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Florian Cajori

AN INDUCTIVE MANUAL

OF

THE STRAIGHT LINE

AND

THE CIRCLE

WITH MANY EXERCISES

BY

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"The better method .... to put it to the test of Experience." Preface to Novum Organum.

"The Syllogism consists of Propositions, propositions of words, words are the signs of Nctions. If therefore the notions (which form the basis of the whole) be confused and carelessly abstracted from things, there is no solidity in the superstructure. Our only hope, then, is in genuine INDUCTION."

Novum Organum, aphorism XIV

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PREFACE.

This manual is published to supply a need which its author has long felt, in the instruction of his own classes, for some other mode of teaching geometry than the purely deductive, which is almost exclusively followed in most of our text-books of to-day. That method seems in many cases to fail to give clear ideas to the young beginner in geometry and to involve the whole subject in a haze which it takes a considerable time to clear away. In too many cases, because he fails to comprehend the reasoning employed, he becomes discouraged; and, putting aside all confidence in his own powers of thought, he attempts to convince himself that such and such things must be true because "the book says so,"—a basis for opinion which is, it must be confessed, even more unsatisfactory in geometry than it is in other things with which we concern ourselves. With such a student in such circumstances, the results obtained through attempting to follow the deductive method of teaching geometry are too apt to be a befogging of any ideas he may previously have had, and the direct discouragement of his powers of imagination, invention, and judgment.

The course of work outlined in the following pages is designed to be actually and accurately worked out by the student, using the draughtsman's instruments and the draughtsman's methods as far as practicable; and each exercise is to be worked out a sufficient number of times and under sufficiently varied conditions for the student to *know* why he comes to such and such conclusions and not merely to succeed in guessing what his instructor wishes to be told. Geometry is one of the first sciences in the historical order of development, and its phenomena are among the simplest of all those that demand man's consideration. It offers, then, one of the best means of training a student to exact thinking and to scientific investigation. It is believed that with young students and with those whose reasoning powers have not received a practical training of a considerable extent, a faithful following-out of an inductive investigation, such as it has been attempted to indicate in the following pages, will yield as extensive and exact a knowledge as will the deductive method in the same length of time, and, besides that, a much greater readiness in the application of the knowledge obtained and a much more thorough training of the invaluable powers before-mentioned,—imagination, invention, and judgment.

It seems hardly necessary to say that the book is not particularly designed for the use of students who have not the benefit of an instructor (although many such will be able to use it to advantage,) and so is much briefer than it might otherwise have been, and shows what may seem to some an alarming paucity of illustrations in connection with the definitions. Diagrams are also purposely few, because it is believed that in most cases the student will be able to supply his own diagrams. Sundry new words have been introduced where there seemed to be need of them. It is presumed that their convenience will offer sufficient justification for their use, either in the forms here given or with such modifications as experience may suggest. The word "sect" is due to Prof. Halsted of the University of Texas.

The manual herewith offered being somewhat of an experiment, criticisms and suggestions are invited from those instructors who may have occasion to examine the work, and especially from those who use it with their classes.

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SYMBOLS AND ABBREVIATIONS.

$+$	Plus.
$-$	Minus.
$\times$	Into.
$\div$	Divided by.
$=$	Equals.
$\doteq$	Equivales.
$\doteq$	Approaches as a limit.
$>$	Is larger than.
$<$	Is smaller than.
$\angle$	Angle.
$\angle_s$	Angles.
$\llcorner$	Right angle.
$\llcorner_s$	Right angles.
$\perp$	Perpendicular.
$\perp_s$	Perpendiculars.
$\parallel$	Parallel.
$\parallel_s$	Parallels.
$\triangle$	Trigon.
$\triangle_s$	Trigons.
▱	Rhomboid.
▱$_s$	Rhomboids.
▭	Rectangle.
▭$_s$	Rectangles.
$\square$	Square.
$\frown$	Arc.
$\bigcirc$	Circumference.
$\odot$	Circle.
$+$	Given point.
$\odot$	Required point.
———————	Given line.
— — —	Hidden line.
—-—-—-	Auxiliary line.
——— ———	Required line.
𝕬	Alternate.
𝕴	Inner.
𝕺	Outer.
adj.	Adjacent.
sup.	Supplementary.
cmp.	Complementary.
exp.	Explementary.
crsp.	Corresponding.

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INSTRUCTIONS

TO STUDENTS USING THIS BOOK.

The exercises called for in this book are to be completely and accurately worked out, and every conclusion to which the student may come should be amply tested in experience before being finally settled upon. Such drawings as are indicated by the instructor should be inked in after approval by him. These when bound together will form an appropriate companion to the student's note-book in geometry, which should contain statements of his conclusions upon the various subjects investigated, and discussions of the various problems proposed, numbered in correspondence with the numbers of the questions and exercises in the text-book. These statements in the note-book should be complete, so as to be intelligible without reference to the text-book, and in making them the student should, as far as he is able, indicate the connections existing among the various relations which he discovers. He will soon find that from the relations he first discovers he can in many cases predict what further relations he will discover. This he should accustom himself to do; but he should in all cases, or at least in all cases except those in which his instructor may deem it unnecessary, submit his prediction to the test of experiment, and his experiments should be extensive enough to cover fairly well all the conditions coming within the range of his statements. By so doing he will discover wherein his statements are too broad and wherein they are unnecessarily narrow, and will gain quickness and accuracy in conceiving and mentally reviewing all the aspects under which any relation may be viewed, and will gain also that valuable practical judgment which depends so largely on experience.

Concerning Drawing Instruments.—The student will not need a large selection of drawing instruments but those he has should be good. He will need at least one 3½" compasses, with fixed needle-point, and pen- and pencil attachments, both legs jointed; one 3½" dividers; one 5" ruling-pen; two triangles, one 45°-45°-90° and the other 30°-60°-90°, each having a hypotenuse about seven inches long; and a T square the length of whose blade is more than that of the diagonal of the sheet of paper on which his drawings are to be made. Besides these he will need a 6H pencil, an eraser, a bottle of drawing-ink (Higgins's American is best,) a finely-divided scale six inches or more in length, a supply of thumb-tacks, etc. His compasses and dividers should not be below the grade of the "Arrow*

* They should be of higher grade if to be used for regular draughting.

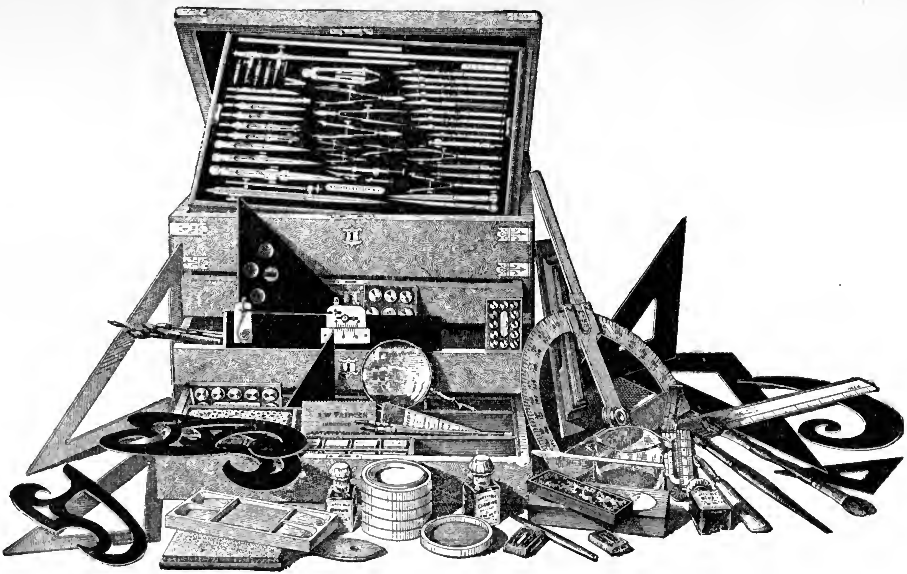
German" brand of the Keuffel and Esser Co., and his ruling-pen should be of the highest grade; a poor ruling-pen is an abomination. In making his purchases the student should bear in mind that poor drawing instruments are an incessant plague to the user and like cats have nine lives. The best economy is to purchase high grade instruments and get along with few if need be. Triangles should be of vulcanite or some material more durable than wood, since that soon warps and twists out of shape, especially in a dry climate. The T square is also preferably of some material more durable than wood, although a wooden T square is much less objectionable than are wooden triangles. Very satisfactory drawing pencils are Dixon's, Faber's, Hardtmuth's, or Guttknecht's. The student's drawing-board may be made of some soft wood into which the thumb-tacks may easily be pressed; but if so made it must be frequently examined to see that its standard edge remains straight. A piece of well-seasoned half-inch poplar or whitewood * will serve fairly well; an edge running across the fiber of the wood should be "trued up" and used as the edge along which to slide the T square head.

A cut of a collection of draughtsman's instruments is given herewith. It is taken (by permission) from the catalogue of the Keuffel and Esser Co., 42 Ann St., New York City, one of the foremost firms in the United

* California redwood boards, one inch in thickness, are found to be very satisfactory in Colorado.

States dealing in and manufacturing such instruments. Their higher grade instruments are very satisfactory, and the student who is not sure of getting their instruments through his stationer, should send directly to them, or buy others only upon the advice of an experienced draughtsman.

A COLLECTION OF DRAUGHTSMAN'S INSTRUMENTS.



DEFINITIONS, CONCEPTS, AUXILIARIES.

INTRODUCTORY CHAPTER. *

1.—Anything is **appreciable** which is **capable of recognition.**

2.—One of two things is said to be **equal** to the other when either may be put for the other and, when so put, will produce all the effects of that other, or effects which are not appreciably different from those of that other.

3.—It is evident that if each single thing in a group is equal to a certain definite thing outside the group, any single thing in the group is equal to any other single thing in it.

4.—The sign of equality is =. When placed between two expressions, it is read "is (or are) equal to," or, more briefly, "equals (or equal)." $a = b$ means that the thing (not necessarily a number) represented by a equals that represented by b.

* This chapter is here inserted because it belongs here logically and not pedagogically. The instructor will select such portions of it as he needs to use for introduction and leave the rest for later discussion or for reference as occasion may arise.

5.—One of two things is said to be **equivalent** to the other when the two can be conceived to be divided into parts such that for every part in one there is an equal part in the other, no part of either being taken to correspond to more than one part of the other.

6.—It is evident that if each of the things in a certain group is equivalent to a certain thing outside the group, any one thing in the group is equivalent to any other one thing in it.

7.—The sign of equivalency is $\mathrel{\ooalign{$=$\cr\hidewidth$\cdot$\hidewidth}}$. When placed between two expressions, it is read "is (or are) equivalent to," or, more briefly, "equivales (or equivale.)" $a \mathrel{\ooalign{$=$\cr\hidewidth$\cdot$\hidewidth}} b$ means that the thing (not necessarily a number) represented by a equivales that represented by b.

8.—The first of two things is said to be **larger** than the second when the first can conceivably be divided into two parts one of which equivales the second thing. Saying that the second of two things is **smaller** than the first means the same as saying that the first is larger than the second.

9.—The signs of non-equivalence are $>$, read "is (or are) larger than,"—and $<$, read "is (or are) smaller than."

10.—Anything which continually maintains all its properties unchanged is called a **constant.**

11.—Anything some or all of whose properties change from time to time is called a **variable** with respect to those properties which change. One variable is said to be **dependent** on another when some or all of its properties are restricted by those of that other. Thus, the variable weight of a growing plant is dependent upon the age of the plant, and the variable amounts and intensities of heat, moisture, light, etc., that it receives. Any variable dependent on another is called a **function** of that other. In any collection of variables, the one whose law of change does not depend on that of any of the others is called the **independent** variable.

12.—Any constant to which a variable may become as nearly equal as we please without, however, becoming exactly equal to it under the conditions imposed, is called a **limit** of that variable. Thus, numerically, 2 is a limit of the variable sum, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ etc., as we take more and more terms, for we can not take a sufficiently great number of terms to make the sum exactly 2, but we may take a sufficiently great number to make the sum as nearly equal to 2 as we please.

13.—**Quantity,** or amount, of anything is that which **tells how much** with respect to that thing. It is evident that quantity is **purely relative**; *i. e.*, the quantity of anything can be told only by the relation between that thing and some other thing of the same kind.

14.—This other thing which is used as **standard** and in terms of which the quantities of all things of the same kind are (or may be) expressed is called the **unit.**

15.—Finding the quantity of anything is called the **measurement** of that thing. The thing measured is called the **metrand.** The unit is frequently called the measure. Measurement is evidently performed essentially by conceiving the metrand to be divided, or cut up, into as many parts as possible, all of which, or all but one of which, shall each equivale the unit,—and then counting the parts so obtained. Should the parts so obtained each equivale the unit, the metrand is said to be a **multiple** of the unit, and the unit is said to be a **sub-multiple,** or an exact measure, of the metrand. Should one of the parts so obtained be smaller than the unit, some sub-multiple of the unit is to be taken and used as an auxiliary unit for the measurement of the part aforesaid. Should this part equivale a sub-multiple of the unit, or should it and the unit have a common sub-multiple, the metrand is said to be **commensurable** (*i. e.*, measurable together) with the unit used; otherwise it is **incommensurable** with (or in terms of) the unit used. The **quantity of any commensurable thing** is evidently the name of the unit preceded by the number telling into how many parts equivalent to the unit and into how many parts equivalent to a designated sub-multiple of the unit the metrand is capable of being divided.

16.—The numerical portion of the quantity of anything is the **enumerator** of that thing relative to the unit used.

17.—**Two things** are said to be **incommensurable** when each is incommensurable in terms of the other as unit. The quantity of any incommensurable thing can not be exactly expressed. We can approximate, however, as closely as we please to the quantity of any incommensurable thing; for, it will be noticed, the remainder or unmeasured portion of the metrand may always be made less than the measure used. Since we may take as small a sub-multiple of the unit as we please for the auxiliary measure, this remainder may evidently be made as small as we please.

18.—The **approximate quantity**, or amount, **of any incommensurable** thing is the quantity of that thing which is most nearly equivalent to the given incommensurable and is yet commensurable in terms of the smallest sub-multiple (of the unit) which it is desired to use. It will be noticed that the error in the approximate quantity of any incommensurable thing is never so large as half of the smallest auxiliary measure used.

