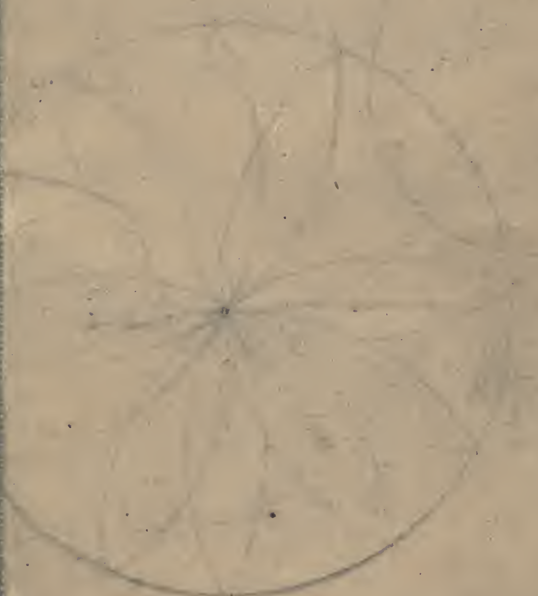
PLANE GEOMETRY.

BY

THOMAS HUNTER.

[illegible]



ELEMENTS

OF

PLANE GEOMETRY.

ELEMENTS

OF

PLANE GEOMETRY.

PART I.

WITH AN APPENDIX ON MENSURATION.

BY THOMAS HUNTER, A.M.,
PRESIDENT OF THE NORMAL COLLEGE OF THE CITY OF NEW YORK.

NEW YORK:
HARPER & BROTHERS, PUBLISHERS,
FRANKLIN SQUARE.
1872.

PREFACE.

THE only apology necessary for adding another to the many text-books with which the market is overstocked is that this little work on Geometry is very much needed. The greater portion of it was prepared while engaged in teaching the subject, and while all the difficulties of presenting it in a practical manner were fresh in the author's mind. Several friends—themselves superior instructors—strongly advised its publication, in the hope that it would render the study of the elementary principles of Geometry more simple and easy of comprehension.

In teaching Geometry, it appeared, at first, singular that students could master the difficulties of Arithmetic and Algebra, and yet fail to comprehend the relations of magnitudes which appealed to the sense of sight. A little closer observation, however, revealed the fact that many students accomplished little more than the committing to memory variations of "A," "B," "C," and "1," "2," "3." Of correct geometrical reasoning they had hardly a conception. Even the intelligent pupils were found unable to apply the principles to new matter; and the solution of problems not in the book was almost an impossibility. Geometry, if properly taught and thoroughly understood, is just as flexible as Arithmetic or Algebra.

Geometry was truly a "rope of sand" to whole classes. The necessary examination completed, the subject was abandoned and forgotten. The principal cause for this was a wide departure by many recent writers from the rigid system of Euclid. For example, Euclid commences with the simple problem, "On a given straight line to construct an equilateral triangle." By means of the postulate or problem, whose solution is self-evident, that "A circle may be described with any point as centre and any length of line as radius," how simple, beautiful, and satisfactory to the mind of the learner the construction becomes! Besides, the compasses and ruler are placed in the hand of the student from the very beginning; he does something for himself; sees its truth, and assimilates it with his intellect. Nearly all the Geometries in use in the schools commence with a theorem. The pupils are told to erect a perpendicular from a given point in a given line. By what authority? By a postulate! Then it is a problem whose solution is self-evident! If so, why after using it to establish, link by link, a chain of truths extending through three books, does the author proceed to demonstrate it? It reminds us of a man who, building a superstructure on a false foundation, is forced to pause in his work when he has completed his third story, and reconstruct a true foundation to prevent the whole edifice from toppling over. It is a problem whose solution is self-evident that "A line may be bisected." Why not, also, that it may be trisected? *Five postulates are subsequently demonstrated.* First, they are self-evident; second, they are not self-evident, and require solution! Is it any wonder that the youthful mind is shocked *ab initio?* Imagine Euclid asking his auditors: "I beg that you will grant that a straight line may be drawn through a given point parallel to a given line." His auditors would have granted no such thing, and would have told him he was begging too much. It is really more "self-evident" that "if one straight line cut another straight line, the opposite or vertical angles are equal." Why not beg this too? or, indeed, beg the whole subject?

Three other works on Geometry contain no postulates whatever! If Geometry be founded on definitions, axioms, and postulates, it is certainly a violation of rigid geometrical reasoning to omit any necessary step in the process. If the first fifteen or twenty propositions are thoroughly taught and perfectly mastered, the subsequent study of geometry is comparatively simple. The aim of the teacher should be to train the scholar rigidly —to take nothing for granted, unless really self-evident or previously demonstrated. The *why* and the *wherefore* of every step must be stated. Each link in the chain must be as strong as any other link.

A great evil has arisen from the attempts to shorten the demonstrations of the propositions. Important omissions are likely to occur, and haste and inaccuracy frequently follow. It is much better to make the proof complete and satisfactory, so that the student will not be obliged to review again and again. In every geometrical demonstration so much and no more is necessary. It is as bad to omit as to add; and great care should be exercised in giving just enough. Circumlocution is tedious; but lack of thoroughness often vitiates the truth. Besides its practical utility, the study of Geometry imparts a love for truth for its own sake; it strengthens the reasoning faculties more than the study of any of the other mathematical sciences; and, unless carried to too great an extent, cultivates clearness, precision, and brevity of expression.

The present volume is intended only for beginners, for those who are preparing for college, and for intermediate and high schools generally. The Geometry of Planes and Solids is omitted. Nearly all the works hitherto published on this subject contain in addition appendices on Plane and Spherical Trigonometry and Logarithms. The vast majority of students rarely advance beyond the geometry of lines, angles, and plane figures. The work in its present form will be cheap, and will exhaust the first and most important department of geometrical study. Any pupil wishing to make further progress can readily do so by taking up any other work on the subject. Should the present volume, however, accomplish its mission, the author will publish a second volume containing the higher departments.

We claim for this little work on Geometry—1. That it commences aright; 2. That it contains more problems, solved and unsolved, than any other volume of its size extant; 3. That it is more practical than the works generally in use; 4. That it contains an appendix on Mensuration of Surfaces, which furnishes a useful application of Arithmetic to the Geometry previously studied.

CONTENTS.

BOOK I.

BOOK II.

BOOK III.

BOOK IV.

BOOK V.

APPENDIX.

ELEMENTS OF PLANE GEOMETRY.

BOOK I.

EXPLANATION OF TERMS.

1. GEOMETRY is that science which treats of the relation and measurement of magnitudes.

2. Magnitudes have three dimensions—length, breadth, and thickness.

3. The Science of Geometry is founded on Definitions, Axioms, and Postulates.

4. A Definition is an explanation of any term or word in a science, showing the sense in which it is employed.

5. An Axiom is a self-evident truth.

6. A Postulate is a self-evident problem.

7. A Theorem is a proposition requiring a demonstration.

8. A Problem is a proposition requiring a solution.

9. A Demonstration is a chain of logical arguments establishing the truth of some proposition.

10. A Direct or Positive Demonstration is one that concludes with certain proof of the proposition.

11. An Indirect or Negative Demonstration is one which proves a proposition to be true by demonstrating that some absurdity must follow if the proposition advanced were false.

12. A Lemma is a preparatory proposition employed for the demonstration of a theorem or the solution of a problem.

13. A Corollary is an obvious consequence deduced from one or more propositions.

14. A Scholium is a remark on one or more preceding propositions, showing their use, their connection, their restriction, or their extension.

15. An Hypothesis is a supposition assumed to be true in the statement of a proposition.

Signs.

1. The sign of Equality is two parallel straight lines of equal length; thus, $A=B$ is read A equals B.

2. The sign of Inequality is an acute angle: the greater quantity is placed at the opening of the angle; thus, $A>B$ is read A greater than B.

3. The sign of Addition is an erect cross; thus, $A+B$ is read A plus B, and means that B is added to A.

4. The sign of Subtraction is a horizontal line; thus, $A-B$ is read A minus B, and means that B is to be taken from A.

5. The sign of Multiplication is an oblique cross; thus, $A\times B$ is read A multiplied by B, and means that A is taken B times. It is also expressed by a point, or by simply writing the letters together; thus, $A.B$ or AB.

6. All the quantities within parentheses, braces, or brackets, or under a vinculum, are considered as one quantity; thus, $(A+B-C)$, $\{A+B-C\}$, $[A+B-C]$, $\overline{A+B-C}$.

7. The sign of Division is a horizontal line, with a dot above and another below; thus, $A\div B$ is read A divided by B; or the division is expressed by making A the numerator of a fraction, and B the denominator; thus, $\frac{A}{B}$.

8. The power of a quantity is expressed by means of a figure or letter placed to the right and a little above the quantity; thus, A^2, A^3, A^4, A^m, is read A squared, A cubed, A raised to the fourth power, A raised to the *m*th power. 2, 3, 4, and *m* are called exponents, or indices.

9. The root of a quantity is expressed by means of a symbol called the radical sign, with a figure or letter to indicate the particular root; thus, $\sqrt{A}$, $\sqrt[3]{A}$, $\sqrt[4]{A}$, $\sqrt[m]{A}$ is read the square root, the cube root, the fourth root, the mth root of A; or these roots may be expressed by fractional exponents; thus $A^{\frac{1}{2}}$, $A^{\frac{1}{3}}$, $A^{\frac{1}{4}}$, and $A^{\frac{1}{m}}$.

10. The sign of *therefore*, or *hence*, is three dots placed in a triangular form; thus $\therefore$

11. A Ratio is a quotient; the ratio of 3 to 4 is $\frac{3}{4}$.

12. A Proportion is an equality of Ratios; and is written thus: $\frac{A}{B}=\frac{C}{D}$, $A \div B = C \div D$, or $A : B :: C : D$, and is read A is to B as C to D.

Definitions.

1. A Point is that which has position only.

2. A Line is length without breadth.

3. A Straight Line is one that does not change its direction at any point, or it is the shortest distance between two points. A straight line can not include a space or a segment.

4. A Curved Line is one that changes its direction at every point.

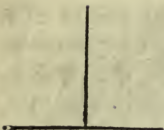
5. A Broken Line is made up of two or more straight lines not lying in the same direction.

6. A Surface, or Superficies, is that which has length and breadth.

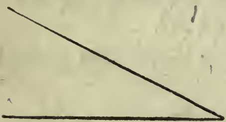
7. A Solid is that which has length, breadth, and thickness.

8. The boundaries of solids are surfaces; of surfaces, lines; and the extremities of lines, points. Imagine a point moved forward: it would generate a line; the line moved forward would generate a surface; and the surface moved forward would generate a solid.

9. An Angle is the difference of direction of two straight lines meeting in a point.



10. When a straight line meets another straight line, and makes the adjacent angles equal, each angle is called a right angle; and the line which meets the other is said to be a perpendicular to it.



11. An Acute Angle is an angle less than a right angle.

12. An Obtuse Angle is an angle greater than a right angle.

plane

13. A Plane Figure is a portion of a surface bounded by straight or curved lines.

14. The area of a figure is the quantity of space contained in it.

15. A Circle is a plane figure bounded by a curved line such that every point upon it is equally distant from a point within it called the centre.

16. The curved line which bounds it is called the Circumference.

17. The Diameter of a circle is a straight line passing through the centre, and terminating both ways in the circumference.

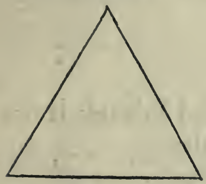
18. A Radius is a straight line drawn from the centre to any part of the circumference of a circle.

19. All Radii of the same circle are equal; all diameters of the same circle are also equal; and each diameter is double the radius.

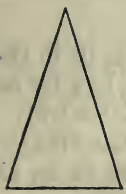
20. A Polygon is a plane figure bounded by straight lines; these lines are called sides, and the broken line forming the boundary is called the perimeter.

21. A Triangle is a plane figure bounded by three straight lines. There are six sorts of triangles; three

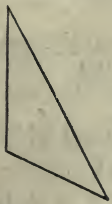
with respect to their sides, and three with respect to their angles. The three with respect to their sides are equilateral,* isosceles,† and scalene;‡ and the three with respect to their angles are right-angled, acute-angled, and obtuse-angled.



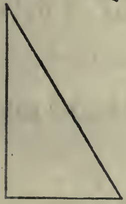
22. An Equilateral Triangle is a triangle that has its three sides equal.



23. An Isosceles Triangle is a triangle that has only two of its sides equal.



24. A Scalene Triangle is a triangle that has no two of its sides equal.

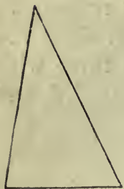


25. A Right-angled Triangle is one that contains a right angle.

* From æquus, equal, and latus, a side; equal-sided.

† From two Greek words meaning equal legs; hence a triangle having two equal sides.

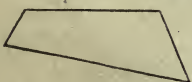
‡ Scalene means squinting.



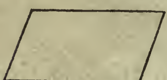
26. An Acute-angled Triangle is one that has all its angles acute.

27. An Obtuse-angled Triangle is one that has an obtuse angle.

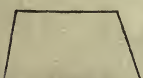
28. Parallel lines are such as lie in the same plane, and can not meet, how far soever they may be produced either way.



29. A Quadrilateral, a Quadrangle, or a Trapezium, is a plane figure of four sides.

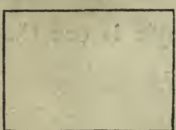


30. A Parallelogram is a quadrilateral whose opposite sides are parallel.



31. A Trapezoid is a quadrilateral which has only two of its sides parallel.

32. A Rhomboid is a parallelogram that has no right angle.



33. A Rectangle is a right-angled parallelogram.

34. A Square is an equilateral rectangle.

35. A Rhombus is an equilateral rhomboid.

36. A Diagonal is a straight line connecting two angles not consecutive.

37. The Hypothenuse of a right-angled triangle is the side opposite the right angle.

38. The Base of a plane figure is that side upon which it is supposed to stand. Any side may be considered the base.

End

Axioms.

1. Things which are equal to the same thing are equal to each other.
2. If equals be added to equals the sums are equals.
3. If equals be taken from equals the remainders are equals.
4. If equals be added to unequals the sums will be unequals.
5. If equals be subtracted from unequals the remainders will be unequals.
6. Things which are double of the same or equal things are equal.
7. Things which are halves of the same or equal things are equal.
8. The whole is equal to the sum of all its parts.
9. The whole is greater than any one of its parts.
10. Things which coincide, or fill the same space, are identical and equal in all their parts.
11. All right angles are equal.
12. Two straight lines which intersect each other can not both be parallel to the same straight line.

End

Postulates.

1. Let it be granted that a straight line may be drawn from one point to another;
2. That a straight line may be produced to any length;

3. And that a circle may be described from any point as centre, and with any length of line as radius.

End

HINTS TO TEACHERS.

Before commencing the study of the Propositions of Geometry it is absolutely necessary that the students should be provided with compasses and rulers. They should draw lines of all sorts and sizes; they should place them in such a manner as to form all kinds of angles, triangles, quadrilaterals; and they should write the names of the different figures. The students must commence with measuring; otherwise they are not studying *Geometry;* they are simply committing to memory variations of A, B, C and 1, 2, 3. One of the chief difficulties encountered hitherto in the study of this beautiful science arose from the fact that the text-books in common use begin with a *theorem*, and that theorem none of the easiest. In the present work the first proposition is a *problem*. But although this is the case the scholars should be compelled to draw accurately and neatly the following:

1. With ruler and pen draw *one* inch* on the horizontal.†
2. Draw four, six, eight, and ten inches on the horizontal.
3. Draw four, six, eight, and ten inches on the vertical.†
4. Draw a line equal to the sum of two other lines.
5. Draw a line equal to the sum of four other lines.
6. Draw a line equal to two other lines of four inches each.
7. Draw a horizontal line equal to two inches, and an

---

* If the student does not thoroughly comprehend distance, the inch, the foot, the yard, the pole, the mile, the league should be carefully taught.

† The horizon is the line that bounds our view. Hence a horizontal line is a line parallel with the horizon. A vertical line is perpendicular to the horizon.

oblique line touching its extremity. Cut off from the oblique line two inches.

8. Draw a vertical line six inches long, and from the lower extremity draw a horizontal line of the same length.

9. Describe a circle with any point as centre, and any length of line as radius.

10. Describe a circle with a radius equal to two inches.

11. Draw a radius, a diameter, a chord.*

12. Draw a tangent,† a secant.‡

13. Draw a triangle, a quadrilateral, a pentagon.§

14. Draw all kinds of angles.

15. Draw two diameters at right angles,‖ and join their extremities.

16. Draw four tangents at right angles, and join their extremities.

17. Lay off the radius on the circumference six times, and join the points; join the alternate points. (What figures have you formed?)

18. Draw a parallelogram, a rectangle, a rhomboid.

19. Make a square, and divide it into four equal parts.

20. Divide a circle into four equal parts.

21. Draw a rectangle twice as long as it is broad.

22. Divide a hexagon¶ into six parts by drawing lines from a point within it.

23. Make a right-angled triangle, each of whose sides shall be double the sides of a given right-angled triangle.

24. Make a square, each side of which shall be three times one inch.

---

* A chord is a line which cuts off a portion of a circle; it terminates in the circumference both ways.

† A tangent is a line which touches the circumference of a circle.

‡ A secant is a line which cuts the circumference.

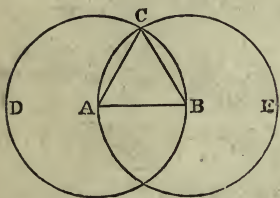
§ A pentagon is a figure of five sides.

‖ Of course, the learner performs these problems to the best of his ability. The object is simply to make him expert in the use of instruments, and to fasten on his mind more thoroughly the definitions already committed to memory.

¶ A hexagon is a figure of six sides.

Proposition I.—Problem.

On a given straight line to construct an equilateral triangle.



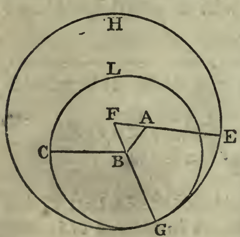
Let AB be a given straight line; it is required to construct upon it an equilateral triangle.

From the point A as centre, and with AB as radius (Postulate 3), describe the circle BCD; and from the point B as centre, and with AB as radius, describe the circle ACE; and from the point C, where the circles cut each other, draw the straight lines (Post. 1) AC and BC to the points A and B; ABC is an equilateral triangle.

Because the point A is the centre of the circle BCD, AC is equal to AB (Definition 19), and because the point B is the centre of the circle ACE, BC is equal to AB. Hence, since AC and CB are each equal to AB, they must be equal to each other (Axiom 1); therefore AB, AC. and CB are equal, and the triangle ABC is equilateral.

Proposition II.—Problem.

From a given point to draw a straight line equal to a given straight line.



Let A be the given point and BC the given line; it is required to draw from the point A a straight line equal to BC.

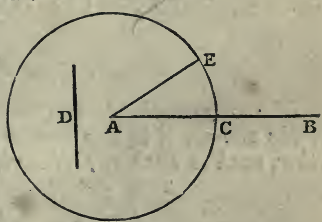
From the point A to B draw the line AB (Post. 1), and upon it construct an equilateral triangle, ABF (Proposition I.); from the point B as centre, and with the radius BC, describe the circle GCL (Post. 3); produce the line FB until it meets the circumference at G; then, from the point F as centre, and with the line FG as radius, describe the circle GEH; produce FA until it meets the circumference at E. The line AE will be equal to BC.

Because the point B is the centre of the circle GCL, BC is equal to BG; and because F is the centre of the circle GEH, FG is equal to FE (Def. 19); but BF and AF are equal; therefore (Ax. 3), if these equals be subtracted, the remainders, AE and BG, are equal; but it was before proved that CB is equal to BG, and since AE and BC are each equal to BG, they must be equal to each other.

PROPOSITION III.—PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and D be two given straight lines, of which AB is the greater; it is required to cut off from AB a part which shall be equal to D.



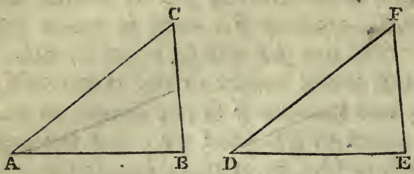
From the point A draw the line AE equal to D (Prop. II.); and from the centre A, and with the radius AE, describe the circle ECF (Post. 3); then will AC be equal to D.

The line AE is equal to the line D by construction, and the line AC is equal to the line AE (Def. 19). Now, since AC and D are each equal to AE, they must be equal to each other (Ax. 1).

PROPOSITION IV.—THEOREM.

If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles are every way equal.

Let ACB and DFE be two triangles which have the two sides AC and CB and the included angle ACB of the one equal to DF and FE and the included angle

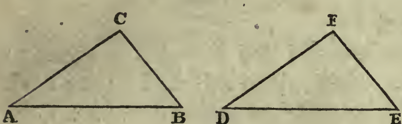


DFE of the other; then will the two triangles ACB and DFE be every way equal.

For let the triangle ACB be applied to the triangle DFE, so that the point C may fall on the point F, and the line AC upon DF; and since AC is equal to DF the point A must coincide with the point D; since AC coincides with DF, and the angle ACB is equal to the angle DFE, the line CB must coincide with the line FE; and since CB is equal to FE the points B and E must coincide; since the points A and D coincide, and the points B and E coincide, the line AB must coincide with the line DE (Ax. 10); therefore the two triangles coincide throughout their whole extent, and are every way equal (Ax. 10).

Proposition V.—Theorem.

If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles are every way equal.



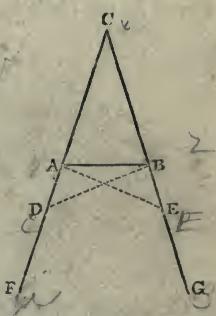
Let the two triangles ABC and DEF have the angles A and B equal to the angles D and E, each to each, and the included side AB equal to the included side DE, then will the two triangles be every way equal.

For apply the triangle ABC to the triangle DEF, in such a manner that the line AB will coincide with the line DE, and since the angle A is equal to the angle D, the side AC will take the direction DF, and the point C will fall somewhere on the line DF; and since the angle

B is equal to the angle E, the side BC must take the direction of EF, and the point C must fall somewhere on the line EF. Now, since the point C falls at the same time upon the lines DF and EF, and since it can not fall in two places at the same time, it must fall at their point of intersection, F. Therefore the points C and F coincide, and the two triangles coincide throughout their whole extent, and are therefore every way equal (Ax. 10).

Proposition VI.—Theorem.

The angles at the base of an isosceles triangle are equal, and, if the equal sides be produced, the angles below the base are also equal.



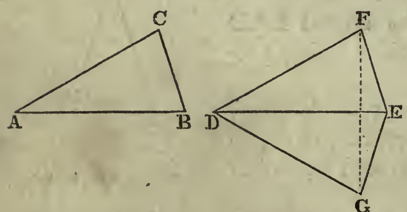
Let ABC be an isosceles triangle; then will the angles CAB and CBA be equal, and also the angles BAD and ABE (the equal sides being produced).

From AF cut off any part AD, and from BG cut off a part, BE, equal to AD; join the points A and E, and D and B.

The lines AC and BC are equal (hypothesis), and AD and BE (construction) are also equal. To AC add AD, and to BC add BE, and CD will be equal to CE (Ax. 2). In the two triangles DCB and ECA, the sides DC and CB are equal to the sides EC and CA, each to each, and the angle C is common to both. Hence the two triangles are every way equal (Prop. IV.); the angle CDB is equal to CEA, and CBD is equal to CAE, and the side DB is equal to the side EA. Again, in the two triangles ADB and BEA, the sides AD and BE are equal; and it was before proved that DB and EA are equal, and that the angle ADB is equal to the angle BEA. Hence the two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each; they are, therefore, equal in all their parts (Prop. IV.); the angle DAB is equal to the EBA, these are the angles below the base, and the angle DBA is equal to the angle EAB; but it was before proved that DBC is equal to EAC; if from these equals the equals DBA and EAB be taken, there will remain (Ax. 3) the angle ABC, equal to the angle BAC; these are the angles at the base.

Proposition VII.—Theorem.

If two triangles have three sides of the one equal to three sides of the other, each to each, the triangles will be equal in all their parts.

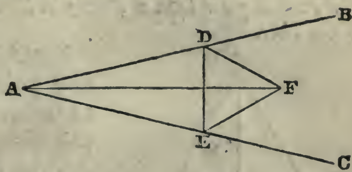


Let the two triangles ABC and DEF have the sides AB, AC, and BC equal to the sides DE, DF, and EF, each to each; then will these triangles be equal in all their parts.

Conceive the triangle ABC applied to the triangle DEF, so that the equal sides AB and DE will coincide, and the sides AC and BC will take the direction of DG and EG. Join the points F and G. The sides DF and DG are equal (hyp.), and the triangle FDG is isosceles; hence the angle DFG is equal to the angle DGF (Prop. VI.). In like manner it can be demonstrated that the angle EFG is equal to EGF. Therefore (adding these equals) the angle DFE is equal to the DGE (Ax. 2). In the two triangles DFE and DGE, the sides DF and FE are equal to DG and GE, each to each (hyp.), and the included angles have just been proved equal; hence the triangles are equal in all their parts (Prop. IV.); but the triangle DGE is equal to the triangle ABC. Therefore ABC and DFE are equal in all their parts.

Proposition VIII.—Problem.

To bisect a given angle.



Let BAC be the given angle; it is required to bisect it—that is, to divide it into two equal parts.

