NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

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Hill's Equation. II. Transformations, Approximation, Examples

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HILL'S EQUATION

Part II. Transformations, Approximation, Examples

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C. 1

TABLE OF CONTENTS

Table of Contents, continued

Introduction

It is the purpose of the present, second, report on Hill's equation to provide the reader with a few ready made transformation and approximation formulas which may be useful to anyone working on a particular problem. Also, we shall discuss some examples in detail, partly because they illustrate the general theory and partly because they appear frequently in applications. The section on the coexistence problem contains a little more than mere examples. It may be considered as a complete discussion of those results which can be obtained from the investigation of three term recurrence relations.

A third part of this report will contain applications of the theory to problems of physics and engineering.

3. Transformations

3.1 Elementary formulas

(i) Transformation into a standard form

The differential equation

$$\frac{d^2z}{dx^2} + a(x)\frac{dz}{dx} + b(x)\,z = 0 \tag{3.1}$$

can be transformed into

$$\frac{d^2y}{dx^2} + Q(x)y = 0 \tag{3.2}$$

by putting

$$y = \left[\exp \tfrac{1}{2} A(x)\right] z \quad , \qquad a(x) = dA/dx \tag{3.3}$$

where

(3.4) $$Q = -\frac{1}{2}\frac{da}{dx} - \frac{1}{4}a^2 + b .$$

(ii) The Liouville Transformation

The differential equation for the interval $0 \leqq t \leqq \omega$:

(3.5) $$\frac{d^2z}{dt^2} + \lambda M^4(t)\, z = 0 \qquad (\lambda = \text{constant},\ M(t) > 0)$$

can be transformed into the differential equation

(3.6) $$\frac{d^2y}{dx^2} + [\lambda\gamma^2 + Q(x)]y = 0$$

for the interval $0 \leqq x \leqq \pi$ by the Liouville Transformation:

(3.7) $$x = \frac{1}{\gamma}\int_0^t M^2(\tau)\, d\tau \quad , \qquad \gamma = \frac{1}{\pi}\int_0^\omega M^2(t)\, dt$$

(3.8) $$y(x) = M(t)\, z(t)$$

(3.9) $$Q(x) = (M(t))^{-1}\frac{d^2M(t)}{dx^2} = -\gamma^2[M(t)]^{-3}\frac{d^2}{dt^2}\left(\frac{1}{M(t)}\right)$$

The inverse of this transformation is given by

(3.10) $$t = \gamma\int_0^x \frac{d\zeta}{[M^*(\zeta)]^2} \quad , \qquad \gamma^{-1} = \omega^{-1}\int_0^\pi \frac{d\xi}{[M^*(\xi)]^2}$$

where $M^*(x) = M(t)$ is a positive solution of the differential equation

(3.11) $$\frac{d^2M^*}{dx^2} = Q(x)\, M^*(x) .$$

It should be noted that the Liouville transformation is not applicable unless M(t) is twice differentiable, at least in the sense that d^2M/dt^2 is bounded and continuous almost everywhere. An example for the Liouville transformation is provided by the following

Lemma 3.1: The function Q(x) corresponding to M(t) under the Liouville transformation is a constant if and only if

$$(3.12) \qquad M(t) = \left[\alpha t^2 + 2\beta t + \delta\right]^{-\frac{1}{2}}, \qquad (\alpha,\beta,\delta \text{ constant}) .$$

In this case the differential equation for y(x) will be

$$(3.13) \qquad \frac{d^2y}{dx^2} + \gamma^2\left[\lambda - (\alpha\delta - \beta^2)\right]y = 0 .$$

If

$$(3.14) \qquad D^2 = \alpha\delta - \beta^2 > 0 ,$$

that is if $\alpha t^2 + 2\beta t + \delta$ does not have any real zeros, we have

$$(3.15) \qquad \pi D\gamma = \tan^{-1}\left[(\alpha\omega + \beta)|D^{-1}\right] - \tan^{-1}(\beta D^{-1}) .$$

Obviously,

$$|D\gamma| \leqq 1 .$$

(iii) Polar Coordinates

Let $y_1(x)$ and $y_2(x)$ be the standard solutions of

$$(3.16) \qquad \frac{d^2y}{dx^2} + Q(x)y = 0$$

which satisfy the initial conditions

$$(3.17) \qquad y_1(0) = 1, \quad y_2(0) = 0, \quad y_1'(0) = 0, \quad y_2'(0) = 1,$$

where a prime denotes the derivative with respect to x. If we interpret y_1 and y_2 as Cartesian coordinates in a plane, a <u>transformation to polar coordinates</u> leads to the following formulas: Let

$$(3.18) \qquad y_1(x) = \rho \cos \phi , \qquad y_2(x) = \rho \sin \phi$$

$$\rho > 0 , \qquad \phi(0) = 0 , \qquad \rho(0) = 1 .$$

Then

$$(3.19) \qquad \phi(x) = \int_0^x \frac{dt}{\rho^2(t)}$$

$$(3.20) \qquad \rho^2(x) = y_1^2(x) + y_2^2(x)$$

$$(3.21) \qquad \frac{d^2\rho}{dx^2} - \rho^{-3} + Q(x)\rho = 0 .$$

It should be noted that $\phi(x)$ is a monotonically increasing function of x. If $|\phi(x)| \to \infty$ as $x \to +\infty$ or $x \to -\infty$, both $y_1(x)$ and $y_2(x)$ will have infinitely many zeros. Otherwise, both of them will have a finite number of zeros.

<u>(iv) Differential equation for the product of two solutions.</u>

Let $y = \eta_1$ and $y = \eta_2$ be any two solutions of

$$(3.22) \qquad \frac{d^2y}{dx^2} + Q(x)y = 0$$

where, for all x,

$$Q(x+\pi) = Q(x) .$$

Let

$$z = \eta_1 \eta_2$$

be the product of these solutions. Then

$$(3.23) \qquad \frac{d^3 z}{dx^3} + 4Q \frac{dz}{dx} + 2 \frac{dQ}{dx} z = 0 .$$

and equation (3.23) has at least one non-trivial periodic solution with period π. The following result holds:

Lemma 3.2. Either all periodic solutions of (3.23) with period π are constant multiples of a single one, or all solutions of (3.23) are periodic with period π. This takes place if and only if all solutions of (3.16) are periodic with period π or 2π.

Lemma 3.2 is an immediate consequence of Floquet's Theorem (see Part I, Section 1.2). If the characteristic equation for (1.16) has the roots ρ_1, ρ_2, and if η_1, η_2 are non-trivial solutions such that

$$(3.24) \qquad \eta_1(x+\pi) = \rho_1 \eta_1(x), \qquad \eta_2(x+\pi) = \rho_2 \eta_2(x),$$

then clearly $\eta_1 \eta_2$ is periodic with period π since $\rho_1 \rho_2 = 1$. In general, all periodic solutions of (3.23) with period π will be multiples of this particular one because of the following rather obvious fact:

If η_1 and η_2 are two linearly independent solutions of (1.16), then

$$\eta_1^2, \quad \eta_1 \eta_2, \; \eta_2^2$$

are linearly independent solutions of (3.23). That all solutions of (3.23) will have period π if and only if all solutions of (1.16) are periodic with period π or 2π can be derived from this fact by choosing for η_1 and η_2 the linearly independent solutions which appear in Floquet's Theorem.

Equation (3.23) can be reduced to a non-linear second order equation:

$$(3.25) \qquad z\frac{d^2z}{dx^2} - \frac{1}{2}\left(\frac{dz}{dx}\right)^2 + 2Qz^2 = C$$

where C is a constant. Putting $z = \eta_1\eta_2$, we find

$$C = -\frac{1}{2}\left(\eta_1\frac{d\eta_2}{dx} - \eta_2\frac{d\eta_1}{dx}\right)^2$$

and this implies

Lemma 3.3. Assume that not all of the solutions of (3.23) are periodic with period π. Then, if z is a periodic non-trivial solution, the constant C in (3.25) will be

negative, if $y'' + Qy = 0$ has an unbounded solution,
positive, if all solutions are bounded,
zero, if a non-trivial solution has period π or 2π.

4. Tests for the existence of oscillating solutions.

The real solutions of Hill's equation with a real λ and Q:

$$(4.1) \qquad \frac{d^2y}{dx^2} + [\lambda + Q(x)]y = 0 , \qquad Q(x+\pi) = Q(x)$$

may or may not have infinitely many zeros. The situation can be described as follows:

Theorem 4.1. Either all non-trivial real solutions of (4.1) have only a finite number of zeros, or all real solutions of (4.1) have infinitely many zeros. Let λ_0 be the smallest value of λ for which (4.1) has a periodic solutions. Then for $\lambda \leqq \lambda_0$, all non-trivial real solutions have only a finite number of zeros, but for $\lambda > \lambda_0$, every real solution has infinitely many zeros.

We shall prove Theorem 4.1 by using part of a result which is due to Hamel, 1912:

Theorem 4.2. There exists a real solution of (4.1) which has only a finite number of zeros if and only if for all continuously differentiable functions w(x) with period π

$$(4.2) \qquad \int_0^\pi [\lambda + Q(x)]\; w^2(x)\; dx \leqq \int_0^\pi \left(\frac{dw}{dx}\right)^2 dx \; .$$

We shall need only the fact that (4.2) must hold if (4.1) has a solution without infinitely many zeros, and we shall not prove the sufficiency of condition (4.2). (See Lemma 4.2).

To prove Theorem 4.1, we observe first that every real solution of (4.1) will have infinitely many zeros if a single non-trivial solution has this property. This follows from the formulas which describe the transformation to polar coordinates. In fact, (3.18) shows that every real solution of (4.1) can be written in the form

$$(4.3) \qquad y(x) = A\rho(x)\cos(\phi - \phi_0)$$

where A and ϕ_0 are constants. If (4.3) has infinitely many real zeros for $A \neq 0$ and a particular value of ϕ_0, then $\phi(x)$ must be unbounded if x goes from $-\infty$ to $+\infty$, and therefore y(x) has infinitely many zeros for every choice of ϕ_0 and A. Now we can prove

<u>Lemma 4.1</u>. If (4.1) has a real, non-trivial, solution with finitely many zeros, then (4.1) must have a real solution y(x) such that

$$(4.4) \qquad y(x+\pi) = \rho y(x)$$

where ρ is real and positive and where y(x) has no zeros.

Proof: If there exists a solution of (4.1) such that ρ is complex, we know from Floquet's theorem that all solutions are bounded. In this case, formulas (3.18) to (3.20) show that $\phi \to \infty$ as $x \to \infty$ and therefore (4.3) shows that there is a real solution with infinitely many zeros and no real solution with finitely many. Therefore we know that ρ in (4.4) must be real and that we may take y in (4.4) to be real also if there exists a solution with finitely many zeros. But in this case, ρ must also be positive since otherwise y would have infinitely many changes of sign, and y itself cannot vanish for any fixed $x=x_0$ because it would also vanish for $x = x_0 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. This proves Lemma 4.1 completely. Now we shall prove

<u>Lemma 4.2</u>. If (4.1) has a real solution with finitely many zeros,

then the inequality (4.2) of Theorem 4.2 holds for all continuously differentiable functions $w(x)$ for which $w(x+\pi) = w(x)$.

Proof: We know from Lemma 4.1 that we have a real solution y satisfying (4.3) with a positive ρ. Putting $\rho = \exp \beta$, where β is real, we can write

$$y(x) = e^{\beta x} p(x) = \exp P(x)$$

where

$$p(x+\pi) = p(x), \qquad P(x+\pi) = \beta + P(x)$$

and where $P(x)$ is two times differentiable since $p(x)$ does not vanish. We find easily that

$$P'' + (P')^2 + Q^*(x) = 0 \tag{4.5}$$

where $Q^* = Q + \lambda$, and therefore

$$\int_0^\pi Q^*(x)\, w^2(x)\, dx = -\int_0^\pi (P'w^2)'\, dx - \int_0^\pi (P'w - w')^2 dx$$

$$+ \int_0^\pi (w')^2$$

Since

$$\int_0^\pi (P'w^2)'\, dx = 0$$

because of the periodicity of P' and w, Lemma 4.2 is true. Now we can prove

Lemma 4.3. Let λ_o be the smallest value of λ for which (4.1) has a periodic, non-trivial solution $y = p_o(x)$ with period π. Then for any $\lambda > \lambda_o$, every solution of (4.1) has infinitely many zeros.

Proof: According to Lemma 4.2, it suffices to show that for $\lambda > \lambda_o$ the inequality (4.2) of Theorem 4.2 can be violated by at least one periodic function $w(x)$. We choose $w = p_o(x)$, and we find

$$(4.8) \qquad \int_0^\pi \left[(\lambda + Q)p_o^2 - (p_o')^2\right] dx$$

$$= (\lambda - \lambda_o)\int_0^\pi p_o^2\, dx + \int_0^\pi \left[(\lambda_o + Q)p_o^2 - (p_o')^2\right] dx.$$

The second integral on the right hand side of (4.6) vanishes because

$$-(p_o')^2 + (Q + \lambda_o)p_o^2 = -(p_o p_o')'$$

and the integral of $(p_o p_o')'$ from zero to π vanishes since $p_o p_o'$ is periodic with period π. This proves Lemma 4.3, since the first integral on the right hand side of (4.6) is positive.

All that remains to do now is an investigation of the case $\lambda \leqq \lambda_o$. We shall reduce this problem to the case $\lambda = \lambda_o$ by proving

Lemma 4.4. If (2.1) has a real, non-trivial solution V, with infinitely many zeros, for $\lambda = \lambda^*$, the same is true for all $\lambda \geqq \lambda^*$.

Proof: If (4.1) should have a solution with finitely many zeros, we could (according to Lemma 4.1) assume that it has a real solution $y(x)$ of type (4.4) where ρ is positive and $y(x)$ has no zeros. We shall show that this leads to a contradiction. For this purpose, we construct the solution y^* of

$$y^{*''} + (\lambda^* + Q)\, y^* = 0$$

which has the same initial conditions at $x = 0$ as $y(x)$. Clearly, y^* is real and, according to our assumption and the remarks made before Lemma 4.1, it will have infinitely many zeros (although it is not identically zero). But then $y(x)$ will also have a zero. For let x_o^* be the smallest positive zero of y^*. We find easily

$$yy^{*'} - y^* y' = (\lambda - \lambda^*) \int_0^x y(t)\, y^*(t)\, dt \; . \tag{4.7}$$

We may assume that at $x = 0$, both y and y^* are positive. Then, at the first zero x_o of y^*, clearly $y^{*'}$ must be negative since a solution of a homogeneous linear second order differential equation cannot vanish together with its first derivative. Therefore, for $x = x_o$, we have

$$y(x_o)\, y^{*'}(x_o) = (\lambda - \lambda^*) \int_0^{x_o} y(x)\, y^*(x)\, dx \; . \tag{4.8}$$

However, if y does not have any zeros, the left hand side in (2.8) is negative but the right hand side is positive. Therefore $y(x_o) < 0$ and $y(x)$ has a zero between 0 and x_o. This proves Lemma 4.4.

Now we shall have completed the proof of Theorem 4.1 once we shall have proven

Lemma 4.5. Let λ_o and $p_o(x)$ be defined as in Lemma 4.3. Then $p_o(x) \neq 0$ for all x.

The combination of Lemma 4.3 and Lemma 4.5, together with the remarks made before Lemma 4.1, contains the full statement of Theorem 4.1. We shall now prove Lemma 4.5. First, we observe that $p_o(x)$ must have at least two zeros in the half open interval $0 \leqq x < \pi$ if $p_o(x)$ has any zeros at all. This follows from $p(o) = p(\pi)$, $p'(o) = p'(\pi)$. If $p(o) \neq 0$, $p(x)$ has two zeros in $o < x < \pi$ since it must change its sign at least twice. If $p(o) = 0$, we have $p'(o) \neq 0$ and if $p'(o) > 0$, it follows that $p(x)$ must be increasing at $x = \pi$. Therefore, $p(\epsilon) > 0$, $p(\pi - \epsilon) < 0$ for a sufficiently small $\epsilon > 0$.