19.—The **real quantity of an incommensurable** thing is the limit of the approximate quantity as the auxiliary measure used is taken smaller and smaller indefinitely. The **enumerator of an incommensurable** is the limit of the enumerator of the approximate commensurable, and evidently can not be expressed in the ordinary arithmetic symbols of number. Such an enumerator is called an incommensurable number.

We become aware of incommensurable things only through theory. In practice they are dealt with by means of their approximate quantities.

20.—The ratio $\left\{\begin{array}{c}\text{of}\\ \text{between}\end{array}\right\}$ the first of two things $\left\{\begin{array}{c}\text{to}\\ \text{and}\end{array}\right\}$ the second is the enumerator of the first when the second is taken as unit. Ratio can exist between two things, evidently, only when they are of the same kind. By the ratio between two numbers is meant the ratio between two things whose enumerators these numbers are.

21.—If the two things whose ratio is to be considered are incommensurable with respect to each other, the ratio is said to be an **incommensurable ratio.** The approximate enumerator of the first thing in terms of the second as unit, when the two are incommensurable, is called the approximate ratio between them. Evidently in such a case the real ratio is the limit which the approximate ratio approaches.

22.—The first of the two things concerned in any ratio is called the **antecedent;** the second is called the **consequent.**

23.—The ratio between a and b is indicated by $a : b$, or by $\frac{a}{b}$, or by a/b.

24.—If the antecedent equivales the consequent, the ratio between them is called **unity** and is said to be a ratio of equality. If the antecedent be larger than the consequent, the ratio is a ratio of **larger inequality**; if smaller, the ratio is one of **smaller inequality.**

25.—A **proportion** is an equality of ratios. Four things are said to be in proportion when the ratio of the first to the second equals the ratio of the third to the fourth.

26.—The first and fourth of four things in proportion are called the **extremes** of the proportion; the other two are the **means.**

27.—Three or more things are said to be in **continued proportion** when the ratio of the first to the second equals the ratio of the second to the third, this ratio equals the ratio of the third to the fourth, and so on. If three things are in continued proportion, the second is called a **mean proportional** between the first and the third.

28.—**Two things are proportional to two others** when the ratio between the first two equals that between the other two taken in the same order.

29.—If a, b, c, and d are in proportion, it is indicated thus; $a : b :: c : d$, read "a is to b as c is to d;" or $\frac{a}{b} = \frac{c}{d}$, or $a/b = c/d$, read "the ratio of a to b equals the ratio of c to d" or, more briefly, "a to b equals c to d" or "a over b equals c over d."

30.—The **place** of anything is that which is indicated by the (true) response to the question "where?" concerning that thing.

31.—The aggregate of all conceivable places is called **Space.** Since there is no limit to our conception of places, space is limitless or infinite.

32.—Anything is said to have **extent** if it can be conceived to be divided into parts no two of which occupy the same place.

33.—**Distance** is quantity of difference of position, or place.

34.—**Motion** is appreciable change of place. Anything is said to move when any part of it changes its place through an appreciable distance.

35.—The aggregate of all the different portions of space occupied by anything during its motion, is called its **path.** The path is said to be **traced** or **generated** by the moving thing, which is said to be the **generator** of its path.

36.—Any body, *i. e.*, any limited portion of matter is said to be more or less **solid** according as (up to the time of yielding suddenly) it offers more or less resistance to pressure from opposite sides when otherwise unconfined. Thus, at ordinary temperatures, a mass of iron is more solid than a mass of lead, and that is much more solid than a mass of paraffine or one of bees-wax.

That property of a body which is more or less permanent according as the body is more or less solid is called the **shape** or form of the body.

37.—Because we consider any bounded part of space that we deal with in geometry to keep the same shape constantly, we call it a **geometric** solid; or, more frequently, merely a solid, the adjective being understood from the nature of the discussion.

38.—Any thing so small that the distance between no two parts is appreciable is called a **point.**

39.—The path of a moving point is called a **line.** If the line is such that the tracing point may return to its initial position without leaving the line or retracing any portion of it, the line is a **closed line**; otherwise it is an **open** one.

40.—The path of a moving line, if other than a line, is called a **surface.** If the surface completely encloses any limited portion of space it is a **closed surface**; otherwise, it is an **open** one.

Q. 1.—May a line ever move so as to generate merely a line? If so, show how.

41.—The path of a moving surface, if other than a surface, is a **geometric body.** *

42.—Any two parts of a line are **contiguous**, or adjacent, if they are separated merely by a point; any

* What is here called a geometric body is usually called a solid; *i. e.*, we understand the word solid to be restricted to mean a solid which is neither line nor surface, unless the contrary is stated or clearly implied. Such will be the usage with respect to this word throughout the remainder of this manual. Lines, and surfaces may, however, be very appropriately classed as solids, since their shapes are supposed to be permanent.

two parts of a surface, if separated merely by a line; any two parts of a solid if separated merely by a surface.

43.—Any two points in a line, a surface, or a solid, are called **consecutive** * if the distance between them is inappreciable. Saying that two points of a line are consecutive does not, of course, mean that there are no other points of the line between them.

44.—Non-consecutive points are called **separate** * points. When speaking of any limited number of points, separate points are always to be understood unless the contrary is stated or clearly implied.

45.—The **size** of a thing is the quantity of its extent. The size of a line is called its **length**; of a surface, its **area**; of a solid, its **volume.**

Q. 2.—What is the size of a point?

46.—The word size, as above defined, is used only with respect to lines, surfaces, and solids. The word **magnitude** is frequently used for the word size, as here defined, but its use is not restricted to the discussion of lines, surfaces, and solids; *e. g.*, we speak of the magnitude of a weight, or of a value, etc.

47.—The four classes of things, solids, surfaces, lines, and points, are called the **geometric concepts,** because they are the things we are continually thinking about in every geometric discussion. They are all purely

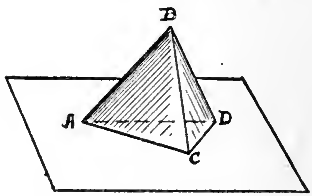
* It will be understood that these words are here used technically.

ideal, or abstract; *i. e.*, as here defined, they exist only as mental conceptions, and there is nothing cognizant to the senses which exactly corresponds to them. Ordinarily, when we speak of a solid we think of some limited portion of matter of almost permanent shape; of a surface, the outermost parts of a body; of a line, a body whose extent in one way is much greater than in others, as, *e. g.*, a fish-line; of a point, a very small body. In geometry, however, the word solid suggests to us merely the idea of a limited portion of space considered with respect to shape and to extent; surface, the boundary of a solid, or that between two contiguous portions of a solid, or that which might serve as such; etc., etc.

48.—**Geometry** is that branch of science which is **concerned in the investigation of the relations of solids, surfaces, and lines** with respect to their two properties, **extent and shape.**

49.—When any one of these three concepts is considered with regard to its extent, it is called a **magnitude;** with regard to its shape, or form, a **figure.** Custom is somewhat variable, however, in the use of the word figure in geometry. When one of the concepts is spoken of as a figure, its shape is always intended to be considered and sometimes its extent also. The context will usually indicate sufficiently clearly whether shape alone or both shape and extent should be considered.

50.—In discussing the geometric properties (extent and form) of anything, the imagination frequently needs some external aid; that is to say, on account of the complexity of the figure considered, there has to be some drawing or other thing to represent to the mind through the eye the thing under discussion. The drawing representing any geometric figure is itself frequently called a figure. When only lines and points are represented, it is usually called a **diagram.** Points are usually represented by means of dots, and named by means of letters (generally Roman capitals,) or other symbols written near them; thus, this figure, .B C. .A, represents the three points, A, B, and C. Lines are represented by narrow marks, and are named by letters (generally lower-case Roman or Italic) written alongside them, or are named by means of their extremities; thus, this figure, $B\,\underset{a}{\overline{\hspace{3em}}}\,C$, represents the line *a*, or BC. Surfaces are represented by their boundary lines usually, and the drawing representing them has its light and its dark parts so distributed, when necessary, as to make a sufficiently complete picture to convey the idea intended. A surface is named by naming its boundary lines, or by naming a sufficient number of points in it to distinguish it from any other surface pictured in the same drawing. Thus the left hand



front surface (or face) of the solid represented in this drawing, would be named the surface (or face) ABC;

the right hand, the surface BCD, etc. Solids are represented and named by means of their surfaces, or they may be named by naming a sufficient number of their points to distinguish them from other solids represented in the same drawing.

51.—The figures of geometry are treated, in the drawings representing them, as being opaque. All lines of the figure which would be in sight when the figure has the position represented in the drawing, are represented (if at all) by full lines (marks); thus, ————. Those which are hidden are represented (if need be) by means of lines composed of dashes of medium length; thus, — — — —; these are drawn in the positions in which the lines they are to represent would appear if the figure were transparent. Lines not appearing in the real figure but drawn in the diagram to aid in the investigation are called auxiliary lines, and are represented by dashes alternating with dots; thus, —·—·—. The lines sought for in any discussion are called resultant lines and are represented by very long dashes; thus, ——— ——— ———.

52.—The instruments which geometers have, through common consent, restricted themselves to the use of in making their diagrams, are the pen or pencil, the ruler, and the compasses (an instrument used in drawing circles, and in transferring lines.) No construction is admitted to be legitimately within the range of elementary geometry, which is composed of lines other than those which may be drawn by aid of these instruments. Besides these three instruments, however, others are frequently used in practical drawing, some of which serve merely to enable the draughtsman to do more conveniently and expeditiously what can also be done by means of those just mentioned. These and their uses will be discussed later. Since they add no new powers, but act simply as time-savers, they may very properly be used in making geometrical constructions.

BOOK I.

RECTILINEAR FIGURES IN A SINGLE PLANE.

CHAPTER I.

LINEAR RELATIONS.

53.—The **straight** (*i. e., stretched*) line is the line whose position is completely determined by the positions of any two of its (separate) points; that is to say, it is such a line that its position can not be changed without changing the position of every point in it but one. A fine fiber of silk suspended with a weight at its lower end suggests an approximate idea of a straight line. Straight lines are also called **right lines**, whence figures made up of straight lines are called **rectilinear**.

Q. 3.—Can you think of any line which is determined in position by the position of only one of its points?

4.—If two or more straight lines could be drawn joining the same pair of points, would it be possible to say that the straight line is determined in position by the positions of two of its points?

5.—Immediately from the definition then, what do we know about the number of straight lines that may be drawn joining the same pair of points?

6.—Why should the line now under discussion be called "straight" (*i. e.*, stretched)?

7.—How does the carpenter get a straight mark from a point at one end of a board to a point at the other end?

8.—What is the shortest path from one point to another?

9.—In what sort of a line does a ray of light travel?

10.—What use of the fact that a straight line is determined by two points does a hunter make in firing a rifle?— a farmer when he wishes to set a row of posts in a straight line, or to plow a straight furrow?

54.—Since between two points but one straight line can be drawn, and since the straight line is the simplest of all lines, we take **the length of the straight line joining two points** to be **the distance between them.**

55.—A is said to be **nearer** to B than C is to D when the **distance** between A and B is **smaller** than that between C and D. Saying that C is farther from D than A is from B is equivalent to saying that A is nearer to B than C is to D.

56.—The **ruler, rule,** or **straight-edge,** is an instrument having an edge along which straight lines may be drawn. Because such an instrument is frequently called a rule, straight lines are sometimes called **ruled lines.**

57.—When the word line is used hereafter, throughout this manual, a straight line is to be understood, unless the contrary is explicitly stated, or unless the context clearly indicates that the most general meaning of the word is intended. Moreover, the line is to be understood to be indefinite in its extent unless the contrary is stated or clearly implied.

58.—A line of definite extent is called a **sect.** The indefinitely extended line of which any sect is a part is the **seat** of that sect. Sects of equal length are equal.

59.—A line of indefinite extent but having **one end definite** is a **semi-sect.** Any given point upon a line evidently divides it into two semi-sects.

60.—Either of the two semi-sects into which an indefinite line is divided by any given point upon it is called the **complement** of the other.

61.—A line not straight, but capable of being divided into parts of appreciable length, each of which is straight, is frequently called a **broken line,** and the straight parts are called its **segments.** We shall here call such a line a **chain,** and its segments we shall call **links.**

62.—Three or more points are said to be **collinear** when they lie on the same straight line. It is evident that collinear points are not independent, since the position of each point is restricted (although not completely determined) by the positions of any other two points.

63.—Two lines passing through the same point and not coinciding are said to **intersect** at that point. Two sects intersect when they have a common point not at an extremity of either and do not have a common seat.

Q. 11.—At how many separate points can two lines intersect at any one time?

12.—If they could intersect at two or more points, could we consistently say that a straight line is determined by two points?

64.—Two or more lines passing through the same point and not coinciding are said to be **concurrent** lines. The common point is called the **center of concurrence.** The lines are called **rays** with respect to this center.

Q. 13.—How many rays may there be through the same center?

65.—If a surface is such that the line through any two points of it lies entirely in it, it is called a **plane surface,** or merely a plane. Such a surface is evidently perfectly flat, and presents the same appearance from one side or face as from the other. Planes are understood to be indefinite in extent unless the contrary is stated or clearly implied.

Q. 14.—How many points are necessary to determine a plane?

15.—Are ten collinear points sufficient?—five?

16.—What is the least number of collinear points necessary to determine a plane?

17.—Why should a certain tool used by carpenters be called a plane?

18.—What kind of a surface has a good drawing board?—the top of a billiard-table?

19.—How does a mechanic work up a set of "surface-plates," *i. e.*, a set of pieces of metal each with a plane face,—and how many of these must he work up in one set in order to work up the set independently of any other set?

20.—If two planes intersect (*i. e.*, cut each other) what sort of a line is the line of intersection?

21.—How does the carpenter test the surface of a piece which he has planed to see whether or not it is perfectly flat?

22.—How does a stone-cutter find out what parts of a stone to dress off if he wants to give the stone a flat face?

66.—The **trace of one figure upon another** is the part common to the two. It is understood when we speak of the trace of one figure upon another that each has parts not included by the other.

Q. 23.—What is the trace of one line upon another?

24.—Of a line upon a plane?

25.—Of one plane upon another?

67.—Any figure all of whose parts lie in a single plane is called a **plane figure.** On account of the greater simplicity of plane figures and of the fact that a knowledge of many of their properties is necessary to a discussion of those of non-plane figures, the study of plane figures is undertaken first.

68.—To make a geometric drawing (*i. e.*, one in which the extent and shape of the various figures must be exact,) the instruments before mentioned* are usually necessary, and a good drawing board also if the drawing is to be made upon a thin sheet of paper.