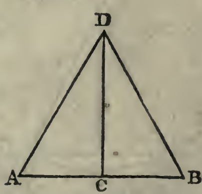
Take any point D on the line AB, and from AC cut off a part AE equal to AD (Prop. III.); join the points E and D; and on the line ED construct an equilateral triangle, EDF; join the points A and F; the line AF bisects the angle BAC.

In the two triangles ADF and AEF the sides AD and AE are equal (const.), FD and EF are also equal (Prop. I.), and the side AF is common to both; hence (Prop. VII.) the two triangles are every way equal, and the angle DAF is equal to the angle EAF.

Scholium. By the same construction DAF and EAF may be divided into two equal parts; and thus, by successive divisions, a given angle may be divided into four equal parts, into eight, into sixteen, and so on.

Proposition IX.—Problem.

To bisect a given straight line.



Let AB be a given straight line; it is required to bisect it.

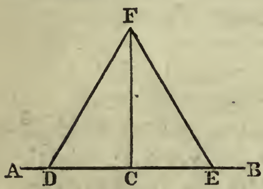
On AB construct an equilateral triangle (Prop. I.); bisect the angle ADB (Prop. VIII.); the line DC will bisect the line AB at the point C.

In the two triangles ACD and BCD, the sides AD and BD are equal, being sides of an equilateral triangle; DC is common to both, and the angle

ADC is equal to the angle BDC (const.). Hence the two triangles are equal in all their parts (Prop. IV.), and the side AC is equal to the side BC.

Proposition X.—Problem.

From a point on a straight line to erect a perpendicular to the line.



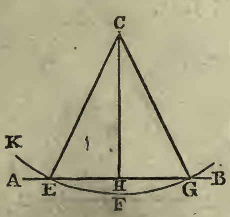
Let AB be a given line and C any point on it; it is required to erect a perpendicular from this point.

Take any point, D, in AC and make (Prop. III.) CE equal to DC; and upon DE construct an equilateral triangle, DFE, and join F and C; the line FC is perpendicular to AB.

In the two triangles DFC and EFC, DF is equal to FE, DC to CE (const.), and FC is common; hence they have three sides of the one equal to three sides of the other, each to each. Therefore the angle DCF (Prop. VII.) is equal to the angle ECF; and, since these angles are equal, each is a right angle (Def. 9). Hence CF is perpendicular to AB.

Proposition XI.—Problem.

From a given point without a straight line to let fall a perpendicular to the line.



Let AB be a given line and C a point without it; it is required to draw a perpendicular line from C to AB.

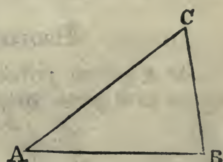
Take any point F on the opposite side of the given line, and from the centre C, with the distance CF as radius, describe (Post. 3) the circle KFG, meeting AB in the points E and G. Join E and C, and G and C; bisect (Prop. IX.) the line EG at the point H; join H and C. The line HC will be perpendicular to AB.

In the two triangles ECH and GCH, EC and GC are equal (Def. 15), EH and HG are also equal (const.), and HC is common to both. Hence the two triangles are every way equal (Prop. VII.), and the angle EHC is equal to the angle GHC; and, since these angles are equal, each is a right angle (Def. 10). Therefore the line HC is perpendicular to AB.

Proposition XII.—Theorem.

*Any two sides of a triangle are together greater than the third.**

Let ABC be a triangle; then will the sum of the sides AC and CB be greater than AB.

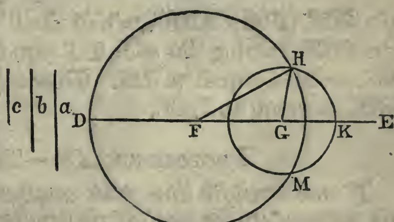


For the straight line AB (Def. 2) is the shortest distance between the two points A and B. Hence the broken line ACB is greater than the straight line AB.

Proposition XIII.—Problem.

Given three straight lines, the sum of any two being greater than the third,† to construct a triangle whose three sides shall be equal to the given lines, each to each.

Let a, b, and c be the given straight lines; it is required to construct a triangle whose three sides shall be equal to these three lines.



Draw an indefinite

* This proposition is an obvious deduction from the definitions of a straight line and of a triangle.

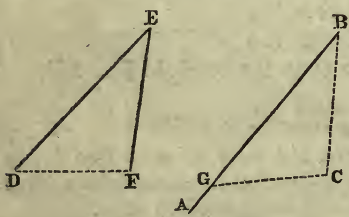
† A straight line is the shortest distance between two points; hence, to form a triangle, the sum of two sides must be greater than the third.

line, DE, and on it lay off DF equal to *a*, FG equal to *b*, and GK equal to *c;* then, from the point F as centre, and with the line DF as radius, describe the circle DHM (Post. 3), and from the point G as centre, and with the line GK as radius, describe the circle HKM (Post. 3). Join the points F and G with the point of intersection, H; then will FGH be the required triangle.

The line *a* is equal to DF (const.), but FH is equal to DF (Def. 19); therefore FH is equal to *a* (Ax. 1); *c* is equal to GK (const.), but GH is equal to GK; therefore GH is equal to *c* (Ax. 1), and the line FG is equal to *b* (const.). Hence the three sides of the triangle are equal to the three given lines.

Proposition XIV.—Problem.

At a given point in a given line to make an angle equal to a given angle.



Let B be the given point in the straight line, AB; it is required to construct an angle at B equal to the given angle DEF.

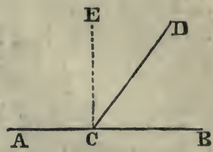
Join the points D and F, and construct a triangle GBC (Prop. XIII.), which shall be equal to the triangle DEF, having the side GB equal to DE, BC equal to EF, and GC equal to DF. Hence (Prop. VII.) the angle GBC is equal to DEF.

Proposition XV.—Theorem.

If one straight line meet another straight line (not at the extremity) the sum of the angles thus formed is equal to two right angles.

Let the line AB meet the line CD at the point C; then will the sum of the angles ACD and DCB be equal to two right angles.

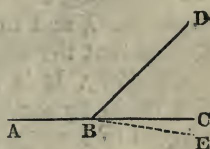
At the point C (Prop. X.) erect the perpendicular CE. The sum of the angles ACE and ECB is equal to two right angles (Def. 10). But the sum of the angles BCD, DCE, and ACE is equal to the sum of the angles ACE and ECB. Therefore the sum of the angles BCD, DCE, and ACE is equal to two right angles (Ax. 1). But the angles DCE and ACE are together equal to ACD (Ax. 8); therefore the sum of the angles BCD and ACD is equal to two right angles.



PROPOSITION XVI.—THEOREM.

If two straight lines meet a third straight line, making the sum of the adjacent angles equal to two right angles, these two straight lines will form one and the same straight line.

Let two straight lines, AB and BC, meet a third straight line, BD, making the sum of the adjacent angles, ABD and CBD, equal to two right angles, then will these two lines form one and the same straight line.



For, suppose that AB and BC do not form one and the same straight line,* then some other line, as BE, must be the continuation of AB; then, if ABE be a straight line, the sum of the angles ABD and EBD is equal to two right angles (Prop. XV.). But (by hypothesis) the sum of the angles ABD and CBD is also equal to two right angles. Hence the sum of the angles ABD and EBD is equal to the sum of the angles ABD and CBD; but the angle ABD is common to both; take it away, and there will remain the angle EBD equal to CBD, which is absurd (Ax. 9). Therefore BE is not a continuation of

* This method of demonstration is called indirect, or the *reductio ad absurdum.*

AB, and in a similar manner it can be demonstrated that no other line than BC can be the continuation of AB.

Proposition XVII.—Theorem.

If one straight line intersect another straight line, the opposite or vertical angles are equal.

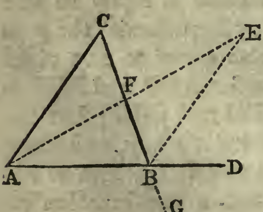
Let the straight line AB intersect the straight line CD in the point G; then will the angle AGD be equal to the angle CGB, and AGC equal to BGD.

For the sum of the angles AGD and AGC is equal to two right angles (Prop. XV.), and the sum of the angles BGC and AGC is also equal to two right angles (Prop. XIV.). Therefore the sum of the angles AGD and AGC is equal to the sum of the angles BGC and AGC. Take away the common angle AGC (Ax. 3), and the remaining angle AGD will be equal to the remaining angle BGC, and in the same manner it can be demonstrated that the angle AGC is equal to the angle BGD.

Corollary. It is manifest that the sum of all the angles made by straight lines meeting at a common point is equal to four right angles.

Proposition XVIII.—Theorem.

If one side of a triangle be produced, the exterior angle is greater than either of the interior and ~~opposite~~ angles.



Let the triangle ABC have one of its sides, AB, produced to D; then will the exterior angle CBD be greater than either of the two interior and opposite angles, CAB and BCA.

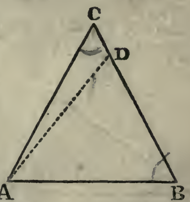
Bisect BC at the point F (Prop. IX.); join the points F and A; produce AF until FE is equal to it; join B and E.

In the two triangles AFC and EFB, the sides CF and FB are equal (const.), the sides AF and FE are also equal (const.), and the angles AFC and EFB are equal (Prop. XVII.). Therefore the two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, and are every way equal (Prop. IV.); the angle EBF is equal to the angle FCA. But DBC is greater than EBF (Ax. 9), and therefore greater than FCA. In the same manner, by producing BC and bisecting AB, it can be demonstrated that ABG, or its equal, DBC, is greater than CAB.

Cor. Any two angles of a triangle are together less than two right angles. Because the exterior angle is greater than either of the interior and opposite angles; to this inequality add the adjacent angle, and the sum of the exterior angle and the adjacent angle will be greater than the sum of the adjacent angle, and either of the other two. But the sum of the adjacent angle and the exterior angle is equal to two right angles (Prop. XV.). Therefore the sum of the two angles of the triangle is less than two right angles.

PROPOSITION XIX.—THEOREM.

If two angles of a triangle be equal, the sides opposite to them are also equal.

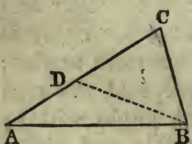


Let ABC be a triangle having the angle BAC equal to the angle ABC; then will the side BC be equal to the side AC.

For, suppose AC and BC are not equal, then one of them must be the greater; let BC be the greater; then from BC (Prop. III.) cut off a part, BD, equal to the less, AC. In the two triangles BAC and ABD, the side AC is equal to BD (const.), the side AB common, and the angles BAC and ABD are equal (hypothesis.) Therefore the two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each. Hence (Prop. IV.) they are equal in all their parts, which is impossible (Ax. 9). Therefore BD is not equal to AC; and in like manner it can be proved that no other line than BC is equal to AC.

Proposition XX.—Theorem.

The greater side of every triangle has the greater angle opposite to it.



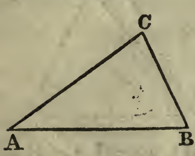
Let ABC be a triangle having the side AC greater than the side BC; then will the angle ABC be greater than the angle CAB.

From the greater line, AC, cut off CD equal to CB (Prop. III.). Join the points D and B. Because the triangle DCB is isosceles, the angle CDB is equal to the angle DBC (Prop. VI.). But the angle CDB is greater than the angle A (Prop. XVIII.), and hence the angle DBC must also be greater than the angle A. But the angle ABC is greater than DBC (Ax. 9), therefore ABC is much greater than A.

end

Proposition XXI.—Theorem.

The greater angle of every triangle has the greater side opposite to it.

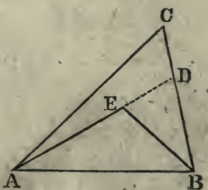


Let ABC be a triangle having the angle ABC greater than the angle BAC; then will the side AC be greater than the side BC.

For if AC be not greater than BC, it must be equal to, or less than it. It can not be equal to it; for then the angle ABC would be equal to the angle BAC (Prop. VI.), which is contrary to the hypothesis. It can not be less, for then (Prop. XX.) ABC would be less than BAC, which is also contrary to the hypothesis. Since AC is neither equal to, nor less than BC, it must be greater.

Proposition XXII.—Theorem.

If, from a point within a triangle, two straight lines be drawn to the extremities of one side, these two lines will be less than the other two sides of the triangle.



Let ABC be a triangle, and the two lines AE and EB drawn from the point E to the extremities A and B; then will the sum of AE and EB be less than the sum of AC and CB.

Produce AE to D. In the triangle BDE the side BE is less than the sum of BD and DE (Prop. XII.). To this inequality add AE; then the sum of BE and AE (Ax. 4) is less than the sum of BD, DE, and AE, or BD and DA. In the triangle ACD, the side AD is less than the sum of AC and CD (Prop. XII.). To this inequality add BD; then the sum of AD and DB is less than the sum of AC, CD, and DB, or than the sum of AC and CB. But it was before proved that the sum of AE and EB is less than the sum of AD and DB; hence the sum of AE and EB is much less than the sum of AC and CB.

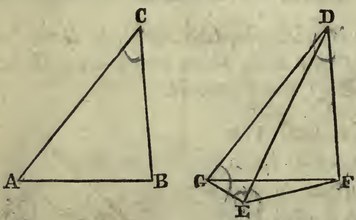
Cor. The two lines drawn from the point E to the extremities of the side AB will contain a greater angle than the other two sides of the triangle.

For the angle AEB is greater than the angle EDB; and EDB is greater than the angle ACD (Prop. XVIII.). Hence AEB is much greater than ACB.

B 2

Proposition XXIII.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angle of the one greater than the included angle of the other, that triangle having the greater included angle will have the greater side opposite to it.



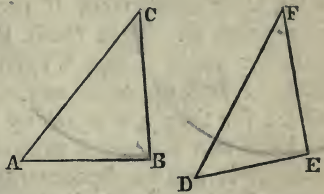
Let ABC and DEF be two triangles having the sides AC and CB equal to ED and DF, each to each, and the included angle ACB greater than the included angle EDF, then will the side AB be greater than the side EF.

At the point D, with the line DF, construct an angle, FDG, equal to the angle ACB (Prop. XIV.); make GD equal to ED; join the points G and F, and G and E.

In the two triangles ACB and GDF, the sides AC and CB are equal to GD and DF, each to each, and the included angle ACB is equal to the included angle GDF (const.); therefore AB is equal to GF (Prop. IV.). The triangle GDE is isosceles, because DG is equal to DE (const.); hence the angle DGE is equal to the angle DEG; but the angle GEF is greater than the angle GED (Ax. 9), and also greater than DGE; but DGE is greater than EGF; hence GEF is much greater than EGF; and, since GEF is greater than EGF, the side GF must be greater than the side EF; but GF is equal to AB. Therefore AB is greater than EF.

Proposition XXIV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the third side of the one greater than the third side of the other, that triangle having the greater third side will have the greater angle opposite to it.



Let the two triangles ABC and DEF have two sides, AC and CB, of the one equal to two sides, DF and FE, of the other, each to each, and let the third side, AB, be greater than the third side, DE; then will the angle ACB opposite the greater side be greater than the angle DFE opposite the less side.

For, if the angle ACB be not greater than the angle DFE, it must be equal to, or less than, it. The angle ACB can not be equal to the angle DFE, because then the side AB would be equal to the side DE (Prop. IV.), which is contrary to hypothesis. It can not be less, for then the side AB would be less than DE (Prop. XXIII.), which is also contrary to hypothesis. Then, since the angle ACB is neither equal to nor less than the angle DFE, it must be greater.

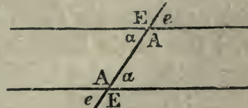
Proposition XXV.—Theorem.

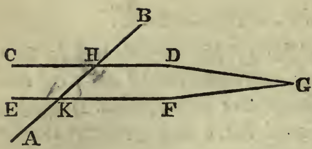
If a straight line intersect two other straight lines, and make the alternate angles equal, these two straight lines will be parallel.*

Let the straight line, AB, intersect the two straight

---

* Alternate literally means by turns; in Geometry alternate angles are the internal angles made by two lines with a third, on opposite sides of it. A, A are alternate angles; so are *a*, *a*. E, E are exterior angles; so are *e*, *e*.



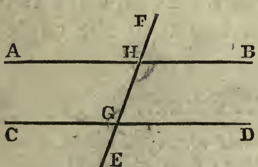


lines, CD and EF, and make the alternate angles EKH and DHK equal; then will these two straight lines CD and EF be parallel.

For, if they are not parallel, they must meet if produced. Let them meet at any point, G. Then HGK becomes a triangle; and the exterior angle (Prop. XVIII.) EKH is greater than the interior and opposite angle KHD; but this is impossible; for they are equal by hypothesis. Therefore CD and EF can not meet towards D or F, and in the same manner it can be demonstrated that they can not meet towards C or E; and, since they can not meet either way, they must be parallel.

Proposition XXVI.—Theorem.

If one straight line intersect two other straight lines, and make the exterior angle equal to the interior and opposite angle upon the same side of the line, or make the sum of the interior angles on the same side equal to two right angles, the two straight lines are parallel to one another.



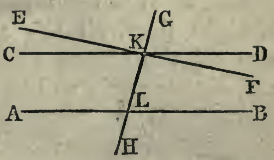
Let the straight line FE intersect the two straight lines AB and CD, and make the exterior angle AHF equal to the interior and opposite angle, CGH, on the same side of the line FE, or make the sum of the interior angles, AHG and CGH, equal to two right angles; then the two straight lines AB and CD are parallel.

For the angle CGH is equal to the angle AHF (hyp.), and the angle BHG is also equal to AHF (Prop. XVII.); hence the angles CGH and BHG are equal to each other (Ax. 1); but these are alternate angles. Therefore, the two straight lines AB and CD are parallel (Prop.

XXV.). Again, the sum of the angles CGH and AHG is equal to two right angles (hyp.), and the sum of the angles AHG and BHG is also equal to two right angles (Prop. XV.). Take away the common angle AHG, and there remain the equal angles CGH and BHG; but these are alternate angles. Therefore AB is parallel to CD (Prop. XXV.).

Proposition XXVII.—Theorem.

If a straight line intersect two parallel straight lines, it makes the alternate angles equal; it makes the exterior angle equal to the interior and opposite angle on the same side, and the sum of the two interior angles on the same side equal to two right angles.



Let the straight line GH intersect the two parallel straight lines, AB and CD; then will the alternate angles CKL and BLK be equal: the exterior angle CKG will be equal to the interior and opposite angle ALK upon the same side, and the sum of the two interior angles CKL and ALK upon the same side be equal to two right angles.

For if the angle CKL be not equal to BLK, make, at the point K, an angle, EKL, equal to it, and produce EK to F; and, since the alternate angles EKL and BLK are equal, the lines EF and AB are parallel; but (by hypothesis) CD is parallel to AB, which is impossible (Ax. 12), for two straight lines can not be drawn through the same point parallel to the same straight line. Therefore CKL must be equal to BLK, and ALK to DKL. Now DKL is equal to CKG (Prop. XVII.); hence ALK is also equal to CKG: to each of these equals add the angle CKL, and the sum of the angles ALK and CKL will be equal to the sum of the angles CKG and CKL; but the sum of CKG and CKL is equal to two right angles (Prop.

XV.); therefore the sum of the angles ALK and CKL is also equal to two right angles.

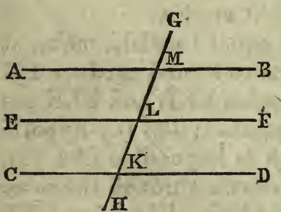
Cor. 1. If a straight line intersect two other straight lines, and make the sum of the two interior angles on the same side less than two right angles, these two straight lines will meet on the side of the secant line on which the sum of the two angles is less than two right angles. For if they did not meet when produced, they would be parallel; and if parallel, the sum of the two interior angles on the same side would be equal to two right angles, which is contrary to hypothesis.

Cor. 2. If a line be perpendicular to one of two parallels, it must be perpendicular to the other. For if one of the alternate angles be a right angle, the other must be a right angle also.

Cor. 3. When a straight line intersects two parallel straight lines, all the acute angles will be equal, and also all the obtuse angles; and the sum of any acute and obtuse angle will be equal to two right angles.

Proposition XXVIII.—Theorem.

Two straight lines which are parallel to the same straight line are parallel to each other.

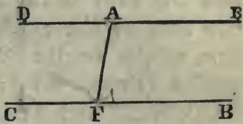


Let AB and CD be parallel to the same straight line EF; then will they be parallel to each other.

Because the parallel lines AB and EF are cut by GH, the angle AML is equal to MLF (Prop. XXVII.); but MLF is equal to ELK (Prop. XVII.); and since the parallel lines CD and EF are cut by GH, the angles ELK and DKL are equal (Prop. XXVII.). Therefore DKL must be equal to AML; but these are alternate angles. Hence AB and CD are parallel (Prop. XXV.).

Proposition XXIX.—Problem.

Through a given point to draw a straight line parallel to a given straight line.



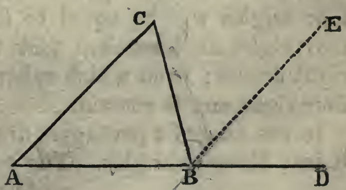
Let A be a given point, and BC a given line; it is required to draw a line through A parallel to BC.

Take any point, F, in the line BC, and connect it with the given point; then at the point A, with the line AF, make an angle, DAF, equal to the angle AFB (Prop. XIV.). Produce DA to E. Then will the straight line DE be parallel to BC.

For, since the alternate angles AFB and DAF are equal, the lines DE and BC are parallel (Prop. XXV.).

Proposition XXX.—Theorem.

If one side of a triangle be produced, the exterior angle is equal to the sum of the two interior and opposite angles, and the three angles of the triangle are together equal to two right angles.



Let ABC be a triangle, and let one of its sides, AB, be produced to D; then will the angle CBD be equal to the sum of the interior and opposite angles A and C; and the sum of the three angles A, C, and ABC will be equal to two right angles.

Through the point B draw the line BE parallel to AC (Prop. XXIX.); and because the lines AC and BE are parallel, and the line BC intersects them, the alternate angle CBE is equal to the alternate angle ACB, and because AD cuts them, the exterior angle DBE is equal to the interior and opposite angle, BAC. Hence the sum of the angles CBE and DBE, or CBD, is equal to the sum of the angles ACB and CAB (Prop. XXVII.). Again, the angle CBD has just been proved equal to the sum of the angles BAC and ACB; to each add the angle ABC, and the sum of the angles CBD and CBA is equal to the sum of the angles BCA, BAC, and ABC; but the sum of the angles CBD and CBA is equal to two right angles (Prop. XV.); therefore the sum of the angles ACB, BAC, and ABC is equal to two right angles.

Cor. 1. Since the three angles of every triangle are together equal to two right angles, it follows that if two triangles have two angles of the one equal to two angles of the other, the remaining angles must be equal.

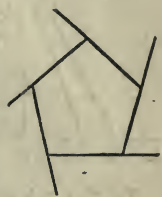
Cor. 2. The sum of all the interior angles of a polygon is equal to twice as many right angles as the figure has sides, minus four right angles.

In the case of the triangle, this corollary has just been demonstrated; for, two right angles taken three times equal six right angles, from which subtract four right angles, and two right angles remain.

In the case of a quadrilateral, draw a diagonal, dividing it into two triangles; it is evident that the sum of the four angles will be equal to four right angles—that is, two right angles taken four times, which make eight right angles; from which subtract four right angles, and four right angles remain.

In the case of a pentagon, draw two diagonals, dividing it into three triangles. The sum of the interior angles of the pentagon is equal to the sum of the angles of the three triangles. But the sum of the angles of the three triangles is equal to six right angles (Prop. XXX.); hence the sum of the interior angles of the pentagon is equal to six right angles—that is, twice as many right angles as the figure has sides, minus four right angles. And in the same manner it may be demonstrated of any polygon.

Cor. 3. All the exterior angles of any polygon are together equal to four right angles. It has just been proved that the sum of all the interior angles of a pentagon is equal to six right angles; but the sum of each interior and exterior angle (Prop. XIV.) is equal to two right angles; therefore the sum of the five interior and exterior angles must be equal to ten right angles; from these ten right angles subtract the six right angles (to which the sum of the interior angles is equal), and there remain four right angles, to which the sum of all the exterior angles must be equal.



Cor. 4. Two angles of a triangle being given, the third may be found by subtracting their sum from two right angles.

Cor. 5. In any triangle there can be but one right angle; for if there were two, the third angle would be nothing.

Cor. 6. In every right-angled triangle the sum of the acute angles is equal to one right angle.

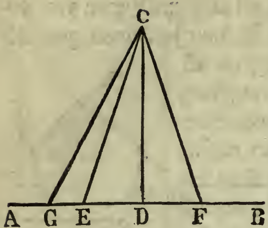
Cor. 7. Every equilateral triangle must be also equiangular (Prop. VII.); and each angle will be one-third of two right angles, or two-thirds of one right angle.

Cor. 8. Since the sum of the angles of a quadrilateral is equal to four right angles, if the angles are all equal each must be a right angle.

Proposition XXXI.—Theorem.

A perpendicular is the shortest line that can be drawn from a given point to a given line; two oblique lines drawn from this point to two points on the line equidistant from the perpendicular are equal, and two oblique lines to points unequally distant from the perpendicular are unequal, and the longer oblique line is farther from the foot of the perpendicular.

Let AB be a given line and C a given point; then will



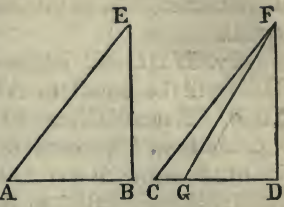
the perpendicular CD be the shortest line that can be drawn to AB; and let the oblique lines CE and CF be equidistant from the point D; then will CE and CF be equal; and also let CE and CG be unequally distant from D; then will CG be greater than CE.

