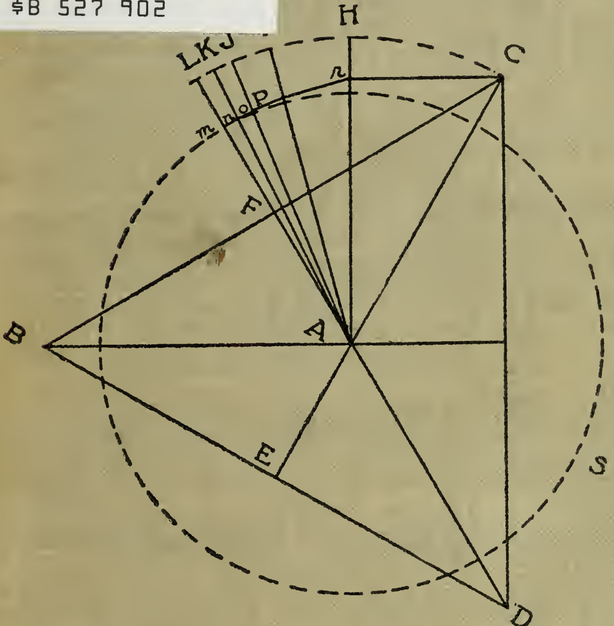
IMPORTANT MATHEMATICAL DISCOVERIES



By P. D. WOODLOCK

IMPORTANT DISCOVERIES

IN PLANE AND SOLID GEOMETRY

CONSISTING OF

THE RELATION OF POLYGONS TO CIRCLES

AND THE

EQUALIZING OF PERIMETERS TO CIRCUMFERENCES

AND

DRAWING CURVED LINES EQUAL TO STRAIGHT LINES

THE

TRISECTION OF AN ANGLE

AND THE

DUPLICATION OF THE CUBE

By P. D. WOODLOCK

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PREFACE.

In publishing this book the author feels confident that he has added something to geometrical science, which has not heretofore been known.

He especially invites geometricians and mathematicians to examine carefully and without bias, the several propositions and problems, and their demonstrations, contained in the book, and he has no doubt that they will find interest in every page.

Some of the problems appearing in this book have occupied the attention of geometricians in all ages since the introduction of geometry as a science, yet all attempts at their solution have been unsuccessful down to the present time. That the author of this little work has been rewarded with the discovery of the true solutions of these problems he confidently leaves to the consideration and candid judgment of geometricians throughout the world.

PART I.

CONSISTING OF

PRELIMINARY DEMONSTRATIONS

LEADING TO THE

EQUALIZING OF CURVED LINES

TO STRAIGHT LINES.

PRELIMINARY DEMONSTRATIONS.

PROPOSITION A.

To form a series of polygons upon a square or equilateral triangle having the same center and equal perimeters, the number of their sides being to each other consecutively in the ratio of two.

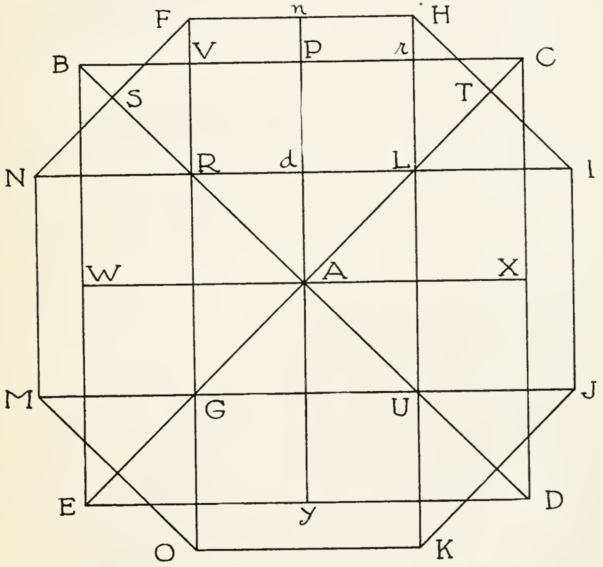

FIGURE A.

In Figure A, let BCDE be a square, and draw the diagonals BD and CE, bisecting each other at A, which is the center of the square, and draw the line AP. Then AP is the radius of the inscribed, and AC is the radius of the circumscribed circles. Then bisect the lines AB and AC at R and L, and draw the lines RF and LH equal to RB and LC respectively, and parallel to AP, and draw the

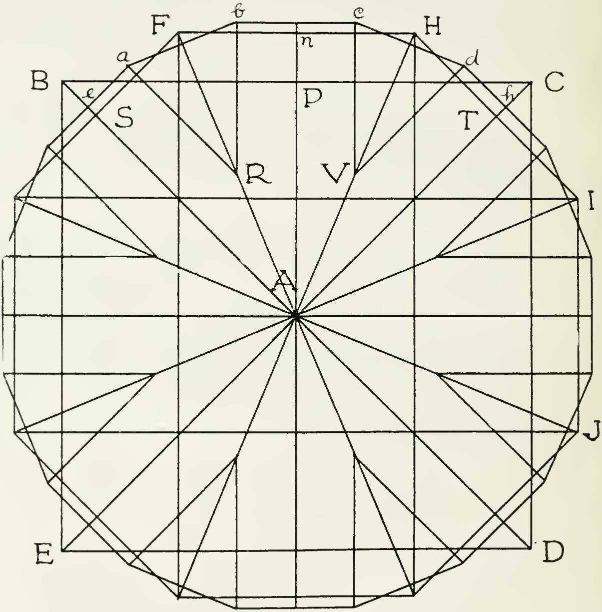

FIGURE B.

lines AW, AX and Ay, bisecting the lines BE, CD, and DE, and draw RN equal to RB, and parallel to AW, and draw LI equal to LC, and parallel to AX. And in like manner bisect the lines AE and AD at G and U, and draw the lines GM and GO, equal to GE, and draw UK and

UJ, equal to UD, and draw the lines RL, LU, UG, and GR, and draw the lines FH, HI, IJ, JK, KO, OM, MN and NF. And the figure thus formed is an octagon, whose perimeter is equal to the perimeter of the square BCDE.