The paper is mounted (or fastened) upon the board usually by tacking down its edges or else by pasting them to the board. In case the edges are to be tacked down it is advisable to use draughtsman's thumb-tacks, which are broad, flat-headed tacks, easily pressed into soft wood by the thumb, and whose heads being flat and thin do not materially interfere with the use of the

* See art. 52, page 13.

draughting instruments. In case the edges are to be pasted to the board, a margin from one-half to three-quarters of an inch wide should be folded up all around the sheet, and the rest of the paper thoroughly dampened. The water for this purpose must be entirely clean, else the paper will be streaked. The dampening being completed, the dry margins are to be pasted to the board, being careful in pasting to stretch the dampened paper sufficiently to make it dry perfectly flat. For pasting the dry margins to the board, strong and quickly drying paste or glue must be used, as the edges must be securely fastened before the paper has dried materially, else it will wrinkle. Great care must be taken not to wet these margins, as wetting them will cause them to tear. In order to stretch the paper to the best advantage the second margin pasted down should be the one opposite to the one first pasted. This second method of mounting the paper upon the board should be used whenever it is to remain for any considerable time on the board, as in making an extensive and complicated drawing; also whenever wet-tints or ink washes are to be used. These would cause the paper to wrinkle in drying if it were mounted without having been previously moistened and stretched. For drawings such as will be required by most of the exercises in this manual, it will be sufficient to secure the paper by thumb-tacks.

For drawing straight lines the pencil should be given a point like a needle point, or else the marking portion should be made wedge-shaped: this latter form we shall, for convenience, call a "wedge-point" although it is not, properly speaking, a point at all. For keeping the pencil sharp a piece of very fine sand-paper or a very fine file is convenient. The pencil used should always be hard enough to make a fine, clean mark, but not so hard as to indent the paper very considerably, else the marks will be difficult to erase should any erasure be necessary. For drawing straight lines, the wedge-point is usually more satisfactory than the needle-point, because it demands less frequent sharpening.

In drawing lines along the ruler, care must be taken that the pencil (or ruling-pen, as the case may be) is not tilted one way or the other. Likewise in drawing lines with the compasses, care must be taken that the portions of the legs next the paper are not tilted one way or the other to it. A good pair of compasses is always provided with a joint in each leg.

For erasing pencil marks, soft, clean, uniform india-rubber should be used. For erasing ink marks, a harsher grade of rubber is sometimes used; more frequently, however, a steel eraser is first used, then india-rubber, and the erased surface finally burnished. Very fine sand-paper often gives very satisfactory results. A dense, tough paper must be used where much erasing is likely to be necessary.

69.—**To test the straightness of the edge of the**

ruler, take two points, A and B, on the mounted sheet, nearly as far apart as the ruler is long, or, if this be impossible, as far apart as the size of the sheet of paper will conveniently permit. Along the edge of the ruler which it is desired to test, draw a line through these two points. Then, calling the point of the ruler nearest A, A′, and that nearest B, B′, turn the ruler over upon its opposite face, keeping A′ at A and B′ at B, and draw another line through A and B along the edge to be tested. If the two lines so drawn coincide throughout, the tested edge of the ruler is straight (?); if they do not, it is not straight (?).

Q. 26.—Why should the points A and B be taken as far apart as is practicable?

27.—In actual drawing can a straight line (mark) be determined so exactly by two points (dots) a fiftieth of an inch apart, as by two five inches apart?

28.—If so, why; if not, why not?

29.—If the ruler in the second position cover the line first drawn between A and B, what portion of the ruler should be cut away in order to straighten it, and how much of it with respect to the distance between the two lines?

70.—One sect a is **added** to another sect b by producing (*i. e.*, drawing out, extending) b until the produced part equals a. It is, of course, understood that the produced part must make with b a single sect and not a two-linked chain.

71.—The sect so obtained, the first part of which is b, the second equal to a, is the **sum** of the two.

72.—Three or more things are added by adding to the sum of the first two the third, to the sum of the first three the fourth, etc.

73.—The sign of addition is the same as in arithmetic and in algebra, and signifies here as there that the thing whose symbol follows the sign is to be added to that whose symbol precedes it.

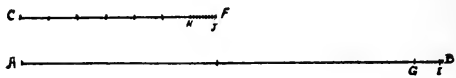
Q. 30.—What relation exists between the length of the sum of two sects and the sum of their lengths?

74.—Sects are transferred from one place in a drawing to another by means of the compasses or dividers. The legs of the dividers are spread until when one point rests on one end of the sect the other point may rest on the other end. The distance between the points is then equal to the length of the sect, and by careful handling of the dividers this length may be set off upon any desired line. In case great accuracy is required, or in case doubt exists concerning the constancy of the distance between the points, the dividers should again be applied to the first sect, noticing whether or not the distance between the points is yet equal to the length of the sect. Opening the legs of the dividers so that the distance between the points equals the length of the sect PQ is called "taking the sect PQ in the dividers."

E. 31.—Take four sects a, b, c, and f, no two being equal and find their sums, taking them in various orders, twenty-four in all. What relation, if any, exists among the sums so found?

75.—The **problem, to find the ratio between two commensurable sects,** evidently consists of two parts; first, to find a common measure or common unit of the two sects, — and second, to find the lengths of these sects in terms of the common measure and take their ratio.

First, to find the common measure. If the sects are unequal, suppose the longer called AB, and the shorter



CF. Taking CF in the dividers, lay it off as many times as possible on AB, beginning, say, at the end A. If the line AB is a multiple of CF, CF is the measure sought; if not, the last point of division will fall somewhere between B and A and nearer to B than C is to F. Call this point G. Apply GB in like manner to CF, beginning, say, at C, and if CF is not a multiple of GB, call the last point of division on CF, H; and in like manner apply HF to GB, and so on back and forth until some remainder is found which is a sub-multiple of the last preceding remainder; this is the common measure sought, for each remainder and each divisor before it is a multiple of it, and so their sums must be.

Second, to find the ratio between the sects, express the length of each divisor in terms of the common measure, after which the lengths of CF and AB may easily be expressed in terms of it. The ratio between these lengths is the ratio required.

Q. and E. 32.—What processes of arithmetic and algebra is this operation essentially similar to?

33.—Draw three sects at random and find their common measure.

76.—It is apparent that, since the successive remainders in the operation just described become smaller, we shall in any given case in practice soon find a remainder so small that we are unable to deal with it on our drawing. All pairs of sects will thus in practice appear to be commensurable. We become aware of incommensurable magnitudes only through calculation. Later this will appear more clearly, especially in studying trigonometry.

77.—In practice, we are most frequently concerned with the ratio a sect bears to a customary standard of length, as the inch in the English system, the centimeter in the French, etc. This is so frequently called for that we find it convenient to have a good collection of the multiples and sub-multiples of the unit marked off on an instrument which is called a **scale.** All that we need do then to find the length of a sect in terms of the customary unit is to take the sect in the dividers and apply it to the graduated (*i. e.*, measured and marked) line of a suitable scale and observe how many units and what fraction of a unit it covers. Practically, too, when we wish to find the ratio between two sects we find their lengths by means of some suitably graduated scale, and then compute the ratio of one of these lengths to the other. This is much more convenient than the method first described, and in most cases at least as nearly accurate.

78.—To $\left\{\begin{array}{l}\textbf{bisect}\\ \textbf{trisect}\end{array}\right\}$ **a sect** is to divide it into $\left\{\begin{array}{l}\text{two}\\ \text{three}\end{array}\right.$ equal parts. That point of a sect which bisects it is its **middle point.** In practice, especially where a sect is to be divided into only a small number of equal parts and where only a moderate degree of accuracy is desired, we frequently divide it into equal parts by trial. The dividers * are opened until it is estimated that they contain between their points the proper distance, and the sect between their points is laid off upon the given sect the desired number of times. Should the sect be more than covered the points of the dividers are too far apart; and contrarily. A little experience will suffice to show that this is a very tedious method when the sect is to be divided into a considerable number of parts. As was said at the beginning, this method is a method "by trial." An exact method will appear later.

79.—A line is said to be **curved** when no part of it having appreciable length is straight. A curved line is usually called by the briefer name, a **curve.**

80.—Any **open curve** is called an **arc.**

81.—A sect joining any two points of a curve is called a **chord.** When *the chord* of any arc is spoken of, the chord joining the extremities of the arc is meant.

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* The use of the dividers in dividing sects into equal parts and in dividing one sect by another as in the process of finding the common measure between two sects shows the reason for the name. Hitherto the words *dividers* and *compasses* have been used interchangeably. Hereafter an instrument for transferring lengths and for dividing sects will be called a pair of dividers. A similar instrument used for drawing will be called a pair of compasses. Both points of the dividers should be needle points. One of the compass-points should be pen, pencil, or crayon.

82.—A plane curve all of whose points are equally distant from a fixed point in the plane is called a **circular arc,** or, if a closed curve, it is called a **circle.** The circular arc being the simplest of all arcs and the most frequently used, the word arc is always to be understood as meaning circular arc unless some other meaning is stated or clearly implied.

Q. 34.—With what instrument are circular arcs usually drawn?

83.—The **fixed point** in the plane of a circular arc or circle from which all points of the curve are equidistant is called the **center** of the arc or circle.

Q. 35.—How many centers can a circle have?

84.—That point of an arc which bisects it (*i. e.*, divides it into two equal parts) is called its **middle point.**

85.—A sect drawn from the center of an arc or circle to any point of it is called a **radius** (pl. radii.) When we speak of *the* radius of any arc we mean *the length* of a radius of that arc.

Q. 36.—What relations, if any, exist between the radii of any arc?

86.—To **strike an arc** with a certain sect as radius is to draw one whose radius is the length of that sect.

87.—A chord of a circle passing through the center is a **diameter** of that circle.

Q. 37.—What relation, if any, exists between the diameters of a circle?—between a diameter and a radius of the same circle?

E. 38.—Draw a circle and show a diameter, a radius, a chord.

39.—Draw an arc with its chord.

88.—The **locus** (pl. loci) of a point satisfying a given condition is the figure all of the points of which and no points outside of which satisfy the given condition. If we are required to find a point satisfying two conditions, we find the points common to the loci of points satisfying those two conditions. Such points being on both loci evidently satisfy both conditions.

89.—To **"construct" a locus** we locate a sufficient number of points upon it to give us the idea of its appearance, and join these points by a suitable line or other figure. Of course the more points that are located exactly, the more accurate will our construction be.

Q. and E. 40.—Construct the locus of a point which shall be at a distance of two inches from a given point.

41.—What kind of figure is the locus above called for, and what relation does the given point bear to it?

42.—Find a point, P, which shall be at a distance of two inches from one of two given points, A, and three inches from the other, B.

43.—How many solutions (*i. e.*, how many points satisfying the requirements) when A and B are six inches apart?—five and a quarter?—five?—two and a half?—one inch?—one half of an inch? Make constructions showing these various cases.

44.—Construct the locus of a point which shall be equi-distant from each of two given points.

45.—What kind of figure is it?

46.—Does the shape or size of this locus vary with the positions of the two points?

47.—Does its position?

48.—Construct the locus of a point which shall be twice as far from one of two given points as from another.

49.—What kind of figure is it?

50.—What relation, if any, exists between its size and the distance between the given points?

51.—Same for a point which shall be three times as far from one of two given points as from the other;—four times;—five times.

52.—What general relation exists among the loci obtained in Ex. 48 and 51?

90.—Two semisects terminating at the same point form a figure called an **angle.** The two semisects are its **sides** or **arms,** and are said to **"inclose"** the angle. The point common to the two sides is called the **vertex** of the angle.

91.—The **essential part of the idea of an angle** is that of its **shape at the vertex.** The lengths of the sides being indefinite are not taken into consideration.

92.—If the two sides of an angle are complements (see art. 60, page 17,) of each other, the angle is called a **straight angle.**

93.—It will be noticed that the two sides of any angle not straight (nor yet a zero angle,—see art. 108, page 32,) are sufficient to determine a plane in position. The plane so determined is called the **plane,** or the **seat, of the angle.**

94.—The plane of an angle is divided into two parts by the sides of the angle. Each of these is frequently called an angle, and this is the meaning with which the word will be used throughout the remainder of this manual unless some other is indicated.

95.—Of the two angles having the same sides, the one which is crossed by the sect joining any point in one side to any point in the other is called a **convex** angle; the other is called a **concave** angle.

96.—An **angle** is **divided into parts** by a semi-sect (or semi-sects) lying in it and drawn from its vertex. It will be noticed that the parts of angles are also angles.

Q. 53.—Of what kind are the parts of a convex angle?—of a straight angle?—of a concave angle?

54.—Which is the larger, a convex angle or a straight angle? —a straight angle or a concave angle?

97.—An **angle** is **named** by naming its vertex or by naming its sides. In case two or more angles have a common vertex, it evidently is insufficient to name the vertex; in this case the two sides are named, or else the vertex is named *between* two other points one of which pertains to one side, the other to the other side. It should be particularly noticed that when an angle is named by means of three points, the vertex is customarily named between the other two. When any angle is named, of the two angles to which the name applies, the convex is always to be understood unless the contrary be indicated or unless the angle be straight.

98.—Two **angles** are **adjacent** when they have such positions that they are the two parts into which a third angle is divided by their common side.

99.—Two **angles** are **vertical** to each other when the sides of one are the complements (see art. 60, page 17,) of those of the other.

100.—Two **angles** are **equal** when upon being applied one to the other so that they have a side and the vertex of one coinciding with a side and the vertex of the other, the other two sides will coincide. It is to be understood that if one is convex the other is convex also.

101.—One **angle** is **added** to another by being placed adjacent to it. The angle of which the two given angles then form parts is the sum of the two.

102.—An **angle** is **bisected** when it is divided into two equal parts; in fact, to bisect any figure is to divide it into two equal parts, to trisect it is to divide it into three equal parts, etc. The line bisecting any angle is called the **bisector of the angle**.

103.—Half a straight angle is called a **right angle**, or an **orthogon.** The right angle is one of the most commonly used units of angle in geometry.

104.—The most commonly used unit of angle, however, is the **degree**, which is the ninetieth part of a right angle. The degree is subdivided into sixty equal parts, each of which is called a **minute**; and each minute is further subdivided into sixty equal parts, each of which is called a **second.** Seconds are sometimes each subdivided into sixty equal parts, each of which is called a third; but the subdivisions of the second are usually decimal. The symbols for degrees, minutes and seconds are °, ′, ″; thus 27° 15′ 42″.3 is read 27 degrees, 15 minutes, and 42.3 seconds.