In the triangle CDE, the angle EDC is a right angle (Def. 10.), and therefore greater than the angle CED (Prop. XXX.). But the greater angle has the greater side opposite to it (Prop. XXI.); hence CE is greater than CD. In the two triangles CDE and CDF, the sides ED and DF are equal (const.), the side CD is common, and the angle EDC is equal to the angle FDC (Def. 10.). Hence the two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, and are equal in all their parts (Prop. IV.). Therefore CE is equal to CF. The angle CED is an acute angle; therefore CEG is obtuse (Prop. XV.); and if CEG is obtuse, CGE must be acute (Prop. XXX.). Hence CG, opposite the greater angle, is greater than CE, opposite the less (Prop. XXI.).

Proposition XXXII.—Theorem.

If two right-angled triangles have the hypothenuse and a side of the one equal to the hypothenuse and a side of the other, each to each, the two right-angled triangles are equal in all their parts.

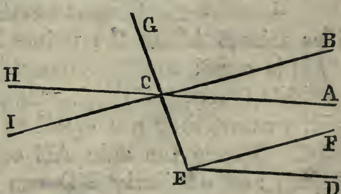
Let the two right-angled triangles ABE and CDF have the hypothenuse AE and the side EB of the one equal to the hypothenuse CF and the side FD of the other; then will the two triangles AEB and CFD be equal.



If AB were equal to CD the equality of the triangles would be manifest. Now suppose CD not equal to BA, and let CD be the greater; then from CD cut off a part, GD, equal to AB (Prop. III.). Join G and F. In the two triangles ABE and GDF, the sides BE and FD are equal (hyp.), and BA and GD are also equal (const.); and the included angles B and D are equal, being right angles. Hence the two triangles are equal in all their parts (Prop. IV.); GF is equal to AE; but FC is equal to AE (hyp.); therefore FC is equal to FG; but this is impossible (Prop. XXXI.). Hence GD can not be equal to AB; and in the same way it can be proved that no other line except CD can be equal to AB. The two triangles ABE and CDF are, therefore, every way equal.

Proposition XXXIII.—Theorem.

If two angles have their sides parallel, each to each, and lying in the same direction, they are equal.



If the straight lines AC and CB be parallel to ED and EF, each to each, and lie in the same direction, then will the angle ACB be equal to the angle DEF.

For, draw the line ECG through their vertices; and since AC is parallel to DE, the exterior angle ACG is equal to the interior and opposite angle, DEC, on the same side (Prop. XXVII.); and since BC is parallel to FE, the exterior angle BCG is also equal to the interior and opposite angle, FEC, on the same side (Prop. XXVII.). If from the equal angles ACG and DEC, the equal angles

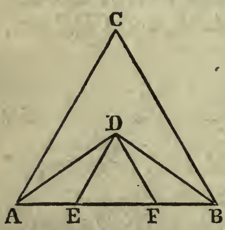
BCG and FEC be subtracted, there remain ACB and DEF equal.

Cor. If AC and BC be produced to H and I, the angle HCI will be equal to ACB; but DEF is also equal to ACB. Hence HCI and DEF are equal.

Scholium. The restriction that the parallel sides must lie in the same direction is necessary; for the angle BCH has its sides parallel to the sides of the angle DEF, but is not equal to it.

The following test examples, involving the First Book only, are solved with the view of giving the learner an insight into the methods of geometrical application of previous principles.

1. *To trisect a given line.**



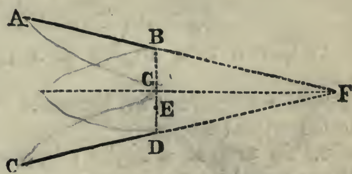
Let AB be the given line; it is required to trisect it.

On AB construct an equilateral triangle, ABC (Prop. I.); bisect the angles BAC and ABC (Prop. VIII.); from the point of intersection, D, draw DE and DF respectively parallel to AC and CB. Then will these lines trisect AB in the points E and F. For the angle DAE is equal to CAD (const.), and ADE is also equal to CAD (Prop. XXVII.). Therefore, since each is equal to the same, they are equal to each other; DAE equals ADE. Hence the side AE equals the side ED (Prop. XIX.), and similarly FB can be proved equal to DF. The angles DEF and DFE are equal respectively to the angles CAB and ABC (Prop. XXVII.). Therefore the angle EDF is equal to the angle C. But the triangle ABC is equilateral and equiangular. Hence the triangle

* This problem is usually and easily solved by the subsequent proposition, that the lines parallel to the base divide the other sides proportionally; but we are limited to the propositions of the First Book.

EDF is also equilateral and equiangular. Therefore EF is equal to ED and DF, or to their equals, AE and FB. Hence AB is trisected.

2. *Through a given point to draw a straight line which shall make equal angles with two straight lines given in position.*

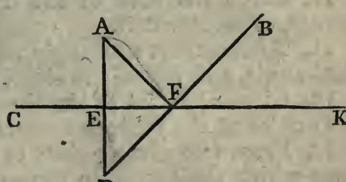


Let E be the given point, and AB and CD the two lines given in position.

Prolong AB and CD until they meet in the point F; bisect the angle BFD;* from the point E draw the perpendicular GE, and produce it both ways to B and D; then will the line DB make equal angles with the lines AB and CD.

For in the two triangles GBF and GDF, the angles GFB and GFD are equal (const.), and the angles BGF and DGF are equal, being right angles. Therefore the remaining angles, GBF and GDF, are equal.

3. *From two given points on the same side of a line given in position to draw two lines which shall meet in that line and make equal angles with it.*



Let A and B be the given points, and CK the given line in position.

From the point A let fall the perpendicular AE; prolong AE until ED is equal to it. Join D and B, and A and F. Then the required angles are AFE and BFK.

* It is supposed that the student is sufficiently familiar with the problems required in this solution without naming them.

For, in the two triangles AFE and DFE, AE is equal to ED (const.), EF is common, and the included angles AEF and DEF are equal, being right angles. Hence the angle AFE is equal to the angle DFE; but the angle DFE is equal to the angle BFK. Therefore AFE is also equal to BFK.

TEST EXAMPLES IN BOOK I.

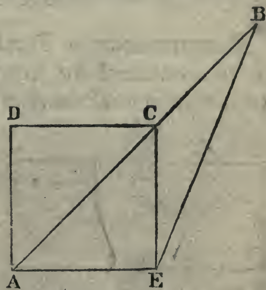
1. Given two angles of a triangle, to find the third angle.
2. To construct an isosceles triangle with a given base and vertical angle.
3. To trisect a right angle—that is, to divide it into three equal parts.
4. Prove that every point of the bisectrix of an angle is equally distant from the sides.
5. The three straight lines drawn from a point within a triangle to the angles are together less than the perimeter, but greater than its half.
6. To construct an isosceles triangle, so that the vertex will fall at a given point and the base fall in a given line.
7. Given the perpendicular of an equilateral triangle, to construct it.
8. Given the diagonal of a square, to construct it.
9. To construct an isosceles triangle, so that the base shall be a given line, and the vertical angle a right angle.
10. Given the sum of the diagonal, and a side of a square, to construct it.
11. To construct a triangle when the altitude, the vertical angle, and one of the sides, are given.
12. Given the sum of the three sides of a triangle, and the angles at the base, to construct it.
13. From two given points to draw two equal straight lines which shall meet in the same point of a line given in position.

A KEY TO THE TEST EXAMPLES IN BOOK I.

1. See Prop. XXX.
2. See Props. XXX., XIV., and XV.
3. See Props. I. and VIII.

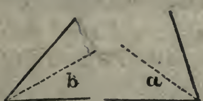
4. See Props. XXXI. and V.
5. See Props. XXII. and XII.
6. See Props. XI., IX., and IV.
7. See Test Ex. III., and Prop. X.
8. See Prop. VIII. Bisect right angles.
9. Bisect right angle Prop. VIII., and see Prop. XIV.

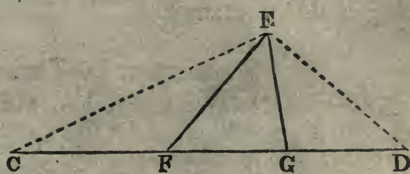
10. See the accompanying figure: AB equals sum of diagonal and side. At B make an angle equal to one-quarter of a right angle. At A make BAE equal to one-half a right angle. The student will readily solve the remainder.



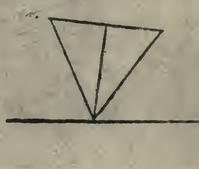
11. First take $\underline{\quad a \quad}$ = altitude, $\underline{\quad s \quad}$ given side, and $\underline{\angle v}$ the given angle; then draw a base line of any length, and by Prop. X. erect perpendicular = to a. With the upper extremity as centre, and the length of s as radius, describe an arc cutting the base. Then, by means of Props. XXXI. and XIV., the solution becomes simple.

12. See Props. VIII., XIV., XIX., and XXX.





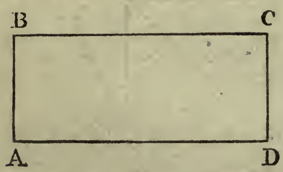
13. See Props. IX. and X., and the accompanying figure:



BOOK II.

DEFINITIONS.

A RIGHT-ANGLED Parallelogram, or Rectangle, is said to be contained by any two of the straight lines which are about one of the right angles.



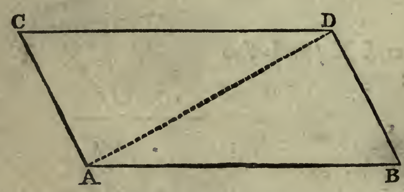
Thus the rectangle ADCB is said to be the rectangle contained by AD and DC, or by AD and AB, etc.

For the sake of brevity, the rectangle ADCB is usually called the rectangle AC or DB; and instead of the rectangle contained by AD and DC, it is simpler to say the rectangle AD.DC, placing a point between the two sides of the rectangle.

In Geometry, the product of two lines means the same thing as their rectangle.

Proposition I.—Theorem.

Two straight lines which join the extremities of two equal and parallel straight lines, towards corresponding parts, are also equal and parallel.



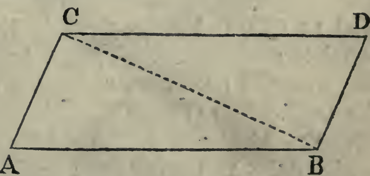
Let AB and CD be two equal and parallel straight lines; and let AC and BD join their corresponding extremities; then will AC and BD be equal and parallel.

Draw AD. And because AB and CD are parallel, and AD intersects them, the alternate angles BAD and ADC are equal (Prop. XXVII., Bk. I.). Now, in the two triangles BAD and ADC, the sides AB and CD are equal (hyp.), the side AD is common, and the included angles BAD and ADC are also equal. Hence the two triangles are every way equal; BD is equal to AC, and the angle CAD is equal to the angle ADB; but these being alternate angles, AC and BD are parallel (Prop. XXVI., Bk. I.).

Cor. If two sides of a quadrilateral are equal and parallel, the figure will be a parallelogram.

Proposition II.—Theorem.

If the opposite sides of a quadrilateral are equal, each to each, the figure is a parallelogram.



Let the opposite sides, AB and CD, be equal, and also AC and BD; then will the quadrilateral ABDC be a parallelogram.

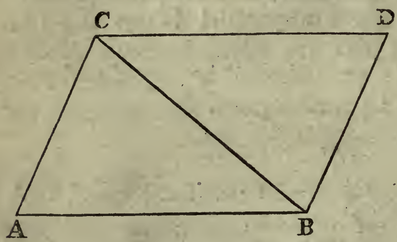
For, having drawn BC, the two triangles ABC and DCB have three sides of the one equal to three sides of the other, each to each; hence they are every way equal (Prop. VII., Bk. I.); and the angle ABC is equal to the angle BCD: these being alternate angles, AB and CD are parallel (Prop. XXVI.); and the angle ACB is equal to the angle CBD; and these being also alternate angles, AC and BD are parallel (Prop. XXVI., Bk. I.). Hence the figure ABDC is a parallelogram.

Cor. If the opposite angles of a quadrilateral be equal, each to each, the figure is a parallelogram.

For all the angles of the figure being equal to four right angles (Cor. I., Prop. XXX., Bk. I.), and the opposite angles being mutually equal, each pair of adjacent angles must be together equal to two right angles; but these adjacent angles become the interior angles on the same side; hence the opposite sides must be parallel (Prop. XXVI., Bk. I.).

Proposition III.—Theorem.

The opposite sides of a parallelogram are equal, each to each, and the diagonal bisects it.



Let ABDC be a parallelogram; then will the opposite sides AB and CD be equal, and also AC and BD; and the diagonal CB will bisect the figure.

Because AB is parallel to CD, and BC intersects them, the angle ABC is equal to the angle BCD; and because AC is parallel to BD, and CB intersects them, the angle ACB is equal to the angle CBD (Prop. XXVII., Bk. I.). Now in the two triangles ABC and DCB, the side CB is common, and the angles ACB and ABC are equal to CBD and BCD, each to each. Hence the triangles are equal in all their parts (Prop. V., Bk. I.)—that is, the diagonal bisects the figure, and the opposite sides are equal.

Cor. 1. The opposite angles of a parallelogram are equal. For the angle A is equal to D, and the sum of ACB and BCD or ACD is equal to the sum of CBD and CBA or ABD.

Cor. 2. Two parallel lines included between two other parallel lines are equal.

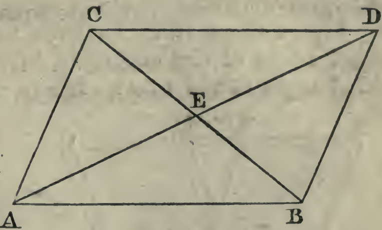
Cor. 3. Hence two parallels are everywhere equally distant.

Proposition IV.—Theorem.

The diagonals of a parallelogram mutually bisect each other.

Let ABDC be a parallelogram; then will its diagonals, AD and CB, mutually bisect each other at the point E.

For, in the two triangles ABE and DCE, the sides AB and DC are equal (by hyp.); the angle ABE is equal to DCE (Prop. XXVII., Bk. I.); and for the same reason the an-



gle BAE is equal to the angle CDE: the two triangles have, therefore, two angles and the included side of the one equal to two angles and the included side of the other, each to each; hence they are equal in all their parts (Prop. V., Bk. I.); the side AE is equal to ED, and BE to EC.

Proposition V.—Theorem.

If the diagonals of a quadrilateral mutually bisect each other, the figure is a parallelogram.

Let ABDC be a quadrilateral whose diagonals mutually bisect each other at the point E; then will the figure be a parallelogram.

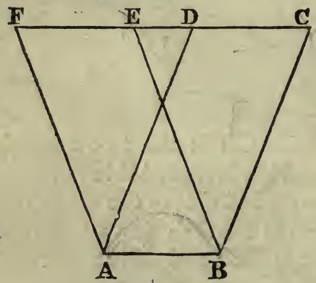
For in the two triangles AEB and CED, AE is equal to ED, and BE to EC (hyp.), and the included angles AEB and CED are equal (Prop. XVII., Bk. I.). Hence they are equal in all their parts (Prop. IV., Bk. I.); AB is equal to CD, and in the same way it may be demonstrated that AC is equal to BD. Therefore ABDC is a parallelogram (Prop. II.).

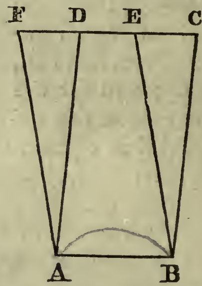
Proposition VI.—Theorem.

Parallelograms upon the same base, and between the same parallels, are equal.

Let the two parallelograms ABCD and ABEF have the same base, AB, and be between the same parallels AB and FC, then will these two parallelograms be equal.

In the two triangles EBC and FAD, the sides BC and AD are equal, and BE and AF are equal (Prop. III.), and the angles CBE and FAD are equal (Prop. XXXIII., Bk. I.). Hence these triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each; they are, there-



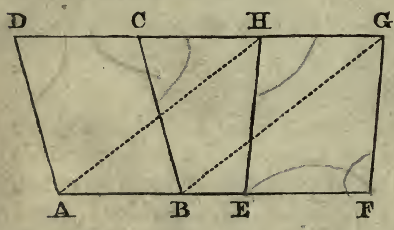


fore, equal in all their parts. If, now, from the whole figure ABCF the triangle BCE be taken, there will remain the parallelogram ABEF, and if the equal triangle AFD be taken from the same figure, there will remain the parallelogram ABCD. Hence the parallelograms are equal (Ax. 3).

Cor. Parallelograms having the same base and the same altitude are equal; for, having the same altitude, or perpendicular height, they must be between the same parallels (Cor. 3, Prop. III.).

Proposition VII.—Theorem.

Parallelograms having equal bases, and between the same parallels, are equal.



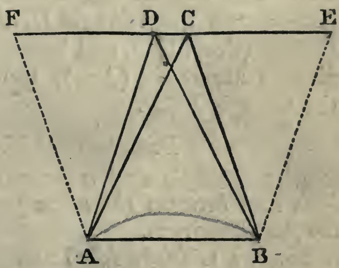
Let the parallelograms ABCD and EFGH have equal bases, AB and EF, and be between the same parallels, AF and DG; then will these parallelograms be equal.

Join the points A and H, and B and G. AB is equal to EF (by hyp.), and HG is equal to EF (Prop. III.); hence AB and HG are equal (Ax. 1). HG and AB being equal and parallel, the lines AH and BG, which join their corresponding extremities, must also be equal and parallel (Prop. I.). Therefore ABGH is a parallelogram; and the parallelogram ABCD is equal to this parallelogram, ABGH (Prop. VI.), and for the same reason EFGH is equal to ABGH. Hence ABCD and EFGH are equal (Ax. 1).

Friday

Proposition VIII.—Theorem.

Triangles upon the same base, and between the same parallels, are equal.



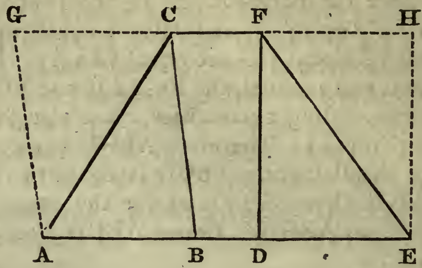
Let the two triangles ABC and ABD have the same base, AB, and be between the same parallels, AB and CD; then will these two triangles be equal.

Produce CD both ways, and make the line AF parallel to BC (Prop. XXIX., Bk. I.); make BE also parallel to AD. The parallelograms ABCF and ABED are equal (Prop. VI.); but the triangle ABC is half the parallelogram ABCF, and the triangle ABD is half the parallelogram ABED (Prop. III.); and because the halves of equal magnitudes are equal (Ax. 7) the triangle ABC is equal to the triangle ABD.

Proposition IX.—Theorem.

Triangles upon equal bases, and between the same parallels, are equal.

Let the triangles ABC and DEF stand upon equal bases, AB and DE, and be between the same parallels, AE and CF; then will these triangles be equal.

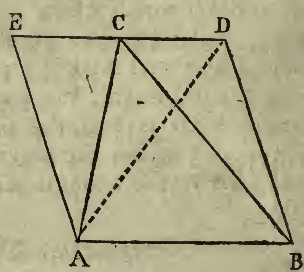


Through the points A and E draw AG and EH parallel to BC and DF, each to each (Prop. XXIX., Bk. I.), and produce CF both ways to G and H; then the figures ABCG and DEHF are parallelograms, and are equal to each other (Prop. VII.), because they are upon equal bases, AB and DE, and between the same parallels, AE and GH. But the triangle ABC is half the parallelogram ABCG, and the triangle DEF is half the parallelogram DEHF (Prop. III.), and because the halves of equal magnitudes are equal (Ax. 7) the triangle ABC is equal to the triangle DEF.

Monday

PROPOSITION X.—THEOREM.

If a triangle and a parallelogram be upon the same base and between the same parallels, the triangle is half the parallelogram.

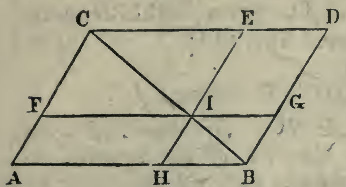


Let the triangle ABC and the parallelogram ABDE stand upon the same base, AB, and be between the same parallels, AB and ED; then will the triangle ABC be half the parallelogram ABDE.

Draw the diagonal AD; then the triangle ABC is equal to the triangle ABD (Prop. VIII.), because they are upon the same base, AB, and between the same parallels, AB and ED: but the triangle ABD is half the parallelogram ABDE (Prop. III.); hence the triangle ABC is also half the parallelogram ABDE.

Proposition XI.—Theorem.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to each other.*

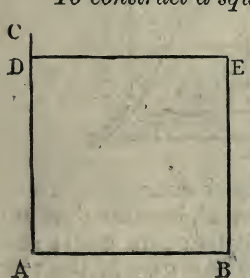


Let AD be a parallelogram, BC its diagonal, FIG parallel to AB or CD, and HIE parallel to AC or BD; FE and HG the parallelograms about BC, and AI and ID the complements of the whole figure AD; then will AI be equal to ID.

The triangle ABC is equal to the triangle CDB, FCI to IEC, and HBI to IGB, because the diagonal bisects the three parallelograms AD, FE, and HG (Prop. III.). If from the triangle ABC the sum of FCI and HBI be taken, and if from the triangle CDB the sum of IEC and IGB be taken, the complements AI and ID remain; they are therefore equal. Tuesday

Proposition XII.—Problem.

To construct a square upon a given straight line.



Let AB be the given straight line; it is required to construct a square upon it.

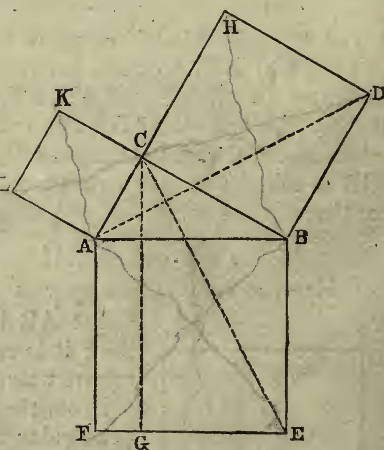
From the point A erect the perpendicular AC (Prop. X., Bk. I.), and make AD equal to AB, and through the point D draw DE parallel to AB, and through B draw BE parallel to AD (Prop. XXIX., Bk. I.). ABED is a square.

* The student will perceive what is meant by "complements" if he examine carefully the application of the enunciation to the figure. FE and HG are about the diagonal, equal portions being on either side of it; and AI and ID are the "complements," which, added to the parallelograms FE and HG, make the whole parallelogram AD.

For, since DE is parallel to AB, and BE to AD, ABED is a parallelogram: but AD is equal to AB (const.); hence the four sides are equal (Prop. III.), and the figure is equilateral. The angle A is a right angle; therefore the other three angles are right angles (Cor. 1, Prop. III.), and the figure is rectangular. The figure ABED is therefore a square, and is constructed upon AB.

Proposition XIII.—Theorem.

In any right-angled triangle, the square described upon the hypothenuse is equal to the sum of the squares described upon the other two sides.



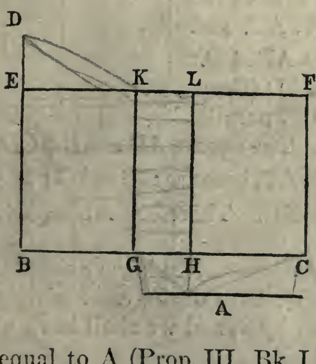
Let ABC be a right-angled triangle, and the angle ACB the right angle; then will the square of AB be equal to the sum of the squares of AC and CB.

Upon AB, AC, and CB describe the squares AE, AK, and BH (Prop. XII.); through C draw CG (Prop. XXIX., Bk. I.) parallel to AF or BE, and join the points A and D, and C and E.

The angles CBD and ABE are equal, because they are right angles (Ax. 11); to each add the angle ABC; and the angle ABD will be equal CBE (Ax. 2). In the two triangles ABD and CBE, the sides AB and BE are equal, because they are sides of the same square; and for a similar reason the sides BD and BC are also equal. Hence these two triangles have two sides, AB and BD, and the included angle, ABD, of the one equal to two sides, EB and BC, and the included angle, EBC, of the other, each to each; they are, therefore, equal in all their parts (Prop. IV., Bk. I.). But the parallelogram BG is double the triangle EBC, because they are upon the same base and between the same parallels (Prop. X.), and the square BH is double the triangle ABD for a similar reason; and because the doubles of equal magnitudes are equal (Ax. 6), the parallelogram BG is equal to the square BH. In like manner it may be demonstrated that the parallelogram AG is equal to the square AK. Hence the sum of these parallelograms, or the square BF, is equal to the sum of the two squares BH and AK; but BF is the square of AB, AK of AC, and BH of BC. Therefore the square of AB is equal to the sum of the squares of AC and CB.

Proposition XIV.—Theorem.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.



Let A and BC be two straight lines, and let BC be divided into any number of parts in the points G and H; the rectangle contained by A . BC is equal to the sum of the rectangles A . BG, A . GH, A . HC.

From the point B erect BD perpendicular to BC (Prop. XI., Bk. I.), make BE equal to A (Prop. III., Bk. I.), through the point E draw EF parallel to BC (Prop. XXIX., Bk. I.), make EF equal to BC, and through the points G, H, C, draw GK, HL, CF, parallel to BE; BF, BK, GL, and HF are rectangles, and BF=BK+GL+HF.

But BF=BE . BC=A . BC, because BE=A.

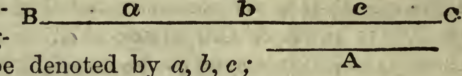
BK=BE . BG=A . BG, for the same reason.

