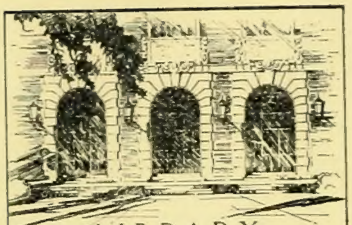

HIGH SCHOOL MATHEMATICS

Teachers' Edition

UNIT 6

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

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Introduction

This unit presents what is, at least nominally, a one-semester course which includes the properly geometrical topics usual to a high school course in geometry. [It can be taught either before or after Unit 5.] Much less time than usual is spent on mechanical drawing [constructions], and on trivial instancing of theorems. None should be spent on rote learning of proofs. Although most of the ''important'' theorems are boxed and numbered, many theorems are represented only by exercises of the hypothesis-conclusion form. [For numbered theorems, see the summary at the end of each section.] Finally, a considerable saving of time results, on the one hand, from the consistent treatment of geometric figures as sets of points, and, on the other, from the use of precise language, and attention to the nature of proof. The appendix on logic, which is intended to be studied concurrently with section 6.01, helps students to become aware of the basic rules of reasoning, some of which they have practiced in earlier units. Just as knowledge of the principles for real numbers supplies meaning to such processes as the simplification of algebraic expressions, so, knowledge of the principles of logic is a prerequisite for understanding the nature of proof. [On this point, see the beginning of the COMMENTARY for page 6-357.] In both cases, the knowledge acquired increases one's chances of being able to apply what he knows in new situations.

Most students in American high schools begin their study of geometry with a totally inadequate knowledge of the facts of physical geometry, and with no idea of the nature of proof. Indeed, one of the professed major aims of geometry courses has been to <u>initiate</u> students into the mysteries of proof -- typically, ''algebra is when you solve problems, and geometry is when you prove theorems''. Consequently, a teacher of geometry has to spend considerable time in what may properly be considered as remedial work. This, of course, leaves him with less than enough time for his proper tasks -- (1) leading students to see geometry as a mathematical theory, abstracted from physical experience, and deductively organized; and, (2) helping students gain, first, <u>more</u> of the kind of insight which will enable them to guess probable consequences of assumptions, and, second, a <u>deeper</u> understanding of logic which will aid them in establishing that their guesses are, indeed, consequences of their assumptions. A more serious result of such remedial work is that it blurs the distinction between physical and ''mathematical'' geometry and, as indicated above, suggests a distinction between branches of mathematics which does not, in fact, exist.

Fortunately, students of earlier UICSM units already have considerable experience in proving theorems, experience which they have gained in the relatively simple process of deducing consequences of the basic principles for real numbers. They have learned the use of '=' to refer to the logical relation of identity [and only for this] and are aware, at least on a nonverbal level, of the basic logical principles which govern its use -- the substitution rule for equations [page 6-359] and the principle of identity [page 6-362]. They also understand the use of variables and quantifiers, and the role of test-patterns as proofs of universal generalizations. Finally, they have had a little experience with conditional sentences and the use of the basic principles which govern the use of 'if. . . then___' -- modus ponens [page 6-367] and conditionalizing, and discharging an assumption [page 6-373]. Thus, they are in large part prepared for understanding, and discovering, the much more complex proofs required for theorems of geometry.

That they have this much preparation is, indeed, fortunate. For it would be difficult to find a branch of mathematics at all accessible to high school students which is <u>less</u> suitable, than geometry, as an <u>introduction</u> to rigorous thinking. Because of its intuitive appeal [to students well-grounded in physical geometry] and the intricacy of the proofs of most of its substantial theorems, it is fairly good as a second experience with proof. However, the great number and variety of geometrical concepts, which, admittedly, adds interest to the subject, results in proofs which are, for the most part, too complex to be accessible to most 16 year-olds. Consequently, the usual high school geometry proofs are full of holes. And, for that matter, so are most of the proofs in this unit. However, there is a difference. In conventional geometry courses, the holes are, for the most part, not apparent to a student, and he is, in consequence, led into habits of sloppy thinking. In contrast, a student's experience in studying this unit should result in his being aware of, at least, most of the gaps in his, and the text's, proofs, and in his knowing, to some extent, how these gaps could, given time and patience, be filled. [At this point, it may be helpful for you to read some of the COMMENTARY for page 6-18, beginning at the middle of TC[6-18]a.]

Sloppy reasoning is not inherently bad -- indeed, in dealing initially with a complicated situation it is almost unavoidable. But, what is unconscionable is failure to be aware of sloppy reasoning when it occurs. Now, it is much more difficult to learn to reason correctly, after one is habituated to reasoning sloppily, than it is to learn to judge the degree of sloppiness which a given occasion justifies, after one has learned at least what it means to reason correctly. Consequently, in section 6.01, for example, proofs are given in considerable detail, and such

gaps as there are, are clearly labelled and discussed in the light of the Introduction section. [To make this possible, the theorems of section 6.01 are of a rather simple-minded kind.] As a student's experience grows, and his sensitivity to non sequiturs increases, he should be allowed to omit more steps from his proofs. [Of course, the teacher should run frequent spot-checks to make sure that students are aware of the gaps they leave, and that they have some reasonable basis for believing that they can be filled.] Routine procedures for omitting steps, in certain circumstances, or of certain kinds, are discussed on page 6-33 ['Introduction', 'algebra'], page 6-42 ['figure'], page 6-43 ['Steps like . . . '], and page 6-72.

It has been said, above, that the theorems of section 6.01 are of a rather simple-minded kind. By this it was meant that they are, for the most part, intuitively obvious. For example, the proof of Theorem 1-1 [page 6-33] shows that Axiom A implies a statement which Axiom A was, with some explicit pains [see the paragraph beginning with the last five lines on page 6-30], framed to imply. It is sometimes asserted that to ask students to study proofs of "obvious" theorems is [necessarily] stultifying. On the contrary, for a student who already has some notion of proof and is in the process of enlarging this notion, such proofs serve as tests of the principles of logic which he is on the verge of accepting. Rather pragmatically, he argues that since the use of these principles enables him to prove some theorems which are intuitively correct, the principles are probably valid. Moreover, the principles will be worth using in cases where the result to be proved is surprising.

One innovation introduced in this unit is the one-column proof. [This is not absolutely an innovation, since it is commonly favored by logicians.] The two-column proof customary in conventional geometry texts gives a false impression of the logical structure of a proof and, in fact, has to be tortured to accomodate "indirect" proofs. [This may be one reason why indirect proofs are considered difficult to understand.] It seems a likely guess that the two-column form grew out of the belief that all reasoning is syllogistic, a conclusion belied by the amazing growth of logical theory since 1840. For a chain of syllogisms, two columns are convenient:

minor premiss (1)	major premiss (1)
conclusion (1)	major premiss (2)
[= minor premiss (2)]	
conclusion (2)	

etc.

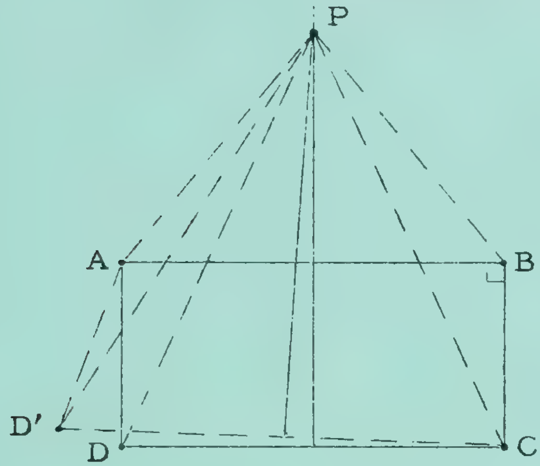
For the more varied modes of reasoning employed, not only in mathematics, but in every-day living, this form is totally inadequate.

If one wishes to display the structure of a proof, one needs something equivalent to the tree-diagrams used in the text. [The proof of Theorem 1-5, given in paragraph form on pages 6-44 and 6-45, analyzed in column-form on pages 6-400 and 6-401, and "treed" on TC[6-400], is an example of a not very complicated argument which could only with difficulty, and great loss of clarity, be put in the standard two-column form.] However, tree-diagrams are very uneconomical of space, especially when the steps of the proof are written in the diagram instead of merely being referred to by number. Fortunately, a one-column proof does not distort the picture, as does a two-column proof, by separating "statements" and "reasons", and can easily be supplemented by marginal comments [see, for example, page 6-33] which convey, less graphically, the structure which is displayed by the "tree".

Although the writing of column proofs, supplemented by marginal comments, and, then, diagramming such proofs by trees, is a good way to learn how the rules of reasoning operate, column proofs, despite conventions for omitting steps, grow to unwieldy length in the case of most "interesting" theorems. Consequently, it is desirable to summarize, or outline, column proofs in the form of paragraph proofs. This is done from the beginning [see pages 6-35 and 6-41], and students are expected, in the later parts of the unit, to give paragraph proofs in preference to column proofs. This, of course, is what they will, in the natural course of events, be expected to do in later courses in mathematics.

It is now high time that something is said about the particular organization of geometry which has been adopted in this unit. Mention has already been made of the complications which are inherent in geometry because of the number and variety of geometrical concepts. The situation can be simplified to some extent, as is done in this unit, by treating all geometric figures as sets of points. This approach has the added advantage of giving UICSM students additional practice in thinking in terms of the concepts of set and operations on sets, which concepts are of fundamental importance in much of present-day mathematics. However, if one is to avoid sloppy thinking, or even to be aware of the degree of sloppiness in his thinking, in geometrical matters, one must take some account of many complications which still remain. As an indication of the kind of point on which more care must be lavished than is usual, if one is to give adequate proofs of geometry theorems, consider the following alleged proof of the statement:

Each right angle is congruent to an obtuse angle.



Let $\angle ABC$ be any right angle, and construct a rectangle ABCD. Choose D′, outside of ABCD so that the segments AD′ and AD have the same length. The perpendicular bisectors of segments DC and D′C intersect, in a point P, as shown in the figure. ΔAPB, $\Delta D'PC$, and ΔDPC are isosceles triangles; so, $\Delta AD'P$ and ΔBCP are congruent. In particular, $\angle PAD'$ and $\angle PBC$ are congruent and, since ΔAPB is isosceles, $\angle PAB$ and $\angle PBA$ are congruent. Since differences of congruent angles are congruent, $\angle BAD'$ and $\angle ABC$ are congruent. But, $\angle BAD'$ is an obtuse angle; so, $\angle ABC$ is congruent to an obtuse angle.

[Before reading further you may wish to discover the error in this reasoning. A carefully drawn figure will help.]

The error in the supposed proof lies in the tacit assumption that the point B is interior to $\angle PAD'$, just as A <u>is</u> interior to $\angle PBC$. It is on the basis of this assumption that one argues from the congruence of $\angle PAD'$ and $\angle PBC$, and the congruence of $\angle PAB$ and $\angle PBA$, to the conclusion that $\angle BAD'$ and $\angle ABC$ are congruent. A "carefully drawn figure" will show, for example, that A and B are on the same side of the line through P and D′, rather than on opposite sides of this line, as suggested by the figure above. But, this should not restore one's feeling of satisfaction (if any) with conventional proofs. A proof of a theorem of geometry should show by logically justifiable steps that the theorem is a consequence of the postulates. When one introduces into his reasoning a conclusion drawn only from a figure, whether the picture is the one above, <u>or a "carefully drawn" one</u>, he has departed from this standard of rigor.

Without recourse to the postualtes, one has no more justification for introducing into a proof his "correct" conclusion as to the relative position of the points A and B than the writer of the proof given above had for assuming that B is interior to $\angle PAD'$.

COURSE CONTENT

In order to give rigorous proofs of theorems of geometry it is essential that one pay attention to questions such as which of three collinear points is between the other two, and whether two points are on the same side of a line, or on opposite sides [or neither]. For many reasons, it is impossible to adhere consistently to such standards of rigor in an elementary course. However, as remarked earlier, it is not so important, at this level, at least, to adhere to such standards as to be aware of when one departs from them. In order to lay the basis for such an awareness [and also to introduce some concepts which will be of continual use in the sequel], the unit begins with an Introduction [pages 6-1 through 6-28] which deals, for the most part, with the notion of betweenness and related concepts. Students become acquainted with fifteen Introduction Axioms [which they are not expected to memorize] and with a few of the theorems which follow from these axioms. To help teachers, who so desire, to appreciate a rigorous exposition of euclidean geometry, proofs of these theorems are given in the COMMENTARY. Later, at appropriate places in the COMMENTARY, other such theorems are proved. Furthermore, the answers in the COMMENTARY for the exercises are usually given in complete enough detail that a teacher can supply such "Introduction material" as is necessary for a rigorous solution of each exercise. Suggestions as to how far one may make use of such material in class will be given later.

The geometrical content of section 6.01 has to do with measures of segments. Three axioms are introduced, and some of their consequences derived. Section 6.02 deals with angles, and their degree-measures, perpendicularity and adjacent angles. Five more axioms on angle-measure and its relation to segment-measure are introduced in this section. [Except for two axioms on area-measure, introduced in section 6.11, this completes the set of axioms.] At this point, special mention should be made of Axiom E, first given on page 6-54. This is an existence axiom which, among other things, guarantees the existence [and uniqueness] of the perpendicular to a line at a point on it and, as is seen later, also guarantees the existence of the parallel to a given line through a given point not on the line. [In this connection, note that lines are sets of points and exist independently of our labors. Properly,

one considers the (already existing) line through two given points, or perpendicular to a given line at a given point, rather than "constructing" or "drawing" it. The usual justifications of constructions are actually proofs of the existence of lines, circles, or what have you, which satisfy given conditions. On the other hand, in classroom discussion of pictures drawn to illustrate geometrical situations, it is perfectly correct to say, for example, 'Draw the circumcircle of the triangle.', meaning thereby that the hearer is to draw a picture of this circle. In writing up the corresponding proof, one might find it convenient to write 'Consider the circumcircle of the triangle.', or 'Let O be the center of the triangle's circumcircle.'.] Axiom F justifies conclusions as to the sums of measures of adjacent angles. Axiom H furnishes a quick, and natural, path to the usual congruence theorems for triangles.

One point of usage introduced in section 6.01 may also require special mention here. In view of the fact that in this as in earlier units '=' always means 'is' in the sense of 'is the same as', one must refrain from speaking of 'equal angles' or 'equal segments' except in cases in which only one angle or segment is being referred to. Two angles, or segments, are, by virtue of their being two, never equal. However, in case they have the same measure, they are congruent.

In section 6.03 the notion of a triangle is introduced, and the congruence theorems s.s.s., s.a.s., and a.s.a. are proved, together with the usual applications to isosceles triangles, etc. In going over this material, one realizes that one seldom is interested in merely proving that two triangles are congruent. What one wants to know is that, for example, two angles are congruent "because they are corresponding parts of congruent triangles". The need to be able to know without referring outside a proof, which are "corresponding parts of congruent triangles" motivates the discussion of matching [page 6-80, et. seq.] and the somewhat unusual form in which the congruence theorems are stated [page 6-86].

Section 6.04 deals with geometric inequations -- the exterior angle theorem is perhaps the most familiar example of a theorem which deals with such matters. A strong case could be made for the statement that, throughout mathematics, inequations occur more frequently than equations. Consequently, inequations deserve a much more extended treatment then has customarily been accorded to them in elementary courses.

Section 6.05 deals with parallel lines, alternate interior angles, etc. The exercises give a preview of the developments in section 6.06 which treats of polygons, with special emphasis, as usual, on various kinds of quadrilaterals. In this latter section students are given an

opportunity to search out, and prove, theorems of their own devising [see the two final paragraphs on page 6-166].

After a short interlude on the notion of necessary and sufficient conditions, which, incidentally, serves as a review of section 6.06, section 6.07 takes up proportionality and the concept of similarity. Section 6.08 applies some of the results to an elementary discussion of trigonometric ratios.

Section 6.09 is a short introduction to analytic geometry. The COMMENTARY for page 6-232 attempts to clarify the role of measure --which, in contrast to conventional treatments, plays a prominent part in this unit--in euclidean geometry.

Section 6.10 introduces circles and related concepts. There are the usual theorems on tangents, inscribed circles, measures of angles inscribed in a circle, etc. The COMMENTARY for page 6-329 contains a rather extensive discussion of the notion of arc-length-measure [as contrasted with arc-degree-measure], for those who wish to go further than does the text into such questions as why the circumference of a circle is given by the formula '$c = 2\pi r$'.

The final section, 6.11, deals with area-measure. As a basis for justifying the conclusions which are drawn, two additional postulates are introduced, and some theorems whose proofs are far beyond the level of this course are introduced without proof.

Following the appendix on logic there are collections of supplementary exercises. Most of these are referred to at appropriate points in the text. [See bottom of page 6-50 for an example of such a reference.] They consist, for the most part, of easy exercises and are meant to supplement, at need, the minimum collections of such exercises in the text proper. However, some contain minor theorems. Certain of the collections of supplementary exercises are <u>not</u> signalled in the text but are noticed at appropriate places in the COMMENTARY. Among these are the ones on sets [pages 6-402 through 6-404] and on square root [pages 6-431 and 6-432]. They will be of help to students who have not studied Unit 5 or who need a review of these subjects.

Finally, there is a collection of review exercises, some easy, others difficult. They are suitable, for example, to use as reminders of geometry at times when students are studying later units. They include [pages 6-451 through 6-453] the only specific mention of the word 'locus' in the course. You may, if you wish, bring up the concept of locus at some earlier point.

In addition, of course, to Units 1 through 5, there are a number of books which can be of help to a teacher who wishes to supplement his mathematical background. Among those which are particularly pertinent to the subject matter of Unit 6, the following are especially worth mentioning:

Euclid's Elements, translated with introduction and commentary by Sir Thomas L. Heath [3 vols.] [Dover reprint]

The Foundations of Geometry, by O. Veblen, in Monographs on Topics of Modern Mathematics, edited by J. W. A. Young [Dover reprint]

How to Solve It, by G. Polya [Anchor Books reprint]

Mathematics and Plausible Reasoning, by G. Polya [2 vols.] [Princeton University Press]

Introduction to Logic, by P. Suppes [Van Nostrand]

PEDAGOGY

A person who reads this unit [or, for that matter, any of the UICSM units] and notices the care we have used in saying things precisely is likely to go away thinking that the teachers and students who use the text must also carry on their classroom conversations with the same kind of precision of spoken language. A visit to the classroom of a teacher who is using these textbooks properly would soon dispel such a notion. Any successful teacher knows that the spoken word conveys only a small portion of the ideas which are exchanged in face-to-face communication. Spoken words are accompanied by paralinguistic devices such as intonations, inflections, and pauses, as well as by kinesic devices such as shrugs, grimaces, and hand movements. Teachers who have learned to recognize the nonverbal awarenesses in their students which are promoted through exploratory exercises have really succeeded in opening more channels of communication between themselves and their students. Since the most cleverly formulated metaphor in written language is probably not as effective as the intonations which any child will pick up from his culture, a textbook author must maintain a high standard of precision when he makes assertions. If he tries to use only examples to get a generalization across [as we do in many places], there must be a skillful teacher somewhere in the picture who can detect nonverbal awareness, and who can invent more examples when necessary. And, of course, the teacher can rely upon spoken language with all of its paralinguistic and kinesic devices to enrich the communication as he gives the examples.

*

The teaching of geometry has a long tradition of excellent pedagogy and practically all of it can be used in teaching this unit. Take, for example, the technique of helping students discover relationships by using deformable figures or having students imagine points moving or lines rotating. Although these things are not part of our formal geometric structure, we fully expect teachers to make ample use of them in the classroom. A few of these techniques are suggested in the COMMENTARY and we urge teachers to familiarize themselves with some of the vast professional literature on the subject. In writing the COMMENTARY we have assumed that either the teacher has had experience in using such devices in conventional courses and will not hesitate to use them in this one or that he has access to additional pedagogical sources.

*

In a similar fashion, we are also assuming that teachers and students will feel free to invent names for the axioms and theorems which are used over and over again or which take so many words to state that to do so in class discussion or even on homeowrk papers would be irksome. For example, Axiom A might be called 'the point-on-segment axiom' or 'the-segment-sum axiom'; Axiom F might be called 'the-angle-sum axiom'; Theorem 3-5 might be called 'the-base-angles-of-an-isosceles-triangle theorem'. [Even in the text itself, we felt it necessary to use the familiar names 's.a.s.', 's.s.s.', etc. for the various triangle-congruence theorems.] The need for such short names will arise naturally in class, and the inventions should come from the students with help from the teacher. Of course, the teacher will want to make occasional checks to be sure students can state the theorems they actually use, but no attempt should be made to compel students to memorize the wording used in the text.

*

As a teaching aid we have included in the COMMENTARY a quiz for each section, a mid-unit quiz, and a unit quiz. For the most part, these quiz items are designed to test the "average" student. They are straight-forward and reasonably routine, probably more so than are the regular exercises in the text. For classes of high ability students, these tests should be augmented with more difficult items. In a few of the tests we have included starred items for this purpose. But, once again, we are assuming that the teacher is responsible for evaluating his course, and that he has access to sources of ideas for test questions.

[In this connection, we call to your attention the excellent problem book <u>Mathematics Review Exercises</u> (3rd Edition) by Smith and Fagan (Boston: Ginn and Company, 1956). Since the problems in this book are designed for the conventional high school curriculum, they are not stated in language which follows our conventions. Nevertheless, the problem ideas are good and varied, and it is not hard to rephrase the problems if you wish to. In fact, after your students have completed section 6.06, they should be able to handle geometry problems stated in conventional language, for the statements of such problems are mostly descriptions of drawings.]

*

As mentioned earlier, many of the solutions given in the COMMENTARY deal with issues which a teacher would not expect to find in solutions submitted by students. These COMMENTARY solutions are not to be regarded as models against which a student's solutions should be graded. They are included to alert the teacher to opportunities to point out to all students that there are gaps in their solutions. For most students, for example, a good job of teaching would consist in having them admit that they did assume, probably without knowing it, that the diagonals of a parallelogram crossed each other. Other students should express their conviction that such an assumption could be predicted from the axioms. And, still other students should feel that given enough time and patience, they could probably carry out the steps in the derivation. Naturally, the teacher does not raise these issues for every exercise in the text. But, he should do it enough times to insure that students do not leave the course thinking that their proofs are complete, and that they know all there is to know about geometry. In fact, the course will have been successful if the students have become sufficiently critical observers to raise these issues themselves.

*

As remarked at the beginning of this introduction, this unit can be taught before Unit 5 or after Unit 5. The choice depends largely on local custom. In some high schools it is expected that students complete their geometry course by the end of the tenth grade because many of them will not study more mathematics in high school. If it is likely that students will not complete Unit 6 at the end of the tenth grade should it be taught after Unit 5, the natural order of the units should be reversed. Unit 5 contains many topics which are customarily taught at the eleventh grade level.

It should be pointed out, however, that Unit 6 is designed for students who have studied Units 1-4. A teacher who wishes to use Unit 6 with students who have not been through Units 1-4 will have at least two major problems. The first of these is to prepare his students for the use of set-notation. Although the supplementary exercises on pages 6-402 through 6-404 will help in this matter, they will not be enough. The second problem is more serious. Students of Units 1-4 have experienced a careful development of deductive proof in algebra. They know what proof does, and many of them have acquired a real taste for it. That is, they feel uncomfortable about leaving provable things unproved, not because they need to be convinced of their correctness, but because they want to show that the things fit in the system. Unit 6 simply continues in this vein. Proving theorems is an accepted thing for these students and they don't have to be motivated by the usual devices found in conventional geometry textbooks. Moreover, the appendix on logic which attempts to call students' attention to some of the principles underlying deductive reasoning includes many examples of proofs of theorems studied in Unit 2. These two problems -- the use of set-notation and the need for motivating proof -- are not insuperable but their solution does demand more instructional material than is provided in Unit 6.

*

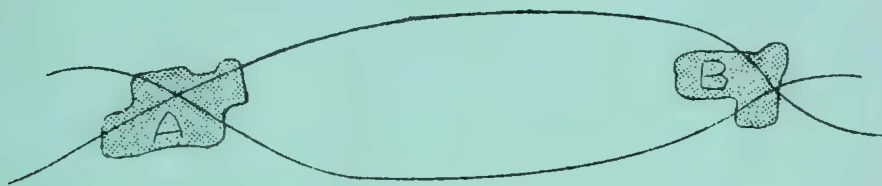
To end this Introduction on a highly practical note, we pass on to you Mr. Howard Marston's suggestion that you use index tabs or looseleaf dividers to mark the various sections of the book.

A VISIT TO THE PLANET GLOX is intended to reacquaint the student with the idea that if he accepts certain "facts", he can deduce new information from these facts. This new information will be as valid as the previous facts. Thus, Jo at Zabranchburg High deduces new information from the five original messages. These new facts were verified by the observations of the second space man but this was an unnecessary expenditure of time.

Pages 6-1 through 6-6 should be completed in one day. Writing Messages 1-5 on the board as a student reads may prove helpful for the discussion on page 6-4. At this stage we do not intend that the student make a verbal identification of cities with points, and highways with lines. Even though you know that these messages will "turn into" the axioms of connection, do not suggest that the student think in these terms.

Message B.

Jo had just deduced that there were at least three highways. Message 3 had told her there were at least three cities. She wondered if there were two highways which met in two cities A and B.



But, Message 5 told her that there was one and only one highway connecting two cities A and B, which convinced her that there could not be two highways running between cities A and B.

Message C.

Jo started thinking about the cities on Glox. Could she get on some highway in city A and travel through every other city on Glox without changing highways? If this were the case, then each Gloxian city had to be on this highway. But, Message 3 said that there were at least three cities not all on the same highway. She deduced that there was at least one city she couldn't reach by staying on this highway.

*

Sentences (1), (2), and (3) concerning the businessmen in Zabranchburg are analogous to Messages 3, 4, and 5 from Glox. The sentence that says that there are at least three partnerships among these businessmen corresponds to Message A, and can be deduced in a manner entirely parallel to Jo's reasoning. We can also deduce sentences corresponding to Messages B and C.

Corresponding to Message B: There are not two partnerships which contain the same two businessmen. If the students do not respond readily with such a statement, accept an instance such as:

If Smith and Jones are two businessmen, there are not two partnerships to which both Smith and Jones belong.

Corresponding to Message C: No single partnership contains all the businessmen.

Answers for questions on page 6-9.

line 12. One.

line 14. No. [Remember that when we talk about two straight lines ℓ and m, we mean that ℓ and m are <u>different</u> straight lines, that is, that $\ell \neq m$.]

line 15. Yes. [parallel straight lines]

line 20. Read '$\ell \cap m = \{P\}$' as 'the intersection of lines ℓ and m is the set consisting of the point P' or as 'ℓ intersects m at P' or as 'ℓ intersects m in P'.

line 4b. $\overleftrightarrow{PR}$ is m; $\overleftrightarrow{PQ} \cap \overleftrightarrow{PR} = \{P\}$ [Read the latter as 'the intersection of lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ is the set consisting of the point P' or as 'lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ intersect at [or : in] P'.]

line 3b. $\overleftrightarrow{PQ} \cap \overleftrightarrow{RQ} = \{Q\}$ [Although $\overleftrightarrow{RQ}$ is not pictured in the figure, it does exist.]

line 1b. Yes; $\overleftrightarrow{AB} \cap \overleftrightarrow{BA} = \overleftrightarrow{AB} = \overleftrightarrow{BA}$

Since we wish to think of geometric figures as sets of points, we do not say that a point is a geometric figure. The <u>set</u> consisting of a point P <u>is</u> a geometric figure. However, avoid any discussion of this matter. On the other hand, the distinction between P and $\{P\}$ must be made clear. The following example may help.

> Suppose that the Zabranchburg High School Music Club has ten members. Five members graduate and three members move away. Mary Smith and Jane Dale are still members of the club. Now, suppose that Mary Smith resigns. Jane Dale is still a member of the club. The Student Council decides to abolish the club, but it does not abolish Jane Dale.

[Chapter III of the 23rd Yearbook [NCTM] is a very good reference if you wish to read more about sets in general, or about the distinction between a singleton set and its single member.]

Observe that a sentence such as:

the intersection of two straight lines is at most a single point

is meaningless. It is correct to say 'The intersection of two straight lines <u>contains</u> at most a single point'. Since a straight line is a set of points, the intersection of two straight lines must be a set. If the lines are parallel, this intersection is the empty set. If the lines are not parallel, the intersection is a set consisting of a single point.

*

The Supplementary Exercises on page 6-402 are designed to acquaint [or reacquaint] the student with set notation. You may wish to use these exercises to facilitate the reading of page 6-9 and subsequent pages. [Also, see section 5.02 of the 1960-61 edition of Unit 5 and the related COMMENTARY.]

*

Answers for questions on page 6-10.

line 1. No.

line 3. One. [Read '$\ell \cap \overleftrightarrow{AB} = \emptyset$' as '$\ell$ and $\overleftrightarrow{AB}$ have no point in common' or as 'ℓ and $\overleftrightarrow{AB}$ intersect in the empty set' or as 'the intersection of ℓ and $\overleftrightarrow{AB}$ is empty' or as 'ℓ and $\overleftrightarrow{AB}$ do not intersect'.]

line 6. Notice the use of 'if and only if'. This is the first occurrence of this sentence connective in Unit 6. [See pages 6-390 and 6-391.] At this time point out to the student that the 'if and only if' in line 6 tells him that whenever he sees a sentence like 's is parallel to t' he can replace it by the sentence '$s \cap t = \emptyset$', and conversely, '$s \cap t = \emptyset$' can be replaced by 's is parallel to t'.

By our definition, which is the one ordinarily used in secondary school mathematics, a line is not parallel to itself [$\forall_\ell\ \ell \cap \ell = \ell$ and $\ell \neq \emptyset$]. In more advanced work, it is sometimes convenient to use the following definition:

$$\forall_\ell \forall_m\ \ell \text{ is parallel to } m \text{ if and only if } \ell \cap m = \emptyset \text{ or } \ell = m$$

line 7. No; yes. [Suppose that s is parallel to t. Then, by definition, $s \cap t = \emptyset$. Since intersection of sets is a commutative operation, $t \cap s = \emptyset$. So, again by definition, t is parallel to s.]

line 8. No. [It could be the case that $\ell = n$. But, it does follow that ℓ is parallel to n or $\ell = n$. If we had started the question by saying that ℓ, m, and n are three lines then the conclusion that ℓ is parallel to n would follow.]

line 11. Yes.

lines 12-13. Here is another occurrence of 'if and only if'. Be certain that students understand that if you are told that M, N, and R are collinear points then you can conclude that there is a line which contains M, N, and R, and if you are told that M, N, and R belong to the same line then you can conclude that M, N, and R are collinear points.

The answers to the exercises on page 6-10 and everywhere else in the Introduction should be informal. Do not expect or insist on polished answers, especially to the tell-why questions. Remember, the purposes of the Introduction are to build up good intuition about problems concerning collinearity, order of points on a line, separation, etc., and to give practice in using set notation.

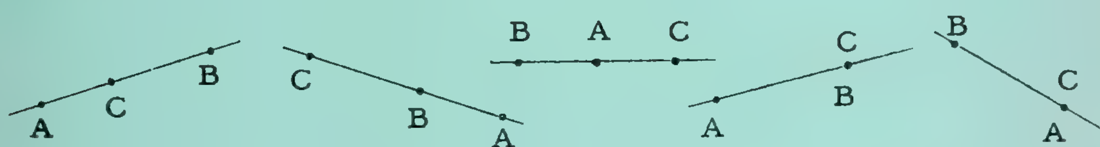
*

line 19. Note the use of a principle about sets in the Solution to Sample 2. If a point belongs to a first set and to a second set then it belongs to their intersection. [See section 5.02 of the 1960-61 edition of Unit 5 for a detailed treatment of the principles of sets.]

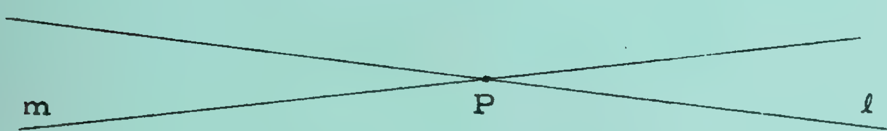
*

Answers for Part A [on pages 6-10 and 6-11].

1. Drawings like any of the following five. Be sure that all five are shown in class.



2.

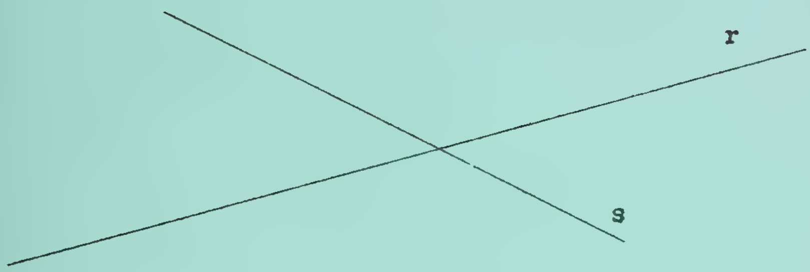


3. Can't occur. Two straight lines, ℓ and m, cannot have more than one point in common. '$\{P, R\} \subseteq \ell \cap m$' and '$P \neq R$', together, say that ℓ and m have two points in common.

4.

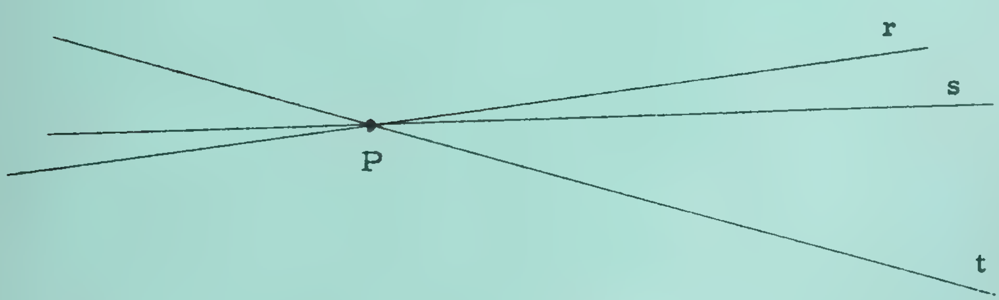
[Any picture which shows A, B, C, and D collinear is correct.]

5.



6.

7.



[Read '$(r \cap s) \cap t$' as 'r intersection s intersection t'. It stands for the set of points which belong to both $r \cap s$ and t. Ask students if $(r \cap s) \cap t = r \cap (s \cap t)$.]

*

Answers for questions at the bottom of page 6-11.

On marking another point V on ℓ, students may produce one of three pictures.

For this picture, statements (3) and (4) are true.

For this picture, statements (5) and (6) are true.

For this picture, statements (1) and (2) are true.

14. Can't occur. If A, B, and C are three collinear points, $\overleftrightarrow{AC} = \overleftrightarrow{BC}$; and, if B, C, and D are three noncollinear points then $\overleftrightarrow{BC} \neq \overleftrightarrow{CD}$. So, if the conditions of the exercise are met, $\overleftrightarrow{AC} \neq \overleftrightarrow{CD}$, and A, C, and D are not collinear.

15. Can't occur. If A, B, C, and D are four points then $B \neq A$. Hence, if $B \in \overleftrightarrow{AC}$ then $C \in \overleftrightarrow{AB}$, and if $B \in \overleftrightarrow{AD}$ then $D \in \overleftrightarrow{AB}$. Consequently, if the conditions of the exercise are met, both C and D belong to $\overleftrightarrow{AB}$, and A, B, C, and D cannot be noncollinear.

*

Answers for Part ☆B [on page 6-11].

1. Yes. If s is empty or consists of a single point [that is, if s is a degenerate set], then s is clearly a set of collinear points. Now, suppose that s contains at least two points, and let A and B be two points of s. If X is any other point of s then, since each three points of s are collinear, X, A, and B are collinear. So, $X \in \overleftrightarrow{AB}$. Since, also, $\{A, B\} \subseteq \overleftrightarrow{AB}$, it follows that each point of s belongs to $\overleftrightarrow{AB}$. Consequently, s is a set of collinear points.

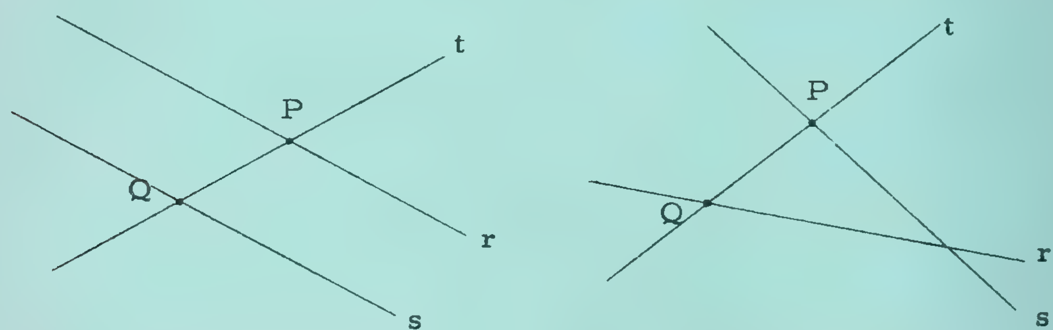
2. No. Since any two points are collinear, any set of noncollinear points is a counter-example.

3. t is a straight line. For, since t is a set of collinear points, there is a straight line ℓ such that each point of t belongs to ℓ--that is, such that $t \subseteq \ell$. Now, if $P \in \ell$, it follows that each point of $t \cup \{P\}$ belongs to ℓ and, hence, that $t \cup \{P\}$ is a set of collinear points. But, we are given that if $P \notin t$ then $t \cup \{P\}$ is not a set of collinear points. So, we know that if $P \in \ell$ then it is not the case that $P \notin t$. That is, each point of ℓ belongs to t. Since each point of t belongs to ℓ and each point of ℓ belongs to t, $t = \ell$.

8. Same answer as for Exercise 7.

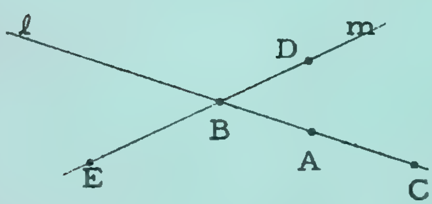
9. Can't occur. A subset of the intersection of two sets is a subset of each of them. So, if $\{P, Q\} \subseteq (r \cap s) \cap t$ then $\{P, Q\} \subseteq r \cap s$. Thus, we have the same situation as in Exercise 3. This provides a chance for some good teaching in showing students how to reduce a new problem to one which has already been solved.

10. [Read '$t \cap (r \cup s)$' as 'the intersection of t and the union of r and s'. Students will find the exercise very easy if they recall, from Unit 5, that $t \cap (r \cup s) = (t \cap r) \cup (t \cap s)$. If they don't recall it, they may discover the generalization as a result of working the exercise.]

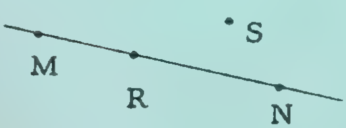


Exercises 1-9 are the build-up for Exercise 10. Although we do not introduce the term 'transversal' at this time, Exercise 10 actually foreshadows the definition given on page 6-142.

11. Any figure showing <u>four</u> collinear points, A, B, C, and D.

12. 

Any two intersecting lines, ℓ and m, such that $\ell \cap m = \{B\}$, $\{A, B, C\} \subseteq \ell$, and $\{B, D, E\} \subseteq m$

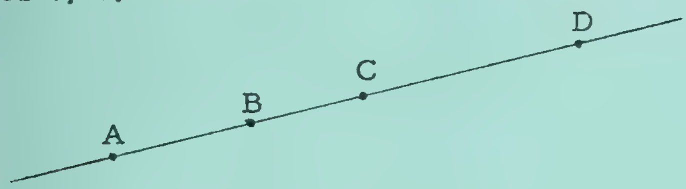
13. 

[To say that points are noncollinear amounts to saying that there does not exist a line which contains all of them.]

Answers for questions on page 6-12.

line 4. Only one.

lines 7, 8.



B is between A and D.

C is between A and D.

*

Read '$\overrightarrow{PQ}$' as 'half-line PQ'.

Notice that $P \notin \overrightarrow{PQ}$. [$P \in \ell$, but P is not on either side of itself.]

*

If the students are familiar with the idea of complementation of sets [see the Supplementary Exercises on pages 6-403 and 6-404], they may appreciate the statement:

> if $P \in \ell$ then the complement of $\{P\}$ with respect to ℓ is the union of two half-lines

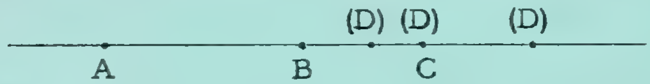
This is a fancy [and correct] way of saying that if you pluck a point out of a line, what's left is a pair of half-lines.

*

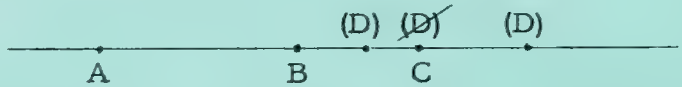
The first paragraph on page 6-12 makes clear that betweenness is a notion that pertains to <u>collinear</u> points.

Suppose that B is also between A and D. Where might D be?

[Answers should lead to concluding that either D is between B and C, or D = C, or D is "to the right of C". Indicate on drawing.]



Suppose, now, that $D \neq C$.



What can we say about B, C, and D? [Discussion, prompted if necessary by the question 'Can we be sure that D is between B and C?', should lead to the conclusion that either C is between B and D or D is between B and C.]

So [writing], if B is between A and C and also between A and D, and $C \neq D$, then either C is between B and D or D is between B and C.

Can we write this a shorter way? What can we write, for example, instead of 'B is between A and C'? [Answer: $B \in \overline{AC}$]

So, we can write [doing so]:

$$\text{if } B \in \overline{AC} \text{ and } B \in \overline{AD} \text{ and } C \neq D \text{ then } [C \in \overline{BD} \text{ or } D \in \overline{BC}]$$

We can use set notation to shorten this still more. [Try to elicit how this can be done, and rewrite, as below.]

$$\text{if } B \in \overline{AC} \cap \overline{AD} \text{ and } C \neq D \text{ then } [C \in \overline{BD} \text{ or } D \in \overline{BC}]$$

So far, we have been writing open sentences. Can we write a generalization which has this [pointing to last sentence] as an instance? [Students should suggest writing '$\forall_A \forall_B \forall_C \forall_D$' in front of last sentence. Do so. Then ask if:

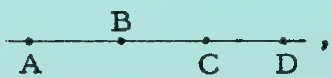
$$\forall_W \forall_X \forall_Y \forall_Z \text{ if } X \in \overline{WY} \cap \overline{WZ} \text{ and } Y \neq Z \text{ then } [Y \in \overline{XZ} \text{ or } Z \in \overline{XY}]$$

"says the same thing".]

As students will learn [but should not be told at this time], proofs are clearer if one uses one set of letters [say, 'W', 'X', 'Y', and 'Z'] with quantifiers when stating generalizations, and another set [say, 'A', 'B', 'C', and 'D'] in formulating instances.

line 16. '$C \in \overline{AB}$' follows from (a), from (b), from (f), and from (g).

(e) describes the intersection of the half-line $\overrightarrow{AB}$ and the half-line consisting of points "to the right of B". Since the second half-line is a subset of the first, the intersection of the two is the second half-line. [Another description of this set is that it is the complement, with respect to ℓ, of the ray $\{B\} \cup \overrightarrow{BA}$.]

line 23. 

, or: 

line 24. A, B, C, and D must be collinear. For, if $B \in \overline{AC} \cap \overline{AD}$ then $B \in \overline{AC}$. Since $\overline{AC} \subseteq \overleftrightarrow{AC}$, it follows that $B \in \overleftrightarrow{AC}$. Since $B \neq A$, it follows that $C \in \overleftrightarrow{AB}$. Similarly, $D \in \overleftrightarrow{AB}$.

In the first figure, above, $C \in \overline{BD}$ and $D \notin \overline{BC}$.

In the second figure, above, $C \notin \overline{BD}$ and $D \in \overline{BC}$.

[So, a student's answers to the last two questions in line 23 depend on which figure he has drawn.]

line 25. No. The second figure above provides a counter-example; the third point, C, is not between the second, B, and the fourth, D.

line 26. Yes.

*

The last five lines on page 6-13 serve to develop an idea which will be stated in one of the Introduction Axioms [Axiom 12 on page 6-21]. To help prepare students to grasp such generalizations more easily, you might try to get them to state this one right now. Here is one possible approach.

Suppose that B is between A and C. [Draw on board.]

So [pointing], C is over here, somewhere.

Since we learned on page 6-12 that no one of three noncollinear points is between the other two, B is not between A and K. So,

$$K \notin \{X: \text{ B is between A and X}\}.$$

It follows that the set in which we are interested is a subset of ℓ.

Let's canvass the points of ℓ to see which of them belong to the set.

Try A. Since B is not between A and A, we reject A.

Try B. Since B is not between A and B, we reject B.

Try C. Since B is between A and C, we accept C; so

$$C \in \{X: \text{ B is between A and X}\}.$$

Now, consider a point R "to the left of A", that is, a point R such that A is between R and B. Since just one of three collinear points is between the other two,

B is not between A and R.

So, we reject R. Consider a point S which is between A and B. For the reason just cited,

B is not between A and S.

So, we reject S.

The only remaining points of ℓ are those "to the right of B". [We have already tried one of these, the point C.] Such points are just those points T such that B is between A and T. Somewhat anticlimatically, these are just the points which satisfy (*) and, also, are just the points which belong to $\overrightarrow{BC}$. So, $\{X: \text{ B is between A and X}\} = \overrightarrow{BC}$.

*

line 8. $\{X: \text{ B is between C and X}\} = \overrightarrow{BA}$

line 9. 

line 10. $\overrightarrow{MN} \neq \overrightarrow{NM}$; $\overrightarrow{MN} \cup \overrightarrow{NM} = \overleftrightarrow{MN} = k$;
$\overrightarrow{MN} \cap \overrightarrow{NM} = \{X: \text{ X is between M and N}\}$

line 14. Yes; $\overline{MN} = \overline{NM}$. [Read '$\overline{MN}$' as 'interval MN'.]

safety's sake, a restriction ['x a real number'] to indicate the domain of the index. The index and the restriction are separated by a colon from a sentence ['2x + 3 = −7'], called <u>the set selector</u>. The whole symbol names the set whose elements are just those members of the domain of the index which satisfy the set selector. In the case in point, the set named is that whose sole member is the real number −5.

In Units 3 and 4 we were most often interested in sets of real numbers. So, for brevity, we adopted the convention that, in the absence of a restriction, the domain of an index was to be understood to be the set of real numbers. Under this convention, '$\{x$, x a real number: $2x + 3 = -7\}$' reduces to '$\{x\colon\ 2x + 3 = -7\}$'.

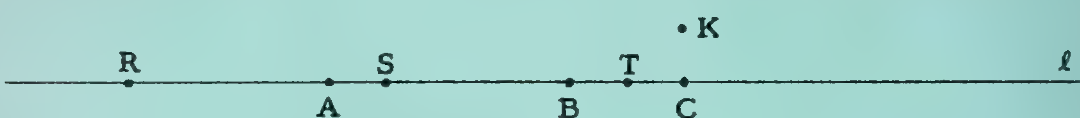
In this unit we shall be interested mainly in sets of points. We shall use capital letters 'W', 'X', 'Y', and 'Z' [and, sometimes others] as indices, and shall adopt the convention that their domain is the plane. So, for example, the symbol:

$$\{X\colon\ X \text{ is between } P \text{ and } Q\}$$

names the set whose members are just those points each of which is

between the point P and the point Q. [So, if P = Q then the set in question is the empty set.]

Consider, now, $\{X\colon\ B$ is between A and $X\}$, where, as on page 6-13, B is between A and C.



To decide whether a given thing is a member of this set, we must decide whether it is a point which satisfies the set selector:

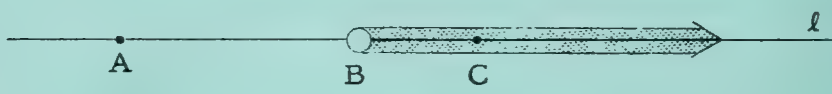
(*) B is between A and X

Evidently, we may restrict our queries to things which are points. [The answer to the question 'Does Johnny belong to this set?' is, trivially: no] Let us begin by considering a point K which does not belong to ℓ. Does K satisfy (*)? This is the case if and only if

B is between A and K.

Answers for questions on page 6-13.

line 2. (a) [It is helpful to use shading or colored chalk to show the half-line $\overrightarrow{BC}$ in the diagram.]



Then, to show that '$D \in \overrightarrow{BC}$' does not follow from (a), a student can mark the point D between A and B. Since D is not in the shaded portion, $D \notin \overrightarrow{BC}$. In that case, D is between A and C but $D \notin \overrightarrow{BC}$. A student might protest and say 'But, what if the point D is between B and C?'. In such a case, point out that when you ask whether '$D \in \overrightarrow{BC}$' follows from 'D is between A and C', what you mean is whether someone can predict [with complete accuracy] that $D \in \overrightarrow{BC}$ from the knowledge that D is between A and C.

(b) Yes (c) No (d) Yes (e) Yes

(f) Yes [Notice that this condition takes account of all cases in which $D \in \overrightarrow{BC}$.]

line 7. The set of all points X such that B is between A and X is $\overrightarrow{BC}$.

*

Word descriptions of sets of points can be quite complicated. We can simplify descriptions of sets by adopting the brace-notation of Units 3, 4, and 5. For example, the symbol:

$$\{x, \text{ x a real number}: 2x + 3 = -7\}$$

is a name for the set whose only member is -5. We use a pair of braces to show that we are naming a set, an index [here, 'x'] and, for line 1. The phrase 'the half-line $\overrightarrow{PQ}$' should be read as 'the half-line PQ' [rather than as 'the half-line the half-line PQ']. It contains a redundancy which is included for emphasis. [Similarly, 'line $\overleftrightarrow{PQ}$' is read as 'line PQ' and 'interval $\overline{PQ}$' is read as 'interval PQ'.]

line 12. Read '$\overset{\bullet\!\!\longrightarrow}{PQ}$' as 'ray PQ'.

line 15. Read '$\overset{\bullet\!\!-\!\!\bullet}{PQ}$' as 'segment PQ'.

*

As a help in remembering the difference between intervals and segments, recall that 'inter' corresponds to 'between', while 'segment' suggests, perhaps, a hunk of material, complete with ends.

*

When summarizing our discussion of intervals, segments, etc. in the axioms at the end of this introduction, it will be convenient to extend the notions, somewhat. For example, although on page 6-14 we do not explicitly assume that $P \neq Q$, students have reason, in view of the preceding development, to expect this to be the case. So, what they have learned upon completing page 6-14 includes, among other things, that if $P \neq Q$ then

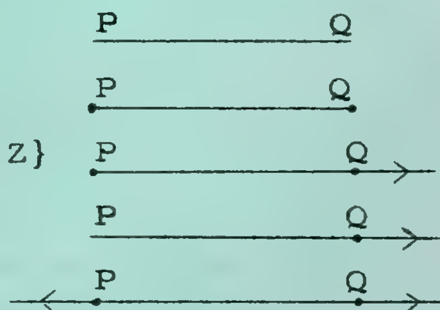
(1) $\overline{PQ} = \{Z: \text{ Z is between P and Q}\}$

(2) $\overset{\bullet\!\!-\!\!\bullet}{PQ} = \overline{PQ} \cup \{P, Q\}$

(3) $\overset{\bullet\!\!\longrightarrow}{PQ} = \overset{\bullet\!\!-\!\!\bullet}{PQ} \cup \{Z: \text{ Q is between P and Z}\}$

(4) $\overrightarrow{PQ} = \{Z: Z \in \overset{\bullet\!\!\longrightarrow}{PQ} \text{ and } Z \neq P\}$

(5) $\overleftrightarrow{PQ} = \overrightarrow{PQ} \cup \overrightarrow{QP}$



Now, for technical reasons, it is inconvenient to use a notation ['$\overline{PQ}$', for example] which is defined only conditionally [that is, subject to the condition '$P \neq Q$']. This inconvenience has already been noted, in connection with division by 0, on TC[2-84]ab. As was pointed out there, since '$\frac{0}{0}$' is meaningless, we cannot, for example, accept the generalization:

$$\forall_x \text{ if } x \neq 0 \text{ then } \frac{0}{x} = 0$$

as true because it has the meaningless "instance":

$$\text{if } 0 \neq 0 \text{ then } \frac{0}{0} = 0$$

So, it is necessary either to give '$\frac{0}{0}$' a meaning or to use a restricted quantifier and write instead:

$$\forall_{x \neq 0}\ \frac{0}{x} = 0$$

Since, in Unit 2, the first way out would have been too confusing to students, we introduced restricted quantifiers.

A similar situation arises here. For example, it follows from (1) and (2), above, that each interval with two end points [say, P and Q] is a subset of the segment with the same end points. If interval and segment are defined only conditionally, we cannot state this consequence of (1) and (2) as:

$$\forall_X \forall_Y \text{ if } X \neq Y \text{ then } \overline{XY} \subseteq \overleftrightarrow{XY},$$

because the "instance":

$$\text{if } P \neq P \text{ then } \overline{PP} \subseteq \overleftrightarrow{PP}$$

is, in this situation, as meaningless as:

$$\text{if } 0 \neq 0 \text{ then } \frac{0}{0} = 0$$

The "instance" is meaningless because, if interval and segment are defined only conditionally, '$\overline{PP}$' and '$\overleftrightarrow{PP}$' are nonsense. We could, as in the case of division by 0, use a restricted quantifier and write:

$$\forall_X \forall_{Y \neq X}\ \overline{XY} \subseteq \overleftrightarrow{XY}$$

However, this solution to the difficulty is unsatisfactory, due in part to the difficulty of writing restricted quantifiers, and in part to certain technical disadvantages to the use of restricted quantifiers. Fortunately, there is another way out. All we need do is to so frame the definitions that '$\overline{PP}$', etc. are meaningful. We do this in Axiom 5 [page 6-19] by accepting (1) - (5) without the restriction '$P \neq Q$'. It follows, now, that

since there is no point which is between a given point P and itself,

$$\overline{PP} = \{Z\colon\ Z \text{ is between P and P}\} = \emptyset .$$

Hence, by (2),

$$\overset{\bullet\!-\!\bullet}{PP} = \overline{PP} \cup \{P, P\} = \emptyset \cup \{P\} = \{P\}.$$

Also, since no point is between itself and a second point,

$$\{Z\colon\ P \text{ is between P and Z}\} = \emptyset.$$

Hence, by (3),

$$\overset{\bullet\!\!\longrightarrow}{PP} = \overset{\bullet\!-\!\bullet}{PP} \cup \{Z\colon\ P \text{ is between P and Z}\} = \{P\} \cup \emptyset = \{P\}.$$

By (4), then, $\overrightarrow{PP} = \emptyset$, and, by (5), $\overleftrightarrow{PP} = \emptyset$.

Now, we can, for example, accept:

$$\forall_X \forall_Y\ \overline{XY} \subseteq \overset{\bullet\!-\!\bullet}{XY}$$

[The previously sticky case, '$\overline{PP} \subseteq \overset{\bullet\!-\!\bullet}{PP}$', is, now, just a long way of writing '$\emptyset \subseteq \{P\}$'. This latter is so because the empty set is a subset of each set.]

Of course, with these conventions, we are no longer entitled to read '$\overleftrightarrow{AB}$' as 'line AB' <u>unless</u> we know that $A \neq B$. [For the empty set is not a line.] Fortunately, this anticipated difficulty does not occur in practice. For, on the one hand, if one wishes to introduce the notation '$\overleftrightarrow{AB}$' into a discussion, as an abbreviation for 'the line determined by the points A and B', one will surely have already proved [or assumed] that A and B are two points--that is, that $A \neq B$. And, on the other hand, if one wants to initiate a discussion about points and lines one may either say, for example:

suppose that ℓ is a line, and P is a point such that $P \notin \ell$

or, in a different situation:

suppose that $Q \in \overleftrightarrow{AB}$

In the first case, if one later finds two points R and S which belong to ℓ,

one may, if it is convenient to do so, assert that $\ell = \overleftrightarrow{RS}$. In the second case, one is supposing that $\overleftrightarrow{AB} \neq \emptyset$; and, from this it follows that $A \neq B$ and, so, that $\overleftrightarrow{AB}$ is indeed a line.

So, in practice, one will never have occasion to use '$\overleftrightarrow{AB}$', say, in cases where it does not refer to a line. Hence, in practice, one will always be justified in reading '$\overleftrightarrow{AB}$' as 'line AB'. A similar discussion applies to symbols such as '$\overrightarrow{AB}$' and '$\overset{\bullet\!\!\rightarrow}{AB}$'. As far as '$\overline{AB}$' and '$\overset{\bullet\!-\!\bullet}{AB}$' are concerned, there is little that is counter-intuitive in accepting these as referring to an interval and a segment, respectively, even if A and B should refer to the same point.

The preceding discussion is largely for your own orientation. However, after completing the exercises on page 6-15, you may find it worthwhile to develop (1) - (5), above, on the board, without mentioning '$P \neq Q$'. Sample:

How can we use set notation to describe the interval joining P and Q?

$$\overline{PQ} = \{Z: [\text{what?}]\}$$

How about the segment joining P and Q?

. . . .

How about the ray from P through Q?

$$\overset{\bullet\!\!\rightarrow}{PQ} = \overset{\bullet\!-\!\bullet}{PQ}\ [\text{and what else?}]$$

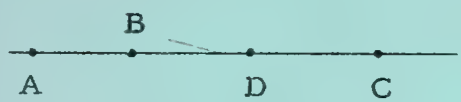
etc.

Then, ask what, in view of (1), one could mean by '$\overline{PP}$'. By '$\overset{\bullet\!-\!\bullet}{PP}$'. By '$\overset{\bullet\!\!\rightarrow}{PP}$'. By '$\overrightarrow{PP}$'. By '$\overleftrightarrow{PP}$'. So, to know that, say, $\overleftrightarrow{AB}$ is really a line, you need to know that $A \neq B$. And, you do know this if you know that $\overleftrightarrow{AB} \neq \emptyset$.

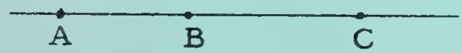
24. 

25. Can't occur. B cannot be between A and A.

26. Occurs only in case A = B.

27. 

[only case]

28. 

[only case]

29. Can't occur. If $B \in \overline{AC}$ then $\overrightarrow{BC} = \{Z: B \in \overline{AZ}\}$. So, since $D \in \overrightarrow{BC}$, it follows that $B \in \overline{AD}$.

14. Can't occur [except in the trivial case in which A = B = C]. If A $\neq$ B then A $\in \overleftrightarrow{AB}$, but A $\notin \overrightarrow{AC}$ and A $\notin \overrightarrow{AB}$. [If B $\neq$ C then either B or C is different from A, and $\overrightarrow{AC} \cup \overrightarrow{AB} \neq \emptyset$. So, if $\overleftrightarrow{AB} = \overrightarrow{AC} \cup \overrightarrow{AB}$ then, if B $\neq$ C, $\overleftrightarrow{AB} \neq \emptyset$. Hence, if B $\neq$ C then A $\neq$ B.]

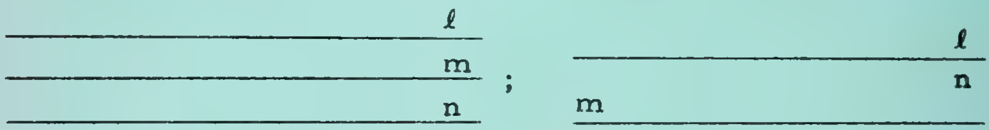
15. 

[only case]

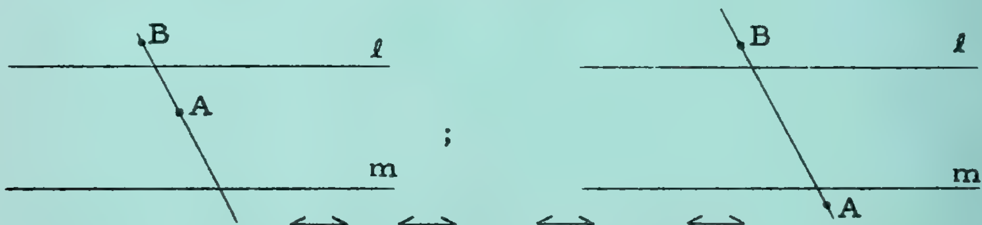
16. 

[only case]

17. Can't occur. A $\notin \overrightarrow{AB}$, but A $\in \overline{AB} \cup \overrightarrow{BC}$ because A $\in \overline{AB}$.

18. 

19. See second solution for Exercise 18.

20. 

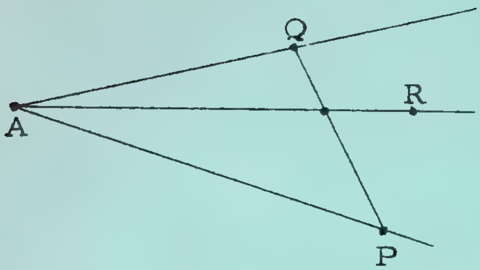
21. Can't occur. If B $\in \overleftrightarrow{AB} \cap \overleftrightarrow{AC}$ then $\overleftrightarrow{AB} \neq \emptyset \neq \overleftrightarrow{AC}$; so, B $\neq$ A $\neq$ C. Hence, A and B are two points which belong to the line $\overleftrightarrow{AC}$ as well as to the line $\overleftrightarrow{AB}$. Since two points determine a line, $\overleftrightarrow{AB} = \overleftrightarrow{AC}$.

22. 

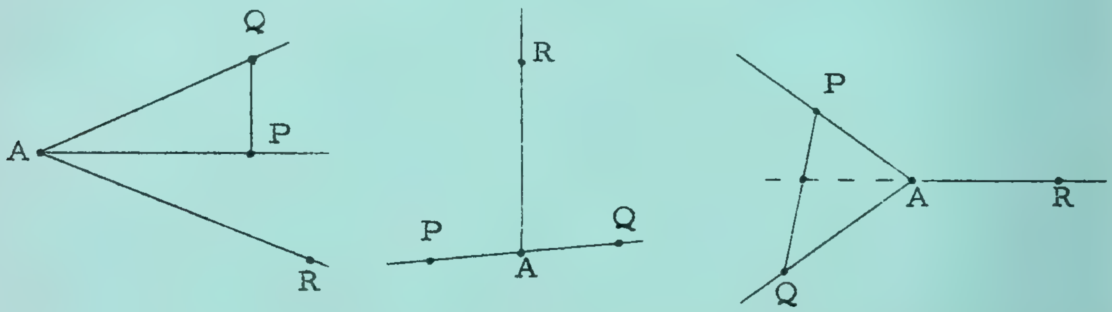
['C $\in$ {Z: Z is between A and B}' is just a long way of saying that C is between A and B.]

23. 

[only case]

9. 

The case shown is the only one. Note that if the intersection of two sets is not empty then there is something which belongs to both of them. Hence, neither set is empty. So, if $\overline{PQ} \cap \overrightarrow{AR} \neq \emptyset$ then $\overline{PQ} \neq \emptyset$; so $P \neq Q$. Also, $\overrightarrow{AR} \neq \emptyset$; so, $A \neq R$. If A, P, and R were collinear then, if $\overline{PQ} \cap \overrightarrow{AR} \neq \emptyset$, the points A, P, Q, and R would be collinear. But, then $\overrightarrow{AP}$, $\overrightarrow{AQ}$, and $\overrightarrow{AR}$ would be the same half-line, or two of them would be the same and the other would be collinear with that one. In neither case could $\overrightarrow{AP}$, $\overrightarrow{AQ}$, and $\overrightarrow{AR}$ be <u>three</u> half-lines. [Exercises 9 and 10 foreshadow the work on adjacent angles starting on page 6-69.]

10. 

[A, P, Q are collinear] [$\overline{PQ} \cap \overleftrightarrow{AR}$ might not be $\emptyset$]

11. Can't occur. If Q is between P and R then P, Q, and R are collinear.

12. 

[If $A \neq B$ then R and S may be any two points of $\overleftrightarrow{AB}$. If $A = B$ then $R = S$ [$\overleftrightarrow{AB} = \emptyset = \overleftrightarrow{RS}$].

But, don't go out of your way to bring up this last unimportant case.]

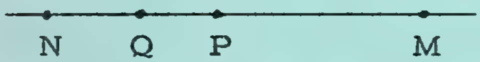
13. Can't occur. $\overleftrightarrow{AC} = \overrightarrow{AC} \cup \overrightarrow{CA}$; so, if $B \in \overleftrightarrow{AC}$ then either $B \in \overrightarrow{AC}$ or $B \in \overrightarrow{CA}$.

Answer for Exercises [which begin on page 6-14].

[When the situation described in an exercise can occur, there are usually several possible cases. In class discussion, sufficiently many of such cases should be brought up to illustrate their variety. In the answers which follow we shall occasionally show more than one correct drawing, but no attempt will be made to picture all cases. To save space, all lines will be drawn horizontally. However, this should be avoided in board work. You should also avoid marking points A, B, and C, say, in alphabetic order from left to right. Go from right to left occasionally, and look for opportunities to mix up the order.]

1. 

; 

2. 

3. 

; 

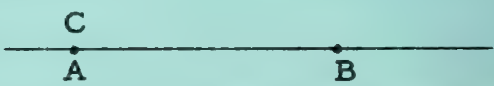
; 

[Note that if A = B then, since there are no points between A and A, $\overline{AB} = \emptyset$. Also, in Exercise 3, if C = D then A = B.]

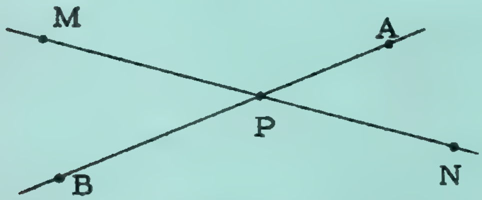
4. 

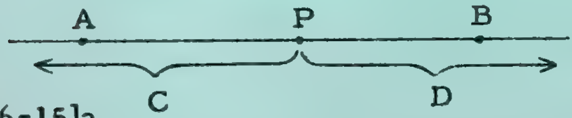
[There are 3 other cases in which $B \neq A$; don't bother with the case in which B = A].

5. 

; 

6. Can't occur. Of three points, at most one is between the other two.

7. 

8. 

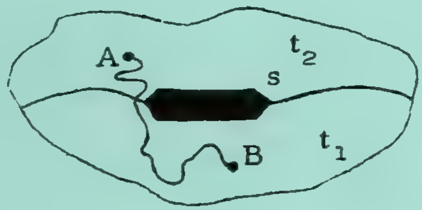
[9 cases]

You may want to assign the Supplementary Exercises dealing with complementation on pages 6-403 and 6-404 before doing the work on page 6-16.

*

The verb 'separates' is commonly used in two quite different senses. One of these is somewhat like the meaning of 'classifies'. Examples of this use are 'John separates the milk [into skim milk and cream].' and 'Lois separated her guests into two groups.'. In this sense, 'separates' refers to an action. The only reason for mentioning this meaning of 'separates' is to alert you to the possibility that students may have trouble in reading page 6-16 through interpreting the word in this sense.

The second meaning of 'separates' refers, initially, to spatial relationships. One says, for example, that two city lots are separated by an alley, or that a city is separated into two parts by a river. In the latter case, a portion of the river will be within the city limits but will not be reckoned as belonging to either part of the city. Similarly, the white stripe painted down the middle of a highway separates the road into two traffic lanes, neither of which contains the white strip. This use of 'separates' has been extended in mathematics, where, generally, a subset s of a set t is said to separate t if the points of t which are not in s fall into two sets, t_1 and t_2 such that a "path" from any point of t_1 to any point of t_2 must intersect s. Here, what one means by a "path" depends to some extent on the branch of mathematics one is developing.



In this sense we say that a point P of a line ℓ separates ℓ into two half-lines, neither of which contains P. [It would correspond better to the general situation described above if we were to say that $\{P\}$, rather than P, separates ℓ.] In this case, a "path" from a point R of one of the half-lines to a point Q of the other may conveniently be thought of as the segment joining R and Q [and this path intersects $\{P\}$].

In an entirely similar way, a line separates the plane into two half-planes. Neither half-plane contains any point of ℓ, but each point

of the plane belongs either to one of the two half-planes or to ℓ. Here again, one may think of the segment joining R and Q as being a "path" between points R and Q of opposite half-planes.

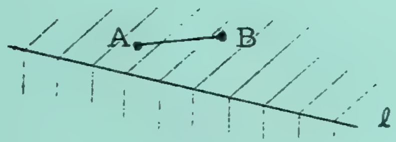
*

Answers to questions on page 6-16.

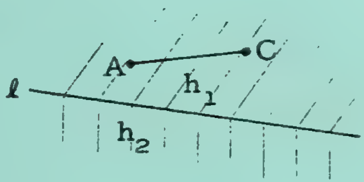
line 7. $h_1 \cup \ell \cup h_2$ is the plane. [By convention, '$h_1 \cup \ell \cup h_2$' is an abbreviation for '$(h_1 \cup \ell) \cup h_2$'.]

line 8. $h_1 \cap \ell = \emptyset$; $h_1 \cap h_2 = \emptyset$

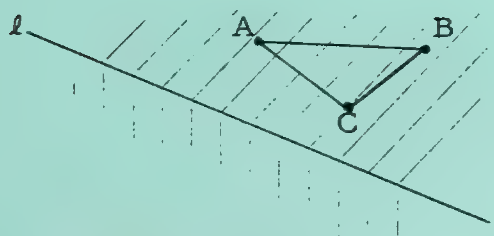
line 9. Yes.



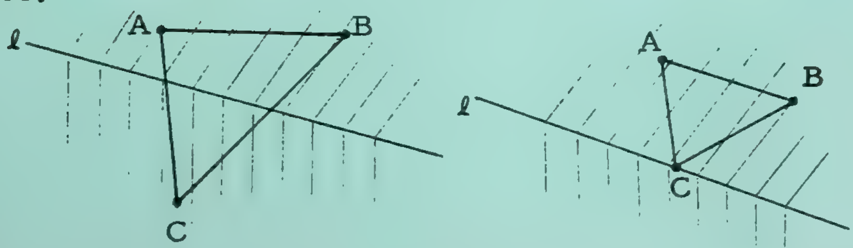
line 10. Yes.



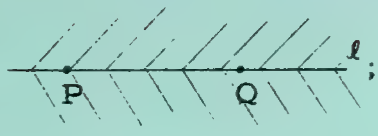
line 12. Yes.



line 14. Yes.



line 16. No.



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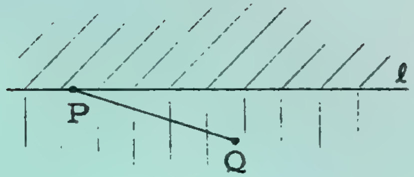
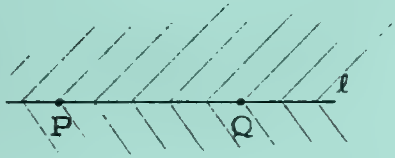
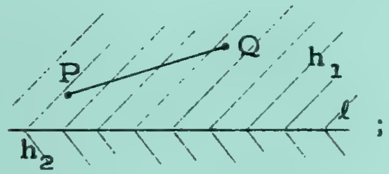


line 18. No.



line 20. Yes.



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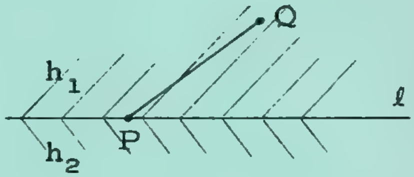
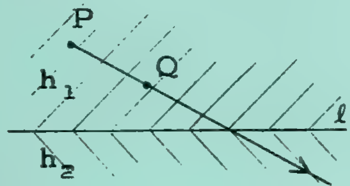
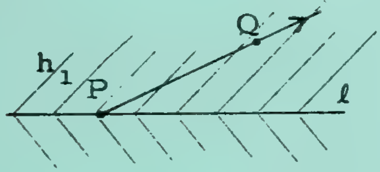


line 21. No.

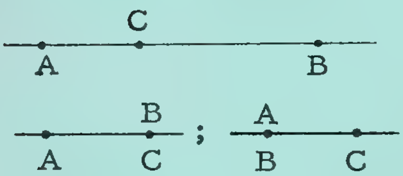


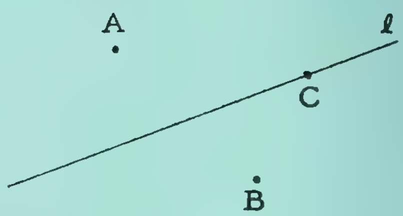
line 22. Yes.



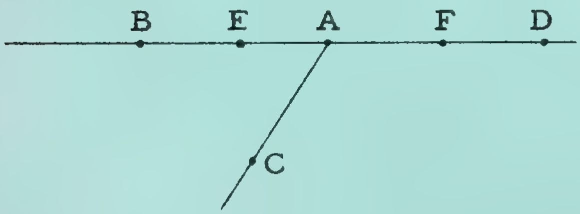
3. 

4. 

5. 

6. 

7. 

8. 

[There are other cases, but for all of them, B, A, and D are collinear.

*

You will be starting section 6.01 very soon. Remind students to bring rulers with English and metric scales.

Quiz.

Draw pictures which illustrate the situations described below.

1. A, B, and C are three collinear points, and D is a point such that $D \notin \overleftrightarrow{AC}$

2. ℓ and m are parallel lines and n is a line such that $n \cap \ell \neq \emptyset$ and $n \cap m = \emptyset$

3. A, B, C, and D are four points and $\overline{AD} \subseteq \overline{BC}$

4. A and B are two points and C is a point such that $C \in \overleftrightarrow{AB}$ and $C \notin \overrightarrow{AB}$

5. A and B are two points and C is a point such that $C \in \{Z: Z \in \overrightarrow{AB} \text{ and } Z \in \overrightarrow{BA}\}$

6. A and B are on opposite sides of a line ℓ and C is a point such that $\overline{AC} \cap \ell \neq \emptyset$ and $\overline{BC} \cap \ell \neq \emptyset$

7. ℓ and m are lines such that $\ell \cap \tilde{m} = \emptyset$ [The symbol '$\tilde{m}$' stands for the complement of m with respect to the plane.]

8. A, B, C, and D are four points and E and F are points such that $E \in \overrightarrow{AB}$, $F \in \overrightarrow{AD}$, $\overline{EF} \cap \overrightarrow{AC} = \emptyset$, and $\overline{EF} \cap \overleftrightarrow{AC} \neq \emptyset$

*

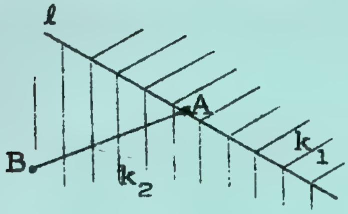
Answers for Quiz.

1.

2.

line 14. $k_1 \cup k_2$ is the plane   line 15. $k_1 \cap k_2 = \ell$

7.



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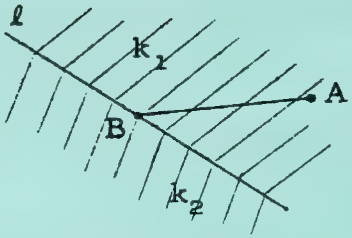

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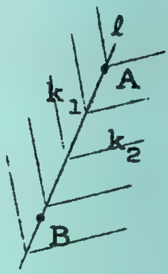
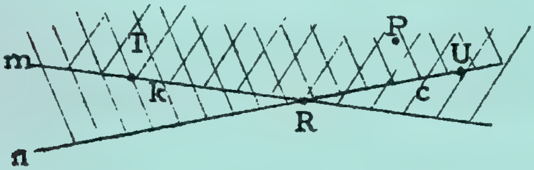


[The word 'crosses' is a good one to use in place of 'intersects' when you mean that the intersection contains exactly one point. The word 'intersects' is used in conventional geometry courses with the meaning of 'crosses'. But, in this course, when one says, for example, that $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$, he could mean either that $\overleftrightarrow{AB}$ crosses $\overleftrightarrow{CD}$ or that $\overleftrightarrow{AB} = \overleftrightarrow{CD}$. Conventional courses take care of this other meaning of 'intersects' by using the word 'coincides'.]

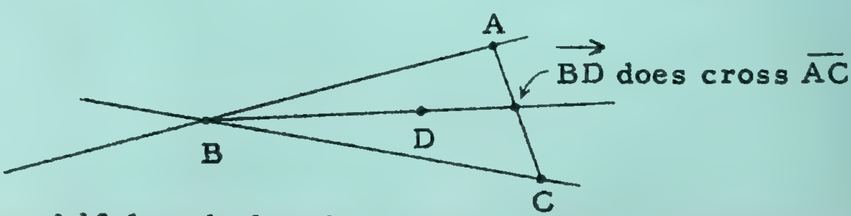
8.



[<u>Note</u>: $c \cap k$ is the union of the set consisting of points "above" both m and n, the set $\overrightarrow{RT}$, and the set $\overrightarrow{RU}$.]

9. Can't occur. If A and B both belong to ℓ then $\overline{AB} \subseteq \ell$. So, $\overline{AB} \subseteq k$. If one of the points does not belong to ℓ [or if neither belongs to ℓ] then, since $A \in k$ and $B \in k$, $\overline{AB}$ is a subset of the half-plane which consists of the points of k not on ℓ. In either case, $\overline{AB} \subseteq k$.

10. Can't occur.

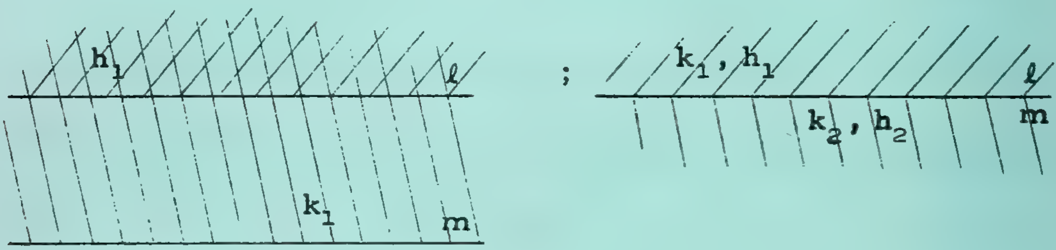


[Exercises 8 and 10 foreshadow the work on interiors of angles. See page 6-55.]

Answers for Exercises.

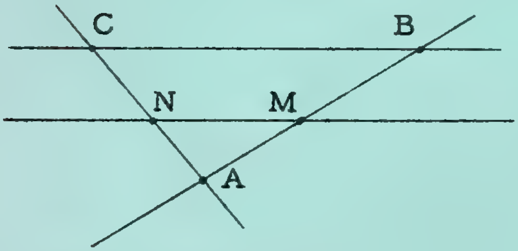
1. Can't occur. If $\overleftrightarrow{AB} \parallel \ell$ then $\overleftrightarrow{AB} \cap \ell = \emptyset$. But, from the work done on page 6-16, if A and B are on opposite sides of ℓ then $\overline{AB} \cap \ell \neq \emptyset$. Since $\overline{AB}$ is a subset of $\overleftrightarrow{AB}$, it follows that any point common to $\overline{AB}$ and ℓ must also be common to $\overleftrightarrow{AB}$ and ℓ. So, since $\overline{AB} \cap \ell \neq \emptyset$, it follows that $\overleftrightarrow{AB} \cap \ell \neq \emptyset$.

2.



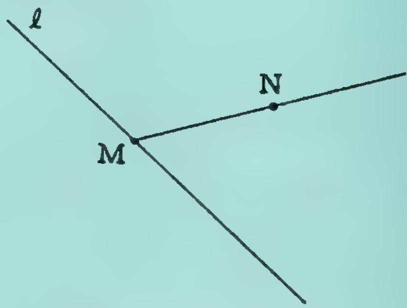
3. Can't occur. If $\overset{\bullet\!-\!\bullet}{AB} \cap \ell = \emptyset$ then A and B are on the same side of ℓ. If $\overset{\bullet\!-\!\bullet}{BC} \cap \ell = \emptyset$ then B and C are on the same side of ℓ. So, under the conditions of the exercise, A, B, and C are on the same side of ℓ. But, if A and C are on the same side of ℓ then $\overset{\bullet\!-\!\bullet}{AC} \cap \ell = \emptyset$.

4.



[Note: $\overleftrightarrow{MN} \parallel \overleftrightarrow{BC}$]

5.



6. Can't occur. If $M \in \ell$ then M cannot belong to either side of ℓ. But [since $M \neq N$], $M \in \overrightarrow{NM}$. Hence, $\overrightarrow{NM}$ is not a subset of either side of ℓ because there is a point of $\overrightarrow{NM}$ which does not belong to either side of ℓ. [If you're wondering about how to show that '$M \neq N$' follows from the premisses '$M \in \ell$' and '$N \notin \ell$', see TC[6-378, 379]c.]

The fifteen Introduction Axioms on pages 6-18 through 6-22 state some of the facts about points and lines, and also characterize some of the concepts which students have discovered while studying the preceding pages. The basic concept is that of betweenness--of one point being between two others. As was pointed out on TC[6-14]a, the notions of interval, segment, ray, half-line, and line can all be characterized in terms of betweenness. [See Axiom 5 on page 6-19, and Theorem 4 on page 6-24.] This is also true of the separation of the plane by a line. [See Axiom 15 on page 6-22.] In fact, in the COMMENTARY for pages 6-23 through 6-28, it is pointed out that the axioms might be modified in such a way that the resulting set of axioms would contain one:

a set ℓ is a line if and only if there are two points, X and Y, such that $\ell = \overleftrightarrow{XY}$

which serves essentially as a definition of the word 'line'. If this procedure were adopted, it would be unnecessary to use the word 'line' in any of the other axioms. In fact, this word would then become excess baggage and could, except for strong pedagogical reasons, be deleted from the text.

The Introduction Axioms, and the Introduction Theorems which can be derived from them [a few of which are given on pages 6-23 through 6-28], deal for the most part with properties of geometric configurations which are, customarily, "seen from the figure". In the succeeding sections of this unit we shall adopt this custom and, so, shall very seldom make explicit reference to this Introduction. Consequently, it is completely unnecessary for students to memorize the Introduction Axioms, or to study proofs of any Introduction Theorems. Indeed, memorizing the Introduction Axioms [or the given Introduction Theorems] would be an intolerable burden.

In view of this, one may well ask 'What is the purpose of giving the Introduction Axioms to students?'. There are two kinds of reasons for doing so. In the first place, reading and discussing the Introduction Axioms and some Introduction Theorems will increase a student's ability to make use easily and with understanding of set notation and of the notation introduced in Axiom 5 for intervals, segments, etc. This is important, because he will use these notations throughout the course. In particular, such reading and discussion will sharpen a student's intuitive feeling for the features of the terrain [actually, of course, the plane] which he will study during the remainder of this unit.

Another reason for giving the Introduction Axioms arises out of the fact that in addition to teaching the "facts of geometry", one of the customary aims of a geometry course is to enlarge a student's understanding

of the nature of proof. Now, a *proof* of a theorem should be an argument which starts from axioms, or previously proved theorems, and shows how the theorem in question follows, by accepted logical principles, from just these premisses. Such an argument may be stated in any number of forms [two-column, one-column, paragraph, or what-have-you]. But, the essential point is that the proof must show that the explicitly stated premisses suffice to yield the desired conclusion. For example, the proofs with which students have become acquainted in Unit 2 fulfill this requirement. Each proof in Unit 2 shows how the algebra theorem which is being proved follows from basic principles, and previously proved theorems, which are explicitly stated in the proof.

Now, it happens to be the case that geometry is much more complicated than algebra. In fact, it is so complicated that it seems to be impossible at the high school level to give really solid proofs of any but a few of the usual theorems. Consider, as one of the simplest examples, the theorem:

the diagonals of a parallelogram bisect each other

Somewhere, in any form of proof of this theorem, there will be a step like:

let P be the point in which the diagonals of ▱ ABCD intersect

Now, before introducing this step one must, in all strictness, have proved that the diagonals of ▱ABCD do indeed intersect, and that they intersect in a single point. This is typical of the sort of thing which, customarily, one "sees from the figure", and, as has already been said, we shall follow this custom. However, if one leaves it at this, the student's notion of proof will be dulled, rather than sharpened. It is important that he realize that the conclusions which he is, for practical reasons, taught to draw from the figure, actually can be derived from his axioms. In other words, he should be aware of the fact that most of the proofs he gives have gaps in them and, in general, he should know where these gaps are. But, he should realize that these gaps exist, not because his axioms do not give a sufficient basis for filling them, but merely because taking the time to fill them would direct his attention away from what, for him now, are more important matters. Reading and discussing the axioms and theorems on pages 6-18 through 6-28 should prepare him to recognize gaps in his proofs, and give him an inkling of how they could be filled. [When he does discover such a gap, he can signal his discovery by writing 'Introduction' as part of the explanation for the resulting step. See the column proof on page 6-33.]

*

The first four axioms deal with simple properties of points and lines. Axiom 1 rules out the possibility that the empty set or singleton sets are lines. Together with later axioms [particularly, Axioms 8 and 11], it ensures that each line contains many points.

Axiom 2 says two things. First, that

each two points are contained in at least one line,

and, second, that

each two points are contained in at most one line.

From the second of these it follows that two lines cannot have more than one point in common--that is, that the intersection of two lines is a degenerate set. [See Theorem 1 on page 6-23.] Furthermore, a line is determined by any two of its points.

Notice that, although Axiom 1 tells us that if there is a line then there are at least two points, and Axiom 2 tells us that if there are two points then there is at least one line, neither of these axioms tells us that there are any points or any lines. Axiom 3 does this for us.

$\overset{\bullet\!-}{AA} \subseteq \overset{\bullet\!\rightarrow}{AA}$, and $A \in \overset{\bullet\!-\!\bullet}{AA}$. So, $\overset{\bullet\!\rightarrow}{AA} = \{A\}$. This takes care of the uninteresting cases. Now, what about $\overline{AA}$ and $\overset{\bullet\!-\!\bullet}{AA}$? Well, Axiom 5 tells us that $\overset{\bullet\!-\!\bullet}{AA} \subseteq \overset{\bullet\!\rightarrow}{AA}$; so, $\overset{\bullet\!-\!\bullet}{AA} \subseteq \{A\}$. That is, A is the only point which can belong to $\overset{\bullet\!-\!\bullet}{AA}$. But, Axiom 5 also tells us that A does belong to $\overset{\bullet\!-\!\bullet}{AA}$[$\overset{\bullet\!-\!\bullet}{AA} = \overline{AA} \cup \{A, A\}$]. So, $\overset{\bullet\!-\!\bullet}{AA} = \{A\}$. As to $\overline{AA}$, we know that $\overline{AA} \subseteq \overset{\bullet\!-\!\bullet}{AA}$ [see preceding bracket]. So, A is the only point which can belong to $\overline{AA}$. However, by Axiom 7, $A \notin \overline{AA}$. Hence, $\overline{AA} = \emptyset$. This result agrees with our intuition; for, there should, surely, be no points between a point and itself.

Axioms 7, 9, and 10 state some of the simple properties of betweenness. Axiom 9, for example, says, in terms of intervals, that the points which are between A and B are just those which are between B and A. Axiom 7 tells us that A is not between A and B. Since, similarly, B is not between B and A, we see by Axiom 9 that B is not between A and B. Finally, Axiom 10 tells us that if C is between A and B then B is not between A and C. Since, if C is between A and B then C is between

B and A, it follows similarly that if C is between A and B then A is not between B and C. So, Axioms 9 and 10 together tell us that, given three points, at most one of them is between the other two.

Axioms 8 and 11 give us additional information about the existence of points. Axiom 8, for example, tells us that between any two points there is at least one more. Axiom 11 tells us that "beyond" any two points, in either direction, there is at least one more.

If students bring up the degenerate cases '$\overline{AA}$, ..., '$\overleftrightarrow{AA}$', delay discussing them until the discussion of Axiom 6.

As pointed out on page 6-20, Axiom 6 tells us two things:

(1) <u>if</u> $A \neq B$ and A, B, and C are collinear <u>then</u> $C \in \overleftrightarrow{AB}$,

and:

(2) <u>if</u> $C \in \overleftrightarrow{AB}$ <u>then</u> $A \neq B$ and A, B, and C are collinear

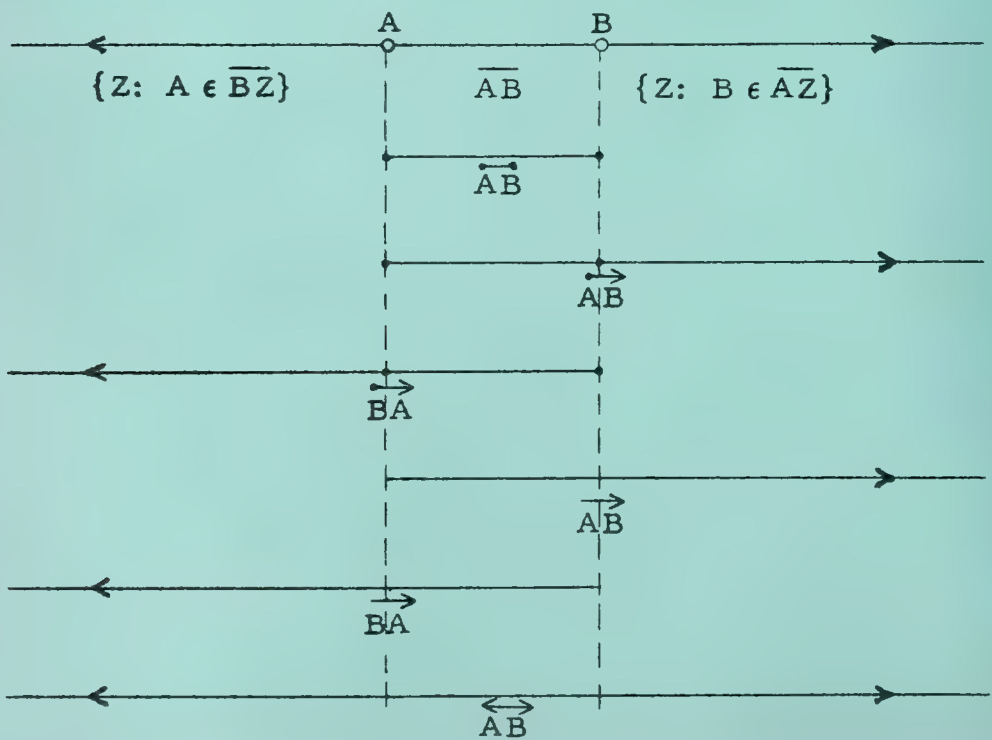
The first of these tells us that if A and B are two points [and, so, by Axiom 2, determine a line] then each point C of the line determined by A and B is, also, a point belonging to $\overleftrightarrow{AB}$. The second tells us that if there is any point which belongs to $\overleftrightarrow{AB}$ then A and B are two points, and each point C which belongs to $\overleftrightarrow{AB}$ also belongs to the line determined by A and B. Combining these results we see that if $A \neq B$ then $\overleftrightarrow{AB}$ is the line determined by A and B. [See <u>Theorem 4</u> on page 6-24.]

As has been pointed out, above, one consequence of (2) is that if there is any point which belongs to $\overleftrightarrow{AB}$, then $A \neq B$. That is, if $A = B$ then there is no point which belongs to $\overleftrightarrow{AB}$. [See Theorem 5(d) on page 6-25.] More simply, for any point A, $\overleftrightarrow{AA} = \emptyset$. Now is the time to discuss the degenerate cases of Axiom 5. [See the COMMENTARY for page 6-14.] Evidently, it is not strictly correct to read '$\overleftrightarrow{AB}$' as 'line AB' unless one knows that $A \neq B$. However, as you can assure your students, the case '$A = B$' will never occur in connection with the notation '$\overleftrightarrow{AB}$' except in this Introduction. In fact, the only other time it will so occur is in the discussion of Theorem 5(d). The same applies to the notations '$\overrightarrow{AB}$' and '$\overset{\bullet\!\!\rightarrow}{AB}$'. However, we shall sometimes want to use '$\overset{\bullet\!-\!\bullet}{AA}$' and '$\overline{AA}$'; so, let's see what Axiom 5 actually tells us about these cases. Since $\overleftrightarrow{AA} = \emptyset$, and since, by Axiom 5, $\overleftrightarrow{AA} = \overrightarrow{AA} \cup \overrightarrow{AA}$, it follows that $\overrightarrow{AA} = \emptyset$. Furthermore, since, by Axiom 5, $\overrightarrow{AA} = \{Z\colon\ Z \in \overset{\bullet\!\!\rightarrow}{AA} \text{ and } Z \neq A\}$, and since $\overrightarrow{AA} = \emptyset$, it follows that A is the only point which can belong to $\overset{\bullet\!\!\rightarrow}{AA}$. But, by Axiom 5,

Axiom 2 tells us that each two points are collinear. Axiom 3, on the other hand, tells us that there are three noncollinear points. In particular, Axiom 3 tells us that there are at least three points. As Jo discovered by analyzing Messages 3 and 5, Axioms 2 and 3 together imply that there are at least three lines. One further immediate consequence of Axiom 3 is that not all points belong to any single line. Equivalently [see Theorem 2 on page 6-23], for each line, there is a point not on it.

Axiom 4 tells us that if neither of two lines intersects a third then they do not intersect each other. Consequently, if the lines do intersect then at least one of them intersects any given third line. In other words [see Theorem 3 on page 6-24], there cannot be two lines through a given point both of which are parallel to a given line. [It will be proved at the beginning of section 6.05 that, given a line ℓ and a point $P \notin \ell$, there is at least one line through P which is parallel to ℓ. So, from this and Axiom 4 it will follow that there is exactly one line through P and parallel to ℓ.]

Axiom 5 introduces notations with which students have already become familiar. The diagram below may be helpful in illustrating Axiom 5.



is interpreted.] Now, draw attention to the theorem stated and proved on TC[6-21, 22]e, and check that it, also, is a true statement no matter which of these interpretations is given to the word 'line'. Elicit from your students that, since the theorem is a consequence of Axioms 2 and 3, it follows that any interpretation of 'point' and 'line' for which Axioms 2 and 3 are true is bound to make the theorem true. Similarly, under any "true interpretation" of the complete set of axioms, all the theorems derived from the postulates will be true statements. While studying "mathematical" geometry, it is no concern of ours what the "undefined terms" 'point', 'line', and 'between' mean. All that does concern us is what sentences are consequences of the sentences we have chosen as axioms. [Of course, our choice of axioms was strongly motivated by consideration of particular interpretations for these three words.] So, this course presents an abstract deductive theory in which all we are concerned with is the logical relationships among certain sentences. However, as pointed out above, students will have in mind preferred models of this deductive theory. There is no harm in this as long as they refrain from making use of properties of their models which are not formulated in the axioms.

A more extensive discussion of the matter treated above is given in Chapter VII of An Introduction to Logic and Scientific Method by Cohen and Nagel [Harcourt Brace and Company, New York]. See, also, the very valuable article: "Geometry and Empirical Science" by Carl G. Hempel, in vol. 52 [1945] of the American Mathematical Monthly.

*

It may interest you or your students to notice that thirteen of the fifteen axioms will be satisfied if one interprets 'point', 'line', and 'between' in such a way that there are just three points, P_1, P_2, and P_3, three lines $\{P_1, P_2\}$, $\{P_2, P_3\}$, and $\{P_3, P_1\}$, and no point is between the other two. In this case, Axioms 1, 2, and 3 are obviously satisfied, and Axiom 4 is satisfied for the reason that there are no parallel lines. The only interval is the empty set. There is no difference between segments and rays, and each is either a line or consists of a single point. The half-lines are the sets which consist of a single point. Axioms 6, 7, 9, and 10 are obviously satisfied, but, since the only interval is the empty set, Axioms 8 and 11 are not. On the other hand, Axioms 12, 13, and 14 are satisfied <u>because</u> the only interval is the empty set. Finally, Axiom 15 is satisfied. For example, the two half-planes determined by the line $\{P_1, P_2\}$ are $\{P_3\}$ and $\emptyset$.

*

Your students will probably have in mind some particular interpretation of the words 'point', 'line', and 'between' for which the axioms are true statements--that is, they will have in mind some <u>model</u> of the deductive theory based on these axioms. More exactly, probably, each student has in mind some vague notions of interpretations for the words 'point', 'line', and 'between' for which the axioms are "nearly true" statements. [You will do well not to inquire too closely into these interpretations.] Point out that, at least as far as Axioms 1, 2, and 3 are concerned [the situation is similar for the complete set of axioms], there are alternative interpretations--theirs, Gloxian cities and highways; businessmen and partnerships [see page 6-6], and various three-point interpretations of the kind explored above. [Since Axioms 1, 2, and 3 do not deal with betweenness, it is immaterial, for your present purposes, how 'between'

Under this correspondence, Message 3 becomes:

> The plane has at least three points
> not all on the same line.

And, of course, this says just what Axiom 3 says.

Message 4 becomes:

> Each line in the plane passes
> through at least two points.

This is equivalent to Axiom 1.

Message 5 tells you that, for each two cities on Glox, there is a highway connecting them, and, moreover, this is the only highway connecting them. This is like saying that each two cities on Glox determine a highway. So, Message 5 can be translated into Axiom 2.

Now, Message A translates readily into a theorem that claims that there are at least three lines. Since Message A follows from Messages 3 and 5, the same reasoning will show that the theorem about lines follows from Axioms 3 and 2. [See below.]

<u>Theorem</u>. There are at least three lines.

<u>Proof</u>. By Axiom 3, there are at least three noncollinear points. Suppose that A, B, and C are three such points. Since $A \neq B$, it follows from Axiom 2 that there is one and only one line, ℓ, which contains A and B. Similarly, there is one and only one line, m, which contains B and C, and one and only one line, n, which contains C and A. Since A, B, and C are noncollinear, neither ℓ, m, nor n contains all three points. So, since A and B belong to ℓ, $C \notin \ell$. But $C \in m$ and $C \in n$. Hence, $\ell \neq m$ and $\ell \neq n$. Similarly, $A \notin m$, but $A \in n$. Hence, $m \neq n$. So, ℓ, m, and n are three lines. Consequently, there are at least three lines.

Two points are said to be <u>on opposite sides</u> of ℓ when the first belongs to one of the two half-planes determined by ℓ and the second belongs to the other of these two half-planes.

Notice that if two points are on the same side of ℓ or on opposite sides of ℓ then neither point belongs to ℓ. [So, if either of two points belongs to ℓ then the two points are not on the same side of ℓ and are not on opposite sides of ℓ.] Conversely, if neither of two points belongs to ℓ then the two points are either on the same side of ℓ or on opposite sides of ℓ.

*

Comments on the bottom paragraph on page 6-22.

If you handle this in class, take the time to write the three messages and the three axioms on the board in this order:

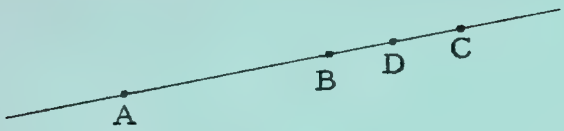
Message 3. ------------
Message 4. ------------
Message 5. ------------

Axiom 1. ------------
Axiom 2. ------------
Axiom 3. ------------

It should be very apparent that Message 3 resembles Axiom 3. Message 3 mentions three cities, and Axiom 3 mentions three points. So, perhaps cities correspond to points. Noncollinear points are points not all on the same line. So, noncollinear points correspond to cities not all on the same highway. Hence, highways correspond to lines. Lines and points are in the plane; cities and highways are on Glox.

Cities $\longleftrightarrow$ Points, Highways $\longleftrightarrow$ Lines, Glox $\longleftrightarrow$ The Plane

Where must such a point be?



What does the picture suggest about A, B, and D, this time?

Now, state Axiom 14.

With the foregoing approach, some of your students may see that Axioms 13 and 14 amount to saying that if $B \in \overline{AC}$ and $D \in \overrightarrow{BC}$ then $B \in \overline{AD}$. In other words,

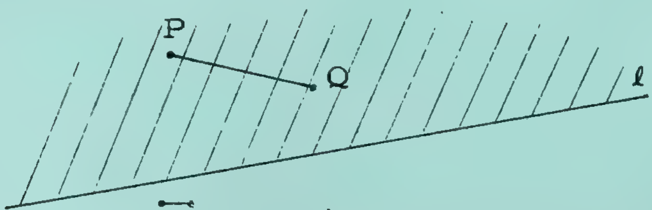
$$\text{if } B \in \overline{AC} \text{ then } \overrightarrow{BC} \subseteq \{Z: B \in \overline{AZ}\}.$$

As a matter of fact, it was brought out earlier [on page 6-13, line 6, part (f)] that

$$\text{if } B \in \overline{AC} \text{ then } \overrightarrow{BC} = \{Z: B \in \overline{AZ}\}.$$

What we have just suggested that students might notice is that "half" of this result is an immediate consequence of Axioms 13 and 14.

Finally, Axiom 15 deals with the separation of a plane by a line. The discussion on page 6-16 pointed out that the complement of a line is the union of two sets called half-planes. Axiom 15(1) says that two points, P and Q, belong to the same half-plane determined by a line ℓ [or: are

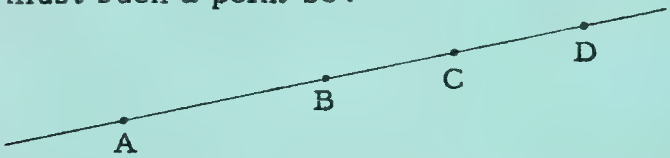


on the same side of ℓ] if $\overleftrightarrow{PQ} \cap \ell = \emptyset$. Axiom 15(2) says that if two points, P and Q, belong to the same half-plane determined by ℓ then $\overleftrightarrow{PQ}$ is a subset of this half-plane.

Does A belong to $\overleftrightarrow{BC}$? [Yes; since $B \neq C$, $\overleftrightarrow{BC}$ is the line containing B and C. But, there is only one such line, and $\overleftrightarrow{AC}$ is it.]

Is there a fourth point D such that $C \in \overline{BD}$?

Where must such a point be?



Could you show that A, B, C, and D are collinear?

[Yes; $D \in \overleftrightarrow{BC}$ for the same reasons that $A \in \overleftrightarrow{BC}$. So, A, B, C, and D all belong to $\overleftrightarrow{BC}$.]

Could you show that A, B, C, and D are four points?

[Well, we showed that A, B, and C are different points, and, for the same reasons, B, C, and D are different. So, that just leaves A and D. I don't see how D <u>could</u> be A. Oh, yes, if $D = A$ then, since $C \in \overline{BD}$, it follows that $C \in \overline{BA}$. But, $B \in \overline{AC}$. And, Axioms 9 and 10 say that these can't both happen. So, $D \neq A$.]

What does the picture suggest about A, B, and D?

[Well, it looks as though $B \in \overline{AD}$. But, I don't see how to prove it.]

[At this point, the student is ready for Axiom 13. So, state it.]

Now, let's go back to where we were when we supposed that $B \in \overline{AC}$. Is there a fourth point D such that $D \in \overline{BC}$?

It seems intuitively obvious that [once we know from Axiom 8 that there is at least one point between any two points] we should be able, given two points A and B, to find as many points between A and B as we wish. For example, if we choose a point C between A and B, we can then find

a point D between C and B, a point E between D and B, etc. However, our axioms so far are not sufficient to guarantee that if C is between A and B, and D is between C and B, then D is between A and B. [Theorem 5(e) says that this is the case, but to prove this theorem, we need additional axioms.] Axioms 12, 13, and 14 contain enough additional information about betweenness to settle questions of this kind. From these three axioms [together with some of the earlier ones], it follows that if you have pictured some points on a line then any other point which is between some two of these is also between any two which it looks to be between. For example, referring to the figure above, any point between C and D is also between A and D, between C and E, and between C and B. [Warning: The proof of this is not altogether easy.]

On TC[6-13]d and e, there is a suggested procedure for clarifying the meaning of Axiom 12. Similar procedures can be used for Axioms 13 and 14. For example:

Suppose that $B \in \overline{AC}$.

Can B = A? [No; Axiom 7.]

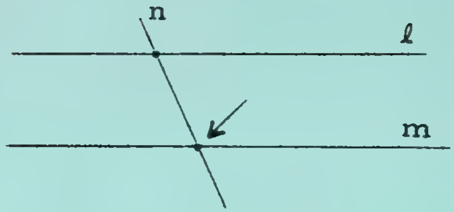
Can B = C? [No; Axioms 9 and 7.]

Can A = C? [No; if A = C then $\overline{AC} = \emptyset$. Therefore, B couldn't belong to $\overline{AC}$.]

So, A, B, and C are three points.

Are they collinear? [Yes; since $A \neq C$, $\overleftrightarrow{AC}$ is the line containing A and C, and since $\overline{AC} \subseteq \overleftrightarrow{AC}$ and $B \in \overline{AC}$, it follows that B is a point of the line containing A and C.]

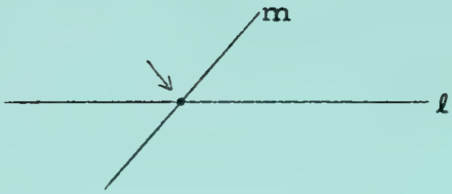
Theorem 20.



[if $\ell \parallel m$ and n crosses ℓ then n crosses m]

Suppose that $\ell \parallel m$ and that n crosses ℓ. Then, $\ell \cap m = \emptyset$ and, by Theorem 19, $\ell \cap n$ consists of a single point, say the point A. So, $n \neq m$ [$A \in n$ but $A \notin m$] and, by Theorem 1, $n \cap m$ consists of at most one point. If $n \cap m = \emptyset$ then ℓ and n are two lines through the point A which are parallel to m. By Theorem 3, this cannot be the case. So, $n \cap m \neq \emptyset$. Hence, $n \cap m$ consists of a single point and, by Theorem 19, n crosses m.

Theorem 19.

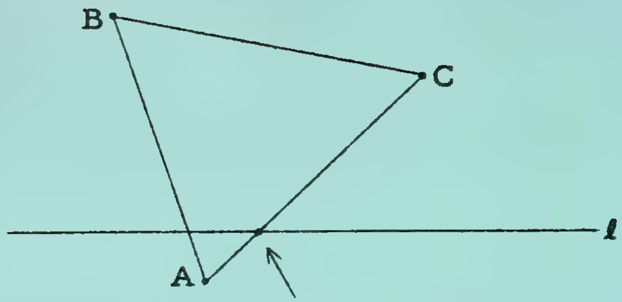


[if $m \cap \ell$ consists of a single point then m crosses ℓ--that is, m contains points on each side of ℓ; if m crosses ℓ then $m \cap \ell$ consists of a single point]

Suppose that $m \cap \ell = \{A\}$. Then, $A \in m$ and, by Axiom 1, there is a point $B \in m$ such that $B \neq A$. Hence, by Theorem 4, $m = \overleftrightarrow{AB}$. By Axiom 11, there is a point C such that $A \in \overline{BC}$ and, by Axiom 9, it follows that $A \in \overline{CB}$. Since $B \in m$ and $B \neq A$, it follows that $B \notin \ell$. So, since $A \in \overline{CB} \cap \ell$, it follows by Theorem 16 that C and B are on opposite sides of ℓ. Since we know that $B \in m$, it will follow [from the definition of 'crosses'--see Exercise 7 on page 6-17] that m contains points on both sides of ℓ once we have shown that $C \in m$. Now, since $A \in \overline{CB}$, it follows from Theorem 14 that $m = \overrightarrow{AC} \cup \{A\} \cup \overrightarrow{AB}$. By Axiom 7, since $A \in \overline{CB}$, $A \neq C$. Hence, by Theorem 5(c), $C \in \overrightarrow{AC}$. So, $C \in m$.

Suppose, now, that m crosses ℓ. Then, there are two points, B and C, of m on opposite sides of ℓ. By Theorem 16, $C \notin \ell$ and $\overline{BC} \cap \ell \neq \emptyset$. Since $C \in m$ and $C \notin \ell$, it follows that $m \neq \ell$ and, by Theorem 1, $m \cap \ell$ consists of <u>at most</u> one point. Since $B \neq C$ and both B and C belong to m, it follows by Theorem 4 that $m = \overleftrightarrow{BC}$. By Theorem 5(b), $\overline{BC} \subseteq \overleftrightarrow{BC}$. So, since $\overline{BC} \cap \ell \neq \emptyset$, $\overleftrightarrow{BC} \cap \ell \neq \emptyset$, --that is, $m \cap \ell \neq \emptyset$. Hence, $m \cap \ell$ consists of <u>exactly</u> one point.

Theorem 17.

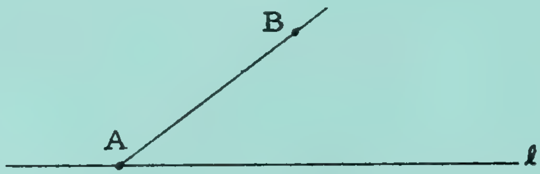


[if $\overleftrightarrow{BC} \cap \ell = \emptyset$ and $\overline{AB} \cap \ell \neq \emptyset$ then $\overline{AC} \cap \ell \neq \emptyset$]

Suppose that $\overleftrightarrow{BC} \cap \ell = \emptyset$ and that $\overline{AB} \cap \ell \neq \emptyset$. From the first assumption, it follows from Theorem 15 that B and C are on the same side of ℓ and, in particular, that $B \notin \ell$. From this last and the second assumption, it follows by Theorem 16 that A and B are on opposite sides of ℓ. Consequently, A and C are on opposite sides of ℓ. So, by Theorem 16, $\overline{AC} \cap \ell \neq \emptyset$.

[One consequence of this theorem is that a line ℓ which intersects one side of $\triangle ABC$ and contains no vertex of this triangle must intersect another side of the triangle.]

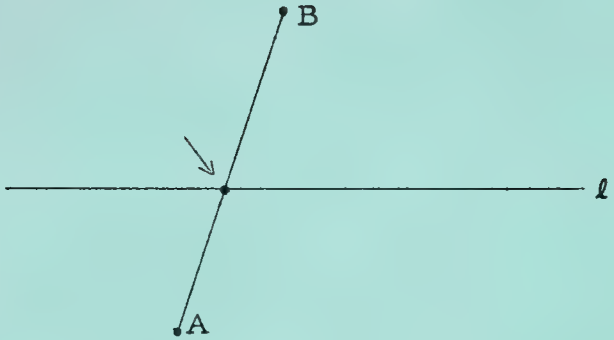
Theorem 18.



[if $A \in \ell$ and $B \notin \ell$ then $\overrightarrow{AB}$ is a subset of one side of ℓ]

Suppose that $A \in \ell$ and $B \notin \ell$. Then, by Theorem 7, $\overrightarrow{AB} \cap \ell = \emptyset$. By Theorem 13, if a point $C \in \overrightarrow{AB}$ then $\overleftrightarrow{BC} \subseteq \overrightarrow{AB}$. Hence, if $C \in \overrightarrow{AB}$ then $\overleftrightarrow{BC} \cap \ell = \emptyset$, and, by Theorem 15, B and C are on the same side of ℓ. Consequently, if $A \in \ell$ and $B \notin \ell$ then each point of $\overrightarrow{AB}$ is on the same side of ℓ as is B.

Theorem 16.

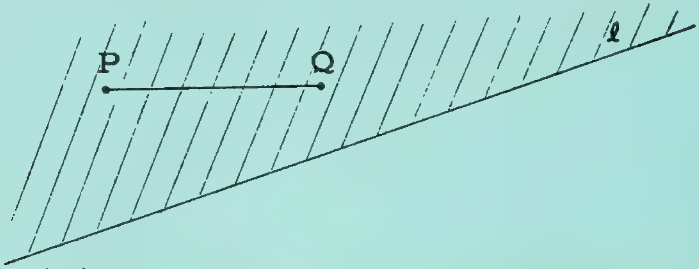


$\left[\begin{array}{l}\text{if } B \notin \ell \text{ and } \overline{AB} \cap \ell \neq \emptyset \text{ then A and B are on opposite sides of } \ell; \\ \text{if A and B are on opposite sides of } \ell \text{ then } B \notin \ell \text{ and } \overline{AB} \cap \ell \neq \emptyset\end{array}\right]$

Suppose that $B \notin \ell$ and that $\overline{AB} \cap \ell \neq \emptyset$. Since, by Theorem 5(a), $\overline{AB} \subseteq \overleftrightarrow{AB}$, it follows that $\overleftrightarrow{AB} \cap \ell \neq \emptyset$. Hence, by Theorem 15, A and B are not on the same side of ℓ. Since the two sides of a line are subsets of its complement, it will now follow that A and B are on opposite sides of ℓ if we can show that neither belongs to ℓ. But, by Theorem 5(b), $\overline{AB} \subseteq \overrightarrow{AB}$. So, since $\overline{AB} \cap \ell \neq \emptyset$, $\overrightarrow{AB} \cap \ell \neq \emptyset$. Hence, by Theorem 7, since $B \notin \ell$, it follows that $A \notin \ell$. So, as was to be shown, neither A nor B belongs to ℓ. Consequently, if $B \notin \ell$ and $\overline{AB} \cap \ell \neq \emptyset$ then A and B are on opposite sides of ℓ.

Suppose, now, that A and B are on opposite sides of ℓ. Then, neither A nor B belongs to ℓ, and A and B are not on the same side of ℓ. Hence, by Theorem 15, $\overleftrightarrow{AB} \cap \ell \neq \emptyset$, and since, by Theorem 5(a), $\overleftrightarrow{AB} = \overline{AB} \cup \{A, B\}$, $\overline{AB} \cap \ell \neq \emptyset$. Consequently, if A and B are on opposite sides of ℓ then $B \notin \ell$ and $\overline{AB} \cap \ell \neq \emptyset$.

The remaining six theorems deal with the separation of the plane by a line. As discussed on page 6-16, the complement of a line is the union of two sets, called <u>half-planes</u>. As stated in Axiom 15, (1) two points, P and Q, belong to the same half-plane determined by ℓ [or: are <u>on</u> <u>the</u>



<u>same</u> <u>side</u> <u>of</u> ℓ] if $\overleftrightarrow{PQ} \cap \ell = \emptyset$; and (2) if two points, P and Q, belong to the same half-plane determined by ℓ then $\overleftrightarrow{PQ}$ is a subset of this half-plane.

Two points are said to be <u>on</u> <u>opposite</u> <u>sides</u> <u>of</u> ℓ when the first belongs to one of the two half-planes determined by ℓ and the second belongs to the other of the two half-planes. Notice that if two points are on the same side of ℓ or on opposite sides of ℓ then neither point belongs to ℓ. [So, if either of two points belongs to ℓ then the two points are not on the same side of ℓ and are not on opposite sides of ℓ.] Conversely, if neither of two points belongs to ℓ then the two points are either on the same side of ℓ or on opposite sides of ℓ.

*

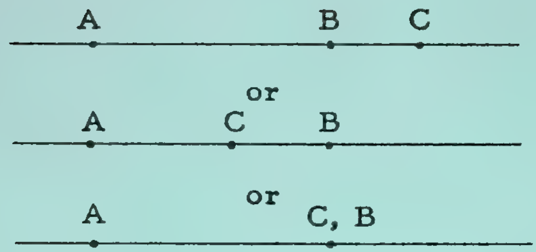
Theorem 15. [See picture above.]

[if $\overleftrightarrow{AB} \cap \ell = \emptyset$ then A and B are on the same side of ℓ;
if A and B are on the same side of ℓ then $\overleftrightarrow{AB} \cap \ell = \emptyset$]

By Axiom 15(1), if $\overleftrightarrow{AB} \cap \ell = \emptyset$ then A and B are in the same half-plane determined by ℓ. That is, A and B are on the same side of ℓ.

By Axiom 15(2), if A and B are in the same half-plane determined by ℓ then $\overleftrightarrow{AB}$ is a subset of this half-plane. Since the half-plane is a subset of the complement of ℓ, $\overleftrightarrow{AB}$ is a subset of the complement of ℓ. So, $\overleftrightarrow{AB} \cap \ell = \emptyset$.

Theorem 13.



[if $C \in \overrightarrow{AB}$ then $\overset{\bullet\!-\!\bullet}{BC} \subseteq \overrightarrow{AB}$]

Suppose that $C \in \overrightarrow{AB}$. By Theorem 5(d), $A \neq B$. So, by Theorem 8, $\overrightarrow{AB} = \overline{AB} \cup \{B\} \cup \{Z: B \in \overline{AZ}\}$. Since $C \in \overrightarrow{AB}$, it follows that either (1) $C \in \overline{AB}$, or (2) $C = B$, or (3) $B \in \overline{AC}$. If (1) $C \in \overline{AB}$ then, by Theorem 5(e), $\overline{CB} \subseteq \overline{AB}$. So, by Axiom 9, $\overline{BC} \subseteq \overline{AB}$. Hence, by Theorem 5(b), $\overline{BC} \subseteq \overrightarrow{AB}$. So, if $C \in \overline{AB}$ then $\overline{BC} \subseteq \overrightarrow{AB}$. If (2) $C = B$ then, by Theorem 5(d), $\overline{BC} = \emptyset \subseteq \overrightarrow{AB}$. If (3) $B \in \overline{AC}$ then, by Theorem 5(e), $\overline{BC} \subseteq \overline{AC}$ and, by Theorem 5(b), $\overline{BC} \subseteq \overrightarrow{AC}$. But, by Theorem 12, since $C \in \overrightarrow{AB}$, it follows that $\overrightarrow{AC} = \overrightarrow{AB}$. So, if $B \in \overline{AC}$ then $\overline{BC} \subseteq \overrightarrow{AB}$. Hence, in any case, if $C \in \overrightarrow{AB}$ then the interval $\overline{BC} \subseteq \overrightarrow{AB}$. But, also, if $C \in \overrightarrow{AB}$ then, as we have seen, $A \neq B$ and, by Theorem 5(c), $B \in \overrightarrow{AB}$. Since $\overset{\bullet\!-\!\bullet}{BC} = \overline{BC} \cup \{B, C\}$, we have, finally, that if $C \in \overrightarrow{AB}$ then the segment $\overset{\bullet\!-\!\bullet}{BC} \subseteq \overrightarrow{AB}$.

Theorem 14.

[if $A \in \overline{CB}$ then $\overleftrightarrow{AB} = \overrightarrow{AC} \cup \{A\} \cup \overrightarrow{AB}$]

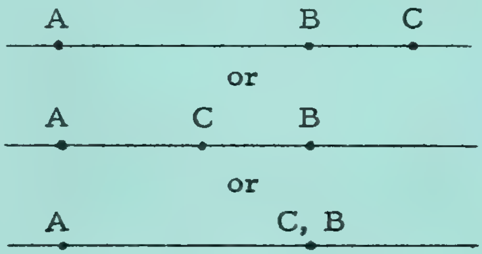
By Axiom 5, $\overleftrightarrow{AB} = \overrightarrow{BA} \cup \overrightarrow{AB}$. Suppose that $A \in \overline{CB}$. By Axioms 7 and 9, $A \neq B$; so, by Theorem 8, $\overrightarrow{BA} = \{Z: A \in \overline{BZ}\} \cup \{A\} \cup \overline{BA}$. By Axiom 9, $\overline{CB} = \overline{BC}$. So, $A \in \overline{BC}$. Hence, by Theorem 11, $\{Z: A \in \overline{BZ}\} = \overrightarrow{AC}$. By Axiom 9, $\overline{BA} = \overline{AB}$, and, by Theorem 5(b), $\overline{AB} \subseteq \overrightarrow{AB}$. Consequently, $\overrightarrow{BA} \cup \overrightarrow{AB} = \overrightarrow{AC} \cup \{A\} \cup \overrightarrow{AB}$.

*

On the other hand, still assuming that $A \in \overline{DB}$, if $C \in \{Z\colon A \in \overline{DZ}\}$, so that $A \in \overline{DC}$, then, by Axiom 12, either $C = B$ or $C \in \overline{AB}$ or $B \in \overline{AC}$. In any case, $C \in \overrightarrow{AB}$. Hence, $\{Z\colon A \in \overline{DZ}\} \subseteq \overrightarrow{AB}$.

Consequently, if $A \in \overline{DB}$ then $\overrightarrow{AB} = \{Z\colon A \in \overline{DZ}\}$.

<u>Theorem 12.</u>



[if $A \neq B$ and $\overrightarrow{AC} = \overrightarrow{AB}$ then $C \in \overrightarrow{AB}$; if $C \in \overrightarrow{AB}$ then $A \neq B$ and $\overrightarrow{AC} = \overrightarrow{AB}$]

Suppose, first, that $A \neq B$ and that $\overrightarrow{AC} = \overrightarrow{AB}$. By Theorem 5(c), $B \in \overrightarrow{AB}$; so, $\overrightarrow{AB} \neq \emptyset$. Hence, $\overrightarrow{AC} \neq \emptyset$ and, by Theorem 5(d), $A \neq C$. Hence, by Theorem 5(c), $C \in \overrightarrow{AC}$. So, $C \in \overrightarrow{AB}$.

On the other hand, suppose that $C \in \overrightarrow{AB}$. By Theorem 5(d), $A \neq B$; so, by Axiom 11, $\{Z\colon A \in \overline{BZ}\} \neq \emptyset$. Let D be a point in $\{Z\colon A \in \overline{BZ}\}$, that is, such that $A \in \overline{BD}$. By Axiom 9, $\overline{BD} = \overline{DB}$; so, $A \in \overline{DB}$. Consequently, by Theorem 11, $\overrightarrow{AB} = \{Z\colon A \in \overline{DZ}\}$. It follows that, since $C \in \overrightarrow{AB}$, $A \in \overline{DC}$. So, by Theorem 11, $\overrightarrow{AC} = \{Z\colon A \in \overline{DZ}\}$. Hence, $\overrightarrow{AC} = \overrightarrow{AB}$.

<u>Corollary of Theorem 12.</u> [Same picture as for Theorem 12.]

[if $C \in \overrightarrow{AB}$ then $B \in \overrightarrow{AC}$]

Suppose that $C \in \overrightarrow{AB}$. By Theorem 12, $A \neq B$ and $\overrightarrow{AC} = \overrightarrow{AB}$. Since $A \neq B$, it follows by Theorem 5(c) that $B \in \overrightarrow{AB}$. So, since $\overrightarrow{AC} = \overrightarrow{AB}$, it follows that $B \in \overrightarrow{AC}$.