Now the line Cr is perpendicular to LH, and HT is perpendicular to LC. And LH being equal to LC, therefore

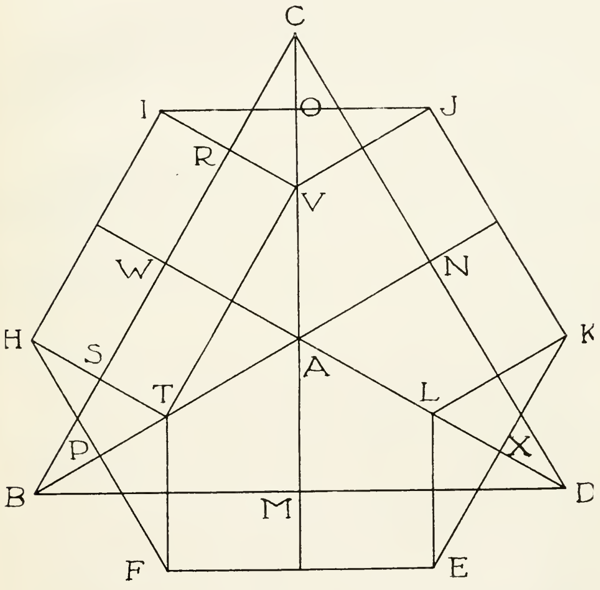

FIGURE C.

the line Cr equals HT. In like manner FS = BV, and FH = Vr. Hence the lines SF + FH + HT = BV + Vr + rC = BC, which is a side of the square. Now the lines SF + FH + HT are equal to one-fourth of the perimeter of the octagon. And BC is one-fourth of the perimeter of the square; therefore the perimeter of the octagon FHIJ is equal to the perimeter of the square BCDE.

In Figure B, let the square BCDE and the octagon FHIJ have equal perimeters, and produce AP to n, which bisects FH, and draw the lines AF and AH. Bisect AF

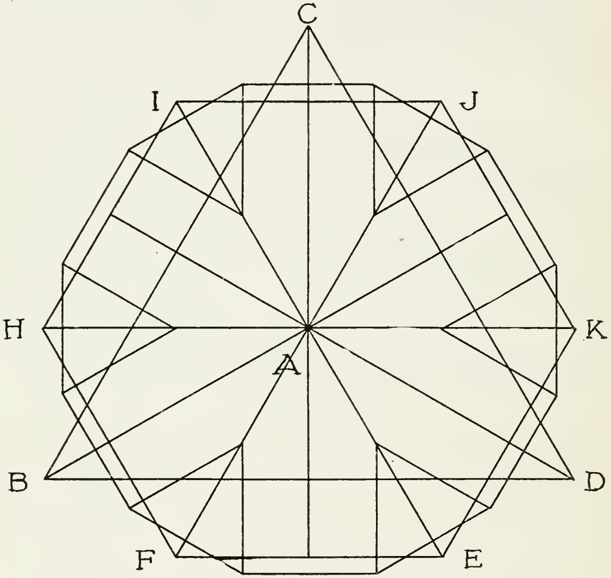

FIGURE D.

at R, and AH at V, and draw the lines Ra and Rb, each equal to RF, and parallel to AB and An respectively. In like manner draw the lines Vc and Vd, each equal to VH, and draw ab, bc, and cd, and draw the line ae, parallel to FS, and dh parallel to HT. Then ea + ab + bc + cd + dh = SF + FH + HT = BC. And as ea + ab + bc + cd + dh form one-fourth of the perimeter of a polygon of 16 sides, and by completing the remaining sides of that polygon, its perimeter is equal to the perimeter of the octagon, and also to that of the square, their sides are to each other consecutively in the ratio of two. Proceed likewise in forming polygons of 32, 64, 128, 256, etc., sides, until they become infinite in number and minuteness.

In Figure C, let BCD be an equilateral triangle, and bisect its sides at W, N and M, and draw the lines CM, BN and DW, intersecting each other at the point A, which is the center of that triangle. And bisect AC at V, AB at T, and AD at L, and draw VI and TH, parallel to AW, making VI = VC, and TH = TB. And draw VJ and LK parallel to AN, and equal to VC and LD. In like manner draw LE and TF, equal to LD and TB, and draw the lines HI, IJ, JK, KE, EF, and FH. And the figure thus formed is a hexagon whose perimeter is equal to that of the triangle BCD.

Now it will be seen that PH = BS, HI = SR, and IO = RC. Hence PH + HI + IO = BC. In like manner OJ + JK + KX = CD, and XE + EF + FP = BD. Therefore their perimeters are equal.

Figure D represents an equilateral triangle, a hexagon, and a polygon of 12 sides, whose perimeters are equal to each other, and whose sides are to each other in the ratio of two. And so on to infinity.

PROPOSITION B.

In a square, an equilateral triangle and in all regular polygons, the sum of the least and greatest radii, divided by two, is equal to the least radius of a polygon of twice the number of sides and equal perimeter.