GL=GK . HG=A . HG, because GK=BE=A.

HF=HL . HC=A . HC, because HL=GK=A.

Hence A . BC=A . BG+A . HG+A . HC.

Scholium. Propositions like the above are easily derived from Algebra. Let the segments of BC be denoted by a, b, c;

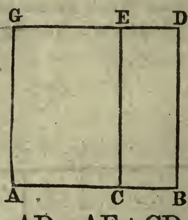


then $\quad A\,(a+b+c)=Aa+Ab+Ac;$

that is, $\quad A . BC \;=A . a+A . b+A . c.$

Proposition XV.—Theorem.

If a straight line be divided into two parts, the square of the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.



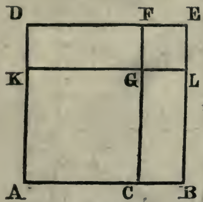
Let the straight line AB be divided into any two parts at the point C; the square of AB is equal to the sum of the rectangles contained by AB . AC, and AB . CB; or $\overline{AB}^2$=AB . AC+AB . CB.

On AB describe the square AD (Prop. XII.), and through C draw CE parallel to AG (Prop. XXIX., Bk. I.). The square AD=AE+CD. But AD=$\overline{AB}^2$, and AE=AB . AC, because AB=AG; and CD=AB . CB, because AB=BD. Therefore $\overline{AB}^2$=AB . AC+AB . CB.

Scholium. The Algebraic demonstration is very simple: Let a=AB, b=AC, and c=CB; then $a=b+c$. Multiply both members of the equation by a, and we shall have $a^2=ab+ac$.

Proposition XVI.—Theorem.

If a straight line be divided into two parts, the square of the whole line is equal to the sum of the squares of the parts and twice the rectangle contained by the parts.



Let AB be a straight line divided into two parts, AC and CB; then will the square of AB be equal to the sum of the squares of AC and CB, and twice the rectangle contained by AC and CB. Or $AB^2 = AC^2 + CB^2 + 2AC \times CB$.

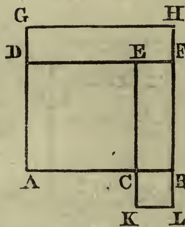
On AB describe the square AE (Prop. XII.), on AC describe the square AG, produce CG and KG to F and L. KL=AB and CF=BE or AB. Hence KL=CF. But KG=GC, being sides of a square. If from KL and CF, KG and GC be subtracted respectively, GL will remain=to GF. But GL and GF are respectively equal to FE and EL. Hence GE is a square, and since LG=CB, it is the square of CB. KG=GC and GF=GL: therefore the rectangles KF and CL are equal. But KG and GC are each equal to AC, and GF and GL are each equal to CB. Hence the rectangles KF and CL are twice the rectangle contained by AC and CB. The whole square AE is equal to the squares AG and GE and the two rectangles KF and CL. Therefore the square of AB=the square of AC, the square of BC and twice the rectangle contained by AC and CB. Or $AB^2 = AC^2 + CB^2 + 2\, AC \,.\, CB$.

Algebraically: Let $AC = a$, and $CB = b$; then $(a+b)^2 = a^2 + b^2 + 2ab$.

Cor. If the line be divided into two equal parts, the square of the whole line equals four times the square of half the line.

Proposition XVII.—Theorem.

The square of the difference of two lines is equal to the sum of their squares diminished by twice the rectangle contained by the lines.



Let AB and BC be two lines whose difference is AC; then will the square of AC be equal to the sum of the squares of AB and BC diminished by twice the rectangle of AB and BC: or $AC^2 = AB^2 + BC^2 - 2AB.BC$.

On AC construct the square AE, on AB construct the square AH, and on BC construct the square CL (Prop. XII.); produce DE to F. AG=AB, and AD=AC (const.); hence (Ax. 3) DG=BC; GH=AB. The rectangle DH is the rectangle of GH and DG, or of AB and BC. AC =AD or BF, and CB=BL: hence, by adding these equals, AB=FL, and, being sides of the same square, KL =BC. But the rectangle KF is the rectangle contained by FL and LK, or by their equals AB and BC. Therefore the two rectangles DH and KF are the rectangles contained by twice AB and BC. If from the whole figure ACKLBHG the two rectangles DH and KF be subtracted, there will remain the square AE. But the whole figure is equal to the two squares AH and CL. Hence $AC^2 = AB^2 + BC^2 - 2AB.BC$.

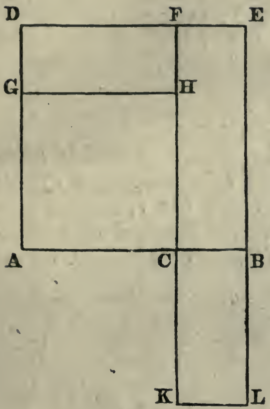
Algebraically: Let $AB = a$, and $BC = b$; then $(a-b)^2 = a^2 + b^2 - 2ab$.

Proposition XVIII.—Theorem.

The rectangle of the sum and difference of two lines is equal to the difference of their squares.

Let AB and AC be two lines; then will the rectangle of their sum and difference be equal to the difference of their squares; or $(AB+AC).(AB-AC) = AB^2 - AC^2$.

On AB and AC construct the squares AE and AH;

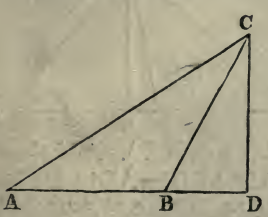


prolong BE until BL is equal to AC, draw CK equal and parallel to BL, produce CH to F, and join KL. AB=AD, and AC=AG; hence (Ax. 3) CB=DG. BL=AC or GH; therefore the rectangle GF = the rectangle CL. EL=EB+BL, or AB+AC and CB or KL=AB−AC. The rectangle EK is the rectangle contained by EL and LK. Hence the rectangle EK is also the rectangle contained by AB+AC, and AB−AC. Now the difference of the squares of AB and AC is the rectangles GF and CE. But GF was before proved equal to CL: hence the difference of the squares of AB and AC is the rectangle EK. But EK is the rectangle under AB+AC, and AB−AC; that is, $(AB+AC).(AB-AC)=AB^2-AC^2$.

Algebraically: Let $AB=a$, and $AC=b$; then $(a+b)(a-b)=a^2-b^2$.

Proposition XIX.—Theorem.

In an obtuse-angled triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides and twice the rectangle contained by the base and the distance from the obtuse angle to the foot of the perpendicular falling on the base produced.



Let ABC be an obtuse-angled triangle; then will the square of AC be equal the sum of the squares of AB and BC and twice the rectangle contained by AB and BD.

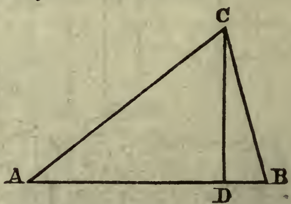
$AD^2=AB^2+BD^2+2AB.BD$ (Prop. XVI.), to each add DC^2;

and $AD^2+DC^2=AB^2+BD^2+DC^2+2AB.BD$. But $AD^2+DC^2=AC^2$ (Prop. XIII.), and $BD^2+DC^2=BC^2$; hence (by substitution) $AC^2=AB^2+BC^2+2AB.BD$.

PROPOSITION XX.—THEOREM.

In an acute-angled triangle the square of a side opposite an angle is equal to the sum of the squares of the other two sides diminished by twice the rectangle contained by the base and the distance between this angle and the foot of a perpendicular let fall on the base.

Let ABC be an acute-angled triangle; then will the square of AC be equal to the sum of the squares of AB and BC diminished by twice the rectangle contained by AB and BD.

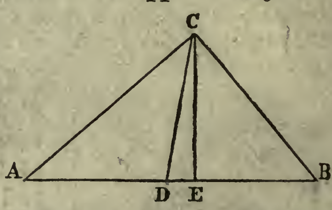


$AD^2=AB^2+BD^2-2AB.BD$ (Prop. XVII.); to both add DC^2 and $AD^2+DC^2=AB^2+BD^2+DC^2-2AB.BD$. But $AD^2+DC^2=AC^2$ (Prop. XIII.), and $BD^2+DC^2=BC^2$. Therefore $AC^2=AB^2+BC^2-2AB.BD$.

PROPOSITION XXI.—THEOREM.

If one side of a triangle be bisected, the sum of the squares of the other two sides is equal to twice the square of half the side bisected and twice the square of the line drawn from the point of bisection to the opposite angle.

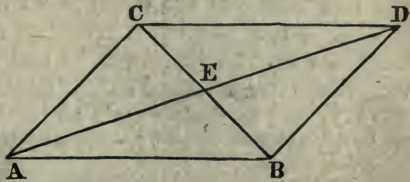
Let AB be bisected in D by the line DC; draw EC perpendicular to AB. Then will AC^2+BC^2 be equal to $2AD^2+2DC^2$.



For $AC^2=AD^2+DC^2+2AD.DE$ (Prop. XIX.), and $BC^2=BD^2+DC^2-2BD.DE$ (Prop. XX.). Add these two equations together, and $AC^2+BC^2=AD^2+DB^2+2DC^2$. But $DB=AD$; hence $AC^2+BC^2=2AD^2+2DC^2$.

Proposition XXII.—Theorem.

The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals.

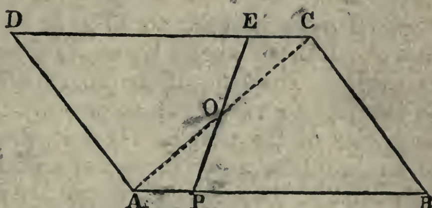


Let ABCD be a parallelogram; then will the sum of the squares of AB, BD, CD, and AC be equal to the sum of the squares of AD and BC, or $AB^2+BD^2+CD^2+AC^2=AD^2+BC^2$.

The diagonals of a parallelogram mutually bisect each other (Prop. IV.); hence $AB^2+AC^2=2EB^2+2AE^2$, and $BD^2+DC^2=2BE^2+2DE^2$ (Prop. XXI.). Add these equations, and $AB^2+AC^2+BD^2+DC^2=4BE^2+2AE^2+2ED^2$; or $AB^2+AC^2+BD^2+DC^2=4BE^2+4AE^2$. But since the square of a line is equal to four times the square of its half, $AB^2+AC^2+BD^2+DC^2=AD^2+BC^2$.

The following are Test Examples involving the First and Second Books solved or demonstrated:

1. *To bisect a parallelogram from a point in one of its sides.*



Let ABCD be the given parallelogram, and P the given point.

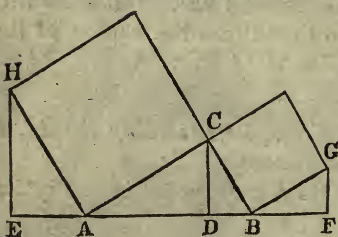
On the line DC lay off a segment EC equal AP, and join the points P and E. The line PE bisects the parallelogram.

For, draw the diagonal AC.* The two triangles ABC

* The student may readily invent other methods of demonstrating this truth by drawing different lines.

and CDA are equal (Prop. III.). In the two triangles APO and CEO the angles OAP and APO are equal to OCE and CEO, each to each (Prop. XXVII., Bk. I.), and the included sides AP and CE are equal (const.). Hence the two triangles are equal in all their parts (Prop. V., Bk. I.). From the equal triangles ABC and CDA subtract the equal triangles APO and CEO, and there will remain the quadrilateral PBCO equal to the quadrilateral EDAO; and to these equals add CEO and APO respectively, and the sums will be the equal quadrilaterals PBCE and EDAP.

2. *If from the angular points of the squares described upon the sides of a right-angled triangle perpendiculars be let fall upon the hypothenuse produced, they will cut off equal segments, and the perpendiculars will be together equal to the hypothenuse.*



Let ABC be the given right-angled triangle, HC and CG the given squares, and HE and GF the perpendiculars upon the hypothenuse produced; then will EA be equal to BF, and HE and GF will be together equal to AB.

Draw CD perpendicular to AB. In the two triangles HEA and ADC, the angles E and ADC are equal, being right angles. The sum of the angles CAD, HAC, and HAE is equal to two right angles, and the sum of the three angles of the triangle HEA is also equal to two right angles: take away the equal angles E and HAC, and the sum of the angles CAD and HAE will be equal to the sum of the angles EHA and EAH; take away the common angle HAE, and the angle CAD will be equal to the angle EHA. Hence the two triangles are equiangular. And since AH is equal to AC, the two triangles are equal in all their parts; EA is equal to CD, and HE is equal to AD. Similarly it can be proved that GF is equal to DB, and BF to CD. Since EA and BF are each equal to CD, they must be equal to each other; and since HE is equal to AD, and GF equal to DB, the sum of HE and GF must be equal to the sum of AD and DB or AB.

TEST EXAMPLES IN BOOK II.

1. To construct a quadrilateral when three sides, one angle, and the sum of two other angles are given.

2. To construct a quadrilateral when three angles and two opposite sides are given.

3. Prove that two parallelograms are equal when they have two sides and the included angle equal, each to each.

4. Prove that the sum of the diagonals of a trapezium is less than the sum of any four lines which can be drawn to the four angles from any point within a figure.

5. Prove that if in the sides of a square four points be taken at equal distances from the angles, the lines joining these points will form another square.

6. To bisect a trapezium by a line drawn from one of its angles.

7. Prove that if lines be drawn from the extremities of one of the sides of a trapezoid to the middle point of the opposite side (these sides not to be the parallel sides), the triangle so formed will be half the trapezoid.

8. If the sides of an equilateral and equiangular pentagon be produced to meet, the angles formed by these lines are together equal to two right angles.

9. If the figure be a hexagon, prove that angles formed as above will be equal to four right angles.

10. Prove that two rhombi are equal when a side and an angle of the one are equal to a side and an angle in the other.

11. Prove that if the diagonals of a quadrilateral bisect each other at right angles, the figure will be a rhombus.

12. Prove that in a trapezoid the line joining the middle points of the sides which are not parallel, is parallel to the parallel sides.

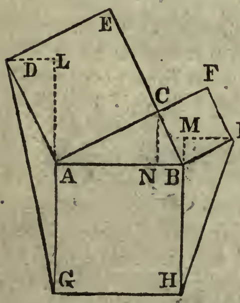
13. Prove that the squares of the diagonals of a trapezium are together less than the squares of the four sides by four times the square of the line joining the points of bisection of the diagonals.

14. Prove that if squares be described on the three sides of a right-angled triangle, and the extremities of the adjacent sides be joined, the triangles so formed are equal to the given triangle and to each other.

15. Prove that if lines be drawn from any point within a rectangle to the four angles, the sums of the squares of those lines drawn to the opposite angles are equal.

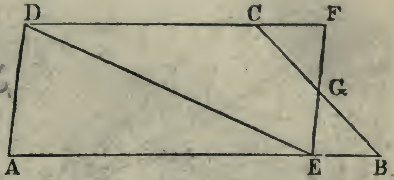
16. Prove that the sum of the squares of the diagonals of a trapezoid is equal to the sum of the squares of the two sides which are not parallel, and twice the rectangle of the parallel sides.

17. Prove that if squares be described upon the three sides of a right-angled triangle, and the extremities of the square on the hypothenuse be joined to the extremities of the adjacent lines of the other two squares, the sum of the squares of the two lines thus formed will be equal to five times the square of the hypothenuse; or, prove that $GD^2 + HK^2 = 5AB^2$. To assist the demonstration, make the triangles GDL and HMK right-angled.

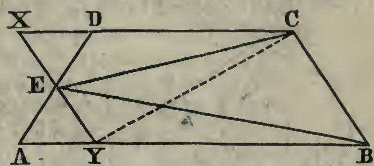


A KEY TO THE TEST EXAMPLES IN BOOK II.

1. See Props. XIV., XXX. (Cor. 1), Bk. I.
2. See same Propositions.
3. See Prop. III. and Cor. 1.
4. See Prop. XII., Bk. I.
5. By constructing the figure the student will have no difficulty in perceiving the equality of the sides; and by observing the relations of the four triangles he can readily demonstrate that each angle of the small square is a right angle.
6. Bisect CB at G, and through G draw FE parallel



to AD. From the angle ADC draw DE: this line DE bisects the trapezium ABCD. See Prop. III. Prove the equality of the triangles EGB and CGF. The student will readily prove the remainder.



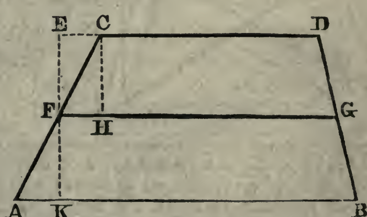
7. Required to prove that the triangle BEC is half the trapezoid. Bisect AD at E, and draw XY parallel to CB. See Prop. IX.

8. See Prop. XXX., Bk. I., Cor. 1. It will be found that each angle of the pentagon is = to 108°; and by Prop. XV. each exterior angle is 72°. The student can now easily finish the demonstration.

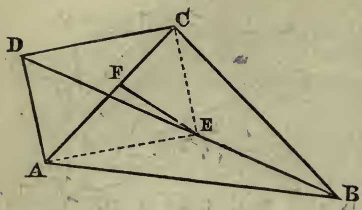
9. Proved in a similar manner by a similar construction.

10. This theorem is easily proved from the definition of a rhombus, and by drawing diagonals and showing the equality of the triangles so formed.

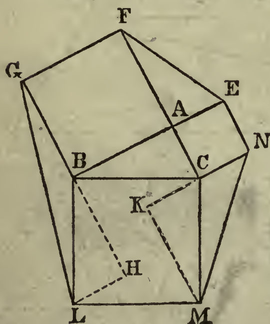
11. Easily demonstrated by means of the propositions in the beginning of Book I. relating to the equality of triangles.



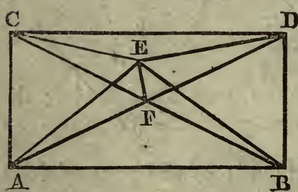
12. Through F, the middle point of AC, draw KE perpendicular to AB: produce CD to E, and draw CH perpendicular to FG. It is easy to prove that the sum of the angles HCD and CHG is equal to two right angles. Hence the truth of the theorem.



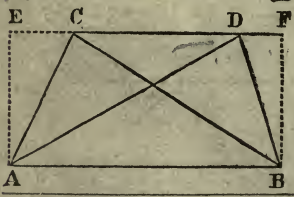
13. Required to prove that $DB^2+AC^2=AB^2+BC^2+DC^2+AD^2-4FE^2$. See Props. XXI. and XVI., Cor. The student commences $DC^2+CB^2=2DE^2+2EC^2$; and, following up this line of demonstration, the truth of the theorem is easily established.



14. It is required to prove that the triangles ABC, AEF, GBL, and CMN are all equal. A glance will show the equality of ABC and AEF. BH is made equal to AB, and KC to AC. The student can now prove that the triangles LHB and KCM are each equal to ABC (Prop. IV., Bk. I.); and by Prop. IX. he can demonstrate that LBH and KCM are respectively equal to LBG and CMN.



15. It is required to prove that the sum of the squares of AE and ED is equal the sum of the squares of CE and EB. Draw the diagonals CB and AD, and join the points E and F. See Props. IV. and XXI.*



16. It is required to prove that $AD^2+BC^2=AC^2+DB^2+2(AB.CD)$.

Erect the perpendiculars AE and BF, and produce CD to E and F.

* The students who can solve or demonstrate without the aid of the Key should be encouraged to do so; and if they can discover other and easier methods, so much the better. The teacher is recommended to make his reviews in Geometry by means of test examples. Indeed, two or three of these, carefully selected, *compel* a general review of the previous principles.

$AD^2 = AC^2 + CD^2 + 2(CD \cdot CE)$, Prop. XIX.;
$BC^2 = DB^2 + CD^2 + 2(CD \cdot DF)$; and, by addition,
$AD^2 + BC^2 = AC^2 + DB^2 + 2CD^2 + 2(CD \cdot CE) + 2(CD \cdot DF)$.
Taking FE as one line divided into three parts, and CD as another line, by Prop. XIV.,

$$EF \cdot CD = CD \cdot EC + CD \cdot CD + CD \cdot DF;$$
$$\therefore AD^2 + BC^2 = AC^2 + DB^2 + 2(EF \cdot CD);$$
$$AD^2 + BC^2 = AC^2 + DB^2 + 2(AB \cdot CD).$$

17. The figure has been already constructed. Prove AN = to AL, and NB=BM.

$$DG^2 = AG^2 + AD^2 + 2(AG \cdot AL), \text{ Prop. XIX.}$$

The student who has attended carefully to the previous demonstration will find no difficulty in establishing the truth of the theorem.

BOOK III.

DEFINITIONS.

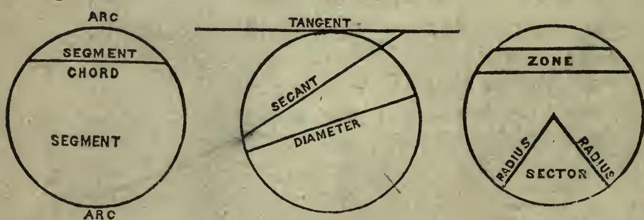
1. An Arc is any portion of the circumference of a circle.

2. A Chord is a line joining the extremities of an arc.

3. A Tangent is a line without a circle, which touches it in one point only. & will not cut the circum. [illegible]

4. A Line is inscribed in a circle when its extremities terminate in the circumference.

5. A Secant is a line which cuts the circumference in two points.



6. A Segment of a circle is that part of it bounded by a chord and the arc subtending it.

7. A Sector of a circle is that part of it bounded by two radii and the intercepted arc.

8. An Angle is inscribed within a circle when the vertex and the extremities of the sides are in the circumference.

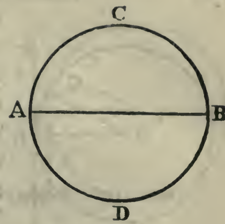
9. A Polygon is inscribed within a circle when all its vertices are in the circumference.

10. The Zone of a circle is a part bounded by two parallel lines and the intercepted arcs.

Proposition I.—Theorem.

Every diameter bisects a circle and its circumference.

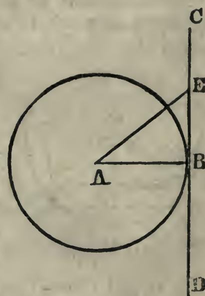
Let ACBD be a circle; then will the diameter AB bisect it and its circumference.



Conceive the part ADB turned over and applied to the part ACB: it must exactly coincide with it; for if it does not, there must be points in either portion of the circumference unequally distant from the centre; which is contrary to the definition of a circle. Therefore the two parts exactly coincide and are equal (Ax. 10).

PROPOSITION II.—THEOREM.

A line perpendicular to a radius at its extremity is tangent to the circumference.



Let the line CD be perpendicular to the radius AB at its extremity; then will CD touch the circumference at one point, B, only.

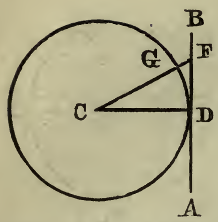
From any point, as E, in the line CD, draw the line AE to the centre A. Since AB is perpendicular to CD, it is shorter than any oblique line, AE (Prop. XXXI., Bk. I.). Hence the point E is without the circle; and in like manner it can be proved that any other point in the line CD is without the circle. Hence CD touches the circumference in one point, B, only.

PROPOSITION III.—THEOREM.

When a line is tangent to the circumference of a circle, a radius drawn to the point of contact is perpendicular to the tangent.

Let the line AB be tangent to the circumference of a circle at the point D; then will the radius CD be perpendicular to the tangent AB.

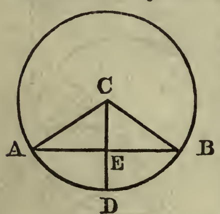
For the line AB being wholly without the circumference, except at the point D, it follows that any line, as



CF, drawn from the centre C to meet the line AB at any point different from D, must have its extremity, F, without the circumference. Hence the radius CD is the shortest line that can be drawn from the centre to meet the tangent AB; and therefore CD is perpendicular to AB (Prop. XXXI., Bk. I.).

Proposition IV.—Theorem.

If a line drawn through the centre of a circle bisect a chord, it will be perpendicular to the chord; or, if it be perpendicular to the chord, it will bisect both the chord and the arc of the chord.



Let the line CD, drawn from the centre of the circle, bisect the chord AB in E; then will CD be perpendicular to AB.

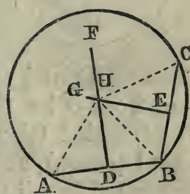
Draw the radii CA and CB. In the two triangles CAE and CBE the sides AC and CB are equal, being radii of the same circle; AE and BE are also equal (hyp.), and CE is common to both. Hence the two triangles have three sides of the one equal to three sides of the other, each to each, and are equal in all their parts (Prop. VII., Bk. I.). Therefore the angle AEC is equal to the angle BEC; and, since they are equal, each must be a right angle, and the line CD must be perpendicular to AB (def. 9).

Again, if CD be perpendicular to AB, then will the chord AB be bisected at the point E, and the arc ADB be bisected at the point D.

For, in the two right-angled triangles AEC and BCE, the hypothenuse AC is equal to the hypothenuse BC (def. 19), and the side CE is common to both; hence the two triangles are equal in all their parts (Prop. XXXII., Bk. I.); the side AE is equal to the side BE, and the angle ACE is equal to the angle BCE. Then apply the sector ACD to the sector DCB, so that DC will be common; and since the angle ACD is equal to the angle DCB, the line AC will coincide with CB; and since the lines AC and CB are equal, the point A must fall upon B. The sectors must, therefore, coincide; for if they did not, some point of the arc AD would fall within or without the circumference, which is contrary to the definition of a circle. Hence the arc AD is equal to the arc DB.

PROPOSITION V.—PROBLEM.