We shall now consider

$$\Delta(\lambda_o) = y_1(\pi,\lambda_o) + y_2'(\pi,\lambda_o)$$

where $y_1(x)$, $y_2(x)$ are the standard solutions of (4.1) for $\lambda = \lambda_o$ defined by the initial conditions (3.17). We know from the general theory of Hill's equation (Part One of this report, Theorem 12.1and Lemmas 2.1, 12.2) that

$$\Delta(\lambda_o) = 2, \qquad \Delta(\lambda) > 2 \quad \text{for } \lambda < \lambda_o .$$

We also know from the proof of Lemma 2.1 in Part I of this report that

$$y_1(x,\lambda^*) > 0 , \qquad y_2(x,\lambda^*) > 0$$

for all $x > 0$ if λ^* is such that

$$\lambda^* + Q(x) < 0$$

for all x. We know that $\lambda^* < \lambda_o$ and that

$$\Delta(\lambda^*) > 2 \quad .$$

We shall prove Lemma 4.5 by showing:

If $p_o(x)$ has at least two zeros in $0 \leqq x < \pi$, then there exists a value λ' of λ such that

$$\Delta(\lambda') < 0, \qquad \lambda^* < \lambda' < \lambda_o \quad .$$

But in this case, λ_o cannot be the smallest value of λ for which (4.1) has a periodic solution of period π since there would exist a value λ_1' of λ such that

$$\Delta(\lambda_1') = -2 , \qquad \lambda_1' < \lambda_o$$

and (4.1) would have a solution of period 2π for $\lambda = \lambda_1'$. (For a proof, see Lemma 2.4 of Part I). But this contradicts Theorem 2.1 of Part I about the arrangement of the characteristic values of Hill's equation.

In order to show that λ' exists, we introduce the quantity

$$\psi(\lambda) = \int_0^\pi \left[y_1^2(x,\lambda) + y_2^2(x,\lambda)\right]^{-1} dx \quad .$$

Then we have according to (3.18) that

$$y_1(x,\lambda) = \rho(x)\cos\phi\ , \qquad y_2(x,\lambda) = \rho(x)\sin\phi$$

where $\phi(x,\lambda)$ is an increasing function of x such that

$$\phi(o,\lambda) = 0\ , \qquad \phi(\pi,\lambda) = \psi(\lambda)\ .$$

We can write $p_o(x)$ in the form

$$p_o(x) = A\rho\cos(\phi - \phi_o)$$

where, for $0 \leqq x \leqq \pi$,

$$0 \leqq \phi \leqq \psi(\lambda_o)\ .$$

Since p_o is supposed to have two zeros in $0 \leqq x < \pi$, we have

$$\psi(\lambda_o) > \pi\ .$$

On the other hand, since for $\lambda = \lambda^*$ neither y_1 nor y_2 have any zeros for $x > 0$, we see that

$$0 < \psi(\lambda^*) < \frac{\pi}{2}\ .$$

Since it is clear that $\psi(\lambda)$ depends continuously on λ, it follows that there exists a value $\lambda = \lambda'$ such that

$$\psi(\lambda') = \pi\ , \qquad \lambda^* < \lambda' < \lambda_o\ .$$

Now we have

$$\Delta(\lambda') = y_1(\pi,\lambda') + y_2'(\pi,\lambda') = \rho(\pi)(1+\psi'(\lambda'))\cos\psi(\lambda') + \rho'(\pi)\sin\psi(\lambda')$$

where $\psi'(\lambda) = d\phi/dx$ for $x=\pi$ and $\rho'(x) = d\rho/dx$. Since $d\phi/dx > 0$, it follows from $\psi(\lambda') = \pi$ that

$$\Delta(\lambda') < 0 .$$

This proves our contention and Lemma 4.5. Therefore, Theorem 4.1 has been proven completely.

We can use the method of proving Lemma 4.3 to derive the following

Theorem 4.3. Let λ_o be the smallest value of λ for which (4.1) has a periodic, non-trivial solution of period π. Let $\lambda < \lambda_o$ and let

$$y = \exp(\beta x)\,p(x) , \qquad p(x+\pi) = p(x) , \quad p \not\equiv 0$$

be a Floquet-type solution of (2.1). Then

$$\beta \geqq (\lambda_o - \lambda)^{1/2} .$$

Proof: Using the fact that $p(x)$ satisfies the differential equation

$$p'' + 2\beta p' + (\lambda + Q + \beta^2)p = 0 ,$$

we find by a calculation similar to the one used in proving Lemma 4.3 that, for any constant μ

$$(4.9) \qquad \int_0^\pi \left[(\mu + Q)p^2 - p'^2\right] dx = (\mu - \lambda - \beta^2) \int_0^\pi p^2 dx \,.$$

Therefore we see from Lemma 4.2 that the real, non-trivial solutions of

$$y'' + (\mu + Q)y = 0$$

must all have infinitely many zeros if

$$\mu > \lambda + \beta^2$$

since then the right hand side in (4.9) is positive. But then it follows from Theorem 4.1 that

$$\lambda + \beta^2 \geqq \lambda_0 \,.$$

Finally, we mention the following

<u>Corollary 4.2:</u> If $\lambda + Q(x) < 0$ for all x, then the non-trivial solutions of (4.1) have only finitely many zeros. If

$$\int_0^\pi \left[\lambda + Q(x)\right] dx > 0 \,,$$

the real solutions of (4.1) will always have infinitely many zeros.

The proof of the first statement of Corollary 4.1 can be given by using the fact that $y_2(x)$ will have only one zero if $\lambda + Q < 0$ for all x. This follows in the same manner in which Lemma 2.1 of Part I of this report has been proven. The second statement of Corollary 4.1 follows if we use Lemma 4.2 with $w(x) = 1$.

5. Intervals of Stability

5.1 Introduction

The general theory, as presented in Part I of this report, has been supplemented by numerous investigations both of special cases and of classes of functions $Q(x)$ for which more precise statements can be obtained. We shall try now to indicate what has been achieved. Since in most cases the results either are complicated or based on numerical computations, we shall confine ourselves to giving a general description of the most important methods together with a few references. Only the results about the so-called "regions of absolute stability" which are due to G. Borg, 1944, will be presented in detail. The notations used will be those of Part 1. We shall always write Hill's equation in the form of Equation 2.1 as

$$(5.1) \qquad y'' + [\lambda + Q(x)]y = 0 , \qquad Q(x+\pi) = Q(x)$$

and we shall use the terminology of Theorem 2.1. For $Q(x)$ we shall assume a convergent Fourier expansion

$$(5.2) \qquad Q(x) = \sum_{r=-\infty}^{+\infty} g_n e^{2inx} , \qquad g_o = 0, \qquad g_{-n} = \overline{g_n} ,$$

where a bar denotes the conjugate complex quantity. Unless otherwise stated, we shall assume that

$$(5.3) \qquad \sum_{n=1}^{\infty} n^2 |g_n|^2 < \infty .$$

The model for the questions to be asked about Hill's equation has been provided by Mathieu's equation which we shall write in the form

$$(5.4) \qquad y'' + [\lambda - 2\theta \cos 2x]y = 0 .$$

Thorough accounts of the theory of this equation and its application have been given, at different times, by Strutt, 1932; McLachlan, 1947; and by Meixner and Schaefke, 1954. It may be considered as a two parameter equation, and the points in the λ,θ-plane where the solutions are stable form certain stability regions which have been studied thoroughly. Strutt, 1944 has considered numerous other two and three parameter equations of Hill's type; we shall report on some of his results in Section 5.3. The intervals of instability of Mathieu's equation (for a fixed θ) are known to decrease exponentially with their order. A generalization of this result found by Hochstadt will be formulated below in Theorem 5.1.

If we consider Hill's equation as an equation with infinitely many parameters g_r, the dependence of the intervals of instability on these parameters can be described approximately by a theorem due to Erdélyi. This will be discussed in section 5.4. If we write $Q(x) = \beta\phi(x)$ and subject $\phi(x)$ to certain normalizing conditions, we obtain two-parameter problems for which Borg has given regions of "absolute" stability. These regions will be described in Section 5.2. Finally, in Section 5.5, we shall describe briefly some results found by an application of the theory of general systems of linear differential equations.

With respect to the question of the actual behavior of the length of n-th interval of instability of Hill's equation we mention

Theorem 5.1. If $Q(x) = Q(-x)$, $Q(x+\pi) = Q(x)$, and if $Q(x)$ is an analytic function of x in a strip of the complex x-plane containing the real x-axis, and if L_n denotes the length of the n-th interval of instability of Hill's equation, then for any positive integer M,

$$(5.6) \qquad \lim_{n \to \infty} L_n \, n^M = 0 .$$

For a proof see Hochstadt, 1961. The question when some intervals of instability will actually disappear can also be answered in certain cases. For this problem, see Chapter 7.

5.2 Regions of absolute stability

We shall now write

$$(5.7) \qquad Q(x) = \beta \psi(x)$$

where, according to (5.2)

$$(5.8) \qquad \int_0^{\pi} \psi(x)\, dx = 0 .$$

We shall consider all "Functions ψ of class p" which are defined by

$$(5.9) \qquad \left\{ \frac{1}{\pi} \int_0^{\pi} |\psi(x)|^p \, dx \right\}^{1/p} = 1$$

where $p=1,2,3,\ldots$ or $p=\infty$. If $p=\infty$, (5.9) means, of course, that

$$\operatorname{Max} |\psi(x)| = 1 .$$

We shall not postulate (5.3) in this section. It suffices to assume that $\psi(x)$ is continuous except for a finite number of points where $\psi(x)$ may have a jump.

A region in the real λ,β plane will be called a <u>region of absolute stability for functions of class p</u>, if, for any point in this region, (5.1) will have stable solutions for all functions $Q = \beta\psi$ where ψ belongs to the class p.

Borg, 1944, proved the following results:

<u>Theorem 5.2</u>. Let n=0,1,2,... . The region of absolute stability for functions $\psi(x)$ of class one is bounded by the curves

$$\beta_{n+1} = \pm \frac{4(n+1)\sqrt{\lambda}}{\pi} \operatorname{ctg} \left[\frac{\pi\sqrt{\alpha}}{2(n+1)}\right], \quad n^2 < \lambda < (n+1)^2$$

$$\beta_n = \pm 2\lambda \left(1 - n\Big/\sqrt{\lambda}\right), \quad \lambda > 1, \qquad n \geqq 1$$

$$\alpha = 0 \qquad \text{for } n = 0,$$

and is such that none of these curves is contained in its interior. The open region bounded by these curves is maximal; for any point outside or on the boundary of this region, there exists a function ψ of class one such that not all solutions of (5.1) are bounded.

<u>Theorem 5.3</u>. Let κ be a real variable, $0 \leqq \kappa^2 < 1$, and let

$$K = \int_0^{\pi/2} \frac{ds}{\sqrt{1-\kappa^2\sin^2 s}}, \qquad E = \int_0^{\pi/2} \sqrt{1-\kappa^2\sin^2 s}\; ds .$$

Then the curves defined for n=0,1,2,... by

$$\beta_{n+1} = \pm 8.3^{-1/2}\pi^{-2}(n+1)^2 K\left[K^2(\kappa^2-1) + 2KE(2-\kappa^2) - 3E^2\right]^{1/2}$$

$$\lambda_{n+1} = 4\pi^{-2}(n+1)^2\left[K^2(\kappa^2-1) + 2KE\right],$$

$$\lambda > 0,$$

bound the region of absolute stability of the functions of class 2. The boundary points do not belong to the region, since for

$$\lambda + Q(x)$$

$$= 4\pi^{-2}(n+1)^2K^2(1+\kappa^2) - 8\pi^{-2}(n+1)^2\kappa^2K^2\,\mathrm{sn}^2\left(\frac{2}{\pi}(n+1)Kx\right)$$

the differential equation (5.1) has only one periodic solution (and, therefore, at least one unbounded solution). The periodic solution (with period π or 2π) is

$$y = \mathrm{sn}\ t, \qquad t = 2(n+1)Kx/\pi$$

where sn t is the Jacobian elliptic function with module κ and period 4K.

<u>Theorem 5.4</u>. For the functions of class ∞, the region of absolute stability is bounded by the curves

$$\left(\lambda_{n+1} + \beta_{n+1}\right)^{\frac{1}{2}} \ \mathrm{tg}\left[\pi\sqrt{\lambda_{n+1} + \beta_{n+1}}\Big/4(n+1)\right]$$

$$= \left(\lambda_{n+1} - \beta_{n+1}\right)^{\frac{1}{2}} \ \mathrm{ctg}\left[\pi\sqrt{\lambda_{n+1} - \beta_{n+1}}\Big/4(n+1)\right],$$

where

$$n = 0,1,2,\ldots$$

and where the region does not contain any one of these curves in its interior. If one of the square roots should be imaginary, the functions tg and ctg have to be replaced by the corresponding hyperbolic functions.

Borg also observed that Liapounoff's Theorem 2.13 can be derived from Theorem 5.2, and that Theorems 5.3 and 5.4 contain, respectively, the following results as special cases:

If $Q(x) + \lambda > 0$ and

$$\int_0^\pi \left[\lambda + Q(x)\right]^2 dx < \frac{64}{3\pi^2} \left\{ \int_0^{\pi/2} \frac{ds}{\sqrt{1+\sin^2 s}} \right\}^4 ,$$

then the solutions of (5.4) are stable. The same is true if

$$\max \left[Q(x) + \lambda\right] < 1 .$$

Finally, Borg indicates how the following result due to A. Beurling (unpublished) can be derived from his arguments:

<u>Theorem 5.5</u>. If a,b are real numbers and

$$a^2 \leqq \lambda + Q(x) \leqq b^2 ,$$

then the solution of (5.1) will be stable for all possible $Q(x) + \lambda$ satisfying this condition if and only if the interval (a^2,b^2) does not contain the square of an integer.

5.3 Equations with two or more parameters

Strutt, 1944, has investigated the differential equations

$$(5.10) \qquad y'' + \left[\lambda + \gamma\,\Phi(x)\right] y = 0$$

and

$$(5.11) \qquad y'' + \left[\lambda + \gamma_1\Phi_1(x) + \gamma_2\Phi_2(x)\right] y = 0$$

where Φ, Φ_1, Φ_2 are real periodic functions of x with period ξ and λ, γ, γ_1, γ_2 are parameters. His approach is the following one: Let σ be a constant such that $|\sigma| = 1$. Then Strutt asks for the values of λ, γ or (in the case of 5.11) of $\lambda, \gamma_1, \gamma_2$ for which (5.10) or (5.11) has a solution $y = w(x)$ such that

$$(5.12) \qquad w(x+\xi) = \sigma w(x) \;, \quad w'(x+\xi) = \sigma w'(x) \;,$$

where w does not vanish identically. In earlier papers (Strutt, 1943), it had been demonstrated that, for given values of γ_1, γ_2, the smallest value of λ for which (5.11) has a solution satisfying (5.12) is the minimum assumed by the expression

$$M = -\int_0^\xi \bar{v}\, v''\, dx - \gamma_1 \int_0^\xi \Phi_1(x)\, \bar{v}\, dx - \gamma_2 \int_0^\xi \Phi_2\, \bar{v}\, dx \;,$$

for the set of all two times differentiable functions v for which

$$v(x+\xi) = v(x) \;, \quad v'(x+\xi) = v'(x)$$

and

$$\int_0^{\xi} v \, \bar{v} \, dx = 1 .$$

Here a bar denotes the conjugate complex quantity. Higher "eigenvalues" λ can be found by Maximum - Minimum conditions of a more complicated nature. Another approach utilized by Strutt, 1944, is the method of linear integral equations. The function $w(x)$ will satisfy the equation

$$(5.13) \qquad w(x) = \int_0^{\rho} \left[\gamma_1 \Phi_1(t) + \gamma_2 \Phi_2(t) \right] G(x,t) \, w(t) \, dt ,$$

when $G(x,t)$ is Green's function for the case $\gamma_1 = \gamma_2 = 0$ which, for

$$\rho \sqrt{\lambda} \neq n\pi , \qquad n = 0, \pm 1, \pm 2, \ldots$$

is given by

$$G(x,t) = \frac{-\sigma \omega^{-1} \sin \omega(x - t + \xi) + \omega^{-1} \sin \omega(x-t)}{\sigma^2 - 2\sigma \cos \omega\xi + 1}$$

for $x \leqq t$,

$$G(x,t) = \frac{\sigma \omega^{-1} \sin \omega(x - t - \xi) - \sigma^2 \omega^{-1} \sin \omega(x-t)}{\sigma^2 - 2\sigma \cos \omega\xi + 1}$$

for $x \geqq t$

where $\omega = \sqrt{\lambda}$. Strutt, 1944, derives from 5.13 the inequality

$$(5.14)\ \left\{\gamma_1^2 + \gamma_2^2\right\}^{-1} \leqq \int_0^{\xi}\int_0^{\xi}\left(\Phi_1^2(x) + \Phi_2^2(x)\right)\ \left|G(x,t)\right|^2\,dx\,dt$$

Other results obtained by Strutt refer to the shape of the surfaces in the $(\lambda,\gamma_1,\gamma_2)$-space belonging to a constant σ, including asymptotic relations.

