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# Hill's Equation. Part I: General Theory

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## Part I: General Theory

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## Introduction

It is the purpose of this report to give a detailed account of the elements of the theory of Hill's equation, to review briefly the more sophisticated recent results and to provide a guide to the extensive literature on special cases and their applications to mechanical and electrical problems.

The first part of this report deals with the general theory of Hill's equation. In Chapter 1 and in the first three sections of Chapter 2 the elementary parts of the theory are developed and complete proofs of all results are given. Prerequisites for these sections are the elementary theory of ordinary linear differential equations and standard results in the theory of functions of a complex variable. The last three sections of Chapter 2 have the character of a review and proofs of the quoted results have been omitted.

A brief historical introduction to Hill's equation can be found in Moulton, 1930.

The references in the present report do not always give the earliest source for a theorem. In particular, certain results usually referred to as ''Haupt's theorem'' have been quoted under this name although they were first derived by Liapounoff, 1907.

For some interesting approaches to the theory of Hill's equation which, so far, have been applied to a special case (Mathieu's equation), see Chapter 20 in Whittaker-Watson, 1927.

## 1. Basic Concepts

### 1.1. Preliminary remarks

Any homogeneous linear differential equation of second order with real periodic coefficients can be reduced to an equation of Hill's type. The specific question which arises in the theory of Hill's equation is the problem of the existence of periodic solutions. This problem has many features in common with the ordinary Sturm-Liouville problems, and in certain cases it can, in fact, be reduced to ordinary boundary value problems of the Sturm-Liouville type, (see Section 1.3). However, in general such a reduction is not possible, and imposing the periodicity requirements on a solution of the differential equation leads to phenomena different from those resulting from the imposition of a homogeneous boundary condition of the Sturm-Liouville type. Thus, the differential equation can have two linearly independent periodic solutions but it cannot have two linearly independent solutions satisfying the same homogeneous boundary conditions. Futhermore, the value of the period of the solution (which is a multiple of the period p of the coefficients) plays an important role in the discussion of periodic solutions. In a certain sense, only the solutions of period p and 2 p are of interest. We shall now proceed with a detailed presentation of some basic theorems and their proofs.

As references to the general theory of Sturm-Liouville (self-adjoint) boundary-value problems we mention Courant and Hilbert, 1953 and Coddington and Levinson, 1955.

### 1.2 Floquet's Theorem

Let $Q(x)$ be a (real or complex valued) function of the real variable x defined for all values of x. We assume that $Q(x)$ is piecewise continuous in every finite interval and that it is periodic with minimal period $\pi$. By this we mean that for all x

$$Q(x+\pi) = Q(x), \tag{1.1}$$

and that if p is a number with $0 < p < \pi$, then there exists at least one real interval I such that $Q(x+p) \neq Q(x)$ for $x \in I$.

If $Q(x)$ has the properties stated above, then the differential equation

$$y'' + Q(x)y = 0 \tag{1.2}$$

has two continuously differentiable solutions $y_1(x)$ and $y_2(x)$ which are uniquely determined by the conditions:

$$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

These solutions are referred to as normalized solutions of (1.2).

Before stating Floquet's theorem we must define the notions of characteristic equation and characteristic exponent associated with (1.2). Thus, the characteristic equation is the equation

$$\rho^2 - \left[y_1(\pi) + y_2'(\pi)\right] \rho + 1 = 0 \tag{1.3}$$

and the characteristic exponent $\alpha$ is a number which satisfies the equations

$$\exp i\alpha\pi = \rho_1 , \quad \exp(-i\alpha\pi) = \rho_2 , \tag{1.4}$$

where $\rho_1$ and $\rho_2$ are the roots of the characteristic equation (1.3).

It is clear that $\alpha$ is defined up to an integral multiple of 2. Also $2 \cos \alpha\pi = y_1(\pi) + y_2'(\pi)$. Finally $\rho_1\rho_2 = 1$.

We can now state Floquet's Theorem:

$1^\circ$. If the roots $\rho_1$ and $\rho_2$ of the characteristic equation (1.3) are different from each other, then Hill's equation(1.2) has two linearly independent solutions

$$f_1(x) = e^{i\alpha x}p_1(x), \; f_2(x) = e^{-i\alpha x}p_2(x),$$

where $p_1(x)$ and $p_2(x)$ are periodic with period $\pi$.

$2^\circ$. If $\rho_1 = \rho_2$, then equation (1.2) has a non-trivial solution which is periodic with period $\pi$ (when $\rho_1 = \rho_2 = 1$) or $2\pi$ (when $\rho_1 = \rho_2 = -1$). Let $p(x)$ denote such a periodic solution and let $y(x)$ be another solution linearly independent of $p(x)$. Then

$$y(x+\pi) = \rho_1 \, y(x) + \theta \, p(x), \qquad \theta \text{ constant},$$

and $\theta = 0$ is equivalent to

$$y_1(\pi) + y_2'(\pi) = \pm 2, \quad y_2(\pi) = 0, \quad y_1'(\pi) = 0.$$

Before starting with the proof of Floquet's theorem it may be appropriate to discuss its significance. Thus, let $\rho_1 \neq \rho_2$. If $\alpha$ is real, then there exists an upper bound M for the absolute value $|y(x)|$ of every solution of (1.2) and M depends on the initial conditions for y and not on x. If $\alpha$ is not real, then there exists a non-trivial unbounded solution y(x) of (1.2). If $\rho_1 = \rho_2$, then for all solutions of (1.2) to be bounded it is necessary and sufficient that

$$y_1(\pi) + y_2'(\pi) = \pm 2, \quad y_2(\pi) = 0, \quad y_1'(\pi) = 0.$$

Whenever all solutions of (1.2) are bounded we say that they are stable. Otherwise we say that they are unstable.

The solutions of period $\pi$ and $2\pi$ play an exceptional role as is seen from the following

<u>Corollary to Floquet's Theorem.</u> If (1.2) has a periodic non-trivial solution with period $n\pi$, $n > 2$, but no solution with period $\pi$ or $2\pi$, then all solutions are periodic with period $n\pi$.

Indeed, our assumption implies that $\rho_1 \neq \rho_2$ so that every solution y of (1.2) is of the form

$$y = \mu f_1(x) + \nu f_2(x).$$

If one such solution is periodic with period $n\pi$, then $y(x+n\pi) = \mu c f_1 + \nu \bar{c} f_2 = y(x)$ where $c = \exp(i\alpha n\pi)$, $\bar{c} = \exp(-i\alpha n\pi)$. Since $f_1$ and $f_2$ are linearly independent, $c = \bar{c} = 1$. Therefore, $n\alpha$ is an even integer, and both $f_1$ and $f_2$ are periodic with period $n\pi$.

<u>Proof of Floquet's Theorem.</u> If y(x) is a solution of (1.2), then, obviously, $y(x+\pi)$ is also a solution of (1.2). In particular, $y_1(x+\pi)$ and $y_2(x+\pi)$ are solutions of (1.2).

Since $y_1(x)$ and $y_2(x)$ form a basis for the set of all solutions of (1.2), it must be possible to express $y_1(x+\pi)$ and $y_2(x+\pi)$ as linear combinations of $y_1(x)$ and $y_2(x)$. We find easily that

$$(1.5) \qquad \begin{cases} y_1(x+\pi) = y_1(\pi)y_1(x) + y_1'(\pi)y_2(x) \\ y_2(x+\pi) = y_2(\pi)y_1(x) + y_2'(\pi)y_2(x) \end{cases} .$$

Assume now that $y(x) \not\equiv 0$ is a solution of (1.2) such that

$$(1.6) \qquad y(x+\pi) = \rho y(x)$$

for some constant $\rho$. If $y(x) = c_1y_1(x) + c_2y_2(x)$, then it follows from (1.6) that $c_1$ and $c_2$ must satisfy the system of linear equations

$$(1.7) \qquad \begin{cases} (y_1(\pi) - \rho)c_1 + y_2(\pi)c_2 = 0 \\ y_1'(\pi)c_1 + (y_2'(\pi) - \rho)c_2 = 0 \end{cases} .$$

Conversely, if (1.7) is satisfied, $y(x)$ satisfies (1.6). Now, the necessary and sufficient condition for (1.7) to have a solution $c_1$, $c_2$ such that not both $c_1$ and $c_2$ vanish is

$$(1.8) \qquad \begin{vmatrix} y_1(\pi)-\rho \ , & y_2(\pi) \\ y_1'(\pi) \ , & y_2'(\pi)-\rho \end{vmatrix} = 0 .$$

Since, for all $x$, the Wronskian

$$y_1(x)y_2'(x) - y_2(x)y_1'(x) = 1,$$

equation (1.8) is identical with the characteristic equation (1.3). Thus, if $\rho = \rho_1$ is a root of (1.8), we can find $c_1$ and $c_2$ such that $y = c_1y_1 + c_2y_2 \not\equiv 0$ and such that $y$ satisfies (1.6). Obviously, if (1.6) is satisfied, we may write

$$y = y(x) = \exp(i\alpha x)\ p_1(x) = f_1(x)$$

where $\exp(i\alpha\pi) = \rho_1$ and where $p_1(x)$ is a periodic function of $x$ with period $\pi$. Suppose now that (1.8) has a second solution $\rho = \rho_2 \neq \rho_1$. We may use $\rho_2$ for the construction of a solution $y = f_2(x) \not\equiv 0$ of (1.3) such that $f_2(x+\pi) = \rho_2 f_2(x)$. We observe that $f_1$ and $f_2$ are linearly independent. Otherwise, we could find constants $\lambda_1$ and $\lambda_2$ not both of which vanish such that

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) \equiv 0.$$

But then we would have

$$\lambda_1 f_1(x+\pi) + \lambda_2 f_2(x+\pi) = \lambda_1\rho_1 f_1(x) + \lambda_2\rho_2 f_2(x) \equiv 0.$$

Since $\lambda_1 f_1$ and $\lambda_2 f_2$ do not both vanish identically, the last two equations are compatible only if $\rho_1 = \rho_2$, which we have excluded. This proves Floquet's theorem in the case where $\rho_1 \neq \rho_2$.

Since $\rho_1\rho_2 = 1$, either $|\rho_1| = |\rho_2| = 1$ or at least one of the numbers $|\rho_1|$, $|\rho_2|$ exceeds 1. In the first case, we have stability, in the second case instability of the solutions of (1.2), provided that $\rho_1 \neq \rho_2$.

If $\rho_1 = \rho_2$, we still can construct one solution $y_1^*(x)$ of (1.2) such that

$$y_1^*(x+\pi) = \rho_1 y_1^*(x).$$

Since $\rho_1 = \rho_2$ and $\rho_1\rho_2 = 1$ implies that $\rho_1 = \pm 1$, $y_1^*$ is obviously periodic with period $\pi$ or $2\pi$. In order to find the properties of a solution $y_2^*(x)$ which is linearly independent of $y_1^*$, assume first that $y_2(\pi) \neq 0$. Then we may choose [cf. (1.7) and (1.3]

$$y_1^*(x) = y_2(\pi)y_1(x) + \left[\rho_1 - y_1(\pi)\right] y_2(x)$$

$$y_2^*(x) = y_2(x)$$

and we find from $2\rho_1 = y_1(\pi) + y_2'(\pi)$ that

$$y_2^*(x+\pi) = \rho_1 y_2^*(x) + y_1^*(x).$$

Similarly, if $y_2(\pi) = 0$, we may choose

$$y_1^*(x) = y_2(x), \; y_2^*(x) = y_1(x).$$

Since $y_1(\pi)y_2'(\pi) - y_1'(\pi)y_2(\pi) = 1$ it follows from $y_2(\pi) = 0$ and $y_1(\pi) + y_2'(\pi) = 2\rho_1$ that $y_1(\pi) = y_2'(\pi) = \rho_1$ and therefore we have from (1.5) that

$$y_1^*(x+\pi) = \rho_1 y_1^*(x)$$

$$y_2^*(x+\pi) = \rho_1 y_2^*(x) + y_1'(\pi)y_1^*(x).$$

This proves Floquet's theorem in all details.