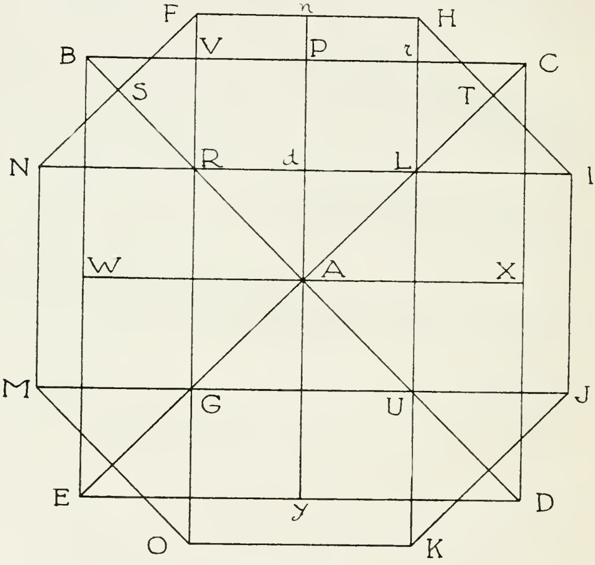

FIGURE E.

In Figure E, let the square BCDE, and the octagon FHIJ have equal perimeters (Proposition A), and we see that the line AP is the radius of the inscribed circle, and AC is the radius of the circumscribed circle, and they are the least and greatest lines that can be drawn from the center A to the perimeter of the square BCDE, and for convenience we will call them the least and greatest radii of that square.

Now the line NI bisects the line AP at d, and AC at L. Hence dL is parallel to PC, and AL = LC, and LC = LH;

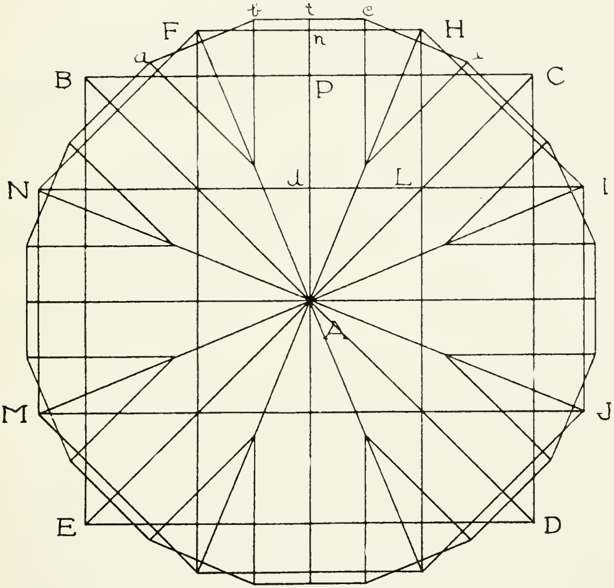

FIGURE F.

therefore AL = LH, and as LH = dn, hence AL = dn. Then (AP + AC) ÷ 2 = Ad + AL = An, which is the least radius of the octagon FHIJ.

Let Figure F represent a polygon of 16 sides drawn upon the square and octagon, their perimeters being equal. Then in the octagon, An and AH are the least and greatest radii, and it will be seen that (An + AH) ÷ 2 = At; and At is the least radius of a polygon of 16 sides, whose perimeter is equal to that of the octagon, or of the square, and so on until the sides become infinite in number and minuteness.

PROPOSITION C.

If a series of polygons be drawn upon each other, having the same center and equal perimeters, and the number of their sides being to each other consecutively in the ratio of two, the greatest radii of each polygon bisect the angles formed at the center, by the least and greatest radii of its next preceding polygon, and the lines joining the outer points of the greatest radii of any one of these polygons, with the outer points of the greatest radii of its next succeeding polygon, are perpendicular to the latter radii.

In Figure J, let the square BCDE and the octagon FHIJ, have equal perimeters, and let the line bc be a side of a polygon of 16 sides, and or, a side of a polygon of 32 sides, whose perimeters are equal to that of the square, or octagon, and produce the line An to X, making AX equal to AB or AC, and draw the arc BXC, and it will be seen that the lines nH, tc and dr, are each half a side of a polygon of 8, 16, and 32 sides respectively; and join the points CH, Hc, and cr.

Now the triangles AnH and ATH are similar, hence the angles nAH and TAH are equal. And the line AH bisects the angle nAC, and in like manner the line AC bisects the angle nAH, and the line Ar bisects the angle tAc, and so on to infinity. And in the triangle AHC, the line Ac is bisected at L, and LH, LC and LA are equal to each other. Hence if a circle be described on the line AC as diameter, its circumference must pass through the point H. Therefore the angle AHC is a right angle.

In like manner the angles AcH, Arc, etc., are right angles, and the lines CH, Hc, and cr are perpendicular to the bisecting lines AH, Ac, and Ar respectively. Hence if the angles be bisected indefinitely, and polygons formed, until the last polygon that can possibly be drawn and re-

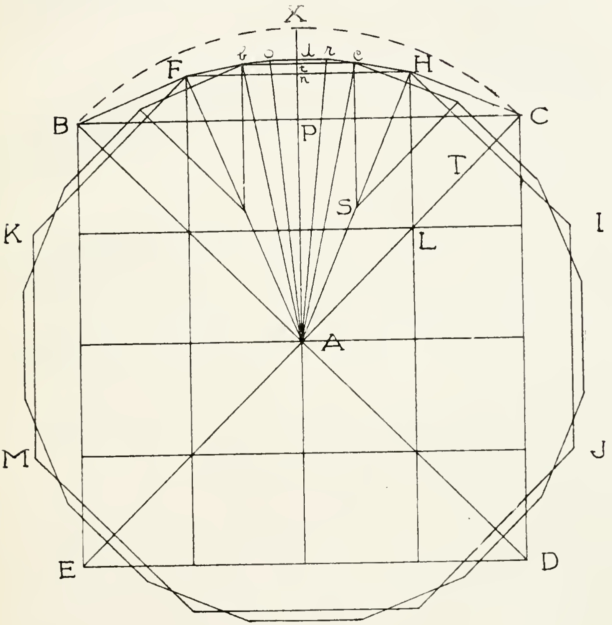

Figure J.

tain distinct sides is reached, the lines joining the outer points of the greatest radii of any one of these polygons, with the outer points of the greatest radii of its next succeeding polygon, are perpendicular to the latter radii, and the rectangle formed by the greatest radii of any one of these polygons with the least radii of its next succeeding polygon, is equal to the square of the greatest radius of the latter polygon.