105.—Any angle smaller than a right angle is called an **acute angle;** one larger than a right angle and smaller than a straight angle is called an **obtuse angle.** Acute and obtuse angles are called **oblique angles.**

106.—If the sum of two angles is a $\left\{\begin{array}{l}\textbf{right angle}\\ \textbf{straight angle}\\ \textbf{full plane}\end{array}\right\}$ either is said to be the $\left\{\begin{array}{l}\textbf{complement}\\ \textbf{supplement}\\ \textbf{explement}\end{array}\right\}$ of the other.

107.—An angle and its adjacent explement are said to be **conjugate.**

108.—Two coincident semi-sects are sometimes said to include a **zero angle.** With this usage agreed to, it may be said that any two semi-sects drawn from the same point include an angle.

109.—If one of the four convex angles formed at the point of intersection of two lines is a right angle, the lines are said to be **perpendicular, normal,** or **orthogonal,** to each other.

110.—Two sects or two semisects or a sect and a semisect are said to be perpendicular to each other when their seats are so.

111.—$\left\{\begin{array}{l}\text{To } \textbf{erect a perpendicular} \text{ to a given line}\\ \text{To } \textbf{drop a perpendicular} \text{ to a given line}\end{array}\right.$

$\left.\begin{array}{l}\text{at a given point upon that line}\\ \text{from a given point without that line}\end{array}\right\}$ is to draw through the given point a perpendicular to the given line. The **foot of a perpendicular** is the point where it meets the **base,** *i. e.*, the line to which it is perpendicular; or it is the point at which it would meet the base if both were sufficiently produced.

112.—Any sect extending from a point outside a line to a point of that line is called an **oblique** if it makes with that line an oblique angle.

113.—The **orthogonal projection** of a point upon a line is the foot of the orthogonal, or perpendicular, from the point to the line. The orthogonal projection of a figure is the aggregate of the orthogonal projections of its points. Other sorts of projection are used as well as orthogonal, but projection is understood to be orthogonal unless some other is stated or implied.

114.—A figure composed of three (independent) points and their connecting sects is called a **triangle,** or a **trigon,** or a **trilateral.**

115.—The three points are called the **vertices** of the trigon; the connecting sects **its sides**; the angles within the figure and having its vertices for their vertices, **its angles.**

116.—The side upon which the trigon is supposed to rest is called the **base.** The other two sides are the **legs.** When we speak of *the* vertex of a trigon, the one opposite the base is understood. By *the vertical angle* of a trigon is meant the one at the vertex.

117.—Customarily when the word trigon, triangle, or trilateral is used we understand it to mean besides the vertices and sides the portion of a plane enclosed by the three sides.

118.—With respect to their sides, trigons are classified as **scalene** (unequal legged) having no two sides equal; **isosceles** (equal legged) having two sides equal; and **equilateral,** having all three sides equal: with respect to their angles,—as **acute,** each angle acute; **right,** one angle right; **obtuse,** one angle obtuse; and **equiangular, isogonic,** or **isogonal,** having the three angles equal. In every isosceles trigon, the **two equal sides** are to be taken as **the legs** unless the contrary is stated or implied; in a right trigon, the two sides including the right angle. The side opposite the right angle in any right trigon is called the **hypotenuse.**

119.—The sides of a trigon are usually represented by the three letters a, b, and c; the vertices opposite by the letters A, B, and C respectively; and the angles at these vertices by α, β, and γ *. When two or more trigons are to be denoted, subscripts or indices are used; thus the sides of the first may be represented by a_1, b_1; c_1, and those of the second by a_2, b_2, c_2, etc.

* α, β, and γ are the lower-case forms of the first three letters of the Greek alphabet. Their names are respectively alpha, beta, and gamma.

120.—The face of any plane figure which is toward the eye of the draughtsman as the figure is being drawn, we shall call the **obverse;** the opposite face, the **reverse.**

121.—In plane geometry, in saying that two figures are equal we mean that the obverse of one is equal to the obverse of the other. If the obverse of one is equal to the reverse of the other, we shall say the figures are opposite. For the sign of opposition we shall use $\supset$; $a \supset b$ will be read, a is the opposite of b. Unequal will be used to mean neither equal nor opposite.

In the exercises of this and the next article the figures may be cut out of the drawing for purposes of comparison, or they may be drawn on thin transparent paper, such as tracing paper, or the so-called "onion-skin" note-paper. The latter mode will be the more expeditious and satisfactory.

Q. and E. 55.—If one angle is the opposite of another, are the angles equal or unequal?

56.—Can an isosceles trigon be the opposite of a scalene trigon?

57.—If one scalene trigon is the opposite of another are the two equal?

58.—Same for isosceles.

59.—Same for equilateral.

60.—If two angles of a trigon are equal, is the trigon equal to its opposite?

61.—Is it if no two angles are equal?

62.—Is it if it is isogonic?

122.—**To construct a trigon whose sides shall**

be equal to given sects. Suppose the given sects *m*, *n*, and *p*. Draw a sect AB equal to one of the given sects, say *m*, and then construct the locus of a point whose distance from A is the length of another, say *n*, and the locus of a point whose distance from B is the length of the third sect, *p*. Any point common to these loci may evidently serve as the third vertex, C, of the required trigon, which is finished by drawing the sects AC and BC.

Q. and E. 63.—What relation must exist among the lengths of the three sides of a trigon?

64.—Can a trigon be constructed the lengths of whose sides shall be 3, 5, and 10 inches? If so, construct one; if not, say why not?

65.—Can one be constructed with sides 3 inches, 5 inches, and 8 inches long respectively?—3 inches, 5 inches and anything less than 8 inches?—3 inches, 5 inches, and 2 inches?—3 inches, 5 inches, and 1½ inches?

66.—Construct a scalene trigon,—an isosceles trigon,—an equilateral trigon.

67.—What relation exists among the angles of the scalene trigon? Are any two of them equal? Which two, if any, are equal? If no two are equal, which side is the smallest angle opposite?—which the largest? Are the angles proportional to the sides opposite?

68.—Same for isosceles trigon.

69.—Same for equilateral trigon.

70.—Try several different and unequal scalene trigons, and see if your answers to question number 67 hold true of these also.

71.—What general relation, if any, seems to hold between the relations among the sides of a trigon and the relations among the angles opposite them?

72.—If the sides of one scalene trigon are respectively equal to those of another, what relation if any exists between the two?

73.—Same for isosceles trigons.

74.—Same for equilateral trigons.

75.—Construct two equilateral trigons in which the sides of one are smaller than those of the other. What relation, if any, exists between the angles of one and those of the other? If the angles are unequal in which are they the larger?

76.—After making the comparisons called for in this set of exercises, find the sum of the angles of each trigon you have used. Is there any uniformity in these various sums? If so, what?

77.—Can a trigon have two right angles?—two obtuse angles?—only one acute angle?

123.—The sect joining any vertex of a trigon to the middle point of the side opposite is called the **median of the trigon** to that side.

124.—The sect bisecting any angle of a trigon and terminating in the side opposite is called the **bisector of the trigon** to that side.

125.—The sect drawn from any vertex of a trigon perpendicular to the side opposite and terminating in that side (or that side produced) is called the **altitude of the trigon** upon that side.

Q. and E. 78.—Draw three or more unequal scalene trigons, and draw the three medians of each. What relation, if any, exists among the three medians of any one of these trigons?

79.—Do the same with isosceles trigons and with equilateral trigons. Do your conclusions in exercise number 78 hold true with these trigons also? If not, what modifications are necessary?

80.—What peculiar relation, if any, does the median to the base of an isosceles trigon bear to the base?

81.—What relation does it bear to the vertical angle?

82.—Devise some means of erecting a perpendicular at a given point of a given line, making use of your conclusion in q. no. 80.

83.—How many different perpendiculars can be erected at the same point and on the same side of the line? How might this have been known from the definition of a perpendicular?

84.—Devise some means of dropping a perpendicular to a line from a point outside the line, making use of your conclusion in q. no. 80.

85.—How many different perpendiculars can be dropped upon the same line from the same point outside the line?

86.—Devise some means of finding exactly the middle point of any given sect.

87.—What relation, if any, exists between obliques drawn from the same point and having equal projections?—unequal projections?

88.—What relation, if any, exists between the projections of equal obliques drawn from the same point?—from different points in the same perpendicular?—of unequal obliques drawn from the same point?—from different points in the same perpendicular?

89.—What kind of angle does the shortest sect from a given point to a given line make with the line?

90.—What is the locus of a point equidistant from the extremities of a given sect?

91.—Find a point equidistant from three given independent points.

92.—Show how to find the center of a circle which will go through the three vertices of a given trigon. Such a circle is said to be **circumscribed** about the trigon through whose vertices it passes, and its center is called the **circum-center** of the trigon.

93.—If the middle point of the base of an isosceles trigon be joined to the middle point of one of the legs, what relation, if any, exists between the length of the resulting sect and the length of the leg of the trigon? Try three or four different trigons before coming to any definite conclusion.

94.—By the aid of your conclusion in q. no. 93 devise some method of erecting a perpendicular when its foot is at or near the end of a given line.

95.—Devise some means of dropping a perpendicular from a point without a given line, when the point lies so that the foot of the perpendicular will be near the extremity of the line.

96.—Devise some means of drawing the bisector of a given angle. How many different bisectors may the same angle have?

126.—By **the distance of a point from** any **line** (straight or curved) is meant the length of the shortest sect that can be drawn from the point to the line. In practical drawing we frequently find the distance of a point from a line by trial, striking arcs of various radii about the point as center until we find the least radius whose arc will reach the given line. This radius then gives the required distance. By the distance of a point from a sect is meant the distance of the point from the seat of the sect.

Q. and E. 97.—What relation, if any, exists between the distances from the sides of an angle of any point on the bisector of the angle?—of any point between the sides of the angle and not on the bisector?

98.—What is the locus of a point equidistant from any two intersecting lines?—twice as far from one as from the other?—three times?—n times?

99.—Devise a method of finding a point within a trigon which shall be equidistant from the three sides of the trigon; or else show that there is no such point.

100.—Within a given trigon draw a circle which shall touch each of the sides. Such a circle is said to be **inscribed** in the trigon and its center is called the **in-center** of the trigon.

101.—Draw several trigons in one of which $a = b$; another, $a = 2b$; $a = 3b$; $a = 4b$; $a = 5b$; etc.,—and in each draw the bisector to the side c. What general relation, if any, exists between the ratio between the segments of the third side and the ratio between the sides to which those segments are adjacent?

102.—What relation, if any, exists among the three bisectors of a trigon?

103.—Draw the three altitudes of each of four different trigons,—one of which is scalene and acute, one isosceles and acute, one right, and one obtuse. What relation, if any, is there among the three altitudes of each of these?

104.—Is it possible to have the three altitudes of a trigon within the trigon?—two only?—one only?—none?

105.—Draw the three altitudes of a trigon in which the sides are proportional to two, three, and four, and find the relation, if any, existing among the lengths of these altitudes.

106.—Same with sides proportional to three, four, and five;—to four, five, and six.

107.—What general conclusion seems probably true, concerning the ratio between any two sides of a trigon and the ratio between the altitudes drawn to them?

108.—In a scalene trigon, which is the longest altitude?—which the shortest?

109.—Same for bisectors.

110.—Same for medians.

111.—What relation does the length of the median to the hypotenuse of a right trigon bear to the length of the hypotenuse?

112.—In case the bisector, the median, and the altitude to the same side of a trigon are all unequal, which is the longest, and which the shortest?

113.—Which two may be equal and yet differ from the third?

127.—Among the most useful of a **draughtsman's** instruments are his **triangles,** which are thin, flat pieces of wood, metal, vulcanite, or other hard and durable substance, whose outlines are those of trigons, or triangles. Each has usually one right angle, and the acute angles are of such sizes as are most frequently needed,—usually, 30° and 60°, and 45° and 45°.

Throughout the rest of this manual, when the word triangle is used a draughtsman's triangle is to be understood, unless some other meaning is clearly indicated.

Q. and E. 114.—Devise some method ot testing the (supposed) right angle of a right triangle, using a ruler or another triangle in connection with the one to be tested. See art. 103, page 31.

115.—Devise a method of drawing through a given point a line perpendicular to a given line, using a right triangle and a ruler or another triangle.

116.—With a triangle whose angles are 30°, 60°, and 90°, and another whose angles are 45°, 45°, and 90° devise means of constructing angles of 15°, 75°, and 105°.

117.—Erect three or more perpendiculars at points taken at equal intervals on any straight line x, constructing the perpendiculars on the same side of x, and call these perpendiculars, taken in order, a, b, c, etc. Take a point, A, on a at a convenient distance from x, and another, B, on b at a trifle less distance from x. Through A and B draw an indefinite straight line, locating C where this line crosses c, D where it crosses d, etc. What relation exists among the distances of A, B, C, D, etc., from x?

118.—Same as No. 117, except that the point A and the second point, to be taken on a perpendicular at a considerable distance from a, say the point M on the perpendicular m, are to be taken equally distant from x and on the same side of it, and B, C, D, etc., are to be located where the lines b, c, d, etc., are crossed by the line through A and M. What relation exists among the distances of the points A, B, C, D, etc., from x?

119.—If a straight line has two of its points $\left\{ \begin{matrix} \text{unequally} \\ \text{equally} \end{matrix} \right\}$ distant from another straight line lying in the same plane (the two points spoken of being on the same side of the second line) can the lines be sufficiently prolonged to cause them to meet?

120.—If the sides of one angle are perpendicular to those of another, what relation exists between the convex angles concerned?

121.—If the sides of one trigon are respectively perpendicular to those of another what relation exists among the angles of the trigons?

128.—**Two points** are said to be **symmetric** about, or with respect to, a $\left\{\begin{array}{l}\text{third point as }\textbf{center}\\ \text{certain line as }\textbf{axis}\end{array}\right\}$ when this $\left\{\begin{array}{c}\text{third point}\\ \text{line}\end{array}\right\}$ is the $\left\{\begin{array}{c}\text{mid-point}\\ \text{perpendicular bisector}\end{array}\right\}$ of the sect joining the two points. This $\left\{\begin{array}{c}\text{third point}\\ \text{line}\end{array}\right\}$ is called the $\left\{\begin{array}{c}\text{center}\\ \text{axis}\end{array}\right\}$ of **symmetry**, or, more briefly, the $\left\{\begin{array}{c}\textbf{sym-center}\\ \textbf{sym-axis}\end{array}\right\}$ for the two points.