From Axiom 6 it follows at once that if A, B, and C are <u>three</u> collinear points then [since $A \neq B$] $C \in \overleftrightarrow{AB}$. Consequently, by Theorem 8, [since $C \neq A$ and $C \neq B$], either $A \in \overline{BC}$ or $C \in \overline{AB}$ or $B \in \overline{AC}$. So we have Theorem 9.

On the other hand, if $C \in \overline{AB}$ then, by Theorem 5(b), $C \in \overleftrightarrow{AB}$. Hence, by Axiom 6, $A \neq B$ and A, B, and C are collinear. Moreover, if $C \in \overline{AB}$ then, by Axioms 7 and 9, $C \neq A$ and $C \neq B$. Consequently, if $C \in \overline{AB}$ then A, B, and C are three collinear points. Hence, Theorem 10.

*

For each of the remaining ten theorems we shall give a picture illustrating the theorem and a terse outline of a proof. The pictures should be used in class discussion of the theorems; the proofs are for you to fall back on in case you are pushed by exceptionally interested students.

*

<u>Theorem 11.</u>

[if $A \in \overline{DB}$ then $\overrightarrow{AB} = \{Z: A \in \overline{DZ}\}$]

Suppose that $A \in \overline{DB}$. By Axioms 7 and 9, $A \neq B$. So, by Theorem 8, $\overrightarrow{AB} = \overline{AB} \cup \{B\} \cup \{Z: B \in \overline{AZ}\}$. By Axiom 14, since $A \in \overline{DB}$, it follows that if a point $C \in \overline{AB}$ then $A \in \overline{DC}$. So, $\overline{AB} \subseteq \{Z: A \in \overline{DZ}\}$. Again, since $A \in \overline{DB}$, $\{B\} \subseteq \{Z: A \in \overline{DZ}\}$. By Axiom 13, since $A \in \overline{DB}$, it follows that if $B \in \overline{AC}$ then $A \in \overline{DC}$. So, $\{Z: B \in \overline{AZ}\} \subseteq \{Z: A \in \overline{DZ}\}$. Combining these results, we see that [assuming that $A \in \overline{DB}$] $\overrightarrow{AB} \subseteq \{Z: A \in \overline{DZ}\}$.

*

Incidentally, Axiom 14, which was used in proving Theorem 5(e), can be derived from some of the earlier Introduction Axioms, including Axioms 12 and 13. However, since this derivation of Axiom 14 is somewhat complicated, we shall not give it here.

*

Theorem 6 is essentially equivalent to (*) on TC[6-23]b, and has been proved on TC[6-23]c.

Theorem 7 says that, given a line ℓ, a second line, which contains a point $A \in \ell$ and a point $B \notin \ell$, contains no point of ℓ other than A. Also, that $\overrightarrow{AB} \cap \ell = \emptyset$. The first part of the theorem follows at once from Theorem 1, once one has noted that, by Theorem 4, $\overleftrightarrow{AB}$ is a line which contains A. The second part now follows from the first, together with the fact that, since $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$, $\overrightarrow{AB} \cap \ell \subseteq \overleftrightarrow{AB} \cap \ell$, and the fact that $A \notin \overrightarrow{AB}$.

Theorem 8 tells us, first, that

$$\text{if } A \neq B \text{ then } \overrightarrow{AB} = \overline{AB} \cup \{B\} \cup \{Z\colon\ B \in \overline{AZ}\}.$$

By Axiom 5, $\overrightarrow{AB}$ is the set consisting of all the points in $\overleftrightarrow{AB}$ except A. Also by Axiom 5, $\overleftrightarrow{AB}$ is the set of all the points which belong to $\overline{AB}$ or to $\{A, B\}$ or to $\{Z\colon\ B \in \overline{AZ}\}$. Now, by Axiom 7, $A \notin \overline{AB}$. And, by Theorem 5(d), $\overline{AA} = \emptyset$; so, since $B \notin \overline{AA}$, $A \notin \{Z\colon\ B \in \overline{AZ}\}$. Hence, if $A \neq B$ then $\overrightarrow{AB}$ is the set of all the points which belong to $\overline{AB}$ or to $\{B\}$ or to $\{Z\colon\ B \in \overline{AZ}\}$. So, we have proved the first part of Theorem 8.

For the second part of the theorem, we need only remark that, by the first part, if $B \neq A$ then $\overrightarrow{BA} = \overline{BA} \cup \{A\} \cup \{Z\colon\ A \in \overline{BZ}\}$, that, by Axiom 5, $\overleftrightarrow{AB} = \overrightarrow{BA} \cup \overrightarrow{AB}$, and that, by Axiom 9, $\overline{BA} = \overline{AB}$. As a consequence, $\overleftrightarrow{AB} = \{Z\colon\ A \in \overline{BZ}\} \cup \{A\} \cup \overline{AB} \cup \{B\} \cup \{Z\colon\ B \in \overline{AZ}\}$.

to $\overleftrightarrow{AB}$. The other part of Axiom 6 can now be derived if we use the theorem:

$$(3) \qquad \forall_X \; \overleftrightarrow{XX} = \emptyset$$

[This we have previously seen to be a consequence of Axiom 6.] In fact, if $C \in \overleftrightarrow{AB}$ then it follows from the theorem just stated that $A \neq B$. So, by definition, $\overleftrightarrow{AB}$ is a line which, by Axiom 5, contains A and B. So, A, B, and C are collinear.

The upshot of all this is that Axioms 1, 2, and 6 might be replaced by the displayed sentences (1), (2) [or (*)], and (3).

*

We return now to parts (e) and (f) of Theorem 5. To prove part (e), we need to show that if $C \in \overline{AB}$ then $\overline{CB} \subsetneq \overline{AB}$. That is, we need to show that if $C \in \overline{AB}$ and $D \in \overline{CB}$ then $D \in \overline{AB}$. We begin by using Axiom 14. This axiom tells us that if $C \in \overline{AB}$ and $D \in \overline{CB}$ then $C \in \overline{AD}$.

By Axiom 9, $\overline{CB} = \overline{BC}$ and $\overline{AD} = \overline{DA}$. So, we know that if $C \in \overline{AB}$ and $D \in \overline{CB}$ then $C \in \overline{DA}$; also, $D \in \overline{BC}$. However, Axiom 13 tells us that if $D \in \overline{BC}$ and $C \in \overline{DA}$ then $D \in \overline{BA}$. Since $\overline{BA} = \overline{AB}$, it follows that if $C \in \overline{AB}$ and $D \in \overline{CB}$ then $D \in \overline{AB}$. That is, if $C \in \overline{AB}$ then $\overline{CB} \subsetneq \overline{AB}$.

The proof of Theorem 5(f) is now easy. We want to show that, if $A \neq B$, there are at least two points in $\overline{AB}$. Axiom 8 tells us at once that there is at least one such point. Suppose, then, that C is a point such that $C \in \overline{AB}$. Axioms 7 and 9 tell us that $C \neq B$. So, again by Axiom 8, there is a point, say D, such that $D \in \overline{CB}$. By Axiom 7, $D \neq C$. By Theorem 5(e), since $C \in \overline{AB}$, $\overline{CB} \subsetneq \overline{AB}$. So, since $D \in \overline{CB}$, $D \in \overline{AB}$. Consequently, there are at least two points in $\overline{AB}$.

This is, in fact, Theorem 6 [on page 6-25]. To prove this theorem, suppose that $C \neq D$ and that $\{C, D\} \subseteq \overleftrightarrow{AB}$. It follows, from the latter assumption, that $\overleftrightarrow{AB} \neq \emptyset$ and, hence [as proved earlier], that $A \neq B$. So, by Theorem 4, $\overleftrightarrow{AB}$ is a line. Hence, $\overleftrightarrow{AB}$ is a line which contains the points C and D. By (*), since $C \neq D$, it follows that there is at most one such line. In fact, by Theorem 4, this line is $\overleftrightarrow{CD}$. So, $\overleftrightarrow{CD} = \overleftrightarrow{AB}$.

We have seen that, from Axiom 1 and Theorem 4, one can derive:

(1) A set ℓ is a line
if and only if
there are two points X and Y such that $\ell = \overleftrightarrow{XY}$

This suggests that we might have defined the word 'line' in this way. Had we done so, Axiom 1 could have been omitted. For, as we have seen, it follows from Axiom 5 that if A and B are two points then $\{A, B\} \subseteq \overleftrightarrow{AB}$. Moreover, the part of Axiom 2 which says that each two points are contained in at least one line could have been omitted. For, if A and B are two points, it would now follow, by definition, that $\overleftrightarrow{AB}$ is a line and, by Axiom 5, that this line contains A and B. The remaining part, (*), of Axiom 2 could, then, be replaced by Theorem 6:

(2) $\forall_W \forall_X \forall_Y \forall_Z$ if $W \neq Z$ and $\{W, Z\} \subseteq \overleftrightarrow{XY}$ then $\overleftrightarrow{WZ} = \overleftrightarrow{XY}$

For, from this and the suggested definition for 'line' it follows that if ℓ is any line which contains two given points, C and D, then ℓ is the line $\overleftrightarrow{CD}$. So, (*) is a consequence of Theorem 6 and the suggested definition. Hence, Axioms 1 and 2 could be replaced by Theorem 6 and the suggested definition of 'line'.

Once these changes are made, part of Axiom 6 becomes superfluous. For, if $A \neq B$ and A, B and C are collinear then $\overleftrightarrow{AB}$ is the unique line containing A and B; and C, being collinear with A and B, must belong

Parts (e) and (f) of Theorem 5 depend on some of the later axioms. But, before taking these up, it will be helpful to get a better idea of how Axioms 1, 2, 5, and 6 hang together. In the process, we shall prove Theorem 4, on page 6-24, and Theorem 6, on page 6-25. And, we shall see that Axioms 1, 2, and 6 could be replaced by a definition of 'line', Theorem 6, and the theorem '$\forall_X \overleftrightarrow{XX} = \emptyset$'.

We can use Axiom 6 to link up the word 'line' with the notation used in Axiom 5. Suppose that A and B are two points. As has just been shown, it follows from Axiom 5 that $\overleftrightarrow{AB} \neq \emptyset$. So, there is a point C in the set $\overleftrightarrow{AB}$. Hence, by Axiom 6 [only-if-part], C belongs to some line containing A and B. So [without using Axiom 2], there is a line which contains A and B. However, Axiom 2 tells us that [assuming that $A \neq B$] there is <u>at most one</u> line which contains A and B. So, if $A \neq B$, each point of $\overleftrightarrow{AB}$ belongs to <u>the</u> line containing A and B. On the other hand, by Axiom 6, if $A \neq B$ and C belongs to the line which contains A and B, then $C \in \overleftrightarrow{AB}$. Hence, if $A \neq B$, each point of the line which contains A and B belongs to $\overleftrightarrow{AB}$. Consequently, if $A \neq B$ then the set $\overleftrightarrow{AB}$ is the line which contains A and B. So, we have proved Theorem 4:

$\forall_X \forall_Y$ if $X \neq Y$ then $\overleftrightarrow{XY}$ is the line determined by X and Y

In the first part of the proof of Theorem 4, we saw that it follows from Axioms 5 and 6, without using Axiom 2, that each two points are contained in <u>at least</u> one line. So, this part of Axiom 2 might be omitted--that is, we might replace Axiom 2 by the weaker sentence:

(*) Each two points are contained in <u>at most</u> one line.

Now, by Axiom 1, each line contains two points, and so, by Theorem 4, each line is some set $\overleftrightarrow{XY}$, for two points X and Y. Hence, (*) and Theorem 4 tell us this:

$\forall_W \forall_X \forall_Y \forall_Z$ if $W \neq Z$ and $\{W, Z\} \subseteq \overleftrightarrow{XY}$ then $\overleftrightarrow{WZ} = \overleftrightarrow{XY}$

line 5b should read 'Theorems 1, 4, 5b, and'.

Theorems 1, 2, 3, and 4 have been discussed in the COMMENTARY for pages 6-18, 6-19, and 6-20. Now, we shall discuss the remaining theorems, beginning with Theorem 5 on page 6-25.

In addition to this we shall give an alternative proof for Theorem 4. This proof is more complicated than that given on TC[6-19, 20]b, but is needed to justify the remarks on TC[6-18]a concerning the possibility of defining the word 'line'.

You will probably find little direct use for this COMMENTARY in the classroom. However, study of it will deepen your understanding of the Introduction Axioms and should help you decide how [and how far] to proceed in discussing the Introduction Theorems. You will find it easier to read and appreciate what follows if you invest some time in trying to prove the Introduction Theorems by yourself.

*

Axiom 5 introduces various notations with which the student has become familiar, and shows how each can be described in terms of the notion of betweenness. Theorem 5(a) on page 6-25 summarizes some of the obvious consequences of Axiom 5. The same holds for Theorem 5(b), except that its first clause ['$\overline{XY} \subseteq \overrightarrow{XY}$'] depends, in part, on Axiom 7. For, by Axiom 5 [or Theorem 5(a)], $\overline{AB} \subseteq \overset{\bullet\!\rightarrow}{AB}$. But, by Axiom 7, $A \notin \overline{AB}$. Therefore, since $\overrightarrow{AB}$ consists of the points of $\overset{\bullet\!\rightarrow}{AB}$ other than A, $\overline{AB} \subseteq \overrightarrow{AB}$.

An argument for Theorem 5(c) goes as follows:

> By Axiom 5, $A \in \overset{\bullet\!-\!\bullet}{BA}$. Hence, again by Axiom 5, $A \in \overset{\bullet\!\rightarrow}{BA}$ and, if $A \neq B$, $A \in \overrightarrow{BA}$. Since $\overset{\bullet\!\rightarrow}{AB} = \{A\} \cup \overrightarrow{AB}$, $\overset{\bullet\!\rightarrow}{AB} \subseteq \overrightarrow{BA} \cup \overrightarrow{AB} = \overleftrightarrow{AB}$.

As a consequence of this, we see that if $A \neq B$ then $\{A, B\} \subseteq \overleftrightarrow{AB}$. In particular, if $A \neq B$ then $\overleftrightarrow{AB} \neq \emptyset$.

Theorem 5(d) depends on Axiom 6. By this axiom, if there is a point C such that $C \in \overleftrightarrow{AB}$ then $A \neq B$. In other words, if $\overleftrightarrow{AB} \neq \emptyset$ then $A \neq B$. So, if $A = B$ then $\overleftrightarrow{AB} = \emptyset$. [Equivalently: $\forall_X\ \overleftrightarrow{XX} = \emptyset$]

*

line 27. about 5.08; about 3.81; about 8.89

[Students will probably give results such as 5.05 or 5.1, 3.8, and 8.85 or 8.9. Accept such results without getting entangled in the subject of approximations. This work and the work at the top of page 6-30 should move rapidly toward Axiom A.]

line 32. Notice the functional notation introduced in line 32. This is not an accident. There exists a function [a variable quantity] which we call 'inch-m'. It is a set of ordered pairs. The first component of each of these pairs is a segment, and the second component is a number of arithmetic. The second component is said to be the inch-measure of the first component. There are many such measure functions. Another is called '$\frac{1}{2}$-inch-m', and another is called 'cm-m'. If the value of inch-m for a given segment is k then the value of $\frac{1}{2}$-inch-m for that segment is 2k and the value of cm-m for that segment is 2.54k.

At the moment, all we know in our formal geometry about these measure functions is that they are functions with the set of all segments as domain and the set of numbers of arithmetic as range. Some of the properties of this undefined concept are expressed by Axioms A, B, and C.

There are measure functions for intervals, but we shall have no need for such functions in our geometry.

Correction. On page 6-29, line 18 should read 'To $\overleftrightarrow{BC}$? To $\overleftrightarrow{AC}$?'.

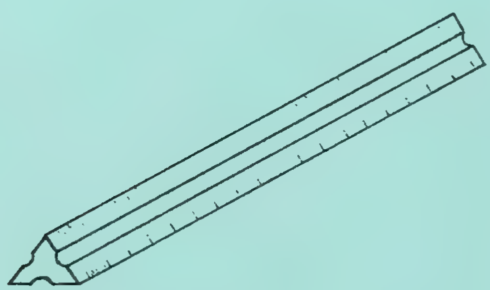
Line 1b should begin '[Read ---'.

line 14. Have students turn their rulers so that the scale numerals are inverted and facing up. Ask if they still get the same scale difference for $\overleftrightarrow{AB}$.

line 17. Since we always subtract the smaller scale number from the larger to obtain the measure of the segment, the same number is assigned to both $\overleftrightarrow{AB}$ and $\overleftrightarrow{BA}$. We should expect this to be the case because $\overleftrightarrow{AB} = \overleftrightarrow{BA}$ and m is a function which maps segments into numbers. So, $m(\overleftrightarrow{AB}) = m(\overleftrightarrow{BA})$. [By Axiom 5, $\overleftrightarrow{AB} = \overline{AB} \cup \{A, B\}$ and $\overleftrightarrow{BA} = \overline{BA} \cup \{B, A\}$. By Axiom 9, $\overline{AB} = \overline{BA}$. So, since $\{A, B\} = \{B, A\}$, $\overleftrightarrow{AB} = \overleftrightarrow{BA}$.]

line 18. $m(\overleftrightarrow{BC}) = 1.5$; $m(\overleftrightarrow{AC}) = 3.5$

line 20. An architect's scale or an engineer's scale would be a handy thing to have in the classroom at this time.



If you don't own one, borrow one from the mechanical drawing teacher or the shop teacher.

line 23. 4; 3; 7

lines 3 - 7.

When the unit is	m($\overline{AB}$) is	m($\overline{BC}$) is	m($\overline{AC}$) is
the $\frac{1}{2}$-inch	4	3	7
the $\frac{1}{8}$-inch	16	12	28
the 2-inch	1	3/4	1 3/4
the centimeter	5.08	3.81	8.89
the og	28	21	49

The United States standard inch is now defined to be exactly 2.54 centimeters. This means that the standard inch today is approximately 0.00000508 centimeters shorter than it was prior to the legislative action.

line 20. Be sure that 'AB' is read 'the measure of segment $\overline{AB}$' often enough that the students think of it properly. Since m($\overline{AB}$) is a number, so is AB.

*

line 6b. Point out that in stating generalizations we use the letters that come late in the alphabet, and in stating instances we use the letters that come early. This distinction will make it easier to follow and write proofs.

*

It is helpful if students read 'Y $\in \overline{XZ}$' as 'Y is between X and Z' rather than as 'Y belongs to the interval $\overline{XZ}$'.

*

Intuitively, (*) says that if you take a segment and break it into two segments at some point between its end points then the measures of the two pieces add up to the measure of the given segment.

Intuitively, Axiom A says that it doesn't matter at what point in a given segment you break it in two--the measures of the two pieces will add up to the measure of the given segment. (*) restricted us to breaking the segment at a point between its end points. Of course, neither (*) nor Axiom A requires that the segment have two end points. However, in practice, one would use (*) only with nondegenerate segments while Axiom A could be used in all cases.

*

Note that Axiom A is somewhat like the conventional axiom about the whole being equal to the sum of its parts. The conventional axiom is ambiguous in many respects. Does the word 'sum' refer to segments or to measures of segments? And, what is meant by 'part'? Is a part of a segment just any subset of it? Or must a part of a segment be a segment? Also, how many parts does a segment have? It is clear that the precise language of Axiom A avoids such ambiguities, and thereby makes it usable in proofs. But, before one can feel at ease with this precise language, he must have the experience provided by the table and discussion on page 6-30. Under ideal circumstances, each class should produce its own statements of axioms, sharpening the statements until they express exactly what is meant.

*

i	AB_i	B_iC	$AB_i + B_iC$	AC
1	1	1.5	2.5	2
2	1.5	1.5	3	2
3	1.5	1	2.5	2
4	2	1	3	2
5	3	1	4	2
6	0.5	2.5	3	2

*

line 2b. $AB_i + B_iC > AC$ [Notice, also, that $B_i \notin \overleftrightarrow{AC}$.]

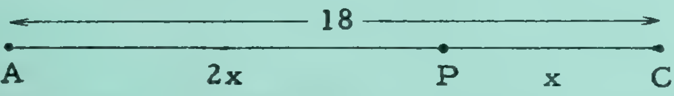
line 1b. No

In accepting Axiom B, be sure students understand that Axiom B holds even if Y is collinear with X and Z. This idea is covered in the cases of points B_5 and B_6 on page 6-31. But, unless students have had the experience of trying to state Axiom B by themselves with these cases clearly in view, the full message of Axiom B may escape them. Hence, it would be good practice to get them to state Axiom B when they reach the bottom of page 6-31 and before turning to page 6-32.

*

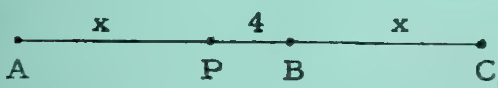
Answers for Exercises.

1.



Suppose that PC is x. Then, AP is 2x and, since $P \in \overleftrightarrow{AC}$ and AC = CA, it follows from Axiom A that $2x + x = 18$. So, x is 6. Hence, AP is 12. [Of course, all that students should be required to submit in answer to this exercise is: 12. If you discuss the exercise in class, you can bring in the role of Axiom A.]

2.

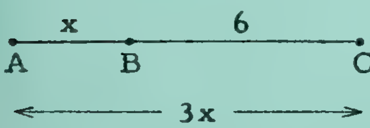


$x + 4 + x = 24$; $x = 10$

So, AB = 14 and PC = 14.

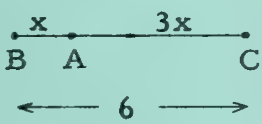
[Actually, there is quite a bit involved in this problem, although it should not be your purpose to dig it out unless students raise questions. For example, how do we get the equation '$x + 4 + x = 24$'? Well, we are given that $B \in \overline{PC}$. From this and Axiom 5, it follows that $B \in \overleftrightarrow{PC}$. So, by Axiom A, since PB = 4 and BC = x, it follows that PC = 4 + x. Now, since $P \in \overline{AB}$ and $B \in \overline{PC}$, it follows from Axiom 13 that $P \in \overline{AC}$. Then, by Axiom 5 and Axiom A, AC = AP + PC. Since AP = x, PC = 4 + x, and AC = 24, it follows that $24 = x + (4 + x)$.]

3.



$x + 6 = 3x$

$x = 3$



$x + 3x = 6$

$x = 1.5$

There is no third case --that is, a case in which $C \in \overline{BA}$. The fact that AC > AB rules this out.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

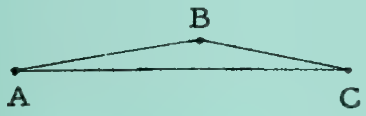
[illegible]

4. Suppose that $B \notin \overleftrightarrow{AC}$. Now, either A, B, and C are collinear or A, B, and C are noncollinear.

Consider the case in which A, B, and C are collinear. Since $B \notin \overleftrightarrow{AC}$, it follows that either $C \in \overline{AB}$ or $A \in \overline{BC}$. Now, since $AC = 9$, $AB = 5$, and BC is a number of arithmetic, it follows that $AC + CB \neq AB$. So, by Axiom A, $C \notin \overleftrightarrow{AB}$. Thus, $C \notin \overline{AB}$. Hence, if $B \notin \overleftrightarrow{AC}$ and A, B, and C are collinear then $A \in \overline{BC}$. In that case, $BA + AC = BC$. That is $5 + 9 = BC$. So, $BC = 14$.

Also, by Axiom B, since $B \notin \overleftrightarrow{AC}$, it follows that $AB + BC > AC$. That is, $5 + BC > 9$. So $BC > 4$.

Now, consider the case in which A, B, and C are noncollinear.



Then, $A \notin \overleftrightarrow{BC}$. So, by Axiom B, $BA + AC > BC$. That is, $5 + 9 > BC$. So, $BC < 14$.

Consequently, if $B \notin \overleftrightarrow{AC}$ then $4 < BC \leq 14$. $BC \neq 16$ and $BC \neq 2$.

[Do not ask for more than an intuitive defense of the correct answers.]

*

[More intuitive problems of this nature are on page 6-405. You may wish to assign these before you reach page 6-36. Students will find it convenient to use compasses in Exercises 2 and 3 on page 6-405.]

As suggested in the box at the bottom of page 6-34, this is an appropriate place to stop the study of geometry and take a look at some of the logical principles used in writing proofs. We have placed our treatment of logic on a semi-optional basis because we have not had an opportunity to test this treatment. We have included it in this revision because of the demand of many cooperating teachers for an explicit treatment of logic. We hope that all teachers will include the Appendix in the course. In fact, if the interest of the class warrants it, you might just as well teach the material from page 6-357 through page 6-398 before returning to page 6-35.

*

The problem of teaching students to write paragraph proofs is a very difficult bit of pedagogy not unlike that the English teacher faces in theme-writing. Our point of view here is that the student will learn to do this by being exposed to good examples of such proofs and by trying to write his own and then comparing his products with the models. We do not want students to drop column proofs in favor of paragraph proofs immediately, but we know that, eventually, column proofs will become so burdensome because of their length and degree of detail that students will need another mode of proof-writing to turn to. They can be preparing for this stage right now by writing paragraph proofs as suggested in pages 6-35 through 6-40.

Another reason for trying to get students to write paragraph proofs is that such proofs are customary in all parts of mathematics [except high school geometry]. Our column proofs are useful as preparation for paragraph proofs because the marginal comments that accompany column proofs make it easy for the student [and the teacher who checks his paper] to see the logical connections between steps. Once a student has a firm grasp of these matters, he is ready to move to paragraph proofs.

There are certain pedagogical objections to paragraph proofs. One of these is the business of checking student work. Column proofs are easier to check, probably because they are easier to read. High school students who cannot write sentences can still produce readable column proofs. What they do with paragraph proofs is almost beyond description. Another objection teachers raise to paragraph proofs is that, since students are not required to state axioms and theorems [as they are in column proofs], they get no opportunity to learn [or memorize] the principles. We have concocted a device which we believe will meet this valid objection. See pages 6-73 and 6-100.

We start the column proof by writing Axiom A since our preceding analysis indicated that this premiss would be fruitful. Then, we state (2), an instance of Axiom A. Now, since we wish to deduce the sentence 'AA + AA = AA', all we need for our next premiss is '$A \in \overleftrightarrow{AA}$' because this affirms the antecedent [modus ponens] of the conditional sentence (2). Now, from where do we get the premiss '$A \in \overleftrightarrow{AA}$'? This is a consequence of one of the Introduction Theorems. In particular, it is a consequence of Theorem 5(d). But, the particular Introduction Theorem is not important. All we want here is that the student recognize that '$A \in \overleftrightarrow{AA}$' follows from the work in the introductory section of the unit.

In view of (2) and (3), one can conclude (4). And, in view of his work in algebra, (4) leads to (5). So, steps (1) - (5) make up a test-pattern for (6).

As indicated in the text at the bottom of page 6-33 and in the second paragraph on page 6-34, the column proof has two gaps in it. One of these gaps can be closed by making explicit use of an Introduction Theorem--that is, by giving the Introduction Theorem as one of the steps in the proof. This is a kind of gap which we must tolerate in proofs in geometry if we are ever "to get on with the subject". All the student needs to know at this point is that there is a gap and that the Introduction Axioms are sufficient to provide the necessary theorems with which to close the gap. [In proofs which occur somewhat later in the text, we shall not bother even to give explicit notice of the gaps. But, we shall try to alert you in the COMMENTARY to their presence, at least in the case of proofs of important theorems.]

A second kind of gap is that in which we omit mention of the algebra theorems used in deriving certain conclusions. Since the algebra theorems are not part of the structure of geometry, we feel they can be omitted. [This coincides with the practice in conventional courses.] On the other hand, with students of this age, it may be pedagogically fitting to ask students to state such algebra theorems when they give an oral presentation. It would be deadening to require them to do it in writing as a standard part of their assignments.

*

> Correction. On page 6-35, line 11 should begin:
>
> column proof. If you ---

We give here a Unit 2-type proof and a column proof of the algebra theorem used in the proof of Theorem 1-1.

Prove: $\forall_x$ if $x + x = x$ then $x = 0$

Proof I. Suppose $a + a = a$.

Then, $a + a = a + 0$. [pa0]

Hence, $a = 0$. [left canc. prin. for add.]

So, if $a + a = a$ then $a = 0$.

Consequently, $\forall_x$ if $x + x = x$ then $x = 0$.

Proof II.

(1)	$a + a = a$	[assumption] *
(2)	$\forall_x\ x + 0 = x$	[basic principle]
(3)	$a + 0 = a$	[(2)]
(4)	$a + a = a + 0$	[(1) and (3)]
(5)	$\forall_x \forall_y \forall_z$ if $x + y = x + z$ then $y = z$	[theorem]
(6)	if $a + a = a + 0$ then $a = 0$	[(5)]
(7)	$a = 0$	[(4) and (6)]
(8)	if $a + a = a$ then $a = 0$	[(7); *(1)]
(9)	$\forall_x$ if $x + x = x$ then $x = 0$	[(1) - (8)]

*

To see how to prove Theorem 1-1 ['1-1' because it is the first theorem of section 6.01], one must first understand what it says. It claims that the measure of a degenerate segment is 0. This is something of which we were aware when we strengthened (*) to produce Axiom A. But, just as (*) arose from intuitive explorations with scales, so did the fact that the measure of a degenerate segment is 0. To make it part of our formal geometric structure, we can state it as a separate axiom or we can try to deduce it from axioms already stated.

Correction. On page 6-37, the last part of line 13 should read:

--- the if-part of step (11) is the
↑

After learning from Theorem 1-1 that each degenerate segment has zero-measure, the next natural question to ask is if each nondegenerate segment has nonzero-measure. Since measures are numbers of arithmetic, this is the same as asking if the measure of each nondegenerate segment is greater than 0. Intuitively, the answer is 'yes'. But, is this something we can deduce from our present axioms and theorems, or is it something we must assume?

In trying to prove Theorem 1-2, we shall start with the assumption [or: supposition]:

$$A \neq B,$$

and try to deduce from this premiss together with axioms and theorems the sentence:

$$AB > 0$$

The next step in the proof would then be the conditional sentence:

$$\text{if } A \neq B \text{ then } AB > 0$$

This conditional sentence is a consequence of <u>just</u> the axioms and theorems used in the proof.

So, the problem is one of finding out how to deduce 'AB > 0' from 'A = B' and other premisses. Since Axiom B deals with inequality of measures, it appears reasonable that we shall want to use Axiom B. Let's look at an instance of Axiom B:

$$\text{if } B \notin \overleftrightarrow{AC} \text{ then } AB + BC > AC$$

If we let A = C, we get the instance:

$$\text{if } B \notin \overleftrightarrow{AA} \text{ then } AB + BA > AA$$

And, if we use Theorem 1-1, we get:

$$\text{if } B \notin \overleftrightarrow{AA} \text{ then } AB + BA > 0$$

Now, since $A \neq B$, it follows from the Introduction Axioms that $B \notin \overleftrightarrow{AA}$. So, the antecedent of the last conditional is affirmed. Hence, $AB + BA > 0$. It follows from the Introduction Axioms that $AB = BA$. So, by algebra, we now have:

$$2 \cdot AB > 0$$

Hence, by algebra again, we have:

$$AB > 0$$

*

Answers for Part B [on pages 6-36 and 6-37].

(2) $\forall_X \forall_Y \forall_Z$ if $Y \notin \overleftrightarrow{XZ}$ then $XY + YZ > XZ$

(6) $\forall_X\ XX = 0$

(7) $AA = 0$

(12) $\forall_X \forall_Y$ if $X \neq Y$ then $XY > 0$

*

If a student were writing his own column proof of Theorem 1-2, his marginal comment for step (2) would just be 'axiom', and that for step (6) would be 'theorem'. It is not necessary to cite axioms and theorems by letter or number. We do this early in the text just for reference purposes, but later [see page 6-99] we omit such references.

*

(a) [These remarks are superfluous for students who have studied the Appendix.]

(b) if $A \neq B$ then $B \notin \overleftrightarrow{AA}$

(c) [See comment above for (a).]

(d) $\forall_x$ if $x + x > 0$ then $x > 0$ [<u>Proof</u>: Suppose that $x + x > 0$. Then, since $x + x = 2x$, it follows that $2x > 0$. By the mtpi, since $\frac{1}{2} > 0$, $2x \cdot \frac{1}{2} > 0 \cdot \frac{1}{2}$. So [by various elementary theorems], $x > 0$.]

2. [See page 6-41.]

Theorem 1-3 is another result which is intuitively obvious from the work students have done in using a ruler. The proof of this theorem should not be presented as a device for convincing students of the correctness of the theorem. Rather, you should ask the question about whether to make the statement of Theorem 1-3 an axiom in our system or whether the statement can be predicted from statements already in the system. [Generally speaking, a proof does add to one's conviction, but not in the case of the theorems of section 6.01.]

Here is an approach which may help students formulate their own column proof.

> We are given the segment joining A and C, and some point B which is between the end points A and C. We want to show that the segment joining A and B has a smaller measure than the segment joining A and C. By Axiom A, we know that AB + BC = AC. We want to show that AB < AC. Now, if you have a first number [AB] and a second number [BC] whose sum is a third number [AC], does it follow that the first number is less than the third? It certainly does not follow if the second number is negative or 0. But, since measures are numbers of arithmetic, BC is not negative. Can BC = 0? Not if B $\neq$ C. For, by Theorem 1-2, if B $\neq$ C then BC > 0. But, since B is between A and C, it follows from the Introduction Axioms that B $\neq$ C. So, BC > 0. And, with the help of a little algebra, we deduce from 'AB + BC = AC' and 'BC > 0' the sentence 'AB < AC'. Let's see how:
>
> $$\left.\begin{array}{r} BC > 0 \\ BC + AB > 0 + AB \end{array}\right\}\text{atpi}$$
> $$\left.\begin{array}{r} BC + AB > 0 + AB \\ AC > AB \end{array}\right\}AC = AB + BC$$
>
> [See, also, Exercise 4 on page 6-39.]

With this approach, the sequence of steps used in the column proof on page 6-38 will be easy to understand.

*

Answers for Part C [on pages 6-38 and 6-39].

1. (2) $\forall_X \forall_Y \forall_Z$ if $Y \in \overleftrightarrow{XZ}$ then XY + YZ = XZ

   (3) if $B \in \overleftrightarrow{AC}$ then AB + BC = AC

   (5) AB + BC = AC [This and (9) are key steps in the proof.]

(7) $\forall_X \forall_Y$ if $X \neq Y$ then $XY > 0$

(8) if $B \neq C$ then $BC > 0$

(11) if $B \in \overline{AC}$ then $AB < AC$

(12) $\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY < XZ$

2. (a) if $B \in \overline{AC}$ then $B \in \overleftrightarrow{AC}$ [Actually, this is not an instance of any of the theorems on pages 6-23 through 6-28. By 'Introduction Theorem' we mean any consequence of the Introduction Axioms. You may want to warn students about this to save them from a fruitless search through pages 6-23 through 6-28.]

   (b) if $B \in \overline{AC}$ then $B \neq C$

5. [See page 6-41.]

*

Here is an approach to Theorem 1-4.

Suppose that I have a segment $\overline{AC}$ which is 10 inches long, and I pick a point B on the segment. What can you predict about the lengths of the segments $\overline{AB}$ and $\overline{BC}$? [They add up to 10 inches.] What is the basis for this prediction? [Axiom A.] Now, let's suppose that I have a segment $\overline{AC}$ which is 10 inches long, and that I pick some point B in such a way that when I measure $\overline{AB}$ and $\overline{BC}$, and add their measures, I get 10. What can you predict about the location of the point B?

The student's work on converses in the Appendix should help him see that the basis for the prediction that $B \in \overline{AC}$ is the converse of Axiom A. Once again, we have a result which is intuitively correct; shall we add it to our list of axioms, or shall we try to deduce it?

Let's try to prove that if $AB + BC = AC$ then $B \in \overline{AC}$. We start by supposing that $AB + BC = AC$. This tells us that $AB + BC \not> AC$. Now, take a look at Axiom B on page 6-32. What do you conclude? [Students who have studied modus tollens in the Appendix should give the answer quickly.] Axiom B tells us that if $B \notin \overline{AC}$ then $AB + BC > AC$. But, we know that $AB + BC \not> AC$. So, it must be the case that $B \in \overline{AC}$.

[It is easy to devise a similar approach which contains the reasoning in the alternate proof of Theorem 1-4 given in Exercise 2 on page 6-40.]

☆3.

$$
\begin{array}{c}
\dfrac{\begin{array}{c} * \\ (1) \\ \cdots \\ (4) \end{array} \quad \dfrac{(2)}{(3)}}{(5)} \qquad \dfrac{\begin{array}{c} * \\ (1) \\ \cdots \\ (6) \end{array} \quad \dfrac{(7)}{(8)}}{(9)} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\begin{array}{c} (10) \\ \hline (11) \\ \hline\hline (12) \end{array} \; *
\end{array}
$$

The dotted bars show very clearly the gaps in the proof. We know that the gaps can be filled by bringing in Introduction Theorems and algebra theorems. So, as we can tell by examining the ends of the branches in the diagram, (12) is a consequence of (2), (7), Introduction Theorems, and algebra theorems. Since we also know that (2) is an axiom and (7) is a theorem, we can see that (12) is a theorem.

4. (a) We infer (9.1) from (5) and an instance of the algebra theorem:

$$\forall_x \forall_y \forall_z \text{ if } x + y = z \text{ then } y = z - x$$

[This theorem is proved in Unit 2 on page 2-89.] Of course, a student might say that (9.1) is obtained by using the addition transformation principle and other theorems for simplifying expressions. In saying this, he is really telling what he would use to prove the theorem displayed above. It is more important at this time that he actually state the displayed theorem rather than tell what would be used to prove it. In fact, the proof of the algebra theorem is unimportant right now.

(b) Step (9.2) is obtained by substituting 'AC - AB' from (9.1) for 'BC' in (9).

(c) Step (9.2) and an instance of the algebra theorem:

$$\forall_x \forall_y \text{ if } x - y > 0 \text{ then } x > y$$

imply step (9.3).

Correction. On page 6-40, line 7 should read ' --- then B $\in \overleftrightarrow{AC}$. Line 12b should read: (8) ______________ [(1) - (7)]

Answers for Part D [on pages 6-40 and 6-41].

1. (1) AB + BC = AC [assumption]*

   (2) AB + BC $\not>$ AC [(1); algebra]

   (3) $\forall_X \forall_Y \forall_Z$ if $Y \notin \overleftrightarrow{XZ}$ then $XY + YZ > XZ$ [Axiom B]

   (4) if $B \notin \overleftrightarrow{AC}$ then $AB + BC > AC$ [(3)]

   (5) B $\in \overleftrightarrow{AC}$ [(2) and (4)]

   (6) if AB + BC = AC then B $\in \overleftrightarrow{AC}$ [(5); * (1)]

   (7) $\forall_X \forall_Y \forall_Z$ if $XY + YZ = XZ$ then $Y \in \overleftrightarrow{XZ}$ [(1) - (6)]

2. (1) B $\notin \overleftrightarrow{AC}$ [assumption]*

   (2) $\forall_X \forall_Y \forall_Z$ if $Y \notin \overleftrightarrow{XZ}$ then $XY + YZ > XZ$ [Axiom B]

   (3) if $B \notin \overleftrightarrow{AC}$ then $AB + BC > AC$ [(2)]

   (4) $AB + BC > AC$ [(1) and (3)]

   (5) AB + BC $\neq$ AC [(4); algebra]

   (6) if B $\notin \overleftrightarrow{AC}$ then $AB + BC \neq AC$ [(5); * (1)]

   (7) if AB + BC = AC then B $\in \overleftrightarrow{AC}$ [(6)]

   (8) $\forall_X \forall_Y \forall_Z$ if $XY + YZ = XZ$ then $Y \in \overleftrightarrow{XZ}$ [(1) - (7)]

3. [See page 6-41.]

4. $\forall_X \forall_Y \forall_Z$ Y $\in \overleftrightarrow{XZ}$ if and only if XY + YZ = XZ

Correction. On page 6-42, in the Hypothesis for the Example, insert a comma after 'AM = AP'.

Step (8) in the proof is inferred from (6), (7), and an instance of the algebra theorem:

$$\forall_u \forall_v \forall_x \forall_y \text{ if } u = v \text{ and } x = y \text{ then } u + x = v + y$$

If we wish to show this explicitly, we could include the following steps in the proof:

(7.1)	$\forall_u \forall_v \forall_x \forall_y$ if $u = v$ and $x = y$ then $u + x = v + y$	[algebra theorem]
(7.2)	if AM = AP and MN = PQ then AM + MN = AP + PQ	[(7.1)]
(7.3)	AM = AP and MN = PQ	[(6) and (7)]
(8)	AM + MN = AP + PQ	[(7.3) and (7.2)]

Then, to show the actual substitution inferences involved in deriving step (9), we could continue as follows:

(8.1)	AN = AP + PQ	[(4) and (8)]
(9)	AN = AQ	[(5) and (8.1)]

Note, by the way, the justification for step (7.3). The logical principle used here is the first of the three logical principles for working with conjunction sentences. [See page 6-392.]

*

One could avoid the use of the algebra theorem mentioned above and derive (9) just by using substitution inferences. This should not be surprising when you realize that the algebra theorem in question is a consequence of logical principles only. [See the proof of Exercise ☆6 on page TC[2-66]b of Unit 2.]

Here is an outline of the derivation in the Example on page 6-42, as expanded on TC[6-42]a:

$$\frac{\dfrac{(1') \qquad \dfrac{(2)}{(3')}}{(5)} \qquad\qquad \dfrac{\dfrac{(1) \qquad \dfrac{(2)}{(3)}}{(4)} \qquad\qquad \dfrac{\dfrac{(6) \qquad (7)}{(7.3)} \qquad\qquad \dfrac{(7.1)}{(7.2)}}{(8)}}{(8.1)}}{(9)}$$

Evidently, (9) is a consequence of premisses [(6) and (7)] stated in the hypothesis, additional premisses [(1), and (1'): $P \in \overleftrightarrow{AQ}$] suggested by the figure, an axiom [(2)], and an algebra theorem [(7.1)]. One might have derived (6), (7), (1), and (1') from a single premiss:

(0) $\quad M \in \overleftrightarrow{AN}$ and $P \in \overleftrightarrow{AQ}$ and $AM = AP$ and $MN = PQ$

[An outline for such a derivation would differ from the one shown above only in having a '$\underline{(0)}$' surmounting each of the symbols '(1')', '(1)', '(6)', and '(7)'.] So, (9) is a consequence of (0), an axiom, and an algebra theorem. Conditionalizing so as to discharge the premiss (0) would result in a derivation of:

(10) if $M \in \overleftrightarrow{AN}$ and $P \in \overleftrightarrow{AQ}$ and $AM = AP$ and $MN = PQ$ then $AN = AQ$

from an axiom and an algebra theorem. So, the generalization displayed on page 6-43 is a theorem.

Any hypothesis-conclusion argument can, in the way just illustrated, be enlarged to a proof of a theorem. Notice that there are two steps. The important step, if one wishes a correctly stated theorem, is making explicit the premisses which are suggested by the figure. The other step, extending the derivations at both ends to obtain a proof of the desired theorem, is purely mechanical and, once understood, can safely be omitted.

inferences have been combined in some of the proofs given above. For example, consider the proof for Exercise 2. Here are the steps which follow (7) in an expanded version:

(7.1)	$AD = CB$	[(6) and (7)]
(7.2)	$CE + EB = AD$	[(7.1) and (5)]
(8)	$AF + FD = CE + EB$	[(7.2) and (4)]
(9)	$AF = CE$	[Hypothesis]
(9.1)	$CE + FD = CE + EB$	[(9) and (8)]
(10)	$FD = EB$	[(9.1); algebra]

To show the algebra involved, we can expand it still further:

(9.2)	$\forall_x \forall_y \forall_z$ if $x + y = x + z$ then $y = z$	[algebra theorem]
(9.3)	if $CE + FD = CE + EB$ then $FD = EB$	[(9.2)]
(10)	$FD = EB$	[(9.1) and (9.3)]

It should be clear that we must permit and even encourage students to combine steps if we expect them to do several proofs in a homework assignment.

2. (1) $F \in \overleftrightarrow{AD}$ [figure]

(2) $\forall_X \forall_Y \forall_Z$ if $Y \in \overleftrightarrow{XZ}$ then $XY + YZ = XZ$ [axiom]

(3) if $F \in \overleftrightarrow{AD}$ then $AF + FD = AD$ [(2)]

(4) $AF + FD = AD$ [(1) and (3)]

(5) $CE + EB = CB$ [Steps like (1) and (3)]

(6) $AD = BC$ [Hypothesis]

(7) $BC = CB$ [Introduction]

(8) $AF + FD = CE + EB$ [(4), (5), (6), and (7)]

(9) $AF = CE$ [Hypothesis]

(10) $FD = EB$ [(8) and (9); algebra]

3. (1) $B \in \overleftrightarrow{AC}$ [figure]

(2) $\forall_X \forall_Y \forall_Z$ if $Y \in \overleftrightarrow{XZ}$ then $XY + YZ = XZ$ [axiom]

(3) if $B \in \overleftrightarrow{AC}$ then $AB + BC = AC$ [(2)]

(4) $AB + BC = AC$ [(1) and (3)]

(5) $BC + CD = BD$ [Steps like (1) and (3)]

(6) $AB = CD$ [Hypothesis]

(7) $AC = BD$ [(4), (5), and (6); algebra]

4. [You can obtain a proof for this exercise just by interchanging steps (6) and (7) in the proof for Exercise 3.]

*

When you discuss these exercises with the class, you may very well wish the students to show in detail the substitutions they made and the algebra they used in deriving various steps. Several substitution

> **Correction.** On page 6-43, line 11
> should read:
>
> --- if $Y \in \overline{XZ}$, $U \in \overline{XV}$, $XY = XU$ and
>
> In the Hypothesis for Exercise 2,
> insert a comma after '$AD = BC$'.

line 1. The missing steps are '$P \in \overline{AQ}$' and 'if $P \in \overline{AQ}$ then $AP + PQ = AQ$'.

*

The word 'hypothesis', as it is used in these geometry "originals", is synonymous with 'assumptions'.

*

As illustrated in the paragraph preceding the exercises, each original provides you with a theorem, that is, the proof of the original is really the major part of the proof of the corresponding theorem. The theorem is a conditional, and the antecedent is the conjunction of the assumptions used in the proof. These assumptions are either stated in the hypothesis or are taken from the figure. Usually, the theorem thus obtained is of so little importance in helping to prove other theorems that we do not bother to take explicit notice of it either by stating it or by giving it a number. However, there will be cases in the text where a theorem proved in one exercise could be used in solving another exercise occurring later in the list.

*

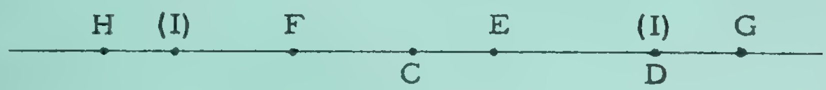
Answers for Exercises.

1.

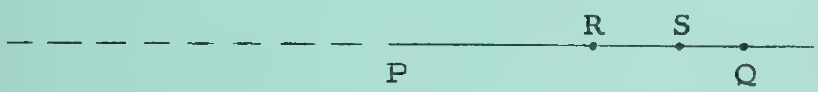
(1)	$E \in \overline{AC}$	[figure]
(2)	$\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY + YZ = XZ$	[axiom]
(3)	if $E \in \overline{AC}$ then $AE + EC = AC$	[(2)]
(4)	$AE + EC = AC$	[(1) and (3)]
(5)	$AE = EC$	[Hypothesis]
(6)	$AE = \frac{1}{2} \cdot AC$	[(4) and (5); algebra]
(7)	$ED = \frac{1}{2} \cdot BD$	[Steps like (1), (3), (4), and (5)]
(8)	$AC = BD$	[Hypothesis]
(9)	$AE = ED$	[(6), (7), and (8); algebra]

Answers for Exploration Exercises [on page 6-44].

A. 1-5.



B. 1, 3.



2. No; no; yes

4. No; no; yes

C. 1, 3.

2. No; no; yes

4. Yes; no; no

*

Answers for Exploration Exercises [on page 6-46].

1. (a) Yes; Theorem 1-2.

(b) Yes; just one. Since $k > 0$, it follows that $\frac{k}{2} > 0$. Also, $A \neq B$. So, Axiom C tells us there is one and only one such point P.

(c) By Theorem 1-5, since $P \in \overrightarrow{AB}$ and $AP < AB$, it follows that $P \in \overline{AB}$.

(d) Yes. [Since $P \in \overleftrightarrow{AB}$, $AP + PB = AB$. Since $AP = \frac{k}{2}$ and $AB = k$, it follows that $PB = \frac{k}{2}$. So, $AP = PB$.]

2. (a) No, $k = 0$. Theorem 1-1.

(b) Yes, the point A.

Using Axiom C and Introduction Axioms one can show that, given two points A and B, there is a one-to-one mapping of the number line onto $\overleftrightarrow{AB}$. That is, it is possible to associate each point of $\overleftrightarrow{AB}$ with a single corresponding real number in such a way that each real number is associated with a unique point. In fact, by Axioms 11 and 9, there is a point C such that $A \in \overline{CB}$; and, by Introduction Theorem 14, $\overleftrightarrow{AB} = \overrightarrow{AC} \cup \{A\} \cup \overrightarrow{AB}$. The desired mapping is then obtained by associating the real number 0 with the point A, and, for each nonzero number x of arithmetic associating the real number ^{+}x with the point $Z \in \overrightarrow{AB}$ such that $AZ = x$ and the real number ^{-}x with the point $Z \in \overrightarrow{AC}$ such that $AZ = x$. [Axiom C tells us that, for each such x, the points in question are unique.] For this correspondence, it is not difficult to prove that the distance between any two points of $\overleftrightarrow{AB}$ is the absolute value of the difference of the real numbers associated with them. [However, the proof is tedious, and we shall not give it here.]

Such a correspondence between the points of a line ℓ and the real numbers is called <u>a coordinate system on ℓ</u>. With respect to a given coordinate system on ℓ, the number associated with any point of ℓ is called <u>the coordinate</u> of that point, and the point of ℓ whose coordinate is 0 is called <u>the origin</u> of the coordinate system. From the discussion above, it is clear that, given two points of a line, there is a coordinate system whose origin is the first of the two points and which is such that the coordinate of the second point is positive.

*

It would be possible, without loss, to omit the words 'and only one' from Axiom C. For it follows from Introduction Axioms and Theorems 1-5 and 1-6 that, if $A \neq B$ then there cannot be two points, C and D, of $\overrightarrow{AB}$ such that $AC = AD$. In fact, by Introduction Theorem 12, if $C \in \overrightarrow{AB}$ then $\overrightarrow{AC} = \overrightarrow{AB}$. So, if $D \in \overrightarrow{AB}$ then $D \in \overrightarrow{AC}$. Hence, by Introduction Theorem 8, $D \in \overline{AC}$ or $D = C$ or $C \in \overline{AD}$. But, by Theorem 1-5, if $D \in \overline{AC}$ then $AD < AC$ and, by Theorem 1-6, if $C \in \overline{AD}$ then $AD > AC$. The assumption that $AC = AD$ contradicts both of these alternatives. So, if $C \in \overrightarrow{AB}$, $D \in \overrightarrow{AB}$, and $AC = AD$, then $D = C$.

[illegible]

page 6-359], eliminate the word 'midpoint', from any context, in favor of the more elementary notions of segment and measure. It turns out to be somewhat simpler to adopt a slightly different record of our agreement:

(3) $\forall_X \forall_Y \forall_Z$ [Y is the midpoint of $\overleftrightarrow{XZ}$ if and only if
[$Y \in \overleftrightarrow{XZ}$ and $XY = YZ$]]

Using (3) we can, by virtue of the substitution rule for biconditional sentences [see page 6-391 and accompanying COMMENTARY], replace such sentences as 'B is the midpoint of $\overleftrightarrow{AC}$' by more basic ones--in this case, by '$B \in \overleftrightarrow{AC}$ and $AB = BC$'.

Notice that the only-if-part of (3) says just that the midpoint of a segment belongs to the segment and is equidistant from its end points while the if-part of (3) says that there is no other such point. Hence, the content of (3) is that there is one and only one point of a segment which is equidistant from the segment's end points--to wit, the midpoint of the segment. So, (3) turns out to be only a restatement of (1), in terms of the word 'midpoint'. Consequently, it is not unreasonable to accept (3) --that is, Theorem 1-7--as a surrogate for (1), and, at the same time, to call it 'the definition of midpoint'.

In general, once we have proved an existence and uniqueness theorem [(1)], we may then introduce an appropriate definite description ['the. . . '], and substitute for the theorem a restatement [(3)] in terms of the definite description. This restatement is, at the same time, a theorem and a definition.

The subject of definition will be discussed further in later parts of this COMMENTARY.

The sentence 'Either $A \neq B$ or $A = B$' is valid [see TC[6-395]d] and, so, can be accepted as a premiss of any argument, without cost. The reason for this is that, as shown on TC[6-394]a, it can be derived from two assumptions, '$A \neq B$' and '$A = B$', both of which are discharged during the derivation. Hence, if it is itself used as an assumption, it can be thought of as being discharged as soon as it is written down.

*

On definitions.--The great number of geometrical concepts, and the complexity of most of them, makes a consistent formal treatment of definitions impractical--at least in a beginning course. Consequently, we limit ourselves to occasional illustrations of formal procedures for the introduction of new terms and, for the most part, introduce such terms in informal discussions. The treatment of 'midpoint' on page 6-47 is an illustration of how a definite description--roughly, a phrase beginning with 'the...'--can be formally "defined".

The procedure begins by proving a theorem:

(1) $\forall_X \forall_Z$ there is one and only one point Y such that $Y \in \overleftrightarrow{XZ}$ and $XY = YZ$

Since this is a theorem, one may speak of the point of a given segment which is equidistant from the end points of the segment and, for brevity, call this point the midpoint of the given segment. This amounts to agreeing that, for example, the phrases 'the midpoint of $\overleftrightarrow{AC}$' and 'the point Y such that $Y \in \overleftrightarrow{AC}$ and $AY = YC$' are to be considered equivalent geometric expressions. [The meaning of 'equivalent' here is completely analogous to its meaning in discussions of equivalent algebraic expressions.] We could record this agreement by writing:

(2) $\forall_X \forall_Z$ the midpoint of $\overleftrightarrow{XZ}$ = the point Y such that $[Y \in \overleftrightarrow{XZ}$ and $XY = YZ]$

Using (2) we could, by virtue of the substitution rule for equations [see

In order to shorten column proofs we shall not introduce definitions as steps in such proofs. Instead, as illustrated in the Example on page 6-48, we shall pass directly from a step containing a defined term to its defining sentence or sentences. And, as is illustrated by the passage from steps (4) and (16) to step (17), we shall also reverse this process. In each case, the marginal comments can be made sufficiently explicit to clarify what is going on. In the case of the example, this procedure saves nine steps. Without it, the example would be supplemented as indicated below.

(0)	$\forall_X \forall_Y \forall_Z$ Y is the midpoint of $\overleftrightarrow{XZ}$ if and only if $Y \in \overleftrightarrow{XZ}$ and $XY = YZ$	[def. of midpoint]
(0.1)	B is the midpoint of $\overleftrightarrow{AC}$ if and only if $B \in \overleftrightarrow{AC}$ and $AB = BC$	[(0)]
(0.2)	$B \in \overleftrightarrow{AC}$ and $AB = BC$	[Hypothesis and (0.1)]
(0.3)	C is the midpoint of $\overleftrightarrow{BD}$ if and only if $C \in \overleftrightarrow{BD}$ and $BC = CD$	[(0)]
(0.4)	$C \in \overleftrightarrow{BD}$ and $BC = CD$	[Hypothesis and (0.3)]
(0.5)	D is the midpoint of $\overleftrightarrow{CE}$ if and only if $D \in \overleftrightarrow{CE}$ and $CD = DE$	[(0)]
(0.6)	$D \in \overleftrightarrow{CE}$ and $CD = DE$	[Hypothesis and (0.5)]
(1)	$B \in \overleftrightarrow{AC}$	[(0.2)]
⋮	[steps (2), (3), and (4)]	⋮
(5)	$AB = BC$	[(0.2)]
⋮	[steps (6) - (16)]	⋮
(16.1)	$C \in \overleftrightarrow{AE}$ and $AC = CE$	[(4) and (16)]
(16.2)	C is the midpoint of $\overleftrightarrow{AE}$ if and only if $C \in \overleftrightarrow{AE}$ and $AC = CE$	[(0)]
(17)	C is the midpoint of $\overleftrightarrow{AE}$	[(16.1) and (16.2)]

Answer for Part ☆C [on pages 6-49 and 6-50].

The problem posed here is the following. Someone tells you that he has marked a point P on the line containing A and B. He measures $\overset{\bullet\!-\!\bullet}{AP}$ and $\overset{\bullet\!-\!\bullet}{PB}$, and reports that AP = PB. With this information, you can conclude that P is the midpoint of $\overset{\bullet\!-\!\bullet}{AB}$, because just knowing that AP = PB and that $P \in \overleftrightarrow{AB}$ is enough to tell you that $P \in \overset{\bullet\!-\!\bullet}{AB}$. Here's why. Suppose that $P \in \overleftrightarrow{AB}$ and that AP = PB. By an Introduction Theorem, it follows that (1) $B \in \overline{PA}$ or (2) $A \in \overline{PB}$ or (3) $P \in \overset{\bullet\!-\!\bullet}{AB}$. If $B \in \overline{PA}$ then [by Theorem 1-3 and some algebra] $PB \neq PA$. Since AP = PB, that is, since PB = PA, it follows [using modus tollens] that $B \notin \overline{PA}$. Similarly, if $A \in \overline{PB}$ then $PA \neq PB$. So, $A \notin \overline{PB}$. Therefore, if $P \in \overleftrightarrow{AB}$ and AP = PB then $P \in \overset{\bullet\!-\!\bullet}{AB}$. [See the discussion on page 6-394 concerning the rule for denying an alternative, and the discussion on pages 6-400 and 6-401.]

Theorem 1-9 is used on page 6-92 in proving that each point equidistant from A and B belongs to the line perpendicular to $\overset{\bullet\!-\!\bullet}{AB}$ at its midpoint. So, even though Part C is starred, all students should know what Theorem 1-9 is about.

Here is a concise translation of Theorem 1-9:

$$\forall_X \forall_Y \forall_Z \text{ if } X \neq Y,\ XZ = ZY, \text{ and } Z \in \overleftrightarrow{XY} \text{ then } Z \text{ is the midpoint of } \overset{\bullet\!-\!\bullet}{XY}$$

*

Students will need protractors for the work starting on page 6-51. Now might be a good time to alert them about bringing protractors to class. You might find it useful to keep a supply of protractors on hand. And, of course, it is very helpful to have a large protractor for blackboard use. [Also, in view of the exploration exercises starting on page 6-297, it might be a good idea to have a few circular protractors in your collection. However, don't use them until students are convinced that there are no such things as reflex angles in our geometry.]

Students will appreciate the concise language of the theorems and axioms preceding Theorem 1-8 after they have had experience writing a step like (2). Some students may ask if a briefer statement can be used such as:

$$\forall_X \forall_Y \forall_Z \text{ if } Y \text{ is the midpoint of } \overleftrightarrow{XZ} \text{ then } YX = \tfrac{1}{2} \cdot XZ = YZ$$

Of course, your answer should be 'yes'. The displayed translation of Theorem 1-8 makes it easier to form an instance of it, something the student has to do in writing step (3). Students may need a bit of help in recognizing that Theorem 1-8 is actually a translation of a conditional sentence. This problem must be faced eventually [see page 6-60]; this might be a good time to do it. Of course, the students could avoid Theorem 1-8 altogether by giving a different proof [see the proof of Exercise 1 on TC[6-43]a]. In that case, you should develop the proof shown above to exemplify the labor-saving aspects of using theorems already proved rather than going back to the axioms.

Paragraph proof of Exercise 2:

Since M is the midpoint of $\overleftrightarrow{AB}$ and N is the midpoint of $\overleftrightarrow{AC}$, it follows from an earlier theorem (1) that $AM = \frac{1}{2} \cdot AB$ and $AN = \frac{1}{2} \cdot AC$. But, by hypothesis, $AB = AC$. So, $AM = AN$.

---

(1) The distance between the midpoint of a segment and either end point of the segment is half the distance between the end points.

*

Answers for Part B [on page 6-49].

1. No. 

M must also belong to $\overleftrightarrow{AB}$.

2. (a) 5 (b) 1 (c) 1; 5

3. (a) 3.5 (b) 2.5 (c) 0.75

*

Answers for Part A.

1.

(1)	B is the midpoint of $\overline{AC}$	[Hypothesis]
(2)	AB = BC	[(1); def. of midpoint]
(3)	AB = CD	[Hypothesis]
(4)	BC = CD	[(2) and (3)]
(5)	$C \in \overline{BD}$	[figure]
(6)	C is the midpoint of $\overline{BD}$	[(4) and (5); def. of midpoint]

[Note the need for step (5). Ask students to draw a figure meeting the conditions in the hypothesis but such that C is not the midpoint of $\overline{BD}$.]

Paragraph proof of Exercise 1:

By hypothesis, B is the midpoint of $\overline{AC}$. So, by definition, AB = BC. But, we are given that AB = CD. So, BC = CD. Also, from the figure, we see that $C \in \overline{BD}$. Hence, by definition, C is the midpoint of $\overline{BD}$.

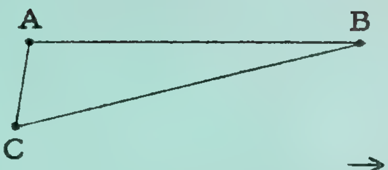
2.

(1)	M is the midpoint of $\overline{AB}$	[Hypothesis]
(2)	The distance between the midpoint of a segment and either end point of the segment is half the distance between the end points.	[theorem]
(3)	if M is the midpoint of $\overline{AB}$ then $AM = \frac{1}{2} \cdot AB$	[(2)]
(4)	$AM = \frac{1}{2} \cdot AB$	[(1) and (3)]
(5)	$AN = \frac{1}{2} \cdot AC$	[Steps like (1) and (3)]
(6)	AB = AC	[Hypothesis]
(7)	AM = AN	[(4), (5), and (6)]

[Ask students to draw a figure for which A, B, C are collinear, which meets the conditions in the hypothesis, and for which the conclusion holds.]

Quiz.

1. State the axiom that tells you in the situation pictured on the right that $AB + BC > AC$.

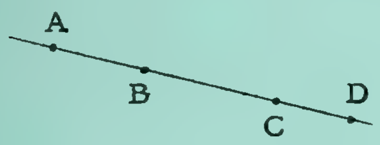


2. Suppose that P, Q, and R are collinear points and that $R \in \overrightarrow{PQ}$. If $PQ = 7$ and $PR = 7.0001$, it follows that $Q \in \overline{PR}$. State the theorem which justifies this.

3. Suppose that $B \in \overline{AC}$ and that $AB = x \cdot BC$. If M is the midpoint of $\overleftrightarrow{AC}$ and $M \in \overline{AB}$ then ______(?)______ .

(A) $0 < x < \frac{1}{2}$ (B) $\frac{1}{2} < x < 1$ (C) $1 < x$

4. Fill in the blanks in the following proof.



Hypothesis: $B \in \overline{AC}$, $C \in \overline{BD}$

Conclusion: $BC < AD$

(1)	$C \in \overline{BD}$	[__________]
(2)	__________________	[theorem]
(3)	__________________	[(2)]
(4)	$BC < BD$	[(1) and (3)]
(5)	$B \in \overline{AC}$	[__________]
(6)	$B \in \overline{DA}$	[(1) and (5); __________]
(7)	__________________	[(2)]
(8)	$DB < DA$	[__________]
(9)	$DB = BD$	[__________]

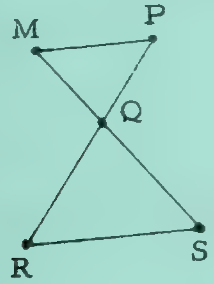
(10) ______________________ [(8) and (9)]

(11) ______________________ [(4) and (10); ____________]

(12) DA = AD [____________]

(13) BC < AD [____________]

5.



Hypothesis: MQ = PQ,
QS = QR

Conclusion: MS = PR

*

Answers for quiz.

1. $\forall_X \forall_Y \forall_Z$ if $Y \notin \overleftrightarrow{XZ}$ then $XY + YZ > XZ$ [Axiom B]

2. $\forall_X \forall_Y \forall_Z$ if $Z \in \overrightarrow{XY}$ then $Y \in \overline{XZ}$ if and only if $XZ > XY$ [Th. 1-6]

[Of course, alphabetic variants of these generalizations should be given full credit. Also, a student should receive full credit if he writes as his answer for item 2:

$$\forall_X \forall_Y \forall_Z \text{ if } Z \in \overrightarrow{XY} \text{ then } Y \in \overline{XZ} \text{ if } XZ > XY$$

or:

$$\forall_X \forall_Y \forall_Z \text{ if } Z \in \overrightarrow{XY} \text{ and } XZ > XY \text{ then } Y \in \overline{XZ}$$

Each of these alternative answers is a logical consequence of just Theorem 1-6. Students should not be expected to memorize the exact wording of the axioms and theorems in the text. What is expected is that they be able to recognize theorems which have been proved and logical consequences of these theorems.]

3. (C) [Since $M \in \overline{AB}$, it follows that $AB > BC$. So, $AB \div BC > 1$.]

4. (1) $C \in \overline{BD}$ [Hypothesis]

(2) $\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY < XZ$ [theorem]

(3) if $C \in \overline{BD}$ then $BC < BD$ [(2)]

(4) $BC < BD$ [(1) and (3)]

(5) $B \in \overline{AC}$ [Hypothesis]

(6) $B \in \overline{DA}$ [(1) and (5); Introduction]

(7) if $B \in \overline{DA}$ then $DB < DA$ [(2)]

(8) $DB < DA$ [(6) and (7)]

(9) $DB = BD$ [Introduction]

(10) $BD < DA$ [(8) and (9)]

(11) $BC < DA$ [(4) and (10); algebra]

(12) $DA = AD$ [Introduction]

(13) $BC < AD$ [(11) and (12)]

5. (1) $MQ = PQ$ [Hypothesis]

(2) $QS = QR$ [Hypothesis]

(3) $MQ + QS = PQ + QR$ [(1) and (2); algebra]

(4) $Q \in \overleftrightarrow{MS}$ [figure]

(5) $\forall_X \forall_Y \forall_Z$ if $Y \in \overleftrightarrow{XZ}$ then $XY + YZ = XZ$ [axiom]

(6) if $Q \in \overleftrightarrow{MS}$ then $MQ + QS = MS$ [(5)]

(7) $MQ + QS = MS$ [(4) and (6)]

(8) $PQ + QR = PR$ [Steps like (4) and (6)]

(9) $MS = PR$ [(3), (7), and 8)]

<u>Paragraph proof for item 5:</u>

By hypothesis, $MQ = PQ$ and $QS = QR$. So, $MQ + QS = PQ + QR$. From the figure, $Q \in \overleftrightarrow{MS}$ and $Q \in \overleftrightarrow{PR}$. Hence, by an axiom (1), $MQ + QS = MS$ and $PQ + QR = PR$. So, $MS = PR$.

---

(1) $\forall_X \forall_Y \forall_Z$ if $Y \in \overleftrightarrow{XZ}$ then $XY + YZ = XZ$

using such formality. Consequently, as in the case of 'an angle' we shall introduce indefinite descriptions quite informally. You should probably point out to students that the definition in the text:

> An angle <u>is</u> the union of two noncollinear rays which have the same vertex.

is short for:

> For each set s, s is an angle <u>if and only if</u> s is the union of two noncollinear rays which have the same vertex.

Consequently [only if-part], each angle is the union of two noncollinear rays which have the same vertex. And [if-part], each union of two noncollinear rays which have the same vertex is an angle.

More on definitions. --In the COMMENTARY for 6-47 we have discussed definitions which introduce definite descriptions. A phrase such as 'an angle' is an indefinite description. Such a phrase can be introduced by a defining postulate like:

> For each set s, s is an angle if and only if there exist three noncollinear points X, Y, and Z such that s = $\overrightarrow{YX} \cup \overrightarrow{YZ}$.

This, by virtue of the substitution rule for biconditional sentences, paves the way for eliminating a phrase such as 's is an angle' in favor of the more "primitive" phrase 'there exist three noncollinear points, X, Y, and Z such that s = $\overrightarrow{YX} \cup \overrightarrow{YZ}$'.

To make convenient use of such a definition, one needs to use quantifying phrases like 'for each angle p', as in:

> For each angle p, for each Y, Y is the vertex of p if and only if there exist points X and Z such that p = $\overrightarrow{YX} \cup \overrightarrow{YZ}$.

For technical reasons, the introduction of such a quantifying phrase requires a preliminary theorem, which, in this case, turns out to be Introduction Axiom 3:

> There are [at least] three noncollinear points.

Briefly, just as, when introducing a definite description, one must first establish the existence and uniqueness of the object which is described, so, in order to introduce variables ['p'] whose domain is a set of objects covered by an indefinite description, one must first establish the existence of such objects. Failure to do so may introduce inconsistency into a previously consistent system.

Obviously, in a beginning course there is no time for developing and

answer is 'no'. Even though, for each X, $\emptyset = \overleftrightarrow{XX}$, the empty set is not a line; neither is a unit set a ray. [Of course, even if we did decide to call unit sets 'rays', this would not modify the concept of angle. For there would not exist two noncollinear rays with a common vertex one of which is a unit set.]

*

In order to completely justify speaking of <u>the</u> sides and <u>the</u> vertex of an angle it would be necessary [see COMMENTARY for page 6-47] to prove that if a set is an angle then there is just one couple of rays whose union is the set, and that if a set is a ray then there is just one of its points which is its vertex [that is, if $\overrightarrow{AB} = \overrightarrow{CD}$ then A = C]. These theorems can be derived from the Introduction Axioms, using, of course, in the case of the first, the definition of angle. However, attention to such theorems would require more time than is available for a beginning course in geometry.

*

Note that, in introducing notations such as '$\angle$EFG', we offer no interpretation for cases in which the points referred to are collinear. So, as in the case of "division by 0" [see TC[6-14]a], we should, strictly, take care to guard against meaningless expressions. We do, in fact, do so when stating Axioms D, E, F, and G on pages 6-54 and 6-56. The restricted quantifiers in these statements preclude the possibility of these axioms having instances in which symbols which refer, ostensibly, to angles are, actually, meaningless. However [as noted on lines 7 and 6 from the foot of page 6-51], we shall not always be so careful. As has been mentioned previously, geometrical notation is too complex to allow for a completely formal beginning treatment.

*

Notice that Theorem 12 on page 6-27 tells us that if $V \in \overrightarrow{PK}$ and $W \in \overrightarrow{PJ}$ then $\overrightarrow{PV} = \overrightarrow{PK}$ and $\overrightarrow{PW} = \overrightarrow{PJ}$. Hence, it follows that $\angle VPW = \angle KPJ$.

*

The definition of an angle as the union of two noncollinear rays with the same vertex is in accord with our stipulation that geometric figures be sets of points. One of the alternative definitions which this stipulation excludes is the one according to which an angle is a pair of rays with a common vertex--that is, a set with two members, each of which is a ray and both of which have the same vertex. There is, of course, nothing "wrong" with this latter definition of angle. Our preference for the former definition is due, in part, to an aesthetic bias toward having lines, angles, triangles, etc. be the same sort of thing, and, in part, to the fact that the habit of thinking of geometric figures as sets of points is good preparation for later work in mathematics. Furthermore, this approach gives students needed practice in thinking in terms of sets and operations on sets.

*

Notice that, since an angle is the union of two noncollinear rays, there are, in this treatment, no "straight angles". One reason for this exclusion is pointed out on pages 6-56 and 6-57. A straight angle would, in a treatment such as ours, be merely a straight line, and would not have a unique vertex, a unique interior, or a unique bisector.

Also, this text does not recognize "reflex angles". We could do so by defining an angle as the union of three sets--two of them being noncollinear rays with a common vertex, and the third being either the interior or the exterior [see page 6-55] of the union of the two rays. In this case, two noncollinear rays with a common vertex would determine two angles, one of which could be called 'a reflex angle'. In consequence, one could not properly speak of the angle whose sides are given rays, $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Instead, one would be forced to speak of the reflex angle with these sides and of the nonreflex angle with these sides. Presumably, the notation '$\angle ABC$' would be used in referring to the latter, and some new notation would be devised for the former. Concepts based on the notion of angle would either have to be revised so as to apply to angles of both kinds, or restricted to nonreflex angles. The first course would introduce additional complexities, on top of the already more complicated notion of angle, and both courses would result in a rash of 'nonreflex's in the statements of definitions and theorems. Clearly, the cost of introducing reflex angles is much too great in comparison with the small advantage which might be gained.

*

If, as seems unlikely, a student brings up the point that, for each point X, $\{X\} = \overrightarrow{XX}$, and asks whether a set consisting of a single point is a ray, the

Angle-measure is, like segment-measure, one of our primitive concepts. Just as in the latter case the concept is developed by leading students to experiment with rulers, so, here, the concept of angle-measure is developed through experiments with protractors.

*

There are many systems of linear measure [inch-measure, yard-measure, etc.] Similarly, there are many systems of angular measure. For simplicity, we concentrate on degree-measure. [The subsidiary units, 1 minute, and 1 second, are introduced in Exercise 2 on page 6-409.]

*

Note that just as the inch-measure of a segment 2 inches long is the number 2, so the degree-measure of an angle 110 degrees "large" is the number 110.

Correction. On page 6-53, in Exercise 2 of Part B, delete the period after '60°'.

Answers for Part A [which begins on page 6-52].

1. $m(\angle B) = 60$; $m(\angle ABC) = 60$; $m(\angle CBA) = 60$; $\angle ABC$ is an angle of 60°; $m(\overrightarrow{BC} \cup \overrightarrow{BA}) = 60$; $m(\overrightarrow{BA} \cup \overrightarrow{BC}) = 60$.

2. 130 3. 90 4. 95

*

Answers for Part B.

3. $m(\angle BAC) = 115$ [Students who recall the appropriate theorem from an earlier course may determine this measure without using a protractor. This is permissible for this kind of exercise. If they mention the theorem at this time, just say that it will be included in our geometry later in the course. See page 6-148.]

4. $m(\angle QOR) = 55$ 5. $m(\angle QOR) = 5$ 6. $m(h \cup \overrightarrow{BA}) = 161$

*

Answers for Part C [on page 6-54].

1. 27 2. 145 3. 35; 37; 72 4. 55; 125; 55; 125

*

Answer for Part E [on page 6-54].

There is only one half-line $h \subseteq s_1$ such that $m(h \cup \overrightarrow{AB}) = 25$.

*

Axiom D tells us that the range of the degree-measure function for angles is a subset of the set of all nonzero numbers of arithmetic between 0 and 180. In view of the definition of angle, this tells us that there is no angle with degree-measure, say, 230. So, there are no "reflex angles" in our geometry.

Notice, however, that Axiom D does not tell us that there exists, say, an angle of 70°. Axiom E tells us this among other things.

Note the similarity between Axiom E and Axiom C. Also, compare Axiom F with Axiom A.

$$\frac{\overline{AB} \cong \overline{CD} \qquad\qquad \overline{AB} \cong \overline{EF}}{\overline{CD} \cong \overline{EF}}$$

is valid because it is an abbreviation for:

$$\frac{m(\overline{AB}) = m(\overline{CD}) \qquad\qquad m(\overline{AB}) = m(\overline{EF})}{m(\overline{CD}) = m(\overline{EF})},$$

and the latter is valid by virtue of the substitution rule for equations. Similarly, the inference:

$$\frac{\overline{AB} \cong \overline{CD} \qquad\qquad m(\overline{AB}) = 2 \cdot m(\overline{EF})}{m(\overline{CD}) = 2 \cdot m(\overline{EF})}$$

is valid [roughly, one can substitute from segment-congruence into segment-measure contexts], but, as is seen by unabbreviating, the inference:

$$\frac{\overline{AB} \cong \overline{CD} \qquad\qquad P = \text{the midpoint of } \overline{AB}}{P = \text{the midpoint of } \overline{CD}}$$

is not valid. [Note that failure to distinguish notationally between identity and congruence would make it difficult to explain why this last inference is invalid.]

In addition to the general consequences of reflexivity and a restricted rule of substitution [see Exercise 1 on page 6-408], particular equivalence relations enjoy special properties. For example, the statement '$\forall_X \forall_Y\ \overline{XY} \cong \overline{YX}$' is an abbreviation for '$\forall_X \forall_Y\ m(\overline{XY}) = m(\overline{YX})$, which is an easy consequence of the Introduction Theorem '$\forall_X \forall_Y\ \overline{XY} = \overline{YX}$'.

Note that, in this development of geometry, '$\overline{AB} \cong \overline{CD}$' is merely another abbreviation for '$m(\overline{AB}) = m(\overline{CD})$'. Similarly, '$\angle ABC \cong \angle PQR$' is just an abbreviation for '$m(\angle ABC) = m(\angle PQR)$'. And, on page 6-83, we introduce '$\Delta ABC \cong \Delta DEF$' as an abbreviation for the much longer sentence 'There is a matching of the vertices of ΔABC with those of ΔDEF such that all corresponding parts are congruent.'.

Each of these notions of congruence corresponds to a way of classifying objects [segments, angles, or triangles] into sets of objects which share a common property. Congruence of segments [or: of angles, or: of triangles] is an example of what is technically called an equivalence relation. [$\overline{AB} \cong \overline{CD}$ if and only if $\overline{AB}$ is "equivalent in length" with $\overline{CD}$.] Another equivalence relation is identity, =, which classifies objects in a trivial way into unit sets. [$\overline{AB} = \overline{CD}$ if and only if $\overline{AB}$ is "equivalent in identity" with $\overline{CD}$--that is, if and only if $\overline{AB}$ is $\overline{CD}$.] As illustrated by identity and congruence for segments, objects of the same kind may, for different purposes, be classified in different ways. For example, four common ways of classifying triangles are in terms of identity, congruence, similarity, and equivalence-in-area. Considerable confusion can be introduced if one fails to distinguish notationally among different equivalence relations. It is for this reason that we insist on using '=' and 'equals' only when referring to the relation of identity. Thus, for example, we distinguish sharply between identity of segments and congruence of segments.

For each equivalence relation, there is an analogue of the principle of identity [technically, each equivalence relation is reflexive] and, for each equivalence relation, there is some, more or less restricted, rule of substitution. For example, '$\forall_X \forall_Y\ \overline{XY} \cong \overline{XY}$' is an abbreviation for '$\forall_X \forall_Y\ m(\overline{XY}) = m(\overline{XY})$' and, so, is a consequence of the principle of identity. Also, the inference:

Answers for Part A.

1. Students are supposed to do this exercise by measuring $\angle A$ and then using their protractors to locate a point in $\overrightarrow{DF}$.

2. By Axiom D, $m(\angle A)$ is a number between 0 and 180. So, Axiom E can now be called into play. It is the axiom which tells you that the required half-line exists.

3. Axiom E.

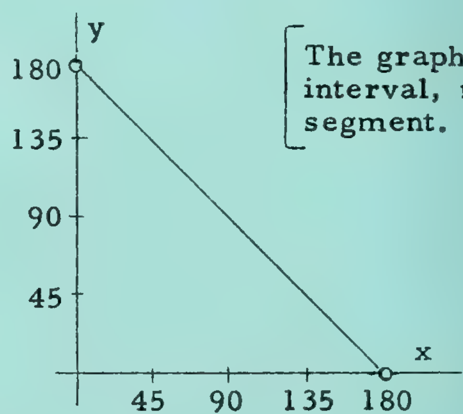
*

Answers for Part B [on pages 6-57 and 6-58].

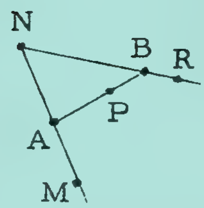
1. $m(\angle PBA) = 40$

2. $m(\overrightarrow{RQ} \cup \overleftrightarrow{RM}) = 70$

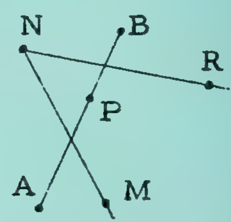
3. (a) 165 (b) 15

4. 

[The graph is an interval, not a segment.]

5. (a) No 

(b) Yes

(c) No 

6. (a) x (b) 360 - x

7. [$\angle DCA$ is an angle of 120°.]

*

Answers for Part C [on page 6-58].

1. 130

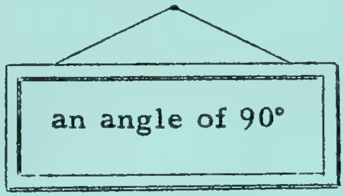
2. 43

3. 133

4. 54

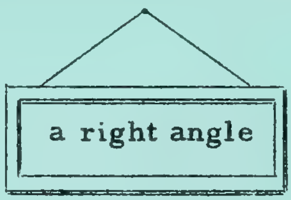
are the corresponding degree-measures. In working with this set of ordered pairs, we become aware of a certain one of its subsets. This subset consists of all the ordered pairs with second component 90. Since the domain of this subset is of interest to us, we decide to name it. The label we use is:

(2)



Let us suppose that we have not yet made the discovery that we are dealing in these two cases with the very same set. In the first case, suppose we decide to shorten the label to:

(3)



just because it is easier to use the shorter label. The act of attaching this label to the set amounts to defining the common noun 'right angle'. The meaning or referent of 'right angle' is the set to which the label is attached. This is the action we took when, on page 6-59, we defined 'a right angle'.

Now, after some thought, we discover that labels (1) and (2) are really attached to the same set. This means that the angles we have been calling 'right angles' are precisely the same things we have been calling 'angles of 90°'. And, this is what Theorem 2-1 tells us.

It is conceivable that we might have decided to use the shorter label (3) in place of (2). This act would have given us a different definition for the common noun 'right angle'.

The intuitive feeling students should develop for a pair of supplementary angles is that you can place the angles in such a position that a side of one coincides with a side of the other and the other sides form a straight line.

*

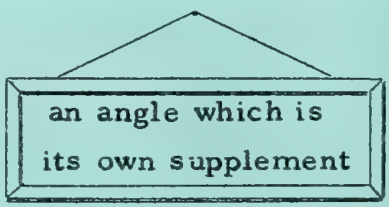
The predicate 'is a supplement of' denotes a relation among angles. The relation is a symmetric one. That is, for each $\angle X$, for each $\angle Y$, if $\angle X$ is a supplement of $\angle Y$ then $\angle Y$ is a supplement of $\angle X$. [This is a consequence of the definition and the commutative principle for addition.] Because the relation is symmetric, it makes sense to say that two angles are supplementary.

The relation of being a supplement of is not reflexive. That is, it is not the case that for each $\angle X$, $\angle X$ is the supplement of $\angle X$. However, the solution set of '$\angle X$ is the supplement of $\angle X$' is of special interest. In fact, we give a special name to this set: the set of all right angles. In view of the definition of supplementary angles, it follows that each such angle is an angle of 90°. [This is Theorem 2-1.] And, so, in view of the definition of congruent angles, it turns out all the angles in this set are congruent to each other. [This is Theorem 2-2.]

*

The discussion following the column proof of Theorem 2-1 on page 6-60 deals with the problem of assigning names to abstract entities. In working with the set of ordered pairs of angles which belong to the relation of being a supplement of, we become aware of a certain subset of this relation. This subset is the set of all such ordered pairs of angles with equal components. Since the domain of this subset is of special interest to us, we hang a label on it:

(1)



Now, let's direct our attention to another set of ordered pairs, this time the degree-measure function. This set consists of all the ordered pairs whose first components are angles and whose second components

Correction. On page 6-61, line 11, change 'steps' to 'Steps'.

You may wish to assign the exercises on page 6-408 before you get to page 6-62. The exercises provide practice with the concepts of right angle and congruent angles, and they foreshadow the work with complementary angles and acute and obtuse angles.

*

Marginal comments for the column proof on page 6-61.

(1) [assumption]* (3) [(2)] (4) [(3)]

(6) [(4) and (5)] (7) [Steps like (3), (4), and (5)]

(10) [(9); *(1)] (11) [(1)-(10)]

*

Note the justification for step (5). Step (5) follows from the conjunction sentence (1) by virtue of the second logical principle for conjunction sentences. [See page 6-392.]

*

Answers for question in the text on page 6-62.

line 7b. $\angle M$ has an infinite number of complements.

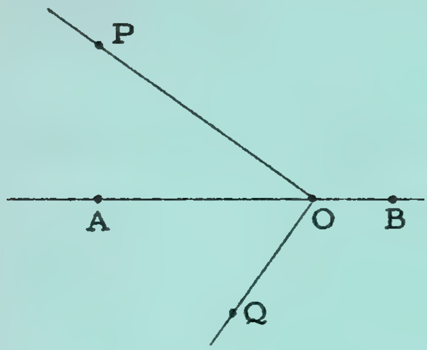
line 6b. The measure of each complement of $\angle M$ is 70.

line 5b. Although each angle has a supplement, it is not the case that each angle has a complement. An angle whose measure is not less than 90 does not have a complement.

line 2b. Suppose $\angle A$ is acute. Then, $m(\angle A) < 90$. So, $90 - m(\angle A) > 0$. Since, by Axiom D, $m(\angle A) > 0$, $90 - m(\angle A) < 180$. So, by Axiom E, there exists an angle, $\angle B$, such that $m(\angle B) = 90 - m(\angle A)$. Hence, $\angle B$ is a complement of $\angle A$. On the other hand, suppose $\angle B$ is a complement of $\angle A$. Then, $m(\angle A) = 90 - m(\angle B)$. But, by Axiom D, $m(\angle B) > 0$. So, $m(\angle A) < 90$. Hence, $\angle A$ is acute.

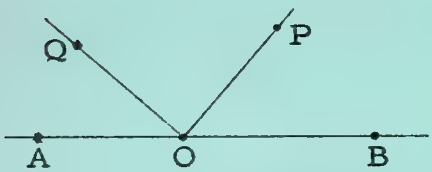
8. 45

9.



$\angle POQ$ is a right angle;
$\angle AOP$ and $\angle AOQ$ are complementary;
$\angle BOQ$ and $\angle QOA$ are supplementary;
$\angle BOP$ and $\angle POA$ are supplementary.

10.



$\angle POQ$ is a right angle;
$\angle POB$ and $\angle QOA$ are complementary;
$\angle POB$ and $\angle POA$ are supplementary;
$\angle QOA$ and $\angle QOB$ are supplementary.

11. From the figure we assume that P is in the interior of $\angle AOQ$, that Q is in the interior of $\angle POR$, and that R is in the interior of $\angle QOB$. Then, since $O \in \overline{AB}$, it follows from Axioms F and G that

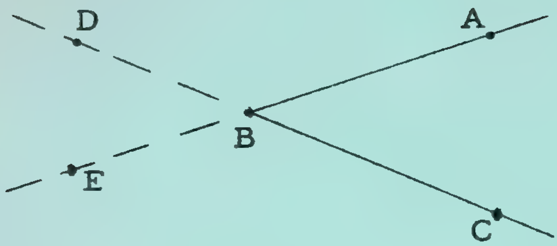
$$m(\angle AOP) + m(\angle POQ) + m(\angle QOR) + m(\angle ROB) = 180.$$

But, we are assuming that $\angle POQ \cong \angle AOP$ and that $\angle QOR \cong \angle BOR$. So, $m(\angle POQ) + m(\angle QOR) = \frac{1}{2} \cdot 180 = 90$. Hence, by Axiom F, $m(\angle POR) = 90$.

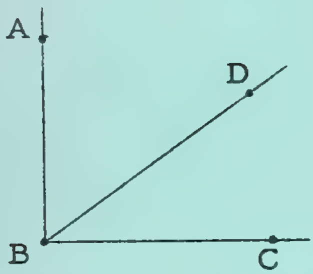
12. Suppose the angle is an angle of $x°$. Then, $x = 8(180 - x)$. So, $x = 160$.

> Correction. On page 6-63, the last part of line 13 should read:
>
> --- measures of a supplement and a com-

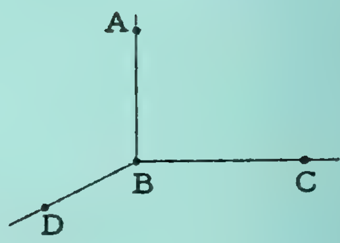
Answers for Exercises.

1. 

$B \in \overline{DC}$ and $B \in \overline{AE}$. $\angle ABD$ and $\angle CBE$ are two supplements of $\angle ABC$.

2. 

$\angle ABC$ is a right angle; $\overrightarrow{BD}$ is a half-line; $\angle ABD$ and $\angle DBC$ are acute angles.

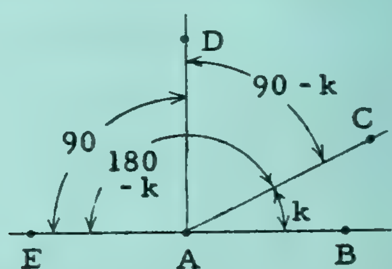
3. 

$\angle ABC$ is a right angle; $\overrightarrow{BD}$ is a half-line; $\angle ABD$ and $\angle DBC$ are obtuse angles.

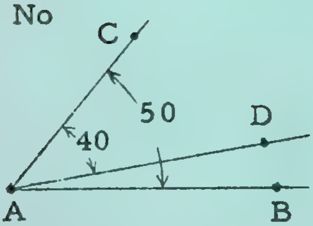
4. No; no. In neither case is the sum of the measures 180.

5. 40; 130

6. 180 - k; 90 - k; 90



7. No



Marginal comments for the column proof on pages 6-64 and 6-65.

(1) [assumption]*

(2) [(1); def. of congruent angles]

(3) [(1); def. of supplementary angles]

(4) [(1); def. of supplementary angles]

(5) [(2), (3), and (4); algebra]

(6) [(5); *(1)]

(7) [(1) - (6); def. of congruent angles]

*

The justification for step (2) involves the use of the inference scheme:

$$\frac{\dfrac{p \text{ and } q \text{ and } r}{p \text{ and } q}}{p}$$

This inference scheme follows from two applications of the second logical principle for conjunction sentences [see page 6-392]. Of course, the sentence '$\angle A \cong \angle B$' is translated into '$m(\angle A) = m(\angle B)$' by using the definition of congruent angles.

The justification for step (3) involves the use of the inference scheme:

$$\frac{\dfrac{p \text{ and } q \text{ and } r}{p \text{ and } q}}{q}$$

which follows from the second and third logical principles for conjunction sentences. The sentence '$\angle C$ is a supplement of $\angle A$' is translated into '$m(\angle A) + m(\angle C) = 180$' by using the definition of supplementary angles [and the commutative principle for addition].

*

A paragraph proof and a column proof for Theorem 2-4 are obtained by a simple paraphrasing of the two forms of proof for Theorem 2-3.

Correction. On page 6-72, the last part of line 14b should read:

[Steps like (1) and (2)]

Answers for Part A.

[The Given-Find format for exercises indicates that the only thing required is a numerical answer. You can ask for justifications during recitation.]

1. $m(\angle BOC) = 50$; $m(\angle EOD) = 40$; $m(\angle EOC) = 130$ [Since $\angle AOC$ is a right angle, $\angle AOB$ and $\angle BOC$ are complementary. So, $m(\angle BOC) = 50$. Since $O \in \overline{AD}$ and $O \in \overline{BE}$, $\angle AOB$ and $\angle EOD$ are vertical angles. So, $m(\angle EOD) = 40$. Since $O \in \overline{EB}$, $m(\angle EOC) + m(\angle BOC) = 180$. Therefore, $m(\angle EOC) = 130$.]

2. $m(\angle FOG) = 25$; $m(\angle GOH) = 25$; $m(\angle HOC) = 130$; $m(\angle HOB) = 155$ [$\angle EOF$ and $\angle COD$ are vertical angles. So, $m(\angle COD) = 20$. Also, $m(\angle EOC) = 160$. But, $m(\angle EOA) = 110$. So, $m(\angle AOC) = 50$. Therefore, $m(\angle BOC) = 25$. Since $\angle FOG$ and $\angle BOC$ are vertical angles, $m(\angle FOG) = 25$. Since $\angle GOH$ and $\angle AOB$ are vertical angles, $m(\angle GOH) = 25$. Since $O \in \overline{AH}$, $m(\angle HOC) = 180 - m(\angle AOC) = 130$. Similarly, $m(\angle HOB) = 180 - m(\angle AOB) = 155$.]

3. $m(\angle AOD) = 90 = m(\angle DOC) = m(\angle COB)$. [This exercise foreshadows Theorem 2-7 on page 6-67.]

*

Answers for Part B.

(1)	For each three noncollinear points X, Y, and Z, and each point W interior to $\angle XYZ$, $°m(\angle XYW) + °m(\angle WYZ) = °m(\angle XYZ)$	[axiom]
(2)	B, O, and D are three noncollinear points	[figure]
(3)	C is interior to $\angle BOD$	[figure]
(4)	$m(\angle BOC) + m(\angle COD) = m(\angle BOD)$	[(1), (2), and (3)]

Correction. On page 6-72, the last part of line 14b should read:

[Steps like (1) and (2)]

Answers for Part A.

[The Given-Find format for exercises indicates that the only thing required is a numerical answer. You can ask for justifications during recitation.]

1. m(∠BOC) = 50; m(∠EOD) = 40; m(∠EOC) = 130 [Since ∠AOC is a right angle, ∠AOB and ∠BOC are complementary. So, m(∠BOC) = 50. Since O ∈ $\overline{AD}$ and O ∈ $\overline{BE}$, ∠AOB and ∠EOD are vertical angles. So, m(∠EOD) = 40. Since O ∈ $\overline{EB}$, m(∠EOC) + m(∠BOC) = 180. Therefore, m(∠EOC) = 130.]

2. m(∠FOG) = 25; m(∠GOH) = 25; m(∠HOC) = 130; m(∠HOB) = 155 [∠EOF and ∠COD are vertical angles. So, m(∠COD) = 20. Also, m(∠EOC) = 160. But, m(∠EOA) = 110. So, m(∠AOC) = 50. Therefore, m(∠BOC) = 25. Since ∠FOG and ∠BOC are vertical angles, m(∠FOG) = 25. Since ∠GOH and ∠AOB are vertical angles, m(∠GOH) = 25. Since O ∈ $\overline{AH}$, m(∠HOC) = 180 - m(∠AOC) = 130. Similarly, m(∠HOB) = 180 - m(∠AOB) = 155.]

3. m(∠AOD) = 90 = m(∠DOC) = m(∠COB). [This exercise foreshadows Theorem 2-7 on page 6-67.]

*

Answers for Part B.

(1)	For each three noncollinear points X, Y, and Z, and each point W interior to ∠XYZ, °m(∠XYW) + °m(∠WYZ) = °m(∠XYZ)	[axiom]
(2)	B, O, and D are three noncollinear points	[figure]
(3)	C is interior to ∠BOD	[figure]
(4)	m(∠BOC) + m(∠COD) = m(∠BOD)	[(1), (2), and (3)]

(5)	$m(\angle AOB) + m(\angle BOC) = m(\angle AOC)$	[Steps like (2) and (3)]
(6)	$\angle BOD$ and $\angle AOC$ are right angles	[Hypothesis]
(7)	All right angles are congruent.	[theorem]
(8)	$m(\angle BOD = m(\angle AOC)$	[(6) and (7); def. of cong. angles]
(9)	$\angle AOB \cong \angle COD$	[(4), (5), and (8); algebra; def. of congruent angles]

[Notice that step (2) could be justified by noting that by hypothesis, $\angle BOD$ is an angle, and then using the definition of angle.]

*

Answer for Part C.

(1)	$\angle A$ and $\angle B$ are supplementary and $\angle A \cong \angle B$	[assumption]*
(2)	$m(\angle A) + m(\angle B) = 180$	[(1); def. of supp. angles]
(3)	$m(\angle A) = m(\angle B)$	[(1); def. of cong. angles]
(4)	$m(\angle A) = 90$	[(2) and (3); algebra]
(5)	An angle is a right angle if and only if it is an angle of 90°.	[theorem]
(6)	if $\angle A$ is an angle of 90° then $\angle A$ is a right angle	[if-part of (5)]
(7)	$\angle A$ is a right angle	[(4) and (6)]
(8)	$\angle B$ is a right angle	[Steps like (4) and (6)]
(9)	$\angle A$ and $\angle B$ are right angles	[(7) and (8)]
(10)	if $\angle A$ and $\angle B$ are supplementary and $\angle A \cong \angle B$ then $\angle A$ and $\angle B$ are right angles	[(9); * (1)]
(11)	If two supplementary angles are congruent, they are right angles.	[(1) - (10)]

Paragraph proof for Part C:

Suppose that $\angle A$ and $\angle B$ are supplementary and congruent. By the definition of supplementary angles, $m(\angle A) + m(\angle B) = 180$, and by the definition of congruent angles, $m(\angle A) = m(\angle B)$. So, $m(\angle A) = 90$ and $m(\angle B) = 90$. Therefore, since an angle is a right angle if it is an angle of 90°, $\angle A$ and $\angle B$ are right angles. Hence, if $\angle A$ and $\angle B$ are supplementary and congruent then $\angle A$ and $\angle B$ are right angles. Consequently, if two supplementary angles are congruent, they are right angles.

*

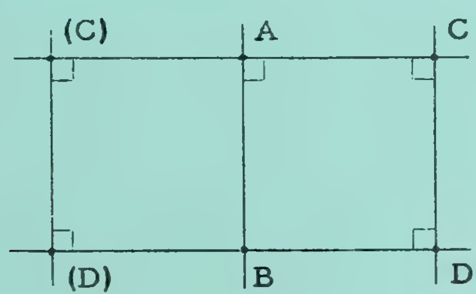
Perpendicularity is a relation among lines. The relation consists of ordered pairs of lines (ℓ, m) such that $\ell \cup m$ contains a right angle. Since $\ell \cup m = m \cup \ell$, perpendicularity is a symmetric relation. But, it is not reflexive [since any angle must contain three noncollinear points].

*

The discussion of Theorem 2-8 on page 6-68 should be carried out at the board. Use a blackboard protractor to find a point of h. That the line m which contains h is perpendicular to ℓ follows from the definition of perpendicular lines.

*

Answers for Exercises [on page 6-69]



After students have examined the three pictures of pairs of adjacent angles $\angle AOB$ and $\angle BOC$ [figures (1), (2), and (4)], and the one picture of nonadjacent angles $\angle AOB$ and $\angle BOC$, ask them to draw a picture of two adjacent angles, $\angle MRS$ and $\angle SRN$, and a picture of two nonadjacent angles $\angle MRS$ and $\angle SRN$.

*

Note the phrase 'closed half-planes' in the definition on page 6-70. Since side $\overrightarrow{OA}$ of $\angle AOB$ is a ray and $O \in \overrightarrow{OA} \cap \overleftrightarrow{OB}$, $\overrightarrow{OA}$ is not a subset of either of the half-planes determined by $\overleftrightarrow{OB}$. But, $\overrightarrow{OA}$ is a subset of one of the <u>closed</u> half-planes determined by $\overleftrightarrow{OB}$. In fact, by Introduction Theorem 18, $\overrightarrow{OA}$ is a subset of the A-side of $\overleftrightarrow{OB}$. So, $\overrightarrow{OA}$ is a subset of the closed half-plane which is the union of $\overleftrightarrow{OB}$ and the A-side of $\overleftrightarrow{OB}$.

Notice that this argument shows that $\angle AOB$ and $\angle BOC$ are adjacent if A and C are on opposite sides of $\overleftrightarrow{OB}$. The converse follows from the fact that $A \in \overrightarrow{OA}$ and $C \in \overrightarrow{OC}$ [and the assumption, made implicitly in speaking of $\angle AOB$ and $\angle BOC$, that neither A nor C belongs to $\overleftrightarrow{OB}$]. Using Introduction Theorem 12, this criterion can be generalized, showing that if $P \in \overrightarrow{OA}$ and $Q \in \overrightarrow{OC}$ then $\angle AOB$ and $\angle BOC$ are adjacent angles if and only if P and Q are on opposite sides of $\overleftrightarrow{OB}$. By Introduction Theorem 16, it then follows that if $Q \in \overrightarrow{OA}$ and $P \in \overrightarrow{OC}$, $\angle AOB$ and $\angle BOC$ are adjacent if and only if $\overline{PQ} \cap \overleftrightarrow{OB} \neq \emptyset$.

The applications of Axiom F in cases (1) and (2) depend on the fact that if $\overline{PQ} \cap \overleftrightarrow{OB} = \{R\}$, where $R \neq O$, then $\overrightarrow{OR}$ is a subset of the interior of $\angle AOC$. This is a consequence of two basic results:

If $P \in \overrightarrow{OA}$ and $Q \in \overrightarrow{OC}$ then $\overline{PQ} \subseteq$ the interior of $\angle AOC$.

If $R \in$ the interior of $\angle AOC$ then $\overrightarrow{OR} \subseteq$ the interior of $\angle AOC$.

For the first of these results, note that, by Introduction Theorem 18, $\overrightarrow{OA} \subseteq$ the A-side of $\overleftrightarrow{OC}$. Hence, if $P \in \overrightarrow{OA}$ then $P \in$ the A-side of $\overrightarrow{OC}$. So, by the same theorem, if $Q \in \overrightarrow{OC}$ then $\overrightarrow{QP} \subseteq$ the A-side of $\overleftrightarrow{OC}$. Similarly, $\overrightarrow{PQ} \subseteq$ the C-side of $\overleftrightarrow{OA}$. Consequently, since $\overline{PQ} = \overrightarrow{QP} \cap \overrightarrow{PQ}$, $\overline{PQ} \subseteq$ the interior of $\angle AOC$.

The second result is deduced from Introduction Theorem 18 by the same kind of argument.

3\. (1) $\overleftrightarrow{EB} \perp \overleftrightarrow{BC}$ [Hypothesis]

(2) $\angle EBC \subseteq \overleftrightarrow{EB} \cup \overleftrightarrow{BC}$ [figure]

(3) [Theorem 2-7 on 6-67] [theorem]

(4) $\angle EBC$ is a right angle [(1), (2), and (3)]

(5) [Theorem 2-1 on 6-60] [theorem]

(6) $m(\angle EBC) = 90$ [(4) and the only-if-part of (5)]

(7) A is interior to $\angle EBC$ [figure]

(8) [Axiom F on 6-56] [axiom]

(9) $m(\angle EBA) + m(\angle ABC) = 90$ [(6), (7), and (8)]

(10) $\angle EBA$ is a complement of $\angle ABC$ [(9); def. of comp. angles]

(11) $\angle DCA$ is a complement of $\angle ACB$ [Steps like (1) - (9)]

(12) $\angle ABC \cong \angle ACB$ [Hypothesis]

(13) [Theorem 2-4 on 6-65] [Theorem]

(14) $\angle EBA \cong \angle DCA$ [(10), (11), (12), and (13)]

<u>Paragraph proof of Exercise 3:</u>

By hypothesis, $\overleftrightarrow{EB} \perp \overleftrightarrow{BC}$. So, $\angle EBC$ is a right angle, or an angle of 90 (1, 2). From the figure, A is interior to $\angle EBC$. So, $m(\angle EBA) + m(\angle ABC) = 90$ (3). Therefore, $\angle EBA$ is a complement of $\angle ABC$. Similarly, $\angle DCA$ is a complement of $\angle ACB$. But, by hypothesis, $\angle ABC \cong \angle ACB$. So, $\angle EBA \cong \angle DCA$ (4).

---

(1) [Theorem 2-7 on 6-67]
(2) [Theorem 2-1 on 6-60]
(3) [Axiom F on 6-56]
(4) [Theorem 2-4 on 6-65]

2. (1)	$B \in \overline{AC}$	[Hypothesis; figure]
(2)	$\angle ABD$ and $\angle DBC$ are adjacent angles whose noncommon sides are collinear	[(1); def. of adj. angles]
(3)	[Theorem 2-9 on 6-71]	[theorem]
(4)	$\angle DBC$ is a supplement of $\angle ABD$	[(2) and the if-part of (3)]
(5)	$\angle ABE \cong \angle DBC$	[Hypothesis]
(6)	$\angle ABE$ is a supplement of $\angle ABD$	[(4) and (5); def. of supp. angles; def. of cong. angles]
(7)	E and D are in opposite sides of $\overleftrightarrow{AB}$	[figure]
(8)	$\angle ABD$ and $\angle ABE$ are adjacent angles	[(8); def. of adj. angles]
(9)	D, B, and E are collinear	[(6), (8), and the only-if-part of (3)]

Paragraph proof of Exercise 2:

By hypothesis, A, B, and C are collinear. Since from the figure, $B \in \overline{AC}$, it follows from the definition of adjacent angles that $\angle ABD$ and $\angle DBC$ are adjacent angles. Also, their noncommon sides are collinear. So, $\angle DBC$ is a supplement of $\angle ABD$ (1). But, by hypothesis, $\angle ABE \cong \angle DBC$. So, by the definitions of supplementary angles and congruent angles, $\angle ABE$ is a supplement of $\angle ABD$. Since, from the figure, E and D are in opposite sides of $\overleftrightarrow{AB}$, it follows that $\angle ABE$ and $\angle ABD$ are supplementary adjacent angles. Hence, D, B, and E are collinear (1).

---

(1) [Theorem 2-9 on 6-71]

Answers for Exercises.

1.	(1)	A, B, E are three noncollinear points and D is interior to $\angle ABE$	[figure]
	(2)	[Axiom F on 6-56]	[axiom]
	(3)	$m(\angle ABE) = w + x$	[(1) and (2)]
	(4)	$m(\angle EBC) = y + z$	[Steps like (1) and (2)]
	(5)	$m(\angle ABE) + m(\angle EBC) = w + x + y + z$	[(3) and (4); algebra]
	(6)	$w + y = 90 = x + z$	[Hypothesis]
	(7)	$m(\angle ABE) + m(\angle EBC) = 180$	[(5) and (6); algebra]
	(8)	$\angle ABE$ and $\angle EBC$ are supplementary	[(7); def. of supp. angles]
	(9)	A and C are in opposite sides of $\overleftrightarrow{BE}$	[figure]
	(10)	$\angle ABE$ and $\angle EBC$ are adjacent angles	[(9); def. of adj. angles]
	(11)	[Theorem 2-9 on 6-71]	[theorem]
	(12)	A, B, and C are collinear	[(8), (10), and the only-if-part of (11)]

<u>Paragraph proof of Exercise 1:</u>

Since, from the figure, A, B, and E are noncollinear and D is interior to $\angle ABE$, $m(\angle ABE) = w + x$(1). Similarly, $m(\angle EBC) = y + z$. But, by hypothesis, $w + y = 90$ and $x + z = 90$. So, by algebra, $m(\angle ABE) + m(\angle EBC) = 180$. Now, since $\overrightarrow{BE}$ is a side common to $\angle ABE$ and $\angle EBC$, and since, from the figure, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are contained in opposite closed half-planes determined by $\overleftrightarrow{BE}$, it follows from the definition of adjacent angles that $\angle ABE$ and $\angle EBC$ are adjacent angles. But, by the definition of supplementary angles, $\angle ABE$ and $\angle EBC$ are supplementary. Hence, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are collinear (2). That is, A, B, and C are collinear.

---

(1) [Axiom F on 6-56] (2) [Theorem 2-9 on 6-71]

4. (1) $M \in \overline{AB}$ [Hypothesis]

(2) $\angle BMP$ and $\angle AMP$ are adjacent angles whose noncommon sides are collinear [(1); def. of adjacent angles]

(3) [Theorem 2-9 on 6-71] [theorem]

(4) $\angle BMP$ and $\angle AMP$ are supplementary [(2) and the if-part of (3)]

(5) $\angle BMP \cong \angle AMP$ [Hypothesis]

(6) [Theorem 2-6 on 6-67] [theorem]

(7) $\angle BMP$ is a right angle [(4), (5), and (6)]

(8) $\overleftrightarrow{PM} \perp \overleftrightarrow{AB}$ [(7); def. of perpendicular lines]

Paragraph proof of Exercise 4:

By hypothesis, $M \in \overline{AB}$. So, $\angle BMP$ and $\angle AMP$ are adjacent angles whose noncommon sides are collinear. Hence $\angle BMP$ and $\angle AMP$ are supplementary (1). But, by hypothesis, $\angle BMP \cong \angle AMP$. So, $\angle BMP$ is a right angle (2). Thus, by the definition of perpendicular lines, $\overleftrightarrow{PM} \perp \overleftrightarrow{AB}$.

---

(1) [Theorem 2-9 on 6-71] (2) [Theorem 2-6 on 6-67]

*

Answers for Exploration Exercises.

1. Axiom C tells you that once B is chosen, there is a unique point B′ such that AB = A′B′. [Of course, in order to put Axiom C into play, we need Theorem 1-2 to assure us that since $A \neq B$, $AB > 0$.] It should turn out to be the case that BC = B′C′.

2. As in Exercise 1.

3. $BC \neq B'C'$

*

Axiom H is the key axiom to be used in our work in the next section with congruent triangles. The exercises on 6-75 and 6-76 foreshadow the proofs of the s.s.s. and the s.a.s. triangle-congruence theorems.

Answers for Exercises [on pages 6-75, 6-76, and 6-77].

1. The relevant instance of Axiom H is:

   if $AB = PT$ and $BC = TQ$ then
   $CA = QP$ if and only if $m(\angle ABC) = m(\angle PTQ)$

   Since $AB = 5 = PT$ and $BC = 10 = TQ$, we have [by modus ponens]:

   $CA = PQ$ if and only if $m(\angle ABC) = m(\angle PTQ)$

   Since $\angle ABC \cong \angle PTQ$, we have [using the if-part of the foregoing biconditional, and modus ponens]:

   $$CA = QP$$

   So, since $CA = 7$, $PQ = 7$.

2. Since C is the midpoint of $\overleftrightarrow{AD}$, we know, by definition, that $AC = DC$ and that $C \in \overleftrightarrow{AD}$. From the figure, $A \neq D$. So, $C \in \overline{AD}$. Similarly, $CB = CE$ and $C \in \overline{BE}$. Since $C \in \overline{AD}$ and $C \in \overline{BE}$, $\angle ACB$ and $\angle DCE$ are vertical angles. So, they are congruent. By Axiom H,

   if $AC = CD$ and $CB = CE$ then
   $BA = ED$ if and only if $m(\angle ACB) = m(\angle DCE)$.

   Since $AC = CD$ and $CB = CE$, and since $\angle ACB \cong \angle DCE$, it follows [from the if-part] that $BA = ED$. But, $BA = 4$. So, $DE = 4$.

   Again, by Axiom H,

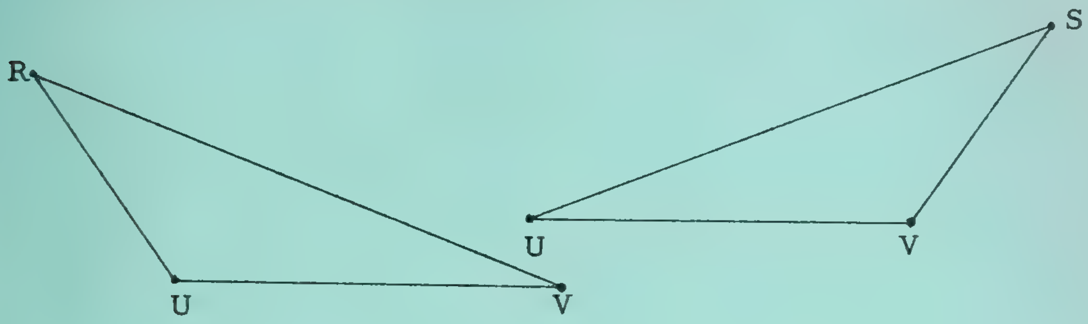
   if $BA = ED$ and $AC = DC$ then
   $CB = CE$ if and only if $m(\angle BAC) = m(\angle EDC)$.

   Since [by the preceding part] $BA = ED$ and $AC = DC$, and since $CB = CE$, it follows [from the only-if-part] that $m(\angle BAC) = m(\angle EDC)$.

Correction. On page 6-76, line 3b should read:

If $\overline{AB} \cong \overline{AC}$ then $\angle C \cong$ ______.

3. In this exercise and in Exercise 4, it helps to think of the "overlapping triangles" as being "moved apart":



or, to mark the congruent sides with colored chalk in the given figure [$\overline{RV}$ and $\overline{SU}$ red, $\overline{RU}$ and $\overline{SV}$ blue, $\overline{VU}$ and $\overline{UV}$ white].

Since RV = SU and RU = SV, and since VU = UV, it follows from Axiom H that $m(\angle R) = m(\angle S) = 35$.

Since RU = SV and UV = VU, and since RV = SU, it follows from Axiom H that $\angle SVU \cong \angle RUV$.

Also, $\angle RTS \cong \angle UTV$ [vertical angles].

4. Since B is the midpoint of $\overline{AC}$, $CB = \frac{1}{2} \cdot AC$, Similarly, $CD = \frac{1}{2} \cdot EC$. Since AC = EC, CB = CD. So, since AC = CE and CD = CB, and since $m(\angle ACD) = m(\angle ECB)$ [because $\angle ACD = \angle ECB$], it follows from Axiom H that AD = EB. So, $\overline{AD} \cong \overline{EB}$. Also, $\overline{AB} \cong \overline{ED}$ [B and D are midpoints and $\overline{AC} \cong \overline{EC}$].

5. Since C is the midpoint of $\overline{AE}$ and $\overline{BD}$, and since $\overline{AE} \cong \overline{BD}$, it follows that BC = CE = AC = CD. Because $\angle ACB$ and $\angle DCE$ are vertical angles, they are congruent. So, since BC = EC and CA = CD, and since $m(\angle ACB) = m(\angle DCE)$, it follows from Axiom H that AB = DE. Again, by Axiom H, since AB = DE and BC = EC,

and since AC = DC, it follows that m(∠ABC) = m(∠DEC). Hence, ∠B ≅ ∠E. [The sentence '∠B ≅ ∠D' was put in the consequent to have students search for another angle congruent to ∠B. Of course, they can put an '∠B' or even an '∠D' in the blank and be correct. But, we hope they will not be so clever.]

6. Since AC = AB and CB = BC, and since AB = AC, it follows from Axiom H that m(∠ACB) = m(∠ABC). So, ∠C ≅ ∠B. [Colored chalk is exceedingly helpful for this problem.]

7. 140; 40; 140; 40; 140; 40; 140

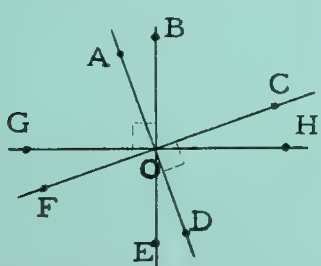
*

Quiz.

1. (a) Draw an acute angle ∠ABC, and an obtuse angle ∠CBD such that ∠ABC and ∠CBD are adjacent angles.
   (b) Repeat (a) but make ∠ABC and ∠CBD nonadjacent angles.
   (c) If m(∠ABC) = x and m(∠CBD) = y, compute m(∠ABD) in part (b).

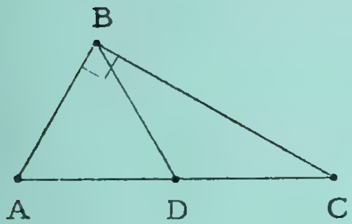
2. If m(∠A) is two thirds the measure of one of its complements, how many degrees are there in ∠A?

3. Suppose that C and D are two points in the interior of ∠AOB such that C is in the interior of ∠AOD. If ∠AOB and ∠COD are supplementary and ∠AOC and ∠BOD are complementary, find the number of degrees in ∠COD.

4. 

Given: $\overleftrightarrow{AD} \perp \overleftrightarrow{CF}$, $\overleftrightarrow{GO} \perp \overleftrightarrow{BE}$, m(∠AOG) = 70

Find: m(∠BOD), m(∠EOF)

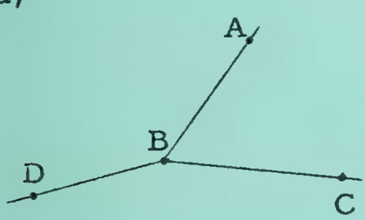
5. 

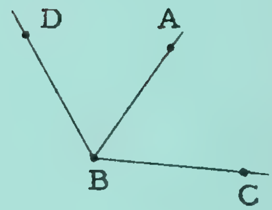
Hypothesis: $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$,
$\angle DBC \cong \angle C$,
$\angle ABD \cong \angle A$

Conclusion: $\angle A$ and $\angle C$ are complementary

*

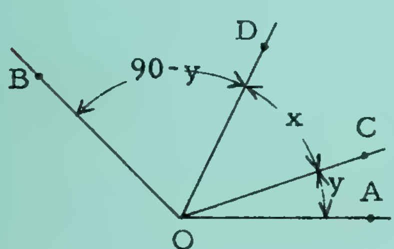
Answers for Quiz.

1. (a) 

(b) 

(c) y - x

2. $x + \frac{3}{2}x = 90$; $x = 36$; $\angle A$ is an angle of $36°$.

3. 

$(90 - y + x + y) + x = 180$
$x = 45$

$\angle COD$ is an angle of $45°$

4. $m(\angle BOD) = 160$; $m(\angle DOG) = 70$

5.	(1) $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$	[Hypothesis]
	(2) [Theorem 2-7]	[theorem]
	(3) $\angle ABC$ is a right angle	[(1) and (2)]
	(4) D is interior to $\angle ABC$	[figure]
	(5) [Axiom F]	[axiom]
	(6) $m(\angle ABD) + m(\angle DBC) = m(\angle ABC)$	[(4) and (5)]
	(7) [Theorem 2-1]	[theorem]
	(8) $m(\angle ABC) = 90$	[(3) and the only-if-part of (7)]
	(9) $m(\angle ABD) + m(\angle DBC) = 90$	[(6) and (8)]
	(10) $\angle ABD \cong \angle A$	[Hypothesis]
	(11) $\angle DBC \cong \angle C$	[Hypothesis]
	(12) $m(\angle A) + m(\angle C) = 90$	[(9), (10), and (11); def. of cong. angles]
	(13) $\angle A$ and $\angle C$ are complementary	[(12); def. of comp. angles]

<u>Paragraph proof of item 5:</u>

By hypothesis, $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$. So, $\angle ABC$ is a right angle (1), that is, an angle of 90° (2). Since, from the figure, D is interior to $\angle ABC$, it follows that $m(\angle ABD) + m(\angle DBC) = 90$ (3). But, by hypothesis $\angle ABD \cong \angle A$ and $\angle DBC \cong \angle C$. So, $m(\angle A) + m(\angle C) = 90$. Hence, $\angle A$ and $\angle C$ are complementary.

---

(1) [Theorem 2-7]

(2) [Theorem 2-1]

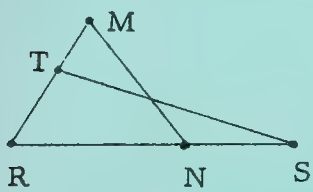
(3) [Axiom F]

In the second paragraph, it is implied that each side of a triangle is a subset of two of its angles. For example, $\overline{CA} \subseteq \angle CAB$ and $\overline{CA} \subseteq \angle ACB$. Also, each side of a triangle is a subset of the triangle. But, it is not the case that an angle of a triangle is a subset of the triangle. For example, suppose that D is a point such that $C \in \overline{AD}$. Then, since $D \in \overrightarrow{AC}$, $D \in \angle CAB$. But $D \notin \overline{AC}$. Also, since A, B, and C are noncollinear, $D \notin \overline{AB}$ and $D \notin \overline{BC}$. So, $D \notin \triangle ABC$. Therefore, $\angle CAB \nsubseteq \triangle ABC$. So, although a triangle <u>has</u> sides and angles and <u>contains</u> its sides, it does not contain its angles.

*

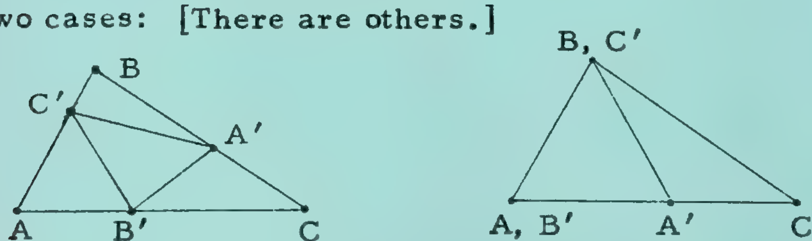
Answers for Exercises [on pages 6-79 and 6-80].

1. $\triangle ABG$, $\triangle ACD$, $\triangle AJE$, $\triangle ACJ$, $\triangle ADE$, $\triangle BCH$, $\triangle CDJ$, $\triangle DFG$, $\triangle DEJ$, $\triangle GJH$

2. 

[More practice with overlapping triangles.] $\angle MRN = \angle SRT$; so, $\angle MRN \cong \angle SRT$. Hence, there is an angle of $\triangle MNR$ which is congruent to an angle of $\triangle RST$.

3. Two cases: [There are others.]



Ask students to draw $\triangle ABC$ and $\triangle A'B'C'$ such that each vertex of $\triangle A'B'C'$ belongs to two sides of $\triangle ABC$. What conclusion can they draw about $\triangle ABC$ and $\triangle A'B'C'$? [$\triangle ABC = \triangle A'B'C'$]

[It is impossible to draw <u>two</u> triangles such that the vertices of each belong to the sides of the other.]

4. [Reading practice]

5. [See the COMMENTARY for Exercise 2 on page 6-75.]

6. $\angle PTK$; $\overleftrightarrow{TK}$; $\angle TPK$

*

Note carefully that in talking about the matching of the vertices of ΔABC with those of ΔFED, nothing is said about congruence of angles [or of sides]. The word 'match' sometimes denotes a comparison. Hence, students may think that to match the vertices, you must compare the angles.