As a rather obvious consequence of Floquet's theorem we mention the following

<u>Stability Test.</u> <u>The solutions of</u> (1.2) <u>are stable if and only if</u> $y_1(\pi) + y_2'(\pi)$ <u>is real and</u>

$$|y_1(\pi) + y_2'(\pi)| < 2$$

<u>or</u>

$$y_1(\pi) + y_2'(\pi) = \pm 2$$

<u>and</u>

$$y_2(\pi) = y_1'(\pi) = 0.$$

<u>Proof.</u> If $\rho_1 \neq \rho_2$, then stability is equivalent to $\alpha \neq 0$, $\alpha$ real, which, in turn, is

equivalent to $y_1(\pi) + y_2'(\pi)$ being real and in absolute value < 2. If $\rho_1 = \rho_2$, then stability is equivalent to $y_1(\pi) + y_2'(\pi) = \pm 2$ and $y_2(\pi) = y_1'(\pi) = 0$.

## 1.3 The symmetric case $Q(x) = Q(-x)$

If in (1.2) the function $Q(x)$ is even, i.e., if

$$Q(x) = Q(-x) \tag{1.9}$$

it is possible to establish relations between the values of $y_1$, $y_2$, $y_1'$, $y_2'$ at $x = \pi/2$ and at $x = \pi$ and these relations allow a more detailed classification of the solutions of period $\pi$ and $2\pi$. We summarize our results by stating

Theorem 1.1. Let $y_1(x)$ and $y_2(x)$ be the normalized solutions of (1.2) and assume that $Q(x)$ satisfies (1.9). Then the following relations hold:

$$y_1(\pi) = 2y_1(\tfrac{\pi}{2})y_2'(\tfrac{\pi}{2}) - 1 = 1 + 2y_1'(\tfrac{\pi}{2})y_2(\tfrac{\pi}{2}) \tag{1.10}$$

$$y_2(\pi) = 2y_2(\tfrac{\pi}{2})y_2'(\tfrac{\pi}{2}) \tag{1.11}$$

$$y_1'(\pi) = 2y_1(\tfrac{\pi}{2})y_1'(\tfrac{\pi}{2}) \tag{1.12}$$

$$y_2'(\pi) = y_1(\pi). \tag{1.13}$$

Theorem 1.2. If the conditions of Theorem 1.1 are satisfied, then there exists a non-trivial periodic solution of (1.2) which is

(1) even and of period $\pi$ if and only if $y_1'(\frac{\pi}{2}) = 0$

(2) odd and of period $\pi$ if and only if $y_2(\frac{\pi}{2}) = 0$

(3) even and of period $2\pi$ if and only if $y_1(\frac{\pi}{2}) = 0$

(4) odd and of period $2\pi$ if and only if $y_2'(\frac{\pi}{2}) = 0$.

Periodic solutions of period $\pi$ or $2\pi$ are necessarily multiples of the normalized solutions $y_1(x)$ and $y_2(x)$.

Proof of Theorem 1.1. If $Q(x)$ is even and if $y(x)$ is a solution of (1.2), then $y(-x)$ is also a solution. Since the initial conditions for $y_1(-x)$ and $y_1(x)$ coincide and, similarly, those for $y_2(x)$ and $-y_2(-x)$ are identical, it follows that $y_1(x)$ is even and $y_2(x)$ is odd. Therefore we find from (1.5) for $x = -\pi/2$:

$$y_1(\tfrac{\pi}{2}) = y_1(\pi)y_1(\tfrac{\pi}{2}) - y_1'(\pi)y_2(\tfrac{\pi}{2}) \tag{1.14}$$

(1.15) $$y_2(\tfrac{\pi}{2}) = y_2(\pi)y_1(\tfrac{\pi}{2}) - y_2'(\pi)y_2(\tfrac{\pi}{2}).$$

Obviously, $y_1'(x)$ is odd and $y_2'(x)$ is even. Using this fact, we find from (1.5) by differentiating both sides with respect to x and by putting $x = -\pi/2$ that

(1.16) $$y_1'(\tfrac{\pi}{2}) = -y_1(\pi)y_1'(\tfrac{\pi}{2}) + y_1'(\pi)y_2'(\tfrac{\pi}{2})$$

(1.17) $$y_2'(\tfrac{\pi}{2}) = -y_2(\pi)y_1'(\tfrac{\pi}{2}) + y_2'(\pi)y_2'(\tfrac{\pi}{2}) .$$

We may treat (1.14) to (1.17) as a system of linear equations for $y_1(\pi)$, $y_2(\pi)$, $y_1'(\pi)$, $y_2'(\pi)$. Solving the equations for these quantities, we arrive at Theorem 1.1 if we observe that

$$y_1(\tfrac{\pi}{2})y_2'(\tfrac{\pi}{2}) - y_1'(\tfrac{\pi}{2})y_2(\tfrac{\pi}{2}) = 1.$$

Proof of Theorem 1.2. It is easily seen that an even solution of (1.2) is a multiple of $y_1(x)$ and that an odd solution must be a multiple of $y_2(x)$.

To prove(1) let us assume that y(x) is a non-trivial, even, periodic solution of (1.2) with period $\pi$. Then $y_1(x)$ is also periodic with period $\pi$ and the same is true about $y_1'(x)$. Thus $y_1'(\frac{\pi}{2}) = y_1'(-\frac{\pi}{2})$. Since $y_1'(x)$ is an odd function it follows that $y_1'(\frac{\pi}{2}) = -y_1'(-\frac{\pi}{2})$. Consequently, $y_1'(\frac{\pi}{2}) = 0$. Conversely, if $y_1'(\frac{\pi}{2}) = 0$, then $y_1'(-\frac{\pi}{2}) = 0$. Also, $y_1(-\frac{\pi}{2}) = y_1(\frac{\pi}{2})$. Therefore $y_1(x)$ satisfies the same conditions at $x = -\frac{\pi}{2}$ and at $x = \frac{\pi}{2}$. From this and from the periodicity of Q(x) it follows that $y_1(x)$ is periodic with period $\pi$.

The proof of (2) is entirely analogous to the proof of (1).

In proving (3) we show that the values of $y_1(x)$ and $y_1'(x)$ at $x = -\frac{\pi}{2}$ differ in sign from the values of $y_1(x)$ and $y_1'(x)$ at $x = \frac{\pi}{2}$. This shows that $y_1(x+\pi) = -y_1(x)$ and therefore $y_1(x+2\pi) = y_1(x)$.

The proof of (4) is analogous to the proof of (3).

## 2. Characteristic Values and Discriminant

### 2.1. Characteristic values and intervals of stability

In this and in the following sections we shall study Hill's equation in its standard form

(2.1) $$y'' + \left[\lambda + Q(x)\right] y = 0,$$

where $\lambda$ is a parameter and where Q(x) is a real periodic function of x with period $\pi$.

Unless otherwise stated we shall assume that $Q(x)$ is two times differentiable for all x. This is a rather strong assumption, and occasionally we shall treat examples where $Q(x)$ is not even continuous.

Let $y_1$ and $y_2$ be the two linearly independent solutions of (2.1) which we defined by simple initial conditions in Section 1.2. To emphasize their dependence on $\lambda$ we shall sometimes write $y_1(x,\lambda)$ and $y_2(x,\lambda)$ instead of $y_1(x)$ and $y_2(x)$. One of our main problems will be the determination of those values of $\lambda$ for which the solutions of (2.1) are stable. This problem is identical with the problem of determining those values of $\lambda$ for which equation (2.1) has a solution of period $\pi$ or $2\pi$.

The following theorem due to O. Haupt (1914, 1918) connects the two problems.

Theorem 2.1. (Oscillation Theorem) To every differential equation (2.1), there belong two monotonically increasing infinite sequences of real numbers

(2.2) $$\lambda_0, \lambda_1, \lambda_2, \ldots$$

and

(2.3) $$\lambda_1', \lambda_2', \lambda_3', \lambda_4', \ldots$$

such that (2.1) has a solution of period $\pi$ if and only if $\lambda = \lambda_n$, $n = 0, 1, 2, \ldots$ and a solution of period $2\pi$ if and only if $\lambda = \lambda_n'$, $n = 1, 2, 3, \ldots$ . The $\lambda_n$, $\lambda_n'$ satisfy the inequalities

(2.3) $$\lambda_0 < \lambda_1' \leq \lambda_2' < \lambda_1 \leq \lambda_2 < \lambda_3' \leq \lambda_4' < \lambda_3 \leq \lambda_4 < \cdots$$

and the relations

(2.5) $$\lim_{n\to\infty} \lambda_n^{-1} = 0 , \qquad \lim_{n\to\infty} (\lambda_n')^{-1} = 0.$$

The solutions of (2.1) are stable in the intervals

(2.6) $$(\lambda_0, \lambda_1'), (\lambda_2', \lambda_1), (\lambda_2, \lambda_3'), (\lambda_4', \lambda_3), \ldots .$$

At the endpoints of these intervals the solutions of (2.1) are, in general, unstable. This is always true for $\lambda = \lambda_0$. The solutions of (2.1) are stable for $\lambda = \lambda_{2n+1}$ or $\lambda = \lambda_{2n+2}$ if and only if $\lambda_{2n+1} = \lambda_{2n+2}$, and they are stable for $\lambda = \lambda_{2n+1}'$ or $\lambda = \lambda_{2n+2}'$ if and only if $\lambda_{2n+1}' = \lambda_{2n+2}'$.

For complex values of $\lambda$ (2.1) has always unstable solutions.

In order to be able to refer briefly to the assertions of Theorem 2.1, we shall use the following definitions:

The real numbers $\lambda_n$ shall be called characteristic values of the first kind of (2.1) and the $\lambda'_n$ shall be called characteristic values of the second kind. The intervals (2.6) on the real $\lambda$-axis shall be called intervals of stability; an end point of such an interval shall belong to it if and only if (2.1) has stable solutions for the corresponding value of $\lambda$. Similarly, we shall talk about intervals of instability. Both the intervals of stability and of instability are ordered in a natural manner. We shall always disregard the interval of instability $(-\infty, \lambda_0)$. Therefore, the first interval of instability will, in general, be the interval $(\lambda'_1, \lambda'_2)$. Observe that, according to Theorem 2.1, neither an interval of stability nor an interval of instability can ever shrink to a point. The intervals of stability can never disappear, but two of them can combine to a single one if $\lambda_{2n+1} = \lambda_{2n+2}$ or $\lambda'_{2n+1} = \lambda'_{2n+2}$. However, the intervals of instability may disappear altogether (e.g., if $Q(x)$ is a constant). It is a deep result due to G. Borg, 1946, that for a non-constant $Q(x)$ there exists at least one interval of instability. As we shall see later (Section 2.4) there may exist only one interval of instability.