Strutt applies his results to differential equations arising from the separation of variables of Laplace's equation in four variables in a coordinate system that is based on confocal paraboloids in a subspace of three dimensions. One of the resulting ordinary differential equations arising from separation of variables is

$$(5.14)\quad y'' + \left\{\eta - \frac{1}{8}\alpha^2 - (p+1)\alpha\cos 2x + \frac{\alpha^2}{8}\cos 4x\right\}y = 0\ ,$$

which is a special case of (5.11). Strutt, 1944, discusses the surfaces in the spaces of the parameters belonging to $\sigma = 1$.

5.4 Remarks on a perturbation method

In a certain sense, most of the methods dealing with Hill's equation may be considered as perturbation methods. The number of papers applying iteration or perturbation methods to technical or physical problems is rather large, and a good survey of the older literature may be found in Erdélyi, 1935. Erdélyi, 1934, 1935 investigates closed electric circuits with time dependent, periodic capacitance, inductance and (Ohmic) resistance, the period being the same for all three quantities. This problem leads (after an application of a Liouville transformation) to an equation of Hill's type, and Erdélyi develops a formal but rather general perturbation theory for it, with a critical commentary on the range of validity of the formulas. We shall indicate briefly what may be found in Erdélyi's papers which are particularly systematic and thorough.

In order to make Erdélyi's papers more accessible, we shall use here the notation that is used there. Consider

(5.15) $$\frac{d^2z}{dx^2} + [\lambda - kr(x)]\, z = 0 , \qquad r(x+2\pi) = r(x) ,$$

define λ' by

$$\lambda' = (n/2)^2 - \lambda , \qquad n = \left[\sqrt{2\lambda}\,\right] ,$$

where $[u]$ means the largest integer not exceedding u. Then, for large values of λ, we may write

$$\lambda' = \lambda_0 + \frac{2}{n}\lambda_1 + \left(\frac{2}{n}\right)^2 \lambda_2 + \cdots$$

$$z(x) = z_0(x) + \frac{2}{n} z_1(x) + \left(\frac{2}{n}\right)^2 z_2(x) + \cdots ,$$

where

$$z_0(x) = \cos\left(\frac{1}{2}nx - \eta\right) , \qquad \eta = \text{constant} ,$$

and where

$$z(x) = z_o(x) + \frac{2}{n} \int_0^x \left[\lambda' + kr(\xi)\right] z(\xi) \sin \frac{n(x-\xi)}{2} d\xi$$

and therefore, for $\nu = 1,2,3,\ldots$

$$z_\nu(x) = \int_0^x \left[\lambda_o + kr(\xi)\right] \sin \frac{1}{2} n(x-\xi) z_{\nu-1}(\xi) d\xi$$

$$+ \sum_{\rho=1}^{\nu-1} \lambda_\rho \int_0^x \sin \frac{1}{2} n(x-\xi) z_{\nu-\rho+1}(\xi) d\xi$$

If λ is a characteristic value, $z(x)$ will be of period 2π or 4π respectively if n is even or odd, and conditions for determining $\eta, \lambda_o, \lambda_1, \lambda_2, \ldots$ may be obtained by postulating that $z_\nu(x)$ is periodic with the corresponding period 2π or 4π. Assume that

$$kr(x) = \sum_{r=0}^{\infty} \gamma_\nu \cos(\nu x - \epsilon_\nu) \ .$$

Then the postulate that $z_1(x)$ shall be periodic leads to the condition

$$\frac{1}{2}(\lambda_o + \gamma_o) \sin\left(\frac{1}{2}nx - \eta\right) + \frac{1}{4}\gamma_n \sin\left(\frac{1}{2}nx + \eta - \epsilon_n\right) \equiv 0 \ ,$$

which gives, as a first approximation

$$(5.16) \qquad \lambda \approx \left(\frac{1}{2}n\right)^2 + \gamma_o + \frac{1}{2}\gamma_n$$

or

$$\lambda \approx \left(\frac{1}{2} n\right)^2 + \gamma_o - \frac{1}{2}\gamma_n . \tag{5.17}$$

The solutions belonging to the λ values in (5.16) and (5.17) are, respectively

$$\cos\left(\frac{1}{2} nx - \frac{1}{2}\epsilon_n\right) , \qquad \sin\left(\frac{1}{2} nx - \frac{1}{2}\epsilon_n\right) .$$

Erdelyi continues by giving an approximate form for the unbounded solutions in the case where λ is in an interval of instability, using a method introduced by Whittaker, 1914. Critical remarks about the method may be found in Erdelyi, 1934, p. 617.

In a subsequent paper (Erdelyi, 1935), the equation

$$y'' + \left[\lambda + \gamma\Phi(x)\right] y = 0 \tag{5.18}$$

with

$$\Phi(x+2\pi) = \Phi(x) , \qquad \int_0^{2\pi} \Phi(x)dx = 0 , \qquad |\Phi(x)| \leqq 1$$

is treated in the two cases where $|\gamma|$ is small compared to 1 and where λ is positive and large compared to 1, $|\gamma|$ being smaller than λ. Conditions for stability are established and approximate formulas for the solutions are given, in the second case (λ large) by using a refined W.K.B. method.

With respect to the approximations given by (5.16) and (5.17), see also Theorem 6.5.

5.5. Application of the Theory of Systems of Differential Equations

Haake, 1952, Gambill, 1954,1955, and Golomb, 1958, applied their results obtained for systems of linear homogeneous differential equations to the theory of Hill's equation which is a special case of a system of two differential equations of the first order. We shall deal here only with the results of Golomb, 1958.

According to Floquet's Theorem (see Section 1.2), Hill's equation

$$y'' + Q(x)y = 0 \tag{5.19}$$

always has a pair of solutions $f_1(x)$, $f_2(x)$ given by

$$f_1(x) = e^{i\alpha x}p_1(x) , \qquad f_2(x) = e^{-i\alpha x}p_2(x) ,$$

where $p_1(x)$ and $p_2(x)$ are periodic with period π. (If $\alpha \neq 0, \pm 1, \pm 2, \ldots$, these solutions are linearly independent). We shall call α and $-\alpha$ the characteristic exponents of (5.19). Assume now that $Q(x)$ is even. Then we may write

$$Q(x) = \omega^2 + 2\theta \sum_{n=1}^{\infty} \gamma_n \cos 2nx \ . \tag{5.20}$$

Equations (5.19) and (5.20) are equivalent to a system of linear differential equations for two functions w_1 and w_2, defined by

$$w_1 = -\omega y + iy' , \qquad w_2 = -\omega y - iy' ,$$

which emerges in the form

$$\frac{\omega}{i}\,\frac{dw_1}{dx} = \omega^2 w_1 + \Theta \sum_{n=1}^{\infty} (\gamma_n \cos 2nx)(w_1 + w_2)$$

$$-\frac{\omega}{i}\,\frac{dw_2}{dx} = \omega^2 w_2 + \Theta \sum_{n=1}^{\infty} (\gamma_n \cos 2nx)(w_1 + w_2) \;.$$

Golomb, 1958 shows that α can be determined from the equation

$$\alpha^2 = \omega^2 + \Theta^2 \sum_{n=-\infty}^{\infty} \frac{\gamma_{|n|}}{(2n+\alpha)^2 - \omega^2} \tag{5.21}$$

$$+ \Theta^3 \sum_{\substack{n,m=-\infty \\ n \neq m}}^{\infty} \frac{\gamma_{|n|}\gamma_{|m|}\gamma_{|n-m|}}{\left[(2n+\alpha)^2 - \omega^2\right]\left[(2m+\alpha)^2 - \omega^2\right]} + \mathcal{O}(\Theta^4) \;,$$

(where $\gamma_o = 0$), provided that Θ is sufficiently small. The resulting value for α, apart from an error of the order of Θ^4, may be written (with the convention $\gamma_o = 0$) in the form

$$\alpha = \pm \left[\omega + \frac{\Theta^2}{4\omega} \sum_{n=1}^{\infty} \frac{\gamma_n^2}{n^2 - \omega^2} \right. \tag{5.22}$$

$$\left. + \frac{\Theta^3}{32\omega} \sum_{\substack{n,m=-\infty \\ n \neq m}}^{\infty} \frac{\gamma_{|n|}\gamma_{|m|}\gamma_{|n-m|}}{nm(n+\omega)(m+\omega)} + \mathcal{O}(\Theta^4) \right]$$

If, in (5.22), α is equated to 0,1,2,..., one obtains relations between ω and Θ which, if satisfied, guarantee that (5.19), (5.20) has a periodic solution of period π or 2π.

Formulas for the solutions of 5.19) and answers to questions of convergence were also given by Golomb, 1958.

6. Discriminant

The discriminant of Hill's equation,

$$(6.1) \qquad y'' + \left[\lambda + Q(x)\right] y = 0$$

where

$$(6.2) \qquad Q(x) = \sum_{n=-\infty}^{\infty} g_n e^{2inx}, \qquad g_{-n} = \bar{g}_n$$

has been introduced in Part I, Section 2. We shall assume that

$$(6.3) \qquad \sum_{n=-\infty}^{\infty} n^2 |g_n|^2 < \infty$$

and that

$$(6.4) \qquad g_o = 0 .$$

We shall use the notations introduced in Section 2.2; in particular, we shall define the quantities Δ_n as in (2.44), (2.45), and we shall use the notation $\sqrt{\lambda} = \omega$ and the definition of the discriminant Δ:

$$\Delta = \sum_{n=0}^{\infty} \Delta_n(\lambda) .$$

We recall that the boundary points $(\neq -\infty)$

$$\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{2n-1}, \lambda_{2n}, \ldots$$

of the even-numbered intervals of instability are the roots of $\Delta - 2 = 0$ and that the boundary points

$$\lambda_1', \lambda_2', \ldots , \quad \lambda_{2n-1}', \lambda_{2n}', \ldots$$

of the odd-numbered intervals of instability of (6.1) are the roots of $\Delta + 2 = 0$. For this reason, a more precise and a more explicit knowledge of the Δ_n seems desirable. The first three theorems of this section will provide information of this type. The last two theorems will be of a more theoretical interest in connection with Borg's Theorem 2.12 and with Erdelyi's approximation formulas (See Formulas 5.16,5.17). The method used in proving Theorem 2.4 (of Part I) can be applied to prove the more general

<u>Theorem 6.1</u>. For real positive $\lambda \to +\infty$,

$$(6.5) \qquad \left| \Delta(\lambda) - \sum_{n=0}^{N} \Delta_n(\lambda) \right| = \mathcal{O}\left(\lambda^{-(N+1)/2}\right) .$$

Theorem 6.1 could also be proven by using the following result which sharpens Theorem 2.5:

<u>Theorem 6.2</u>. Let

$$c(\ell_1, \ell_2, \ldots, \ell_n)$$

be the coefficient of

$$\varepsilon_{\ell_1} \varepsilon_{\ell_2} \cdots \varepsilon_{\ell_n}$$

in Δ_n. Then

$$c(\ell_1, \ldots, \ell_n) = A(\lambda) \cos \pi \sqrt{\lambda} + B(\lambda) \frac{\sin \pi \sqrt{\lambda}}{\sqrt{\lambda}}$$

where A,B are rational functions of λ such that the degree of denominator exceeds that of the numerator

in $A(\lambda)$ by at least $n - \frac{1}{2}\left[n/2\right]$

in $B(\lambda)$ by at least $n - \frac{1}{2}\left[n/2\right] - \frac{1}{2}$,

where $\left[n/2\right]$ is the largest integer not exceeding $n/2$.

It seems that the only way to prove Theorem 6.2 is the one that uses the definition of Δ as an infinite determinant. Using Theorem 2.9 of Part I, and inspecting the determinant $D_o(\lambda)$, we see that Δ_n can be written as an infinite sum of determinants with n rows in columns, each of which is multiplied by the reciprocal of a polynomial in λ. Instead of writing down the general formula, we shall illustrate the situation by writing down the expressions for Δ_3 and Δ_4 which will make apparent the general law for the formation of Δ_n. We have

$$(6.6) \qquad -\Delta_3 \left[4 \sin^2\left(\pi \sqrt{\lambda}/2\right)\right]^{-1} = \sum_{\ell,m=1}^{\infty} \sum_{t=-\infty}^{\infty} \frac{\begin{vmatrix} 0 & g_{\ell} & g_{\ell+m} \\ g_{-\ell} & 0 & g_m \\ g_{-\ell-m} & g_{-m} & 0 \end{vmatrix}}{\left(\lambda - 4t^2\right)\left[\lambda - 4(t+m)^2\right]\left[\lambda - 4(t+\ell+m)^2\right]}$$

and

$$(6.7) \qquad -\Delta_4 \left[4 \sin^2\left(\pi \sqrt{\lambda}/2\right)\right]^{-1} = \sum_{\ell,m,k=1}^{\infty} \sum_{t=-\infty}^{\infty} \frac{1}{P(\lambda;t,k,\ell,m)} \begin{vmatrix} 0 & g_{\ell} & g_{\ell+m} & g_{\ell+m+k} \\ g_{-\ell} & 0 & g_m & g_{m+k} \\ g_{-\ell-m} & g_{-m} & 0 & g_k \\ g_{-\ell-m-k} & g_{-m-k} & g_{-k} & 0 \end{vmatrix}$$

where

$$(6.8) \qquad P(\lambda;t,k,\ell,m) = \left(\lambda - 4t^2\right)\left[\lambda - 4(t+k)^2\right]\left[\lambda - 4(t+k+m)^2\right]\left[\lambda - 4(t+k+m+\ell)^2\right]$$

We know from Theorem 2.5 the general behavior of the coefficient c of a particular product of degree n in the g_ℓ. If we make ω purely imaginary and $\lambda = \omega^2$ negative, i.e. if we put

$$\omega = i\theta, \qquad \lambda = -\theta^2$$

then, for $\theta \to \infty$,

$$(6.9) \qquad c(\ell_1, \ldots, \ell_n)\left[\sin^2 \frac{\pi}{2}\omega\right]^{-1} = \mathcal{O}(\theta^{-d})$$

where d is the smaller of the differences between the degrees of the denominators and the numerators in $A(\omega)$ and $B(\omega)$. (Special attention must be given to the case where these two differences are equal and where d may be greater than either of these two differences because in $A \cos \pi\omega + B \sin \pi\omega$, the asymptotic behavior for $\omega = i\theta$, $\theta \to +\infty$ may be different from that of $A \cos \pi\omega$ and of $B \sin \pi\omega$. This case has to be settled by putting $\omega = (1+i)\theta$, and letting $\theta \to +\infty$.) If we wish to compute d by using (6.9), we may apply the following.