Intuitively, you can think of matching the vertices as follows. Take three strings, and tack an end of one string at A, an end of a second at B, and an end of the third at C. Now, take the string which is fastened at A and tack its other end at one of the three vertices, F, E, or D. As soon as this is accomplished, you have indicated a matching of A with one of the vertices of ΔFED. Similarly, tack the other end of the second string at one of the two remaining vertices of ΔFED. Thus, you have a matching of B with one of the vertices of ΔFED. There is only one vertex of ΔFED left. This is the one at which you tack the other end of the third string. Now, you have one matching of the vertices of ΔABC with those of ΔFED. [Segments drawn with colored chalk can be used instead of string.]

Students of Unit 5 should recognize a matching of the vertices of ΔABC with those of ΔFED as a mapping or function whose domain is $\{A, B, C\}$ and whose range is $\{D, E, F\}$. Each such mapping has an inverse. [Hence, the '$\leftrightarrow$' notation.] Incidentally, the symbol '$ABC \leftrightarrow FED$' is a noun, not a sentence. It can be thought of as an abbreviation for

$$'\{(A, F), (B, E), (C, D)\}'$$

*

The idea underlying the top paragraph on page 6-81 is that it makes no sense to talk about corresponding sides and corresponding angles unless you have in mind some matching of the vertices. For each pair of sides, one from ΔABC and the other from ΔFED, there are two matchings of the vertices of ΔABC with those of ΔFED with respect to which the pair of sides is a pair of corresponding sides. And, there are four matchings with respect to which this pair of sides is not a pair of corresponding sides.

The exercises on page 6-81 and the exercises on pages 6-412 and 6-413 are very important. You might have students do those on page 6-81 in class, and assign those on 6-412 and 6-413 for homework. Students should feel at home with matchings and the procedure for picking out pairs of corresponding parts before they attempt the work on congruent triangles starting on page 6-82.

Answers for Exercises.

1. $(\overleftrightarrow{KQ}, \overleftrightarrow{JM})$, $(\overleftrightarrow{QG}, \overleftrightarrow{ML})$, $(\overleftrightarrow{GK}, \overleftrightarrow{LJ})$, $(\angle GKQ, \angle LJM)$, $(\angle KQG, \angle JML)$, $(\angle QGK, \angle MLJ)$

2. $(\overleftrightarrow{QK}, \overleftrightarrow{ML})$, $(\overleftrightarrow{KG}, \overleftrightarrow{LJ})$, $(\overleftrightarrow{GQ}, \overleftrightarrow{JM})$, $(\angle GQK, \angle JML)$, $(\angle QKG, \angle MLJ)$, $(\angle KGQ, \angle LJM)$

   [A good question to ask following Exercise 2 is:

   Now, list the pairs of corresponding parts of ΔKQG and ΔJML with respect to the matching $QGK \longleftrightarrow MJL$.

   Those who missed the point of the question at the very bottom of page 6-80 will get another chance to see that a matching can have several names.]

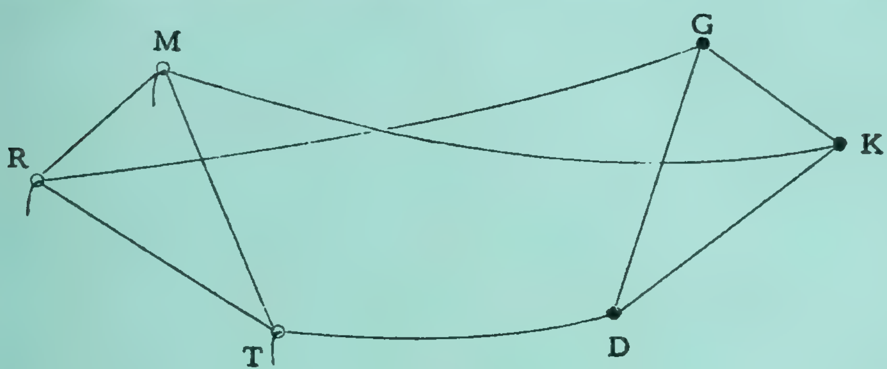
   [We hope that students are discovering how to pick out names of corresponding parts just by using a name of the matching.]

3. There are two matchings [of the vertices of ΔABC with those of ΔPQR] with respect to which $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are corresponding sides. These are the matchings $ABC \longleftrightarrow PQR$ and $ABC \longleftrightarrow QPR$. So, it is only with respect to the first of these matchings that $\overleftrightarrow{AC}$ and $\overleftrightarrow{PR}$ are corresponding sides. On the other hand, $\angle C$ and $\angle R$ are corresponding angles with respect to each of the two matchings. So, the answer to (a) is 'no' and the answer to (b) is 'yes'.

4. (a) $ABC \longleftrightarrow FDE$, $ABC \longleftrightarrow DFE$
   (b) $ABC \longleftrightarrow DFE$, $ABC \longleftrightarrow EFD$
   (c) $ABC \longleftrightarrow EDF$
   (d) $ABC \longleftrightarrow DEF$
   (e) $ABC \longleftrightarrow FED$
   (f) $ABC \longleftrightarrow EDF$, $ABC \longleftrightarrow FDE$

5. $ABC \longleftrightarrow ABC$, $ABC \longleftrightarrow ACB$, $ABC \longleftrightarrow BCA$, $ABC \longleftrightarrow CAB$, $ABC \longleftrightarrow CBA$

One can get a good intuitive feeling for what a pair of congruent triangles is by imagining one of the triangles being picked up and rotated or turned over in such a way that it can be superposed on the other with all the parts fitting "just right". In order to see what the matching of vertices has to do with congruence of triangles, imagine that a triangle ΔMRT is drawn on a flat level board with holes drilled through the board at M, R, and T. Suppose that another triangle ΔKGD, made of coat hanger wire, is placed on the board. Strings are fastened at G, K, and D. Now, to



indicate the matching $GKD \leftrightarrow RMT$, we pass the strings through the holes at R, M, and T. To say that $GKD \leftrightarrow RMT$ is a congruence is to say that if the strings are grasped under the board and pulled away from the board, ΔGKD will eventually come to rest right on top of ΔRMT with no string showing above the board [except for knots at G, K, and D]. By 'eventually' we mean that the wire triangle may have to be flipped over one or more times. It is easy to see that even though $GKD \leftrightarrow RMT$ is a congruence, there may be other matchings which are not. In fact, if ΔRMT is scalene, $GKD \leftrightarrow RMT$ is the only congruence.

*

The definition given in the first paragraph on page 6-82 replaces the corresponding-parts-of-congruent-triangles-are-equal-refrain of many conventional courses. The definition is very easy to use. Since there are six matchings of the vertices of a first triangle with those of a second, all one needs to do to show that the triangles are congruent is to test each matching. If at least one of the matchings is such that all six pairs of corresponding parts with respect to this matching are pairs of congruent parts then the triangles are congruent. Each such matching of the vertices is called a <u>congruence</u> [or: a congruence of the vertices --see line 3 on page 6-84]. On the other hand, if you are told that a first triangle is congruent to a second, then the definition tells you that there must be at least one matching of the vertices which is a congruence. It

is convenient to indicate one such matching when you state that the triangles are congruent. This is the burden of the discussion on the lower half of page 6-83. We try to adhere to this convention of indicating the congruence in the assertion that the triangles are congruent. But, this convention is not observed widely and students should not depend on it when taking standardized tests or using other textbooks. On the other hand, the double-arrow notation for naming a matching which is a congruence, for example:

$$ABC \leftrightarrow EDF \text{ is a congruence}$$

is most useful since the names of congruent corresponding parts can be picked out of the sentence mechanically.

$$\overline{AB} \cong \overline{ED}$$
$$\overline{BC} \cong \overline{DF}$$
$$ABC \longleftrightarrow EDF \text{ is a congruence}$$
$$\overline{CA} \cong \overline{FE}$$

$$\angle A \cong \angle E$$
$$\angle B \cong \angle D$$
$$ABC \longleftrightarrow EDF \text{ is a congruence}$$
$$\angle C \cong \angle F$$

> Correction. On page 6-83, line 3b
> should begin 'is more helpful because --- '.

Answers for Exercises [on pages 6-83 and 6-84].

A. A triangle is congruent to itself. For each triangle ΔABC, $ABC \leftrightarrow ABC$ is a congruence because an angle is congruent to itself and a segment is congruent to itself.

A triangle with two congruent sides and with the angles opposite these sides congruent also is a triangle for which two matchings are congruences. [This foreshadows the work on pages 6-103 and 6-104.]

A triangle with three congruent sides and with three congruent angles is a triangle for which more than two matchings are congruences. In fact, in that case, all six matchings are congruences.

B. No. Some other matching of the vertices might be a congruence. In fact, $ABC \leftrightarrow EDF$ is a congruence.

C. $\Delta A'B'C' \cong \Delta WZY$ [$\overline{A'B'} \cong \overline{WZ}$, $\overline{B'C'} \cong \overline{ZY}$, $\overline{C'A'} \cong \overline{YW}$, $\angle A' \cong \angle W$, $\angle B' \cong \angle Z$, $\angle C' \cong \angle Y$]

$\Delta GHI \cong \Delta CDX$ [$\overline{GH} \cong \overline{CD}$, $\overline{HI} \cong \overline{DX}$, $\overline{IG} \cong \overline{XC}$, $\angle G \cong \angle C$, $\angle H \cong \angle D$, $\angle I \cong \angle X$]

$\Delta JKL \cong \Delta PRQ \cong \Delta AST$ [$\overline{JK} \cong \overline{PR} \cong \overline{AS}$, $\overline{KL} \cong \overline{RQ} \cong \overline{ST}$, $\overline{LJ} \cong \overline{QP} \cong \overline{TA}$, $\angle J \cong \angle P \cong \angle A$, $\angle K \cong \angle R \cong \angle S$, $\angle L \cong \angle Q \cong \angle T$]

$\Delta NMO \cong \Delta UVB$ [$\overline{NM} \cong \overline{UV}$, $\overline{MO} \cong \overline{VB}$, $\overline{ON} \cong \overline{BU}$, $\angle N \cong \angle U$, $\angle M \cong \angle V$, $\angle O \cong \angle B$]

*

Alert students to the need for compasses in doing the work on page 6-87.

Answers to questions in the text.

(1) $\angle TQR$ (2) $\overleftrightarrow{TQ}$ and $\overleftrightarrow{TR}$ (3) $\overleftrightarrow{RT}$ (4) $\angle RTQ$ and $\angle TRQ$

*

Answers for Part A.

1. Let students experiment a bit in this exercise. Most of them should remember the drawing technique from their 7th or 8th grade work. Note that we are not interested in this exercise in a Euclidean construction problem. All we want students to do is know how to use drawing instruments. See pages 6-293 and 6-294.

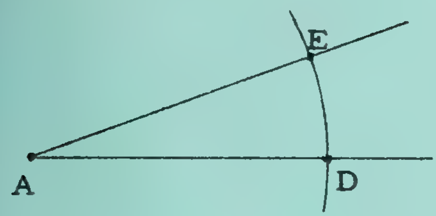
   Many such triangles can be drawn. Be sure that students see that the vertex opposite the longest side can be on either side of ℓ. All such triangles are congruent by virtue of the s.s.s. theorem and the definition of congruence.

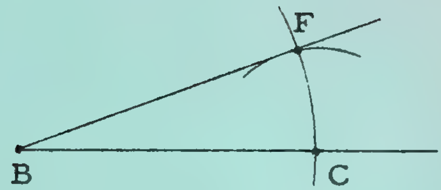
2. If there were such a triangle then, since the vertices are noncollinear, Axiom B tells us that $GH + IJ > KL$. But, $GH + IJ \not> KL$. So, there is no triangle whose sides are congruent to $\overleftrightarrow{GH}$, $\overleftrightarrow{IJ}$, and $\overleftrightarrow{KL}$.

3. (a) triangle (b) AC (c) CA (d) BA

   (e) It is greater than the measure of the third side. Axiom B. [See Theorem 4-1 on page 6-112.]

Answers for Part B.





Method: With the compass, draw part of the circle with center A and radius BC so that it intersects the sides of $\angle A$ in D and E. Then, find the point F in one of the half-planes determined by $\overleftrightarrow{BC}$ such that F is in the intersection of the circle with center B and radius BC and the circle with center C and radius DE. Since AD = BC, AE = BF, and DE = CF, it follows from s. s. s. that $ADE \leftrightarrow BCF$ is a congruence; so, $\angle A \cong \angle B$.

*

The exercises on pages 6-414, 6-415, and 6-416 will help prepare students to use the congruence theorems in proofs.

Answers for Part A.

1.	(1)	M is the midpoint of $\overline{AB}$	[Hypothesis]
	(2)	$\overline{AM} \cong \overline{BM}$	[(1); def. of midpoint]
	(3)	$\overline{MD} \cong \overline{MC}$	[Hypothesis]
	(4)	$\overline{DA} \cong \overline{CB}$	[Hypothesis]
	(5)	A, M, D and B, M, C are vertices of triangles	[figure]
	(6)	s.s.s.	[theorem]
	(7)	AMD $\leftrightarrow$ BMC is a congruence	[(5), (2), (3), (4), and (6)]
	(8)	$\angle D \cong \angle C$	[(7); def. of congruence]
	(9)	$\angle DAE$ and $\angle DAM$ are adjacent angles with their noncommon sides collinear	[figure]
	(10)	[Theorem 2-9 on page 6-78]	[theorem]
	(11)	$\angle DAE$ is a supplement of $\angle DAM$	[(9) and the if-part of (10)]
	(12)	$\angle CBG$ is a supplement of $\angle CBM$	[Step like (9)]
	(13)	[Theorem 2-3 on page 6-78]	[theorem]
	(14)	$\angle DAM \cong \angle CBM$	[(7); def. of congruence]
	(15)	$\angle DAE \cong \angle CBG$	[(11), (12), (14), and (13)]

Paragraph proof for Exercise 1:

We are given that M is the midpoint of $\overline{AB}$. Hence, $\overline{AM} \cong \overline{BM}$. Also by hypothesis, $\overline{MD} \cong \overline{MC}$ and $\overline{DA} \cong \overline{CB}$. So, by s.s.s., for triangles $\triangle AMD$ and $\triangle BMC$, AMD $\leftrightarrow$ BMC is a congruence. Therefore, $\angle D \cong \angle C$.

From the figure, we see that $\angle DAE$ and $\angle DAM$ are adjacent angles with their noncommon sides collinear. So, these angles are supplementary (1). Similarly, $\angle CBG$ and $\angle CBM$ are supplementary. But, since $AMD \longleftrightarrow BMC$ is a congruence, $\angle DAM \cong \angle CBM$. So, $\angle DAE \cong \angle CBG$ (2).

---

(1) [Theorem 2-9 on page 6-78.]

(2) [Theorem 2-3 on page 6-78.]

2.	(1)	$\overline{AC}$ and $\overline{BD}$ bisect each other at E	[Hypothesis]
	(2)	E is the midpoint of $\overline{AC}$	[(1); def. of bisect]
	(3)	$\overline{AE} \cong \overline{EC}$	[(2); def. of midpoint]
	(4)	$\overline{BE} \cong \overline{ED}$	[Step like (2)]
	(5)	$\angle AEB$ and $\angle CED$ are vertical angles	[figure]
	(6)	[Theorem 2-5 on page 6-78]	[theorem]
	(7)	$\angle AEB \cong \angle CED$	[(5) and (6)]
	(8)	A, E, B and C, E, D are vertices of triangles	[figure]
	(9)	s.a.s.	[theorem]
	(10)	$AEB \longleftrightarrow CED$ is a congruence	[(8), (3), (7), (4), and (9)]
	(11)	$\overline{AB} \cong \overline{CD}$	[(10); def. of congruence]
	(12)	$\overline{BC} \cong \overline{AD}$	[Steps like (2) - (5) and (7), (8), and (10)]

[Note. Actually, the derivation of (12) amounts to nothing more than an alphabetic variant of the derivation of (11). Just interchange 'A' and 'C'.]

Paragraph proof of Exercise 2:

By hypothesis, $\overline{AC}$ and $\overline{BD}$ bisect each other at E. So, $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$. The vertical angles, $\angle AEB$ and $\angle CED$, are congruent. So, by s.a.s., $AEB \longleftrightarrow CED$ is a congruence. Hence, $\overline{AB} \cong \overline{CD}$. Similarly, $\overline{BC} \cong \overline{DA}$.

*

You might try having students give plans or "oral proofs" for Exercises 3-6 on page 6-417.

*

Answers for Part B [on page 6-91].

(1) [Hypothesis]

(2) [Hypothesis]

(5) [assumption]*

(6) M is the midpoint of $\overline{AB}$

(7) AM = MB

(8) AP = PB

(9) [(8); *(5)]

(10) [assumption]†

(12) [Identity; def. of cong. segments]

(13) [Theorem 2-7 on page 6-78]

(14) $\angle PMB$ and $\angle PMA$ are right angles

(15) [Theorem 2-2 on page 6-78]

(16) $\angle PMB \cong \angle PMA$

(19) [(11), (12), (16), (17), and (18)]

(21) [(20); † (10)]

Corrections. On page 6-92, line 7b should read 'vertices of triangles'.

On page 6-93, line 9b should read:
---such that JDL $\leftrightarrow$ JDE is a congruence.
and line 1b should read:
---such that MRS $\leftrightarrow$ MNS is a congruence.

Answers for Part C.

(2) [assumption]*

(3) [Hypothesis]

(4) [Theorem 1-9 on page 6-50]

(5) P is the midpoint of $\overleftrightarrow{AB}$

(6) [Hypothesis]

(7) P = M

(9) $P \in \ell$

(10) [(9); *(2)]

(11) [assumption]†

(12) [(11); Hypothesis]

(13) [Hypothesis]

(14) $\overline{AM} \cong \overline{BM}$

(15) $\overline{MP} \cong \overline{MP}$, [Identity; def. of cong. segments]

(17) AMP $\leftrightarrow$ BMP is a congruence

(18) [(17); def. of congruence]

(20) Theorem 2-9 on page 6-78

(21) ∠PMA and ∠PMB are supplementary; [(19) and the if-part of (20)]

(22) [Theorem 2-6 on page 6-78]

(23) ∠PMA and ∠PMB are right angles

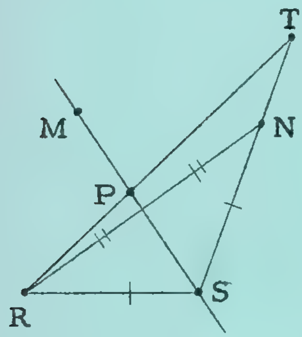
(25) [Hypothesis]

(26) Theorem 2-8 on page 6-79

(29) [(28); † (11)]

Answers for Part E [on page 6-93].

1. No. This result is intuitively obvious and, since we shall make no use of it later, we shall not show how it can be derived from the Introduction Axioms.

2. 

As in Exercise 2 of Part D, $MRS \leftrightarrow MNS$ is a congruence if and only if $RS = NS$ and $MR = MN$. So, since $N \in \overrightarrow{ST}$, N is the point of $\overrightarrow{ST}$ such that $RS = NS$. And, as before, M may be any point of the perpendicular bisector of $\overline{RN}$ which is exterior to $\triangle RST$. Since the midpoint of $\overline{RN}$ is interior to $\angle RST$, the half-line with vertex S which contains this midpoint intersects $\overline{RT}$ in a single point P. Any point M such that $P \in \overline{MS}$ will satisfy the requirements of the problem. [So will any point M such that $S \in \overline{MP}$.]

(2) if D is interior to $\angle CAB$ then B and C are on opposite sides of $\overleftrightarrow{AD}$.

To do so, consider a point B′ such that A is between B′ and B. By Theorem 16, B′ and B are on opposite sides of $\overleftrightarrow{AD}$. So, to establish (2) it is sufficient to show that B′ and C are on the same side of $\overleftrightarrow{AD}$--that is [Theorem 15], that $\overset{\bullet\!-\!\bullet}{B'C} \cap \overleftrightarrow{AD} = \emptyset$. Since $\overset{\bullet\!-\!\bullet}{B'C} = \overset{\bullet\!\!\rightarrow}{CB'} \cap \overset{\bullet\!\!\rightarrow}{B'C}$ and $\overleftrightarrow{AD} = \overset{\bullet\!\!\rightarrow}{AD} \cup \overset{\bullet\!\!\rightarrow}{AD'}$, this will follow if we show that $\overset{\bullet\!\!\rightarrow}{CB'} \cap \overset{\bullet\!\!\rightarrow}{AD} = \emptyset$ and that $\overset{\bullet\!\!\rightarrow}{B'C} \cap \overset{\bullet\!\!\rightarrow}{AD'} = \emptyset$--for, if so, no point of $\overset{\bullet\!-\!\bullet}{B'C}$ can belong to either $\overset{\bullet\!\!\rightarrow}{AD}$ or to $\overset{\bullet\!\!\rightarrow}{AD'}$.

Now, since B and B′ are on opposite sides of $\overleftrightarrow{AC}$ and B and D are on the same side of $\overleftrightarrow{AC}$, it follows that B′ and D are on opposite sides of $\overleftrightarrow{AC}$. So, by Theorem 18, $\overrightarrow{CB'}$ and $\overrightarrow{AD}$ are subsets of opposite sides of $\overleftrightarrow{AC}$. In particular, $\overrightarrow{CB'} \cap \overrightarrow{AD} = \emptyset$. Since $C \neq A$, and since neither C nor A is on either side of $\overleftrightarrow{AC}$, it follows that $\overset{\bullet\!\!\rightarrow}{CB'} \cap \overset{\bullet\!\!\rightarrow}{AD} = \emptyset$. Similarly, since D and D′ are on opposite sides of $\overleftrightarrow{AB}$ and D and C are on the same side of $\overleftrightarrow{AB}$, it follows that D′ and C are on opposite sides of $\overleftrightarrow{AB}$. So [arguing as before], $\overset{\bullet\!\!\rightarrow}{B'C} \cap \overset{\bullet\!\!\rightarrow}{AD'} = \emptyset$. This completes the argument for (2).

Combining (1) and (2), and the first result in the COMMENTARY for page 6-71:

if $P \in \overrightarrow{AB}$ and $Q \in \overrightarrow{AC}$ then $\overline{PQ}$ is a subset of the interior of $\angle CAB$,

we can now show that

(3) if D is interior to $\angle CAB$ then $\overrightarrow{AD} \cap \overline{BC}$ consists of a single point.

For, by (2), B and C are on opposite sides of $\overleftrightarrow{AD}$, whence, by Theorem 19, $\overleftrightarrow{AD} \cap \overline{BC}$ consists of a single point. Consequently, since $\overline{BC}$ is a subset of the interior of $\angle CAB$ and, by (1), no point of $\overset{\bullet\!\!\rightarrow}{AD'}$ is interior to $\angle CAB$, it follows that $\overleftrightarrow{AD} \cap \overline{BC} = \overrightarrow{AD} \cap \overline{BC}$. Hence, $\overrightarrow{AD} \cap \overline{BC}$ consists of a single point.

*

$\overline{LK}$--that is, to the fact that the apparently unlikely event suggested at the end of the solution for Exercise 2, does not occur. To see that this is the case, we shall now prove some additional Introduction Theorems.

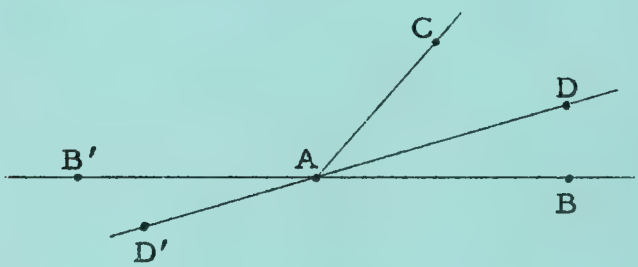
To begin with, recall that, according to the second result given in the COMMENTARY for page 6-71,

if D is interior to $\angle CAB$ then $\overrightarrow{AD}$ is a subset of the interior of $\angle CAB$.

We can strengthen this result by showing that

(1) if D is interior to $\angle CAB$ then the intersection of $\overleftrightarrow{AD}$ and the interior of $\angle CAB$ is $\overrightarrow{AD}$.

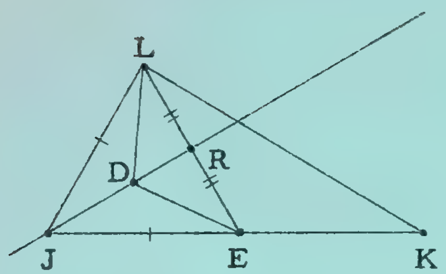
To do so, since A is not interior to $\angle CAB$, it is sufficient, by Theorem 14 of page 6-27, to show that if A is between D and D′ [and D is interior to $\angle CAB$] then no point of $\overrightarrow{AD'}$ is interior to $\angle CAB$. Now, using Theorem 16 of page 6-27, D′ and D are on opposite sides of $\overleftrightarrow{AB}$ and, since C and



D are on the same side of $\overleftrightarrow{AB}$, it follows that C and D′ are on opposite sides of $\overleftrightarrow{AB}$. Hence, using Theorem 18, each point of $\overrightarrow{AD'}$ is on the side of $\overleftrightarrow{AB}$ opposite C. So, no point of $\overrightarrow{AD'}$ is on the C-side of $\overleftrightarrow{AB}$; whence, by definition, no point of $\overrightarrow{AD'}$ is interior to $\angle CAB$. Consequently, (1) is established.

Next, we need to show that

2.



The matching JDL $\leftrightarrow$ JDE is a congruence if and only if JE = JL and LD = ED. So, to satisfy the conditions of the problem, E must be the point of $\overrightarrow{JK}$ such that JE = JL. The problem can now be solved by locating a point D interior to ΔJKL and equidistant from L and E. The point most obviously equidistant from L and E is the midpoint, R, of $\overline{LE}$ and [see the COMMENTARY for page 6-71] this, like any point of $\overline{LE}$, is at least interior to $\angle$LJK. So, if R is on the J-side of $\overleftrightarrow{LK}$, the problem can be solved by taking D to be R. In general, however, this is not the case. But, since JRL $\leftrightarrow$ JRE is, in any case, a congruence, $\angle$LJR $\cong$ $\angle$EJR. From this, together with the fact that LJ = EJ, it follows that each point of $\overleftrightarrow{JR}$ is equidistant from L and E. Since [see the COMMENTARY for page 6-71], because R is interior to $\angle$LJK, each point of $\overrightarrow{JR}$ is interior to $\angle$LJK, it suffices to choose for D any point of $\overrightarrow{JR}$ which is on the J-side of $\overleftrightarrow{LK}$. If $\overrightarrow{JR}$ intersects $\overleftrightarrow{LK}$ in a point F then, using Introduction Theorem 18, any point between J and F will do for D. And, in the apparently unlikely event that $\overrightarrow{JR}$ and $\overleftrightarrow{LK}$ do not intersect, then, using Introduction Theorem 15, D might be chosen anywhere on $\overrightarrow{JR}$.

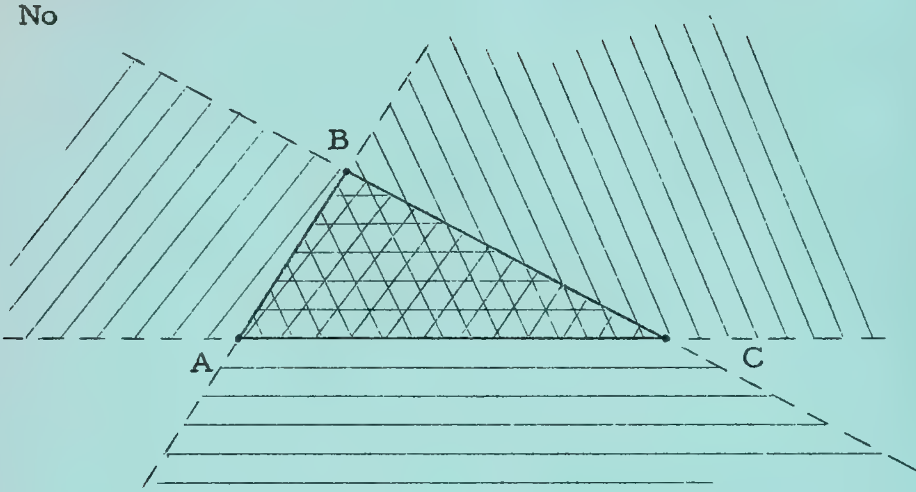
Note that this exercise introduces ideas relating to the notions both of perpendicular bisector [$\overleftrightarrow{JR}$ is the perpendicular bisector of $\overline{LE}$] and of angle bisector [$\overrightarrow{JR}$ is the angle bisector of $\angle$LJE]. It may be thought of as exploration for Theorem 3-3 on page 6-94, Theorem 3-7 on page 6-107, and Theorem 4-17 on page 6-133.

*

In a later COMMENTARY we shall want to make use of the fact that a half-line which, like $\overrightarrow{JR}$, is interior to an angle, $\angle$LJK, does intersect

Answers for Part D

1. No

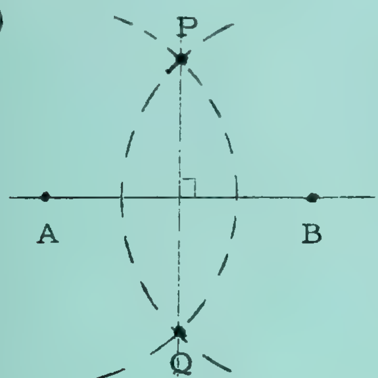


Suppose P belongs to $i(\angle BAC) \cap i(\angle BCA)$. Then, $P \in i(\angle BAC)$ and $P \in i(\angle BCA)$. By the definition of the interior of angle, since $P \in i(\angle BAC)$, it follows that P belongs to the C-side of $\overleftrightarrow{AB}$ [and to the B-side of $\overleftrightarrow{AC}$]. Also, since $P \in i(\angle BCA)$, P belongs to the A-side of $\overleftrightarrow{BC}$ [and to the B-side of $\overleftrightarrow{AC}$]. So, again by definition, since P belongs to the C-side of $\overleftrightarrow{AB}$ and to the A-side of $\overleftrightarrow{BC}$, it follows that $P \in i(\angle CBA)$. Therefore, if $P \in i(\angle BAC) \cap i(\angle BCA)$ then $P \in i(\angle CBA)$. So, if P belongs to the intersection of the interiors of two angles of a triangle, it also belongs to the interior of the third angle. That is, it belongs to the intersection of all three interiors. So, by definition, it belongs to the interior of the triangle.

The foregoing argument proves the theorem that the interior of a triangle is the intersection of the interiors of any two angles of the triangle. Similar arguments show that the interior of $\triangle ABC$ is the intersection of the A-side of $\overleftrightarrow{BC}$, the B-side of $\overleftrightarrow{CA}$, and the C-side of $\overleftrightarrow{AB}$, and, so, that it is the intersection of the interior of $\angle C$ and the C-side of $\overleftrightarrow{AB}$.

Answers for Part A [on pages 6-94 and 6-95].

1. (a)



Since $AP = AQ = BP = BQ$, the points P and Q are equidistant from A and B. So, P and Q determine the perpendicular bisector of $\overline{AB}$.

2. Find the midpoint by drawing the perpendicular bisector.

3. Locate two points A and B on the given line such that the given point is the midpoint of $\overline{AB}$. Then, since the given point is on the perpendicular bisector of $\overline{AB}$, just find one more point on the perpendicular bisector. The perpendicular bisector of $\overline{AB}$ is the perpendicular to the given line at the given point.

*

Answers for Part B [on page 6-95].

1. Since M is the midpoint of $\overline{BC}$, M is equidistant from B and C. Also, by hypothesis, A is equidistant from B and C. From the figure, $B \neq C$. So, by the only-if-part of Theorem 3-3, A and M are points on the perpendicular bisector of $\overline{BC}$. From the figure we see that A and M are two points. So, they determine the perpendicular bisector of $\overline{BC}$. Hence, $\overleftrightarrow{AM} \perp \overline{BC}$. Since $\overline{AM} \subseteq \overleftrightarrow{AM}$, $\overline{AM} \perp \overline{BC}$. Finally, since $\{M\} = \overleftrightarrow{AM} \cap \overleftrightarrow{BC}$ and $M \in \overline{BC}$, $\overline{AM} \perp \overline{BC}$ at M.

2. We see from the figure that $B \neq C$ and $A \neq D$. By hypothesis, A and D are each equidistant from B and C. So, $\overleftrightarrow{AD}$ is the perpendicular bisector of $\overline{BC}$. Since $\overline{AD} \subseteq \overleftrightarrow{AD}$, $\overline{AD} \perp \overline{BC}$.

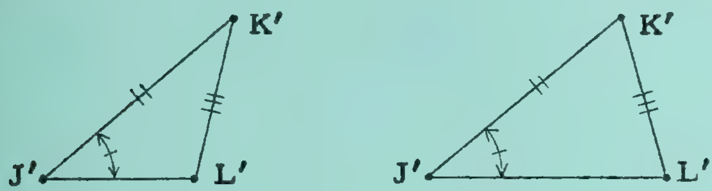
Answers for Exploration Exercises [on pages 6-96 and 6-97].

A. 2. Yes [s.s.s.]; no

B. 2. Yes [s.a.s.]; no

C. 2. Yes; no [intuition]

D. 2. Yes [or: no]; yes



E. 2. Yes; no [intuition]

F. 2. Yes [or: no]; yes

G. 2. Yes; no [intuition]

*

Parts C, D, and E of the Exploration Exercises, in addition to providing foils for the a.s.a. congruence theorem suggested in Part F, call student's attention to the "ambiguous case" and foreshadow the resolution of this ambiguity given in Theorem 4-13 and Theorem 4-14 on page 6-129. In Parts C, D, and E, students are asked to consider pairs of matched triangles for which two pairs of corresponding sides are congruent and the angles opposite the members of one of these pairs of sides are also congruent. Part C may suggest the conclusion that this is sufficient in order that the triangles be congruent, but Part D should correct this error. Reconsideration of these exercises may suggest [Part C] that two such triangles <u>are</u> congruent if the angles which are specified as being congruent are obtuse, but [Part D] that two such triangles need not be congruent if the angles in question are acute. Further consideration of Part D may suggest that the ambiguity of the case of acute angles can be resolved by specifying that the angles opposite the other two congruent sides be both obtuse or both acute. Part E suggests another resolution--the triangles are congruent if the sides opposite the angles specified to be congruent are longer than the other congruent sides.

<u>Corrections.</u> On page 6-99, line 15 should read: (4) --- [Step like (1)] and line 3b should read: (19) --- [Step like (16)]

You may wish to motivate the proof on page 6-98 by an intuitive superposition argument. If you pick up ΔABC and place it on $\Delta A'B'C'$ so that A fits on A', B on B', and C and C' are on the same side of $\overleftrightarrow{A'B'}$, then $\overrightarrow{AC}$ will fit on $\overrightarrow{A'C'}$ and $\overrightarrow{BC}$ will fit on $\overrightarrow{B'C'}$. The problem, then, is to show that C fits on C'.

$*$

line 8 on page 6-98. Since A', B', and C' are noncollinear, it follows that $\overleftrightarrow{A'C'} \cap \overleftrightarrow{B'C'} = \{C'\}$. So, since $\overrightarrow{A'C'} \subseteq \overleftrightarrow{A'C'}$ and $\overrightarrow{B'C'} \subseteq \overleftrightarrow{B'C'}$, and since $C' \in \overrightarrow{A'C'} \cap \overrightarrow{B'C'}$, we know that C' is the only point which belongs to $\overrightarrow{A'C'}$ and $\overrightarrow{B'C'}$. Hence, $P = C'$.

'$C'A'B' \leftrightarrow CAB$ is a congruence' follows from '$P = C'$' and (*) by substitution.

Answers for Part A [on pages 6-100 and 6-101].

1.

(1)	$\angle AED$ and $\angle AEB$ are adjacent angles with their noncommon sides collinear	[figure]
(2)	[Theorem 2-9 on page 6-78]	[theorem]
(3)	$\angle AED$ is a supplement of $\angle AEB$	[(1) and the if-part of (2)]
(4)	$\angle CED$ is a supplement of $\angle CEB$	[Step like (1)]
(5)	$\angle AEB \cong \angle CEB$	[Hypothesis]
(6)	[Theorem 2-3 on page 6-78]	[theorem]
(7)	$\angle AED \cong \angle CED$	[(3), (4), (5), and (6)]
(8)	$\overline{ED} \cong \overline{ED}$	[Identity; def. of congruent segments]
(9)	$\angle ADE \cong \angle CDE$	[Hypothesis]
(10)	A, E, D and C, E, D are vertices of triangles	[figure]
(11)	a. s. a.	[theorem]
(12)	$AED \longleftrightarrow CED$ is a congruence	[(10), (7), (8), (9), and (11)]
(13)	$\overline{AE} \cong \overline{CE}$	[(12); def. of congruence]

[Note step (9). Actually, the hypothesis tells us that $\angle ADB \cong \angle CDB$. But, since $E \in \overrightarrow{DB}$, we can use an Introduction Theorem to prove that $\overrightarrow{DE} = \overrightarrow{DB}$. Then, by using the definition of an angle and substitution, we can show that $\angle ADE \cong \angle CDE$. This is the kind of gap in the proof that we can ignore.]

Paragraph proof for Exercise 1:

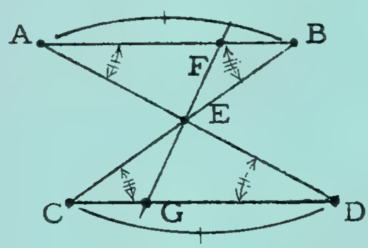
From the figure, we see that $\angle AED$ and $\angle AEB$ are adjacent angles with their noncommon sides collinear. So, they are supplementary (1). Similarly, $\angle CED$ and $\angle CEB$ are supplementary. Since, by hypothesis, $\angle AEB \cong \angle CEB$, it follows from an earlier theorem (2) that $\angle AED \cong \angle CED$. Now, $\overline{ED} \cong \overline{ED}$, and, by hypothesis, $\angle ADE \cong \angle CDE$. So, by a.s.a., $AED \leftrightarrow CED$ is a congruence, Hence, $\overline{AE} \cong \overline{CE}$.

---

(1) [Theorem 2-9 on page 6-78]

(2) [Theorem 2-3 on page 6-78]

4. Students can organize the data in a problem of this type into Hypothesis-Conclusion format, listing all of the suppositions in the hypothesis. But, to save time and writing, it is good practice to put as many of the suppositions into the figure as possible. [Students should be encouraged to write very informal paragraph proofs for problems of the "show that" variety].

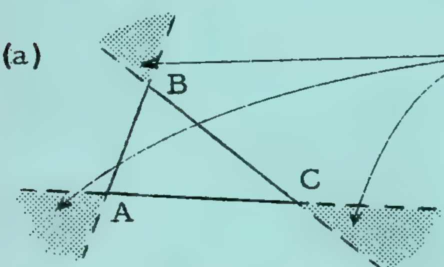


$ABE \leftrightarrow DCE$ is a congruence by a.s.a. So, $\overline{AE} \cong \overline{ED}$. Also, $\angle AEF \cong \angle DEG$ since they are vertical angles. Therefore, by a.s.a., $AEF \leftrightarrow DEG$ is a congruence. So, $\overline{FE} \cong \overline{GE}$.

[It is interesting to note that if the line $\overleftrightarrow{FG}$ pivots about the point E with $F \in \overleftrightarrow{AB}$ and $G \in \overleftrightarrow{CD}$ then E is the midpoint of $\overline{FG}$.]

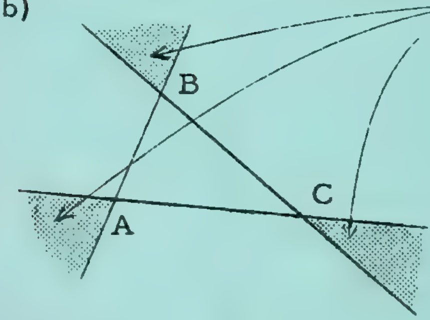
5. Since $\overleftrightarrow{MN} \perp \overleftrightarrow{BD}$, $\angle MDB$ and $\angle NDB$ are right angles. So, they are congruent. Also, $\overline{BD} \cong \overline{BD}$. So, since $\angle DBA \cong \angle DBC$, it follows from a.s.a. that $BDM \leftrightarrow BDN$ is a congruence. Therefore, $\angle DMB \cong \angle DNB$.

6. (a)



This is the intersection of the exteriors of the three angles of ΔABC. It is not the same thing as the exterior of ΔABC.

(b)



This is not the same set as in part (a). This set includes the sides of the vertical angles of the angles of ΔABC. [Ask students to sketch the union of the exteriors of the angles of ΔABC. Is this the same thing as the exterior of ΔABC?]

(2) $\angle A \cong \angle C$	[Hypothesis]
(3) $\overline{AB} \cong \overline{CB}$	[Hypothesis; def. of mid-point]
(4) G is interior to $\angle CBF$	[figure]
(5) [Axiom F on page 6-78]	[axiom]
(6) $m(\angle CBF) = m(\angle CBG) + m(\angle GBF)$	[(4) and (5)]
(7) $m(\angle ABG) = m(\angle ABF) + m(\angle GBF)$	[Step like (4)]
(8) $m(\angle ABF) = m(\angle CBG)$	[Hypothesis; def. of cong. angles]
(9) $\angle ABG \cong \angle CBG$	[(6), (7), and (8); def. of cong. angles]
(10) a.s.a	[theorem]
(11) $ABG \longleftrightarrow CBF$ is a congruence	[(1), (2), (3), (9), and (10)]
(12) $\overline{AG} \cong \overline{CF}$	[(11); def. of congruence]

Paragraph Proof of Exercise 3:

For the triangles ΔABG and ΔCBF, we are given that $\angle A \cong \angle C$ and $\overline{AB} \cong \overline{CB}$. Since F is interior to $\angle ABG$ and G is interior to $\angle CBF$, it follows from an axiom (1) that $m(\angle ABG) = m(\angle ABF) + m(\angle FBG)$ and that $m(\angle CBF) = m(\angle CBG) + m(\angle FBG)$. But, by hypothesis, $m(\angle ABF) = m(\angle CBG)$. So, $m(\angle ABG) = m(\angle CBF)$. Hence, by a.s.a., $ABG \longleftrightarrow CBF$ is a congruence; so, $\overline{AG} \cong \overline{CF}$.

---

(1) [Axiom F on page 6-78]

2\.

(1)	$\angle BCA$ and $\angle DCE$ are vertical angles	[figure]
(2)	[Theorem 2-5 on page 6-78]	[theorem]
(3)	$\angle BCA \cong \angle DCE$	[(1) and (2)]
(4)	$\overline{AD}$ bisects $\overline{BE}$	[Hypothesis]
(5)	$\{C\} = \overline{AD} \cap \overline{BE}$	[figure]
(6)	$\overline{BC} \cong \overline{CE}$	[(4) and (5); defs. of bisect and midpoint]
(7)	$\angle B \cong \angle CED$	[Hypothesis]
(8)	A, B, C and D, E, are vertices of triangles	[figure]
(9)	a.s.a.	[theorem]
(10)	$ABC \longleftrightarrow DEC$ is a congruence	[(8), (3), (6), (7), and (9)]
(11)	$\overline{AC} \cong \overline{CD}$	[(10); def. of congruence]
(12)	$\overline{BE}$ bisects $\overline{AD}$	[(5) and (11); defs. of midpoint and bisect]

Paragraph proof of Exercise 2:

For the triangles $\triangle ABC$ and $\triangle DEC$, we are given that $\angle B \cong \angle DEC$. Since $\angle BCA$ and $\angle ECD$ are vertical angles, they are congruent (1). Since, by hypothesis, $\overline{AD}$ bisects $\overline{BE}$, and since C is the common point of these segments, it follows that $\overline{BC} \cong \overline{EC}$. So, by a.s.a, $ABC \longleftrightarrow DEC$ is a congruence. Hence, $\overline{AC} \cong \overline{DC}$. So, since $C \in \overline{AD}$, $\overline{BE}$ bisects $\overline{AD}$.

---

(1) [Theorem 2-5 on page 6-78.]

3\. Plan. Show that $ABG \longleftrightarrow CBF$ is a congruence.

(1) A, B, G and C, B, F are vertices of triangles [figure]

Answers for Part B.

1. (1) $\angle A \cong \angle B$ [Hypothesis]

   (2) $\overline{AB} = \overline{BA}$ [Identity; def. of cong. segments]

   (3) $\angle B \cong \angle A$ [(1); def. of cong. angles]

   (4) A, B, C and B, A, C are vertices of triangles [figure]

   (5) a.s.a. [theorem]

   (6) $ABC \longleftrightarrow BAC$ is a congruence [(4), (1), (2), (3), and (5)]

2. (7) $\overline{BC} \cong \overline{AC}$ [(6); def. of congruence]

*

Answers for Part C [on page 6-102].

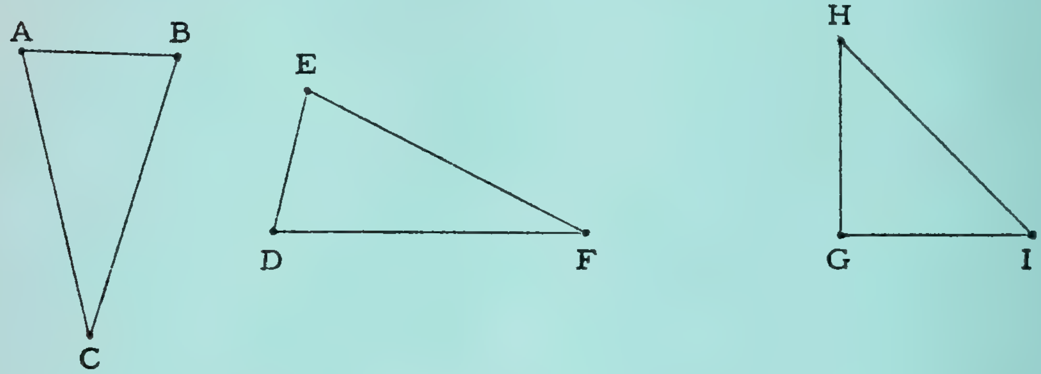
1. In triangles ΔACB and ΔBCA, $\overline{AC} \cong \overline{BC}$ by hypothesis. So, $\overline{CB} \cong \overline{CA}$. Also, by identity, $\overline{AB} \cong \overline{BA}$. Hence, by s.s.s., $ACB \longleftrightarrow BCA$ is a congruence. [Students might also note that $\angle ACB \cong \angle BCA$ and use s.a.s.]

2. Since, by Exercise 1, $ACB \longleftrightarrow BCA$ is a congruence, it follows that $\angle B \cong \angle A$.

*

Note that the substitution rule for biconditional sentences justifies inferring that a triangle is isosceles if and only if two of its angles are congruent from the definition of an isosceles triangle and Theorem 3-5. If one wanted to, he could use this inferred result as the definition and then derive from it and Theorem 3-5 what is now the definition. Which of the two statements is actually called 'the definition of an isosceles triangle' is a matter of custom.

In checking a student's understanding of the terms 'legs', 'base', 'base angle', and 'vertex angle', it is a good idea to draw pictures of isosceles triangles in various positions.



Then, ask questions such as the following:

(1) If you are told that ΔABC is isosceles, can you tell which of its sides are the legs? [Answer: no]

(2) If $\angle C$ is the vertex angle of the isosceles triangle ΔABC, which sides are the legs? Which angles are the base angles?

(3) If $\angle E \cong \angle D$, is ΔEDF isosceles? What tells you this? [Answer: the if-part of Theorem 3-5 and the definition of isosceles triangle]

(4) If $HG = GI$, is ΔHIG isosceles? What tells you this? [Answer: the definition of isosceles triangle]

*

The familiar "base angles of an isosceles triangle are congruent" theorem is a consequence of Exercise 2 of Part C on page 6-102. It follows from the only-if-part of Theorem 3-5 [and the definition of base angles of an isosceles triangle].

Correction. On page 6-105, line 11b should read: ---, $\angle DCA \cong \angle DCB$. ↑

For a definition of 'corollary' see page 6-27.

*

The word 'equilateral' refers to the fact that the three sides of the triangle have equal measures, not that the sides are equal. Similarly, 'equiangular' refers to the fact that the three angles of the triangle have equal measures.

Since two sides of an equilateral triangle are congruent, it follows that the triangle is isosceles.

*

Proof of Theorem 3-6:

By definition, ΔLMN is equilateral if and only if $LM = MN = NL$. By Theorem 3-5, $LM = MN = NL$ if and only if $\angle N \cong \angle L \cong \angle M$. By definition, $\angle N \cong \angle L \cong \angle M$ if and only if ΔLMN is equiangular. So, ΔLMN is equilateral if and only if ΔLMN is equiangular. Consequently, a triangle is equilateral if and only if it is equiangular.

*

Notice that in the column proof on page 6-105, step (1) is justified by the two assumptions 'ΔABC is isosceles' and '$\angle C$ is its vertex angle' together with the definitions of isosceles triangle and vertex angle of an isosceles triangle.

*

Answers for Part A [on page 6-105].

1. six
2. two or six
3. one

> Corrections. On page 6-106, the figure for Exercise 1 should include a '┐' to show that $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ [as in the figure for Exercise 2].
>
> On page 6-108, line 2b should read:
> (8) --- [Step like (5)]
> ↑ ↑ ↑

Answers for Part B.

[We give just brief outlines of proofs.]

1. $\angle ACD \cong \angle BCD$, $CD = CD$, and $\angle CDA \cong \angle CDB$. So, $ACD \leftrightarrow BCD$ is a congruence. Hence, $AC = BC$.

2. $CD = CD$, $\angle CDA \cong \angle CDB$, and $DA = DB$. So, $CDA \leftrightarrow CDB$ is a congruence. Hence, $AC = BC$.

   Alternative: Just use Theorem 3-3.

3. Since $AB = BC$, $\angle A \cong \angle C$. Also, since $DE = EC$, $\angle EDC \cong \angle C$. So, by the definition of congruent angles [and substitution], $\angle A \cong \angle EDC$.

4. $\angle MRP \cong \angle NSP$ because they are supplements of congruent angles. By Theorem 3-5, $RP = SP$. Since $\angle RPM \cong \angle SPN$ by hypothesis, it follows from a.s.a. that $MRP \leftrightarrow NSP$ is a congruence. So, $MP = NP$.

☆5. $\angle A \cong \angle B \cong \angle C$. Since $AM = MB = BN = NC = CP = PA$ it follows that $PAM \leftrightarrow MBN$, $MBN \leftrightarrow NCP$, and $NCP \leftrightarrow PAM$ are congruences. So, $PM = MN = NP$.

intersect in a single point, it is enough to establish the more general result that

> if, in $\triangle ABC$, $D \in \overline{AC}$ and $E \in \overline{AB}$ then $\overline{BD} \cap \overline{CE}$ consists of a single point.

To show this, we note that, since $D \in \overline{AC}$, D is interior to $\angle ABC$ and that since $E \in \overline{AB}$, $\angle EBC = \angle ABC$. So, D is interior to $\angle EBC$ and, by the previously mentioned result, $\overrightarrow{BD} \cap \overline{EC}$ consists of a single point. Similarly, $\overrightarrow{CE} \cap \overline{BD}$ consists of a single point. Now, since $A \notin \overleftrightarrow{BC}$, $\overleftrightarrow{BD} \cap \overleftrightarrow{CE}$ consists of at most one point. [Otherwise, $\overleftrightarrow{BD} = \overleftrightarrow{CE}$, from whence it follows, first, that $D \in \overleftrightarrow{BC}$, and, then, that $A \in \overleftrightarrow{BC}$.] But, since $\overrightarrow{BD} \subseteq \overleftrightarrow{BD}$ and $\overline{EC} \subseteq \overleftrightarrow{CE}$, the point of intersection of $\overrightarrow{BD}$ and $\overline{EC}$ belongs to $\overleftrightarrow{BD} \cap \overleftrightarrow{CE}$, and this latter, then, consists of this single point. It results from a similar argument that, also, $\overrightarrow{CE} \cap \overline{BD} = \overleftrightarrow{BD} \cap \overleftrightarrow{CE}$. Hence, the unique point of intersection of $\overleftrightarrow{BD}$ and $\overleftrightarrow{CE}$ belongs to both $\overline{EC}$ and $\overline{BD}$. So, $\overline{EC} \cap \overline{BD}$ consists of this single point.

Answers for Part B.

1. By Theorem 3-8, $m(\angle FBC) = \frac{1}{2} \cdot m(\angle ABC)$ and $m(\angle FCB) = \frac{1}{2} \cdot m(\angle ACB)$. But, by Theorem 3-5, $\angle ABC \cong \angle ACB$. So, $\angle FBC \cong \angle FCB$. Hence, by Theorem 3-5, $FC = FB$.

2. $FB = FC$, $\angle FBA \cong \angle FCA$, and $BA = CA$. So, $FBA \leftrightarrow FCA$ is a congruence. Hence, $\angle FAB \cong \angle FAC$. Assuming from the figure that F is in the interior of $\angle A$, it follows from the definition of angle bisector that $\overrightarrow{AF}$ is the bisector of $\angle A$.

*

In order to avoid awkward questions in class, we have included a gratuitous assumption in the Hypothesis of Exercise 1 of Part B. This assumption is that the bisectors of $\angle B$ and $\angle C$ intersect in a single point, F. It is gratuitous in that it can be proved, using the Introduction Axioms and the definition of angle bisector, that the bisectors of each two angles of a triangle intersect in a single point. [One needs this preliminary result in the proof [see page 6-134] that the angle bisectors of a triangle are concurrent.]

In fact, we have already shown [see result (3) in the COMMENTARY for page 6-93] that

if D is interior to $\angle ABC$ then $\overrightarrow{BD} \cap \overline{AC}$ consists of a single point.

From this, and the definition of angle bisector, it follows that

the bisector of $\angle B$ of $\triangle ABC$ intersects $\overline{AC}$ in a single point

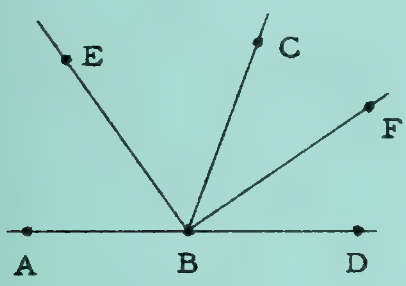
and that

the bisector of $\angle C$ of $\triangle ABC$ intersects $\overline{AB}$ in a single point.

So, in order to show that the bisectors of each two angles of a triangle

Answers for Part C.

1.



Hypothesis: $\angle ABC$ and $\angle CBD$ are adjacent supplementary angles,

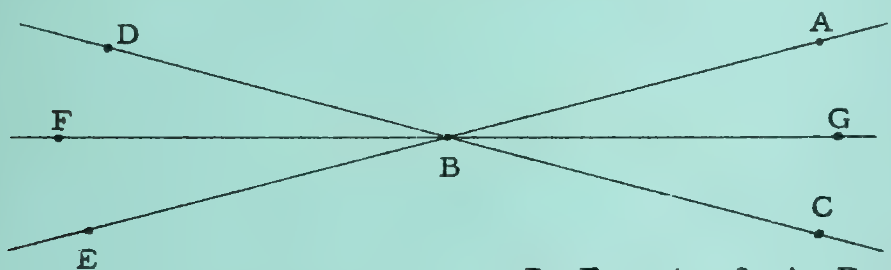
$\overrightarrow{BE}$ is the bisector of $\angle ABC$,

$\overrightarrow{BF}$ is the bisector of $\angle CBD$

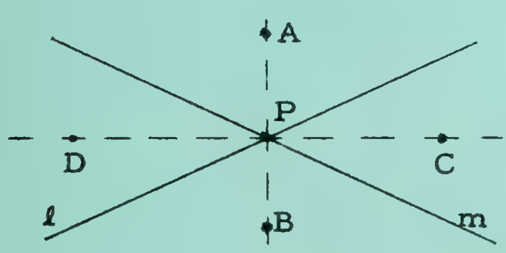
Conclusion: $\overrightarrow{BE} \perp \overrightarrow{BF}$

[See COMMENTARY for Exercise 11 on page 6-63.]

2. Since $\angle ABC \cong \angle DBE$, it follows from Theorem 3-8 that $\angle ABG \cong \angle FBE$. So, as in Exercise 2 on page 6-73, F, B, and G are collinear.



3.



By Exercise 2, A, P, and B are collinear and C, P, and D are collinear. By Exercise 1, $\overrightarrow{PA} \perp \overrightarrow{PC}$. So, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are perpendicular lines.

*

Answer for Part D.

$\angle AC'B$ is a supplement of $\angle BC'C$. $\angle BC'C \cong \angle C$.

*

Answers for Quiz.

1. 2.5; 90

2. 9

3. $\overline{JK}$; $\overline{MK}$

4. ADB $\leftrightarrow$ CBE

5. BAD $\leftrightarrow$ CDA; s.a.s.

6. By Theorem 3-3, BA = BC and CA = CB. So, $\overline{AB} \cong \overline{AC}$.

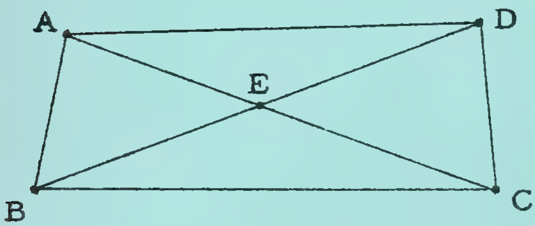
7. BAD $\leftrightarrow$ CAD is a congruence by s.a.s. So, BD = CD. Similarly, BAE $\leftrightarrow$ CAE is a congruence. So, BE = CE. But, BD = BE. So, CD = CE.

Quiz.

1. Suppose that A, B, and C are three collinear points and that $B \in \overline{AC}$. If D is a point such that $DA = 10 = DC$ and $\angle ADB \cong \angle CDB$, and if $AC = 5$ then $AB =$ _____ and $m(\angle DBA) =$ _____.

2. Suppose that $\angle RPT \cong \angle SQU$, that $TR = PT = 9 = QS$, and that $UQ = 4 = RP$. Then, $US =$ _____.

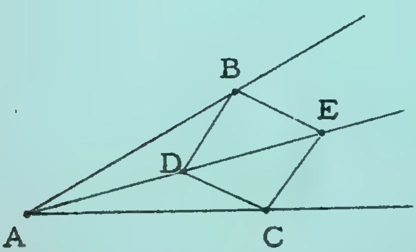
3. If $\triangle MJK$ is isosceles and $\angle K$ is the vertex angle then the legs are _____ and _____.

4. 

Consider the triangles $\triangle ADB$ and $\triangle EBC$. Give a matching of the vertices for which $\overline{AD}$ and $\overline{BC}$, $\overline{AB}$ and $\overline{CE}$, are pairs of corresponding sides.

5. [Refer to the diagram in Exercise 4.] If $AB = DC$ and $\angle BAD \cong \angle ADC$, give a matching of the vertices of $\triangle BAD$ with those of $\triangle ADC$ which is a congruence. What triangle-congruence theorem tells you that this matching is a congruence?

6. Suppose that D is a point on side $\overrightarrow{AC}$ of $\angle BAC$ such that $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$ and that E is a point on $\overrightarrow{AB}$ such that $\overleftrightarrow{CE}$ is the perpendicular bisector of $\overline{AB}$. Show that $\overline{AB} \cong \overline{AC}$.

7. 

Hypothesis: $\overrightarrow{AE}$ is the bisector of $\angle BAC$,

$\triangle BDE$ is isosceles with vertex angle $\angle DBE$,

$AB = AC$

Conclusion: $\triangle CED$ is isosceles

Answers for Part ☆C [on page 6-113].

1. By Theorem 3-3, AD = FD. So, AD + DB = FD + DB. Since D $\in \overleftrightarrow{FB}$, FD + DB = FB. Hence, AD + DB = FB.

2. As in Exercise 1, AP + PB = FP + PB. Since F, B, and P are non-collinear, it follows from Theorem 4-1 that FP + PB > FB. So, since FB = AD + DB, AP + PB > AD + DB.

3. In view of Exercises 1 and 2, the "minimizing" point is the point of intersection of $\overleftrightarrow{BF}$ and $\overleftrightarrow{CE}$ where F is the point such that AC = CF and $\overleftrightarrow{AF} \perp \overleftrightarrow{CE}$. If Q is this point of intersection then, for each point X $\in \overleftrightarrow{CE}$ other than Q, AX + XB > AQ + QB. Since Q $\in \overline{FB}$, $\angle$FQC $\cong$ $\angle$BQE. Since, by s.s.s., AQC $\longleftrightarrow$ FQC is a congruence, $\angle$AQC $\cong$ $\angle$FQC. So, $\angle$AQC $\cong$ $\angle$BQE.

Part C is the basis of an interesting application. Suppose that $\overrightarrow{AP}$ is a ray of light and that $\overrightarrow{PB}$ is the ray of light reflected by the mirror $\overleftrightarrow{CE}$. Since light travels in such a way that the path it takes is always a minimum path, then AP + PB is a minimum. That is, P = Q. Let $\overrightarrow{RQ}$ be the half-line on the A-side of $\overleftrightarrow{CE}$ and perpendicular to $\overleftrightarrow{CE}$ at Q. Then, $\angle$AQR is called 'the angle of incidence' of the light ray $\overrightarrow{AQ}$, and $\angle$BQR is called its 'angle of reflection'. Since, $\angle$AQC $\cong$ $\angle$BQE, it follows that the angle of incidence of a ray of light is congruent to the angle of reflection of the ray by a plane mirror.

> Corrections. On page 6-113, line 14, delete the 'inter' which occurs at the end of the line. In line 1b, insert a period after '$\angle BQE$'.

The exercises on page 6-422 provide a brief review of inequations.

*

Answers for Part A.

1. no	2. no	3. no	4. yes	5. yes
6. yes	7. no	8. yes	9. no	

*

Answers for Part B.

1. Suppose the side-measures of a triangle are a, b, and c. Now, either $a \geq b$ or $a < b$. In the first case, since, by Theorem 4-1, $a < b + c$, it follows that $a - b < c$. In the second case, since $b < a + c$, it follows that $b - a < c$. So, in either case, $|a - b| < c$.

2. By Theorem 4-1, $BC < BD + DC$. But, by hypothesis, $AD = BD$. So, $BC < AD + DC$. From the figure, $D \in \overleftrightarrow{AC}$. So, by Axiom A, $AC = AD + DC$. Therefore, $BC < AC$.

☆3. [Note the implicit use, in the Hint, of result (3) of the COMMENTARY for page 6-93.]

Let E be the point of intersection of $\overrightarrow{AC}$ and $\overleftrightarrow{BD}$.

For ΔADE, $AE < AD + DE$. For ΔBCE, $CB < CE + EB$.

So, $$AE + CB < AD + DE + CE + EB.$$

But, $C \in \overleftrightarrow{AE}$ and $E \in \overleftrightarrow{DB}$.

So, $$(AC + CE) + CB < AD + (DE + EB) + CE$$

and $$AC + CB + CE < AD + DB + CE.$$

Hence, $$AC + CB < AD + DB.$$

*

CAM $\leftrightarrow$ C′BM is a congruence by s.a.s.

*

B $\notin \overleftrightarrow{CC'}$ because if B did belong to $\overleftrightarrow{CC'}$, so would A. [Since M is the midpoint of $\overline{AB}$, M $\in \overline{AB}$; so, B, M, and A are collinear.] But, we are given that A, B, and C are vertices of a triangle. Hence, they are non-collinear, and A does not belong to the line determined by B and C.

*

Answers for Exercises [on page 6-115].

A. Suppose that α and β are the measures of two angles of a triangle. Then, by Theorem 4-2, $\alpha + \beta < 180$. Hence, by the definition of supplementary angles, the angles whose measures are α and β are not supplementary.

B. Suppose that $\angle A$ of $\triangle ABC$ is a right angle or an obtuse angle. Then, $m(\angle A) \geq 90$. So, by Theorem 4-2, $m(\angle B) < 180 - m(\angle A) \leq 90$. Hence, $\angle B$ is acute. Similarly, $\angle C$ is acute.

C. Suppose that α, β, and γ are the measures of the angles of a triangle. Then, by Theorem 4-2,

$$\alpha + \beta < 180, \quad \beta + \gamma < 180, \quad \text{and} \quad \gamma + \alpha < 180.$$

So, $(\alpha + \beta + \gamma) + (\beta + \gamma + \alpha) < 540$. Hence, $\alpha + \beta + \gamma < 270$.

D. Since $\delta + \beta = 180$ and $\alpha + \beta < 180$, it follows that $\alpha + \beta < \delta + \beta$. So, $\alpha < \delta$. Similarly, $\gamma < \delta$.

*

A triangle has six exterior angles, two at each vertex. The two at each vertex are congruent.

*

Note the predicate 'is larger than' in Theorem 4-5. $\angle A$ is larger than $\angle B$ if and only if $m(\angle A)$ is greater than $m(\angle B)$.

Answers for Part E.

4. two; one

5. It must have at least two acute angles. It can have at most one right angle, and at most one obtuse angle.

6. An exterior angle of a triangle is a supplement of one of the angles of the triangle. Since each angle is acute, it follows that each exterior angle is obtuse.

7. Suppose $\angle C$ of $\triangle ABC$ is a right angle. Then, each of the exterior angles at A and at B is larger than $\angle C$. Hence, each such exterior angle is obtuse.

8. [As in Exercise 7.]

*

Answer for Part F.

No, since these two exterior angles may have the same vertex. If the exterior angles have different vertices, then the angles of the triangle to which they are adjacent and supplementary are also congruent. Hence, the triangle is isosceles by Theorem 3-5. [A triangle which has three congruent exterior angles is isosceles !]

*

Answer for Part G.

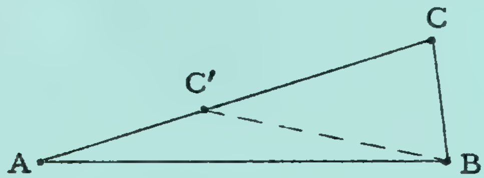
Since $B \in \overline{AC}$, $\angle ABD$ is an exterior angle of $\triangle BDC$. So, by Theorem 4-5, $\beta_1 > \gamma$. Since B is interior to $\angle ADC$, it follows from Axiom F that $m(\angle ADC) = \delta_1 + \delta_2$. By Axiom D, $\delta_2 > 0$. So, $m(\angle ADC) > \delta_1$. But, since $AD = AC$, $m(\angle ADC) = \gamma$. So, $\gamma > \delta_1$. Hence, $\beta_1 > \delta_1$.

*

Answer for Part ☆H.

By Theorem 4-1, $CC' < C'B + CB$. Since $CAM \longleftrightarrow C'BM$ is a congruence, $CA = C'B$. So, $CC' < CA + CB$. But, M is the midpoint of $\overleftrightarrow{CC'}$. Hence, $CM < \frac{1}{2}(CA + CB)$. [This exercise tells us that a median of a triangle is shorter than the average of the sides which "include" the median.]

Here is an approach to the proof of Theorem 4-6 which might help students discover the isosceles triangle gimmick. Consider the isosceles triangle ΔABC with $AB = AC$. If the side $\overleftrightarrow{AC}$ is shortened by sliding C



toward A, what happens to $\angle C$? It gets larger. Why? Because $\angle AC'B$ is an exterior angle of $\Delta BC'C$, $\angle AC'B$ is larger than $\angle C$. What happens to $\angle ABC$? It gets smaller. Why? Since C' is interior to $\angle ABC$, Axiom F tells us that $m(\angle ABC') + m(\angle C'BC) = m(\angle ABC)$. But, by Axiom D, $m(\angle C'BC) > 0$. So [by algebra--see Exercise 2(e) on page 6-422], $\angle ABC'$ is smaller than $\angle ABC$. Now, since $\angle C \cong \angle ABC$, it follows that $\angle AC'B$ is larger than $\angle ABC'$. So, by shortening one leg of an isosceles triangle, you change the base angles in such a way that the one opposite the longer leg is larger than the one opposite the shorter leg.

This suggests that given a triangle with one side longer than the other, you can tell which of the opposite angles is the larger by considering the isosceles triangle from which the given triangle was generated.

*

line 5. [See the COMMENTARY for Exercise 6(c) on page 6-421.]

line 9. $\angle ACB$ is an exterior angle of ΔBCD and $\angle D$ is one of the angles opposite it. So, by Theorem 4-5, $\angle ACB$ is larger than $\angle D$.

line 10. The base angles of an isosceles triangle are congruent.

line 8b. [See the COMMENTARY for Exercise 6(c) on page 6-421.]

line 7b. Since $\angle B$ is not larger than $\angle C$, it follows from Theorem 4-6 [and modus tollens] that $\overleftrightarrow{AC}$ is not longer than $\overleftrightarrow{AB}$.

line 5b. If $AC \not> AB$ then either $AC = AB$ or $AC < AB$. But, $AC \neq AB$. So, $AC < AB$. That is, $AB > AC$.

Answers for Part A.

1. $\angle R$; $\angle M$

2. $\overleftrightarrow{UB}$; $\overleftrightarrow{TB}$

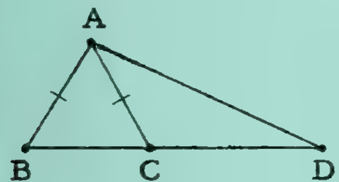
3. $AR \leq CR$

4. The smallest angle of ΔABC is $\angle BCA$. The smallest angle of ΔCDE is $\angle D$. Since $\angle BCA \cong \angle DCE$ [by Theorem 2-5], it follows that $\angle D$ is smaller than $\angle BCA$. So, $\angle D$ is the smallest of the six angles of ΔABC and ΔCDE.

*

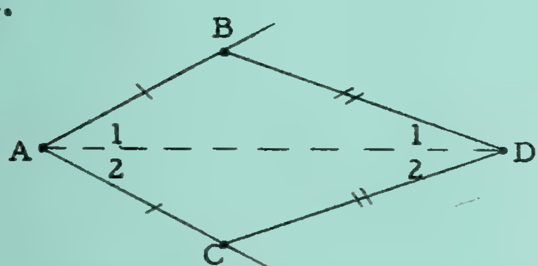
Answers for Part B.

1.



Since $\angle ACB$ is an exterior angle of ΔACD, it follows from Theorem 4-5 that $\angle ACB$ is larger than $\angle D$. But, by Theorem 3-5, $\angle ACB \cong \angle B$. So, $\angle B$ is larger than $\angle D$. Hence, by Theorem 4-7, $AD > AB$.

2.



Since D is in the interior of $\angle BAC$, $m(\angle BAC) = m(\angle A_1) + m(\angle A_2)$; and, since A is in the interior of $\angle BDC$, $m(\angle BDC) = m(\angle D_1) + m(\angle D_2)$. Now, since $AB < BD$ and $AC < CD$, it follows from Theorem 4-6 that $m(\angle A_1) > m(\angle D_1)$ and $m(\angle A_2) > m(\angle D_2)$. So, $m(\angle A_1) + m(\angle A_2) > m(\angle D_1) + m(\angle D_2)$. Hence, $\angle BAC$ is larger than $\angle BDC$.

[An interesting variation of Exercise 2 arises from stipulating that A is a point in the exterior of $\angle BDC$ rather than in the interior.]

Answers to questions in the text on page 6-119.

line 10. Suppose m and n are two lines through P and perpendicular to ℓ. Let the two points of intersection with ℓ be M and N, respectively. Then, $PM > PN$ and $PN > PM$; so, by algebra, $PM > PM$. But, $PM \not> PM$. So, there cannot be two lines through P and perpendicular to ℓ.

line 8b. $\overline{PP'}$ intersects ℓ because P and P′ are on opposite sides of ℓ.

line 7b. If $R \in \overrightarrow{QT}$ then $\angle PQT = \angle PQR$ and $\angle P'QT = \angle P'QR$; if $Q \in \overline{RT}$ then $\angle PQT$ is a supplement of $\angle PQR$ and $\angle P'QT$ is a supplement of $\angle P'QR$. In either case [and, since $R \neq Q$, there is no other], $\angle PQT \cong \angle P'QT$ because $\angle PQR \cong h \cup \overrightarrow{QR} = \angle P'QR$.

line 6b. Theorem 2-6 line 2b. QT line 1b. RT

*

Note that the distance between a point and a line ℓ has been defined conditionally: If $P \notin \ell$ then the distance between P and ℓ is PT where $\overleftrightarrow{PT}$ is the perpendicular to ℓ through P and $\overleftrightarrow{PT} \cap \ell = \{T\}$. It is natural to ask about the distance between P and ℓ if $P \in \ell$. The natural extension of the definition is to say that in such a case the distance is 0.

*

Answers for Part A [which begins on page 6-120].

1. (a) 6 (b) 4; 8 (c) 0; 4 (d) 12; 10

Corrections. On page 6-121, line 15 should read:
[Choose P_3 $\underbrace{\text{so that } T \in \overline{P_2P_3} \text{ and}}$ $P_1T = TP_3$. Then ---

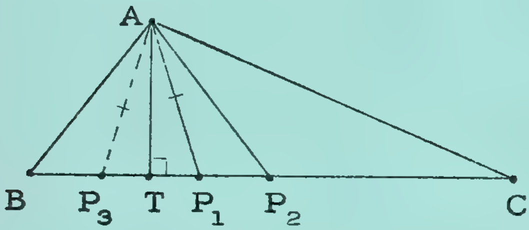
On page 6-122, line 7, delete the period after 'BISECTOR'.

2. (b) We assume from the figure that the perpendicular to $\overleftrightarrow{BC}$ through A intersects $\overline{BC}$. The measure of the perpendicular segment is the smallest value of the variable quantity y. So, since there is a perpendicular segment from A to $\overline{BC}$, y has a smallest value.

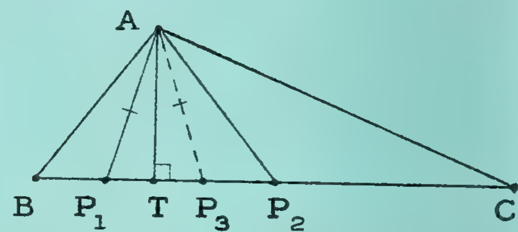
3. (a) 4

(b) $y(P_1) = y(P_2)$
[See the COMMENTARY for Exercise 2 on page 6-106.]

(c) Case I



Case II



There are two cases. In either case, see the COMMENTARY for Exercise 1 of Part B on page 6-118 and for Exercise 7 on page 6-138. In each case, $y(P_2) > y(P_1)$. [Theorem 4-9 takes care of the trivial case in which $P_1 = T$.]

4. 4

5. Since AC > AB, it follows from Theorem 4-6 that $\angle B$ is larger than $\angle C$.
(a) Yes. $\angle APC$ is an exterior angle of $\triangle ABP$; so, it is larger than $\angle B$. Since $\angle B$ is larger than $\angle C$, $\angle APC$ is larger than $\angle C$.

(b) Yes. Theorem 4-7.

6. Yes. By the same argument as in Exercise 5(a), $\angle APC$ is larger than $\angle B$. But, $\angle B \cong \angle C$; so, $\angle APC$ is larger than $\angle C$. Hence, by Theorem 4-7, AC > AP.

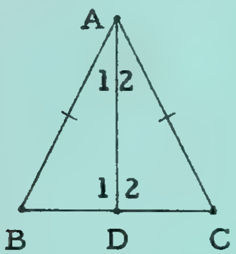
is isosceles with vertex angle at A. Consequently, if the altitude of $\triangle ABC$ from A is the angle bisector of $\triangle ABC$ from A then $\triangle ABC$ is isosceles with vertex angle at A.

3. Suppose that $\overleftrightarrow{AD}$ is the altitude of $\triangle ABC$ from A and $\overleftrightarrow{AD}$ is the median of $\triangle ABC$ from A. Then, $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$ and $BD = DC$. Since $\angle D_1$ and $\angle D_2$ are right angles, they are congruent. Also, $AD = AD$. So, by s.a.s., it follows that $BAD \longleftrightarrow CAD$ is a congruence. Therefore, $BA = CA$. So, $\triangle ABC$ is isosceles with vertex angle at A. Consequently, if the altitude of $\triangle ABC$ from A is the median of $\triangle ABC$ from A then $\triangle ABC$ is isosceles with vertex angle at A.

The assertion in the bracket at the top of page 6-123 is proved in the COMMENTARY for page 6-109.

*

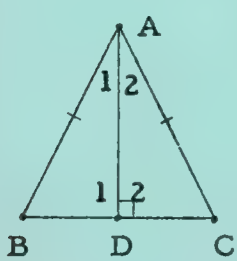
Answers for Part B.

1. 

Suppose that AB = AC and $\angle A_1 \cong \angle A_2$. Then, since AD = AD, it follows from s.a.s. that BAD $\leftrightarrow$ CAD is a congruence. So, $\angle D_1 \cong \angle D_2$. Since $\angle D_1$ and $\angle D_2$ are supplementary, it follows that they are right angles. Now, since $\overline{AD}$ is a subset of the interior of $\angle BAC$, and since $\angle A_1$ and $\angle A_2$ are congruent, it follows that $\overline{AD}$ is a subset of the bisector of $\angle BAC$. So, $\overline{AD}$ is the angle bisector of $\triangle ABC$ from A. Since $\overleftrightarrow{AD}$ is the perpendicular to $\overleftrightarrow{BC}$ through A, and since $\overline{AD} \subseteq \overleftrightarrow{AD}$, it follows that $\overline{AD}$ is the altitude of $\triangle ABC$ from A. Consequently, the angle bisector of $\triangle ABC$ from A is the altitude of $\triangle ABC$ from A.

Now, suppose that AB = AC and BD = DC. Then, since AD = AD, it follows from s.s.s. that BAD $\leftrightarrow$ CAD is a congruence. So, as above, $\overline{AD}$ is the altitude of $\triangle ABC$ from A. Since $\overline{AD}$ is the median of $\triangle ABC$ from A, it follows that the median of $\triangle ABC$ from A is the altitude of $\triangle ABC$ from A.

So [by substitution], the angle bisector, the median, and the altitude of an isosceles triangle from the vertex of the vertex angle are the same segment.

2. 

Suppose that $\overline{AD}$ is the altitude of $\triangle ABC$ from A and $\overline{AD}$ is the angle bisector of $\triangle ABC$ from A. Then $\overline{AD} \perp \overline{BC}$ and $\angle A_1 \cong \angle A_2$. Since $\angle D_1$ and $\angle D_2$ are right angles, they are congruent. Also, AD = AD. So, by a.s.a., BAD $\leftrightarrow$ CAD is a congruence. Therefore, BA = CA. So, $\triangle ABC$

Corrections. On page 6-124, line 7 should begin '$AC = A'C'$, $AB = A'B'$, and ---'.

Line 5b should read:
we conclude that $\underbrace{C'' \text{ is interior to } \angle BAC.}$

Answers for Part ☆C.

line 11. $\overline{AB}$ is not longer than $\overline{AC}$ if and only if $AB \leq AC$, and $\overline{AC}$ is not longer than $\overline{AB}$ if and only if $AC \leq AB$. So, these two cases are the only ones necessary to consider, since $\forall_x \forall_y\ x \geq y$ or $y \geq x$.

1. By Axioms D and E, there is a half-line h in the C-side of $\overleftrightarrow{AB}$ with vertex A such that $m(h \cup \overrightarrow{AB}) = m(\angle A')$. By Axiom C, there is a point C'' on h such that $AC'' = A'C'$. Since $C'' \in h$, C'' is in the C-side of $\overleftrightarrow{AB}$.

   So, $\angle C''AB \cong \angle A'$ and $\overline{AC''} \cong \overline{A'C'}$, and since $\overline{AB} \cong \overline{A'B'}$, it follows by s.a.s. that $ABC'' \longleftrightarrow A'B'C'$ is a congruence.

   So, there exists a point C'' in the C-side of $\overleftrightarrow{AB}$ such that $ABC'' \longleftrightarrow A'B'C'$ is a congruence.

2. If $C'' \in \overrightarrow{AC}$ then $m(\angle C''AB) = m(\angle CAB)$. But, $m(\angle CAB) > m(\angle A') = m(\angle C''AB)$; so, $m(\angle C''AB) \neq m(\angle CAB)$. Hence, $C'' \notin \overrightarrow{AC}$.

3. Since $m(\angle CAB) > m(\angle C''AB)$, $m(\angle CAB) + m(\angle CAC'') > m(\angle C''AB)$; so, $m(\angle CAB) + m(\angle CAC'') \neq m(\angle C''AB)$. Hence, by Axiom F, the point C is not interior to $\angle BAC''$.

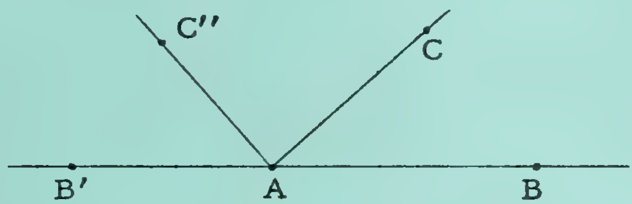
*

The conclusion in Exercise 3 follows from the results of Exercises 2 and 3, and the fact that

> if C and C'' are on the same side of $\overleftrightarrow{AB}$ then either $C'' \in \overrightarrow{AC}$ or C'' is interior to $\angle BAC$ or C is interior to $\angle BAC''$.

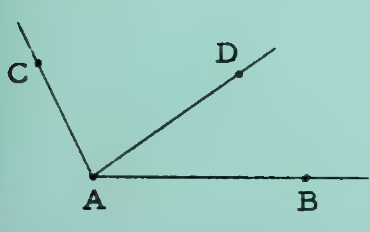
To establish this fact, we note that, since $A \neq B$ and $C \notin \overleftrightarrow{AB}$, $B \notin \overleftrightarrow{AC}$.

Also, since C'' is on the C-side of $\overleftrightarrow{AB}$, it follows that if $C'' \notin \overrightarrow{AC}$ then $C'' \notin \overleftrightarrow{AC}$. Consequently, if $C'' \notin \overrightarrow{AC}$ then either B and C'' are on the same side of $\overleftrightarrow{AC}$ or B and C'' are on opposite sides of $\overleftrightarrow{AC}$. Under the first alternative, since C'' is on the B-side of $\overleftrightarrow{AC}$ and on the C-side of $\overleftrightarrow{AB}$, it follows that C'' is interior to $\angle BAC$. Under the second alternative, we have C'' and C on the same side of $\overleftrightarrow{AB}$ and B and C'' on opposite sides of $\overleftrightarrow{AC}$. To conclude that, in this case, C is in the interior of $\angle BAC''$, we need to deduce that B and C are on the same side



of $\overleftrightarrow{AC''}$. To do so, choose B' so that $A \in \overline{B'B}$. Then, B' and B are on opposite sides of $\overleftrightarrow{AC''}$ and on opposite sides of $\overleftrightarrow{AC}$. Since, by hypothesis, B and C'' are on opposite sides of $\overleftrightarrow{AC}$, it follows that B' and C'' are on the same side of $\overleftrightarrow{AC}$. But, by hypothesis, C'' and C are on the same side of $\overleftrightarrow{AB'}$. So, C'' is interior to $\angle B'AC$. Consequently, by result (2) in the COMMENTARY for page 6-93, B' and C are on opposite sides of $\overleftrightarrow{AC''}$. Since, as noted above, B' and B are on opposite sides of $\overleftrightarrow{AC''}$, it follows that B and C are on the same side of $\overleftrightarrow{AC''}$.

The result (2) in the COMMENTARY for page 6-93 and the result obtained, above, in treating the second alternative can be combined:



If C and D are on the same side of $\overleftrightarrow{AB}$ then B and C are on opposite sides of $\overleftrightarrow{AD}$ if and only if B and D are on the same side of $\overleftrightarrow{AC}$.

*

4. As in the case of the angle bisector [see note at top of page 6-123, and, more explicitly, result (3) in the COMMENTARY for page 6-93], since C'' is interior to $\angle BAC$, $\overline{BC} \cap \overrightarrow{AC''}$ consists of one point.

5. By Theorem 4-10, since $\overset{\bullet\!-\!\bullet}{AB}$ is not longer than $\overset{\bullet\!-\!\bullet}{AC}$, $\overset{\bullet\!-\!\bullet}{AC}$ is longer than $\overset{\bullet\!-\!\bullet}{AD}$. So, since $AC'' = AC$, $\overset{\bullet\!-\!\bullet}{AD}$ is shorter than $\overset{\bullet\!-\!\bullet}{AC''}$.

6. Since $D \in \overrightarrow{AC''}$ and $AD < AC''$, by Theorem 1-5, $D \in \overline{AC''}$.

7. Since $D \in \overline{C''A}$, it follows from Axiom 5 that $A \in \overrightarrow{C''D}$. Since $D \in \overline{BC}$, D is interior to $\angle CC''B$. So [see result (1) of the COMMENTARY for page 6-93], $\overrightarrow{C''D}$ is a subset of the interior of $\angle CC''B$. Hence, A belongs to the interior of $\angle CC''B$. By a similar argument, B is interior to $\angle ACC''$.

8. Since $AC = AC''$, it follows from Theorem 3-5 that $\angle ACC'' \cong \angle AC''C$.

9. Exercise 7 and Axioms F and D.

10. Theorem 4-7.

11. $\overset{\bullet\!-\!\bullet}{BC''} \cong \overset{\bullet\!-\!\bullet}{B'C'}$, and Exercise 10.

*

12. <u>Theorem.</u> If two triangles agree in two pairs of sides but not in the third pair of sides then the triangle with the longer third side has the larger angle opposite the third side.

<u>Proof.</u> Suppose that, in ΔABC and $\Delta A'B'C'$, $AB = A'B'$, $AC = A'C'$, and $BC > B'C'$. It follows that $ABC \leftrightarrow A'B'C'$ is not a congruence and so, by s.a.s., that $m(\angle A) \neq m(\angle A')$. So, either $m(\angle A) > m(\angle A')$ or $m(\angle A') > m(\angle A)$. In the latter case it follows, by Theorem 4-11, that $B'C' > BC$ and, since $BC > B'C'$, that $BC > BC$. But, $BC \not> BC$. Hence, $m(\angle A') \not> m(\angle A)$. Consequently, $m(\angle A) > m(\angle A')$.

Answers for Part A.

1. If $\angle Q$ and $\angle Q'$ are acute angles then $m(\angle Q) + m(\angle Q') < 180$. So, $\angle Q$ and $\angle Q'$ are not supplementary, and, by the Sample, we conclude that $RPQ \longleftrightarrow R'P'Q'$ is a congruence.

2. [Similar to Exercise 1.]

3. If $\angle P$ is a right angle then, since $\angle P \cong \angle P'$, so is $\angle P'$. Hence, by Theorem 4-4, $\angle Q$ and $\angle Q'$ are acute angles. Consequently, by Exercise 1, $RPQ \longleftrightarrow R'P'Q'$ is a congruence.

4. [Similar to Exercise3.]

5. [Similar to Exercise 3.]

*

Another condition [you might make this Exercise 6] which leads to the conclusion that $RPQ \longleftrightarrow R'P'Q'$ is a congruence is '$RQ > PR$'. For, from this and Theorem 4-6, it follows that $\angle P$ is larger than $\angle Q$. Hence, $\angle Q$ is an acute angle. For, if $\angle Q$ were not acute it would follow that $\angle P$ was not acute, and that ΔPQR would have two nonacute angles. Since $PR = P'R'$ and $RQ = R'Q'$, it follows from the assumption that $RQ > PR$, that $R'Q' > P'R'$. So, in a similar manner, $\angle Q'$ is an acute angle. Consequently, by Exercise 1, $RPQ \longleftrightarrow R'P'Q'$ is a congruence.

This result yields the following theorem [see Part E on page 6-97]:

> If, for some matching of the vertices of one triangle with those of a second, two pairs of corresponding sides are congruent, and the angles opposite the members of the pair of longer sides are congruent, then the matching is a congruence.

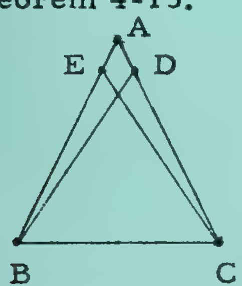
Notice that Theorem 4-14 on page 6-129 follows from the theorem just proved and Theorems 4-7 and 4-4. For, if $\angle P$ is not acute then, by Theorem 4-4, $m(\angle P) > m(\angle Q)$ and, by Theorem 4-7, $RQ > PR$. So, by the theorem just proved, $RQP \longleftrightarrow R'P'Q'$ is a congruence.

*

After discussing the exercises of Part A, it may be helpful to discuss again Parts C, D, and E of Exploration Exercises on pages 6-96 and 6-97. As mentioned on page TC[6-96, 97], Part C suggests Theorem 4-14, Part D suggests Theorem 4-13, and Part E suggests the theorem proved above.

*

Here is an exercise which your class might discuss in order to elucidate Theorem 4-13.

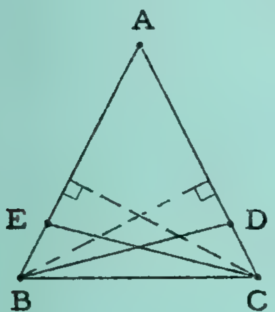


Hypothesis: $AB = AC$,
$CE = BD > BC$

Conclusion: $EB = DC$

Solution. In ΔEBC and ΔDCB, $BC = CB$, $CE = BD$ and, by Theorem 3-5, since $AB = AC$, $\angle EBC \cong \angle DCB$. So, we can use Theorem 4-13 to show that $EBC \longleftrightarrow DCB$ is a congruence and, hence, that $EB = DC$, if we can show that $\angle BEC$ and $\angle CDB$ are either both acute or both obtuse. Now, since $BD > BC$, it follows from Theorem 4-6 that $\angle DCB$ is larger than $\angle CDB$. But, $\angle DCB$ is acute, for, if it were not, ΔABC would have two nonacute angles. Hence, $\angle CDB$ is acute. Similarly, $\angle BEC$ is acute. So, by Theorem 4-13, $EBC \longleftrightarrow DCB$ is a congruence, and $EB = DC$.

When first suggesting this exercise to your students, omit the part '> BC' of the hypothesis, and elicit from them the content of the first two sentences of the preceding solution. Have students attempt to show that $\angle$BEC and $\angle$CDB are, as the figure suggests, both acute. Then, draw another figure, say, the one below, in which the two angles are both obtuse, noting that, by Theorem 4-7, this can happen only if $\overleftrightarrow{CE}$ and $\overleftrightarrow{BD}$ are shorter than $\overleftrightarrow{BC}$. Now, ask whether there is another point, D′, on $\overline{AC}$ such that BD′ = BD. [Of course, the point D′ ϵ $\overline{AC}$ such that the foot of the altitude to $\overleftrightarrow{AC}$ is the midpoint of $\overleftrightarrow{DD'}$ is such a point.] Draw $\overleftrightarrow{BD'}$, and point out that CE = BD′ but EB $\neq$ D′C. So, the conclusion of the exercise does not follow from the hypothesis. Can we strengthen the hypothesis so that the conclusion will follow? It should, now, be easy to elicit the information that if $\overleftrightarrow{BD}$ is longer than $\overleftrightarrow{BC}$ then, by Theorem 4-6, $\angle$DCB is larger than $\angle$CDB. So [see Solution], $\angle$CDB is acute. Hence, adding the part '> BC' to the hypothesis is sufficient to guarantee the desired conclusion.



This should be a good point at which to bring in the discussion suggested on TC[6-128, 129]a. The proof of the theorem given at the foot of that page of the COMMENTARY duplicates the last part of the Solution of the exercise, and the exercise serves as motivation for stating and proving the theorem in question. Once this is done, the Solution of the exercise can be shortened. All we need is the first sentence of the given Solution and a second sentence: Since, by hypothesis, $\overleftrightarrow{CE}$ and $\overleftrightarrow{BD}$ are longer than $\overleftrightarrow{BC}$ and $\overleftrightarrow{CB}$, it follows, by the theorem of TC[6-128, 129]a, that EBC $\longleftrightarrow$ DCB is a congruence and, hence, that EB = DC.

Answers for Part B.

1. Since AC = AD, $\angle ADC \cong \angle ACD$. By Theorem 4-2, each of these angles is acute. So, since a supplement of an acute angle is not acute, $\angle ADE$ and $\angle ACB$ are not acute. Also, since $\angle ADC \cong \angle ACD$, $\angle ADE \cong \angle ACB$. So, in the triangles ΔADE and ΔACB, AE = AB, AD = AC, and the angles opposite $\overleftrightarrow{AE}$ and $\overleftrightarrow{AB}$ are congruent and not acute. Hence, by Theorem 4-14, AED $\leftrightarrow$ ABC is a congruence. So, ED = BC.

2. In the triangles ΔACB and ΔDCE, AB = DE, AC = DC, and the angles opposite $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ are right angles. So, they are congruent and not acute. Hence, by Theorem 4-14, ACB $\leftrightarrow$ DCE is a congruence. So, CB = CE, and by definition, ΔBCE is isosceles.

3. In the triangles ΔACD and ΔACB, AC = AC and CD = CB, and the angles opposite $\overleftrightarrow{AC}$ are $\angle D$ and $\angle B$, respectively. But, these are right angles. Hence, they are congruent and not acute. So, by Theorem 4-14, ACD $\leftrightarrow$ ACB is a congruence, and $\angle DAC \cong \angle BAC$.

☆4. Since BA = BC, $\angle A \cong \angle C$, and BD = BD, it follows from the Sample on page 6-128 that if $\angle BDA$ and $\angle DBC$ are not supplementary then BAD $\leftrightarrow$ BCD is a congruence. But, by hypothesis, BAD $\leftrightarrow$ BCD is not a congruence. So, $\angle BDA$ and $\angle BDC$ are supplementary. But, they are adjacent angles. So, A, D, and C are collinear.

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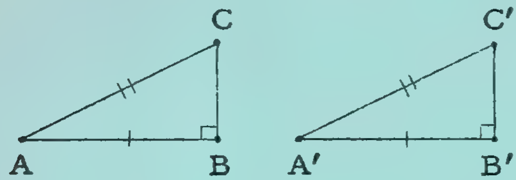
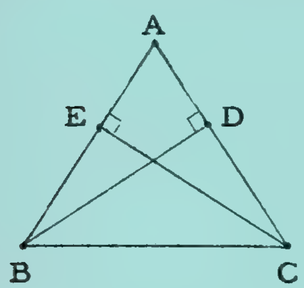
[A right triangle can be isosceles. In that case, the right angle is the vertex angle.]

Answers for Part D [on page 6-131].

1. 

Suppose that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are hypotenuses of the right triangles ΔABC and $\Delta A'B'C'$, respectively. Further, suppose that $AC = A'C'$ and $AB = A'B'$. Now, by the definition of hypotenuse, $\angle B$ and $\angle B'$ are right angles. Hence, they are congruent and not acute. So, by Theorem 4-14, $ABC \leftrightarrow A'B'C'$ is a congruence. Therefore, if $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are congruent hypotenuses of the right triangles ΔABC and $\Delta A'B'C'$, and $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are congruent legs, then $ABC \leftrightarrow A'B'C'$ is a congruence.

2. If the hypotenuse of a first right triangle is congruent to a leg of a second then, by Exercise 1 of Part C, the hypotenuse of the second triangle is longer than each side of the first. So, there can exist no matching of the vertices of the triangles for which the hypotenuse of the second triangle and a side of the first are congruent corresponding parts. Hence, the two right triangles are not congruent.

3. 

ΔACE is right-angled at E. So, $\overleftrightarrow{AC}$ is its hypotenuse and $\overleftrightarrow{CE}$ is one of its legs. Similarly, $\overleftrightarrow{AB}$ is the hypotenuse and $\overleftrightarrow{BD}$ is a leg of ΔABD. So, since $\overleftrightarrow{AC} \cong \overleftrightarrow{AB}$ and $\overleftrightarrow{CE} \cong \overleftrightarrow{BD}$, it follows from Theorem 4-15 that $ACE \leftrightarrow ABD$ is a congruence.

[That the altitudes to the legs of an isosceles triangle <u>are</u> congruent is Exercise 2 of Part E on page 6-134.]

> Correction. On page 6-132, line 2 should read:
>
> --- in the interior of an angle and

The right triangles pictured at the top of page 6-131 are ΔEAB, ΔEBC, ΔECD, ΔEDA, ΔABC, ΔBCD, ΔCDA, ΔDAB; ΔFGH, ΔFHI, ΔFGI.

*

Answers for Part C.

1. Suppose that, in right triangle ΔABC, $\overline{AB}$ is the hypotenuse. Then, $\angle C$ is a right angle, and, by Theorem 4-4, $\angle A$ and $\angle B$ are acute angles. So, since $\angle C$ is larger than $\angle A$ and $\angle B$, it follows from Theorem 4-7 that $\overline{AB}$ is longer than $\overline{BC}$ and $\overline{AC}$. Hence, if $\overline{AB}$ is the hypotenuse of the right triangle ΔABC, $\overline{AB}$ is longer than $\overline{BC}$ and $\overline{AC}$.

*

[A set of concurrent lines is a set of lines which intersect in a single point. Similarly, a set of concurrent segments is a set of segments which intersect in a single point].

*

2. The vertex of the right angle of a right triangle is the foot of each of the altitudes from the vertices of the acute angles. It is also one end point of the altitude to the hypotenuse. So, the point of concurrence of the three altitudes of a right triangle is the vertex of the right angle.

3. Each exterior angle of a triangle is a supplement of the angle of the triangle which is not opposite the exterior angle. So, for a right triangle, each exterior angle is a supplement of either a right angle or an acute angle. Hence, each exterior angle is either a right angle or an obtuse angle.

*

Correction. On page 6-133, line 6 should begin:

$\overleftrightarrow{BA}$, and, by Theorem 1-6, $A \in \overline{BA''}$. ---

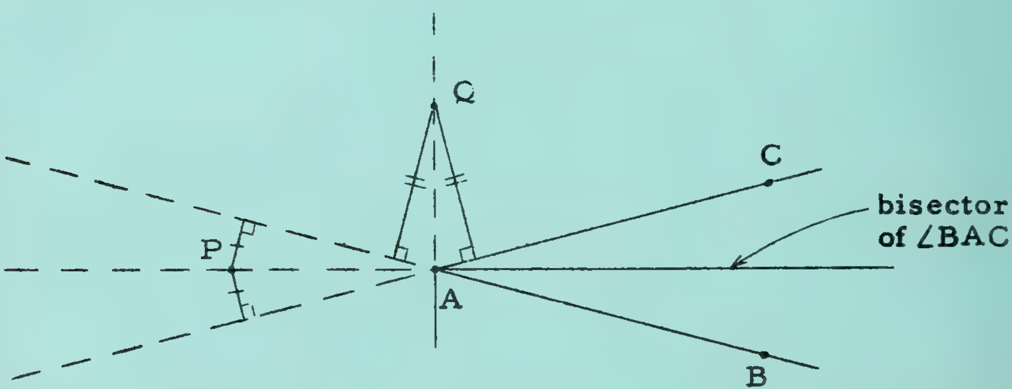
The last sentence in the first paragraph might be clearer if it were rewritten as: Hence, $\angle A'' \cong \angle A'$, and, since, by hypothesis, $\angle CAB \cong \angle A'$, $\angle CAB \cong \angle A''$.

The second paragraph contains a nice example of the use of modus tollens and double denial. We show that if $\overleftrightarrow{B'A'}$ is longer than $\overleftrightarrow{BA}$ then $\angle CAB \ncong \angle A''$. But, in the preceding paragraph, we show that $\angle CAB \cong \angle A''$, that is, that it is not the case that $\angle CAB \ncong \angle A''$. So, applying modus tollens, we conclude that $\overleftrightarrow{B'A'}$ is not longer than $\overleftrightarrow{BA}$.

The 'Similarly' in the third paragraph may need expanding. If we suppose that $\overleftrightarrow{BA}$ is longer than $\overleftrightarrow{B'A'}$ then, by Theorem 1-5, $A'' \in \overline{BA}$. In this case, $\angle CA''B$ is an exterior angle of $\Delta CA''A$; so, $\angle CAB \ncong \angle CA''B$. Hence, $\overleftrightarrow{BA}$ is not longer than $\overleftrightarrow{B'A'}$.

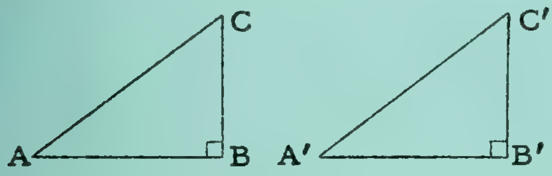
*

Note in Theorem 4-17 the phrase 'interior to the angle'. If [as we have said in line 3 on page 6-132] the distance between a point and a side of an angle [that is, a ray] is the distance between the point and the line containing the side, the point can be equidistant from the sides of an angle and not belong to the angle bisector.



[See Part D on page 6-134.]

Answer for Part A [on page 6-133].



Suppose that $\angle B$ and $\angle B'$ are right angles, that the hypotenuses $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are congruent, and that $\angle A \cong \angle A'$. Now, consider the matching $ABC \longleftrightarrow A'B'C'$. Since $\angle B \cong \angle B'$, $\angle A \cong \angle A'$, and the sides opposite $\angle B$ and $\angle B'$ are congruent, it follows from Theorem 4-16 that this matching is a congruence.

Now, suppose that the legs $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are congruent and the acute angles $\angle A$ and $\angle A'$ are congruent. Since $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and the sides opposite $\angle A$ and $\angle A'$ are congruent, it follows from Theorem 4-16 that $ABC \longleftrightarrow A'B'C'$ is a congruence.

*

Answers for Part B.

1. Since P belongs to the bisector $\overrightarrow{BD}$ of $\angle ABC$, it follows from Theorem 4-17 that PH = PF. Similarly, PH = PG. So, PH = PF = PG. That is, P is equidistant from the sides of $\triangle ABC$. [In this context, we are defining the distance between a point and a segment to be the distance between the point and the line containing the segment.]

2. Since PF = PG and P is in the interior of $\angle CAB$ [because it is in the interior of the triangle and the interior of the triangle is the intersection of the interiors of its angles], it follows from Theorem 4-17 that P belongs to the bisector of $\angle CAB$.

*

Answer for Part C.

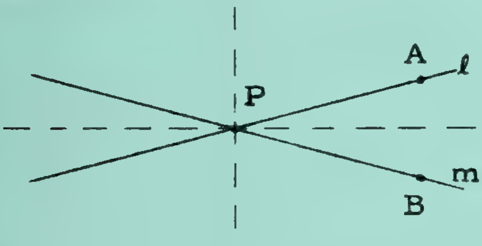
Suppose that lines ℓ and m are the perpendicular bisectors of sides $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$, respectively, of ΔABC, and suppose that P is the point of intersection of ℓ and m. By Theorem 3-3, PA = PB and PB = PC. So, PA = PC. Therefore, by Theorem 3-3, P is a point on the perpendicular bisector of $\overleftrightarrow{AC}$. Hence, P is the point of intersection of all three perpendicular bisectors of the sides of ΔABC. [As an application of this theorem, ask students to locate the center of a circle which contains the vertices of a given triangle. See page 6-282.]

*

Recall that the result needed in Part B, that two angle bisectors of a triangle intersect at a point interior to the triangle, can be derived from the Introduction Axioms. [See the COMMENTARY on Part B on page 6-109.] In contrast, the proof that the perpendicular bisectors of two sides of a triangle intersect depends on properties of parallel lines which, in turn, depend on some of our measure axioms.

*

Answer for Part D.

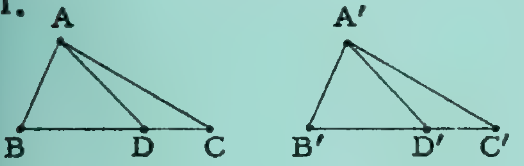


The set of points which are equidistant from ℓ and m is the union of the line containing the bisector of $\angle APB$ and the perpendicular to this line through P.

[See Part C on page 6-110.]

*

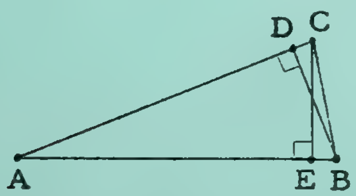
Answers for Part ☆E.

1. 

Suppose that $ABC \longleftrightarrow A'B'C'$ is a congruence. Then, medians $\overline{AD}$ and $\overline{A'D'}$ are corresponding medians. Since $ABC \longleftrightarrow A'B'C'$ is a congruence, $AB = A'B'$, $\angle B \cong \angle B'$, and $BC = B'C'$. Since D and D' are midpoints of $\overline{BC}$ and $\overline{B'C'}$, respectively, it follows that $BD = B'D'$. So, by s.a.s., $ABD \longleftrightarrow A'B'D'$ is a congruence, and $AD = A'D'$.

Now, suppose that $\overline{AD}$ and $\overline{A'D'}$ are corresponding angle bisectors. Since $\angle B \cong \angle B'$, $BA = B'A'$, and $\angle BAD \cong \angle B'A'D'$, it follows from a.s.a. that $BAD \longleftrightarrow B'A'D'$ is a congruence. So, $AD = A'D'$.

Finally, suppose $\overline{AD}$ and $\overline{A'D'}$ are corresponding altitudes. Now, either $\angle B$ and $\angle B'$ are obtuse or not obtuse. If they are obtuse, $B \in \overline{DC}$ and $B' \in \overline{D'C'}$. If they are not obtuse, $D \in \overrightarrow{BC}$ and $D' \in \overrightarrow{B'C'}$. Consider the first case. Since $\angle ABC \cong \angle A'B'C'$, $\angle ABD \cong \angle A'B'D'$ because supplements of congruent angles are congruent. Also, the right angles $\angle D$ and $\angle D'$ are congruent. Since the sides of ΔADB and $\Delta A'D'B'$ opposite $\angle D$ and $\angle D'$ are congruent [$ABC \longleftrightarrow A'B'C'$ is a congruence], it follows from a.a.s. that $ADB \longleftrightarrow A'D'B'$ is a congruence. So, $AD = A'D'$. Now, consider the case in which $D \in \overrightarrow{BC}$ and $D' \in \overrightarrow{B'C'}$. If $D = B$ then $m(\angle B) = 90 = m(\angle B')$; so, $D' = B'$ and $AD = A'D'$. If $D \in \overrightarrow{BC}$ then $D' \in \overrightarrow{B'C'}$ and $ABD \longleftrightarrow A'B'D'$ is a congruence by a.a.s. Hence, $AD = A'D'$.

2. 

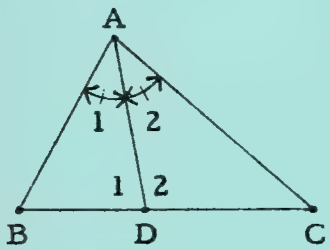
$\angle AEC \cong \angle ADB$, $\angle A \cong \angle A$, and $AC = AB$. So, by a.a.s., $AEC \longleftrightarrow ADB$ is a congruence. Hence, $CE = BD$.

Answers for Miscellaneous Exercises [on pages 6-137 and 6-138].

1. $m(\angle Q) < 60$

2. XZ is between 5 and 13

3. [Since $\angle B_1$ and $\angle B_2$ are supplementary and $\angle B_1$ is larger than $\angle B_2$, it follows that $\angle B_1$ is obtuse.]
$\angle B_1$ is larger than $\angle M$, $\angle R_3$, $\angle R_1$, $\angle S$, $\angle N$;
$\angle R_4 \cong \angle R_2$ and larger than $\angle T_3$, $\angle T_1$, $\angle S$, $\angle N$, $\angle M$, $\angle B_2$;
$\angle T_2 \cong \angle T_4$ and larger than $\angle S$, $\angle R_1$, $\angle R_3$;
$\angle T_1 \cong \angle T_3$ and larger than $\angle M$, $\angle N$;
$\angle B_2$ is larger than $\angle S$, $\angle N$;
$\angle R_1 \cong \angle R_3$

4. 

Since $D \in \overline{BC}$, $\angle D_2$ is an exterior angle of ΔABD. So, $\angle D_2$ is larger than $\angle A_1$. But, $\angle A_1 \cong \angle A_2$. So, $\angle D_2$ is larger than $\angle A_2$. Therefore, $AC > CD$. Similarly, $AB > BD$.

5. (a) $\angle QPM$, $\angle QMP$, $\angle ONQ$, $\angle NOQ$

(b) $\angle QOP$, $\angle QPO$, $\angle QNM$, $\angle NMQ$

(c) $MN > OQ$ [Theorem 4-11]

6. Suppose that $\angle A$ and $\angle B$ are the base angles of an isosceles triangle. Then, by Theorem 3-5, $m(\angle A) = m(\angle B)$, and, by Theorem 4-2, $m(\angle A) + m(\angle B) < 180$. So, $m(\angle A) < 90$. Hence, $\angle A$ is acute. Similarly, $\angle B$ is acute. Consequently, if $\angle A$ and $\angle B$ are the base angles of an isosceles triangle, $\angle A$ and $\angle B$ are acute.

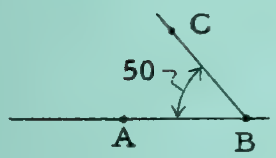
7. By Axiom C, there is a point $P \in \overrightarrow{TC}$ such that $TP = TA$. Since $TA < TC$, $TP < TC$. So, by Theorem 1-5, $P \in \overline{TC}$. Now, since $\overleftrightarrow{BT}$ is the altitude and median from B of ΔABP, it follows from Theorem 4-12(b) that ΔABP is isosceles with $\angle ABP$ as vertex angle. Hence, by Theorem 3-5, $\angle BPA \cong \angle BAP$. Since $P \in \overline{TC}$, $\angle BPA$ is an exterior angle of ΔBPC. So, by Theorem 4-5, $\angle BPA$ is larger than $\angle C$. Hence, $\angle BAC$ is larger than $\angle C$. So, by Theorem 4-7, $BC > AB$.

8. Since $BD = BD$ and $CB = AB$, it follows from h.ℓ. that $BDC \longleftrightarrow BDA$ is a congruence. So, $DC = AD$.

9. Since the hypotenuse is the longest side, its measure is 50.

*

Quiz.

1. 

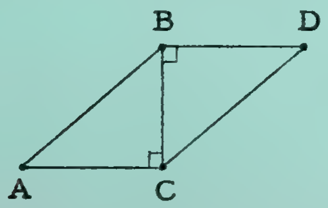
Is there a point P on line $\overleftrightarrow{AB}$ such that $B \in \overline{AP}$ and $m(\angle CPB) = 50$? Justify your answer.

2. If ΔABC is an obtuse isosceles triangle with vertex angle at B then $\angle A$ cannot be an angle of __?__

(A) 20° (B) 32° (C) 39.9° (D) 44.8° (E) 45°

3. In ΔABC, $AB = 2.5$, $BC = 7.5$, and $CA = 5.5$. Name the largest angle of ΔABC.

4. Prove that an altitude of a triangle is shorter than two of the sides.

5. 

Hypothesis: $\angle DBC$ and $\angle ACB$ are right angles, $AB = CD$

Conclusion: $\angle A \cong \angle D$

☆6. Suppose that ΔABE is isosceles and that C and D are points on the base $\overleftrightarrow{BE}$ such that $D \in \overline{CE}$ and $BC = CD = DE$. Do you think the angles $\angle BAC$, $\angle CAD$, and $\angle DAE$ are congruent? Prove your conjecture.

*

Answers for Quiz.

1. No. If there were such a point P then $\angle ABC$ would be an exterior angle of ΔBCP and $m(\angle CPB)$ would be less than 50. [Students might also note that $m(\angle CBP)$ would be 130 and then use Theorem 4-2 or Theorem 4-3.]

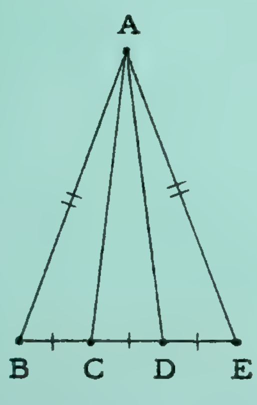
2. (E) 45°

3. $\angle A$

4. Suppose that $\overleftrightarrow{AD}$ is the altitude of ΔABC from A. Then, by Theorem 4-9, $\overleftrightarrow{AD}$ is shorter than $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

5. By hypothesis [and definition], $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are the hypotenuses of the right triangles ΔABC and ΔDCB, respectively. Also, $BC = BC$. So, by h.ℓ., $ABC \leftrightarrow DCB$ is a congruence. Hence, $\angle A \cong \angle D$.

☆6.



$\overleftrightarrow{AC}$ is the median of ΔABD from A. So, by Theorem 4-12(b), if $\overleftrightarrow{AC}$ is the angle bisector of ΔABD from A then $AB = AD$. But, $\angle ADB$ is an exterior angle of ΔADE; so, $\angle ADB$ is larger than $\angle E$. Since $AB = AE$, $\angle E \cong \angle B$. So, $\angle ADB \ncong \angle B$. Hence, $AB \neq AD$. Therefore, $\overleftrightarrow{AC}$ is not the angle bisector of ΔABD from A. So, $\angle BAC \ncong \angle CAD$.

Note that, although, using Axioms D and E, we can also show that there is only one line through P for which [see figure at foot of page 6-139] $\alpha = \beta$, this does not, in itself, establish the uniqueness of the parallel to ℓ through P. The argument shows only that, for lines through P which are not parallel to ℓ, alternate interior angles are not congruent. But, it does not preclude the possibility that there may also be lines through P which are parallel to ℓ and for which alternate interior angles are not congruent. So, the argument based on Axioms D and E shows that the line through P for which $\alpha = \beta$ is a parallel to ℓ. Axiom 4 is needed to show that there is no other parallel to ℓ through P.

*

The omission of Axiom 4 would allow interpretations in which, for each line ℓ, and for each point $P \notin \ell$, there is more than one line through P which does not intersect ℓ. [Thus showing that Axiom 4 cannot be derived from the other axioms.] A somewhat more radical revision of the axioms would allow interpretations in which each two lines intersect in a single point [but in which a substantial number of Euclidean theorems still hold]. These noneuclidean interpretations are models of two kinds of noneuclidean geometry, hyperbolic geometry and elliptic geometry, respectively. The first of these was developed during the first half of the 19th century by a Russian, Lobachevsky, a Hungarian, Bolyai, and a German, Gauss. The second was developed by another German mathematician, Riemann. Although these geometries are of great interest, both for their own sakes and as examples of the development of mathematical thought, it seems inappropriate to discuss them in this course. An adequate discussion from either of the points of view just mentioned would be too lengthy--an inadequate one would be likely to create confusion. [At the very least, we would have to make one Glox-type visit for each interpretation.]

Correction. On page 6-139, in the figure at the bottom of the page, insert an 'ℓ' below the right end of $\overleftrightarrow{AC}$.

There are two competing uses of the word 'parallel'. According to one [the one adopted in the text], a line ℓ is parallel to a line m if and only if $\ell \cap m = \emptyset$. According to the other case, ℓ is parallel to m if and only if either $\ell \cap m = \emptyset$ or $\ell = m$. This second use has considerable technical advantage over the first. For example, using 'parallel' in this sense, parallelism is transitive: if ℓ is parallel to m and m is parallel to n then ℓ is parallel to n [with the first use, the consequent must be replaced by 'ℓ is parallel to n or $\ell = n$']. Also, in the second sense, such definite descriptions as 'the line parallel to ℓ through P' are always meaningful, but, with the first use, we must first establish that a point is not on a line before we may speak of the line through this point and parallel to the given line.

Despite these advantages, we have adopted the first use of 'parallel' as being more in accord with common speech.

*

For handy reference, here is a proof of Theorem 3:

> Suppose that $P \notin \ell$, that $P \in m \cap n$, that $m \parallel \ell$, and that $n \parallel \ell$. By Axiom 4, if $m \neq n$ then $m \parallel n$. But, since $P \in m \cap n$, $m \nparallel n$. Hence, $m = n$. So, if $P \notin \ell$, there are not two lines which contain P and are parallel to ℓ--that is, there is at most one line which contains P and is parallel to ℓ.

*

The proof in the text, that, through a point $P \notin \ell$, there is at least one parallel to ℓ, amounts to showing that if $P \in m$ and $m \nparallel \ell$ then a pair of alternate interior angles [see page 6-142] are not congruent. So, if $P \in m$ and a pair of alternate interior angles are congruent them $m \parallel \ell$. Axioms D and E guarantee the existence of such a line m. So, among the lines through P, there is at least one which is parallel to ℓ.

Answer for Part A.

[Part A makes use of a theorem which was proved on pages 6-139 and 6-140 but which is first stated on page 6-144 as Theorem 5-2.]

Since $\beta = 60$, it follows that $m_{60} \parallel \ell$. So, $m_{60} \cap \ell = \emptyset$. Hence, since, also, $P \in m_{60}$ [and $P \notin \ell$], it follows from Theorem 5-1 that m_{60} is parallel to ℓ through P. Since, by Axiom E, $m_{59.9} \neq m_{60}$, $m_{59.9}$ is not parallel to ℓ through P. So, by Theorem 5-1, either $P \notin m_{59.9}$ or $m_{59.9} \cap \ell \neq \emptyset$. Since $P \in m_{59.9}$, it follows that $m_{59.9}$ intersects ℓ.

[More briefly: Since $\beta = 60$, it follows that $m_{60} \parallel \ell$. Since $P \in m_{60}$ and $P \notin \ell$, it follows from Theorem 3 that no other line through P is parallel to ℓ. But, $P \in m_{59.9}$ and, by Axiom E, $m_{59.9} \neq m_{60}$. Hence, $m_{59.9} \nparallel \ell$. Consequently, $m_{59.9}$ intersects ℓ.]

Answer for Part B.

The line $\overleftrightarrow{RP}$ is parallel to ℓ and contains P. So, by Theorem 5-1, $\overleftrightarrow{RP}$ is the line m_{60} of Part A. Therefore, R is any point of m_{60} on the A-side of $\overleftrightarrow{PC}$.

Answers for Part C.

(a) 60	(b) 130	(c) 50	(d) 50	(e) 60
(f) 130	(g) 120	(h) 70	(i) 180	(j) 110
(k) 60	(ℓ) 110			

The careful descriptions of the various pairs of angles associated with two lines and a transversal are given to show students that it is not necessary to point to a picture in order to describe these pairs of angles. But, such descriptions should not be memorized or belabored.

*

Answers for Part A.

(1) $\angle A_2$ and $\angle B_4$, $\angle A_3$ and $\angle B_1$, $\angle C_2$ and $\angle D_1$, $\angle C_3$ and $\angle D_4$

(2) $\angle A_1$ and $\angle B_1$, $\angle A_2$ and $\angle B_2$, $\angle A_3$ and $\angle B_3$, $\angle A_4$ and $\angle B_4$, $\angle C_2$ and $\angle D_3$, $\angle C_3$ and $\angle D_2$, $\angle C_4$ and $\angle D_1$, $\angle C_1$ and $\angle D_4$

(3) $\angle B_2$ and $\angle A_4$, $\angle B_3$ and $\angle A_1$, $\angle C_4$ and $\angle D_3$, $\angle C_1$ and $\angle D_2$

(4) $\angle A_2$ and $\angle B_1$, $\angle A_3$ and $\angle B_4$, $\angle C_2$ and $\angle D_4$, $\angle C_3$ and $\angle D_1$

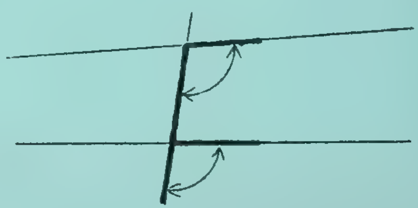
(5) $\angle A_1$ and $\angle B_2$, $\angle A_4$ and $\angle B_3$, $\angle C_1$ and $\angle D_3$, $\angle C_4$ and $\angle D_2$

(6) $\angle A_1$ and $\angle A_2$, $\angle A_2$ and $\angle A_3$, $\angle A_3$ and $\angle A_4$, $\angle A_4$ and $\angle A_1$, $\angle B_1$ and $\angle B_2$, $\angle B_2$ and $\angle B_3$, $\angle B_3$ and $\angle B_4$, $\angle B_4$ and $\angle B_1$, $\angle C_1$ and $\angle C_2$, $\angle C_2$ and $\angle C_3$, $\angle C_3$ and $\angle C_4$, $\angle C_4$ and $\angle C_1$, $\angle D_1$ and $\angle D_2$, $\angle D_2$ and $\angle D_3$, $\angle D_3$ and $\angle D_4$, $\angle D_4$ and $\angle D_1$

(7) $\angle A_1$ and $\angle A_3$, $\angle A_2$ and $\angle A_4$, $\angle B_1$ and $\angle B_3$, $\angle B_2$ and $\angle B_4$, $\angle C_1$ and $\angle C_3$, $\angle C_2$ and $\angle C_4$, $\angle D_1$ and $\angle D_3$, $\angle D_2$ and $\angle D_4$

*

By now, some of your students may have invented the following "mnemonic" devices for picking out a pair of alternate interior angles and a pair of corresponding angles:



Correction. On page 6-145, line 17 should read:

(9) --- [(6), (7), and (8)]

Answer for Part B.

By hypothesis, $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$ and $\angle A_2$ and $\angle B_2$ are alternate interior angles; so, by Theorem 5-3, $\angle A_2 \cong \angle B_2$. But, $\angle B_1$ and $\angle B_2$ are adjacent angles whose noncommon sides are collinear; so, by Theorem 2-9, they are supplementary. Hence, $\angle B_1$ and $\angle A_2$ are supplementary.

*

Theorems 5-2 and 5-4 give us two ways of showing that lines are parallel, and Theorems 5-3 and 5-5 tell us two of the things that follow from assuming that the lines are parallel. Schematically:

$$\text{if } \left\{ \begin{array}{l} \text{(1) alt. int. } \angle\text{s are } \cong \\ \text{(2) cons. int. } \angle\text{s are supp.} \end{array} \right\} \text{ then lines are } \parallel$$

$$\text{if lines are } \parallel \quad \text{then } \left\{ \begin{array}{l} \text{(1) alt. int. } \angle\text{s are } \cong \\ \text{(2) cons. int. } \angle\text{s are supp.} \end{array} \right.$$

In Part C the student is asked to discover, state, and prove another pair of theorems about corresponding angles and parallel lines. These new theorems should be added to the scheme shown above.