It will be seen from the foregoing demonstrations that if an infinite series of polygons be thus formed, the perimeter of any one of them intersects each side of all its preceding polygons, in two places, the points of intersection being equally distant from the middle of the side, no matter how minute the sides may become, and as a circle is the ultimate form of a polygon whose sides by the process of bisection become infinitely minute, the circumference of that circle will also intersect each side of all preceding polygons in like manner.

PART II.

PERIMETERS AND CIRCUMFERENCES MADE EQUAL

AND

CURVED LINES MADE EQUAL TO STRAIGHT LINES.

PERIMETERS AND CIRCUMFERENCES.

PROPOSITION I.

To form a circle whose circumference is equal to the perimeter of a given square.

Let BCDE be a given square (the quadrature of the circle) and AR its least radius, and AC its greatest radius,

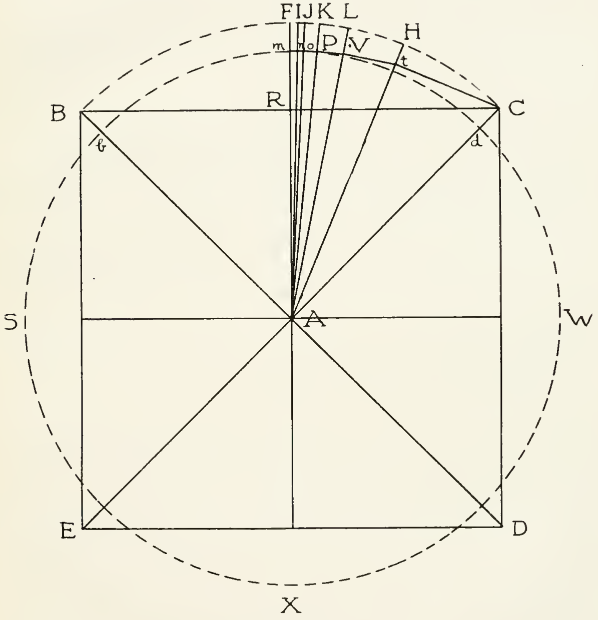

FIGURE I.

as in Figure I. And with A as center and AC as radius, draw the arc BFC, and produce the line AR to F, bisecting the arc BFC at F. Then bisect the arc FC at H, and FH at L, and FL at K, and FK at J, and FJ at I, and so on, bisecting to infinity, or as far as it is within our means to bisect. And draw the lines AH, AL, AK, AJ, and AI.

Then from point C draw Ct perpendicular to AH, and from the point t draw tV perpendicular to AL, and draw VP, Po, and on, perpendicular to AK, AJ, and AI, respectively, the perpendicular always falling on each bisecting line, until the least possible bisecting line is reached, which is represented by AI. Then draw the final perpendicular nm to the line AF, and it will be seen that nm is half the side of a polygon, whose perimeter is equal to that of the square BCDE. (Prop. C, Preliminary Demonstrations.) Then through the point m, with radius Am, draw the circle SmWX, and the circumference of that circle is equal to the perimeter of the square BCDE.

Now let the perpendicular nm be the least possible part or division of a straight line, a mere point, and we see that it is half the side of a polygon whose perimeter is equal to that of the square, and whose sides are capable of only one division or bisection, and as the circumference of a circle, which is equal to the perimeter of a polygon, both having the same center, must intersect each side of that polygon in two places, the points of intersection being equally distant from the middle of the side, therefore, the circumference SmWX must intersect the perpendicular nm, but nm being a mere point, cannot be either bisected, intersected, or divided further. Therefore the circumference SmWX must pass over and coincide with the perpendicular or point nm, which represents half a side of a polygon, and hence it must pass over and coincide with the remaining half of that side, consequently it must pass over and coincide with each and every side comprising the whole perimeter of that polygon, and the point is reached where the final figure loses its identity as a polygon, and assumes and contains all the properties of a complete circle, as represented by the circle SmWX. Therefore, the circumference of the circle SmWX is equal to the perimeter of the square BCDE, and the arc bmd equals the line BC.

Now if the line BC = 2, the perimeter of the square equals 8. Hence the circumference of the circle SmWX = 8. And as Am, or the radius of a circle whose circumference is 8, is equal to the ratio of a circle to its circumscribing square, which is 1.2732395 +. Therefore Am = 1.2732395 +.

The following table gives the radii of each polygon, from the square to a polygon of 524288 sides, the perimeter of each being 8, and the number of their sides being to each other consecutively in the ratio of 2.