Lemma 6.1. For $\theta \to +\infty$,

$$(6.10) \qquad \sum_{m_1,\ldots,m_r=1}^{\infty} \; \sum_{t=-\infty}^{+\infty} \prod_{\nu=1}^{n}\left[\theta^2 + (t + m_1 + \ldots + m_\nu)^2\right]^{-1} \cdot (\theta^2 + t^2)^{-1}$$

$$= (\theta^{r-2n-1}) .$$

The proof of (6.10) is based on a comparison of the multiple sum in (6.10) with the multiple integral

$$(6.11)\quad \int_0^\infty \cdots \int_0^\infty d\mu_1 \cdots d\mu_r \int_{-\infty}^\infty d\tau (\theta^2+\tau^2)^{-1} \prod_{\nu=1}^n \left[\theta^2 + (\tau+\mu_1+\ldots+\mu_\nu)^2\right]^{-1}$$

By substituting $\theta\tau$, $\theta\mu_1, \ldots, \theta\mu_n$ for $\tau, \mu_1, \ldots, \mu_n$, this integral is seen to be equal to

$$\theta^{r-2n-1} \int_0^\infty \cdots \int_0^\infty d\mu_1 \cdots d\mu_r \int_{-\infty}^\infty d\tau \left(1+\tau^2\right)^{-1} \prod_{\nu=1}^n \left[1 + (\tau+\mu_1+\ldots+\mu_\nu)^2\right]^{-1}$$

and here the integral is a constant independent of θ. On the other hand, it is easily seen that (6.11) is a majorant of the sum in (6.10).

Using Lemma 6.1 we can prove Theorem 6.2 as follows: Looking at the representation of Δ_n as a sum of determinants which are divided by certain products of the type appearing in (6.10) and picking a particular product

$$P = g_{\ell_1} \cdots g_{\ell_n}$$

appearing in these determinants, we see that P can appear in infinitely many of these determinants only if several of the subscripts $\ell_1, \ldots, \ell_n$ are coupled in pairs, say ℓ_1, ℓ_2 and ℓ_3, ℓ_4 etc., such that $\ell_1 = -\ell_2$, $\ell_3 = -\ell_4$, and so on. If we have r such pairs, the coefficient of P in Δ_n is essentially an $(r+1)$-fold infinite sum of the type (6.10). But in a determinant of the type appearing in (6.7) (which illustrates the case $n = 4$) at most $[n/2]$ such pairs can appear, and this proves Theorem 6.2.

Since a skew symmetric determinant of odd order always vanishes, we have as a by-product of our determinantal expression for Δ_n:

Corollary 6.2. If $Q(x)$ in (6.1) is an odd function (that is, if $g_{-n} = -g_n$), then $\Delta_n \equiv 0$ for $n = 1,3,5,\ldots$.

As an aid for computational purposes we state now

Theorem 6.3. For $n = 0,1,2,3,4$, the values of Δ_n are given by the following formulas:

$$\Delta_0(\lambda) = 2\cos\pi\sqrt{\lambda}\ , \qquad \Delta_1(\lambda) = 0$$

$$\Delta_2(\lambda) = \frac{\pi\sin\pi\sqrt{\lambda}}{2\sqrt{\lambda}} \sum_{r=1}^{\infty} \frac{g_r g_{-r}}{\lambda - r^2}$$

$$\Delta_3(\lambda) = \frac{\pi\sin\pi\sqrt{\lambda}}{8\sqrt{\lambda}} \sum_{r,s=1}^{\infty} \frac{(g_r g_s g_{-r-s} + g_{-r} g_{-s} g_{r+s})\,(r^2+s^2+rs-3\lambda)}{(\lambda - r^2)(\lambda - s^2)[\lambda - (r+s)^2]}$$

$$\Delta_4(\lambda) = -\pi^2 \frac{\cos\pi\sqrt{\lambda}}{16\lambda} \left\{ \sum_{r=1}^{\infty} \frac{g_r g_{-r}}{\lambda - r^2} \right\}^2$$

$$- \frac{\pi\sin\pi\sqrt{\lambda}}{64\sqrt{\lambda}} \sum_{r=1}^{\infty} \left(g_r g_{-r}\right)^2 \frac{30\lambda^2 - 70\lambda r^2 - 16r^4}{\left(\lambda - r^2\right)^3\left(\lambda - 4r^2\right)}$$

$$- \frac{\pi\sin\pi\sqrt{\lambda}}{\sqrt{\lambda}} \sum_{\substack{r,s=1 \\ r>s}}^{\infty} g_r g_{-r} g_s g_{-s} R_{r,s}(\lambda)$$

$$+\sum_{k,\ell,m=1}^{\infty}\left(g_k g_\ell g_m g_{-k-\ell-m} + g_{-k}g_{-\ell}g_{-m}g_{k+\ell+m}\right) \quad S_{k,\ell,m}(\lambda)$$

$$+\sum_{k,\ell,m=1}^{\infty}\left(g_{k+m}g_{\ell+m}g_{-m}g_{-k-\ell-m} + g_{-k-m}g_{-\ell-m}g_m g_{k+\ell+m}\right) \quad S_{k,\ell,m}(\lambda)$$

where the functions $R_{r,s}(\lambda)$ and $S_{k,\ell,m}(\lambda)$ are defined by

$$R_{r,s}(\lambda) = \frac{5\lambda^2 - 3\lambda(r^2 + s^2) + r^2 s^2}{\lambda(\lambda - r^2)^2(\lambda - s^2)^2}$$

$$+ \frac{10\lambda - 2r^2 - 2s^2}{(\lambda - r^2)(\lambda - s^2)\left[\lambda - (r-s)^2\right]\left[\lambda - (r+s)^2\right]}$$

and by

$$4S_{k,\ell,m}(\lambda) = \frac{1}{km}\,\sigma(k+m,\ell) - \frac{1}{k(k+m)}\,\sigma(m,\ell) - \frac{1}{m(k+m)}\,\sigma(k,\ell+m)$$

with

$$\sigma(\ell,m) = \frac{\pi \sin \pi\sqrt{\lambda}}{8\sqrt{\lambda}} \quad \frac{3\lambda - \ell^2 - m^2 - \ell m}{(\lambda - \ell^2)(\lambda - m^2)\left[\lambda - (\ell+m)^2\right]}$$

The proof of Theorem 6.3 is based on some juggling with infinite sums. The basic information required is the following one: By expanding $\cos(x\omega)/2$ in a Fourier series in the interval $-\pi < x < \pi$ we find

$$(6.12) \quad \sum_{t=-\infty}^{\infty} \frac{(-1)^t e^{itx}}{\omega^2 - 4t^2} = \frac{t \cos(x\omega/2)}{2\omega \sin(\pi\omega/2)} \quad .$$

By substituting $-(t+\ell)$ for t in the left hand side, the right hand side of (6.12) does not change, and we find thus

$$(6.13) \quad S(\ell) = \sum_{t=-\infty}^{\infty} \left\{ \left[\lambda - 4t^2\right]\left[\lambda - 4(t+\ell)^2\right] \right\}^{-1}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{t=-\infty}^{\infty} \frac{(-1)^t e^{itx}}{\omega^2 - 4t^2} \sum_{t=-\infty}^{\infty} \frac{(-1)^t e^{-itx}}{\omega^2 - 4(t+\ell)^2} \, dx$$

$$= \frac{\pi^2(-1)^\ell}{8\pi\omega^2 \sin^2(\pi\omega/2)} \int_{-\pi}^{\pi} \cos^2(x\omega/2) e^{-i\ell x} dx$$

$$= \frac{\pi \cos(\omega\pi/2)}{4\omega \sin(\omega\pi/2) \left[\omega^2 - \ell^2\right]} \quad .$$

Formula (6.13) allows us to compute the value of $\Delta_2(\lambda)$. For the computation of $\Delta_3(\lambda)$ we need

$$S(m,\ell) = \sum_{t=-\infty}^{\infty} \left\{ \left[\lambda - 4t^2\right]\left[\lambda - 4(t+m)^2\right]\left[\lambda - 4(t+m+\ell)^2\right] \right\}^{-1} ,$$

and we find after an elementary calculation that

$$S(m,\ell) = \frac{S(m)}{4\ell(m+\ell)} + \frac{S(\ell+m)}{4\ell m} - \frac{S(\ell)}{4m(m+\ell)}$$

where $S(\ell)$ is defined by (6.13).

Finally, we need several sums for the computation of D_4, all of which can be obtained through linear combinations from the formulas already used and from their derivatives with respect to λ. For details see Magnus, 1959.

The following theorems are of some interest because of their relation to Borg's Theorem 2.12 and to Erdelyi's approximation formulas.

The method used in the proofs was established by Schaefke, 1954. We have

Theorem 6.4. Let the roots of $\Delta(\lambda) + 2$ and of $\Delta(\lambda) - 2$ be denoted as the beginning of Chapter 6. Then

$$(6.14) \qquad \sum_{n=1}^{\infty} \left[\lambda'_{2n-1} + \lambda'_{2n} - 2(2n-1)^2\right] = 0$$

$$(6.15) \qquad \lambda_0 + \sum_{n=1}^{\infty} \left[\lambda_{2n-1} + \lambda_{2n} - 2(2n)^2\right] = 0 .$$

Whereas Theorem 2.12 shows that for large n, λ'_{2n-1} and λ'_{2n} exceed $(2n-1)^2$ and λ_{2n-1} and λ_{2n} exceed $(2n)^2$, it follows from Theorem 6.4 that the same statement cannot be true for all n.

The next result refers to the case where $g(x)$ is an even function of x, i.e. where

$$Q(x) = Q(-x) \; ; \qquad g_{-n} = g_n \qquad (n=1,2,3,\dots) \; .$$

In this case, the periodic solutions of (6.1) belonging to the characteristic values λ'_{2n-1} and λ'_{2n} are either even or odd periodic functions of x with period 2π. Let

$$\gamma'_1 < \gamma'_2 < \dots < \gamma'_n < \dots$$

be the ordered sequence of those numbers of the set λ'_{2n-1}, λ'_{2n} to which there belongs an even periodic function of period 2π, and

$$\sigma'_1 < \sigma'_2 < \dots < \sigma'_n < \dots$$

be the ordered sequence of the remaining numbers of the λ'_{2n-1}, λ'_{2n} to which there belongs an odd periodic solution of (6.1). Similarly, let

$$\gamma_1 < \gamma_2 < \dots < \gamma_n < \dots$$

be the characteristic values belonging to the set of numbers λ_{2n-1} and λ_{2n} to which there belong even solutions of (2.1) which are of period π and

$$\sigma_1 < \sigma_2 < \dots < \sigma_n < \dots$$

the sequence of characteristic values corresponding to odd solutions of period π. Then we have

Theorem 6.5

$$(6.16) \qquad \sum_{n=1}^{\infty} (\gamma_n' - \sigma_n') = -2 \sum_{n=1}^{\infty} g_{2n-1}$$

$$(6.17) \qquad \lambda_o + \sum_{n=1}^{\infty} (\gamma_n - \sigma_n) = -2 \sum_{n=1}^{\infty} g_{2n} .$$

The following comment should be made concerning Theorem 6.5. For large n,

$$(6.18) \qquad \begin{aligned} |\gamma_n' - \sigma_n'| &= \lambda_{2n}' - \lambda_{2n-1}' \\ |\gamma_n - \sigma_n| &= \lambda_{2n} - \lambda_{2n-1} . \end{aligned}$$

Therefore, the absolute values of the terms in the sums on the left-hand sides of the equations in Theorem 6.5 represent the lengths of the intervals of instability, at least for large values of n. On the other hand, A. Erdélyi, 1934, has shown that, for sufficiently small values of

$$\sum_{n=1}^{\infty} |g_n|^2 ,$$

the n-th interval of instability is approximately of length $2|g_n|$. Theorem 6.5 shows that, although Erdélyi's result may not be exact for the individual intervals of instability, there exist a weaker but exact substitute for it in the form of a relation between sums.

We shall now prove Theorem 6.4, and we shall confine ourselves to a proof of (6.14). We know from Borg's Theorem (2.12) that

$$(6.19) \qquad (2n-1)^2 \left[\lambda'_{2n-1} - (2n-1)^2\right]$$

and

$$(6.20) \qquad (2n-1)^2 \left[\lambda'_{2n} - (2n-1)^2\right]$$

are bounded for $n \to \infty$. We also know that $\Delta + 2$ is a function of order of growth $\frac{1}{2}$ and that therefore for a suitable value of the constant c,

$$(6.21) \qquad \Delta + 2 = c \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda'_{2n-1}}\right)\left(1 - \frac{\lambda}{\lambda'_{2n}}\right)$$

unless one of the λ'_n vanishes, in which case we would have to use a slight modification of this formula which will not affect the proof. We know from Theorem 6.3 that, for large positive values of $\mu = -\lambda$

$$(6.22) \quad \Delta + 2 = 2\cosh^2 \frac{\pi}{2}\sqrt{\mu} \left[1 - \frac{\pi \sinh \pi\sqrt{\mu}/2}{2\sqrt{\mu}\cosh \pi\sqrt{\mu}/2} \sum_{n=1}^{\infty} \frac{|g_n|^2}{\mu + n^2} + \mathcal{O}(\mu^{-5/2})\right]$$

Now consider the behavior of

$$(6.23) \qquad L(\mu) = \frac{d}{d\mu} \log \frac{\Delta + 2}{\cosh^2(\pi\sqrt{\mu}/2)}$$

for $\mu \to \infty$.

From the product representation of $\Delta + 2$ and of $\cosh(\pi\sqrt{\mu}/2)$ we find

$$(6.24) \qquad L(\mu) = \sum_{n=1}^{\infty} \left\{ \frac{1}{\mu + \lambda'_{2n-1}} + \frac{1}{\mu + \lambda'_{2n}} - \frac{2}{\mu + (2n-1)^2} \right\}$$

$$= \sum_{n=1}^{\infty} \frac{\lambda'_{2n-1} - (2n-1)^2}{(\mu + \lambda'_{2n-1})\left[\mu + (2n-1)^2\right]} + \frac{\lambda_{2n} - (2n-1)^2}{(\mu + \lambda'_{2n})\left[\mu + (2n-1)^2\right]}$$

$$= - \sum_{n=1}^{\infty} \left[\lambda'_{2n-1} + \lambda'_{2n} - 2(2n-1)^2\right] \mu^{-2}$$

$$+ \sum_{n=1}^{\infty} \left[\lambda'_{2n-1} - (2n-1)^2\right] \frac{\mu\left[\lambda'_{2n-1} + (2n-1)^2\right] + \lambda'_{2n-1}(2n-1)^2}{\mu^2\left[\mu + (2n-1)^2\right]\left[\mu + \lambda'_{2n-1}\right]}$$

$$+ \sum_{n=1}^{\infty} \left[\lambda'_{2n} - (2n-1)^2\right] \frac{\left[\lambda'_{2n} + (2n-1)^2\right]\mu + \lambda'_{2n}(2n-1)^2}{\mu^2\left[\mu + (2n-1)^2\right]\left[\mu + \lambda'_{2n}\right]} \quad .$$

We wish to show that

$$(6.25) \qquad L(\mu) = \sum_{n=1}^{\infty} \left[\lambda'_{2n-1} + \lambda'_{2n} - 2(2n-1)^2\right]\mu^{-2} + \mathcal{O}(\mu^{-5/2})$$

for $\mu \to \infty$. For this purpose, we have to estimate the last two sums in (6.24), proving that they are of the order of $\mu^{-5/2}$. Using the boundedness of the expression (6.19) we find that

$$(6.26) \qquad \sum_{n=1}^{\infty} \left[\lambda'_{2n-1} - (2n-1)^2\right] \frac{\mu\left[\lambda'_{2n-1} + (2n-1)^2\right] + \lambda'_{2n-1}(2n-1)^2}{\left[\mu + (2n-1)^2\right]\left[\mu + \lambda'_{2n-1}\right]}$$

can be majorized by

$$S = M \sum_{n=1}^{\infty} \frac{\mu + (2n-1)^2}{\left[\mu + (2n-1)^2\right]^2}$$

where M is a suitable constant. By using a standard procedure we see that S can be written in the form

$$(6.27) \qquad 2M \int_0^{\infty} \frac{\mu + t^2}{(\mu + t^2)^2}\, dt + \mathcal{O}(\mu^{-1})$$

Now the integral in (6.27) equals $\pi/2\sqrt{\mu}$ and this proves (6.25).

Next, we shall derive an asymptotic expansion for $L(\mu)$ from (6.22). From some calculations and by using an argument similar to the one we employed in estimating (6.26) we find that

$$(6.28) \qquad L(\mu) = \mathcal{O}(\mu^{-5/2})$$

provided that

$$\sum_{n=1}^{\infty} n^2 |g_n|^2 < \infty .$$

A comparison of (6.25) and (6.28) proves Theorem 6.4.

