have shown that for each real number $t \neq 0$, the parametric equations (2) give a point (x, y) which satisfies equation (1), and is, therefore, a point on ℓ. What about the point (x_0, y_0)? Clearly, this

point is on ℓ. You can get this point from equations (2) by assigning to 't' the value 0.

Summing up,

> For each two points (x_0, y_0) and (x_1, y_1), the line which contains (x_0, y_0) and (x_1, y_1) is
>
> $$\{(x, y): \text{For some real number } t, \quad x = x_0 + t(x_1 - x_0) \quad \text{and} \quad y = y_0 + t(y_1 - y_0)\}.$$

[Compare this boxed statement with the one on page 1-35.]

If we have a line ℓ which contains the two points (x_0, y_0) and (x_1, y_1), then the boxed statement above tells us that, for each real number t, the corresponding point on ℓ is

$$\left(x_0 + t(x_1 - x_0),\ y_0 + t(y_1 - y_0)\right),$$

and, for each (x, y) on ℓ, there is a corresponding real number t such that

$$(x, y) = \left(x_0 + t(x_1 - x_0),\ y_0 + t(y_1 - y_0)\right).$$

Since either $x_1 - x_0 \neq 0$ or $y_1 - y_0 \neq 0$, different values of 't' give either different values of 'x' or different values of 'y'. So, the correspondence between real numbers, t, and the points, (x, y), of ℓ, which is established by the parametric equations, is (1-1).

The job of writing a pair of parametric equations for the line which contains two given points is quite easy. For example, given the points

(3, 9) and (5, -2), a pair of parametric equations for the line through these points is:

$$\begin{cases} x = 3 + t(5 - 3) \\ y = 9 + t(-2 - 9), \end{cases}$$

or, more simply:

$$\begin{cases} x = 3 + 2t \\ y = 9 - 11t . \end{cases}$$

[Refer to the first paragraph of Part B on page 1-101. Do you see that the parametric equations:

$$\begin{cases} x = m_1 + n_1 t \\ y = m_2 + n_2 t , \end{cases} \quad [(n_1, n_2) \neq (0, 0)]$$

have as locus the line containing the <u>two</u> points (m_1, m_2) and $(m_1 + n_1, m_2 + n_2)$?]

EXERCISES

A. Find parametric equations for the line through the given points. [Graph each line and save the pictures for Parts B and C.]

1. (3, 2), (-1, 1)
2. (-4, 3), (0, 0)
3. (-5, -2), (-7, 8)
4. (-7, 8), (-5, -2)
5. (3, 1), (3, -5)
6. (1, 1), (-7, -7)

B. For each exercise of Part A, choose four values of the parameter and find the corresponding points on the given line.

C. For each exercise of Part A, choose four points on the given line [use the picture], different from the points which you found in answering Part B, and find the corresponding values of the parameter.

D. You should have noticed that the same line is referred to in Exercises 3 and 4 of Part A. You may have gotten different pairs of parametric equations for this line. Use the same method you used in Part A to find three other pairs of parametric equations for this line.

(1-1)-CORRESPONDENCES AND PARAMETRIC EQUATIONS

Each pair of parametric equations for a line specifies a (1-1)-correspondence between the points of the line and the real numbers. The two-point form for a pair of parametric equations:

$$\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \end{cases}$$

indicates a (1-1)-correspondence. With what value of 't' does the point (x_0, y_0) correspond? If $x_1 \neq x_0$, we can find this value by replacing 'x' by 'x_0' in the first parametric equation.

$$\begin{aligned} x_0 &= x_0 + t(x_1 - x_0) \\ 0 &= t(x_1 - x_0) \\ 0 &= t\,. \end{aligned}$$

If $x_1 = x_0$ then $y_1 \neq y_0$, and we find from the second parametric equation that the corresponding value of 't' is still 0. Similarly, if $(x_0, y_0) \neq (x_1, y_1)$, the parametric equations tell us that the point (x_1, y_1) corresponds with the real number 1.

In general,

$$\text{if } x_1 \neq x_0, \qquad t = \frac{x - x_0}{x_1 - x_0},$$

and

$$\text{if } y_1 \neq y_0, \qquad t = \frac{y - y_0}{y_1 - y_0}.$$

[We know that $x_1 \neq x_0$ or $y_1 \neq y_0$. Why?]

So, the (1-1)-correspondence c specified by the given pair of parametric equations is given by:

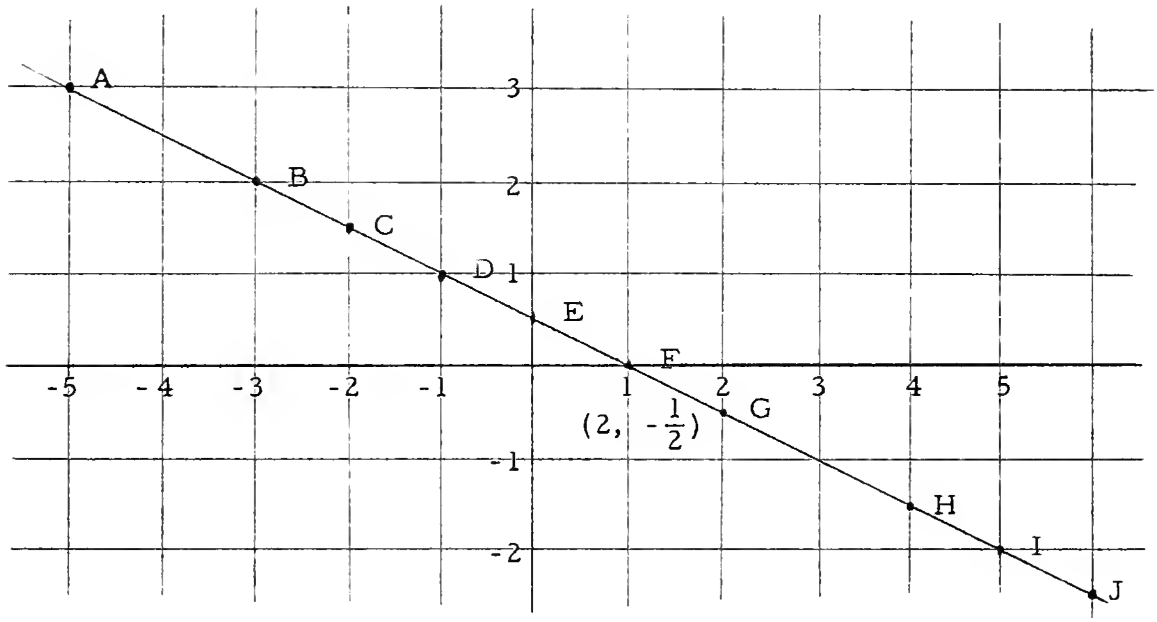
$$(1) \qquad c((x, y)) = \frac{x - x_0}{x_1 - x_0} \qquad [\text{if } x_1 \neq x_0],$$

$$(2) \qquad c((x, y)) = \frac{y - y_0}{y_1 - y_0} \qquad [\text{if } y_1 \neq y_0].$$

Do (1) and (2) give 0 for 't' when $(x, y) = (x_0, y_0)$, and 1 for 't' when $(x, y) = (x_1, y_1)$?