129.—**Two figures** are **symmetric** about a certain $\left\{\begin{array}{c}\text{point as center}\\ \text{line as axis}\end{array}\right\}$ when for every point in one there is a symmetric point in the other about the $\left\{\begin{array}{c}\text{center}\\ \text{axis}\end{array}\right\}$ aforesaid.

130.—A **single figure** is **symmetric** about a certain $\left\{\begin{array}{c}\text{point as center}\\ \text{line as axis}\end{array}\right\}$ when it can be divided into two parts which are symmetric about that $\left\{\begin{array}{c}\text{point as center}\\ \text{line as axis.}\end{array}\right\}$

131.—For the phrase, symmetric about $\left\{\begin{array}{c}\text{a center}\\ \text{an axis}\end{array}\right\}$ we shall use the shorter one, $\left\{\begin{array}{c}\textbf{sym-centric.}\\ \textbf{sym-axic.}\end{array}\right\}$

Q. and E. 122.—Is a scalene trigon sym-centric? If so, about what point?

123.—Is it sym-axic? If so, about what line?

124.—Same for isosceles trigon.

125.—Same for equilateral trigon.

126.—Same for acute trigon,—right trigon,—obtuse trigon.—isogonic trigon.

127.—Is any sect sym-centric? If so, about what point?

128.—Is any sect sym-axic? If so, about what line?

129.—Is there any angle which is sym-centric? If so, what angle and about what point?

130.—Are there any open lines which are sym-centric? If so, draw three or four such, curved if possible.

131.—At what point of the sect joining the extremities of a sym-centric curve is it crossed by the curve if at all? Need it be crossed by the curve if the curve is sym-centric? In case it is crossed, what relation between the point of intersection and the sym-center?

132.—Is any angle sym-axic? If so, about what line?

133.—Are there any open lines which are sym-axic? If so draw three or four, curved if possible.

134.—If the extremities of one sect are sym-centric with those of another, are the sects sym-centric? Are their seats?

135.—Same for axic symmetry.

136.—Can two sym-centric lines meet?

137.—Must they lie in the same plane?

138.—Same for two sym-axic lines.

139.—If two sym-axic lines meet, where does their point of intersection lie?

140.—How many sym-centers may one figure have?—sym-axes?

141.—Draw, if possible, a figure having three or more sym-centers;—three or more sym-axes.

142.—If a sym-centric figure is also sym-axic, is or is not its sym-center necessarily on its sym-axis?

143.—If a sym-axic figure has two axes, is or is not their intersection necessarily a sym-center for the figure?

144—If a sym-axic figure has three or more sym-axes, are or are they not necessarily concurrent?

132.—Any figure is said to **revolve** (in the geometric sense of the word) **about** a given fixed **point** when it moves so that each point in it maintains a fixed distance from that given point. The given point is the **center of revolution.**

133.—Any figure is said to **revolve about** the **line** through any two fixed points when it revolves about each of those two points at the same time. The line aforesaid is called the **axis of revolution.**

134.—A thing **revolves through a full revolution** about any axis when it revolves about that axis until it reaches its original position, without having retraced any portion of its path. It **revolves through a given angle** when a perpendicular from any point of it to the axis sweeps over the given angle.

135.—It must be remembered that these uses of the word *revolve* are the only ones which it has in geometry. Outside of geometry it is used in other and wider senses.

Q. and E. 145.—If one figure sym-axic with another be revolved through half a revolution about the sym-axis, what relation will it then bear to the second figure?

146.—If one plane figure sym-centric with another in the same plane be revolved through half a revolution (keeping always in the plane) about the sym-center, what relation will it then bear to the second figure?

147.—If one of two indefinite perpendicular lines be revolved about the other as axis, what will its path be?

148.—Can a line meet a plane in such a way that it shall be perpendicular to every line lying in the plane and drawn through the trace of the given line upon the plane?

136.—The **centric angle** of any arc is the angle whose vertex is at the center of the arc, whose sides pass through the extremities of the arc, and which is swept across by the arc. The arc is said to **subtend** its centric angle and to be **intercepted** by it.

137.—The **chord to a given radius of any angle** is the chord of the arc intercepted by the angle and struck with the given radius from the vertex of the angle as center. When no length of radius is specified, the radius is understood to be of unit length.

Q. and E. 149.—If two angles are equal, what relation exists between their chords to the same radius?—what if they are explementary?

150.—If two angles have equal chords to the same radius, what relation exists between the angles? What if the chord of one is larger than the chord of the other?

151.—Devise a method of constructing an angle which shall have a given side and a given vertex, and which shall be equal to a given angle.

152.—By the method called for in ex. 151, find the sum of the angles of a scalene trigon;—of an isosceles trigon;—of an equilateral trigon.

153.—Same for trigons with sides twice as large as in those just used;—thrice.

154.—What general statement, if any, seems to hold true in the results of the nine exercises just called for in ex. 152 and 153?

155.—How many right angles may any trigon have?—obtuse? —concave?

156.—If in two trigons $a_1 = a_2$, $b_1 = b_2$, and $\gamma_1 = \gamma_2$, what relation exists between the trigons?

157.—What, if $a_1 = a_2$, $\beta_1 = \beta_2$, and $\gamma_1 = \gamma_2$?

158.—What, if $a_1 = a_2$, $b_1 = b_2$, and $\beta_1 = \beta_2$?

159.—Does the relation between the lengths of a and b in ex.

158 have anything to do with the relation between the trigons?—Does the size of β?

160.—If $b_1 = b_2$, $\beta_1 = \beta_2$, and $\gamma_1 \doteq \gamma_2$, what relation exists between the trigons?

161.—What if $a_1 = a_2$, $\beta_1 = \beta_2$, and $\gamma_1 = \gamma_2$?

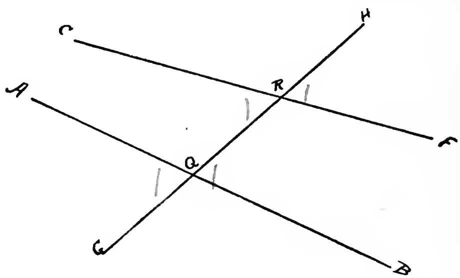
162.—Can there be two trigons in which $a_1 = a_2$, $\beta_1 = \beta_2$, and $\gamma_1 > \gamma_2$?

163.—The sides and angles of any trigon are customarily called the **parts of** the **trigon.** What three parts of one trigon must be respectively equal to three parts of another, in order that the two trigons may be equal or opposite?

164.—If in two trigons $a_1 = a_2$, $b_1 = b_2$, but $\gamma_1 > \gamma_2$, what relation exists between c_1 and c_2? Try several different pairs of trigons before announcing any general conclusion.

165.—If $a_1 = a_2$, $b_1 = b_2$, but $c_1 > c_2$, what relation exists between γ_1 and γ_2?—what between β_1 and β_2?—what between a_1 and a_2? Try several different pairs before announcing any general conclusion.

138.—When one line intersects another it is said to **traverse** that other at the point of intersection and is called a **traverser** with respect to that other line.



When a traverser traverses two lines anywhere else than at their point of intersection eight angles are formed, which, on account of the frequency with which the case occurs, are divided into groups, and the groups named. Thus if AB and CF are the traversees (*i. e.*, the things traversed,) and GH is the traverser, crossing AB at Q and CF at R, the four angles between the traversees, AQR, BQR, CRQ, and FRQ, are called **inner angles;** the other four are **outer angles.** Any two angles lying on opposite sides of the traverser, and not adjacent, are called **alternate.** The eight angles are divided into pairs of **corresponding angles** in these three ways:—any pair of alternate inner angles; any pair of alternate outer angles; any pair of non-adjacent angles lying on the same side of the traverser and being, one of them inner, and the other outer. Either of a pair of corresponding angles is said to be the **correspondent** of the other.

Q. and E. 166.—If, when two lines are traversed by a third, some one angle equals its correspondent, what relation, if any, exists between any other angle and its correspondent?

139.—Two **lines** are said to be **parallel** when they lie in the same plane and have such positions that they can never meet, no matter how far produced. Two **sects** are **parallel** when their seats are parallel. The sign for "parallel" is ‖.

Q. and E. 167.—Through one point how many lines can be drawn parallel to a given line?

168.—If each of two given lines is parallel to a third given line, are or are not the first two parallel to each other?

169.—If a straight line lying in the plane of two parallels cross one of them, will or will not it cross the other?

170.—At least how many of the seats of the sides of a trigon will be crossed by any line lying in their plane?

171.—If two lines lying in the same plane be crossed by a traverser, and their positions be such that corresponding angles are equal, on which side of the traverser will the traversees meet?

172.—If they meet on one side, must they not also meet on the other side?

173.—Of what kind are the two lines mentioned in q. 171,—parallel, perpendicular, or oblique?

174.—If two parallel lines be crossed by a traverser, what relation in size exists between corresponding angles?

175.—If a line is perpendicular to one of two parallels, what relation does it bear to the other?

176.—If one pair of parallels intersect another pair of parallels, what relation exists between the segments of one pair intercepted by the other pair?

177.—What may be said of the distances of the various points of a given line from another line to which the first is parallel?

178.—What is the locus of a point at a given distance from a given line?

179.—If two parallels cut a circle what relation exists between the chords of the arcs intercepted by the parallels?

180.—Devise as many methods as you can for drawing through a given point a line parallel to a given line.

181.—If three parallels make equal intercepts on one traverser, what relation is there between the intercepts they make on any other traverser?

182.—What relation exists between any side of a trigon and the line joining the middle points of the other two sides? See q. 181, also q. 93.

183.—If from any point whose distances from two intersecting lines are in a given ratio, $\frac{m}{n}$, sects be drawn parallel to each until they meet the other of the two given lines, what is the ratio between these sects, and what is that between the distances of their extremities in the given lines from the point of intersection of the given lines?

184.—Devise an easier method than that developed in q. 98, for constructing the locus of a point whose distances from two intersecting lines shall be in a given ratio.

185.—Find the locus of a point equi-distant from two non-parallel lines whose intersection is off the drawing.

186.—Devise a method for drawing through a given point a traverser which shall cross two non-parallel lines so as to make the inner angles on the same side of it equal.

187.—Devise a method by which to divide a given line into any desired number of equal parts.

188.—If the sides of one angle are parallel to those of another, what relation is there between the convex angles concerned?

189.—If the sides of one trigon are parallel to those of another, what relation exists among the angles of the trigons?

140.—If a **collection of traversers** be drawn across a **collection of parallels,** the segments of the traversers intercepted between any pair of parallels are called **corresponding segments.**

Q. and E. 190.—Draw a series of parallels, and call them m, n, o, p, etc.; then draw any pair of traversers, t_1 and t_2. The segment of t_1 which is between m and n call b_1, that between n and o call c_1, etc.; corresponding segments of t_2 call b_2, c_2, d_2, etc. If the ratio of b_1 to b_2 equals $1\frac{1}{2}$, what is the ratio of c_1 to c_2,—of d_1 to d_2,—of any pair of corresponding segments of t_1 and t_2? If $\frac{b_1}{b_2} = 3$, what are the values of the ratios $\frac{c_1}{c_2}$, $\frac{d_1}{d_2}$, etc.? What when $\frac{b_1}{b_2} = 4\frac{1}{4}$? What when the ratio $\frac{b_1}{b_2} = r$ (r meaning any number)?

191.—Complete the following proposition:—"If three or more parallels are crossed by two traversers, the ratio between any two corresponding segments .... that between any other two, taken in the same order."

192.—When these traversers are concurrent lines, call the point of concurrence C, and the segments of m, n, etc. between t_1 and t_2 denote by $m_{1,2}$, $n_{1,2}$, etc.,—those between t_2 and t_3 by $m_{2,3}$, $n_{2,3}$, etc. What relation, if any, exists between the ratios

$\frac{m_{1,2}}{n_{1,2}}$, $\frac{m_{2,3}}{n_{2,3}}$, etc., and the ratio between the distances of m and n from C?

193.—Complete the following proposition:—"The ratio between the segments of two parallels intercepted by two concurrent traversers .... the ratio between the distances of these parallels from the center of concurrence."

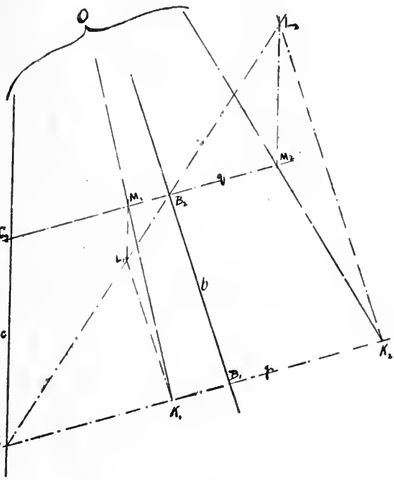
194.—Calling the point where t_1 crosses m, M_1, where it crosses n, N_1, etc., what relation, if any, exists between the ratios $\frac{m_{1,2}}{n_{1,2}}$, $\frac{m_{2,3}}{n_{2,3}}$, etc., and the ratios $\frac{CM_1}{CN_1}$, $\frac{CM_2}{CN_2}$, etc.?

195.—From the principles developed in the preceding exercises, deduce a method for finding the fourth proportional to three given sects; *i. e.*, having given a, b, and c, find a sect d so that $\frac{a}{b} = \frac{c}{d}$.

196.—Deduce a method of finding the third proportional to two given sects; *i. e.*, having given a and b, find c so that $\frac{a}{b} = \frac{b}{c}$.

197.—Deduce a method of dividing a given sect into parts which shall be proportional to given sects.