The proof of Theorem 6.5 may be based on the following remarks:

First, we can show that for large n the pair of numbers γ_n', σ_n' is identical with the pair $\lambda_{2n-1}', \lambda_{2n}'$, apart from the ordering of the numbers. To prove this statement, we will have to consider the equation

$$y'' + \left[\lambda + \epsilon g(x)\right] y = 0 \qquad \left(0 \leqq \epsilon \leqq 1\right)$$

For $\epsilon = 0$, $\lambda_{2n-1}' = \lambda_{2n}' = (2n-1)^2$. If ϵ increases, the difference $\lambda_{2n-1}' - \lambda_{2n}'$ will, in general, be different from zero, but it will stay small and one of the periodic solutions belonging to these two numbers will be odd, the other, even. Since both λ_{2n-1}' and λ_{2n}' will not leave a certain neighborhood of $(2n-1)^2$ bounded by, say, $(2n-1)^2 - \frac{1}{2}$ and $(2n-1)^2 + \frac{1}{2}$, these two numbers will always be the characteristic values belonging respectively to the n-th even and to the n-th odd periodic solution of period 2π, although we do not know whether λ_{2n} belongs to an even or to an odd solution.

From this remark and from Borg'd Theorem 2.12 we see that the series (6.16) and (6.17) converge absolutely and can be majorized by the series $M \sum n^{-2}$ where M is a constant. Furthermore, we know that the numbers

$$\gamma_n', \ \sigma_n', \ \gamma_n, \ \sigma_n \qquad\qquad (\gamma_o = \lambda_o)$$

are respectively the zeros of

$$y_1(\pi/2,\lambda), \ y_2'(\pi/2,\lambda), y_1'(\pi/2,\lambda), \ y_2(\pi/2,\lambda)$$

where y_1, y_2 are the normalized solutions of (6.1) described in Chapter 1. (See Theorem 1.1) From the method of solving (6.1) by iteration, we find that, for $\lambda = -\mu$, μ large and positive

$$(6.29) \qquad y_1(\pi/2,-\mu) = \cosh(\pi\sqrt{\mu}/2)\left[1 - \sum_{n=1}^{\infty} \frac{g_{2n+1}}{\mu + (2n-1)^2} + \mathcal{O}(\mu^{-3/2})\right]$$

$$(6.30) \qquad y_2'(\pi/2,-\mu) = \cosh(\pi\sqrt{\mu}/2)\left[1 + \sum_{n=1}^{\infty} \frac{g_{2n+1}}{\mu + (2n+1)^2} + \mathcal{O}(\mu^{-3/2})\right]$$

$$(6.31) \qquad y_1'(\pi/2,-\mu) = \sqrt{\mu}\ \sinh(\pi\sqrt{\mu}/2)\left[1 - \sum_{n=1}^{\infty} \frac{g_{2n}}{\mu + 4n^2} + \mathcal{O}(\mu^{-3/2})\right]$$

$$(6.32)\qquad y_2(\pi/2,-\mu) = \frac{\sinh \pi\sqrt{\mu}/2}{\sqrt{\mu}}\left[1 + \sum_{n=1}^{\infty} \frac{g_{2n}}{\mu + 4n^2} + \mathcal{O}(\mu^{-3/2})\right]$$

where the $\mathcal{O}(\mu^{-3/2})$-terms may be differentiated with respect to μ, having a derivative of the order of $\mu^{-5/2}$.

Again, the left-hand sides in (6.24) to (6.31) are functions of μ of order of growth $\frac{1}{2}$, and will admit product representations of the type (6.21), e.g. (for $\lambda_o \neq 0$, $\gamma_n \neq 0$, $\sigma_n \neq 0$)

$$(6.33)\qquad y_1'(\pi/2,-\mu) = c(1+\mu/\lambda_o)\prod_{n=1}^{\infty}(1+\mu/\gamma_n)$$

$$(6.34)\qquad y_2(\pi/2,-\mu) = c^*\prod_{n=1}^{\infty}(1+\mu/\sigma_n) \qquad ,$$

where c, c^* are constants. Now we can calculate an asymptotic expansion for

$$\frac{d}{d\mu}\log\frac{y_1'(\pi/2,-\mu)}{y_2(\pi/2,-\mu)}$$

in two different ways, using (6.31) and (6.32) or using (6.33) and (6.34) By equating these two expansions, we find

$$\frac{1}{\mu} + 2\sum_{n=1}^{\infty}\frac{g_{2n}}{\left[\mu + 4n^2\right]^2} + \mathcal{O}(\mu^{-5/2}) = \frac{1}{\mu+\lambda_o} + \sum_{n=1}^{\infty}\frac{\sigma_n - \gamma_n}{(\mu+\gamma_n)(\mu+\sigma_n)}$$

or

$$2 \sum_{n=1}^{\infty} g_{2n} \mu^{-2} + \mathcal{O}(\mu^{-5/2}) = - \left[\lambda_o + \sum_{n=1}^{\infty} (\gamma_n - \sigma_n) \right] \mu^{-2} + \mathcal{O}(\mu^{-5/2})$$

which proves (6.17). Equation (6.16) can be proved in the same manner.

7. Coexistence

7.1. Introduction.

According to Floquet's Theorem (Section 1.2), Hill's equation will, in general, have only one periodic solution (and its constant multiples) of period π or 2π. If it should happen that two linearly independent (and therefore all) solutions of Hill's equation are of period π or 2π, we shall say that two such solutions coexist or we shall call this an instance of coexistence. It should be noted that, according to the Corollary to Floquet's Theorem in Section 1.2, the coexistence problem never arises for solutions of a period $n\pi$ where $n=3,4,5,\ldots$. Also, we should recall here that coexistence of periodic solutions of period π or 2π is equivalent, respectively, to the occurrence of a double root of the equation $\Delta(\lambda) - 4 = 0$ or $\Delta(\lambda) + 4 = 0$. If, say, the n-th root of $\Delta(\lambda) = 4$ is a double root, i.e. if

$$\lambda_{2n-1} = \lambda_{2n} \qquad (n=1,2,3,\ldots)$$

we may say that the 2n-th interval of instability disappears; similarly, if

$$\lambda'_{2n-1} = \lambda'_{2n} , \qquad (n=1,2,3,\ldots)$$

we may say that the (2n-1)-st interval of instability disappears, and these statements are equivalent with the statement that coexistence of periodic solutions of period π or 2π occurs. It may be useful to recall here that λ_0 can never be a double root of $\Delta - 4 = 0$, and that the zero-th interval of instability from $-\infty$ to λ_0 cannot disappear at all.

It had been noted early and proven repeatedly that for Mathieu's equation

$$(7.1) \qquad y'' + (\lambda + a \cos 2x)y = 0 ,$$

no interval of instability can ever disappear unless $\alpha = 0$, in which case only the zero-th interval of instability remains. For a proof see, e.g. MacLachlan, 1947. The methods developed in the following sections would also lead to an easy proof of this fact.

Meissner, 1918, studied the equation

$$(7.2) \qquad y'' + \lambda g(x)y = 0$$

where $g(x)$ is a piecewise constant function assuming two different values in the interval $0 \leqq x \leqq \pi$. For this case, the coexistence problem can be answered completely; a recent discussion of the situation can be found in a paper by Hochstadt (1961) and also in section 8.2. However, it is impossible to transform (7.2) into the standard form of Hill's equation used throughout the present monograph, since the Liouville Transformation as defined by equations (3.6) to (3.9) cannot be applied to non-differentiable functions.

For various cases of Hill's equation with an analytic coefficient $Q(x)$ the coexistence problem has been investigated. Of these, the case of Lamé's equation is particularly important; it offers the simplest example of an equation of Hill's type for which all but a finite number k of intervals of instability disappear. In view of the generalized Fourier Theorem 2.16, these cases provide particularly simple analogs to the ordinary Fourier theorem, since the integration has to be extended over a finite number of intervals only. Lame's equation will be discussed in Section 7.3.

It has been shown by S. Winkler (1958) that all known cases of equations of Hill's type with analytic coefficients and with a decidable coexistence problem are special cases of a four parametric equation that was called "Ince's Equation" by Winkler. It is the most general equation to which Ince's method of three term recurrence relations can be applied. The theory of Ince's equation will be developed in Section 7.2. The equation can be transformed into the standard form (3.2) by using the substitution (3.3) with a properly defined $A(x)$.

It is necessary to solve a transcendental equation if we wish to decide whether an equation of Hill's type has a periodic solution of period π or 2π. However, once it is known that a given equation of Ince's type has a solution of period π or 2π, in general merely the solution of a problem of linear algebra is required in order to decide whether all solutions of Ince's equation are periodic with the same period. In addition, there exists a very simple necessary condition for the parameters of the equation that must be satisfied if coexistence can occur, regardless of the existence or non-existence of at least one periodic solution with period π or 2π.

7.2 Ince's Equation

We shall discuss the coexistence problem for Ince's equation that will be written in the form

$$(7.3) \qquad (1 + a \cos 2x)y'' + b(\sin 2x)y' + (c + d \cos 2x)y = 0 ,$$

where a,b,c,d are real parameters and $|a| < 1$. The transformation

$$y = (1 + a \cos 2x)^{b/(4a)} z$$

carries (7.3) into the equation

$$(7.4) \qquad z'' + \frac{\alpha + \beta \cos 2x + \gamma \cos 4x}{(1 + a \cos 2x)^2} z = 0 ,$$

(provided that $a \neq 0$), where

$$\alpha = c - ab - b^2/8 + ad/2$$

$$\beta = d + ac - b$$

$$\gamma = ad/2 + b^2/8 \quad ;$$

and (7.4) has the form of Hill's equation. The case where $a = 0$ can be dealt with by the substitution

$$y = w \exp\left[(b \cos 2x)/4\right]$$

which leads to

$$(7.5) \qquad w'' + \left[c - b^2/8 + (d - b) \cos 2x + (b^2/8) \cos 4x\right] w = 0 ,$$

and this is the most general equation of Hill's type where the coefficient of w is of the form

$$\alpha + \beta \cos 2x + \gamma \cos 4x$$

with $\gamma \geqq 0$.

However, we shall base our discussion on Equation (7.3). The first result we shall prove is

Theorem 7.1. If Ince's equation (7.3) has two linearly independent solutions of period π, then the polynomial

$$Q(\mu) = 2a\mu^2 - b\mu - d/2$$

has a zero at one of the points

$$\mu = 0, \pm 1, \pm 2, \ldots .$$

If (7.3) has two linearly independent solutions of period 2π, then

$$Q^*(\mu) = 2Q(\mu - \tfrac{1}{2}) = a(2\mu - 1)^2 - b(2\mu - 1) - d$$

vanishes for one of the values of $\mu = 0, \pm 1, \pm 2, \ldots$.

To prove Theorem 7.1, we need the following

Lemma 7.1. Let $P(\mu)$ be a polynomial of first or second degree in μ and let D_n $(n=0,1,2,\ldots)$ be elements of a sequence which satisfy the recurrence relations.

$$(7.6) \qquad P(n)D_n = P(-n-2)D_{n+1}, \qquad n=0,1,2,\ldots$$

and the limit relations

$$(7.7) \qquad \lim_{n \to \infty} n^p D_n = 0 \qquad \text{for } p=1,2,3,\ldots .$$

Then either all D_n vanish or $P(\mu)$ has an integral root.

Proof: If $P(\mu)$ does not have an integral root then either all of the D_n are zero or none of them vanishes. (See (7.6)). Assume that none of the D_n vanishes and that $P(n)$ and $P(-n-1)$ are different from zero for all integers n. Then we can obtain a contradiction to (7.7) and therefore prove our lemma. To do this, we choose a fixed integer $k > 0$ and derive from (7.6) that

$$(7.8) \qquad D_k = \frac{P(-k-2)P(-k-3)\ \ldots\ P(-k-r-2)}{P(k)\ P(k+1)\ \ldots\ P(k+r)}\ D_{k+r+1}$$

$$(r=0,1,2,\ldots) .$$

Assume that $P(\mu)$ is of second degree and that

$$P(\mu) = A(\mu-\lambda_1)(\mu-\lambda_2) .$$

Then (7.8) can be written as

$$(7.9) \qquad \frac{\Gamma(k+2+\lambda_1)\ \Gamma(k+2+\lambda_2)}{\Gamma(k-\lambda_1)\ \Gamma(k-\lambda_2)}\ D_k = \frac{\Gamma(k+3+\lambda_1+r)\ \Gamma(k+3+\lambda_2+r)}{\Gamma(k-\lambda_1+1+r)\ \Gamma(k-\lambda_2+1+r)}\ D_{k+r+1} ,$$

and if $P(\mu)$ is of first degree and

$$P(\mu) = A(\mu - \lambda) ,$$

then

$$(7.10) \qquad \frac{\Gamma(k+2+\lambda)}{\Gamma(k - \lambda)} D_k = (-1)^{r+1} \frac{\Gamma(k+3+\lambda+r)}{\Gamma(k-\lambda+1+r)} D_{k+r+1} .$$

Here Γ denotes the Gamma function. As a simple consequence of Stirling's Formula we have the asymptotic relations

$$\lim_{t \to +\infty} \left| \frac{\Gamma(t+\rho)}{\Gamma(t)} t^{-\rho} \right| = 1 ,$$

$$(7.11) \qquad \lim_{t \to +\infty} \left| \frac{\Gamma(t+\rho)}{\Gamma(t-\rho)} t^{-2\rho} \right| = 1 .$$

where ρ is any fixed real or complex number. Putting $t = k+r+1$, $\rho = \lambda+1$, we find from (7.10) and (7.7) by letting $r \to \infty$ that

$$(7.12) \qquad \frac{\Gamma(k+1+\lambda)}{\Gamma(k-\lambda)} D_k = 0$$

Similarly, we find that the left hand side in (7.9) must vanish, and this proves Lemma 7.1. Incidentally, we can use the vanishing of the left hand sides of (7.9) and (7.10) to prove

<u>Lemma 7.2</u>. If the numbers D_n satisfy the recurrence relations (7.6) and the limit relation (7.7), then $D_n = 0$ for $n > k_o$, where k_o is the largest non-negative integer such that $P(k_o) = 0$. If no such integer exists, all D_n vanish.

We shall apply Lemma 7.2 to the proof of a later theorem. To prove Theorem 7.1, we shall derive now

Lemma 7.3. If Ince's equation has two linearly independent solutions of period π or 2π, then two solutions y_1, y_2 can be found such that either

$$(7.13)\qquad y_1 = \sum_{n=0}^{\infty} A_{2n} \cos 2nx , \qquad y_2 = \sum_{n=1}^{\infty} B_{2n} \sin 2nkx$$

or

$$(7.14)\qquad y_1 = \sum_{n=0}^{\infty} A_{2n+1} \cos(2n+1)x , \quad y_2 = \sum_{n=0}^{\infty} B_{2n+1} \sin(2n+1)x$$

where, for every positive exponent p,

$$(7.15)\qquad \lim_{n\to\infty} n^p A_{2\lambda} = \lim_{n\to\infty} n^p B_{2n} = \lim_{n\to\infty} n^p A_{2n+1} = \lim_{n\to\infty} n^p B_{2n+1} = 0.$$

Proof: We know from Theorem 2.1 that an equation of Hill's type cannot have a solution of period π and a solution of period 2π (which is not of period π). Since Ince's equation can be transformed into Hill's equation by multiplication of y with a function of period π, the same is true of Ince's equation. Also, we can apply to Ince's equation (7.3) the results of Theorem 1.1, since (7.3) can be transformed into a symmetric equation of Hill's type. Therefore, (7.3) has an even and an odd solution, both with the same period, and if the period is 2π, the solutions must multiply by (-1) if x is increased by π.