Consider the pair of parametric equations for the line ℓ which contains the points $(-1, 1)$ and $(2, -\frac{1}{2})$:

$$\begin{cases} x = -1 + t(-1 - 2) \\ y = 1 + t(-\frac{1}{2} - 1) \end{cases} \quad \text{or:} \quad \begin{cases} x = -1 - 3t \\ y = 1 - \frac{3}{2}t\,. \end{cases}$$



A description of a (1-1)-correspondence c which pairs points on this line with the real numbers can be obtained by solving either of the parametric equations for 't':

$x = -1 - 3t$	$y = 1 - \frac{3}{2}t$
$x + 1 = -3t$	$y - 1 = -\frac{3}{2}t$
$t = \frac{x + 1}{-3}$	$t = \frac{2(y - 1)}{-3}$

Then, there is a (1-1)-correspondence c between the real numbers and the points of ℓ such that if $(x, y) \in \ell$, then

$$c((x, y)) = \frac{x + 1}{-3} \qquad \left[\text{or, } c((x, y)) = \frac{2(y - 1)}{-3}\right].$$

Some of the elements in the set c are (A, c(A)), (B, c(B)), (C, c(C)), etc. You can compute the elements in c by using either the formula '$c((x, y)) = \frac{x + 1}{3}$' or the formula '$c((x, y)) = \frac{2(y - 1)}{-3}$'. For example,

$$(A, c(A)) = \left((-5, 3), \frac{-5 + 1}{-3}\right) = \left((-5, 3), \frac{4}{3}\right).$$

$$(A, c(A)) = \left((-5, 3), \frac{2(3 - 1)}{3}\right) = \left((-5, 3), \frac{4}{3}\right).$$

[Of course, the value of 'c(A)' is precisely that value of the parameter 't' which corresponds with point A.]

EXERCISES

A. Refer to the line pictured on page 1-107, and compute for the set c the elements

1. (B, c(B))	2. (C, c(C))	3. (D, c(D))
4. (E, c(E))	5. (F, c(F))	6. (G, c(G))
7. (H, c(H))	8. (I, c(I))	9. (J, c(J))

B. Note that for the line pictured on page 1-107, B is between A and C, that is, [ABC]. By our interpretation of '[. . . ___ _ _]' given on page 1-84, we know that [ABC] because $\{A, B, C\} \subseteq \ell$ and $-5 < -3 < -2$ [as well as $3 > 2 > \frac{1}{2}$]. Since $c(A) = \frac{4}{3}$, $c(B) = \frac{2}{3}$, and $c(C) = \frac{1}{3}$, we have, along with the fact that

[ABC] ,

the fact that

$$c(A) > c(B) > c(C).$$

(continued on next page)

That is, the point B is between the points A and C, and the parameter value corresponding with the point B is between the parameter values corresponding with the points A and C.

Check to see if these two facts [betweenness for points and betweenness for parameter values] go together for other triples of points among the points A, B, C, D, E, F, G, H, I, and J.

PARAMETRIC EQUATIONS AND BETWEENNESS

Consider the pair of parametric equations for the line ℓ through the points (x_0, y_0) and (x_1, y_1):

$$\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0). \end{cases}$$

For each point (x_p, y_p) in ℓ, there is a real number, t_p, such that

$$x_p = x_0 + t_p(x_1 - x_0).$$

Now, consider points and their corresponding parameter values:

$$(x_a, y_a) \longleftrightarrow t_a$$

$$(x_b, y_b) \longleftrightarrow t_b$$

$$(x_c, y_c) \longleftrightarrow t_c\ .$$

Part B of the preceding Exercises suggested that

$$[(x_a, y_a)(x_b, y_b)(x_c, y_c)]$$

(*) if and only if

$$t_a < t_b < t_c \quad \text{or} \quad t_a > t_b > t_c\ .$$

Let us try to establish this.

From our interpretation of '[... __ _ _]' on page 1-84 and from Part E on page 1-85, we know that since (x_a, y_a), (x_b, y_b), and (x_c, y_c) are collinear,

$$[(x_a, y_a)(x_b, y_b)(x_c, y_c)]$$

if and only if

$$(x_c - x_b)(x_b - x_a) > 0 \quad \text{or} \quad (y_c - y_b)(y_b - y_a) > 0.$$

[illegible] that [illegible] between the points [illegible] C, and the parameter [illegible] corresponding [illegible] the point P [illegible] in the parameter [illegible] from [illegible], with the [illegible] A and [illegible]

[illegible] and [illegible] for other triples [illegible] F, G, H, I, and J.

[illegible]

[illegible] the line [illegible] the p[illegible] 5 [illegible]

[illegible]

Now [illegible] value

[illegible]

Let us try to [illegible]

From our [illegible] from Part II, page 1-85, we know that [illegible] are collinear,

[illegible]

if and only if

[illegible]

5-4-57, second C[illegible]

Since

$$x_a = x_0 + t_a(x_1 - x_0),$$
$$x_b = x_0 + t_b(x_1 - x_0),$$

and

$$x_c = x_0 + t_c(x_1 - x_0),$$

then

$$x_c - x_b = t_c(x_1 - x_0) - t_b(x_1 - x_0)$$
$$= (t_c - t_b)(x_1 - x_0),$$

and

$$x_b - x_a = (t_b - t_a)(x_1 - x_0).$$

So,

$$(1)\quad (x_c - x_b)(x_b - x_a) = (t_c - t_b)(t_b - t_a)(x_1 - x_0)^2.$$

In the same way, you can show that

$$(2)\quad (y_c - y_b)(y_b - y_a) = (t_c - t_b)(t_b - t_a)(y_1 - y_0)^2.$$

We know that either $x_1 - x_0 \neq 0$, or $y_1 - y_0 \neq 0$. Suppose $x_1 - x_0 \neq 0$. Then, since $(x_1 - x_0)^2 > 0$, it follows from (1) that

if $(t_c - t_b)(t_b - t_a) > 0$ then $(x_c - x_b)(x_b - x_a) > 0$.

Hence, in this case, if t_b is between t_a and t_c then

$$[(x_a, y_a)(x_b, y_b)(x_c, y_c)].$$

If $y_1 - y_0 \neq 0$, you can establish the same conclusion by using (2) instead of (1). So, we have established the "if part" of (*).

Again suppose $x_1 - x_0 \neq 0$. It follows from (1) that

if $(x_c - x_b)(x_b - x_a) > 0$ then $(t_c - t_b)(t_b - t_a) > 0$.

Moreover, if $(y_c - y_b)(y_b - y_a) > 0$ then, by Part H on page 1-86,

either $(x_c - x_b)(x_b - x_a) > 0$ or $x_a = x_b = x_c$.

But, if $(y_c - y_b)(y_b - y_a) > 0$ then (x_a, y_a), (x_b, y_b), and (x_c, y_c) are different points. Since (x_0, y_0) and (x_1, y_1) are collinear with these

three points and have different first components, it follows from Part I on page 1-86, that x_a, x_b, and x_c are different. So, if $(y_c - y_b)(y_b - y_a) > 0$ then $(x_c - x_b)(x_b - x_a) > 0$ and, as shown above, $(t_c - t_b)(t_b - t_a) > 0$. Hence [in case $x_1 - x_0 \neq 0$], if $[(x_a, y_a)(x_b, y_b)(x_c, y_c)]$ then t_b is between t_a and t_c. Again, if $y_1 - y_0 \neq 0$, you can establish this same conclusion by using (2) instead of (1). So, we have also established the "only if" part of (*).