NO. OF SIDES.	SHORT RADIUS.	LONG RADIUS.
4..........	1.	1.414213562373095 +
8..........	1.2071067811865475 +	1.3065629648763765 +
16..........	1.25683487303146201 +	1.28145768485268807 +
32..........	1.26914627894207004 +	1.27528713444730576 +
64..........	1.2722167067075638 +	1.2737509956132868 +
128..........	1.2729838511604253 +	1.2733673656153701 +
256..........	1.2731756083078977 +	1.2732714833914617 +
512..........	1.2732235458896797 +	1.2732475148077608 +
1024..........	1.27323553034872029 +	1.2732415224659654 +
2048..........	1.2732385264073428 +	1.2732400244357728 +
4096..........	1.2732392754215578 +	1.2732396499325372 +
8192..........	1.2732394626770475 +	1.27323955630262909 +
16384..........	1.27323950948983831 +	1.2732395328962334 +
32768..........	1.27323952119303589 +	1.2732395254738383 +
65536..........	1.2732395233334371 +	1.2722395247570668 +
131072..........	1.273239524045252 +	1.2732395244109769 +
262144..........	1.2732395242281144 +	1.2732395243195456 +
524288..........	1.27323952427383 +	
262144..........	1.2732395242281144 +	1.2732395243195456 +
524288..........	1.27323952427383 +	

From the above we find the ratio between diameter and circumference is 3.1415927 +.

PROPOSITION II.

To form a circle whose circumference is equal to the perimeter of a given equilateral triangle.

In Figure II, let BCD be a given equilateral triangle, and draw the perpendicular CE, and let A be the center

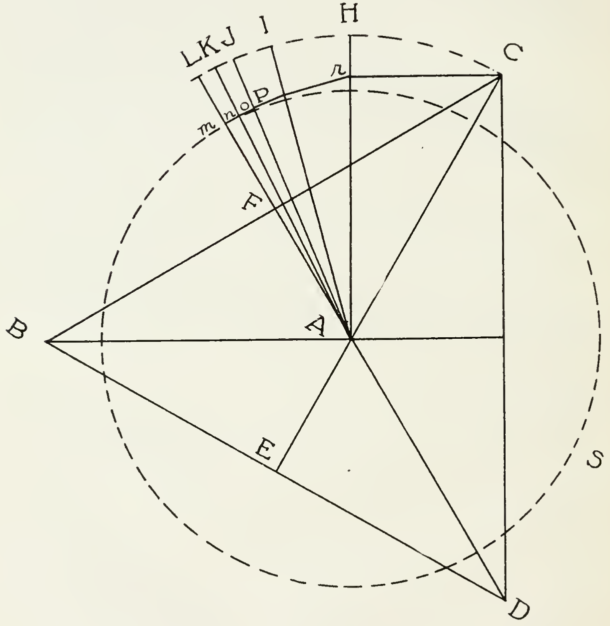

FIGURE II.

of that triangle. Then AC is its greatest radius. Then draw AF perpendicular to BC, and the line AF is its least radius. Then produce AF to L, making AL = AC. Then with A as center draw the arc LC, and draw the bisecting lines AH, AI, AJ, AK, etc., to infinity, or as far as it is possible to bisect. Then from the point C draw Cr perpendicular to AH, and rP perpendicular to AI, and po to AJ, and on to AK, etc., and draw the final perpendicular to the point m, on the line AL. And with A as center and Am as radius draw the circle mS, and the circunference of that circle is equal to the perimeter of the triangle BCD. (See Demonstration of Proposition I, Quadrature of Circle.)

PROPOSITION III.

To determine the Quadrature of the Circle independently of all infinite or unlimited bisections.

In Figure III, let BCDE be a given square and A its center, and draw the lines AF and AD, and draw DH forming a right angle with AD, and produce AF to H. Then on the line AH, as diameter, draw the circle HDAC. And bisect the line HD at P, and draw FJ, which intersects HD at P, and draw PA. And on the line AH, make AO = AP, and draw PL at right angles with AH. Then on the line AO draw the semicircle OnA, and draw the lines On and nA. Then from the point D draw the line Dm perpendicular to AP, and draw mR parallel to FD. And on the line On make nX = FR, and draw AX. Then with A as center and AX as radius, draw the circle XWYS, and the circumference of that circle is equal to the perimeter of the square BCDE.

Now let the line CD = 2. Then FD = 1. And AF = 1, and FH = 1, and AD and DH are equal to each other, and each is equal to $\sqrt{2}$, and Dp = half the $\sqrt{2}$ = .70710678+ and $AD^2 + DP = AP^2 = 2.5$. Hence $AO^2 = 2.5$. Therefore $AO = \sqrt{2.5} = 1.58113883+$. Now $AO \times AF = An^2$. But $AO \times AF = AO \times 1 = AO$. Therefore, $An^2 = AO = 1.58113883+$. Now in the right-angled triangle ADP, $PA \times Am = AD^2 = 2$. Hence PA^2

$\times Am^2 = 4$. And as $PA^2 = 2.5$, then $4 \div 2.5 = Am^2$. Hence $Am^2 = 1.6$, and $AR^2 = 1.44$, and $AR = 1.2$. Then $FR = .2$, and hence $nX = .2$. Now $An^2 = 1.58113883+$,

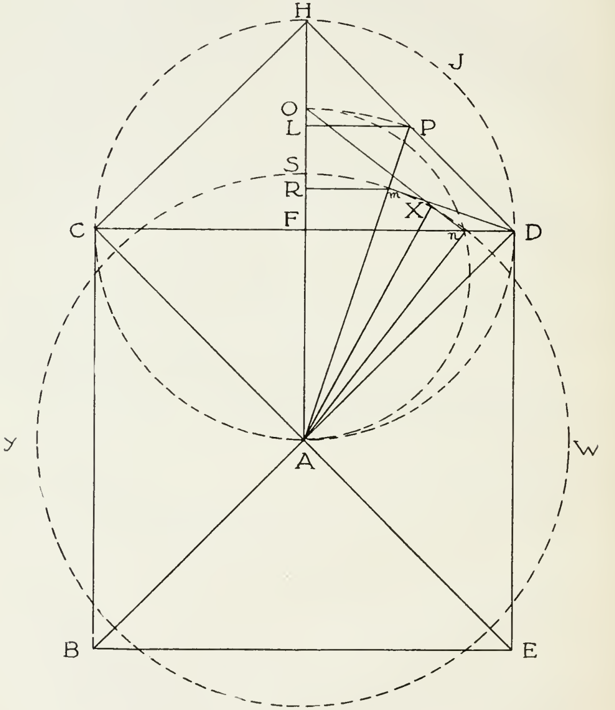

FIGURE III.