These remarks prove (7.13) and (7.14). To prove (7.15), we observe that the solutions of (7.3) must be analytic in a strip of constant width in the complex x-plane that contains the real axis. Therefore, the series in (7.13) and (7.14) must converge for $x = x_1 + ix_2$, where x_1, x_2 are real, x_1 is arbitrary, and x_2 ranges over a sufficiently small interval defined by

$$|1 + a \cos (x_1 + ix_2)| > 0 .$$

Therefore, there exists a constant $M > 1$ such that, for instance,

$$\lim_{n \to \infty} M^n A_{2n} = 0 ,$$

and this fact implies immediately the first limit relation in (7.15). This completes the proof of Lemma 7.3. Now we need

<u>Lemma 7.4</u>. If Ince's Equation (7.3) has the solutions y_1, y_2 defined by 7.13, then the $A_{2\lambda}, B_{2\lambda}$ satisfy the recurrence relations

$$(7.16) \quad -cA_0 + Q(-1)A_2 = 0$$

$$(7.17) \quad Q(n-1)A_{2n-2} + (4n^2-c)A_{2n} + Q(-n-1)A_{2n+2} = 0, \quad (n=1,2,3,\dots)$$

$$(7.18) \quad (2^2 - c)B_2 + Q(-2)B_4 = 0$$

$$(7.19) \quad Q(n-1)B_{2n-2} + (4n^2 - c)B_{2n} + Q(-n-1)B_{2n+2} = 0, \quad (n=2,3,4,\dots)$$

and if (7.3) has the solutions defined by (7.14), then the recurrence relations

(7.20) $\left[Q^*(o) - 2(c-1)\right] A_1 + Q^*(-1)A_3 = 0$

(7.21) $Q^*(n)A_{2n-1} + 2\left[(2n+1)^2 - c\right] A_{2n+1} + Q^*(-n-1)A_{2n+3} = 0$

$(n=1,2,3,\ldots)$

and

(7.22) $\left[-Q^*(o) - 2(c-1)\right] B_1 + Q^*(-1)B_3 = 0$

(7.23) $Q^*(n)\, B_{2n-1} + 2\left[(2n+1)^2 - c\right] B_{2n+1} + Q^*(-n-1)B_{2n+3} = 0$

hold for $n=1,2,3,\ldots$.

The proof is obvious if we substitute the series in question into the differential equation. Our last step in proving Theorem 7.1 may be formulated as

Lemma 7.5. Let $D_0, D_1, D_2, \ldots$ and $D_0^*, D_1^*, D_2^*, \ldots$ be defined respectively by

$$D_0 = A_0B_2 , \qquad D_n = -A_{2n+2}B_{2n} + B_{2n+2}A_{2n} ,$$

$(n=1,2,3,\ldots)$

and

$$D_0^* = A_1B_1 , \qquad D_n^* = -A_{2n+1}B_{2n-1} + A_{2n-1}B_{2n+1}.$$

$(n=1,2,3,\ldots)$.

Then the following recurrence relations hold:

$$(7.24) \qquad Q(n)D_n = Q(-n-2)D_{n+1}, \quad (n=0,1,2,\ldots)$$

$$(7.25) \qquad 2Q^*(o)\, D_o^* = Q^*(-1)D_1^*$$

$$(7.26) \qquad Q^*(n)D_n^* = Q^*(-n-1)\, D_{n+1}^* \quad (n=1,2,3,\ldots)\,.$$

The proof is immediate from the recurrence relations of Lemma 7.4.

Now the proof of Theorem 7.1 is easy. Lemmas 7.1 and 7.3 show that all D_n of Lemma 7.5 must vanish if $Q(\mu)$ does not have a root which is an integer. But then $D_o = A_oB_2 = 0$ and either $B_2 = 0$ or $A_0 = 0$. If $B_2 = 0$, the recurrence relations of Lemma 7.4 show that all B_{2n} must vanish (since $Q(n-1) \neq 0$), and therefore the series y_2 in (7.13) is identically zero and y_1, y_2 are not linearly independent. If $A_o = 0$, it follows that y_1 vanishes identically.

Since the proof of Lemma 7.1 would also go through if, in (7.6), we would replace $P(-n-2)$ by $P(-n-1)$, it follows from a modified Lemma 7.1 and from Lemma 7.3 that all the D_n^* in Lemma 7.5 must vanish when $Q^*(\mu)$ has an integral root. Again $D_o^* = A_1B_1 = 0$ implies that either all A_{2n+1} or all B_{2n+1} vanish which, in turn, would contradict Lemma 7.3. This completes the proof of Theorem 7.1.

We have now a necessary condition for coexistence of two periodic solutions of (7.3) with periods π or 2π, and this condition can be stated even without knowing whether (7.3) has at least <u>one</u> periodic solution or not. In order to obtain sufficient conditions for the coexistence of periodic solutions of Ince's equation we have to assume that the values of the parameters are such that at least one solution with period π or 2π exists. We shall settle here first the question of the existence of such values of the parameters by proving

<u>Theorem 7.2</u>. For any given real values of the parameters $\underline{a}$,b,d (with $|a| < 1$), there exist infinitely many values of the parameter c such that Ince's equation has an even or an odd periodic solution of period π or 2π.

Before proving Theorem 7.2 we note that the parameter c does not enter into the necessary conditions (as stated in Theorem 7.1) for Ince's equation to have periodic solutions of period π or 2π. To prove Theorem 7.2, we shall write (7.4) in the form

$$(7.27) \qquad z'' + \left[\frac{c}{1 + a\cos 2x} + \frac{\rho + \sigma\cos 2x + \tau\cos 4x}{(1 + a\cos 2x)^2}\right] z = 0 ,$$

where ρ, σ, τ do not depend on c. Let us denote the coefficient of z by $H(x)$, and let λ^2 be the minimum and Λ^2 the maximum of H in $0 \leqq x \leqq \pi$. Obviously, both λ^2 and Λ^2 increase as c increases, and they tend to infinitey as $c \to +\infty$. Let z_1 and z_2 be the standard solutions of (7.27) defined by $z_1(o) = z_2'(o) = 1$ and $z_1'(o) = z_2(o) = 0$. The number of zeros of z_1, z_2, z_1', z_2' in the interval $0 \leqq x \leqq \pi/2$ is then majorized or minorized respectively by the number of zeros of the corresponding solutions of $z'' + \Lambda^2 z = 0$ or $z'' + \lambda^2 z = 0$.

Since the zeros of the solutions depend continuously on c, it follows that, with increasing c, we must obtain infinitely many values for c such that, for the solutions of (7.27), one of the quantities

$$z_1(\pi/2),\ z_1'(\pi/2),\ z_2(\pi/2),\ z_2'(\pi/2)$$

vanishes. Now an application of Theorem 1.1 will prove Theorem 7.2.

We shall now try to find out under which circumstances the necessary conditions of Theorem 7.1 for coexistence are also sufficient. We shall need the following

Definition: A solution of Ince's equation (7.3) that is given by a series of type (7.13) is called finite of order k if A_{2k} or B_{2k} is different from zero, but if all A_{2n} or B_{2n} with $n > k$ vanish. Similarly, y_1 (or y_2) in (7.14) will be called finite of order k if A_{2k+1} (or B_{2k+1}) is $\neq 0$, but $A_{2n+1} = 0$ (or $B_{2n+1} = 0$) for $n > k$.

Our results will be stated by formulating several theorems.

Theorem 7.3. If $Q(\mu)$, as defined in Theorem 7.1, has a non-negative integral root, and if k_o is the largest such root, then Ince's equation will have two linearly independent solutions of period π provided that one such solution (of type (7.13)) exists that is either infinite or finite of an order $k > k_o$. Similarly, two linearly independent solutions of period 2π will exist if $Q^*(\mu)$ has an integral non-negative root k_o^* (and no larger one), provided that one solution of type (7.14) exists that is infinite or of finite order $k^* > k_o^*$.

Proof. Consider the case where Ince's equation has a solution

$$y_2 = \sum_{n=1}^{\infty} B_{2n} \sin 2nx$$

such that $B_{2k} \neq 0$ for a $k > k_o$. In general, we shall define the solution

$$y_1 = \sum_{n=0}^{\infty} A_{2n} \cos 2nx$$

in the following manner: Let $A_{2n} = B_{2n}$ for $k > k_o$. Our assumptions then guarantee that y_1 is not identically zero and that the series defining it converges everywhere. If y_1 is to be a solution of (7.3), the recurrence relations (7.16), (7.17) must be satisfied. For $n > k_o$ this will be automatically true since the B_{2n} satisfy the same recurrence relations as the A_{2n}. Note that this is true even if $k_o = 0$, because then (7.18) has the same shape as (7.17) for $n = 1$. For $n \leqq k_o$, we must determine the A_{2n} from the following system of linear equations

$$(7.28) \qquad -cA_o + Q(-1)A_2 = 0$$

and, if $k_o > 0$,

$$(7.29) \qquad Q(n-1)A_{2n-2} + (4n^2 - c)A_{2n} + Q(-n-1)A_{2n+2} = 0 \qquad (n=1,\ldots,k_o-1)$$

and finally

$$(7.30) \qquad Q(k_o-1)A_{2k_o-2} + (4k_o^2 - c)A_{2k_o} = -Q(-k_o-1)B_{2k_o+2} \quad .$$

These are exactly $k_o + 1$ linear equations for the unknowns $A_o, \ldots, A_{2k_o}$. In general, they will have a unique solution, namely when the determinant of the matrix K of the coefficients of the unknowns is different from zero. In this case we have constructed y_1 in the required manner. However, it may happen that K has determinant zero. Then we shall define y_1 as follows: Determine $A_o, \ldots, A_{2k_o}$ as a non-trivial solution of the homogeneous equations (7.28), (7.29), (7.30), (replacing B_{2k_o+2} by zero). For $n > k_o$, put $A_{2n} = 0$. Since $Q(k_o) = 0$, this leads to a non-trivial solution of our system of recurrence relations and to a solution y_1 of finite order $\leqq k_o$.

A similar argument can be used to prove Theorem 7.3 in the remaining three cases where another one of the four series in (7.13) and (7.14) is assumed to be a solution of (7.3). The case where $Q(o) = 0$ and a series of type y_1 in (7.13) is a given periodic solution requires special attention but is perfectly trivial.

Having disposed of the case where Q or Q^* has a non-negative integral zero, we can now discuss the situation described in the following

Theorem 7.4. Assume that $Q(\mu)$ (as defined in Theorem 7.1) has one or two negative integral zeros but none that is $\geqq 0$. Let $-k' - 1$ denote this zero or one of them if there are two, where $k' = 0,1,2,\ldots$. Then Ince's equation will never have a solution of period π that is of finite order. If it has one (infinite) solution y of period π, it will have two linearly independent ones if, and only if, the coefficient of $\cos 2kx$ (for an even y) or of $\sin 2k'x$ (for an odd y) in the expansion (7.13) vanishes. In this case all Fourier coefficients of the periodic solution with an index less than $2k'$ also vanish.

Proof: Since $Q(\mu) \neq 0$ for $\mu=0,1,2,\ldots$, it follows from the recurrence relations (7.16) to (7.19) that all A_{2n} (or all B_{2n}) vanish for $n \leqq \ell$ if $A_{2\ell+2}$ and $A_{2\ell+4}$ (or $B_{2\ell+2}$ and $B_{2\ell+4}$) vanish. Therefore, periodic solutions of finite order cannot exist. Assume now that (7.3) has solutions y_1 and y_2 of type (7.13). According to Lemma 7.2, all D_n vanish. In particular, $A_0B_2 = 0$. Assume $A_0 = 0$, and let $A_{2\ell}$ be the first one of the A_{2n} that does not vanish. A glance at (7.17) shows that then $\ell = k' + 1$, where $Q(-k' - 1) = 0$. Since, according to Lemma 7.2,

$$A_{2\ell-2} B_{2\ell} = A_{2\ell} B_{2\ell-2} ,$$

and since $A_{2\ell-2} = 0$, it follows that $B_{2\ell-2} = 0$. A similar argument is valid if $B_2 = 0$. Therefore the conditions for coexistence stated in Theorem 7.4 are necessary ones. But they are also sufficient. For suppose again that

$$A_0 = A_2 = \ldots = A_{2\ell-2} = 0, \quad B_{2\ell-2} = 0 ,$$

and that (7.3) has a solution y_2 of type (7.13). Then we may choose y_1 to be defined by

$$y_1 = \sum_{n=k'+1}^{\infty} B_{2n} \cos 2nx$$

and the Fourier coefficients of y_1 will satisfy (7.16) and (7.17), since those of y_2 satisfy (by assumption)(7.18) and (7.19). We see easily that $B_{2\ell-2} = 0$ implies that the previous B_{2n} also vanish.

A result similar to that of Theorem 7.4 holds for solutions of period 2π. We have:

<u>Theorem 7.4*</u> Assume that $Q^*(\mu)$ (as defined in Theorem 7.1) has

one or two negative integral roots but none that is $\geqq 0$. Let $-k^* - 1$ denote one of these roots, where $k^* = 0,1,2,\ldots$. Then Ince's equation will never have a solution of finite order that is of period 2π. If it has at least one (infinite) solution y of period 2π, it will have two such solutions that are linearly independent if and only if the coefficient of $\cos(2k^*+1)x$ (if y is even) or if $\sin(2k^*+1)x$ (if y is odd) in the Fourier expansion of y vanishes.

The proof follows exactly the line of the proof of Theorem 7.4 and will be omitted here.

Theorems 7.1 to 7.4* allow us to decide for (7.3) whether two linearly independent solutions of period π or 2π exist provided that one such solution is known, unless the known solution is of finite order and there do not exist two linearly independent solutions of finite order.

Because of the importance of solutions of finite order for the coexistence problem we shall prove

<u>Theorem 7.5</u> A necessary condition for Ince's equation to have two linearly independent solutions of finite order is that $Q(\mu)$ (for period π) or $Q^*(\mu)$ (for period 2π) has two integral roots, at least one of which is positive The order of the finite solutions cannot exceed the largest positive root of $Q(\mu)$ or $Q^*(\mu)$.

<u>Proof</u>. As in the proof of Theorem 7.4 it follows from the recurrence relations (7.16) to (7.22) that $A_{2\ell}$ vanishes if $A_{2\ell+2}$ and $A_{2\ell+4}$ vanish, unless $Q(2\ell) = 0$. Similar arguments hold for the 3 other solutions

of type (7.13), (7.14), and therefore a solution of finite order can exist only if $Q(\mu)$ or $Q^*(\mu)$ have a non-negative root, and the order cannot exceed the largest root of this type. An inspection of the recurrence relations shows immediately that at most one solution of finite order can exist if $\mu = 0$ is the largest integral root of $Q(\mu)$ or of $Q(\mu^*)$. Therefore $Q(\mu)$ or $Q^*(\mu)$ must have at least one positive integral root. The rest of Theorem 7.5 will be proven if we can show that $b/(2a)$ must be an integer. Because in this case,

$$\mu^2 - \beta\mu - d/(4a) = 0 , \qquad (\beta = b/(2a))$$

must have an integral root. Since β is integral, it follows that $d/(4a)$ is also an integer, and therefore $Q(\mu)$ has two integral roots. A similar argument works for $Q^*(\mu)$. Therefore, all that remains to be shown is

Lemma 7.6. If (7.3) has two linearly independent solutions of finite order, then $b/(2a)$ is a non-negative integer.

Proof: If y_1, y_2 are solutions of finite order of (7.3), then

$$w = y_1 y_2' - y_2 y_1'$$

is also of finite order. If y_1, y_2 are linearly independent, w does not vanish identically. Also

$$(1 + a \cos 2x) w' + b(\sin 2x) w = 0 ,$$

and therefore we have, with a constant $w_0 \neq 0$,

$$w = w_0 (1 + a \cos 2x)^{b/(2a)} .$$

Obviously, W will have a finite Fourier expansion if and only if $b/(2a) = 0,1,2,3,\ldots$.

In our discussion of Ince's equation, none of the parameters a,b,c,d plays exactly the role of the eigenvalue parameter λ of the standard Hill's equation used in the first chapter. However, in special cases c may play this role, and we then can talk about intervals of stability and instability. The following result prepares the way for the investigations of this type in later sections. We have:

Theorem 7.6. Let a,b,d be such that $Q(\mu)$ or, (for period 2π) $Q^*(\mu)$ has an integral non-negative root k or (for period 2π) k^*. If there are two such solutions, k or k^* shall denote the larger one of the two. Let C be the (infinite) set of all values of c for which, with the given values of a,b,d, (7.3) has at least one periodic solution of period π or 2π. Then there will be at most k+1 or (for period 2π) k^*+1 values $c_o,\ldots,c_k$ or $c_o^*,\ldots,c_{k^*}^*$ in C for which (7.3) has a solution of finite order. For all other values of $c \in C$, Ince's equation will have two linearly independent solutions of period π or 2π.