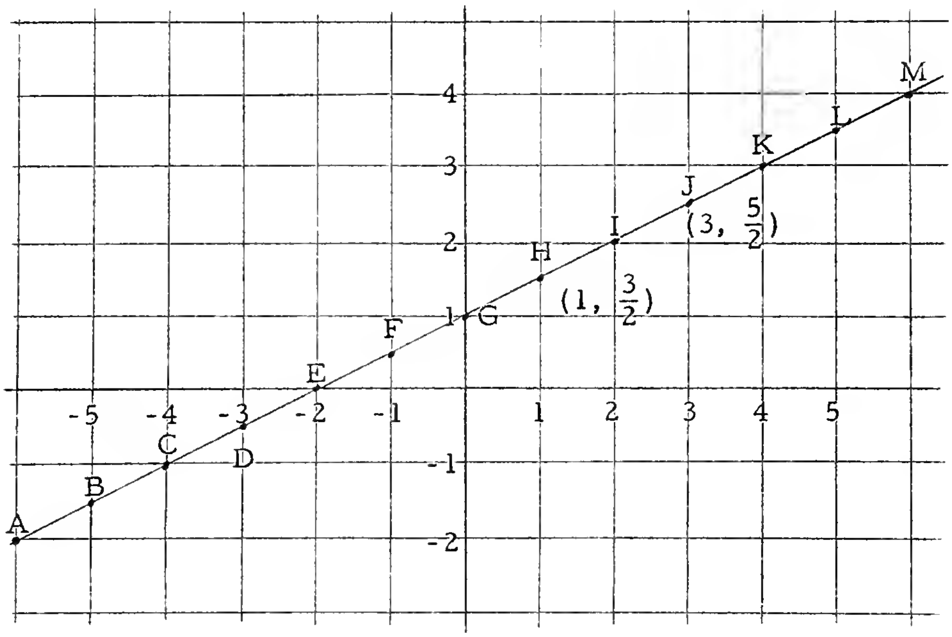
Thus, we know that if c is the (1-1)-correspondence between the real numbers and the points of a line ℓ which is specified by a pair of parametric equations for ℓ, then, if $\{A, B, C\} \subseteq \ell$,

$$\boxed{\begin{array}{c}[ABC] \\ \text{if and only if} \\ c(A) < c(B) < c(C) \ \text{ or } \ c(A) > c(B) > c(C).\end{array}}$$

EXPLORATION EXERCISES

The line pictured in the diagram has the pair of parametric equations:

$$\begin{cases} x = 1 + 2t \\ y = \frac{3}{2} + t. \end{cases}$$



A. Complete the following table.

P	A	B	C	D	E	F	G	H	I	J	K	L	M
t_p	$-\frac{7}{2}$												

B. Complete the following tables.

$\{P, Q\}$	$\{A, C\}$	$\{C, E\}$	$\{E, G\}$	$\{I, G\}$	$\{I, K\}$	$\{M, K\}$	$\{J, L\}$
$d(P, Q)$	$\sqrt{5}$						
$\lvert t_q - t_p \rvert$							

$\{P, Q\}$	$\{A, E\}$	$\{G, C\}$	$\{E, I\}$	$\{K, G\}$	$\{D, H\}$
$d(P, Q)$					
$\lvert t_q - t_p \rvert$					

$\{P, Q\}$	$\{A, G\}$	$\{G, M\}$	$\{A, K\}$	$\{M, C\}$	$\{B, L\}$
$d(P, Q)$					
$\lvert t_q - t_p \rvert$					

C. Study the tables in Part B and complete the following statement.

For collinear points A, B, C, and D

with parameter values t_a, t_b, t_c, and t_d, respectively,

$$\{A, B\} \cong \{C, D\}$$

if and only if

____________________________________.

Try to derive this statement before reading the next section.

PARAMETRIC EQUATIONS AND CONGRUENCE

Your work in the preceding Exploration Exercises suggested that congruence of point couples on a line is related to the absolute value of the difference between the parameter values of the points in the couple. Let us state this idea more precisely.

Suppose A, B, C, and D are points on the line ℓ which passes through the two points P_0 and P_1. If $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ then:

$$\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \end{cases}$$

are parametric equations of ℓ. In the associated correspondence between the real numbers and the points of ℓ, 0 corresponds with P_0 and 1 with P_1. Suppose t_a corresponds with A, t_b with B, t_c with C, and t_d with D. Then we want to establish that

$$\{A, B\} \cong \{C, D\} \text{ if and only if } |t_b - t_a| = |t_d - t_c|.$$

In order to establish this, suppose $\{P, P'\} \subseteq \ell$. If $P = (x, y)$ and $P' = (x', y')$, then there are real numbers t and t' such that

$$\begin{matrix} x = x_0 + t(x_1 - x_0) & & x' = x_0 + t'(x_1 - x_0) \\ & \text{and} & \\ y = y_0 + t(y_1 - y_0) & & y' = y_0 + t'(y_1 - y_0). \end{matrix}$$

By the distance formula,

$$d(P, P') = \sqrt{(x' - x)^2 + (y' - y)^2}\ .$$

Since $x' - x = (t' - t)(x_1 - x_0)$ and $y' - y = (t' - t)(y_1 - y_0)$,

$$\begin{aligned} d(P, P') &= \sqrt{\left[(t' - t)(x_1 - x_0)\right]^2 + \left[(t' - t)(y_1 - y_0)\right]^2} \\ &= \sqrt{(t' - t)^2(x_1 - x_0)^2 + (t' - t)^2(y_1 - y_0)^2} \\ &= \sqrt{(t' - t)^2[(x_1 - x_0)^2 + (y_1 - y_0)^2]} \\ &= \sqrt{(t' - t)^2} \cdot \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= |t' - t| \; d(P_0, P_1) . \end{aligned}$$

[Return now to the Exploration Exercises, find the points whose parameter values are 0 and 1, call them 'P_0' and 'P_1', and check the formula just derived:

$$d(P, P') = |t' - t| \; d(P_0, P_1),$$

for several pairs of points contained in the line through H and I.]

Coming back now to the result we are trying to establish, we recall that, by the explanation of congruence for the number plane model,

$$\{A, B\} \cong \{C, D\} \text{ if and only if } d(A, B) = d(C, D).$$

But, by the formula we derived above,

$$d(A, B) = |t_b - t_a| \; d(P_0, P_1)$$

and

$$d(C, D) = |t_d - t_c| \; d(P_0, P_1).$$

Since $d(P_0, P_1) \neq 0$ [Why?],

$$d(A, B) = d(C, D) \text{ if and only if } |t_b - t_a| = |t_d - t_c|.$$

Hence, as we wished to show,

$$\{A, B\} \cong \{C, D\} \text{ if and only if } |t_b - t_a| = |t_d - t_c|.$$

Thus, we know that if c is the (1-1)-correspondence between the real numbers and the points of a line ℓ which is specified by a pair

of parametric equations for ℓ , then if $\{A, B, C, D\} \subseteq \ell$,

$$\{A, B\} \cong \{C, D\}$$

if and only if

$$|c(B) - c(A)| = |c(D) - c(C)|.$$

COORDINATES ON LINES

You have seen that for each two points P_0 and P_1, if $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$, the correspondence c such that, for each real number t,

$$c^{-1}(t) = \left(x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0)\right),$$

is a (1-1)-correspondence between the points of the line ℓ through P_0 and P_1 and the real numbers; $c(P_0) = 0$ and $c(P_1) = 1$. Moreover, this correspondence has the two properties described in the dashed boxes on pages 1-111 and 1-116. (1-1)-correspondences with these two properties are very useful and deserve a special name. Such a (1-1)-correspondence will be called <u>a coordinate system on ℓ</u> .

Definition

A coordinate system on a line ℓ is a (1-1)-correspondence c between the points of ℓ and the real numbers such that

(1) if $\{A, B, C\} \subseteq \ell$ then

$$[ABC]$$

if and only if

$$c(A) < c(B) < c(C) \quad \text{or} \quad c(A) > c(B) > c(C),$$

and

(2) if $\{A, B, C, D,\} \subseteq \ell$ then

$$\{A, B\} \cong \{C, D\}$$

if and only if

$$|c(B) - c(A)| = |c(D) - c(C)|.$$

The point $c^{-1}(0)$ is the origin of the coordinate system c. The point $c^{-1}(1)$ is the unit-point of the coordinate system c.
For each $A \in \ell$, c(A) is the coordinate of A with respect to the coordinate system c.