141.—The relation developed in example 192 leads to a very useful and convenient method for drawing through a given point a line which if sufficiently produced would pass through the point of concurrence of two lines, this point of concurrence being off the drawing. Suppose the two lines are b and c, and the given point is K. Call the center of concurrence O. Draw through K any convenient traverser, p, which shall cross b at B_1 and c at C_1, and draw another traverser, q, which shall be parallel to p, and at as considerable distance from it as is convenient. Suppose q crosses b at B_2 and c at C_2. Draw B_2C_1. Draw through K a line parallel to b meeting B_2C_1 at L, and draw from L a line parallel to c, meeting q at M. A line through K and M,



if sufficiently prolonged will pass through O. The student may give the reasons. *

142.—In drawing, very frequent use is made of the relation between corresponding angles when two parallels are crossed by a traverser. One of the most common instruments whose uses depend wholly or partially on the relation just mentioned is the **T square** (symbol, T□.) It consists of a long, thin, straight-edged blade, fastened (or capable of being fastened) rigidly to a "head," which is a thicker and usually a much shorter block than the blade; one face of this head is intended to be plane, and in using the instrument, this face is

* See Eagles's Const. Geom. of Plane Curves, page 6, prob. 4.

slid along the straight edge of the drawing-board. The appropriateness of the name appears when the resemblance of the instrument to the letter T is noticed. If the edge of the drawing-board along which the T□ head is slid be truly straight, and lines be drawn along the same edge of the blade when the straight edge of the head is put in contact with the straight edge of the board at different places, these lines will be parallel. (Why?) Other very common instruments used frequently in drawing parallels are the triangles, mentioned in Art. 127, page 40. They are used in pairs or else one is used singly in connection with a ruler. The principle upon which depends their use in drawing parallels is the same as that upon which depends the use of the T□ for like purpose.

Q. and E. 198.—With two triangles or with a triangle and a ruler, devise a method of drawing through a given point a line parallel to a given line.

199.—Devise a method of dropping a perpendicular from a given point to a given line when the distance of the point from the line exceeds the length of any side of the right triangles at hand, and is also larger than the largest spread of the compass legs.

143.—Any plane figure bounded by sects is called a **polygon**; the bounding sects are its **sides**, their extremities the corners or **vertices**, and the interior angles at the vertices are the **angles of the polygon**.

144.—The sum of the sides of a polygon is called its **perimeter**.

145.—Any sect joining any two vertices not pertaining to the same side is called a **diagonal.**

146.—A polygon is **convex** when all its angles are convex; otherwise, it is **concave.**

147.—A concave angle of a polygon is sometimes called a **re-entrant angle.**

148.—If any two sides of a polygon intersect, it is called a **crossed polygon;** otherwise it is **non-crossed.** Polygons are to be understood to be non-crossed, unless the context clearly indicates that the most general meaning of the word is intended.

149.—With respect to the number of their angles, polygons are classified as **trigons, tetragons, pentagons, hexagons, heptagons,** or septagons, **octagons, nonagons, decagons, undecagons, dodecagons,** etc., according as they have three, four, five, six, seven, eight, nine, ten, eleven, twelve, etc., angles. Those of twenty angles are called **icosagons**. Those whose number of angles is represented by the algebraic number n we shall call enagons; *i. e.*, instead of the phrase "polygon of n sides" we shall use the word "enagon."

150.—Any polygon whose angles are all equal is called an **isogon**; one whose sides are all equal is called an **equilateral**. One both isogonic and equilateral is said to be **regular**.

151.—The ends of the links of a chain are its vertices; the angles at its vertices are its angles.

152.—A **chain** is **convex** when, of every four consecutive vertices, the sect joining the first to the third is crossed by the sect joining the second to the fourth; otherwise it is **concave.**

153.—Any **chain** is **isogonic** when it is convex and all its convex angles are equal. It is **equilateral** when its links are equal. When both isogonic and equilateral it is **regular.**

Q. and E. 200.—What is the sum of the angles of a pentagon? —of a hexagon?—of a heptagon?—of an octagon?—of an enagon?

201.—Compute the size of an angle in an isogonic trigon,—tetragon,—pentagon,—enagon.

202.—Draw an isogonic trigon,—tetragon,—hexagon,—octagon,—dodecagon.

203.—What regular polygon has each angle equal to $1 3/7$ times a right angle?—$1 5/9$?—$1 11/15$?

204.—How many re-entrant angles may a trigon have?—a tetragon?—a pentagon?—an enagon?

205.—Draw an equilateral trigon,—tetragon,—pentagon,—hexagon.

206.—Can an equilateral trigon be concave?—an equilateral tetragon?—an equilateral pentagon?

207.—How many convex angles must there be in an equilateral tetragon?—pentagon?—hexagon?—heptagon?—enagon?

208.—What is the least number of convex angles which it is possible to have in any polygon?

154.—A **line** is **tangent** to a circle when it just touches the circle without crossing it. A tangent line thus has no points within the circle to which it is tangent.

155.—A **sect** is **tangent** to a circle when it has a point in common with the circle and its seat is tangent.

156.—Any **polygon** or **chain** is **circumscribed** about a circle when all its sides or links are tangent to the circle. It is **inscribed** in a circle when all its vertices are points on the circle.

157.—If any figure be $\left\{\begin{matrix}\text{inscribed in}\\ \text{circumscribed about}\end{matrix}\right\}$ a circle, the circle is $\left\{\begin{matrix}\text{circumscribed about}\\ \text{inscribed in}\end{matrix}\right\}$ the first-mentioned figure.

158.—The center of the circle $\left\{\begin{matrix}\text{inscribed in}\\ \text{circumscribed about}\end{matrix}\right\}$ any figure is called the $\left\{\begin{matrix}\textbf{in-center}\\ \textbf{circumcenter}\end{matrix}\right\}$ of the figure.

Q. and E. 209.—Draw the perpendicular bisectors of the links of a regular three-linked chain. What relation, if any, exists among them?

210.—Is a regular chain inscriptible; *i. e.*, capable of being inscribed in a circle?

211.—If so, show how to find its circum-center; if not, show why the circle through any three consecutive vertices will not also pass through the fourth.

212.—Draw the bisectors of the angles of a regular four-linked chain. What relation, if any, exists among them?

213.—May a circle be inscribed in any regular chain?

214.—If so, show how to find the in-center of any such chain; if not, show why the circle touching any three consecutive links will not also touch the fourth.

215.—If a circle may be circumscribed about any regular chain, and another may be inscribed in such chain, what relation exists between their centers?

216.—Same as ex. 209 to ex. 215, for regular polygons.

217.—Is an inscribed equilateral chain regular or not?

218.—Same for isogonic inscribed chain.

219.—Is a circumscribed equilateral chain regular or not?

220.—Same for circumscribed isogonic chain.

221.—Same as q. 217 to q. 220, for polygons.

222.—Are any tetragons inscriptible?

223.—Are any capable of being circumscribed about a circle?

224.—If some tetragons are inscriptible, and others are not, state the determining conditions so far as you can.

225.—Same for those capable of being circumscribed about a circle, if there are any such.

226.—Are any regular polygons sym- $\left\{ \begin{array}{c} \text{centric} \\ \text{axic} \end{array} \right\}$?

227.—If some are, are all?

228.—If some are and others are not, state the determining conditions for each sort of symmetry.

229.—If there are any sym-centric regular polygons, what relation exists between the sym-center and the in-center?

230.—How many sym-axes, if any, has a regular enagon?

231.—Same as q. 226 to q. 230 for regular chains.

159.—A perpendicular let fall from the **in-center** of any regular polygon or chain upon any side or link is called an **apothem.**

160.—A sect drawn from the **circumcenter** of any regular polygon or chain to any vertex is called a **radius.**

Q. and E. 232.—What relation, if any, exists among the various apothems of a regular polygon or chain?—radii?

233.—At what point does an apothem meet the side or link to which it is drawn?

234.—What relation exists between the radii of a regular polygon or chain and the angles to whose vertices they are drawn?

161.—When we speak of *the radius* or *the apothem* of a regular polygon or chain, we mean *the length* of a radius or of an apothem as the case may be.

162.—The angle included by any two radii drawn to consecutive vertices of a regular polygon or of a regular chain is the **centric angle** of such **polygon** or **chain.**

Q. and E. 235.—What relation, if any, exists among the various centric angles of any regular polygon or regular chain?

236.—What is the size of a centric angle of a regular trigon?—tetragon?—pentagon?—enagon?

163.—Tetragons are sometimes called **quadrilaterals,** also **quadrangles;** the reasons for these names are readily apparent. Tetragons are usually classified with respect to their sides.

164.—Any tetragon in which **no two sides** are **parallel** is called a **trapezium.**

165.—Any one in which **two sides** are **parallel** and **unequal** in length is called a **trapezoid.** The parallel sides of a trapezoid are called its **bases**; the other two its **legs.** When the two legs are equal the trapezoid is **isosceles**; otherwise, it is **scalene.** The line joining the middle points of the legs of a trapezoid is called the **median** of the trapezoid. The perpendicular distance between the bases of a trapezoid is the **altitude** of it.

166.—Any tetragon in which **two sides** are **parallel** and **equal** in length, is a **parallelogram.** The two parallel sides, upon one of which the parallelogram is supposed to rest, are the **bases.** The other two sides are the **legs** When the two legs are equal, the parallelogram is **isosceles**; otherwise, it is **scalene.**

167.—Parallelograms whose angles are right angles are called **rectangles**, others are called **oblique parallelograms.** When the word parallelogram is used hereafter, an oblique non-equilateral parallelogram, or **rhomboid** as it is frequently called, is to be understood unless the context indicates that the general meaning is intended.

168.—Equilateral parallelograms are called **rhombuses** when **oblique, squares** when **rectangular.**

Q. and E. 237.—Draw a trapezium,—a trapezoid,—an isosceles trapezoid,—a crossed trapezoid. Are isosceles crossed trapezoids possible?—are scalene?

238.—Draw a parallelogram,—an isosceles parallelogram,—a crossed parallelogram,—a scalene parallelogram. Are isosceles non-crossed parallelograms possible?—scalene? Are isosceles crossed?—scalene?

239.—Draw a rhombus,—a crossed rhombus.

240.—Draw a rectangle,—a square.

241.—Draw the diagonals of each of the figures above called for. How many in each?

242.—If two consecutive sides of a trapezium are equal, and the other two are also equal, what relation, if any, exists between the diagonals?

243.—In such a figure, in what way do the diagonals divide the angles from whose vertices they are drawn?

244.—Can such a trapezium be a crossed trapezium?

245.—How many sym-axes, if any, has such a trapezium? If there be any, show how to draw them (or it, if there be but one).

246.—How many sym-centers, if any, has such a trapezium?

247.—What relation, if any, exists between the angles at a base of an isosceles trapezoid?—scalene?

248.—What relation, if any, exists between the angles adjacent to a leg of a trapezoid?

249.—How many sym-axes has an isosceles trapezoid, if any? —sym-centers?

250.—Same for scalene trapezoid.

251.—Are your conclusions in q. 247 to q. 250 true for crossed trapezoids?

252.—What relation exists between the length of a median and the lengths of the two bases of a trapezoid?

253.—How far would the median of a trapezoid have to be produced to meet the longer base, produced if necessary?

254.—At which of its points is a diagonal of a trapezoid crossed by the median?

255.—Are the two diagonals and the median of a trapezoid concurrent lines?

256.—If not, on which side of the median is the intersection of the diagonals?

257.—What relation is there between the lengths of the two diagonals of a scalene trapezoid?—of an isosceles?

258.—What relation exists between any two consecutive angles of a parallelogram?—any two alternate angles?

259.—What relation exists between any pair of alternate sides in a non-crossed parallelogram? The answer to this shows the reason for the name.

260.—Does it make any difference which two opposite sides of a parallelogram are taken for the bases?

261.—What relation does the point of intersection of the two diagonals bear to each?

262.—How do the diagonals divide the angles from whose vertices they are drawn?

263.—What relation exists between the two trigons into which a diagonal of a parallelogram divides it?

264.—What relation exists between the lengths of the diagonals of a parallelogram?

265.—Are the diagonals ever perpendicular?

266.—How many sym-axes has a parallelogram?—sym-centers?

267.—Same as q. 258 to q. 266 for rhombus.

268.—Same as q. 258 to q. 266 for rectangle.

269.—Same as q. 259 to q. 266 for square.

270.—Representing the consecutive sides of a tetragon by a, b, c, and d, the angle between any two, a and b for instance, by $\angle ab$, and the diagonal joining the vertices of $\angle ab$ and $\angle cd$ by g_1, the other by g_2, state the conditions necessary to the equality of two trapeziums;—of two trapezoids;—of two parallelograms;—of two rhombuses;—of two rectangles;—of two squares.

169.—Two **rectilinear figures** are **similar** when the angles of one are respectively equal to those of the other and arranged in the same order, while the sides of one are proportional to the sides of the other and arranged in the same order with respect to the angles.

170.—Any $\left\{\begin{matrix}\text{angle}\\ \text{side}\end{matrix}\right\}$ of one and its $\left\{\begin{matrix}\text{equal}\\ \text{proportional}\end{matrix}\right\}$ in the other of two similar figures are said to be **homologous** when their relative positions in the figures are the same. The vertices of any two homologous angles are homologous. If two things are homologous, either is the **homologue** of the other.

171.—Two **similar figures** are **similarly placed** when the lines through any three vertices of one and their respective homologues are concurrent (the center of concurrence not being between any two homologous vertices,) and the distances of the three vertices of one figure from the center of concurrence are proportional to those of their homologues.

172.—Any point of one of two similar figures is homologous to a certain point of the other if when the two figures are similarly placed the sects joining the point in one to any two of its vertices are parallel to the sects joining the point in the other to the homologous vertices.

173.—A sect in one of two similar figures is homologous to one in the other if the extremities of the first are homologous to those of the second.

174.—Two **chains** are **similar** when upon drawing their chords (*i. e.*, the lines connecting their extremities) two similar polygons result, of which the chords are homologous sides. Any two things of the chains are homologous if they are homologous things in the similar polygons which result when the chords are drawn.

175.—The **ratio of similitude,** or the **linear ratio,** between two similar figures is the ratio of any side of the first to its homologue of the second.

176.—Shape being dependent upon the relative positions of the different parts of a figure with respect to each other, and these relative positions being the same in any two similar figures, it is frequently said that **similar figures have the same shape.**

Q. and E. Make at least six different comparisons in each case before announcing any conclusion concerning the relations below inquired about.

271.—If two unequal trigons are mutually isogonic (*i. e.*, have the angles of one respectively equal to those of the other), are they similar or not?

272.—Same for two tetragons.

273.—Same for two pentagons.

274.—Same for any two polygons of more than three sides.

275.—Same for two trigons having two angles of one respectively equal to two of the other. If so, show why; if not, give the conditions which determine in which of the two trigons the third angle is the larger.

276.—If two trigons have one angle of one equal to one angle of the other, and the two sides including that angle in the first proportional to the two sides including its equal in the second, are or are not the trigons similar?