Proof: According to Theorem 7.3, Ince's equation will always have two linearly independent solutions of period π or 2π, if the assumptions of Theorem 7.6 are satisfied and if one solution of period π or 2π is given that is of infinite order. According to Theorem 7.6, the order of a finite solution cannot exceed k or k^*. All we have to show now is that there are at most $k+1$ (or k^*+1) different values of c for which such a finite solution can exist. By inspecting the recurrence relation (7.16) to (7.22) we see that the existence of such a finite solution is equivalent to the existence of a non-trivial solution for a system of at most $k+1$ (or k^*+1) linear homogeneous equations with an equal number of unknowns.

In turn, the existence of such a non-trivial solution is equivalent to the vanishing of the determinant of the system. Since every row of the determinant in question involves c linearly, the determinant is a polynomial on c of a degree $\leqq k+1$ (or k^*+1) and does not have more zeros than its degree indicates. This proves Theorem 7.6.

We can prove an analog to Theorem 7.6 which shows that, in general, coexistence takes place if $Q(\mu)$ or $Q^*(\mu)$ has a negative integral root. We have:

<u>Theorem 7.7</u>. Let a,b,d be such that $Q(\mu)$ or, (for period 2π), $Q^*(\mu)$ has a negative integral root. Let $-k_o-1$ or (for roots of $Q^*(\mu)$), $-k_o^*-1$ be the smallest of these roots. Then there exist at most k_o+1 or, (for period 2π), k_o^*+1 values of c such that Ince's equation has one periodic solution of period π or 2π but not two linearly independent ones.

<u>Proof</u>: We observe that, in the proof of Theorem 7.4, $B_2,\dots,B_{2\ell-2}$ will satisfy a homogeneous system of linear equations. This can have a non-trivial solution for only a finite number of values of c, as may be seen by the argument used in proving Theorem 7.3.

7.3. Lamé's Equation and Generalizations

Lamé's differential equation may be written in the form

$$(7.31) \qquad y'' + \left[\lambda - m(m+1)k^2 \operatorname{sn}^2 x\right] y = 0 ,$$

where sn x is Jacobi's elliptic function defined by

$$(7.32) \qquad x = \int_0^{\operatorname{sn} x} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} .$$

Here k^2 is called the module of sn x. The basic periods of sn x are denoted by 2K and 2iK; we are only interested in the real period 2K which is given by

$$(7.33) \qquad K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} .$$

Lamé's equation arises from the theory of the potential of an ellipsoid. It is discussed in detail in Chapter 15 of "Higher Transcendental Functions, Vol. III, by A. Erdèlyi et al., 1955. Obviously, Lamé's equation is of the standard type introduced for Hill's equation in Chapter I. We can apply the terminology used there and, in particular, we can talk about intervals of instability on the λ-axis. Our main result will be the following

Theorem 7.8. If and only if m is an integer can periodic solutions of period 2K or 4K of Lamé's equation coexist. If ℓ is defined by $\ell = m$ if m is a non-negative integer and by $\ell = -m - 1$ if m is a negative integer, then Lamé's equation will have at most $\ell+1$ intervals of instability (including the zero-th interval which starts at $\lambda = -\infty$).

Proof: We shall transform (7.31) into an equation of Ince's type by substituting

$$x = \mathrm{am}(t,k) , \tag{7.34}$$

where the function am(t,k) is defined by

$$\frac{dt}{dx} = \left(1 - k^2 \sin^2 t\right)^{1/2} \qquad (k^2 < 1) .$$

Then Lamé's equation takes the form

$$\left(1 - k^2 \sin^2 t\right) \frac{d^2y}{dt^2} - \frac{1}{2} k^2 (\sin 2t) \frac{dy}{dt} + \left[\lambda - m(m+1) k^2 \sin^2 t\right] y = 0 \tag{7.35}$$

which is Ince's equation with

$$a = -b = k^2/(2 - k^2)$$

$$c = \left[2\lambda - m(m+1)k^2\right] / (2 - k^2)$$

$$d = m(m+1)k^2 / (2 - k^2)$$

The period 2K of (7.31) corresponds to π for (7.35). The roots of the polynomial $Q(\mu)$ belonging to (7.35) are

$$\mu_1 = m/2 , \qquad \mu_2 = -(m+1)/2$$

and the roots of $Q^*(\mu)$ are

$$\mu_1^* = (m+1)/2 , \qquad \mu_2^* = -m/2 .$$

None of the numbers $\mu_1, \mu_2, \mu_1^*, \mu_2^*$ can be an integer unless m is an integer. We may assume that $m \geqq 0$, since Lamé's equation remains unchanged if m is replaced by $-m-1$. If m is an even integer, $m = 2\ell'$, then

$$\mu_1 = \ell' , \qquad \mu_2^* = -\ell' .$$

According to Theorem 7.6, there will not exist more than $\ell + 1$ values of c (and, therefore, of λ) for which one but not two linearly independent solutions of period π exist. Similarly, it follows from Theorem 7.7 that not more than ℓ' values of c and of λ exist for which (7.35) has one but not two linearly independent solutions of period 2π. Therefore, at most $2\ell' + 1 = m + 1$ intervals of instability may remain. The case where m is odd can be dealt with in the same manner.

A. Erdélyi, 1941, showed also that not fewer than $m+1$ intervals of instability remain if the parameter m in Lamé's equation is a non-negative integer.

There are several generalizations of Lamé's equation which can be treated in the same manner. We shall list here the results briefly, following Winkler, 1958:

<u>The Hermite elliptic equation</u> can be written in the form

$$(7.36) \qquad y'' + \frac{2(r+1)k^2 \operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x} y' + \left[\lambda - (m-r)(m+r+1)k^2 \operatorname{sn}^2 x\right] y = 0$$

where k^2 and sn x are defined as above and

$$\operatorname{cn} x = \left(1 - \operatorname{sn}^2 x\right)^{1/2}, \quad \operatorname{dn} x = \left(1 - k^2 \operatorname{sn}^2 x\right)^{1/2} .$$

the transformation $x = \operatorname{am}(t,k)$ carries (7.36) into

$$(7.37) \quad \left(1 - k^2 \sin^2 t\right) \frac{d^2 y}{dt^2} + \left[(2r+1)k^2 \sin t \cos t\right] \frac{dy}{dt}$$

$$+ \left[\lambda - (m-r)(m+r+1)k^2 \sin^2 t\right] y = 0 .$$

This is Ince's equation (7.3) with

$$a = k^2/(2-k^2)$$

$$b = (2r+1)k^2 \Big/ (2-k^2)$$

$$c = \left[2\lambda - (m-r)(m+r+1)k^2\right] \Big/ (2-k^2)$$

$$d = (m-r)(m+r+1)k^2 \Big/ (2-k^2) .$$

The roots of $Q(\mu)$ are given in this case by

$$\mu_1 = \tfrac{1}{2}(r+m+1) \qquad \mu_2 = \tfrac{1}{2}(r-m)$$

and the roots of $Q^*(\mu)$ are given by

$$\mu_1^* = \tfrac{1}{2}(m+r) + 1 \qquad \mu_2^* = \tfrac{1}{2}(r-m+1).$$

Necessary and sufficient condition for one of them to be an integer is that $r+m$ or $r-m$ is an integer.

<u>The Picard elliptic equation</u> may be written in the form

$$(7.38) \qquad y'' + \frac{k^2 \operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x} y' + \lambda y = 0 .$$

The transformation $x = \operatorname{am}(t,k)$ carries it into Ince's equation with

$$a = b = k^2/(2-k^2)$$

$$c = \lambda/(2-k^2), \quad d = 0 .$$

In this case, $Q(\mu)$ always has the root zero and therefore there exists at most one value of λ for which (7.38) has one but not two linearly independent solutions of period 2K.

Equation (7.36) can be transformed into an equation of the form

$$u'' + \left[H + p \operatorname{sn}^2 x + q(\operatorname{cn}^2 x/\operatorname{dn}^2 x) \right] u = 0$$

with constant H,p,q by putting

$$y = u(\operatorname{dn} x)^{-r-1} .$$

This equation for u was called the "Associated Lamé Equation" by Ince.

7.4. The Whittaker-Hill Equation

The differential equation

$$(7.39) \qquad y'' + \left[\lambda + 4mq \cos 2x + 2q^2 \cos 4x \right] y = 0$$

appeared first in the work of Liapounoff, 1902. It was studied extensively by Whittaker, 1915 in a paper on differential equations whose solutions satisfy homogeneous integral equations. Whittaker remarked that (7.34) is related to the Mathieu equation in the same manner in which the equation for the associated Legendre functions is related to Bessel's equation. Ince, 1923, pointed out that Whittaker's equation (7.39) has the same relation to the confluent hypergeometric equation as the Mathieu equation has to the Bessel equation. Ince, 1926, discussed the real zeros of the solution of Whittaker's equations. Klotter and Kotowski, 1943, conducted extensive numerical calculations in connection with (7.34) which resulted in a stability chart and examples for coexistence of periodic solutions. Humbert, 1926a, obtained the Whittaker equation from a separation of the Laplace equation (in four dimensions) by introducing hypercylinders formed by three-dimensional confocal paraboloids.

The Whittaker equation (7.34) is not the most general Hill's equation with three real parameters. The assumption that q be real enforces that the coefficient of cos 4x is non-negative. It has been shown by Winkler, 1958, that coexistence of periodic solutions of (7.34) cannot occur if the coefficient of cos 4x is replaced by $-2q$ (and if λ m, q are real).

The substitution

$$y = u \exp(q \cos 2x)$$

carries (7.34) into a special case of Ince's equation, namely

$$(7.40) \qquad u'' - 4q(\sin 2x)u' + \left[\lambda + 2q^2 + 4(m-1)q \text{ cox } 2x\right]u = 0 .$$

The polynomials $Q(\mu)$ and $Q^*(\mu)$, as defined in the theory of Ince's equation (Section 7.2), are linear for (7.40); we have

$$Q(\mu) = 4q\mu - 2(m-1)q$$

$$Q^*(\mu) = 4q(2\mu - 1) - 4(m-1)q .$$

If $q \neq 0$ (the case $q = 0$ is trivial), we see that $Q(\mu)$ has the root

$$\mu = \frac{1}{2}(m-1)$$

and $Q^*(\mu)$ has the root

$$\mu^* = m/2 .$$

Now the theory of Ince's equations gives us the following result:

<u>Theorem 7.9</u>. Whittaker's equation (7.39) can have two linearly independent solutions of period π or 2π if and only if m is an integer. If $m = 2\ell$ is even, then the odd intervals of instability on the λ-axis disappear with at most $|\ell| + 1$ exceptions, but no even interval of instability disappears. If $m = 2\ell + 1$ is odd, then at most $|\ell| + 1$ even intervals of instability remain but no odd interval of instability disappears.

7.5. Finite Hill Equations

The solutions of the Ince equation in the case of coexistence are such that the coefficients of the sine series and the coefficients of the cosine series are equal except for at most the first N coefficients, which sometimes all vanish identically. However, for the Whittaker equation, as a result of the transformation required to put it into the form of an Ince equation, these Ince-type solutions do not occur. Instead, the solutions of the Whittaker equation are Ince-type multiplied by an exponential in cos 2x. That Ince-type solutions cannot coexist for a Finite Hill equation is the content of the following theorem.

Theorem 7.10. In the differential equation

$$y'' + (C_0 + C_2 \cos 2x + \dots + C_{2m} \cos 2mx)y = 0 \tag{7.41}$$

where $m > 0$ is any integer, two solutions with period π of the form

$$y_1 = \sum_{0}^{N} A_{2n} \cos 2nx + \sum_{N+1}^{\infty} A_{2n} \cos 2nx \tag{7.42a}$$

and

$$y_2 = \sum_{1}^{N} B_{2n} \sin 2nx + \sum_{N+1}^{\infty} A_{2n} \sin 2nx \tag{7.42b}$$

or two solutions with period 2π of the form

$$y_1 = \sum_{0}^{N} A_{2n+1} \cos(2n+1)x + \sum_{N+1}^{\infty} A_{2n+1} \cos(2n+1)x \tag{7.43a}$$

and

$$y_2 = \sum_{0}^{N} B_{2n+1} \sin(2n+1)x + \sum_{N+1}^{\infty} A_{2n+1} \sin(2n+1)x \tag{7.43b}$$

cannot exist simultaneously.

We assume, of course, $C_{2m} \neq 0$.

<u>Proof.</u> Assume that equations (7.42) coexist for $N = 0$. Then by direct substitution in the differential equation we obtain the following pair of identities:

$$(7.44a) \quad \sum_{0}^{\infty} \Big[(2C_o - 8n^2)A_{2n}\cos 2nx + C_2A_{2n}\cos(2n+2)x + \ldots + C_{2m}A_{2n}\cos(2n+2m)x$$
$$+ C_2A_{2n}\cos(2n-2)x + \ldots + C_{2m}A_{2n}\cos(2n-2m)x\Big] = 0$$

$$(7.44b) \quad \sum_{1}^{\infty} \Big[(2C_o - 8n^2)A_{2n}\sin 2nx \quad C_2A_{2n}\sin(2n+2)x + \ldots + C_{2m}A_{2n}\sin(2n+2m)x$$
$$+ C_2A_{2n}\sin(2n-2)x + \ldots + C_{2m}A_{2n}\sin(2n-2m)x\Big] = 0.$$

The recurrence relations obtained from both (7.44a) or (7.44b) will be identical for $n > m$. Hence

$$(7.45) \quad C_{2m}A_{2n+2m} + \ldots + C_2A_{2n+2} + (2C_o - 8n^2)A_{2n} + C_2A_{2n-2} + \ldots + C_{2m}A_{2n-2m} = 0; \quad (n > m).$$

Now consider the recurrence relations for $n = m$.

$$(7.46a) \quad C_{2m}A_{4m} + \ldots + C_2A_{2m+2} + (2C_o - 8m^2)A_{2m} + C_2A_{2m-2} + \ldots + C_{2m-2}A_2 + C_{2m}A_o = 0$$

$$(7.46b) \quad C_{2m}A_{4m} + \ldots + C_2A_{2m+2} + (2C_o - 8m^2)A_{2m} + C_2A_{2m-2} + \ldots + C_{2m-2}A_2 = 0 .$$

It is clear that $A_o = 0$.

Comparing the relationships for $n = m - 1$ yields $A_2 = 0$, since $\sin(-2x) = -\sin 2x$ and $\cos(-2x) = \cos 2x$. Thus

$$(7.47a)\quad C_{2m}A_{4m-2} + \dots + C_2A_{2m} + \left[2C_o - 8(m-1)^2\right] A_{2m-2} + C_2A_{2m-4} + \dots +$$

$$C_{2m-4}A_2 + C_{2m-2}A_o + C_{2m}A_2 = 0$$

$$(7.47b)\quad C_{2m}A_{4m-2} + \dots + C_2A_{2m} + \left[2C_o - 8(m-1)^2\right] A_{2m-2} + C_2A_{2m-4} + \dots +$$

$$C_{2m-4}A_2 - C_{2m}A_2 = 0 .$$

But $A_o = 0$, and therefore, subtracting (7.47b) from (7.47a) gives $2C_{2m}A_2 = 0$.

Continuing this process we get, upon reaching the recurrence relationship for $n = 0$,

$$(7.48)\qquad 0 = A_o = A_2 = A_4 = \dots = A_{2m} .$$

Then with (7.48) and the recurrence relationship for $n = 1$, $A_{2m+2} = 0$ and so on, until the recurrence relationship for $n = m$, $A_{4m} = 0$ is reached. Therefore

$$(7.49)\qquad 0 = A_o = A_2 = \dots = A_{2m} = \dots A_{4m} .$$

Now consider equation (7.45) for $n = m + 1$

$$(7.50)\qquad C_{2m}A_{4m+2} + C_{2m-2}A_{4m} + \dots + C_{2m-2}A_4 + C_{2m}A_2 = 0$$

and applying equation (7.49)

$$C_{2m}A_{4m+2} = 0 .$$

Continuing with $n = m+2,\ m+3,\ \ldots,$ it is clear that all the A's must vanish. Hence the theorem is true for $N = 0$.