As noted above, we have derived the following statement from properties of the number plane model.

For each two points P_0 and P_1 there is a coordinate system on the line through P_0 and P_1 such that its origin is P_0 and its unit-point is P_1.

Since it is true of the number plane model we could add this statement to our set of postulates. You can see that if we added this to our set of postulates it would help us prove theorems about betweenness and congruence. From the earlier postulates alone we could not prove such theorems because these postulates do not mention these two notions. For this reason, it is impossible to deduce the boxed statement itself from our postulates.

Another way to see that the boxed statement cannot be deduced from postulates I - V is to describe a model of the postulates for which the boxed statement is false. One such model consists of the three businessmen (points), A, B, and C, and their three partnerships (lines), $\{A, B\}$, $\{B, C\}$, and $\{C, A\}$. We might describe '[... ___ ---]' by saying that each sentence of this form is false; and describe '... ≅ ---' by saying that each sentence of this form is true. But, whatever interpretation is put on these primitive terms, it is clear that the statement in the dashed box on page 1-117 would be false. For there is no (1-1)-correspondence between the real numbers and the members of the line $\{A, B\}$.

However, rather than adding the boxed statement on page 1-117 to our postulate system, we choose to add the following statement.

VI. For each two points P_0 and P_1, there is one and only one coordinate system on the line through P_0 and P_1 such that its origin is P_0 and its unit-point is P_1.

We have shown for the number plane model that there is one coordinate system, c, which has the given points P_0 and P_1, as origin and unit-point, respectively. To complete the derivation of postulate VI from properties of the number plane model, we should have to show that there is no other such coordinate system. That is, we should have to show that

> If c and g are coordinate systems on the line through P_0 and P_1 such that
>
> $$c(P_0) = 0 = g(P_0)$$
>
> and
>
> $$c(P_1) = 1 = g(P_1)$$
>
> then, for each point P on the line,
>
> $$g(P) = c(P).$$

This can be done, but it requires a greater knowledge of algebra than you now have, or will need for the remainder of this course. We ask you to accept the fact that the boxed statement can be derived from the properties of the number plane model. Postulate VI is a consequence of the two statements in the dashed boxes, the one above and the one on page 1-117.

EXERCISES

The points A, B, C, D, and E are collinear, and f is a coordinate system on the line containing them such that

$$f(A) = 4, \quad f(B) = 1, \quad f(C) = 0, \quad f(D) = 3, \quad f(E) = -1.$$

<u>True</u> <u>or</u> <u>false</u>:

1. [ABC]
2. [BCD]
3. [EAB]
4. [CDE]
5. [ECD]
6. [BAC]
7. $\{A, B\} \cong \{C, D\}$
8. $\{B, C\} \cong \{D, C\}$
9. $\{D, E\} \cong \{E, B\}$
10. $\{A, C\} \cong \{D, E\}$

* * *

We introduce a new notation.

<u>Definition</u>

For all points A, B, C, and D,

[ABCD]

if and only if

[ABC] and [BCD].

* * *

<u>True</u> <u>or</u> <u>false</u>:

11. [ABCD]
12. [ABCE]
13. [CBDA]
14. [ECBA]

A BETWEENNESS POSTULATE

In solving the preceding exercises you were able to use postulate VI to check on numerous examples of betweenness. The reason you could do this using only postulate VI is that you were given that all the points you dealt with were collinear. Suppose you are given three points, A, B, and C, which are not collinear. Could you decide <u>from the postulates alone</u> whether [ABC] or not? From our interpretation of '[... ___ ---]' for the number plane model, you would say that '[ABC]' is false. But we don't have a postulate which would enable us to do so. What we want is a postulate which tells us that

> For every A, B, and C,
> if A, B, and C are not collinear
> then it is not the case that [ABC].

However, from your study of statements and their contrapositives, this is equivalent to

> VII. For every A, B, and C,
>
> [ABC]
> only if
> A, B, and C are collinear.

Postulate VII is simpler than the equivalent statement which was displayed above it.

With postulates I - VII, and the red-boxed definition, we are ready to prove several interesting theorems.

Example. Although, in the number plane model, it is clear that if [ABC], then A, B, and C must be different points, we have not yet seen that it is a consequence of our postulates. [If it were not, we would certainly want to add it (or something which would imply it) to our set of postulates.] We shall show you that it can be deduced from the postulates already chosen by giving a proof for it. In the exercises which follow, we give several statements which you are to show to be theorems by deducing them from the postulates, that is, by proving them.

Theorem. If [ABC] then A, B, and C are different points.

Proof. By postulate VII, if [ABC] then there is a line ℓ such that $\{A, B, C\} \subseteq \ell$. By postulate I, there are at least two points on ℓ. By postulate VI, there is a coordinate system c on ℓ. By the definition of 'coordinate system on a line', we know that if [ABC] then the coordinate c(B) is between c(A) and c(C), that is,

$$(1) \quad \Big(c(C) - c(B)\Big)\Big(c(B) - c(A)\Big) > 0.$$

But, if (1), $c(C) - c(B) \neq 0$ and $c(B) - c(A) \neq 0$. Hence, $c(A) \neq c(B)$ and $c(B) \neq c(C)$. We also need to show that $c(C) \neq c(A)$. Suppose $c(C) = c(A)$. Then

$$(2) \quad \Big(c(C) - c(B)\Big)\Big(c(B) - c(A)\Big) = -\Big(c(B) - c(A)\Big)^2.$$

Since $-\Big(c(B) - c(A)\Big)^2 \leq 0$, (2) contradicts (1). So, $c(C) \neq c(A)$.

Since we have shown that if [ABC] then c(A), c(B), and c(C) are different coordinates, and since c is a (1-1)-correspondence between the real numbers and the points on ℓ, it follows that if [ABC] then A, B, and C are different points.

Since this proof is probably more complicated than any you have yet given, we shall talk about it a bit.

By 'proving a theorem' we mean showing that it is a consequence of our postulates and our red-boxed definitions. In this case, we set out to prove:

(3) if [ABC] then A, B, and C are different points.

We recognized that in order to prove this it would be sufficient to prove:

(4) if [ABC] then in some coordinate system, A, B, and C have different coordinates.

[We knew that this would be sufficient because, by definition, a coordinate system is a (1-1)-correspondence.] Now, in order to show that (4) follows from our postulates and definitions, it is sufficient to show that:

in some coordinate system, A, B, and C have different coordinates

follows from our postulates, definitions, and [ABC]. [In this proof, as throughout the course, we take for granted facts concerning sets (such as (1-1)-correspondences) and facts concerning real numbers (such as, $-\left(c(B) - c(A)\right)^2 \leq 0$).]

Since A, B, and C might be any points whatever, we are justified in claiming to have proved the generalization:

For every A, B, and C,
if [ABC] then A, B, and
C are different points.

Compare this statement with the statement of the theorem given on page 1-122. Frequently, theorems will be stated without the beginning phrase 'For every . . .'. Just as in the present case, you may always add such a phrase.

EXERCISES

Show that each of the following can be deduced from the postulates and red-boxed definitions. [Since the previous theorems have been deduced from postulates and red-boxed definitions, you may also use these in your proofs. For the same reason, in solving each of the following exercises, you may use the results of the previous exercises.]

1. If [ABC] then [CBA].

2. If [ABC] then not [BAC].

3. If not [ABC] then not [CBA].

(continued on next page)

4. If A, B, and C are three collinear points, then either [ABC], or [ACB], or [CAB].

5. If $A \neq B$ then there is a point P such that [APB]. [This theorem tells you that between each two points there is a third point.]