and $nX^2 = .04$. Hence $AX^2 = 1.58113883+, + .04 = 1.62113883+$, and $Ax = \sqrt{1.62113883+} = 1.2732395+$, which is the ratio of a circle to its circumscribing square, and is the radius of the circle XWYS. Hence the circumference of that circle is equal to the perimeter of the square BCDE. And $4 \div 1.2732395+ = 3.141592+$, which is the ratio of diameter to circumference.

From the foregoing demonstration (Figure III) the following rule is obtained, viz.:

Multiply the square of the radius of the inscribed circle by the $\sqrt{2\frac{1}{2}}$, and add to the product the one-hundredth part of the area of the square, and the square root of the sum is the radius of the circle whose circumference is equal to the perimeter of the given square.

It will also be seen that to draw a circle whose circumference is equal to the perimeter of a given polygon, bring that perimeter into the form of a square, and proceed as in Figure III or Figure I, Part II.

PROPOSITION IV.

To find the number of degrees in any angle of a given triangle.

In Figure IV, let the angle BAC be a given angle, and it is required to determine the number of degrees in it. Draw the lines AB and AC equal to each other and draw BC, and draw the arc mHn equal to the line BC. Then bisect

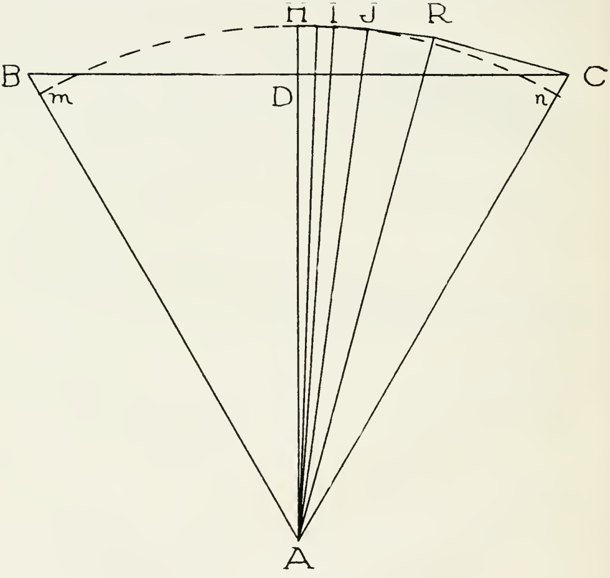

FIGURE IV.

angle BAC by the line AH, which is the radius of the circle of which the arc mHn is a part. Then find the circumference of that circle, and divide by the arc mHn or the line BC, and then divide the result into 360 for the degrees in said angle.

PART III.

THE TRISECTION OF AN ANGLE.

THE TRISECTION OF AN ANGLE

PROPOSITION I.

In any triangle, the sum of any two of its sides, is to the third side as either one of those two sides is to that corresponding part of the third, cut off by a line bisecting the angle which the said two sides contain.

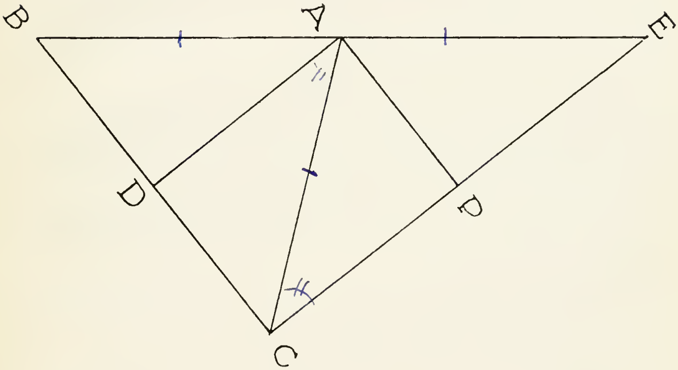

FIGURE A.

Let BAC (Figure A) be a triangle, and produce BA to E, making AE = AC, and draw EC, and draw AP parallel to BC. Then draw AD parallel to EC, and AD bisects the angle BAC, as will be seen by the following demonstration:

AE = AC, then BE = BA + AC, and the angle AEC = ACE, and as the angle BAC = angles AEC + ACE, therefore the angle BAC = 2 ACE, and as AD is parallel to EC, the angles DAC and ACE are equal. Hence the

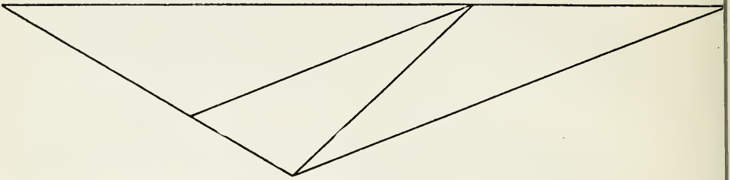

FIGURE B.

the angle BAC = 2 DAC, and the line AD bisects the angle BAC. Now BE = BA + AC, and in triangles BEC and BAD, BE : BC : : BA : BD, and BE : BC : : AE : AP. Hence BE : BC : : AE : DC. Therefore, BA + AC : BC : : BA : BD, and BA + AC : BC : : AC : CD.