Proof of Theorem 2.1. We shall exclude first the possibility of stable solutions of (2.1) in the case of a complex value of $\lambda$. Assume that $\lambda = \mu + i\nu$, where $\mu$, $\nu$ are real and $\nu \neq 0$. Let $y = u + iv$ be a solution of (2.1) which is of the type

$$y(x) = e^{i\alpha x}\, p(x) = u + iv \tag{2.7}$$

where $\alpha$ is real and where $p(x)$ is periodic with period $\pi$. According to Floquet's Theorem, such a solution $y(x)$ exists if we have stability for the solutions of (2.1). By splitting (2.1) into its real and imaginary parts, we find

$$\begin{aligned} u'' + \left[\mu + Q(x)\right] u &= \nu v \\ v'' + \left[\mu + Q(x)\right] v &= -\nu u \quad . \end{aligned} \tag{2.8}$$

If we multiply the first of the equations (2.8) by v and the second by u and form the difference of the resulting equations we find that

$$u''\,v - v''u = \nu\,(u^2 + v^2)\ ,$$

or, upon integrating, that

$$u'v - v'u = \nu \int_0^x \left[u^2(t) + v^2(t)\right] dt + c, \tag{2.9}$$

where c is a constant. Now we see from (2.7) that all of the functions $|u|$, $|v|$, $|u'|$, $|v'|$ must be bounded for all values of x (since p(x) is a differentiable periodic function of x). Therefore, there exists an upper bound for $|uv'-vu'|$ which is independent of x. According to (2.9), the same must be true for the absolute value of

$$I(x) = \int_0^x \left[u^2(t) + v^2(t)\right] dt .$$

However, $|I(x)| \to \infty$ as $x \to \infty$ since $u^2 + v^2 = |p|^2$ and therefore, for n = 1, 2, 3, ...,

$$I(n\pi) = \int_0^{n\pi} \left[u^2(t) + v^2(t)\right] dt = n \int_0^{\pi} |p(t)|^2 dt.$$

Therefore if $\lambda$ is not real we cannot have a solution of type (2.7).

Next we wish to show that there exists a real number $\lambda^*$ such that for any $\lambda \leq \lambda^*$ the solutions of (2.1) are unstable. For this purpose, select a $\lambda^*$ such that for all x

$$\lambda^* + Q(x) < 0.$$

This is certainly possible since Q(x), being periodic, is a bounded function of x. We shall show that if $\lambda \leq \lambda^*$ then $y_1(x,\lambda) \to \infty$ as $x \to \infty$. To this end we shall write (2.1) in the form

$$y'' = D(x)y \tag{2.10}$$

where $D(x) = -\lambda - Q(x) > 0$ for all x. Since $y_1(0) = 1$, $y_1''(0) > 0$ it follows from $y_1'(0) = 0$ that $y_1'(x) > 0$ for all sufficiently small positive x. Therefore, if the set S of positive zeros of $y_1'(x)$ is not empty it has a greatest lower bound $\varepsilon > 0$.

We shall show that $\varepsilon$ does not exist and that therefore $y_1'(x) > 0$ for all $x > 0$. For this purpose we observe that the continuity of $y_1'(x)$ implies that $y_1'(\varepsilon) = 0$. Now we have from (2.10) that

$$y_1'^2(\varepsilon) = 2\int_0^{\varepsilon} D(x)\, y_1(x) y_1'(x) dx. \tag{2.11}$$

Since $y_1(0) = 1$ and $y_1'(x) \geq 0$ for $0 \leq x \leq \varepsilon$ we have $y_1(x) > 0$, $D(x) > 0$, and $y_1'(x) > 0$ for $0 < x < \varepsilon$. Therefore, the right hand side of (2.11) is positive whereas we had assumed that $y_1'(\varepsilon) = 0$. This shows that $y_1'(x) > 0$ if $x > 0$. Therefore $y_1(x)$ is monotonically increasing for $x > 0$ which means that $y_1(x) \geq 1$ for $x > 0$. Since (2.10) implies that

$$y_1'^2(x) = 2\int_0^x D(t) y_1(t) y_1'(t) dt,$$

we see that $y_1'(x)$ is also monotonically increasing with x, and since $\delta = y_1'(x_0) > 0$ for a certain $x_0 > 0$, we see that

$$y_1(x) \geq 1 + (x-x_0)\delta \qquad (x \geq x_0).$$

Therefore $y_1(x) \to \infty$ as $x \to \infty$. By a similar argument, we can show that $y_2'(x) > 1$ for $x > 0$. We have thus proved incidentally

Lemma 2.1. If $\lambda \leq \lambda^*$, then, for $x > 0$,

$$y_1(x,\lambda) + y_2'(x,\lambda) > 2.$$

Since this result implies that $\rho_1 \neq \rho_2$, we conclude that $\alpha$ in $y_1(x) = A\, e^{i\alpha x} p_1(x) + B\, e^{-i\alpha x} p_2(x)$ cannot be real. We thus find that if $\lambda$ is complex or $\lambda \leq \lambda^*$, then there exists no solution $y(x)$ of (2.1) which is of type (2.7) with real $\alpha$.

We shall now examine closely the properties of the functions $\Delta(\lambda) - 2$ and $\Delta(\lambda) + 2$, where

$$\Delta(\lambda) = y_1(\pi,\lambda) + y_2'(\pi,\lambda). \tag{2.12}$$

We fist note that $\Delta(\lambda) = 2$ is equivalent to $\rho_1 = \rho_2 = 1$ and that $\Delta(\lambda) = -2$ is equivalent to $\rho_1 = \rho_2 = -1$. Hence, if $\Delta(\lambda) = \pm 2$, (2.1) will have a solution of type (2.7) with real $\alpha$ (cf. Floquet's theorem). Since $\lambda$ complex or $\lambda \leq \lambda^*$ implies that (2.1) has no such solution it follows that

Lemma 2.2. All roots of the equations $\Delta(\lambda) - 2 = 0$ and $\Delta(\lambda) + 2 = 0$ are real and $> \lambda^*$.

In Section 2.2 we shall prove that both $\Delta(\lambda) - 2$ and $\Delta(\lambda) + 2$ are entire analytic functions of $\lambda$ which have infinitely many zeros. According to Lemma 2.2, all of these zeros are real and greater than $\lambda^*$. This establishes the existence of the two sequences (2.2) and (2.3) of Theorem 2.1. In fact, we have immediately from Floquet's theorem that the following assertion is true:

Lemma 2.3. Equation (2.1) has a periodic solution of period $\pi$ if and only if $\Delta(\lambda) = 2$ and a periodic solution of period $2\pi$ if and only if $\Delta(\lambda) = -2$.

To prove Lemma 2.3, we merely have to use once more the fact that the condition $\rho_1 = \rho_2 = 1$ of Floquet's theorem is identical with the condition $\Delta(\lambda) = 2$, and that $\rho_1 = \rho_2 = -1$ is equivalent to $\Delta(\lambda) = -2$.

Since $\Delta(\lambda)$ is an entire analytic function, the limit relations (2.5) are obviously true. We now turn to a proof of the inequalities (2.4). For this purpose, we need

> Lemma 2.4. Let $\mu$ be a root of the equation $\Delta(\lambda) - 2 = 0$ such that the derivative $\Delta'(\lambda)$ of $\Delta(\lambda)$ with respect to $\lambda$ is negative or zero for $\lambda = \mu$. Then $\Delta'(\lambda) < 0$ in any open interval $\mu < \lambda < \mu_1^*$ in which $\Delta(\lambda) > -2$. Similarly, let $\mu'$ be a root of $\Delta(\lambda) + 2 = 0$ and let $\Delta'(\mu') \geq 0$. Then $\Delta'(\lambda) > 0$ in any open interval $\mu' < \lambda < \mu_1'$ in which $\Delta(\lambda) < 2$.

Before proving Lemma 2.4, we may observe that it proves both the inequalities (2.4) and the assertion that the open intervals (2.6) are intervals of stability. In fact, we see from Lemma 2.1 that if $\lambda \leq \lambda^*$, then $\Delta(\lambda) > 2$. Among the infinitely many real zeros of the function $\Delta(\lambda) - 2$ there must be a smallest one which we call $\lambda_o$. We shall prove later (see Lemma 2.6) that $\Delta'(\lambda_o) > 0$. Therefore Lemma 2.4 shows that, for $\lambda > \lambda_o$, $\Delta(\lambda)$ must be a decreasing function until $\Delta(\lambda) = -2$. This must actually happen for a certain $\lambda = \lambda_1' > \lambda_o$, since $\Delta(\lambda) + 2$ has infinitely many real zeros without a finite limit point. Now, either $\Delta'(\lambda_1') = 0$, or $\Delta'(\lambda_1') < 0$. If $\Delta'(\lambda_1') = 0$, $\lambda_1'$ is a double root of $\Delta(\lambda) + 2 = 0$ and will be listed as $\lambda_1'$ and as $\lambda_2'$. (According to Lemma 2.5 proved below $\Delta(\lambda) + 2 = 0$ cannot have roots of multiplicity higher than two.) $\Delta'(\lambda_1') = 0$ implies (cf. Lemma 2.4) that $\Delta(\lambda)$ increases for $\lambda > \lambda_2' = \lambda_1'$ until it reaches the value 2. On the other hand, if $\Delta'(\lambda_1') < 0$, then $\Delta(\lambda) < -2$ for $\lambda_1' < \lambda < \lambda_2'$, where $\lambda_2'$ is the smallest zero of $\Delta(\lambda) + 2$ which is $> \lambda_1'$. Since $\Delta(\lambda) < -2$ in the interval $(\lambda_1', \lambda_2')$ this is an interval of instability for the solutions of (2.1) (cf. the stability test of Section 1.2). Now, $\Delta'(\lambda_2') \geq 0$ as can be seen from the fact that $\Delta(\lambda) < \Delta(\lambda_2')$ for all $\lambda < \lambda_2'$ and sufficiently close to $\lambda_2'$. Using Lemma 2.4 we may therefore conclude that $\Delta(\lambda)$ is an increasing function of $\lambda$ in any interval $\lambda_2' < \lambda < \hat{\lambda}$ in which $\Delta(\lambda) < 2$. The largest interval of this kind is the interval $(\lambda_2', \lambda_1)$, where $\lambda_1$ denotes the smallest root of $\Delta(\lambda) - 2$ which is $> \lambda_2'$. The stability test of Section 1.2 shows that this interval is an interval of stability for the solutions of (2.1).

Continuing in this manner we find that the inequalities (2.4) hold and that the open intervals (2.6) are the only intervals of stability for the solutions of (2.1).