6. If $A \neq B$ then there is a point P such that [ABP].

7. If [ABCD] then A, B, C, and D are different.

8. If [ABCD] then [ABD] and [ACD].

9. If [ABC] and [ACD] then [ABCD].

10. Show by a counter-example that:

    if [ABD] and [ACD] then [ABCD]

    is not a theorem.

11. Is:

    if [ABD] and [BCD] then [ABCD]

    a theorem?

SUBSETS OF LINES

Using the notion of betweenness we can define several kinds of subsets of lines which we shall use in the next unit. [In stating these definitions, we use the expression ' $\{P: [APB]\}$ '. Can you tell what it means?]

<u>Definition</u>

For every A and B,

(1) $\overline{AB} = \{P: [APB]\}$

[read ' $\overline{AB}$ ' as 'the interval ay bee'],

(2) $\overset{\circ\!-\!\circ}{AB} = \overline{AB} \cup \{A, B\}$

[read ' $\overset{\circ\!-\!\circ}{AB}$ ' as 'the segment ay bee'],

(continued on next page)

(3) $\overrightarrow{AB} = \overline{AB} \cup \{B\} \cup \{P: [ABP]\}$

[read '$\overrightarrow{AB}$' as 'the half-line from ay through bee'],

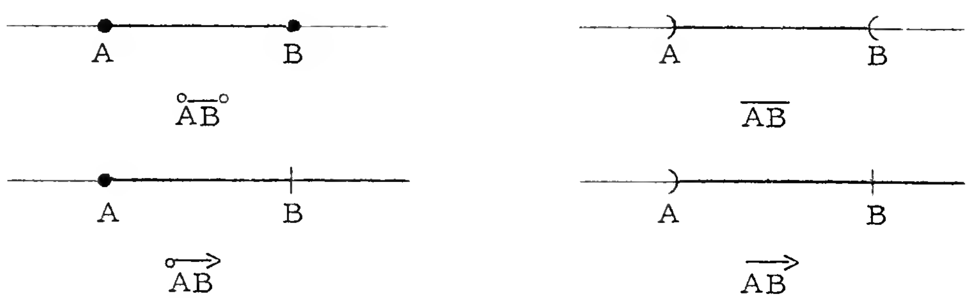
(4) $\overset{\circ\!\!\longrightarrow}{AB} = \{A\} \cup \overrightarrow{AB}$

[read '$\overset{\circ\!\!\longrightarrow}{AB}$' as 'the ray from ay through bee'].

[The end-points of $\overline{AB}$, $\overset{\circ\!-\!\circ}{AB}$ are A and B; the vertex of $\overrightarrow{AB}$ and of $\overset{\circ\!\!\longrightarrow}{AB}$ is A.]

The expression 'the line containing the two points A and B' will usually be abbreviated '$\overleftrightarrow{AB}$'.

The following pictures illustrate the definitions given in the box.



EXERCISES

A. Suppose A and B are different points and c is the coordinate system on the line through A and B such that is origin is A and its unit-point is B. Then, for each point P, P ϵ $\overline{AB}$ if and only if $0 < c(P) < 1$, [by postulates VI and VII].

Make similar statements for cases (2), (3), and (4) of the red-boxed definition.

B. To give you practice in using these new symbols, read each of the following statements, and tell whether it is true or false. [The true ones are provable as theorems, and you may prove them if you want to. In later exercises you will be asked to prove theorems like these.]

1. $\overline{AB} = \overline{BA}$

2. No interval contains its end-points.
[To prove this, it would be sufficient to prove: $A \notin \overline{AB}$. Why?]

3. No segment contains its end-points.

4. $\overrightarrow{AB} = \overrightarrow{BA}$ 5. $\overset{\circ\!\longrightarrow}{AB} = \overset{\circ\!\longrightarrow}{BA}$ 6. $\overleftrightarrow{AB} = \overleftrightarrow{BA}$

7. Given any two points on a ray, there is a point between them.

8. Given any point on a ray, there are two points belonging to the ray such that the first point is between them.

9. Given any point on a half-line, there are two points belonging to the half-line such that the first point is between them.

10. Given any two points, there is one and only one ray containing them.

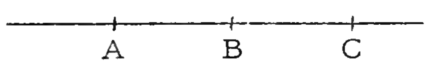
11. Given an ordered pair of two points, there is one and only one ray for which the first point is the vertex and which contains the second point.

12. If [ABC] then $\overrightarrow{BC} \cup \overrightarrow{BA} = \overleftrightarrow{AC}$.

13. If $B \in \overleftrightarrow{AC}$ and $B \neq C$, then $A \in \overleftrightarrow{BC}$.

C. Prove the following theorems.

Sample 1. If [ABC] then $\overline{AB} \subseteq \overline{AC}$.

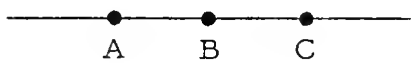


(continued on next page)

<u>Proof</u>. If [ABC] then $A \neq B$, so, as in Part A, there is a coordinate system c such that $c(A) = 0$, $c(B) = 1$, and $c(C) > 1$. For each point P, $P \in \overline{AB}$ if and only if $0 < c(P) < 1$, and $P \in \overline{AC}$ if and only if $0 < c(P) < c(C)$. Since $c(C) > 1$, each point which belongs to $\overline{AB}$ also belongs to $\overline{AC}$. Thus, we have shown that

For every A, B, and C,
if [ABC] then $\overline{AB} \subseteq \overline{AC}$.

<u>Sample 2</u>. If [ABC] then $\overline{AB} \cap \overset{\circ\!\!\longrightarrow}{BC} = \emptyset$ and $\overline{AB} \cup \overset{\circ\!\!\longrightarrow}{BC} = \overrightarrow{AB}$.



<u>Discussion</u>. In referring to the conditional sentence which we are trying to prove, it will be convenient to refer to certain parts of it. The sentence

'[ABC]'

is called <u>the antecedent of the conditional</u>, and the sentence

'$\overline{AB} \cap \overset{\circ\!\!\longrightarrow}{BC} = \emptyset$ and $\overline{AB} \cup \overset{\circ\!\!\longrightarrow}{BC} = \overrightarrow{AB}$'

is called <u>the consequent of the conditional</u>.

Since in this case the consequent is a <u>conjunction</u>, in order to prove the theorem, we must prove two conditional sentences:

(1) If [ABC] then $\overline{AB} \cap \overset{\circ\!\!\longrightarrow}{BC} = \emptyset$,

and:

(2) If [ABC] then $\overline{AB} \cup \overset{\circ\!\!\longrightarrow}{BC} = \overrightarrow{AB}$.

<u>Proof of (1)</u>.

[Choose a coordinate system c as in Sample 1.] If $P \in \overline{AB} \cap \overset{\circ\!\!\longrightarrow}{BC}$ then $P \in \overline{AB}$ and $P \in \overset{\circ\!\!\longrightarrow}{BC}$. If $P \in \overline{AB}$ then $0 < c(P) < 1$. If $P \in \overset{\circ\!\!\longrightarrow}{BC}$ then $1 \leq c(P)$ [Why?]. Since no number can be both less than 1 <u>and</u> greater than or equal to 1, there is no such point P. Thus, $\overline{AB} \cap \overset{\circ\!\!\longrightarrow}{BC} = \emptyset$. In choosing

(continued on next page)

a coordinate system, we made use of the fact that [ABC]. Hence, we have proved that

For every A, B, and C,

if [ABC] then $\overline{AB} \cap \overset{\circ\!\!\longrightarrow}{BC} = \emptyset$.