PROPOSITION II.*

In Figure C, let ABC be a right-angled isoceles triangle and it is required to trisect the right angle ABC. Bisect the angle ABC by the line BD, and bisect and angle BAC by the line AP, and draw BH = BP, and with A as center and AH as radius, draw the arc HS, meeting the line DB

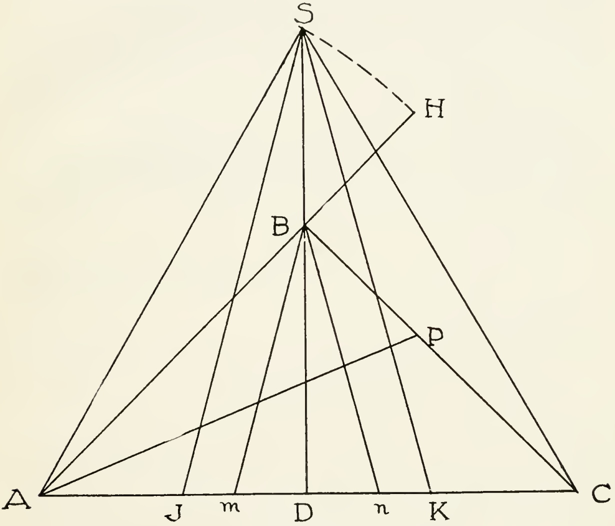

FIGURE C.

produced, at the point S, and draw AS and CS. Then draw the line SJ bisecting the angle DSA, and draw SK bisecting the angle DSC, and draw Bm and Bn parallel to SJ and SK respectively, and the lines Bm and Bn trisect the angle ABC.

*The trisection of acute and obtuse angles will appear in the next issue of book.

Now in triangle ABC, let the line Ac = 2. Hence DB, DA and DC each equals 1, and AB and CB each equals $\sqrt{2}$. And as the line AP bisects the angle BAC, it will be seen that the line BP = .58578+, and as BP = BH, therefore the line AH = ($\sqrt{2}$ + .58578+) = 2. Hence the line AS = 2, and the line SD = $\sqrt{3}$, and the triangle ASC is equilateral, and the angle ASC is 60 degrees, the angle ASD is 30 degrees, and the angle JSD = 15 degrees. And as the line Bm is parallel to SJ, hence the angle mBD is 15 degrees. And as the angle ABD = 45 degrees, therefore the angle mBD = one-third of angle ABD. In like manner the angle nBD = one-third of angle CBD. Hence the angle mBN is one-third of the angle ABC.

In all cases an isoceles triangle must be formed, the angle to be trisected being the vertical angle, as in Figure C.

PART IV.

THE QUADRATURE AND DUPLICATION OF THE CUBE.

QUADRATURE AND DUPLICATION OF THE CUBE.

PROPOSITION I.

In Figure I, let ABCD be a square side of a given cube, and on the line AB draw the semicircle BOA, and let BE $= 1$, and draw the lines EFP, FB and FA. Then $AB \times$

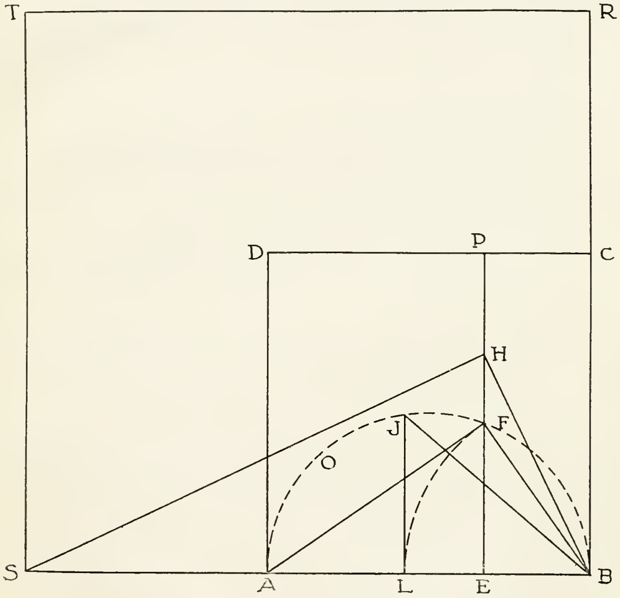

FIGURE I.

$BE = FB^2 = AB$. Then make $BL = BF$, and draw LJ. Then $AB \times BL = BJ^2 = BH^2$, and draw HS at right angles with BH, and draw the square SBRT, and the square

SBRT = AB^3, of which the square ABCD is one of the square sides.

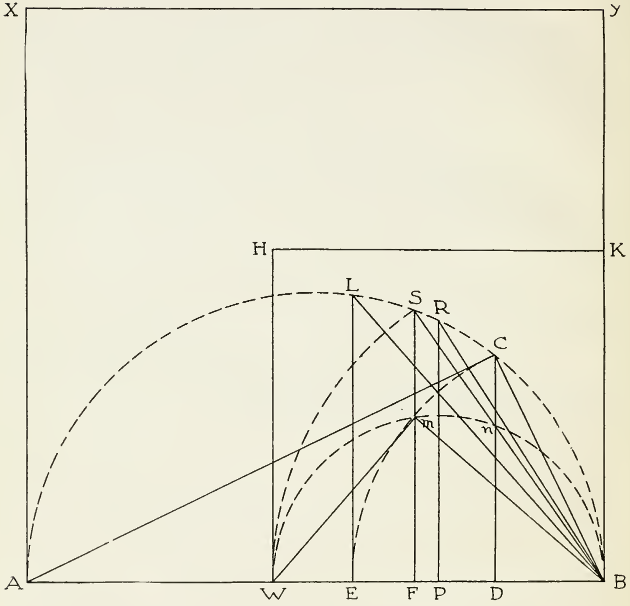

FIGURE II.