To describe a circle which shall pass through three given points not in the same straight line.



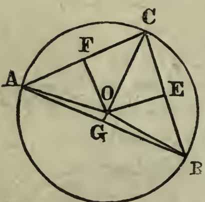
Let A, B, and C be three points not in the same straight line; it is required to describe a circle passing through these three points.

Join the points A and B, and B and C; bisect AB at the point D, and BC at the point E (Prop. IX., Bk. I.). From D erect the perpendicular DF, and from E erect the perpendicular EG (Prop. X., Bk. I.); the point of intersection, H, is the centre of the circle whose circumference will pass through A, B, and C.

The perpendiculars DF and EG must intersect; for if they do not, they are parallel, which is impossible, because, if DF and EG were parallel, since the angles HDB and HEB are right angles, the lines AB and BC must be the secant line, and form one and the same straight line, which is contrary to hypothesis. In the two triangles ADH and BDH, AD and BD are equal (const.). DH is common, and the angles ADH and BDH are equal, being right angles; hence the two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, and are therefore equal in all their parts (Prop. IV., Bk. I.). Hence AH is equal to BH; and, by a similar demonstration, HC can be proved equal to BH; and, since AH and CH are each equal to HB, they are equal to each other (Ax. 1). Therefore AH, BH, and CH are radii of the circle ABC, which passes through the three given points A, B, and C.

Proposition VI.—Problem.

To circumscribe a circle about a given triangle.



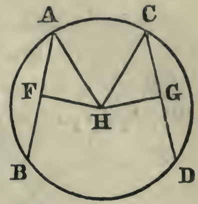
Bisect the sides AB and BC in the points G and E, and from these points erect perpendiculars GO and EO. These perpendiculars must intersect, because, if they do not, they must be parallel, and if parallel, AB and BC, being perpendicular to them, would also be parallel, which is impossible. Hence GO and OE must meet. Join AO, BO, and CO, and from O let fall the perpendicular OF. In the two right-angled triangles, AGO and BGO, AG and BG are equal (const.), and GO is common. Hence the triangles are equal in all their parts (Prop. [illegible], Bk. I.); AO is equal to OB. By comparing the two triangles BOE and COE, by a similar demonstration CO can be proved equal to OB. Hence AO and OC, each being equal to OB, must be equal to each other. Therefore the point O is equidistant from the points A, B, and C. If, then, with the point O as centre, and OB as radius, a circumference be described, it will pass through the points A, B, and C.

Proposition VII.—Theorem.

Equal chords are equally distant from the centre; and, conversely, chords equally distant from the centre are equal.

If the chord AB be equal to the chord CD, then will they be equally distant from the centre.

For since AB is equal to CD, half of AB must be equal to half of CD. Hence AF is equal to CG. We also have



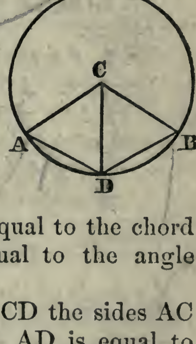
HA and HC equal, and the angle HFA equal to the angle HGC, each being a right angle. Therefore the triangle HAF is equal to the triangle HGC (Prop. XXXII., Bk. I.), and consequently HF is equal to HG.

Conversely. Let AB and CD be any two chords equally distant from the centre H: then will these two chords be equal to each other.

Draw the two radii HA and HC, and the two perpendiculars HF and HG, which are the equal distances of the chords from the centre H. The two right-angled triangles HAF and HCG have the sides HA and HC equal, being radii, the side HF equal to HG, and the angles HFA and HGC right angles. Therefore the two triangles are every way equal (Prop. XXXII., Bk. I.), and AF is equal to CG. But AB is double of AF, and CD is double of CG. Hence AB is equal to CD (Ax. 6).

Proposition VIII.—Theorem.

Equal angles at the centre are subtended by equal chords; and, conversely, equal chords are subtended by equal angles at the centre.



Let the angles ACD and BCD at the centre be equal; then will the chords AD and DB be equal.

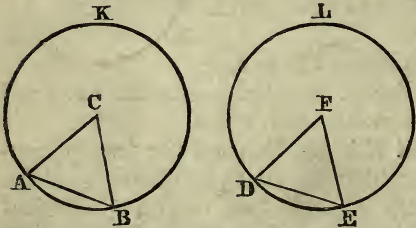
For in the two triangles ACD and BCD, AC and CB are equal, CD is common, and the included angles ACD and BCD are equal (by Hyp.). Therefore AD is equal to BD (Prop. IV., Bk. I.).

Conversely. Let the chord AD be equal to the chord DB; then will the angle ACD be equal to the angle BCD.

For in the two triangles ACD and BCD the sides AC and CB are equal, CD is common, and AD is equal to DB (by Hyp.). Therefore the angle ACD is equal to the angle BCD (Prop. VII., Bk. I.).

Proposition IX.—Theorem.

In the same, or in equal circles, equal angles at the centre intercept equal arcs.



Let ABK and DEL be two equal circles, and let the angle ACB be equal to the angle DFE; then will the arc AB be equal to the arc DE. Join A and B, and D and E.

AC is equal to DF, and CB is equal to FE, and the included angles ACB and DFE are equal. Therefore, if the sector ACB be applied to the sector DFE, the point A will fall upon D, and B upon E, and the arc AB must coincide with the arc DE, for, if it do not, some point of AB must fall within or without the arc DE, which is contrary to the definition of a circle. Hence the arcs AB and DE are equal.

Cor. 1. Equal chords are subtended by equal arcs, and, conversely, equal arcs by equal chords.

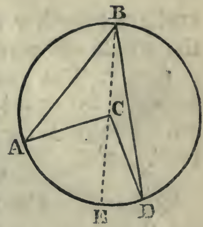
Cor. 2. Equal arcs are subtended by equal angles at the centre.

Cor. 3. An angle at the centre is measured by the subtending arc. All the angles at the centre are together equal to four right angles, or to 360°; and 360° is the measure of the circumference. A quadrant, or 90°, is the measure of a right angle; a sextant, or 60°, is the measure of an angle of an equilateral triangle.

Proposition X.—Theorem.

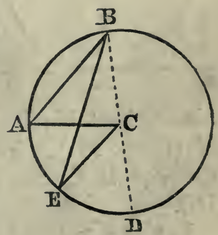
An angle at the centre is double an angle at the circumference when both angles stand on the same arc.

Let the angle ABD at the circumference stand on the arc AD, and the angle ACD at the centre stand on the



same arc AD; then will ACD be double ABD.

For, draw the diameter BE. AC and BC being equal, the angle A is equal to the angle ABC; and BC and CD being equal, the angle D is equal to the angle CBD. But the angle ACE is equal to the sum of the angles A and ABC, and the angle ECD is equal to the sum of the angles D and DBC (Prop. XXX., Bk. I.). Since A and ABC are equal, ACE is double of ABC; and, in like manner, ECD is double of CBD. Therefore the sum of ACE and ECD is double the sum of ABC and CBD; or, ACD is double of ABD.



Cor. 1. Since ACD is measured by the arc AD, ABD is measured by half the arc AD; that is, an inscribed angle is measured by half the subtending arc.

Scholium. The angle might have both its sides on either side of the diameter. In this case it may be readily shown that ACD is double of ABD, and ECD is double of EBC. Then, by subtraction, ACE is double of ABE.

Cor. 2. All angles inscribed in a semicircle are right angles, because each is measured by half the semi-circumference; that is, 90°.

Cor. 3. All angles inscribed in an arc less than the semi-circumference are obtuse angles, because each is measured by half of an arc greater than 180°.

Cor. 4. All angles inscribed in an arc greater than the semi-circumference are acute angles, because each is measured by half of an arc less than 180°.

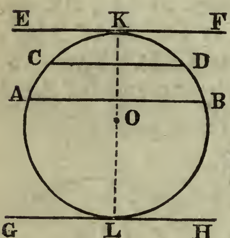
Cor. 5. All angles inscribed in the same arc are equal, because each has the same measure.

Cor. 6. The opposite angles of an inscribed quadrilateral are together equal to two right angles, because half the whole circumference, or 180°, is their measure.

Cor. 7. The sum of the three angles of an inscribed triangle is equal to two right angles, because they are measured by half the whole circumference, or by 180°.

Proposition XI.—Theorem.

Parallel chords intercept equal arcs.



There may be three cases: 1st, both may be secants; 2d, both may be tangents; and, 3d, one may be secant, and the other tangent.

Let AB and CD be parallel secants; then will the intercepted arcs AC and DB be equal.

Draw the radius OK perpendicular to EF. Then the arc AKB is bisected at K; and the arc CKD is also bisected at K (Prop. IV.). From the equals AK and KB subtract the equals CK and KD, and AC will remain equal to DB.

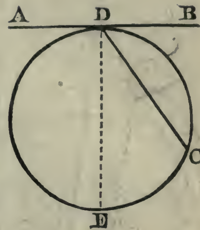
Let the tangent EF be parallel to the tangent GH; then will the arc LAK be equal to LBK. The arc AK is equal to the arc BK, and AL is equal to BL (Prop. IV.); and by adding these equals LAK is equal to LBK.

Let AB be a secant, and EF tangent. The radius OK bisects the arc AKB. Therefore AK is equal to KB.

Proposition XII.—Theorem.

An angle formed by a tangent and chord is measured by half the arc of that chord.

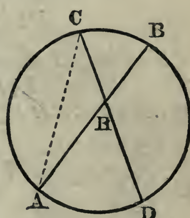
Let AB be a tangent, and CD a chord drawn to the point of contact; then will the angle BDC be measured by half the intercepted arc DC. Draw the diameter ED.



The angle EDB being a right angle is measured by half the semi-circumference—that is, by the arc DCE; and EDC is measured by half the arc EC (Prop. X., Cor. 1). Now if the angle EDC be taken from the angle EDB, and the arc EC from the arc DCE, there will remain the angle BDC measured by half the arc CD; and, by addition, the angle ADC is measured by half DEC.

Proposition XIII.—Theorem.

The angle formed by the intersection of two chords is measured by half the sum of the two intercepted arcs.



Let the two chords AB and CD intersect each other; then will the angle AHD be measured by half the sum of the intercepted arcs AD and CB.

Join the points A and C. The angle AHD is equal to the sum of the angles HAC and HCA (Prop. XXX., Bk. I.). The angle HAC is measured by half the arc CB, and the angle ACH is measured by half the arc AD (Prop. X. Cor. 1). Therefore the angle AHD, which is equal to the sum of HAC and HCA, is measured by half the sum of BC and AD.

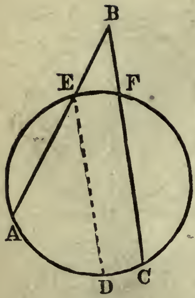
The angle CHB is vertical to AHD; therefore CHB is also measured by half the sum of the arcs CB and AD.

By joining AD it may be proved in a similar manner that AHC, or its equal BHD, is measured by half the sum of the arcs AC and BD.

Proposition XIV.—Theorem.

The angle formed outside a circle *by two secants is measured by half the difference of the intercepted arcs.*

Let the angle B be formed by the two secants AB

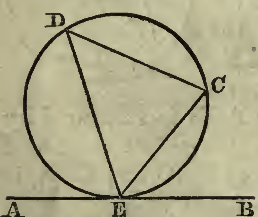


and BC; then will the angle B be measured by half the difference of the arcs AC and EF.

Draw ED parallel to BC (Prop. XXIX., Bk. I.). The angle AED is equal to the angle B (Prop. XXVII., Bk. I.). But the angle AED is measured by half the arc AD. Therefore the angle B is measured by half the arc AD. AD is equal to the difference of AC and DC. But DC is equal to EF (Prop. XI.). Hence AD is the difference of the arcs AC and EF. Therefore the angle B is measured by half the difference of the arcs AC and EF.

Proposition XV.—Theorem.

The angle formed by a tangent and a chord drawn from the point of contact is equal to the angle inscribed in the alternate segment of the circle.



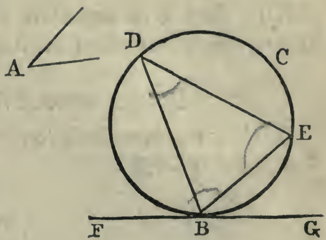
Let AB be a tangent and EC a chord, forming the angle CEB; then will the angle CEB be equal to any angle, as D, in the alternate segment EDC.

The angle CEB is measured by half the arc EC (Prop. XII.), and the angle D is measured by half the same arc (Prop. X., Cor. 1). Therefore the angle CEB is equal to the angle D (Ax. 1).

Proposition XVI.—Problem.

To cut off a segment from a given circle which shall contain an angle equal to a given angle.

Let A be the given angle, and BECD the given circle; then it is required to cut off a segment which shall contain an angle equal to A.



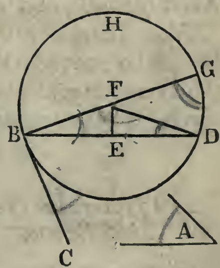
Draw the tangent FG. At the point of contact, B, make the angle EBG equal to A (Prop. XIV., Bk. I.). The line BE cuts off the required segment, BDE.

For the angle D is equal to the angle EBG, each being measured by half the arc BE. But the angle A is also equal to EBG (const.). Therefore the angle D is equal to A; and BDE is the required segment.

If A were a right angle, a semicircle would be the required segment.

Proposition XVII.—Problem.

On a given line to construct a segment of a circle that shall contain an angle equal to a given angle.



Let A be the given angle, and BD the given line.

At the point B, and with the line BD, make an angle, CBD, equal to A (Prop. XIV., Bk. I.). Bisect BD at E (Prop. IX., Bk. I.), and at E erect the perpendicular EF, and at B erect the perpendicular BG (Prop. X., Bk. I.). F is the centre of the circle, and BHD the required segment.

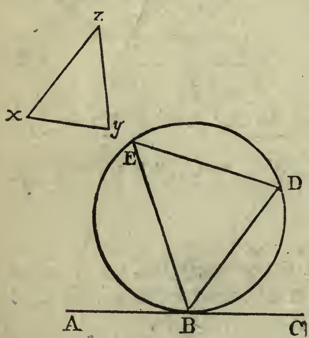
For, in the two triangles BEF and DEF, BE is equal to DE (const.), and EF is common, and the included angles BEF and DEF are equal. Hence BF is equal to DF (Prop. IV., Bk. I.), and the circle described with F as centre and BF as radius will pass through the point D.

Because BG is perpendicular to BC (const.), BC is tangent to the circle at the point B (Prop. II.); the angle CBD, made by a tangent and a chord, is equal to any angle in the alternate segment, BHD, of the circle (Prop.

XV.). But A is equal to CBD. Therefore A is equal to any angle inscribed in the segment BHD.

Proposition XVIII.—Problem.

Within a given circle to inscribe a triangle equiangular to a given triangle.



Let xyz be the given triangle, and BED the given circle.

Draw AC tangent at B: at the point B and with the line BC make the angle CBD equal to the angle y, and at the same point make the angle ABE equal to x. Join ED. BED is the required triangle.

For the angle x is equal to ABE (const.), and the angle D is equal to ABE (Prop. XV.). Therefore the angle x is equal to the angle D. It can be proved similarly that y is equal to E. Hence the remaining angles z and EBD are equal, and the two triangles are equiangular.

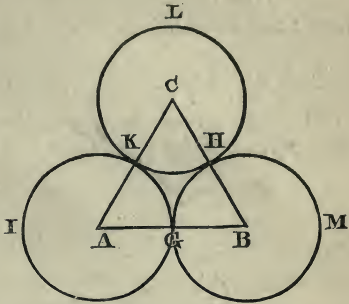
The following Test Examples involve the First, Second, and Third Books.

1. *Describe three circles of equal diameters which shall touch each other.*

Take any line, AB, and bisect it at the point G;* then, with A as centre and AG as radius, describe the cir-

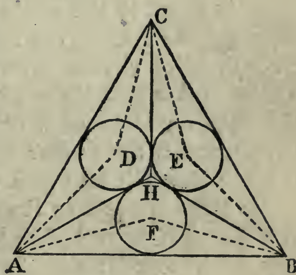
* The student should accurately perform every construction with ruler and compasses. Unless the teacher insists upon the constant use of the instruments and upon accurate measurements, the study of Geometry will be, to a great extent, in vain.

cle GKI; and with B as centre and BG as radius, describe the circle MHG. On AB describe the equilateral triangle ABC; then, with C as centre, and half of AC as radius, describe the circle KHL. It is evident that the three circles, having equal radii, have also equal diameters, and that they touch each other in the points G, K, and H.



2. *In an equilateral triangle to inscribe three equal circles which shall touch each other and the three sides of the triangle.*

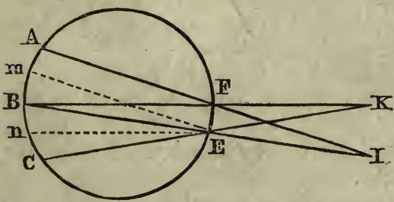
Bisect the angles A and B,* and produce the bisecting lines from A and B until they meet in H. Join the points H and C. It is evident that AHB is an isosceles triangle (Prop. V., Bk. I.); and the two triangles AHC and BHC have two sides and the included angles equal. Therefore (Prop. IV., Bk. I.) the angle ACH is equal to BCH, and the triangles are isosceles. Again, bisect angles at A, B, and C, of the three isosceles triangles, by the lines AD, DC, CE, EB, BF, and FA. The points of intersection, D, E, and F, will be the centres of the three equal circles. From the previous propositions of the First and Third Books the student will have no difficulty in proving the



* The student will perceive that every operation has been accurately performed. This doing the work frequently makes that clear which was before obscure.

equality of the inscribed circles. He has only to prove that their radii are equal.

3. *If from each extremity of any number of equal arcs lines be drawn through two given points in the opposite part of the circumference and produced till they meet, the angles formed by these lines will be equal.*



Let AB and BC be equal arcs, and F and E two points in the opposite part of the circumference, through which let the lines AFI, BEI, BFK, and CEK be drawn; the angles at I and K will be equal.

Through the point E draw E*m* parallel to AF, and E*n* parallel to BF. Since E*m* is parallel to AFI, the angle BE*m* is equal to the angle I (Prop. XXVII., Bk. I.), and, for a similar reason, the angle CE*n* is equal to the angle K. Parallel chords intercept equal arcs (Prop. XI.). The arcs A*m* and B*n* are both parallel to FE; therefore they are equal to each other. But AB is equal to BC (hyp.), and if the former equals be taken from the latter, the arc B*m* will be equal to the arc C*n*. Therefore the angle BE*m* is equal to the angle CE*n* (Prop. IX., Cor. 2). But I and K were before proved equal to these two angles, each to each; therefore they are equal to each other.

TEST EXAMPLES IN BOOK III.

1. Through two given points to draw a circumference of given radius. (The radius must be greater than half the distance between the two points.)

2. Draw a tangent to a given circle parallel to a given line.

3. Describe a circle of given radius tangent to a given line at a given point.

4. Describe a circle of given radius touching the two sides of a given angle.

5. Prove that if a circle be described on the radius of another circle, any straight line drawn from the point where they meet to the outer circumference is bisected by the interior one.

6. Prove that if two circles cut each other, and from either point of intersection diameters be drawn, the extremities of these diameters and the other point of intersection shall be in the same straight line.

7. Describe a circumference which shall be embraced between two parallels and pass through a given point within the parallels.

8. Find in one side of a triangle the centre of a circle which shall touch the other two sides.

9. Through a given point on a circumference, and another given point without, to describe a circle touching the given circumference.

10. In the diameter of a circle produced, to determine a point from which a tangent drawn to the circumference shall be equal to the diameter.

11. Prove that in a quadrilateral circumscribing a circle the opposite sides are equal to half the perimeter.

12. Prove that if the opposite angles of a quadrilateral be equal to two right angles a circle may be described about it.

13. Describe a circle of given radius touching two given circles.

14. In a given circle to inscribe a right angle, one side of which is given.

15. In a given circle to construct an inscribed triangle of given altitude and vertical angle.

16. Prove that if an equilateral triangle be inscribed in a given circle, the square described on a side is equal to three times the square described on the radius.

17. Inscribe a square in a given right-angled isosceles triangle.

18. Inscribe a square in a given quadrant of a circle.

19. Find the centre of a circle in which two given lines meeting in a point shall be a tangent and a chord.

20. Inscribe a square in a circle, and circumscribe a circle with a square.

21

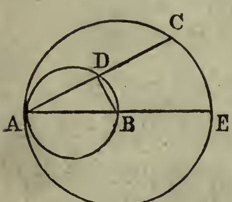
21. Inscribe in a circle a regular hexagon; also an equilateral triangle.

22. Describe a circle the circumference of which shall pass through a given point and touch a given straight line in a given point.

23. If a circle be inscribed in a right-angled triangle, the difference between the sum of the two sides containing the right angle and the hypothenuse is equal to the diameter of the circle.

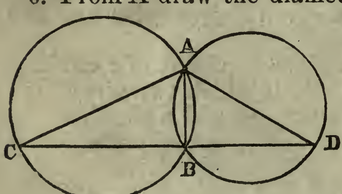
A KEY TO THE TEST EXAMPLES IN BOOK III.

1. See Props. IX. and X., Bk. I.
2. See Props. XI., XXIX., Bk. I. and III., Bk. III.
3. See Props. X., Bk. I., and III., Bk. III.
4. See Prop. VIII., Bk. I.: every point of the bisectrix of an angle is equally distant from the sides. If perpendiculars be let fall from any point, two right-angled triangles will be formed having two angles and the included side of the one equal to two angles and the included side of the other, each to each.



5. The circle ABD is described upon the radius AB of the circle AEC: it is required to prove that the line AC is bisected at the point D. Produce AB to E and draw DB. The angle ADB being inscribed in a semicircle is a right angle (Prop. X.). Hence DB, drawn from the centre, B, of the large circle, is perpendicular to the chord AC, and bisects it (Prop. IV.).

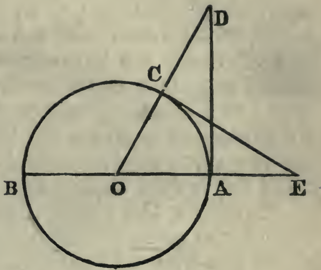
6. From A draw the diameters AC and AD, and from B draw BC and BD. Join AB. See Prop. X.



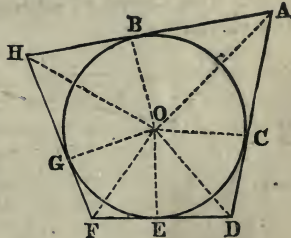
7. Take two points in the parallel lines, which, with the given point, shall not be in the same straight line. See Prop. V.

8. See Test Example 4.

9. Join the two points. See Props. IX. and I., Bk. I.



10. From A draw AD perpendicular and equal to AB. From the centre O draw OD; and from C draw CE perpendicular to OD, meeting BA produced at E. By comparing the two triangles the student can readily prove that CE $=$AD$=$BA.



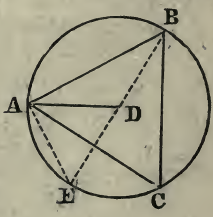
11. Draw the lines OB, OC, OE, and OG perpendicular to the four sides of the quadrilateral; and join OA, OD, OF, and OH. By comparing the triangles there will be no difficulty in proving AC$+$CD$=$ AB$+$DE, and HG$+$GF$=$FE $+$HB.

12. See Prop. X., Cor. 1. The sum of the two angles will require (when they are inscribed) the whole circumference to measure them. See also Prop. V.

13. Draw two circles, respectively equal to the given circles, in such a manner that the distance between them shall be double the given radius. Bisect this distance, and with the point of bisection as centre, and half the distance as radius, describe the required circle.

14. See Prop. X., Cor. 2.

15. Draw an inscribed angle equal to the given angle. It is evident that the altitude is always limited by the measure of the inscribed angle.



16. The angle ADE is double ABE, or is $=$ to ABC or ACB. But the angles C and E are equal (Prop. X. and Cor. 1). Hence the angle ADE $=$ the angle E. Therefore AE$=$AD. The square of EB is

18) 26

equal to AE^2+AB^2, or 4 times $ED^2=AE^2+AB^2$. Subtract AE^2 from one side, and its equal DE^2 from the other, and 3 times $DE^2=AB^2$.

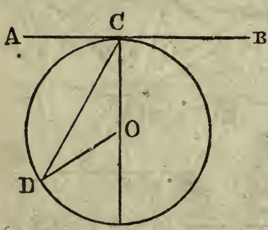
17. Trisect the hypothenuse (see Test Ex., Bk. I.), and from the points of trisection erect perpendiculars to the other two sides. Connect the points in the two sides cut by these perpendiculars. The middle portion of the hypothenuse, the perpendiculars, and the connecting line form the square required.

18. Bisect the right angle to the quadrant. The remainder is quite simple.

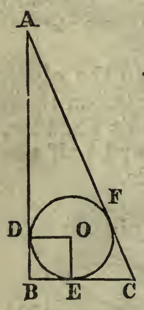
19. See Props. IV. and XII., Bk. III.

20. The student should solve this without any help.

21. The chord of 60 degrees is equal to the radius of the circle.



22. Let AB be the given straight line, C the given point in which the circle is to touch it, and D the point through which it must pass. Draw CO perpendicular to AB. Join CD; and at the point D make the angle CDO=DCO: the intersection of the lines CO and DO is the centre of the circle required. Since the angle DCO=CDO, CO=DO. Therefore a circle described from the centre O, at the distance OD, will pass through C and touch the line AB in C, because OC is perpendicular to AB.