<u>Proof</u> <u>of</u> <u>(2)</u>.

[Again, choose a coordinate system c as in <u>Sample</u> <u>1.</u>] A point $P \in \overline{AB}$ if and only if $0 < c(P) < 1$, and belongs to $\overset{\circ\!\!\longrightarrow}{BC}$ if and only if $1 \leq c(P)$. Therefore, for each point P, $P \in \overline{AB} \cup \overset{\circ\!\!\longrightarrow}{BC}$ if and only if $0 < c(P)$. But this is the case if and only if $P \in \overrightarrow{AB}$. [Complete the proof of (2), and of the given theorem.]

1. If $A \neq B$ then $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$, and $\overleftrightarrow{AB} = \overrightarrow{BA} \cup \overrightarrow{AB}$.
2. If [ACB] then $\overrightarrow{CA} \cap \overset{\circ\!\!\longrightarrow}{CB} = \emptyset$, and $\overleftrightarrow{AB} = \overrightarrow{CA} \cup \overset{\circ\!\!\longrightarrow}{CB}$.
3. If [ABCD] then $\overline{BC} = \overline{AC} \cap \overline{BD}$.
4. If $C \in \overrightarrow{AB}$ then $\overrightarrow{AC} = \overrightarrow{AB}$.
5. If $C \in \overrightarrow{AB}$ then $\overline{AC} \cup \overline{BC} \subseteq \overrightarrow{AB}$.

SUMMARY

Part of what you did in this unit was to study properties of the number plane. For example, you saw that

(1) each linear equation is satisfied by at least two ordered pairs of real numbers,

(2) there are three ordered pairs of real numbers which do not satisfy any one linear equation,

(3) each two ordered pairs of real numbers satisfy some linear equation,

and (4) if two ordered pairs of real numbers satisfy each of two linear equations then these two equations are equivalent.

We decided to interpret the words 'point' and 'line' for the number plane as follows:

(a) a point in the number plane is an ordered pair of real numbers,

and (b) a line in the number plane is the solution set of a linear equation [page 1-27].

In this language of points and lines, statements (1)-(4) can be reformulated as:

I. Each line is a set of points, and contains at least two points [page 1-28].

II. There are three points which do not belong to the same line [page 1-28].

III. For each two points, there is a line which contains them [page 1-36].

IV. For each two points, there is at most one line which contains them [page 1-49].

Statements (a) and (b) describe what we called 'the number plane model for postulates I - IV'. You saw that there are other possible models for I - IV [Part D on page 1-51].

From I - IV, without thinking of any model, you were able to deduce other statements, for example:

Theorem 1. For each line, there is a point not on the line [page 1-30],

Theorem 2. The intersection of two lines consists of at most one point [page 1-50].

[There are other examples on pages 1-50 and 1-51.] Each such statement tells you something about each model of I - IV. For example, Theorem 2 tells you the following fact about the number plane model:

Two non-equivalent linear equations have at most one common solution.

We called statements I, II, III, and IV postulates, and called the statements that can be deduced from them theorems. The words 'point' and 'line' we called primitive terms. You can describe a model for a set of postulates by assigning meanings to their primitive terms in such

a way that the postulates became true statements. No matter how you do this, the theorems will also become true statements. This fact makes it possible to study many subjects at the same time. [For example, parts of the real world are "approximate" models for our postulates, so the theorems provide us with useful approximations in working with these parts of the real world.]

In addition to the primitive terms, it is often convenient to use other terms which can be defined from the primitive terms. For example, we defined '$||$' by:

<u>Definition</u>: $m \parallel n$ if and only if $m \cap n = \emptyset$ [page 1-71].

Using this, we were able to state an additional postulate.

V. For each three lines, ℓ, m, and n, if $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.

This postulate tells us something about the number plane model. In fact, like the earlier postulates, we got it by translating into the language of points and lines a fact which we had discovered concerning solution sets of linear equations. [What is this fact?]. From postulates I - V, you were able to deduce additonal theorems which you could not deduce from postulates I - IV. [Three such theorems are stated in Exercises 1, 2, and 4 on page 1-77.] Each such theorem tells you something about each model of I - V. [Because it does, you saw a way of proving that some statements <u>cannot</u> be deduced from I - V. How?]

Sometimes, in order to deal with new situations, it is necessary to introduce additional primitive terms. We did this in section 1.04 where we discussed the notions of betweenness and congruence. We first explained these notions in connection with the number plane model and then, after learning more about algebra [parametric equations of lines], were able to formulate two new postulates.

VI. For each two points P_0 and P_1, there is one and only one coordinate system on the line through P_0 and P_1 such that its origin is P_0 and its unit-point is P_1 [page 1-119].

VII. For every A, B, and C, [ABC] only if A, B, and C are collinear [page 1-121].

[Before stating postulate VI, we had to define [page 1-117] 'coordinate system', 'origin', and 'unit-point'. We did this by using the new primitive terms '[... ——— ---]' and '... ≅ ---'. The word 'collinear' in postulate VII was defined [page 1-81], using the primitive terms 'point' and 'line'.] Using the primitive terms, we defined additional terms [page 1-120 and pages 1-124 and 125], and, using the new postulates, we were able to deduce new theorems.

In later units we shall choose additional postulates [there will be fifteen altogether], and deduce additional theorems.

* * *

In order to show that the number plane model is actually a model for I - VII, we needed to know some algebra beyond that which you learned in FIRST COURSE. You learned something about the algebra of sets [pages 1-18 through 1-22] and about (1-1)-correspondences (section 1.05)]. You learned how to solve systems of linear equations [pages 1-53 through 1-57] and, by using the notion of proportionality [pages 1-58 through 1-64], how to determine whether or not such a system has a solution [pages 1-65, 1-66, and 1-68 through 1-70]. You also studied parametric equations of lines [pages 1-97 through 1-116]. In particular, you learned the two-point form for parametric equations of lines [page 1-104], which is comparable with the two-point form for linear equations (of lines) [page 1-35].

* * *

You also learned something about logic. On pages 1-66 through 1-68 you studied conditional sentences and their converses and contrapositives, and on page 1-123 you studied how a statement which is a generalization of a conditional sentence can be proved. Later we shall tell you more about the rules of logic by means of which you can construct proofs.

REVIEW EXERCISES

A. In each of the following exercises, you are given a system of equations [in some cases, not linear]. Graph each equation in each system, and tell all the common solutions of each system. [A member of the solution set of an equation is called a solution of the equation. The common solutions of a system are the members of the intersection of the solution sets of the equations.]

1. $\begin{cases} 3x + 2y = 7 \\ y = x - 9 \end{cases}$

2. $\begin{cases} 5x - 2y = 4 \\ 3y + 7x = -2 \end{cases}$

3. $\begin{cases} 8x = 7y - 2 \\ 14x = 16x + 5 \end{cases}$

4. $\begin{cases} x + 3y - 2 = 0 \\ 4 - 6y = 2x \end{cases}$

5. $\begin{cases} x^2 + y^2 = 25 \\ x + y = 7 \end{cases}$

6. $\begin{cases} y = 1 \\ x^2 + y^2 = 16 \end{cases}$

7. $\begin{cases} x^2 + y^2 = 9 \\ x + y = 6 \end{cases}$

8. $\begin{cases} |x| = |y| \\ 2y + x = 6 \end{cases}$

9. $\begin{cases} |x| + |y| = 10 \\ x^2 + y^2 = 36 \end{cases}$

10. $\begin{cases} y = x^2 \\ y = 2 - x^2 \end{cases}$

B. Solve these systems of equations by an algebraic procedure.

1. $\begin{cases} 7x - 2y = 5 \\ 3x + 4y = 9 \end{cases}$

2. $\begin{cases} 6x - 5y = 9 \\ 7x + 13y = 6 \end{cases}$

3. $\begin{cases} x = 4y + 9 \\ x = 3y - 7 \end{cases}$

4. $\begin{cases} 2y = 6 - 3x \\ 5x = 12 \end{cases}$

5. $\begin{cases} 3x + 5y = 7 \\ 2(x - 3) + 4(y + 7) = 9 \end{cases}$

6. $\begin{cases} 2 - 7(x + y) = 3 - 9x \\ 8 - 5(x - y) = 7 + 2y \end{cases}$

C. Give a linear equation whose locus is a line which contains the given points.

1. $(3, 5), (-2, 9)$

2. $(6, -1), (0, -3)$

3. $(-8, -8), (7, -8)$

4. $(6, \frac{1}{2}), (\frac{3}{2}, -\frac{5}{3})$

5. $(3, 9), (6, 13), (0, 5)$

6. $(2, 5), (7, 6), (8, 7)$

D. For each of the exercises in Part C, give a pair of parametric equations for the line which contains the given points.