Correction. On page 6-146, line 8 should read:

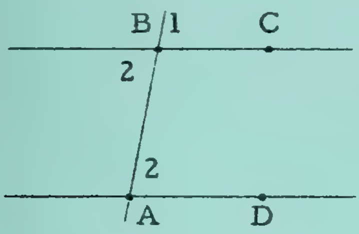
--- to Theorems 5-2 and 5-4. Then,

↑

Answers for Part C.

[Theorems 5-6 and 5-7 are stated on page 6-158.]

1.



Hypothesis: $\angle B_1$ and $\angle A_2$ are congruent corresponding angles

Conclusion: $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$

Solution.

By hypothesis, $\angle B_1 \cong \angle A_2$. Since $\overleftrightarrow{BC}$ and $\overleftrightarrow{BA}$ are straight lines, the vertical angles $\angle B_1$ and $\angle B_2$ are congruent. So, $\angle B_2 \cong \angle A_2$. But, from the figure, $\angle B_2$ and $\angle A_2$ are alternate interior angles. So, by Theorem 5-2, $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$.

2.

[Same figure as in Exercise 1.]

Hypothesis: $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$, $\angle B_1$ and $\angle A_2$ are corresponding angles

Conclusion: $\angle B_1 \cong \angle A_2$

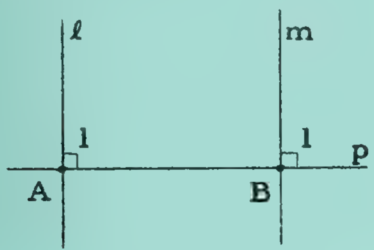
Solution.

Since, by hypothesis, $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$, it follows from Theorem 5-3 that each two alternate interior angles are congruent. From the figure, $\angle B_2$ and $\angle A_2$ are alternate interior angles; so, they are congruent. Since $\overleftrightarrow{BC}$ and $\overleftrightarrow{BA}$ are straight lines, the vertical angles $\angle B_2$ and $\angle B_1$ are congruent, also. So, $\angle B_1 \cong \angle A_2$.

*

Answers for Part D.

1.



Hypothesis: $\ell \perp p$ at A,
$m \perp p$ at B,
$\ell \neq m$

Conclusion: $\ell \parallel m$

Solution.

By hypothesis, $\ell \perp p$ and $m \perp p$. So, the corresponding angles $\angle A_1$ and $\angle B_1$ are right angles. Hence, they are congruent, and, by Theorem 5-6, $\ell \parallel m$.

2.

[Same figure as in Exercise 1.]

Hypothesis: $\ell \parallel m$,
$\ell \perp p$ at A

Conclusion: $m \perp p$

Solution.

By an Introduction Theorem [Theorem 3 on page 6-24], m intersects p. [Otherwise, ℓ and p would be two lines parallel to m through A.] Let B be the point of intersection. Since p is a transversal of the parallel lines ℓ and m, it follows from Theorem 5-7 that the corresponding angles $\angle A_1$ and $\angle B_1$ are congruent. But, since $\ell \perp p$, $\angle A_1$ is a right angle. So, $\angle B_1$ is a right angle, also. Therefore, $m \perp p$.

Answers for Part F [on pages 6-147 and 6-148].

1. [Suppose that D is on the A-side of $\overleftrightarrow{BC}$. Since, by hypothesis, $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$, $\overrightarrow{CD} \cap \overline{AB} = \emptyset$. Hence, D is not interior to $\angle ACB$. Since D is on the A-side of $\overleftrightarrow{BC}$, it follows that D is not on the B-side of $\overleftrightarrow{AC}$. Also, since $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$, $D \notin \overleftrightarrow{AC}$. Consequently, D and B are on opposite sides of $\overleftrightarrow{AC}$. Since $C \in \overline{BE}$, it follows that D is on the E-side of $\overleftrightarrow{AC}$. And, since D is on the A-side of $\overleftrightarrow{BC}$, it follows that D is interior to $\angle ACE$.]

   Since A and D are on the same side of $\overleftrightarrow{BC}$ and $C \in \overline{BE}$, it follows that $\angle B$ and $\angle C_1$ are corresponding angles. Hence, since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, it follows, by Theorem 5-7, that $\angle B \cong \angle C_1$. Since [as shown above] B and D are on opposite sides of $\overleftrightarrow{AC}$, it follows that $\angle A$ and $\angle C_2$ are alternate interior angles. Hence, since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, it follows, by Theorem 5-3, that $\angle A \cong \angle C_2$. Consequently, $m(\angle A) + m(\angle B) = m(\angle C_1) + m(\angle C_2)$. But, since [as shown above] D is interior to $\angle ACE$, it follows, by Axiom F, that $m(\angle C_1) + m(\angle C_2) = m(\angle ACE)$. So, $m(\angle A) + m(\angle B) = m(\angle ACE)$.

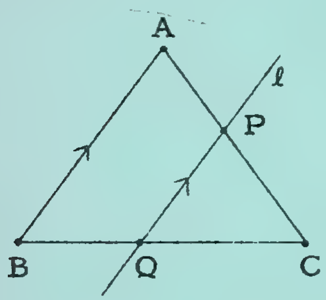
2. It follows from Axiom G that $m(\angle ACE) + m(\angle ACB) = 180$. So, by the result of Exercise 1, $m(\angle A) + m(\angle B) + m(\angle ACB) = 180$.

3. (a) 70 (b) 40; 20 (c) 90; complementary (d) 60

   (e) 45, 90 or $67\frac{1}{2}$, $67\frac{1}{2}$ (f) 120 (g) $90 + \frac{a}{2}$

Answers for Part E.

1. Since $\angle D_3 \cong \angle E_1$ and $\angle E_1 \cong \angle F_1$, $\angle D_3 \cong \angle F_1$. So, by Theorem 5-2, $\overleftrightarrow{AD} \parallel \overleftrightarrow{CF}$. Hence, by Theorem 5-7, $\angle A_3 \cong \angle C_3$. But, $\angle A_3$ and $\angle A_2$ are supplementary. Therefore, so are $\angle C_3$ and $\angle A_2$.

2. 

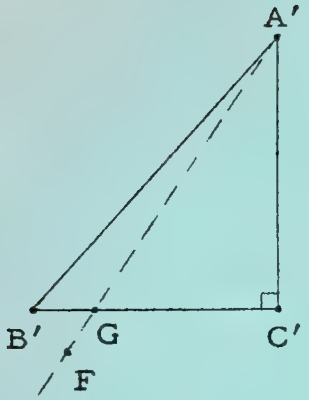
Since $\ell \parallel \overleftrightarrow{AB}$, it follows from Theorem 5-7 that the corresponding angles $\angle B$ and $\angle PQC$ are congruent. Since AB = AC, it follows from Theorem 3-5 that $\angle B \cong \angle C$. So, $\angle PQC \cong \angle C$. Hence, by Theorem 3-5, PC = PQ, and, by definition, ΔPQC is isosceles. [If $\angle C$ is the vertex angle of the isosceles triangle ΔABC, then $\angle B \cong \angle A$. Since, by Theorem 5-7, $\angle PQC \cong \angle B$ and $\angle QPC \cong \angle A$, it follows that $\angle PQC \cong \angle QPC$. Hence, by Theorem 3-5, PC = QC. So, by definition, ΔPQC is isosceles, but this time with vertex angle at C rather than at P.]

[It is interesting to vary the problem by replacing '$\overline{AC}$' by '$\overleftrightarrow{AC}$' and '$\overline{BC}$' and '$\overleftrightarrow{BC}$' and assuming that $C \notin \ell$.]

[Note the arrowheads in the figure to show that lines $\overleftrightarrow{AB}$ and ℓ are given to be parallel. You may want to use this marking device in other problems.]

3. Consider the triangles ΔABC and ΔMNP. By hypothesis, AB = MN. Also, BN = CP. So, since, from the figure $N \in \overline{BC}$ and $C \in \overline{NP}$, and since BN + NC = NC + CP, it follows from Axiom A that BC = NP. By Theorem 5-7, since $\overleftrightarrow{AB} \parallel \overleftrightarrow{MN}$, the corresponding angles $\angle B$ and $\angle MNP$ are congruent. Thus, by s.a.s., $ABC \leftrightarrow MNP$ is a congruence. So, $\angle ACB \cong \angle MPN$, and by Theorem 5-6, $\overleftrightarrow{AC} \parallel \overleftrightarrow{MP}$.

*



the problem, it is sufficient to show that F is in the interior of $\angle B'A'C'$. For, in that case, the result (3) on TC[6-93]d would allow us to conclude that $\overrightarrow{A'F}$ crosses $\overleftrightarrow{B'C'}$ in a point between B' and C'. Once this is established, the problem is solved in a manner entirely analogous to that used in the alternative described above. So, let us show that F is in the interior of $\angle B'A'C'$.

The result stated at the bottom of TC[6-124, 125]a is what we need. Since B' and F are on the same side of A'C', it follows that either $F \in \overrightarrow{A'B'}$ or B' is in the interior of $\angle FA'C'$ or F is in the interior of $\angle B'A'C'$. If $F \in \overrightarrow{A'B'}$ then $\angle FA'C' = \angle B'A'C'$. But, $30 \neq 35$. So, $F \notin \overrightarrow{A'B'}$. If B' is in the interior of $\angle FA'C'$ then, by Axiom F, $m(\angle FA'B') + m(\angle B'A'C') = m(\angle FA'C')$.

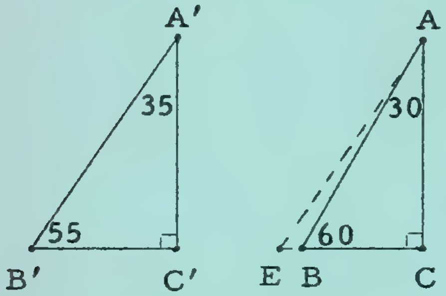
Since, by Axiom D, $m(\angle FA'B') > 0$, $m(\angle B'A'C') < m(\angle FA'C')$. But, $35 \not< 30$. So, B' is not in the interior of $\angle FA'C'$. Therefore, F is in the interior of $\angle B'A'C'$.

The foregoing discussion suggests the following theorem:

$\forall_X \forall_Y \forall_U \forall_V$

if $X \neq Y$ and U and V are on the same side of $\overleftrightarrow{XY}$
then $m(\angle UXY) < m(\angle VXY)$ if and only if
U is in the interior of $\angle VXY$

There is an interesting alternative justification of this result which does not make use of Theorem 5-12. [Thus, it could be used in a problem with a weaker hypothesis, say, '$m(\angle B) > m(\angle B') > m(\angle A')$'.]



There is a point $E \in \overrightarrow{CB}$ such that $CE = C'B'$. As above, $ACE \longleftrightarrow A'C'B'$ is a congruence. So, $AE = A'B'$, $\angle E$ is an angle of 55°, and $\angle EAC$ is an angle of 35°. Now, either $E = B$ or $E \in \overline{BC}$ or $B \in \overline{EC}$. If $E = B$ then $\angle EAC = \angle BAC$. But, by Theorem 5-11, $\angle BAC$ is an angle of 30°. Since $35 \neq 30$, $\angle EAC \neq \angle BAC$. So, $E \neq B$. Now, suppose that $E \in \overline{BC}$. Then, $\angle AEC$ is an exterior angle of ΔBAE. So, by Theorem 4-5, if $E \in \overline{BC}$ then $m(\angle AEC) > m(\angle ABC)$. But, $55 \not> 60$. So, $E \notin \overline{BC}$. Therefore, $B \in \overline{EC}$.

So, since $E \in \overrightarrow{CB}$ and $B \in \overline{EC}$, it follows from Theorem 1-6 that $EC > BC$. Since $EC = B'C'$, $B'C' > BC$.

By Theorem 4-7, since $m(\angle B') > m(\angle A')$, $A'C' > B'C'$. So, since $AC = A'C'$, $AC > B'C' > BC$.

By Theorem 4-7 again, since $m(\angle C) > m(\angle CBA)$, $AB > AC$. So, $AB > AC > B'C' > BC$.

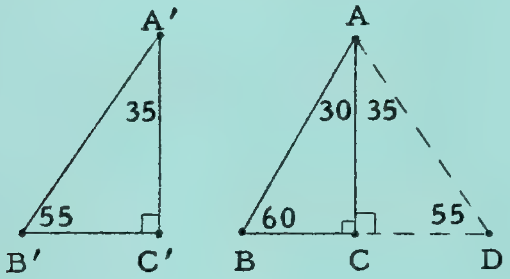
Finally, since $\angle EBA$ is a supplement of the acute angle $\angle ABC$, it is an obtuse angle and is, therefore, larger than $\angle E$. So, Theorem 4-7 tells us that $AE > AB$. Since $AE = A'B'$, $A'B' > AB > AC > B'C' > BC$.

Some students might approach this problem by first drawing the half-line $\overrightarrow{A'F}$ in the B'-side of $\overleftrightarrow{A'C'}$ such that $\angle FA'C'$ is an angle of 30°. To solve

Theorem 3-5, $\angle D$ is an angle of 60°. So, by Theorem 5-11, $\angle BAD$ is an angle of 60°. By Theorem 3-5, AB = DB. So, since, by Theorem 1-8, BC = $\frac{1}{2} \cdot$ DB, it follows that BC = $\frac{1}{2} \cdot$ AB.

2. $\overline{A'B'}$, $\overline{AB}$, $\overline{AC}$, $\overline{B'C'}$, $\overline{BC}$

The following justification of this result is an interesting variation of the one used for Exercise 1.



There is a point D ϵ $\overrightarrow{BC}$ such that CD = C'B' and C ϵ $\overline{BD}$. $\angle ACD$ is a right angle since it is a supplement of the right angle $\angle ACB$. So, $\angle ACD \cong \angle C'$. Since AC = A'C', it follows from s.a.s. that ACD $\longleftrightarrow$ A'C'B' is a congruence.

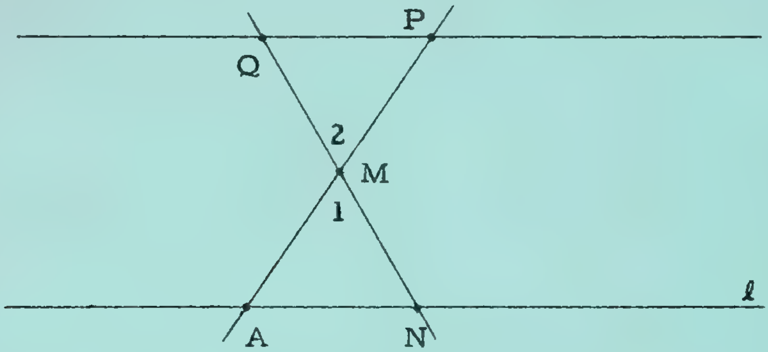
So, AD = A'B' and $\angle D$ is an angle of 55°. Now, in $\triangle ABD$, since $\angle B$ is larger than $\angle D$, it follows from Theorem 4-7 that AD > AB. Hence, A'B' > AB.

By Theorem 4-9, the perpendicular segment $\overline{AC}$ is shorter than $\overline{AB}$. So, A'B' > AB > AC.

By Theorem 5-11, $\angle BAD$ is an angle of (180 - 60 - 55)°, that is, an angle of 65°. So, by Theorem 4-7, BD > AB. But, since $\angle BAC$ is an angle of 30° [Theorem 5-11], it follows from Theorem 5-12 that AB = 2 $\cdot$ BC. Now, since C ϵ $\overline{BD}$, BC + CD = BD. So, BC + CD > 2 $\cdot$ BC. Hence, CD > BC. But, since ACD $\longleftrightarrow$ A'C'B' is a congruence, CD = C'B'. So, B'C' > BC.

Finally, since $\angle A'$ is an angle of 35° [Theorem 5-11], it follows from Theorem 4-7 that A'C' > B'C'. But, AC = A'C'. So, AC > B'C'. Therefore, A'B' > AB > AC > B'C' > BC.

Answer for Part G [which begins on page 6-148].



Since AM = MP, NM = MQ, and the vertical angles $\angle M_1$ and $\angle M_2$ are congruent, it follows from s.a.s. that AMN $\longleftrightarrow$ PMQ is a congruence. So, $\angle NAM \cong \angle QPM$. Since N and Q are on opposite sides of $\overleftrightarrow{AP}$, $\angle NAM$ and $\angle QPM$ are alternate interior angles. Hence, by Theorem 5-2, $\overleftrightarrow{PQ} \parallel \ell$.

*

Here is another interesting way to locate a point on the line parallel to ℓ through P. Find a point N on ℓ such that AP = AN. Next, find the point Q on the non-A-side of $\overleftrightarrow{PN}$ such that QP = PA = QN. The line $\overleftrightarrow{PQ}$ is parallel to ℓ.

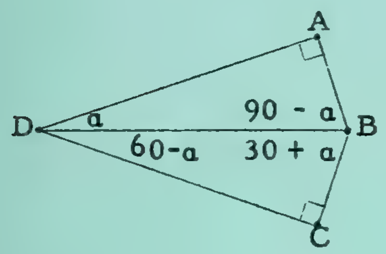
Students may enjoy an opportunity to discover other constructions.

*

Answers for Part H [on pages 6-149, 6-150, and 6-151].

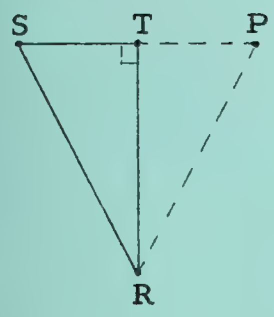
1. Axiom C tells us that there is a point D in $\overrightarrow{BC}$ such that BC = CD. [Since $B \notin \overrightarrow{BC}$, $D \neq B$. So, by Theorem 1-9, C is the midpoint of $\overline{BD}$.] Hence, $\overline{AC}$ is the median of $\triangle ABD$ from A. By Theorem 5-11, $m(\angle ACB) = 180 - 30 - 60 = 90$. So, $\angle ACB$ is a right angle, and $\overline{AC}$ is the altitude of $\triangle ABD$ from A. Therefore, by Theorem 4-12(b), $\triangle ABD$ is isosceles, with vertex angle at A, and, by

3. Suppose that $\Delta DAB \cong \Delta DCB$. Then, there is some matching of the vertices of ΔDAB with those of ΔDCB such that the corresponding parts are congruent. Since $\angle A$ and $\angle C$ are right angles and since no triangle can have more than one right angle, the only matchings which can be congruences are $DAB \leftrightarrow BCD$ and $DAB \leftrightarrow DCB$. If $DAB \leftrightarrow BCD$ is a congruence then $\angle ADB \cong \angle CBD$. From the figure, B is interior to $\angle ADC$. So, if $m(\angle ADB) = a$ then $m(\angle CDB) = 60 - a$, and by Theorem 5-11, $m(\angle CBD) = 30 + a$. Hence, $a = 30 + a$, that is, $0 = 30$. So, assuming that B is interior to $\angle ADC$, if $DAB \leftrightarrow BCD$ is a congruence then $0 = 30$. But, $0 \neq 30$. So, $DAB \leftrightarrow BCD$ is not a congruence. Hence, $DAB \leftrightarrow DCB$ is a congruence. Therefore $\angle ADB \cong \angle CDB$. So, $\angle ADB$ is an angle of $30°$ and $\angle ABD$ is an angle of $60°$. By Theorem 5-12, $AB = \frac{1}{2} \cdot DB$. But, $AB = BC$. So, $AB + BC = DB$.

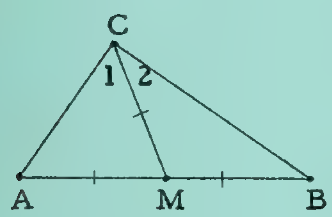


If, despite the figure, we assume that B is not interior to $\angle ADC$, it turns out that $DAB \leftrightarrow BCD$ is a congruence and that $DAB \leftrightarrow DCB$ is not. But, in that case, $AB + BC > DB$. [When students have had the Pythagorean Theorem you may wish to give them the problem of computing DB given AB.]

4. Since $NQ = \frac{1}{2} \cdot MN$, and since $PN = \frac{1}{2} \cdot NQ$, it follows that $PN = \frac{1}{4} \cdot MN$. [You can extend the problem by drawing the altitude $\overleftrightarrow{PR}$ of ΔPQN from P, the altitude $\overleftrightarrow{RS}$ of ΔRPN from R, and asking what fraction SN is of MN.]

5. 

There is a point $P \in \overrightarrow{ST}$ such that $ST = TP$. Since $P \neq S$, it follows that T is the mid-point of $\overleftrightarrow{SP}$. Hence, $\overleftrightarrow{RT}$ is the median of $\triangle RPS$ from R. Since $\overleftrightarrow{RT} \perp \overleftrightarrow{ST}$, $\overleftrightarrow{RT}$ is the altitude of $\triangle RPS$ from R. So, $\triangle RSP$ is an isosceles triangle with $RS = RP$. By hypothesis, $ST = \frac{1}{2} \cdot SR$. Since $ST = TP$, $SP = SR$. So, $\triangle SPR$ is equilateral. Hence, it is equiangular, and, by Theorem 5-11, $\angle S$ is an angle of $60°$.

6. (a) 

Since $CM = AM$, $\angle C_1 \cong \angle A$. Since $CM = BM$, $\angle C_2 \cong \angle B$. Since M is interior to $\angle ACB$, $m(\angle C) = m(\angle C_1) + m(\angle C_2) = m(\angle A) + m(\angle B)$. But, by Theorem 5-11, $m(\angle C) + [m(\angle A) + m(\angle B)] = 180$. So, $\angle C$ is an angle of $90°$.

(b) No. In view of Exercise 6(a), such a triangle would have two right angles.

7. Since $AM > CM$, $\angle C_1$ is larger than $\angle A$. Since $BM > CM$, $\angle C_2$ is larger than $\angle B$. So, since M is in the interior of $\angle ACB$, $m(\angle C) > m(\angle A) + m(\angle B)$. Hence, by Theorem 5-11, $2 \cdot m(\angle C) > 180$. Therefore, $\angle C$ is obtuse.

8. [Similar to Exercise 7.]

9. Since, from the figure, B is in the interior of $\angle ACE$, $m(\angle ACE) = m(\angle ACB) + m(\angle BCE)$. Since A is in the interior of $\angle DCB$, $m(\angle DCB) = m(\angle ACD) + m(\angle ACB)$. But, $\angle BCE \cong \angle ACD$. So, $\angle ACE \cong \angle DCB$. Also, by hypothesis, $CE = CB$ and $CA = CD$. So, by s.a.s., $ECA \leftrightarrow BCD$ is a congruence. Hence, $\Delta ECA \cong \Delta BCD$.

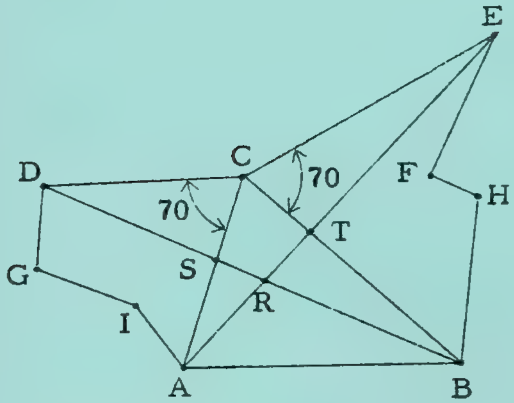
$m(\angle ETB) = \alpha + \beta$ [Theorem 5-10]

$m(\angle CBD) = \beta$ [$ECA \leftrightarrow BCD$ is a congruence]

$m(\angle TRB) = \alpha$ [Theorem 5-10]

10. Since $\angle ETB$ is an exterior angle of both ΔTCE and ΔTRB, $90 + m(\angle AEC) = m(\angle TRB) + m(\angle DBC)$. As in Exercise 9, $ECA \leftrightarrow BCD$ is a congruence. So, $\angle AEC \cong \angle DBC$. Therefore, $\angle TRB$ is a right angle. So, $m(\angle TRS) = 90$.

☆ 11.



$m(\angle ECB) = 70$,

$m(\angle ACD) = 70$,

$AC = DC$,

$CE = CB$.

Find $m(\angle TRS)$.

[Students should make the discovery that the number of sides of the figures adjoined to ΔABC is irrelevant. The points F and G play no essential role in Exercise 10, and, as shown in the second part of Exercise 9, $\angle TRB \cong \angle BCE$. So, in any case, $\angle TRS$ is a supplement of $\angle BCE$. (However, the proof is slightly different if $\angle BCE$ is supplementary to, or larger than, $\angle ACB$. In the first case $C \in \overline{AE}$, and $\overrightarrow{CA} \cup \overrightarrow{CE}$ is not an angle, while in the second, B is not interior to $\angle ACE$.)]

> Correction. On page 6-152, line 4 should read:
>
> --- parallel if $\underbrace{\text{and only if}}_{\uparrow}$ they are ---

line 6. No, because $\overline{MN}$ and $\overline{PQ}$ could, for example, be subsets of the same line.

*

Answers for Part I [on pages 6-152, 6-153, and 6-154].

[$\overrightarrow{AB}$ and $\overrightarrow{DC}$ are parallel because $\overrightarrow{AB} \subseteq \overleftrightarrow{MT}$, $\overrightarrow{DC} \subseteq \overleftrightarrow{GJ}$, and $\overleftrightarrow{MT} \parallel \overleftrightarrow{GJ}$.]

1. $\overrightarrow{GD}$, $\overrightarrow{DC}$, $\overrightarrow{JC}$, $\overrightarrow{CJ}$

2. $\overrightarrow{GD}$, $\overrightarrow{DJ}$, $\overrightarrow{AB}$, $\overrightarrow{AT}$, $\overrightarrow{BT}$; [$\overrightarrow{AT} = \overrightarrow{AB}$]

3. $\overrightarrow{DG}$, $\overrightarrow{AM}$, $\overrightarrow{BA}$

4. (a) yes (b) yes
   (c) They are congruent. [Each is congruent to $\angle RBT$.]

5. (a) yes (b) yes
   (c) They are congruent. [Each is congruent to $\angle CBT$.]

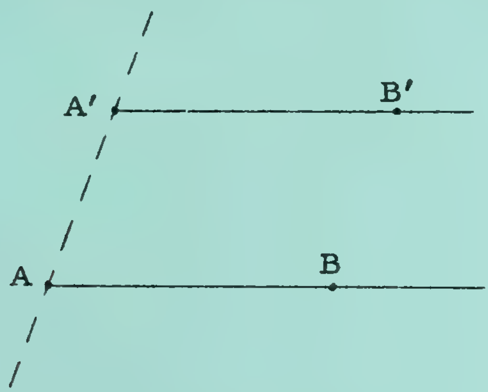
6. (a) $\angle GDE$ and $\angle GCL$ [or $\angle GDA$ and $\angle GCB$, $\angle CDE$ and $\angle LCJ$, $\angle CDN$ and $\angle JCB$]
   These angles are congruent.
   (b) $\angle GDE$ and $\angle JCB$ [or $\angle GDA$ and $\angle LCJ$, $\angle CDE$ and $\angle GCB$, $\angle CDN$ and $\angle GCL$]
   These angles are congruent.

7. $\angle MAN$ [or $\angle MBR$, $\angle DCR$, $\angle GDN$]
   They are supplementary.

*

For each two points A and B, $\overrightarrow{AB}$ and $\overrightarrow{AB}$ are similarly directed, and $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are oppositely directed.

The proof of Theorem 5-13 is somewhat complicated by the need to consider several cases. Suppose that $\angle CAB$ and $\angle C'A'B'$ are two angles such that $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are similarly directed, and $\overrightarrow{AC}$ and $\overrightarrow{A'C'}$ are similarly directed. We note, first, that it follows that $A \neq A'$. For, if $A = A'$ then, for any points P and P′ such that $\overrightarrow{AP}$ and $\overrightarrow{A'P'}$ are similarly directed, $\overrightarrow{AP} = \overrightarrow{A'P'}$. So, if $A = A'$ then $\angle CAB = \angle C'A'B'$. However, since $\angle CAB$ and $\angle C'A'B'$ are two angles, $A \neq A'$. Next, we note that, since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are noncollinear, they are not both subsets of $\overleftrightarrow{AA'}$. So, we may simplify the discussion by assuming that $\overrightarrow{AB}$ is not a subset of $\overleftrightarrow{AA'}$. Now, since $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are similarly directed, it follows that $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$ and that $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are subsets of the same side of $\overleftrightarrow{AA'}$. [If $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ were collinear, $\overrightarrow{AB}$ would be a subset of $\overleftrightarrow{AA'}$.]



There are now three cases to consider:

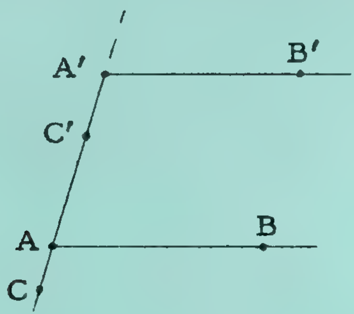
(1) $\overrightarrow{AC}$ is a subset of $\overleftrightarrow{AA'}$

(2) $\overrightarrow{AC}$ is on the non-B-side of $\overleftrightarrow{AA'}$

(3) $\overrightarrow{AC}$ is on the B-side of $\overleftrightarrow{AA'}$

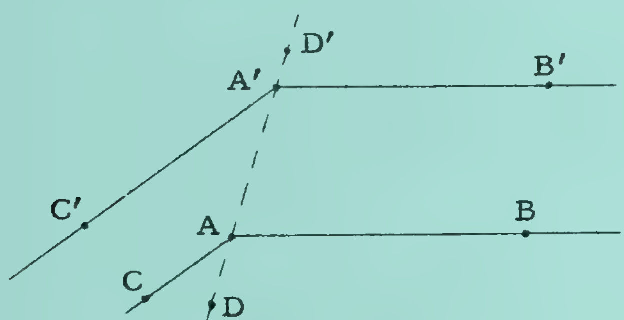
In case (1), since $\overrightarrow{AC}$ and $\overrightarrow{A'C'}$ are similarly directed, one is a subset of the other, and we may assume that $\overrightarrow{AC} \subseteq \overrightarrow{A'C'}$. It follows that $A \in \overrightarrow{A'C'}$ and, since $A \neq A'$, that $A' \notin \overrightarrow{AC}$. Hence, $\angle CAB$ and $\angle C'A'B'$ are

corresponding angles [exterior and interior, respectively] and, since



$\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$, it follows by Theorem 5-7 that $\angle CAB \cong \angle C'A'B'$.

In case (2), since $\overrightarrow{AC}$ and $\overrightarrow{A'C'}$ are similarly directed, $\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$ and $\overrightarrow{AC}$ and $\overrightarrow{A'C'}$ are both subsets of the non-B-side of $\overleftrightarrow{AA'}$. Let D be a



point such that $A \in \overline{DA'}$. Since A, B, and C are not collinear, and since B and C are on opposite sides of $\overleftrightarrow{AD}$, it follows by Theorem 2-9 that $\angle CAD$ is not a supplement of $\angle DAB$. So, either:

(2′) $\angle CAD$ is smaller than a supplement of $\angle DAB$,

or:

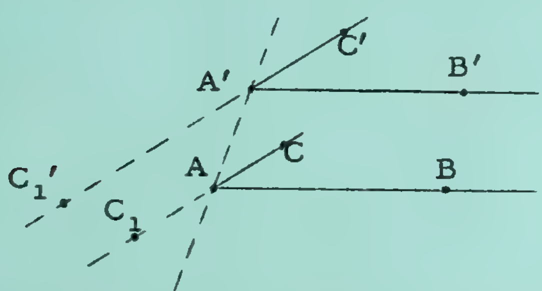
(2″) $\angle CAD$ is larger than a supplement of $\angle DAB$

In the second of these cases, since $\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$, it follows by Theorem 5-7 that $\angle C'A'D$ is larger than a supplement of $\angle DA'B'$. Hence, a supplement of $\angle C'A'D$ is smaller than $\angle DA'B'$. Consequently, if D′ is a point such that $A' \in \overline{D'D}$, $\angle C'A'D'$ is smaller than a supplement of $\angle D'A'B'$.

Comparing this condition with (2′), we see that it is sufficient to consider the first of the two cases. So, we assume (2′). From this it follows, using Theorem 5-7, that $\angle C'A'D$ is, also, smaller than a supplement of $\angle DA'B'$. Since, as we shall see, it follows on the assumption (2′) that D is interior to $\angle CAB$, it also follows that D is interior to $\angle C'A'B'$. From this, Theorem 5-7, and Axiom F, it follows that $\angle CAB \cong \angle C'A'B'$.

So, to settle case (2), all that remains is to show that, assuming (2′), D is interior to $\angle CAB$. To do so, we note that, since B and C are on opposite sides of $\overleftrightarrow{AA'}$, $\overline{BC}$ crosses $\overleftrightarrow{AA'}$ at a point P interior to $\angle CAB$. So, by Axioms F and D, $m(\angle CAP) + m(\angle PAB) = m(\angle CAB) < 180$. Hence, $\angle CAP$ is smaller than a supplement of $\angle PAB$. Now, either $A \in \overline{PD}$ or $D \in \overrightarrow{AP}$ and, so, is interior to $\angle CAB$. But, if $A \in \overline{PD}$ then $\angle CAP$ and $\angle CAD$ are supplementary, as are $\angle PAB$ and $\angle DAB$. So, since $\angle CAP$ is smaller than a supplement of $\angle PAB$, a supplement of $\angle CAP$, such as $\angle CAD$, is larger than $\angle PAB$. Hence, $\angle CAD$ is larger than a supplement of $\angle DAB$. Consequently, if $\angle CAD$ is smaller than a supplement of $\angle DAB$ then $A \notin \overline{PD}$, $D \in \overrightarrow{AP}$, and D is interior to $\angle CAB$.

Case (3) is, now, settled very easily. For, if C_1 and C_1' are points such that $A \in \overline{C_1C}$ and $A' \in \overline{C_1'C'}$ then $\overrightarrow{AC_1}$ and $\overrightarrow{A'C_1'}$ are subsets of the



non-B-side of $\overleftrightarrow{AA'}$. So, by case (2), $\angle C_1AB \cong \angle C_1'A'B'$. But, also, $\angle CAB$ and $\angle C_1AB$ are supplementary, as are $\angle C'A'B'$ and $\angle C_1'A'B'$. So, by Theorem 2-3, $\angle CAB \cong \angle C'A'B'$.

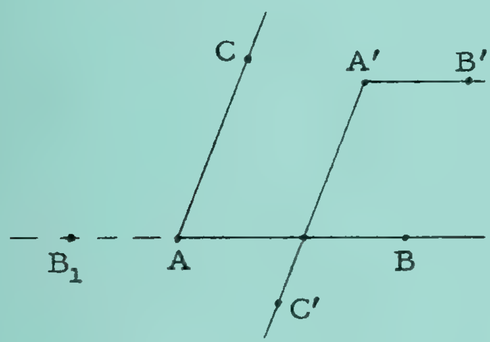
*

8. Theorem 5-14 is:

If the sides of two angles can be matched in such a way that corresponding sides are oppositely directed then the angles are congruent.

This theorem follows immediately from Theorem 5-13 and Theorem 2-5. There are two cases: If the angles have the same vertex then they are vertical angles and are congruent by Theorem 2-5. If the angles do not have the same vertex then, by Theorem 5-13, either is congruent to the vertical angle of the other. So, by Theorem 2-5, the angles are congruent.

9.



The missing word is 'supplementary'. To prove Theorem 5-15 it is sufficient [referring to the figure] to consider a point B_1 such that $A \in \overline{B_1B}$. By Theorem 5-14, $\angle CAB_1$ and $\angle C'A'B'$ are congruent. Since $\angle CAB$ and $\angle CAB_1$ are supplementary, it follows that $\angle CAB$ and $\angle C'A'B'$ are supplementary. [Theorem 5-15 can also be deduced from Theorem 5-13. This procedure requires the consideration of two cases, $A = A'$, and $A \neq A'$.]

2. By the Example [foot of page 6-154], $\angle C$ is a right angle and $\angle B \cong \angle D$. By Theorem 5-9, $\overleftrightarrow{CD} \perp \overleftrightarrow{AD}$; so, $\angle D$ is a right angle. Hence, $\angle B$ is a right angle.

3. From the figure, A is in the interior of $\angle BCD$ and C is in the interior of $\angle BAD$. So,

$$m(\angle BAD) = m(\angle BAC) + m(\angle DAC),$$

and $$m(\angle BCD) = m(\angle BCA) + m(\angle DCA).$$

By Theorem 5-11,

$$m(\angle BAC) + m(\angle BCA) = 180 - m(\angle B),$$

and $$m(\angle DAC) + m(\angle DCA) = 180 - m(\angle D).$$

So, $$m(\angle BAD) + m(\angle BCD) = 360 - m(\angle B) - m(\angle D).$$

Therefore, $$m(\angle BAD) + m(\angle B) + m(\angle BCD) + m(\angle D) = 360.$$

In connection with the Example [which begins on page 6-154], since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, C and D are on the same side of $\overleftrightarrow{AB}$. So, since $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$, it follows that $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are similarly directed rays. So, $\overrightarrow{BC}$ and $\overrightarrow{DA}$ are oppositely directed. This argument shows that it is not necessary, as is done in the text, to appeal to the figure.

The Example can also be solved by using Theorem 5-5: Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, C and D are on the same side of $\overleftrightarrow{AB}$. So, $\angle A$ and $\angle B$ are consecutive interior angles and, since $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$, are supplementary. Similarly, $\angle A$ and $\angle D$ are supplementary. Consequently, by Theorem 2-3, $\angle B \cong \angle D$.

*

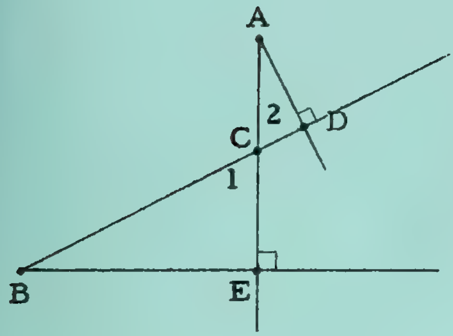
The problems of Part J are important in two respects. For one thing, the strategies used will be helpful in deriving theorems about quadrilaterals in section 6.06. In fact, some of the exercises will suggest theorems. For example, the Example leads to the theorem about the opposite angles of a parallelogram being congruent. These problems are also important in that the results of exercises solved early in the list can be used to solve problems which occur later in the list. For example, the result of the Example can be used to good advantage in Exercises 1 and 2, and Exercise 3 can be used in Exercise 5. Students should be encouraged to look for connections like these. This outlook will be helpful to the student in the next section.

*

Answers for Part J [on pages 6-155, 6-156, and 6-157].

1. Since $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$, $\overleftrightarrow{DC} \perp \overleftrightarrow{BC}$, and [from the figure] $\overleftrightarrow{AB} \neq \overleftrightarrow{DC}$, it follows from Theorem 5-8 that $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$. Similarly, $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$. So, by the Example [foot of page 6-154], $\angle BAD \cong \angle DCB$. Hence, since $\angle DCB$ is a right angle, so is $\angle BAD$. Therefore, $\overleftrightarrow{BA} \perp \overleftrightarrow{DA}$.

   [Alternative proof. After showing, as above, that $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, use Theorem 5-9 and the hypothesis that $\overleftrightarrow{DC} \perp \overleftrightarrow{AD}$ to show that $\overleftrightarrow{BA} \perp \overleftrightarrow{DA}$.]

4. 

By Theorem 5-11,

$$m(\angle A) + m(\angle C_2) = 90$$

and $m(\angle B) + m(\angle C_1) = 90$.

But, $\angle C_1$ and $\angle C_2$ are vertical angles.

So, $\angle A \cong \angle B$.

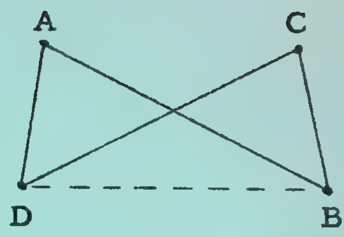
5. From the figure, B is interior to $\angle A$ and A is interior to $\angle B$. So, by Exercise 3, $m(\angle A) + 90 + m(\angle B) + 90 = {}^-360$. Hence, $\angle A$ and $\angle B$ are supplementary.

6. From the figure, $\angle ABD$ and $\angle CDB$ are alternate interior angles and, since $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, it follows by Theorem 5-3 that $\angle ABD \cong \angle CDB$. Similarly, $\angle ADB \cong \angle CBD$. So, by a.s.a., $ABD \leftrightarrow CDB$ is a congruence. Hence, $AB = CD$ and $AD = CB$.

The reference to the figure is not needed. To show that $\angle ABD$ and $\angle CDB$ are alternate interior angles, we need to know that A and C are on opposite sides of $\overleftrightarrow{BD}$. We do so by showing that $\overline{AC} \cap \overrightarrow{BD} \neq \emptyset$ [that is, that the diagonals of a parallelogram intersect]. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, B is on the A-side of $\overleftrightarrow{DC}$. Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, B is on the C-side of $\overleftrightarrow{AD}$. Hence, B is interior to $\angle ADC$. Consequently, $\overrightarrow{DB} \cap \overline{AC}$ consists of a single point. Similarly, C is interior to $\angle DAB$. Consequently, $\overline{DB} \cap \overrightarrow{AC}$ consists of a single point. It follows that $\overleftrightarrow{DB} \cap \overleftrightarrow{AC}$ consists of a single point, and that this point belongs to both $\overline{DB}$ and $\overline{AC}$.

7. By s.s.s., $ABD \leftrightarrow CDB$ is a congruence. So, $\angle ABD \cong \angle CDB$ and $\angle ADB \cong \angle CBD$. From the figure, these are pairs of alternate interior angles. Hence, by Theorem 5-2, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$.

Note that, for Exercise 7, reference must be made to the figure. That this is so can be seen by considering the figure consisting of two legs and the diagonals of an isosceles trapezoid.



To avoid reference to the figure [of Exercise 6], it is sufficient to add to the Hypothesis of Exercise 7 the condition: A and C are on opposite sides of $\overleftrightarrow{BD}$.

8. No. Since $ABD \leftrightarrow CBD$ is a congruence, it follows that $\angle ABD \cong \angle CBD$. It may not be the case that $\angle ABD \cong \angle CDB$. If not, $\overleftrightarrow{AB} \nparallel \overleftrightarrow{CD}$.

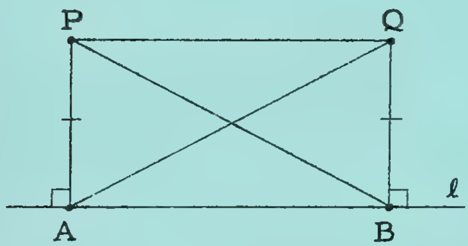
9. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, it follows from Theorem 5-3 that $\angle ABD \cong \angle CDB$. So, $ABD \leftrightarrow CDB$ is a congruence [by s.a.s.]. Hence, $AD = CB$. Also, $\angle ADB \cong \angle CBD$. So, by Theorem 5-2, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

Note that, as in Exercise 7, the conclusion that $\angle ABD$ and $\angle CDB$ are alternate interior angles [so that Theorem 5-3 is applicable] is one which must be drawn from the figure. As in Exercise 7, one assumes that A and C are on opposite sides of $\overleftrightarrow{BD}$. This also insures that $\angle ADB$ and $\angle CBD$ are alternate interior angles, so that, in showing that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, one can apply Theorem 5-2.

10. By Exercise 6, $\overline{BA} \cong \overline{CD}$. By hypothesis, $\angle BAD$ is a right angle, and, by Exercise 2, $\angle CDA$ is a right angle. Also, $\overline{AD} \cong \overline{DA}$. Hence, by s.a.s., $BAD \leftrightarrow CDA$ is a congruence. Consequently, $\overline{AC} \cong \overline{DB}$.

7. $\angle ABE \cong \angle DEF$ by Theorem 5-7. Since $m(\angle CBE) = \frac{1}{2} \cdot m(\angle ABE)$ and $m(\angle GEF) = \frac{1}{2} \cdot m(\angle DEF)$, it follows that $\angle CBE \cong \angle GEF$. So, by Theorem 5-6, $\overleftrightarrow{CB} \parallel \overleftrightarrow{GE}$.

[Since from the figure, A and D are on the same side of $\overleftrightarrow{BE}$ and $E \in \overline{FB}$, $\angle ABE$ and $\angle DEF$ are corresponding angles. By the definition of angle bisector, C is on the A-side of $\overleftrightarrow{BE}$ and G is on the D-side of $\overleftrightarrow{BE}$. So, again, $\angle CBE$ and $\angle GEF$ are corresponding angles.]

8. 

Let A and B be the feet of the perpendiculars to ℓ from P and Q. Since $\overleftrightarrow{PA}$ and $\overleftrightarrow{QB}$ are perpendicular to ℓ, it follows from Theorem 5-8 that $\overleftrightarrow{PA} \parallel \overleftrightarrow{QB}$. So, by Theorem 5-3, $\angle PAQ \cong \angle BQA$. By hypothesis, $PA = BQ$. Also, $AQ = QA$. So, by s.a.s., $PAQ \leftrightarrow BQA$ is a congruence. Hence, $\angle PQA \cong \angle BAQ$. Therefore, by Theorem 5-2, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{AB}$.

*

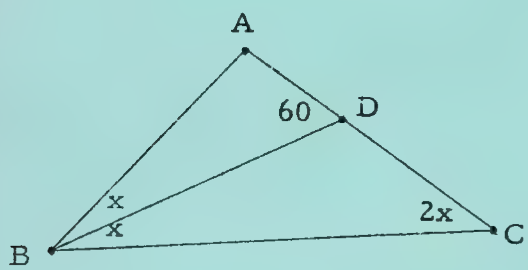
Answers for Quiz.

1. 57°

2. $\angle C$

3. $m(\angle C) = 180 - 2\alpha$, $m(\angle A) = \alpha - \beta$

4.

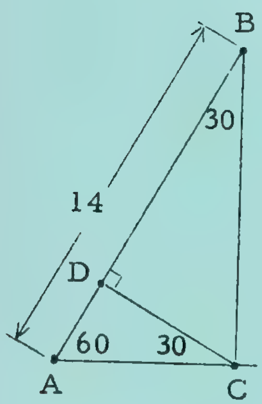


$x + 2x = 60$

$x = 20$

$m(\angle A) = 180 - 20 - 60 = 100$

5.



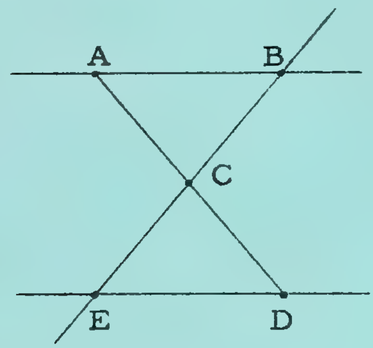
$AC = 7$, $AD = 3.5$

So, $DB = 10.5$

6. Since $\angle ACB \cong \angle DCE$, it follows by s.a.s. that $ACB \longleftrightarrow DCE$ is a congruence. So, $\angle BAC \cong \angle EDC$. Hence, by Theorem 5-2, [since A and D are on opposite sides of $\overleftrightarrow{BE}$], $\overleftrightarrow{AB} \parallel \overleftrightarrow{ED}$.

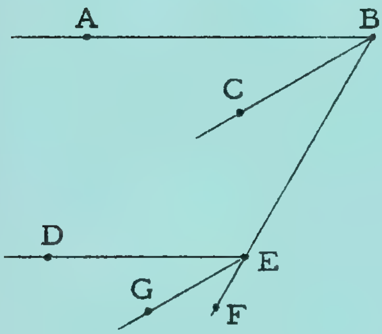
4. Suppose that $\triangle ABC$ is isosceles with $AB = AC$. If $\overline{BD}$ is the angle bisector of $\triangle ABC$ from B and $m(\angle ADB)$ is 60, what is $m(\angle A)$?

5. In $\triangle ABC$, $\angle A$ is an angle of $60°$, $\angle C$ is an angle of $90°$, and $AB = 14$. If $\overline{CD}$ is the altitude of $\triangle ABC$ from C then $DB =$ ____.

6. 

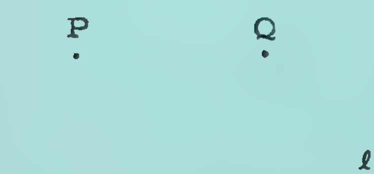
Hypothesis: C is the midpoint of $\overline{AD}$, C is the midpoint of $\overline{BE}$

Conclusion: $\overleftrightarrow{AB} \parallel \overleftrightarrow{ED}$

7. 

Hypothesis: $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$, $\overrightarrow{BC}$ bisects $\angle ABF$, $\overrightarrow{EG}$ bisects $\angle DEF$

Conclusion: $\overleftrightarrow{CB} \parallel \overleftrightarrow{GE}$

☆ 8. 

Hypothesis: P and Q are two points on the same side of ℓ and equidistant from ℓ

Conclusion: $\overleftrightarrow{PQ} \parallel \ell$

11. [As shown in the discussion of Exercise 6, the third part of the Hypothesis of Exercise 11 is unnecessary. The diagonals of a parallelogram do intersect in a single point, and one is at liberty to call the point 'E'.] Since $\overline{BD}$ and $\overline{AC}$ cross at E, A and C are on opposite sides of $\overleftrightarrow{BD}$. So, $\angle CBD$ and $\angle ADB$ are alternate interior angles. Since, by hypothesis, $\overline{BC} \parallel \overline{AD}$, it follows by Theorem 5-3 that $\angle CBD \cong \angle ADB$. Similarly, $\angle BCA \cong \angle DAC$. Also, since $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$, it follows from Exercise 6 that BC = AD. So, by a.s.a., BCE $\longleftrightarrow$ DAE is a congruence. Consequently, DE = BE and CE = AE. Since, by hypothesis, $E \in \overline{BD} \cap \overline{AC}$, it follows that E is the midpoint of $\overline{BD}$ and, also, the midpoint of $\overline{AC}$. So, by definition, $\overline{AC}$ and $\overline{BD}$ bisect each other.

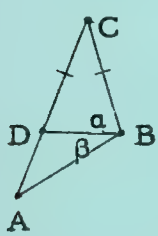
12. By Exercise 11, ED = EB. By hypothesis, DA = BA. So, by Theorem 3-3, $\overleftrightarrow{AE}$ is the perpendicular bisector of DB. Consequently, $\overline{AC} \perp \overline{BD}$ at E.

*

Quiz.

1. In $\triangle ABC$, if $\angle A$ is an angle of 50° and $\angle B$ is an angle of 73° then $\angle C$ is an angle of _____.

2. In $\triangle ABC$, if AB > BC and $\angle A$ is an angle of 60°, which is the largest angle of the triangle?

3. 

Suppose that D is a point of $\overline{AC}$ such that CD = CB, $m(\angle CBD) = \alpha$, and $m(\angle DBA) = \beta$. Use α and β to compute the measures of $\angle C$ and $\angle A$.

Answers for Part A.

Starting at A, he can walk, first, to any of the four points B, C, D, and E; next, to any of the remaining three points; next, to any of the remaining two points; and, finally, from there to A. So, there are $4 \times 3 \times 2$, or 24, possible trips. However, with each trip there corresponds another in which he traverses the same path in the opposite direction. For example, the trips $A \to B \to C \to D \to E \to A$ and $A \to E \to D \to C \to B \to A$ are taken over the same path. So, there are 12 possible paths.

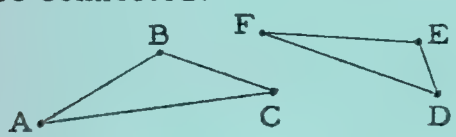
*

Answer for Part B.

Yes; the only simple closed polygonal path is that which connects A and B, B and D, D and C, C and E, and E and A.

*

Properties (1) and (2) are not [contrary to the statement at the foot of page 6-159] quite sufficient to characterize what are usually called 'simple closed polygonal paths'. In addition to these, it is customarily assumed that a simple closed polygonal path is connected--that is, "all one piece". Each of the twelve paths of Part A has this property, but a union of six or more segments may have properties (1) and (2) but not be connected:



In the figure, each of the six lettered points is an end point of just two segments and no two segments intersect except at an end point. [Incidentally, (2) should be interpreted as saying that if two segments intersect then their intersection consists of a single point which is an end point of both segments. You may wish to make this more explicit, here and on page 6-160, either by replacing the word 'an' in (2) by 'a common', or by adding 'of both' after the word 'point'. It is good practice to encourage students to correct the wording.]

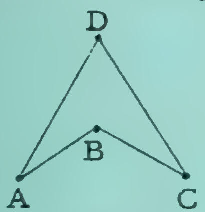
In discussing the notion of simple closed polygonal path, it will be best to bring out, by examples such as that pictured above, the notion of connectedness, and agree with your students that simple closed polygonal paths should be connected. Then, in the definition of 'polygon' on page 6-160, supplement line 13 to read:

> A connected set which is the union of a finite number of segments satisfying the conditions:

Answers for Part C.

1. Yes. Each triangle is the union of its three sides, each vertex is an end point of just two sides, no two sides intersect except at a common end point, and a triangle is connected.

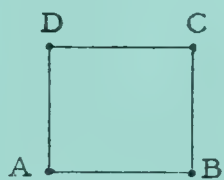
2. There are two possibilities.



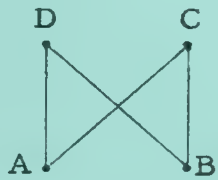
simple

[B interior to ΔACD]

[no nonsimple path]



simple



nonsimple

3. Yes. If A, B, and C are collinear and B, say, is between A and C then $\overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$ is a simple closed polygonal path from A back to A.

*

line 8 from bottom: An angle of a polygon is an angle which contains two adjacent sides of the polygon.

*

Answers for Part D.

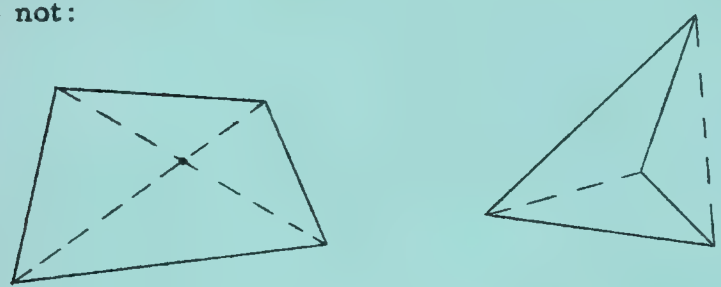
1. Yes. It is a triangle, ΔACD, which is the union of $\overline{AC}$, $\overline{CD}$, and $\overline{DA}$. [It is <u>not</u> polygon ABCD. See the note on TC[6-161]a.]

2. 'triangle'

Answers for Part F.

1. The diagonals are five in number; $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CE}$. For each n, an n-gon has $n(n - 3)/2$ diagonals. [There are $n - 3$ diagonals with a given vertex as end point, and each diagonal has 2 end points.]

2. Two. Some four-sided polygons have intersecting diagonals, others do not:

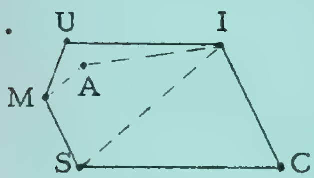


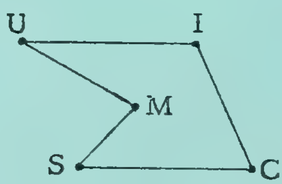
3. 0; 2; 5; 9; 14; 20; 170; 4850; 49985000

The notation 'ABCD' is an abbreviation for '$\overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$'. In the case pictured on page 6-161, ABCD is a polygon and, in particular, is a four-sided polygon, or quadrilateral. Since CDAB = $\overline{CD} \cup \overline{DA} \cup \overline{AB} \cup \overline{BC}$ [and since unioning is associative and commutative], CDAB = ABCD. On the other hand, BDAC, while it is a closed polygonal path, is not a simple one. So, BDAC is not a polygon and, in particular, BDAC $\neq$ ABCD. If the figure on page 6-161 were modified in such a way that D = C, ABCD would still be a polygon [$\overline{AB} \cup \overline{BC} \cup \overline{CC} \cup \overline{CA} = \overline{AB} \cup \overline{BC} \cup \overline{CA}$]. However, it is convenient to adopt the convention that when one uses such phrases as [see Exercise 1 of Part E] 'polygon UICSM', one implies that the points referred to are different and are the vertices of the polygon; and that in questions like that of Exercise 3(a), the word 'polygon' signals that A, B, C, and D are the <u>four</u> vertices of the polygon in question. Thus, the polygon described in the solution for Exercise 3 of Part C on page 6-160 can properly be spoken of as 'the polygon ACD', but not as 'the polygon ABCD'.

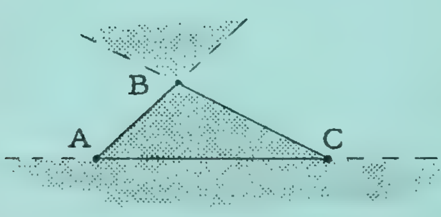
*

Answers for Part E.

1. 

2. 

[I, S, and M are collinear.]

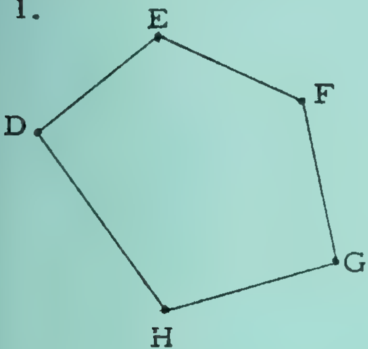
3. (a) 

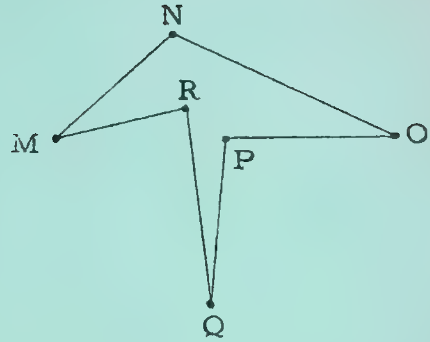
D is either on the non-B-side of $\overleftrightarrow{AC}$, or interior to the angle vertical to $\angle B$ or interior to $\angle B$ and not on $\overline{AC}$

(b) The set in question is the complement of $\overleftrightarrow{AB} \cup \overleftrightarrow{BC} \cup \overleftrightarrow{CA}$.

*

Answers for Part G.

1. 

2. 

By definition, a polygon is convex if and only if, for each of its sides, all of its points not on this side belong to the same one of the two half-planes whose common edge contains the side in question. For example, the polygon MNOPQR, pictured above, is not convex because N and O are on opposite sides of $\overleftrightarrow{MR}$ and, also, because $M \in \overleftrightarrow{OP}$. On the other hand, DEFGH is convex. For a more explicit example, it is easy to show that a parallelogram is convex. For, if $\overline{CD} \parallel \overline{AB}$ then, by Theorem 15, C and D are on the same side of $\overleftrightarrow{AB}$. Consequently, $\overrightarrow{AD}$, $\overline{DC}$, and $\overrightarrow{BC}$ are all subsets of this same side of $\overleftrightarrow{AB}$. [See, for $\overrightarrow{AD}$ and $\overrightarrow{BC}$, Theorem 18, and, for $\overline{DC}$, Theorem 15.] So, for parallelogram ABCD, all points not on $\overline{AB}$ are on one side of $\overleftrightarrow{AB}$. Similarly, all points not on $\overline{BC}$ are on one side of $\overleftrightarrow{BC}$, all points not on $\overline{CD}$ are on one side of $\overleftrightarrow{CD}$, and all points not on $\overline{DA}$ are on one side of $\overleftrightarrow{DA}$. So, ABCD is convex. A similar argument proves that to show any given quadrilateral [four-sided polygon] ABCD to be convex, it is sufficient to show that C and D are on the same side of $\overleftrightarrow{AB}$, that D and A are on the same side of $\overleftrightarrow{BC}$, that A and B are on the same side of $\overleftrightarrow{CD}$, and that B and C are on the same side of $\overleftrightarrow{DA}$. To establish the convexity of quadrilateral ABCD it is, in fact, sufficient to show merely that C and D are on the same side of $\overleftrightarrow{AB}$ and A and B are on the same side of $\overleftrightarrow{CD}$. [Among other

consequences, this implies that trapezoids are convex.] To see that this is so, we shall first establish that

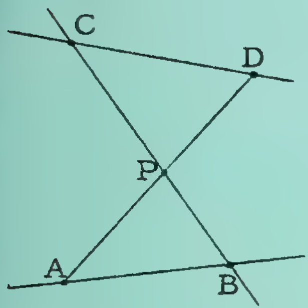
if A, B, C, and D are four points such that

(1) C and D are on the same side of $\overleftrightarrow{AB}$,

(2) A and B are on the same side of $\overleftrightarrow{CD}$, and

(3) A and D are on opposite sides of $\overleftrightarrow{BC}$,

then $\overline{BC} \cap \overline{AD}$ consists of a single point.



For, from (3) it follows that $\overleftrightarrow{BC} \cap \overline{AD}$ consists of a single point P. By (1), D is on the C-side of $\overleftrightarrow{AB}$. So, since $P \in \overline{AD} \subseteq \overrightarrow{AD}$, P is on the C-side of $\overleftrightarrow{AB}$. Hence, since $P \in \overleftrightarrow{BC}$, it follows that $P \in \overrightarrow{BC}$. Similarly, using (2), $P \in \overrightarrow{CB}$. Since $\overrightarrow{BC} \cap \overrightarrow{CB} = \overline{BC}$, $P \in \overline{BC}$. Consequently, since $P \in \overline{AD}$, $P \in \overline{BC} \cap \overline{AD}$. Finally, since $\overline{BC} \subseteq \overleftrightarrow{BC}$ and P is the <u>only</u> member of $\overleftrightarrow{BC} \cap \overline{AD}$, it follows that P is the only member of $\overline{BC} \cap \overline{AD}$.

Now, if ABCD [that is, $\overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$] is a quadrilateral then, by definition, $\overline{BC} \cap \overline{AD} = \emptyset$. So, from the result just proved, if, in quadrilateral ABCD, C and D are on the same side of $\overleftrightarrow{AB}$ and A and B are on the same side of $\overleftrightarrow{CD}$, then A and D are <u>not</u> on opposite sides of $\overleftrightarrow{BC}$. Since, by definition, neither A nor D belongs to $\overleftrightarrow{BC}$, it follows that A and D are on the same side of $\overleftrightarrow{BC}$. Similarly [interchanging the roles of A and C and those of B and D], it follows that B and C are on the same side of $\overleftrightarrow{AD}$. So, since each two adjacent vertices of quadrilateral ABCD are on the same side of the line containing the other two, quadrilateral ABCD is convex.

Note that if A, B, C, and D are <u>any</u> points such that A and B are on the same side of $\overleftrightarrow{CD}$, B and C are on the same side of $\overleftrightarrow{DA}$, and C and D are

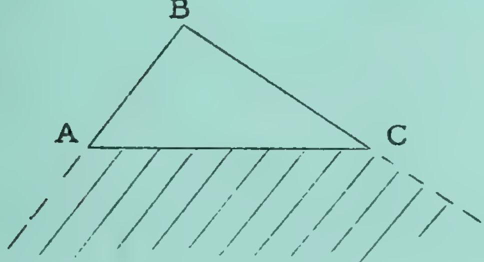
on the same side of $\overleftrightarrow{AB}$, then ABCD is a quadrilateral and, by the result just established, is convex. For, to show that ABCD is a quadrilateral, it is sufficient to show that each of the sets $\{B, C, D\}$, $\{A, C, D\}$, $\{A, B, D\}$, and $\{A, B, C\}$ is a set of noncollinear points, and that each of the sets $\overline{AB} \cap \overline{CD}$ and $\overline{BC} \cap \overline{DA}$ is empty. But since, by hypothesis, A and B are on the same side of $\overleftrightarrow{CD}$, it follows that B, C, and D are noncollinear, that A, C, and D are noncollinear, and that $\overline{AB} \cap \overline{CD} = \emptyset$. And, since C and D are on the same side of $\overleftrightarrow{AB}$, it follows that A, B, and D are noncollinear and that A, B, and C are noncollinear. Finally, since B and C are on the same side of $\overleftrightarrow{DA}$, $\overline{BC} \cap \overline{DA} = \emptyset$.

*

3.

The diagonals of the convex polygon intersect; those of the nonconvex polygon do not.

4. (a)



(b)

The points in question are those which are interior to $\angle B$ and on the non-B-side of $\overleftrightarrow{AC}$.

*

Exercise 3 of Part G suggests that

> for each four-sided polygon, the diagonals of the polygon intersect if and only if the polygon is convex.

To establish this, suppose, first, that ABCD is a convex four-sided polygon. By definition, C is on the D-side of $\overleftrightarrow{AB}$ and on the B-side of $\overleftrightarrow{AD}$--that is, C is interior to $\angle DAB$. Hence, $\overline{BD}$ intersects $\overrightarrow{AC}$ in a single point. Similarly, A is interior to $\angle BCD$, and, so, $\overline{BD}$ intersects $\overrightarrow{CA}$ in a single point. From each of these results it follows that $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$ intersect in a single point, and, from both together, it follows that this point belongs both to $\overline{BD}$ and to $\overline{AC}$.

On the other hand, suppose that ABCD is a four-sided polygon whose diagonals intersect. Since, by definition, no three of the four vertices are collinear, it follows that $\overline{AC} \cap \overline{BD}$ consists of a single point P, and that this point P is not collinear with any side of the quadrilateral. Since $C \in \overrightarrow{AP}$ and $D \in \overrightarrow{BP}$, it follows that C and D are on the same side of $\overleftrightarrow{AB}$ [in fact, the P-side of $\overleftrightarrow{AB}$]. Similarly, D and A are on the same side of $\overleftrightarrow{BC}$, A and B are on the same side of $\overleftrightarrow{CD}$, and B and C are on the same side of $\overleftrightarrow{DA}$. Consequently, ABCD is convex.

Note that, since we have shown that parallelograms and trapezoids are convex, it follows that the diagonals of a quadrilateral of either of these kinds intersect. [Most elementary texts assume, tacitly, that the diagonals of a parallelogram intersect, as a basis for proving that they bisect each other.]

(3) Quadrilateral RSTU is such that RS = ST = TU = UR. By definition, what kind of quadrilateral is this? [Answer: a rhombus. It is also the case that this quadrilateral is a parallelogram, but to justify this one requires more than the definition of a parallelogram.]

(4) Quadrilateral ABCD is a trapezoid. What follows from this by definition? [Answer: either $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ or $\overleftrightarrow{BC} \parallel \overleftrightarrow{DA}$, but not both.]

*

Our definitions could be improved. For example, the definition of parallelogram does not tell us that a parallelogram is a quadrilateral. Although the phrase 'various types of quadrilaterals' indicates that we are talking about quadrilaterals, and that the boxed statements should be read in this context, the boxed statements by themselves are not definitions in the sense that they single out from the universe in which we are working those things which are parallelograms or rectangles or rhombuses, etc. Statements which would serve as definitions of a parallelogram are:

> A set [of points] is a parallelogram if and only if it is a quadrilateral whose opposite sides are parallel.

and:

> A parallelogram is a quadrilateral whose opposite sides are parallel.

In the latter statement, the 'is' is the 'is' of definition.

The definitions of the various types of quadrilaterals given on pages 6-163 and 6-164 may differ from some of those which students learned in earlier grades or from those in conventional textbooks on high school geometry. For that matter, conventional textbooks differ among themselves. All of this serves to point out that definitions involve a certain element of arbitrariness and that there is no universal agreement on the meaning of terms like 'rhombus' or 'trapezoid'. In some treatments, a square is not a rhombus; in ours, it is. In some treatments, a parallelogram is a trapezoid; in ours, it is not.

The purpose of these definitions is to single out certain subsets of the set of all quadrilaterals. It is conceivable, for example, that someone might define a parallelogram to be a quadrilateral with opposite sides congruent. In that case, he would have as a theorem what is now our definition. Or, someone might define a parallelogram to be a quadrilateral with opposite sides both parallel and congruent. In that case, he would have as theorems that a quadrilateral is a parallelogram if and only if its opposite sides are parallel and if and only if its opposite sides are congruent. It is sometimes said in conventional textbooks that one should not include in a definition a property which can be derived from other properties stated in the definition. For example, such textbooks would object to our definition of a rectangle. They would claim that all we should say is that a quadrilateral is a rectangle if and only if three of its angles are right angles. For, then, we could prove that the fourth angle is a right angle. [See Exercises 1 or 3 on page 6-155.] But, such textbooks do not observe this principle when they define, say, congruent triangles. They say that congruent triangles are triangles which "agree in all their parts", and then they obtain a theorem which shows that congruent triangles are triangles which agree in their sides. Clearly, their definition included more than was necessary.

*

You can give students some preliminary practice in learning these definitions by giving them exercises like the following:

(1) I know that quadrilateral ABCD is a parallelogram.
What follows from this by definition?
[Answer: $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{DA}$.]

(2) Quadrilateral MNOP is a rectangle.
What follows from this by definition?
[Answer: $\angle M$, $\angle N$, $\angle O$, and $\angle P$ are right angles.]

Correction.

On page 6-165, line 8 should read:
(6) ...[Steps like (2) and (3)]

Notice that Example 1 is, except for the introduction of the word 'parallelogram', the same as Exercise 6 on page 6-156. As pointed out on TC[6-42]b, the transformation of a hypothesis-conclusion argument such as Exercise 6 into a proof of the corresponding theorem is a relatively standardized procedure. Roughly, one treats the hypothesis as an assumption and, on reaching the conclusion, conditionalizes, thereby discharging this assumption; and generalizes.

As pointed out in the COMMENTARY for Exercise 6 on page 6-156, step (3) of the column proof on page 6-165 can be derived from step (1) [so, in this case, reference to the figure is unnecessary]. The argument is, in part, similar to that given on TC[6-162]d to show that the diagonals of a convex quadrilateral intersect. Also, step (8) is a consequence of part (3) of the definition of polygon on page 6-160. So, in sum, Theorem 6-1 [step (13)] is, indeed, a theorem.

Step (4) is, of course, Theorem 5-3, and step (9) is a.s.a.

Correction.

On page 6-166, line 11 should read:

(4)...[Step like (2)]

and line 14 should read:

(7)...[Steps like (3) and (4)]

In the column proof on page 6-166, step (5) is Theorem 5-8. Strictly, in order to use this theorem one should establish a step '$\overleftrightarrow{AD} \nparallel \overleftrightarrow{BC}$'.

*

The class project mentioned at the bottom of page 6-166 should prove to be a delightful experience for students. You will find students to be quite productive. The practice they get in making conjectures, trying to prove them, and then trying to state theorems in unequivocal language cannot be matched by any set of textbook exercises. Students will probably obtain ideas for possible theorems by making drawings. You may also find it worthwhile to introduce mechanical aids consisting of sticks which can be pivoted together at various points to form deformable models of quadrilaterals. [See the 18th Yearbook of the National Council of Teachers of Mathematics.]

Some of the theorems which your students should discover are given on pages 6-176 through 6-178. [These include the unnumbered boxed theorems on pages 6-165 through 6-172.] Students should, before going on to page 6-179, prove any of the first 29 of these theorems which they have not already proved. [Theorems 6-30 through 6-33 should be thoroughly understood, but their <u>proofs</u> require mathematical induction.]

*

Examples 1 and 2 are Theorems 6-1 and 6-2 of page 6-176. Students are already acquainted, from the Example on pages 6-154 and 6-155, with Theorem 6-4 and can, as in the Example, easily prove Theorem 6-3. They have, essentially, proved Theorem 6-5 when solving Exercise 11 on page 6-157. Part G on pages 6-148 and 6-149 requires, basically, the proof of Theorem 6-7. Theorems 6-6 and 6-8 are suggested by Exercises 7 and 9 on page 6-156.

*

As noted in the COMMENTARY, one must, for completeness, strengthen the hypothesis in each of Exercises 7 and 9 on page 6-156 in order to be able to show that A and C are on opposite sides of $\overleftrightarrow{BD}$. As shown in the COMMENTARY for page 6-162, if ABCD is a quadrilateral such that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ then ABCD is convex. So, as shown later in the same COMMENTARY, the diagonals of ABCD intersect. Hence, if ABCD is a quadrilateral with $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ then A and C are on opposite sides of $\overleftrightarrow{BD}$. Consequently, in the case of Exercise 9, sufficient strength is gained by assuming that ABCD is a quadrilateral. This takes care of Theorem 6-8. A different argument is required to boost the solution for Exercise 7 to a complete proof of Theorem 6-6. Here is such a proof:

Suppose that ABCD is a quadrilateral such that $AB = CD$ and $DA = BC$. Since $BD = DB$, it follows by s.s.s. that $ABD \leftrightarrow CDB$ is a congruence. Hence, $\angle ABD \cong \angle CDB$ and $\angle ADB \cong \angle CBD$. If A and C are on opposite sides of $\overleftrightarrow{BD}$ then these are pairs of congruent alternate interior angles and, by Theorem 5-2, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. So, to show that ABCD is a parallelogram it only remains to be shown that A and C are on opposite sides of $\overleftrightarrow{BD}$. Since ABCD is a quadrilateral, neither A nor C belongs to $\overleftrightarrow{BD}$. Suppose, now, that A and C are on the same side of $\overleftrightarrow{BD}$. Since ABCD is a quadrilateral, $C \notin \overrightarrow{DA}$. Hence, either A is interior to $\angle CDB$ or C is interior to $\angle ADB$. Suppose that A is interior to $\angle CDB$. Then, $\overrightarrow{DA} \cap \overline{CB} \neq \emptyset$, and $m(\angle ADB) < m(\angle CDB)$. Since $\angle ADB \cong \angle CBD$ and $\angle CDB \cong \angle ABD$, it follows that $m(\angle CBD) < m(\angle ABD)$. Hence, by the assumption that A and C are on the same side of $\overleftrightarrow{BD}$, it follows that C is interior to $\angle ABD$. So, $\overline{AD} \cap \overrightarrow{BC} \neq \emptyset$. Since, as shown earlier, $\overrightarrow{DA} \cap \overline{CB} \neq \emptyset$, it follows that $\overline{AD} \cap \overline{BC} \neq \emptyset$. But, since ABCD is a quadrilateral, this is not the case. Consequently, A is not interior to $\angle CDB$. Similarly, C is not interior to $\angle ADB$. Hence, A and C are not on the same side of $\overleftrightarrow{BD}$. Since, as shown earlier, neither A nor C belongs to $\overleftrightarrow{BD}$, it follows that A and C are on opposite sides of $\overleftrightarrow{BD}$.

*

Note that Theorem 6-13 ["A rhombus is a parallelogram."] is a corollary of Theorem 6-6. Also, Theorem 6-14 is a corollary of Theorem 6-1.

*

Theorem 6-11 ["The diagonals of a rectangle are congruent."] may be suggested by Theorem 6-1 and Theorem 6-2. If ABCD is a parallelogram then $\overline{DA} \cong \overline{CB}$ and $\overline{AB} \cong \overline{BA}$. So, by Theorem 4-11, if the diagonal $\overline{BD}$ is longer than the diagonal $\overline{AC}$ then $\angle A$ is larger than $\angle B$; and, if $\overline{AC}$ is longer than $\overline{BD}$ then $\angle B$ is larger than $\angle A$. So, if $\angle A \cong \angle B$ then neither $\overline{AC}$ nor $\overline{BD}$ is longer than the other; that is, if $\angle A \cong \angle B$ then $\overline{AC} \cong \overline{BD}$. Now, if ABCD is a rectangle, $\angle A \cong \angle B$ and, by Theorem 6-2, ABCD is a parallelogram. So, the diagonals of a rectangle are congruent.

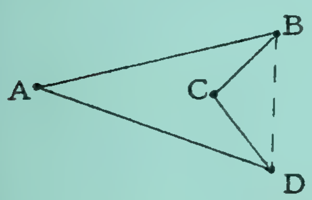
A similar argument suggests, and proves, Theorem 6-12 ["If the diagonals of a parallelogram are congruent then the parallelogram is a rectangle."] One argues, on the basis of Theorem 4-11 that if $\angle A$ is larger than $\angle B$ then $\overline{BD}$ is longer than $\overline{AC}$, etc. Hence, if $\overline{AC} \cong \overline{BD}$ it follows that $\angle A \cong \angle B$. But, by Theorem 6-3, $\angle A$ and $\angle B$ are supplementary. Hence, by the definition of right angle, both are right angles. Still, by Theorem 6-3, $\angle C$ and $\angle D$ are right angles. Hence, ABCD is a rectangle.

*

If, in Theorem 6-9, one inserts 'convex' before the first 'quadrilateral', then the weakened theorem which results is easy to prove. For if ABCD is a <u>convex</u> quadrilateral then C and D are on the same side of $\overleftrightarrow{AB}$, and $\angle A$ and $\angle B$ are consecutive interior angles. So, by Theorem 5-4, if $\angle A$ and $\angle B$ are supplementary, $\overline{AD} \parallel \overline{BC}$. Similarly, if $\angle A$ and $\angle D$ are supplementary then $\overline{AB} \parallel \overline{CD}$. Hence, a convex quadrilateral such that each two adjacent angles of it are supplementary is a parallelogram.

Similarly, if one inserts 'convex' in Theorem 6-10 one obtains a theorem which is not difficult to prove. One way is to make use of Theorem 5-11 and the weak form of Theorem 6-9 which has just been proved. In fact, if ABCD is convex then the sum of the measures of its angles is 360 [See Exercise 3 on page 6-155]. So, if $\angle A \cong \angle C$ and $\angle B \cong \angle D$ then $m(\angle A) + m(\angle B) = 180$ and $m(\angle A) + m(\angle D) = 180$. So, as before, ABCD is a parallelogram.

However, to establish Theorems 6-9 and 6-10 as stated [without 'convex'], it is easier to begin by proving Theorem 6-10. Once this is done, Theorem 6-9 follows at once. For, if each two adjacent angles are supplementary then, by Theorem 2-3, each two opposite angles are congruent. Theorem 6-10 can be proved by, first, proving that each quadrilateral whose opposite angles are congruent is convex and, then, proving that each convex quadrilateral whose opposite angles are congruent is a parallelogram. We shall not give a complete proof of the first result. However, the basic idea for such a proof is that if a quadrilateral is not convex then two of its vertices, say A and C, are on the same side of the line containing the other two, and one of A and C, say C, is interior to the angle which has the other, A, as vertex.

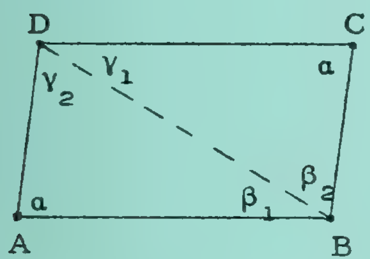


Now it follows easily that $\angle C$ is larger than $\angle A$. For, since C is interior to $\angle BAD$, $\overrightarrow{AC} \cap \overline{BD}$ consists of a single point P. Since $P \in \overline{BD}$, P is interior to $\angle BCD$. So, $m(\angle C) = m(\angle BCP) + m(\angle PCD)$, and $m(\angle A) = m(\angle BAP) + m(\angle PAD)$. Since $\angle BCP$ is an exterior angle of ΔABC, $m(\angle BCP) > m(\angle BAP)$. Similarly, $m(\angle PCD) > m(\angle PAD)$. Hence, $m(\angle C) > m(\angle A)$. [You may want to use this result in the form: if C is interior to ΔBAD then $\angle BCD$ is larger than $\angle BAD$, as a review question for section 6.04.] So, a quadrilateral whose opposite angles are congruent is convex.

There are now two ways to proceed:

Method I. Suppose that each two opposite angles of ABCD are congruent. Then, ABCD is convex and, as shown earlier, each two adjacent angles are supplementary. From this, by the weak form of Theorem 6-9, it follows that ABCD is a parallelogram. So, a quadrilateral whose opposite angles are congruent is a parallelogram.

Method II. Suppose that each two opposite angles of ABCD are congruent. Then, ABCD is convex. So, [see figure], $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$. But, by



Theorem 5-11, $\alpha + \beta_1 + \gamma_2 = \alpha + \gamma_1 + \beta_2$; whence, $\beta_1 + \gamma_2 = \gamma_1 + \beta_2$. So [subtracting], $\beta_2 - \gamma_2 = \gamma_2 - \beta_2$. Hence, $\beta_2 = \gamma_2$. Consequently, since A and C are on opposite sides of $\overleftrightarrow{BD}$, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Similarly, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. So, ABCD is a parallelogram.

Having proved Theorem 6-10, we can now proceed to derive Theorem 6-9 from it and Theorem 2-3. Note that if we use the first procedure to prove Theorem 6-10, we are, ultimately, using the weak form of Theorem 6-9 to prove the strong form. Of course, this involves no circularity.

Answers for Exercises.

2. $m(\angle A) = 120 = m(\angle C)$; $m(\angle B) = 60 = m(\angle D)$ [By Theorem 5-5, $\angle A$ and and $\angle B$ are supplementary.]

3. (a) $m(\angle A') = m(\angle A)$; $m(\angle B') = m(\angle B)$; $m(\angle C') = m(\angle C)$ [Students may justify these answers by referring to the Example on page 6-154. Alternatively, they may first solve part (b) of the present exercise.]

   (b) $ABC \longleftrightarrow A'CB$ is a congruence by Theorem 5-3 and a.s.a. Etc.

   (c) Since $ABC \longleftrightarrow BAC'$ and $ABC \longleftrightarrow CB'A$ are congruences, so is $BAC' \longleftrightarrow CB'A$. Hence, $B'A = AC'$. Since $A \in \overline{B'C'}$, it follows that A is the midpoint of $\overline{B'C'}$. Etc.

   (d) The altitude from B of $\triangle ABC$ is the segment whose end points are B and the foot of the perpendicular from B to $\overleftrightarrow{AC}$. Since $\overleftrightarrow{A'C'} \parallel \overleftrightarrow{AC}$, it follows by Theorem 5-9 that this segment is perpendicular to $\overleftrightarrow{A'C'}$. So, it is a subset of the perpendicular at B to $\overleftrightarrow{A'C'}$. But, since B is the midpoint of $\overline{A'C'}$ [part (c)], the perpendicular at B to $\overleftrightarrow{A'C'}$ is, by definition, the perpendicular bisector of $\overline{A'C'}$.

   (e) [Note that the boxed statement of part (e) involves a colloquialism. The proper interpretation is that the lines which contain the altitudes of a triangle are concurrent. Be sure that students see that the actual altitudes of an obtuse triangle are not concurrent.]

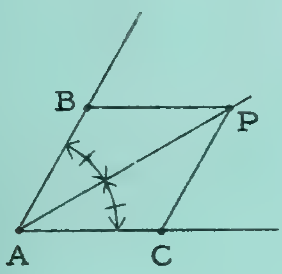
   From part (d), the lines which contain the altitudes of a triangle are the perpendicular bisectors of the sides of another triangle. Hence, by Theorem 4-19, these lines are concurrent.

Correction.

On page 6-168, line 9 should begin:

(b) Show that the segment ---

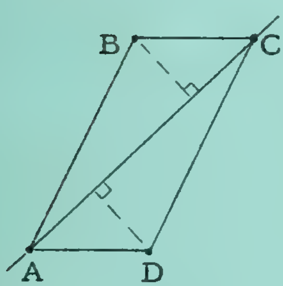
4. (a)



By Theorem 5-3 and a.s.a., $ABP \leftrightarrow PCA$ is a congruence. [See solution for Exercise 6 on page 6-156.] Hence, $\angle BPA \cong \angle CAP$. But, by hypothesis, $\angle CAP \cong \angle BAP$. So, $\angle BPA \cong \angle BAP$ and, by Theorem 3-5, $\overline{AB} \cong \overline{BP}$. But, by hypothesis and definition, ABPC is a parallelogram. Hence, by Example 1 on page 6-164, $\overline{AB} \cong \overline{PC}$ and $\overline{BP} \cong \overline{CA}$. So, ABPC is a quadrilateral whose sides are congruent--that is, ABPC is a rhombus. [Note that we did not use the hypothesis that $\angle A$ is an angle of 60°.]

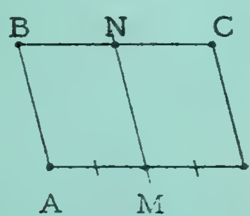
(b) ΔABC is an isosceles triangle whose vertex angle at A is an angle of 60°. By Theorem 5-11, the sum of the measures of its base angles is 120. Since, by Theorem 3-5, these angles are congruent, each is an angle of 60°. Hence, ΔABC is equiangular and, by Theorem 3-6, is equilateral. So, $\overline{BC} \cong \overline{AB}$.

5.



$ABC \leftrightarrow CDA$ is a congruence. So, by Exercise 1 of Part ☆E on page 6-134, the altitude from B of ΔABC and the altitude from D of ΔADC are congruent. Hence, by definition, B and D are equidistant from $\overleftrightarrow{AC}$.

6. (a)



Suppose that M is the midpoint of $\overline{AD}$. The line through M parallel to $\overleftrightarrow{AB}$ intersects $\overline{BC}$ at a point N. [For proof of this, see below.] Since $\overline{AB} \parallel \overline{MN}$ and $\overline{AM} \parallel \overline{BN}$, ABNM is a parallelogram and, by Example 1 on page 6-164, $\overline{AM} \cong \overline{BN}$. Similarly, $\overline{MD} \cong \overline{NC}$. Hence, since $N \in \overline{BC}$, it follows that N is the midpoint of $\overline{BC}$.

[To show that the line through M parallel to $\overleftrightarrow{AB}$ does intersect $\overline{BC}$ at a point N, one may argue as follows: First, since A, B, and D are not collinear, $M \notin \overleftrightarrow{AB}$. So, there is a unique line through M parallel to $\overleftrightarrow{AB}$. This line is not $\overleftrightarrow{AD}$, since $\overleftrightarrow{AD} \cap \overleftrightarrow{AB} \neq \emptyset$ [that is $\overleftrightarrow{AD} \nparallel \overleftrightarrow{AB}$]. So, the line in question crosses $\overleftrightarrow{AD}$. Consequently, by Theorem 20 on page 6-28, the line crosses $\overleftrightarrow{BC}$ at some point N. Since $\overleftrightarrow{MN} \parallel \overleftrightarrow{AB}$, N is on the M-side of $\overleftrightarrow{AB}$. Since $M \in \overrightarrow{AD}$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$, M and C are on the same side of $\overleftrightarrow{AB}$. So, N is on the C-side of $\overleftrightarrow{AB}$. Similarly, N is on the B-side of $\overleftrightarrow{CD}$. Since $N \in \overleftrightarrow{BC}$, it follows that $N \in \overline{BC}$.

Actually, it is not necessary for the purposes of the exercise to show that $N \in \overline{BC}$. If one uses Theorem 1-9, rather than the definition of midpoint, it is enough to know that $N \in \overleftrightarrow{BC}$.]

(b) Suppose that M is the midpoint of $\overline{AD}$ and that N is the midpoint of $\overline{BC}$. Since, by part (a), the line through M parallel to $\overline{AB}$ contains N, it follows that this line is $\overleftrightarrow{MN}$. So, $\overline{MN}$ is parallel to $\overline{AB}$.

Since $\overline{MN} \parallel \overline{AB}$ and $\overline{AM} \parallel \overline{BN}$, ABNM is a parallelogram and, by Example 1, $\overline{MN} \cong \overline{AB}$.

(c) By part (a), the line through the midpoint M of $\overline{AB}$ and parallel to $\overline{AC}$ intersects $\overline{CD}$ at its midpoint E. By Example 1, $\overline{AB} \cong \overline{CD}$. So, $\overline{MB} \cong \overline{EC}$. As shown in Exercise 4(a), $ABC \leftrightarrow DCB$ is a congruence. So, $\angle MBN \cong \angle ECN$. $\angle MNB$ and $\angle ENC$ are vertical angles and, so, are congruent. Hence, by Theorem 4-16 [a.a.s.], $MNB \leftrightarrow ENC$ is a congruence. Consequently, $\overline{BN} \cong \overline{NC}$. Since $N \in \overline{BC}$, N is the midpoint of $\overline{BC}$.

[That the line through M parallel to $\overleftrightarrow{AC}$ does, as shown in the figure, intersect $\overline{BC}$, may be proved as follows: As shown in the solution for Exercise 6 on page 6-156, A and D are on opposite sides of $\overleftrightarrow{BC}$. So, since $M \in \overrightarrow{BA}$ and $E \in \overrightarrow{CD}$, M and E are on opposite sides of $\overleftrightarrow{BC}$. So, $\overline{ME}$ crosses $\overleftrightarrow{BC}$ at some point N. Since $\overline{MN}$ is parallel to $\overleftrightarrow{AC}$ it follows that M and N are on the same side of $\overleftrightarrow{AC}$. Since $M \in \overrightarrow{AB}$, this is the B-side of $\overleftrightarrow{AC}$. Similarly, N is on the C-side of $\overleftrightarrow{BD}$. So, since $N \in \overleftrightarrow{BC}$, it follows that $N \in \overline{BC}$.]

(d) By part (c), the parallel through M to $\overleftrightarrow{AC}$ contains N. So, this line is $\overleftrightarrow{MN}$. Hence, $\overline{MN} \parallel \overline{AC}$. Furthermore, it was shown in part (c) that [M being the midpoint of $\overline{AB}$ and, as just shown, $\overline{MN}$ being parallel to $\overline{AC}$] $MNB \longleftrightarrow ENC$ is a congruence. So, $\overline{MN} \cong \overline{EN}$. Since $N \in \overline{ME}$, it follows that $MN = \frac{1}{2} \cdot ME$. But, since AMEC is a parallelogram, $ME = AC$. Hence, $MN = \frac{1}{2} \cdot AC$.

*

To establish the boxed theorems, the work in parts (c) and (d) needs to be supplemented by a proof that, given ΔABC, there is a point D such that ABDC is a parallelogram. To show this, let ℓ be the parallel to $\overleftrightarrow{AC}$ through B and let m be the parallel to $\overleftrightarrow{AB}$ through C. Now, if $\ell \parallel m$ then, since $\ell \parallel \overleftrightarrow{AC}$, it follows that $m \parallel \overleftrightarrow{AC}$ or $m = \overleftrightarrow{AC}$. And, since $m \parallel \overleftrightarrow{AB}$, it then follows that $\overleftrightarrow{AC} \parallel \overleftrightarrow{AB}$ or $\overleftrightarrow{AC} = \overleftrightarrow{AB}$. Since this is not the case, $\ell \nparallel m$. Hence, $\ell \cap m \neq \emptyset$, and one can choose for D any point of this intersection. [Of course, with a little more trouble, one can prove that the intersection consists of a single point.]

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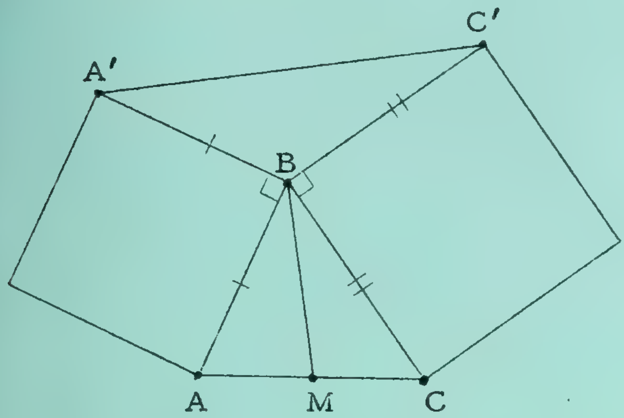
Here is an interesting exercise which can be solved by using the second of the boxed theorems:



Hypothesis: $\angle A'BA$ and $\angle C'BC$ are right angles, $BC' = BC$, $BA' = BA$, M is the midpoint of $\overline{AC}$

Conclusion: $MB = \frac{1}{2} \cdot A'C'$

Solution. $\angle ABC$ and $\angle A'BC'$ are supplementary. So, if $\Delta A'BC'$ is rotated about B so that C' coincides with C, the points A', B, and A become collinear. Now, $\overline{MB}$ is the segment joining the midpoints of two sides of a triangle whose third side has measure $A'C'$. So, $MB = \frac{1}{2} \cdot A'C'$.

The loose talk, above, about rotating $\Delta A'BC'$ can be replaced by:

If A'' is the point on the non-A-side of $\overleftrightarrow{BC}$ such that $\angle A''BC \cong \angle A'BC'$ and $A''B = A'B$, then by s.a.s., $A''BC \longleftrightarrow A'BC'$ is a congruence. And, since $\angle A''BC$ and $\angle ABC$ are adjacent supplementary angles, $B \in \overline{A''A}$. So, B is the midpoint of $\overline{A''A}$ and, since M is the midpoint of $\overline{AC}$, it follows that $\overline{MB}$ is parallel to $\overleftrightarrow{A''C}$ and that $MB = \frac{1}{2} \cdot A''C$. But, $\overline{A''C} \cong \overline{A'C'}$. So, $MB = \frac{1}{2} \cdot A'C'$.

However, the point of such an exercise as this is to generate the insight that $\Delta A'BC'$ and ΔABC can be "put together" into a larger triangle, and the rotation-language expresses this clearly enough.

on page 6-177. So, this exercise may suggest Theorem 6-21 [and, also, Theorem 6-20]. Another method for proving (4) is, first, to prove that the diagonals of an isosceles trapezoid are congruent, and, then, use Theorem 6-24. [This is the second theorem at the foot of page 6-168.] [Alternatively, if one has proved (4) in another way, one may use it and Theorem 6-24 to prove that the diagonals of an isosceles trapezoid are congruent.]

To prove Theorem 6-21, suppose that ABCD is an isosceles trapezoid with $\overline{AB}$ and $\overline{DC}$ as bases. Since ABCD is, by definition, not a parallelogram, it follows by Theorem 6-8 that $\overline{AB} \not\cong \overline{CD}$. For simplicity, suppose that $\overline{CD}$ is longer than $\overline{AB}$. The parallel to $\overline{AD}$ through B will intersect $\overleftrightarrow{CD}$ at a point R such that, by Theorem 6-1, $\overline{DR} \cong \overline{AB}$. Since B, C, and R are on the same side of $\overleftrightarrow{AD}$, $R \in \overrightarrow{DC}$. So, since $DR = AB < CD$, $R \in \overline{DC}$. Since ABRD is a parallelogram, $\angle D$ and $\angle BRD$ are supplementary. So, $\angle D$ and $\angle BRC$ are congruent. But, since ΔBRC is isosceles, $\angle BRC$ and $\angle C$ are congruent. So, $\angle D \cong \angle C$. By Theorem 5-5, and the convexity of ABCD, $\angle A$ and $\angle B$ are supplements of the congruent angles $\angle D$ and $\angle C$. So, by Theorem 2-3, $\angle A \cong \angle B$.

To prove Theorem 6-20, suppose that ABCD is a nonisosceles trapezoid. Proceeding as in the proof, above, for Theorem 6-21, $\angle D$ and $\angle BRC$ are congruent. But, since $BR \neq BC$, $\angle BRC \not\cong \angle C$. So, $\angle C \not\cong \angle D$. Hence, by Theorem 5-5 and Theorem 2-3, $\angle A \not\cong \angle B$. Hence, if either pair of base angles of a trapezoid are congruent then the trapezoid is isosceles.

Theorem 6-21 can be used to prove that the diagonals of an isosceles trapezoid are congruent. For [see figure], it follows by s.a.s. that $ADC \longleftrightarrow BCD$ is a congruence.

(5) For this, see the discussion of (2).

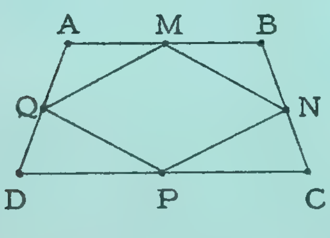
10. Since $ABA'B'$ is a parallelogram, $\overline{AA'}$ and $\overline{BB'}$ bisect each other. Since $ACA'C'$ is a parallelogram, $\overline{AA'}$ and $\overline{CC'}$ bisect each other. Consequently, $\overline{BB'}$ and $\overline{CC'}$ bisect each other. So, $BCB'C'$ is a parallelogram.

a rhombus. To establish the if-part of this guess, suppose that ABCD is a rhombus. Then, by s.s.s., ABC $\longleftrightarrow$ ADC is a congruence. Hence, $\angle BAC \cong \angle DAC$. Moreover, since each rhombus is convex, each point of $\overrightarrow{AC}$ is interior to $\angle A$. Consequently, $\overrightarrow{AC}$ is the bisector of $\angle A$. Similarly, $\overrightarrow{CA}$ is the bisector of $\angle C$. So, the diagonal $\overline{AC}$ is a subset of the bisector of each of the angles $\angle A$ and $\angle C$. So [Theorem 6-18), the diagonals of a rhombus are contained in the bisectors of its angles.

Suppose, now, that ABCD is a quadrilateral whose diagonal $\overline{AC}$ is a subset of the bisectors of $\angle A$ and $\angle C$. [Then, the bisector of $\angle A$ is $\overrightarrow{AC}$ and that of $\angle C$ is $\overrightarrow{CA}$.] Then, since C is interior to $\angle A$, it follows that B and C are on the same side of $\overleftrightarrow{AD}$ and that C and D are on the same side of $\overleftrightarrow{AB}$; and, since A is interior to $\angle C$, it follows that A and B are on the same side of $\overleftrightarrow{CD}$ and that A and D are on the same side of $\overleftrightarrow{BC}$. Consequently, ABCD is convex. Moreover, by a.s.a., ABC $\longleftrightarrow$ ADC is a congruence. Hence, ABCD is a kite, with $\angle A$ and $\angle C$ as "vertex angles". So, if a diagonal of a quadrilateral is a subset of the bisectors of each of two of its angles, then the quadrilateral is a kite. From this it is only a step to prove that if <u>each</u> diagonal of a quadrilateral is a subset of the bisector of each of two angles of the quadrilateral, then the quadrilateral is a rhombus. This is Theorem 6-19. [To establish the only-if part of the guess--that a <u>parallelogram</u> two of whose angle bisectors are collinear is a rhombus--it is sufficient to note that a parallelogram which is, also, a kite is a rhombus.]

*

(4)



Here one can use the same method of solution as for case (1) [that is--prove that MAQ $\longleftrightarrow$ MBN is a congruence], once one has proved that each pair of base angles of an isosceles trapezoid are congruent. This is Theorem 6-21

(2) Since the sides of MNPQ are parallel to the diagonals of ABCD, (2) will follow if we show that the diagonals of a rhombus are perpendicular to each other, and that lines parallel to two perpendicular lines are, also, perpendicular to each other. This procedure will also take care of (5), once we have proved that the diagonals of a kite are perpendicular. [In fact, since each rhombus is a kite, (2) is a consequence of (5).] But, by definition, and Theorem 3-3, the line containing one of the diagonals of a kite is the perpendicular bisector of the other diagonal. So, the diagonals of a kite are perpendicular to each other [this is Theorem 6-15] and, in particular, the diagonals of a rhombus are perpendicular bisectors of each other [this is Theorem 6-16]. All that remains to the proof of (2) and of (5) is to show that if $\ell \perp m$, $\ell_1 \parallel \ell$, and $m_1 \parallel m$, then $\ell_1 \perp m_1$. By definition, if $\ell \perp m$ then ℓ and m intersect at a unique point E. Since $m_1 \parallel m$, ℓ also crosses m_1, say at F. Since $\ell_1 \parallel \ell$ and m_1 and m cross ℓ, they also cross ℓ_1, say at G and H, respectively. It follows that EFGH is a parallelogram such that $\angle E$ is a right angle. So, by Theorem 6-4 and Theorem 2-1, $\angle G$ is a right angle. Hence, $\ell_1 \perp m_1$.

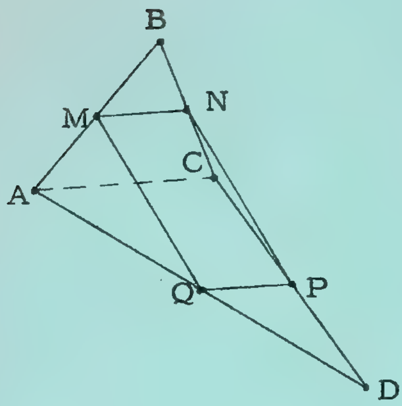
It is, of course, not a theorem that if the diagonals of a quadrilateral are perpendicular then the quadrilateral is a kite. However [Theorem 6-17], if the diagonals of a quadrilateral are perpendicular bisectors of each other then the quadrilateral is a rhombus. For, by Theorem 6-7, such a quadrilateral ABCD is a parallelogram and, in addition, if the point of intersection of its diagonals is P then, by s.a.s., APB $\longleftrightarrow$ CPB is a congruence. So, AB = CB and, by Theorem 6-14, ABCD is a rhombus.

(3) Since a square is, by definition, a rectangle and a rectangle is, by Theorem 6-2, a parallelogram, a square is a parallelogram. Since two adjacent sides of a square are congruent, it follows by Theorem 6-14 that a square is a rhombus. So, each square is a rhombus which is a rectangle. On the other hand, each rhombus which is a rectangle is, by definition, a square. Hence, (3) follows at once from (1) and (2).

*

While discussing rhombuses, students may discover Theorem 6-18 and Theorem 6-19. It is easy to see that the bisectors of two opposite angles of a parallelogram are either parallel or collinear; and it is a natural guess that they are collinear if and only if the parallelogram is

9. (a)



By the second boxed theorem on page 6-168, $\overleftrightarrow{MN} \parallel \overleftrightarrow{AC}$ and $\overleftrightarrow{PQ} \parallel \overleftrightarrow{AC}$. Hence, $\overleftrightarrow{MN} \parallel \overleftrightarrow{PQ}$ [or, $\overleftrightarrow{MN}$ and $\overleftrightarrow{PQ}$ are collinear]. Similarly, $\overleftrightarrow{NP} \parallel \overleftrightarrow{QM}$ [or $\overleftrightarrow{NP}$ and $\overleftrightarrow{QM}$ are collinear]. Consequently, MNPQ is a parallelogram. [The bracketed remarks are usually omitted. However, to justify the conclusions, one must show that M, N, P, and Q are not collinear. But, if M, N, P, and Q are collinear then, since $\overleftrightarrow{AC} \parallel \overleftrightarrow{MN}$ and $\overleftrightarrow{BD} \parallel \overleftrightarrow{NP}$, it follows that $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$ or A, B, C, and D are collinear. The second alternative is ruled out by the assumption that ABCD is a quadrilateral, and, for the same reason, neither C nor D can be on $\overleftrightarrow{AB}$. Now, under the first alternative, if C and D are on the same side of $\overleftrightarrow{AB}$ then ACDB is a trapezoid [or a parallelogram] and its diagonals $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect; while, if C and D are on opposite sides of $\overleftrightarrow{AB}$, ACBD is a trapezoid [or a parallelogram], and $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect. But, both these possibilities are, again, ruled out by the assumption that ABCD is a quadrilateral. So, the midpoints of the sides of a quadrilateral are noncollinear.]

(b) Using the notation of part (a),

(1) if ABCD is a rectangle then MNPQ is a rhombus,
(2) if ABCD is a rhombus then MNPQ is a rectangle,
(3) if ABCD is a square then MNPQ is a square,
(4) if ABCD is an isosceles trapezoid then MNPQ is a rhombus
and (5) if ABCD is a kite then MNPQ is a rectangle.

In case (1), MA = MB, $\angle A \cong \angle B$, and [because a rectangle is a parallelogram, whence AD = BC] AQ = BN. So, by s.a.s., MAQ $\longleftrightarrow$ MBN is a congruence. Hence, MQ = MN. Since, by part (a), MNPQ is a parallelogram, it follows, using Example 1 on page 6-164, that all four sides of MNPQ are congruent. So, MNPQ is a rhombus.

7. 15

8. (a) The line through M parallel to $\overline{AD}$ contains the midpoint P of $\overline{BD}$. Since $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$, this line is the line through P parallel to $\overleftrightarrow{BC}$. But, this line contains the midpoint N of $\overline{CD}$. So, the line through M parallel to $\overline{AD}$ is $\overleftrightarrow{MN}$.

This argument, as remarked in part (b), shows that the median [$\overline{MN}$] of a trapezoid is parallel to the bases of the trapezoid. It also proves:

> The line which bisects one leg of a trapezoid and is parallel to the bases bisects the other leg.

(b) From the figure, $P \in \overline{MN}$. So, by Axiom A, $MN = MP + PN$. But, from the second boxed theorem on page 6-168, $MP = \frac{1}{2} \cdot AD$ and $PN = \frac{1}{2} \cdot BC$. So, $MN = \frac{1}{2}(AD + BC)$. [You can relate these results and those of Exercise 6 by pretending that a triangle is a trapezoid with one base of measure 0.] [For completeness, we should show, without reference to the figure, that $P \in \overline{MN}$. (Although we know, from part (a), that $P \in \overleftrightarrow{MN}$, this is not sufficient for part (b).) As shown in the COMMENTARY for page 6-162, a trapezoid is convex and, hence, the diagonals of a trapezoid intersect. Referring, for notation, to the figure, it follows that $\overline{AC} \cap \overline{BD} \neq \emptyset$ and, since B, C, and D are noncollinear, it further follows that A and C are on opposite sides of of $\overleftrightarrow{BD}$. Since $N \in \overrightarrow{DC}$, N is on the C-side of $\overleftrightarrow{BD}$. Since $M \in \overrightarrow{BA}$, M is on the A-side of $\overleftrightarrow{BD}$. So, M and N are on opposite sides of $\overleftrightarrow{BD}$. Hence, $\overline{MN}$ intersects $\overleftrightarrow{BD}$ in a single point. Since $P \in \overleftrightarrow{MN} \cap \overline{BD}$ it must be the point in question. So, $P \in \overline{MN}$.]

11. [The bracketed remarks at the beginning of the solution of each part show how one can avoid reference to the figures. From your students' viewpoint, the unbracketed parts of the solutions will probably suffice.

(a) [Suppose that ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are parallel lines and that m and m′ are parallel lines such that $m \cap \ell_1 = \{A\}$ and $m' \cap \ell_1 = \{A'\}$. Then, m crosses ℓ_2, ℓ_3, and ℓ_4 at points B, C, and D, and m′ crosses ℓ_2, ℓ_3, and ℓ_4 at points B′, C′, and D′. Since $m \parallel m'$, it follows that $A \neq A'$, $B \neq B'$, $C \neq C'$, and $D \neq D'$. So, $\ell_1 = \overleftrightarrow{AA'}$, $\ell_2 = \overleftrightarrow{BB'}$, $\ell_3 = \overleftrightarrow{CC'}$, and $\ell_4 = \overleftrightarrow{DD'}$. Since $\ell_1 \parallel \ell_4$, $A \neq D$ and $A' \neq D'$. Hence, $m = \overleftrightarrow{AD}$ and $m' = \overleftrightarrow{A'D'}$.] Since $\overline{AA'} \parallel \overline{BB'}$ and $\overline{AB} \parallel \overline{A'B'}$, it follows that AA′B′B is a parallelogram. Hence, by Example 1 on page 6-164, $\overline{AB} \cong \overline{A'B'}$. Similarly, $\overline{BC} \cong \overline{B'C'}$, and $\overline{CD} \cong \overline{C'D'}$. Hence, assuming that $AB = BC = CD$, it follows that $A'B' = B'C' = C'D'$.

(b) [Suppose that ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are parallel lines and that m and m′ are nonparallel lines such that $m \cap \ell_1 = \{A\}$ and $m' \cap \ell_1 = \{A'\}$. Then, m crosses ℓ_2, ℓ_3, and ℓ_4 at points B, C, and D, and m′ crosses ℓ_2, ℓ_3, and ℓ_4 at points B′, C′, and D′. Suppose that $B \in \overline{AC}$ and that $C \in \overline{BD}$. Suppose that $\overline{A'C'} \cap \overline{AC} = \emptyset$ and that $\overline{B'D'} \cap \overline{BD} = \emptyset$. It follows that $A \neq A'$, $B \neq B'$, $C \neq C'$, and $D \neq D'$. Hence, $\ell_1 = \overleftrightarrow{AA'}$, $\ell_2 = \overleftrightarrow{BB'}$, $\ell_3 = \overleftrightarrow{CC'}$, and $\ell_4 = \overleftrightarrow{DD'}$. Since $\ell_1 \parallel \ell_4$, $A \neq D$ and $A' \neq D'$. Hence, $m = \overleftrightarrow{AD}$ and $m' = \overleftrightarrow{A'D'}$.] Since $\overline{AA'} \parallel \overline{CC'}$, $\overline{A'C'} \cap \overline{AC} = \emptyset$, and $\overline{A'C'} \nparallel \overline{AC}$, it follows that AA′C′C is a trapezoid with legs $\overline{AC}$ and $\overline{A'C'}$. Since $B \in \overline{AC}$, and assuming that $AB = BC$, it follows that B is the midpoint of $\overline{AC}$. Consequently, $\overleftrightarrow{BB'}$ is the line which bisects the leg $\overline{AC}$ of trapezoid AA′C′C and is parallel to its bases, $\overline{AA'}$ and $\overline{CC'}$. Hence, by a previous theorem [see COMMENTARY for part (a) of Exercise 8], $\overleftrightarrow{BB'}$ bisects the other leg, $\overline{A'C'}$. So, B′ is the midpoint of $\overline{A'C'}$, and $A'B' = B'C'$. Similarly, assuming that $BC = CD$, it follows that $B'C' = C'D'$.

(c) [Suppose that ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are parallel lines and that m and m′ are any two transversals intersecting ℓ_1, ℓ_2, ℓ_3, and ℓ_4 in the points A, B, C, D, and A′, B′, C′, D′, respectively. Suppose that $B \in \overline{AC}$ and that $C \in \overline{BD}$. Let m″ be a line parallel to m′, intersecting ℓ_1, ℓ_2, ℓ_3, and ℓ_4 at A″, B″, C″, and D″, such that $\overleftrightarrow{A''C''} \cap \overleftrightarrow{AC} = \emptyset$ and $\overleftrightarrow{B''D''} \cap \overleftrightarrow{BD} = \emptyset$. Such a transversal can be found by considering, first, the transversal through A parallel to m′. This intersects ℓ_4 at a point P. It is sufficient to take for m″ the line parallel to m′ through any point D″ such that $P \in \overline{D''D}$ and $P \neq D'$.] By part (b), assuming that AB = BC = CD, it follows that A″B″ = B″C″ = C″D″. So, by part (a), A′B′ = B′C′ = C′D′.

*

The boxed statement at the foot of page 6-170 is a somewhat imprecise statement of the result established in Exercise 11. What has actually been shown is that

> if ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are parallel lines which intersect a transversal m in points A, B, C, and D, respectively, such that $B \in \overline{AC}$, $C \in \overline{BD}$, and AB = BC = CD, then they intersect any other transversal m′ in points A′, B′, C′, and D′ respectively, such that $B' \in \overline{A'C'}$, $C' \in \overline{B'D'}$, and A′B′ = B′C′ = C′D′.

A result which is sometimes more useful is that

> if ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are parallel lines which intersect a transversal m at points A, B, C, and D, such that AB = CD, then they intersect any other transversal m′ in points A′, B′, C′, and D′, respectively, such that A′B′ = C′D′.

For this result, the case in which $m \parallel m'$ can be handled just as in part (a). The case in which $m \nparallel m'$ is somewhat more complicated because of the number of subcases which must be treated.

(b) If $\overleftrightarrow{PQ} \parallel \ell$ and A and B are the feet of the perpendiculars to ℓ from P and Q, respectively, then, by Theorem 5-8, since $P \neq Q$ it follows that $\overline{PA} \parallel \overline{QB}$. So, by definition, APQB is a parallelogram. Hence, P and Q are on the same side of $\overleftrightarrow{AB}$. Also, by Theorem 6-1, $PA = QB$. So, by definition, P and Q are equidistant from $\overleftrightarrow{AB}$.

*

Theorem 6-29 is a brief way of saying that if $\ell \parallel m$ then each two points of ℓ are equidistant from m, and each two points of m are equidistant from ℓ. This is justified by part (b) of Exercise 15.

*

It is a triviality that the sum of the measures of the angles of a rectangle is 360. If ABCD is a parallelogram then, since $\overline{AD} \parallel \overline{BC}$, B and C are on the same side of $\overleftrightarrow{AD}$. Hence, $\angle A$ and $\angle D$ are consecutive interior angles and, since $\overline{AB} \parallel \overline{DC}$, are supplementary. Similarly, $\angle B$ and $\angle C$ are supplementary. So, the sum of the measures of the angles of a parallelogram is 360. The case of a trapezoid ABCD, with bases $\overline{AB}$ and $\overline{CD}$, can be handled similarly once it is known that B and C are on the same side of $\overleftrightarrow{AD}$ [and that A and D are on the same side of $\overleftrightarrow{BC}$]. Your students are not likely to question this, but, in the COMMENTARY for page 6-162, it has been shown to be the case.

*

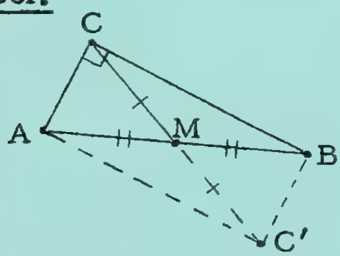
The suggestion that the sum of the measures of the angles of any quadrilateral is 360 is, of course, misleading. The argument given on page 6-173 for the quadrilateral pictured at the foot of page 6-172 uses the fact that C is interior to $\angle DAB$ and that A is interior to $\angle BCD$--that is, that C and B are on the same side of $\overleftrightarrow{AD}$, C and D are on the same side of $\overleftrightarrow{AB}$, A and B are on the same side of $\overleftrightarrow{CD}$, and A and D are on the same side of $\overleftrightarrow{CB}$. In other words, the argument depends on the fact that ABCD is convex.

13. By Exercise 6(b) on page 6-168, $\overline{NM} \parallel \overline{AD}$. So, by Theorem 6-23, $\overline{NM}$ bisects $\overline{AC}$. Let R be the midpoint of $\overline{AC}$. Then, since P and Q are trisection points of $\overline{AC}$, AP = QC. But, AR = CR. So, by Axiom A, PR = QR. Hence, $\overline{NM}$ bisects $\overline{PQ}$. By Theorem 6-24, NR = $\frac{1}{2} \cdot$ AD and RM = $\frac{1}{2} \cdot$ BC. But, since ABCD is a parallelogram, AD = BC; so, NR = RM. Hence, $\overline{PQ}$ bisects $\overline{NM}$. Thus, by Theorem 6-7, MPNQ is a parallelogram.

14. (a) 5

(b) <u>Theorem.</u> The measure of the median to the hypotenuse of a right triangle is half the measure of the hypotenuse.

<u>Proof.</u>



Let C′ be the point of $\overrightarrow{CM}$ such that CM = MC′. Then, by Theorem 6-7, ACBC′ is a parallelogram. Since $\angle C$ is a right angle, it follows [for example, using Theorem 6-3] that ACBC′ is a rectangle. Hence, by Theorem 6-11, CC′ = AB. Consequently, CM = $\frac{1}{2} \cdot$ AB.

(c) [See Theorem 6-28 on page 6-178.]

15. (a) If A and B are the feet of the perpendiculars to ℓ from P and Q, respectively, then by Theorem 5-8, $\overleftrightarrow{PA} \parallel \overleftrightarrow{QB}$, or $\overleftrightarrow{PA} = \overleftrightarrow{QB}$. If P and Q are equidistant from ℓ then PA = QB. So, if P and Q are, also, on the same side of ℓ and P $\neq$ Q, it follows that A $\neq$ B; whence, $\overleftrightarrow{PA} \neq \overleftrightarrow{QB}$. So, if P and Q are two points on the same side of ℓ and equidistant from ℓ then $\overline{PA} \parallel \overline{QB}$ and $\overline{PA} \cong \overline{QB}$. Hence, by Theorem 6-8, APQB is a parallelogram. Consequently, $\overleftrightarrow{PQ} \parallel \ell$.

Theorem 5-11, to show, systematically, that the sum of the measures of the angles of a convex quadrilateral is $180 + 180$, or $180 \cdot 2$, that of a convex pentagon is $180 \cdot 2 + 180$, or $180 \cdot 3$, that of a convex hexagon is $180 \cdot 3 + 180$, or $180 \cdot 4$, etc. [The <u>proof</u> that, for each whole number $n > 2$, the sum of the measures of the angles of a convex n-gon is $180(n - 2)$, requires a procedure known as <u>mathematical induction.</u>]

Suppose that A, B, and C are successive vertices of a convex polygon [not a triangle]. We shall show that the figure which results on replacing the sides $\overset{\bullet\!-\!\bullet}{AB}$ and $\overset{\bullet\!-\!\bullet}{BC}$ by the diagonal $\overset{\bullet\!-\!\bullet}{AC}$ is also a convex polygon. To establish this, it is sufficient to show that each vertex Q of the original polygon, other than A, B, and C, is on the non-B-side of $\overleftrightarrow{AC}$. This we proceed to do. Since the given polygon is convex, A and Q are on the same side of $\overleftrightarrow{BC}$ and C and Q are on the same side of $\overleftrightarrow{AB}$. So, Q is interior to $\angle ABC$. Let P be a vertex adjacent to Q. Then, $P \neq B$. Suppose that $Q \in \overleftrightarrow{AC}$. Then, since Q is interior to $\angle ABC$, $Q \in \overline{AC}$. But, if this were so, A and C could not be on the same side of $\overleftrightarrow{PQ}$. So, $Q \notin \overleftrightarrow{AC}$. Suppose that Q is on the B-side of $\overleftrightarrow{AC}$. Then, Q is interior to $\angle CAB$ and is, also, interior to $\angle ACB$. Hence, $\overrightarrow{AQ} \cap \overline{CB} \neq \emptyset$ and $\overrightarrow{CQ} \cap \overline{AB} \neq \emptyset$. So, since B and C are on the same side of $\overleftrightarrow{PQ}$, $P \notin \overleftrightarrow{AQ}$. Similarly, $P \notin \overleftrightarrow{BQ}$. On the other hand, if P is interior to $\angle AQB$ or to its vertical angle then $\overrightarrow{QP}$ or $\overrightarrow{PQ}$ intersects $\overline{AB}$. Since A and B are on the same side of $\overleftrightarrow{PQ}$, this is impossible. Finally, if P is interior to one of the adjacent supplements of $\angle AQB$ then $\overrightarrow{QP}$ or $\overrightarrow{PQ}$ intersects $\overline{CB}$ [at a point between B and the point where $\overrightarrow{AQ}$ intersects $\overline{BC}$]. Since B and C are on the same side of $\overleftrightarrow{PQ}$, this is impossible. So, Q is not on the B-side of $\overleftrightarrow{AC}$. Since, as shown previously, $Q \notin \overleftrightarrow{AC}$, it follows that Q is on the non-B-side of $\overleftrightarrow{AC}$.

*

Answers for Part A.

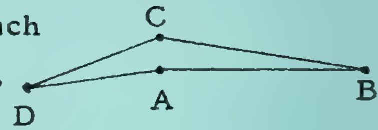
1. $180 \cdot 4$ 2. $180 \cdot 10$ 3. $180 \cdot 1000$ 4. $180(n - 2)$

Correction.

On page 6-173, line 2b should read:

...angles of a convex polygon of

line 6. Any nonconvex quadrilateral ABCD such that C is not the "re-entrant" vertex.



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Since the pentagon ABCDE is convex, A and B are on the same side of $\overleftrightarrow{CD}$ and A and D are on the same side of $\overleftrightarrow{BC}$. So, A is interior to $\angle BCD$, and $m(\angle C) = m(\angle BCA) + m(\angle ACD) = \gamma_1 + \gamma_2$. Similarly, $m(\angle D) = \delta_1 + \delta_2$. Also, C and E are on the same side of $\overleftrightarrow{AB}$, and C and B are on the same side of $\overleftrightarrow{AE}$. So, C is interior to $\angle EAB$, and

$$m(\angle EAB) = m(\angle EAC) + m(\angle CAB) = m(\angle EAC) + \alpha_3.$$

If it can be shown that D is interior to $\angle EAC$, it will follow that $m(\angle EAC) = \alpha_1 + \alpha_2$; so, $m(\angle A) = \alpha_1 + \alpha_2 + \alpha_3$. Then, the sum of the measures of the angles of ABCDE is

$$\begin{aligned}
&(\alpha_1 + \alpha_2 + \alpha_3) + \beta + (\gamma_1 + \gamma_2) + (\delta_1 + \delta_2) + \epsilon \\
&= (\alpha_3 + \beta + \gamma_1) + (\alpha_2 + \gamma_2 + \delta_1) + (\alpha_1 + \delta_2 + \epsilon) \\
&= 180 \cdot 3,
\end{aligned}$$

by Theorem 5-11. Now, by a result proved in the COMMENTARY for page 6-162, since, because of the convexity of ABCDE, C and D are on the same side of $\overleftrightarrow{EA}$ and A and C are on the same side of $\overleftrightarrow{DE}$ and E and A are on the same side of $\overleftrightarrow{CD}$, it follows that CDEA is convex. So, D and E are on the same side of $\overleftrightarrow{AC}$, and, as was to be shown, D is interior to $\angle EAC$.

*

More generally, one can show that "cutting a corner off any convex polygon" [replacing ABCDE by ACDE] "leaves it convex". [Of course, the given polygon must have more than three sides.] We shall prove this shortly. Once this is known, one can proceed, starting with

Corrections.

On page 6-175, line 10b should read:
5. ...angles of a convex polygon ...

and line 2b should read:
9. ...angles of a convex polygon of ___

Answers for Part B.

[Be sure that students see, by examples, that a convex polygon which is equiangular need not be equilateral (for example, a nonsquare rectangle), and that a convex polygon which is equilateral need not be equiangular (for example, a nonsquare rhombus).]

1. $\frac{180(5 - 2)}{5}$, or 108   2. 60; 90; 108; 120; 144; $\frac{180000}{1002}$; $\frac{180(n - 2)}{n}$

[Students may enjoy considering what floor patterns can be laid out using tiles in the shapes of regular polygons. Restricting yourself to tiles of one size and shape, it can be seen from the table of Exercise 2 that the only usable regular shapes are equilateral triangles, squares, and regular hexagons. If one allows tiles of two shapes one can combine squares and regular octagons.]

*

Answers for Part C.