23. Find the centre O. Join OD and OE. The angles B, D, and E are right angles; OD and OE are equal. Hence OB is a square. In Test Example 11 it is shown that FC=CE, and that AF=AD. Hence AC=EC+AD, or AB+BC−BE+BD. But BE+BD=the diameter.

BOOK IV.

DEFINITIONS.

1. A LESS magnitude or quantity is a measure of a greater magnitude or quantity when the less is exactly contained a certain number of times in the greater.

2. A greater magnitude is a multiple of a less when the greater is measured by the less; that is, when the greater contains the less an exact number of times.

3. *Ratio* is the relation of two magnitudes of the same kind, the one to the other; or it is the quotient arising from the division of the one by the other.

4. *Proportion* is an equality of ratios.

5. Three quantities are in proportion when the first is to the second as the second to the third. Four quantities are in proportion when the first is to the second as the third to the fourth.

For example, $A:B::B:C$. Let $A=2$, $B=4$, and $C=8$. Then $2:4::4:8$; that is,

$$\frac{2}{4}=\frac{4}{8};\text{ the equality of ratios.}$$

In this example B, or 4, is a mean proportional between A and C, or between 2 and 8.

Again, let $A=3$, $B=4$, $C=6$, and $D=8$.
Then $A:B::C:D$, or, $3:4::6:8$; that is,

$$\frac{3}{4}=\frac{6}{8};\text{ an equality of ratios.}$$

The first term is called the antecedent, the second the consequent, the third the antecedent, and the fourth the consequent, and so on.

6. Magnitudes are in continued proportion when they have a common ratio; as, 1 is to 2 as 2 to 4, as 4 to 8, and so on. Here the common ratio is $\frac{1}{2}$.

7. The first and fourth terms are called the *extremes*, and the second and third are called the *means*.

8. *An Inverse Proportion* is where the antecedent is made the consequent, and the consequent the antecedent. Thus, if $2:3::6:9$, then, by inversion, $3:2::9:6$.

9. *Alternate Proportion* is where antecedent is compared with antecedent, and consequent with consequent. Thus, if $2:3::6:9$, then, by alternation, it will be $2:6::3:9$.

10. *A Compound Proportion* is where the sum of the antecedent and consequent is compared either with the antecedent or consequent. Thus, if $2:3::6:9$, then, by composition, $2+3:2::6+9:6$.

11. *A Divided Proportion* is where the difference of the antecedent and consequent is compared with either the antecedent or consequent. Thus, if $2:3::6:9$, $3-2:3::9-6:9$.

Proposition I.—Theorem.

Equimultiples of any two magnitudes have the same ratio as the magnitudes themselves.

Let A and B be any two magnitudes, and mA and mB any equimultiples of them (m being any quantity whatever), then will mA and mB have the same ratio as A and B, or $A:B::mA:mB$.

For $$\frac{mA}{mB}=\frac{A}{B}.$$

Let $A=2$ B4 and m any number, say 6:
$$\frac{6\times2}{6\times4}=\frac{2}{4};\text{ that is, }\frac{12}{24}=\frac{2}{4}.$$

Proposition II.—Theorem.

If four magnitudes are proportional, the product of the extremes is equal to the product of the means.

Let the four magnitudes A, B, C, and D form the proportion $A:B::C:D$, then will $A\times D=B\times C$.

Since $A:B::C:D, \frac{A}{B}=\frac{C}{D}$ (Def. 4).

Multiply these equals by BD, and we have

$$AD=BC.$$

Let $A=2, B=3, C=6$, and $D=9$,

$$2:3::6:9, \text{ and } 2\times9=3\times6.$$

Proposition III.—Theorem.

If the product of two magnitudes be equal to the product of two other magnitudes, they will constitute a proportion in which either two magnitudes will be the extremes, and the other two the means.

Let $A\times B=C\times D$. Dividing both members by $B\times D$, and we have $\frac{A\times B}{B\times D}=\frac{C\times D}{B\times D}$; striking out the common factors, and $\frac{A}{D}=\frac{C}{B}$, or $A:D::C:B$.

Let $A=2, B=9, C=6$, and $D=3$; $2\times9=3\times6$,

$$\frac{2\times\cancel{9}}{\cancel{9}\times3}=\frac{\cancel{3}\times6}{9\times\cancel{3}}=\frac{2}{3}=\frac{6}{9}, \text{ or } 2:3::6:9.$$

Proposition IV.—Theorem.

If four magnitudes are proportional, they will be in proportion by alternation.

Let four magnitudes, A, B, C, and D, be in proportion, as $A:B::C:D$, then will they be alternately in proportion as $A:C::B:D$.

For since $A:B::C:D, \frac{A}{B}=\frac{C}{D}$. Multiplying both members by $\frac{B}{C}$, we have $\frac{A\times B}{B\times C}=\frac{C\times B}{C\times D}$. Striking out common factors, we have $\frac{A}{C}=\frac{B}{D}$, or $A:C::B:D$.

Let $A=3, B=6, C=4$, and $D=8$,

$$3:6::4:8, \text{ or } 3:4::6:8.$$

Proposition V.—Theorem.

If four magnitudes are in proportion, they will be in proportion inversely.

Let four magnitudes, A, B, C, and D, be in proportion, as A : B :: C : D, then will they be in proportion inversely, as B : A :: D : C.

Since A : B :: C : D, $\frac{A}{B}=\frac{C}{D}$, or B : A :: D : C.

Let A=4, B=8, C=6, and D=12; 4 : 8 :: 6 : 12, or 8 : 4 :: 12 : 6.

Proposition VI.—Theorem.

If four magnitudes be proportional, and four other magnitudes proportional, having the antecedents the same in both, the consequents will be proportional.

Let A : B :: C : D be four proportional magnitudes, and A : m :: C : x four other proportional magnitudes, having the antecedents A and C the same in both, then will the consequents be proportional, B : m :: D : x.

$$A : B :: C : D \text{ gives } \frac{A}{B}=\frac{C}{D},$$

and

$$A : m :: C : x \text{ gives } \frac{A}{m}=\frac{C}{x}.$$

Divide the second equation, member by member, by the first, and we have $\frac{AB}{Am}=\frac{CD}{Cx}$, or $\frac{B}{m}=\frac{D}{x}$, or

$$B : m :: D : x.$$

Let A=2, B=4, C=6, D=12, m=3, and x=9; 2 : 4 :: 6 : 12, and 2 : 3 :: 6 : 9. Hence 4 : 3 :: 12 : 9.

Proposition VII.—Theorem.

If four magnitudes be proportional, they will be in proportion by composition and division.

Let A : B :: C : D be a proportion; then will A+B : A :: C+D : C, or A−B : A :: C−D : C be also proportional.

For $B \times C = A \times D$; by adding both members of this equation to, and subtracting them from, $A \times C$, we have

$$A \times C \pm B \times C = A \times C \pm A \times D;$$
$$(A \pm B) \times C = (C \pm D) \times A;$$
$$A \pm B : A :: C \pm D : C.$$

Let $A = 2, B = 4, C = 5$, and $D = 10$;

$$2 + 4 : 2 :: 5 + 10 : 5;$$
$$2 - 4 : 2 :: 5 - 10 : 5;$$
$$6 : 2 :: 15 : 5;$$
$$-2 : 2 :: -5 : 5.$$

Proposition VIII.—Theorem.

Of four proportional magnitudes, if any equimultiples of the antecedents whatever, and any equimultiples of the consequents be taken, the resulting magnitudes will be proportional.

Let A, B, C, and D be in proportion; as $A : B :: C : D$, and r be an equimultiple of the antecedents, and s an equimultiple of the consequents; then will rA, rB, sC, and sD be in proportion, as $rA : rB :: sC : sD$.

For, since $A : B :: C : D$, $A \times D = B \times C$. Multiply both members by $r \times s$, and we have

$$rA \times sD = rB \times sC;$$
$$rA : rB :: sC : sD.$$

Let $A = 3, B = 6, C = 8$, and $D = 16, r = 4$, and $s = 3$;

$$3 : 6 :: 8 : 16;$$
$$3 \times 4 : 6 \times 3 :: 8 \times 4 : 16 \times 3;$$
$$12 : 18 :: 32 : 48.$$

Proposition IX.—Theorem.

If there be four proportional magnitudes, and the two consequents be either augmented or diminished by magnitudes that have the same ratio as the respective antecedents, the results and the antecedents will still be proportional.

Let $A : B :: C : D$, and

$$A : C :: m : n;$$

then will $A : C :: B \pm m : D \pm n$.

For, since $A:B::C:D$, $A\times D=B\times C$;
and, since $A:C::m:n$, $A\times n=C\times m$.

By addition and subtraction of equals,

$$A\times D\pm A\times n=B\times C\pm C\times m.$$

By factoring, $A\times(D\pm n)=C\times(B\pm m)$.

Therefore $A:C::B\pm m:D\pm n$.

Let $A=2$, $B=4$, $C=3$, and $D=6$. Then increase the two consequents of the proportion $2:4::3:6$ by 1 and $1\frac{1}{2}$, and we have $2:3::5:7\frac{1}{2}$; or diminish, and we have $2:3::3:4\frac{1}{2}$.

Proposition X.—Theorem.

If any magnitudes be proportional, any like powers or roots of them will be proportional.

Let $A:B::C:D$;
then will $A^2:B^2::C^2:D^2$,
or $A^3:B^3::C^3:D^3$,
or $\sqrt{A}:\sqrt{B}::\sqrt{C}:\sqrt{D}$.

$A\times D=B\times C$; square and $A^2\times D^2=B^2\times C^2$; cube and $A^3\times D^3=B^3\times C^3$; extract the square root, and $\sqrt{A\times D}=\sqrt{B\times C}$; that is, $A^2:B^2::C^2:D^2$;

$$A^3:B^3::C^3:D^3;\ \text{or}\ \sqrt{A}:\sqrt{B}::\sqrt{C}:\sqrt{D}.$$

Let $A=2$, $B=4$, $C=3$, and $D=6$,
and $2:4::3:6$, or $2^2:4^2::3^2:6^2$;
$4:16::9:36$, or $2^3:4^3::3^3:6^3$;
$8:64::27:216$, or $\sqrt{2}:\sqrt{4}::\sqrt{3}:\sqrt{6}$.

It will be found that by substituting numerical values for the magnitudes the propositions of the Fourth Book can be demonstrated with great facility; and young students can be taught to comprehend them without difficulty. Beginners do not readily perceive that A and B, m and n, and x and y have a geometrical meaning. However, when A is 2 and B 4, they see instantly the relation of 2 to 4, although the relation of A to B may be very vague.

BOOK V.

DEFINITIONS.

1. SIMILAR polygons are those which have their angles equal, each to each, and the sides about the equal angles proportional.

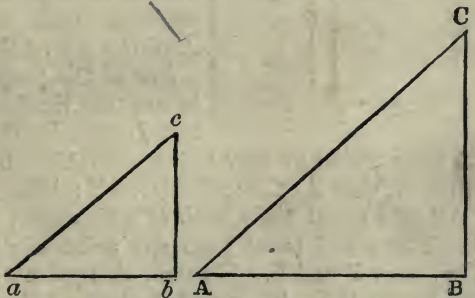
2. In similar polygons the sides adjacent to the equal angles are called homologous sides, and the angles themselves are called homologous angles.

3. Two sides of one polygon are said to be reciprocally proportional to two sides of another when one of the sides of the first is to one of the sides of the second as the remaining side of the second is to the remaining side of the first.

4. The altitude of a triangle is the perpendicular from the vertex to the base, or the base produced. The altitude of a parallelogram is the perpendicular between two opposite sides. The altitude of a trapezoid is the perpendicular between the two parallel sides.

5. Area denotes the superficial contents of a figure.

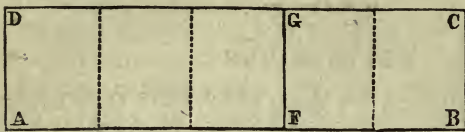
A and *a*, B and *b*, C and *c*, are homologous, each to



each. AB is homologous to *ab*, BC to *bc*, and AC to *ac*. The triangles are similar, and the sides are proportional.

Proposition I.—Theorem.

Two rectangles of the same altitude are to each other as their bases.



Let ABCD and AFGD be two rectangles having the common altitude AD; then will they be to each other as their bases AB and AF.

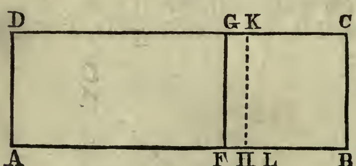
First. Suppose the bases AB and AF to be commensurable; as, for example, suppose they are to each other as 5 to 3. If AB be divided into 5 equal parts, AF will contain 3 of these parts. At each point of division draw lines perpendicular to the base, forming 5 rectangles, which will be equal, since they have equal bases and altitudes (Prop. VII., Bk. II.).

Since the rectangle ABCD contains 5 of these rectangles, and the rectangle AFGD contains 3, it follows that

ABCD : AFGD :: 5 : 3;

but AB : AF :: 5 : 3;

therefore ABCD : AFGD :: AB : AF.

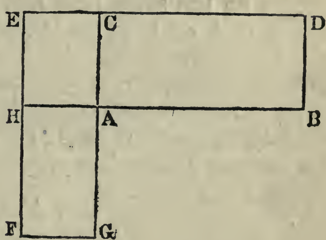


Second. Suppose AB and AF to be incommensurable; still ABCD : AFGD :: AB : AF.

For if this proposition be not true, the first three terms remaining the same, the fourth term will be greater or less than AF. Suppose it to be greater, then ABCD : AFGD :: AB : AL. Divide the line AB into equal parts, each of which shall be less than FL. There will be at least one point of division, as at H, between F and L. Through this point H draw the perpendicular HK. Then will the bases AB and AH be commensurable, and ABCD : AHKD :: AB : AH. But, by supposition, ABCD : AFGD :: AB : AL. Since the antecedents in both proportions are the same, the consequents form a new proportion, AHKD : AFGD :: AH : AL. But AL is greater than AH; therefore AFGD is greater than AHKD, which is absurd. Therefore ABCD can not be to AFGD as AB is to a line greater than AF. And, similarly, it can be proved that the fourth proportional can not be less than AF. Hence, ABCD : AFGD :: AB : AF.

Proposition II.—Theorem.

Any two rectangles are to each other as the product of their bases and altitudes.



Let ABDC and AGFH be two rectangles; then will ABDC be to AGFH as $AB \times AC : AG \times AH$.

Place the two rectangles so that the angles at A may be vertical; produce DC and FH until they meet at E.

The two rectangles ABDC and ACEH, having the same altitude, AC, are to each other as their bases, AB and AH. In like manner, the two rectangles AGFH and ACEH, having the same altitude, AH, are to each other as their bases, AG and AC. Hence there are two proportions:

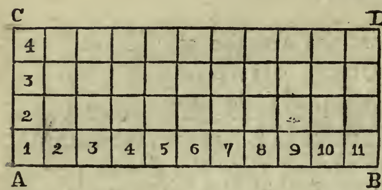
ABDC : ACEH :: AB : AH,

and ACEH : AGFH :: AC : AG.

By multiplying these two proportions, observing to strike out ACEH, which appears both in the antecedent and consequent, there will be a new proportion:

$$ABDC : AGFH :: AB \times AC : AH \times AG.$$

Scholium. Suppose AB divided into 11 equal parts, and AC into 4 equal parts, and suppose the unit of measure be one yard; then the area of the rectangle ABCD will be 44 square yards; that is, $11 \times 4 = 44$ square yards. By actual count, there will be found 44 little squares,

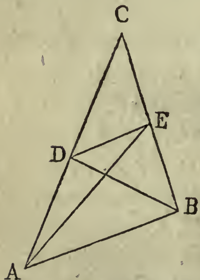


each of which is a square yard. Hence, *to find the area of a rectangle or of a parallelogram, multiply the base by the altitude;* and, to find the area of a triangle, multiply the base by one-half the altitude, or the altitude by one-half the base; or multiply the base by the altitude and take half the product.

Cor. Since a triangle is half a rectangle or parallelogram, having the same ~~base and~~ + betw. the same para altitude; therefore triangles on the same base and between the same parallels are to each other as their bases.

PROPOSITION III.—THEOREM.

A line drawn parallel to one side of a triangle divides the other two sides into proportional parts.



Let ABC be any triangle, and DE a line drawn parallel to AB; then will DE divide AC and BC into proportional parts; that is, DC : AD :: CE : EB.

For, draw the lines AE and BD. The triangles ADE and BDE are equal, because they have the same base, DE, and are between the same parallels, DE and AB (Prop. VIII., Bk. II.). But the triangles ADE and DEC have their bases in the same line, AC, and have the same altitude. Therefore

ADE : CDE :: AD : DC (Prop. II., Cor.);

and, for the same reason,

BDE : CDE :: BE : EC.

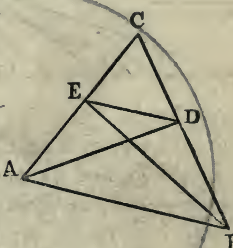
And, since ADE is equal to BDE, and CDE common in both proportions, by equality of ratios, AD : DC :: BE : EC.

Cor. 1. By composition (Prop. VII., Bk. IV.), AD+DC : DC :: BE+EC : EC, or AC : DC :: BC : EC, or AC : AD :: BC : BE.

Cor. 2. If any number of lines be drawn parallel to a side of a triangle, each parallel cuts the other two sides proportionally.

Proposition IV.—Theorem.

If a straight line cut two sides of a triangle proportionally, it will be parallel to the other side.

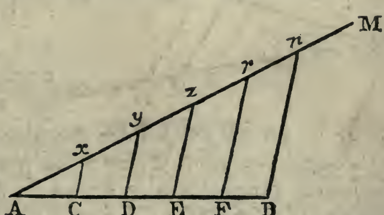


Let the straight line DE cut the two sides of the triangle ABC proportionally; then will DE be parallel to AB.

Draw AD and BE. The triangle ADE : DEC :: AE : EC (Prop. II., Cor.), and BDE : DEC :: BD : DC. But, by hypothesis, AE : EC :: BD : DC. Hence, by equality of ratios, ADE : DEC :: BDE : DEC. But ADE and BDE have the same ratio to DEC. Hence they are equal. Now if the triangles ADE and BDE are equal, and have the same base, AB, it is evident that they must have the same altitude; that is, AB and ED are parallel.

Proposition V.—Problem.

To divide a line into any number of equal parts.



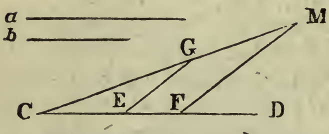
Let AB be the given line: it is required to divide it into any number of equal parts, say five.

Draw the indefinite line AM, making any angle with AB. On AM cut off five equal parts, A*x*, *xy*, *yz*, *zr*, and *rn*. Join

B and n. Then, through the points x, y, z, r, draw the lines Cx, Dy, Ez, and Fr parallel to Bn. These parallel lines divide AB and An proportionally (Prop. III., Cor. 2). But An is divided into five equal parts (const.). Therefore AB is divided in five equal parts.

Proposition VI.—Problem.

To find a third proportional to two given lines.

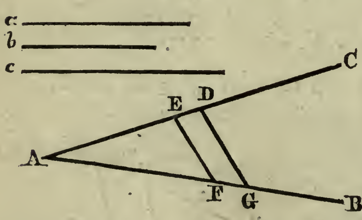


Let a and b be two given lines: it is required to find a third proportional.

Draw CD and CM, making any angle. On CD and CM cut off parts CF and CG equal to a, and on CD cut off a part CE equal to b. Join E and G, and through the point F draw FM parallel to EG. CM is the third proportional. For CE : CF :: CG : CM (Prop. III.). But CF and CG are each equal to a, and CE is equal to b. Therefore $b : a :: a :$ CM.

Proposition VII.—Problem.

To find a fourth proportional to three given lines.



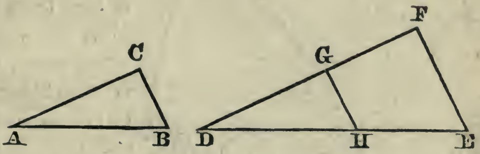
Let a, b, and c be three given lines: it is required to find a fourth proportional.

Draw two lines AC and AB, making any angle. On AC cut off AD equal to a, and AE equal to b; on AB cut off AG equal to c. Join D and G, and through the point E draw EF parallel to DG. AF is the fourth proportional. For AD : AE :: AG : AF; but AD is equal to a, AE to b, and AG to c. Hence $a : b :: c :$ AF.

Proposition VIII.—Theorem.

Equiangular triangles are similar, or have their homologous sides proportional.

Let the two triangles ABC and DEF be equiangular, the angle A equal



to the angle D, the angle B equal to the angle E, and the angle C equal to the angle F; then will these two triangles be similar, and AB will be to AC as DE to DF, and so on.

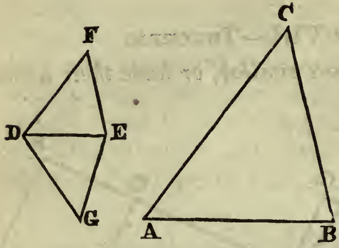
For from DE cut off the part DH equal to AB, and from DF cut off the part DG equal to AC. Join G and H. The two triangles ABC and DHG have two sides of the one equal to two sides of the other (const.), and the included angles A and D equal (hyp.); therefore they are equal in all their parts (Prop. IV., Bk. I.), and the angle DHG is equal to the angle B; but the angle E is also equal to the angle B. Therefore the angle DHG is equal to the angle E (Ax. 1); and since DHG is equal to E, HG must be parallel to EF (Prop. XXVI., Bk. I.). Because HG is parallel to EF, DH : DE :: DG : DF (Prop. III.). By substituting for DH and DG their equals, AB and AC, then AB : AC :: DE : DF.

Proposition IX.—Theorem.

Two triangles which have their homologous sides proportional are equiangular and similar.

Let the two triangles ABC and DEF have their homologous sides proportional, DF : DE :: AC : AB, and DE : EF :: AB : BC; then will the two triangles be equiangular and similar.

For at the point D make the angle GDE equal to A,



and at the point E make the angle DEG equal to B; and since the sum of the angles GDE and DEG is less than two right angles, the lines DG and EG must meet, and the angle G must equal the angle C (Prop. XXX., Cor., Bk. I.). Hence the triangles ABC and DGE are equiangular, and

$$AB : BC :: DE : EG$$

(by previous Prop.), but, by hypothesis,

$$AB : BC :: DE : EF;$$

since the three first terms are identical, it follows that EG is equal to EF. Again,

$$AB : AC :: ED : DG$$

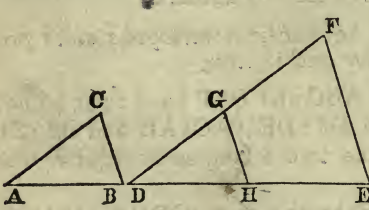
(by previous Prop.), but, by hypothesis,

$$AB : AC :: ED : DF;$$

since the three first terms are identical, DG is equal to DF. The two triangles DEG and DEF have their three sides equal, each to each; they are, therefore, equal in all their parts, and quiangular (Prop. VII., Bk. I.). But ABC is equiangular to DEG. Hence ABC is also equiangular to DEF.

Proposition X.—Theorem.

If two triangles have an angle in each equal, and the sides about these equal angles proportional, they will be equiangular.



Let the two triangles ABC and DEF have the angle A equal to the angle D, and the side AB to the side AC as DE to DF; then will these triangles be equiangular.

For from DE cut off the part DH equal to AB, and from DF cut off the part DG equal to AC. Join G and H. The two triangles ABC and DHG have two sides of the one equal to two sides of the other, each to each (const.), and the included angles A and D equal. They are, therefore, equal in all their parts (Prop. IV., Bk. I.); the angle DGH is equal to C, and DHG is equal to B.

By hypothesis,

$$AB : AC :: DE : DF;$$

but DH and DG are equal to AB and AC (const.). Therefore

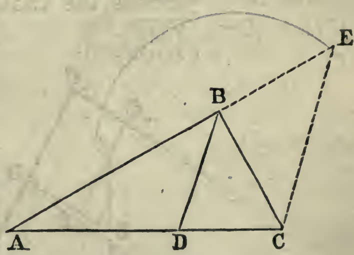
$$DE : DF :: DH : DG.$$

Hence GH is parallel to FE (Prop. IV.). Consequently the angle F is equal to the angle DGH, and the angle E is equal to the angle DHG (Prop. XXVII., Bk. I.). But the angles C and B were before proved equal to DGH and DHG. Therefore the angles C and B are equal to the angles F and E, each to each, and the triangles ABC and DEF are equiangular.

Proposition XI.—Theorem.

A line which bisects any angle of a triangle, divides the opposite side into segments proportional to the other two sides.

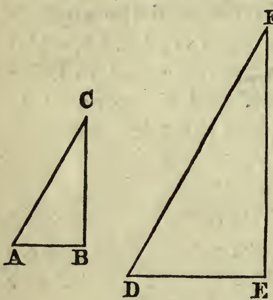
Let the line DB bisect the angle ABC of the given triangle ACB; then will the segments AD and DC be proportional to the other two sides AB and BC, or AD : DC :: AB : BC.



For, from the point C draw CE parallel to DB and produce AB, until it meets CE at the point E. Since DB and CE are parallel, the angle BCE is equal to the angle DBC (being alternate angles); and the angle E is equal to the angle ABD [an exterior angle equal to an interior and opposite angle on the same side of the secant line] (Prop. XXVII., Bk. I.); and since the angles E and BCE are each equal to half of the angle ABC, they are equal to each other, and the triangle BCE is isosceles (Prop. XIX., Bk. I.). Hence CB is equal to BE. Since DB is parallel to CE, AD : DC :: AB : BE (Prop. III.), but BC is equal to BE. Therefore AD : DC :: AB : BC.