We now prove Lemma 2.4. For this purpose, we introduce the following notations:

$$\frac{\partial}{\partial\lambda} y_1(x,\lambda) = z_1(x,\lambda), \qquad \frac{\partial}{\partial\lambda} y_2(x,\lambda) = z_2(x,\lambda),$$

$$\frac{\partial}{\partial\lambda} y_1'(x,\lambda) = z_1'(x,\lambda), \qquad \frac{\partial}{\partial\lambda} y_2'(x,\lambda) = z_2'(x,\lambda)$$

where obviously $z_1' = \frac{\partial}{\partial x} z_1$ and $z_2' = \frac{\partial}{\partial x} z_2$. Also, we shall write $\eta_1$, $\eta_2$, $\eta_1'$, $\eta_2'$ respectively for $y_1(\pi,\lambda)$, $y_2(\pi,\lambda)$, $y_1'(\pi,\lambda)$, $y_2'(\pi,\lambda)$. As before, we shall write $\Delta$ for $\eta_1 + \eta_2'$ and $\Delta'$ for the derivative of $\Delta$ with respect to $\lambda$, that is:

$$\Delta' = z_1(\pi,\lambda) + z_2'(\pi,\lambda).$$

Next, we shall derive the formula

$$(2.13) \qquad \Delta'(\lambda) = (\eta_1 - \eta_2') \int_0^\pi y_1(x)y_2(x)dx - \eta_2 \int_0^\pi y_1^2(x)dx + \eta_1' \int_0^\pi y_2^2(x)dx.$$

To prove (2.13), we differentiate (2.1) with respect to $\lambda$ and we obtain (for $y = y_1$ and $y = y_2$ respectively)

$$(2.14) \qquad \begin{aligned} z_1'' + (\lambda + Q)z_1 &= -y_1 \\ z_2'' + (\lambda + Q)z_2 &= -y_2 \end{aligned}$$

The general formula for the solution of an inhomogeneous linear differential equation of the second order in terms of the solution of the homogeneous equation yields

$$(2.15) \qquad \begin{aligned} z_1(x) &= y_1(x) \int_0^x y_2(t)y_1(t)dt - y_2(x) \int_0^x y_1^2(t)dt \\ z_1'(x) &= y_1'(x) \int_0^x y_2(t)y_1(t)dt - y_2'(x) \int_0^x y_1^2(t)dt \\ z_2(x) &= y_1(x) \int_0^x y_2^2(t)dt - y_2(x) \int_0^x y_1(t)y_2(t)dt \\ z_2'(x) &= y_1' \int_0^x y_2^2(t)dt - y_2'(x) \int_0^x y_1(t)y_2(t)dt \quad . \end{aligned}$$

The functions $z_1$ and $z_2$ in (2.15) are those solutions of (2.14) which satisfy the initial conditions $z_1(0) = z_1'(0) = 0$, $z_2(0) = z_2'(0) = 0$. Since $z_1 = (\frac{\partial}{\partial\lambda})y_1$ etc., and since the initial conditions for $y_1$ and $y_2$ are independent of $\lambda$, the solutions (2.15) are the correct ones. Equation (2.13) follows immediately from (2.15) by putting $x = \pi$.

Since the Wronskian $y_1 y_2' - y_2 y_1' = 1$ for all x, we find for $x = \pi$ that

(2.16) $$\eta_1 \eta_2' - \eta_2 \eta_1' = 1.$$

Hence

$$\Delta^2 - 4 = (\eta_1 + \eta_2')^2 - 4(\eta_1 \eta_2' - \eta_2 \eta_1') = (\eta_1 - \eta_2')^2 + 4\eta_1' \eta_2 .$$

Putting sgn $\eta_1' = +1$ if $\eta_1' > 0$, sgn $\eta_1' = -1$ if $\eta_1' < 0$ and sgn $\eta_1' = 0$ if $\eta_1' = 0$, and assuming that $\eta_1' \neq 0$, we find from (2.13):

(2.17) $$\Delta'(\lambda) = \operatorname{sgn} \eta_1' \left\{ \int_0^\pi \left( \sqrt{|\eta_1'|}\; y_2 + \operatorname{sgn} \eta_1' \frac{\eta_1 - \eta_2'}{2\sqrt{|\eta_1'|}} y_1 \right)^2 dx - \frac{\Delta^2 - 4}{4|\eta_1'|} \int_0^\pi y_1^2 \, dx \right\} .$$

Equation (2.17) shows that $\Delta'(\lambda)$ has the same sign as $\eta_1'$ in any interval in which $\eta_1' \neq 0$ and $\Delta^2 \leq 4$. Consider now a value $\mu$ of $\lambda$ such that $\Delta(\mu) = 2$ and $\Delta'(\mu) \leq 0$. We wish to establish the fact that, for a sufficiently small $\delta$, $\Delta(\lambda)$ is decreasing in the interval $\mu < \lambda < \lambda + \delta$. If $\Delta'(\mu) > 0$, this is obvious. Assume now that $\Delta(\mu) = 2$, $\Delta'(\mu) = 0$. In this case, according to (2.17), we must have $\eta_1'(\mu) = 0$. Since we also have

$$\Delta^2 - 4 = (\eta_1 - \eta_2')^2 + 4\eta_1' \eta_2 = 0,$$

we find that $\eta_1(\mu) - \eta_2'(\mu) = 0$ and, since $\eta_1 \eta_2' \eta_2 \eta_1' = 1$, we have $\eta_1(\mu) = \eta_2'(\mu) = 1$. Then (2.13) reduces to

$$\Delta'(\mu) = -\eta_2 \int_0^\pi y_1^2(x) \, dx,$$

and therefore $\Delta'(\mu) = 0$ implies $\eta_2(\mu) = 0$.

Now we shall compute

$$\Delta''(\lambda) = \frac{d\,\Delta'(\lambda)}{d\lambda}$$

for $\lambda = \mu$, where $\mu$ is such that

$$\eta_1'(\mu) = \eta_2(\mu) = 0, \quad \eta_1(\mu) = \eta_2'(\mu) = 1. \tag{2.18}$$

We shall do this by differentiating (2.13) with respect to λ and by using (2.15) for $x = \pi$ in order to obtain the derivatives with respect to λ of $\eta_1'(\lambda)$, etc., for $\lambda = \mu$. A straightforward computation shows that, if (2.18) holds,

$$\Delta''(\mu) = 2\left\{\int_0^\pi y_1(x)y_2(x)dx\right\}^2 - 2\int_0^\pi y_1^2(x)dx \int_0^\pi y_2^2(x)dx \ . \tag{2.19}$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent functions, we see from an application the Schwartz inequality to (2.19) that

$$\Delta''(\mu) < 0 \ . \tag{2.20}$$

Thus $\Delta'(\lambda)$ is again found to be decreasing in an interval $\mu < \lambda < \mu+\delta$. Assume now that 2.4 is false. Then there would exist a smallest number $\mu^* > \mu$ such that $\Delta'(\lambda) < 0$ for $\mu < \lambda < \mu^*$ but $\Delta'(\mu^*) = 0$ although $\Delta(\mu^*) > -2$. We would then have

$$\Delta^2(\mu^*) - 4 = (\eta_1 - \eta_2')^2 + 4\,\eta_1'\,\eta_2 < 0 \tag{2.21}$$

and therefore $\eta_1'\,\eta_2 < 0$ for $\lambda = \mu^*$. But then, $\eta_1'(\mu^*) \neq 0$ and so, according to (2.17), $\Delta(\mu^*) \neq 0$, which produces a contradiction. This proves Lemma 2.4 in the case where $\Delta(\lambda) = 2$.

If $\Delta(\lambda) = -2$, the proof is almost literally the same. Incidentally, our proof of Lemma 2.4 shows that the following is true:

<u>Lemma 2.5.</u> The roots of the equation

$$\Delta^2(\lambda) - 4 = 0$$

are either simple or double roots. If, for a particular value of $\lambda = \mu$,

$$\Delta^2(\mu) = 4, \quad \Delta'(\mu) = 0, \tag{2.22}$$

then $\Delta''(\mu) < 0$ if $\Delta(\mu) = 2$ and $\Delta''(\mu) > 0$ if $\Delta = -2$. Necessary and sufficient conditions for $\Delta^2(\mu) - 4$ and $\Delta'(\mu)$ to vanish simultaneously are

$$\eta_1(\mu) - \eta_2'(\mu) = \eta_1'(\mu) = \eta_2(\mu) = 0. \tag{2.23}$$

In order to complete the proof of Theorem 2.1 we need

Lemma 2.6. Let $\lambda_o$ be the smallest root of the equation $\Delta^2(\lambda) - 4 = 0$. Then $\lambda_o$ is a simple root and $\Delta'(\lambda_o) < 0$.

Proof. We know from Lemma 2.1 that $\Delta(\lambda) > 2$ for $\lambda < \lambda_o$. Therefore $\lambda = \lambda_o$ cannot be a maximum of $\Delta(\lambda)$ if $\Delta(\lambda_o) = 2$. But Lemma 2.5 shows that $\Delta(\lambda)$ would have a maximum at $\lambda = \lambda_o$ if $\Delta'(\lambda_o) = 0$. This is a contradiction, and therefore Lemma 2.6 is true.

A comparison of Lemma 2.5 and of Floquet's theorem in Chapter I shows immediately that we can supplement Theorem 2.1 by the following

Corollary 2.1. Hill's equation (2.1) has two linearly independent periodic solutions of period $\pi$ or $2\pi$ if and only if the equation $\Delta^2(\lambda) - 4 = 0$ has a double root.

The stability test at the end of Section 1.2, Corollary 2.1 and Lemma 2.5 show that the solutions of (2.1) are stable for $\lambda = \lambda_{2n+1}$ (or $\lambda = \lambda'_{2n+1}$) if and only if $\lambda_{2n+1}$ (or $\lambda'_{2n+1}$) is a double root of $\Delta^2(\lambda) - 4$, that is, if and only if $\lambda_{2n+1} = \lambda_{2n+2}$ (or $\lambda'_{2n+1} = \lambda'_{2n+2}$). This proves Theorem 2.1 completely.

## 2.2. Analytic properties of the discriminant

The function $\Delta(\lambda)$ as defined by (2.12) will be called the discriminant of Hill's equation (2.1). In this section, we shall prove a result which we used already in Section 2.1 and which we may state as follows:

Theorem 2.2. The function

$$\Delta(\lambda) = y_1(\pi, \lambda) + y_2'(\pi, \lambda)$$

is an entire analytic function of the complex variable $\lambda$. Its order of growth for $|\lambda| \to \infty$ is exactly $\frac{1}{2}$; i.e., there exists a positive constant M such that

$$|\Delta(\lambda)| \exp(-M\sqrt{|\lambda|}) \tag{2.24}$$

is bounded for all $\lambda$ and a positive constant m such that, $\lambda$ real and $\lambda \to -\infty$ implies

$$|\Delta(\lambda)| \exp(-m\sqrt{|\lambda|}) \to \infty . \tag{2.25}$$

Corollary 2.2. The functions $\triangle(\lambda) + 2$ and $\triangle(\lambda) - 2$ have infinitely many zeros.

Note that Corollary 2.2 follows immediately from Theorem 2.2 which permits us to conclude that $\triangle(\lambda) + 2$ and $\triangle(\lambda) - 2$ are functions of order of growth $\frac{1}{2}$. According to a well known theorem on entire functions, (see Nevanlinna, 1936 or Titchmarsh, 1938) any entire function of order of growth $\frac{1}{2}$ has infinitely many zeros.