1. An exterior angle of a convex polygon is one which is adjacent and supplementary to one of the angles of the polygon.

2, 3, 4, 5. 360   6. 360 $[n(180 - \frac{180(n - 2)}{n})]$ 7. 36   8. $\frac{360}{n}$

*

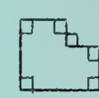
Answers for Part D [on pages 6-174 and 6-175].

1. rhombus   2. rectangle   3. square

4. (a), (b), (e), and (f) are theorems. Counter-examples for the others:

(c) any nonsquare rectangle   (d) any nonsquare rhombus

(g)    (h) any sat-upon regular pentagon

(i) any not too sat-upon regular pentagon   (j)    (k) 

Answers for Part E [on page 6-175].

1. 30   2. 540   3. 8   4. 36   5. 1800   6. no

7. 120, if convex; 90, if not convex [See Exercise 4(k) of Part D.]   8. 6   9. 4

Proofs of Theorems 6-1 through 6-14 are discussed in the COMMENTARY for page 6-166; Theorems 6-15 through 6-21 in the COMMENTARY for page 6-169; Theorem 6-22--page 6-167; Theorem 6-23 and Theorem 6-24--page 6-168; Theorem 6-25 and Theorem 6-26--page 6-169; Theorem 6-27--page 6-170; Theorem 6-28 and Theorem 6-29--pages 6-171, 172; Theorem 6-30--page 6-173; Theorem 6-31 through 6-33--pages 6-174, 175.

*

The following is a quiz covering pages 6-159 through 6-178. A quiz over pages 6-1 through 6-185 is given in the COMMENTARY for page 6-185.

*

Quiz.

1. Suppose that quadrilateral ABCD is a parallelogram and that the measure of $\angle A$ is three times the measure of $\angle B$. How many degrees are there in $\angle C$?

2. Find the number of degrees in each exterior angle of a regular 12-sided polygon.

3. If an exterior angle of a regular polygon is an angle of $10°$, what is the sum of the measures of the angles of the polygon?

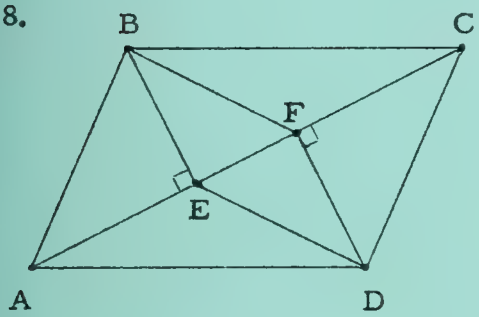
4. Suppose that quadrilateral ABCD is a parallelogram, that E is a point in $\overline{BC}$ such that $\overrightarrow{AE}$ bisects $\angle BAD$, and that F is a point in $\overline{AD}$ such that $\overrightarrow{CF}$ bisects $\angle BCD$. If $\angle BAE$ is an angle of $x°$, what are the measures of the angles of the quadrilateral AECF?

5. Suppose that quadrilateral ABCD is a rhombus and that $\angle ABC$ is an angle of $x°$. If the diagonals $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E, what are the measures of $\angle EAB$, $\angle ABE$, $\angle EBC$, and $\angle BCE$?

6. One of the base angles of an isosceles trapezoid is an angle of 60°. If the bases of the trapezoid are 10 inches and 16 inches long, how long is each leg?

7. Suppose that $\triangle ABC$ is a right triangle with $\angle B$ an angle of 60° and $\angle C$ a right angle. If D and E are the midpoints of $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$, respectively, and $AB = 8$, what is ED?

8.



Hypothesis: quadrilateral ABCD is a parallelogram, E and F are two points on $\overleftrightarrow{AC}$ such that $\overleftrightarrow{BE} \perp \overleftrightarrow{AC}$ and $\overleftrightarrow{DF} \perp \overleftrightarrow{AC}$

Conclusion: quadrilateral BFDE is a parallelogram

9. Suppose that quadrilateral ABCD is a square. Let A' be the point on $\overrightarrow{AB}$ such that $AB = BA'$, B' be the point on $\overrightarrow{BC}$ such that $BC = CB'$, C' be the point on $\overrightarrow{CD}$ such that $CD = DC'$, and D' be the point on $\overrightarrow{DA}$ such that $DA = AD'$. Prove that the quadrilateral $A'B'C'D'$ is a square.

*

Answers for Quiz.

1. Since $m(\angle A) + m(\angle B) = 180$ and since $m(\angle A) = 3 \cdot m(\angle B)$, $m(\angle B) = 45$ and $m(\angle A) = 135$. Since $\angle C \cong \angle A$, $m(\angle C) = 135$.

2. $\frac{360}{12} = 30$

3. $\frac{360}{10} = 36$. So, the polygon has 36 sides. Each angle is a supplement of an angle of 10°; so, each angle is an angle of 170°. $170 \times 36 = 6120$

4. m($\angle$FAE) = x, m($\angle$AEC) = 180 - x, m($\angle$ECF) = x, m($\angle$CFA) = 180 - x

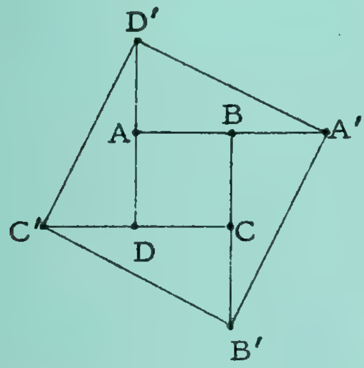
5. m($\angle$EAB) = $90 - \frac{x}{2}$, m($\angle$ABE) = $\frac{x}{2}$, m($\angle$EBC) = $\frac{x}{2}$, m($\angle$BCE) = $90 - \frac{x}{2}$

6. Each leg is 6 inches long.

7. 2

8. Since quadrilateral ABCD is a parallelogram, AB = CD and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\angle$BAE $\cong$ $\angle$DCF. Also, since $\angle$BEA and $\angle$DFC are right angles, they are congruent. So, by a. a. s., BAE $\leftrightarrow$ DCF is a congruence. Hence, BE = DF. Since $\overleftrightarrow{BE}$ and $\overleftrightarrow{DF}$ are two lines perpendicular to $\overleftrightarrow{AC}$, $\overleftrightarrow{BE} \parallel \overleftrightarrow{DF}$. Hence, by Theorem 6-8, quadrilateral BFDE is a parallelogram.

9.



Consider the triangles ΔA′AD′ and ΔB′BA′. A′A = B′B, $\angle$A′AD′ $\cong$ $\angle$B′BA′, and AD′ = BA′. So, A′AD′ $\leftrightarrow$ B′BA′ is a congruence. Therefore, $\angle$D′A′A $\cong$ $\angle$A′B′B. But, since $\angle$B′BA′ is a right angle, $\angle$A′B′B is a complement of $\angle$BA′B′. Consequently, $\angle$D′A′A is a complement of $\angle$BA′B′. Since D′ and B′ are on opposite sides of $\overleftrightarrow{AA'}$, A is interior to $\angle$D′A′B′. So, since the sum of the measures of complementary angles is 90, $\angle$D′A′B′ is an angle of 90°; that is, it is a right angle. Similarly, $\angle$A′B′C′, $\angle$B′C′D′, and $\angle$C′D′A′ are right angles. So, by definition, quadrilateral A′B′C′D′ is a rectangle. Since A′AD′ $\leftrightarrow$ B′BA′ is a congruence, D′A′ = A′B′. So, quadrilateral A′B′C′D′ is a rectangle with two adjacent sides congruent. Hence, by definition, it is a square.

The material covered in pages 6-179 through 6-185 serves two purposes. For one thing, it acquaints students with a mode of speech found in many mathematics textbooks. Secondly, it provides the students with a fairly comprehensive review of the first half of the course. A mid-unit examination is given in the COMMENTARY for 6-185.

*

Answers for Exploration Exercises.

(1) No; Yes	(2) Yes; No	(3) No; No	(4) Yes; No
(5) Yes; Yes	(6) Yes; Yes	(7) No; No	(8) Yes; No
(9) Yes; Yes	(10) Yes; No	(11) Yes; Yes	(12) Yes; Yes
(13) No; No	(14) No; Yes		

[Note that if we interpret 'ABCD' merely as an abbreviation for '$\overleftrightarrow{AB} \cup \overleftrightarrow{BC} \cup \overleftrightarrow{CD} \cup \overleftrightarrow{DA}$' then neither (6) nor (12) implies (*). On the other hand, if one reads 'ABCD' as 'quadrilateral ABCD' then both (6) and (12) do imply (*).]

line 3. (1), (5), (6), (9), (11), (12), and (14) are sufficient conditions for (*).

line 7. (*) is a sufficient condition for (2), (4), (5), (6), (8), (9), (10), (11), and (12).

line 14. (2), (4), (5), (6), (8), (9), (10), (11), and (12) are necessary conditions for (*). In short, the sentences which are necessary conditions for (*) are exactly the sentences for which (*) is a sufficient condition.

line 16. (*) is a necessary condition for (1), (5), (6), (9), (11), (12), and (14). In short, the sentences for which (*) is a necessary condition are exactly the sentences which are sufficient conditions for (*).

*

The expressions 'if...then---', '---if...' and '...only if---' have been discussed in the COMMENTARY for page 6-384.

*

Note that the scheme in the box on page 6-180 [likewise, the scheme in the box on page 6-182] is not quite technically adequate. If the 'p's and 'q's to the left of the brace are replaced by sentences then [see line 2 below the box] each of the three resulting compound sentences should be enclosed in semiquotes, and [see line 4 below the box] the 'p's and 'q's to the right of the dashed lines should be replaced by names of the given component sentences. Part of what the scheme is intended to convey can be said more correctly as follows:

If one replaces 'p' and 'q' in:

'if p then q' is a theorem

by <u>sentences</u>, and replaces 'P' and 'Q' in:

P is a sufficient condition for Q

by <u>names of these sentences</u>, then the two statements which result say the same thing.

Answers for Part A.

1. (a) 'ΔABC is equilateral' is a sufficient condition for 'ΔABC is isosceles'.

   (b) 'ΔABC is isosceles' is a necessary condition for 'ΔABC is equilateral'.

   (c) 'ΔABC is isosceles if ΔABC is equilateral' is a theorem.

   (d) 'ΔABC is equilateral only if ΔABC is isosceles' is a theorem.

The solutions for Exercises 2, 3, and 4, are similar to that for 1.

*

Answers for Part B [on page 6-182].

The sentences on page 6-179 which are both necessary and sufficient for (*) are (5), (6), (9), (11), and (12). The fact that sentence (5) is necessary and sufficient for sentence (*) is expressed by each of the following statements:

(I) 'the diagonals of ABCD are perpendicular bisectors of each other' is a necessary and sufficient condition for 'ABCD is a rhombus'.

(II) 'ABCD is a rhombus' is a necessary and sufficient condition for 'the diagonals of ABCD are perpendicular bisectors of each other'.

(III) 'the diagonals of ABCD are perpendicular bisectors of each other if and only if ABCD is a rhombus' is a theorem.

(IV) 'ABCD is a rhombus if and only if the diagonals of ABCD are perpendicular bisectors of each other' is a theorem.

Answers for Part C [on pages 6-183, 6-184, and 6-185].

1. (a) Yes; Yes (b) No; Yes (c) Yes; Yes (d) No; Yes

   (e) Yes; Yes (f) Yes; No (g) Yes; No

   In connection with sentence (f), note that, from the figure, $\angle A_2$ and $\angle A_4$ are vertical angles as are $\angle B_2$ and $\angle B_4$. So, by Theorem 2-5, sentence (f) is a theorem. So, by conditionalizing, 'if ℓ is parallel to m then $\angle A_2 \cong \angle A_4$ and $\angle B_2 \cong \angle B_4$' is a theorem. Hence, (f) is a necessary condition for (*).

*

Note that whether one sentence is a necessary or a sufficient condition for another is relative to the postulates whose consequences are being developed. For example, to say that 'ABCD is a square' is a sufficient condition for 'ABCD is a rhombus' is to say, in the context of this book, that from our postulates and definitions, together with the assumption 'ABCD is a square' one can derive the conclusion 'ABCD is a rhombus'. In another context, say with weaker postulates or different definitions, it might not be possible to carry out such a derivation. In such a case, 'ABCD is a square' would not be a sufficient condition for 'ABCD is a rhombus'.

In exercises which, like those of Part C, require a figure for their interpretation, the words 'necessary' and 'sufficient' are used in a somewhat looser sense. For example, that (a) is a necessary condition for (*) means that from postulates, definitions, and the premiss '$\angle A_1$ and $\angle B_1$ are corresponding angles' [which is suggested by the figure], together with the assumption (*), one can derive the conclusion (a). Similarly, in showing that (d) is sufficient for (*), and that (g) is necessary for (*), one "takes for granted" not only the postulates and definitions but also the premiss '$\ell \neq m$'.

*

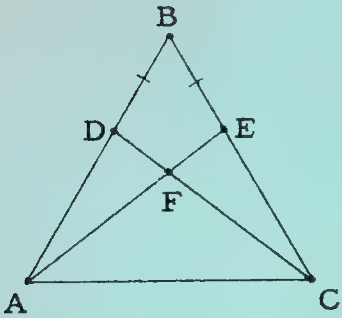
2. (a) No; Yes (b) No; No (c) No; No (d) No; No

   (e) Yes; No (f) No; Yes

3. (a) Yes; No (b) No; Yes (c) Yes; No (d) Yes; No

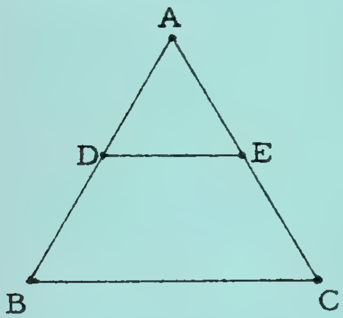
   (e) Yes; No (f) Yes; No (g) Yes; Yes (h) Yes; No

48.



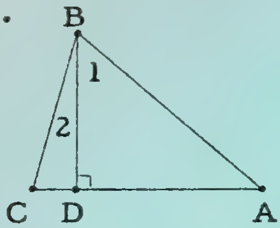
Since the base angles of an isosceles triangle are congruent, $\angle ECA \cong \angle DAC$. Since $BC = BA$, $BE = BD$, $D \in \overline{BA}$, and $E \in \overline{BC}$, it follows from an axiom that $EC = DA$. Finally, $CA = AC$. So, by s.a.s., $ECA \leftrightarrow DAC$ is a congruence. So, $\angle EAC \cong \angle DCA$. Therefore, since two congruent angles of a triangle are opposite congruent sides, $FC = FA$ and, by definition, $\triangle AFC$ is isosceles.

49.

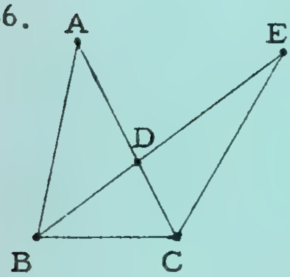


Since the sum of two sides of a triangle is greater than the third, $AB + AC > BC$. By hypothesis, $AB = AC$. So, $2 \cdot AB > BC$. Now, since D and E are midpoints of two sides of a triangle, it follows that DE is half the measure of the third side. So, $2 \cdot DE = BC$. Hence, $2 \cdot AB > 2 \cdot DE$ and, so, $AB > DE$.

50. Let P be the midpoint of $\overline{AB}$. Then, since M is the midpoint of $\overline{BD}$, $\overline{PM} \parallel \overline{AD}$. Since $\overline{AD} \parallel \overline{BC}$ and $\overleftrightarrow{PM} \neq \overleftrightarrow{BC}$, $\overleftrightarrow{PM} \parallel \overleftrightarrow{BC}$. Since N is the midpoint of $\overline{AC}$, $N \in \overleftrightarrow{PM}$. Now, $PM = \frac{1}{2} \cdot AD$ and $PN = \frac{1}{2} \cdot BC$. But, $MN = PN - PM$. So, $MN = \frac{1}{2}(BC - AD)$.

45. 

Since the sum of the measures of the angles of a triangle is 180 and a right angle is an angle of 90°, it follows that $m(\angle A) = 90 - m(\angle B_1)$ and $m(\angle C) = 90 - m(\angle B_2)$. But, by hypothesis, $m(\angle B_1) > m(\angle B_2)$. So, $m(\angle A) < m(\angle C)$. Therefore, since the longer of two sides of a triangle is opposite the larger of the two opposite angles, $AB > BC$.

46. 

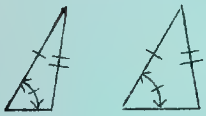
Since D is interior to $\angle ABC$, it follows from an axiom that $\angle ABC$ is larger than $\angle EBC$. Similarly, since D is interior to $\angle BCE$, $\angle BCE$ is larger than $\angle ACB$. But, by hypothesis, ΔABC is isosceles with vertex angle at A. So, since the base angles of an isosceles triangle are congruent, $\angle ABC \cong \angle ACB$. Hence, $\angle BCE$ is larger than $\angle EBC$. Therefore, since the longer of two sides of a triangle is opposite the larger of the two opposite angles, $BE > CE$.

47. Since quadrilateral ABCD is a parallelogram, $\overleftrightarrow{AE} \parallel \overleftrightarrow{DF}$. So, $\angle DFA \cong \angle FAE$. But, by hypothesis, $\overrightarrow{AF}$ is the bisector of $\angle DAE$. So, $\angle DAF \cong \angle FAE$. Hence, $\angle DFA \cong \angle DAF$. Therefore, since two sides of a triangle are congruent if they are opposite congruent angles, $DA = DF$. Similarly, $DA = AE$. Therefore, $DF = AE$. So, since a pair of opposite sides of quadrilateral AEFD are both parallel and congruent, it is a parallelogram. But, two of its adjacent sides are congruent, also. So, by definition, it is a rhombus.

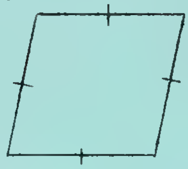
18. 16 19. 2 20. 1080 21. 15 22. (C) 23. (B)

24. (B) 25. (C) 26. (A)

27.



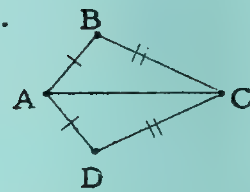
28.



29. theorem

30. theorem

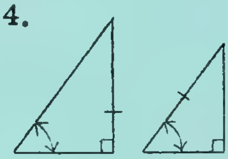
31.



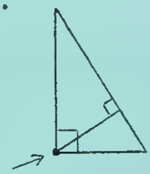
32. theorem

33. theorem

34.



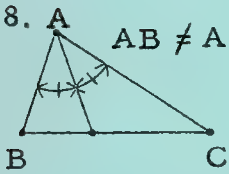
35.



36. theorem

37. theorem

38.

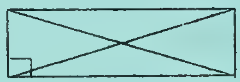


AB $\neq$ AC

39. theorem

41. theorem

40.

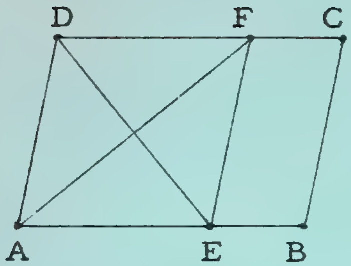


42.

43. Since AD > DC, $\angle$ACD is larger than $\angle$CAD. Since quadrilateral ABCD is a parallelogram, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; so, $\angle ACD \cong \angle CAB$. Hence, $\angle$CAB is larger than $\angle$CAD. So, $\angle CAB \ncong \angle CAD$ and, therefore, $\overrightarrow{AC}$ is not the bisector of $\angle$BAD.

44. Since $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$ and $\overleftrightarrow{DB} \parallel \overleftrightarrow{CE}$, it follows that quadrilateral BCED is a parallelogram. So, since the opposite sides of a parallelogram are congruent, BD = CE. But, by hypothesis, BD = AC. Therefore, AC = CE. Since the base angles of an isosceles triangle are congruent, $\angle CAD \cong \angle CED$. But, since $\overleftrightarrow{BD} \parallel \overleftrightarrow{CE}$, it follows that the corresponding angles $\angle$BDA and $\angle$CED are congruent. So, $\angle CAD \cong \angle BDA$. Finally, AD = DA. So, by s.a.s., ACD $\longleftrightarrow$ DBA is a congruence; whence, $\overleftrightarrow{CD} \cong \overleftrightarrow{BA}$.

47.



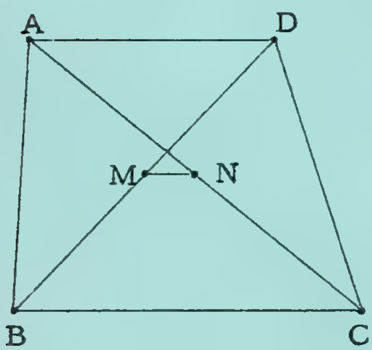
<u>Hypothesis:</u> quadrilateral ABCD is a parallelogram, $\overrightarrow{AF}$ bisects $\angle BAD$, $\overrightarrow{DE}$ bisects $\angle FDA$

<u>Conclusion:</u> quadrilateral AEFD is a rhombus

48. Suppose that $\triangle ABC$ is isosceles with $AB = BC$. Let D be a point on $\overline{AB}$ and E be a point on $\overline{BC}$ such that $BD = BE$. If $\overline{AE} \cap \overline{CD} = \{F\}$, prove that $\triangle$ AFC is isosceles.

49. Suppose that, in the isosceles triangle $\triangle ABC$, D and E are the midpoints of the congruent sides $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, respectively. Prove that $2 \cdot AB > BC$ and that $AB > DE$.

50.



<u>Hypothesis:</u> quadrilateral ABCD is a trapezoid with $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, M and N are the midpoints of $\overleftrightarrow{BD}$ and $\overleftrightarrow{CA}$, respectively, $BC > AD$

<u>Conclusion:</u> $MN = \frac{1}{2}(BC - AD)$

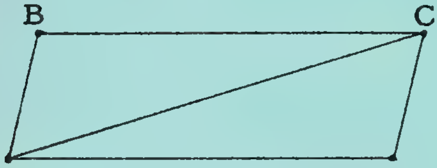
*

Answers for Quiz.

1.	$90 - x$ or $270 - x$	2.	one	3.	60	4.	35	5.	20
6.	55	7.	24	8.	15	9.	50	10.	92.5	11.	120
12.	24	13.	6	14.	18	15.	5 in.	16.	7.5 in.	17.	30

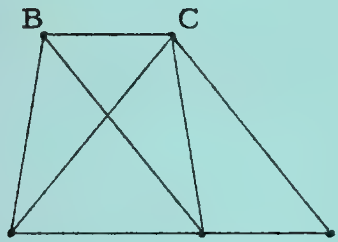
42. For each triangle $\triangle XYZ$, if $\triangle XYZ$ is isosceles then the median of $\triangle XYZ$ from Y is the angle bisector of $\triangle XYZ$ from Y.

<u>Part III.</u>

43. 

<u>Hypothesis:</u> quadrilateral ABCD is a parallelogram, $AD > DC$

<u>Conclusion:</u> $\overleftrightarrow{AC}$ is not the bisector of $\angle BAD$

44. 

<u>Hypothesis:</u> quadrilateral ABCD is a trapezoid with $\overline{BC} \parallel \overline{AD}$, $D \in \overline{AE}$, $\overline{AC} \cong \overline{BD}$, $\overline{CE} \parallel \overline{BD}$

<u>Conclusion:</u> $AC = CE$, $ACD \leftrightarrow DBA$ is a congruence, $\overline{AB} \cong \overline{CD}$

45. Suppose that $\triangle ABC$ is an acute triangle and that $\overline{BD}$ is the altitude from B. If $\angle ABD$ is larger than $\angle CBD$, prove that $AB > BC$.

46. Suppose that $\triangle ABC$ is isosceles with vertex angle at A. Let D be a point on $\overline{AC}$ and E be a point on $\overrightarrow{BD}$ such that $D \in \overline{BE}$. Prove that $BE > CE$.

29. For all angles $\angle X$, $\angle Y$, and $\angle Z$, if $\angle X$ is an acute angle and $\angle Y$ is a supplement of $\angle X$ and $\angle Z$ is a complement of $\angle X$ then $m(\angle Y) - m(\angle Z) = 90$.

30. If two parallel lines are cut by a transversal, the bisectors of two consecutive interior angles are perpendicular.

31. If diagonal $\overleftrightarrow{AC}$ of quadrilateral ABCD divides it into two congruent triangles then the quadrilateral is a parallelogram.

32. If the diagonals of a quadrilateral are not congruent and bisect each other at right angles, the quadrilateral is a rhombus.

33. If two segments join the midpoints of the opposite sides of a quadrilateral, the segments bisect each other.

34. If two triangles have a side and two angles of one congruent to a side and two angles of the other then the triangles are congruent.

35. If the point of concurrence of the altitudes of a triangle is not in the interior of the triangle then the triangle is an obtuse triangle.

36. The perimeter of the triangle formed by joining the midpoints of the sides of a given triangle is one half the perimeter of the given triangle.

37. The bisectors of two supplementary adjacent angles are perpendicular to each other.

38. The bisector of an angle of a triangle bisects the side opposite.

39. Two isosceles triangles are congruent if their vertex angles are congruent and their bases are congruent.

40. If the diagonals of a parallelogram are congruent, the parallelogram is a square.

41. If the diagonals of a parallelogram are congruent and perpendicular, the parallelogram is a rectangle.

22. If $\overleftrightarrow{CD}$ is the median of ΔABC from C and $AB = 2 \cdot CD$ then ΔADC and ΔBDC are __(?)__ triangles.

(A) congruent (B) right (C) isosceles

23. Two opposite angles of an isosceles trapezoid are __(?)__.

(A) congruent (B) supplementary (C) complementary

24. The point which is equidistant from the three vertices of a triangle is the point of concurrence of __(?)__.

(A) the angle bisectors (B) the perpendicular bisectors of the sides
(C) the altitudes

25. An exterior angle at one vertex of a triangle, and an exterior angle at another vertex of the triangle may both be __(?)__.

(A) acute (B) right (C) obtuse

26. Suppose that A, B, and C are vertices of a triangle and that D is a point on $\overrightarrow{BC}$ such that $C \in \overline{BD}$. From this it follows that __(?)__.

(A) $m(\angle ACD) > m(\angle A)$ (B) $m(\angle ACD) < m(\angle A)$
(C) $m(\angle ACD) > m(\angle ACB)$ (D) $m(\angle ACD) < m(\angle ACB)$

Part II.

Each of the following sentences is a generalization. If you think the generalization is a theorem, say so. If you think the generalization is not a theorem, draw a counter-example.

27. If two triangles have two sides and an angle of one congruent to two sides and an angle of the other, the triangles are congruent.

28. If a polygon is equilateral, it is equiangular.

13. Suppose that N and P are the midpoints of the diagonals $\overline{AC}$ and $\overline{BD}$, respectively, of quadrilateral ABCD. If M is the midpoint of $\overline{AD}$ and MN = 5 and MP = 3, what is AB?

14. If the measures of the diagonals of a quadrilateral are 8 and 10, what is the perimeter of the new quadrilateral whose adjacent vertices are the midpoints of the adjacent sides of the given quadrilateral?

15. The median of a trapezoid is 7 inches long and one base is 9 inches long. How long is the other base?

16. If the hypotenuse of a right triangle is 15 inches long, how long is the median to the hypotenuse?

17. In $\triangle ABC$, $\angle C$ is a right angle, AB = 6, and AC = 3. Find the number of degrees in $\angle B$.

18. Suppose that each of a pair of base angles of an isosceles trapezoid is an angle of 45°, the smaller base is 10 inches long, and the bases are 3 inches apart. How long is the longer base?

19. Suppose that quadrilateral ABCD is a parallelogram with AB = 10 and AD = 4. If $\angle A$ is an angle of 30°, what is the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$?

20. If the average of the measures of the exterior angles of a convex polygon is 45, what is the sum of the measures of the angles of that polygon?

21. An angle of a regular polygon is an angle of 156°. Find the number of sides of the polygon.

2. Points P and Q are 7 inches apart. How many points are there which are 12 inches from P and 5 inches from Q?

3. What is the measure of an angle whose supplement is four times its complement?

4. Two angles are complementary and one is 20° larger than the other. Find the number of degrees in the smaller angle.

5. The measures of the three angles of a triangle are in the ratio 1:3:5. What is the measure of the smallest angle of the triangle?

6. The vertex angle of an isosceles triangle is an angle of 70°. Find the number of degrees in a base angle.

7. Suppose that, in $\triangle ABC$, $m(\angle C) = 90$ and $m(\angle B) = 33$. If $\overline{CM}$ is the median and $\overline{CD}$ is the altitude of $\triangle ABC$ from C, what is $m(\angle MCD)$?

8. In $\triangle ABC$, $m(\angle B) = 3 \cdot m(\angle A)$ and an exterior angle at C is an angle of 60°. How many degrees are there in the smallest angle of the triangle?

9. If, in $\triangle ABC$, $m(\angle A) = x + 5$, $m(\angle B) = x + 15$, and $m(\angle C) = 2x - 20$, what is the measure of the smallest angle of the triangle?

10. Suppose that quadrilateral ABCD is convex and that $m(\angle A) = 85$ and $m(\angle B) = 100$. If E is a point such that $\overrightarrow{CE}$ bisects $\angle BCD$ and $\overrightarrow{DE}$ bisects $\angle CDA$, find the number of degrees in $\angle CED$.

11. Suppose that quadrilateral ABCD is a trapezoid with $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $AB = 2 \cdot DC$, and $AD = DC = CB$. What is $m(\angle ADC)$?

12. In $\triangle ABC$, D and E are midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. If $DE = 12$, what is AC?

(i) Yes; Yes (j) Yes; Yes (k) No; Yes (ℓ) Yes; Yes

(m) Yes; Yes (n) Yes; Yes

*

In Exercise 2, the only role played by the figure is to identify $\angle A$ as $\angle CAB$, $\angle B'$ as $\angle A'B'C'$, etc. In Exercise 3, the figure plays no essential role except in part (i) where to show that (i) is sufficient for (*) one must take for granted that C and D are on the same side of $\overleftrightarrow{AB}$. In Exercise 4, different people may reasonably give different answers according as to what each takes from the figure. For example, one who accepts the figure's suggestion that $\overline{AD} \nparallel \overline{BC}$ will say that (a) is necessary for (*), while one who does not accept this suggestion will say that (a) is not necessary for (*) [but that '$\overline{AB} \parallel \overline{DC}$ or $\overline{AD} \parallel \overline{BC}$' is necessary for (*)]. Similarly, one who accepts the suggestion that $\overline{AD} \cap \overline{BC} = \emptyset$ will say that (b) is sufficient for (*), while one who does not accept this suggestion will say that (b) is not sufficient for (*). The answers given below are based on the figure's suggestion that $\overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$ is a quadrilateral, and [for part (d)] that its diagonals intersect at E.

*

4. (a) No; No (b) No; Yes (c) No; Yes (d) No; Yes

*

Quiz. [Covering pages 6-1 through 6-185].

[There probably are more items here than you might wish to include even in a mid-unit examination. Perhaps you will find use for some of them in review assignments to be given prior to the examination.]

Part I.

1. Suppose that B is a point in $\overline{ED}$ and A and C are two points in the same side of $\overleftrightarrow{ED}$ such that $\angle ABC$ is a right angle. If $m(\angle CBD)$ is x then $m(\angle ABE) =$ __?__

Corrections.

On page 6-187, delete 'also' from the last part of line 7b:

..., and ___ are in pro-

Line 2b should end with:

..., and $\overleftrightarrow{BC}$ are in

Answers for Part A [on pages 6-186 and 6-187].

1. (a) $\frac{1}{2}$ (b) 2 (c) 1 (d) $\frac{2}{3}$ [See Part B on page 6-187.]

   (e) 2 (f) congruent (g) $\frac{1}{2}$

2. (a) 8; 12 (b) 1; $\frac{1}{2}$ (c) $\frac{3}{2}$

3. (a) $\overleftrightarrow{PN}$; $\overleftrightarrow{QS}$ [or: $\overleftrightarrow{MN}$; $\overleftrightarrow{RS}$] (b) 2

*

Answers for Part B [on pages 6-187 and 6-188].

['nondegenerate ones' signifies that none of the ratios we shall consider is 0. The 'of course' signifies the restriction about not dividing by 0.]

In Unit 5, we said that nonzero <u>numbers</u> u, v, x, and y are in proportion if and only if $u/v = x/y$. In Unit 6, we extend the meaning of 'in proportion' to segments. Similarly, the word 'ratio' has its meaning extended to include segments. These extensions in meaning are not too great. In fact, if we wished to be pedantic, we could stick with the Unit 5 meaning completely just by talking about the ratio of the <u>measure</u> of a first segment to the <u>measure</u> of a second, etc.

$\overleftrightarrow{AB}$, $\overleftrightarrow{EF}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{GH}$ are in proportion because $\frac{AB}{EF} = \frac{CD}{GH}$.

$\overleftrightarrow{AB}$, $\overleftrightarrow{EF}$, $\overleftrightarrow{GH}$, and $\overleftrightarrow{CD}$ are not in proportion because $\frac{AB}{EF} \neq \frac{GH}{CD}$.

1. $\overleftrightarrow{CD}$; $\overleftrightarrow{EF}$; $\overleftrightarrow{GH}$; $\overleftrightarrow{CD}$; $\overleftrightarrow{GH}$; $\overleftrightarrow{EF}$

2. Since $\frac{5}{10} = \frac{DE}{4}$, $DE = 2$.

3. Suppose that $\overleftrightarrow{AD}$, $\overleftrightarrow{DE}$, $\overleftrightarrow{AB}$, and $\overleftrightarrow{BC}$ are in proportion. Then, by definition,

$$\frac{AD}{DE} = \frac{AB}{BC}.$$

So,

$$\frac{AD}{DE}(DE \cdot BC) = \frac{AB}{BC}(DE \cdot BC),$$

$$AD \cdot BC = AB \cdot DE,$$

$$\frac{AD \cdot BC}{AD \cdot AB} = \frac{AB \cdot DE}{AD \cdot AB},$$

$$\frac{BC}{AB} = \frac{DE}{AD},$$

$$\frac{BC}{AB} + 1 = \frac{DE}{AD} + 1,$$

$$\frac{BC + AB}{AB} = \frac{DE + AD}{AD},$$

$$\frac{AC}{AB} = \frac{AE}{AD},$$

$$\frac{AC}{AB}\left(\frac{AB \cdot AD}{AC \cdot AE}\right) = \frac{AE}{AD}\left(\frac{AB \cdot AD}{AC \cdot AE}\right),$$

$$\frac{AD}{AE} = \frac{AB}{AC}.$$

Therefore, by definition, $\overleftrightarrow{AD}$, $\overleftrightarrow{AE}$, $\overleftrightarrow{AB}$, and $\overleftrightarrow{AC}$ are in proportion.

4. Since $\frac{AB}{BC} = \frac{CD}{EF}$ and $\frac{CD}{EF} = \frac{FG}{GH}$, it follows that $\frac{AB}{BC} = \frac{FG}{GH}$. Therefore, by definition, $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{FG}$, and $\overleftrightarrow{GH}$ are in proportion.

*

Answers for Part C [on pages 6-188, 6-189, and 6-190].

1. 4 : 5; 5 : 4

2. Since $\frac{4}{N_1N_2} = \frac{N_1N_2}{9}$ and $N_1N_2 > 0$, $N_1N_2 = 6$.

3. (a) 10 (b) 4 (c) 10 (d) 20 (e) 10, −10 (f) $-\frac{46}{5}$

4. 10, $\frac{50}{3}$, 30 or 6, 10, 18 or $\frac{10}{3}$, $\frac{50}{9}$, 10

5. ak, bk, ck

6. d, $\frac{bd}{a}$, $\frac{cd}{a}$

7. Yes, because $\frac{a}{ak} = \frac{b}{bk} = \frac{c}{ck} = \ldots$.

8. kx_1; kx_2; kx_3

<u>Proof.</u> Suppose $x_1, x_2, x_3, \ldots$ is proportional to $y_1, y_2, y_3, \ldots$.

Then, by definition, $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = \ldots$. Let $k = \frac{y_1}{x_1}$. $k \neq 0$ because $y_1 \neq 0$. Also, $y_1 = \frac{y_1}{x_1} \cdot x_1$, $y_2 = \frac{y_2}{x_2} \cdot x_2 = \frac{y_1}{x_1} \cdot x_2$, ...

So, $y_1 = kx_1$, $y_2 = kx_2$, ...

On the other hand, suppose there is a nonzero number k such that $y_1 = kx_1$, $y_2 = kx_2$, $y_3 = kx_3$, ... Then, since none of the numbers $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ is 0, $\frac{x_1}{y_1} = \frac{1}{k} = \frac{x_2}{y_2} = \frac{x_3}{y_3} \ldots$.

So, by definition, $x_1, x_2, x_3, \ldots$ is proportional to $y_1, y_2, y_3, \ldots$.

9. Suppose that a, b is proportional to c, d. Then, a/c = b/d. That is, b/d = a/c. So, b, a is proportional to d, c. Hence, if a, b is proportional to c, d then b, a is proportional to d, c.

10. (a) Suppose that a, b is proportional to c, d. Then, by definition, $\frac{a}{c} = \frac{b}{d}$. So, $\frac{a}{c}(cd) = \frac{b}{d}(cd)$ and $ad = bc$. Hence, if a, b is proportional to c, d then $ad = bc$.

(b) $ad = bc$; $ad \cdot \frac{1}{cd} = bc \cdot \frac{1}{cd}$; $\frac{a}{c} = \frac{b}{d}$. So, a, b is proportional to c, d.

(c)

a, b is proportional to c, d	
$ad = bc$	[Part (a)]
$ad = cb$	
a, c is proportional to b, d	[Part (b)]

(d)

a, b is proportional to c, d	
$ad = bc$	[Part (a)]
$ad + ab = bc + ab$	
$a(b + d) = b(a + c)$	
a, b is proportional to a + c, b + d	[Part (b)]

[Also, see the COMMENTARY for Exercise 3 of Part B on 6-187.]

(e)

a, b is proportional to c, d	
$ad = bc$	[Part (a)]
$ad + ac = bc + ac$	
$a(c + d) = (a + b)c$	
a, a + b is proportional to c, c + d	[Part (b)]

11. <u>Given:</u> $\frac{HA}{IA} = \frac{AB}{AC} = \frac{BD}{CE} = \frac{DF}{EG}$

(1) Since $\frac{AB}{AC} = \frac{BD}{CE}$, by Exercise 10(e), $\frac{AB}{AC} = \frac{AB + BD}{AC + CE}$. But, $AB + BD = AD$ and $AC + CE = AE$. So, $\frac{AB}{AC} = \frac{AD}{AE}$.

(2) Since $\frac{AB}{AC} = \frac{BD}{CE}$, it follows from Exercise 11(1) that $\frac{AB}{AC} = \frac{AD}{AE}$.
So, by Exercise 10(c), $\frac{AB}{AD} = \frac{AC}{AE}$.

(3) Since $\frac{HA}{IA} = \frac{AB}{AC}$, it follows from Exercise 10(e) that $\frac{HA}{IA} = \frac{HB}{IC}$.
So, by Exercise 10(c), $\frac{HA}{HB} = \frac{IA}{IC}$; that is, $\frac{AH}{HB} = \frac{AI}{IC}$.

(4) Since $\frac{BD}{CE} = \frac{DF}{EG}$, it follows from Exercise 10(e) that $\frac{BD}{CE} = \frac{BF}{CG}$.
By Exercise 11(1), $\frac{AB}{AC} = \frac{AD}{AE}$. So, since $\frac{AB}{AC} = \frac{BD}{CE}$, it follows that $\frac{AD}{AE} = \frac{BF}{CG}$. So, by Exercise 10(c), $\frac{AD}{BF} = \frac{AE}{CG}$.

(5) By Exercise 11(3), $\frac{HA}{IA} = \frac{HB}{IC}$. So, since $\frac{HA}{IA} = \frac{DF}{EG}$, it follows that $\frac{DF}{EG} = \frac{HB}{IC}$. Hence, by Exercise 10(c), $\frac{DF}{HB} = \frac{EG}{IC}$.

(6) Since $\frac{BD}{CE} = \frac{DF}{EG}$, it follows from Exercise 10(e) that $\frac{BF}{CG} = \frac{DF}{EG}$.

(7) Since $\frac{AB}{AC} = \frac{BD}{CE}$, it follows from Exercise 10(a) that $AB \cdot CE = BD \cdot AC$; that is, that $AB \cdot CE = AC \cdot BD$.

(8) Since $\frac{AB}{AC} = \frac{BD}{CE} = \frac{DF}{EG}$, it follows from Exercise 10(e) that
$\frac{AB}{AC} = \frac{AB + BD}{AC + CE} = \frac{AB + BD + DF}{AC + CE + EG} = \frac{AF}{AG}$. So, by Exercise 10(a), $AB \cdot AG = AF \cdot AC$; that is, $AF \cdot AC = AG \cdot AB$.

(9) Since $\frac{HA}{IA} = \frac{BD}{CE} = \frac{DF}{EG}$, it follows from Exercise 10(e) that $\frac{HA}{IA} = \frac{BF}{CG}$.
So, by Exercise 10(a), $HA \cdot CG = BF \cdot IA$; that is $CG \cdot AH = BF \cdot IA$.

(10) Since $\frac{HA}{IA} = \frac{AB}{AC}$, it follows from Exercise 10(a) that

$HA \cdot AC = AB \cdot IA$; that is, that $AB \cdot AI = HA \cdot AC$.

12. (I) 'a, b is proportional to c, d' is a necessary and sufficient condition for 'ad = bc'.

(II) 'ad = bc' is a necessary and sufficient condition for 'a, b is proportional to c, d'.

(III) 'a, b is proportional to c, d if and only if ad = bc' is a theorem.

(IV) 'ad = bc if and only if a, b is proportional to c, d' is a theorem.

13. (a) sufficient [See Exercise 10(c).]

(b) only if [See Exercise 10(d).]

(c) necessary [See Exercise 10(e).]

[Actually, 'necessary' is also correct for part (a). The converse of the theorem in Exercise 10(c) is just an alphabetic variant of the theorem itself.]

14. Ex. 10(c): if a, c is proportional to b, d then a, b is proportional to c, d

This is a theorem because it is just an alphabetic variant of the given conditional. [Interchange 'b' and 'c'.]

Ex. 10(d): if a, b is proportional to a + c, b + d, then a, b is proportional to c, d

This is a theorem. Just reverse the order of the steps in the proof for Exercise 10(d).

Ex. 10(e): if a, a + b is proportional to c, c + d then a, b is proportional to c, d

This is a theorem. Just reverse the order of the steps in the proof for Exercise 10(e).

> **Correction.**
>
> On page 6-192, the last part of line 6 should be: FGHE $\longleftrightarrow$ ABCD

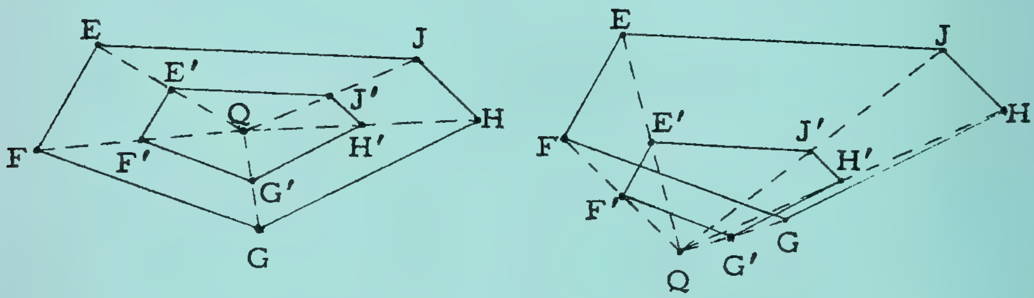
line 3: Theorem 6-24

line 6: Theorem 5-13

line 8: $\frac{MN}{AB} = \frac{1}{2} = \frac{NR}{BC} = \frac{RS}{CD} = \frac{SM}{DA}$

*

Answers for Part A.



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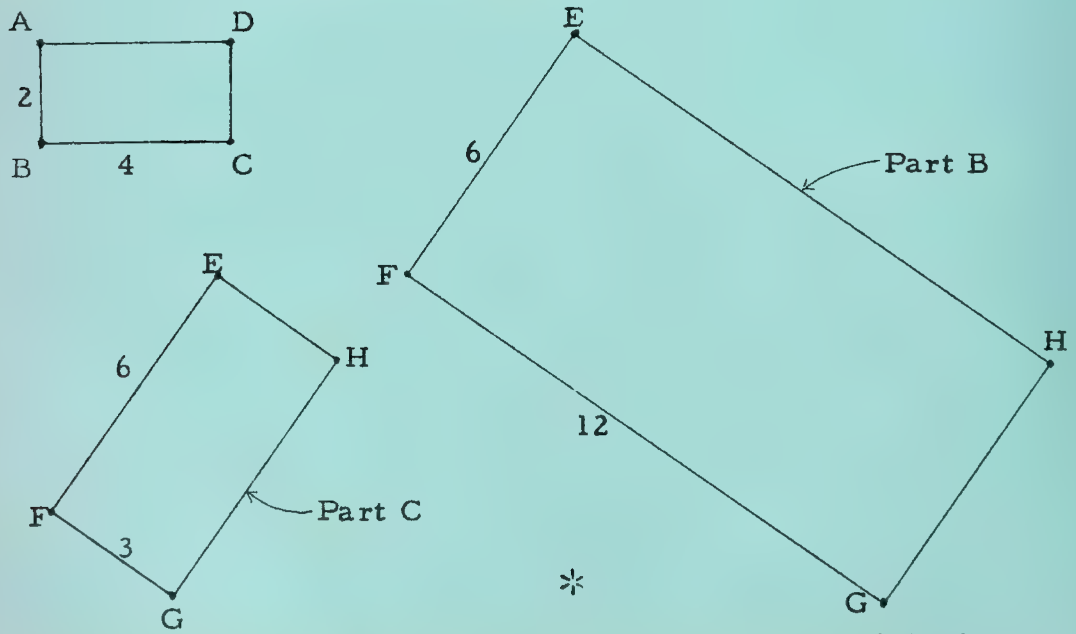
Answers for Parts B and C [on page 6-192].



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The exercises on page 6-430 are important for the work in the next subsection.

Correction.

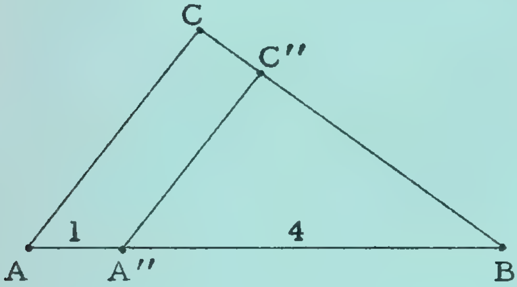
On page 6-193, line 3, change '$\overrightarrow{A'B''}$' to '$\overset{\bullet\!\!\longrightarrow}{A'B''}$'.
In line 4, change '$\overrightarrow{B'A''}$' to '$\overset{\bullet\!\!\longrightarrow}{B'A''}$'.

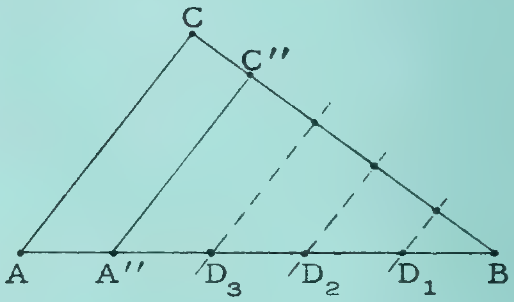
Answers to questions on page 6-193.

line 7. Theorem 5-11 line 12. s.a.s.

*

The argument referred to between brackets on line 11 of page 6-194 can be obtained by starting on line 10 of page 6-193 and systematically interchanging 'A' and 'B' throughout the text up to and including line 6 on page 6-194. Also, replace the two figures on page 6-193 as indicated.





*

Referring to the figure on page 6-195, note that each of the p congruent segments is congruent to each of the q congruent segments.

That $\sqrt{2}$ is not rational can be proved as follows [see Unit 4, page 4-48]:

Suppose that $\sqrt{2}$ were rational. Then, there would be many nonzero whole numbers whose products by $\sqrt{2}$ would be whole numbers. Let q be the least such, and suppose that $q\sqrt{2} = p$. Since $1 < \sqrt{2} < 2$, $1 < p/q < 2$. Hence, $q < p < 2q$, and $0 < p - q < q$. Since $p - q$ is a nonzero whole number smaller than q, it follows that $(p - q)\sqrt{2}$ is not a whole number. But, $(p - q)\sqrt{2} = (p - q)(p/q) = (p^2/q^2) \cdot q - p = 2q - p$. Since $2q - p$ is a nonzero whole number, it follows that $(p - q)\sqrt{2}$ is a whole number. It results from this contradiction that there is no whole number whose product by $\sqrt{2}$ is a whole number. That is, $\sqrt{2}$ is not rational.

If [line 3 from foot of page 6-195] it were possible to find a point $B \in \overline{AC}$ such that the two segments could be divided into p segments and q segments, respectively, all congruent, then $\sqrt{2} = AB/BC = p/q$; whence, $\sqrt{2}$ would be rational. Since $\sqrt{2}$ is irrational, there is no such point B.

Correction.

On page 6-199, line 8 b should read:
(10) . . . [Steps like (2), (4), (6), (7), and (8)]

Answers to questions on page 6-199.

line 5. Theorem 2-1 and Theorem 5-11

last line.

(16)	$\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are proportional to $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{A'C'}$	[(14); def. of triangle-similarity]
(17)	$AB/A'B' = AC/A'C'$	[(16); def. of proportionality]
(18)	$AB \cdot A'C' = A'B' \cdot AC$	[(17)]

*

Answers for Part A [on page 6-200].

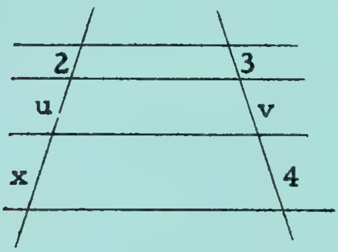
1. By Theorem 7-2, $\frac{10}{5} = \frac{x}{6} = \frac{y}{9}$. So, $x = 12$ and $y = 18$.

2. By Theorem 7-2, $\frac{3}{4} = \frac{x}{28/3} = \frac{5}{y}$. So, $x = 7$ and $y = \frac{20}{3}$.

3. By Theorem 7-1, $\frac{5}{3} = \frac{7}{x}$. So, $x = \frac{21}{5}$.

4. By Theorem 7-1, $\frac{2}{4} = \frac{x}{5}$. So, $x = 2.5$.

5. This problem requires a double use of Theorem 7-1.



By Theorem 7-1, $\frac{2}{u} = \frac{3}{v}$ and $\frac{u}{x} = \frac{v}{4}$.

So, $\frac{2}{u} \cdot \frac{u}{x} = \frac{3}{v} \cdot \frac{v}{4}$.

Therefore, $\frac{2}{x} = \frac{3}{4}$. So, $x = \frac{8}{3}$.

6. By Theorem 7-1 [and Axiom A], $\frac{5}{7} = \frac{7}{y}$. So, $y = \frac{49}{5}$. So, $x = \frac{84}{5}$.
[Or, use Exercise 10(d) on page 6-189 to get '$5/12 = 7/x$'.]

7. By Theorem 7-1, $\frac{2}{x} = \frac{3}{3}$. So, $x = 2$. But, $\frac{3}{6} = \frac{x}{y}$. So, $y = 4$.

8. Use various transformations to get '$7/35 = x/30$'. Then, $x = 6$.

Since $MNP \leftrightarrow SQR$ is a similarity, it follows that $\frac{QR}{NP} = \frac{SQ}{MN}$. So, $\frac{QR}{NP} = \frac{26}{13}$; and $QR = 2 \cdot NP$. [Now, if $K \in \overrightarrow{NP}$ then either $K \in \overline{NP}$ or $K = P$ or $P \in \overline{NK}$. But, if $K = P$ then $KP = 0$ and if $P \in \overline{NK}$ then $KP < NK$. Since $KP = 10$ and $NK = 5$, it follows that $KP \neq 0$ and that $KP \not< NK$. So, $K \neq P$ and $P \notin \overline{NK}$. Hence, if $K \in \overrightarrow{NP}$ then $K \in \overline{NP}$.] Since $K \in \overline{NP}$, $NP = NK + KP = 5 + 10 = 15$. Since $QR = 2 \cdot NP$, $QR = 30$. [Since (assuming that $K \in \overrightarrow{NP}$) $T \in \overrightarrow{QR}$, and since $QT = 10 < 30 = QR$, it follows that $T \in \overline{QR}$.] Since $T \in \overline{QR}$, $TR = QR - QT = 30 - 10 = 20$. [On the other hand, if $N \in \overline{KP}$ and $Q \in \overline{TR}$ then $KP = KN + NP$ and $TR = TQ + QR$. Hence, $NP = 10 - 5 = 5$ and, since $QR = 2 \cdot NP$, $QR = 10$. So, $TR = 20$.]

*

Answer for Part D.

By Theorem 3-6, the triangles are equiangular. By Theorem 5-11, each angle is an angle of 60°. So, $\angle A \cong \angle D$ and $\angle B \cong \angle E$. Hence, by the a. a. similarity theorem, $ABC \leftrightarrow DEF$ is a similarity. Therefore, by the definition of similar triangles, $\Delta ABC \sim \Delta DEF$. [Ask students if each two squares are similar. How about each two rectangles? Each two rhombuses?]

*

Answer for Part E.

By Theorem 2-2, $\angle A \cong \angle A'$, and, by hypothesis, $\angle B \cong \angle B'$. So, by the a. a. similarity theorem, $ABC \leftrightarrow A'B'C'$ is a similarity. Hence, $\Delta ABC \sim \Delta A'B'C'$.

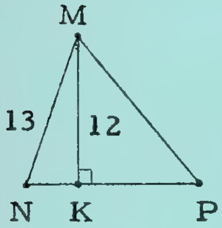
*

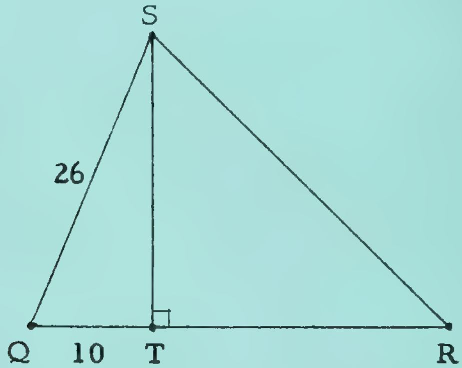
Answers for Part F [on pages 6-201 and 6-202].

[The exercises in Part F foreshadow Theorems 10-30, 10-31, and 10-32.]

1. The vertical angles $\angle APD$ and $\angle BPC$ are congruent. By hypothesis, so are $\angle D$ and $\angle C$. Hence, by the a. a. similarity theorem, $APD \leftrightarrow BPC$ is a similarity. By the definition of triangle-similarity, $\frac{AP}{BP} = \frac{PD}{PC}$. Finally, by algebra, $AP \cdot PC = BP \cdot PD$.

[Since $MN \neq MK$, $K \neq N$. So, either $K \in \overrightarrow{NP}$ or $N \in \overline{KP}$. In either case, since $\angle MKN$ is a right angle, $\angle MNK$ is an acute angle. If $K \in \overrightarrow{NP}$ then $\angle MNK = \angle MNP$, and $\angle MNP$ is acute. If $N \in \overline{KP}$ then $\angle MNK$ and $\angle MNP$ are supplementary, and $\angle MNP$ is obtuse. Since $MNP \leftrightarrow SQR$ is a similarity, $\angle MNP \cong \angle SQR$. So, if $K \in \overrightarrow{NP}$, $\angle SQR$ is acute and, if $N \in \overline{KP}$, $\angle SQR$ is obtuse. Hence, since $\overleftrightarrow{ST} \perp \overleftrightarrow{QR}$, it follows, in both cases, that $T \neq Q$. So, either $T \in \overrightarrow{QR}$ or $Q \in \overline{TR}$. Since if $T \in \overrightarrow{QR}$ then $\angle SQT = \angle SQR$ and $\angle SQT$ is acute, it follows that if $T \in \overrightarrow{QR}$ then $N \notin \overline{KP}$. Hence, if $N \in \overline{KP}$ then $T \notin \overrightarrow{QR}$; so, $Q \in \overline{TR}$. Similarly, if $K \in \overrightarrow{NP}$ then $T \in \overrightarrow{QR}$.]





Now, if $K \in \overrightarrow{NP}$ and $T \in \overrightarrow{QR}$ then $\angle MNK = \angle MNP$ and $\angle SQR = \angle SQT$. Since $MNP \leftrightarrow SQR$ is a similarity, it follows that $\angle MNP \cong \angle SQR$. Hence, $\angle MNK \cong \angle SQT$. [On the other hand, if $N \in \overline{KP}$ and $Q \in \overline{TR}$ then $\angle MNK$ and $\angle SQT$ are supplements of the congruent angles $\angle MNP$ and $\angle SQR$. So, in this case, also, $\angle MNK \cong \angle SQT$.] Since $\angle MKN$ and $\angle STQ$ are right angles, they are congruent. Hence, by the a.a. similarity theorem, $MKN \leftrightarrow STQ$ is a similarity. Therefore, $\frac{MK}{ST} = \frac{KN}{TQ} = \frac{NM}{QS}$. So, $\frac{12}{ST} = \frac{KN}{10} = \frac{13}{26}$. Consequently, $ST = 24$ and $KN = 5$.

The exercises on pages 6-431 and 6-432 give a developmental review of work with radicals. Students will need such skills in connection with the Pythagorean Theorem.

*

Answers for Part B.

[Notice that as in the case of triangle-congruence, one cannot depend upon a sentence like '$\Delta ABC \sim \Delta DEF$' to tell him which of the six matchings of the vertices is a similarity. Hence, the bracketed sentence for safety's sake.]

1. Since $ABC \leftrightarrow DEF$ is a similarity, it follows from the definition of triangle-similarity that $AB/DE = BC/EF = CA/FD$. Therefore, $2/DE = 7/14 = 6/FD$. Hence, $FD = 12$ and $DE = 4$.

2. $\frac{4}{7} = \frac{10}{EF}$, $EF = \frac{35}{2}$

3.
$$\frac{DE - 3/2}{DE} = \frac{4}{6} = \frac{AC}{AC + 5/2}$$

$4 \cdot DE = 6 \cdot DE - 9$	$4 \cdot AC + 10 = 6 \cdot AC$
$DE = \frac{9}{2}$	$AC = 5$
$AB = \frac{9}{2} - \frac{3}{2} = 3$	$FD = 5 + \frac{5}{2} = \frac{15}{2}$

*

Answer for Part C.

For this exercise there are two cases to be considered, according as $K \in \overrightarrow{NP}$ and $T \in \overrightarrow{QR}$ or $N \in \overline{KP}$ and $Q \in \overline{TR}$. A complete solution includes a proof that these are the only cases. You may wish your students to concentrate on the solution of the first case. If so, omit the bracketed portion of the solution which follows. If you want to make the exercise still simpler, assume that $K \in \overline{NP}$ and $T \in \overline{QR}$ and omit the portions of the third paragraph which are enclosed in bold-face brackets.

2. By hypothesis, $\angle A \cong \angle D$. Also, $\angle P \cong \angle P$. So, by the a.a. similarity theorem, $APC \leftrightarrow DPB$ is a similarity. So, $\frac{AP}{DP} = \frac{PC}{PB}$. Hence, $AP \cdot PB = DP \cdot PC$; that is, $PA \cdot PB = PD \cdot PC$.

3. By hypothesis, $\angle A \cong \angle PBC$. Also, $\angle P \cong \angle P$. So, by the a.a. similarity theorem, $PBC \leftrightarrow PAB$ is a similarity. So, $\Delta PBC \sim \Delta PAB$. Also, $\frac{PB}{PA} = \frac{PC}{PB}$. Hence, $(PB)^2 = PA \cdot PC$.

*

Answers for Part G.

[This part foreshadows Part H.]

1. Since $\angle C$ and $\angle D$ are right angles, ΔABC and ΔFED are right triangles. By Theorem 5-11, $\angle A$ is a complement of $\angle B$, and, by hypothesis, $\angle F$ is a complement of $\angle B$. So, $\angle A \cong \angle F$. Hence, by Theorem 7-3, $\Delta ABC \sim \Delta FED$.

2. Since $\angle C \cong \angle D$ and $\angle A \cong \angle F$, $ABC \leftrightarrow FED$ is a similarity. So, $\frac{AB}{FE} = \frac{BC}{ED} = \frac{CA}{DF}$. That is, $\frac{5}{10} = \frac{3}{ED} = \frac{4}{DF}$. Therefore, $ED = 6$ and $DF = 8$.

*

Answers for Part H.

1. By Theorem 5-11, $\angle ACD$ is a complement of $\angle A$, and $\angle B$ is a complement of $\angle A$. Hence, $\angle ACD \cong \angle B$. So, by Theorem 7-3, $\Delta ADC \sim \Delta CDB$.

2. Since $\angle ADC \cong \angle CDB$ and $\angle ACD \cong \angle B$, $ADC \leftrightarrow CDB$ is a similarity. So, $\frac{AD}{CD} = \frac{DC}{DB}$. That is, $(CD)^2 = AD \cdot DB$. So, $CD = \sqrt{3 \cdot 12} = 6$.

3. Since $\angle A \cong \angle A$, it follows from Theorem 7-3 that $\Delta ACD \sim \Delta ABC$.

4. Since $\angle ADC \cong \angle ACB$ and $\angle A \cong \angle A$, $ACD \leftrightarrow ABC$ is a similarity. So, $\frac{AD}{AC} = \frac{AC}{AB}$. That is, $(AC)^2 = AD \cdot AB$. So, $AC = \sqrt{5 \cdot 20} = 10$.

5. Since $\angle B \cong \angle B$, it follows from Theorem 7-3 that $\Delta BCD \sim \Delta BAC$. [Alternatively, one can prove that triangle-similarity is a transitive relation and a symmetric relation and use Exercises 1 and 3 to do Exercise 5.]

6. Since $\angle CDB \cong \angle ACB$ and $\angle B \cong \angle B$, $BCD \leftrightarrow BAC$ is a similarity. So, $\frac{BC}{BA} = \frac{BD}{BC}$. That is, $(BC)^2 = BD \cdot BA$. But, $BA = AD + DB$. Hence, $BC = \sqrt{9(7 + 9)} = 12$.

7. From Exercise 4, $(AC)^2 = AD \cdot AB$. So, $AC = \sqrt{xc}$.

8. From Exercise 6, $(BC)^2 = BD \cdot BA$. So, $BC = \sqrt{yc}$.

9. From Exercises 7 and 8, $(AC)^2 + (BC)^2 = xc + yc = (x + y)c$. Since $D \in \overleftrightarrow{AB}$, $AD + DB = AB$; that is, $x + y = c$. So, since $AB = c$, $(AC)^2 + (BC)^2 = (AB)^2$.

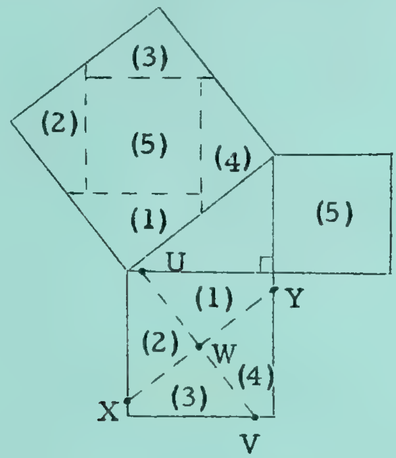
*

A good visual aid for Part H can be made by cutting two enlarged copies of ΔABC from cardboard. Color one of these red. Then, draw $\overleftrightarrow{CD}$ in the other, and cut along $\overleftrightarrow{CD}$ to obtain the second and third triangles [ΔACD and ΔBCD]. Color the larger of these yellow and the other blue. The smaller ones can then be manipulated and compared both with the larger and with each other.

In Exercises 6 and 9 of Part H one assumes, from the figure, that $D \in \overline{AB}$. Since the figure is part of these exercises, this is a legitimate assumption. However, Theorem 7-4 speaks of "segments into which the foot of the altitude divides the hypotenuse", and in the proof of Theorem 7-5 given at the foot of page 6-203 it is assumed that $D \in \overset{\bullet\!-\!\bullet}{AB}$ $[y + x = c]$. In these situations it is essential that we be able to prove that $D \in \overset{\bullet\!-\!\bullet}{AB}$, without any reference to the figure. Here is a proof that if, in ΔABC, $\angle C$ is a right angle then the foot D of the altitude from C belongs to $\overline{AB}$:

> In ΔABC, $\angle C$ is larger than $\angle A$. So, $AB > BC$. In ΔBCD, $\angle CDB$ is larger than $\angle DCB$. So, $BC > BD$. Hence, $AB > BD$. Consequently, $A \notin \overset{\bullet\!-\!\bullet}{BD}$. Similarly, $AB > AD$. So, $B \notin \overset{\bullet\!-\!\bullet}{AD}$. Since $D \in \overleftrightarrow{AB}$, it follows that $D \in \overline{AB}$. [Note that this argument applies more generally than to the case in which $\angle C$ is a right angle. It is sufficient that neither $\angle A$ nor $\angle B$ be larger than $\angle C$.]

Encourage students to carry out the dissection problem suggested at the bottom of page 6-204. Putting the pieces (1) - (5) together to form (6) is an interesting jigsaw puzzle. Here is a solution to the puzzle:



Another area approach to the Pythagorean Theorem is the following:

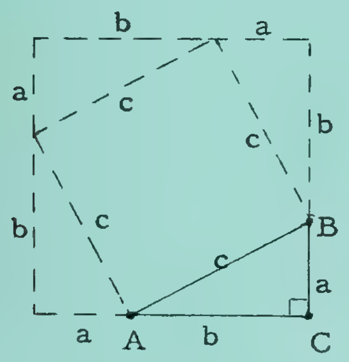


Figure 1.

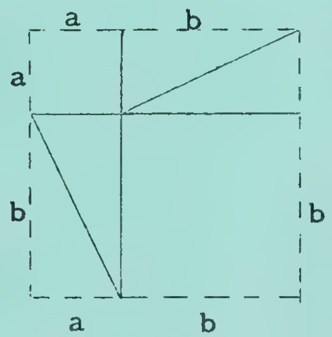


Figure 2.

Figure 1 shows the right triangle ΔABC with a square built on it. The area-measure of the total square is $c^2 + 4t$, where t is the area-measure of one of the right triangles. Figure 2 shows the same total square but dissected in such a way that its area-measure is $a^2 + b^2 + 4t$. So, since $a^2 + b^2 + 4t = c^2 + 4t$, it follows that $a^2 + b^2 = c^2$.

Answers for Part A.

1. $\frac{5}{3} = \frac{4 + x}{4}$, $x = \frac{8}{3}$; $\frac{4}{8/3} = \frac{5}{y}$, $y = \frac{10}{3}$

2. $x^2 = 5^2 + 12^2$, $x = 13$; $5^2 = 12z$, $z = \frac{25}{12}$; $5^2 + \left(\frac{25}{12}\right)^2 = y^2$, $y = \frac{65}{12}$

*

Answers for Part B.

1. 10, $10\sqrt{2}$ 2. 7, $7\sqrt{2}$ 3. $5\sqrt{2}$ 4. $\frac{9\sqrt{2}}{2}$ 5. $s\sqrt{2}$ 6. $\frac{h\sqrt{2}}{2}$

*

Answers for Part C [on page 6-207].

(1) 5 (2) 12 (3) 41 (4) $\sqrt{21}$

(5) $2\sqrt{6}$ (6) $5\sqrt{58}$ (7) $7\sqrt{5}$ (8) no such triangle

*

Answers for Part D [on page 6-207].

1. 16, $8\sqrt{3}$ 2. 170, $85\sqrt{3}$ 3. $2x$, $x\sqrt{3}$

4. 50, $50\sqrt{3}$ 5. $\frac{9}{2}$, $\frac{9}{2}\sqrt{3}$ 6. $\frac{x}{2}$, $\frac{x\sqrt{3}}{2}$

7. 10, 20 8. $\frac{10\sqrt{3}}{3}$, $\frac{20\sqrt{3}}{3}$ 9. $\frac{x\sqrt{3}}{3}$, $\frac{2x\sqrt{3}}{3}$

*

Answers for Part E [on pages 6-207 and 6-208].

1. Since $AC = BD$, and $(AC)^2 = 7^2 + 9^2$, it follows that $AC \cdot BD = 130$.

2. $25\sqrt{2}$ [Use the theorem proved in Exercise 5 of Part B.]

3. $16\sqrt{3}$ [Use the theorem proved in Exercise 9 of Part D.]

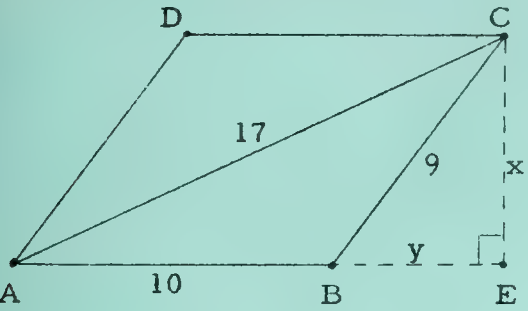
4. $AB = \sqrt{13}$ 5. $13^2 - x^2 = 15^2 - (14 - x)^2$; $AD = 5$, $DC = 9$, $BD = 12$

Correction.

On page 6-208, Exercise 12 should begin:

12. In $\triangle ABC$, if $m(\angle A)$ ...

6.

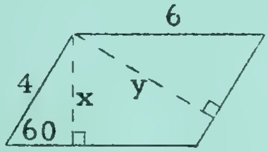


$$y^2 + x^2 = 9^2$$
$$(10 + y)^2 + x^2 = 17^2$$
$$100 + 20y + y^2 + x^2 = 289$$
$$100 + 20y + 81 = 289$$
$$y = 5.4$$
$$x = 7.2$$

7. 10

8. $16\sqrt{3}$ inches

9.



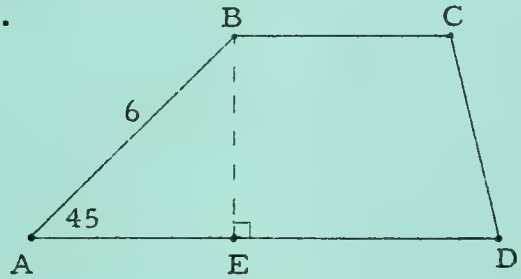
$x = 2\sqrt{3}$

$y = 3\sqrt{3}$

10. $BD = 10\sqrt{3}$

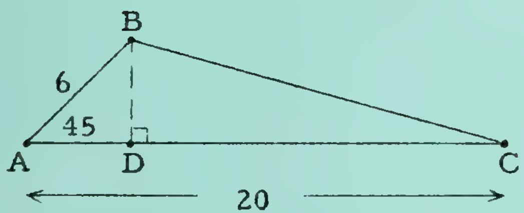
$BC = 20$

11.



$BE = 3\sqrt{2}$

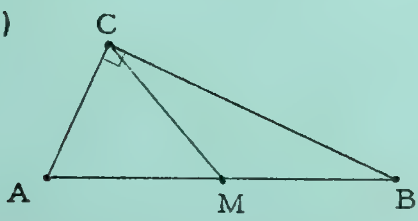
12.



$$BD = AD = 3\sqrt{2}\,,$$
$$DC = 20 - 3\sqrt{2}\,,$$
$$(BC)^2 = (BD)^2 + (DC)^2$$
$$= 18 + 400 - 120\sqrt{2} + 18$$
$$BC = 2\sqrt{109 - 30\sqrt{2}}$$

13. (a) 12 (b) s

(c)



The measure of a mean proportional between $\overleftrightarrow{AM}$ and $\overleftrightarrow{MB}$ is $\sqrt{AM \cdot MB}$. But, $AM = MB$. So, this measure is AM. But, by Theorem 6-28, $CM = AM$. So, $\overleftrightarrow{CM}$ is a mean proportional between $\overleftrightarrow{AM}$ and $\overleftrightarrow{MB}$.

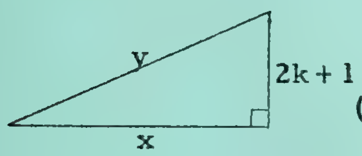
*

Answers for Part F [on pages 6-208 and 6-209].

1.

a	3	5	7	9	11	13	15	17	19	21	23	25
b	4	12	24	40	60	84	112	144	180	220	264	312
c	5	13	25	41	61	85	113	145	181	221	265	313

2.



$$y^2 = x^2 + (2k + 1)^2$$
$$y^2 - x^2 = (2k + 1)^2$$
$$(y + x)(y - x) = (2k + 1)^2$$

We notice from the table that $y - x = 1$.

So, $$y + x = (2k + 1)^2$$

Also, $y + x = (x + 1) + x = 2x + 1$.

So, $$2x + 1 = (2k + 1)^2$$
$$= 4k^2 + 4k + 1$$

Therefore, $$x = 2k^2 + 2k,$$

and $$y = 2k^2 + 2k + 1.$$

3. $(x + 1)^2 - x^2 = 2x + 1 = (\sqrt{2x + 1})^2$

*

line 13. $A'C' = 5$ because, by Theorem 7-5, $A'C' = \sqrt{3^2 + 4^2}$.

line 14. Yes.

line 15. $ABC \leftrightarrow A'B'C'$ is a congruence by s.s.s.

line 16. $\angle B$ and $\angle B'$ are corresponding angles with respect to this matching. So, $\angle B \cong \angle B'$ and, since $\angle B'$ is a right angle, so is $\angle B$. Hence, by definition, ΔABC is a right triangle.

Answers for Part G: 1, 2, 4, 5, 6, 7, 8

*

Answer for Part H.

Since, for each nonzero number k of arithmetic $(3k)^2 + (4k)^2 = (5k)^2$, it follows from Theorem 7-6 that any triangle with sides 3k, 4k, and 5k is a right triangle.

*

Answers for Part ☆I.

1. Since $(p^2 - q^2)^2 + (2pq)^2 = p^4 - 2p^2q^2 + q^4 + 4p^2q^2 = (p^2 + q^2)^2$, it follows from Theorem 7-6 that the triangle whose sides measure $p^2 - q^2$, 2pq, and $p^2 + q^2$ is a right triangle.

2.

p	2	3	4	5	6	7	8	9	3	4
q	1	1	1	1	1	1	1	1	2	2
$p^2 - q^2$	3	8	15	24	35	48	63	80	5	12
$2pq$	4	6	8	10	12	14	16	18	12	16
$p^2 + q^2$	5	10	17	26	37	50	65	82	13	20

*

Answers for Part J.

1. By hypothesis, $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$. So, by Theorem 5-7, $\angle ADE \cong \angle ABC$. Also, $\angle A \cong \angle A$. So, by the a. a. similarity theorem, $ADE \longleftrightarrow ABC$ is a similarity. Hence, $\triangle ADE \sim \triangle ABC$.

2. Suppose that $\triangle ABC \cong \triangle A'B'C'$. Let $ABC \longleftrightarrow A'B'C'$ be a congruence. Then, $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$. So, by the a.a. similarity theorem, $ABC \longleftrightarrow A'B'C'$ is a similarity. Hence, $\triangle ABC \sim \triangle A'B'C'$.

3. Suppose that $\triangle ABC \sim \triangle GHI$ and $\triangle DEF \sim \triangle GHI$. Let $ABC \longleftrightarrow GHI$ and $DEF \longleftrightarrow GHI$ be similarities. Then, $\angle A \cong \angle G$ and $\angle D \cong \angle G$. So, $\angle A \cong \angle D$. Similarly, $\angle B \cong \angle E$. Hence, by the a.a. similarity theorem, $ABC \longleftrightarrow DEF$ is a similarity. So, $\triangle ABC \sim \triangle DEF$.

Correction.

On page 6-215, lines 7b and 6b should read:

... [Steps like (1) - (3) and (5) - (7)]

lines 9, 10. Yes

lines 17-20. $\Delta R'S'T' \cong \Delta RMN$ because, by s.a.s., $R'S'T' \longleftrightarrow RMN$ is a congruence. Now, since $\overline{MN}$ joins the midpoints of sides $\overline{RS}$ and $\overline{RT}$ of ΔRST, $\overline{MN}$ is parallel to $\overline{ST}$. So [by Exercise 1 of Part J on page 6-210], $\Delta RST \sim \Delta RMN$. Since $\Delta R'S'T' \cong \Delta RMN$, it follows from Theorem 7-7 that they are similar. And, since $\Delta RST \sim \Delta RMN$, it follows from Theorem 7-8 that $\Delta R'S'T' \sim \Delta RST$.

*

line 7 on page 6-212. Theorems 7-7 and 7-8

line 9 on page 6-212. Theorem 5-7

*

line 10 on page 6-213. Substitution

*

Query on page 6-214. If $\angle A$ is a right angle, $\overline{CD} = \overline{CA}$ and $\overline{BE} = \overline{BA}$. So, of course, $CD \cdot AB = BE \cdot CA$. Otherwise, whatever the sizes of $\angle A$, $\angle B$, and $\angle C$ [as long as $\angle A$ is not a right angle], C, D, and A are noncollinear and B, E, and A are noncollinear. So, the proof given in the solution continues to apply. Students should sketch other cases [for example, one in which $\angle A$ is obtuse, and one in which $\angle C$ is obtuse] and see that the positions of D and E on $\overleftrightarrow{AB}$ and on $\overleftrightarrow{AC}$, respectively, are irrelevant to the argument.

Note on page 6-214. $3 \cdot 4 = 5x$. So, $x = 2.4$.

Answers for Part A.

1. $\frac{ac}{b}$ 2. $\frac{ac}{a + b}$ 3. $\frac{ab}{b + c}$ 4. $\frac{bd}{a + b + c}$

*

Answer for Part ☆B.

Since $\frac{PD + c}{PD} = \frac{x}{a}$, $\frac{c}{PD} = \frac{x - a}{a}$. So, $PD = \frac{ac}{x - a}$.

Also, $\frac{PD + (c + d)}{PD} = \frac{b}{a}$. So, $PD = \frac{a(c + d)}{b - a}$.

Therefore, $\frac{ac}{x - a} = \frac{a(c + d)}{b - a}$; whence, $\frac{x - a}{c} = \frac{b - a}{c + d}$.

So, $x = a + \frac{c(b - a)}{c + d} = \frac{ad + bc}{c + d}$.

*

Answer for Part C.

Suppose that A is the point of intersection. Measure $\angle BAC$ [with a magnetic compass]. Then, draw a triangle $\Delta A'B'C'$ such that $\angle A' \cong \angle BAC$ and $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{A'C'}$ are proportional to $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. [Of course, we are assuming that the roads are straight.] Then, by the s.a.s. similarity theorem, $A'B'C' \longleftrightarrow ABC$ is a similarity. So, by algebra,

$$BC = \frac{AB}{A'B'} \cdot B'C'.$$

Measure $\overleftrightarrow{B'C'}$, and then compute BC. To find the direction from B to C, measure $\angle B'$. Then, use this measure together with your knowledge of the direction from A to B to compute the direction from B to C.

$$\frac{PA}{PA'} = \frac{AB}{A'B'} \quad \text{and} \quad \frac{PA}{PA'} = \frac{AC}{A'C'}.$$

So, $\frac{AB}{A'B'} = \frac{AC}{A'C'}$. Also, $\angle BAC \cong \angle B'A'C'$. So, by the s.a.s. similarity theorem, $ABC \longleftrightarrow A'B'C'$ is a similarity. So, $\angle ABC \cong \angle A'B'C'$. But, $\angle PBA \cong \angle PB'A'$. So, $\angle PBC \cong \angle PB'C'$; whence, $\overleftrightarrow{BC} \parallel \overleftrightarrow{B'C'}$.

2. As in Exercise 1 of Part D, $GKE \longleftrightarrow HKF$ is a similarity. So, $\frac{GK}{HK} = \frac{KE}{KF}$. Hence, $GK \cdot KF = EK \cdot KH$.

3. By Theorem 5-13, $\angle E \cong \angle B$. So, by the a.a. similarity theorem, $EDF \longleftrightarrow BCA$ is a similarity. Therefore, $\frac{ED}{BC} = \frac{DF}{CA} = \frac{EF}{BA}$. So, $ED = EF \cdot \frac{BC}{BA}$ and $DF = EF \cdot \frac{CA}{BA}$.

[Note that this exercise deals with the problem of the inclined plane. The force needed to move a body along the incline and the weight of the body are proportional to ED and EF, respectively. The moving force is BC/BA of the weight. Clearly, the steeper the incline, the larger the force required to move the body. If you wish to move the body through a vertical distance (BC), you can increase the mechanical advantage by using a longer inclined plane (BA).]

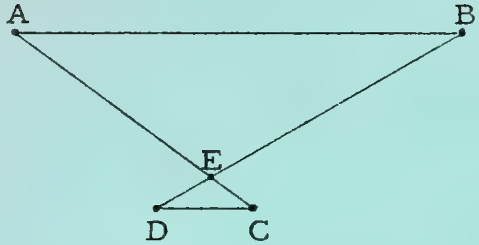
4. Let D be the point of intersection of the three segments. Then, as in Exercise 1 of Part D, $ABD \longleftrightarrow A'B'D$ and $ACD \longleftrightarrow A'C'D$ are similarities. So,

$$\frac{AB}{A'B'} = \frac{AD}{A'D} \quad \text{and} \quad \frac{AC}{A'C'} = \frac{AD}{A'D}.$$

Therefore, $\frac{AB}{A'B'} = \frac{AC}{A'C'}$. So, $\frac{AB}{AC} = \frac{A'B'}{A'C'}$.

Answers for Part D.

1.



ABE $\longleftrightarrow$ CDE is a similarity.

So, $\frac{AB}{CD} = \frac{BE}{DE}$. Hence, $\frac{BE}{DE} = 7$.

Therefore, $\frac{BE + DE}{DE} = 7 + 1$.

So, $BD = 8 \cdot ED$.

[Vary the problem so that $\overleftrightarrow{AC} \cap \overleftrightarrow{BD} = \{E\}$ while $\overline{AC} \cap \overline{BD} = \emptyset$. Then, show that $BD = 6 \cdot ED$.]

2. Yes; ABC $\longleftrightarrow$ FED

3. We recognize this triangle as one which is similar to a 3-4-5 triangle. So, it is a right triangle. Hence, if x is the measure of the altitude to the longest side, $60x = 36 \cdot 48$ [by Example 1 on page 6-214]. So, $x = 144/5$.

4. Since ABC $\longleftrightarrow$ A′B′C′ is a similarity, $\angle B \cong \angle B'$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'}$. Now, $BM = \frac{1}{2} \cdot BC$ and $B'M' = \frac{1}{2} \cdot B'C'$. So, $\frac{BM}{B'M'} = \frac{BC}{B'C'}$. Hence, $\frac{BM}{B'M'} = \frac{AB}{A'B'}$. So, by the s.a.s. similarity theorem, ABM $\longleftrightarrow$ A′B′M′ is a similarity. Hence, $\frac{AM}{A'M'} = \frac{AB}{A'B'}$.

*

Answers for Part E.

1. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$, $\angle PAB \cong \angle PA'B'$. So, since $\angle P \cong \angle P$, it follows from the a.a. similarity theorem that PAB $\longleftrightarrow$ PA′B′ is a similarity. Similarly, PAC $\longleftrightarrow$ PA′C′ is a similarity. Therefore,

Answers for Part ☆F.

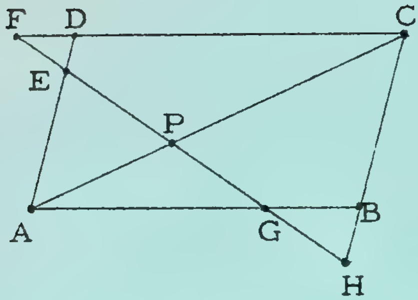
1. $\frac{r}{p} = \frac{n}{m + n}$ 2. $\frac{r}{q} = \frac{m}{m + n}$

3. $\frac{r}{p} + \frac{r}{q} = \frac{n}{m + n} + \frac{m}{m + n} = \frac{n + m}{m + n} = 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$

4. Draw $\overline{AB}$ so that its length is 3 inches, and draw $\overline{CF}$ so that its length is 5 inches. Then, regardless of the length of $\overline{AC}$, $\overline{ED}$ will be 15/8 inches long. Moreover, it is not necessary that $\overline{AB}$, $\overline{DE}$, and $\overline{CF}$ be perpendicular to $\overleftrightarrow{AC}$. They need only be parallel to each other. This latter point is made in Exercise 5.

5. By Exercise 3, $\frac{1}{FE} = \frac{1}{x} + \frac{1}{y}$ and $\frac{1}{EG} = \frac{1}{y} + \frac{1}{x}$. So, $FG = 2 \cdot FE$ and, since $FE = \frac{xy}{x + y}$, $FG = \frac{2xy}{x + y}$.

[Notice that $\overline{FG}$ is the segment with end points on the legs of the trapezoid, parallel to the bases, and containing the point of intersection of the diagonals. If the distance between the bases is increased, the distance between E and $\overleftrightarrow{AB}$ is increased; but, no change takes place in the length of $\overline{FG}$. Another parallel segment whose length does not change when the distance between the bases is changed is the median of the trapezoid. Notice that the median's measure is the arithmetic mean of the measures of the bases, and that $\overline{FG}$'s measure is the harmonic mean of the measures of the bases. An interesting problem is to find the parallel segment whose measure is the geometric mean of the measures of the bases. The arithmetic mean segment contains the midpoints of the legs, and the harmonic mean segment contains the intersection of the diagonals. What special property does the geometric mean segment have?]

10. 

<u>Hypothesis:</u> quadrilateral ABCD is a parallelogram,

$D \in \overline{FC}$,

$B \in \overline{CH}$

<u>Conclusion:</u> $PE \cdot PF = PG \cdot PH$

*

Answers for Quiz.

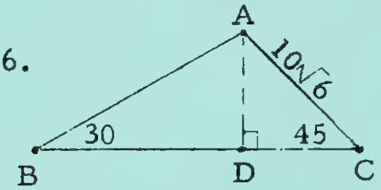
1. 63

2. $ABC \longleftrightarrow ADB$ or $ABC \longleftrightarrow DAB$

3. 20/3

4. 30

5. 6.5

6. 

$AD = 10\sqrt{3}$,

$AB = 20\sqrt{3}$

7. 1/2

8. 15; 20

9. Each side of the smaller triangle is half as long as the parallel side of the larger triangle. So, the triangles are similar by the s.s.s. similarity theorem.

10. $APE \longleftrightarrow CPH$ is a similarity. So, $\frac{AP}{CP} = \frac{PE}{PH}$.

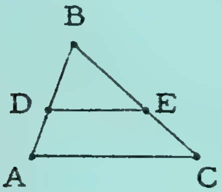
$APG \longleftrightarrow CPF$ is a similarity. So, $\frac{AP}{CP} = \frac{PG}{PF}$.

Therefore, $\frac{PE}{PH} = \frac{PG}{PF}$. So, $PE \cdot PF = PG \cdot PH$.

Quiz.

1. Suppose that $ABC \longleftrightarrow DEF$ is a similarity. If $AB = 3 \cdot DE$ and the perimeter of $\triangle DEF$ is 21, what is the perimeter of $\triangle ABC$?

2. Suppose that $\triangle ABC$ is isosceles with vertex angle at C. If D is a point in $\overline{AC}$ such that $AB = BD$, give a matching of the vertices of $\triangle ABC$ with those of $\triangle ABD$ which is a similarity.

3. 

If $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$, $AD = 2$, $DB = 3$, and $DE = 4$, then $AC =$ __?__

4. If the measure of an altitude of an equilateral triangle is $5\sqrt{3}$, what is the perimeter of the triangle?

5. If the measures of two legs of a right triangle are 5 and 12, what is the measure of the median from the vertex of the right angle?

6. Suppose that, in $\triangle ABC$, $\angle A$ is an angle of $105°$, $\angle C$ is an angle of $45°$, and $AC = 10\sqrt{6}$. Find the measure of $\overleftrightarrow{AB}$.

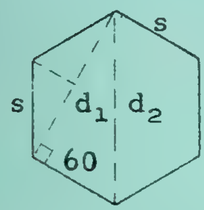
7. Suppose that quadrilateral ABCD is a parallelogram and that M is the midpoint of $\overleftrightarrow{AB}$. If $\overline{DM} \cap \overline{AC} = \{P\}$, what is the ratio of $\overleftrightarrow{AP}$ to $\overleftrightarrow{PC}$?

8. Suppose that $\triangle ABC$ is a right triangle with $\angle C$ the right angle. If $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ at D, $AD = 9$, and $BD = 16$, then $AC =$ __?__ and $BC =$ __?__.

9. Prove that the triangle whose vertices are the midpoints of the sides of a given triangle is similar to the given triangle.

Answers for Exploration Exercises.

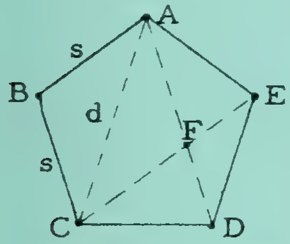
A. By Exercise 5 of Part B on page 6-206, $d = s\sqrt{2}$.

B. 

1. By Exercise 6 of Part D on page 6-207, $d_1 = 2\left(\frac{s}{2}\sqrt{3}\right) = s\sqrt{3}$.

2. By the same exercise, $d_2 = 2s$.

C. Parts A and B are readily solved, since the ratios of the sides of 45-45-90 and 30-60-90 triangles are well-known. A similar procedure for finding the measure of the diameter of a regular pentagon would require finding, first, the ratios of the sides of a 36-54-90 triangle. The purpose of this exercise, indeed, is to point out the utility of the ratios of the sides of a right triangle [that is, the trigonometric ratios] by placing students in a position where they will wish to know them. It is not likely that many students will find the formula asked for, nor is it worth much of their time to search for it. However, here is a simple derivation of the formula from theorems about isosceles triangles and similarity of triangles: Since each angle of a regular pentagon is an angle of 108°, the base angles of the isosceles triangle ΔABC [see figure], whose base has measure d and whose legs have measure s, are angles of 36°. Since $108 = 36 \cdot 3$, the two diagonals from a vertex of a regular pentagon trisect the angle at that vertex. It now follows, by a. s. a., that $ABC \longleftrightarrow AFC$ is a congruence. Hence, by the s. a. s. similarity theorem, $ACD \longleftrightarrow CDF$ is a similarity. Consequently, $AC/CD = CD/DF$--that is, $d/s = s/(d - s)$. Hence, $d^2 - sd - s^2 = 0$, and $d = \frac{1}{2}(s + \sqrt{s^2 + 4s^2})$ or $d = \frac{1}{2}(s - \sqrt{s^2 + 4s^2})$. Since d is a number of arithmetic, only the former makes sense. Hence, $d = \frac{1}{2}s(1 + \sqrt{5})$.



☆ D. $d_2 = s + 2s\frac{\sqrt{2}}{2} = s(1 + \sqrt{2})$; $\quad d_3 = \sqrt{d_2^{\,2} + s^2} = s\sqrt{4 + 2\sqrt{2}}$;

$d_1 = d_3\frac{\sqrt{2}}{2} = s\sqrt{2 + \sqrt{2}}$

Approximations asked for in last three paragraphs on page 6-222.

The measure of a diagonal of a regular pentagon whose side-measure is 10 is 16.18 correct to the nearest 0.01. [In fact, $16.179 \le d < 16.181$. So, $|d - 16.18| \le 0.001$.]

The corresponding result for a regular pentagon whose side-measure is 8 is 12.94. [In fact, $12.9432 \le d < 12.9448$. So, $|d - 12.944| \le 0.0008$.]

If the measure of a diagonal of a regular pentagon is 162 then the side-measure is approximately 100, and the perimeter is approximately 500.

An approximation to ST correct to the nearest 0.01 is 16.38. [In fact, $16.383 \le ST < 16.385$. So, $|ST - 16.384| \le 0.001$.]

*

Since each two congruent acute angles have the same sine ratio, we can think of a sine ratio as pertaining to the class of all angles of a given size, rather than to individual angles. So, for example, if $\angle A$ is an angle of 36°, we can speak of the sine ratio of 36°, rather than the sine ratio of $\angle A$. Each of the expressions 'sin 36°' and 'sin $\angle A$' is useful in various contexts, and, at this level, it will usually not be necessary to distinguish between the somewhat different meanings which 'sin' has in the two expressions. However, if one wishes to talk about trigonometric <u>functions</u>, the 'sin' in 'sin 36°' names a function whose domain is the set of congruence classes of acute angles, while the 'sin' in 'sin $\angle A$' names a function whose domain is the set of acute angles. One argument of the first function is, for example, the class of all 36°-angles, while one argument of the second function is some particular 36°-angle. In this unit we make little use of the function concept, and both 'sin 36°' and 'sin $\angle A$' [when $m(\angle A) = 36$] can be thought of merely as new numerals for a certain number which is, approximately, 0.5878.

*

Answers for Part A [on page 6-223].

1. 0.6691	2. 5.6713	3. 0.9659	4. 1	5. 0.7071
6. 0.5	7. 0.866	8. 0.866	9. 0.0175	10. 0.7431
11. 0.7431	12. 0.3249	13. 0.3502	14. 0.6177	15. 0.4970

Exercises 7 and 8 and Exercises 10 and 11 may call students' attention to the theorem according to which if $\angle A$ and $\angle B$ are complementary then

cos $\angle A = \sin \angle B$. This theorem follows at once from Theorem 5-11 and the definitions on page 6-223.

The answers given above for Exercises 13, 14, and 15 have been obtained by linear interpolation in the table on page 6-231. Alternative acceptable answers are:

13. 0.3420 [or: 0.3584] 14. 0.6249 15. 0.5

The word 'approximations' covers a multitude of alternatives. We do not attempt to teach the method of linear interpolation in the text, since we are not, here, particularly interested in refined computations. However, as you know, the method of linear interpolation is a good application of Theorem 7-3, and you may wish to touch on it in your class.

*

Note that, since there are [in this treatment of geometry] no angles of 0°, 'sin 0°', 'cos 0°' and 'tan 0°' are meaningless. Students who have heard otherwise should be told that the conventional values [0 for the first and third, and 1 for the second] are useful in other contexts than this, and that they will be made sensible in a later unit. Similar remarks pertain to 'sin 90°' and 'cos 90°', but, of course, 'tan 90°' is never defined.

*

Answers for Part B [on page 6-224].

1. 64 2. 78 3. 34 4. 56 5. 18

6. 72 7. 28 8. 44 [or: 43.7] 9. 70 [or: 69.8]

10. 45 11. 30 12. 30 13. The solution set is $\{\beta : 0 < \beta < 90\}$.

Exercises 7, 8, and 9 do not quite fit the instructions for Part B. A root of the equation '$\sin \angle A = 0.4695$' is, strictly, an <u>angle</u> whose sine ratio is 0.4695, and each such angle is a root of the equation. However, the conclusion which students will want to draw, later, from '$\sin \angle A = 0.4695$' is '$\angle A$ is an angle of approximately 28°'.

An alternative form for answering Exercise 7 is suggested by the introduction of the symbol '$\doteq$' in Example 1 on page 6-224: $m(\angle A) \doteq 28$

Note well that, as indicated in the sample for Part B, although the tabular entry for tan 51° is '1.2349', all this tells us is that tan 51° is 1.2349 correct to the nearest 0.0001. So, 51 is only an approximation to the root of '$\tan x° = 1.2349$'. In contrast, the solutions for Exercises 10, 11, and 12 are "exact".

Answers for Part C.

1. $\beta = 48$; $a \doteq 13.4$; $b \doteq 14.9$

2. $\alpha = 62$; $m \doteq 5.3$; $s \doteq 11.3$

3. $\alpha = 40$; $t \doteq 22.5$; $u \doteq 26.8$

4. $\beta = 20$; $g \doteq 9.1$; $k \doteq 26.6$

5. $c = 5$; $\alpha \doteq 36.9$; $\beta = 53.1$

6. $s = 10\sqrt{3} \doteq 17.3$; $\alpha = 60$; $\beta = 30$

7. $\alpha = 56$; $x = y \doteq 32$

8. $\alpha = \beta = 65$; $c \doteq 42.3$

*

Answers for Part D [on page 6-226].

1. 46.6 2. 83.9 3. 37.3 [$z = y - x$]

[Be sure that all three of these exercises are assigned since Exercise 3 capitalizes on Exercises 1 and 2.]

*

Answers for Part E [on pages 6-226 and 6-227].

1. [Explanation: a.a.s. congruence theorem]
   $BD = 10 \cos 70° \doteq 3.42$;
   $AD = 10 \sin 70° \doteq 9.4$, $m(\angle DAC) = 62$, $DC \doteq 9.4 \tan 62° \doteq 17.67$;
   $BC = BD + DC \doteq 3.42 + 17.67 \doteq 21.1$

2. (a) 17.4 (b) 20.5 (c) 12.3 (d) 17 [Note that $\angle R$ is a right angle.]

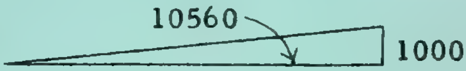
3. (a) $\beta \doteq 24$; $\alpha \doteq 46$; $x \doteq 18.3$ (b) $\beta \doteq 72$; $\alpha \doteq 33$; $x \doteq 14.2$

4. $AB \doteq 100 \cos 40° + 80 \cos 53°$, or $AB \doteq 100 \cos 40° - 80 \cos 53°$. So, $AB \doteq 125$ or $AB \doteq 28$.

   [Interpolation gives 53.5 in place of 53, and 124 and 29 as approximations to AB.]

Answers for Part F [on pages 6-228, 6-229, and 6-230].

1. This exercise has two interpretations. 'travelling 2 miles' might mean actual distance traveled [hypotenuse] or ground distance [horizontal leg].



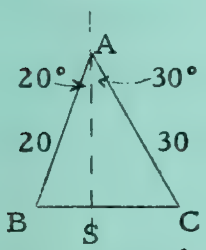
In the first case, the angle of climb is an angle of approximately 5.43°. In the second case, it is an angle of approximately 5.41°. Within the limits of accuracy which we are using, the answer is the same in both cases. Students should discover this by solving the exercise both ways.

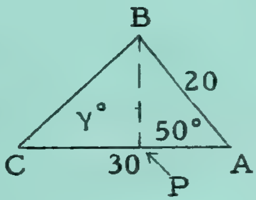
2. $x = 500 \tan 38° \doteq 390.7$. So, the monument is about 391 feet tall.

3. $\tan (90 - \alpha)° = 2.47$. So, the measure of the angle of elevation is approximately 22.

4. $x = 120 (\tan 75° - \tan 70°) \doteq 118.152$. So, the distance is about 118 feet.

5.





$$\tan \gamma° = \frac{BP}{30 - AP} = \frac{20 \sin 50°}{30 - 20 \cos 50°}$$

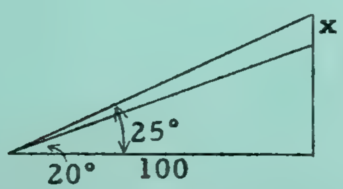
$$BC = \frac{BP}{\sin \gamma°} = \frac{20 \sin 50°}{\sin \gamma°}$$

$20 \sin 50° \doteq 15.32$; $20 \cos 50° \doteq 12.96$; $\tan \gamma° \doteq 0.8991$; $\gamma \doteq 42$;

$\sin 42° \doteq 0.6691$; $BC \doteq \dfrac{15.32}{0.6691} \doteq 22.9$. So, the ships are about

23 miles apart.

6.

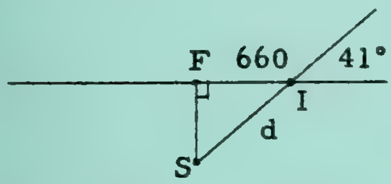


$x = 100(\tan 25° - \tan 20°) \doteq 10.23$

So, the antenna is about

10 feet 3 inches tall.

7. $3000 \tan 57° \doteq 4620$. So, the cloud is about 4600 feet high.

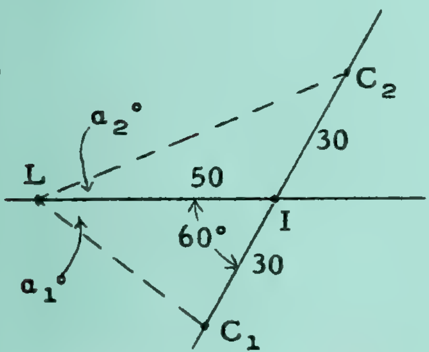
8. 

$d = \frac{660}{\cos 41°} \doteq 875$

45 miles per hour = 66 feet per second

60 miles per hour = 88 feet per second

Since $\frac{660}{66} = 10 > \frac{875}{88}$, the second car will reach the intersection before the first car does. [They'll probably collide.]

9. 

As indicated in the figure, there are two locations possible for the car. The distance between the locomotive and the car is either LC_1 yards or LC_2 yards. If P_1 and P_2 are the feet of the perpendiculars to $\overleftrightarrow{LI}$ from C_1 and C_2, respectively, then $IP_1 = IP_2 = 15$ and $C_1P_1 = C_2P_2 = 15\sqrt{3} \doteq 26$. So, $LP_1 = 35$ and $LP_2 = 65$. Hence, $\tan a_1° \doteq \frac{26}{35} \doteq 0.7429$ and $\tan a_2° \doteq \frac{26}{65} = 0.4$. Consequently, $a_1 \doteq 37$ and $a_2 \doteq 22$. So, $LC_1 = \frac{LP}{\cos a_1°} \doteq \frac{35}{0.7986} \doteq 43.8$ and $LC_2 \doteq \frac{65}{0.9272} \doteq 70.1$. Hence, the distance between the locomotive and the car is either about 44 yards or about 70 yards.

Answers for Part G.

1.

	5°	85°	12°	78°	89°	1°
sin	.0872	.9962	.2079	.9781	.9998	.0175
cos	.9962	.0872	.9781	.2079	.0175	.9998

2. For each a such that $0 < a < 90$, there is a right triangle ΔABC such that $m(\angle A) = a$ and $m(\angle B) = 90 - a$. By definition,

$$\sin a° = \frac{BC}{AC} = \cos(90 - a)°.$$

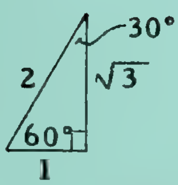
3. With the notation of Exercise 2, for each a such that $0 < a < 90$,

$$\frac{\sin a°}{\cos a°} = \frac{BC/AC}{AB/AC} = \frac{BC}{AB} = \tan a°.$$

4. With the notation of Exercise 2, for each a such that $0 < a < 90$,

$$\begin{aligned} &[\sin a°]^2 + [\cos a°]^2 \\ &= [\frac{BC}{AC}]^2 + [\frac{AB}{AC}]^2 \\ &= \frac{(BC)^2 + (AB)^2}{(AC)^2} \\ &= 1, \text{ by the Pythagorean Theorem.} \end{aligned}$$

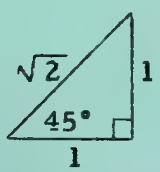
5.



$\sin 30° = \frac{1}{2}$ $\qquad$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\qquad$ $\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{\sqrt{3}}{3}$ $\qquad$ $\tan 60° = \sqrt{3}$



$\sin 45° = \frac{\sqrt{2}}{2} = \cos 45°$

$\tan 45° = 1$

*

Quiz.

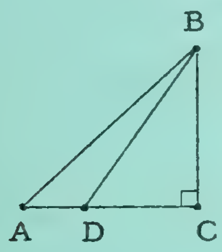
1. Suppose that ΔABC is a right triangle with $\overleftrightarrow{AB}$ as hypotenuse. If $AC = 5$ and $BC = 12$, what are $\sin \angle A$, $\cos \angle A$, and $\tan \angle A$?

2. If $0 < x < 90$ and $\sin x^\circ = \cos x^\circ$ then $x =$ __?__

3. Suppose that, in ΔABC, $\angle C$ is a right angle, $\angle B$ is an angle of 54°, and $BC = 8$. Find AC correct to the nearest unit.

4. Suppose that quadrilateral $ABCD$ is a rectangle, AC is 11, and AB is 9. Find $m(\angle CAB)$ correct to the nearest degree.

5.



Given: $m(\angle A) = 43$,
$m(\angle BDC) = 54$,
$m(\angle C) = 90$,
$DC = 170$

Find: AB, correct to nearest unit

6. Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ are the bases of trapezoid $ABCD$. If $AB > CD$, $\angle B$ is a right angle, $\angle A$ is an angle of 67°, $AD = 8$, and $DC = 12$, find the distance between the bases, and AB, each correct to the nearest unit.

*

Answers for Quiz.

1. $\sin \angle A = \frac{12}{13}$; $\cos \angle A = \frac{5}{13}$; $\tan \angle A = \frac{12}{5}$

2. 45

3. $AC = 8 \cdot \tan 54^\circ \doteq 8 \cdot 1.3764 \doteq 11$

4. $\cos \angle CAB = \frac{9}{11} \doteq 0.8182$; $m(\angle CAB) \doteq 35$

5. $\sin 43^\circ = \frac{BC}{AB}$; $AB = \frac{BC}{\sin 43^\circ} = \frac{DC \cdot \tan 54^\circ}{\sin 43^\circ} = \frac{170 \cdot \tan 54^\circ}{\sin 43^\circ}$

$$\doteq \frac{170 \cdot 1.3764}{0.682} = \frac{233.988}{0.682} \doteq 343$$

6. distance between bases $= 8 \cdot \sin 67^\circ \doteq 8 \cdot 0.9205 \doteq 7$;
$AB = 12 + 8 \cdot \cos 67^\circ \doteq 12 + 8 \cdot 0.3907 \doteq 15$

On pages 6-29 and 6-30, in the introductory remarks to section 6.01, it was pointed out that, "in real life", given a system of linear measure, one can obtain another such system by replacing the given measures by numbers proportional to them. That this is also the case for the abstract segment-measures dealt with in our axioms can be seen by examining Axioms A-H. Of these, only Axioms A, B, D, and H refer to measures of segments. Suppose, for the moment, that 'k' denotes some nonzero number of arithmetic, and define, for each X and Y,

$$d(\overline{XY}) = k \cdot XY.$$

If, now, one replaces, in Axioms A, B, C, and H,

'XY' by '$d(\overline{XY})$', 'YZ' by '$d(\overline{YZ})$', etc.,

each of the resulting statements is [by virtue of the definition] equivalent, by algebra, to the corresponding axiom. The fact that each is derivable from the corresponding axiom means that, given any theorem, the statement obtained from it by making the substitutions indicated above is also a theorem.

As an application of this relativity of segment-measure, if O and U are two points and we define, for each X and Y, $d(\overline{XY}) = XY/OU$, then d is a segment-measure function and, whatever system of measures expressions like 'AB' refer to, $d(\overline{OU}) = 1$. In other words, given any nondegenerate segment $\overline{OU}$, there is a segment-measure function with respect to which $\overline{OU}$ is a unit segment. [As a matter of fact, there is only one such measure function; but, this is rather difficult to prove.]

On TC[6-44, 45, 46]b we showed how, in terms of a segment-measure function, one can assign a coordinate to each point <u>of a line</u>. The procedure described on page 6-232 for assigning a pair of coordinates to <u>every</u> point amounts to using the earlier procedure to assign a coordinate to each point of each of two perpendicular lines, in terms of the segment-measure function d defined in the preceding paragraph. Then, one defines the corresponding coordinate pair of each point as the pair of coordinates of its projections on the two lines. That this procedure assigns a unique pair of coordinates to each point follows from the uniqueness of the perpendicular to a line from a point. That each pair of real numbers is the coordinate pair of a unique point follows from the theorem that lines which are, respectively, perpendicular to two perpendicular lines are perpendicular to each other and, so, intersect in a unique point.

*

Note that the introduction of a coordinate system does not, by some magic, turn a point into an ordered pair of real numbers. The number plane--whose points are ordered pairs of real numbers--is a unique plane. The discussion of coordinate systems shows how this one plane can be mapped, in many ways, on any given plane. It turns out that such mappings can be used in proving theorems about subsets [geometric figures] of the given plane. This is because the introduction of coordinates opens the way for the use of algebraic techniques based on the properties of the real number system. Since algebraic techniques are, in some ways, more simple and powerful than geometric techniques, this is sometimes an advantage. [However, the advantage often lies with the "synthetic" rather than with the "analytic" approach.]

*

On measures. --Throughout our development of geometry, our fundamental assumption [aside from the assumptions stated in the Introduction Axioms] has been that there is a measure function for segments, and a measure function for angles, which satisfy Axioms A-H. In contrast to this apparent preoccupation with measures, the classical development of euclidean geometry says nothing about measures. There, instead of the concept of measures, one deals with concepts of congruence, and of ratio, of segments and angles. In our treatment, congruency of segments, or of angles means equality of their measures, and, when we speak of ratios of segments, or of angles, this is merely another way of referring to the ratios of their measures. Note that, for us, congruency can also be defined in terms of measure-ratios. For congruency means equality of measures and [setting aside the trivial case of degenerate segments] to say that two segments, or two angles, have the same measure is merely to say that the ratio of their measures is 1.

The question now arises, is our geometry essentially different from Euclid's? More specifically, do we in terms of our measure functions, have theorems which are not merely restatements of theorems of classical euclidean geometry? The answer can be discovered by examining Axioms A-H. If, to set aside formally the case of degenerate segments, we adopt a consequence of Introduction Axioms and Axioms A and B:

$$\forall_X \forall_Y\ [XY = 0 \text{ if and only if } X = Y]$$

as an axiom then Axiom A can be replaced by:

$$\forall_X \forall_Y \forall_{Z \neq X} \text{ if } Y \in \overleftrightarrow{XZ} \text{ then } (XY/XZ) + (YZ/XZ) = 1$$

in which measures occur only in ratios. For, from these two [and

Introduction Axioms] it is easy to infer Axiom A. Now, Axiom B can be treated in a similar manner. And Axiom C can be replaced by:

$$\forall_X \forall_{Y \neq X} \forall_{x > 0} \text{ there is one and only one point Z such that } Z \in \overrightarrow{XY} \text{ and } XZ/XY = x$$

For, the point $Z \in \overrightarrow{AB}$ such that $AZ = c$ is the point $Z \in \overrightarrow{AB}$ such that $AZ/AB = c/AB$. [And, assuming that $B \neq A$, $c > 0$ if and only if $c/AB > 0$.] So, Axiom C really refers, not to measures per se, but only to measure-ratios. The only other axiom which refers to measures of segments is Axiom H, and, here, only equality of measures is in question. As we have seen, equality of measures amounts to one-ness of their ratio. So, Axiom H can be rewritten in such a way that segment-measures, and also angle-measures, occur only in ratios. Clearly, since Axiom D implies that angle measures are different from 0, Axiom F can be rewritten so that measures occur only in ratios. This leaves us with Axioms D, E, and G, which deal only with angle-measure. All axioms which refer to segment-measure can be replaced by axioms which refer to measure-ratios. Consequently, all our theorems which deal only with segment-measure are [if we exclude degenerate segments] essentially euclidean.

When we include angle-measure, the situation is slightly different, but not significantly so. Axiom G restricts us to <u>degree</u>-measure for angles, and allows us, for example, to prove Theorem 2-1, which is foreign to Euclid's own development of geometry. In place of Theorem 2-1, Euclid had Theorem 2-2. [As a matter of fact, this was one of his postulates.] Still, this difference is a minor one. For, suppose that we choose any nonzero number of arithmetic, k, and define, for each three noncollinear points X, Y, and Z,

$$^{*}m(\angle XYZ) = \frac{k}{180} \cdot {}^{\circ}m(\angle XYZ).$$

Then, replacing, in Axioms D, E, F, and G, '180' by 'k', '$^{\circ}m(\angle XYZ)$' by '$^{*}m(\angle XYZ)$', etc., we obtain statements equivalent to the original axioms. Using these new axioms we should derive the same theorems as before, except that, for example, Theorem 2-1 would read 'An angle is a right angle if and only if its *-measure is k/2.'. Other extra-euclidean theorems would undergo similar modifications. The fact that replacing degree-measures of angles by numbers proportional to them makes no significant change in our axioms shows that, except for the rather fortuitous singling out of the number 180, our axioms prescribe only properties of angle-measure which have to do with measure-ratios.

Consequently, in spite of the introduction of measure, in itself foreign to Euclid's geometry, our geometry is richer than his only to the extent

that it contains theorems which [like Theorems 1-1 and 1-2] deal with degenerate segments and theorems which [like Theorem 2-1] specify the degree-measures of angles of particular kinds.

[Theorems like Theorem 5-11 and Theorem 6-30 are replaced in Euclid's treatment by theorems dealing with sums of angles. Euclid's theorem corresponding with Theorem 5-11 is, when translated from the Greek, and slightly paraphrased: A sum of the angles of a triangle is a sum of two right angles. Similarly, in place of speaking of a 30°-angle, Euclid would refer to an angle which is a third of a right angle. As for us, so for Euclid, a right angle is one which is congruent to one of its supplements. Angles are <u>supplementary</u> if they are congruent, respectively, to adjacent supplementary angles; and adjacent supplementary angles are adjacent angles whose noncommon sides are collinear. (Incidentally, Euclid did not countenance "straight angles".)]

*

Answers for questions (1) - (4) on page 6-233.

(1) 1 and 0 (2) 0 and 0 (3) positive; neither (4) Don't know

The additional information about P enables one to conclude that $x(P) = 2$ and $y(P) = -3$.

As pointed out in the COMMENTARY for page 6-232, changing the measure function has no effect on coordinates of points. So, using the measure function m for which $m(\overline{OU}) = 8$, one still finds that $x(P) = 2$ and $y(P) = -3$. And this continues to be the case if one uses, as measure function, one for which the measure of $\overline{OU}$ is 2. In this case, the measures of $\overline{OL}$ and $\overline{OM}$ are 4 and 6, respectively, but, again, $x(P) = 2$ and $y(P) = -3$.

The choice of a different unit point will result in a change in the assignment of coordinates. If the midpoint W of $\overline{OU}$ is chosen as unit point, then the coordinates of each point are doubled. So, in this case, $x(P) = 4$ and $y(P) = -6$ [independently of the measures of $\overline{OU}$ and $\overline{OW}$]. Doubling the unit segment--that is, choosing for unit point the point V such that U is the midpoint of $\overline{OV}$--halves the coordinates of all points.

Answers for Part A.

1.

	P_1	P_2	P_3	P_4	P_5	A	B	C	D	E
x(P)	3	−1	−3	−3	4	1	2	3	4	5
y(P)	5	2	0	−1	−3	0	0	0	0	0

2.

x(P)	1	$-\frac{1}{3}$	−1	−1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$
y(P)	$\frac{5}{3}$	$\frac{2}{3}$	0	$-\frac{1}{3}$	−1	0	0	0	0	0

3.

x(P)	3	−1	−3	−3	4	1	2	3	4	5
y(P)	5	2	0	−1	−3	0	0	0	0	0

4.

x(P)	3	−1	−3	−3	4	1	2	3	4	5
y(P)	5	2	0	−1	−3	0	0	0	0	0

*

Answers for Part B [on page 6-235].

1.

	P_3	0	A	B	C	D	E
$m(\overleftrightarrow{OP})$	9	0	3	6	9	12	15
$\lvert x(P) \rvert$	3	0	1	2	3	4	5

2.

	P_3	0	A	B	C	D	E
$d(\overleftrightarrow{OP})$	3	0	1	2	3	4	5

3. (a) 2, 2 (b) 1, 1 (c) 3, 3 (d) 3, 3

(e) 4, 4 (f) 4, 4 (g) 3, 3 (h) 3, 3

Correction.

On page 6-238, line 9b should read:

$$d(\overleftrightarrow{PQ}) = \sqrt{|x(Q) - x(P)|^2 + |y(Q) - y(P)|^2}$$

Answers for Part C [on pages 6-236 and 6-237].

2. (a) 1, 1, 0 (b) 3, 3, 0 (c) 4, 4, 0 (d) 2, 2, 0

(e) 3, 3, 0 (f) 4, 4, 0 (g) 3, 0, 3 (h) 6, 6, 0

(i) 4, 4, 0 (j) 2, 0, 2 (k) 3, 0, 3 (ℓ) 2, 0, 2

(m) 3, 0, 3 (n) 0, 0, 0 (o) 0 (p) 8

3. (b) 5 (c) $\sqrt{5}$, 2, 1 (d) 6, 6, 0 (e) $3\sqrt{2}$ (f) 4 (g) $\sqrt{17}$

4. (a) the set consisting of the point with coordinates (2, 0)

(b) the line perpendicular to the x-axis at the point with coordinates (2, 0)

(c) the line perpendicular to the x-axis at the point with coordinates (−3, 0)

(d) the line perpendicular to the y-axis at the point with coordinates (0, 3)

(e) the x-axis

5. (a) the line perpendicular to the x-axis and containing the point P

(b) the line perpendicular to the y-axis and containing the point P

(c) (1) perpendicular (2) parallel

(3) $x(Z) = x(P)$ (4) $\{Z: x(Z) = x(P)\}$

*

Students will need cross-section paper for the exercises from page 6-239 through page 6-245 and from page 6-257 through page 6-268.

Answers for Part A.

$d(\overline{AB})$: $2\sqrt{2}$, 5, 5, 13, $3\sqrt{2}$, 10, 13, $\sqrt{(m-2)^2 + (n-3)^2}$

*

Answers for Part B.

1. $d(\overline{AB}) = 5$, $d(\overline{BC}) = 5$, $d(\overline{CA}) = 5\sqrt{2}$.

   So, since $d(\overline{AB}) = d(\overline{BC})$, ΔABC is isosceles with vertex angle at B.

2. perimeter of $\Delta ABC = 5 + 5 + 5\sqrt{2} = 10 + 5\sqrt{2}$

*

Answers for Part C.

2. $d(\overline{AB}) = 15$, $d(\overline{BC}) = 10$, $d(\overline{AC}) = 5$.

   So, since $15 + 10 \neq 5$, it follows from Axiom A that $B \notin \overline{AC}$.

3. Since $d(\overline{AC}) + d(\overline{CB}) = 5 + 10 = 15 = d(\overline{AB})$, it follows from Theorem 1-4 that $C \in \overline{AB}$. So, A, B, and C are collinear. That is, $B \in \overleftrightarrow{AC}$.

4. Yes, because $10 > 5$. [Ask students to graph the set mentioned in this exercise. It is the set of all points "outside" the circle with center C and radius $d(\overline{AC})$.]

Correction. On page 6-240, line 11 should read:
--- of $\overleftrightarrow{AB}$ [A(2, 4), B(−1, 7)].
↑

Answers for Part D.

2. $d(\overline{AB}) = 2\sqrt{5}$, $d(\overline{AC}) = 4\sqrt{5}$, $d(\overline{BC}) = 10$.
So, since $(2\sqrt{5})^2 + (4\sqrt{5})^2 = 10^2$, it follows from Theorem 7-6 that ΔABC is a right triangle with right angle at A.

3. Let M be the midpoint of $\overline{BC}$. Then, the coordinates of M are (2, 1). So, $d(\overline{AM}) = \sqrt{9 + 16} = 5 = \frac{1}{2} \cdot d(\overline{BC})$. So, since $\overline{AM}$ is the median to $\overline{BC}$, it follows from Theorem 6-28 that ΔABC is a right triangle with right angle at A.

4. $d(\overline{DE}) = \sqrt{26}$, $d(\overline{EF}) = \sqrt{650}$, $d(\overline{FG}) = \sqrt{26}$, $d(\overline{GD}) = \sqrt{650}$.
So, quadrilateral DEFG is a parallelogram since its opposite sides are congruent [Theorem 6-6].

$d(\overline{FD}) = \sqrt{676}$, $d(\overline{EG}) = \sqrt{676}$.
So, parallelogram DEFG is a rectangle since its diagonals are congruent [Theorem 6-12].

5. The point of intersection of the diagonals is their common midpoint. So, find the coordinates of the midpoint of either diagonal. They are (3, 3).

The work on finding the coordinates of midpoints in Exercises 3 and 5 foreshadows the formal development starting with the Exploration Exercises on page 6-241. In Exercises 3 and 5 students can compute the coordinates on the basis of intuition or they can find the coordinates by inspecting the figure. They can then prove that the point whose coordinates they have found is the midpoint of the segment. They can do this by appealing to the definition of midpoint. For example, in Exercise 3, they should just use Theorem 1-4 to show that the point with coordinates (2, 1) belongs to the segment whose end points have coordinates (7, 1) and (−3, 1). Then, use the distance formula to show that the distance between the alleged midpoint M and one end point is the distance between M and the other end point.

*

Answers for Part E.

1. Let M be the point with coordinates $(-2, 5)$, N be the point with coordinates $(-5, 1)$, P be the point with coordinates $(-1, -2)$, and Q be the point with coordinates $(2, 2)$. Now, $d(\overline{MN}) = 5$ and $d(\overline{QP}) = 5$. Also, $d(\overline{MQ}) = 5$ and $d(\overline{NP}) = 5$. So, by Theorem 6-6, MNPQ is a parallelogram. Since $d(\overline{MP}) = \sqrt{50} = d(\overline{NQ})$, it follows from Theorem 6-12 that parallelogram MNPQ is a rectangle. Finally, since $d(\overline{MN}) = 5 = d(\overline{NP})$, it follows from the definition that rectangle MNPQ is a square.

2. Let P be the point with coordinates $(2, 7)$. Then, $d(\overline{PA}) = 3 = d(\overline{PB})$. So, by Theorem 3-3, P belongs to the perpendicular bisector of $\overline{AB}$.

3. Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, and $\overleftrightarrow{BC}$ is parallel to the y-axis, and since $A \notin \overleftrightarrow{BC}$, $y(D) = y(A) = 2$. Since $d(\overline{AD}) = d(\overline{BC})$ and $d(\overline{BC}) = 5$, $x(D)$ is 6 or -4. Since $\overrightarrow{AD}$ and $\overrightarrow{BC}$ must be similarly directed and since $x(C) > x(B)$, it follows that $x(D) > x(A)$. So, $x(D) = 6$ and the coordinates of D are $(6, 2)$.