If $N \neq 0$, compare the recurrence relationships for $n = N+m$. Since by hypothesis $A_{2N} \neq B_{2N}$ but all $A_{2n} = B_{2n}$ for $n > N+1$, we obtain

$$(7.51a) \qquad C_{2m}A_{2N+4m} + \ldots + C_{2m-2}A_{2N+2} + C_{2m}A_{2N} = 0$$

$$(7.51b) \qquad C_{2m}A_{2N+4m} + \ldots + C_{2m-2}A_{2N+2} + C_{2m}B_{2N} = 0 .$$

Subtracting (7.51b) from (7.51a)

$$(7.52) \qquad C_{2m}(A_{2N} - B_{2N}) = 0 \qquad \text{or } A_{2N} = B_{2N}$$

contrary to hypothesis, and the theorem holds for solutions with period π.

If solutions (7.43) coexist, then for $N = 0$ and $n \geq m$, the recurrence relations are the same for (7.43a) and (7.43b), or

$$(7.53) \qquad C_{2m}A_{2n+2m+1} + \ldots + 2\left[C_o - (2n+1)^2\right]C_{2n+1} + \ldots + C_{2m}A_{2n-2m+1} = 0 .$$

Compare recurrence relationships for $n = m-1$ and find, by the same reasoning as before, that

$$C_{2m}A_1 = 0 .$$

By continuing with the relationships for $n = m-2, m-3, \ldots$ we obtain

(7.54) $$0 = A_1 = \ldots = A_{2m-1} .$$

Now, from the recurrence relations for $n = 1$, we get $A_{2m+1} = 0$. Examination of the recurrence relations as n increases makes it clear that all the A's must vanish.

For $N \neq 0$, by comparing the recurrence relationships for $n = N + m$, we find the contradiction

(7.55) $$C_{2m}(A_{2N+1} - B_{2N+1}) = 0 ,$$

which completes the proof of the theorem.

The impossibility of the coexistence of finite trigonometric polynomial solutions, which is not included in Theorem 21.1, could be demonstrated by using the same procedure as in Section 21. However, it is possible to prove the stronger result:

Theorem 7.11. Finite Hill equations cannot have finite trigonometric polynomial solutions.

Proof: Assume a solution of equation (7.41) exists in the form

(7.56) $$y = \sum_{0}^{N} A_{2n} \cos 2nx \qquad \text{or} \qquad y = \sum_{1}^{N} A_{2n} \sin 2nx$$

where A_{2N} is the last non-zero coefficient in either of the series (7.56).

For $n > m$ the recurrence relations (7.45) hold. Consider the recurrence relation for $n = N+m$, since all $A_{2n} = 0$ for $n \geq N+1$

$$C_{2m}A_{2N} = 0 \qquad \text{or} \qquad A_{2N} = 0 \; . \tag{7.57}$$

This contradicts the hypothesis that A_{2N} is the last non-zero coefficient and proves the theorem for solutions with period π.

Assume that a solution of equation (7.41) exists in the form

$$y = \sum_{0}^{N} A_{2N+1}\cos(2N+1)x \quad \text{or} \quad y = \sum_{0}^{N} A_{2N+1}\sin(2N+1)x \;, \tag{7.58}$$

where A_{2N+1} is the last non-zero coefficient of either of the series (3). Consider the recurrence relation (7.53) for $n = N+m$. Since all $A_{2n+1} = 0$ for $n > N+1$

$$C_{2m}A_{2N+1} = 0 \qquad \text{or} \qquad A_{2N+1} = 0 \;, \tag{7.59}$$

contradicting the hypothesis that A_{2N+1} is the last non-zero coefficient. This proves the theorem.

Further results on finite Hill equations have been found by S. Winkler, 1958.

8. Examples

In this chapter, we shall discuss briefly a few examples of equations of Hill's type which are particularly easily approachable and which may be useful for the purpose of a first orientation. The first three cases are distinguished by the fact that they admit an explicit integration of Hill's equation in terms of well known functions. The fourth example (the equation for frequency modulation) offers the simplest illustration available for the results on coexistence in Chapter 7.

8.1. Impulse functions

In connection with a problem in quantum theory (electrons in a one-dimensional conductor), Kronig and Penney, 1931, studied the special equation of Hill's type:

$$(8.1) \qquad y'' + [\lambda + Q(x)]y = 0$$

where

$$Q(x) = -v_o \quad \text{for } -b < x < 0$$

$$Q(x) = 0 \quad \text{for } 0 < x < a$$

$$Q(x+c) = Q(x) \quad \text{where } c = a+b .$$

The problem is to find intervals of stability, that is those values of λ for which (8.1) has a non-trivial solution y with the property

$$y(x+c) = e^{ikc}y(x)$$

where k is real. Such a solution will exist if and only if the equation

$$(8.2) \qquad \cos a\sqrt{\lambda}\,\cos b\sqrt{\lambda - v_o} + \frac{v_o - 2\lambda}{\sqrt{\lambda}\,\sqrt{\lambda - v_o}}\sin a\sqrt{\lambda}\,\sin b\sqrt{\lambda - v_o} = \cos kc$$

can be satisfied for a real value of k. Of course, this means that the left hand side of (8.2) must lie between -1 and +1.

Equation (8.2) is still somewhat hard to discuss numerically. A further simplification can be obtained if we replace Q (which so far has the shape of a "well") by a Dirac Delta function. This can be done by letting

$$b \to 0, \qquad v_o \to \infty, \qquad bv_o \to P/a,$$

where P is a positive constant. Then (8.2) reduces to

$$\cos a\sqrt{\lambda} + \frac{P}{a\sqrt{\lambda}} \sin a\sqrt{\lambda} = \cos kc \quad , \tag{8.3}$$

and the condition that the left hand side of (8.3) should have values between -1 and +1 can be discussed readily by drawing the curve

$$u = \cos v + \frac{P}{v} \sin v$$

in the (u,v)-plane. It follows that the intervals of instability (where (8.3) cannot hold for a real value of k) will tend towards zero like $\lambda^{-1/2}$ as $\lambda \to +\infty$.

We followed here the exposition given by Sommerfeld and Bethe, 1933, where many more details may be found. See also Strutt, 1932.

8.2. Piecewise constant functions

Meissner, 1918, investigated the equation

$$y'' + \omega^2 q(x) y = 0 \tag{8.4}$$

where q(x) assumes a finite number of different values in the interval $0 \leq x \leq 2\pi$ and is periodic with period 2π. Although (8.4) cannot be transformed into an equation of type (8.1) since the Liouville transformation is not applicable to discontinuous functions, equation (8.4) is relevant to the theory of Hill's equation and we shall now report briefly on it.

Assume that the interval $0 \leq x \leq 2\pi$ has been divided into n parts of length τ_i, $i=1,\dots,n$, $\tau_1 + \dots + \tau_n = 2\pi$, and assume that

$$q(x) = v_i^2 \Big/ (4\pi^2)$$

in the i-th interval. We shall use the notations

$$C_i = \cos\left(\omega v_i \tau_i / 2\pi\right), \qquad S_i = \sin\left(\omega v_i \tau_i / 2\pi\right)$$

$$V_{i,k} = \frac{1}{2}\left(\frac{v_i}{v_k} + \frac{v_k}{v_i}\right), \qquad (i,k=1,2,\dots,n) .$$

$$J_3 = C_1C_2C_3 - v_{12}S_1S_2C_3 - v_{13}S_1S_3C_2 - v_{23}S_2S_3C_1$$

$$J_2 = C_1C_2 - v_{12}S_1S_2 .$$

Then the necessary and sufficient condition for (8.4) to have a periodic solution of period 2π is, for $n=3$, $J_3 = 1$ and for $n=2$, $J_2 = 1$. Similarly, (8.4) will have a periodic solution of period 4π if and only if $J_3 = -1$ (for $n=3$) or $J_2 = -1$ (for $n=2$). If, in particular,

$$\tau_1 = \tau_2 = \pi ,$$

and if we put

$$x_1 = \omega v_1/2 , \qquad x_2 = \omega v_2/2 ,$$

we find

$$(8.5) \qquad J_2 = J_2(x_1,x_2) = \cos x_1 \cos x_2 - \frac{1}{2}\left(\frac{x_1}{x_2} - \frac{x_2}{x_1}\right) \sin x_1 \sin x_2 .$$

Meissner draws the curves $J_2 = \pm 1$ in the (x_1,x_2)-plane. They consist of infinitely many separate branches bounding the regions of instability for (8.4) in the case $n = 2$.

Hochstadt, 1960, analysed more thoroughly an equation which is equivalent to (8.4). Considering

$$(8.6) \qquad y'' + \omega^2 Q(x) y = 0$$

with

$$Q(-x) = Q(x), \qquad Q(x+2L) = Q(x),$$

$$Q(x) = 1 \quad \text{for } |x| \leqq 1, \qquad Q(x) = a \quad \text{for } 1 < |x| \leqq L,$$

he shows that (8.6) will have a periodic solution of period 2L for ω_m, $m=0,1,2,\ldots$, where, for large $m = 2n-1$ or $m = 2n$,

$$\omega_{2n} = \left[\frac{2n}{1+a(L-1)}\right]\frac{\pi}{2} + \Theta\,\frac{\pi}{2}$$

$$\omega_{2n-1} = \left[\frac{2n}{1+a(L-1)}\right]\frac{\pi}{2} + \Theta\,\frac{\pi}{2} .$$

Here $[u]$ denotes the largest integer not exceeding u, and Θ is an undetermined quantity

$$0 \leqq \Theta < 1 \ .$$

Similarly, (816) will have solutions of period $4L$ if $\omega = \omega'_m$, where, for large $m = 2n$ or $m = 2n - 1$,

$$\omega'_{2n} = \left[\frac{2n-1}{1+a(L-1)}\right]\frac{\pi}{2} + \Theta\frac{\pi}{2}$$

$$\omega'_{2n-1} = \left[\frac{2n-1}{1+a(L-1)}\right]\frac{\pi}{2} + \Theta\frac{\pi}{2} \ .$$

It can be shown that the lengths of the intervals of instability, that is, the differences

$$\left|\ \omega^2_{2n} - \omega^2_{2n-1}\right| , \quad \left|\omega'^2_{2n} - \omega'^2_{2n-1}\right|$$

will, in general, tend to infinity as $n \to \infty$. Also, coexistence of periodic solutions of period $2L$ or $4L$ can be shown to take place if and only if $a(L-1)$ is a rational number.

8.3. Piecewise linear reciprocal function.

Schwerin, 1931, also investigated the case where $Q(x)$ is piecewise constant and, in addition, the case where $1/Q(x)$ is a piecewise linear function, and where $\lambda = 0$. In this case, (8.1) can be integrated explicitly in terms of Bessel functions. Using the notation of Schwerin's paper, we shall write the differential equation in the form

$$(8.7) \qquad \frac{d^2\eta}{dt^2} + \theta^2 \frac{\eta}{\epsilon(t)} = 0$$

where $\epsilon(t)$ is either a triangular function:

$$\epsilon(t) = \epsilon_o + 2(\epsilon_1 - \epsilon_o)t \qquad \text{for } 0 \leqq t \leqq \frac{1}{2}$$

$$\epsilon(t) = \epsilon_1 - 2(\epsilon_1 - \epsilon_o)t \qquad \text{for } \frac{1}{2} \leqq t \leqq 1 \ ,$$

$$\epsilon(t+1) = \epsilon(t) \ , \qquad 0 < \epsilon_o < \epsilon_1 \ , \quad \epsilon_o + \epsilon_1 = 2 \ .$$

or a wedge function: (with a jump at $t = 0, \pm 1, \pm 2, \ldots$)

$$\epsilon(t) = \epsilon_o + (\epsilon_1 - \epsilon_o)t \qquad \text{for } 0 < t < 1 \ .$$

$$\epsilon(t+1) = \epsilon(t), \qquad 0 < \epsilon_o < \epsilon_1 \ , \quad \epsilon_o + \epsilon_1 = 2 \ .$$

In the case of the triangular function, the values of θ for which (8.7) has a periodic solution of period 1 are the roots of a transcendental function which is defined as follows:

Let J_m, Y_m $(m = 0,1)$ be respectively the Bessel and the Neumann function of order m. Then θ must satisfy one of the equations

$$J_1\left(\frac{\theta\sqrt{\epsilon_o}}{\epsilon_1 - \epsilon_o}\right) Y_1\left(\frac{\theta\sqrt{\epsilon_1}}{\epsilon_1 - \epsilon_o}\right) = J_1\left(\frac{\theta\sqrt{\epsilon_1}}{\epsilon_1 - \epsilon_o}\right) Y_1\left(\frac{\theta\sqrt{\epsilon_o}}{\epsilon_1 - \epsilon_o}\right)$$

or

$$J_o\left(\frac{\theta\sqrt{\epsilon_o}}{\epsilon_1 - \epsilon_o}\right) \quad Y_o\left(\frac{\theta\sqrt{\epsilon_{1-}}}{\epsilon_1 - \epsilon_o}\right) = J_o\left(\frac{\theta\sqrt{\epsilon_1}}{\epsilon_1 - \epsilon_o}\right) \quad Y_o\left(\frac{\theta\sqrt{\epsilon_o}}{\epsilon_1 - \epsilon_o}\right)$$

if (8.7) has a solution of period 1. Similarly, (8.7) will have a solution of period 2 if and only if one of the following equations is satisfied:

$$J_o\left(\frac{\theta\sqrt{\epsilon_o}}{\epsilon_1 - \epsilon_o}\right) \quad Y_1\left(\frac{\theta\sqrt{\epsilon_1}}{\epsilon_1 - \epsilon_o}\right) = J_1\left(\frac{\theta\sqrt{\epsilon_1}}{\epsilon_1 - \epsilon_o}\right) \quad Y_o\left(\frac{\theta\sqrt{\epsilon_o}}{\epsilon_1 - \epsilon_o}\right)$$

or

$$J_1\left(\frac{\theta\sqrt{\epsilon_o}}{\epsilon_1 - \epsilon_o}\right) \quad Y_o\left(\frac{\theta\sqrt{\epsilon_1}}{\epsilon - \epsilon_o}\right) = J_o\left(\frac{\theta\sqrt{\epsilon_1}}{\epsilon_1 - \epsilon_o}\right) \quad Y_1\left(\frac{\theta\sqrt{\epsilon_o}}{\epsilon_1 - \epsilon_o}\right) .$$

Similar equations can be obtained if $\epsilon(t)$ is a wedge-shaped function. Schwerin, 1931, draws stability charts showing the regions of stability and instability in a plane with the Cartesian coordinates Θ and ϵ_o. Also, numerical tables and a large variety of comments (in part empirical) on the regions of stability may be found in this paper.

8.4. The frequency modulation equation

The equation

$$(8.8) \qquad (1 + a \cos 2x)y'' + \lambda y = 0$$

has been studied by Cambi, 1948, 1949. Because of its applications, it may be called the equation of frequency modulation. We shall assume that $|a| < 1$ and, of course, that a and λ are real. Equation (8.8) is not in the form of (8.1), but it can be transformed into it by using the Liouville transformation of Section 3.1. Therefore, we may talk about intervals of stability and instability on the λ-axis. According to Corollary 4.2, the zero-th interval of instability contains all values $\lambda < 0$. Obviously, $\lambda_0 = 0$, since for $\lambda = 0$, (8.8) has the periodic solution $y = 1$. We shall now prove that, except for the zero-th one, all even intervals of instability disappear for (8.8) but no odd interval of instability disappears if $a \neq 0$. If we look at Theorem 7.1, we see at once that $Q^*(\mu) = a(2\mu - 1)^2$ will not vanish for any integral value of μ, unless $a = 0$, in which case (8.8) is trivial. This proves our statement about the odd-numbered intervals of instability. With respect to the even ones, we find that $Q(\mu)$, as defined in Theorem 7.1, is simply $\underline{a}\,\mu^2$ and therefore has the integral root zero and no other one (if $\underline{a} \neq 0$). Now Theorem 7.3 shows that (8.8) must have two linearly independent solutions of period π whenever it has one such solution unless this solution is a constant. However, a constant solution of (8.8) can occur only for $\lambda = 0$, a case already considered.

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