In order to prove Theorem 2.2, we apply Picard's method of iteration to the differential equation (2.1). Let $\omega = \sqrt{\lambda}$ and let

$$u_o = \cos \omega x, \qquad v_o = \frac{\sin \omega x}{\omega}$$

and define $u_n$, $v_n$ recursively for $n = 1, 2, \ldots$ by

$$u_n(x,\omega) = -\frac{1}{\omega}\int_o^x \sin \omega(x-\xi)\, Q(\xi)\, u_{n-1}(\xi)d\xi \tag{2.26}$$

$$v_n(x,\omega) = -\frac{1}{\omega}\int_o^x \sin \omega(x-\xi)\, Q(\xi)\, v_{n-1}(\xi)d\xi \quad . \tag{2.27}$$

Then

$$y_1(x,\omega^2) = \sum_{n=o}^{\infty} u_n(x,\omega) \tag{2.28}$$

$$y_2(x,\omega^2) = \sum_{n=o}^{\infty} v_n(x,\omega) \tag{2.29}$$

and

$$\triangle(\lambda) = \triangle(\omega^2) = \sum_{n=o}^{\infty} \left[u_n(\pi,\omega) + v_n'(\pi,\omega)\right] \tag{2.30}$$

where $v_o'(x,\omega) = \cos \omega x$ and

$$v_n'(x,\omega) = -\int_o^x \cos \omega(x-\xi)\, Q(\xi) v_{n-1}(\xi,\omega)d\xi \ . \tag{2.31}$$

It is easy to see that for all real values of $x \geq 0$:

$$|u_o| \leq e^{|\omega|x}, \quad |v_o| \leq x\, e^{|\omega|x} \ .$$

Now let $M^*$ be a positive constant such that, for all real values of x, $|Q(x)| \le M^*$. Then we find by induction from (2.26), (2.27) and from

$$|\sin \omega(x-\xi)| \le |\omega| (x-\xi) \, e^{|\omega| (x-\xi)} \quad (0 \le \xi \le x)$$

that

$$(2.32) \qquad |u_n(x,\omega)| \le e^{|\omega| x} (M^* x^2)^n \Big/ (2n)!$$

$$(2.33) \qquad |v_n(x,\omega)| \le x \, e^{|\omega| x} (M^* x^2)^n \Big/ (2n+1)! \ .$$

Equations (2.33) and (2.31) show that

$$(2.34) \qquad |v_n'(x,\omega)| \le e^{|\omega| x} (M^* x^2)^n \Big/ (2n)!$$

and we therefore conclude from (2.30) that

$$(2.35) \qquad |\triangle(\omega^2)| \le 2 \, e^{|\omega| \pi} \cosh (\sqrt{M^*} \, \pi) \ .$$

This proves that the expression (2.24) is bounded if we choose $M = \pi$. In order to complete the proof of Theorem 2.2, we have to prove (2.25). For this purpose, we may assume that $Q(x) \le -1$ for all x. Otherwise, we could replace Q by $Q-M^*-1$ and $\lambda$ by $\lambda + M^* + 1$ without changing the differential equation for y. Putting $\sqrt{\lambda} = i\phi = \omega$, where $\phi$ is real and positive and $\phi \to \infty$ as $\lambda \to -\infty$, we find

$$u_o \ge \frac{1}{2} e^{\phi x} \ge \frac{1}{2} e^{\phi x/2}$$

$$\frac{\sin \omega(x-\xi)}{\omega} = \frac{\sinh \phi(x-\xi)}{\phi} = \frac{1}{2} \int_{\xi-x}^{x-\xi} e^{\phi s} \, ds \ge$$

$$\frac{1}{2} \int_{(x-\xi)/2}^{x-\xi} e^{\phi s} \, ds \ge \frac{1}{4} (x-\xi) e^{\phi(x-\xi)/2} \ .$$

From the preceding inequalities and from (2.26) we find by induction with respect to n that

$$(2.36) \qquad u_n(x,i\phi) \geq \frac{1}{2} e^{\phi x/2}(x/2)^{2n} \Big/ (2n)! \ .$$

By an even simpler argument, we can prove that $v_n'(x,i\phi) \geq 0$, and therefore we find form (2.36) and (2.30) that

$$(2.37) \qquad \triangle(\lambda) = \triangle(-\phi^2) \geq \sum_{n=o}^{\infty} u_n(\pi,i\phi) \geq \frac{1}{2} e^{\phi\pi/2} \cosh(\pi/2) \ .$$

This proves that (2.25) is true for any m between 0 and $\pi/2$.

Obviously, all the functions $u_n$ and $v_n'$ are entire analytic functions and $\lambda$ and, since we have shown that their sums converge uniformly in any finite part of the $\lambda$-plane, it follows that $\triangle(\lambda) = y_1(\pi,\lambda) + y_2'(\pi,\lambda)$ is also an entire function of $\lambda$. The inequalities (2.35) and (2.37) establish the truth of our assertion about the order of growth of $\triangle(\lambda)$ and this completes the proof of Theorem 2.2.

For later purposes we note here a result which can be proved by exactly the same method as Theorem 2.2.

**Theorem 2.3.** *Let $y(x,\lambda)$ be any real solution of Hill's equation (2.1) with initial conditions independent of $\lambda$. Let x be fixed and real, and consider $y(x,\lambda)$ and $y'(x,\lambda)$ as functions of $\lambda$. Then the order of growth of these two functions of $\lambda$ is at most $\frac{1}{2}$.*

The following theorem gives an idea about the asymptotic behavior of $\triangle(\lambda)$ for positive values of $\lambda$.

**Theorem 2.4.** *The absolute value of the function*

$$\sqrt{\lambda}\left[\triangle(\lambda) - 2\cos\pi\sqrt{\lambda}\right]$$

*is bounded for all real, positive values of $\lambda$.*

**Proof.** If $\lambda = \omega^2 > 0$, we find from (2.26) and (2.27) (by induction) that for $x \geq 0$ and $n = 1, 2, 3, \ldots$:

$$|u_n(x,\omega)| \leq (M^*)^n x^n \omega^{-n} / n!$$

$$|v_n(x,\omega)| \leq (M^*)^n x^n \omega^{-n-1} / n!$$

The estimate for $|v_n|$ together with (2.31) shows that

$$|v_n'(x,\omega| \leq (M^*)^n x^n \omega^{-n} / n!$$

Therefore we see from (2.28) and (2.29), that, for $x > 0$,

(2.38) $$|y_1(x,\omega^2) + y_2'(x,\omega^2) - 2\cos\omega x| \leq 2\exp(xM^*/\omega) - 2$$

and, in particular, for $x = \pi$:

$$|\Delta(\lambda) - 2\cos\sqrt{\lambda}\,\pi| \leq 2\exp\frac{\pi M^*}{\omega} - 2\,.$$

Since for any real $t > 0$

$$\exp t - 1 = \int_0^t \exp s\, ds \leq t\exp t$$

we find that

(2.39) $$|\Delta(\lambda) - 2\cos\pi\sqrt{\lambda}\,| \leq \frac{e\pi M^*}{\sqrt{\lambda}}\exp\frac{\pi M^*}{\sqrt{\lambda}}$$

which proves Theorem 2.4.

Obviously, Theorem 2.4 is true for all bounded, square integrable functions $Q(x)$. In order to obtain better results pertaining to the asymptotic behavior of $\Delta(\lambda)$ for $\lambda > 0$, $\lambda \to \infty$, we shall assume that $Q(x)$ can be expanded in a Fourier series

(2.40) $$Q(x) = \sum_{n=-\infty}^{+\infty}{}' g_n e^{2inx}\,,$$

where the prime at the summation symbol indicates that the sum is extended over values of $n \neq 0$ only. This does not impose restriction on $Q(x)$, since a constant term in (2.40) could be combined with the parameter $\lambda$ in Hill's equation (2.1). Equivalently, we could say that $Q(x)$ has been normalized so that

(2.41) $$\int_0^\pi Q(x)\,dx = 0$$

Since $Q(x)$ is real, the constants $g_n$ in (2.40) satisfy the condition

(2.41*) $$g_{-n} = \bar{g}_n$$

for all n, where a bar denotes the conjugate complex quantity. We shall always assume that

(2.42) $$\sum_{n=-\infty}^{+\infty} |g_n| > \infty$$

and in many cases that

(2.43) $$\lim_{n \to \pm\infty} n^2 g_n = 0;$$

(this will be true if Q(x) has a continuous second derivative).

If we put

(2.44) $$\Delta_n(\lambda) = u_n(\pi, \sqrt{\lambda}) + v_n'(\pi, \sqrt{\lambda}) ,$$

where $u_n$, $v_n$ are defined by (2.26) and (2.27), then

$$\Delta(\lambda) = \sum_{n=0}^{\infty} \Delta_n(\lambda) .$$

Obviously, $\Delta_n(\lambda)$ is a homogeneous form of degree n in the infinitely many variables $g_n$, that is

(2.45) $$\Delta_n(\lambda) = \sum_{\ell_1, \ldots, \ell_n = -\infty}^{\infty} c(\ell_1, \ldots, \ell_n)\, g_{\ell_1} \cdots g_{\ell_n}$$

where we may sum without restrictions if we assume that $g_0 = 0$. Now we shall prove:

Theorem 2.5. Let $\omega = \sqrt{\lambda}$. Then

$$c(\ell_1, \ldots, \ell_n) = A(\omega) \cos \pi\omega + B(\omega) \frac{\sin \pi\omega}{\omega} ,$$

where $A(\omega)$ and $B(\omega)$ are even rational functions of $\omega$ such that

(1) In each of the functions $A(\omega)$ and $B(\omega)$ the degree of the denominator exceeds the degree of the numerator by at least n.

(2) The poles of $A(\omega)$ and $B(\omega)$ are at some of the points $\omega = 0$ and

$$\omega = \pm (\ell_r + \ell_{r+1} + \ldots + \ell_s) ,$$

where $1 \leq r \leq s \leq n$.

Proof of theorem 2.5. We shall use the expression for $\Delta_n$ in terms of $u_n(\pi, \lambda)$ and $v_n'(\pi, \lambda)$. However, the expressions of $u_n$ and $v_n'$ as given by (2.26) and (2.31) involve n integrations, and direct evaluation of these integrals leads to results which are cumbersome in form. We shall therefore make use of the fact that the integrals in (2.26) and (2.31) are convolution integrals the Laplace transforms of which can be easily computed. We find for the coefficient of

$$g_{\ell_1} g_{\ell_2} \cdots g_{\ell_n}$$

in

$$\int_0^\infty e^{-px} \left[u_n(x,\lambda) + v_n'(x,\lambda)\right] dx$$

the expression

$$(2.46) \qquad k(\ell_1,\ldots,\ell_n,p) = \frac{2p - 2i \sum_{v=1}^{n} \ell_v}{\left[\omega^2+p^2\right] \prod_{v=1}^{n} \left[\omega^2 + (p-2i\ell_1 - \cdots - 2i\ell_v)^2\right]} .$$

By an application of the inversion formula for the Laplace transformation we find that $c(\ell_1,\ldots\ell_n)$ is the sum of the residues of

$$U = e^{xp} k(\ell_1,\ldots,\ell_n,p).$$

Obviously, the poles of U are located at

$$p = i\omega,\ i\omega + 2i\ell_1,\ i\omega + 2i\ell_1 + 2i\ell_2, \ldots$$

and Theorem 2.5 follows now from an inspection of U in the neighborhood of its poles and from the remark that $c(\ell_1,\ldots,\ell_n)$ must be an even function of $\omega$ (since it is an entire function of $\lambda$).