Note that one consequence of our procedure for introducing coordinates is that parallel rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ which are not perpendicular to the x-axis are similarly directed if and only if $x(B) - x(A)$ and $x(D) - x(C)$ are both positive or both negative. Similarly, parallel rays $\overrightarrow{MN}$ and $\overrightarrow{PQ}$ which are not perpendicular to the y-axis are similarly directed if and only if $y(N) - y(M)$ and $y(Q) - y(P)$ are both positive or both negative.

4. $(9/2, 7/2)$

Answers for Part F.

1. $\frac{1}{6}\sqrt{9m^2 + 100n^2}$ 2. $\frac{1}{2}\sqrt{b^2 + a^2}$ 3. $\sqrt{(a - 3)^2 + (b + 7)^2}$

4. $\sqrt{(x - 4y)^2 + (x - 9y)^2}$ [or: $\sqrt{2x^2 - 26xy + 97y^2}$]

5. $\frac{1}{2}\sqrt{(a - 2)^2 + b^2}$ 6. $|2 - a|$

*

Answers for Part G.

1. $d(\overleftrightarrow{OC}) = a$; $d(\overleftrightarrow{AC}) = \sqrt{2a(a + c)}$; $d(\overleftrightarrow{BC}) = \sqrt{2a(a - c)}$; $d(\overleftrightarrow{AB}) = 2a$;
$[d(\overleftrightarrow{AC})]^2 + [d(\overleftrightarrow{BC})]^2 = 2a(a + c) + 2a(a - c) = 4a^2 = d(\overleftrightarrow{AB})^2$;
so, $m(\angle ACB) = 90$.

2. Same answer as for Exercise 1, but replace 'c' by 'd'.

3. $d(\overleftrightarrow{OE}) = a$; $d(\overleftrightarrow{AE}) = a\sqrt{3}$; $d(\overleftrightarrow{BE}) = a$; $d(\overleftrightarrow{AB}) = 2a$.
Since $[d(\overleftrightarrow{AE})]^2 + [d(\overleftrightarrow{BE})]^2 = 3a^2 + a^2 = 4a^2 = [d(\overleftrightarrow{AB})]^2$, $\angle AEB$ is a right angle. Since $d(\overleftrightarrow{BC}) = \frac{1}{2} \cdot d(\overleftrightarrow{AB})$, $\angle EAB$ is an angle of $30°$.

*

Answers for Exploration Exercises.

1. $(\frac{7}{2}, 0)$ 2. $(2, 0)$ 3. $(0, 8)$ 4. $(0, -\frac{7}{2})$

5. $(2, \frac{13}{2})$ 6. $(-\frac{1}{2}, 7)$ 7. $(5, 5)$ 8. $(-1, 1)$

9. $(3, 4)$ 10. $(2, \frac{3}{2})$ 11. $(7, 5)$ 12. $(3, \frac{11}{2})$

13. $(5, 1)$ 14. $(\frac{7}{2}, -4)$

Correction. On page 6-242, line 4b should begin:

Axiom A and properties ---
$\uparrow \uparrow$

last paragraph on page 6-242.

We know that $A \neq B$ since we have assumed that $\overleftrightarrow{AB}$ is perpendicular to the y-axis. Hence, since M is the midpoint of $\overleftrightarrow{AB}$ and $A \neq B$, $M \in \overline{AB}$. Since $y(A) = y(M) = y(B)$ and A, M, and B are three points, $x(A) \neq x(M) \neq x(B)$. Since $M \in \overline{AB}$,

$$(*) \qquad d(\overleftrightarrow{AM}) + d(\overleftrightarrow{MB}) = d(\overleftrightarrow{AB});$$

that is,

$$|x(M) - x(A)| + |x(B) - x(M)| = |x(B) - x(A)|.$$

Now, suppose that $x(M) - x(A)$ is positive and $x(B) - x(M)$ is negative. Then, it follows that

$$\begin{aligned} |x(M) - x(A)| + |x(B) - x(M)| &= |x(M) - x(A) + x(M) - x(B)| \\ &= |2 \cdot x(M) - [x(A) + x(B)]| \end{aligned}$$

and that $2 \cdot x(M) - [x(A) + x(B)]$ is positive. Now, if $x(B) > x(A)$, $x(B) - x(A)$ is positive. So, from (*),

$$2 \cdot x(M) - [x(A) + x(B)] = x(B) - x(A).$$

From this it follows that $x(M) = x(B)$. Consequently, since $x(M) \neq x(B)$, it follows that $x(B) \not> x(A)$--that is, $x(B) \leq x(A)$. But, if $x(B) \leq x(A)$ then $x(A) - x(B)$ is nonnegative; so, from (*),

$$2 \cdot x(M) - [x(A) + x(B)] = x(A) - x(B).$$

From this we see that $x(M) = x(A)$. So, since $x(M) \neq x(A)$, $x(B) \not\leq x(A)$. Consequently, it is not the case that $x(M) - x(A)$ is positive <u>and</u> $x(B) - x(M)$ is negative. Hence, recalling that $x(B) \neq x(M)$, if $x(M) - x(A)$ is positive then $x(B) - x(M)$ is positive. Similarly, if $x(B) - x(M)$ is positive then $x(M) - x(A)$ is positive. It follows, since $x(A) \neq x(M) \neq x(B)$, that $x(M) - x(A)$ and $x(B) - x(M)$ are both positive or both negative.

lines 4 and 5 on page 6-243. The argument is precisely the same as that given in the case of $\overleftrightarrow{AB}$ perpendicular to the y-axis except that 'x' and 'y' should be interchanged.

line 8 from foot of page 6-243. Theorem 6-23

line 2 on page 6-244. The formulas do hold for A = B. For, $x(M) = \frac{2 \cdot x(A)}{2} = x(A)$ and $y(M) = \frac{2 \cdot y(A)}{2} = y(A)$; so, M = A. And, the midpoint of $\overline{AA}$ is A.

*

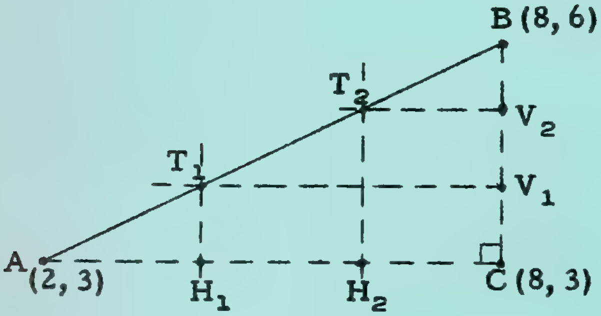
Answers for Part A [on page 6-244].

1. (5, 4) 2. $(\frac{13}{2}, -1)$ 3. (10, 5) 4. (4, -4)

5. $(a + \frac{c}{2},\ b + \frac{c}{2})$ 6. $(a + \frac{c+e}{2},\ b + \frac{d}{2})$

*

Answer for Part B [on page 6-245].



Intuitively, we see that the coordinates of H_1 are (4, 3) and those of H_2 are (6, 3). So, $x(T_1) = 4$ and $x(T_2) = 6$. Similarly, since the coordinates of V_1 are (8, 4) and those of V_2 are (8, 5), $y(T_1) = 4$ and $y(T_2) = 5$. Therefore, the coordinates of T_1 are (4, 4) and those of T_2 are (6, 5). We can prove that our answer is correct by showing that T_1 and T_2 belong to $\overleftrightarrow{AB}$ (Theorem 1-4) and that $\overline{AT_1} \cong \overline{T_1T_2} \cong \overline{T_2B}$ (use the distance formula). That H_1 and H_2 are the trisection points of $\overline{AC}$ and that V_1 and V_2 are the trisection points of $\overline{CB}$ follow from Theorem 6-27 [or Theorem 7-1].

After noting the corresponding result for a segment perpendicular to the x-axis, one can, with the aid of Theorem 7-1, combine the two results to obtain the desired conclusion.

2. By simple algebra, $q \cdot x(A) + p \cdot x(B) = (p + q) \cdot x(A) + p[x(B) - x(A)]$.

   So, $\dfrac{q \cdot x(A) + p \cdot x(B)}{p + q} = x(A) + \dfrac{p}{p + q}\,[x(B) - x(A)]$.

3. For the midpoint, substitute '1' for 'p' and for 'q'. For the trisection points, substitute, first, '2' for 'q' and '1' for 'p', and, next, '1' for 'q' and '2' for 'p'.

4. (a) $\left(\dfrac{3 \cdot 2 + 4 \cdot 9}{7}, \dfrac{3 \cdot 5 + 4 \cdot 17}{7}\right)$ --that is $\left(6, \dfrac{83}{7}\right)$

   (b) The point in question divides $\overline{AB}$ from A to B in the ratio $r : 1 - r$. [$(AP/AB) + (PB/AB) = 1$, by Axiom A.] So, using the result of Exercise 2, $x(P) = x(A) + r[x(B) - x(A)]$

   and $y(P) = y(A) + r[y(B) - y(A)]$.

*

Answers for Part E [on page 6-245].

1. median from A : 10; median from B : 10; median from C : $2\sqrt{10}$

2. $d(\overline{AB}) = 2\sqrt{5}$, $d(\overline{BC}) = \sqrt{65}$, and $d(\overline{AC}) = \sqrt{89}$. So, since no two of these three measures add up to the third measure, A, B, and C are not collinear. Hence, $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$ consists of at most one point. Now, the midpoint of $\overline{AC}$ has coordinates $(2, -3/2)$, and these are also the coordinates of the midpoint of $\overline{BD}$. So, $\overline{AC}$ and $\overline{BD}$ bisect each other. Hence, by Theorem 6-7, quadrilateral ABCD is a parallelogram.

3. Although the midpoint of $\overline{AD}$ is the midpoint of $\overline{BC}$, it does not follow that ABDC is a parallelogram. Actually, ABDC is not a quadrilateral. It is a segment.

4. $(-1, 4)$

5. $(-1, 4)$ or $(1, -4)$ or $(7, 6)$

Correction. On page 6-245, line 5b should begin:

D (924, $\underbrace{725}$) the ---

Answer for Part C.

The derivation here is essentially the same as that of the midpoint formulas. As there, one begins by considering a segment $\overleftrightarrow{AB}$ which is perpendicular to the y-axis. If T is the point which divides $\overleftrightarrow{AB}$ from A to B in the ratio 2:1--that is, if T is the point of $\overleftrightarrow{AB}$ such that $AT/TB = 2$--then, as before, $y(A) = y(T) = y(B)$, and $|x(T) - x(A)| = 2\,|x(B) - x(T)|$. Again, it can be shown [see COMMENTARY for page 6-242] that, since $T \in \overline{AB}$, $x(T) - x(A)$ and $x(B) - x(T)$ are either both positive or both negative. So, $x(T) - x(A) = 2[x(B) - x(T)]$, and

$$(1) \qquad x(T) = \frac{x(A) + 2 \cdot x(B)}{3}.$$

The formula for the y-coordinate of the corresponding trisection point of a segment which is perpendicular to the x-axis is obtained in a similar fashion. Finally, the case of an oblique segment is treated just as on page 6-243 except that Theorem 7-1 must be used instead of Theorem 6-23. The resulting formulas are (1) and:

$$(2) \qquad y(T) = \frac{y(A) + 2 \cdot y(B)}{3}.$$

In order to obtain formulas for the coordinates of the other trisection point, one need merely interchange 'A' and 'B' in formulas (1) and (2).

*

Answers for Part D.

1. One can derive the formulas implied by this exercise by a procedure which does not differ essentially from that used in Part C. The only real difference is that, if P is the point in question, then

$$q\,|x(P) - x(A)| = p\,|x(B) - x(P)|.$$

As before, since $P \in \overline{AB}$, both differences are either positive or negative. Hence, the absolute value bars may be replaced by brackets, and elementary algebra yields:

$$x(P) = \frac{q \cdot x(A) + p \cdot x(B)}{p + q}$$

According to one author, the term 'analytic geometry' came about "because the science of calculating with letters, introduced by Vieta, was termed analysis". [See page 3 of F.D. Murnaghan's Analytic Geometry (New York: Prentice-Hall, Inc., 1946).]

*

Comment on last sentence on 6-246.

As we have seen, one of the freedoms one has in setting up a coordinate system is the choice of the unit point. This freedom is due to the fact that, in euclidean geometry, only ratios of measures are significant. This fact is frequently overlooked by writers of analytic geometry texts. Such writers would claim that the solution given on pages 6-247 and 6-248 is not adequate, since it only applies to a trapezoid one of whose bases has measure 2. The answer to this objection is that any [non-degenerate] segment has measure 2 with respect to some system of measurement.

*

Note 1 on page 6-248.

The midpoint of $\overleftrightarrow{QN}$ has coordinates $(1, \frac{b}{2})$ and so does the midpoint of $\overleftrightarrow{PM}$. Since M and P have the same first coordinate, $\overleftrightarrow{MP}$ is perpendicular to the x-axis. Since Q and N have the same second coordinate, $\overleftrightarrow{QN}$ is perpendicular to the y-axis. Since the x-axis and y-axis are perpendicular to each other, so are $\overleftrightarrow{MP}$ and $\overleftrightarrow{QN}$. So, by Theorem 6-17, MNPQ is a rhombus.

Note 2 on page 6-248.

The coordinates of D are (2a, 2b) and the coordinates of C are (2 - 2a, 2b). So, the coordinates of N are (2 - a, b), those of P are (1, 2b), and those of Q are (a, b).

*

last line on page 6-249.

$d(\overleftrightarrow{AD}) = 1$. So, '$a^2 + b^2 = 1$' is the additional condition.

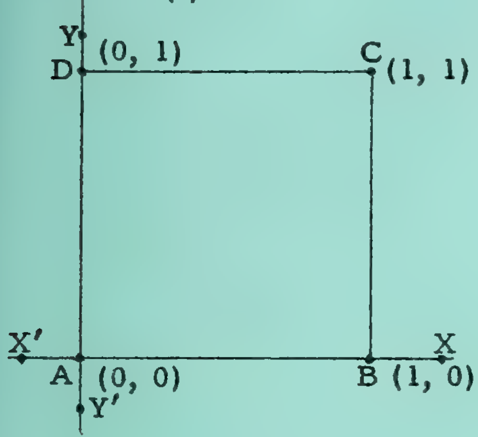
Cross-section paper should not be used for these exercises.

*

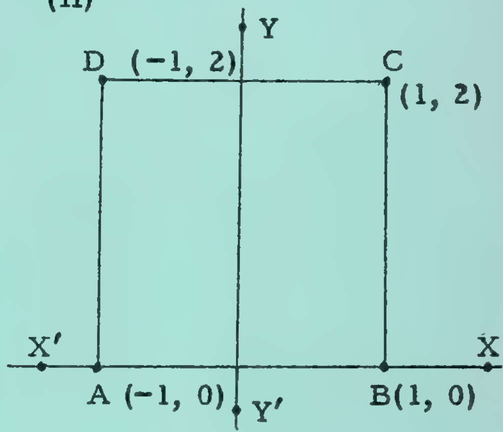
Answers for Part A [on pages 6-250 and 6-251].

1.

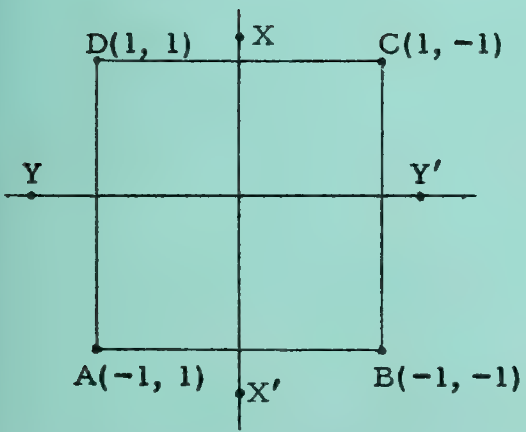
(I)



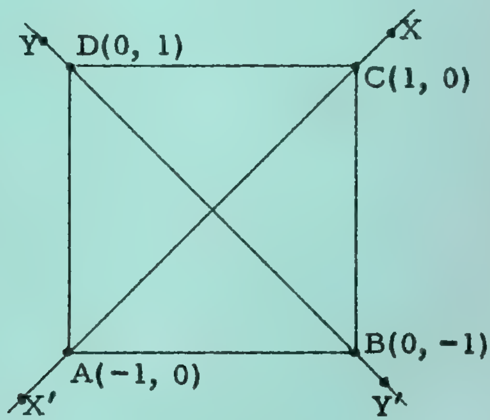
(II)



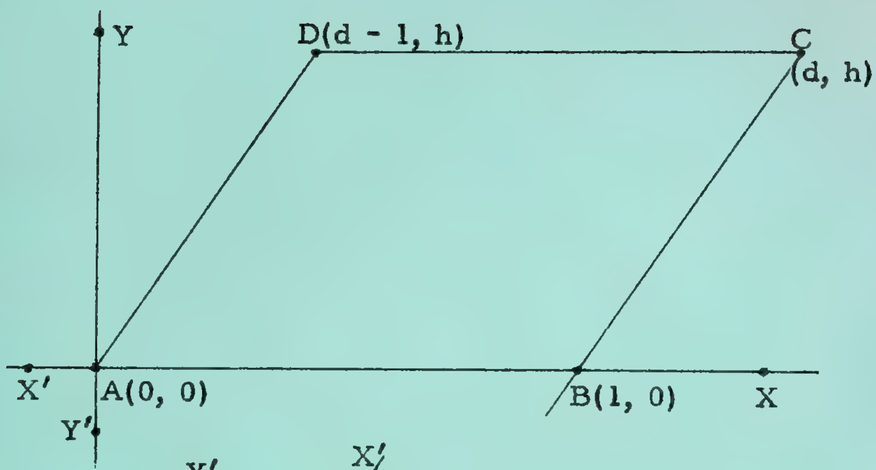
(III)



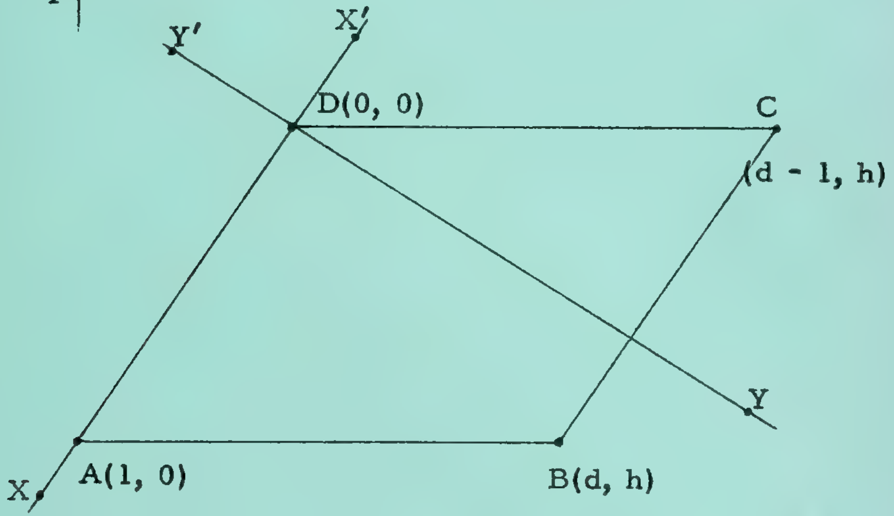
(IV)



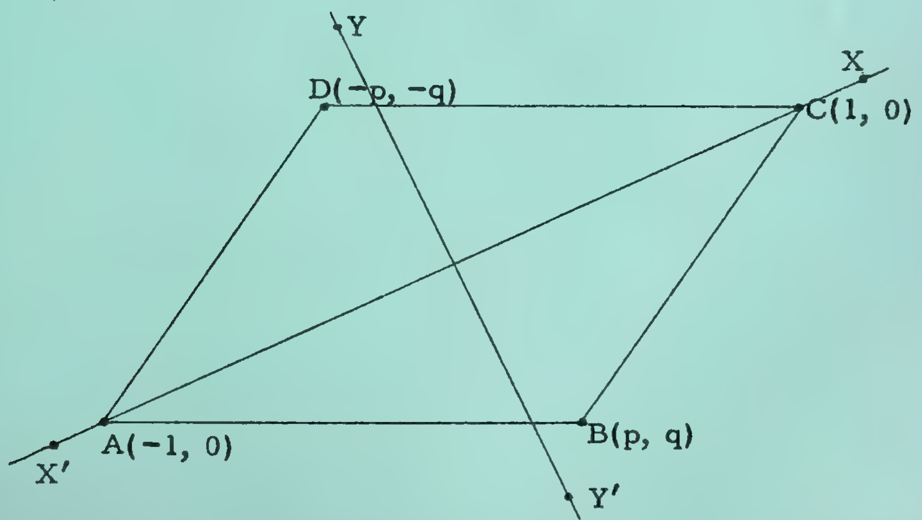
2. (I)


Y
D(d - 1, h)
C
(d, h)
X'
A(0, 0)
B(1, 0)
X
Y'

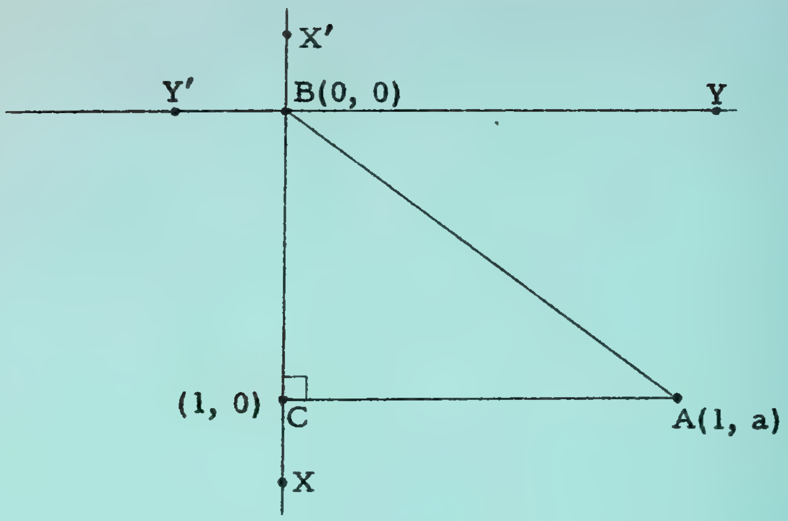
(II)


Y'
X'
D(0, 0)
C
(d - 1, h)
Y
X
A(1, 0)
B(d, h)

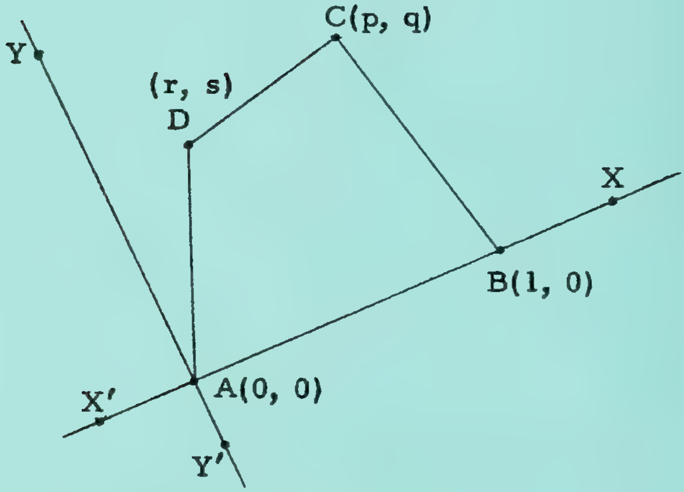
(III)


Y
X
D(-p, -q)
C(1, 0)
A(-1, 0)
B(p, q)
X'
Y'

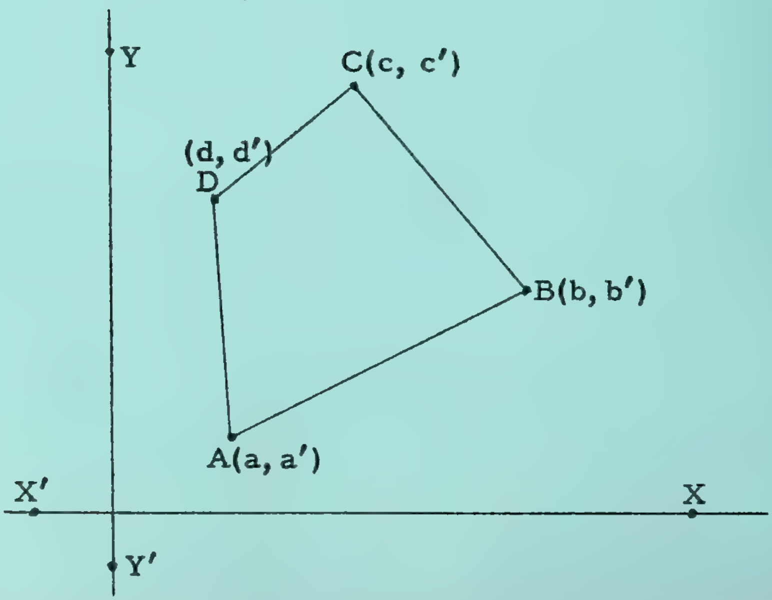
(II)



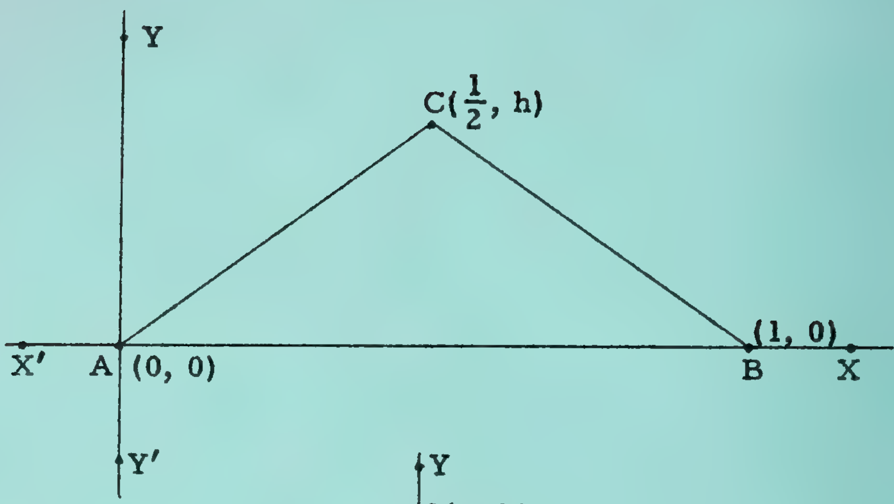
5. (I)



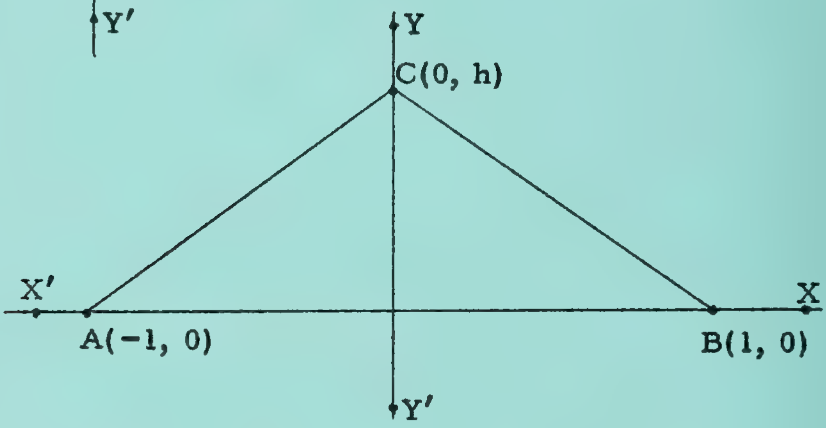
(II)



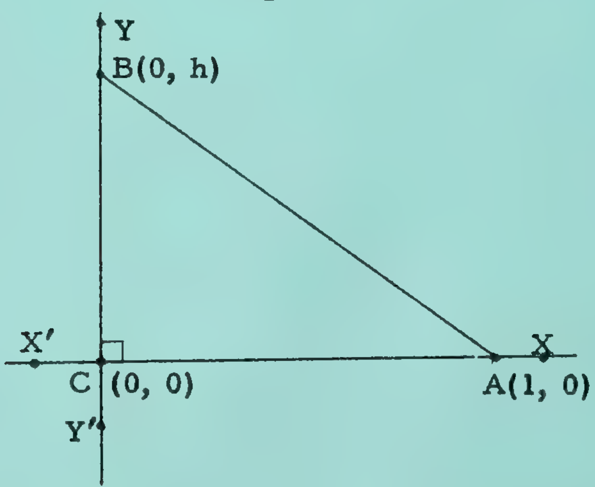
3. (I)



(II)



4. (I)



Answers for Part B [on pages 6-252 and 6-253].

1. Assume that ABCD is a trapezoid with $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $DC/AB = p < 1$. Let M be the midpoint of $\overleftrightarrow{BD}$ and let N be the midpoint of $\overleftrightarrow{AC}$. Choose coordinates so that the coordinates of A are (0, 0), those of B are (2, 0), and those of C are (2r, 2s). Since $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are similarly directed, and since $d(\overleftrightarrow{AB}) = 2$ and $DC/AB = p$, the coordinates of D are (2r - 2p, 2s). So, by the midpoint formula, the coordinates of M are (r - p + 1, s) and those of N are (r, s). Since $y(M) = y(N)$, $\overleftrightarrow{MN} \parallel \overleftrightarrow{AB}$. By the distance formula, $d(\overleftrightarrow{MN}) = |p - 1|$, $d(\overleftrightarrow{AB}) = 2$, and $d(\overleftrightarrow{CD}) = 2p$. Since $DC/AB = p < 1$, it follows that $d(\overleftrightarrow{MN}) = 1 - p = \frac{1}{2}[d(\overleftrightarrow{AB}) - d(\overleftrightarrow{CD})]$.

2. Suppose that A, B, and C are the vertices of a triangle and that $\angle ACB$ is a right angle. Let the coordinates of C be (0, 0), those of B be (2, 0), and those of A be (0, 2p). If M is the midpoint of the hypotenuse then the coordinates of M are (1, p). Now, $d(\overleftrightarrow{MA}) = \sqrt{1 + p^2} = d(\overleftrightarrow{MB}) = d(\overleftrightarrow{MC})$. So, M is equidistant from A, B, and C.

3. The coordinates of Q are $\left(\frac{a+b}{2}, \frac{a'+b'}{2}\right)$, those of M are $\left(\frac{b+c}{2}, \frac{b'+c'}{2}\right)$, those of N are $\left(\frac{c+d}{2}, \frac{c'+d'}{2}\right)$, and those of P are $\left(\frac{d+a}{2}, \frac{d'+a'}{2}\right)$. The coordinates of the midpoint of $\overleftrightarrow{PM}$ are $\left(\frac{d+a+b+c}{4}, \frac{d'+a'+b'+c'}{4}\right)$, and those of the midpoint of $\overleftrightarrow{QN}$ are $\left(\frac{a+b+c+d}{4}, \frac{a'+b'+c'+d'}{4}\right)$. So, $\overleftrightarrow{PM}$ and $\overleftrightarrow{QN}$ have the same midpoint. Hence, since $\overleftrightarrow{PM}$ and $\overleftrightarrow{QN}$ are not collinear, they bisect each other.

4. [Use the coordinate system of Exercise 3(I) on page 6-251.] The coordinates of N are $\left(\frac{1}{4}, \frac{h}{2}\right)$ and those of M are $\left(\frac{3}{4}, \frac{h}{2}\right)$. So, $d(\overleftrightarrow{BN}) = \sqrt{\frac{9}{16} + \frac{h^2}{4}} = d(AM)$. Hence, $\overleftrightarrow{AM} \cong \overleftrightarrow{BN}$.

5. Let the coordinates of A be (0, 0), those of B be (2, 0) and those of C be (2p, 2q). Then, the coordinates of N are (p, q) and those of M are (p + 1, q). Now,

$$d(\overline{AM}) = \sqrt{(p + 1)^2 + q^2} \text{ and } d(BN) = \sqrt{(p - 2)^2 + q^2}\,.$$

But, by hypothesis, $d(\overline{AM}) = d(\overline{BN})$. So,

$$\sqrt{(p + 1)^2 + q^2} = \sqrt{(p - 2)^2 + q^2}\,,$$

$$(p + 1)^2 + q^2 = (p - 2)^2 + q^2\,,$$

$$(p + 1)^2 = (p - 2)^2\,,$$

$$p + 1 = p - 2 \text{ or } p + 1 = 2 - p,$$

$$p = \frac{1}{2}.$$

Therefore, the coordinates of C are (1, 2q). Since the midpoint of $\overline{AB}$ has the coordinates (1, 0), C belongs to the perpendicular bisector of $\overline{AB}$. So, $\overline{CA} \cong \overline{CB}$.

6. Let the coordinates of A be (0, 0), of B be (1, 0), of C be (1, a), and of D be (0, a). Then, assuming that the coordinates of P are (p, q),

$$[d(\overline{PA})]^2 + [d(\overline{PC})]^2 = p^2 + q^2 + (1 - p)^2 + (a - q)^2$$

and

$$[d(\overline{PB})]^2 + [d(\overline{PD})]^2 = (1 - p)^2 + q^2 + p^2 + (a - q)^2\,.$$

[P can be any point at all.]

7. Let the coordinates of A be (2a, 0), those of B be (2b, 0) and those of C be (0, 2c). Then, the coordinates of Q are (a + b, 0), those of M are (b, c), and those of N are (a, c). So

$$[d(\overline{AM})]^2 + [d(\overline{BN})]^2 + [d(\overline{CQ})^2$$

$$= (b - 2a)^2 + c^2 + (a - 2b)^2 + c^2 + (a + b)^2 + 4c^2 = 6(a^2 + b^2 + c^2 - ab).$$

Also, $[d(\overline{AB})]^2 + [d(\overline{BC})]^2 + [d(\overline{CA})]^2$

$$= 4(a - b)^2 + 4b^2 + 4c^2 + 4a^2 + 4c^2 = 8(a^2 + b^2 + c^2 - ab).$$

8. [Use the figure of Exercise 1 on page 6-252.] Let the coordinates of A be (0, 0), those of B be (1, 0), those of D be (p, q), and those of C be (p + b, q) where $0 < b < 1$. Now, $d(\overleftrightarrow{AC}) = \sqrt{(p + b)^2 + q^2}$ and $d(\overleftrightarrow{BD}) = \sqrt{(p - 1)^2 + q^2}$. Since, by hypothesis, $d(\overleftrightarrow{AC}) = d(\overleftrightarrow{BD})$, $(p + b)^2 = (p - 1)^2$. So, since $0 < b$, $p + b = 1 - p$. Hence, $b = 1 - 2p$.

Again, by the distance formula,

$$d(\overleftrightarrow{AD}) = \sqrt{p^2 + q^2} \text{ and } d(\overleftrightarrow{BC}) = \sqrt{(p + b - 1)^2 + q^2}\,.$$

But, since $b = 1 - 2p$, $(p + b - 1)^2 = (-p)^2 = p^2$. So, $d(\overleftrightarrow{AD}) = d(\overleftrightarrow{BC})$.

*

Answers for Part C.

1. By Theorem 7-4, $y^2 = 64$. So, $y = 8$ or $y = -8$.

2. By Theorem 7-4, $|x(A)| \cdot |x(B)| = [y(C)]^2$. So, $|x(A)| \cdot 1 = b^2$. Hence, $x(A) = b^2$ or $x(A) = -b^2$. Since the foot of the altitude to the hypotenuse is between A and B, and since the foot is the origin, $x(A) = -b^2$. Hence, the coordinates of A are $(-b^2, 0)$.

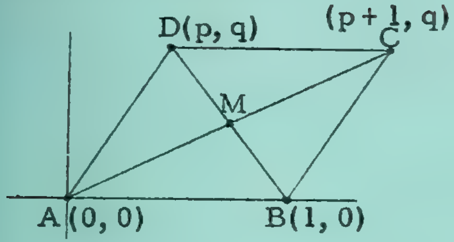
3. By the distance formula, $d(\overleftrightarrow{CB}) = \sqrt{1 + b^2}$. Since $\angle C$ is the vertex angle, $d(\overleftrightarrow{CA}) = \sqrt{1 + b^2}$. Since C and A are on the x-axis, $d(\overleftrightarrow{CA}) = |x(A) - x(C)|$. So, either

$$x(A) = x(C) + \sqrt{1 + b^2} \text{ or } x(C) = x(A) + \sqrt{1 + b^2}\,.$$

Since $x(C) = 1$, either $x(A) = 1 + \sqrt{1 + b^2}$ or $x(A) = 1 - \sqrt{1 + b^2}$.

4. $C(\frac{3}{2}, \frac{\sqrt{3}}{2})$, $D(1, \sqrt{3})$, $E(0, \sqrt{3})$, $F(-\frac{1}{2}, \frac{\sqrt{3}}{2})$

Answer for Part A.



Let AM/AC = r and BM/BD = s. Then, the coordinates of M are

$(x(A) + r[x(C) - x(A)],\ y(A) + r[y(C) - y(A)])$

They are, also,

$(x(B) + s[x(D) - x(B)],\ y(B) + s[y(D) - y(B)])$.

So, $(r(p + 1),\ rq) = (1 + s(p - 1),\ sq)$. Therefore,

(1) $r(p + 1) = 1 + s(p - 1)$ and (2) $rq = sq$.

Since $q \neq 0$, (2) tells us that $r = s$. From this and (1) it follows that $r = \frac{1}{2}$.

*

Answer for Part B.

[Use the coordinate system of Part A.] Suppose that AP/AC = r and BP/BR = s. Then, $(r(p + 1),\ rq) = (1 + s(\frac{p}{2} - 1),\ s\frac{q}{2})$. So, $s = 2r$ and $r = 1/3$. [Note that P and Q are the trisection points of $\overleftrightarrow{AC}$.]

By synthetic geometry, since RPA $\longleftrightarrow$ BPC is a similarity, $\frac{PA}{PC} = \frac{RA}{BC}$. But, $RA = \frac{1}{2} \cdot BC$. So, $PA = \frac{1}{2} \cdot PC$. Hence, $PA = \frac{1}{3} \cdot AC$.

*

Answers for Part C.

1. $CG = \frac{2}{3} \cdot \frac{1}{2} \cdot AB = \frac{1}{3} \cdot 30 = 10$

2. $d = \frac{2}{3} \cdot \frac{s}{2}\sqrt{3} = \frac{s\sqrt{3}}{3}$

3. Let P be the point of concurrence of the medians. Thus, since the measure of the hypotenuse is $s\sqrt{2}$, the measure of the median to the hypotenuse is $\frac{s\sqrt{2}}{2}$. So, $CP = \frac{2}{3} \cdot \frac{s\sqrt{2}}{2} = \frac{s\sqrt{2}}{3}$. The measure of the median to each leg is $\sqrt{s^2 + \frac{s^2}{4}}$, or $\frac{s\sqrt{5}}{2}$. So, $AP = \frac{s\sqrt{5}}{3} = BP$.

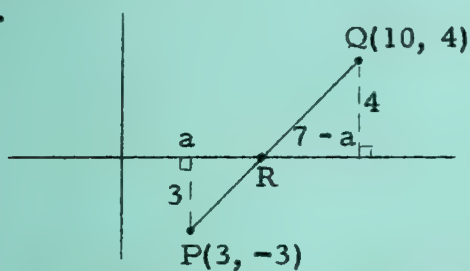
4. (a) $\left(\frac{a_1 + b_1 + c_1}{3}, \frac{a_2 + b_2 + c_2}{3}\right)$ [Note that the x-coordinate of the point of concurrence is the arithmetic mean of the x-coordinates of the vertices. Similarly, for the y-coordinate.]

(b) (4, 5)

*

Answers for Part D.

1.



$\frac{3}{4} = \frac{a}{7 - a}$; $a = 3$

So, the coordinates of R are (6, 0).

2. As in Exercise 1, use similar triangles to deduce that y(S) = 6.

*

Answers for Part A [which begins at the foot of page 6-256].

1. 45

2. 135 [45]

3.

60	120	30	150	74	149
120	60	150	30	106	31

4. 45

5. 135

*

Answers for Part B [on page 6-257].

1. 63.4

2. 63.4

3. 50.2

4. 135

5.

60	63.4	145	135	101.3
120	116.6	35	45	78.7

6. 45

7. 123.7

Answers to questions in the text on page 6-258.

line 7. Axiom E line 8. a

*

In inferring that $a = m(\angle QPD)$, we have tacitly assumed that $x(Q) > x(P)$ and $y(Q) > y(P)$. But, since, for each two points P and Q of an oblique line, $\frac{y(Q) - y(P)}{x(Q) - x(P)} = \frac{y(P) - y(Q)}{x(P) - x(Q)}$, the results obtained are independent of this assumption.

*

Answers for Part A [on page 6-260].

1. -1; 135 2. $-\frac{1}{8}$; 173 3. $\frac{1}{4}$; 14 4. 0; — 5. —; 90 6. $\frac{4}{5}$; 38.7

*

Answers for Part B [on page 6-260].

1. Yes; No [They might be perpendicular to the y-axis]

2. Yes; No 3. Yes; Yes

*

Answers for Part C [on page 6-260].

1. slope of $\overleftrightarrow{AB} = -\frac{1}{4}$ = slope of $\overleftrightarrow{CD}$. So, either $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ or $\overleftrightarrow{AB} = \overleftrightarrow{CD}$. But, the slope of $\overleftrightarrow{AC} = -1$; so, $\overleftrightarrow{AC} \neq \overleftrightarrow{AB}$, and, hence, $\overleftrightarrow{CD} \neq \overleftrightarrow{AB}$. Therefore, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

2. slope of $\overleftrightarrow{AB} = -1$ = slope of $\overleftrightarrow{BC}$. So, either $\overleftrightarrow{AB} = \overleftrightarrow{BC}$ or $\overleftrightarrow{AB} \parallel \overleftrightarrow{BC}$. But, $B \in \overleftrightarrow{AB} \cap \overleftrightarrow{BC}$. So, $\overleftrightarrow{AB} \cap \overleftrightarrow{BC} \neq \emptyset$. Hence, $\overleftrightarrow{AB} \nparallel \overleftrightarrow{BC}$. Thus, $\overleftrightarrow{AB} = \overleftrightarrow{BC}$, and A, B, and C are collinear.

3. $\frac{y(D) - 6}{x(D) - 4} = \frac{4 - 5}{-3 - 2} = \frac{1}{5}$. This is the case if and only if $x(D) \neq 4$ and $y(D) = \frac{1}{5} \cdot x(D) + \frac{26}{5}$. Hence, D is any point except C whose coordinates are such that $y(D) = \frac{1}{5} \cdot x(D) + \frac{26}{5}$.

4. 36

Answers for Part A.

1. $y = 2x + 3$
2. $y = x + 2$
3. $x = 5$
4. $y = -x + 11$
5. $y = \frac{7}{6}x$
6. $y = -\frac{3}{4}x + 3$
7. $y = -\frac{4}{5}x + \frac{83}{10}$; slope of $\overleftrightarrow{AC} = -\frac{4}{5}$.
8. $y = -\frac{1}{3}x + \frac{16}{3}$
9. $y = 4x - 13$
10. $y = \frac{2}{3}x - \frac{20}{3}$
11. $y = x + 4$
12. $x = 0$
13. $y = -x - 2$

*

Answers for Part B.

1. 3; $(0, -5)$; $(\frac{5}{3}, 0)$
2. 3; $(0, -5)$; $(\frac{5}{3}, 0)$
3. 3; $(0, -5)$; $(\frac{5}{3}, 0)$
4. 2; $(0, 5)$; $(-\frac{5}{2}, 0)$
5. 8; $(0, -16)$; $(2, 0)$
6. $-\frac{1}{2}$; $(0, 0)$
7. $\frac{1}{3}$; $(0, -5)$; $(15, 0)$
8. 10; $(0, 6)$; $(-\frac{3}{5}, 0)$
9. 10; $(0, 6)$; $(-\frac{3}{5}, 0)$
10. 5; $(0, 4)$; $(-\frac{4}{5}, 0)$
11. -7; $(0, 2)$; $(\frac{2}{7}, 0)$
12. 3; $(0, -2)$; $(\frac{2}{3}, 0)$
13. $-\frac{2}{5}$; $(0, 2)$; $(5, 0)$
14. $\frac{6}{5}$; $(0, -6)$; $(5, 0)$
15. $-\frac{1}{3}$; $(0, 3)$, $(9, 0)$
16. $-\frac{7}{4}$; $(0, 7)$; $(4, 0)$
17. $\frac{2}{3}$; $(0, -2)$, $(3, 0)$
18. slope is not defined; $(-\frac{10}{3}, 0)$

Answers for Part A.

1. $\ell_1 : y = -2x + 10$; $\ell_2 : y = \frac{1}{2}x + \frac{5}{2}$

2. $\ell_1 : y = 3x - 5$; $\ell_2 : y = -\frac{1}{3}x + \frac{5}{3}$

3. $\ell_1 : y = -\frac{4}{7}x - \frac{50}{7}$; $\ell_2 : y = \frac{7}{4}x - \frac{5}{2}$

4. $\ell_1 : x = 2$; $\ell_2 : y = 5$

5. $\ell_1 : y = -7$; $x = 4$

6. $\ell_1 : y = x - 1$; $\ell_2 : y = -x - 11$

*

Answers for Part B.

1. slope of $\overleftrightarrow{AB} = -2$, slope of $\overleftrightarrow{CD} = -2$, slope of $\overleftrightarrow{AD} = 1/2$, and slope of $\overleftrightarrow{BC} = 1/2$. So, $\overleftrightarrow{AB} \perp \overleftrightarrow{CB}$, $\overleftrightarrow{CD} \perp \overleftrightarrow{BC}$, $\overleftrightarrow{AD} \perp \overleftrightarrow{CD}$, and $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$. Hence, the angles of quadrilateral ABCD are right angles. So, by definition, it is a rectangle.

2. slope of $\overleftrightarrow{AB} = 7/4$, slope of $\overleftrightarrow{CD} = 7/4$, slope of $\overleftrightarrow{AD} = -1/2$, and slope of $\overleftrightarrow{BC} = -1/2$. Since slope of $\overleftrightarrow{AB} \neq$ slope of BC, A, B, and C are not collinear; so, $\overleftrightarrow{AB} \neq \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \neq \overleftrightarrow{BC}$. Hence, $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$. Therefore, ABCD is a parallelogram. Since neither line is perpendicular to an axis and since $(-1/2)(7/4) \neq -1$, it follows from Theorem 9-6 that $\overleftrightarrow{AB} \not\perp \overleftrightarrow{BC}$. So, $\angle ABC$ is not a right angle. Hence, by definition, quadrilateral ABCD is not a rectangle.

3. If A, B, C, and D are the vertices of a square then four of the segments $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, and $\overline{CD}$ are congruent. But, $AB = \sqrt{90}$, $AC = \sqrt{130}$, $AD = \sqrt{40}$, $BC = \sqrt{40}$, $BD = \sqrt{130}$, and $CD = \sqrt{90}$. No four of these are congruent. Hence, A, B, C, and D are not the vertices of a square.

4. from A: $y = \frac{7}{9}x - \frac{1}{3}$

from B: $y = \frac{2}{7}x + \frac{32}{7}$

from C: $y = \frac{5}{2}x - \frac{35}{2}$

5. of $\overleftrightarrow{BC}$: $y = \frac{7}{9}x - \frac{5}{3}$

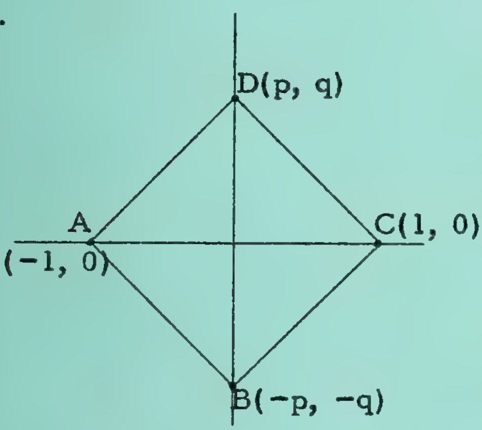
of $\overleftrightarrow{CA}$: $y = \frac{2}{7}x - \frac{37}{14}$

of $\overleftrightarrow{AB}$: $y = \frac{5}{2}x + \frac{7}{4}$

6. $\frac{5}{2}x + \frac{7}{4} = \frac{7}{9}x - \frac{5}{3}$; $x = -\frac{123}{62}$; $y = -\frac{615}{124} + \frac{7}{4} = -\frac{199}{62}$.

The coordinates of the point of concurrence are $\left(-\frac{123}{62}, -\frac{199}{62}\right)$.

7.



Parallelogram ABCD is a rhombus if and only if AD = DC; that is, if and only if

$$\sqrt{(p+1)^2 + q^2} = \sqrt{(p-1)^2 + q^2},$$

or $$p = 0.$$

But, $p = 0$ if and only if $\overline{DB}$ is perpendicular to $\overleftrightarrow{AC}$. So, parallelogram ABCD is a rhombus if and only if the diagonals are perpendicular.

8. $-\frac{4}{3}$; $y = -\frac{4}{3}x + 4$; $\frac{x}{3} + \frac{y}{4} = 1$

9. (a) $\frac{5}{2}$; $y = \frac{5}{2}x + 5$; $\frac{x}{-2} + \frac{y}{5} = 1$

(b) $\frac{3}{2}$; $y = \frac{3}{2}x - 6$; $\frac{x}{4} + \frac{y}{-6} = 1$

(c) $-\frac{1}{3}$; $y = -\frac{1}{3}x - 2$; $\frac{x}{-6} + \frac{y}{-2} = 1$

(d) $-\frac{b}{a}$; $y = -\frac{b}{a}x + b$; $\frac{x}{a} + \frac{y}{b} = 1$

Answers for quiz.

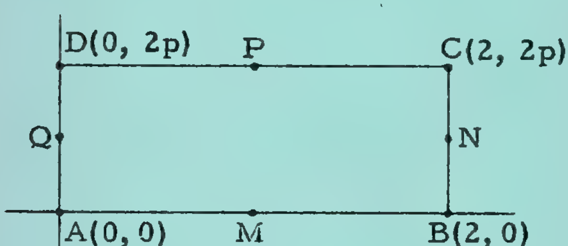
1. $(7, 1)$ 2. (a) $d(\overline{AB}) = \sqrt{10} = d(\overline{CB})$ (b) $(3, 6) \neq (4, 5)$

3. $d(\overline{AB}) = 13$, $d(\overline{BC}) = 5$, $d(\overline{CA}) = \sqrt{68}$. Since $5 < \sqrt{68} < 13$, $\angle C$ is the largest angle of the triangle.

4. $y = -\frac{1}{3}x$

5. $d(\overline{AB}) = 10$, $d(\overline{BC}) = 5\sqrt{2}$, $d(\overline{CA}) = 5\sqrt{2}$. So, the triangle is isosceles. $[d(\overline{AB})]^2 = [d(\overline{BC})]^2 + [d(\overline{CA})]^2$. So, the triangle is a right triangle.

6. (a) slope of $\overleftrightarrow{DC} = \frac{2 - 5}{-2 - 4} = \frac{1}{2} = \frac{-5 - 5}{-5 - 15}$ = slope of $\overleftrightarrow{AB}$.
   (b) $\overleftrightarrow{BC}$ is perpendicular to the y-axis. An equation of $\overleftrightarrow{BC}$ is '$y = 5$'.

7. 

The coordinates of M are $(1, 0)$, of N are $(2, p)$, of P are $(1, 2p)$, and of Q are $(0, p)$. So, $d(\overline{MN}) = \sqrt{1 + p^2}$, $d(\overline{NP}) = \sqrt{1 + p^2}$, $d(\overline{PQ}) = \sqrt{1 + p^2}$, and $d(\overline{QM}) = \sqrt{1 + p^2}$. So, all four sides of the quadrilateral PQMN are congruent. Hence, it is a rhombus.

☆ 8. Suppose the coordinates of A are $(0, 0)$ and those of B are $(2, 0)$, Let the coordinates of C be $(1, \sqrt{3})$. Then, the coordinates of A' are $(4, 0)$, those of B' are $(0, 2\sqrt{3})$, and those of C' and $(-1, -\sqrt{3})$.

So, $d(\overline{A'B'}) = \sqrt{(0 - 4)^2 + (2\sqrt{3} - 0)^2} = \sqrt{28}$,

$d(\overline{B'C'}) = \sqrt{(-1 - 0)^2 + (-\sqrt{3} - 2\sqrt{3})^2} = \sqrt{28}$,

and $d(\overline{C'A'}) = \sqrt{(4 + 1)^2 + (0 + \sqrt{3})^2} = \sqrt{28}$

Hence, $\Delta A'B'C'$ is equilateral.

Quiz.

1. Find the coordinates of the midpoint of a segment if the coordinates of its end points are (8, −3) and (6, 5), respectively.

2. Given the points A(2, 3), B(3, 6), and C(6, 7). Prove
   (a) that B is on the perpendicular bisector of $\overline{AC}$, and
   (b) that B is not the midpoint of $\overline{AC}$.

3. Which is the largest angle of the triangle whose vertices are A(3, −2), B(8, 10), and C(5, 6)?

4. Write an equation of a line which passes through the origin and is perpendicular to the line an equation of which is '$3y - 9x + 4 = 0$'.

5. Prove that the segments joining the points A(5, 3), B(15, 3), and C(10, 8) are the sides of an isosceles right triangle.

6. The vertices of quadrilateral ABCD are A(−5, −5), B(15, 5), C(4, 5), and D(−2, 2).
   (a) Use slopes to prove that $\overleftrightarrow{DC}$ is parallel to $\overleftrightarrow{AB}$.
   (b) Write an equation of $\overleftrightarrow{BC}$.

7. Use the method of analytic geometry to prove that the midpoints of the sides of a rectangle are the vertices of a rhombus.

☆ 8. Suppose that ΔABC is equilateral. Let A' be a point on $\overrightarrow{AB}$ such that $AB = BA'$, B' be a point on $\overrightarrow{BC}$ such that $BC = CB'$, and C' be a point on $\overrightarrow{CA}$ such that $CA = AC'$. Use the method of analytic geometry to prove that $\Delta A'B'C'$ is equilateral.

*

Correction. On page 6-271, the label at the center of the circle should be 'C(h, k)'.

line 3. The alternative use of 'radius', to refer to a segment, is introduced on page 6-277.

The development on pages 6-274 and 6-275, which culminates in Theorem 10-1 on page 6-276, is a good example of the power of the method of analytic geometry. A synthetic treatment would be much more difficult.

*

Solutions for (1) - (4) on page 6-271.

(1) $x^2 + y^2 = 49$

(2) 10; (0, 0)

(3) $\sqrt{2}$; (0, 0)

(4) $(x - 3)^2 + (y - 5)^2 = 4$

*

Note that not only can one derive the equation [see middle of page 6-272]:

$$(*) \qquad [x(P) - 4]^2 + [0 - 4]^2 = 25$$

from (1) and (2), but one can derive (1) from (*) and (2). That is, the system consisting of (1) and (2) is equivalent to [has the same solution set as] the system consisting of (*) and (2). Now, (*) is, as shown, equivalent to:

$$(**) \qquad x(P) = 7 \text{ or } x(P) = 1$$

So, the system consisting of (1) and (2) is equivalent to the system consisting of (**) and (2). Hence, both systems have the same solution set, $\{(1, 0), (7, 0)\}$. Consequently, there is no need to carry out the checking procedure given at the foot of page 6-272. [Nevertheless, you may wish to require your students to carry out such checks for the purpose of discovering errors in computation.]

Answers for Part A.

1. $(x - 2)^2 + (y - 1)^2 = 9$ [or: $x^2 + y^2 - 4x - 2y - 4 = 0$]

2. $(x - 1)^2 + (y - 2)^2 = 9$

3. $(x + 5)^2 + (y - 6)^2 = 5$

4. $(x + 3)^2 + (y + 5)^2 = 121$

5. $(x - 5)^2 + (y - 7)^2 = 29$

6. $(x - 3)^2 + (y - 4)^2 = 25$ [or: $x^2 + y^2 - 6x - 8y = 0$]

*

Answers for Part B.

1. (0, 0); 9
2. (0, 0); 12
3. (1, 2); 13
4. (−1, 2); 15
5. (3, 0); 10
6. (−3, −4); 1
7. (0, −9); 3
8. $(\sqrt{2}, -\sqrt{3})$; $\sqrt{5}$
9. $\{P: x(P) > 0 \text{ and } y(P) < 0\}$; The circle of Exercise 6 is the only one whose center belongs to Quadrant III, so, none of the other circles can be subsets of this quadrant. The circle of Exercise 6 is a subset of Quadrant III.

*

Answers for Part C.

1. (3, 4), (3, −4)
2. (4, 3), (4, −3)
3. (5, 0)
4. none
5. (3, 4), (−3, −4)
6. (0, 0), (0, 8)

Correction. On page 6-279, line 9b should read:

(8) --- [Steps like (3) and (5)]

intersect a noncircular curve in exactly one point and still not be tangent to the curve. But, [for any sufficiently smooth curve] one can define, through each point of the curve, a "normal line", and the tangent to the curve at a given point is the line which contains the point and is perpendicular to the normal line at that point.

*

Answers to questions after Theorem 10-2 on page 6-277.

(a) 1 (b) none (c) two

*

Theorem 10-3 is not quite correct, since there are two radii perpendicular to a given chord. It can be corrected by replacing 'radius' by 'diameter'.

By Theorem 10-1, the distance between the center of a circle and a chord is less than the radius of the circle. So, by sentence (4) on page 6-275, the line through the center and perpendicular to the chord intersects the chord at its midpoint. So, the distance between the center of the circle and the midpoint of the chord is less than the radius of the circle. Again, by Theorem 10-1, the line through the center and perpendicular to the chord intersects the circle in two points. These points are on opposite sides of the given chord, so the chord whose end points they are contains the midpoint of the given chord. Consequently, the diameter perpendicular to a chord bisects the chord.

Since there is only one line perpendicular to a chord at its midpoint, Theorem 10-3 also tells us that the perpendicular bisector of a chord of a circle contains the center of the circle.

*

As in the case of 'radius', one who speaks of <u>the</u> diameter of a circle must mean to refer to the common measure [see Exercise 7 of Part F on page 6-280] of chords which contain the center of the circle, while one who speaks of <u>a</u> diameter refers to such a chord.

*

Answers for Part E [on page 6-278].

1. 13 2. 3 3. 24

Corrections. On page 6-276, change lines 6b and 5b to read:

---, since the perpendicular segment from C to ℓ is the only segment from C to ℓ which has this measure, it follows ---

In line 2b, insert a comma between 'because' and 'since'.

As pointed out in the COMMENTARY for page 6-272, the substitution check described just after equation (4) on page 6-275 is gratuitous.

From (2) and (3) [near the foot of page 6-275] it follows that a line whose distance from the center of a circle is equal to, or greater than, the radius of the circle does not intersect the circle in two points. Hence, if a line intersects a circle in two points then the distance between the line and the center of the circle is less than the radius.

Similarly, it follows from (1) and (3) that if a line intersects a circle in exactly one point then the distance between the line and the center of the circle is the radius of the circle.

From (1) and (2) it follows that if a line does not intersect a circle then the distance between the line and the center of the circle is greater than the radius of the circle.

In general, from three conditional sentences:

(1) if p then s   (2) if q then t   (3) if r then u

and four sentences:

p or q or r   not (s and t)   not (t and u)   not (u and s)

one can infer the converses of (1), (2), and (3):

if s then p   if t then q   if u then r

*

Note that when one speaks of the radius of a circle, he must be using 'radius' as it was introduced on page 6-270. One who speaks of a radius of a circle is using the word with the meaning of 'radial segment'.

*

Note that one could take Theorem 10-2 on page 6-277 as a definition:

A line is tangent to a circle if and only if it contains a point of the circle and is perpendicular to the radius at that point.

In this case, one would have, in place of Theorem 10-2, the theorem:

A line is tangent to a circle if and only if it intersects the circle at a single point.

Such a rearrangement might accord better with more advanced mathematics courses. For a tangent to a noncircular curve may intersect the curve at other points beside the point of tangency, and a line may

Answers for Part F.

1. Suppose that $\overline{AB}$ and $\overline{DE}$ are congruent chords of a circle with center C and that M and N are, respectively, the feet of the perpendiculars from C to $\overline{AB}$ and $\overline{DE}$. We also assume from the figure that $M \neq C \neq N$. By Theorem 10-3, M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{DE}$. Hence, since $AB = DE$, it follows that $MB = NE$. Since B and E are points of the circle, $BC = EC$. Since $\overline{CM} \perp \overline{AB}$ and $\overline{CN} \perp \overline{DE}$, both $\triangle CMB$ and $\triangle CNE$ are right triangles. Hence, by h.l., $CMB \leftrightarrow CNE$ is a congruence, and $CM = CN$.

2. Suppose that $\overline{AB}$ and $\overline{DE}$ are chords of a circle with center C and that the feet M and N of the perpendiculars from C to $\overline{AB}$ and $\overline{DE}$, respectively, are equidistant from C. Also, from the figure, we assume that $M \neq C \neq N$. Since M and N are equidistant from C, $\overline{CM} \cong \overline{CN}$. Since B and E are points of the circle, $CB = CE$. Since $\overline{CM} \perp \overline{AB}$ and $\overline{CN} \perp \overline{DE}$, both $\triangle CMB$ and $\triangle CNE$ are right triangles. Hence, by h.ℓ., $CMB \leftrightarrow CNE$ is a congruence, and $MB = NE$. But, by Theorem 10-3, M and N are the midpoints of $\overline{AB}$ and $\overline{DE}$. Consequently, $AB = 2 \cdot MB = 2 \cdot NE = DE$.

3. Since $\overline{AB}$ is a diameter of the circle, the center C of the circle belongs to $\overline{AB}$. So, $\overleftrightarrow{CA} = \overleftrightarrow{CB}$. Since ℓ is tangent to the circle at A and m is tangent to the circle at B, it follows from Theorem 10-2 that $\ell \perp \overleftrightarrow{CA}$ and that $m \perp \overleftrightarrow{CB}$. Since $\overleftrightarrow{CA} = \overleftrightarrow{CB}$, and $A \neq B$, it follows from Theorem 5-8 that $\ell \parallel m$.

4. Since ℓ and m are tangents at T and S, respectively, it follows by Theorem 10-2 that $\ell \perp \overleftrightarrow{CT}$ and $m \perp \overleftrightarrow{CS}$. Since $\ell \perp \overleftrightarrow{CT}$ and $\ell \parallel m$, it follows by Theorem 5-4 that $m \perp \overleftrightarrow{CT}$. Since $m \perp \overleftrightarrow{CS}$, $\overleftrightarrow{CS}$ is the

perpendicular to m from C; since m $\perp \overleftrightarrow{CT}$, $\overleftrightarrow{CT}$ is the perpendicular to m from C. Hence, $\overleftrightarrow{CT} = \overleftrightarrow{CS}$ and S, C, and T are collinear. Since $\ell \parallel m$, $T \in \ell$, and $S \in m$, it follows that $S \neq T$. Since S and T are points of the circle, CS = CT. Consequently, using Axiom C, it follows that $C \in \overline{ST}$. So, by definition, $\overline{ST}$ is a diameter.

5. By Theorem 3-3, each point equidistant from the end points of a chord belongs to the perpendicular bisector of the chord. So, since the center of a circle is equidistant from the end points of each chord, the center belongs to the perpendicular bisector of every chord. [This also follows at once from Theorem 10-3. See COMMENTARY for page 6-277.]

6. By Theorem 10-3, the line through the center of a circle and perpendicular to a chord contains the midpoint of the chord. If the chord is not a diameter then its midpoint is not the center of the circle, and, since the center and the midpoint are two points on the perpendicular through the center to the chord, the center and midpoint determine this perpendicular to the chord.

7. By Axiom A and the definitions of chord, diameter, and radius, the measure of each diameter of a circle is twice the radius of the circle. So, each two diameters have the same measure. Hence, each two diameters are congruent.

Proof of Theorem 10-4: By Exercises 1 and 2 of Part F on page 6-280, it follows that two chords of a circle, *neither of which is a diameter*, are congruent if and only if they are equidistant from the center. If two chords are equidistant from the center of a circle and one is a diameter then so is the other; and, by Exercise 7, the chords are congruent. Finally, if two chords are congruent, and one is a diameter, then so is the other. For, if $\overline{AB}$ is a chord which is not a diameter and $\overline{AC}$ is the diameter which contains A, then it follows by Theorem 6-28 that $\angle ABC$ is a right angle. So, by Theorems 4-4 and 4-7, $\overline{AC}$ is longer than $\overline{AB}$. So, by Exercise 7, $\overline{AB}$ is shorter than *each* diameter. Consequently, each chord congruent to a diameter is a diameter. So, two congruent chords, one of which is a diameter, are both diameters and, so, are equidistant from the center. This completes the proof of Theorem 10-4.

Theorem 10-5 follows from Exercises 3 and 4 of Part F.

Theorem 10-6 follows from Exercise 5.

Theorem 10-7 follows from Exercise 6.

Theorem 10-8 follows from the Example on pages 6-278 and 6-279.

Theorem 10-9 follows from Exercise 7. It happens to be a corollary of Theorem 10-4, but it was used as a lemma in the above proof of Theorem 10-4.

Correction. On page 6-283, line 5b should begin:

2. Show that each ---

Answer for Part C.

By Theorem 6-28, the midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle. Since the circumcenter of the triangle is the only such point, it follows that the circumcenter of a right triangle is the midpoint of the hypotenuse of the triangle. Now, if ABCD is a rectangle then ΔABC and ΔBCD are right triangles with a common hypotenuse. Hence, they have the same circumcircle.

*

Answer for Part D.

By Axiom A, the perimeter of ΔABC is AM + MB + BN + NC + CQ + QA. By Theorem 10-8, AM = QA, MB = BN, and NC = CQ. Hence [by substitution], the perimeter of ΔABC is AM + BN + BN + CQ + CQ + AM--that is, is 2(AM + BN + CQ).

*

Answers for Part E.

1. By Theorem 4-12, the perpendicular bisectors of the sides of an equilateral triangle contain its medians. By Theorem 9-3, the point of concurrence of the medians is 2/3 the length of each median from the corresponding vertex. By Theorem 4-12, the medians of equilateral triangle are its altitudes. By Example 3 on page 6-205, the measure of an altitude of an equilateral triangle of side measure s is $s\sqrt{3}/2$. So, the radius of the circumcircle of such a triangle is $s\sqrt{3}/3$.

2. By Theorem 4-12, the angle bisectors of an equilateral triangle are its medians. Arguing as in Exercise 1, the radius of the incircle of an equilateral triangle of side measure is $s\sqrt{3}/6$.

3. This has been established in Exercises 2 and 3.

*

Answers for Part ☆F.

See COMMENTARY for page 6-423. A triangle has three excircles.

Corrections. On page 6-285, line 13 should begin:

1. $\underbrace{\text{Each}}_{\uparrow}$ hoop is ---

and line 14 should begin:

2. $\underbrace{\text{Each}}_{\uparrow}$ hoop is ---

The orthocenter of a triangle is the circumcenter of a second triangle each of whose sides contains a vertex of the first triangle and is parallel to the side of the first triangle which is opposite this vertex.

Answers for Part G.

1. [Since, by Theorem 10-8, $AB' = AC'$, it follows by Theorem 4-12 that the bisector $\overrightarrow{AD}$ of $\angle A$ is perpendicular to $\overleftrightarrow{B'C'}$. So, $\overleftrightarrow{B'C'}$ contains the altitude from the vertex of the right angle $\angle B'$ of $\Delta AB'D$. As shown in the COMMENTARY for page 6-203, it follows that $\overline{B'C'} \cap \overline{AD}$ consists of a single point. So, $AC'DB'$ is a convex quadrilateral.] Since, by definition, the convex quadrilateral $AB'DC'$ has right angles at C' and B', it follows from Theorem 6-30 that $m(\angle A) + m(\angle D) = 180$. So, $\angle A$ and $\angle D$ are supplementary.

2. Since P is, by hypothesis, the center of the circle containing A, B, and C, ΔBPC and ΔCPA are isosceles with vertex angles at P. So, by Theorem 3-5, $\angle PBC$ and $\angle PCB$ have the same measure, β, and $\angle PCA$ and $\angle PAC$ have the same measure, γ. Since, by hypothesis, P is interior to ΔABC, $m(\angle ACB) = \beta + \gamma$. Moreover, since P is interior to ΔABC, it follows that $\overrightarrow{CP}$ intersects $\overline{AB}$ at a point E such that $P \in \overline{CE}$. Hence, $\angle EPB$ is an exterior angle of ΔBPC, and $m(\angle EPB) = 2\beta$. Similarly, $m(\angle EPA) = 2\gamma$. Since $\overrightarrow{PE}$ intersects $\overline{AB}$, E is interior to $\angle APB$. Consequently, $m(\angle APB) = m(\angle EPB) + m(\angle EPA) = 2\beta + 2\gamma$. So, $m(\angle ACB) = \frac{1}{2} \cdot m(\angle APB)$.

   [Here is an alternative solution for Exercise 2:

   Since, by hypothesis, P is the center of the circle containing A, B, and C, ΔAPB, ΔBPC, and ΔCPA are isosceles with vertex angles at P. So, by Theorem 3-5, $\angle PAB$ and $\angle PBA$ have the same measure, α; $\angle PBC$ and $\angle PCB$ have the same measure, β; and $\angle PCA$ and $\angle PAC$ have the same measure, γ. Since, by hypothesis, P is interior to ΔABC, it follows that, in ΔABC, $m(\angle A) = \gamma + \alpha$, $m(\angle B) = \alpha + \beta$, and $m(\angle C) = \beta + \gamma$. Hence, using Theorem 5-11, it follows that $\alpha + \beta + \gamma = 90$. Using the same theorem, $\alpha + \frac{1}{2} \cdot m(\angle APB) = 90$. Hence, $\beta + \gamma = \frac{1}{2} \cdot m(\angle APB)$--that is, $m(\angle ACB) = \frac{1}{2} \cdot m(\angle APB)$.]

3. If the circumcenter P of ΔABC belongs to $\overleftrightarrow{AB}$ then P is the midpoint of $\overleftrightarrow{AB}$, and $CP = \frac{1}{2} \cdot AB$. So, by Theorem 6-28, ΔABC is a right triangle.

☆ 4. Since, by Theorem 6-24, $\overleftrightarrow{M_2M_3}$ is parallel to the side [of the given triangle] whose midpoint is M_1, it follows by Theorem 5-9 that the perpendicular bisector of this side is the line which contains the altitude from M_1 of $\Delta M_1M_2M_3$. Similarly, each of the other altitudes of $\Delta M_1M_2M_3$ is contained in a perpendicular bisector of a side of the given triangle. So, by definition, the orthocenter of $\Delta M_1M_2M_3$ is the circumcenter of the given triangle.

[Compare Exercise 4 with Exercise 3 on page 6-167.]

*

Answers for Exploration Exercises [on pages 6-284 and 6-285].

1. Each hoop is outside the other [and they are not in contact].

2. The hoops are in contact at one point and, disregarding this point, each hoop is outside the other.

3. If $r = s$ then one hoop will be on top of the other; otherwise, the smaller hoop is inside the larger.

4. The hoops could be in contact, but need not be. If $r = 10$, $s = 5$ and $d = 4$ then the smaller hoop is inside the larger, and they are not in contact. If $r = 10$, $s = 5$, and $d = 5$ then the hoops are in contact at a single point. If $r = 10$, $s = 5$, and $d = 8$, they are in contact at just two points.

5. The hoops could be in contact, but need not be. The possibilities are illustrated by situations in which $s = 3$, $r = 4$, and $d = 6$, or 7, or 8.

6. The hoops are in contact at exactly two points.

*

Answers for Part B [on page 6-285].

1. $d > r + s$ 2. $d = r + s$ 3. $d = r - s$ 4. $d < r - s$

5. $d = 0$ and $r = s$ 6. $r - s < d < r + s$

Corrections. On page 6-290, line 3 should begin 'the circles. If the center of the circles ---'.

Line 4 should read '---tangent. If the centers of the circles ---'.

Line 8 should read '---other. If the centers of the circles---'.

Comment on last paragraph on page 6-286.

If d = 0 then C = D and $\overleftrightarrow{CD}$ is not a line. Nevertheless, in this case (1) and (2) are still equations for the circles, with respect to any coordinate systems whose origin is C. If d = 0 and r = s, there is only one circle. Algebraically, the solution set of the last displayed equation [fourth line from foot of page 6-286] is the set of all real numbers. So, the solution set of the system consisting of (1) and (3) is just the solution set of (1).

*

Strictly speaking, not (2′) on page 6-289, but '$x = (r^2 - s^2 + d^2)/(2d)$' is an equation of the line in question. [We have been similarly sloppy in lines 5 and 6 on page 6-287.] As shown on page 6-286, a point belongs to both circles if and only if its coordinates satisfy both this equation and equation (1). So, in the two-point case, both points of intersection belong to this line and, so, determine it. In particular, the two points have the same x-coordinate and, as is seen by substitution in (1), opposite y-coordinates. So, the midpoint of the segment joining the two points of intersection is on the x-axis. Since the segment is perpendicular to the x-axis, it follows that the x-axis is the perpendicular bisector of the segment. [Theorem 10-13 also follows readily from Theorem 3-3.]

*

With Theorem 10-14 now available, it is easy to establish another necessary and sufficient condition on the measures of the sides of a triangle:

> For all nonzero numbers of arithmetic x, y, and z, there is a triangle whose side measures are x, y, and z, respectively if and only if $x + y > z$ and $y + z > x$ and $z + x > y$.

For, suppose that A, B, and C are three noncollinear points, and that a, b, and c are the measures of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Then, a, b and c are nonzero numbers of arithmetic and, by Axiom B, since $C \notin \overline{BA}$, $a + b > c$. Similarly, since $A \notin \overline{CB}$, $b + c > a$ and, since $B \notin \overline{AC}$, $c + a > b$.

On the other hand, suppose that a, b, and c are nonzero numbers of arithmetic such that $a + b > c$, $b + c > a$, and $c + a > b$. Either $a \geq b$ or $b \geq a$. In the first case, since $a + b > c$ and $b + c > a$, it follows that $a - b < c < a + b$. In the second case, since $a + b > c$ and $c + a > b$, it follows that $b - a < c < a + b$. So, by Theorem 10-14, in each case, a, b, and c are measures of the sides of a triangle.

Correction. On page 6-291, in line 2b, change 'radii' to 'radius'.

Answers for Part A [which begins on page 6-290].

1. The circles have the same radius.

2. (a) 12 (b) 36/5 and 96/5 (c) $4\sqrt{3}$

3. Suppose that the circles have centers C and D and that the points of tangency are S and T, respectively. By Theorem 10-2, $\overline{CS}$ and $\overline{DT}$ are both perpendicular to ℓ. Assuming that $S \neq T$ [the case in which $S = T$ is treated in Exercise 1 of Part B, below], it follows by Theorem 5-8 that $\overline{CS} \parallel \overline{DT}$. Since the circles have the same radius, $CS = DT$. Since ℓ is a common internal tangent, C and D are on opposite sides of $\overleftrightarrow{ST}$. Since [because $\overline{CS} \parallel \overline{DT}$] C and S are on the same side of $\overleftrightarrow{DT}$ and D and T are on the same side of $\overleftrightarrow{CS}$, it follows [by a result in the COMMENTARY for page 6-162] that $\overline{CD} \cap \overline{ST}$ consists of a single point. Hence, CTDS [rather than CSTD] is a quadrilateral and, since it has two sides parallel and congruent, is, by Theorem 6-8, a parallelogram. Consequently, by Theorem 6-5, $\overleftrightarrow{CD}$ bisects $\overline{ST}$.

4. There are six possible arrangements: four in which all three circles have a common tangent, one in which the two smaller circles form a figure-eight inside the largest, and one in which the centers are vertices of a triangle with side measures 5, 7, and 8.

*

Answers for Part B [on pages 6-291 and 6-292].

1. By Theorem 10-2, both $\overline{PT}$ and $\overline{P'T}$ are perpendicular to $\overleftrightarrow{MN}$. So, by Theorem 2-8, $\overleftrightarrow{PT} = \overleftrightarrow{P'T}$. Now, $P \neq P'$. For, if $P = P'$ then $PT = P'T$, and there would not be two circles. So, $\overleftrightarrow{PT} = \overleftrightarrow{PP'}$, and $T \in \overleftrightarrow{PP'}$.

2. [Same as Exercise 1.]

3. By Theorem 10-8, $AM = ME$, and $EM = MB$. Hence, $AM = MB$.

4. If the circles have the same radius then ABCD is a rectangle. So, AB = CD. If the circles do not have the same radius then $\overrightarrow{AB}$ and $\overrightarrow{DC}$ intersect at a point P. By Theorem 10-8, AP = DP and BP = CP. Hence, AD = DC.

5. By Theorem 10-8, AE = DE and EB = EC. Since $E \in \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$, it follows that AB = AE + EB = DE + EC = DC.

6. By Theorem 10-8, PA = PT = PB.

*

Answers for Part C.

1. Use Theorem 6-28.

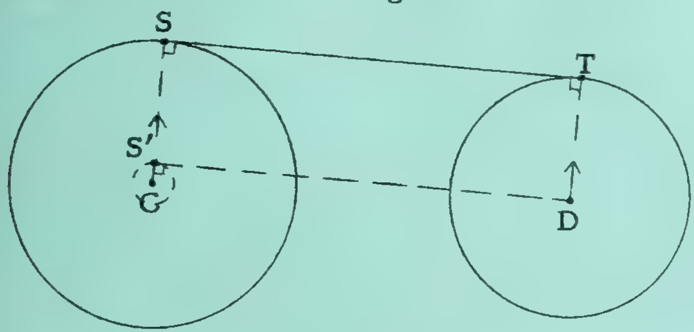
2. Use Exercise 1 and Theorem 10-2.

*

P is an internal point with respect to a circle if and only if $CP < r$.

*

Students who have learned [see Exercise 3] to draw tangents to a circle from an external point, may be interested in the construction of common external and internal tangents to two circles.



Given circles of radius r and s with $r \geq s$, draw tangents from the center of the smaller circle to the circle of radius $r - s$ which is concentric with the larger circle. If S' is one of the points of tangency then $\overrightarrow{CS'}$ intersects the larger circle in the point at which one of the common external tangents is tangent. The point of tangency, for this tangent, on the smaller circle is the point at which $\overrightarrow{DT}$, directed similarly to $\overrightarrow{CS}$, intersects the smaller circle. [For internal tangents, use a similar construction, but begin by drawing tangents from C to the circle of radius $r + s$ and center D.]

Correction. On page 6-294, in line 14, delete the comma after 'Once'.

On page 6-295, line 11b should begin 'problem provides ---', and

line 6 should begin '--- half of BA ---'.

Answers to questions in the text on page 6-294.

line 12: Since the common radius r of the circles is greater than $\frac{1}{2} \cdot CP$, and CP is the distance between the centers of the circles, it follows that $r - r < CP < r + r$. So, by Theorem 10-12, the circles intersect in exactly two points.

line 17: If s is the radius of the given circle, then, since $CP > s$, the radius MC of the second circle is a number $r > s/2$. The distance, d, between the centers of the two circles is r. Now, if $r \geq s$ then $r - s < r < r + s$; and, if $s/2 < r < s$ then $s - r < r < s + r$. Hence, in either case, it follows from Theorem 10-12 that the circles intersect in exactly two points.

line 18: $P \neq T$, since T belongs to the given circle and P does not.

last line: Since B and C belong to a circle with center A, AB = AC. Since A and C belong to a circle with center B, BC = BA.

*

Answers to questions in the text on page 6-295.

line 7: Since both circles have radius r, and the distance between their centers is r, and since $r - r < r < r + r$, Theorem 10-12 tells us that the circles intersect in exactly two points.

line 24: The "other proof" referred to is not, in the present development, of any probative value. For [see Solution.], it makes use of Theorem 10-2, which was proved by analytic methods. And Theorem 2-8 was used in showing that, for each coordinate system, each pair of real numbers is the coordinate-pair of some point.

line 29: Proof asked for is similar to that asked for in line 12 on page 6-294.

*

[For an interesting discussion of the problem of possible euclidean constructions, see Chapter 3 of Courant and Robbins, What Is Mathematics? (New York: Oxford University Press, 1941).]

Answers for Part A.

(1) 90 (2) 60 (3) 90 (4) 150 (5) 40 (6) 10 [or: 90]

(7) 250 [or: 350] (8) 90 [or: 270] (9) 110 [or: 10]

*

Answers for Part B. 1. 40 2. 359

*

Note that a semicircular arc is the intersection of a circle with a closed half-plane whose edge contains a diameter of the circle. In general, it can be proved that an arc is the intersection of a circle with a closed half-plane whose edge contains a chord of the circle. This follows from the definition on page 6-298 and, principally, Theorem 4-10.

What is required is to show that if AB is a chord of a circle with center C, which is not a diameter of the circle, then a point P of the circle which is interior to $\angle ACB$ is on the non-C-side of $\overleftrightarrow{AB}$ and a point P of the circle which is exterior to $\angle ACB$ is on the C-side of $\overleftrightarrow{AB}$. Suppose that P is a point of the circle which is interior to $\angle ACB$. Then, $\overrightarrow{CP}$ crosses $\overline{AB}$ at some point Q. Since $Q \in \overline{AB}$ and $CA = CB$, it follows from Theorem 4-10 that $CQ < CB$. Since $CP = CB$, it follows that $CQ < CP$, and P is on the non-C-side of $\overleftrightarrow{AB}$. Suppose that P is a point of the circle which is exterior to $\angle ACB$. If $\overrightarrow{CP} \cap \overleftrightarrow{AB} = \emptyset$, then P is in the C-side of $\overleftrightarrow{AB}$. Suppose, on the other hand, that $\overrightarrow{CP}$ crosses $\overleftrightarrow{AB}$ at a point Q. Since P is exterior to $\angle ACB$, $P \notin \overline{AB}$. Hence, if M is the midpoint of $\overline{AB}$, either $A \in \overline{QM}$ or $B \in \overline{QM}$. In either case, by Theorem 3-10 and Theorem 4-9, $CM < CQ$. So, by Theorem 4-10, either $CA < CQ$ or $CB < CQ$. Since $CA = CP = CB$, it follows that, in either case, $CQ > CP$. Hence, $\overline{CP} \cap \overleftrightarrow{AB} = \emptyset$, and P is on the C-side of $\overleftrightarrow{AB}$.

*

It follows from the preceding discussion that if A, B, M, and N are points of a circle such that M and N are on opposite sides of $\overleftrightarrow{AB}$ then the union of $\overset{\frown}{AMB}$ and $\overset{\frown}{ANB}$ is the circle, and their intersection consists of A and B. Also, either both are semicircles or one is a minor arc and the other a major arc.

Answers for Part A.

1. (a) 60 (b) 300 2. (a) 130 (b) 160 3. (a) 120 (b) 240

*

Answers for Part B [on pages 6-300 and 6-301].

1. 150 2. 72 3. 240 4. $\overset{\frown}{AB}$ 5. $\overset{\frown}{FAD}$ 6. {B}

7. the circle 8. the circle 9. can't tell 10. can't tell

11. 90; 90 12. 200; 160 13. 80; 150

*

$\angle RTS$ is not inscribed in $\overset{\frown}{TU}$ because the vertex, T, of $\angle RTS$ is an end point of $\overset{\frown}{TU}$.

*

Answers for Part C [on page 6-302].

1. $\overset{\frown}{ELB}$ 2. $\overset{\frown}{EFB}$ 3. $\overset{\frown}{HJ}$ and $\overset{\frown}{ELB}$ 4. $\overset{\frown}{BK}$

5. $\angle EFG$, $\angle EGB$, $\angle EHM$, $\angle EIJ$, $\angle EBD$

6. $\overset{\frown}{BKJ}$ 7. $\overset{\frown}{BKE}$ 8. $\overset{\frown}{HJB}$ 9. $\overset{\frown}{EF}$

10. $\overset{\frown}{EBJ}$ 11. $\overset{\frown}{FBH}$ 12. $\overset{\frown}{LB}$ and $\overset{\frown}{EHB}$

Answers for Part D.

1. [If $\overset{\frown}{AB}$ is a minor arc of a circle with center O then both $\overset{\frown}{AB}$ and its chord $\overline{AB}$ are said to subtend the central angle $\angle AOB$.]

   Since, by definition, minor arcs of the same or congruent circles are congruent if and only if the central angles which they subtend are congruent, it is sufficient to prove that chords of minor arcs of the same or congruent circles are congruent if and only if the central angles which they subtend are congruent. But, the if-part of this last is an immediate consequence of s.a.s., and the only if-part is an immediate consequence of s.s.s.

2. Suppose that $\overline{AB}$ is longer than $\overline{CD}$. Suppose that $\overline{AB}$ is a diameter. It follows, by Theorem 10-9, that $\overline{CD}$ is not a diameter. In this case, $m(\overset{\frown}{AB}) = 180$ and $m(\overset{\frown}{CD}) = m(\angle COD) < 180$ [by Axiom D]. Moreover, the distance between P and $\overline{AB}$ is 0, and [using Theorem 1-2] the distance between P and $\overline{CD}$ is greater than 0. Now, suppose that $\overline{AB}$ is not a diameter. Then, by Exercise 1 of Part C on page 6-292, Exercise 1 of Part C on page 6-131, and Theorem 10-9, it follows that $\overline{AB}$ is shorter than each diameter. Since $\overline{CD}$ is shorter than $\overline{AB}$, it follows that $\overline{CD}$ is shorter than each diameter. So, $\overline{CD}$ is not a diameter. Consequently, since $\overline{AB}$ is longer than $\overline{CD}$, it follows, by Theorem 4-11, that $\angle CPD$ is not larger than $\angle APB$, and, by s.a.s., that $\angle CPD$ is not congruent to $\angle APB$. Hence, $\angle APB$ is larger than $\angle CPD$. So, by definition $m(\overset{\frown}{AB}) > m(\overset{\frown}{CD})$. Finally, if d is the distance between P and $\overline{AB}$, and e is the distance between P and $\overline{CD}$, then, by Theorem 10-1 and the Pythagorean Theorem, $d^2 + [\frac{1}{2} \cdot AB]^2 = e^2 + [\frac{1}{2} \cdot CD]^2$. So, $e^2 - d^2 = \frac{1}{4} \cdot [(AB)^2 - (CD)^2] > 0$ and [since e and d are numbers of arithmetic] $e > d$.

3. Suppose that $\overline{AB}$ is closer to P than $\overline{CD}$ is. Suppose that $\overline{AB}$ is a diameter. Then $\overline{CD}$ is not, and, as shown in the answer, above, for Exercise 2, $CD < AB$. Suppose that $\overline{AB}$ is not a diameter. Then, again, neither is $\overline{CD}$, and $CD < AB$ for the same reasons set forth in the final two sentences of the answer for Exercise 2, above.

4. ΔABC is isosceles with vertex angle at B. By Theorem 4-12, the bisector of $\angle B$ is a subset of the perpendicular bisector of $\overline{AC}$. By Theorem 10-3, this line contains O. So, the bisector of $\angle B$ is a subset of $\overleftrightarrow{BO}$. To prove that the bisector of $\angle B$ is $\overrightarrow{BO}$, we still must show that O belongs to the bisector of $\angle B$ [rather than to the other ray with vertex B which is a subset of $\overleftrightarrow{BO}$]. To do so, let M be the midpoint of $\overline{AC}$. As previously shown, the bisector of $\angle B$ is $\overrightarrow{BM}$, and our problem is to show that $O \in \overrightarrow{BM}$. Now, since $\overline{OM}$ is perpendicular to $\overline{AC}$ at M, and $A \neq M$, it follows that $OM < OA = OB$. But, if $O \notin \overrightarrow{BM}$ then, since $O \in \overleftrightarrow{BM}$, $B \in \overline{OM}$ and, by Axiom A, $OB \leq OM$. Since, as just noted, this is not the case, it follows that $O \in \overrightarrow{BM}$. [As with many of the exercises in this unit, most students will be satisfied to assume from a figure that $O \in \overrightarrow{BM}$. It is to be hoped, however, that there are students who will wonder if such an assumption can be derived.]

5. By Theorem 10-8 and s.s.s., $CTP \longleftrightarrow CSP$ is a congruence. Hence [as in Exercise 4], $\overrightarrow{PC}$ is the bisector of $\angle TPS$. So, $m(\angle TPC) = 30$ and, by Theorem 10-2 and Theorem 5-11, $m(\angle TCM) = 60$. Now, since ΔPCT is 30-60-90, $TC = \frac{1}{2} \cdot CP$. Since $TC = CM$, $CM = MP$.

Answer for Part E.

Suppose that A and B are two points on a circle with center C. By Theorem 10-3, the diameter $\overline{PQ}$ perpendicular to $\overline{AB}$ intersects $\overline{AB}$ at its midpoint M. The points P and Q are on opposite sides of $\overleftrightarrow{AB}$. If $\overline{AB}$ is a diameter then $\angle PCA$ and $\angle PCB$ are right angles and $\overset{\frown}{AP}$ and $\overset{\frown}{PB}$ are both arcs of 90°. So, $\overline{PQ}$ bisects the semicircle $\overset{\frown}{APB}$. Similarly, $\overline{PQ}$ bisects the semicircle $\overset{\frown}{AQB}$. If $\overline{AB}$ is not a diameter then ΔACB is an isosceles triangle with vertex angle C and, by Theorem 4-12, $\overrightarrow{CM}$ is the bisector of $\angle ACB$. If, say, $M \in \overrightarrow{CP}$ then $\overrightarrow{CP}$ is the bisector of $\angle ACB$. So, $\angle ACP \cong \angle PCB$ and, by definition, $\overset{\frown}{AP} \cong \overset{\frown}{PB}$. In this case, $\angle ACQ$ and $\angle BCQ$ are congruent since they are supplements of congruent angles. So, $\overset{\frown}{AQ} \cong \overset{\frown}{QB}$. [The case in which $M \in \overrightarrow{CQ}$ is treated similarly.]

*

Answers for Part F.

1. By hypothesis and Theorem 3-3, both A and O belong to the perpendicular bisector of $\overline{BC}$. So, by Theorem 4-12, $\overrightarrow{AO}$ is the bisector of $\angle A$, and $m(\angle OAB) = 30$. Similarly, $m(\angle OBA) = 30$. Hence, by Theorem 5-11, $m(\angle AOB) = 120$. So, $m(\overset{\frown}{AMB}) = 120$ and, by hypothesis, $m(\overset{\frown}{AM}) = 60$. Similarly, $m(\overset{\frown}{MB}) = m(\overset{\frown}{BN}) = m(\overset{\frown}{NC}) = m(\overset{\frown}{CP}) = m(\overset{\frown}{PA}) = 60$. Hence, by Theorem 10-19, $AM = MB = BN = NC = CP = PA$. Hence, by s.s.s. $\Delta OAM \cong \Delta OMB \cong \ldots \cong \Delta OPA$. Since these are isosceles triangles with vertex angles at O, $\angle OAM \cong \angle OMA \cong \angle OMB \cong \ldots \cong \angle OPA \cong \angle OAP$. Hence, $\angle PAM \cong \angle AMB \cong \ldots \cong \angle CPA$. Consequently, since AMBNCP is both equilateral and equiangular, it is regular.

2. 10

3. ΔCAD is an isosceles triangle with vertex angle at A. Since one of its angles is an angle of 60°, ΔCAD is equilateral.

[The idea of having students use an opaque circular protractor to measure an angle and thus leading them to discover Theorems 10-22 through 10-26 was given to us by Mr. Harry Schor of Abraham Lincoln High School in New York City.] Since the written directions for these Exploration Exercises are fairly intricate, we suggest that the exercises be done in class under the teacher's supervision.

Answers for Part A.

1. 60; 70; 90; 105; 140; 150; 160; 170; doesn't cross [but, $\overleftrightarrow{BC}$ is, in this case, tangent to the circle at B]

2. 30

3. If the scale mark for O is not interior to the angle, then m(∠ABC) is one half the scale difference. [Otherwise, m(∠ABC) is 180 minus one half the scale difference.]

4. (a) 80; 40

   (b) 120; 60

*

Answers for Part B.

1. The values read off the scale should be approximately 52 and 188.

2. $\frac{1}{2}[(52 - 0) + 188 - 180)] = 30$

3. $\frac{1}{2}[(50 - 0) + (190 - 180)] = 30$

*

Answers for Part D [on page 6-308].

1. 35 2. 65 3. 55; 110 4. 55; 160; 35

*

Answer for Part E [on page 6-308].

Yes. [See Theorem 10-26.]

*

Answer for Part F [on page 6-308].

The measure of the angle is at least 90. [This answer is based on the assumption that the vertex is invisible if it is at the edge of the protractor.]

$\overset{\frown}{ABC}$ is a major arc. So, taking account of the definition of degree-measure for arcs, it follows that, in each case, $m(\overset{\frown}{AB}) + m(\overset{\frown}{BC}) = m(\overset{\frown}{ABC})$.

In case (iv), since B is exterior to $\angle AOC$, $\overset{\frown}{ABC}$ is a major arc, and $m(\overset{\frown}{ABC}) = 60 - m(\angle AOC)$. Since $\overset{\frown}{BA'C}$ is a semicircle, $m(\overset{\frown}{BA'C}) = 180$. Since $\angle AOB$ and $\angle AOC$ are supplementary, $m(\overset{\frown}{AB}) = m(\angle AOB) = 180 - m(\angle AOC)$. Combining these results, we find that, in case (iv), $m(\overset{\frown}{AB}) + m(\overset{\frown}{BA'C}) = m(\overset{\frown}{ABC})$.

In case (v), since A′ and C are on opposite sides of $\overleftrightarrow{OB}$, it follows that A′ is exterior to $\angle BOC$, and $\overset{\frown}{BA'C}$ is a major arc. Similarly, B is exterior to $\angle AOC$, and, hence, $\overset{\frown}{ABC}$ is a major arc. So, to show that $m(\overset{\frown}{AB}) + m(\overset{\frown}{B'AC}) = m(\overset{\frown}{ABC})$, we need to show that

$$m(\angle AOB) + [360 - m(\angle BOC)] = 360 - m(\angle AOC),$$

that is, that $m(\angle AOB) + m(\angle AOC) = m(\angle BOC)$. But, since, in case (v), A is on the C-side of $\overleftrightarrow{OB}$ and B and C are on opposite sides of $\overleftrightarrow{OA}$, it follows that A is interior to $\angle BOC$. So, the desired result follows from Axiom F.

This completes the proof of Theorem 10-21.

Similarly, each point of $\overset{\frown}{AB}$, except B, is on the A-side of $\overleftrightarrow{OB}$. So, in the first three cases, of the two arcs with end points B and C, $\overset{\frown}{BC}$, alone, intersects $\overset{\frown}{AB}$ only at B. Hence, in these cases, we need to show that $m(\overset{\frown}{AB}) + m(\overset{\frown}{BC}) = m(\overset{\frown}{ABC})$.

In the fourth case, of the two semicircles, $\overset{\frown}{BAC}$ and $\overset{\frown}{BA'C}$ with end points B and C, it is only the second which intersects $\overset{\frown}{AB}$ only at B. So, in this case, we need to show that $m(\overset{\frown}{AB}) + m(\overset{\frown}{BA'C}) = m(\overset{\frown}{ABC})$.

In the fifth case, A is on the C-side of $\overleftrightarrow{OB}$ and B and C are on opposite sides of $\overleftrightarrow{OA}$. So, A is interior to $\angle BOC$ and, so, belongs to $\overset{\frown}{BC}$. On the other hand, no point of $\overset{\frown}{AB}$ other than B belongs to the major arc $\overset{\frown}{BA'C}$. For, since A is interior to $\angle BOC$, each point interior to $\angle AOB$ is interior to $\angle BOC$. Hence, no point interior to $\angle AOB$ is exterior to $\angle BOC$. So, in the fifth case, of the two arcs with end points B and C, the major arc $\overset{\frown}{BA'C}$, alone, intersects $\overset{\frown}{AB}$ only at B. Hence, in this case, we need to show that $m(\overset{\frown}{AB}) + m(\overset{\frown}{BA'C}) = m(\overset{\frown}{ABC})$.

Now, since, in the first three cases, A and C are on opposite sides of $\overleftrightarrow{OB}$, $\angle AOB$ and $\angle BOC$ are adjacent angles. In each case, $\overline{AC}$ crosses $\overleftrightarrow{OB}$. In case (i), since C is on the B-side of $\overleftrightarrow{OA}$, so is the crossing point. Hence, the crossing point is on $\overrightarrow{OB}$. In case (ii), the crossing point is O. In case (iii), since C is on the B'-side of $\overleftrightarrow{OA}$, the crossing point is on $\overrightarrow{OB'}$. It follows, now, from the work on sums of measures of adjacent angles on page 6-71, that

in case (i), $\quad m(\angle AOB) + m(\angle BOC) = m(\angle AOC)$,

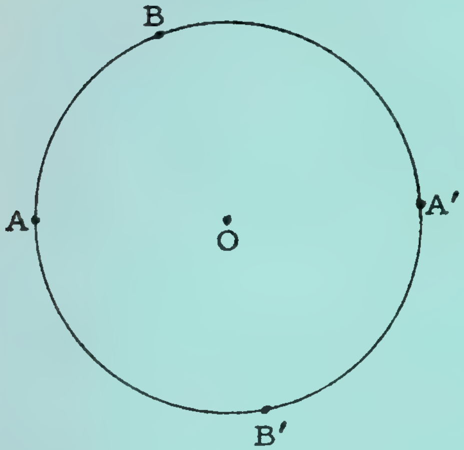
in case (ii), $\quad m(\angle AOB) + m(\angle BOC) = 180$, and

in case (iii), $\quad m(\angle AOB) + m(\angle BOC) = 360 - m(\angle AOC)$

In case (i), since B is interior to $\angle AOC$, $\overset{\frown}{ABC}$ is a minor arc. In case (ii), $\overset{\frown}{ABC}$ is a semicircle. In case (iii), since B is exterior to $\angle AOC$,

Correction. On page 6-309, line 3b should end:
--- which $\overset{\frown}{AKB}$ and $\overset{\frown}{BLC}$ are minor
↑

For completeness, here is a proof of Theorem 10-21 on page 6-310.



If B and C are two points of a circle with center O then, if $O \notin \overline{BC}$, the major arc with end points B and C contains the point B′ such that $O \in \overline{BB'}$. For, since B and B′ are on opposite sides of $\overleftrightarrow{OC}$ and $B' \notin \overrightarrow{OB}$, B′ is exterior to $\angle BOC$. On the other hand, B′ belongs to both semicircular arcs which have B as an end point. Consequently, if the intersection of of two arcs which have B as a common end point consists of the point B, alone, then one of these is a minor arc.

Suppose, now, that $\overset{\frown}{AB}$ is a minor arc, and that A′ and B′ are the opposite end points of the diameters from A and B, respectively. Each point C of the circle such that $C \notin \overset{\frown}{AB}$ is exterior to $\angle AOB$. So, there are five cases:

(i) C is on the A′-side of $\overleftrightarrow{OB}$ and on the B-side of $\overleftrightarrow{OA}$.

(ii) $C \in \overrightarrow{OA'}$ [so, C = A′],

(iii) C is on the A′-side of $\overleftrightarrow{OB}$ and on the B′-side of $\overleftrightarrow{OA}$,

(iv) $C \in \overrightarrow{OB'}$ [so, C = B′],

(v) C is on the A-side of $\overleftrightarrow{OB}$ and on the B′-side of $\overleftrightarrow{OA}$.

In each of the first three cases, A and C are on opposite sides of $\overleftrightarrow{OB}$; so, A is exterior to $\angle BOC$ and belongs to the major arc with end points B and C. On the other hand, each point of $\overset{\frown}{BC}$, with the exception of B and C, is interior to $\angle BOC$ and, hence, is on the C-side of $\overleftrightarrow{OB}$. In the first three cases, this is the A′-side of $\overleftrightarrow{OB}$. Also, C is on the A′-side of $\overleftrightarrow{OB}$.

Theorem 10-22 on page 6-310 is established by the results of Exercises 1, 2, and 3 of Part A, below.

*

Answers for Part A [which begins on page 6-310].

1. $\angle ACB$ is an exterior angle adjacent to the vertex angle $\angle ACD$ of isosceles triangle ΔACD. So, $m(\angle ACB) = 2 \cdot m(\angle ADB)$. But, by definition, $m(\angle ACB) = m(\overset{\frown}{AB})$. Hence, $m(\angle ADB) = \frac{1}{2} \cdot m(\overset{\frown}{AB})$.

2. B' is interior to $\angle ADB$, and $\overset{\frown}{AB'} \cap \overset{\frown}{B'B} = \{B'\}$. So, $m(\angle ADB) = m(\angle ADB') + m(\angle B'DB)$, and $m(\overset{\frown}{AB'B}) = m(\overset{\frown}{AB'}) + m(\overset{\frown}{B'B})$. But, by Exercise 1, $m(\angle ADB') = \frac{1}{2} \cdot m(\overset{\frown}{AB'})$ and $m(\angle B'DB) = \frac{1}{2} \cdot m(\overset{\frown}{B'B})$. So, $m(\angle ADB) = \frac{1}{2} \cdot m(\overset{\frown}{AB'B})$.

3. As in Exercise 2, $m(\angle ADB) + m(\angle BDB') = m(\angle ADB') = \frac{1}{2} \cdot m(\overset{\frown}{AB'}) = \frac{1}{2} \cdot m(\overset{\frown}{AB}) + \frac{1}{2} \cdot m(\overset{\frown}{BB'})$. But, from Exercise 1, $m(\angle BDB') = \frac{1}{2} \cdot m(\overset{\frown}{BB'})$. So, $m(\angle ADB) = \frac{1}{2} \cdot m(\overset{\frown}{AB})$.

4. $m(\angle STA) + m(\angle ATN) = 90 = \frac{1}{2} \cdot m(\overset{\frown}{TAN}) = \frac{1}{2} \cdot m(\overset{\frown}{TA}) + \frac{1}{2} \cdot m(\overset{\frown}{AN})$. Since $m(\angle ATN) = \frac{1}{2} \cdot m(\overset{\frown}{AN})$, it follows that $m(\angle STA) = \frac{1}{2} \cdot m(\overset{\frown}{TA})$.

5. $m(\angle STA) = 180 - m(\angle RTA) = \frac{1}{2} \cdot [360 - m(\overset{\frown}{TA})] = \frac{1}{2} \cdot m(\overset{\frown}{TKA})$

6. $a = m(\angle AB'D) + m(\angle B'AD) = \frac{1}{2} \cdot [m(\overset{\frown}{AKB}) + m(\overset{\frown}{A'K'B'})]$

7. $m(\angle P) = m(\angle BAD) - m(\angle ADE) = \frac{1}{2} \cdot [m(\overset{\frown}{BKD}) - m(\overset{\frown}{AE})]$

8. Since $m(\angle AED) = m(\angle BAE)$, it follows that $m(\overset{\frown}{AKD}) = m(\overset{\frown}{BLE})$. So, $\overset{\frown}{AKD} \cong \overset{\frown}{BLE}$.

9. Since $m(\angle TBA) = m(\angle STB)$, it follows that $m(\overset{\frown}{AKT}) = m(\overset{\frown}{TLB})$. So, $\overset{\frown}{AKT} \cong \overset{\frown}{TLB}$.

10. [as in Exercise 9]

11. [as in Exercise 8]

12. Let P′ be a point such that $T \in \overleftrightarrow{PP'}$.
Since $m(\angle P) = m(\angle P'TB) - m(\angle TBA)$, $m(\angle P) = \frac{1}{2}[m(\overset{\frown}{TKB}) - m(\overset{\frown}{TA})]$.

13. Let $\overrightarrow{PC}$ intersect $\overset{\frown}{TS}$ at M and $\overset{\frown}{TKS}$ at N. Then, $m(\angle P) = m(\angle TPC) + m(\angle CPS) = \frac{1}{2}[m(\overset{\frown}{TN}) - m(\overset{\frown}{TM})] + [m(\overset{\frown}{NS}) - m(\overset{\frown}{MS})] = \frac{1}{2}[m(\overset{\frown}{TKS}) - m(\overset{\frown}{TS})]$.
Also, $m(\angle P) = 360 - m(\angle T) - m(\angle S) - m(\angle C) = 180 - m(\angle C) = 180 - m(\overset{\frown}{TS})$.

*

Theorem 10-23 follows from the results of Exercises 4 and 5 on page 6-311 together with a third case in which [see figure] A = N. In this case, by Theorem 10-2, $m(\angle STA) = 90 = \frac{1}{2} \cdot m(\overset{\frown}{TKA})$.

Theorem 10-24 follows from the result of Exercise 6 on page 6-312.

Theorem 10-25 follows from the result of Exercise 7.

Theorem 10-26 follows from one of the results of Exercise 13, the definition of arc-measure, and the definition of supplementary angles.

Theorem 10-27 follows from the results of Exercises 8 and 9 and a third case of parallel tangents. [This third case is settled by Theorem 10-5 and the remark that semicircular arcs of the same circle are congruent.] [Exercises 10 and 11 suggest that there may be additional cases. But, Exercises 8 and 11 have a common solution, as do Exercises 9 and 10. So, the suggestion is misleading.]

Here is a graphic device which may help students recall whether to add or subtract in using Theorems 10-24 and 10-25. Let $\angle APB$ be an inscribed angle which intercepts $\overset{\frown}{AKB}$. Then, $m(\angle APB) = \frac{1}{2} \cdot m(\overset{\frown}{AKB})$. Let P move into the interior of the circle. The new angle is larger, and, so, its measure is greater than $\frac{1}{2} \cdot m(\overset{\frown}{AKB})$. If P moves into the exterior of the circle, the new angle is smaller. Therefore, its measure is less than $\frac{1}{2} \cdot m(\overset{\frown}{AKB})$.

Correction. On page 6-315, in Exercise 2 of Part C, insert 'γ = ___,' after 'β = ___,'.

Answers for Part B.

1. Since the arcs are congruent, they have the same measure. Since the measure of each angle is the difference between 180 and half the measure of the corresponding arc, the angles have the same measure. Since the angles have the same measure, they are congruent.

2. Since the measure of a semicircle is 180, the measure of an angle inscribed in a semicircle is 90. Hence, such an angle is a right angle.

Answers for Part C [on pages 6-315 and 6-316].

1. 90; 45; 45
2. 40; 50; 180
3. 35; 60; not determined
4. 120; 240; not determined, unless, as the figure may suggest, A, C, and N are collinear. In this case, m($\angle$ASN) = 105.
5. 70; 60; 110
6. 160; 200; 100

*

Exercise 6 suggests the easily established result that opposite angles of an inscribed quadrilateral are supplementary. [See Theorem 10-33 on page 6-323.] One can also prove that if two opposite angles of a convex quadrilateral are supplementary, then the vertices of the quadrilateral are concyclic. [Hint: Suppose that $\angle$B and $\angle$D are supplementary, and consider the circumcircle of ΔABC. Consider the consequences of assuming, first, that D is inside the circle and, second, that D is outside the circle.]

Correction. On page 6-320, line 3 should begin:

$\overleftrightarrow{PT}$ is a mean ---

Proof of Theorem 10-31 [stated on page 6-318].

$\angle EAB$ and $\angle BDE$ are congruent, since the measure of each is $\frac{1}{2} \cdot m(\overset{\frown}{BE})$. Consequently, their supplements, $\angle EAP$ and $\angle BDP$ are congruent. Hence, by the a.a. similarity theorem, $AEP \leftrightarrow DBP$ is a similarity. Consequently, $AP/DP = PE/PB$, and $PA \cdot PB = PD \cdot PE$.

*

Answers for Part A [on page 6-318].

1. 6 2. 4 3. 4 4. 23/4

*

Answers for Part B [on page 6-319].

1. 15/4 2. 17

*

Proof of Theorem 10-32 [stated on page 6-320].

$\angle ATP$ and $\angle TBP$ are congruent, since the measure of each is $\frac{1}{2} \cdot m(\overset{\frown}{AT})$. Consequently, by the a.a. similarity theorem, $ATP \leftrightarrow TBP$ is a similarity. So $PT/PB = PA/PT$, and $(PT)^2 = PA \cdot PB$.

*

Answers for Part A [on page 6-320].

1. 6 2. 16/3 3. 6 4. 8; 9; 12 5. 4 6. 7

Answers for Part B.

1. 8 inches; 12 inches

2. 16/5

3. If s is the measure of a secant segment from P, c is the measure of its chord, and t is the measure of the tangent segment from P, then $t^2 = (s - c)s$, and $s = \frac{c + \sqrt{c^2 + 4t^2}}{2}$ and $c = s - \frac{t^2}{s}$. So, the measure of a secant segment from P is determined by the measure of its chord, and vice versa.

4. As in Exercise 3, $t^2 = (s - c)s$. So, $t^2/(s - c) = s > s - c$, and $t^2 > (s - c)^2$. Hence, $t > s - c$. [Similarly, $t < s$.]

*

A polygon which is inscribed in a circle is convex. For, suppose A and B are adjacent vertices of a polygon inscribed in a circle of center O and radius r. If M is a third vertex then $M \notin \overleftrightarrow{AB}$. For, if $M \in \overleftrightarrow{AB}$ then, since A and B are adjacent, either $A \in \overline{MB}$ or $B \in \overline{MA}$. Since neither A nor B is at a distance less than r from O, this is impossible. Furthermore, there cannot be two vertices on opposite sides of $\overleftrightarrow{AB}$. For, if there were such vertices then there would be adjacent vertices, M and N on opposite sides of $\overleftrightarrow{AB}$. If so, $\overline{MN}$ would cross $\overleftrightarrow{AB}$ at a point P. Since $P \in \overline{MN}$, $PO < r$. Since $P \in \overleftrightarrow{AB}$ and $PO < r$, $P \in \overline{AB}$. But, since A and B, and M and N are adjacent vertices, $\overline{MN} \cap \overline{AB} = \emptyset$. Consequently, all vertices other than A and B are on one side of $\overleftrightarrow{AB}$. Since this is the case for each pair of adjacent vertices, the polygon is convex.

A polygon which is circumscribed about a circle is, also, convex. The proof will not be given here.

Correction. On page 6-322, line 5b should read:

(5) --- [Step like (2)]

The second sentence of Exercise 2 of Part A is intentionally misleading. Unless a student has drawn a square, in answer to the first part of the exercise, the description 'the circle which is inscribed in the rectangle' is nonsense.

*

In the Example, the hypothesis '$\widehat{ADC}$ is a minor arc' is unnecessary. Since, as shown in the COMMENTARY for page 6-321, ABCD is convex, it follows that B and D are on opposite sides of $\overleftrightarrow{AC}$. So, either $\widehat{ADC}$ and $\widehat{ABC}$ are both semicircles or [see COMMENTARY for page 6-298] one is a minor arc and the other a major arc. In either case, $m(\widehat{ADC}) = 360 - m(\widehat{ABC})$.

*

For a further comment relating to Theorem 10-33 on page 6-323, see the COMMENTARY for page 6-315.

Answers for Part B.

1. Since opposite angles of a parallelogram are congruent, and opposite angles of an inscribed quadrilateral are supplementary, it follows that opposite angles of an inscribed parallelogram are right angles. So, such a parallelogram is a rectangle. [Square]

2. Adjacent angles of a trapezoid which are not base angles are supplementary. Opposite angles of an inscribed quadrilateral are supplementary. So, adjacent base angles of an inscribed trapezoid are congruent. Hence, by Theorem 6-20, an inscribed trapezoid is isosceles. [Alternatively, this result can be obtained from Theorem 10-27 and a stronger form of the only if-part of Theorem 10-19: Chords of congruent arcs are congruent. In proving the latter, note that congruent arcs are both minor, both semicircular, or both major.]

3. By Theorem 10-8 and Axiom A, $AD = w + z$, $BC = x + y$, $AB = w + x$, and $DC = y + z$. So, $AD + BC = x + y + w + z = AB + DC$.

4. Suppose ABCD is a parallelogram circumscribed about a circle. By Theorem 6-1, $AB = CD$ and $BC = AD$. By Exercise 3, $AD + BC = AB + DC$. Substituting, $2 \cdot AD = 2 \cdot CD$. Hence, by Theorem 6-14, ABCD is a rhombus.

5. To have an incenter is to have an inscribed circle. The necessity of the condition has been established in Exercise 4. That the condition is sufficient--that is, that each rhombus has an incenter--follows from Theorems 6-18, 6-13, 6-5, 4-17, 4-9, and 10-2.

6. Rectangle. The necessity of the condition has been established in Exercise 1. The sufficiency--that each rectangle has a circumcenter--follows from Theorems 6-2, 6-5, and 6-11.

line 8. definition of regular polygon

line 10. Theorem 10-21

line 11. Theorem 10-19

line 13. Theorem 10-28

line 16. We didn't.

line 21. Equiangular triangles [inscribed, or not] are regular; since each rectangle is inscribable, an inscribed equiangular quadrilateral need not be regular. Consider, now, an inscribed equiangular pentagon ABCDE. [See figure on page 6-324.] Since $\angle EAB \cong \angle ABC$, it follows that $\overset{\frown}{EAB} \cong \overset{\frown}{ABC}$. So, $m(\overset{\frown}{EA}) + m(\overset{\frown}{AB}) = m(\overset{\frown}{AB}) + m(\overset{\frown}{BC})$, and $\overset{\frown}{EA} \cong \overset{\frown}{BC}$. Similarly, $\overset{\frown}{BC} \cong \overset{\frown}{DE}$, $\overset{\frown}{DE} \cong \overset{\frown}{AB}$, $\overset{\frown}{AB} \cong \overset{\frown}{CD}$, and $\overset{\frown}{CD} \cong \overset{\frown}{EA}$. By Theorem 10-19, ABCDE is equilateral.

If one tries the above procedure in the case of an equiangular hexagon, he sees that alternate sides of an inscribed equiangular hexagon are congruent; and, it is easy to draw inscribed equiangular hexagons which are not equilateral. A short meditation on evenness <u>vs.</u> oddness leads one to the conclusion that inscribed equiangular polygons with an odd number of sides are regular and that inscribed equiangular polygons with an even number of sides have alternate sides congruent. As a matter of fact, an equiangular polygon with an odd number of sides is inscribable if and only if it is regular [see above, and Theorem 10-35], and one with an even number of sides is inscribable if and only if alternate sides are congruent. Corresponding results hold concerning the circumscribability of <u>equilateral</u> polygons.

Correction. On page 6-325, line 2 should begin:

not?] for a ---

$\uparrow$

The proof of Theorem 10-35 is invalid. [The difficulty is hidden in the word 'similarly' on the 13th line from the bottom of the page.] To correct the proof, replace the 18th line from the bottom by:

this circle by showing that OD = OB.,

and replace the 14th line from the bottom through the 10th line by:

sector of $\angle ABC$, and $m(\angle OBC) = \frac{1}{2}\cdot m(\angle ABC)$.

Now, since ΔOBC is isosceles, $\angle OBC \cong \angle OCB$. Hence, $m(\angle OCB) = \frac{1}{2}\cdot m(\angle ABC)$. But, by hypothesis, $\angle ABC \cong \angle BCD$. Hence, $m(\angle OCB) = \frac{1}{2}\cdot m(\angle BCD)$. So, $\angle OCB \cong \angle OCD$. Since OC = OC and, by hypothesis, CB = CD, it follows that OCB $\longleftrightarrow$ OCD is a congruence [Why?]. Consequently, OD = OB.

*

The answer to the query in the last line of the correction, above, is 's.a.s.'.

The "theorem about numbers" mentioned in the 6th line from the bottom of the page, is the principle of mathematical induction.

*

line 3 on page 6-326. Theorem 10-3.

line 4. By Theorem 10-2, $\overleftrightarrow{AB}$ is tangent at M to the circle in question.

line 6. By the Pythagorean Theorem [or by the "corresponding medians of congruent triangles" theorem (see Exercise 1, Part E, page 6-134)], the midpoints of the sides of the polygon are equidistant from O. Consequently, by the argument referred to in lines 3 and 4, all sides are tangent, at their midpoints, to the same circle.

of each angle of the polygon is $\left(\frac{n-2}{n}\right)180$. By Exercise 3, this is also the sum of the measures of the base angles of each of the isosceles triangles. So, by Theorem 5-11, each central angle is an angle of $[1 - \frac{n-2}{n}]180°$ --that is, of $\frac{360°}{n}$.

5. By Exercise 4 and Theorem 6-33.

6. Let s_1 be the side-measure of an equilateral triangle inscribed in a circle of radius r, and let s_2 be the side-measure of an equilateral triangle circumscribed about a circle of radius r. Since the perpendicular bisectors of the sides of an equilateral triangle contains its medians, it follows from Theorem 9-3 that the radius of the circumscribed circle of an equilateral triangle is 2/3 the common measure of its medians. For an equilateral triangle of side s_1, this is $(2/3)(s_1\sqrt{3}/2)$, or $s\sqrt{3}/3$. Since the angle bisectors of an equilateral triangle are its medians and its medians are its altitudes, it follows by Theorem 9-3 that the radius of the inscribed circle of an equilateral triangle is 1/3 the common measure of its medians. For an equilateral triangle of side s_2, this is $(1/3)(s_2\sqrt{3}/2)$, or $s_2\sqrt{3}/6$. Consequently, $s_1\sqrt{3}/3 = r = s_2\sqrt{3}/6$, and $s_2 = 2s_1$. Since the ratio of the sides of the triangles is 1:2, the ratio of their perimeters is 1:2.

*

Answers for Part C [on pages 6-327 and 6-328].

1. (a) The vertices of an inscribed square are the end points of two perpendicular diameters.
   (b) $16\sqrt{2}$ (c) $\sqrt{2}$
   (d) Bisect the angles contained in the union of lines containing the diagonals of the square.
   (e) $32\sqrt{2 - \sqrt{2}}$ (f) $2\sqrt{2 + \sqrt{2}}$

2. $80 \cdot \sin 18°$; $4 \cdot \cos 18°$ [= 24.7; = 3.8]

Correction. On page 6-327, in line 8, change the colon after 'following' to a period.

In line 9, change 'equilateral' to 'equiangular'.

Answers for Part A.

1. 16/3; 8/3

2. $15\sqrt{2}$; 15

3. 10; 5

4. $2\sqrt{3}$; $\sqrt{3}$

*

Answers for Part B.

1. Suppose that ABCD. . .is an equiangular polygon circumscribed about a circle with center O, and let α be the measure of each of its angles. Let S and T be the points at which $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are tangent to the circle and let M be the midpoint of $\overline{ST}$. Since ΔBST and ΔOST are isosceles triangles with vertex angles at B and O, respectively, $\overleftrightarrow{BM} \perp \overleftrightarrow{ST}$ and $\overleftrightarrow{OM} \perp \overleftrightarrow{ST}$. Hence, $\overleftrightarrow{BM} = \overleftrightarrow{OM}$. Since ΔBST is isosceles, $\overrightarrow{BM}$ is the bisector of $\angle ABC$. Hence, $m(\angle ABO) = \alpha/2 = m(\angle CBO)$. Similarly, $m(\angle BAO) = \alpha/2$, and $m(\angle BCO) = \alpha/2$. Since OB = OB, it follows by a.a.s. that $BAO \longleftrightarrow BCO$ is a congruence. Consequently, AB = BC. So, each two adjacent sides of the polygon are congruent, and the polygon is equilateral. Since, by hypothesis, the polygon is equiangular, it follows that it is regular.

2. An apothem intersects its regular polygon only at a point where one of the sides of the polygon is tangent to the inscribed circle. Since an apothem is a radius of this circle, an apothem which intersects a side of the polygon is, by Theorem 10-2, perpendicular to that side. That it bisects the side follows by h.ℓ. and the fact that a regular polygon's incenter is its circumcenter.

3. [The solution is contained in the answer for Exercise 1, above.]

4. Each central angle is the vertex angle of an isosceles triangle whose base is a side of the polygon. By Theorem 6-31, the degree-measure

3. $4 \cdot \cos (11\frac{1}{4})°$; $128 \cdot \sin (11\frac{1}{4})°$; $4 \cdot \cos (5\frac{5}{8})°$; $256 \cdot \sin (5\frac{5}{8})°$

4. $256 \cdot \tan (5\frac{5}{8})°$

5. Consider the case in which the vertices of the n-gon are among those of the 2n-gon and apply Axiom B.

6. Consider the case in which the vertices of the inscribed polygon are the points of tangency of the circumscribed polygon, and use Axioms A and B.

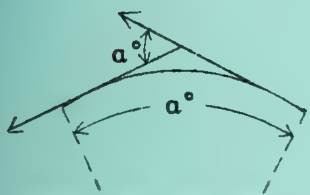
*

It is important that students realize that although they probably have a clear idea of the measure of a segment, and of the perimeter of a polygon, the question as to what is the length-measure of a circle is of an entirely different sort. One cannct, rationally, at least, imagine a circle to be a ''regular polygon with infinitely many, infinitely short sides'' and, so, treat the problem of finding its circumference as one does that of finding the perimeter of a polygon. Even in the case of a polygon, we need a definition of 'perimeter'--the perimeter of a polygon is [<u>by definition</u>] the sum of the measures of its sides. Similarly, in the case of a circle, we need a definition of 'circumference'--the circumference of a circle is [<u>by definition</u>] the least upper bound of the perimeters of inscribed polygons. To justify this choice of definition, one must show that the perimeters of inscribed polygons <u>do</u> have a least upper bound.

The following COMMENTARY goes into more detail on these matters, and on the general subject of length-measure for arcs. The latter subject is glossed over [intentionally] in Example 2 on page 6-329. There it is assumed that arcs [like circles] do have length-measures, and that since a circle is a union of six 60°-arcs, the length-measure of any 60°-arc will be one-sixth the circumference of its circle. The hidden assumptions are (1) that each arc has a length-measure, (2) that congruent arcs have the same length-measure, (3) that the length-measure of an arc which is the union of two arcs with only an end point in common is the sum of the length-measures of the two arcs, and (4) that the sum of the length-measures of the two arcs determined by two points of a circle is the circumference of the circle.