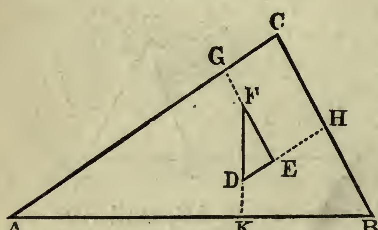
Proposition XII.—Theorem.

Two triangles which have their homologous sides parallel or perpendicular, are similar.



1. Let the two triangles ABC and DEF have AB parallel to DE, AC parallel to DF, and BC parallel to EF; then will they be similar.

For, since AC and CB are parallel to DF and FE, each to each, the angles C and F are equal (Prop. XXXIII., Bk. I.). Similarly it can be proved that the angles A and D, and B and E are equal. Hence the triangles are equiangular and similar (Prop. VIII.).

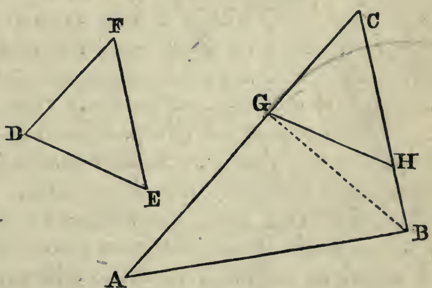


2. Let the triangle DEF have its three sides DE, EF, and FD, perpendicular respectively to CB, CA, and AB; then will these two triangles be similar.

For, prolong DE, FE, and FD, until they meet the sides in H, G, and K. In the quadrilateral AKFG, the angles AKF and AGF are right angles; hence the sum of the angles A and GFK is equal to two right angles (Prop. XXX., Bk. I., Cor. 1.). But the sum of the angles DFE and GFK is equal to two right angles (Prop. XV., Bk. I.). Take away the common angle GFK from these two equations, and there will remain the angle A equal to the angle DFE; and similarly it can be proved that the angle C is equal to the angle DEF, and that the angle B is equal to the angle EDF. Hence the two triangles are equiangular and similar.

Proposition XIII.—Theorem.

Two triangles having an angle in each equal, are to each other as the rectangles of the sides which contain the equal angles.

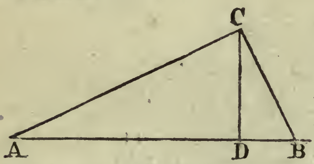


Let the two triangles ABC and DEF have the angles C and F equal; then will ABC be to DEF as $AC \times CB : DF \times FE$.

For, make CG equal to DF, and CH equal to FE. The triangles DEF and CGH are equal (Prop. IV., Bk. I.), and in all respects identical. Join G and B. The triangles ABC and BCG have the same altitude and their bases in the same line. Hence the triangle ABC : the triangle CBG :: AC : GC (Prop. II., Cor.), and the triangle CBG : the triangle CGH :: CB : HC for the same reason. By multiplying these two proportions, and striking out CBG, which is both an antecedent and a consequent, a new proportion is obtained; $ABC : CGH :: AC \times CB : GC \times HC$, or, substituting for GC and CH their equals DF and FE, and for the triangle CGH its equal DEF, $ABC : DEF :: AC \times CB : DF \times FE$.

Proposition XIV.—Theorem.

If a perpendicular be drawn from the right angle of a right-angled triangle to the hypothenuse, it divides the triangle into segments similar to the whole triangle and to each other: the perpendicular is a mean proportional between the segments of the hypothenuse, and either side is a mean proportional between the hypothenuse and the adjacent segment.



1. Let ABC be the right-angled triangle, and CD the perpendicular; then will the triangle ADC be similar to the triangle ABC; DBC will also be similar to ABC, and ADC and DBC will be similar to each other.

For, in the two triangles ABC and ADC, the angles ACB and ADC are equal (each being a right angle), the angle A is common. Hence the remaining angles ABC and ACD are equal (Prop. XXX., Bk. I., Cor.), and the triangles are similar. In like manner it can be proved that ABC and DBC are also similar. In the two triangles ADC and DBC, the angles at D are right angles, and the angle B has just been shown equal to ACD, and A equal to DCB. Hence the triangles ADC and DBC are similar.

2. DC will be a mean proportional between AD and DB. For, since the triangles ADC and DBC are similar,

$$AD : DC :: DC : DB;$$

and, since ABC and ADC are similar,

$$AB : AC :: AC : AD;$$

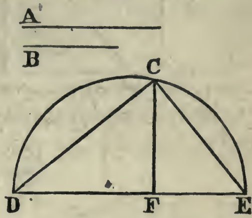
and, in like manner,

$$AB : CB :: CB : DB.$$

Cor. 1. $DC^2 = AD \times DB$; $AC^2 = AB \times AD$, and $CB^2 = AB \times DB$ (Prop. II., Bk. IV.).

PROPOSITION XV.—PROBLEM.

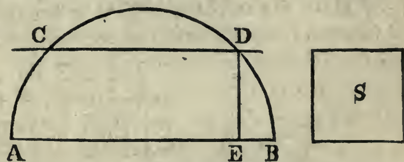
To find a mean proportional between two given lines.



Let A and B be the two given lines: it is required to find a mean proportional between them. Place the two lines A and B so that they will form one continued line DE, with DF equal to A, and FE equal to B. Describe the semicircle of which DE is the diameter. At the point F erect the perpendicular FC. Join D and C, and E and C. The triangle DCE is a right-angled triangle, because DCE is an angle inscribed in a semicircle. Hence DF : CF : : CF : FE.

PROPOSITION XVI.—PROBLEM.

To construct a rectangle equivalent to a given square, and having the sum of its adjacent sides equal to a given line.



Let S be the given square, and AB equal to the sum of the sides of the rectangle.

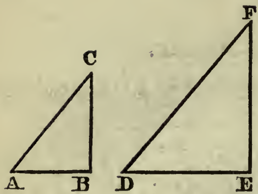
Upon the diameter AB describe the semicircle ACDB, and draw CD parallel to AB, CD being distant from AB the side of the given square; then from the point D draw the perpendicular DE. AE and EB will be the sides of the required rectangle.

For DE is a mean proportional between AE and EB; that is (Prop. XIV., Cor.), $DE^2 = AE \times EB$.

PROPOSITION XVII.—THEOREM.

Similar triangles are to each other as the squares of their homologous sides.

Let the two triangles ABC and DEF be similar, the



angle A equal to the angle D, the angle B equal to the angle E, and the angle C equal to the angle F; then will the triangles be to each other as the squares of their homologous sides.

For the triangle ABC : the triangle DEF : : $AC \times CB : DF \times FE$ (Prop. XIII.), and AC : DF : : CB : FE (Prop. VIII.); multiply the terms of this proportion by the identical proportion,

$$CB : FE :: CB : FE,$$

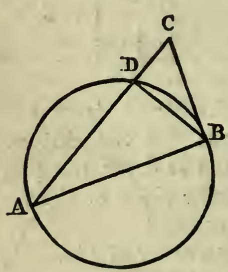
and the result is

$$AC \times CB : DF \times FE :: CB^2 : FE^2.$$

By substituting for the two first terms their equivalent ratio, the triangle ABC : the triangle DEF : : $CB^2 : FE^2$, and in like manner it can be proved of any other two homologous sides.

Proposition XVIII.—Theorem.

If from a point without a circle a tangent and a secant be drawn, the tangent will be a mean proportional between the secant and its external segment.



Let AC be a secant, and CB a tangent, both drawn from the point C; then will CB^2 be equal to $AC \times DC$.

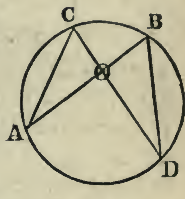
For, joining the points A and B, and B and D, the triangles ABC and CBD will have the angle C common, and the angle CBD equal to the angle A (Prop. XV., Bk. III.). Hence the remaining angles ABC and CDB are equal, and the two triangles are equiangular and similar.

Therefore $\qquad AC : CB :: CB :: DC,$

or $\qquad CB^2 = AC \times DC.$

PROPOSITION XIX.—THEOREM.

If two chords intersect each other, they are reciprocally proportional.



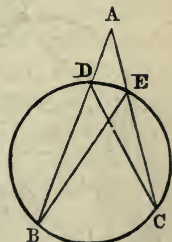
Let the two chords AB and CD intersect each other; then will AO be to DO as CO to BO.

For, joining A and C, and B and D, the two triangles AOC and BOD are equiangular, because the angles AOC and BOD are equal, being vertical, and A is equal to D, and C is equal to B (Prop. X., Cor. 5, Bk. III.). Hence AO : DO : : CO : OB.

Cor. By making the product of the extremes equal to the product of the means, $AO \times OB = DO \times OC$; that is, the rectangle of the two segments of one chord is equal to the rectangle of the two segments of the other chord.

PROPOSITION XX.—THEOREM.

If from a point without a circle two secants be drawn terminating in the concave arc, the whole secants will be reciprocally proportional to their external segments.



Let AB and AC be two secants terminating in the concave arc; then will AB be to AC as AE to AD.

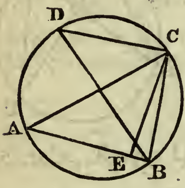
For, drawing BE and DC, the triangles ABE and ACD will have the angle A common, and the angle at B equal to the angle at C, because each is measured by half the arc DE (Prop. X., Cor. 5, Bk. III.). Hence these triangles are equiangular and similar. Therefore, AB : AC : : AE : AD.

Cor. By making the product of the extremes equal to the product of the means,

$$AB \times AD = AC \times AE.$$

PROPOSITION XXI.—THEOREM.

In every triangle the rectangle contained by any two sides is equal to the rectangle contained by the diameter of the circumscribing circle and the perpendicular drawn to the third side from the opposite angle.



Let ABC be the given triangle, CE the perpendicular on AB, and BD the diameter of the circle ABCD; then will $AC \times CB = DB \times CE$.

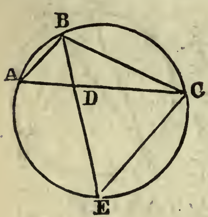
For, drawing DC, the triangles DCB and ACE are right angled, ACB being inscribed in a semicircle, and AEC by construction, and the angle A is equal to the angle D (Prop. X., Cor. 5, Bk. III.). Hence the triangles are equiangular and similar;

and $AC : DB :: EC : CB$;

therefore $AC \times CB = DB \times EC$.

PROPOSITION XXII.—THEOREM.

If a line be drawn bisecting any angle of a triangle and terminating in the opposite side, the rectangle contained by the sides of this angle will be equal to the rectangle of the segments of the third side, together with the square of the bisecting line.



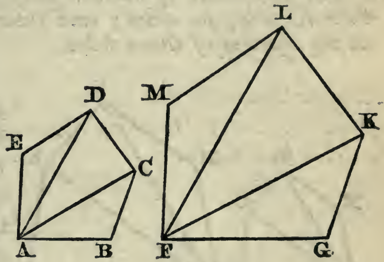
Let BD bisect the angle ABC; then will $AB \times BC = AD \times DC + BD^2$.

Describe a circumference which shall pass through the points A, B, and C (Prop. V., Bk. III.), produce BD to E, and join EC.

In the two triangles ADB and BCE, the angles ABD and CBE are equal (const.), and the angle E is equal to the angle A (Prop. X., Cor. 5, Bk. III.). Hence the two triangles are similar. $AB : BE :: BD : BC$; making the product of the extremes equal to the product of the means, $BA \times BC = BE \times BD$; by substituting $BD + DE$ for BE, then $AB \times BC = BD \times (BD + DE)$; that is, $AB \times BC = BD^2 + BD \times DE$; but $BD \times DE$ is equal to $AD \times DC$ (Prop. XIX.); and, by substitution, $AB \times BC = AD \times DC + BD^2$.

Proposition XXIII.—Theorem.

Two similar polygons are composed of the same number of triangles, similar to each other, and similarly situated.



Let ABCDE and FGKLM be two similar polygons; from any angle, A, draw AD and AC, and from the angle F, homologous to A in the other polygon, draw FL and FK.

Since the polygons are similar, the angle E must be equal to the angle M; and the sides which contain these equal angles are proportional, $AE : ED :: FM : ML$. Therefore the two triangles AED and FML, having an angle in each equal, and the sides containing the angles proportional, are similar (Prop. X.), and, since they are similar, the angle EDA is equal to the angle MLF; but the angle EDC is equal to the angle MLK, and if the former equals be taken from the latter equals, ADC will remain equal to FLK. Since the triangles are similar, AD is proportional to FL; and, since the polygons are similar, DC is proportional to LK. The two triangles ADC and FLK have the angles ADC and FLK equal, and the sides which contain these angles proportional. Therefore they are similar (Prop. X.). In the same manner it can be proved that ABC is similar to FGK.

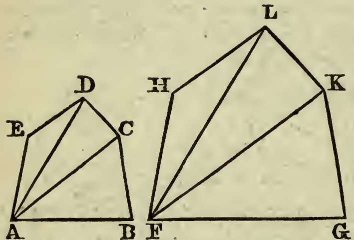
Scholium. The converse of this proposition is also true.

If two polygons are composed of the same number of similar triangles similarly situated, these polygons will be similar.

For the similarity of the respective triangles will give the angle E=to M, B=G, and, by adding the equal angles at A and F, EAB will be equal to MFG, and, in like manner, EDC and DCB=MLK and LKG.

Proposition XXIV.—Theorem.

The perimeters of similar polygons are to each other as their homologous sides; and their areas are to each other as the squares of those sides.



1. Since the polygons are similar, AB : FG :: BC : GK :: DC : LK, etc. Now, as the sum of all the antecedents is to the sum of all the consequents as any one antecedent is to any one consequent, AB+BC+DC +ED+AE : FG+GK+KL+LH+FH :: AB : FG; or, since the sum of these antecedents and of these consequents will be the perimeters of the polygons,

ABCDE : FGKLH :: AB : FG,

and so of any other two homologous sides.

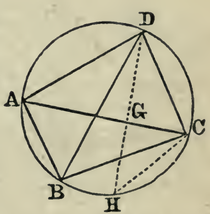
2. Since the triangles ABC and FGK are similar, the triangle ABC : the triangle FGK :: $AC^2 : FK^2$ (Prop. XVII.), and, for a like reason, the triangle ACD : the triangle FLK :: $AC^2 : FK^2$. But in these two proportions the last couplets are identical; hence the triangle ABC : FGK :: ACD : FLK. In like manner it can be proved that ACD : FLK :: AED : FHL, and so on. By adding the antecedents and consequents ABC+ACD+AED : FGK+FLK+FHL :: ABC : FGK; that is, the polygon ABCDE : FGKLH :: ABC : FGK. But these triangles are

to each other as the squares of their homologous sides. Hence $ABCDE : FGKLH :: AB^2 : FG^2$.

Cor. If three similar figures are constructed on the three sides of a right-angled triangle, the figure on the hypothenuse will be equal to the sum of the other two; for the three figures are to each other as the squares of their homologous sides. But the square of the hypothenuse is equal to the sum of the squares of the other two sides. Hence the similar figure described on the hypothenuse is equal to the sum of the two similar figures described on the other two sides.

Proposition XXV.—Theorem.

If a quadrilateral be inscribed in a circle, the rectangle of the two diagonals is equal to the sum of the rectangles of the opposite sides, taken two by two.



Let ABCD be a quadrilateral inscribed in a circle, and AC and BD its diagonals; then will $AC \times BD$ be equal to $AD \times BC$ plus $AB \times DC$.

For, making the arc HC equal to AB, and joining HD, the triangles ADB and GDC are equiangular and similar, because the angles ADB and GDC are equal (Prop. X., Cor. 5, Bk. III.), each being measured by the half of equal arcs; and the angle ABD is equal to the angle GCD, because each is measured by half the arc AD. Hence the two remaining angles BAD and CGD are equal. Therefore,

$$AB : GC :: DB : DC,$$

and, making the rectangle of the extremes equal to the rectangle of the means,

$$AB \times DC = GC \times DB.$$

Again, the two triangles BDC and AGD are equiangular and similar, because the angle BDC is equal to the angle ADG, each being measured by the half of equal arcs BC and AH, for to the equal arcs AB and HC, the arc BH is added, and the angle DBC is equal to the angle DAG, each being measured by half the arc DC. Hence the two remaining angles BCD and AGD are equal. Therefore

$$AD : DB :: AG : BC;$$

and hence $$AD \times BC = DB \times AG.$$

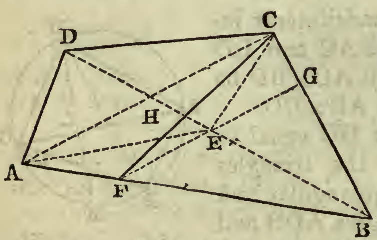
By adding the two equations,

$$AB \times DC + AD \times BC = GC \times DB + AG \times DB;$$

$$AB \times DC + AD \times BC = (GC + AG) \times DB, \text{ or} = AC \times DB.$$

The following are Test Examples involving Book Fifth:

1. *To bisect a quadrilateral by a line drawn from one of its angles.*



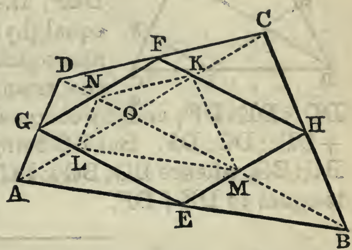
Let DCB be the angle from which the line shall be drawn bisecting the given quadrilateral ABCD.

Draw the diagonals DB and AC; bisect DB in the point E, and through E draw FEG parallel to AC. Join AE and EC, and from C draw CF. CF bisects the quadrilateral.

The triangles DEC and BEC are equal (Prop. VIII., Bk. II.), and also AED and AEB, for the same reason. By adding AED and EDC, and AEB and ECB, two by two, the quadrilateral AECD will be equal to the quadrilateral AECB. The triangles AFE and FEC are equal (Prop. VIII., Bk. II.). From each take away the common triangle FHE, and there will remain AFH equal to HEC. Now if from the quadrilateral AECD the triangle HEC be taken away, and its equal AFH added, we shall have the quadrilateral AFCD; and if from AECB AFH be taken away, and its equal HEC added, we shall have the triangle FCB. Hence FCB is equal to AFCD, and FC bisects the quadrilateral.

2. *To determine the figure formed by joining the points of bisection of the sides of a trapezium, and its ratio to the trapezium.*

Let ABCD be the given trapezium, and let the sides AB, CB, DC, and AD be bisected in the points E, H, F, and G. By joining these points of bisection, a parallelogram, GEHF, is formed.

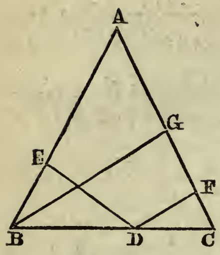


For, since AD and DC are bisected in G and F, AD : GD :: DC : DF, and hence GF is parallel to AC; and in the same manner EH is parallel to AC. Therefore GF and EH are parallel to each other. In like manner, it can be proved that FH and GE are parallel to each other. Hence GEHF is a parallelogram.

Again, the triangle ADC is composed of the triangles ALG, GND, DNF, FKC (without the parallelogram), and of the triangles GLN, LNO, ONK and NFK (within the parallelogram). Now it is evident that ALG and GND are equal to LNG and LNO, and also that DNF and FKC are equal to NFK and ONK. By adding these, it will be found that the parallelogram GLKF is equal to the sum of the triangles ALG, GND, NDF, and FKC. In the same manner, it can be proved that the parallelogram LEHK is equal to the sum of the triangles ALE, EMB, MBH, and HKC. Hence the whole parallelogram is one half the trapezium.

3. *If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides, these together shall be equal to a perpendicular drawn from either extremity of the base to the opposite side.*

Let ABC be an isosceles triangle; and from any point,



D, draw the perpendiculars DE and DF; also the perpendicular BG. It is required to prove that BG is equal to the sum of DE and DF.

In the two triangles BED and DCF, the angles EBD and C are equal (hyp.), and BED and DFC are also equal, being right angles. Hence the triangles are similar, and BD : DC :: DE : DF; and, by composition, BD+DC, or BC : DE+DF :: DC : DF. But BG being parallel to DF, DC : DF :: BC : BG, whence BC : BG :: BC : DE+DF. Therefore BG is equal to DE+DF.

TEST EXAMPLES IN BOOK V.

1. Through a given point situated between the sides of an angle, to draw a line terminating at the sides of the angle, and in such a manner as to be equally divided at the point.

2. Construct a quadrilateral similar to a given quadrilateral, the sides of the latter having to the sides of the former the ratio of 2 to 3.

3. To draw a line parallel to the base of a triangle in such a manner as to divide the triangle into two equal parts.

4. To construct a square when the difference between the diagonal and a side is given.

5. Prove that, if a line touching two circles cut another line joining their centres, the segments of the latter will be to each other as the diameters of the circles.

6. Prove that, if from the extremities of any chord in a circle, perpendiculars be drawn meeting a diameter, the points of intersection are equally distant from the centre.

7. Prove that, if from the extremities of the diameter of a semicircle, perpendiculars be let fall on any line cutting the semicircle, the parts intercepted between those perpendiculars and the circumference are equal.

8. Prove that if two circles touch each other externally or internally, any straight line drawn through the point of contact will cut off similar segments.

9. To determine the point in the circumference of a circle from which lines drawn to two other given points shall have a given ratio.

10. Prove that in any right-angled triangle the straight line joining the right angle and the point of bisection of the hypothenuse is equal to half the hypothenuse.

11. To construct a polygon similar to a given polygon, and bearing to it a given ratio.

12. Prove that, if from any point within an equilateral triangle, perpendiculars be drawn to the sides, they are together equal to a perpendicular drawn from any of the angles to the opposite side.

13. Prove that if the points of bisection of the sides of a given triangle be joined, the triangle so formed will be one-fourth the given triangle.

14. Prove that of all triangles having the same vertical angle and whose bases pass through a given point, the least is that whose base is bisected in the given point.

15. To bisect a given triangle by a line drawn from one of its angles.

16. To bisect a given triangle by a line drawn from a given point in one of its sides.

17. To determine a point within a given triangle from which lines drawn to the several angles will divide the triangle into three equal parts.

18. To trisect a given triangle from a given point within it.

19. Prove that if the three sides of a triangle be bisected, the perpendiculars drawn to the sides at the three points of bisection will meet in the same point.

20. Prove that if, from the three angles of a triangle, lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.

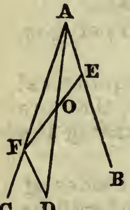
21. Prove that every trapezium is divided by its diagonals into four triangles proportional to each other.

22. To describe a triangle which shall be equal to a given equilateral and equiangular pentagon, and of the same altitude.

23. Prove that if an equilateral triangle be inscribed in a circle, and through the angular points another be circumscribed, the inscribed will be one-fourth the circumscribed.

24. Prove that the ratio of the side of a square to the diagonal is that of 1 to the $\sqrt{2}$.

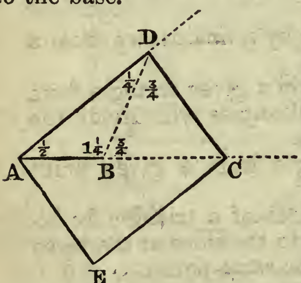
A KEY TO THE TEST EXAMPLES IN BOOK V.



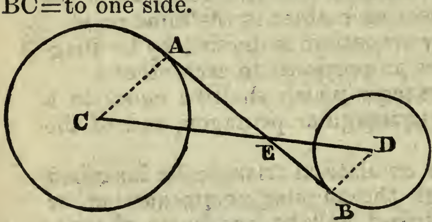
1. A is the given angle, and O the given point. Join AO, and produce it until OD is equal to AO; then through D draw DF parallel to AB. The line FE is the line required. The student will have no difficulty in proving FO=OE.

2. Trisect the sides of the former: two of these parts respectively will be the sides of the latter: then see Prop. XIV., Bk. I.

3. See Prop. XVII. The triangle is given; also its sides. The student will readily discover the proportion whose fourth term will be the portion of the side through the extremity of which the line must be drawn parallel to the base.

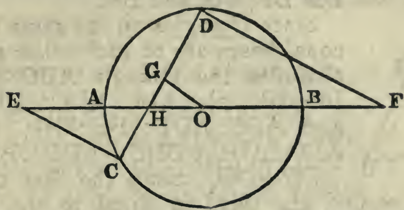


4. Let AB be the given difference: at the point A make DAB=$\frac{1}{2}$ right angle; at B make ABD $1\frac{1}{4}$ right angles: produce these lines until they meet: at D make BDC=$\frac{3}{4}$ right angle: produce DC until it meets AB prolonged. Through C draw EC parallel to AD, and AE, through A, parallel to DC. The student can now readily prove DE a square, and BC=to one side.

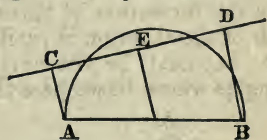


5. AB is tangent to both circles: CD joins their centres. The student will have little difficulty in proving the similarity of the triangles ACE and EBD.

6. At C and D let fall perpendiculars CE and DF, meeting the diameter produced. Required to prove that

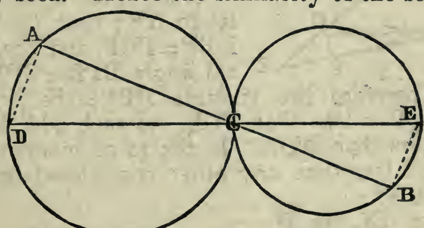


E and F are equally distant from O. Draw OG perpendicular to, and therefore bisecting CD. By the law of similar triangles the student will have no difficulty in proving the theorem.