Since the product $g_{\ell_1} \cdots g_{\ell_n}$ does not change if we permute the variables $g_{\ell_1},\ldots,g_{\ell_n}$, we find that $\Delta_n(\lambda)$ can also be written in the form

$$(2.47) \qquad \Delta_n(\lambda) = \sum_{\ell_1 \le \ell_2 \le \cdots \le \ell_n} \gamma(\ell_1,\ldots,\ell_n) g_{\ell_1} \cdots g_{\ell_n}$$

where $-\infty < \ell_1 \le \ell_2 \cdots \le \ell_n < \infty$. The coefficients g in (2.47) are defined by the relation

$$(2.48) \qquad \gamma(\ell_1, \ldots, \ell_n) = \sum c(K_1, \ldots, K_n),$$

where the sum in (2.48) is to be extended over all distinct sets of integers

$K_1, \ldots, K_n$ which can be made to coincide with the set $\ell_1, \ldots, \ell_n$ by a suitable permutation of the indices 1, ..., n. Concerning the coefficients $\gamma$ we have

Theorem 2.6. The coefficients $\gamma$ defined by (2.48) satisfy the relations

$$\gamma(\ell_1, \ldots, \ell_n) = 0$$

whenever

$$\ell_1 + \ldots + \ell_n \neq 0.$$

Proof. If we replace Q(x) in (2.1) by $Q(x + \phi)$, where $\phi$ is a constant, the resulting differential equation will have the same discriminant $\Delta(\lambda)$ as the original one. In fact, it is clear that if Q(x) is replaced by $Q(x + \phi)$, Hill's equation continues to have periodic solutions for the same values of $\lambda$ and in each case their number is unaffected by the change. Since $\Delta(\lambda) - 2$ is an analytic function of $\lambda$ whose order of growth is $\frac{1}{2}$, $\Delta(\lambda) - 2$ is determined by its zeros up to a multiplicative constant. Assume now that $\Delta(\lambda) - 2$ belongs to the function Q(x) and that $\Delta^*(\lambda) - 2$ belongs to the function $Q(x + \phi)$. Then

$$\Delta(\lambda)-2 = C\left[\Delta^*(\lambda)-2\right],$$

where C is a constant. On the other hand, $\Delta(\lambda)+2$ and $\Delta^*(\lambda)+2$ also have the same zeros (with the same multiplicity), and therefore

$$\Delta(\lambda)+2 = C'\left[\Delta^*(\lambda)+2\right].$$

By subtracting the first of these equations from the second we find that

$$(C' - C)\,\Delta^*(\lambda) = 2C' + 2C - 4.$$

Therefore $\Delta^*(\lambda)$ would be a constant unless $C' = C = 1$. However, Theorem 2.4 shows that $\Delta^*(\lambda)$ can never be a constant. Hence $\Delta(\lambda) = \Delta^*(\lambda)$ for all $\lambda$.

It is obvious that $\Delta^*(\lambda)$ arises from $\Delta(\lambda)$ by substituting $g_n e^{2in\phi}$ for $g_n$, and therefore the coefficient of $g_{\ell_1} \cdots g_{\ell_n}$ in $\Delta^*$ will be

$$g(\ell_1, \ldots, \ell_n)\, e^{2iL\phi},$$

where $L = \ell_1 + \ldots + \ell_n$. Assume now that all of the $g_n$ are zero except for $g_{\ell_1}, g_{\ell_2}, \ldots, g_{\ell_n}$. The same argument which we used in showing that $\Delta(\lambda)$ is an analytic function of $\lambda$ (Theorem 2.2) can be used to show that $\Delta$ depends analytically on $g_{\ell_1}, \dots, g_{\ell_n}$ and the same is true for $\Delta^*$. Hence $\Delta(\lambda) = \Delta^*(\lambda)$, implies that the coefficients of $g_{\ell_1} \dots g_{\ell_n}$ in both functions must be the same, so that for all real values of $\phi$:

$$\gamma(\ell_1, \dots, \ell_n) = e^{2iL\phi} \gamma(\ell_1, \dots, \ell_n).$$

If $L \neq 0$, this is possible only if $\gamma = 0$, as stated in Theorem 2.6. We may note here

Corollary 2.6. The first terms in the expansion (2.44) are

$$\Delta_0(\lambda) = 2 \cos \pi \sqrt{\lambda}, \quad \Delta_1(\lambda) = 0$$

$$\Delta_2(\lambda) = \frac{\pi \sin \pi \sqrt{\lambda}}{2\sqrt{\lambda}} \sum_{n=1}^{\infty} \frac{|g_n|^2}{\omega^2 - n^2}$$

Of these relations, the first one is trivial, the second one follows from Theorem 2.6, and the third one is a result of a somewhat tedious computation.

## 2.3 Infinite Determinants

Hill, 1886, used infinite determinants for the investigation of the characteristic values of $\lambda$ in (2.1). Whittaker and Watson, 1927, showed that the value of Hill's determinant can be expressed in terms of $\Delta(\lambda)$. In this section we shall reproduce the results of Hill and Watson and supplement them by some relations of the type obtained by Whittaker and Watson.

We shall write a determinant in the form

$$\| a_{n,m} \|_k^{\ell}$$

where n and m vary over all integers from k to $\ell$. In particular, we shall consider the determinants where $k = -\infty$, $\ell = \infty$ or where $k = 0$, $\ell = \infty$. These we shall call two sided infinite and one sided infinite determinants, respectively.

We shall always use the first subscript n to denote the rows and the second subscript m to denote the columns of the determinant.

We shall say that the infinite determinants

$$\| a_{n,m} \|_0^{\infty}, \quad \| a_{n,m} \|_{-\infty}^{\infty}$$

exist or converge if the limits

$$\lim_{\ell\to\infty} \|a_{n,m}\|_0^\ell \quad , \quad \lim_{\ell\to\infty} \|a_{n,m}\|_{-\ell}^\ell$$

exist. The value of the limit is then called the value of the determinant. We shall not be concerned here with a general theory of infinite determinants; instead, we shall introduce a special class of infinite determinants and indicate briefly the proof of a few theorems which hold for this class.

We shall say that a determinant is of Hill's type if it satisfies the condition

$$\sum_{n,m} |a_{n,m} - \delta_{n,m}| < \infty \tag{2.49}$$

where $\delta_{n,m} = 1$ for $n = m$ and $\delta_{n,m} = 0$ otherwise, and where the sum in (2.49) is to be taken over all values of n and m. Obviously, every finite determinant is of Hill's type. We shall show now:

Theorem 2.7. An infinite determinant of Hill's type converges.

It suffices to prove Theorem 2.7 in the case of a determinant $\|a_{n,m}\|_0^\infty$. According to a theorem due to Hadamard (see Hardy, Littlewood, Polya, 1934) the absolute value of the square of any finite determinant

$$\|a_{n,m}\|_k^\ell$$

does not exceed the value of the product

$$\prod_{n=k}^{\ell}\left(\sum_{m=k}^{\ell} |a_{n,m}|^2\right)$$

From this we derive the following

Lemma 2.7. Let $\|a_{n,m}\|_0^\infty$ be a determinant of Hill's type. Let

$$a'_{n,m} = a_{n,m} \text{ if } n \neq m$$

$$a'_{n,n} = a_{n,n} \text{ if } |a_{n,n}| \geq 1$$

$$a'_{n,n} = 1 \text{ if } |a_{n,n}| < 1,$$

and let

$$H = \left\{ \prod_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} |a'_{n,m}|^2 \right) \right\}^{1/2}$$

Then $H < \infty$, and the absolute value of any finite subdeterminant of $\| a_{n,m} \|_0^{\infty}$ is at most equal to H.

Proof. The only difficulty is to show that $H < \infty$. For this purpose, let

$$\varepsilon_n = \sum_m{}' |a_{n,m}|$$

where the sum is taken over all $m \neq n$. Also, let

$$\gamma_n = |a'_{n,n}| - 1$$

It follows from (2.49) that

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty, \quad \sum_{n=0}^{\infty} \gamma_n < \infty .$$

Finally, let

$$p_n = \left\{ \sum_{m=0}^{\infty} |a'_{n,m}|^2 - 1 \right\}^{1/2} .$$

Then

$$p_n^2 \leq |a'_{n,n}|^2 - 1 + \left( \sum_m{}' |a_{n,m}| \right)^2 = |a'_{n,n}|^2 - 1 + \varepsilon_n^2 \leq A|a_{nn} - 1| + \varepsilon_n^2$$

where $A = \max|a_{nn}| + 1$.

In view of (2.49)

$$\sum_{n=0}^{\infty} |a_{nn} - 1| < \infty .$$

Also, we know that the convergence of a sum of positive terms implies the convergence of the sum of the squares of the terms. Hence

$$\sum_{n=0}^{\infty} p_n^2 < \infty$$

and

$$H = \left[ \prod_{n=0}^{\infty} (1 + p_n^2) \right]^{1/2} < \infty .$$

This proves Lemma 2.7.

Now we can prove Theorem 2.7. We can expand

$$\| a_{n,m} \|_0^{\ell+1} = D_{\ell+1}$$

in terms of the elements of the last row and their subdeterminants which can be majorized by the quantity H defined in Lemma 2.7. The result is

$$D_{\ell+1} = a_{\ell+1,\ell+1} D_{\ell} + \theta_{\ell} H$$

where $|\theta_{\ell}| \leq \varepsilon_{\ell+1}$. Since $|D_{\ell}| \leq H$, it follows that

$$|D_{\ell+1} - D_{\ell}| \leq H|a_{\ell+1,\ell+1} - 1| + H\varepsilon_{\ell+1}$$

$$= H \sum_{m=0}^{\infty} |a_{\ell+1,m} - \delta_{\ell+1,m}|$$

Therefore

$$\sum_{\ell=1}^{\infty} |D_{\ell+1} - D_{\ell}| < \infty$$

and

$$\lim_{\ell \to \infty} D_{\ell+1} = D$$

exists. This proves Theorem 2.7.

The last result about infinite determinants which we need is

Theorem 2.8. *Let $\| a_{n,m} \|$ be an infinite determinant of Hill's type, and assume that there exist numbers $x_m$ not all of which vanish such that $|x_m| \leq M$ (M fixed) for all m and*

$$\sum_{m} a_{n,m} x_m = 0$$

*for all n. Then*

$$\| a_{n,m} \| = 0.$$

Proof. Since the set of subdeterminants of $\| a_{n,m} \|$ is bounded, it follows that for each n

$$\| a_{n,m} \| x_n = 0.$$

This relation is obtained in exactly the same manner in which the corresponding relation is obtained in the case of a finite system of linear equations. The inequality (2.49) and the condition $|x_m| \leq M$ guarantee the absolute convergence of all sums involved. This proves Theorem 2.8.

We shall now express the discriminant $\Delta(\lambda)$ of Hill's equation in terms of an infinite determinant. For this purpose we shall write (2.1) in the form

(2.50) $$y'' + \left(\sum_{n=-\infty}^{\infty} g_n e^{2inx}\right) y = 0$$

where $\lambda = g_0$ and $Q(x)$ is given by (2.40). We know from Floquet's Theorem that (2.50) has a solution $\not\equiv 0$ of the type

(2.51) $$y = e^{i\alpha x} p(x)$$

where $p(x)$ is a function of period $\pi$ and where

$$2 \cos \pi\alpha = y_1(\pi) + y_2'(\pi).$$

If $Q(x)$ is sufficiently smooth, e.g. if $\sum |g_n| < \infty$, the function $y$ in (2.51) can be expanded in a twice termwise differentiable series

(2.52) $$y(x) = \sum_{n=-\infty}^{\infty} p_n e^{i(\alpha+2n)x}$$

and the left hand side of (2.50) takes the form

(2.53) $$\sum_{-\infty}^{\infty} C_n \, e^{i(\alpha+2n)x} .$$

Since (2.53) must vanish identically, we have $C_n = 0$ for all n. If we write $C_n$ explicitly in terms of the $p_n$ and $g_n$, we have, for $-\infty < n < \infty$,

(2.54) $$\sum_{m=-\infty}^{\infty} \left[g_{n-m} - (\alpha+2n)^2 \, \delta_{n,m}\right] p_m = 0,$$

or, after multiplication by $\left[g_0 - (\alpha+2n)^2\right]^{-1}$ :

(2.55) $$\sum_{m=-\infty}^{\infty} \left[\frac{g_{n-m}}{\lambda-(\alpha+2n)^2} + \delta_{n,m}\right] p_m = 0,$$

where we have replaced $g_0$ by $\lambda$ and where now, just as in (2.41*)

(2.56) $$g_{n-m} = \bar{g}_{m-n} , \quad g_0 = 0 .$$

Obviously, the determinant

$$D(\alpha,\lambda) = \left\| \frac{g_{n-m}}{\lambda-(\alpha+2n)^2} + \delta_{n,m} \right\|_{-\infty}^{\infty} \tag{2.57}$$

converges if $\sum |g_n| < \infty$, except for such values of $\lambda$ and $\alpha$ for which one of the denominators $\lambda-(\alpha+2n)^2$ vanishes.