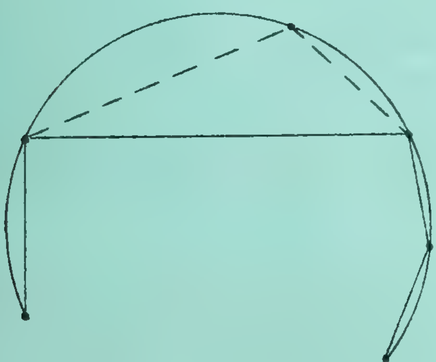
*

On arc-measure. --The degree-measure of an arc is simply the degree-measure of "the angle between the directions of the forward tangents" at the end points of the arc--that is, it is the degree-measure of "the angle through which the tangent rotates as the point of tangency moves from one end of the arc to the other". In still other words, the degree-measure of an arc is a measure of amount the arc bends. It is for this reason that arc-degree-measure is convenient for measuring angles associated with an arc. Since the "rate of bending" is the same for arcs of the same or congruent circles, arc-degree-measure is also suitable for comparing "lengths" of arcs of the same or congruent circles. But, before we can make sense of either "rate of bending" or "length (of arc)", we need to know what is meant by the length-measure [or: linear measure] of an arc.



What we want is a concept of arc-measure which will grow naturally out of the concept of segment measure. In this COMMENTARY, we shall develop such a length-measure concept for circular arcs. [As a matter of fact the same concept applies to a much larger class of sets, called rectifiable arcs. These include segments, polygonal lines, circular arcs, and many other sets.]

Given a circular arc, the first question we need to answer is: What do we mean by its length-measure? At the moment, we have no ready answer, and our first problem is to frame a suitable definition. As a first step, let us choose some points, in order, on the arc including the end points among them, and join successive ones by segments. It is natural to define the length-measure of the inscribed polygonal line so obtained as the sum of the measures of its "sides". And, it is also natural to require that, however we may come to define the length-measure of a circular arc, this length-measure should, in some sense, be approximated by the length-measures of the polygonal lines which are inscribed in the arc. As to the length-measures of such polygonal lines, we can at once make two observations. First, there is a shortest such polygonal line--the segment whose end points are those of the given arc. This observation is no more



important than it probably appears to be. But, second, there is no longest polygonal line inscribed in the given arc. For, given any inscribed polygonal line, we can find a longer one by replacing one of its sides by two segments which, together with it, are the three sides of a triangle [see dotted lines in figure]. We may now guess that, the longer an inscribed polygonal line is, the better its length-measure should approximate the still-to-be-defined length-measure of the given arc. For this to be so, the length-measure of the arc must be a number which is not less than the length-measure of any inscribed polygonal line. If there is a <u>least</u> such <u>upper</u> <u>bound</u> to the set of length-measures of inscribed polygonal lines, then it will have the further property that it can be approximated as closely as one desires by the length-measures of an inscribed polygonal lines. Such a number, if there is one, would be an ideal candidate for the tital of the length-measure of the given circular arc. As, by now, you probably suspect, it can be proved that, given any circular arc, the set of numbers which are length-measures of polygonal lines inscribed in the arc does have a least upper bound; and this least upper bound is, by definition, the length-measure of the given arc.

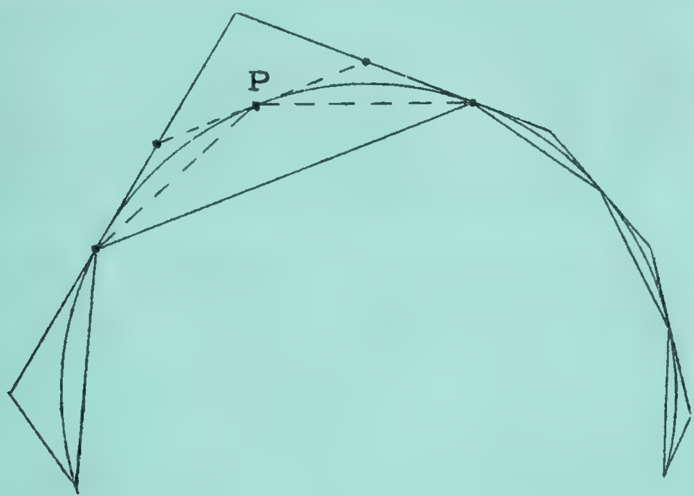
For the proof we need to use a basic principle for numbers of arithmetic [there is an analogous one for real numbers] which we shall call the <u>least upper bound principle</u>. To prepare for stating this principle we note, first, that, given any set S of numbers, any number which is not less than each member of S is called <u>an upper bound of S</u>. For example, each number which is greater than or equal to 3 is an upper bound for the set of all numbers of arithmetic between 2 and 3. On the other hand, the set of whole numbers has no upper bound. Now, the least upper bound principle is simply this:

> Each set of numbers of arithmetic which has an upper bound has a least upper bound.

In other words, if a set of numbers of arithmetic has an upper bound, then the set of all its upper bounds has a smallest member.

In view of this principle, to show that the set of length-measures of polygonal lines inscribed in a given arc has a least upper bound, it is sufficient to show that it has <u>some</u> upper bound. This is easy to do. We begin by noting that, given any polygonal line inscribed in an arc, there is a corresponding polygonal line circumscribed about the arc which

consists of segments of the tangents to the arc at the vertices of the given inscribed polygonal line.



It follows from Axiom B that an inscribed polygonal line is shorter than the corresponding circumscribed polygonal line. And it follows from the same axiom that introducing a new vertex [P] results in an inscribed polygonal line which is longer than the given one, and a corresponding circumscribed polygonal line which is shorter than that corresponding to the given inscribed polygonal line.

From these two remarks it follows that each inscribed polygonal line is shorter than each circumscribed polygonal line. For, given any inscribed polygonal line ℓ and any circumscribed polygonal line L, we can take the vertices of ℓ together with the points of tangency of L as the vertices of an inscribed polygonal line ℓ_1 and, also, as the points of tangency of a circumscribed polygonal line L_1. Since the vertices of ℓ are among those of ℓ_1, ℓ is shorter than ℓ_1. Since ℓ_1 and L_1 are corresponding, ℓ_1 is shorter than L_1. Since the points of tangency of L_1 include those of L, L_1 is shorter than L. So, ℓ is shorter than L.

One consequence of the result just established is that the set of numbers which are length-measures of polygonal lines inscribed in a given arc does have upper bounds. For, the length-measure of each polygonal line circumscribed about the arc is one such upper bound. Consequently, by the principle of least upper bounds, given any arc, there is a number which is the least upper bound of the length-measures of polygonal lines inscribed in the arc.

As we said before, the length-measure of the given arc is defined to be this number. Arc-length-measure has a number of important properties. We shall mention six of them.

In the first place, a change in the unit for segment-measure has the expected effect on length-measures of arcs. For example, since doubling the unit segment halves the length-measure of each segment, it also halves the length-measure of each polygonal line. So, doubling the unit segment halves the length-measure of each arc.

In the second place, congruent arcs have the same length-measure. For, given any polygonal line inscribed in one of two congruent arcs, there is [by s.a.s.] a polygonal line inscribed in the other which has the same length-measure. Hence, the measure of each of the two arcs is the least upper bound of the same set of numbers.

Third [by the same argument, but using the s.a.s. similarity theorem], length-measures of arcs which have the same degree-measure are proportional to the radii of the arcs. In particular, the length-measure of an arc of $\alpha°$ of a circle of radius r can be found by multiplying the length-measure of an $\alpha°$-arc of a circle of radius 1 by r.

In the fourth place, length-measure for arcs is <u>additive</u>, in the sense that the length-measure of an arc which is the union of two arcs which have only an end point in common is the sum of the length-measures of the two arcs. In contrast to the corresponding theorem on arc-degree-measure [Theorem 10-21], this is very easy to prove. For, if A, B, and C are three points on a circle then each polygonal line inscribed in $\overset{\frown}{ABC}$ which does not have B as one vertex is shorter than some inscribed polygonal line which does have B as a vertex. So, the length-measure of $\overset{\frown}{ABC}$ is the least upper bound of the length-measures of those inscribed polygonal lines which have B as one vertex. But each of these is the union of a polygonal line inscribed in the portion of $\overset{\frown}{ABC}$ "between" A and B and a polygonal line inscribed in the portion between B and C. So, the length-measure of $\overset{\frown}{ABC}$ is the sum of the length-measure of these subarcs.

The fifth property is that, for arcs of the same radius, length-measure is proportional to degree-measure. This follows in a fairly straightforward way, using the additivity property. For example, if $\overset{\frown}{ABC}$ is an arc of $(2a)^\circ$ which is bisected at B then each of the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are arcs of a° and, since they are congruent, have the same length-measure. So, by additivity, the length-measure of an arc of $(2a)^\circ$ is twice the length-measure of an arc of a°.

Finally, the length-measure of an arc is greater than the length-measure of each inscribed polygonal line, and is less than the length-measure of each circumscribed polygonal line. For, the length-measure of an arc is, by definition, the least upper bound of the length-measures of inscribed polygonal lines, and, as we have seen, the length-measure of each circumscribed polygonal line is an upper bound of these numbers.

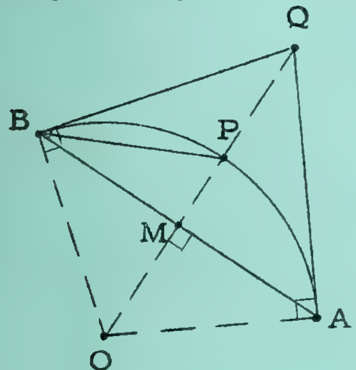
The next question is this: Having defined length-measure for arcs, and established some of its properties, how can we compute the length-measure of a "given" arc? Specifically, given the degree-measure and the radius of an arc, how can we compute its length-measure? This turns out not to be difficult. Recall that the length-measure of arcs which have the same degree-measure are proportional to their radii, and that the length-measures of arcs which have the same radii are proportional to their degree-measures. It follows that there is a number k such that the length-measure s of any arc of degree-measure a and radius r is kar. So, to relate length-measure to degree-measure, all we need is to know the number k. We can determine this number by finding the length-measure of some one arc of given degree-measure and radius. For simplicity, we shall choose, for this arc, a semicircle of radius 1. It is customary to denote the length-measure of a semicircle of radius 1 by the Greek letter 'π'. Adopting this convention, $\pi = k \cdot 180 \cdot 1$, and $k = \pi/180$. So, the formula relating length-measure, s, degree-measure, a, and radius, r, is:

$$s = \frac{\pi}{180} ar$$

[However, until we have computed π, this is no more useful than the formula '$s = kar$'.]

In order to compute π--that is, to compute the length-measure of a semicircle of radius 1--we shall find the length-measures of some polygonal lines inscribed in such a semicircle.

To begin with, we need some results on lengths of chords and segments of tangents of circles of radius 1. Suppose that A and B are end points of an arc of a circle of radius 1, and that AB = c. Let M be the mid-point of $\overline{AB}$, P the bisection point of $\overset{\frown}{AB}$, and Q the point of intersection of the tangents to the circle at A and B. The points O, M, P, and Q are collinear, in that order; $\overleftrightarrow{OM} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{OB} \perp \overleftrightarrow{BQ}$, OB = 1 = OP, BM = c/2, and QA = BQ.



Using these results, the Pythagorean Theorem, and Axiom A, we see that

$$(OM)^2 = (OB)^2 - (BM)^2 = 1 - c^2/4,$$

$$PM = OP - OM = 1 - \sqrt{1 - c^2/4}\,,$$

$$(BP)^2 = (BM)^2 + (PM)^2 = c^2/4 + [1 - \sqrt{1 - c^2/4}]^2$$

$$= c^2/4 + [1 - 2\sqrt{1 - c^2/4} + 1 - c^2/4]$$

$$= 2 - 2\sqrt{1 - c^2/4} = 2 - \sqrt{4 - c^2}\,,$$

(1) $$BP = \sqrt{2 - \sqrt{4 - c^2}}\,.$$

By the a.a. similarity theorem, OBQ ⟷ OMB is a similarity. Hence, BQ/MB = OB/OM. Consequently, $BQ = c/[2\sqrt{1 - c^2/4}\,] = c/\sqrt{4 - c^2}$ Finally,

(2) $$BQ + QA = 2c/\sqrt{4 - c^2}\,.$$

Since, by (1), $(BP)^2 = 2 - \sqrt{4 - c^2}$, it follows that $(BP)^2 \leq 2$ and $2 - (BP)^2 = \sqrt{4 - c^2}$. From this last it follows that $4 - 4 \cdot (BP)^2 + (BP)^4 = 4 - c^2$. So, $(BP)^2[4 - (BP)^2] = c^2$. But, since $(BP)^2 \leq 2$, $4 - (BP)^2 > 2$. Hence,

(3) $$(BP)^2 \leq c^2/2.$$

We now consider a sequence $\ell_1, \ell_2, \ell_3, \ldots$ of polygonal lines inscribed in a semicircular arc of radius 1, and the sequence $L_1, L_2, L_3, \ldots$ of corresponding polygonal lines circumscribed about this semicircle. The

vertices of ℓ_1 [and the points of tangency of L_1] are the two end points and the bisection point of the semicircle. For each n, the vertices of ℓ_{n+1} [and the points of tangency of L_{n+1}] are the vertices of ℓ_n together with the bisection points of the arcs whose end points are adjacent vertices of ℓ_n. For each n, ℓ_n is the union of 2^n congruent segments, and L_n is the union of $2^n + 1$ segments--2 congruent ones tangent at the ends of the semicircle and $2^n - 1$ which are tangent at intermediate points and are twice as long as those at the ends. So, if c_n is the measure of a side of ℓ_n then the length-measure, s_n, of ℓ_n is $2^n c_n$; and if C_n is the measure of one of the longer sides of L_n, the length-measure, S_n, of L_n is $2^n C_n$.

Since the diameter of the semicircle is 2, it follows from (1) that $c_1 = \sqrt{2 - \sqrt{4 - 2^2}} = \sqrt{2}$. Similarly, $c_2 = \sqrt{2 - \sqrt{4 - c_1{}^2}} = \sqrt{2 - \sqrt{2}}$, $c_3 = \sqrt{2 - \sqrt{2 + \sqrt{2}}}$, etc. So, $s_1 = 2\sqrt{2}$, $s_2 = 4\sqrt{2 - \sqrt{2}}$, $s_3 = 8\sqrt{2 - \sqrt{2 + \sqrt{2}}}$, etc. The numbers s_1, s_2, s_3, etc. are successively better approximations to the length-measure s of the semicircle--that is, to the number π. With a considerable amount of labor one can find that $s_{10} = 3.141591$, and that $S_{10} = 3.141595$. [$\pi = 3.14159265358979323846264338 3 \ldots$]

Since the polygonal lines ℓ_1, ℓ_2, ℓ_3, etc. are only a relative few of the polygonal lines inscribed in the semicircle, we do not, as yet, know that, for n sufficiently large, s_n is an arbitrarily good approximation to π. To prove that it is, we recall that the length-measure π of the semicircle is not only greater than s_n, but is also smaller than S_n. Now, by (2),

$$C_n = \frac{2}{\sqrt{4 - c_n{}^2}}\, c_n .$$

Hence,

$$S_n = \frac{2}{\sqrt{4 - c_n{}^2}}\, s_n .$$

Consequently,

$$s_n < \pi < \frac{2}{\sqrt{4 - c_n^2}} s_n ;$$

so,

$$0 < \pi - s_n < \left[\frac{2}{\sqrt{4 - c_n^2}} - 1\right] s_n$$

$$< \left[\frac{2}{\sqrt{4 - c_n^2}} - 1\right] \pi$$

Hence, in order to show that, for n sufficiently large, s_n differs from π by as little as we wish, it is sufficient to show that, for n sufficiently large, $2/\sqrt{4 - c_n^2}$ differs from 1 by as little as we wish. Intuitively [and we shall take the matter no further here], this will be the case if $\sqrt{4 - c_n^2}$ is arbitrarily close to 2. And, again on an intuitive basis, this will be the case if c_n is arbitrarily small. Now, as we know, $c_1 = \sqrt{2}$ and, by (3), for each n, $c_{n+1} \leq c_n/\sqrt{2}$. So [as can be proved by mathematical induction], for each n, $c_n \leq 1/[\sqrt{2}]^{n-2}$. Hence, it seems likely [but still requires proof] that, for n sufficiently large, c_n is arbitrarily small. [Of course, it is "obvious from the figure" that the side measures of the successive polygonal lines ℓ_1, ℓ_2, ℓ_3, etc. approach 0.]

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As a generalization of the property of additivity of arc-measure, it is natural to define the length-measure of a circle [its circumference] to be the sum of the length-measures of any two of its arcs which have the same end points. Due to additivity, it makes no difference which two points of the circle one chooses as end points of the arcs; and it turns out that, according to this definition, the circumference of a circle of radius r is $2\pi r$.

*

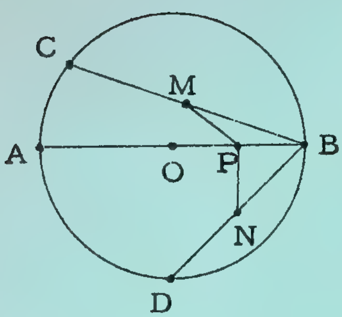
Answers for Exercises [on page 6-330].

1. 4π 2. π 3. 3π; 6π; 72; 216; 18; —; $180/\pi$; 30

[To drive home the fact that π is an honest number, students should be required to give rational approximations to 3π, 6π, and $180/\pi$ $3\pi \doteq 12.42$, $6\pi \doteq 24.85$, and $180/\pi \doteq 57.296$.]

4. $80\pi/3$ [$\doteq 83.78$] 5. insufficient data 6. $k\pi$

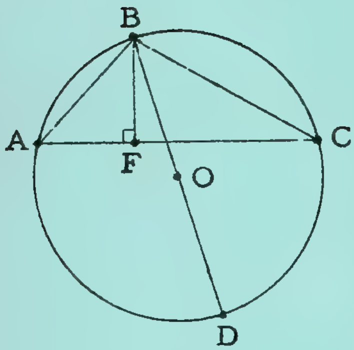
11.



<u>Hypothesis:</u> O is the center of the circle, BM = MC, BN = ND, BP = PO

<u>Conclusion:</u> MP = PN

12.



<u>Hypothesis:</u> $\overline{BD}$ is a diameter of the circle with center O,

$\overline{BF}$ is the altitude of $\triangle$ABC from B

<u>Conclusion:</u> AB $\cdot$ BC = BD $\cdot$ BF

*

Answers for Quiz.

1. 5	2. 130	3. 42	4. 13	5. 75
6. 3	7. 8	8. 150	9. 9π	10. 30

11. By theorem 10-7, $\triangle$OMB is right-angled at M. Hence, median $\overline{MP}$ is half as long as $\overline{OB}$. Similarly, median $\overline{NP}$ of right triangle $\triangle$ONB is half as long as $\overline{OB}$. So, MP = PN.

12. By Theorem 10-29, $\angle$BAD is a right angle. By hypothesis, so is $\angle$BFC. By Theorem 10-28, $\angle BDA \cong \angle BCA$. So, by the a.a. similarity theorem, BAD $\longleftrightarrow$ BFC is a similarity. So, BA/BF = BD/BC. Therefore, AB $\cdot$ BC = BD $\cdot$ BF.

Correction. On page 6-331, line 16 should begin:
circumscribed polygon [6-321]

On page 6-334, line 11b should read:
--- tangent segment is a mean ---
↑

Quiz.

1. An 8-inch chord is 3 inches from the center of a circle. Find the length of a radius of the circle.

2. Suppose that $\overline{AB}$ is a diameter of a circle and is perpendicular to a chord $\overline{CD}$. If $\overset{\frown}{BD}$ is an arc of 50°, how many degrees are there in $\overset{\frown}{AC}$?

3. A regular hexagon is inscribed in a circle whose radius is 7. What is the perimeter of the hexagon?

4. What is the radius of a circle whose center has coordinates (0, 0) and which passes through a point with coordinates (−12, 5)?

5. Suppose that the lines $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents to a circle at A and B, respectively. If $\overset{\frown}{AB}$ is an arc of 105°, what is $m(\angle APB)$?

6. Suppose that $\overline{PQ}$ and $\overline{MN}$ are chords of a circle and that $\overline{PQ} \cap \overline{MN} = \{R\}$. If PR = 5, QR = 6, and MR = 10, what is NR?

7. Suppose that AB = 10 and that A is the center of a circle with radius 6. What is the measure of a tangent segment from B to the circle?

8. Suppose that A and B are points on a circle with center O such that $\angle AOB$ is an angle of 60°. If C is a point on the minor arc $\overset{\frown}{AB}$, what is $m(\angle ACB)$?

9. What is the circumference of a circle inscribed in a square whose perimeter is 36?

10. Suppose that quadrilateral ABCD is inscribed in a circle and that $\overrightarrow{AB} \cap \overrightarrow{DC} = \{P\}$. If $m(\overset{\frown}{AB}) = 50$, $m(\overset{\frown}{BC}) = 70$, and $m(\overset{\frown}{CD}) = 110$, what is $m(\angle BPC)$?

Answers for Part A.

1. 24   2. 24   3. 3   4. $25\sqrt{3}/2$

*

Answers for Part B [on pages 6-336 and 6-337].

1. $K(\blacktriangle A'B'C') = \frac{1}{2} \cdot B'C' \cdot A'D' = \frac{1}{2} r \cdot BC \cdot AD = r \cdot K(\blacktriangle ABC)$
   [Note that the figure for Exercise 1 is misleading. It does not suggest, as it should, that $A'D' = AD$; and it does suggest, as it should not, that $B'D'/D'C' = BD/DC$.]

2. This follows from the definition of area-measure and the theorem on corresponding altitudes of congruent triangles. [See Exercise 1 of Part E on page 6-134.]

3. No. [See Exercises 1 and 2 of Part A, above.]

4. Same computation as for Exercise 1. [The figure is, again, misleading.]

5. The set is the union of two lines parallel to and equidistant from $\overleftrightarrow{AB}$. [The distance of each line from $\overleftrightarrow{AB}$ is the measure of the altitude of $\triangle ABC$ from C.

6. 2:1   7. 3:7

*

The "filled-in-triangle" symbol used in naming a triangular region and the other filled-in symbols are introduced as a device for making sure the student's attention is called to the fact that the domain of the area-measure function consists of regions rather than, for example, polygons. A student who has been exposed to this kind of careful treatment at the beginning will not be confused by colloquialisms such as those found in Part B on pages 6-341 through 6-343.

Obviously either Axiom I or Theorem 11-1 can be taken as an axiom and the other derived from it by use of Axiom J. [The derivation of Axiom I from Theorem 11-1 and Axiom J goes in three steps. First, derive the case of Axiom I in which the triangle there referred to is a right triangle and the altitude and base are its legs--then extend this result to arbitrary triangles, taking the longest side as base [so the corresponding altitude intersects the base]--then use the result already referred to from page 6-214.] Since each polygon region can be split up into triangular regions, Axiom I is a more convenient starting point than is Theorem 11-1.

*

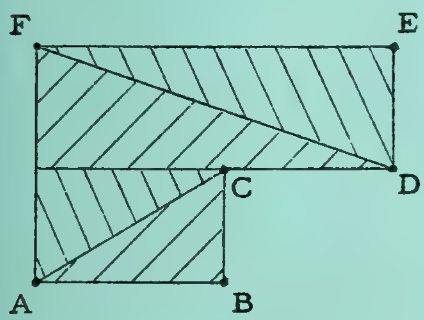
The measure of the polygonal region ABCDEF [page 6-339] is 115.

*

line 2 on page 6-340. The region ABCDEF can be cut up into 4, but no fewer, triangular regions.

*

Theorem 11-2 is intuitively obvious, in view of Axiom J. However, its proof requires considerable attention to "Introduction matters", as well as a careful definition of 'boundary'. Consequently, it is best illustrated by examples. For instance:



The measure of the polygonal region ABCDEF is the sum of the measures of the four triangular regions. But, the sum of the measures of two of these is the measure of the upper rectangular region, and the sum of the measures of the other two is the measure of the lower rectangular region. So, the measure of the polygonal region ABCDEF is the sum of the measures of the two rectangular regions.

Answers for Part A.

1. $K = bh$ 2. $K = \frac{1}{2}(b_1 + b_2)h$ 3. $K = \frac{d_1 d_2}{2}$ 4. $K = \frac{s^2\sqrt{3}}{4}$

*

Answers for Part B [on pages 6-341, 6-342, and 6-343].

1. \$99.84 2. 112.5 3. 112.5 square inches

4. $375\sqrt{3}/2$ 5. $200\sqrt{21}$ square feet 6. $K = b^2 - a^2$

7. The two triangles have congruent bases and the same altitude. So, the two triangular regions have the same area-measure.

8. One diagonal divides the region into two triangular regions with congruent bases [Theorem 6-1] and congruent altitudes [Theorem 6-29]. By Theorem 6-5, the other diagonal contains a median of each of these two triangles and, so, by Exercise 7, divides each into two triangular regions having the same measure.

9. $6\sqrt{91}$ 10. 10 inches 11. $25\sqrt{3}$ 12. 13

13. The set is the union of two lines parallel to $\overline{AB}$ and each at a distance 4 from $\overleftrightarrow{AB}$.

14. $2\sqrt{5}$, $2\sqrt{15}$, $4\sqrt{5}$ 15. 16

Correction. On page 6-343, line 4b, insert a '☆' to the left of '22.' .

16. 72 17. 75/2 18. 800/3

19. By Theorem 6-24 and Theorem 7-1, the measure of the altitudes, from M and N, of $\triangle AMP$ and $\triangle PNC$, respectively, is half the measure of the altitude of $\triangle ABC$ from P. Hence, $K(\triangle AMP) + K(\triangle PNC) = \frac{1}{2} \cdot K(\triangle ABC)$. So, by Theorem 11-2, the area-measure of MPNB is, also, half that of $\triangle ABC$.

20. $K = \frac{1}{2}bc \cdot \sin a°$ 21. $K = \frac{5s^2}{4} \cdot \tan 54°$

☆22. By the Pythagorean Theorem, $c^2 - (b - x)^2 = h^2 = a^2 - x^2$.

So, $x^2 - (b - x)^2 = c^2 - a^2$,

that is, $(2x - b)b = a^2 - c^2$.

Hence, $x = (a^2 + b^2 - c^2)/(2b)$.

Consequently, $h^2 = a^2 - [(a^2 + b^2 - c^2)/(2b)]^2$,

and $4b^2h^2 = 4a^2b^2 - (a^2 + b^2 - c^2)^2$.

So,
$$\begin{aligned} 4b^2h^2 &= [2ab + (a^2 + b^2 - c^2)][2ab - (a^2 + b^2 - c^2)] \\ &= [(a + b)^2 - c^2][c^2 - (a - b)^2] \\ &= [(a + b + c)(a + b - c)][(a - b + c)(-a + b + c)] \\ &= 2s \cdot 2(s - a) \cdot 2(s - b) \cdot 2(s - c). \end{aligned}$$

Hence, $b^2h^2/4 = s(s - a)(s - b)(s - c)$,

and $K(\triangle ABC) = \sqrt{s(s - a)(s - b)(s - c)}$.

Answers for Exploration Exercises.

1. Since DB/AB = EB/CB, and $\angle B \cong \angle B$, it follows by the s.a.s. similarity theorem that ABC $\longleftrightarrow$ DBE is a similarity.

2. By Theorem 6-24, DE = 7/2. So, the perimeter of $\triangle DBE = 2 + 3 + \frac{7}{2}$ = half the perimeter of $\triangle ABC$.

3. By Theorem 7-1, the ratio of the altitudes from B is 2:1.

4. So, since the ratio of the bases is 2:1, the ratio of the area-measures is 4:1.

*

Note that the ratio of similitude is defined in terms of a given similarity between $\triangle ABC$ and $\triangle A'B'C'$. For completeness, we should show that if there are two matchings of the vertices of the triangles which are both similarities, then one gets the same ratio of similitude in both cases. This is easy to do. For, suppose besides the matching ABC $\longleftrightarrow$ A'B'C', some other matching is a similarity. This second similarity will match one of the vertices, say, A, with another vertex than before, say, B'. Since both matchings are similarities, $\angle A' \cong \angle A \cong \angle B'$. [The first congruence stems from the first similarity, the second congruence from the second similarity.] Since $\angle A' \cong \angle B'$, it follows that $a' = b'$. Since, for the first similarity, the ratio of similitude is a/a' and, for the second, it is a/b', it follows that the ratio of similitude is independent of which similarity one uses.

Answers for Part A [on page 6-345]. 4:9

*

Answers for Part B.

1. 9:100 2. 4:9 3. 1:16

[In Exercise 3, since the ratio of the distances to the left-hand vertices is 1:4, it follows by Theorem 7-1 that the ratio of the distances to the top vertices is 1:4. So, again by Theorem 7-1, the ratio of the distances to the right-hand vertices is 1:4. Hence, if the measures of the bases of the triangles are x and y, $(1 + x)/(4 + y) = 1/4$. So, $x/y = 1/4$. Since the ratio of similitude is 1/4, the ratio of the area-measures is 1/16.]

*

Answers for Part C.

$MC/AC = 1/\sqrt{2}$. So, $\frac{AM + MC}{MC} = \sqrt{2}$, $\frac{AM}{MC} = \sqrt{2} - 1$,

and $\frac{MC}{MA} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$.

*

Answers for Part D.

1. Since AB = BC = CD = DE and A′B′ = B′C′ = C′D′ = D′E′, A′B′/AB = B′C′/BC = C′D′/CD = D′E′/DE. So, since all right angles are congruent, the squares are similar.

2. Suppose that the ratio of similitude of ABCDE to A′B′C′D′E′ is k. Since AB = k·A′B′, AE = k·A′E′, and $\angle A \cong \angle A'$, it follows by the s.a.s. similarity theorem that BE = k·B′E′. Similarly, BD = k·B′D′. Since, by hypothesis, ED = k·E′D′, it follows by the s.s.s. similarity theorem that BED $\longleftrightarrow$ B′E′D′ is a similarity.

3. As shown in Exercise 2, the ratio of corresponding diagonals of similar quadrilaterals is the ratio of similitude. [Now, any quadrilateral has at least one diagonal which divides the corresponding quadrangular region into two triangular regions. (Either diagonal of a <u>convex</u> quadrilateral will do this). And, for similar quadrilaterals, corresponding diagonals both have, or both fail to have, this property.] So, by the s.s.s. similarity theorem, each of the

boundaries of the triangular regions into which a diagonal of a quadrilateral divides its quadrangular region is similar to the boundary of the corresponding one of the triangular regions into which the corresponding diagonal of a similar quadrilateral divides its quadrangular region. Hence, the ratio of the area-measures of such corresponding triangular regions is the square of the ratio of similitude. Since the area-measure of each quadrangular region is the sum of the area-measures of the two triangular regions into which it is divided, [and since multiplication is distributive with respect to addition], it follows that the ratio of the area-measures of the two quadrangular regions is also the square of the ratio of similitude.

4. The ratio of the area-measure of [the regions bounded by] two similar polygons is the square of the ratio of similitude.

*

Answers for Part E [on page 6-347].

1. 3/7; 3/7; 9/49 2. 1/5 3. 1225, 400

4. As should, by now, be well-known, all angles shown in the figure are angles of 30°, 60°, 90°, or 120°. Each side of the boundary of the shaded region occupies the middle third of a short diagonal of the regular hexagon ABCDEF, and the measure of each short diagonal is $\sqrt{3}\cdot AB$. So, the measure of each side of the shaded region is $(\sqrt{3}/3)\cdot AB$. Hence, the boundary of the shaded region is similar to the regular hexagon ABCDEF, the ratio of similitude being $\sqrt{3}/3$. So, by the theorem stated in answer to Exercise 4 of Part D, the ratio of the area-measure of the shaded region to that of the region bounded by ABCDEF is $(\sqrt{3}/3)^2$, or 1/3.

Note that, as in the text on page 6-347, and in the exercises on page 6-348, the word 'apothem' is used, not only to refer to a radius of an inscribed circle, but also to the radius of such a circle.

*

Answers for Part A.

1. $1/2$

2. $1/\sqrt{2}$

3. $\sqrt{3}/2$

4. The region bounded by a regular n-gon is divided, by the radii to its vertices, into n congruent triangular regions. The measure of each triangular region is half the product of the measure of a side of the polygon and the measure of an apothem. So, the area-measure of the polygonal region is n times this product. Since the perimeter of the polygon is n times the measure of one of its sides, it follows that the area-measure of the polygonal region is half the product of its perimeter by its apothem.

The text on page 6-349 overlooks a very important point which you should make clear to your students. As in the similar situation concerning the circumference, or length-measure, of a circle [see the COMMENTARY for page 6-329], nothing up to this point gives any meaning to the phrase 'area-measure of a circular region'. We need a definition. And, although we shall treat exercises such as those on pages 6-350 and 6-351 quite informally, our definition should be such as to apply to regions like those pictured there.

The treatment leading up to Theorem 11-4 suggests that one define the area-measure of a "region" to be the least upper bound [if such exists] of the area-measures of polygonal regions whose boundaries are inscribed in the boundary of the given region. As a matter of fact this would be adequate for circular regions or, indeed, for any convex region. But, this definition clearly assigns an area-measure larger than one wishes to regions such as those pictured in Exercise 3 on page 6-350. It turns out that an adequate definition is: The area-measure of a region is the least upper bound of the area-measures of the polygonal regions which are subsets of the given region. [Of course, to make clear sense of this, one must have a satisfactory definition of 'region'. This we refrain from giving.] With this definition, a theorem like Theorem 11-2, but with the word 'polygonal' omitted [both times], can be proved. The previously rejected definition, when modified to apply only to convex regions, also becomes a theorem. And, on the basis of this latter theorem, the argument on page 6-349 is at least fair evidence for Theorem 11-4. [What, principally, is missing is an argument to show that the area-measure of any polygonal region inscribed in a circular region is less than that of some inscribed <u>regular</u> polygonal region.]

*

Answers for Part B.

1. (a) 24π (b) 6 (c) $8\pi\sqrt{2}$ (d) 51.84π

2. (a) 36π (b) $81/\pi$ (c) 192π (d) 1616.04π

3. (a) 8 (b) $2\sqrt{2}$ (c) $10/\sqrt{\pi}$ (d) 6.3

4. $13/\sqrt{\pi}$

5. 144π [Note that, here, we use the analogue, mentioned above, of Theorem 11-2.] Ask students if they could have answered the exercise had the word 'concentric' been omitted.

Correction. On page 6-350, change the sentence in Exercise 3 to read:

Prove that if $\overleftrightarrow{AB}$, $\overleftrightarrow{AO}$, and $\overleftrightarrow{OB}$ are diameters, the ---

Answers for Part C.

1. $\pi(r/2) + \pi(r/2) = \pi r$

2. $\pi[(r + x)/2] + \pi[(r - x)/2] = \pi r$ $[x = OP]$

3. [By the previously mentioned analogue of Theorem 11-2], the area-measure of each of the shaded regions is half the area of the circular region which is their union. [Ask whether there is a generalization of Exercise 3, like the generalization of Exercise 1 which is given in Exercise 2.]

4. $(2\pi R)/(2\pi r) = R/r$; $(\pi R^2)/(\pi r^2) = R^2/r^2$. [One may, reasonably, call the ratio of the radii of two circles their ratio of similitude, and speak of the circles as similar. Then, as for polygons, the ratio of the area-measures of two (similar) circular regions is the square of this ratio of similitude. See below, for further discussion.]

5. $\pi R^2 = \pi D^2/4$, $\pi r^2 = \pi d^2/4$, $(\pi D^2/4)/(\pi d^2/4) = D^2/d^2$

6. 16

*

The concept of similarity can be extended to arbitrary sets of points. One procedure is to say that two sets are similar if and only if all the points of one can be matched, one-to-one, with all the points of the other in such a way that corresponding angles are always congruent. Such a matching is called a similarity. It follows, now, from the a.a. similarity theorem that, in any similarity, the distance between each two points in one set is proportional to the distance between the corresponding points of the other set. [Conversely, by the s.s.s. similarity theorem, each matching which has this proportionality property is a similarity.] Consequently, this extended notion of similarity agrees, for polygons, with the usual more elementary notion. [In fact, two polygons are similar, in either the usual or the extended sense, if and only if there is a matching of their vertices such that corresponding angles of the polygons are congruent and angles containing pairs of corresponding diagonals are congruent.] The ratio between corresponding distances is called 'the ratio of similitude', and if this ratio is 1 then the sets are said to be congruent. As in the elementary case, the ratio of the length-measures of two similar "curves" is the ratio of similitude, and the ratio of the area-measures of two similar "regions" is the square of the ratio of similitude.

Correction. On page 6-351, line 4 should read: area-measure of a circular sector.

A circular sector is a region whose boundary is the union of an arc and the radii to the end points of the arc. [Since two points of a circle determine two arcs, they also determine two sectors, a minor sector and a major sector, or two semicircular sectors.]

Formula (1) for the area-measure of a circular sector applies to all circular sectors; formula (2) applies only to minor ones. The simplest proof of formula (1) is like that of Theorem 11-4. Perhaps the best thing to do is to approximate the proof by drawing pictures showing polygonal lines inscribed in arc $\overset{\frown}{AB}$ and triangular regions based on the sides of these polygonal lines and having O as one vertex. Point out [as you no doubt did in discussing Theorem 11-4] that the length-measures of such polygonal lines approximate the length-measure of $\overset{\frown}{AB}$, and that the unions of the triangular regions tend to fill up the sector. Repeat, with $\overset{\frown}{AB}$ replaced by a major arc. This should be sufficient.

For formula (2), remind students of Exercise 3 on page 6-330. In doing this exercise they must have become aware of the fact that the length-measure of an arc is $\pi/180$ times its degree-measure. Hence, the length-measure of a minor arc of radius r is $\pi r/180$ times the degree-measure of its central angle. So, in the case of a minor arc, $\frac{1}{2}rs = \frac{1}{2}r \cdot \frac{\pi r}{180}\theta$. [For a mnemonic, point out that a sector of $\theta°$ is theta-three-sixtieths of a circular region.]

*

Answers for Part A [on pages 6-351 and 6-352].

1. 25π
2. $15\pi/4$
3. $1250\pi/3$ $[\pi(25)^2 - \frac{1}{3}\pi(25)^2]$
4. $50\pi/3 - 25\sqrt{3}$ [The shaded region is a circular segment--that is, a region whose boundary is the union of an arc and its chord.]
5. $100\pi/3 - 25\sqrt{3}$ [The shaded region has the same area-measure as does a segment of a circle of radius 10 which subtends a central angle of 120°.]

Correction. On page 6-352, the last line of Exercise 8 should be 'of ΔABC.' . In line 2b, change the period after 'others' to a question mark.

6. $(4 - \pi) \cdot 100$

7. $(\pi - 2) \cdot 100$

8. What we want amounts to showing that the sum of the area-measures of the partially shaded semicircular sectors, plus the area-measure of ΔABC, minus the area-measure of the totally unshaded semicircular section is the area-measure of ΔABC. In other words, we want to show that the sum, $\frac{1}{2}\pi c^2 + \frac{1}{2}\pi a^2$, of the area-measures of the partially shaded semicircular sectors is the area-measure, $\frac{1}{2}\pi b^2$, of the totally unshaded one. But, this is obvious. For, ΔABC is right-angled at B, and, by the Pythagorean Theorem, $c^2 + a^2 = b^2$.

*

Answers for Part B [on pages 6-352 and 6-353].

1. By formula (2) on page 6-351, the ratio of the area-measures of minor circular sectors which have congruent central angles is the square of the ratio of their radii. [This is, in a more general setting, a consequence of the theorem according to which the ratio of the area-measures of two similar sets is the square of their ratio of similitude.] Since ΔOCB is an isosceles right triangle, $OC/OB = 1/\sqrt{2}$. Hence, the ratio of the area-measures is 1/2.

2. 4275π

3. $12(5 + \pi)$ inches

4. $5(3 + \pi)/2$ feet

Correction. On page 6-353, in Exercise 6, add this line to the Hypothesis: $\overline{AC}$ and $\overline{CB}$ are diameters

5. 16

6. Let AB = d. Then, $AC = \frac{kd}{k+1}$ and $CB = \frac{d}{k+1}$. So, the ratio of the area-measures is $[1 + \left(\frac{k}{k+1}\right)^2 - \left(\frac{1}{k+1}\right)^2]/[1 - \left(\frac{k}{k+1}\right)^2 + \left(\frac{1}{k+1}\right)^2]$. Simplifying, the ratio is k. [This is an example of the fact that multiplying lengths in one direction by k multiplies areas by k. Doing this in two directions results in multiplying areas by k^2.]

7. $\overline{AB}$ bisects $\square$ APBU, $\overline{BC}$ bisects $\square$ BQCV, etc. So, twice the area-measure of $\square$ABCD is the sum of the area-measures of the four regions at the corners of the figure plus twice the area-measure of $\square$STUV. But, this is the sum of the area-measures of $\square$ STUV and $\square$ MPQR. Hence, the conclusion.

8. By s.a.s., $ADM_3 \leftrightarrow BAM_4$ is a congruence. Since $\angle BAM_4$ is a right angle, $\angle AM_4B$ is a complement of $\angle ABM_4$. Hence, it is a complement of $\angle DAM_3$. So, ΔAEM_4 is right-angled at E. Consequently, $\angle FEH$ is a right angle. Similarly, $\angle EHG$, $\angle HGF$, and $\angle GFE$ are right angles. So, EFGH is a rectangle.

   Since $\angle DAH \cong \angle ABE$ and $\angle AHD \cong \angle BEA$, and $\overline{AD} \cong \overline{BA}$, it follows by a.a.s. that $DAH \leftrightarrow ABE$ is a congruence. So, AH = BE. Since M_4 is the midpoint of $\overline{AD}$ and $\overleftrightarrow{M_4E} \parallel \overleftrightarrow{DH}$, it follows that E is the midpoint of $\overline{AH}$. Similarly, F is the midpoint of $\overline{BE}$. Consequently, EH = FE. So, rectangle EFGH is a square.

   ☆ Since ΔADM_3 is right-angled at D and $\overleftrightarrow{DH} \perp \overleftrightarrow{AM_3}$, $AHD \leftrightarrow ADM_3$ is a similarity. So, $AH/AD = AD/AM_3$. But, since $AD = 2 \cdot DM_3$, $AM_3 = (\sqrt{5}/2) \cdot AD$. Hence, $AH = (2/\sqrt{5}) \cdot AD$. Since $AH = 2 \cdot EF$, $EF = AD/\sqrt{5}$. So, the area-measure of EFGH is one fifth that of ABCD.

Quiz.

1. What is the area-measure of an equilateral triangle whose side-measure is 2?

2. Suppose that quadrilateral ABCD is a parallelogram, AB = 10, AD = 6, and $m(\angle A) = 30$. What is the area-measure of ABCD?

3. The ratio of the corresponding sides of two similar triangles is 2:3. If the area-measure of the larger triangle is 36, what is the area-measure of the smaller?

4. If the central angle of a circular sector is an angle of 80° and the area-measure of the circle is 72π, what is the area-measure of the sector?

5. The area of a trapezoid is 60 square inches. If the bases are 7 and 17 inches long, respectively, how many inches apart are the bases?

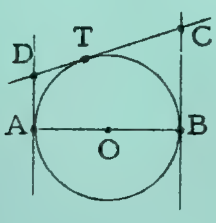
6. A circle with radius k is inscribed in a square. What is the area-measure of the square?

7. Find the area-measure of a regular polygon whose perimeter is 60 and whose apothem is 5.

8. If the radius of a circle is tripled, what change takes place in the area-measure?

9. Two squares have a side of one congruent to a diagonal of the other. What is the ratio of their area-measures?

10.



Hypothesis: $\overleftrightarrow{DA}$, $\overleftrightarrow{CB}$, and $\overleftrightarrow{CD}$ are tangents to the circle with center O,

$\overleftrightarrow{DC} \nparallel \overleftrightarrow{AB}$

Conclusion: the area-measure of ABCD is $\frac{1}{2} \cdot AB \cdot CD$

11. Suppose that two of the medians of a triangle are congruent, each with measure 30. If the third median of the triangle has measure 36, what is the area-measure of the triangle?

☆12. Suppose that the coordinates of the vertices of a triangle are (2, 3), (5, 7), and(4, 10), respectively. What is the area-measure of the triangle?

*

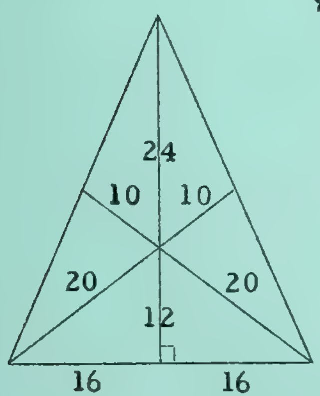
Answers for Quiz.

1. $\sqrt{3}$ 2. 30 3. 16 4. 16π 5. 5 6. $4k^2$

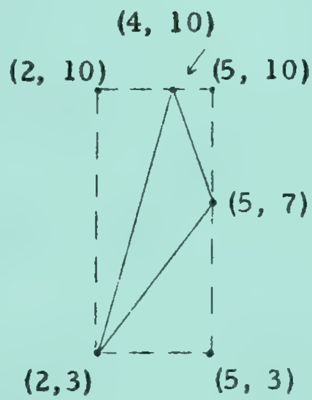
7. 150 8. multiplied by 9 9. 2:1

10. Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ and $\overleftrightarrow{DC} \nparallel \overleftrightarrow{AB}$, quadrilateral ABCD is a trapezoid. So, since $\overleftrightarrow{AB}$ is perpendicular to the bases, the area-measure of ABCD is $\frac{1}{2} \cdot AB(AD + BC)$. But, since $\overleftrightarrow{CT}$ and $\overleftrightarrow{CB}$ are tangents, $TC = BC$. Similarly, $DT = AD$. So, the area-measure of ABCD = $\frac{1}{2} \cdot AB(DT + TC) = \frac{1}{2} \cdot AB \cdot DC$.

11.



☆12.



$3 \cdot 7 - (\frac{1}{2} \cdot 4 \cdot 3 + \frac{1}{2} \cdot 1 \cdot 3 + \frac{1}{2} \cdot 2 \cdot 7) = 6.5$

Quiz [covering pages 6-1 through 6-356].

[As in the case of the mid-unit quiz, you may wish to choose items from the following list for a unit examination. The remaining items can be used for review.]

Part I.

1. The altitude of $\triangle ABC$ from A is ______ to side $\overleftrightarrow{BC}$.

2. Diagonal $\overleftrightarrow{AC}$ of parallelogram ABCD ______ diagonal $\overleftrightarrow{BD}$.

3. The sum of the measures of two supplementary angles is ______.

4. The bisectors of two complementary adjacent angles are the sides of an angle of ______ degrees.

5. If a line is perpendicular to one of two parallel lines, it is ______ to the other.

6. If the midpoints of two adjacent sides of a rhombus are joined by a segment, the triangle thus formed is ______.

7. In two concentric circles, all chords of the larger circle which are tangent to the smaller circle are ______.

8. The segment whose end points are the midpoints of two sides of a triangle is ______ to the third side.

Part II.

9. If the sum of the measures of two angles is the measure of an obtuse angle then one of the two angles must be

   (A) an acute angle (B) a right angle (C) an obtuse angle

10. Two angles that are congruent and supplementary are

    (A) adjacent angles (B) right angles (C) acute angles

11. If $\angle A$ is a complement of $\angle B$ and $\angle C$ is a supplement of $\angle B$ then

(A) $m(\angle A) > m(\angle C)$ (B) $m(\angle A) = m(\angle C)$

(C) $m(\angle A) < m(\angle C)$

12. If, in quadrilateral ABCD, AB = BC and CD = DA then the diagonals AC and BD

(A) bisect each other (B) are perpendicular (C) are congruent

13. The circumcenter of a triangle is interior to the triangle if the triangle is

(A) acute (B) right (C) obtuse

Part III.

14. What is the sum of the measure of the angles of a convex polygon of 5 sides?

15. Suppose that in $\triangle ABC$ a line parallel to $\overleftrightarrow{AC}$ intersects $\overline{AB}$ at D and $\overline{CB}$ at E. If AB = 8, BC = 12, and BD = 6, find BE.

16. A tangent segment and a secant segment to a circle from an external point are 6 inches and 12 inches long, respectively. How long is the external segment of the secant segment?

17. What is the distance between the point with coordinates (3, 3) and (8, 8)?

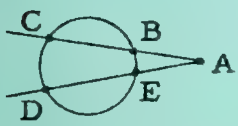
18. Find the length-measure of an arc of 45° in a circle whose radius is 8.

19. Suppose that $\overline{AB}$ is a chord of a circle and that $\widehat{AB}$ is an arc of 50°. Find the measure of the acute angle one of whose sides is $\overrightarrow{BA}$ and the other of whose sides is tangent to the circle at B.

20. The measure of the altitude to the hypotenuse of a right triangle is 4, and the measure of one of the segments of the hypotenuse made by the altitude is 2. Find the measure of the other segment of the hypotenuse.

21. What are the measures of the angles of a parallelogram in which one of the angles is three times as large as another?

22. Suppose that in ΔABC, $\angle A$ is an angle of 60° and an exterior angle at B is an angle of 130°. Which is the longest side of the triangle?

23. 

If $\angle A$ is an angle of 18° and $\overset{\frown}{BE}$ is an arc of 24°, what is the number of degrees in $\overset{\frown}{CD}$?

24. A circle is tangent to each of the sides of an angle of 72°. Find the measures of the arcs determined by the points of tangency.

25. Two tangent segments to a circle from an external point are each 6 inches long and are contained in an angle of 60°. How long is the chord joining the points of tangency?

26. Corresponding sides of two similar triangles are in the ratio 1 : 4. What is the ratio of a pair of corresponding altitudes?

27. What is the area-measure of an equilateral triangle whose side-measure is 5?

28. Suppose that in ΔABC, AB = BC. Find, correct to the nearest unit, the measure of the altitude from B if $m(\angle B) = 96$ and AB = 10.

29. If the diagonals of a rhombus are 10 inches and 20 inches long, respectively, how many square inches are there in its area?

30. The side-measure of a right triangle are 3, 4, and 5, respectively. What is the cosine of the smallest angle?

31. How far from the center of 3-inch circle should you choose a point so that the tangent segments to the circle are 4 inches long?

32. Chord $\overleftrightarrow{AB}$ is bisected at M by chord $\overleftrightarrow{CD}$. If CM = 8 and MD = 18, what is AB?

33. If a radius of a circle is 13 inches long, how long is the shortest chord which contains a point 5 inches from the center?

34. Suppose that, in ΔABC, AB = 15 = AC and BC = 24. What is the diameter of the circumcircle of ΔABC?

35. The sum of the area-measures of two similar triangles is 78. If a pair of corresponding sides measure 6 and 9, respectively, what is the area-measure of the smaller triangle?

36. If the bases of an isosceles trapezoid measure 10 and 14, respectively, and the measure of a diagonal is 13, what is the area-measure of the trapezoid?

Part IV.

37. Suppose that A(9, 10), B(−3, −6), and C(13, 2) are the vertices of a triangle.

    (a) Prove that the triangle is a right triangle.

    (b) Find its area-measure.

    (c) Find the radius of its circumcircle.

    (d) Write an equation of its circumcircle.

38. Suppose that A, B, and C are points on a circle such that $\widehat{ABC}$ is a major arc and B is its midpoint. If D is a point on the circle such that $\overline{AC} \cap \overline{BD} = \{E\}$, show that $(AB)^2 = BD \cdot BE$.

39. Suppose that, in quadrilateral ABCD, AB = BC and $m(\angle A) > m(\angle C)$. Prove that CD > DA.

40. Suppose that ΔABC is right angled at A. Let $\overleftrightarrow{AM}$ be the median from A, $\overleftrightarrow{AH}$ be the altitude from A, and $\overleftrightarrow{AT}$ be the angle bisector from A.

Assuming that $T \in \overline{HM}$, show that $\overrightarrow{AT}$ bisects $\angle HAM$.

[Unless ΔABC is isosceles, $T \in \overline{HM}$. For, if ΔABC is not isosceles, $\angle B$ is smaller than each of the congruent angles, $\angle TAB$ and $\angle TAC$. But, $\angle MAB$ and $\angle HAC$ are both congruent to $\angle B$.]

*

Answers for Quiz.

1. perpendicular
2. bisects
3. 180
4. 45
5. perpendicular
6. isosceles
7. congruent
8. parallel
9. (A)
10. (B)
11. (C)
12. (B)
13. (A)
14. 540
15. 9
16. 3
17. $5\sqrt{2}$
18. 2π
19. 25
20. 8
21. 45 and 135
22. $\overline{AB}$
23. 60
24. 108 and 252
25. 6 inches
26. 1:4
27. $25\sqrt{3}/4$
28. 7
29. 100
30. 0.8
31. 5 inches
32. 24
33. 24
34. 25
35. 24
36. 60
37. (a) $[d(\overline{AB})]^2 = 400$, $[d(\overline{BC})]^2 = 320$, $[d(\overline{CA})]^2 = 80$. So, the triangle is right-angled at C.

    (b) $d(\overline{BC}) = 8\sqrt{5}$ and $d(\overline{CA}) = 4\sqrt{5}$. Therefore, the area-measure is $\frac{1}{2} \cdot 8\sqrt{5} \cdot 4\sqrt{5} = 80$.

    (c) radius of circumcircle = $\frac{1}{2} \cdot$ measure of hypotenuse = 10

    (d) $(x - 3)^2 + (y - 2)^2 = 100$

38. Since $\overleftrightarrow{AB} \cong \overleftrightarrow{BC}$, it follows that $\angle BAC \cong \angle BDA$. Also, $\angle B \cong \angle B$. So, by the a.a. similarity theorem, $ABE \longleftrightarrow DBA$ is a similarity. Hence, $(AB)^2 = BD \cdot BE$.

39. Since $AB = BC$, $m(\angle BAC) = m(\angle BCA)$. Since $m(\angle BAD) > m(\angle BCD)$, $m(\angle DAC) > m(\angle ACD)$. So, in ΔACD, $CD > DA$.

40. $ABH \longleftrightarrow CBA$ is a similarity; so, $\angle BAH \cong \angle BCA$. Since $\overleftrightarrow{AM}$ is the median to the hypotenuse, $AM = MC$. So, $\angle BCA \cong \angle CAM$. Hence, $\angle BAH \cong \angle CAM$. But, by hypothesis, $\angle BAT \cong \angle CAT$. So, $\angle HAT \cong \angle MAT$.

Here is a slightly more complicated column proof which still depends only on universal instantiation and the test-pattern principle:

(1)	$\forall_x \forall_y$	$xy = yx$	[basic principle]
(2)	$\forall_y$	$(a + 2)y = y(a + 2)$	[(1)]
(3)		$(a + 2)a = a(a + 2)$	[(2)]
(4)	$\forall_x$	$(x + 2)x = x(x + 2)$	[(1)-(3)]

Here, each of steps (2) and (3) follows from the preceding step by universal instantiation; and steps (1)-(3) constitute a test-pattern for the conclusion. So, the proof shows that

'$\forall_x\ (x + 2)x = x(x + 2)$'

is a logical consequence of

'$\forall_x \forall_y\ xy = yx$'.

Since this premiss is a basic principle, the conclusion is a theorem. In practice, one would probably omit step (2), and consider that (3) follows from (1) by instantiation. [On this point, see the COMMENTARY for page 2-30.]

As a final example, consider:

(1)	$\forall_x \forall_y$	$xy = yx$	[basic principle]
(2)	$\forall_y$	$(a + 2)y = y(a + 2)$	[step (1)]
(3)		$(a + 2)b = b(a + 2)$	[step (2)]
(4)	$\forall_y$	$(y + 2)b = b(y + 2)$	[steps (1)-(3)]
(5)	$\forall_x \forall_y$	$(y + 2)x = x(y + 2)$	[steps (1)-(4)]

In this proof there are two uses of universal instantiation, followed by two cases of the test-pattern principle. In practice, the proof would probably be abbreviated to three steps, steps (1), (3), and (5).

There is a third form of proof--tree-form proofs--which has been illustrated at various places in the COMMENTARY for Unit 1 and Unit 2. [See, for example, TC[2-31, 32]b.] Using this form, we should write:

$$\cfrac{\cfrac{\forall_x\ x\cdot 1 = x}{(2a)\cdot 1 = 2a}}{\forall_x\ (2x)\cdot 1 = 2x}$$

Here, the single horizontal bar indicates that the second line is a consequence of the first, while the double bar indicates that the two lines above it constitute a test-pattern for the universal generalization sentence written below it. In Unit 6 we shall not write proofs in tree-form, but we shall use a similar device to diagram the structure of a proof. For example, we would diagram the preceding column proof by:

$$\cfrac{\cfrac{(1)}{(2)}}{(3)}$$

Notice that, although in Unit 2 we would probably have used '$(2x)\cdot 1 = 2x$' as step (2) in the preceding proof, the use of a different letter, say 'a', instead of 'x' makes for a clearer understanding of the different roles of the 'x's in

'$\forall_x\ x\cdot 1 = x$'

and the 'a's in

'$(2a)\cdot 1 = 2a$'.

The latter are, in the strict sense of the word, pronumerals--it makes sense to replace them by numerals. The former, on the other hand, serve merely as indices which link the quantifier '$\forall$' with the two argument places in the predicate '$\ldots \cdot 1 = \_\_\_$'. For a more complete discussion of this, see TC[2-27]. As far as your students are concerned, it may be sufficient to tell them that forming instances of a generalization by using letters other than those associated with the quantifiers makes proofs easier to follow. As remarked earlier [TC[6-13]b], we shall, for the most part, use 'W', 'X', 'Y', and 'Z' with quantifiers in geometry theorems, and use alphabetically earlier letters in forming instances. [In technical terms, we use 'W', 'X', 'Y', and 'Z' as apparent variables, and the other capital letters as variables.]

By universal instantiation, the pml implies each of its instances. So, in particular, it implies

$$`(2a)\cdot 1 = 2a\text{'}.$$

Since each instance of

$$`\forall_x\ (2x)\cdot 1 = 2x\text{'}$$

can be obtained by making suitable substitution for the 'a's in this instance of the pml, we can construct a test-pattern for this universal generalization sentence. The form used in Unit 2 is:

$$(2a)\cdot 1 = 2a \qquad [\forall_x\ x\cdot 1 = x]$$

Since this is a test-pattern for '$\forall_x\ (2x)\cdot 1 = 2x$', the test-pattern principle justifies our adding:

$$\text{therefore,}\quad \forall_x\ (2x)\cdot 1 = 2x$$

The two displayed lines just above constitute a proof which shows that the conclusion '$\forall_x\ (2x)\cdot 1 = 2x$' is a consequence of the premiss '$\forall_x\ (x\cdot 1) = x$'.

In Unit 6 we adopt a different form for writing proofs. Using it here, we should write:

(1)	$\forall_x\ x\cdot 1 = x$	[pml]
(2)	$(2a)\cdot 1 = 2a$	[(1)]
(3)	$\forall_x\ (2x)\cdot 1 = 2x$	[(1) and (2)]

The proof consists of the three sentences in the middle column. Proofs of this form will be called <u>column proofs</u>. [Each sentence is a <u>step</u> in the proof.] The parenthesized numerals in the left-hand column merely furnish an easy way to refer to the steps of the proof. The bracketed remarks in the right-hand column indicate the source of each step. They are aids to following the proof, but are not part of it. The comment 'pml' for step (1) identifies this step as the principle for multiplying by 1. A less explicit, but probably adequate, comment which might have been used is 'basic principle'. The comment for step (2) indicates that this step is a consequence of step (1). Inspection of the two steps shows that (2) follows from (1) by virtue of the rule of universal instantiation. The comment for step (3) draws attention to steps (1) and (2), where one sees that these form a test-pattern for (3).

One of the purposes allegedly served by the study of geometry is the development of an understanding of the nature of proof. Something concerning this has already been said in the COMMENTARY for page 6-18. As was remarked there, a proof shows how its conclusion follows, step-wise, from its premisses, by the application of principles of logic. So, one can scarcely understand the nature of proof unless he is acquainted with at least some of the logical principles which justify his inferring of later steps in the proof from earlier ones. This appendix furnishes an introduction to some of the more commonly used logical principles, and contains illustrations of their use in proofs of theorems from algebra. Since its illustrations are drawn from algebra, the Appendix can be studied independently of the remainder of Unit 6. However, it will probably be of more help to students of Unit 6 if it is studied piecewise, as is suggested at various places in section 6.01. The MISCELLANEOUS NOTES, beginning on page 6-398, illustrate this use of logical principles in some proofs of theorems from geometry.

*

Note the word 'premiss' [plural: premisses]. This, rather than the legal term 'premise', is the historically proper word to use in referring to the sentences which one takes as the initial ones of a proof. [The usage dates back at least to 1599.]

*

Students of Unit 2 have already become acquainted with some principles of logic. For example, they have learned that one can prove a universal generalization sentence such as

$$`\forall_x \; x(x + 1) = xx + x'$$

by constructing a testing pattern for its instances. [See pages 2-31 through 2-33.] Moreover, they are aware that a universal generalization sentence implies each of its instances. The first of these two rules may be called <u>the test-pattern principle</u>, the second, [<u>the rule of</u>] <u>universal instantiation</u>. Together, they explain the meaning of universal generalization sentences.

As an example of the use of the test-pattern principle and universal instantiation, consider the derivation of the universal generalization sentence

$$`\forall_x \; (2x) \cdot 1 = 2x'$$

from the principle for multiplying by 1,

$$`\forall_x \; x \cdot 1 = x'.$$

Although universal instantiation and the test-pattern principle are of fundamental importance, it is obvious that no very startling conclusions can be justified by virtue of these rules of reasoning alone. In fact, most of the proofs in Unit 2 appeal to two other logical principles--<u>the substitution rule for equations</u> and <u>the principle of identity</u>. [Both are illustrated in the tree-form proof on TC[2-31, 32]b. Three of the four principles of logic mentioned so far are illustrated in the discussion, beginning on page 6-357, of the theorem

$$\text{`}\forall_x\ x(x + 1) = xx + x\text{'}.$$

The principle of identity,

$$\text{`}\forall_x\ x = x\text{'},$$

is first mentioned on page 6-362.] The use of the substitution rule for equations is best discussed in connection with the column proof on page 6-358. The first five steps of this proof constitute a test-pattern for the sixth. Let's consider a particular instance of the sixth line, say, '$3(3 + 1) = 3 \cdot 3 + 3$' and see how the test-pattern yields a proof of this instance. What we must do, of course, is substitute '3's for the 'a's.

(1′)	$\forall_x \forall_y \forall_z\ x(y + z) = xy + xz$	[theorem]
(2′)	$3(3 + 1) = 3 \cdot 3 + 3 \cdot 1$	[(1′)]
(3′)	$\forall_x\ x \cdot 1 = x$	[basic principle]
(4′)	$3 \cdot 1 = 3$	[(3′)]
(5′)	$3(3 + 1) = 3 \cdot 3 + 3$	[(2′) and (4′)]

Step (2′) follows from step (1′) by the principle of universal instantiation [actually, by three applications of this principle], and the same principle justifies inferring step (4′) from step (3′). Step (5′) is obtained by substituting the right-hand side of step (4′) for the '$3 \cdot 1$' in step (2′). The substitution rule for equations says that, since the left-side of (4′) is '$3 \cdot 1$', it follows that (5′) is a consequence of (4′) and (2′):

$$\frac{3 \cdot 1 = 3 \qquad\qquad 3(3 + 1) = 3 \cdot 3 + 3 \cdot 1}{3(3 + 1) = 3 \cdot 3 + 3}$$

Just as the test-pattern principle and universal instantiation embody the meaning of universal generalization sentences, so the substitution rule

and the principle of identity embody the meaning of equation sentences. The acceptability of the inference of (5′) from (4′) and (2′) [and the acceptability of the substitution rule, generally] is due entirely to our interpretation of equation sentences. Step (5′) is a consequence of (4′) and (2′) just because we have agreed to understand '=' in such a way that (4′) means that '3 · 1' and '3' are names for the same thing.

Returning, now, to the proof on page 6-358, step (5) follows from steps (4) and (2):

$$\frac{\overset{(4)}{a1 = a} \qquad \overset{(2)}{a(a + 1) = aa + a1}}{\underset{(5)}{a(a + 1) = aa + a}}$$

because the same argument given above for the case in which the 'a's were replaced by '3's applies equally well no matter what numeral replaces the 'a's. [Notice that we cannot, here, parody the argument by saying that the inference is valid because (4) means that 'a1' and 'a' are names for the same thing. For, neither 'a1' nor 'a' is a name for anything. Only the numerical expressions, which result when numerals are substituted for the 'a's, are names.]

*

Diagramming proofs is one of the best ways of learning to appreciate the role of logical principles. In discussing the diagram on page 6-360, it will be helpful to have the column proof on page 6-358 on the board, and to write the diagram, as is done in the text [beginning on page 6-359] one inference at a time. Before writing an inference in the diagram, locate the corresponding marginal comment in the column proof. When reading the three sentences preceding the exercises, point to the appropriate inferences in the diagram. [Notice that the premisses of which the last conclusion is a consequence can be located by going to the very top of the branches in the diagram.]

of the principle of identity and the substitution rule for equations, that equality is symmetric.

In view of this result, we have another substitution rule for equations. The statement of this rule can be obtained from that in the box on page 6-359 by interchanging the words 'left' and 'right'. This second substitution rule can be applied in Exercise 7 on page 6-363 where the first is not applicable. In Exercise 9, the missing premiss can be supplied in either of two ways, to illustrate either of the substitution rules. For Exercise 10, there is 1 solution which illustrates the first substitution rule and there are 31 solutions each of which illustrates the second rule.

Answers for Part A [which begins on page 6-360].

1. if $B \in \overline{AC}$ and $C \in \overline{BD}$ then $C \in \overline{AD}$
[or: if $P \in \overline{QR}$ and $R \in \overline{PF}$ then $R \in \overline{QF}$]

[For later exercises, we shall refrain from giving alternative answers which are, like the bracketed one above, merely alphabetic variants of previously given answers.]

2. $(3 + 5)^3 = 3^3 + 3 \cdot 3^2 \cdot 5 + 3 \cdot 3 \cdot 5^2 + 5^3$
[or: $(a + 2b)^3 = a^3 + 3a^2(2b) + 3a(2b)^2 + (2b)^3$]

3. $\forall_u \forall_v (u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$

4. $\forall_x \forall_y \forall_z$ if $x + y = z$ then $x = z - y$

*

Answers for Part B [on pages 6-361 and 6-362].

[The two other possible conclusions for the Sample are '$7 \cdot 8 > 5(6 + 2)$' and '$7(6 + 2) > 5(6 + 2)$'.]

1. $8(2 + 1) > 23$

2. $8 \cdot 3 > 7 \cdot 3$

3. $(b + c)^2 - d^2 = (b + c - d)(b + c + d)$, [or: $(b + c)^2 - d^2 = (a - d)(a + d)$, or: ... There are 7 possible answers.]

4. The length of $\overleftrightarrow{AB}$ = the length of $\overleftrightarrow{BA}$ [There are two other answers.]

5. if $a > b$ and $b > c$ then $a > c$ [There are two other answers.]

6. $a = a$

*

The principle of identity, 'A thing is itself.', is a logical principle which justifies accepting premisses such as '$2 = 2$', '$\overleftrightarrow{AB} = \overleftrightarrow{AB}$', etc. As is illustrated in Exercise 6, the use of such premisses opens the way for applications of the substitution rule for equations which justify interchanging the sides of an equation. In other words, it is a consequence

When diagramming a proof it is helpful to indicate inferences which depend on the substitution rule for equations by placing the reference to the equation from which the substitution is made on the left and the reference to the sentence in which the substitution is made on the right. Thus, above:

$$\frac{(6) \qquad (5)}{(7)}, \quad \text{not:} \quad \frac{(5) \qquad (6)}{(7)}$$

*

Students, having seen that the substitution rule for equations and the principle of identity justify turning an equation around--that is, that symmetry of equality is a consequence of the substitution rule for equations and the principle of identity--may try to use Exercise 12 as an argument to show that the principle of identity follows from the substitution rule and symmetry of equality. Such an argument may proceed as follows:

$$\frac{a = b \qquad b = a}{b = b}$$

> So, if you have 'b = a', then, since equality is symmetric, you have 'a = b'. And, from these you get 'b = b', by substitution. [Q.E.D.]

The fallacy here is that, even if you grant the symmetry of equality, you still need the assumption 'b = a' in order to draw the conclusion 'b = b'. What has been shown is that it follows from the substitution rule and the symmetry of equality that <u>if a thing is anything</u>, it is itself. [If one adopts an additional principle to the effect that everything is something, then one can use the above argument together with this <u>additional</u> principle to prove that everything is itself.]

7. $3 \cdot 2 + 7 = 13$

8. $a = c$

9. $a = 5$ [or: $5 = a$]

10. $aa = aa + a0$

[One can also obtain a correct conclusion for Exercise 10 by replacing any (at least one) of the five 'a's in the second premiss by '(a + 0)'.]

11. $b = a$ [or: $a = b$, or: $a = a$]

12. $b = b$ [or: $a = a$]

13. $ac = ac$ [or: $bc = bc$]

14. $a + b > 0$

15. $c = d$ [or: $d = c$]

16. if $B \in \overrightarrow{BC}$ then $\overleftrightarrow{BC} \cap \ell = \{A\}$ [15 solutions]

17. James lives in the capital of California

18. $M = N$ [or: $N = M$]

19. M is the midpoint of $\overline{AB}$

*

Answers for Part C [on pages 6-363 and 6-364].

1. apm; instance of (1); cpm; instance of (3); substitution from (4) into (2); instance of (1); substitution of (6) into (5); instance of (3); substitution of (8) into (7); instance of (1); substitution of (10) into (9); (1)-(11) form a test-pattern [Eventually you should accept less explicit marginal comments: basic principle; (1); basic principle; (3); (2) and (4); (1); (5) and (6); (3); (7) and (8); (1); (9) and (10); (1) - (11)]

2.

$$\begin{array}{c} \dfrac{\dfrac{(1)}{(10)} \qquad \dfrac{\dfrac{(3)}{(8)} \qquad \dfrac{\dfrac{(1)}{(6)} \qquad \dfrac{\dfrac{(3)}{(4)} \qquad \dfrac{(1)}{(2)}}{(5)}}{(7)}}{(9)}}{(11)} \\ \hline\hline (12) \end{array}$$

[Note that (12) is a theorem because it is a consequence of (1) and (3), and (1) and (3) are basic principles.]

The Exploration Exercises lead to the principle of logic called modus ponens [or, sometimes, the rule of detatchment]. This is one of the two basic principles of logic which deals with conditional [or: if-then] sentences. The other such rule, conditionalizing, and discharging an assumption, is discussed on page 6-372 et seq.

*

Answers for Exploration Exercises.

Exercises 1, 2, 3, 6, 8, and 10 are valid inferences. Exercise 11 is a scheme which exhibits the form common to each of these. If copies of any sentence are written in the two rectangles, and copies of any sentence are written in the two ovals, the result will be a valid inference.

*

One sometimes hears it said that "anything follows from a false premiss" or "a false statement implies anything". There is a possibility that one of your students may have heard this and may bring it up in connection with Exercise 9 to support a claim that the conclusion of this exercise does follow from the premisses. [If this does not happen, you will be wise not to mention it.] If someone does, explain that one thing which leads people to make such misleading statements as those quoted above is the fact that any sentence does follow from any two premisses, one of which is the denial of the other. So, as you will later be in a position to show [see TC[6-386]a], '2 = 3', or any other sentence, does follow logically from the two premisses '7 = 8' and '7 ≠ 8'. But, '2 = 3' does not follow from '7 = 8' alone, or from '7 = 8' together with 'if 2 = 3 then 7 = 8'. Another thing which may lead a person to make one of the questionable statements is that it is conventional to label 'if 7 = 8 then 2 = 3' true on the grounds that '7 = 8' is false. However, this convention has nothing to do with the problem of what sentences follow, logically, from the premiss '7 = 8'. In fact, as we are developing the concept here, the notion of logical consequence depends in no way on notions of truth and falsity. When students do the exercises using modus ponens on pages 6-368 and 6-369, you will probably have to point out, again, that the truth or falsity of the premisses is entirely irrelevant to the problem of completing the inferences. Stress the fact that determining whether or not a given inference is an example of modus ponens [or of any logical principle] is a purely mechanical task, something a machine could do.

Correction. On page 6-367, line 10 should read:

$$(8) \quad --a = a$$

In the proof on page 6-366, step (1) is the principle of opposites, step (3) is the commutative principle for addition, and step (6) is the 0-sum theorem. Steps (2), (4), and (7) follow from (1), (3), and (6), respectively, by universal instantiation. Step (5) follows by substitution from (4) into (2). [(5) could also be obtained by substitution from (2) into (4), yielding '0 = −a + a', followed by an application of the principle of symmetry of equality. So, following step (4), one might have:

(4.1)	$0 = -a + a$	[substitution from (2) into (4)]
(5)	$-a + a = 0$	[(4.1), by symmetry of equality]

or:

(4.1)	$0 = -a + a$	[substitution from (2) into (4)]
(4.2)	$0 = 0$	[principle of identity]
(5)	$-a + a = 0$	[substitution from (4.1) into (4.2)]]

Step (8) follows from (5) and (7) by the new rule, modus ponens. This rule, like the earlier ones, is acceptable by virtue of the logical terms in question--in this case, the sentence connective 'if ... then ___'. One who, for example, claims that bats are birds, and admits [perhaps on the basis of a belief that all birds lay eggs] that <u>if</u> bats are birds <u>then</u> bats lay eggs, is stuck with the conclusion that bats lay eggs.

*

Answers to questions near the bottom of page 6-367.

Of the Exploration Exercises on page 6-365, Exercises 1, 2, 3, 6, 8, and 10 illustrate modus ponens. For these, the replacements for 'p' and 'q' in the scheme on page 6-367 are:

1. $2 = 2$; $2 + 3 = 2 + 3$
2. $a = b$; $a + c = b + c$
3. bats are birds; bats lay eggs
6. $A \in \overleftrightarrow{BC}$; $m(\overleftrightarrow{BA}) + m(\overleftrightarrow{AC}) = m(\overleftrightarrow{BC})$
8. $2 = 3$; $2 + 5 = 3 + 5$
10. $a \neq b$; $a > b$ or $a < b$

One can see that Exercise 4 does not fit this scheme by noticing that to obtain the first premiss, 'p' must be replaced by 'Ed lives in Iowa', while to obtain the second premiss, 'p' must be replaced by 'Ed lives in Ames'.

The fallacy of affirming the consequent is committed when one judges an inference like those of Exercises 4, 5, 7, and 9 on page 6-365 to be valid. This happens more often than one might think. Arguments like the following one:

if $3x + 5 = 2$ then $x = -1$; so, the root of '$3x + 5 = 2$' is -1

are not uncommon. [Of course, all that one is entitled to conclude from 'if $3x + 5 = 2$ then $x = -1$' is that the equation '$3x + 5 = 2$' has no root other than -1.] Probably, one's failure to recognize the invalidity of such arguments is due, in part, to prior knowledge, gained by inspection of the equation, that -1 is, indeed, the root of '$3x + 5 = 2$'. And, in part, it may be due to confusing the argument quoted with a more complex bit of valid reasoning:

'$3x + 5 = 2$' has a root, and if $3x + 5 = 2$ then $x = -1$; so the root of '$3x + 5 = 2$' is -1

That the simpler argument is invalid, despite the correctness of its conclusion, is easily seen by considering a parallel argument which leads to a false conclusion:

if $x + 1 = x$ then $(x + 1)x = x^2$, $x^2 + x = x^2$, and $x = 0$; so the root of '$x + 1 = x$' is 0

Here, one can carry the argument preceding the semicolon further, to obtain a valid result, as follows:

so, if $x + 1 = x$ then $0 + 1 = 0$, whence $1 = 0$; hence, since $1 \neq 0$, '$x + 1 = x$' has no root

*

Answers for Part A [on pages 6-368 and 6-369].

[Answers consist of 3 parts:

the missing sentence; the antecedent of if-then sentence; the consequent of if-then sentence]

1. $c = d$; $a = b$; $c = d$

2. A ϵ $\overline{BC}$; A ϵ $\overline{BC}$; A ϵ $\overset{\bullet\!\!-\!\!\bullet}{BC}$ [For this exercise, some student may supply 'if [if A ϵ $\overline{BC}$ then A ϵ $\overset{\bullet\!\!-\!\!\bullet}{BC}$] then A ϵ $\overset{\bullet\!\!-\!\!\bullet}{BC}$' as the missing sentence. This is a correct answer. However, in writing illustrations of modus ponens, we shall customarily follow the form displayed in the box on page 6-367.]

3. if $a + b = 0$ then $b = -a$; $a + b = 0$; $b = -a$

4. $ab = c$; $ab = c$; $b = c \div a$

5. if $a - b \neq 0$ then $a \neq b$; $a - b \neq 0$; $a \neq b$

6. $A \in m$; $A \in \ell$; $A \in m$

7. if $A \notin \ell$ then $A \notin m$; $A \notin \ell$; $A \notin m$

8. if $\ell \,||\, m$ then $\ell \cap m = \emptyset$; $\ell \,||\, m$; $\ell \cap m = \emptyset$

9. Bill lives in Texas; Bill lives in Dallas; Bill lives in Texas

10. if Bill lives in Texas then Bill lives in Dallas; Bill lives in Texas; Bill lives in Dallas

11. if Bill does not live in Texas then Bill does not live in Dallas; Bill does not live in Texas; Bill does not live in Dallas

*

Answers for Part B.

1.
$$\dfrac{\dfrac{a = b \qquad a \neq c}{b \neq c} \qquad \text{if } b \neq c \text{ then } b \neq d}{b \neq d}$$
[substitution]
[modus ponens]

[Notice that our convention concerning the writing of substitution inferences precludes the possibility that the second of the two inferences be of this kind. For, its first premiss is clearly '$b \neq c$', and this is not an equation.]

2.
$$\dfrac{\dfrac{a = b \qquad a = a}{b = a} \qquad b = c}{a = c}$$
[substitution]
[substitution]

[In place of '$a = a$', one might have '$b = b$', or 'if $a = b$ then $b = a$'. In place of '$b = c$' one might have 'if $b = a$ then $a = c$'. With either of these last alternatives, the inference whose premiss is involved would be an example of modus ponens.]

3.
$$\dfrac{\ell = n \qquad \dfrac{\ell \parallel m \qquad \text{if } \ell \parallel m \text{ then } m \parallel \ell}{m \parallel \ell}}{m \parallel n}$$
[modus ponens]
[substitution]

4.
$$\dfrac{C \in \overrightarrow{BC} \qquad \dfrac{A = C \qquad \text{if } A \in \overrightarrow{BC} \text{ then } \overleftrightarrow{BC} \cap \ell = \{A\}}{\text{if } C \in \overrightarrow{BC} \text{ then } \overleftrightarrow{BC} \cap \ell = \{A\}}}{\overleftrightarrow{BC} \cap \ell = \{A\}}$$
[substitution]
[modus ponens]

[In place of '$A = C$', one might have '$C = A$'.]

Answers for Part C.

1. No. Mr. Jones has committed the fallacy of affirming the consequent. [Incidentally, even though his reasoning is invalid, his conclusion might be true.]

2. Steve's reasoning is not correct. Still, he did carry out the teacher's instructions. Notice that, in his reasoning, Steve committed the fallacy of affirming the consequent, but that he can defend his answer by a use of modus ponens:

$$\frac{\angle A \text{ and } \angle B \text{ are two right angles} \qquad \text{if } \angle A \text{ and } \angle B \text{ are two right angles then } \angle A \text{ and } \angle B \text{ have the same number of degrees}}{\angle A \text{ and } \angle B \text{ have the same number of degrees}}$$

He would use universal instantiation to infer the second premiss of this inference from his theorem:

if two angles are right angles then they have the same number of degrees

*

Answers for Part D [on pages 6-370 and 6-371].

1. (1) $\forall_x\ x + 0 = x$ [basic principle]

(2) $a + 0 = a$ [(1)]

(3) $\forall_x \forall_y \forall_z$ if $x = y$ then $xz = yz$ [theorem]

(4) if $a + 0 = a$ then $(a + 0)a = aa$ [(3)]

(5) $(a + 0)a = aa$ [by modus ponens from (2) and (4)]

(6) $\forall_x \forall_y \forall_z (x+y)z = xz + yz$ [basic principle]

(7) $(a + 0)a = aa + 0a$ [(6)]

(8) $aa + 0a = aa$ [(5) and (7)]

(9) $aa + 0 = aa$ [(1)]

(10) $aa + 0a = aa + 0$ [(8) and (9)]

(11)	$\forall_x \forall_y \forall_z$ if $x+y=x+z$ then $y=z$	[theorem]
(12)	if $aa+0a=aa+0$ then $0a=0$	[(11)]
(13)	$0a = 0$	[(10) and (12)]
(14)	$\forall_x \forall_y\ xy = yx$	[basic principle]
(15)	$a0 = 0a$	[(14)]
(16)	$a0 = 0$	[(13) and (15)]
(17)	$\forall_x\ x0 = 0$	[(1) - (16)]

2.

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{(1)}{(9)} \quad \dfrac{\dfrac{(6)}{(7)} \quad \dfrac{\dfrac{(1)}{(2)} \quad \dfrac{(3)}{(4)}}{(5)}}{(8)}}{(10)} \quad \dfrac{(11)}{(12)}}{(13)} \quad \dfrac{(14)}{(15)}}{(16)} \\ \hline\hline (17)\end{array}$$

*

Answers for Part E [on page 6-371].

1. $\forall_x \forall_y \forall_z$ if $y = z$ then $xy = xz$ [or: the left uniqueness principle for multiplication]

2. The left uniqueness principle for multiplication, and the principle for multiplying by 0.

*

Part E is exploration for a new principle of logic--conditionalizing, and discharging an assumption.

Just as the rule of universal instantiation justifies the basic procedure for inferring conclusions from universal generalization sentences, so modus ponens justifies the basic procedure for inferring conclusions from premisses, one of which is a conditional sentence. Now, the test-pattern principle complements universal instantiation by justifying the basic procedure for arriving at conclusions which are universal generalization sentences. What is still needed is a principle of logic which similarly complements modus ponens. How do we infer conclusions which are conditional sentences? The means is well-known and has been illustrated in Part E on page 6-371. One adopts the antecedent of the desired conditional conclusion as an additional premiss, often calling it an assumption, or a supposition. Then, one attempts to derive the consequent of the desired conclusion from this and other premisses. If this can be done, the conditional sentence whose antecedent is the assumption is said to follow from just these other premisses. For example, if, from an assumption 'it will rain this afternoon' and other premisses, one can infer the conclusion 'the grass will need to be cut', then the other premisses, alone, imply the conclusion '*if* it will rain this afternoon *then* the grass will need to be cut'. One keeps the chain of reasoning going by inferring the conditional sentence:

(1) if it will rain this afternoon then the grass will need to be cut

from its previously derived consequent:

the grass will need to be cut

[this is called *conditionalizing*], and then *discharges* the assumption 'it will rain this afternoon' which is the antecedent of (1).

In the test-pattern on page 6-372, the two premisses are the assumption '$a = b$', and the principle of identity. From these, by universal instantiation [to obtain '$a + c = a + c$'] and substitution, one derives '$a + c = b + c$'. Conditionalizing, one can infer from '$a + c = b + c$', the conditional sentence 'if $a = b$ then $a + c = b + c$'. Since the assumption is the antecedent of this conditional sentence, one can discharge the assumption. That is, one recognizes that although '$a + c = b + c$' depends on both '$a = b$' and the principle of identity, the conditional sentence depends only on the latter.

*

The same kind of situation occurs in Part E on page 6-371. Here, '$ab = 0$' is derived from '$b = 0$' and '$\forall_x\ x0 = 0$'. From '$ab = 0$' one can infer the weaker conditional conclusion 'if $b = 0$ then $ab = 0$'.

Having done so, the assumption 'b = 0' can be discharged. So, the conditional sentence is a consequence of the pm0. The proof is simple enough to use as an illustration of a tree-form proof:

$$\cfrac{\cfrac{\cfrac{\overset{*}{b = 0} \qquad \cfrac{\forall_x\ x0 = 0}{a0 = 0}}{ab = 0}}{\text{if } b = 0 \text{ then } ab = 0}\ *}{\forall_x \forall_y \text{ if } y = 0 \text{ then } xy = 0}$$

[universal instantiation]
[substitution]
[conditionalizing, and discharging]
[test-pattern principle]

Now, notice how we can use the principle of identity and modus ponens to derive the pm0 from the theorem we have just proved.

$$\cfrac{\cfrac{\cfrac{\forall_x\ x = x}{0 = 0} \qquad \cfrac{\forall_x \forall_y \text{ if } y = 0 \text{ then } xy = 0}{\text{if } 0 = 0 \text{ then } a0 = 0}}{a0 = 0}}{\forall_x\ x0 = 0}$$

[universal instantiation]
[modus ponens]
[test-pattern principle]

Since the principle of identity is a principle of logic, we are free to use it as a premiss and, then, to forget about it. You can think of it as having been discharged as soon as it is used. So, the only undischarged premiss in the above derivation is '$\forall_x \forall_y$ if y = 0 then xy = 0', and the derivation shows that this premiss implies the pm0.

*

Answers for Part A [on page 6-374].

1. Al's reasoning is valid. His premisses are:

   the Queen of England lives in Chicago; anyone who lives in Chicago lives in Illinois; anyone who lives in Illinois lives in the United States

2. if the Queen of England lives in Chicago then the Queen of England lives in the United States

Correction. On page 6-375, the first two lines [Exercises 1 and 2] should follow the last line of the proof.

Answers for Part B [which starts on page 6-374].

1. [basic principle]; [(2)]; [(1) and (3)]; [basic principle]; [(5)]; [(4) and (6)]; [theorem]; [(8)]; [(7) and (9)]; [conditionalizing (10); discharge (1)]; [(1) - (11)]

2.

$$\begin{array}{c} \dfrac{\dfrac{\dfrac{(5)}{(6)} \qquad \dfrac{\overset{*}{(1)} \qquad \dfrac{(2)}{(3)}}{(4)}}{(7)} \qquad \dfrac{(8)}{(9)}}{(10)} \\ \hline (11) \quad * \\ \hline\hline (12) \end{array}$$

*

Answers for Part C.

1. (1) $a = b$ [assumption]

(2) $\forall_x\, x = x$ [logical principle]

(3) a = a [(2)]

(4) $b = a$ [(1) and (3)]

(5) if $a = b$ then $b = a$ [conditionalizing (4); discharge (1)]

(6) $\forall_x \forall_y$ if $x = y$ then $y = x$ [(1) - (5)]

2.

$$\begin{array}{c} \dfrac{\overset{*}{(1)} \qquad \dfrac{(2)}{(3)}}{(4)} \\ \hline (5) \quad * \\ \hline\hline (6) \end{array}$$

Answers for Part ☆D.

<u>Theorem:</u> $\forall_x \forall_y$ if $x = y$ then $x^2 = y^2$

<u>Proof:</u>

(1)	$a = b$	[assumption]
(2)	$\forall_x\, x = x$	[logical principle]
(3)	$a^2 = a^2$	[(2)]
(4)	$a^2 = b^2$	[(1) and (3)]
(5)	if $a = b$ then $a^2 = b^2$	[conditionalizing (4); discharge (1)]
(6)	$\forall_x \forall_y$ if $x = y$ then $x^2 = y^2$	[(1)-(5)]

*

Answers for Part E [on pages 6-376 and 6-377].

1. if $a = 2$ then $2^2 + 3 \cdot 2 - 6 = 0$

2. if $c = d$ then $b \neq d$

3. if $a \neq c$ then $b \neq d$

*

Exercises 2 and 3 make the important point that one can often conditionalize in different ways, and, so, discharge different premisses. The answer to the questions concerning how you know which you should do must be answered in the same way as are similar questions relating to factoring: In context, one knows what conclusion he wants, and, if possible, conditionalizes in such a way as to obtain it. It may help to call to students' attention the fact that both Exercise 2 and Exercise 3 can be extended by two more conditionalizing steps, resulting in the discharge of all three premisses. For example:

$$\dfrac{\dfrac{\dfrac{c = d \qquad \dfrac{\overset{\dagger}{a = b} \qquad \overset{*}{a \neq c}}{b \neq c}}{b \neq d}}{\text{if } a \neq c \text{ then } b \neq d}\;{*}}{\text{if } a = b \text{ then [if } a \neq c \text{ then } b \neq d]}\;\dagger$$

etc.

*

4. $\overset{*}{a} = b$ $\quad ac = ac$

$$\frac{\dfrac{a = b \qquad ac = ac}{ac = bc}\;*}{\text{if } a = b \text{ then } ac = bc}$$

5. Ed lives in Miami* — if Ed lives in Miami then Ed lives in Florida

Ed lives in Florida — if Ed lives in Florida then Ed lives in the U.S.

Ed lives in the U.S. *

if Ed lives in Miami then Ed lives in the U.S.

6. $\overset{*}{p}$ — if p then q

q — if q then r

r *

if p then r

7. $\overset{*}{a = b}$ — if a = b then c = d

c = d *

if a = b then c = d

*

Exercise 6 shows how modus ponens, and conditionalizing, and discharging an assumption can be used to justify an important logical principle--<u>the hypothetical syllogism</u>. Schematically:

$$\frac{\text{if p then q} \qquad \text{if q then r}}{\text{if p then r}}$$

Exercise 7 points out that using, first, modus ponens and then, second, conditionalizing, and discharging a premiss, leads you back to the conditional premiss you started from.

*

The statement, on page 6-373, of the rule for discharging an assumption is over-simplified. For completeness, we shall point out here how the use of the rule, as it is stated in the fifth, sixth, and seventh lines above the box on page 6-373, <u>can</u> lead to incorrect conclusions, and how the statement of the rule can be modified to avoid this. However, unless your students have a very good grasp of the subject matter of this

appendix, it will probably be best to let the rule stand as it is on page 6-373.

Consider the following argument in tree-form:

$$\dfrac{\dfrac{\forall_x\ x=x}{0=0} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{b\overset{*}{=}0 \qquad \dfrac{\forall_x\ x\cdot 0=0}{1\cdot 0=0}}{1\cdot b=0}}{\forall_y\ 1\cdot y=0}\ *}{\text{if } b=0 \text{ then } \forall_y\ 1\cdot y=0}}{\forall_x \text{ if } x=0 \text{ then } \forall_y\ 1\cdot y=0}}{\text{if } 0=0 \text{ then } \forall_y\ 1\cdot y=0}}{\forall_y\ 1\cdot y=0}$$

Since the premisses [the pm0 and the principle of identity] are acceptable and the conclusion is not, something is wrong with the argument. And, it is not difficult to see what is wrong: The first three lines form a test-pattern for the fourth line ['$\forall_y\ 1\cdot y = 0$']. However, the assumption '$b = 0$' is not discharged in this test-pattern but is discharged after the test-pattern has been invoked in obtaining the fourth line. Now, once the test-pattern principle has been applied, the test-pattern must be held inviolate; for, if it is changed, it is no longer a test-pattern for the same generalization. The error in the above argument consists in discharging the assumption '$b = 0$' after it has been "blocked off" by an application of the test-pattern principle. [Alternatively, one might claim that the error is due to applying the test-pattern principle before discharging the premiss '$b = 0$' which contains the pattern-variable 'b'.]

So, the rule for discharging an assumption must not be applied in a case where the assumption to be discharged contains the pattern-variable of a test-pattern which has already been blocked off by an application of the test-pattern principle.

In practice, one is not likely to violate this restriction. If one makes

an assumption in order to obtain a test-pattern for a generalization, he will ordinarily be careful to see that this assumption is discharged, before stating the generalization as his conclusion. For this reason, it has seemed better to omit discussion of this somewhat complicated restriction in the students' text.

*

The rules discussed so far have been the basic rules for dealing with universal generalizations, equations, and conditionals. Now, we need to consider rules which are concerned principally with denial sentences. In English, one forms a denial of a given sentence, for example, of 'Bill lives in Honolulu' by introducing the word 'not', and making various other changes dictated by rules of grammar. One denial of 'Bill lives in Honolulu' is 'Bill does not live in Honolulu'; another is 'It is not the case that Bill lives in Honolulu.'. The grammatical vagaries of English being of no present concern to us, we shall sometimes form a denial of a sentence by merely prefixing the word 'not'. Thus: not Bill lives in Honolulu. And, we shall call the sentence so obtained the denial of the given sentence. So, the denial of '$3 = 4$' is 'not $3 = 4$' and, of course, this is often abbreviated to '$3 \neq 4$'.

*

It is possible, and aesthetically pleasing, to base the discussion of denial sentences on a single basic principle of logic, the reverse rule of contraposition, given on page 6-386, or an equivalent rule. However, for a beginner, it is simpler to start with three basic rules: modus tollens [page 6-377] and the two rules of double denial [page 6-383]. On the basis of these three rules, together with the two basic rules for conditional sentences, one can readily justify various forms of contraposition, and procedures for indirect proof.

*

Just as modus ponens has associated with it a fallacious kind of reasoning [affirming the consequent], so there is a fallacy, [denying the antecedent], which is sometimes confused with modus tollens. [An example of this fallacy can be obtained from Exercise 4 of Part A on page 6-378 by supplying the conclusion '$A \notin \overleftrightarrow{BC}$'.] This fallacy is pointed out on page 6-379. As indicated on page 6-387, both of these fallacies arise from confusion of a conditional sentence with its converse.

Answers for Part A.

Exercises 1, 2, 3, 6, and 8 can be completed to give illustrations of modus tollens. Here are the missing sentences:

1. $a \ne b$ 2. $-a \ne -b$ 3. $A \notin \overline{BC}$

6. Tom does not live in Atlanta 8. it is not raining

[Exercise 4, as noted on TC[6-376, 377]c, can be completed to yield an illustration of the fallacy of denying the antecedent. Exercise 5 can be completed, by supplying the conclusion '$A \in \overleftrightarrow{BC}$', to an illustration of modus ponens. Exercise 7 is similar to Exercise 4.]

*

Answers for Part B [on page 6-379].

1.
$$\cfrac{\cfrac{\cfrac{a \overset{*}{=} 2 \qquad a^2 - 3a - 6 = 0}{2^2 - 3\cdot 2 - 6 = 0}\,{}^{*}}{\text{if } a=2 \text{ then } 2^2-3\cdot 2-6=0} \qquad 2^2 - 3\cdot 2 - 6 \ne 0}{a \ne 2}$$

2.
$$\cfrac{\cfrac{\cfrac{\text{if } a = 0 \text{ then } ab = 0 \qquad ab \overset{*}{\ne} 0}{a \ne 0} \qquad \text{if } a \ne 0 \text{ then } a^2 > 0}{a^2 > 0}\,{}^{*}}{\text{if } ab \ne 0 \text{ then } a^2 > 0}$$

3.
$$\cfrac{\cfrac{\text{if } A \in \overline{BC} \text{ then } A \in \overleftrightarrow{BC} \qquad A \overset{*}{\notin} \overleftrightarrow{BC}}{A \notin \overline{BC}}\,{}^{*}}{\text{if } A \notin \overleftrightarrow{BC} \text{ then } A \notin \overline{BC}}$$

[Exercise 3 illustrates how modus tollens and conditionalizing, and discharging an assumption can be used to justify inferring the contrapositive [see page 6-381] of a given conditional sentence from the

given sentence. Exercise 4 leads students to construct a scheme for this procedure, thus showing that it can always be carried out.]

4. if p then q $\quad$ not q*

not p

if not q then not p*

*

Another typical application of modus tollens occurs in the argument by which one infers, from two premisses, '$A \in \ell$' and '$A \notin m$', the conclusion '$\ell \neq m$'. You might try writing the given premisses near the top of the board, and ask students what conclusion they can draw concerning ℓ and m. Having obtained some such response as '$\ell \neq m$' or, even, 'they're different', write '$\ell \neq m$' near the bottom of the board, and ask for suggestions on how to obtain this conclusion from the given premisses. Someone [perhaps you] should eventually suggest that if you had 'if $\ell = m$ then $A \in m$', you could use modus tollens to get the desired conclusion from this and the premiss '$A \notin m$'. Start filling in the derivation so that the board looks like this:

$$A \in \ell, \; A \notin m$$

$$\frac{\text{if } \ell = m \text{ then } A \in m \qquad A \notin m}{\ell \neq m}$$

Now, how to get this conditional sentence? How else than by conditionalizing? So, we need to get '$A \in m$' from somewhere, and we can use

'$\ell = m$' as an assumption which will be discharged when we conditionalize. The board should now look like this:

$$A \in \ell, \quad A \notin m, \qquad \ell = m \leftarrow [\text{assumption}]$$

$$\cfrac{\cfrac{A \in m}{\text{if } \ell = m \text{ then } A \in m} \qquad A \notin m}{\ell \neq m}$$

Now, how can we use the assumption to get '$A \in m$'? Well, we haven't yet used the given premiss '$A \in \ell$'. From the assumption and this premiss we can infer '$A \in m$'.

$$\cfrac{\cfrac{\cfrac{\overset{*}{\ell = m} \qquad A \in \ell}{A \in m}}{\text{if } \ell = m \text{ then } A \in m}\,{}^{*} \qquad A \notin m}{\ell \neq m}$$

As a variant of this problem, students can now discover how to infer '$A \neq B$' from '$A \in \ell$' and '$B \notin \ell$'.

$$\cfrac{\cfrac{\cfrac{A \overset{*}{=} B \qquad A \in \ell}{B \in \ell}}{\text{if } A = B \text{ then } B \in \ell}\,{}^{*} \qquad B \notin \ell}{A \neq B}$$

Through such exercises, students can begin to get an understanding of "indirect proofs".

Answers for Part C.

[Note the use of '*'s in the column of comments to indicate [step (5)] where an assumption is discharged, and [step (1)] that it has been discharged. In writing proofs, students should <u>not</u> star assumptions (hopefully) until they have arrived at the step at which they actually are discharged.]

1.
$$\begin{array}{c} \dfrac{(2)}{(3)} \qquad \overset{*}{(1)} \\ \hline (4) \\ \hline (5) \;\; * \\ \hline\hline (6) \end{array}$$

2. (a) universal instantiation (b) modus tollens
   (c) conditionalizing (d) test-pattern principle

3. (a) (1) and (2) (b) (2) (c) (2)
   (d) Because it is a consequence of the theorem (2).

4. If, in the answer for Exercise 1, one blocks out '$\underline{(2)}$' and '$\overline{\overline{(6)}}$', and takes account of the forms of sentences (3), (1), (4), and (5), he should see that this part of the proof illustrates the scheme developed in Exercise 4 of Part B. As pointed out on page 6-381, steps (1) and (4) of the proof could be omitted, and step (5) inferred from (3) by virtue of the rule of contraposition which is justified by the scheme of Exercise 4 of Part B.

We can justify this new rule schematically by:

$$\dfrac{\dfrac{\begin{array}{c}\overset{*}{\underline{p}}\\ \vdots\\ q\end{array}}{\text{if } p \text{ then } q}\,*}{\text{if not } q \text{ then not } p}$$

This scheme shows how, in a proof, the effect of using the new rule can be obtained by, first, conditionalizing, and discharging an assumption, and, then, using the rule of contraposition.

On the other hand, if we take the new rule as basic, we can justify the rule of contraposition as follows, using, first, modus ponens and, then, the new rule:

$$\dfrac{\dfrac{\overset{*}{p} \qquad \text{if } p \text{ then } q}{q}\,*}{\text{if not } q \text{ then not } p}$$

Summarizing, the three rules expressed schematically by:

$$\dfrac{\text{if } p \text{ then } q \qquad \text{not } q}{\text{not } p} \qquad\qquad \dfrac{\text{if } p \text{ then } q}{\text{if not } q \text{ then not } p} \qquad\qquad \dfrac{\begin{array}{c}[p]\\ q\end{array}}{\text{if not } q \text{ then not } p}$$

are equivalent, in the sense that each of them can be used, together with our two basic rules for conditional sentences, to justify any inference which can be justified by using either of the other.

When 'p' and 'q' are replaced in 'if p then q' and in 'if not q then not p' by sentences, then the second of the resulting sentences is the contrapositive of the first. Note that the first is not the contrapositive of the second--contrapositing is a one-way street. The contrapositive of the second sentence is obtained by making the same replacements for 'p' and 'q' in 'if not not p then not not q'. The third sentence so obtained can be shown, by using the rules of double denial [see page 6-383], to be equivalent to the first sentence--each of the two sentences is a consequence of the other--but, they are different sentences.

The rule of contraposition, which says that each conditional sentence implies its contrapositive, was justified, on the basis of modus tollens, and conditionalizing, and discharging an assumption, in Exercise 4 of Part B on page 6-379. This justification is given again on page 6-381. The reverse rule of contraposition, which says that each conditional sentence is implied by its contrapositive, will be justified in Exercise 3 of Part B on page 6-385. As hinted at in the preceding paragraph, the justification of this rule makes use of the rules of double denial.

We have chosen to take modus tollens as one of our basic rules, and have justified the rule of contraposition by applying modus tollens, and conditionalizing, and discharging an assumption. Note that we could as easily have taken the rule of contraposition as basic, and used it and modus ponens to justify modus tollens:

$$\dfrac{\text{not q} \qquad \dfrac{\text{if p then q}}{\text{if not q then not p}}\ \text{[contraposition]}}{\text{not p}}\ \text{[modus ponens]}$$

Another alternative to choosing modus tollens as a basic rule is to take as basic a rule which combines conditionalizing, and discharging an assumption, with the rule of contraposition. Using '[p]' as on page 6-373, this rule can be expressed schematically by:

$$\dfrac{\begin{array}{c}[\text{p}]\\ \text{q}\end{array}}{\text{if not q then not p}}$$

For example, using this rule, we can infer step (6) of the column proof on page 6-382 directly from (4), and in doing so, discharge (1). So, step (5) could be omitted, and the comment for (6) be replaced by that for (5).

The five-step argument on page 6-383 illustrates the scheme:

$$\dfrac{\dfrac{\text{if p then not q} \qquad \dfrac{\overset{*}{q}}{\text{not not q}}\ \text{[double denial]}}{\text{not p}}\ \text{[modus tollens]}}{\text{if q then not p}}\ *\ \text{[conditionalizing and discharging]}$$

This justifies the symmetric rule of contraposition given on page 6-384. [This rule is called 'symmetric' because it is its own reverse--two successive applications of it bring one back to where he started.]

*

Answers for Part A [on page 6-384].

1. if $A \notin \overleftrightarrow{BC}$ then $A \notin \overline{BC}$
2. if $A \notin \overline{BC}$ then $A \notin \overleftrightarrow{BC}$
3. if $a^2 \neq 4$ then $a \neq 2$
4. $a \neq 3$ if $a^2 \neq 9$
5. $a \neq 3$ only if $a^2 \neq 9$

*

Exercises 4 and 5 of Part A are preparation for biconditional sentences. [See page 6-390.] That '$a^2 = 9$ if $a = 3$' is another way of saying 'if $a = 3$ then $a^2 = 9$' should not be hard to see. The corresponding fact about Exercise 5 can be brought out by noting that '$a^2 = 9$ only if $a = 3$' says the same thing as does 'if $a \neq 3$ then $a^2 \neq 9$'. But this last, is the contrapositive of 'if $a^2 = 9$ then $a = 3$', and a sentence and its contrapositive do say the same thing.

Answers for Part B [on pages 6-384 and 6-385].

1. (a)

$$\cfrac{\text{if } A \in \overline{BC} \text{ then } B \neq C \qquad \cfrac{B \overset{*}{=} C}{\text{not } (B \neq C)}\ [\text{d.d.}]}{\cfrac{A \notin \overline{BC}}{\text{if } B = C \text{ then } A \notin \overline{BC}}\ *\ [\text{c.d.}]}\ [\text{m.t.}]$$

(b)

$$\cfrac{\text{if } p \text{ then not } q \qquad \cfrac{\overset{*}{q}}{\text{not not } q}\ [\text{d.d.}]}{\cfrac{\text{not } p}{\text{if } q \text{ then not } p}\ *\ [\text{c.d.}]}\ [\text{m.t.}]$$

2. (a)

$$\cfrac{\cfrac{\text{if } A \neq B \text{ then } B \in \overrightarrow{AB} \qquad B \overset{*}{\notin} \overrightarrow{AB}}{\text{not } A \neq B}\ [\text{m.t.}]}{\cfrac{A = B}{\text{if } B \notin \overrightarrow{AB} \text{ then } A = B}\ *\ [\text{c.d.}]}\ [\text{r.d.d.}]$$

(b)

$$\cfrac{\cfrac{\text{if not } p \text{ then } q \qquad \overset{*}{\text{not } q}}{\text{not not } p}\ [\text{m.t.}]}{\cfrac{p}{\text{if not } q \text{ then } p}\ *\ [\text{c.d.}]}\ [\text{r.d.d.}]$$

3. (a)

$$\frac{\dfrac{\text{if } A \notin \overleftrightarrow{BC} \text{ then } A \notin \overline{BC} \qquad \dfrac{A \overset{*}{\in} \overline{BC}}{\text{not } A \notin \overline{BC}}\ [\text{d.d.}]}{\dfrac{\text{not } (A \notin \overleftrightarrow{BC})}{A \in \overleftrightarrow{BC}}\ [\text{r.d.d.}]}\ [\text{m.t.}]}{\text{if } A \in \overline{BC} \text{ then } A \in \overleftrightarrow{BC}}\ * \quad [\text{c.d.}]$$

(b)

$$\frac{\dfrac{\text{if not } q \text{ then not } p \qquad \dfrac{\overset{*}{p}}{\text{not not } p}\ [\text{d.d.}]}{\dfrac{\text{not not } q}{q}\ [\text{r.d.d.}]}\ [\text{m.t.}]}{\text{if } p \text{ then } q}\ * \quad [\text{c.d.}]$$

4. (a)

$$\frac{\dfrac{\text{if } A \in \overline{BC} \text{ then } A \in \overleftrightarrow{BC} \qquad A \overset{*}{\notin} \overleftrightarrow{BC}}{A \notin \overline{BC}}\ [\text{m.t.}]}{\text{if } A \notin \overleftrightarrow{BC} \text{ then } A \notin \overline{BC}}\ * \quad [\text{c.d.}]$$

(b)

$$\frac{\dfrac{\text{if } p \text{ then } q \qquad \text{not } \overset{*}{q}}{\text{not } p}\ [\text{m.t.}]}{\text{if not } q \text{ then not } p}\ * \quad [\text{c.d.}]$$

*

Exercises 1(b) and 2(b) justify two symmetric rules of contraposition; Exercise 3(b) justifies the reverse rule of contraposition; Exercise 4(b) is the by now familiar justification of the rule of contraposition.

Correction. On page 6-386, at the bottom of the page, each of the lines beginning 'is an instance of' should be moved down to line up with the boxes.

Fill-ins for bottom of page 6-386.

if $w \neq t$ then $r \neq s$; if $w = t$ then $r = s$.

*

<u>Proofs</u> <u>by</u> <u>contradiction</u>.--By conditionalizing, and using modus tollens, one can show that from a sentence and its denial one can infer the denial of any sentence:

$$\dfrac{\dfrac{q}{\text{if } p \text{ then } q} \qquad \text{not } q}{\text{not } p}$$

By replacing 'p', in the scheme above, by 'not p', and then using the reverse rule of double denial:

$$\dfrac{\dfrac{\dfrac{q}{\text{if not } p \text{ then } q} \qquad \text{not } q}{\text{not not } p}}{p}$$

one sees that, as already noted in TC[6-365], any sentence follows from any two sentences one of which is the denial of the other.

Schematically:

$$\boxed{\dfrac{q \qquad \text{not } q}{p}}$$

This <u>rule</u> <u>of</u> <u>contradiction</u> is at the root of proofs by contradiction. Such proofs, in which one establishes a conclusion by showing that its denial leads to a contradiction, are, schematically, of this form:

$$\dfrac{\begin{matrix} \overset{*}{\text{not } p} \\ \vdots \\ q \end{matrix} \qquad\qquad \begin{matrix} \overset{*}{\text{not } p} \\ \vdots \\ \text{not } q \end{matrix}}{p}\;{*}$$

That is, if, using the denial of the desired conclusion as an assumption, one can derive some sentence and can, also, derive its denial, then one can infer the desired conclusion and discharge the assumption. Here, one infers the desired conclusion by use of this rule of contradiction, which we have just now justified. But, it remains to be shown that, on using the rule of contradiction, one is entitled to discharge an assumption which is the denial of the conclusion of the inference.

Now, we do know that we are entitled to discharge an assumption after conditionalizing. So, we might proceed as indicated below:

$$\dfrac{\dfrac{\begin{matrix}\overset{*}{\text{not p}}\\ \vdots\\ \text{q}\end{matrix} \qquad \begin{matrix}\overset{*}{\text{not p}}\\ \vdots \\ \text{not q}\end{matrix}}{\text{p}}\ \text{[rule of contradiction]}}{\text{if not p then p}}\ {*}\ \text{[c. d.]}$$

Then, we will have justified proof by contradiction once we have shown that:

$$\dfrac{\text{if not p then p}}{\text{p}}$$

is a valid scheme of inference. This we now proceed to do.

$$\dfrac{\text{if not p then p} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{\text{not p}} \qquad \text{if not p then p}}{\text{p}}\ \text{[m. p.]} \qquad \overset{*}{\text{not p}}}{\text{not [if not p then p]}}\ \text{[r. of c.]}}{\text{if not p then not [if not p then p]}}\ {*}\ \text{[c. d.]}}{\text{if [if not p then p] then p}}\ \text{[rev. contrap.]}}{\text{p}}\ \text{[m. p.]}$$

Correction. On page 6-390, line 5 should begin:

$$\forall_x \forall_y \forall_z \underbrace{\text{if}}_{\uparrow}\ x + z = ---$$

Answers for Part A [on pages 6-388 and 6-389].

1. if $A \notin \overset{\bullet\!\!\rightarrow}{BC}$ then $A \notin \overrightarrow{BC}$
   if $A \in \overset{\bullet\!\!\rightarrow}{BC}$ then $A \in \overrightarrow{BC}$

2. if $A \notin \overrightarrow{BC}$ then $A \notin \overset{\bullet\!\!\rightarrow}{BC}$
   if $A \in \overrightarrow{BC}$ then $A \in \overset{\bullet\!\!\rightarrow}{BC}$

3. if $\overline{AB} \neq \emptyset$ then $A \neq B$
   if $\overline{AB} = \emptyset$ then $A = B$

4. $a \neq c$ if $ac \neq bc$ [or: $ac \neq bc$ only if $a \neq c$]
   $a = c$ if $ac = bc$ [or: $ac = bc$ only if $a = c$]

5. $a \neq c$ only if $ac \neq bc$ [or: $ac \neq bc$ if $a \neq c$]
   $a = c$ only if $ac = bc$ [or: $ac = bc$ if $a = c$]

6. $\forall_X \forall_Y \forall_Z$ if X, Y, and Z are not collinear then $Z \notin \overleftrightarrow{XY}$
   $\forall_X \forall_Y \forall_Z$ if X, Y, and Z are collinear then $Z \in \overleftrightarrow{XY}$

7. $\forall_x \forall_y$ if $-x \neq y$ then $x + y \neq 0$
   $\forall_x \forall_y$ if $-x = y$ then $x + y = 0$

8. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}$ if $xv \neq uy$ then $\frac{x}{y} \neq \frac{u}{v}$
   $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}$ if $xv = uy$ then $\frac{x}{y} = \frac{u}{v}$

9. $\forall_X \forall_Y$ if $X \neq Y$ then $\{Z\colon\ Y \in \overline{XZ}\} \neq \emptyset$
   $\forall_X \forall_Y$ if $X = Y$ then $\{Z\colon\ Y \in \overline{XZ}\} = \emptyset$

*

Answers for Part B [on page 6-389].

denying the antecedent: 1
inferring the inverse: 6
inferring the converse: 7, 8
affirming the consequent: 9

Correction. On page 6-391, in line 7, delete the comma after '0'.

The substitution rule for biconditional sentences.

Given a biconditional sentence and another sentence, if the left side of the biconditional sentence is replaced by its right side somewhere in the other sentence, the new sentence thus obtained is a consequence of the given sentences.

In the column proof, step (8) is the conclusion of a dilemma whose premisses are steps (1), (3), and (7). Here is a diagram of the column proof:

$$\begin{array}{c} \dfrac{\dfrac{\overset{*}{(1)} \qquad \dfrac{(2)}{(3)} \qquad \dfrac{\dfrac{(5)}{(6)} \qquad \dfrac{(2)}{(4)}}{(7)}}{(8)}}{(9)}\,{}^{*} \\ \hline\hline (10) \end{array}$$

*

The law of the excluded middle can be justified by using a method of proof called <u>proof by cases</u>. Schematically, proof by cases can be indicated by:

$$\dfrac{\begin{array}{c}[p]\\ \\ q\end{array} \qquad \begin{array}{c}[\text{not } p]\\ \\ q\end{array}}{q}$$

In words: If q can be derived from the assumption p, and other premisses, and can also be derived from the assumption not p, and other premisses, then q is a consequence of the other premisses, alone.

Before justifying proof by cases, let's use it to establish the law of the excluded middle:

$$\dfrac{\dfrac{\overset{*}{p}}{p \text{ or not } p} \qquad \dfrac{\overset{*}{\text{not } p}}{p \text{ or not } p}}{p \text{ or not } p}\,{}^{*}$$

The foregoing scheme shows how any sentence of the form 'p or not p' can be derived, using two of the rules for alternation sentences and proof by cases. In deriving such a sentence, all the premisses in the derivation are discharged. Hence, such a sentence can, itself, be used as a premiss in any derivation and be treated as a discharged premiss.

The following scheme shows how proof by cases can be carried out by using only rules which have previously been discussed:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} * \\ \underline{\text{p}} \\ \vdots \\ \text{q} \end{array}}{\text{if p then q}}\,*}{\text{if not q then not p}} \qquad \dfrac{\begin{array}{c} \dagger \\ \underline{\text{not p}} \\ \vdots \\ \text{q} \end{array}}{\text{if not p then q}}\,\dagger}{\text{if not q then q}}}{\text{q}}
$$

The kinds of inference used above are conditionalizing, discharging an assumption, contraposition, the hypothetical syllogism [Exercise 6 on page 6-377], and the kind of inference justified on TC[6-386]b.

The rules for denying an alternative can also be justified on the basis of earlier rules. For example, the first form is justified by the scheme:

$$
\dfrac{\text{p or q} \qquad \dfrac{\overset{*}{\text{p}}}{\text{if p then p}}\,* \qquad \dfrac{\dfrac{\overset{\dagger}{\text{q}} \quad \text{not q}}{\text{p}}}{\text{if q then p}}\,\dagger}{\text{p}}
$$

The inferences used are a contradiction, conditionalizing, and discharging an assumption [twice], and a dilemma.

But, the truth-table tells us that, for this to be so, the antecedent of the premiss must be true and its consequent must be false. Now, in order to conclude that the denial [not q] of a true statement [q] is false, all we need is to be assured that there is some false statement which we can substitute for 'p'. We are assured of this by (iii). Similarly, in order to conclude that the denial [not p] of a false statement [p] is true, all we need is assurance that there is some true statement which can be substituted for 'q'. But, as we have seen, any statement of the form 'if p then p' is true.

In a similar [but simpler] fashion, one can use (i), together with the rules which specify inferences of the forms:

$$\frac{p \text{ and } q}{p} \qquad \frac{p \text{ and } q}{q} \qquad \frac{p \qquad q}{p \text{ and } q}$$

as valid, to deduce the truth-table for conjunction statements [see below]. And, one can use (i), together with the rules that specify inferences of the forms:

$$\frac{p}{p \text{ or } q} \qquad \frac{q}{p \text{ or } q} \qquad \frac{p \text{ or } q \quad \text{if } p \text{ then } r \quad \text{if } q \text{ then } r}{r}$$

as valid, and the rule that statements of the form 'if p then p' are true, to deduce the truth-table for alternation statements. [Hint: In the scheme for the dilemma, replace 'r' by 'p'.]

p	q	p and q	p or q
T	T	T	T
F	T	F	T
T	F	F	T
F	F	F	F

the fact that each statement of the form:

(**) $\qquad$ if p then p

is valid--that is, is a consequence of the empty set. That this is so is shown by the scheme:

$$\dfrac{\dfrac{\overset{*}{p} \qquad \dfrac{\overset{*}{p}}{\text{if } p \text{ then } p}}{p}}{\text{if } p \text{ then } p}\;{*}$$

This being the case, it follows from (i) that each statement of the form (**) is true. In particular, a statement obtained by replacing 'p' in (**) by a false statement must be true. Such a statement is a conditional statement whose antecedent and consequent are both false. Now, by (ii), it follows that since this conditional statement is true, <u>each</u> conditional statement whose antecedent and consequent are both false must, also, be true. So, we have the entry in the fourth line of the truth table.

Now that we have deduced the truth-table for conditional statements from (i) and (ii), and the three rules of reasoning for conditional sentences, we can use this truth-table, (i), (ii), and the reverse rule of contraposition to deduce the usual truth-table for denial sentences. This table is:

p	not p
T	F
F	T

In words: the denial of each true statement is false; the denial of each false statement is true. To establish this, consider an inference of the form:

$$\frac{\text{if not } p \text{ then not } q}{\text{if } q \text{ then } p}$$

obtained by replacing 'q' by a true statement and 'p' by a false statement. The truth-table for conditional statements tells us that the conclusion of this inference is false. So, by (i), the premiss of the inference is false.

For all these reasons, we consider the approach to validity through truth-tables to be an unsatisfactory one, especially for beginning students. For one who adopts our point of view concerning validity, any attempt to define validity in terms of truth appears to be putting the cart before the horse. For, as we shall now show, the rules which we have adopted to prescribe what inferences are valid, when supplemented by three general rules concerning truth, force us to adopt the usual truth-tables. In other words, one can define truth for compound statements [conditionals, denials, conjunctions, and alternations] in terms of validity. The three general rules concerning truth are:

(i) If some consequence of a set of premisses is false then at least one of the premisses is false.

(ii) Whether a compound statement is true is determined by which of its components are true.

(iii) Not all statements are true.

From these and the rules modus ponens, conditionalizing, and discharging an assumption, we shall now derive the truth-table for conditional statements. To begin, since each inference of the form:

$$\frac{q}{\text{if } p \text{ then } q}$$

is valid, it follows from (i) that if a conditional statement [if p then q] is false, then its consequent [q] must be false. Equivalently, if the consequent of a conditional statement is true, then the statement itself is true. This gives us the entries under 'if p then q' in the first two lines of the truth-table.

Similarly, since each inference of the form:

$$\frac{p \qquad \text{if } p \text{ then } q}{q}$$

is valid, it follows, again from (i), that if the consequent of a conditional is false then the conditional statement and its antecedent cannot both be true. So, if the consequent of a conditional statement is false and its antecedent is true, then the conditional statement must be false. This gives us the entry in the third line of the table. Finally, we make use of

There is an alternative procedure for justifying the acceptance of, say, modus ponens-type inferences as valid. This procedure is based on the truth-table for conditional statements:

p	q	if p then q
T	T	T
F	T	T
T	F	F
F	F	T

This table asserts, that [see third line] a conditional statement is false if its antecedent is true and its consequent is false, and [see first, second, and fourth lines] is true if its consequent is true or its antecedent is false. In particular, it asserts that if the antecedent of a conditional is true [first and third lines] and the conditional is itself true [ruling out the third line], then the conditional's consequent is true. So, [the truth-table asserts], a modus ponens-type inference can never lead one from true premisses to a false conclusion. So far, so good, assuming that one has accepted the truth-table. But, as pointed out earlier, one must also be convinced of the validity of modus ponens-type inferences whose premisses are not both true. The argument based on the truth-table does nothing to satisfy this need. Consequently, this approach is inadequate. Moreover, if, as has been urged, one grants that a valid inference is valid irrespective of the truth or falsity of its premisses or conclusion, then the truth-table approach appears to be irrelevant.

Pedagogically, the approach to validity through truth-tables has disadvantages besides those mentioned in the preceding paragraph. To begin with, one needs to give some sort of argument for accepting the truth-tables. It is not easy to give a satisfactory reason for accepting a conditional sentence as true when its antecedent and consequent are both false. In fact, a teacher is likely to encounter resistance against even considering conditional sentences whose antecedents are false. In overcoming these difficulties, so much emphasis is likely to be placed on truth as to make it even harder to convince students that the essence of a proof is not that it shows that its conclusion is true if its premisses are true, but, rather, that it shows that its conclusion necessarily follows from its premisses, whether these are true or false. The relevance of this to the understanding of proofs by contradiction has been pointed out earlier.

(2) If John is poor then John is happy.

from:

(3) John is happy.

For that matter, from (3) one is entitled, by conditionalizing, to infer:

(2′) If grass is blue then John is happy.

The ground for accepting any such inference again lies in the meaning intended for 'if...then...'. Because of this meaning, statement (3) is a "stronger" statement than each statement like (2) or (2′). [In some sense, it says what is said by all such statements, taken together. The flat statement 'John is happy.' tells us that, whatever conditions may exist, John is happy.]

The rule for discharging assumptions on conditionalizing [page 6-37, lines 3 through 11; page 6-372 et seq.] completes the explanation of what 'if...then...' means.

We have said earlier that, in addition to inferences, we would call certain sentences valid. To illustrate this use of the word, consider the scheme:

$$\dfrac{\dfrac{\overset{*}{q}}{\text{if } p \text{ then } q}}{\text{if } q \text{ then (if } p \text{ then } q)}\,*$$

This shows that, since conditionalizing and discharging an assumption are valid, each sentence of the form:

(*) if q then (if p then q)

is a consequence of the empty set ["of sentences"]. Since such a sentence is, trivially, a consequence of a set of sentences no one of which is false, it follows from the relationship, previously pointed out, between validity and truth, that each statement of the form (*) is true. Since the truth of such statements, like the validity of conditionalizing, and discharging an assumption, stems just from the meaning of 'if...then...', it is natural to call these statements valid. Furthermore, for the purposes of test-patterns, it is convenient to call any sentence of the form (*) 'valid', since substituting names for its variables will produce a valid statement.