E. In each of the following exercises, you are given two pairs of points. Find a linear equation of the line which passes through the points in one pair, and parametric equations for the line which passes through the points in the other pair. Then use the linear equation and the parametric equations to find the intersection of the lines.

Sample. $\{(3, 4), (2, 7)\}$ and $\{(5, 2), (6, 9)\}$

Solution. Linear equation:

$$\begin{aligned}(7 - 4)(x - 3) - (2 - 3)(y - 4) &= 0\\ 3(x - 3) + (y - 4) &= 0\\ 3x + y - 13 &= 0\end{aligned}$$

Parametric equations:

$$\begin{cases} x = 5 + t(5 - 6)\\ y = 6 + t(2 - 9)\end{cases}$$

or:

$$\begin{cases} x = 5 - t\\ y = 6 - 7t\end{cases}$$

$$\begin{aligned}3(5 - t) + (6 - 7t) - 13 &= 0\\ 15 - 3t + 6 - 7t - 13 &= 0\\ 8 - 10t &= 0\\ t &= \frac{4}{5}\end{aligned}$$

$$x = 5 - \frac{4}{5} = \frac{21}{5}$$

$$y = 6 - 7\left(\frac{4}{5}\right) = \frac{2}{5}$$

The intersection is $\{(\frac{21}{5}, \frac{2}{5})\}$.

(continued on next page)

1. $\{(8, 3), (7, 1)\}$ and $\{(-4, 9), (-2, 5)\}$
2. $\{(2, -4), (-3, 5)\}$ and $\{(-6, 0), (0, -6)\}$
3. $\{(17, 12), (8, -13)\}$ and $\{(9, 16), (-4, -15)\}$
4. $\{(a, 5), (7, a)\}$ and $\{(b, 6), (3, b)\}$
5. $\{(a, b), (c, d)\}$ and $\{(e, f), (g, h)\}$

F. Solve the following exercises just as you did those in Part E, but in each case, use parametric equations for the line which passes through the second pair of points. State your answer by using '[. . . ——— ---]' for the points in the second pair and the point of intersection. For example, in stating the answer to the Sample in Part E, since $0 < \frac{4}{5} < 1$, we would write:

$$\left[(5, 2)(\tfrac{21}{5}, \tfrac{2}{5})(6, 9)\right].$$

1. $\{(9, 7), (2, 5)\}$ and $\{(6, 1), (3, -2)\}$
2. $\{(-1, 3), (5, 8)\}$ and $\{(-4, -3), (5, 7)\}$
3. $\{(2, 1), (7, 11)\}$ and $\{(6, 3), (10, -1)\}$
4. $\{(10, -1), (6, 3)\}$ and $\{(7, 11), (2, 1)\}$

G. In each of the following exercises you are given two points, P_0 and P_1. Find the coordinate system c on the line $\overleftrightarrow{P_0P_1}$ such that $c(P_0) = 0$ and $c(P_1) = 1$. Then compute as indicated.

1. $P_0 = (3, 5)$, $P_1 = (5, 8)$.

   $c((7, 11)) = ?$, $c((2, 2)) = ?$, $c^{-1}(5) = ?$, $c^{-1}(204) = ?$

2. $P_0 = (7, 8)$, $P_1 = (-6, -4)$.

   $c((7, 8)) = ?$, $c^{-1}(2) = ?$, $c^{-1}(7) = ?$, $c^{-1}(-6) = ?$

3. $P_0 = (a, b)$, $P_1 = (c, d)$.

   $c^{-1}(\frac{1}{2}) = ?$, $c^{-1}(\frac{3}{4}) = ?$, $c^{-1}(1) = ?$, $c^{-1}(10) = ?$

4. $P_0 = (a, b)$, $P_1 = (a + m, b + n)$.

   $c^{-1}(\frac{1}{2}) = ?$, $c^{-1}(\frac{3}{2}) = ?$, $c^{-1}(-\frac{1}{2}) = ?$, $c^{-1}(-1) = ?$

H. True or false?

1. $[(-10, 12)(-4, 9)(4, 5)]$
2. $[(8, 3)(7, 6)(2, 4)]$
3. $[(6, -2)(8, 6)(10, 14)]$
4. $[(3, 9)(7, 2)(12, -4)]$
5. $[(3, 5)(7, 11)(5, 8)(11, 17)]$
6. $[(-2, 8)(3, 9)(13, 11)(18, 12)]$
7. $\{(10, -10), (12, -5)\} \cong \{(15, -12), (10, -14)\}$
8. $\{(12, 7), (15, 1)\} \cong \{(15, 10), (17, 17)\}$
9. $\{(6, 3), (-8, 4)\} \cong \{(9, 1), (7, -3)\}$

I. 1. For each of the pairs of points, $\{P, Q\}$, given below, find a point M such that [PMQ] and $\{P, M\} \cong \{M, Q\}$.

(a) $\{(0, 0), (10, 0)\}$ (b) $\{(0, 0), (0, -8)\}$

(c) $\{(1, 1), (7, 7)\}$ (d) $\{(-6, -6), (9, 9)\}$

(e) $\{(3, 4), (9, 8)\}$ (f) $\{(-6, 2), (8, 16)\}$

[Hint: Use condition (2) in the definition of 'coordinate system' given on page 1-117.]

2. Prove the following theorem [that is, deduce it from postulates I - VII].

If $P \neq Q$, there is just one point M such that $M \in \overleftrightarrow{PQ}$ and $\{P, M\} \cong \{M, Q\}$.

[Hint: Suppose $c(P) = 0$ and $c(Q) = 1$. Show that there is just one number x such that $|x - 0| = |1 - x|$.]

* * *

Definition

If $P \neq Q$ then the point M such that $M \in \overleftrightarrow{PQ}$ and $\{P, M\} \cong \{M, Q\}$ is the mid-point of $\{P, Q\}$.

* * *

3. Prove:

If $P \neq Q$, M is the mid-point of $\{P, Q\}$, and c is <u>any</u> coordinate system on $\overleftrightarrow{PQ}$, then

$$c(M) = \frac{1}{2}[c(P) + c(Q)] .$$

[Hint: Show that $\left|\frac{1}{2}[c(P) + c(Q)] - c(P)\right| = \left|c(Q) - \frac{1}{2}[c(P) + c(Q)]\right|$.]

4. Show that, for each two points, (x_0, y_0) and (x_1, y_1), of the number plane, the mid-point of $\{(x_0, y_0), (x_1, y_1)\}$ is $\left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$.