It is easy to see that

Lemma 2.8. $D(\alpha,\lambda)$ regarded as a function of $\alpha$ is single valued and analytic for all values of $\alpha$ other than the values

$$\alpha = \pm\sqrt{\lambda} - 2n, \quad n = 0, \pm 1, \pm 2, \dots , \tag{2.58}$$

at which the function may have poles. If $\lambda \neq 0$, these poles are (at most) of order one. $D(\alpha,\lambda)$ is periodic (in $\alpha$) with period 2 and for $\alpha \to i\infty$, $D(\alpha,\lambda) \to 1$.

The proof of Lemma 2.8 is mostly routine except for the statement about the periodicity of $D(\alpha,\lambda)$. This follows from the remark that D remains unchanged if we replace $\alpha$ by $\alpha+2$ and at the same time replace n by n-1 and m by m-1. Since both n and m run from $-\infty$ to $+\infty$, the same is true for n-1 and m-1, and therefore D does not change if we replace $\alpha$ by $\alpha+2$. The rest of the proof of Lemma 2.8 is left to the reader.

Since the residues of

$$\frac{g_{n-m}}{\lambda-(\alpha+2n)^2}$$

at the values $\alpha = \sqrt{\lambda} - 2n$ and $\alpha = -\sqrt{\lambda} - 2n$ add up to zero, it follows from the periodicity of $D(\alpha,\lambda)$ that (for $\lambda \neq 0$) all of its residues have the same value K for $\alpha = \sqrt{\lambda} - 2n$ (independent of n) and the value -K for $\alpha = -\sqrt{\lambda} - 2n$. Therefore

$$E(\alpha) = D(\alpha,\lambda) - K \left\{ \operatorname{ctg} \frac{\pi}{2}(\alpha - \sqrt{\lambda}) - \operatorname{ctg} \frac{\pi}{2}(\alpha + \sqrt{\lambda}) \right\} \tag{2.59}$$

is an entire function of $\alpha$ with period 2. Since $E(\alpha)$ is bounded in the strip

$$-1 \le \operatorname{Re} \alpha \le 1 ,$$

$E(\alpha)$ is a constant E. We wish to determine E and K. By letting $\alpha \to i\infty$, we find from Lemma 2.8 that $E = 1$. We cannot determine $K$ explicitly, but we can express it in terms of $D(0,\lambda)$ by putting $\alpha = 0$. The result is

$$K = \frac{1}{2}\left[1 - D(0,\lambda)\right] \operatorname{tg} \frac{\pi}{2}\sqrt{\lambda} . \tag{2.60}$$

We can now apply Theorem 2.8 to (2.55). The infinitely many equations (2.55) are not always the equivalent of (2.54), since $\alpha$ may have one of the exceptional values (2.58). However, we may multiply (2.55) by

$$\left(1 - \frac{\alpha - \sqrt{\lambda}}{2n}\right)\left(1 + \frac{\alpha + \sqrt{\lambda}}{2n}\right) = \frac{(2n+\alpha)^2 - \lambda}{4n^2}$$

for $n \neq 0$ and by $\alpha^2 - \lambda$ for $n = 0$. The determinant of the resulting system must vanish if not all of the $p_m$ vanish. Since

$$\frac{\pi}{4}(\alpha^2 - \lambda) \prod_{n\neq 0} \left(1 - \frac{\alpha-\sqrt{\lambda}}{2n}\right)\left(1 + \frac{\alpha+\sqrt{\lambda}}{2n}\right) = \sin \frac{\pi}{2}(\alpha-\sqrt{\lambda}) \sin \frac{\pi}{2}(\alpha+\sqrt{\lambda}) ,$$

it follows that the existence of a solution of type (2.52) implies the relation

$$\sin \frac{\pi}{2}(\alpha-\sqrt{\lambda}) \sin \frac{\pi}{2}(\alpha+\sqrt{\lambda})\, D(\alpha,\lambda) = 0. \tag{2.61}$$

Therefore, we find from (2.59) and (2.60), by a simple calculation,

$$4 \sin^2 \frac{\pi}{2}\sqrt{\lambda}\, D(0,\lambda) = 2 - 2\cos \pi\alpha = 2 - y_1(\pi,\lambda) - y_2'(\pi,\lambda) . \tag{2.62}$$

Alternatively, we could have computed $K$ in terms of $D(1,\lambda)$. The same argument as above would have given us the relation

$$4 \cos^2 \frac{\pi}{2}\sqrt{\lambda}\, D(1,\lambda) = 2 + y_1(\pi,\lambda) + y_2'(\pi,\lambda) . \tag{2.63}$$

Summarizing, we have

Theorem 2.9. The discriminant $\Delta(\lambda)$ of Hill's equation (2.1) can be expressed in two ways as an infinite determinant involving the Fourier coefficients $g_n$ of $Q(x)$, (which are normalized so that $g_0 = 0$ and $g_{-n} = \bar{g}_n$); namely, with

$$D_0(\lambda) = \left\| \frac{g_{n-m}}{\lambda - 4n^2} + \delta_{n,m} \right\|_{-\infty}^{\infty}$$

and

$$D_1(\lambda) = \left\| \frac{g_{n-m}}{\lambda-(2n+1)^2} + \delta_{n,m} \right\|_{-\infty}^{\infty} ,$$

We have:

$$2 - \Delta(\lambda) = 4 \sin^2 \left(\frac{\pi}{2}\sqrt{\lambda}\right) D_o(\lambda)$$

$$2 + \Delta(\lambda) = 4 \cos^2 \left(\frac{\pi}{2}\sqrt{\lambda}\right) D_1(\lambda) .$$

In the case where $Q(x) = Q(-x)$, the determinants $D_o$ and $D_1$ can be factored into the product of two infinite determinants, each of which can be expressed in terms of the factors $y_1(\frac{\pi}{2})$, ..., $y_2'(\frac{\pi}{2})$ of $\Delta - 2$ and $\Delta + 2$ (see Section 1.3, Theorem 1.3 and equation (2.17)). We have

Theorem 2.10. Let $\varepsilon_n = 2$ for $n = 1,2,3,$ ... and $\varepsilon_o = 0$. Let $\operatorname{sgn} n = +1$ for $n = 1,2,3,$ ..., $\operatorname{sgn}(-n) = -\operatorname{sgn} n$ and $\operatorname{sgn} 0 = 0$. Then the four infinite determinants

$$C_o(\lambda) = \left\| \delta_{n,m} + \frac{(g_{n-m}+g_{n+m})(1+\operatorname{sgn} n \operatorname{sgn} m)}{\sqrt{\varepsilon_n \varepsilon_m}\,(\lambda - 4n^2)} \right\|_o^{\infty}$$

$$S_o(\lambda) = \left\| \delta_{n,m} + \frac{g_{n-m} - g_{n+m}}{\lambda - 4n^2} \right\|_1^{\infty}$$

$$C_1(\lambda) = \left\| \delta_{n,m} \frac{(g_{n-m} + g_{n+m+1})\,[1+\operatorname{sgn} n \operatorname{sgn}(m+1)]}{\sqrt{\varepsilon_n \varepsilon_m}\,[\lambda-(2n+1)^2]} \right\|_o^{\infty}$$

$$S_1(\lambda) = \left\| \delta_{n,m} + \frac{g_{n-m} - g_{n+m+1}}{\lambda-(2n+1)^2} \right\|_o^{\infty}$$

(where the $g_n$ are real and $g_n = g_{-n}$; $g_o = 0$) satisfy the relations

$$C_o(\lambda)\, S_o(\lambda) = D_o(\lambda)$$

$$C_1(\lambda)\, S_1(\lambda) = D_1(\lambda)$$

$$\sqrt{\lambda} \sin\left(\frac{\pi}{2}\sqrt{\lambda}\right) C_o(\lambda) = -y_1'\left(\frac{\pi}{2}, \lambda\right)$$

$$\frac{\sin \frac{\pi}{2}\sqrt{\lambda}}{\sqrt{\lambda}} S_0(\lambda) = y_2(\tfrac{\pi}{2}, \lambda)$$

$$\cos(\pi\sqrt{\lambda})\, C_1(\lambda) = y_1(\tfrac{\pi}{2}, \lambda)$$

$$\cos(\pi\sqrt{\lambda})\, S_1(\lambda) = y_2'(\tfrac{\pi}{2}, \lambda)$$

For a proof of Theorem 2.10 see Magnus, 1955.

Theorems 2.9 and 2.10 are useful for the computation of the first characteristic values of Hill's equation. Theorem 2.9 could also be used for the computation of the first terms $\Delta_n(\lambda)$ (see 2.45) in the expansion of $\Delta(\lambda)$. However, the higher term will then appear in a different form. For example, we find from Theorem 2.9 that

$$\Delta_2(\lambda) = 4\sin^2 \frac{\pi}{2}\sqrt{\lambda} \sum_{n,m=-\infty}^{\infty} \frac{g_{n-m}\, g_{m-n}}{(\lambda-n^2)(\lambda-m^2)} \quad (2.64)$$

and it requires a considerable number of calculations to derive from this the result in Corollary 2.6.

## 2.4. Asymptotic behavior of the characteristic values

In this section we shall be concerned with estimates (upper and lower bounds) for the characteristic values $\lambda_n (n = 0,1, \ldots)$ and $\lambda'_m (m = 1,2,3, \ldots)$ or, alternatively, with the location of the intervals of stability as defined in Theorem 2.1 of Section 2.1.