*

A similar situation arises in proofs by contradiction, where one attempts to show that a given statement is a consequence of certain premisses by showing that the premisses and the denial of the given statement yield a contradiction. Hence, if the premisses happen to be true, the given statement will be true, also. This is "in spite of the fact" that in using the denial of this true statement as an assumption, one has "accepted", at least temporarily, an additional premiss which is false. Probably one difficulty which some have in granting validity to proofs by contradiction is due to their failure to distinguish between accepting a statement as a premiss and accepting it as being true. In the preceding derivation of (3) from (1) and (4) we "accepted (4) as a premiss". But, the derivation formed part of the evidence on which we based our decision <u>not</u> to accept (4) as true. This distinction is confused in the less formal statement of the argument which begins 'Suppose <u>that</u> everyone who is poor is happy.', and ends 'But, John is not happy. So, not everyone who is poor is happy.'

The preceding discussion shows the inadequacy of the frequently given argument that modus ponens-type inferences should be ranked valid because, by using such inferences, one will never infer a false conclusion from true premisses. That such should never happen is, as pointed out earlier, a necessary condition for the validity of any inference. But, it obviously is not a <u>sufficient</u> condition for accepting the validity of such inferences as that of (3) from (1) and (2), in which the premisses are <u>not</u> both true.

Beyond this, we contend that the real ground for accepting an inference as valid has nothing to do with truth. Our ground for rating modus ponens-type inferences valid--that is, for accepting what such an inference "says"--lies in the meaning which we intend the phrase 'if... then...' to have. Conversely, when we tell someone that such inferences are valid, we are giving him a partial explanation of the meaning of 'if...then...'. [Imagine that, through some system of language reform, 'if...then...' came to have the meaning which we now associate with '... or ...'. Then, we should no longer agree with what modus ponens-type inferences say, and would no longer rank them valid.]

Similar remarks apply to the other rules of reasoning discussed in this appendix. For example, take conditionalizing--the rule according to which inferences of the form:

$$\frac{q}{\text{if p then q}}$$

are valid. As an instance of this, it is correct to infer:

But this, although it follows from _the validity of_ the inference:

$$\frac{\text{John is poor.} \qquad \text{If John is poor then John is happy.}}{\text{John is happy.}}$$

is quite different from the inference itself. The inference "says" that (3) is a consequence of (1) and (2), and makes no reference to the truth of any of these three statements. When _we_ say that the inference is _valid_, we are saying that we accept what it "says". As an example of what such acceptance means, suppose that we believe (1) and (2) to be true and, in consequence of the validity of the inference, accept (3) as true. Now, if we discover that (3) is false, we should not say that we reasoned incorrectly--that is, that the inference is not valid. Rather, we should conclude that we were incorrect in believing that both (1) and (2) were true.

[Just as the validity of an inference does not guarantee the truth of its premisses, or of its conclusion, so, the truth of premisses and conclusion does not guarantee the validity of the inference. For example, even if (1), (2), and (3) are all true, it is incorrect to infer (1) from (2) and (3). Doing so is to commit the fallacy of affirming the consequent.]

As a matter of fact, we often find occasion to reason from premisses which we know [or, at least, believe] to be false. As an example, consider the universal generalization:

(4) Everyone who is poor is happy.

We may believe this to be false, and attempt to establish its falsity by finding a counter-example. We succeed in finding John who, we discover, is certainly poor, and certainly unhappy. Using universal instantiation and modus ponens:

$$\frac{\text{John is poor.} \qquad \dfrac{\text{Everyone who is poor is happy.}}{\text{If John is poor then John is happy.}}}{\text{John is happy.}} \qquad \left[\frac{(1) \qquad \dfrac{(4)}{(2)}}{(3)}\right]$$

we see that (3) is a consequence of (4) and (1). But, by observation, we have seen that (3) is false. So, either (4) or (1) must be false. Since we have seen that (1) is true, (4) must be false. [And, for that matter since (3) is a consequence of (1) and (2), (2) must also be false.]

Correction. On page 6-396, the inference scheme for Double denial should be:

$$\frac{p}{\text{not not } p}$$

On page 6-397, in line 6b, change 'An' to 'A'.

On validity and truth.--Truth is a property of statements--some statements are true, some are not. [Statements which are not true are called false. Sentences which are not statements--that is, open sentences such as 'a = b + 3', are neither true nor false.] For example, 'Grass is green.', '2 + 2 = 4', and, as we shall see, 'If 5 = 7 then grass is blue.' are true statements, while 'Grass is blue', '5 = 7', and, as we shall see, 'If grass is green then 5 = 7.' are false statements.

Validity is, at first mention, a property of inferences whose premisses and conclusions are statements. [In constructing test-patterns, we rate an inference valid if each inference which is obtained by substituting names for the variables which occur in the given inference is valid. Later, we shall also speak of certain sentences as being valid.] For example, any inference of the form:

$$\frac{p \qquad \text{if } p \text{ then } q}{q}$$

in which 'p' and 'q' are replaced by statements, is valid. Specifically, from the premisses:

(1) John is poor.

(2) If John is poor then John is happy.

one is justified in inferring the conclusion:

(3) John is happy.

Notice that one's justification for drawing the conclusion (3) on the basis of the premisses (1) and (2) comes merely from the meaning of the phrase 'if...then...'. In other words, the validity of the inference in question derives solely from the fact that (2) is a conditional sentence, (1) is its antecedent, and (3) is its consequent. Which, if any, of the three statements are true, and which are false has no bearing on the validity of the inference.

Validity and truth are related in that consequences of true premisses are also true. [We shall see that this makes it possible to use the notion of validity to explain the circumstances under which, say, a conditional statement is true.] Because of this relationship between validity and truth and because the inference from (1) and (2) to (3) is valid, it follows that

if 'John is poor.' is true, and if 'If John is poor then John is happy.', is true,

then 'John is happy' is true.

1. (a) universal instantiation

   (b) modus ponens

   (c) universal instantiation

   (d) substitution rule for equations [substitution from (7) into (5)]

   (e) conditionalizing, and discharging an assumption

   (f) the test-pattern principle

2. BA = AB; substitution rule for equations

3.

$$\begin{array}{c}
\dfrac{\dfrac{(6)}{(7)} \qquad \dfrac{\begin{array}{c}\overset{*}{(1)}\\ \cdots \\ (4)\end{array} \qquad \dfrac{(2)}{(3)}}{(5)}}{\begin{array}{c}(8)\\ \cdots \\ (9)\\ \cdots \\ (10)\end{array}} \\
\underline{}\ * \\
(11) \\
\hline\hline
(12)
\end{array}$$

4. (a) modus tollens

   (b) modus ponens

Here is a diagram of the proof of Theorem 1-5:

$$\begin{array}{c} \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\dagger}{(1)} \qquad \overset{\cdots}{(2)}}{(3)} \qquad \dfrac{\overset{\text{Th. 1-3}}{\overset{\cdots}{(6)}} \qquad \overset{\overset{*}{(4)}}{\overset{\cdots}{(5)}}}{(7)}}{(8)} \qquad \dfrac{\overset{\cdots}{(10)} \qquad \overset{\overset{*}{(4)}}{\overset{\cdots}{(9)}}}{(11)}}{(12)}}{(13)}\,* \qquad \dfrac{\text{Th. 1-3}}{(14)}}{(15)}}{(16)} \\ \hline\hline \text{Th. 1-5} \end{array}$$

Answers for Supplementary Exercises for <u>Page 6-9</u>.

1. (a) T (b) F (c) T (d) T (e) T (f) T
   (g) F (h) T (i) F (j) T (k) T (ℓ) T
   (m) T (n) F (o) T (p) T (q) T

2. (a) $\{1, 2, 3, 4, 5\}$ (b) $\{3, 4\}$
   (c) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (d) $\emptyset$
   (e) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (f) $\{3, 4, 5, 6, 7, 8, 9\}$
   (g) $\{5\}$ (h) $\emptyset$
   (i) $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (j) $\emptyset$

*

Answers for Supplementary Exercises for <u>Page 6-16</u>.

1. (a) $\{6, 7, 8, 9, 10\}$ (b) $\{1, 2, 3, 4, 5, 10\}$
   (c) $\{10\}$
   (d) c[$a \cap b = \emptyset$; $\forall_x$ the complement of $\emptyset$ with respect to $x = x$]
   (e) $\{6, 7, 8, 9, 10\}$ (f) $\{1, 2, 3, 4, 5, 10\}$

2. (a) Yes (b) h (c) the complement of h with respect to k (d) $\emptyset$

3. (a) $\overrightarrow{CD}$ (b) $\overrightarrow{DC}$ (c) $\overline{CD}$ (d) $\overline{BE}$
   (e) $\overline{AB} \cup \overrightarrow{BC}$ (f) $\overleftrightarrow{AC}$ (g) $\emptyset$ (h) $\{C\}$
   (i) $\overrightarrow{CA}$ (j) $\overrightarrow{CA}$ (k) $\overrightarrow{CE}$ (ℓ) $\overrightarrow{CA} \cup \overrightarrow{DE}$
   (m) $\emptyset$ (n) $\{C\}$ (o) $\overrightarrow{BA} \cup \overrightarrow{BC}$
   (p) $\overrightarrow{DC} \cup \overrightarrow{DE}$ (q) $\overrightarrow{BA} \cup \overline{BD} \cup \overrightarrow{DE}$
   (r) $\{X\colon A \in \overline{BX}\} \cup \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DE} \cup \{X\colon E \in \overline{DX}\}$

Answers for Supplementary Exercises for Page 6-32.

1. (a) 6 (b) 5 (c) 6

   (d) No [Axiom B]

   (e) Since $F \notin \ell$, $F \notin \overleftrightarrow{AC}$. So, by Axiom B, $AF + FC > AC$. Since $B \in \overleftrightarrow{AC}$, it follows from Axiom A that $AB + BC = AC$. Hence, $AF + FC > AB + BC$.

2. (a) By Axiom A, if $P \in \overleftrightarrow{AB}$ then $AP + PB = AB$. By Axiom B, since $C \notin \overleftrightarrow{AB}$, $AC + CB > AB$. Hence, $AP + PB < AC + CB$. So, there is no point P such that $P \in \overleftrightarrow{AB}$ and $AP + PB = AC + CB$.

   (b) There are two such points, P_1 and P_2, where $A \in \overline{P_1B}$ and $B \in \overline{AP_2}$.

   (c) $P_1A = P_2B$

☆3. (c) $\{X: AX + XB = 3\}$ is an ellipse with A and B as foci.

Answers for Supplementary Exercises for <u>Page 6-50</u>.

[Notice that 'Given' is used as a synonym for 'Hypothesis'.]

1. (a) AB = AC

From the figure, $B \in \overleftrightarrow{AD}$ and $C \in \overleftrightarrow{AE}$. So, by Axiom A, AB + BD = AD and AC + CE = AE. By hypothesis, AD = AE. Therefore, AB + BD = AC + CE. But, by hypothesis, BD = CE. So, by algebra, AB = AC.

(b) MN = PQ

Since, by hypothesis, MN = QM and PQ = QM, it follows [by substitution] that MN = PQ.

(c) DB = AB [See argument for(a).]

(d) AC = AF, AD = AG, BC = EF, BD = EG

Since B is the midpoint of $\overleftrightarrow{AC}$, it follows from Theorem 1-8 that $AC = 2 \cdot AB$. Also, since C is the midpoint of $\overleftrightarrow{AD}$, $AD = 2 \cdot AC$. So, $AD = 4 \cdot AB$. Similarly, $AF = 2 \cdot AE$ and $AG = 4 \cdot AE$. But, we are given that AB = AE. Hence, AC = AF and AD = AG.

Since B is the midpoint of $\overleftrightarrow{AC}$, BC = AB. Similarly, EF = AE. So, BC = EF.

Finally, since C is the midpoint of $\overleftrightarrow{AD}$, $CD = 2 \cdot AB$. From the figure, $C \in \overleftrightarrow{BD}$. So, by Axiom A, $BD = BC + CD = AB + 2 \cdot AB = 3 \cdot AB$. Similarly, $EG = 3 \cdot AE$. So, BD = EG.

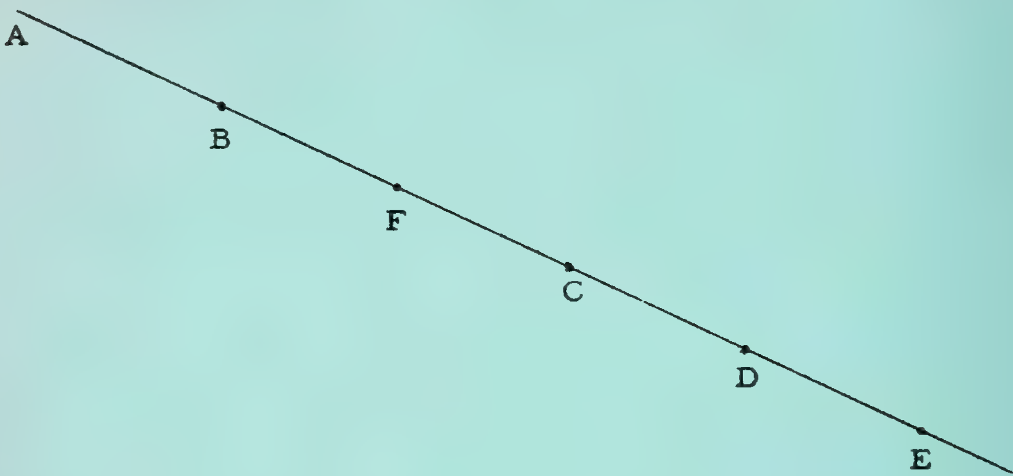
(e) D is the midpoint of $\overleftrightarrow{AC}$

By hypothesis, AD = DB and BD = DC. Since DB = BD, AD = DC. From the figure, $D \in \overleftrightarrow{AC}$. So, by definition, D is the midpoint of $\overleftrightarrow{AC}$.

Correction: The first line of part (d) on page 6-407 should read:

---, for each $k > 0$, there ---

2. (a), (b), (c)



(d) Yes; Axiom C; no

(e) Two. [One of the points belongs to $\overrightarrow{AB}$ and the other to $\overrightarrow{BA}$.]

*

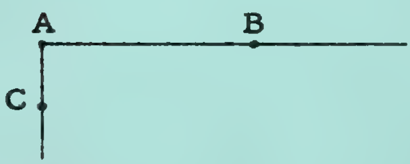
Answers for Supplementary Exercises for Page 6-61.

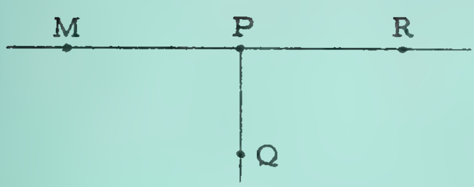
1. (a) By the logical principle of identity, $m(\angle A) = m(\angle A)$. Hence, by the definition of congruent angles, $\angle A \cong \angle A$. [Therefore, angle-congruence is a reflexive relation.]

   (b) Suppose that $\angle A \cong \angle B$. Then, $m(\angle A) = m(\angle B)$. So, $m(\angle B) = m(\angle A)$. Hence, $\angle B \cong \angle A$. [Therefore, angle-congruence is a symmetric relation.]

   (c) Suppose that $\angle A \cong \angle B$ and $\angle B \cong \angle C$. Then, $m(\angle A) = m(\angle B)$ and $m(\angle B) = m(\angle C)$. So, $m(\angle A) = m(\angle C)$. Hence, $\angle A \cong \angle C$. [Therefore, angle-congruence is a transitive relation.]

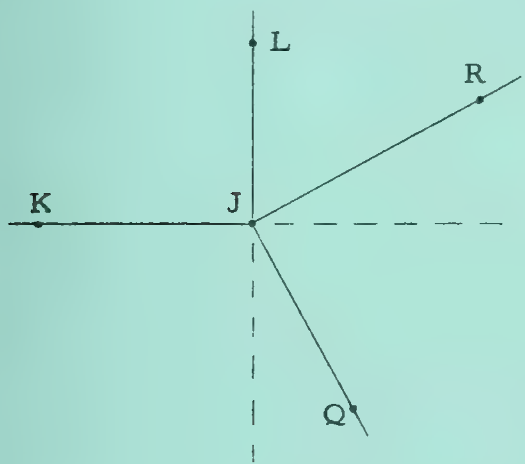
2.

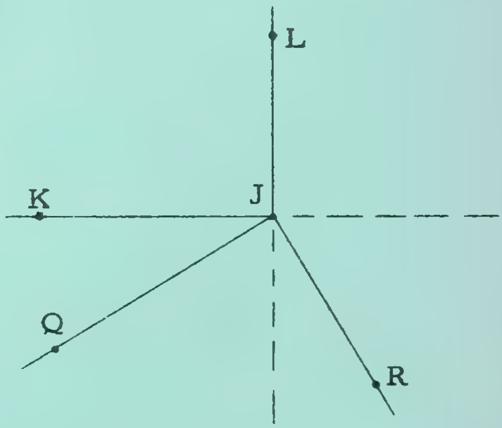


3. 

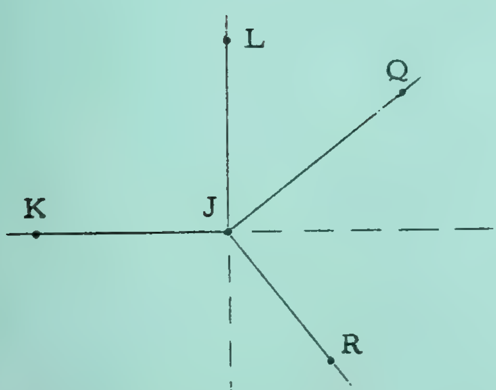
Since $Q \notin \overleftrightarrow{MR}$ and $P \in \overline{MR}$, it follows from Axiom G that $m(\angle MPQ) + m(\angle RPQ) = 180$. But, since $\angle MPQ$ is a right angle, it follows from Theorem 2-1 that $m(\angle MPQ) = 90$. Hence, $m(\angle RPQ) = 90$. So, by definition, $\angle RPQ \cong \angle MPQ$.

4.

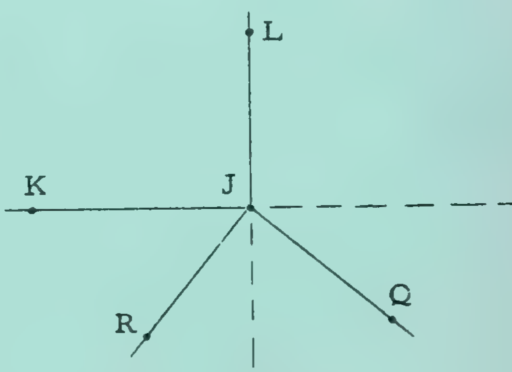




$m(\angle RJL) + m(\angle QJK) = 180$

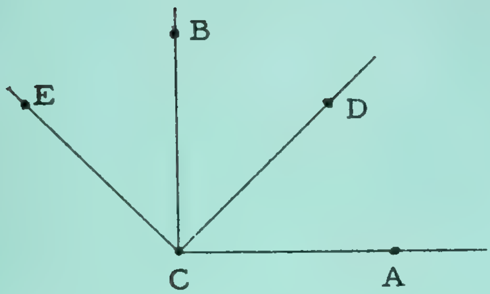


$m(\angle RJL) + m(\angle QJK) =$
$180 + 2 \cdot m(\angle QJL)$



$m(\angle RJL) + m(\angle QJK) =$
$180 + 2 \cdot m(\angle RJK)$

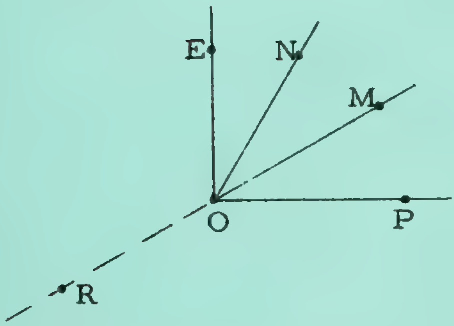
5.



m(∠ACD) = 45

m(∠ACE) = 135

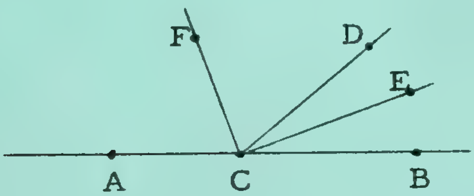
6.



m(∠NOR) = 150

7. (a) m(∠ABC) > 90 (b) m(∠MNP) ≤ 90

8.



m(∠ECF) = 90

Answers for Supplementary Exercises for Page 6-63.

1.

$\angle A$	37°	24°	81°	71°	$(2y)°$	$(90-x)°$	$[180-(x+y)]°$
supplement of $\angle A$	143°	156°	99°	109°	$(180-2y)°$	$(90+x)°$	$(x+y)°$
complement of $\angle A$	53°	66°	9°	19°	$(90-2y)°$	$x°$	$[(x+y)-90]°$
					$y<45$	$x<90$	$90<x+y<180$

2. (a) 43° 20′ 23″ (b) 37° 41′ 15″ (c) 63° 16′ 42″

3. (a) $\angle AXC \cong \angle DYF$

Since B is interior to $\angle AXC$, it follows from Axiom F that

$$m(\angle AXC) = m(\angle AXB) + m(\angle BXC).$$

Similarly, $m(\angle DYF) = m(\angle DYE) + m(\angle EYF)$.

By hypothesis, $\angle AXB \cong \angle EYF$ and $\angle BXC \cong \angle EYD$.

So, $m(\angle AXB) = m(\angle EYF)$ and $m(\angle BXC) = m(\angle EYD)$.

Therefore, $m(\angle AXC) = m(\angle DYF)$. Hence, $\angle AXC \cong \angle DYF$.

(b) $m(\angle M) + m(\angle N) + m(\angle P_3) = 180$

By hypothesis, $\angle P_1 \cong \angle NMP$ and $\angle P_2 \cong \angle PNM$.

So, $m(\angle P_1) = m(\angle M)$ and $m(\angle P_2) = m(\angle N)$.

But, by hypothesis, $m(\angle P_1) + m(\angle P_2) + m(\angle P_3) = 180$.

So, $m(\angle M) + m(\angle N) + m(\angle P_3) = 180$.

(c) $\angle CDE \cong \angle CED$

From the figure, C is interior to $\angle ADE$. So, by Axiom F,

$$m(\angle CDE) = m(\angle ADE) - m(\angle ADC).$$

Similarly, $m(\angle CED) = m(\angle BED) - m(\angle BEC)$.

But, by hypothesis, $\angle ADE \cong \angle BED$ and $\angle ADC \cong \angle BEC$.

So, $m(\angle ADE) = m(\angle BED)$ and $m(\angle ADC) = m(\angle BEC)$.

Therefore, $m(\angle CDE) = m(\angle CED)$. Hence, $\angle CDE \cong \angle CED$.

(d) $\angle MBC \cong \angle MCB$

From the figure, M is interior to $\angle ABC$. So, by Axiom F, $m(\angle ABC) = m(\angle ABM) + m(\angle MBC)$. But, by hypothesis, $\angle ABM \cong \angle MBC$. So, $m(\angle ABM) = m(\angle MBC)$. Hence, $m(\angle MBC) = \frac{1}{2} \cdot m(\angle ABC)$. Similarly, $m(\angle MCB) = \frac{1}{2} \cdot m(\angle ACB)$. By hypothesis, $\angle ABC \cong \angle ACB$; so, $m(\angle ABC) = m(\angle ACB)$. Therefore, $m(\angle MBC) = m(\angle MCB)$. Hence, $\angle MBC \cong \angle MCB$.

(e) complementary

From the figure, D is interior to $\angle BAC$. So, by Axiom F, $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$. By hypothesis, $\angle B \cong \angle CAD$; so, $m(\angle B) = m(\angle CAD)$. Hence, $m(\angle BAC) = m(\angle BAD) + m(\angle B)$. Since $\angle BAC$ is a right angle, it follows from Theorem 2-1 that $m(\angle BAC) = 90$. Therefore, $m(\angle BAD) + m(\angle B) = 90$. Hence, by definition, $\angle B$ and $\angle BAD$ are complementary.

(f) $\angle B \cong \angle C$

By hypothesis, $\angle A_1 \cong \angle B$. Since angle-congruence is a symmetric relation, $\angle B \cong \angle A_1$. But, by hypothesis, $\angle A_1 \cong \angle A_2$. So, since angle-congruence is a transitive relation, $\angle B \cong \angle A_2$. Again by hypothesis, $\angle A_2 \cong \angle C$. So, $\angle B \cong \angle C$.

(g) $\angle AEB \cong \angle DEC$

From the figure, B is interior to $\angle AEC$. So, by Axiom F, $m(\angle AEB) = m(\angle AEC) - m(\angle BEC)$. Similarly, $m(\angle DEC) = m(\angle DEB) - m(\angle BEC)$. By hypothesis, $\angle AEC \cong \angle DEB$; so, $m(\angle AEC) = m(\angle DEB)$. Therefore, $m(\angle AEB) = m(\angle DEC)$. Hence, $\angle AEB \cong \angle DEC$.

hypothesis, $\angle EGC \cong \angle FGB$; so, they have the same measure. Therefore, $m(\angle AGE) = m(\angle DGF)$. Hence, $\angle AGE \cong \angle DGF$.

5. $\angle AOC \cong \angle BOD$

   From the figure, B is interior to $\angle AOC$. So, by Axiom F,

   $$m(\angle AOB) + m(\angle BOC) = m(\angle AOC).$$

   Similarly, $m(\angle BOD) = m(\angle BOC) + m(\angle COD)$.

   But, by hypothesis, $\angle AOB \cong \angle DOC$; so, $m(\angle AOB) = m(\angle COD)$.

   Therefore, $m(\angle AOC) = m(\angle BOD)$. Hence, $\angle AOC \cong \angle BOD$.

*

Answers for Supplementary Exercises for Page 6-81.

1. (a), (b), (e), (f), (h)

2. (a) $\triangle ABE$ [or $\triangle ABC$]; $\triangle DEC$ [or $\triangle DBC$]
   (b) $ABE \leftrightarrow DCE$, $ABE \leftrightarrow DEC$

3. (a) $\triangle ABE$, $\triangle ABC$
   (b) $ABE \leftrightarrow BAC$ [or $ABE \leftrightarrow CAB$, $ABE \leftrightarrow BCA$, $ABE \leftrightarrow CBA$]

4. (a) $ACD \leftrightarrow BCD$ [or $ACD \leftrightarrow BDC$]
   (b) $ACD \leftrightarrow ABC$ [or $ACD \leftrightarrow CBA$]
   (c) $ABC \leftrightarrow CBD$ [or $ABC \leftrightarrow DBC$]
   (d) $ABC \leftrightarrow ACD$ [or $ABC \leftrightarrow CAD$]

Answers for Supplementary Exercises for Page 6-74.

1. $\angle A_2 \cong \angle B_4$

   From the figure, $\angle A_1$ and $\angle A_2$ are adjacent angles whose non-common sides are collinear. So, by Theorem 2-9, $\angle A_2$ is a supplement of $\angle A_1$. Similarly, $\angle B_4$ is a supplement of $\angle B_3$. Since, by hypothesis, $\angle A_1 \cong \angle B_3$ it follows from Theorem 2-3 that $\angle A_2 \cong \angle B_4$.

2. $\angle B \cong \angle D$

   By hypothesis, $\overleftrightarrow{EF} \perp \overleftrightarrow{BE}$. So, by Theorem 2-7, $\angle AEF$ is a right angle, and by Theorem 2-1, $m(\angle AEF) = 90$. Since, from the figure, D is interior to $\angle AEF$, it follows from Axiom F that $m(\angle DEA) + m(\angle E_2) = m(\angle AEF) = 90$. By hypothesis, $\angle B \cong \angle DEA$. So, $m(\angle B) = m(\angle DEA)$. Hence, $m(\angle B) + m(\angle E_2) = 90$. Thus, by definition, $\angle B$ is a complement of $\angle E_2$. Similarly, $\angle D$ is a complement of $\angle C_1$. But, by hypothesis, $\angle C_1 \cong \angle E_2$. So, by Theorem 2-4, $\angle B \cong \angle D$.

3. $\angle A_1$ and $\angle A_2$ are complementary

   Since $n \perp p$, it follows from Theorem 2-7 that $\angle A_3$ [not marked] is a right angle, and by Theorem 2-1, $m(\angle A_3) = 90$. By Axioms G and F and the hypothesis that m is a straight line, $m(\angle A_1) + m(\angle A_3) + m(\angle A_2) = 180$. So, $m(\angle A_1) + m(\angle A_2) = 90$. Hence, by definition, $\angle A_1$ and $\angle A_2$ are complementary.

4. $\angle AGE \cong \angle DGF$

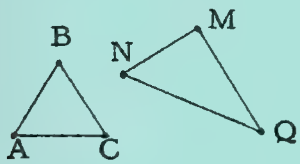
   Since $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{G\}$, $\angle AGC$ and $\angle DGB$ are vertical angles; so, by Theorem 2-5, they are congruent. From the figure, E is in the interior of $\angle AGC$. So, by Axiom F, $m(\angle AGE) + m(\angle EGC) = m(\angle AGC)$. Similarly, $m(\angle DGF) + m(\angle FGB) = m(\angle DGB)$. By

Correction. On page 6-413, the last part of Exercise 6(b) should read '$\overleftrightarrow{TR}$ and $\overleftrightarrow{PK}$'.

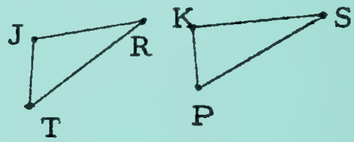
5. (a) ΔEBF and ΔDCF; $EBF \leftrightarrow DCF$ [or $EBF \leftrightarrow CDF$]
ΔEBF and ΔDCB; $EBF \leftrightarrow DCB$ [or $EBF \leftrightarrow CDB$]
ΔEBC and ΔDCF; $EBC \leftrightarrow DCF$ [or $EBC \leftrightarrow CDF$]
ΔEBC and ΔDCB; $EBC \leftrightarrow DCB$ [or $EBC \leftrightarrow CDB$]

(b) ΔBDA and ΔECA; $BDA \leftrightarrow ECA$ [or $BDA \leftrightarrow CEA$]
ΔBDA and ΔECB; $BDA \leftrightarrow ECB$ [or $BDA \leftrightarrow CEB$]
ΔBDC and ΔECA; $BDC \leftrightarrow ECA$ [or $BDC \leftrightarrow CEA$]
ΔBDC and ΔECB; $BDC \leftrightarrow ECB$ [or $BDC \leftrightarrow CEB$]

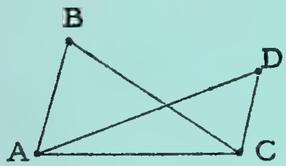
6. (a) $ABC \leftrightarrow MNQ$



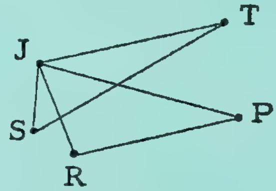
(b) $TJR \leftrightarrow KSP$



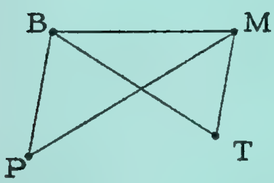
(c) $ABC \leftrightarrow DAC$



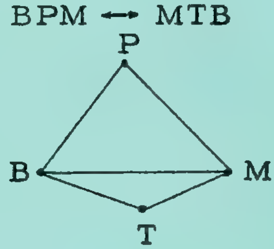
(d) $JTS \leftrightarrow RJP$



(e) $PBM \leftrightarrow TBM$



(f) $BPM \leftrightarrow MTB$



Answers for Supplementary Exercises for Page 6-88.

1. (a) $BF = EC$, $\angle F \cong \angle C$, $AF = DC$
   (b) s.a.s.
   (c) $BFA \longleftrightarrow ECD$ is a congruence
   (d) $\angle A \cong \angle D$, $\angle ABF \cong \angle DEC$, $AB = DE$

2. (a) $FC = CF$, $\angle AFC \cong \angle FCD$, $AF = DC$
   (b) s.a.s.
   (c) $FCA \longleftrightarrow CFD$ is a congruence
   (d) $\angle ACF \cong \angle DFC$, $\angle A \cong \angle D$, $AC = DF$

3. (a) $BC = ED$, $CD = DC$, $BD = CE$
   (b) s.s.s.
   (c) $BCD \longleftrightarrow EDC$ is a congruence
   (d) $\angle BCD \cong \angle EDC$, $\angle CDB \cong \angle DCE$, $\angle DBC \cong \angle CED$

4. (a) $AG = DF$, $\angle BGA \cong \angle EFD$, $BG = EF$
   (b) s.a.s.
   (c) $AGB \longleftrightarrow DFE$ is a congruence
   (d) $AB = DE$, $\angle GAB \cong \angle FDE$, $\angle ABG \cong \angle DEF$

   [The hypothesis that $\angle C$ is a right angle is not used in this problem. But, students will meet figures of this type in the work on similar triangles.]

5. (a) $CE = CA$, $\angle ECB \cong \angle ACD$, $CB = CD$

(b) s.a.s.

(c) $CEB \longleftrightarrow CAD$ is a congruence

(d) $EB = AD$, $\angle CEB \cong \angle CAD$, $\angle CBE \cong \angle CDA$

[$BAD \longleftrightarrow DEB$ is a congruence, also.]

6. (a) $AF = DC$, $\angle F \cong \angle C$, $FE = CB$

(b) s.a.s.

(c) $AFE \longleftrightarrow DCB$ is a congruence

(d) $AE = DB$, $\angle FAE \cong \angle CDB$, $\angle FEA \cong \angle CBD$

[$ABE \longleftrightarrow DEB$ is a congruence, also.]

7. (a) $BE = DE$, $\angle BEC \cong \angle DEC$, $EC = EC$

(b) s.a.s.

(c) $BEC \longleftrightarrow DEC$ is a congruence

(d) $BC = DC$, $\angle BCE \cong \angle DCE$, $\angle EBC \cong \angle EDC$

[$AEB \longleftrightarrow AED$ and $ABC \longleftrightarrow ADC$ are congruences, also.]

8. (a) $AC = DC$, $\angle ACB \cong \angle DCE$, $BC = EC$

(b) s.a.s.

(c) $ACB \longleftrightarrow DCE$ is a congruence

(d) $AB = DE$, $\angle ABC \cong \angle DEC$, $\angle BAC \cong \angle EDC$

[$DCE \longleftrightarrow GFE$ and $ACB \longleftrightarrow GFE$ are congruences, also.]

Answers for Supplementary Exercises for Page 6-90.

1. Since RS = RM, $\angle S \cong \angle M$, and ST = MQ, it follows from s.a.s. that RST $\leftrightarrow$ RMQ is a congruence. So, by the definition of triangle-congruence, $\Delta RST \cong \Delta RMQ$.

2. Since $\angle B$ and $\angle P$ are right angles, it follows from Theorem 2-2 that $\angle B \cong \angle P$. Also, by hypothesis, AB = MP and BC = PQ. Hence, by s.a.s., ABC $\leftrightarrow$ MPQ is a congruence. So, by the definition of triangle-congruence, AC = MQ.

3. DC = DB, $\angle C \cong \angle B$, CF = BE. So, by s.a.s., DCF $\leftrightarrow$ DBE is a congruence. Therefore, $\angle FDC \cong \angle EDB$.

4. $\angle ADB$ and $\angle CDB$ are supplements of the congruent angles $\angle ADE$ and $\angle CDE$, respectively. So, $\angle ADB \cong \angle CDB$. Also, AD = CD and DB = DB. So, by s.a.s., ADB $\leftrightarrow$ CDB is a congruence. Hence, $\angle ABD \cong \angle CBD$. Since $\overrightarrow{BD} = \overrightarrow{BE}$, it follows that $\angle ABE \cong \angle CBE$.

5. Since AF = CD, it follows that AC = FD. Also, CB = DE and BA = EF. So, by s.s.s., ACB $\leftrightarrow$ FDE is a congruence. Hence, $\angle B \cong \angle E$.

6. AB = CD, BC = DA, and CA = AC. So, ABC $\leftrightarrow$ CDA is a congruence. Hence, $\angle ABC \cong \angle CDA$. Similarly, ABD $\leftrightarrow$ CDB is a congruence, and $\angle DAB \cong \angle BCD$.

Answers for Supplementary Exercises for Page 6-95.

1. $\angle ABD \cong \angle CBE$ because they are complements of congruent angles. Also, $AB = BC$ and $DB = EB$. So, by s.a.s., $DBA \leftrightarrow EBC$ is a congruence. Hence, $\angle D \cong \angle E$.

2. $\angle CEA \cong \angle BED$ since they are right angles. Also, $CE = BE$ and $EA = ED$. So, by s.a.s., $CEA \leftrightarrow BED$ is a congruence. Hence, $AC = DB$.

3. $QP = 4$

4. $\angle ADC \cong \angle ABC$ since they are supplements of the same angle. $AD = AB$ and $DF = BE$. So, by s.a.s., $ADF \leftrightarrow ABE$ is a congruence. Hence, $\angle BAE \cong \angle DAF$.

5. $\angle DBE \cong \angle FEC$ since they have the same measure. $BE = EC$ since E is the midpoint of $\overline{BC}$. Also, $EF = DA$ and, since D is the midpoint of $\overline{BA}$, $BD = DA$. So, $EF = BD$. Therefore, by s.a.s., $DBE \leftrightarrow FEC$ is a congruence. So, $\angle BDE \cong \angle EFC$.

Correction. On page 6-419, change lines 2 and 3 to read:
1. --- below, use a definition to restate the property expressed by the sentence.

Answers for Supplementary Exercises for Page 6-111.

1. (a) $\overrightarrow{BC}$ is in the interior of $\angle ABD$ and $\angle CBA \cong \angle CBD$

   (b) $AB = CD$

   (c) $m(\angle A) + m(\angle B) = 90$

   (d) $\angle K$ is its own supplement

   (e) $\ell \cup m$ contains a right angle

   (f) $\overleftrightarrow{AB}$ is perpendicular to $\overline{CD}$ at its midpoint

   (g) $MTR \leftrightarrow SPQ$ is a matching of the vertices of ΔMTR with the vertices of ΔSPQ for which corresponding parts of the triangles are congruent

   (h) ΔABC is not isosceles [$AB \neq BC \neq CA$, and $AB \neq CA$]

   (i) ΔABC has three congruent angles

   (j) ΔABC has three congruent sides

   (k) there is a matching of the vertices of ΔABC with the vertices of ΔDEF which is a congruence

   (ℓ) $\angle E$ and $\angle D$ are supplementary

   (m) ℓ is parallel to m [See page 6-10]

   (n) $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{CD}$

   (o) A, B, and C are vertices of an isoceles triangle

   (p) in ΔMNR, $MR = RN$

   (q) $\angle T$ is a right angle

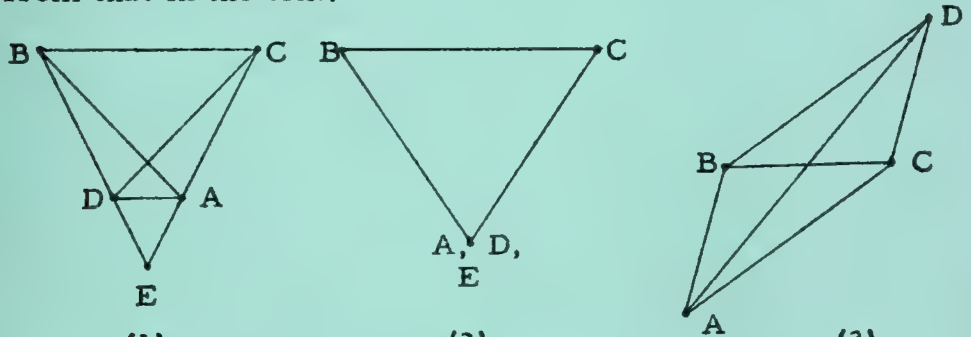
   (r) $m(\angle JKL) < 90$

3. (a) $AB = BC$, $\angle ABC \cong \angle BCD$, $BC = CD$. So, by s.a.s., $ABC \longleftrightarrow BCD$ is a congruence. Hence, $AC = BD$.

(b) $AC = DB$, $CD = BA$, $DA = AD$. So, by s.s.s., $ACD \longleftrightarrow DBA$ is a congruence. Hence, $\angle CDA \cong \angle BAD$.

(c) Since $ACD \longleftrightarrow DBA$ is a congruence, $\angle CAD \cong \angle BDA$. So, by Theorem 3-5, $ED = EA$. Hence, by definition, ΔEAD is isosceles.

It is instructive to note that the hypothesis of Exercise 3 is consistent with each of three additional figures, essentially different from that in the text.



(1) (2) (3)

The solution for part (a) makes no reference to the figure and, for each of the four situations pictured, $AC = BD$. By convention, the conclusion for part (b) implies that we are to assume B, A, and D to be noncollinear, and this makes figure (2), above, inappropriate. However, the solution given for part (b) applies, as well, to the situations pictured in figures (1) and (3).

Finally, the solution for part (c) makes use of the assumption that $\overline{AC}$ and $\overline{BD}$ intersect at a point E, not collinear with A and D. This

is not the case in any of the situations indicated in figures (1), (2), and (3). Still, for figure (1), in which $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect in a point E such that $A \in \overline{EC}$ and $D \in \overline{EB}$, the solution can be modified to give the desired conclusion. In the case of figure (3), in which A and D are on opposite sides of $\overleftrightarrow{BC}$, $\overleftrightarrow{AC} \cap \overleftrightarrow{BD} = \emptyset$.

4. (a) 3

   (b) 6 [assuming no three are collinear]

   [For each whole number of arithmetic $n > 2$, n points, no three of which are collinear, determine $n(n - 1)/2$ lines.]

   (c) A, B, and C are collinear and $A \in \overrightarrow{BC}$

   (d) A, B, and C are noncollinear

5. The three triangles are isosceles and congruent.
   Since ΔABC is equilateral, it follows from Theorem 3-6 that $\angle BAC \cong \angle CBA \cong \angle ACB$. Since $\overrightarrow{AG}$ is the bisector of $\angle BAC$, it follows from Theorem 3-8 that $m(\angle GAB) = \frac{1}{2} \cdot m(\angle BAC)$. Similarly, $m(\angle GBA) = \frac{1}{2} \cdot m(\angle CBA)$. Since $m(\angle BAC) = m(\angle CBA)$ $m(\angle GAB) = m(\angle GBA)$. So, by Theorem 3-5, ΔGAB is isosceles. Similarly, ΔGAC and ΔGCB are isosceles. Also, since $\overrightarrow{AG}$ is the bisector of $\angle BAC$, $\angle GAB \cong \angle GAC$; and, as above, since $\angle CBA \cong \angle ACB$, it follows that $\angle GBA \cong \angle GCA$. So, since $AB = AC$, it follows from a.s.a. that $AGB \longleftrightarrow AGC$ is a congruence. Similarly, $AGC \longleftrightarrow CGB$ and $CGB \longleftrightarrow BGA$ are congruences. Therefore, $\Delta AGC \cong \Delta CGB \cong \Delta BGA$.

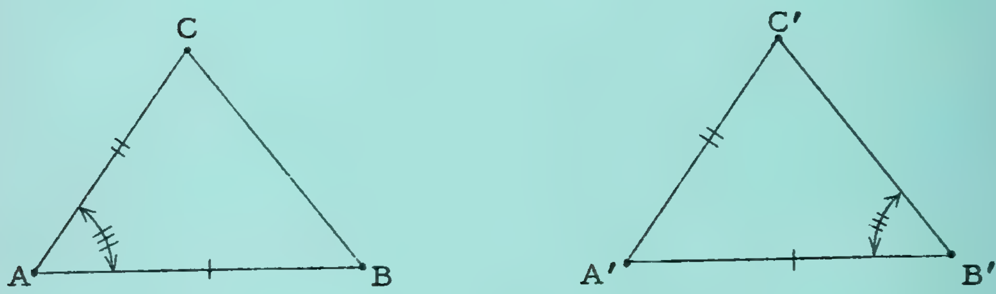
6. (a) Perhaps AB = BC or AC = BC.

(b) By Theorem 3-5, if $\angle C \cong \angle A$ then AB = BC. But, we are given that AB $\neq$ BC. So [by modus tollens], $\angle C \ncong \angle A$.

(c) This follows from Theorem 3-5 by the reasoning displayed below:

$$\cfrac{\cfrac{\cfrac{\text{p if and only if q}}{\text{if p then q}}}{\text{if not q then not p}} \qquad \cfrac{\cfrac{\text{p if and only if q}}{\text{if q then p}}}{\text{if not p then not q}}}{\text{not p if and only if not q}}$$

7. (a)



(b)

8. By s.a.s., EAD $\longleftrightarrow$ BAD is a congruence. Hence, ED = BD. Since, by hypothesis, AE = BD, it follows that AE = ED. Also, by hypothesis, AB = BD. So, since EB = EB, it follows from s.s.s. that AEB $\longleftrightarrow$ DEB is a congruence. Hence, $\angle EAB \cong \angle EDB$.

Correction. On page 6-422, part (e) of Exercise 2 should read:

$$\forall_x \forall_{y > 0}\ x + y > x$$

Answers for Supplementary Exercises for Page 6-112.

1. (a) $\{x: x > 17\}$ (b) $\{x: x < 1\}$ (c) $\{x: x > 0\}$
   (d) $\{x: x > 1\}$ (e) $\{x: x < 4\}$ (f) $\{x: x < 3\}$

2. (a) True

   $a > b$ if and only if $a - b > 0$. But, $a - b = (a + c) - (b + c)$. So, $a - b > 0$ if and only if $(a + c) - (b + c) > 0$; and $(a + c) - (b + c) > 0$ if and only if $a + c > b + c$. Hence, $a > b$ if and only if $a + c > b + c$. Consequently, $\forall_x \forall_y \forall_z\ x > y$ if and only if $x + z > y + z$.

   (b) False $[2 > 5 + -6$ but $2 \not> 5]$

   (c) True

   Suppose that $c > 0$ and suppose that $a > b + c$. Then, $a - b > c$ and $a - b > 0$. So, $a > b$. Hence, if $c > 0$ then if $a > b + c$ then $a > b$. Consequently, $\forall_x \forall_y \forall_z$ if $z > 0$ then if $x > y + z$ then $x > y$. In other words, $\forall_x \forall_y \forall_{z > 0}$ if $x > y + z$ then $x > y$.

   (d) False $[2 + 0 \not> 2]$

   (e) True

   Suppose that $b > 0$. Then, $a + b > a + 0 = a$. So, if $b > 0$ then $a + b > a$. Consequently, $\forall_x \forall_y$ if $y > 0$ then $x + y > x$. In other words, $\forall_x \forall_{y > 0}\ x + y > x$.

   (f) True

   Suppose that $a > b$ and $c > d$. Then, since $a > b$, $a - b > 0$, and since $c > d$, $c - d > 0$. So, $(a - b) + (c - d) > 0$. That is, $(a + c) - (b + d) > 0$. Hence, $a + c > b + d$. Thus, if $a > b$ and $c > d$ then $a + c > b + d$. Consequently, $\forall_x \forall_y \forall_u \forall_v$ if $x > y$ and $u > v$ then $x + u > y + v$.

   (g) False $[3 > 1$ and $5 > 1$ but $3 \not> 5]$

(h) True

Suppose that $a > b$ and $b > c$. Then, $a + b > b + c$. So, $a + b > c + b$. Hence, $a > c$. Consequently, $\forall_x \forall_y \forall_z$ if $x > y$ and $y > z$ then $x > z$. [The relation $>$ is a transitive relation.]

3. (a) Yes (b) No [Perhaps $B \notin \overleftrightarrow{AC}$.]

4. (a) Yes (b) Yes (c) No [Perhaps $AB = A'B'$.]

*

Answers for Supplementary Exercises for Page 6-138.

1. By definition, each of the altitude, angle bisector, and median from a vertex of a triangle is a segment one of whose end points is that vertex and whose other end point is on the line containing the side of the triangle opposite that vertex. Also by definition, the altitude is perpendicular to the line containing the base. Hence, by Theorem 4-9, the altitude is not longer than the angle bisector or the median. [Note that it would be incorrect to change 'not longer' to 'shorter' since in an isosceles triangle one of the altitudes is one of the medians. Hence, in that case, an altitude and a median have the same measure.]

2. $\overleftrightarrow{AB}$ [The figure is deliberately misleading.]

Let D and E be points on the non-C-side of $\overleftrightarrow{AB}$ and belonging to the lines containing the bisectors of the exterior angles at A and B, respectively. It can be shown that $\overrightarrow{AD}$ and $\overrightarrow{BE}$ intersect at a point G interior to $\angle ACB$. Since $G \in \overrightarrow{AD}$, it follows, by Theorem 4-17, that G is equidistant from $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$. Similarly, G is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$. Hence, G is equidistant from $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$. So, since G is interior to $\angle ACB$, it follows, by Theorem 4-17, that G belongs to the angle bisector of $\angle ACB$. Hence [F being a point, other than C, on this angle bisector], $\overleftrightarrow{AD}$, $\overleftrightarrow{CF}$, and $\overleftrightarrow{BE}$ are concurrent.

The proof that $\overrightarrow{AD}$ and $\overrightarrow{BE}$ intersect at a point G interior to $\angle ACB$ depends on theorems on parallel lines [see section 6.05]. Briefly, since $m(\angle DAB)$ is half that of another angle, $\angle DAB$ is acute. Similarly, $\angle EBA$ is acute. Since $\angle DAB$ and $\angle EBA$ are both acute, they are not supplementary. Hence, $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ intersect. In fact since $\angle DAB$ and $\angle EBA$ are acute, the half-lines $\overrightarrow{AD}$ and $\overrightarrow{BE}$ intersect. It remains to be shown that the point G at which they intersect is interior to $\angle ACB$--that is, that G is on the B-side of $\overleftrightarrow{AC}$ and the A-side of $\overleftrightarrow{BC}$. To establish this, it is sufficient, because of Theorem 18, to show that D is on the B-side of $\overleftrightarrow{AC}$ and that E is on the A-side of $\overleftrightarrow{BC}$. Now, since D belongs to the line containing the bisectors of the two exterior angles at A [and since $D \neq A$], D is interior to one of these two exterior angles. That is, either D is on the C-side of $\overleftrightarrow{AB}$ and on the non-B-side of $\overleftrightarrow{AC}$, or D is on the non-C-side of $\overleftrightarrow{AB}$ and on the B-side of $\overleftrightarrow{AC}$. Since, by hypothesis, D is on the non-C-side of $\overleftrightarrow{AB}$, it follows that it is on the B-side of $\overleftrightarrow{AC}$. Similarly, E is on the A-side of $\overleftrightarrow{BC}$.

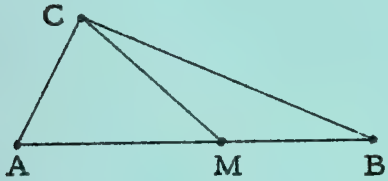
Correction. On page 6-423, change line 6b to:

[See Exercise ☆3, $\underbrace{\text{Part B,}}$ on 6-113.]

3. By Theorem 4-1, $PX + PY > XY$, $PY + PZ > YZ$, and $PZ + PX > ZX$. So, $(PX + PY) + (PY + PZ) + (PZ + PX) > XY + YZ + ZX$. But, $(PX + PY) + (PY + PZ) + (PZ + PX) = 2(PX + PY + PZ)$. Hence, $PX + PY + PZ > \frac{1}{2}(XY + YZ + ZX)$.

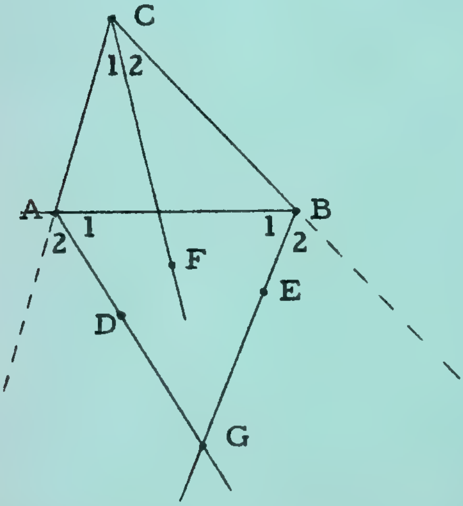
4. By Exercise 3 of Part B on page 6-113, $PX + PY < XZ + ZY$, $PY + PZ < XY + XZ$, and $PZ + PX < YZ + YX$. So, $(PX + PY) + (PY + PZ) + (PZ + PX) < (XZ + ZY) + (XY + YZ) + (YZ + YX)$. Therefore, $2(PX + PY + PZ) < 2(XZ + ZY + YX)$. Hence, $PX + PY + PZ < XZ + ZY + YX$.

[Exercises 3 and 4 tell us that the sum of the distances from a point in the interior of a triangle to the three vertices is between the semiperimeter and the perimeter.]

5. 

By Theorem 4-1, $CM < CA + AM$ and $CM < CB + MB$. So, $2 \cdot CM < CA + CB + (AM + MB)$. Therefore, $CM < \frac{1}{2}(CA + CB + AB)$.

[You can get a simpler proof if you use Part H on page 6-116.]

6. 

<u>Hypothesis:</u> $\angle A_1 \cong \angle A_2$,
$\angle B_1 \cong \angle B_2$,
$\angle C_1 \cong \angle C_2$,

<u>Conclusion:</u> $\overleftrightarrow{AD}$, $\overleftrightarrow{CF}$, and $\overleftrightarrow{BE}$ are concurrent

Correction. On page 6-424, line 1b should read:

Conclusion: m(∠CEB) = ____

Line 9 should begin:
of two consecutive ---

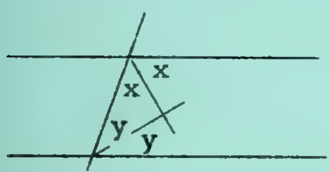
Answers for Supplementary Exercises for Page 6-151.

1. $2x + (x + 30) + 3x = 180$, $x = 25$; 75

2. $2x + 70 = 180$; 55

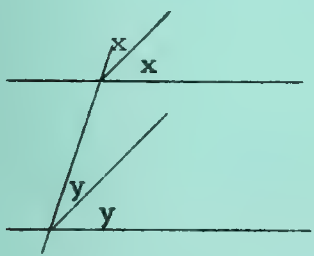
3. $x + (x + 32) = 180$; 74

4. $x + (x + 30) = 180$; 75

5. 

$$2x + 2y = 180$$
$$x + y = 90$$

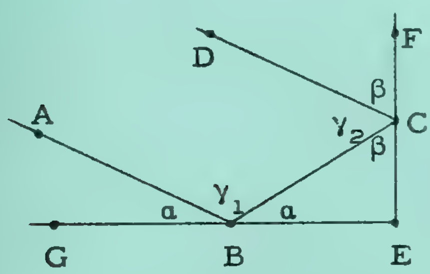
The lines which contain the bisectors are perpendicular to each other.

6. 

$$2x = 2y$$
$$x = y$$

The lines containing the bisectors are parallel to each other.

7. From the figure, C is interior to $\angle DBE$. $\angle DBC \cong \angle EBC$ since they are supplements of congruent angles. So, $\overrightarrow{BC}$ bisects $\angle DBE$.

8. 

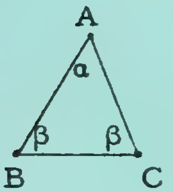
Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\gamma_1 + \gamma_2 = 180$. But, $\gamma_1 + 2\alpha = 180$. So, $\gamma_2 = 2\alpha$. Since $\gamma_2 + 2\beta = 180$, $2\alpha + 2\beta = 180$. Hence, $\alpha + \beta = 90$. Therefore, $m(\angle CEB) = 90$.

[See pages 82-83 of M. Kline's Mathematics and the Physical World (New York: Thomas Y. Crowell Company, 1959) for a discussion of this problem and others involving successive reflections of light rays.]

9. $m(\angle B) = 180 - 60 - 65 = 55$. So, $\angle A$ is the largest angle of the triangle. Hence, $\overleftrightarrow{BC}$ is the longest side.

10. $m(\angle D) + m(\angle E) > 135$. So, $m(\angle F) < 45$. Hence, $\angle F$ is the smallest angle. So, $\overleftrightarrow{DE}$ is the shortest side.

11.



Suppose that $AB > BC$. Then, $\beta > \alpha$. So, since $2\beta = 180 - \alpha$, $2\alpha < 180 - \alpha$. Hence, $\alpha < 60$.

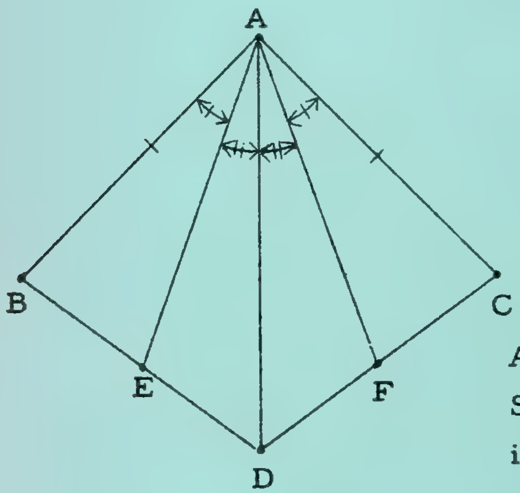
12. $p \perp n$

13. $p \parallel n$

14. False. [It might be the other line itself.]

15. 45

16.



By s.a.s., $BAD \leftrightarrow CAD$ is a congruence. So, $\angle B \cong \angle C$. By Theorem 5-10, since $\angle AED$ is an exterior angle of $\triangle ABE$,

$$m(\angle AED) = m(\angle B) + m(\angle BAE).$$

Also, $m(\angle AFD) = m(\angle C) + m(\angle FAC)$. Since $\angle BAE \cong \angle FAC$ and $\angle B \cong \angle C$, it follows that $\angle AED \cong \angle AFD$.

17. By Theorem 5-10, $m(\angle CAE) = 110$. So, $m(\angle EAD) = 55$. Since $m(\angle DBA) = 30$, it follows from Theorem 5-10 that $m(\angle BDA) = 25$. [In general, $m(\angle BDA) = \frac{1}{2} \cdot m(\angle ACB)$.]

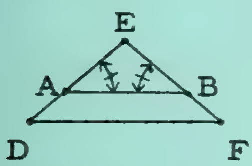
Correction. On page 6-427, line 1b should begin:

Find m($\angle$D), ---
$\uparrow\uparrow$ $\uparrow$

Answers for Supplementary Exercises for Page 6-157.

1. By Theorem 5-10, m($\angle$AOC) = m($\angle$B) + m($\angle$C). By hypothesis, CO = AO and AO = OB; so, CO = OB. Hence, $\angle$B $\cong$ $\angle$C. So, m($\angle$B) = $\frac{1}{2}\cdot$ m($\angle$AOC).

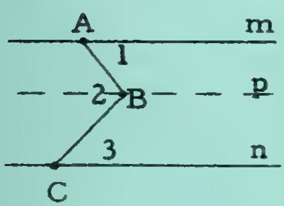
2. [See Exercise 8 on page 6-424.]

3. 

By Theorem 5-11, m($\angle$A) = 90 - $\frac{1}{2}\cdot$m($\angle$E). Since ΔDEF is isosceles with vertex angle $\angle$E, $\angle$D $\cong$ $\angle$F. So, m($\angle$D) = 90 - $\frac{1}{2}\cdot$m($\angle$E). Hence, $\angle$A $\cong$ $\angle$D. So, by Theorem 5-6, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DF}$.

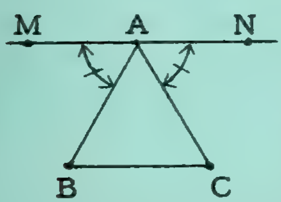
4. (a) 70 (b) 130 (c) 100

5. m($\angle B_2$) = 65, m($\angle C_2$) = 115, m($\angle C_3$) = 65, m($\angle D_3$) = 65

6. 

Suppose that p is the line parallel to m and to n through B. Then, m($\angle A_1$) = α and m($\angle C_3$) = β. But, $\alpha + \beta$ = m($\angle B_2$). So,

m($\angle B_2$) = m($\angle A_1$) + m($\angle C_3$).

7. 

Since $\overleftrightarrow{MN} \parallel \overleftrightarrow{BC}$, it follows from Theorem 5-3 that $\angle$MAB $\cong$ $\angle$ABC and that $\angle$NAC $\cong$ $\angle$ACB. But, by hypothesis, $\angle$MAB $\cong$ $\angle$NAC. So, $\angle$ABC $\cong$ $\angle$ACB. So, by Theorem 3-5, ΔABC is isosceles.

8. Since $\overleftrightarrow{AE} \perp \overleftrightarrow{AD}$, $\overleftrightarrow{BF} \parallel \overleftrightarrow{AE}$, and $\overleftrightarrow{BF} \parallel \overleftrightarrow{CG}$, it follows from Theorem 5-9 that $\angle$FBC and $\angle$GCD are right angles. So, by Theorem 5-11, $\angle$BCF is an angle of 50°. Since $\overleftrightarrow{EB} \parallel \overleftrightarrow{FC}$, it follows from Theorem 5-7 that $\angle$ABE is an angle of 50°. Since $\overleftrightarrow{EB} \parallel \overleftrightarrow{GD}$, it follows from Theorem 5-7 that $\angle$D is an angle of 50°.

Answers for Supplementary Exercises for <u>Page 6-178</u>.

1. $5x - 1 = \frac{1}{2}(9x + 3)$; 5

2. Since MN = 15 and AC = $2 \cdot$ MN, AC = 30. But, BD = AC. So, BD = 30.

3. Since AB = DC, $x + 4 = 3x - 36$ and $x = 20$. So, BC = $2 \cdot 20 - 16 = 24$. Since BC = DA, DA = 24. Also, AB = 20 + 4 and DC = $3 \cdot 20 - 36$. So, AB = 24 = DC = BC = DA. Hence, ABCD is a rhombus.

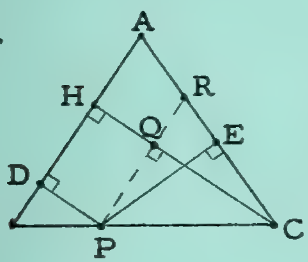
4. Since RS = $\frac{1}{2} \cdot$ NP and RS = 10, NP = 20. But, Δ MNP is a 30-60-90 triangle. So, MP = $\frac{1}{2} \cdot$ NP. That is, MP = 10.

5. Since ΔABC is isosceles and $\overrightarrow{BT}$ is the angle bisector from B, it is also the median from B. But, $\angle$B is a right angle. So, by Theorem 6-28, AC is 20.

6. $\frac{1}{2}(17 + 22) = 19.5$

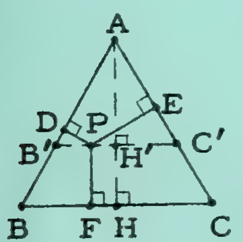
7. [See Exercise 1 of Part C on page 6-131.]

☆8.



Let $\overleftrightarrow{PR}$ be the line parallel to $\overleftrightarrow{AB}$ through P. Then, by Exercise 2 of Part E on page 6-147, ΔPRC is isosceles with vertex angle at R. So, by Exercise 2 of Part ☆E on page 6-134, PE = CQ. Also, by Theorem 6-29, PD = QH. So, PD + PE = CQ + QH = CH.

☆9.



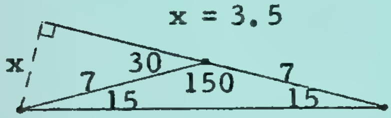
Let $\overleftrightarrow{B'C'}$ be the line through P parallel to $\overleftrightarrow{BC}$. Then, since $\Delta AB'C'$ is isosceles with vertex angle at A, PD + PE is the measure of the altitude of $\Delta AB'C'$ from C'. But, since $\angle$A is an angle of 60°, $\Delta AB'C'$ is equilateral. So, the altitude of $\Delta AB'C'$ from C' is congruent to the one from A. Hence, PD + PE = AH'. By Theorem 6-29, PF = H'H. So, PD + PE + PF = AH' + H'H = AH.

10. Let X $\in \overline{AB}$ and Z $\in \overline{DC}$. Since $\overleftrightarrow{ZC} \parallel \overleftrightarrow{AX}$, $\angle ZCY \cong \angle XAY$. Also, since quadrilateral ABCD is a parallelogram, CY = AY. Finally, $\angle XYA \cong \angle ZYC$. So, XYA $\leftrightarrow$ ZYC is a congruence. Hence, XY = ZY. But, XY = 3.5. So, ZY = 3.5, and XZ = 7.

11. 60 12. 30[See Exercise 5 on page 6-150.]

13. Since $\overleftrightarrow{RS} \parallel \overleftrightarrow{AC}$, $\angle RDA \cong \angle DAC$. But, since $\overrightarrow{AD}$ bisects $\angle RAC$, $\angle RAD \cong \angle DAC$. So, $\angle RAD \cong \angle RDA$. Hence, AR = RD. Similarly, CS = SD. So, RS = AR + CS. Hence, CS = 32.

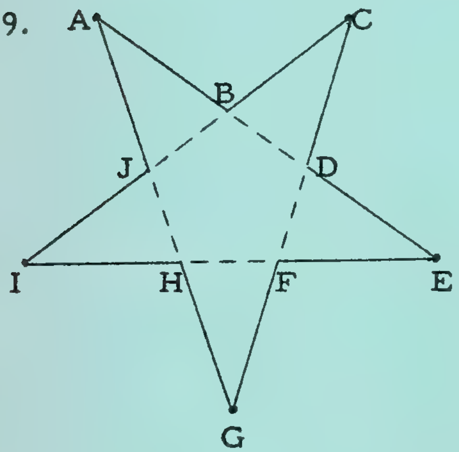
14. Since, $\angle ABC$ is an angle of 120°, it follows that $\angle A$ is an angle of 60°. So, ΔABD is equilateral. Hence, BD = 7.

15. 

16. $m(\angle U) = 2[180 - m(\angle U)]$;
$m(\angle U) = 120$;
$m(\angle W) = 120$

17. [Draw an equilateral triangle and bisect one of its angles.]

18. Each exterior angle is an angle of 60°. So, for example, $\angle E$ is an angle of 60°. Similarly, $\angle A$ is an angle of 60°. So, ΔAIE is equiangular. Hence, it is equilateral.

19. 

Draw a regular pentagon and extend the sides so that each side is the base of an isosceles triangle. Each angle of the regular pentagon BDFHJ is an angle of 108°. Each exterior angle is an angle of 72°. So, $\angle A$ is an angle of 36°, $\angle ABC$ is an angle of 108°, $\angle C$ is an angle of 36°, etc.

*

Answers for Supplementary Exercises for Page 6-192 [on page 6-430].

1. Theorem 6-27

2. (a) 2/3 (b) 3/5 (c) 2/3 (d) 1; 1

3. (a) 10.5; 2/3 (b) $2\sqrt{2}$; 1/2 (c) 7/9

*

Correction. On page 6-432, line 8b should begin:

(a) --- [Ans: $3\sqrt{34}$]
$\uparrow$

Answers for Supplementary Exercises for Page 6-202.

1. two 2. one 3. one 4. 0 5. 25 6. 9 7. 144

8. 81 9. 36 10. 30 11. 2 12. 783 13. 5163

14. 835 15. 589 16. 347; 698 17. 1873; 6952

18. 1873; 6952 19. 1873; 6952 20. 189; 189

21. 64; 36 22. 64; 36 23. 5; 7 24. 5; 7 25. one

26. 5; 7 27. 18; 2 28. p 29. ab 30. ab

31. ab 32. ab 33. ab 34. one; ab

35. 25; 3; 5; 3 36. 500; 100; 10 37. 49; 2; 49; 2; 7; 2

38. 28; 4; 7; 4; 7 39. 99; 9; 11; 9; 11

40. 16; 10; 16; 10; 4 41. 36; 3; 36; 3; 3

42. 9; 7; 3; 7 43. 363 44. 18 45. 10; $2\sqrt{10}$

46. (a) $3\sqrt{34}$ (b) $2\sqrt{26}$ (c) $7\sqrt{5}$ (d) $\sqrt{5}$ (e) 4 (f) 5 (g) 12
(h) $3\sqrt{5}$ (i) $4\sqrt{30}$ (j) $2\sqrt{2}$ (k) $4\sqrt{6}$ (ℓ) $|\frac{s}{2}\sqrt{5}|$ (m) $|\frac{x}{3}\sqrt{10}|$
(n) $|\frac{y}{2}\sqrt{3}|$ (o) $|t|\sqrt{2}$ (p) $|k|$ (q) $|k|\sqrt{3}$

47. (a) 12 (b) 30 (c) 90 (d) 8 (e) 20 (f) 66

*

Answers for Supplementary Exercises for Page 6-219.

1. (a) s (b) $p + r$ (c) $\frac{p}{q}$ (d) $\frac{s}{r}$

2. 6 3. 12 4. Yes; $ABC \leftrightarrow FDE$ is a similarity

5. $\frac{AE + 20}{AE} = \frac{40}{30}$, $AE = 60$ 6. $\frac{x}{x + 4} = \frac{5}{10}$, $x = 4$

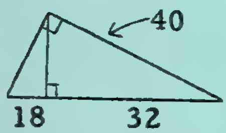
7. 4 8. $\frac{BD}{AD} = \frac{BE}{EC}$, $\frac{AD + 3}{AD} = \frac{20}{15}$, $AD = 9$

9. $6^2 = 4 \cdot DC$, $DC = 9$ 10. $x(20 - x) = 36$; 2 and $20 - 2$, or 18

11. $\sqrt{29}$ 12. $4\sqrt{3}$ 13. $2\sqrt{15}$ 14. $15\sqrt{2}$

15. $15\sqrt{3}$ 16. $3k\sqrt{2}$ 17. 10 18. $\sqrt{15}/2$ 19. 4

20.



21. No. $\sqrt{8^2 + 3^2} = \sqrt{73} < 9$

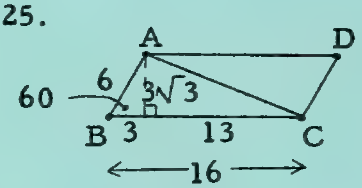
[How about a table top 8.5 feet in diameter?]

22. $AC = 2 \cdot BM = 20$; $BC = 2 \cdot MP = 16$; $AB = 2 \cdot PB = 12$; perimeter $= 48$

23. 5 in.

24. 10

25.



$AC = \sqrt{27 + 169}$
$= 14$

26. k and $k\sqrt{3}$

27. 3

28. By the a.a. similarity theorem, $PBM \leftrightarrow QDM$ and $SBM \leftrightarrow RDM$ are similarities. So, we have the first two sentences in the Conclusion. From them it follows that $\frac{PM}{MQ} = \frac{SM}{MR}$. Now, the vertical angles $\angle SMP$ and $\angle RMQ$ are congruent. So, by the s.a.s. similarity theorem, $PSM \leftrightarrow QRM$ is a similarity.

Corrections. On page 6-436, line 15b should read:

[Hint. Use $\underbrace{\text{Theorem 6-27}}_{\uparrow}$.]

and line 14b should read:

(b) --- such that $B \notin \overleftrightarrow{AC}$. ---

*29. (a) Since $\overline{AT} \parallel \overline{DB}$, it follows from Theorem 7-1 that $\frac{AC}{AD} = \frac{CT}{TB}$. It also follows that $\angle DBA \cong \angle BAT$ and $\angle D \cong \angle CAT$. But, by hypothesis, $\angle BAT \cong \angle CAT$. So, $\angle DBA \cong \angle D$. Hence, $AD = AB$. So, $\frac{AC}{AB} = \frac{CT}{TB}$.

(b) By part (a), $\frac{QN}{NP} = \frac{QM}{MP}$. But, $QM = \frac{\sqrt{3}}{2} \cdot MP$. So, $\frac{QN}{NP} = \frac{\sqrt{3}}{2}$.

*30. (a) Let ℓ be the line parallel to $\overleftrightarrow{BE}$ through A. Then, by Theorem 6-27, since $\ell \parallel \overleftrightarrow{BE} \parallel \overleftrightarrow{CG} \parallel \overleftrightarrow{ID}$ and $AE = EG = GI$, $AB = BC = CD$.

(b) D and E are the midpoints of $\overline{BC}$ and $\overline{AB}$, respectively.

*31. (a) $AD'E' \leftrightarrow ADE$ and $AF'D' \leftrightarrow AFD$ are similarities.

So, $\frac{AD'}{AD} = \frac{D'E'}{DE}$ and $\frac{AD'}{AD} = \frac{F'D'}{FD}$. Therefore, $\frac{D'E'}{DE} = \frac{F'D'}{FD}$.

That is, $\frac{FD}{DE} = \frac{F'D'}{D'E'}$. But, $F'D' = D'E'$. So, $FD = DE$.

Hence, rectangle FDEG is a square.

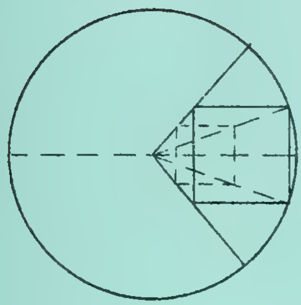
(b) Choose a point $A' \in \overline{MN}$, and draw the perpendicular segment from A' to $\overline{MP}$. Let B' be the foot of this perpendicular. Then, construct a square $A'B'C'D'$ with $C' \in \overline{B'P}$. The half-line $\overrightarrow{MD'}$ intersects $\overline{NP}$ in one vertex of the required square.

*

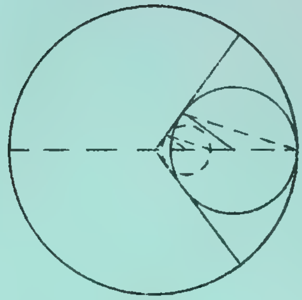
There are many problems which can be solved in a manner similar to Exercise 31(b). Here are two:

(a) Draw a square two of whose vertices belong to a given arc, while each of its other vertices belongs to one of the radii to the end points of the arc.

(b) Draw a circle which is internally tangent to a given arc and is also tangent to the rays which contain the radii to the end points of the arc.



(a)

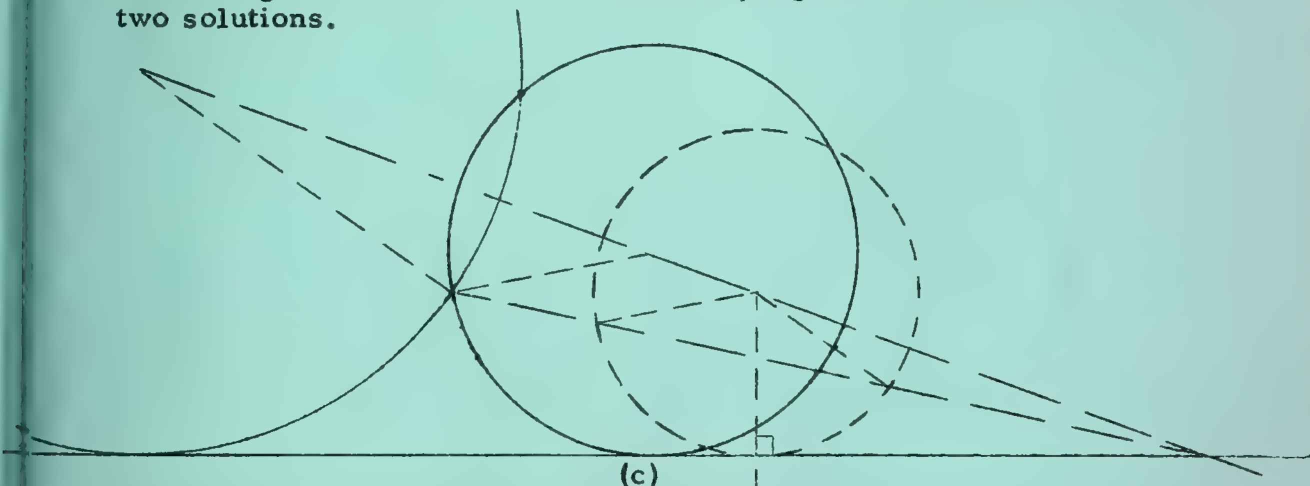


(b)

In each picture, the dotted lines are "construction lines". One obtains an interesting modification of problem (b) by replacing 'internally' by 'externally'. A slightly more difficult problem of the same kind:

(c) Draw a circle which is tangent to a given line and which contains two given points.

There are several cases which must be treated for a complete solution, but the "general case" is illustrated by figure (c). Note that there are two solutions.

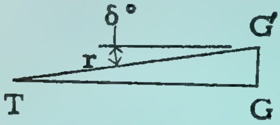


(c)

Although part (b) of Exercise 31 and each of the problems (a), (b), and (c), above, has been stated as a problem in mechanical drawing, each of them corresponds to an existence theorem. The justification for the solution of problem (a) can, for example, be taken as a proof of the theorem:

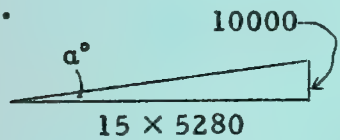
> For each minor arc, there exists a circle which is internally tangent to it and tangent to the radii to its end points.

The range is the length of the hypotenuse of a right triangle in a vertical plane through T and G.

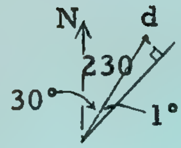


(5)

G′ is the actual position of the gun and, by figure (3) is [6000 - 3000 cos 50°] feet above its projection G. So, $CG' \doteq 6000 - 1928 = 4072$. GT can be computed from figure (4). Since $\alpha \doteq 52$, $\beta \doteq 8$, and $GT \doteq 12867 \cos 8° + 2298 \cos 52° \doteq 12742 + 1415 = 14157$. G′T can, with considerable labor, be found by using the Pythagorean Theorem. However, it is less work to find the angle of depression of T from G′, and use this to compute G′T. [The gunners will want to know this angle, anyway.] From figure (5), $\tan \delta° = \frac{GG'}{TG} \doteq \frac{4072}{14157} \doteq 0.2876$. So, $\delta \doteq 16$. Finally, $G'T = \frac{GT}{\cos \delta°} \doteq \frac{14157}{0.9613} \doteq 14727$. So, the desired range is about 14730 feet.

7. 

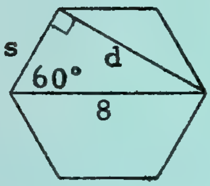
$\tan(90 - \alpha)° = \frac{15 \times 5280}{10000} = 7.92$ So, $90 - \alpha$ is approximately 83°, and the minimum angle of climb is an angle of about 7°.

3. 

$d = 230 \sin 1° \doteq 4.025$

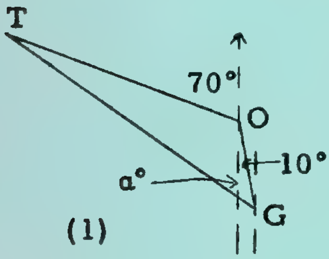
So, at closest, he will be about 4 miles from Zilchville.

4. (a) 3.6 feet (b) about 78° (c) $8 \sin 79°$ feet; so, about 7.85 feet

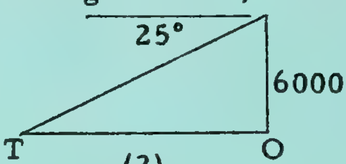
5. 

$s = 4$. So, the perimeter is 24 and $d = 4\sqrt{3}$.

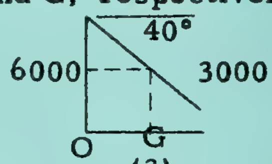
6. This is a rather complex exercise. Projecting the observer, tanks, and guns on a horizontal plane we obtain figure (1). Evidently, the desired bearing is $N(a + 10)° W$. Since $\angle TOG$ is an angle of 120° we can find a once we have computed OT and OG. To find OT and OG, we consider vertical planes containing O and T, and O and G, respectively.



(1)

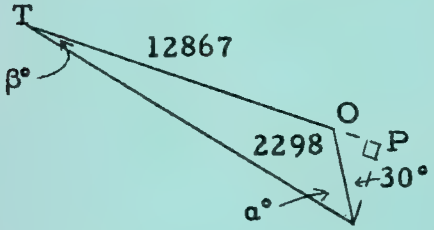


(2)



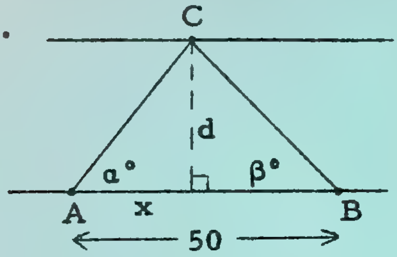
(3)

From figure (2), $OT = 6000 \tan 65° \doteq 12867$. From figure (3), $OG = 3000 \sin 50° \doteq 2298$. Since [see figure (4)] $OP = \frac{1}{2} \cdot OG$, it follows that $TP \doteq 12867 + 1149 = 14016$. Since, also, $PG = \frac{\sqrt{3}}{2} \cdot OG$, $PG \doteq 1990$. Hence, $\tan (a + 30)° \doteq \frac{14016}{1990} \doteq 7$ and $a + 30 \doteq 82$. Consequently, the desired bearing is N62°W.



(4)

Answers for Supplementary Exercises for <u>Page 6-230</u>.

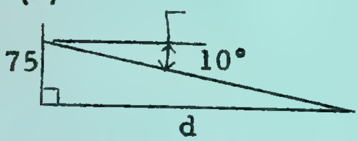
1. 

Choose a mark C on the opposite side of the river and two positions A and B which are 50 feet apart and such that $\angle CAB$ and $\angle CBA$ are both acute.

Since $\frac{d}{x} = \tan \alpha°$ and $\frac{d}{50 - x} = \tan \beta°$, it follows that $x \tan \alpha° = (50 - x) \tan \beta°$. So, $x = \frac{50 \tan \beta°}{\tan \alpha° + \tan \beta°}$ and $d = \frac{50 \tan \alpha° \cdot \tan \beta°}{\tan \alpha° + \tan \beta°}$.

[If the river is very wide, the surveyor can obtain a more accurate measure of its width by choosing a longer base line, say, one which is 100 yards long. He may also save computation by choosing B, say, so that $\angle CBA$ is approximately a right angle, and using the formula '$d = 50 \tan \alpha°$']

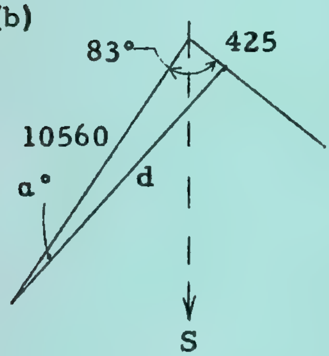
2. (a)



$d = 75 \tan 80° \doteq 425$.

So, the smoke is about 425 feet from the base of the tower.

(b)



A sensible answer for part (b) is that the fire is about 2 miles from headquarters. However, for computational practice, the solution may be carried out in the style of Exercise 5 on page 6-229. On doing so, it turns out that α is about 2.3 and that d is approximately 10517. So, the fire is about 43 feet less than 2 miles from headquarters. [If one accepts 2 as an approximation for α, one obtains 10515 as an approximation for d.]

Answers for Supplementary Exercises for Page 6-316.

1. 37   2. 60   3. 140   4. 40   5. 101

6. 80   7. 30   8. 117   9. 75; 105

10. One of the angles, $\angle PAC$ and $\angle PAE$, is an angle of 60°, and the other is an angle of 120°; $m(\angle PAD) = 90$.

11. 130   12. 25

*

Answers for Supplementary Exercises for Page 6-334.

1. About 48.   2. 7/3   3. 6   4. 18   5. 7.5

6. By Theorem 10-3 and Theorem 10-29, half the measure of the chord is the measure of the altitude to the hypotenuse of a right triangle whose hypotenuse is the given diameter. So, the desired result follows by Theorem 7-4.

7. 10 inches   8. 24   9. $4\sqrt{3}$ inches; 60

10. By s.a.s., $ADB \leftrightarrow BDC$ is a congruence. So, $\overleftrightarrow{AB} \cong \overleftrightarrow{BC}$.

11. Since $\angle ODA$ is inscribed in a semicircle, it is, by Theorem 10-29, a right angle. So, by Theorem 10-20, $\widehat{BC} \cong \widehat{AC}$.

12. The measure of each of the arcs into which the bisector of the angle divides the intercepted arc is, by Theorem 10-22 and the definition of angle bisector, twice half the measure of the given angle. Since they have the same measure, the arcs are congruent.

13. (a) 100   (b) 55   (c) 90   (d) $\beta = 80$; $x = 80$

14. Since the base angles of an isosceles triangle are congruent, it follows, by Theorem 10-22, that the arcs intercepted by the base angles of an inscribed isosceles triangle are congruent. So [unless the tangents at the vertices of the base angles are parallel] the points of intersection of the tangents are the vertices of a triangle which, by Theorem 10-26 has two congruent angles.

15. About 20 inches.

Corrections. On page 6-441, line 9 should read:

--- of radius $\frac{1}{2}\underbrace{[\text{inch}]}_{\uparrow}$ which ---

and line 7b should end:

--- in feet $\underbrace{\text{per minute,}}_{\uparrow}$ of

16. A radius of the rope circle is $\frac{40}{\pi}$ feet longer than a radius of the earth. Since $\frac{40}{\pi} > 6.36$, a 6 foot 4 inch person could walk under the rope without stooping. Since the record for the high-jump is 7 feet $3\frac{1}{4}$ inches [and $\frac{40}{\pi} < 6.37$], some people would find it possible to jump over the rope.

17. The center of the arc is at the intersection of the bisector of $\angle CBA$ and the line parallel to $\overleftrightarrow{AB}$ which is on the C-side of $\overleftrightarrow{AB}$ and $\frac{1}{2}$ inch from $\overleftrightarrow{AB}$.

18. $2\pi(6 - 1) \doteq 31.4$. So, the second man runs about 31.4 feet further than the first.

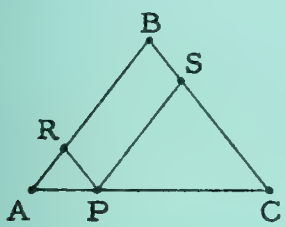
19. 500π feet per minute

20. By Exercise 1 of Part B on Page 6-291, and Axiom A, the centers of the circles are the vertices of an equilateral triangle whose sides contain the points of tangency. Since the degree-measure of each angle of an equilateral triangle is 60, the degree-measure of each of the arcs $\overset{\frown}{SR}$, $\overset{\frown}{RT}$, and $\overset{\frown}{TS}$ is 60. So, the sum of these measures is 180.

Answers for Review Exercises.

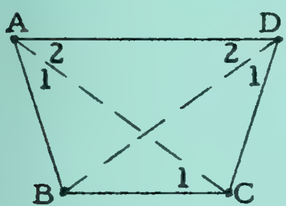
The Review Exercises are designed to give the students additional practice in solving "numericals", "originals", and construction problems. They may be used while the students are studying later units to help maintain the students' efficiency in geometry and also as a "break" from some of the algebraic work.

1. 2100 [$2(3x + 7x) = 200$]

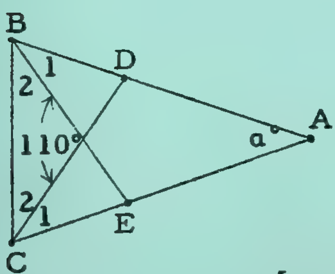
2. 

By hypothesis, BRPS is a parallelogram. So, $RP = BS$. Since $BA = BC$, $\angle A \cong \angle C$ and, since $\overleftrightarrow{AB} \parallel \overleftrightarrow{PS}$, $\angle A \cong \angle SPC$. Thus, $\angle SPC \cong \angle C$. Therefore, $SC = SP$. Hence, $RP + PS = BS + SC = BC$.

3. 1 : 4 [See Exercise 1 of Part B on page 6-336.]

4. 

By hypothesis, $AB = BC = CD$, and $\overline{BC} \parallel \overline{AD}$. Since $AB = BC$, $\angle C_1 \cong \angle A_1$. Also, since $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$, $\angle C_1 \cong \angle A_2$. Hence, $\angle A_1 \cong \angle A_2$. Similarly, $\angle D_1 \cong \angle D_2$. [Ask your students what type of quadrilateral ABCD is if the diagonals bisect the angles at C and B as well as those at A and D.]

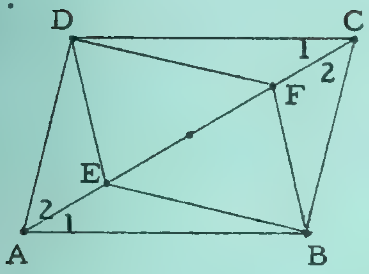
5. 

$a = 40$

Since $\overrightarrow{BE}$ and $\overrightarrow{CD}$ are angle bisectors, $a = 180 - 2[m(\angle B_2) + m(\angle C_2)]$. Also, $m(\angle B_2) + m(\angle C_2) = 180 - 110 = 70$. Thus, $a = 180 - 2(70) = 40$.

[Note that the condition '$\overline{AB} \cong \overline{AC}$' is not necessary.]

6. $\frac{r_1}{r_2} = \sqrt{2}$

7. 

Since ABCD is a parallelogram, $\overline{AD} \parallel \overline{BC}$, $\overline{AD} \cong \overline{BC}$, and $\angle C_2 \cong \angle A_2$. Since E and F are the first and third quadrisection points, respectively, of $\overline{AC}$, $\overline{AE} \cong \overline{FC}$. So, by s.a.s., ADE $\leftrightarrow$ CBF is a congruence, and $\overline{ED} \cong \overline{FB}$. Similarly, $\overline{FD} \cong \overline{EB}$. Hence, EDFB is a parallelogram.

8. By hypothesis, $\overleftrightarrow{EP} \parallel \overleftrightarrow{AF} \parallel \overleftrightarrow{BC}$. Hence, $\overleftrightarrow{EP} \parallel \overleftrightarrow{BC}$. Thus, $\angle AEP \cong \angle B$. Since $\angle EAP \cong \angle BAC$ and $\angle AEP \cong \angle B$, EAP $\leftrightarrow$ BAC is a similarity. Thus, AE/AB = EP/BC. Since ABCD and EPFA are parallelograms, AE = PF, EP = AF, AB = CD, and BC = AD. So, $\frac{AE}{AB} = \frac{PF}{CD} = \frac{EP}{BC} = \frac{AF}{AD}$. Also, since $\angle AEP \cong \angle B$, $\angle EAF \cong \angle BAD$, $\angle EPF \cong \angle BCD$ [each is congruent to $\angle EAF$], and $\angle PFA \cong \angle CDA$ [$\overleftrightarrow{PF} \parallel \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$], AEPF $\leftrightarrow$ ABCD is a similarity. [See definition on page 6-192.]

14. $AB = AC$, $AC' = AB'$, and $\angle BAC' \cong \angle CAB'$. Hence, $ABC' \leftrightarrow ACB'$ is a congruence, and $BC' = B'C$. [It is sufficient that the triangles be isosceles with congruent vertex angles.]

15. an angle of $45°$

16. $\cos \angle A = \frac{4}{5}$; $\tan \angle A = \frac{3}{4}$

17. Suppose that $m(\angle AEB) = \gamma$, $m(\angle BDC) = \beta$, $m(\angle BAC) = \alpha$, and $m(\angle BCA) = \delta$. Since $BC = CD$ and $BD = DE$, $m(\angle DBC) = \beta$ and $m(\angle DBE) = \gamma$. So, $\beta = 2\gamma$ and $\delta = 2\beta$, by the exterior angle theorem. Thus, $\delta = 4\gamma$. That is, $m(\angle AEB) = \frac{1}{4} \cdot m(\angle BCA)$.

Since $CA = CB$ and $CB = CD$, $CA = CD$. So, $\overleftrightarrow{BC}$ is a median to $\overleftrightarrow{AD}$. Hence, since $CB = CA$, ABD is a right angle. So, $\angle A$ and $\angle ADB$ are complementary.

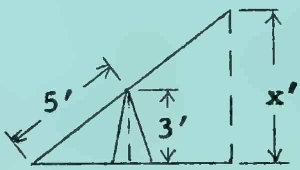
9. By hypothesis, $\overleftrightarrow{T'S'} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{T'S'} \parallel \overleftrightarrow{P'R'}$, and $\overleftrightarrow{T'P'} \parallel \overleftrightarrow{BA}$. Thus, $P'T'S'R'$ is a rectangle, Since PRST is a square, $\overleftrightarrow{PT} \parallel \overleftrightarrow{AB}$. Since, $\overleftrightarrow{T'P'} \parallel \overleftrightarrow{AB}$, $\overleftrightarrow{PT} \parallel \overleftrightarrow{P'T'}$. So, $APT \leftrightarrow A'P'T'$ is a similarity.

Hence, $\frac{AP}{AP'} = \frac{PT}{P'T'}$. Also, $APR \leftrightarrow AP'R'$ is a similarity. So,

$\frac{AP}{AP'} = \frac{PR}{P'R'}$. Hence, $\frac{PT}{P'T'} = \frac{PR}{P'R'}$. Thus, $\frac{PT}{PR} = \frac{P'T'}{P'R'}$.

But, $\frac{PT}{PR} = 1$. [PTSR is a square.] Hence, $\frac{P'T'}{P'R'} = 1$. That is, $P'T' = P'R'$. Thus, since $P'R'S'T'$ is a rectangle and $P'T' = P'R'$, $P'R'S'T'$ is a square.

10. 6 ft; 6 ft



$$\frac{3}{5} = \frac{x}{10}$$

[Your students may come up with a statement something like this:

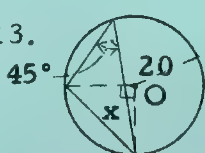
> No matter how long the see-saw is, one end of the board will rise 6 feet above the ground.

This is incorrect. Try a see-saw 5 feet long.]

11. $200(.1736) \leq$ Area-measure $\leq 200(.9848)$
$200(.1736) \leq$ Area-measure ≤ 200

[In general, if two sides of a parallelogram measure a and b, respectively, and the degree-measure θ of the included angle is a number between α and β, $\beta \leq 90$, then $ab \cdot \sin \alpha° \leq$ area-measure $\leq ab \cdot \sin \beta°$. If $\beta > 90$, then $ab \cdot \sin \alpha° \leq$ area-measure $\leq ab$.]

12. 110

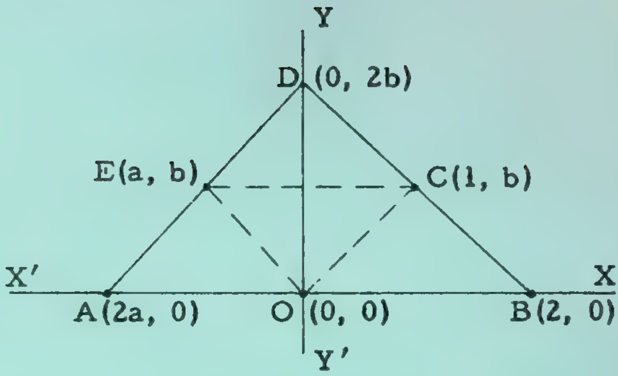
13. 

$20\sqrt{2}$

18. (a) Analytic proof:

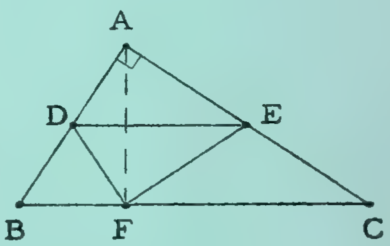
slope $(\overleftrightarrow{CO}) = \frac{b}{1}$

slope $(\overleftrightarrow{EO}) = \frac{b}{a}$



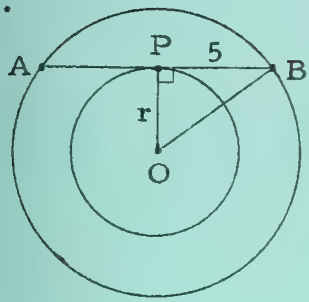
Since ΔADB is a right triangle, $[d(\overline{AD})]^2 + [d(\overline{DB})]^2 = [d(\overline{AB})]^2$. Thus, $4a^2 + 4b^2 + 4 + 4b^2 = 4 - 8a + 4a^2$. Hence, $b^2 = -a$. Therefore, slope $(\overleftrightarrow{CO}) \cdot$ slope $(\overleftrightarrow{EO}) = \frac{b^2}{a} \quad \frac{-a}{a} = -1$. So, $\overleftrightarrow{CO} \perp \overleftrightarrow{EO}$ and ΔEOC is a right triangle.

(b) Synthetic proof:



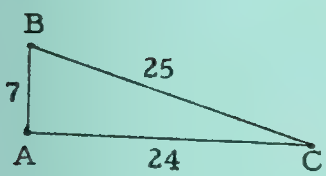
By hypothesis, D and E are midpoints, respectively, of $\overline{AB}$ and $\overline{AC}$. So, $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$. Thus, $m(\angle DFB) = m(\angle FDE)$ [Alt. int. angles]. Since $\overline{AF}$ is an altitude, ΔAFB is a right triangle with hypotenuse $\overline{AB}$. Since D is the midpoint of $\overline{AB}$, FD = DB. [The measure of the median to the hypotenuse of a right triangle is half the measure of the hypotenuse.] Thus, $m(\angle DBF) = m(\angle DFB)$. So, since $m(\angle DFB) = m(\angle FDE)$, $m(\angle DBF) = m(\angle FDE)$. Similarly, $m(\angle FCE) = m(\angle FED)$. Now, since $\angle BAC$ is a right angle, it follows that $m(\angle DBF) + m(\angle FCE) = 90$. So, $m(\angle FDE) + m(\angle FED) = 90$. Hence, $m(\angle DFE) = 90$ [Sum of the angle measures of a triangle is 180.]. That is, ΔDFE is a right triangle.

19.



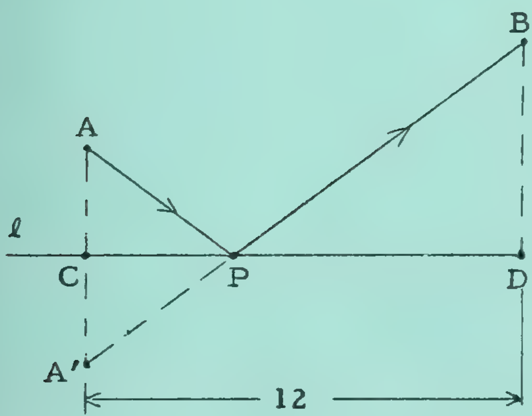
Since PB = 5, OB = $\sqrt{r^2 + 25}$. So, the area-measure of the larger circle is $\pi(r^2 + 25)$. Hence, the area-measure of the circular ring is $\pi(r^2 + 25) - \pi r^2$, that is, 25π.

20.



$\sin \angle B = \frac{24}{25} = .96$	$\sin(\angle C) = .28$
$\cos \angle B = \frac{7}{25} = .28$	$\cos \angle C = .96$
$\tan \angle B = \frac{24}{7} \doteq 3.4286$	$\tan \angle C \doteq .2917$
$m(\angle B) \doteq 74$	$m(\angle C) \doteq 16$

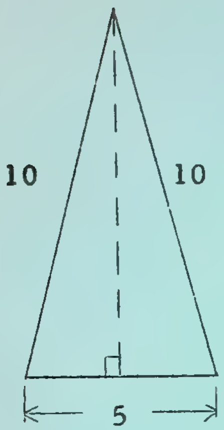
☆21.



[See Part C on page 6-113.]

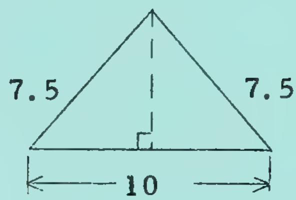
CP = 4 [$\Delta CPA' \sim \Delta DPB$]

24.



Case 1. $A = \frac{1}{2} \cdot 5 \cdot \sqrt{100 - \frac{25}{4}}$

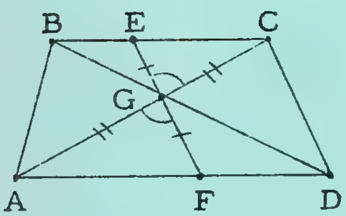
$= \frac{25}{4}\sqrt{15}$



Case 2. $A = \frac{1}{2} \cdot 10 \cdot \sqrt{\frac{225}{4} - 25}$

$= \frac{25}{2}\sqrt{5}$

25.



GAF $\leftrightarrow$ GCE is a congruence.
Hence, $\angle GAF \cong \angle GCE$.
So, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Thus, if AD = BC, ABCD is a parallelogram. If AD $\neq$ BC, ABCD is a trapezoid.

26. $\frac{9\pi}{2}$, 5π

27. 1440

28. $51\frac{3}{7}$

29. $(n - 2)180 = 12 \cdot 360$

$n - 2 = 24$

$n = 26$

30. 110

31. $\frac{100}{3}[2\pi - 3\sqrt{3}]$

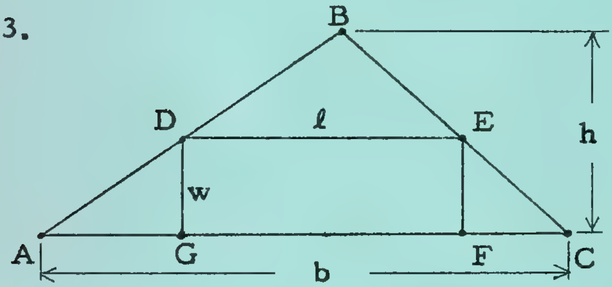
Correction. On page 6-445, line 11b should end
'...a trapezoid or a parallelogram'.
Line 9b should end '...and B?'.

☆22. Suppose w is the width of the rectangle [which has perimeter p] of largest area-measure. Then $\ell = \frac{p}{2} - w$, and the area-measure is $w(\frac{p}{2} - w)$. So,

$$\begin{aligned} A &= w(\tfrac{p}{2} - w) \\ &= -w^2 + \tfrac{p}{2}w \\ &= -[w^2 - \tfrac{p}{2}w + \tfrac{p^2}{16}] + \tfrac{p^2}{16} \\ &= -[w - \tfrac{p}{4}]^2 + \tfrac{p^2}{16}. \end{aligned}$$

Hence, A is a maximum when $w = \frac{p}{4}$. Since $\ell = \frac{p}{2} - w$, the length is also $\frac{p}{4}$. [There is no rectangle with perimeter p and smallest area-measure.]

☆23.



Consider the similar triangles, ΔBDE and ΔBAC.

$$\frac{h - w}{h} = \frac{\ell}{b}.$$

$$\ell = \frac{b(h - w)}{h}$$

So, the area-measure of rectangle DEFG is $\frac{wb(h - w)}{h}$, or $-\frac{b}{h}w^2 + bw$.

$$\begin{aligned} A &= -\tfrac{b}{h}w^2 + bw \\ &= -\tfrac{b}{h}(w^2 - hw + \tfrac{h^2}{4}) + \tfrac{bh}{4} \\ &= -\tfrac{b}{h}(w - \tfrac{h}{2})^2 + \tfrac{bh}{4} \end{aligned}$$

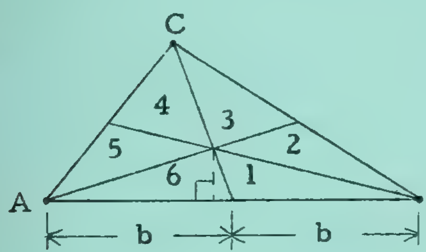
Hence, the maximum area is obtained when $w = \frac{h}{2}$. Thus, D and E are the midpoints of $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ respectively.

32. AEB $\leftrightarrow$ CFD is a congruence [h. ℓ]. Thus, AE = CF. Since $\overleftrightarrow{AE} \parallel \overleftrightarrow{CF}$ [ABCD is a parallelogram] and AE = CF, AECF is a parallelogram. So, $\overleftrightarrow{GF} \parallel \overleftrightarrow{EH}$. Also, $\overleftrightarrow{GE} \parallel \overleftrightarrow{FH}$ [Theorems 5-9 and 5-8]. Since $\overleftrightarrow{GF} \parallel \overleftrightarrow{EH}$ and $\overleftrightarrow{GE} \parallel \overleftrightarrow{FH}$, EHFG is a parallelogram.

33. Consider $\triangle BGC$. $KN = \frac{1}{2} \cdot GC$ and $\overleftrightarrow{KN} \parallel \overleftrightarrow{GC}$ [midpoint theorem for triangles]. Similarly, in $\triangle AGC$, $LM = \frac{1}{2} \cdot GC$ and $\overleftrightarrow{LM} \parallel \overleftrightarrow{GC}$. Hence, KN = LM and $\overleftrightarrow{KN} \parallel \overleftrightarrow{LM}$. So, KNML is a parallelogram. Consequently, $\overline{LN}$ and $\overline{KM}$ bisect each other.

34. By definition, the centroid of a triangle is the intersection of the medians of the triangle. But, in an equilateral triangle, the median from a vertex is the angle bisector from that vertex. Thus, the intersection of the medians <u>is</u> the intersection of the angle bisectors; that is, the centroid of an equilateral triangle is the incenter of the equilateral triangle. Now, use Exercise 3 of Part E on page 6-283.

35.



$$\left.\begin{aligned} K(\triangle_1) &= K(\triangle_6), \\ K(\triangle_2) &= K(\triangle_3), \\ \text{and } K(\triangle_4) &= K(\triangle_5). \end{aligned}\right\}\ \text{congruent bases and same altitude}$$

Also, $K(\triangle_1) + K(\triangle_6) + K(\triangle_5) = K(\triangle_2) + K(\triangle_3) + K(\triangle_4)$. Hence,

$2 \cdot K(\triangle_1) + K(\triangle_5) = 2 \quad K(\triangle_2) + K(\triangle_4)$. But, $K(\triangle_5) = K(\triangle_4)$.

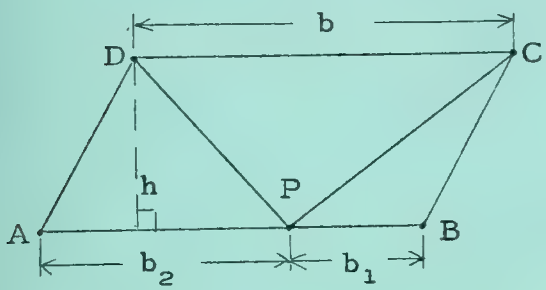
So, $K(\triangle_1) = K(\triangle_2)$. Consequently, $K(\triangle_6) = K(\triangle_1) = K(\triangle_2) = K(\triangle_3)$.

Similarly, we can establish that $K(\triangle_1) = K(\triangle_2) = K(\triangle_3) = K(\triangle_4) = K(\triangle_5)$

$= K(\triangle_6)$.

36. A, C, and D belong to the circle with center B. Thus, $\angle DAC$ is an inscribed angle which intercepts the same arc as central angle $\angle CBD$. Since $\triangle BCD$ is equilateral, $m(\angle CBD) = 60$. Consequently, $m(\angle DAC) = 30$.

37.



$K(\triangle PCD) = \frac{1}{2}bh$

$K(\triangle APD) = \frac{1}{2}b_2h$

$K(\triangle PBC) = \frac{1}{2}b_1h$

Thus, $K(\triangle APD) + K(\triangle PBC) = \frac{1}{2}b_2h + \frac{1}{2}b_1h = \frac{1}{2}h(b_1 + b_2)$

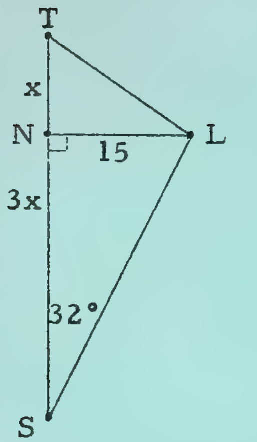
$= \frac{1}{2}hb = K(\triangle PCD)$.

38. Ratio of their perimeters is $\frac{1}{4}$.

Ratio of the area-measures is $\frac{1}{16}$.

The area of the smaller is 100 square inches.

43.

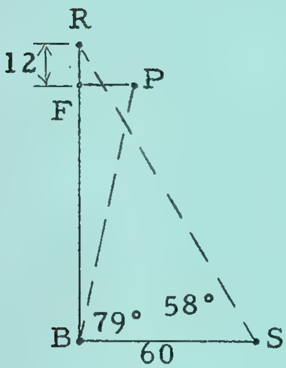


Distance: 17 miles; Bearing: S62°E

$\frac{3x}{15} = \tan 58°$ ‖ $\frac{15}{8} = \tan(\angle NTL)$

$x \doteq 8$ ‖ $62 \doteq m(\angle NTL)$

44.



The length of the flag pole is approximately 16.3 feet.

$\frac{BR}{60} = \tan 58°$

$BR \doteq 96$

Hence, $BF \doteq 84$. Thus, $\frac{FP}{84} \doteq \tan 11°$.

So, $FP \doteq 16.3$.

45. 80

46. 10, 11, and 12. [Suppose x is the measure of the edge of the center cube. Then, $5(x+1)^2 + 4x^2 + 4(x-1)^2 = 1604$.]

47. (a) $x = 80$; $m(\overset{\frown}{DF}) = 80$; $m(\overset{\frown}{FG}) = 108$; $m(\overset{\frown}{GC}) = 122$

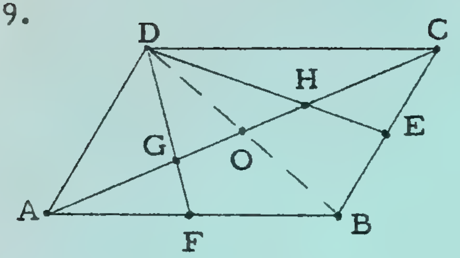
(b) $m(\angle CHG) = 101$; $m(\angle E) = 21$; $m(\angle ACG) = 61$; $m(\angle GFK) = 94$

Correction. On page 6-447, line 2 should read:

---a diagonal. Prove it.

On page 6-448, line 5b should begin 'an arc ---'.

39.

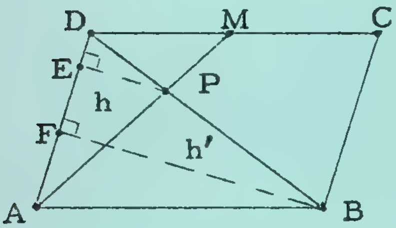


Since ABCD is a parallelogram, O is the midpoint of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$. So, $\overleftrightarrow{CO}$ is a median of $\triangle DBC$. Since $\overleftrightarrow{DE}$ is also a median of $\triangle DBC$ [E is the midpoint of $\overleftrightarrow{BC}$], it follows that $CH = 2 \cdot HO$. Similarly, $AG = 2 \cdot GO$. Hence, $CO = 3 \cdot HO$ and $AO = 3 \cdot GO$. Since $CO = AO$, $HO = GO$. Therefore, $CH = 2 \cdot HO = AG$; also, $2 \cdot HO = HG$. So, $CH = HG = GA$.

40. $50\sqrt{3}$ [$GC = GA = CA = 10\sqrt{2}$. $K(\triangle ACG) = \frac{(10\sqrt{2})^2}{4} \cdot \sqrt{3}$]

41. $\frac{12}{5}$ [$x^2 + r^2 = 9$ and $(5 - x)^2 + r^2 = 16$; so, $9 - x^2 = 16 - (5 - x)^2$]

42.



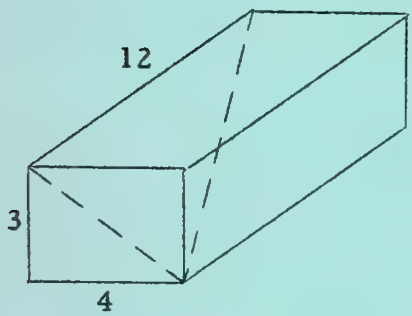
Suppose that E is the foot of the perpendicular from P to $\overleftrightarrow{DA}$, and F is the foot of the perpendicular from B to $\overleftrightarrow{DA}$. Then, $\triangle DEP$ and $\triangle DFB$ are right triangles, and, since $\angle PDE = \angle BDF$, $DEP \longleftrightarrow DFB$ is a similarity. By Exercise 39, $DP = \frac{1}{3} DB$; so, $EP = \frac{1}{3} FB$, that is, $h = \frac{1}{3} h'$. Since $\triangle APD$ and $\triangle ABD$ have the same base, it follows that $K(\triangle APD) = \frac{1}{3} \cdot K(\triangle ABD)$. Now, since $K(\triangle ABD) = \frac{1}{2} \cdot K(\square ABCD)$, $K(\triangle APD) = \frac{1}{6} \cdot K(\square ABCD)$. Hence, the ratio of the area-measure of $\triangle APD$ to the area-measure of $\square ABCD$ is $\frac{1}{6}$.

Summarizing, if the conditions of this exercise be supplemented in any one of three ways: (1) $\angle A$ is not acute: (2) $BD \geq BA$; (3) $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, then it follows that ABCD is a parallelogram.

51. 26

52. By Theorem 6-17, ABCD is a rhombus. Thus, by Theorem 6-12, ABCD is a rectangle. Hence, by definition, each of the angles is a right angle.

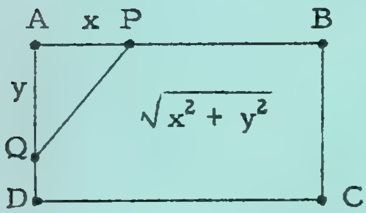
53.



13 inches

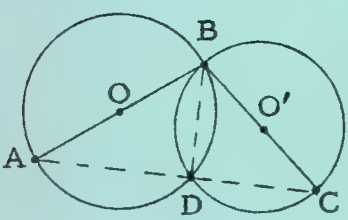
$\sqrt{3^2 + 4^2 + 12^2} = 13$

54.



He could measure any convenient distance along $\overleftrightarrow{AB}$ and along $\overleftrightarrow{AD}$. Then, he could use Theorem 7-6 to determine the length of a piece of wood to nail at P and Q.

55.



$\angle ADB$ and $\angle BDC$ are right angles [inscribed in a semicircle]. Thus, $\angle ADB$ and $\angle BDC$ are supplementary. Hence, by Theorem 2-9, $\overrightarrow{DA}$ and $\overrightarrow{DC}$ are collinear; that is, A, D, and C are collinear.

Correction. On page 6-449, line 12 should end with '---a parallelogram?'.

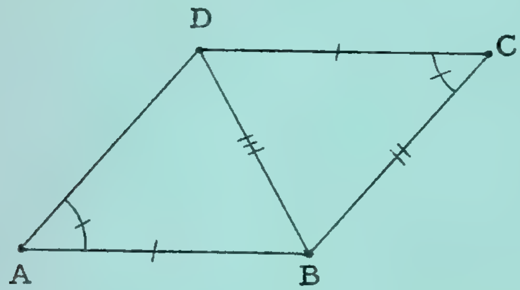
48. 112; 128; 120; 82

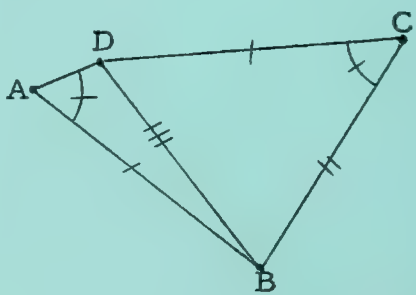
49. 200 ft. [The grade of the highway is the slope of the highway.]

50. [This exercise furnishes a good opportunity to review pages 6-128 and 6-129.]

The given conditions are not sufficient to insure that ABCD be a parallelogram. The natural procedure to use in attempting to show that quadrilateral ABCD is a parallelogram is to prove that ABD $\longleftrightarrow$ CDB is a congruence, and then use Theorem 6-6. Now, since $\angle A \cong \angle C$, AB = CD, and BD = DB, we can argue by Theorem 4-14 that if $\angle A$ is not acute then ABD $\longleftrightarrow$ CDB is a congruence. Or, using a slight extension of the theorem at the foot of TC[6-128, 129]a, we can argue that if BD $\geq$ BA then ABD $\longleftrightarrow$ CDB is a congruence. [This includes the case in which $\angle A$ is not acute, for it $\angle A$ is not acute then $m(\angle A) > m(\angle D)$, and BD $\geq$ BA.]

However, if BD < BA, then $\triangle ABD$ and $\triangle CDB$ need not be congruent. [If they are not congruent, $\angle ADB$ and $\angle CBD$ are supplementary.]





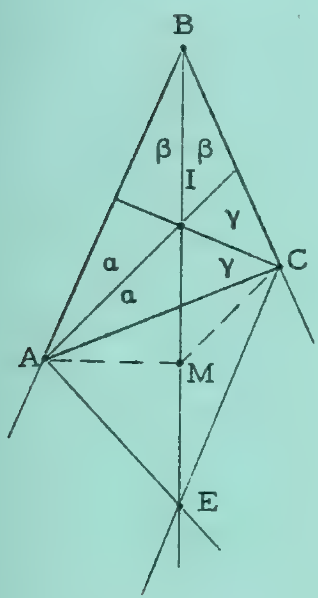
So, ABCD need not be a parallelogram. However, if one knows that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ then $\angle ADB$ and $\angle CBD$ are known to be congruent, and ABD $\longleftrightarrow$ CDB is a congruence. So, one has an additional theorem on parallelograms, reminiscent of Theorems 6-6, 6-8, and 6-10:

> If two sides of a quadrilateral are parallel, two sides are congruent, and two opposite angles are congruent, then the quadrilateral is a parallelogram.

Correction. On page 6-450, line 6 should begin '☆57. ...'.

56. $\alpha + m(\angle BAD) + \gamma + m(\angle BCD) = 360$. By Theorem 6-30, $\beta + m(\angle BAD) + \delta + m(\angle BCD) = 360$. Hence, $\alpha + \gamma = \beta + \delta$.

☆57.



Let I be the incenter, E the excenter, and M the midpoint of $\overleftrightarrow{IE}$. Our job is to show that A, B, C, and M are concyclic. We can do this by showing that $\angle ABC$ and $\angle AMC$ are supplementary. [See COMMENTARY for Exercise 6 on page 6-316.] Since $\overrightarrow{AE}$ and $\overrightarrow{AI}$ are bisectors of supplementary angles, $\angle IAE$ is a right angle [Exercise 1 of Part C on page 6-110]. Similarly, $\angle ICE$ is a right angle. Consequently, since M is the midpoint of $\overleftrightarrow{IE}$, $MA = MI = MC = ME$ [Theorem 6-28]. Thus, M is the circumcenter of AICE. So, $m(\angle AMC) = 2 \cdot m(\angle AEC)$.

Now, $m(\angle AEC) + 90 + \alpha + 2\beta + \gamma + 90 = 360$ [Theorem 6-30]. Thus,

$$
\begin{aligned}
m(\angle AMC) &= 2[360 - 180 - \alpha - 2\beta - \gamma] \\
&= 360 - (2\alpha + 2\beta + 2\gamma) - 2\beta \\
&= 360 - 180 - 2\beta \\
&= 180 - 2\beta
\end{aligned}
$$

So, $m(\angle AMC) + m(\angle ABC) = 180 - 2\beta + 2\beta = 180$.

58. See Davis, D. R., Modern College Geometry, (Cambridge, Mass.: Addison-Wesley Publishing Company, Inc., 1949) on the "nine-point circle".

Corrections. On page 6-452, line 6, change 'endpoints' to 'end points'. Also, delete parts (g) and (k) of Exercise 59.

On page 6-543, line 2 should read:

(a) at a distance $\frac{1}{2}$ from a point Q;

Line 3b should read:

(a) at a given distance from a point C;

Line 2b should read:

(b) at a given distance from a line ℓ;

59. (a) The intersection of two circles with radius 3 and centers A and B.

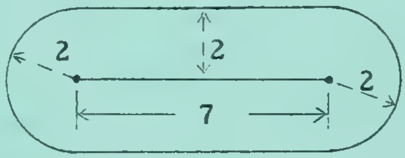
[Note: 'two points' means 'two particular but unspecified points'. A similar convention applies to the rest of these locus problems.]

(b) A line parallel to the given lines and "halfway" between them.

(c) A circle with radius one half that of the given circle and concentric with the given circle.

(d) The median to the given side less its end points.

(e)



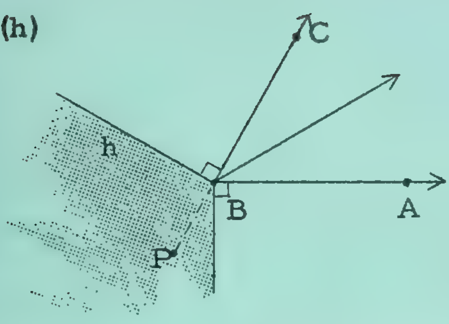
*

Distance from a point P to a line ℓ has been defined as the measure of the the shortest segment from P to ℓ. It seems reasonable to define distance from a point P to any set of points in a similar way. Hence, by 'the distance from P to s' we mean the measure of the shortest segment $\overline{PQ}$ where $Q \in s$.

*

(f) A circle concentric with the given circle and with radius equal to the distance between the center of the given circle and one of the congruent chords.

(h)



The locus is the union of the angle bisector and the intersection of two closed half-planes whose edges are perpendicular to the sides of the angle and do not contain the respective sides. For each point $P \in h$, the distance from P to $\overrightarrow{BA}$ is $m(\overleftrightarrow{PB})$, which is equal to the distance from P to $\overrightarrow{BC}$. [Ask your students to consider the locus of points which are equidistant from the lines containing the sides of an angle. This locus is the union of two perpendicular lines, one of which contains the bisector of the angle, and one which contains the bisector of the supplementary angle adjacent to the given angle.]

(i) The set consisting of the incenter of the triangle. [Have the students describe the locus of points which are equidistant from the lines containing the sides of a triangle. This locus is a set consisting of four points, the incenter of the triangle and its three excenters.]

(j) Relative to the circle which has the hypotenuse as a diameter, the locus is the complement of the set consisting of the end points of the common hypotenuse. [In other words, a circle which has the common hypotenuse as a diameter, less the end points of this hypotenuse, is the locus in question.]

60. See Courant, R and Robbins, H., What is Mathematics? (New York: Oxford University Press, 1941), pp. 152-155.

61. (a) a circle with radius $\frac{1}{2}$ (b) a parabola
(c) an ellipse (d) an hyperbola

62. (a) a sphere (b) a cylindrical surface
(c) a torus--the surface of a doughnut