277.—If two trigons have the sides of one respectively proportional to those of the other, are the trigons similar or not similar?

278.—Are two tetragons similar or not, under like conditions? —two pentagons?—any two polygons of more than three sides?

279.—If the altitude upon the hypotenuse of a right trigon be drawn, into what sort of trigons does it divide the original trigon?

280.—What relation does each bear to the other and to the original trigon?

281.—Of the two segments into which this altitude divides the hypotenuse, what relation does the ratio of one segment to the altitude bear to the ratio of the altitude to the other segment?

282.—Knowing that the three vertices of a right trigon are equi-distant from the middle point of the hypotenuse, devise some means of constructing a mean proportional between two given sects.

283.—What relation does the ratio between the perimeters of two similar trigons bear to their ratio of similitude?

284.—Same for homologous altitudes.

285.—Same for any two homologous sects.

286.—Same as q. 283 to q. 285, for similar polygons of any number of sides.

287.—If any two similar polygons be divided into trigons by drawing homologous diagonals, what relation exists between the trigons of one polygon and those of the other?

288.—What relation exists between the arrangement in one set and that in the other?

289.—If any two polygons be made up of similar trigons, those in one being respectively similar to those in the other with the same ratio of similitude throughout, and occupying the same relative positions, what relation exists between the polygons?

290.—What relation does the map of a plane surface bear to the surface of which it is a map?

291.—If the distance between two places be $24\frac{1}{2}$ miles, what will be the number of inches between their representatives on a map drawn to a scale of 1 to 10000?

177.—If through any center of concurrence, O, an indefinite number of rays be drawn, *a*, *b*, *c*, etc., and on each ray a pair of points be taken, A_1 and A_2 on *a*, B_1 and B_2 on *b*, etc. so that the ratios $\frac{OA_1}{OA_2}$, $\frac{OB_1}{OB_2}$, etc. are all equal, and so that if O is between one pair it shall be between every other pair, the figure composed of the points A_1, B_1, C_1, etc. is said to be **homothetic** to that composed of the points A_2, B_2, C_2, etc.; the center, O, is called the **homothetic center,** or the **center of homothesy**; the rays *a*, *b*, *c*, etc. are called **homothetic rays**; and the ratio $\frac{OA_1}{OA_2}$ is called the **homothetic ratio** or the **ratio of homothesy.** The figures are **directly homothetic** when the center is not between any pair of corresponding or homologous points such as A_1 and A_2. When it is between such a pair, the figures are **inversely homothetic.** The relation existing between two figures by virtue of which they are homothetic is called **homothesy.**

Q. and E. 292.—If one of two homothetic figures is a straight line, what is the other figure? What relation does it bear to the first in size?—in position?

293.—If one of two homothetic figures is a polygon, what kind of figure is the other?

294.—How is it placed with respect to the first when the two are directly homothetic?—inversely?

295.—What relation does the ratio of any side of the first to its homologue in the second bear to the homothetic ratio?

296.—Making use of the relations existing between homothetic figures, devise a method of drawing a polygon similar to a given polygon, and having a given ratio of similitude to it, using a center of homothesy.

297.—If F′ and F″ are two figures, both homothetic to a third figure F, but about different centers, O″ and O′ respectively, are or are not F′ and F″ homothetic to each other? Try half a dozen different cases before announcing a conclusion.

298.—If so, what relation does their center of homothesy, O, bear to the other two centers, O″ and O′?

299.—If not, show what is lacking to their being homothetic to each other.

PROBLEMS.

178.—A **problem** is a proposition or statement of something proposed or commanded to be done.

179.—The **solution** of a problem is the actual accomplishment of the thing commanded, or else the method of the accomplishment.

180.—The **discussion** of a problem is the examination of the reasons underlying the solution and of the cases in which more than one solution may be possible or in which no solution may be possible.

181.—In solving problems, **loci should be used whenever possible.** Occasionally a solution is indicated through a consideration of the relations which would hold between given things and required things if the solution had actually been obtained. The use of one or more of these relations in connection with known, or given, things may lead to the desired solution.

Give full solutions and discussions of the following problems.

Find a point X which shall be

300.—Equidistant from three given points, A, B, and C.

301.—Equidistant from two given points, A and B, and at a given distance from a third point, C.

302.—At given distances from two given points, A and B.

303.—Equidistant from two given points, A and B, and on a given line *m*.

304.—Equidistant from two given points A and B, and at a given distance from a given line *m*.

305.—Equidistant from two given points A and B, and also from two given lines *m* and *n*,—(i), parallel; (ii), intersecting.

306.—At a given distance from a given point A, and on a given line *m*.

307.—At a given distance from a given point A, and at a given distance from a given line *m*.

308.—At a given distance from a given point A, and equidistant from two given lines, *m* and *n*,—(i), parallel; (ii), intersecting.

309.—Equidistant from three given lines,—(i), three lines parallel; (ii), two lines parallel; (iii), no two lines parallel; (iv), three lines concurrent.

310.—On a given line and equidistant from two others,—(i), parallel; (ii), intersecting.

311.—At a given distance from one given line and equidistant from two others,—(i), parallel; (ii), intersecting.

312.—At given distances from two given lines,—(i), parallel: (ii), intersecting.

313.—Draw the complete locus of a point which shall be two times as far from one of two given lines as from the other, (i) lines parallel, (ii) lines intersecting;—three times;—four times;—five times. See ex. 183 and 184, page 48.

314.—Two times as far from *a* and three times as far from *b* as from *c*,—(i) *a*, *b*, and *c* parallel,—(ii) *a* and *b* parallel,—(iii) no two lines parallel,—(iv) *a*, *b*, and *c* concurrent.

315.—In one side of a trigon and equidistant from the other two;—twice as far from the second as from the third;—three times.

316.—Find a point which shall divide one side of a trigon into segments proportional to the other two sides. See ex. 101, page 39.

317.—Draw a line across a trigon so that it shall be parallel to the base and the segment included between the legs (or the legs produced) shall have a given length;—

318.—shall be equal to the sum of the segments of the legs included between it and the base.

319.—Draw a sect which shall be parallel to a given line *a*, have its extremities in two other given lines *b* and *c*, and have a given length,—(i), *a*, *b*, and *c* parallel; (ii), only *b* and *c* parallel; (iii), only *a* and *b* parallel; (iv), no two lines parallel, and lines non-concurrent; (v), *a*, *b*, and *c* concurrent.

320.—Find the path of the middle point of a sect whose ends move in right lines that are perpendicular to each other. See q. 111, page 40.

321.—If P and Q are two points on the same side of a given line x, draw the shortest two-linked chain which shall connect P and Q and touch x. What relation exists between the acute angles its links make with x?

322.—Given any point P between the sides of an angle, to draw a sect terminating in those sides and bisected at P. See q. 182, page 48.

In the exercises following, a, b, and c stand for the sides of a trigon,—α, β, and γ for the angles opposite them respectively,—h_a, h_b, and h_c for the altitudes to them respectively,—m_a, m_b, m_c, the respective medians,—d_a, d_b, d_c, the respective bisectors,—p, the perimeter,—r, the radius of the inscribed circle,—R, the radius of the circumscribed circle.

Construct the trigon which shall have given values for

323. a, b, and c, or a, b, and p.

324. a, b, and m_a.

325. a, b, and h_a.

326. a, b, and α.

327. a, b, and γ.

328. a, m_a, and m_b.

329. a, m_a, and h_a.

330. a, m_a, and β.

331. a, m_b, and m_c.

332. a, m_b, and h_a.

333. a, m_b, and γ.

334. a, h_a, and β.

335. a, d_b, and β.

336. a, d_b, and γ.

337. a, α, and β.

338. a, β, and γ.

339. a, β, and p.

340. m_a, m_b, and m_c;—consider the trigon formed by drawing through the extremities of one median of a trigon lines parallel to the other two.

341. m_a, m_b, and h_a.

342. m_a, m_b, and h_c;—consider the relation between the altitude upon any side of a trigon and the distance from that side to the point of concurrence of two medians.

343. m_a, h_a, and β.

344. m_a, h_b, and α.

345. m_a, α, and β.

346. m_a, β, and γ.

347. h_a, h_b, and γ.

348. h_a, d_a, and α.

349. h_a, d_a, and β.

350. h_a, d_b, and β.

351. h_a, α, and β, or h_a, β, and γ.

352. h_a, β, and p.
353. d_a, α, and β, or
354. d_a, β, and γ.
355. α, β, and p.
356. a, $b-c$, and β.
357. a, $b-c$, and γ.
358. a, $b-c$, and p.
359. a, $b-c$, and $\beta-\gamma$.
360. a, $\alpha+\beta$, and $\alpha-\beta$.
361. a, $\alpha+\beta$, and $\beta+\gamma$.
362. a, $\alpha+\beta$, and $\beta-\gamma$.
363. a, $\alpha-\beta$, and $\beta+\gamma$.
364. $b-c$, α, and β.
365. $b-c$, α, and γ.
366. $b-c$, β, and γ.
367. $b-c$, α, β (or γ), and $\alpha\pm\beta$, (or $\beta\pm\gamma$).
368. m_a, (or m_b or m_c), $m_a\pm m_b$, and $m_b\pm m_c$.

In the isosceles trigons below called for, b represents a leg c the base.

Construct the isosceles trigon which shall have given values for

369. b and c or b and p.
370. b and m_b.
371. b and m_c.
372. b and β
373. b and γ.
374. m_b and h_c.
375. m_b and β.
376. m_b and γ.
377. m_b and m_c.
378. h_b and β.
379. h_b and γ.
380. h_b and h_c.
381. d_b and β.

382. d_b and γ.

383. β and p.

384. γ and p.

385. $b-c$ and γ.

386. $b-c$ and β.

387. $b-c$ and p

388. c and h_c.

389. c and γ.

390. c and β.

391. c and p.

392. c and m_b.

In the right trigons below called for, a and b are the legs, c is the hypotenuse.

Construct the right trigon which shall have given values for

393. a and b.

394. a and c.

395. a and p.

396. a and $c-b$.

397. a and h_c.

398. a and m_a.

399. a and m_b.

400. a and m_c.

401. a and d_b.

402. a and d_c.

403. a and α.

404. a and β.

405. a and $\alpha-\beta$.

406. c and p.

407. c and $b-a$.

408. c and α.

409. c and $\alpha-\beta$.

410. p and α.

411. $c-a$ and α.

412. $c-a$ and β.

413. $c-a$ and $\alpha-\beta$.

414. $b-a$ and α.

415. $b-a$ and β.

416. h_c and m_a.

417. h_c and m_c.

418. h_c and d_c.

419. h_c and α.

420. h_c and $\alpha-\beta$.

421. m_a and m_c.

422. m_a and α.

423. m_a and β.

424. d_a and α.

425. d_a and β.

426. d_c and α.

427. d_c and $\alpha-\beta$.

Construct the isogonic trigon which shall have a given value for

428. a.

429. p.

430. h.

431. $a-h$.

432. $a+h$.

433. $p-h$.

434. $p+h$.

If a, b, c, and d are consecutive sides of a tetragon, p is the perimeter, $\angle ab$ is the angle between the sides a and b, g_1 is the diagonal joining the vertex of $\angle ab$ to that of $\angle cd$, and g_2 the other diagonal, construct the trapezium having given values for

435. a, b, c, d, and g_1,

436. a, b, c, d, and $\angle ab$,

437. a, b, c, g_1, and g_2,

438. a, b, c, g_1, and $\angle ab$,

439. a, b, c, g_1, and $\angle cd$,

440. a, b, c, g_1, and $\angle da$,

441. $a, b, c, \angle ab$, and $\angle bc$,

442. $a, b, c, \angle ab$, and $\angle cd$,

443. $a, b, c, \angle ab$, and $\angle da$,

444. $a, b, c, \angle cd$, and $\angle da$,

445. a, b, p, g_1, and $\angle bc$,

446. a, b, p, g_2, and $\angle bc$,

447. $a, b, p, \angle ab$, and $\angle bc$,

448. $a, b, g_1, \angle ab$, and $\angle bc$,

449. $a, b, g_1, \angle bc$, and $\angle cd$,

450. $a, b, g_2, \angle bc$, and $\angle cd$,

451. $a, b, \angle ab, \angle bc$, and $\angle cd$,

452. a, c, p, g_1, and $\angle bc$,

453. a, c, p, g_2, and $\angle cd$,

454. $a, c, g_1, \angle ab$, and $\angle bc$,

455. $a, c, g_1, \angle bc$, and $\angle cd$,

456. $a, c, g_2, \angle bc$, and $\angle cd$,

457. $a, c, \angle ab, \angle bc$, and $\angle cd$,

458. $a, p, g_1, \angle ab$, and $\angle da$,

459. $a, p, \angle ab, \angle bc$, and $\angle cd$,

460. $a, p, \angle ab, \angle cd$, and $\angle da$,

461. $a, g_1, g_2, \angle ab$, and $\angle bc$,

462. $a, g_1, g_2, \angle ab$, and $\angle da$,

463. $a, g_1, \angle ab, \angle bc$, and $\angle cd$.