For large n or m, Theorem 2.5 and Corollary 2.6 yield some information about the $\lambda_n$ and $\lambda'_m$ which, according to Floquet's Theorem, are the roots of the equations $\Delta(\lambda) - 2 = 0$ and $\Delta(\lambda) + 2 = 0$. However, much better results are available, although these are rather difficult to obtain. Here we shall merely state without proofs a few results. (For a detailed account of the results available and for references to the extensive literature we refer the reader to Starzinskii, 1955 and Krein, 1955.) We have, according to Borg, 1944:

<u>Theorem 2.11.</u> Let Q(x) in (2.1) be normalized so that

$$\int_0^\pi Q(x)\, dx = 0$$

and let

$$\frac{1}{\pi}\int_0^{\pi} |Q(x)|\, dx = A.$$

Let n be an integer such that

$$\frac{1}{2} A < n .$$

Then the (n+1)-st interval of stability for (2.1) contains the interval defined by

$$n + \frac{A}{2n} < \sqrt{\lambda} < n + 1 - \frac{A}{2n}$$

or equivalently (for n = 1,2,3, ...):

$$\sqrt{\lambda'_{2n-1}} > 2n-1 - \frac{A}{4n-2} \quad , \quad \sqrt{\lambda'_{2n}} < 2n-1 + \frac{A}{4n-2}$$

$$\sqrt{\lambda_{2n-1}} > 2n - \frac{A}{4n} \quad , \quad \sqrt{\lambda_{2n}} < 2n + \frac{A}{4n}$$

A result much stronger than Theorem 2.11 is known in the symmetric case (see Section 1.3), where the characteristic values are the eigenvalues of an ordinary boundary value problem of the Sturm-Liouville type. We have (Borg, 1946):

<u>Theorem 2.12.</u> <u>Let</u> $Q(x) = Q(-x)$, $Q(x+\pi) = Q(x)$, <u>and assume that</u> $Q(x)$ <u>has continuous first and second derivatives.</u> <u>Let</u> $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda'_1, \lambda'_2, \ldots$ <u>be defined as in Theorem</u> 2.1. <u>Let</u> n <u>denote a positive integer.</u> <u>Then for</u> $n \to \infty$

$$\lambda'_{2n-1} = (2n-1)^2 + \frac{C}{(4n)^2} + o(n^{-2})$$

$$\lambda'_{2n} = (2n-1)^2 + \frac{C}{(4n)^2} + o(n^{-2})$$

$$\lambda_{2n} = 4n^2 + \frac{C}{(4n)^2} + o(n^{-2})$$

$$\lambda_{2n-1} = (4n)^2 + \frac{C}{(4n)^2} + o(n^{-2})$$

<u>where</u>

$$C = \frac{1}{\pi} \int_0^{\pi} \left[Q(x)\right]^2 dx \quad .$$

Theorem 2.12 shows that, in the symmetric case where $Q(x) = Q(-x)$, the intervals of instability tend to zero like $o(\lambda^{-1})$ as $\lambda \to \infty$. No equivalent theorem seems to be known in the case of a general $Q(x)$. In fact, a result due to Borg, 1946, makes it appear unlikely for such a theorem to hold. Borg's Theorem states that two sequences $\lambda_n$ and $\lambda'_n$ ($n = 1,2,3, \ldots,$ excluding $\lambda_0$) determine uniquely an even function $Q(x)$ such that the $\lambda_n$, $\lambda'_n$ are characteristic values for (2.1) with this particular $Q(x)$, provided that the $\lambda_n$ and $\lambda'_n$ satisfy (2.4) and certain asymptotic conditions as $n \to \infty$. If the characteristic values belonging to (2.1) for any $Q(x)$ would satisfy the same asymptotic conditions as those belonging to an even $Q(x)$, then it would be possible to assign an even function $Q_0(x)$ to every $Q(x)$ such that (2.1) would have the same characteristic values for either $Q_0$ or $Q$. It seems very unlikely that assertion of this type should be true.

Theorems 2.11 and 2.12 are due to Borg, 1944, 1946. For various related theorems see Starzinskii, 1955 and Putnam, 1954. Of these we shall formulate here only the following result which seems to have been the first of its type and which is due to Liapounoff, 1907:

Theorem 2.13 *Let $p(x) \not\equiv 0$ be a non-negative piecewise continuous periodic function with period $\pi$. Then all solutions of*

$$y'' + p(x)\, y = 0$$

*are bounded for all values of x if*

$$\pi \int_0^{\pi} p(x)\, dx \leq 4.$$

*This condition is best possible in the sense that, for any $\varepsilon > 0$, there exists a non-negative piecewise continuous function $p_0(x)$ satisfying*

$$p_0 \not\equiv 0 \quad , \quad p_0(x+\pi) = p_0(x)$$

*and*

$$\pi \int_0^{\pi} p_0(x)\, dx < 4 + \varepsilon ,$$

such that at least one solution of

$$y'' + p_o(x)\, y = 0$$

is unbounded as $x \to \pm\infty$.

Generalizations of Theorem 2.13 which characterize the n-th interval of stability of (2.1) and are best possible results in the same sense as Theorem 2.13 are due to Borg, 1944.

## 2.5. Properties of the solutions

In this and in the following section, we shall review some non-elementary results. Terms belonging to the general theory of linear differential equations will be used without further explanation.

The following result has been proved by Haupt, 1914:

Theorem 2.14. Let $y(x,\lambda)$ be a non-trivial, real periodic solution of (2.1) with period $\pi$ or $2\pi$. If $\lambda = \lambda'_{2n+1}$ or $\lambda = \lambda'_{2n}$, then y has exactly 2n+1 zeros in the half open interval $0 \leq x < 2\pi$. If $\lambda = \lambda_{2n-1}$ or $\lambda = \lambda_{2n}$, then y has exactly 2n zeros in $0 \leq x < \pi$.

If $\lambda$ belongs to an interval of stability, any solution of (2.1) has infinitely many zeros. This conclusion can be derived from Floquet's Theorem and from the fact that the general solution of (2.1) can be written in the form

$$(2.65) \qquad y = A\sqrt{y_1^2 + y_2^2}\ \cos\left\{\int_0^x \left[y_1^2(t) + y_2^2(t)\right]^{-1} dt + \alpha\right\}$$

where A, $\alpha$ are constants and $y_1$, $y_2$ are the normalized solutions introduced in Section 1.2. Yelchin, 1946, showed how one can decide whether a solution of an equation of Hill's type oscillates or not. For results and references see Starzinskii, 1955.

Periodic solutions of period $\pi$ belonging to different characteristic values $\lambda_n$, $\lambda_m$ of (2.1) are orthogonal in the interval $(0,\pi)$. Any two periodic solutions of period $\pi$ or $2\pi$ are orthogonal in the interval $(0,2\pi)$ if they belong to different characteristic values.

The following result is the analog of the well known theorem about the expansion of a function in a Fourier series. (See Weyl, 1910 or Coddington and Levinson, 1955). We have

Theorem 2.15. Let $\mu_m$, $m = 0,1,2, \ldots$ be the characteristic values $\lambda_0, \lambda_1', \lambda_2', \lambda_1, \ldots$ in their natural order and let $z_m(x)$ be an orthonotmal set of periodic solutions of (2.1) such that $z_m$ satisfies the equation

$$z_m'' + (\mu_m + Q(x)\, z = 0 \ .$$

Then every continuous periodic function of period $2\pi$ whose second derivative is square integrable in every finite interval can be explained in a uniformly and absolutely convergent series

$$\sum_{m=0}^{\infty} c_m \, z_m(x)$$

with constant coefficients $c_m$.

The theory of Weyl, 1910, can also be applied to the differential equation (2.1) for the interval $-\infty < x < \infty$. There always exists a solution of (2.1) which is not square integrable in $(-\infty, 0)$. An analogous statement is valid for the interval $(0, \infty)$. We thus have the limit point case at both end points $-\infty$ and $+\infty$. Since every non-trivial solution of (2.1) fails to be square integrable in $(-\infty, \infty)$ (cf. Floquet's theorem), the spectrum is purely continuous and can be seen to coincide with the union of the intervals of stability (see Hartman and Wintner, 1949). Thus, according to Weyl, 1910, we have the following

Theorem 2.16. Let $f(x)$ be a continuous and two times differentiable function of $x$ which is defined for $-\infty < x < \infty$ and for which

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \quad , \quad \int_{-\infty}^{+\infty} |f''(x)|^2 \, dx < \infty \quad .$$

Let $S$ be the set on the real $\lambda$-axis which consists of the union of the open intervals of stability of (2.1) and their endpoints. Let $y_1(x,\lambda)$ and $y_2(x,\lambda)$ be the normalized solutions of (2.1). Then there exist functions $M_1(\lambda)$ and $M_2(\lambda)$ defined on $S$ and such that

$$f(x) = \int_S \left\{ y_1(x,\lambda)\, dM_1(\lambda) + y_2(x,\lambda)\, dM_2(\lambda) \right\} \quad .$$

According to Borg, 1946, the only case in which the set $S$ is connected occurs if $Q(x) \equiv 0$. This case leads to the ordinary Fourier theorem.

The method used in proving Theorem 2.2 can also be used to show that for every fixed value of x a non-trivial solution $y(x,\lambda)$ of (2.1) is of order of growth $\frac{1}{2}$ with respect to $\lambda$. From the estimates (2.32) and from the theorem due to Paley and Wiener, 1934 (Theorem $\overline{x}$, p. 13) the following fact can be derived: Let x be real and let $y(x,\lambda)$ be a solution of (2.1) for which

$$y(0,\lambda) = a \quad , \quad y'(0,\lambda) = b \quad .$$

Then there exists a function $G(x,\theta)$ of the real variables x, $\theta$ which is defined for $|\theta| \leq |x|$ and is such that for all real values of x and for $\lambda > 0$

$$y(x,\lambda) = a \cos(x\sqrt{\lambda}\,) + \int_{x}^{x} G(x,\theta)\, e^{i\theta\sqrt{\lambda}}\, d\theta \quad .$$

The function $G(x,\theta)$ satisfies the partial differential equation

$$\frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial \theta^2} + Q(x)\, G = 0$$

For details see Magnus, 1955, and for applications, see Gelfand and Levitan, 1955, (p. 296).

## 2.6. The coexistence problem

In the case of an ordinary homogeneous boundary value problem for a linear differential equation

$$y'' + [\lambda + Q(x)]\, y = 0 \, ,$$

(where $Q(x)$ need not be periodic) the characteristic values (eigenvalues) of $\lambda$ are all simple, which means that there cannot exist two linearly independent solutions satisfying the same homogeneous boundary conditions for a finite interval. However, the condition that a non-trivial solution should be periodic (of period $\pi$ or $2\pi$ if Q is of period $\pi$) is of a different nature. Thus, we may have double characteristic values and two linearly independent solutions of period $\pi$ or $2\pi$ associated with the same eigenvalue. The conditions for this to happen have been discussed in Section 2.1. They can be stated by saying that the equation $\Delta^2(\lambda) - 4 = 0$ must have a double root or, more vaguely, by asserting that an interval of instability must disappear. In this case there exists an interval of stability which contains a double root of $\Delta^2(\lambda) - 4 = 0$. It should be emphasized that there always exist infinitely many characteristic values

$\lambda_0, \lambda_1, \lambda_2, \ldots$ and $\lambda_1', \lambda_2', \lambda_3', \ldots$ and that the set of these contains the set of boundary points of the intervals of instability, but that (with the exception of $\lambda_0$) any number of characteristic values may lie within intervals of stability. The question of the conditions which Q must satisfy to insure the existence of two linearly independent solutions for a certain set of characteristic values shall be called the coexistence problem. The following two important results bearing on this problem are due to G. Borg, 1946:

Theorem 2.17. All of the roots of the equation

$$\Delta(\lambda) + 2 = 0$$

are double roots (and two linearly independent solutions of (2.1) with period $2\pi$ exist whenever one such solution exists) if and only if $Q(x)$ has period $\pi/2$.

Corollary 2.17. All of the roots of the equation

$$\Delta^2(\lambda) - 4 = 0$$

with the exception of $\lambda_0$, the smallest one, are double roots (and no interval of instability exists for $\lambda > \lambda_0$) if and only if $Q(x)$ is a constant.

One half of Theorem 2.17 is a consequence of the Corollary to Floquet's Theorem in Section 1.2. In fact, if $Q(x)$ is of period $\pi/2$, then $2\pi$ equals four times the smallest period of $Q(x)$ and therefore all of the non-trivial solutions of (2.1) are of period $2\pi$ if one of them has this property. It is much more difficult to prove the converse of this statement, i.e., to prove the other half of Theorem 2.17, and we must refer the reader to Borg's paper for the proof of this result.

Corollary 2.17 follows from Theorem 2.17 if we note that $Q(x)$ would be periodic with periods $\pi/2, \pi/4, \pi/8, \ldots$ and therefore, a constant.

Examples for equations of Hill's type with a finite number of intervals of stability will be given in the section on Lamé's (Ince's) equation.

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