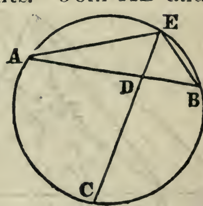


7. The demonstration of this theorem, by means of the law of similar figures, is very simple.

8. The similarity of the triangles DAC and CBE can be readily seen. Hence the similarity of the segments.

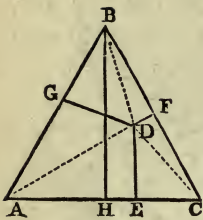


9. A and B are the two given points. Join AB and divide it in D, so that AD : DB may be in the given ratio; bisect the arc ACB in C: join CD and produce it to E. E is the point required. Join AE and EB. Since the arc AC = the arc CB, the angle AED = the angle BED. Then see Prop. XXI.



10. This theorem is very simple. The student has only to inscribe a right-angled triangle in a circle, whence the proof becomes apparent.

11. See Props. XVII. and XXII.

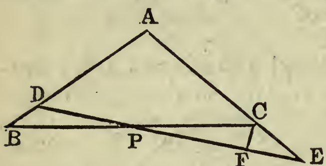


12. Draw the four perpendiculars DE, DF, DG, and BH.

Since triangles on the same base or equal bases are to each other as their altitudes, the triangle ABC : ADC :: BH : DE; also ABC : BDC :: BH : DF; also ABC : ADB :: BH : DG; whence ABC : ADC+BDC+ADB :: BH : DE +DF+DG. Since the first term is equal to the second the third is equal to the fourth. Hence BH=DE+DF+DG.

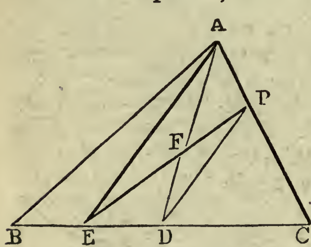
13. If the student bisects the sides, he will readily perceive the formation of three parallelograms; whence the truth of the theorem is easily established.

14. Let BAC be the vertical angle of any number of triangles whose bases pass through a given point, P; and BC be bisected in P; ABC is less than any other triangle, ADE.



From C draw CF parallel to AB; then the angle DBP=PCF, and the vertical angle DPB=CPF, and BP=PC. Therefore the triangle DBP=the triangle PCF; and DBP is less than PCE; to each add ADPC, and ABC is less than ADE. In the same manner ABC may be proved less than any other triangle whose base passes through P.

15. See Prop. IX., Bk. II.

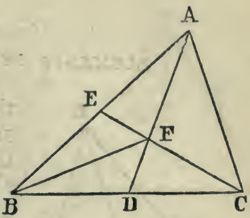


16. Bisect BC in D; join AD and PD, and from A draw AE parallel to PD; join PE; PE bisects ABC.

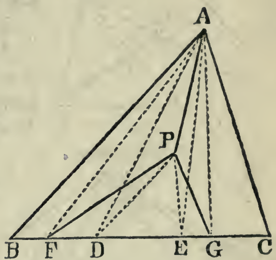
The triangle ADB=ADC, and proving the equality of the smaller triangles, and by subtracting and adding equals, the student can easily prove that APEB=PEC.

17. Bisect AB and BC in E and D; join AD and CE; also B and F. F is the point. Since BD=DC, the triangles BAD and CAD are equal, and, for the same reason, BDF=DFC. Therefore ABF=AFC. Again, since BE

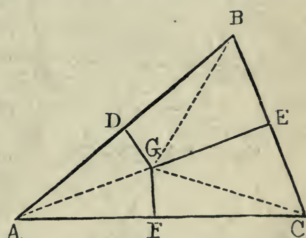
$=AE$, the triangle $BEC=$ the triangle AEC; also $BEF=AEF$. Therefore $BFC=AFC$. Hence the three parts, BFC, BFA, and AFC, are equal.



18. Trisect BC in D and E; join DP and PE, and from A draw AF and AG respectively parallel to them. Join PF, AP, and PG. These three lines divide the triangle into three equal parts. Join AD and AE. Since AF is parallel to PD, the triangle $APF=ADF$; to each of them add ABF; and $ABFP=ADB$. In the same manner $ACGP=AEC$. Hence $FPG=ADE$. But the triangles ABD, AEC, and ADE are equal, each having an equal base and altitude. Therefore $ABFP$, $ACGP$, and FPG are also equal.



19. Let the sides be bisected in D, E, and F. Draw the perpendiculars EG and GF, meeting in G. The perpendicular at D also passes through G. Join GD, GA, GB, and GC. Since $AF=FC$, and FG is common, and the angles at F right angles, $AG=GC$. In the same manner it may be shown that $GC=GB$. Therefore $AG=GB$. But $AD=DB$, and DG is common to the triangles ADG and BDG. Hence the angles at D are equal, and each must be a right angle; or the perpendicular at D passes through G.

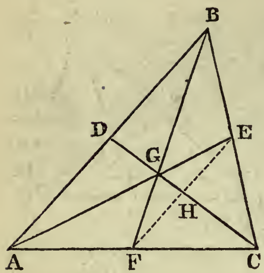


20. Let the sides be bisected in D, E, and F. Join AE, CD, meeting each other in G. Join BG and GF; BGF is a straight line. Join EF, meeting CD in H. Then FE is parallel to AB, and therefore the triangles DAG and GEH are equiangular. Hence

$$DA:DG::HE:HG;$$

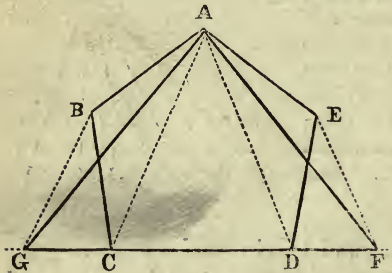
or

$$DB:DG::HF:HG;$$

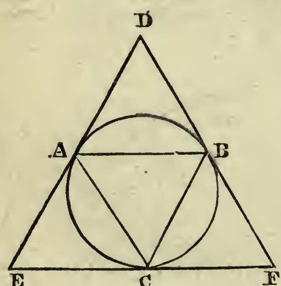


that is, the sides about the equal angles are proportional. Hence the triangles BDG and GHF are similar; and the angle DGB=HGF. Therefore BG and GF are in the same straight line.

21. See Prop. XIII.



22. ABCDE is the given pentagon. Produce CD both ways; join AC and AD. Through B and E draw BG and EF parallel to AC and AD, each to each. Join AG and AF. AGF is the required triangle. The student can readily perceive that, by adding certain equal triangles to a common triangle, the truth of the problem will be established.

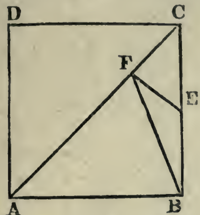


23. ABC is the equilateral triangle inscribed; and DEF is the equilateral circumscribed. Prove that ABC is one-fourth of DEF.

The angle DAB = ACB (Prop. XII., Bk. III.). But ACB=ABC. Therefore DAB =ABC; these are alternate angles. Hence DE is parallel to BC. In the same manner it may be shown that DF is parallel to AC. Therefore ACBD is a parallelogram; and the triangle ABC=ABD; and in like manner it can be proved that ABC=AEC or BCF. Hence ABC=one-fourth of DEF.

24. ABCD is the square, and AC its diagonal. From the diagonal cut off AF=AB; the remainder, CF, must be compared with CB. Join FB, and draw FE perpendicular to AC. The angle ABE=AFE; and

ABF=AFB; and from the former equals subtract the latter, and EFB remains = to EBF. Hence the triangle BEF is isosceles, and BE=EF. The angle FCE is half a right angle. But CFE is a right angle. Therefore FEC is half a right angle. Hence FC=FE; and CE is the diagonal of a square whose side is FC.



Hence, after CF has been taken from CB, it remains to take CF from CE; that is, to compare the side of a square with its diagonal: and we shall find precisely the same difficulty in the next step of the process; so that, continue as far as we please, we shall never arrive at a term in which there will be no remainder. Therefore there is no common measure for the diagonal and a side of a square. If the side of the square be represented by 1, the diagonal will be the square root of 2. But this is only an approximate value.

[The teacher should be very careful not to show the pupils too much in the study of the Test Examples; neither should they be permitted to consume too much time in vain efforts to demonstrate or solve questions too difficult for their comprehension. The teacher should exercise a nice discretion.]

$AH = AF$; and therefore [illegible] equals subtracting them, and CH remains [illegible]. [illegible] the triangle HFE [illegible] and $HE = EF$. The angle HFE is a right angle. But CFE is a right angle. Therefore FEC is half a right angle. Hence $FC = FE$; and CE is the diagonal of a square whose side is FC.

Hence, after CH has been taken from CB, it remains to take CE from CF, that is, to compare the side of a square with its diagonal; and we shall find precisely the same difficulty in every step of the process; so that continue as we may, we shall never arrive at a term in which [illegible] [illegible]. Therefore there is no common [illegible] [illegible] the diagonal and side of a square. [illegible] [illegible] of the square [illegible] [illegible] [illegible]

[illegible]

[illegible]

APPENDIX.

MENSURATION OF SURFACES.

1. THE area of a figure is the measure of its surface.

2. A square whose side is one inch, one foot, or one yard, is called the *unit of measure*, and the area of any figure is computed by the number of those squares contained in that figure.

PROBLEM I.

To find the area of a parallelogram, whether it be a square, a rectangle, a rhombus, or a rhomboid.

RULE.—*Multiply the length by the perpendicular height, and the product will be the area.**

EXAMPLES.

1. To find the area of a parallelogram whose length is 12 feet 3 inches, and height 8 feet 6 inches:

3 inches = .25 foot and 6 inches .5 foot.

Then 12.25 feet × 8.5 feet = 104.125 feet area.

2. Find the area of a square whose side is 35.25 chains.†

Ans. 124 acres 1 rood 1 pole.

3. Find the area of a rectangular board whose length is $12\frac{1}{2}$ feet, and breadth 9 inches. *Ans.* $9\frac{3}{8}$ feet.

4. To find the content of a piece of land in the form of a rhombus, its length being 6.2 chains, and height 5.45 chains.

Ans. 3 acres 1 rood 20 poles.

5. Required the area of a rhomboid whose length is 10.51 chains, and breadth 4.28 chains. *Ans.* 4 acres 1 rood 39.7 poles.

PROBLEM II.

To find the area of a triangle when the base and perpendicular height are given.

RULE.—*Multiply the base by the perpendicular height, and take half the product; or, multiply the base by half the perpendicular height, or the perpendicular height by half the base.*‡

---

* See Proposition II., Bk. V., Scholium.

† The student must know the Table of Measures.

‡ See Proposition II., Bk. V., Cor.

Examples.

1. Find the area of a triangle whose base is 49 feet, and height $25\frac{1}{4}$ feet.

$$\tfrac{1}{4}=.25.$$

$$\frac{49\times 25.25}{2}=618.625 \text{ square feet.}$$

2. How many square yards in the triangle whose base is 40 feet, and perpendicular height 30? *Ans.* $66\frac{2}{3}$ square yards.

3. To find the area of a triangle whose base is 18 feet 4 inches, and height 11 feet 10 inches. *Ans.* 108 feet $5\frac{2}{3}$ inches.

4. Required the area of a triangle whose base is 12.25 chains, and height 8.5 chains. *Ans.* 5 acres 0 rood 36 poles.

Problem III.

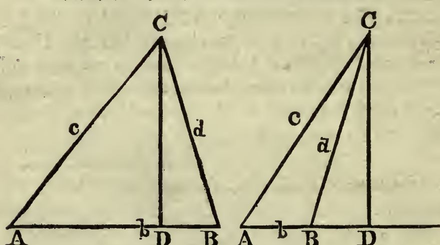
To find the area of a triangle whose three sides only are given.

Rule.—*From half the sum of the three sides subtract each side separately: multiply the half sum and the three remainders continually together, and the square root of the product will be the area required.**

---

* *Demonstration.*—Let $AC=c$, $CB=d$, and $AB=b$; and

$$c^2=AD^2+DC^2;$$
$$d^2=DB^2+DC^2;$$
$$c^2-d^2=AD^2-DB^2;$$
$$(c+d)\,(c-d)=(AD+DB)\,(AD-DB);$$
$$(c+d)\,(c-d)=b(AD-DB) \text{ from Fig. 1};$$
$$(c+d)\,(c-d)=b(AD+DB) \text{ from Fig. 2};$$



$$AD-DB=\frac{(c+d)\,(c-d)}{b}, \text{ and } AD+DB=\frac{(c+d)\,(c-d)}{b};$$

$$AD-DB=\frac{c^2-d^2}{b}, \text{ and } AD+DB=\frac{c^2-d^2}{b};$$

adding half to half diff., $\frac{c^2-d^2}{2b}+\frac{b}{2}=AD$, the greater segment.

$$CD=\sqrt{c^2-AD^2}=\sqrt{c^2-\left(\frac{c^2-d^2+b^2}{2b}\right)^2};$$

$$CD=\frac{\sqrt{4b^2c^2-(c^2-d^2+b^2)^2}}{2b}$$

Examples.

1. Required the area of a triangle whose three sides are 24, 36, and 48 chains respectively:

$$\frac{24+36+48}{2}=\frac{108}{2}=54, \text{ which is the } \tfrac{1}{2} \text{ sum.}$$

Then $54-24=30$, first diff.; $54-36=18$, second diff.; and $54-48=6$, third diff.

(Multiplying), $54\times30\times18\times6=174960$;

(Extracting square root), $\sqrt{174960}=418.282$, area required.

2. Required the area of a triangle whose three sides are 13, 14, and 15 feet. *Ans.* 84 sq. feet.

3. How many acres are there in a triangle whose three sides are 49, 50.25, and 25.69 chains? *Ans.* 61.498 acres.

4. Required the area of a right-angled triangle whose hypothenuse is 50, and the other two sides 30 and 40. *Ans.* 600.

5. Required the area of an equilateral triangle whose side is 25. *Ans.* 270.6328.

6. Required the area of an isosceles triangle whose base is 20, and each of its sides 15. *Ans.* 111.803.

7. Required the area of a triangle whose three sides are 20, 30, and 40 chains. *Ans.* 29 acres 7 poles.

Problem IV.

Any two sides of a right-angled triangle being given to find the third side.

Rule.—*When two sides are given to find the hypothenuse.* Add the squares of the two sides together and extract the square root of the sum.*

When the hypothenuse and one side are given, to find the other. From the square of the hypothenuse subtract the square of the given side, and the square root of the remainder will be the required side.

$$56^2+33^2=3136+1089=4225;\ \text{and } \sqrt{4225}=65,\ Ans.$$

$$CD=\frac{\sqrt{(b^2+2bc+c^2-d^2)\,(d^2-b^2+2bc-c^2)}}{2b};$$

$$CD=\frac{\sqrt{[(b+c)^2-d^2]\,[d^2-(b-c)^2]}}{2b};$$

$$CD=\frac{\sqrt{(b+c+d)\,(b+c-d)\,(b-c+d)\,(-b+c+d)}}{2b}.$$ Multiply by $\frac{b}{2}$, and the

$$\text{area}=\frac{\sqrt{(b+c+d)\,(b+c-d)\,(b-c+d)\,(-b+c+d)}}{4}.$$

$$\text{Area}=\left\{\frac{b+c+d}{2}\times\frac{b+c-d}{2}\times\frac{b-c+d}{2}\times\frac{-b+c+d}{2}\right\}^{\frac{1}{2}}.$$

[In this demonstration the student must understand very clearly the principles of factoring in Algebra.]

---

* See Prop. XIII., Bk. II.

1. Given the base of a right-angled triangle 56, and the perpendicular 33, what is the hypothenuse?

2. If the hypothenuse be 53, and the base 45, what is the perpendicular? *Ans.* 28.

3. The base of a right-angled triangle is 77, and the perpendicular 36, what is the hypothenuse? *Ans.* 85.

4. The hypothenuse of a right-angled triangle is 109, and the perpendicular 60, what is the base? *Ans.* 91.

5. The height of a precipice standing close by the side of a river is 103 feet, and a line of 320 feet will reach from the top of it to the opposite bank: required the width of the river. *Ans.* 302.97 feet.

6. A ladder 50 feet long being placed in a street reached a window 28 feet from the ground on one side; and by turning it over, without removing the foot, it reached another window 36 feet high on the other side: required the width of the street. *Ans.* 23.3238 feet.

Problem V.

To find the area of a trapezium.

Rule.—*Divide the trapezium (an irregular quadrilateral) into two triangles by a diagonal; then to this diagonal let perpendiculars fall from the opposite angles. Find the area of each triangle by* Problem II.

1. Find the area of a trapezium whose diagonal is 84, and the two perpendiculars 21 and 28 respectively.

$$(21+28)\times 84=49\times 84=4116;\ \text{and}\ \frac{4116}{2}=2058,\ \textit{Ans.}$$

2. Required the area of a trapezium whose diagonal is 80.5, and the two perpendiculars 24.5 and 30.1. *Ans.* 2197.65.

3. What is the area of a trapezium whose diagonal is 108 feet 6 inches, and the perpendiculars 56 feet 3 inches and 60 feet 9 inches. *Ans.* 6347 feet 3 inches.

4. How many square yards of paving are there in the trapezium whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet. *Ans.* $222\frac{1}{12}$ yards.

Problem VI.

To find the area of a trapezoid.

Rule.—*Multiply the sum of the parallel sides by the perpendicular distance between them, and half the product will be the area.**

1. The parallel sides of a trapezoid are 750 and 1225, and the perpendicular distance between them 1540 links: required the area.

$$(1225+750)\times\frac{1540}{2}=1975\times 770=152075\ \text{links}=15\ \text{acres}\ 33\ \text{poles},\ \textit{Ans.}$$

2. How many square feet are contained in the plank whose length is

* This is simply finding the areas of the two triangles.

12 feet 6 inches, the breadth at the greater end being 15 inches, and at the less end 11 inches. *Ans.* $13\frac{13}{24}$ feet.

3. The parallel sides of a trapezoid are 12.41 and 8.22 chains, and the perpendicular distance 5.15 chains: required the area.

Ans. 5 acres 120.995 poles.

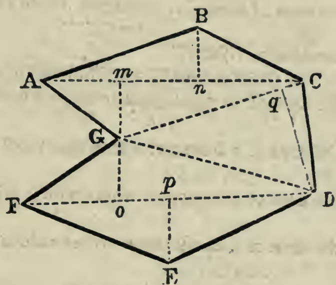
Problem VII.

To find the area of an irregular polygon.

Rule.—*Divide the polygon into triangles and trapeziums; then find the area of all these (by preceding problems), and their sum will be the area required.*

1. Find the area of the irregular figure ABCDEFG, in which are given the following diagonals and perpendiculars: AC=55, FD=52, GC=54, Gm=13, Bn=18, Go=12, Ep=8, and Dq=23.

Ans. $1878\frac{1}{2}$.



Problem VIII.

To find the area of a regular polygon.

Rule.—*Multiply the perimeter of the polygon, or sum of its sides, by the perpendicular drawn from its centre on one of its sides, and take half the product for the area.**

1. Find the area of a regular pentagon, each of whose sides is 25 feet, and the perpendicular from the centre on each side 17.2047737.

$$25\times5=125, \text{ the perimeter,}$$

and $$125\times17.2047737=2150.5967125.$$

The half of this is 1075.298356, the area required.

2. Required the area of a hexagon whose side is 14.6 feet, and perpendicular 12.64. *Ans.* 553.632 feet.

---

* This is simply resolving the polygon into as many equal triangles as the figure has sides, finding the area of each, and taking the sum.

3. Find the area of a heptagon whose side is 19.38, and perpendicular 20. *Ans.* 1356.6.

Problem IX.

To find the area of a regular polygon when a side only is given.

Rule.—*Multiply the square of the side by the number standing opposite to its name in the following table, and the product will be the area.**

No. of sides.	Names.	Multipliers.
3.	Trigon, or equilateral Δ......	0.433013 −
4.	Square...........................	1.000000
5.	Pentagon........................	1.720477 +
6.	Hexagon	2.598076 +
7.	Heptagon........................	3.633912 +
8.	Octagon	4.828427 +
9.	Nonagon.........................	6.181824 +
10.	Decagon.........................	7.694209 −
11.	Undecagon......................	9.365640 −
12.	Duodecagon....................	11.196152 −

1. Find the area of a regular pentagon whose side is 25 feet.

$$25^2=625.$$

The tabular area is 1.720477.

$$625\times1.720477=1075.298375, \textit{ Ans.}$$

2. Find the area of an equilateral triangle whose side is 20. *Ans.* 173.20508.

3. Find the area of a hexagon whose side is 20. *Ans.* 1039.23048.

4. Find the area of an octagon whose side is 16. *Ans.* 1236.0773.

5. Find the area of a duodecagon whose side is 125. *Ans.* 174939.875.

Problem X.

The diameter of a circle being given to find the circumference, or the circumference being given to find the diameter.

Rule.—*Multiply the diameter by* 3.1416, *and the product will be the circumference; or, divide the circumference by* 3.1416, *and the quotient will be the diameter.*†

---

* The multipliers in the table are the areas of the polygons to which they belong when the side is unity, or 1. All similar figures are to each other as the squares of their like sides. See Prop. XVII., Bk. V. Hence 1^2 : given side squared :: given area : area required; or given side squared × given area = area required.

† The proportion of the diameter to the circumference has never yet been exactly obtained: nor can a square or other straight-lined figure be found that shall be equal to a given circle. This is the celebrated problem called squaring the circle, which has never been solved.

Also, as 7 : 22 :: the diameter : the circumference;
or, as 22 : 7 :: the circumference : the diameter.
As 113 : 355 :: the diameter : the circumference;
or, as 355 : 113 :: the circumference : the diameter.

1. Find the circumference of a circle whose diameter is 20.

As 7 : 22 :: 20 : the circumference,

$$\frac{22\times 20}{7}=62\tfrac{6}{7},\ \textit{Ans.}$$

2. If the circumference of a circle is 354, what is the diameter? *Ans.* 112.681.

3. If the circumference of the earth be 25,000 miles, what is the diameter? *Ans.* 7958, nearly.

4. What is the circumference of a circle whose diameter is 40 feet? *Ans.* 125.6640.

Problem XI.

To find the area of a circle.

Rule.—1. *Multiply half the circumference by half the diameter.**

2. *Square the diameter, and multiply that square by the decimal* .7854.†

3. *Square the circumference, and multiply that square by the decimal* .07958.

1. Find the area of a circle whose diameter is 10, and its circumference 31.416.

By Rule 1.

$$\frac{31.416}{2}\times\frac{10}{2}=78.54.$$

By Rule 2.

$$10^2\times .7854=78.54.$$

By Rule 3.

$$31.416^2\times .07958=78.54.$$

2. Find the area of a circle whose diameter is 7, and circumference 22. *Ans.* $38\frac{1}{2}$.

3. How many square yards in a circle whose diameter is $3\frac{1}{2}$ feet? *Ans.* 1.069.

4. Find the area of a circle whose circumference is 12 feet. *Ans.* 11.4595.

5. How many square feet are there in a circle whose circumference is 10.9956? *Ans.* 86.5933.

Problem XII.

To find the area of a circular ring, or of the space included between

---

* Rule I.—In this rule the circumference is made up of an infinite number of straight lines, each of which becomes the base of a triangle whose perpendicular is the radius, or half the diameter. Hence ½ the circumference × ½ the diameter would be the area of an infinite number of triangles, or the area of the circle.

† Rule II.—In this rule the area of a circle whose diameter is 1=.7854; and, by the law of similar figures, $1^2 : \text{diam.}^2 :: .7854 : \text{area required}$.

the circumferences of two circles, the one being contained within the other.

RULE.—*Find the areas of the two circles by the last problem, and subtract the less from the greater.*

1. The diameters of two concentric circles being 10 and 6: required the area of the ring contained between their circumferences.

Ans. 50.2656.

2. What is the area of the ring the diameters of whose bounding circles are 10 and 20? *Ans.* 235.62.

PRACTICAL EXAMPLES.

1. What will the glazing of a triangular sky-light come to at 20 cents per foot, the base being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches? *Ans.* \$16.875.

2. From a mahogany plank 26 inches broad, a yard and a half is to be sawed off; what distance from the end must the line be struck?

Ans. 6.23 feet.

3. How many square yards in a ceiling which is 43 feet 3 inches long and 25 feet 6 inches broad? *Ans.* 122½.

4. What is a marble slab worth whose length is 5 feet 7 inches, and breadth 1 foot 10 inches, at 80 cents per foot? *Ans.* \$8.188.

5. What is the difference between a floor 48 feet long and 30 feet broad, and two others each of half the dimensions? *Ans.* 720 feet.

6. The two sides of an obtuse-angled triangle are 20 and 40 poles; what must be the length of the third side, so that the triangle may contain just an acre? *Ans.* 58.876 poles.

7. A circular fish-pond is to be dug in a garden that shall take up just half an acre; what must the length of the cord be that describes the circle? *Ans.* 27.75 yards.

8. Suppose a ladder 100 feet long placed against a perpendicular wall 100 feet high; how far would the top of the ladder move down by pulling out the bottom thereof 10 feet? *Ans.* .5012563 foot.

9. A telegraph pole was so nearly broken through by a blast of wind that the top fell to the ground 15 feet from the base of the pole: what was the height of the whole telegraph pole, supposing the length of the piece that fell to be 39 feet? *Ans.* 75 feet.

10. Three persons, whose residences are at the vertices of a triangular area, the sides of which are severally 10, 11, and 12 chains, wish to dig a well which shall be at the same distance from the residence of each. Find the point for the well, and its distance from their residences.

Ans. 6.405 chains.

THE END.

$$\begin{array}{r}132\\ 88\\ \hline 44\end{array}$$