[Hint: Apply the result of Exercise 3 for the coordinate system c on $\overleftrightarrow{(x_0, y_0), (x_1, y_1)}$ such that $c((x_0, y_0)) = 0$ and $c((x_1, y_1)) = 1$.]

J. Show that for all a, b, c, a′, b′, and c′,

(a, b, c) is proportional to (a′, b′, c′)
if and only if
(a, b) is proportional to (a′, b′),
and (b, c) is proportional to (b′, c′),
and (c, a) is proportional to (c′, a′).

K. The subject of proportionality at one time occupied an important place in the study of arithmetic. There were many special rules which students were required to memorize. These rules were just shortcuts for applying the equation transformation principles.

Before discussing these rules, we should define a few terms.

(1) <u>The ratio of a first number to a second number</u> is the quotient of the first by the second.

(2) A <u>proportion</u> is an equation whose members are ratios. The numbers a, b, c, and d (in this order) are said to be in proportion if

$$\frac{a}{b} = \frac{c}{d} .$$

[This was sometimes written, 'a:b::c:d'.]

Our work on proportionality in this unit is related to the notions of ratio and of proportion. In fact, if $\frac{a}{b} = \frac{c}{d}$, then (a, b) is proportional to (c, d) [and if $bd \neq 0$ then the converse is true].

In the exercises which follow, assume that the domains of the pronumerals are such that division by 0 does not occur.

Show that

1. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

   [This result was often referred to by saying that in a proportion, the product of the means ($b \times c$) equals the product of the extremes ($a \times d$).]

2. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{b}{a} = \frac{d}{c}$.

3. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$.

4. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{b} = \frac{c+d}{d}$.

   [Hint: If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{b} + 1 = \frac{c}{d} + 1$.]

5. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a-b}{b} = \frac{c-d}{d}$.

6. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a-b}{a+b} = \frac{c-d}{c+d}$.

7. If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+c}{b+d} = \frac{a}{b}$.

L. Another topic related to proportion is that of variation. Probably the most important feature of that topic is the special language used in connection with it. Here are several examples of sentences which deal with variation.

(1) y varies (directly) as x [or: y is proportional to x].

(2) y varies inversely as x [or: y is inversely proportional to x].

(3) y varies jointly as x and z.

Statement (1) refers to a $\{(x, y): y = kx\}$, for some number $k \neq 0$. [The number k is called the constant of proportionality,

or: <u>the constant of variation</u>.]

If $(x_1, y_1) \in \{(x, y): y = kx\}$

and $(x_2, y_2) \in \{(x, y): y = kx\}$ $[k \neq 0]$

then, since

$$y_1 = kx_1, \text{ and } y_2 = kx_2,$$

(y_1, y_2) is proportional to (x_1, x_2).

Several proportions can be derived from this. For example:

$$\frac{y_1}{y_2} = \frac{x_1}{x_2} .$$

Statement (2) refers to a $\{(x, y): xy = k\}$, for some number $k \neq 0$. If $\{(x_1, y_1), (x_2, y_2)\}$ is contained in $\{(x, y): xy = k\}$, $k \neq 0$, then since,

$$x_1 y_1 = k \quad \text{and} \quad x_2 y_2 = k,$$

$$\left(y_1, y_2\right) \text{ is proportional to } \left(\frac{1}{x_1}, \frac{1}{x_2}\right).$$

So, one proportion which is derivable from this is:

$$\frac{y_1}{y_2} = \frac{x_2}{x_1} .$$

Statement (3) refers to a $\{(x, z, y): y = kxz\}$, for some number $k \neq 0$. If (x_1, z_1, y_1) and (x_2, z_2, y_2) belong to this set, a proportion which can be derived is:

$$\frac{y_1}{y_2} = \frac{x_1 z_1}{x_2 z_2} .$$

Most of the situations in which this "language of variation" is used are problems in applied mathematics. We solve one of these problems as a Sample.

(continued on next page)

Sample. The circumference, C, of a circle varies directly as its radius, r. A circle of radius 3.8 has a circumference of 7.6π. What is the circumference of a circle of radius 9.3?

Solution. Since "C varies directly as r", there is a number $k \neq 0$ such that, for each circle,

$$C = kr.$$

We can find the number k as follows. For one circle, $C = 7.6\pi$ and $r = 3.8$. So, $7.6\pi = 3.8k$, and, therefore, $k = 2\pi$. Hence, since C varies directly as r, for every circle, $C = 2\pi r$. So, a circle whose radius is 9.3, has circumference $2\pi(9.3)$, or 18.6π.

A second method of solving this problem is to use the proportion:

$$\frac{C_1}{C_2} = \frac{r_1}{r_2}.$$

Let $C_1 = 7.6\pi$, $r_1 = 3.8$, and $r_2 = 9.3$. Then

$$\frac{7.6\pi}{C_2} = \frac{3.8}{9.3}.$$

So,

$$\frac{7.6\pi}{3.8} = \frac{C_2}{9.3},$$

and

$$C_2 = 2\pi(9.3) = 18.6\pi.$$

[In the Sample, the equation 'C = kr' is called the equation of variation.]

1. The perimeter of an equilateral triangle is directly proportional to its altitude. Write an equation of variation to express this fact. Use it to find the perimeter of an equilateral triangle whose altitude is 7 when the perimeter of another equilateral triangle with altitude 9 is $18\sqrt{3}$.

(continued on next page)

2. At a constant temperature the pressure of a gas varies inversely as its volume. Write an equation of variation. If the pressure in pounds per square inch is 18 and the volume in cubic inches is 250, what would be the pressure if the volume is reduced 20%?

[You can find many more problems like these in high school and college algebra textbooks.]

M. Show that postulate VII cannot be deduced from postulates I - VI.

N. Prove each of the following theorems.

1. If $A \neq B$ and $C \notin \overleftrightarrow{AB}$ then $B \neq C$ and $A \notin \overleftrightarrow{BC}$.

2. For each two points, there is just one interval of which the given points are end-points.

3. If $A \neq B$ and $\overset{\circ\!\!\longrightarrow}{AB} \subseteq \overset{\circ\!\!\longrightarrow}{AC}$ then $\overset{\circ\!\!\longrightarrow}{AB} = \overset{\circ\!\!\longrightarrow}{AC}$.

4. If $\{A, B\} \subseteq \overset{\circ\!-\!\circ}{CD}$ then $\overset{\circ\!-\!\circ}{AB} \subseteq \overset{\circ\!-\!\circ}{CD}$.

5. If $[ABC]$ then $\overset{\circ\!\!\longrightarrow}{BC} \cup \overset{\circ\!\!\longrightarrow}{BA} = \overleftrightarrow{AC}$.

6. If $[ABCD]$ then $\overline{AD} = \overline{AC} \cup \overline{BD}$, and $\overset{\circ\!-\!\circ}{BC} = \overset{\circ\!-\!\circ}{AC} \cap \overset{\circ\!-\!\circ}{BD}$.

O. Prove each of the following theorems.

1. Each line contains just as many points as there are real numbers. [Compare with postulate I.] [In proving this theorem you must have used postulates I and VI, and the fact that a coordinate system is a (1-1)-correspondence.

   How many people are in your classroom?
   How many heads do they have?
   Are there just as many people in your classroom as there are heads?

   Do you need to know the answers to the first two questions before you can answer the third question? Describe a way of determining whether two sets have the same number of members which does not require knowing how many members each set has. Do you

know an answer to the question 'How many real numbers are there?'? (Compare this question with the first question in this paragraph.) Show that $\{x:\ x \text{ is a real number and } 0 < x < 1\}$ and $\{x:\ x \text{ is a real number and } 0 < x < 2\}$ have the same number of members.]

2. Each point is contained in at least as many lines as there are real numbers.