Now BE = 1, then $AB \times BE = BF^2 = AB$. And $BF = \sqrt{AB}$, and therefore $BL = \sqrt{AB}$. Then $AB \times BL = \sqrt{AB^3}$. But $AB \times BL = BJ^2 = BH^2 = BS$. Hence $BS = \sqrt{AB^3}$, consequently $BS^2 = AB^3$. Hence the square $SBRT = AB^3$.

PROPOSITION II.

To bring a slab one inch thick into the form of a cube, whose solid contents are equal to the solid contents of said cube.

In Figure II, let ABYX be a square slab one inch thick, whose solid contents = 27. Then the line AB = $\sqrt{27}$ =

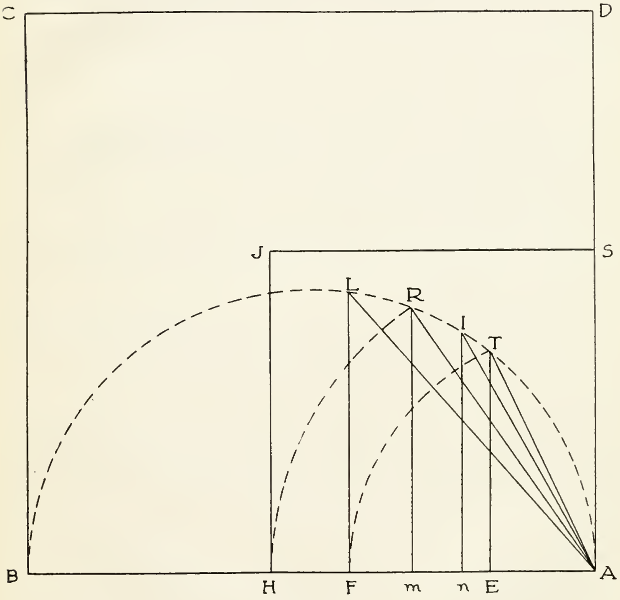

FIGURE III.

5.196+. Now let BD = 1, then AB $\times$ BD = BC^2 (the semicircle BSA being drawn). Then make BE = BC, and draw the lines EL, LB and CA. Then make BP = $\sqrt{BE}$, and it is also the 4th root of AB. Then AB $\times$ BP = BR^2. And make the angle LBS = two-thirds of the angle LBR, and draw SF, then the line BS is the cube root of the slab ABYX, and is therefore the cube root of 27, which is 3.

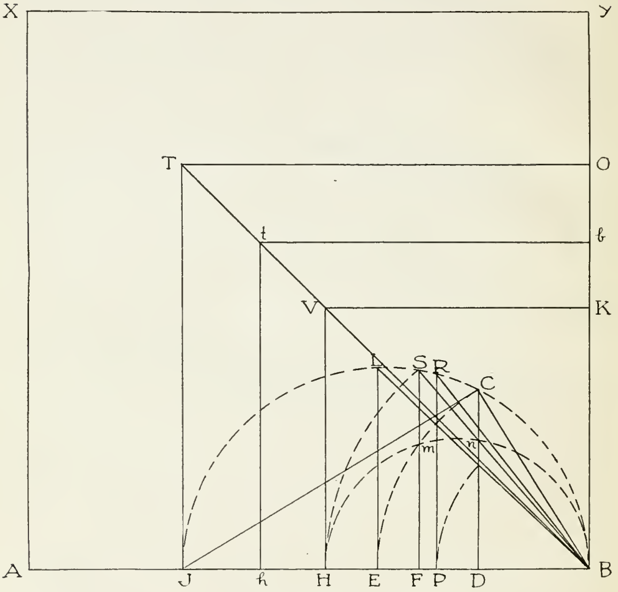

FIGURE IV.

Then make BW = BS, and on BW draw the semi-circle BmW, and draw the lines Bm and mW. Now AB $\times$ BF = BS^2 = BW^2 = 9. Hence 9 $\div$ 5.196+ = $\sqrt{3}$,+ then BF = 1.732+ and = $\sqrt{BW}$. Hence BW is the required line representing the edge of the cube which is equal to the given slab; and BF^3 = AB. Hence any rectangular slab may be brought into a cube.

Or, let Figure III be a reproduction of Figure II, then BA multiplied by its square root AF = AL^2. Then make An = the 8th root of BA. Then BA $\times$ An = AI^2. And the difference between the angles thus formed at A, which is the angle LAI, bisect by line AR, and draw the line Rm. Then Am is the cube root of AB.

This method is different from that of Figure II, and has the advantage of dispensing with trisections.

PROPOSITION III.

To form a cube whose solid contents are twice the solid contents of a given cube, and vice versa.

In Figure IV, let the square BHVK be a side of a given cube, and let BJTO represent a square slab one inch thick, and equal to the contents of the given cube. Then produce BJ to A, making BA = to the diagonal of the square BJTO, and form the square ABYX, and it will be seen that the square ABYX is twice the square BJTO. Then bring the square ABYX into a cube as represented by the square side BHtb.

Now the square ABYX is twice the square BJTO. It is therefore twice the cube BHVK, and the square ABYX being equal to the cube represented by the square side BHtb, hence the cube BHtb is twice the cube BHVK.

It will be seen that a cube may be formed whose solid contents are any given number of times greater than a given cube.