In the exercises below given in the construction of trapezoids, a and c are the two bases, and $a > c$. The symbols have the meanings assigned in the exercises in the construction of tetragons, so far as they correspond. Of the other symbols, m represents the median, h the altitude. Construct the trapezoid which shall have given values for

464. a, b, c, and g_1,

465. a, b, c, and g_2,

466. a, b, c, and $\angle ab$,

467. a, b, c, and h,

468. a, b, d, and $\angle ab$,

469. a, b, d, and h,
470. a, b, p, and g_2,
471. a, b, p, and $\angle ab$,
472. a, b, p, and $\angle da$,
473. a, b, p, and h,
474. a, b, g_1, and g_2,
475. a, b, g_1, and m,
476. a, b, g_1, and $\angle ab$,
477. a, b, g_1, and $\angle da$,
478. a, b, g_1, and h,
479. a, b, g_2, and $\angle da$,
480. a, b, g_2, and m,
481. a, b, $\angle ab$, and $\angle da$,
482. a, b, $\angle ab$, and m,
483. a, b, $\angle da$, and m,
484. a, b, $\angle da$, and h,
485. a, c, d, and g_1,
486. a, c, d, and $\angle ab$,
487. a, c, g_1, and $\angle ab$,
488. a, c, g_1, and h,
489. a, c, $\angle ab$, and $\angle da$,
490. a, c, $\angle ab$, and h,
491. a, p, g_1, and h,
492. a, p, $\angle ab$, and $\angle da$,
493. a, p, $\angle ab$, and h,
494. a, p, m, and h,
495. a, g_1, g_2, and $\angle ab$,
496. a, g_1, g_2, and h,
497. a, g_1, $\angle ab$, and $\angle da$,
498. a, g_1, $\angle ab$, and m,
499. a, g_1, $\angle ab$, and h,
500. a, g_1, $\angle da$, and m,
501. a, $\angle ab$, $\angle da$, and m,

502. a, $\angle ab$, $\angle da$, and h,

503. a, $\angle ab$, m, and h,

504. b, d, g_1, and $\angle ab$,

505. b, d, g_1, and h,

506. b, d, $\angle ab$, and m,

507. b, d, m, and h,

508. b, p, g_1, and $\angle ab$,

509. b, p, g_1, and h,

510. b, p, $\angle ab$, and $\angle da$,

511. b, p, $\angle ab$, and m,

512. b, p, $\angle da$, and h,

513. b, p, m, and h,

514. b, g_1, g_2, and $\angle ab$,

515. b, g_1, g_2, and h,

516. b, g_1, $\angle ab$, and $\angle da$,

517. b, g_1, $\angle ab$, and m,

518. b, g_1, m, and h,

519. b, $\angle ab$, $\angle da$, and m,

520. b, $\angle da$, m, and h,

521. p, g_1, g_2, and h,

522. p, $\angle ab$, $\angle da$, and m,

523. p, $\angle ab$, $\angle da$, and h,

524. p, $\angle ab$, m, and h,

525. g_1, $\angle ab$, $\angle da$, and h,

526. g_1, $\angle ab$, m, and h,

527. $\angle ab$, $\angle da$, m, and h.

Construct the rhomboid which shall have given values for

528. a, b, and g_1,

529. a, b, and $\angle ab$,

530. a, b, and h,

531. a, p, and g_1,

532. a, p, and $\angle ab$,

533. a, p, and h,

534. a, g_1, and g_2,

535. a, g_1, and $\angle ab$,

536. a, g_1, and h,

537. a, $\angle ab$, and h,

538. a, $\angle ab$, and g_2-b,

539. a, $\angle ab$, and g_1-b,

540. p, g_1, and h,

541. p, g_1, and $a-b$,

542. p, $\angle ab$, and h,

543. p, $\angle ab$, and $a-b$,

544. p, h, and $a-b$,

545. p, $a-b$, and g_1-a,

546. g_1, g_2, and h,

547. g_1, $\angle ab$, and h,

548. g_1, h, and $a-b$,

549. $\angle ab$, h, and g_1-a.

Construct the rhombus which shall have given values for

550. a and $\angle ab$,

551. a and g_1,

552. a and g_1-g_2,

553. a and h,

554. $\angle ab$ and g_1,

555. $\angle ab$ and g_1-g_2

556. $\angle ab$ and h,

557. g_1 and g_2,

558. g_1 and h.

Construct the rectangle which shall have given values for

559. a and b,

560. a and p,

561. a and g_1,

562. b and g_1-a.

563. p and g_1,

564. p and $a-b$,

565.—g_1 and $a-b$.

Construct the square which shall have a given value for

566. a,

567. g,

568. $a+g$,

569. $p+g$,

570. $g-a$,

571. $p-g$.

A **polygon** is said to be **inscribed in another** when its vertices are points on the sides of that other. Make use of the principles of homothesy in solving following problems.

572.—Devise a method by which to inscribe in a given trigon one similar to another given trigon, so that a designated side of the inscribed trigon shall be parallel to a given line. How many solutions? Does the number of solutions depend upon the characters of the given trigons? Give a complete discussion.

573.—Devise a method by which to inscribe a square in a given trigon. How many solutions? Does the character of the trigon have anything to do with the number of solutions? Do the inscribed squares differ in size, if more than one may be inscribed?

574.—Same for inscribing rectangle, and similar questions.

575.—Same for inscribing parallelogram, with similar questions. What relation must exist between the angles of the trigon and those of the given parallelogram, in order that there may be a solution?

576.—Same for inscribing trapezoid, with similar questions.

577.—Same for inscribing trapezium, with similar questions.

578.—Is it always possible to draw a tetragon which shall be similar to a given tetragon and have its vertices on three given lines,—(i), lines all parallel; (ii), only two lines parallel; (iii), non-concurrent, and no two parallel; (iv), concurrent? If possible in some of these cases and impossible in others, distinguish those in which it is possible from those in which it is not, and give reasons for the distinction.

579.—Same for three given lines and given trigon, with the restriction that a designated side of resulting trigon shall be parallel to a fourth given line parallel to none of first three.

580.—Similar to q. 578, but for pentagon and four given lines, —(i), four lines parallel; (ii), only three lines parallel; (iii), two lines parallel to each other, and other two lines parallel to each other but not to first two; (iv), only two lines parallel, no three lines concurrent; (v), two lines parallel, three lines concurrent; (vi), no two lines parallel, no three concurrent; (vii), no two parallel, only three concurrent; (viii), four lines concurrent. Make distinctions between possible and impossible cases, if there are such, and give reasons.

CHAPTER II.

AREAL RELATIONS.

182.—As has before been said, things are measured only by comparing them with others of the same character. The area or extent of any surface is measured, then, only by comparing it with the extent of some other surface whose extent is taken as unit.

183.—In practice, we almost always take for **unit of area** that of a **square** whose side has **unit length**; this is done because, for most purposes, such a unit is the most convenient one that could be employed, as will appear later. When no other unit of area is expressed, the unit of area is to be understood to be that of a square the length of one side of which is equal to the unit of length used.

184.—Although, as has just been said, **areas** are found by comparison, they are **found always by indirect comparison**. Our direct comparisons are always of one line with another, and from the results of these we compute the area in any case. As will appear later, whatever our direct measurements may be, they reduce ultimately to finding the lengths of two lines perpendicular to each other. These two lengths are called the

dimensions (*i. e.*, **measurements**) of the figure to which they pertain. It is true that in some figures of special shape, *e. g.*, squares and circles, one measurement seems sufficient, but in reality besides the one direct measurement we need to know the additional fact that the second direct measurement, if made, would give a result bearing a definite known relation to that of the first.

185.—The dimensions of a surface are named **length** and **breadth.** If they are unequal, the name length is to be applied to the larger unless the contrary is stated or implied. The word **width** sometimes takes the place of the word breadth in geometric usage.

186.—When we speak of the **products** or of the **quotients** of sects, limited surfaces, etc., or of lengths, breadths, areas, etc., we mean the products or the quotients of their respective enumerators. See Art. 16, page 4.

187.—The dimensions of a trigon or of a parallelogram are the length of its base and that of its altitude; those of a trapezoid are the length of its median and that of its altitude. Other figures than those of these three classes have their areas found by being divided into parts coming under one or more of these classes, the sum of the areas of which gives the area desired.

188.—By the **rectangle upon two given sects** is meant a rectangle whose base and altitude are equal to the given sects respectively. By the rectangle *of* two given sects is meant the product of those two sects. Similarly for the **square upon any sect**, and the square *of* any sect.

Q. and E. 581.—What is the ratio between the areas of two rectangles having equal altitudes, if their bases are equal?—if the base of the first is two times that of the second?—three times?—seven and a half times?—$26\frac{1}{3}$ times?—n times?

582.—What relation exists between the area of a rectangle and the product of its dimensions?

583.—In order that your answer to the preceding question may be true, is it necessary that both dimensions be expressed in the same unit, or not? If so, what relation does the unit of area in this case bear to the unit of length?

584.—Find the dimensions of ten different rectangles, the area of each of which shall be 360 square feet.

585.—If the area of a rectangle and one dimension of it be given, how may the other dimension be found?

586.—Why should the product of a number by itself be called the square of the number?

587.—Why should the product of two numbers be called the rectangle of those numbers?

588.—Find the relation existing between the square upon the sum of two sects and the rectangle and squares upon those two sects.

589.—Same for the square upon the difference between two sects.

590.—Find the relation existing between the rectangle upon the sum of two sects and their difference, and the squares upon those two sects.

591.—Compare the results of the last three exercises with the algebraic formulæ for $(a+b)^2$, $(a-b)^2$, and $(a+b)(a-b)$.

592.—Draw a rectangle and an oblique parallelogram, having equal bases and equal altitudes. What relation exists between their areas? Which is the larger? Make half a dozen comparisons before announcing any definite conclusion.

593.—How may the area of a parallelogram be found when its dimensions are known?

594.—What relation exists between the ratio of the areas of two parallelograms and the ratio of their altitudes when their bases are equal?—the ratio of their bases when their altitudes are equal?

595.—What relation exists between the area of a trigon and that of a parallelogram having two sides in common with the trigon?—between the altitudes of the two upon their common base?

596.—How may the area of a trigon be found when its two dimensions are known? Give the reason for your answer.

597.—If one side of a trigon is twice as large as another, what is the ratio of the altitudes upon those sides?—if n times?

598.—In a scalene trigon can two altitudes be equal?—three?

599.—In any trigon in which no two altitudes are equal, to which side does the longest altitude pertain?—the shortest?

600.—If two altitudes of a trigon are equal what kind of a trigon is it?—what, if all three are equal?

601.—What is the ratio between the areas of two trigons having equal bases and equal altitudes upon those bases?—equal bases and unequal altitudes upon them?—equal altitudes and unequal bases?—the same base and their vertices on a line parallel to the base?—the same vertex and their bases upon the same line? Give your reason for your answer in each case.

602.—In two trigons, if $a_1 = a_2$, and $\gamma_1 = \gamma_2$, but $b_1 >$ or $< b_2$, what relation exists between the ratio of the area of the first to that of the second and the ratio of b_1 to b_2?—which ratio is invariably the larger?

603.—Extend your conclusion in the preceding to the case where $\gamma_1 = \gamma_2$ but $a_1 >$ or $< a_2$, and $b_1 >$ or $< b_2$;—also to that in which $\gamma_1 + \gamma_2 = 180°$, and $a_1 >$ or $< a_2$ and $b_1 >$ or $< b_2$.

604.—If two equivalent trigons have a common base and lie on the same side of it, what relation does the sect joining their two vertices bear to the common base? Give your reason.

605.—If through the middle point of one leg of a trapezoid a sect be drawn parallel to the other leg, and extended until it meets the longer base and the shorter base produced, thus cutting off a trigon in one place and adding one in another, what kind of figure will result?

606.—What relation does its base bear to the median of the trapezoid?—its altitude to the altitude of the trapezoid? —its area to the area of the trapezoid? Give your reasons for thinking so.

607.—How may the area of a trapezoid be found when its two dimensions are known?—when its altitude and its two bases are known? Give reasons.

608.—If all the radii of a regular polygon be drawn, into what sort of figures is the polygon divided?

609.—How may the area of a regular polygon be found when its apothem and its perimeter are known?

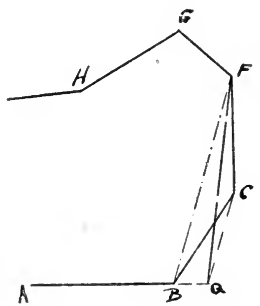
610.—How may the area of a polygon circumscribed about a circle be found when the perimeter of the polygon and the radius of the inscribed circle are known?

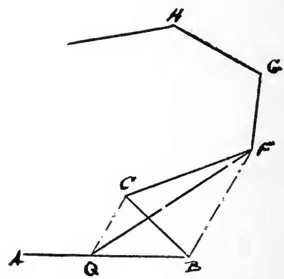
611.—If the two diagonals of a tetragon are perpendicular to each other, how may the area of the figure be found when the lengths of these two diagonals are known?

612.—Does your answer to the preceding question hold true in the case of a re-entrant tetragon with perpendicular diagonals?

189.—The area of a trapezium or of any irregular polygon of five or more sides may be found by drawing a sufficient number of diagonals to divide the polygon into trigons, and taking the sum of the areas of these trigons. Usually however it is more convenient to draw the longest diagonal and drop perpendiculars upon it from the various vertices, thus dividing the polygon into trapezoids and right trigons, the sum of whose areas is the area required.

190.—Occasionally it is desired to reduce a polygon to an equivalent trigon by a purely graphical method. The method followed in such case will now be considered. Suppose A, B, C, F, G, H, etc., the consecutive vertices of the polygon to be reduced and suppose that AB is taken as the sect whose seat is to be the seat of the base of the required trigon. From B draw the diagonal to the second consecutive vertex after it, F in this case, thus dividing the polygon into the trigon BCF and the polygon ABFGH etc. Now if the trigon BCF can be replaced by an equivalent trigon which shall still have BF for one side but shall have its third vertex, Q, say, in AB or AB produced and on the same side of BF as C is, it is evident that the polygon AQFGH etc., will be equivalent to the original and will have one less side.



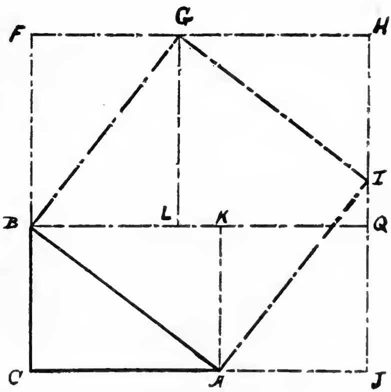


By a consideration of his reply to question No. 604, page 80, the student will see how to draw the sect CQ so that the desired relation between the trigons BCF and BQF shall be obtained. The process is precisely the same when the angle C is concave as when it is convex. Evidently the same sort of an operation may be performed on the polygon AQFGH etc., by which a polygon equivalent to the original and having two less sides will be obtained. By a sufficient number of repetitions of this operation, any polygon may be reduced to an equivalent trigon.

Q. and E. 613.—Reduce a convex heptagon to an equivalent trigon, and then reduce the resulting trigon to an equivalent rectangle.

614.—Reduce a hexagon with two concave angles to an equivalent rectangle.

191.—There is a very important relation existing among the three sides of a right trigon which will now be considered. Suppose ABC any right trigon, the



right angle having its vertex at C. Prolong CA to J making AJ equal to CB, and thus having CJ equal to the sum of CA and CB. Also prolong CB to F making BF equal to CA, and CF equal to CJ. Through F and J draw sects parallel to CJ and CF respectively, to meet at a point which call H. On the sect FH take G, and on the sect HJ take I, so that FG and HI shall each equal CB. Draw the three sects AI, IG, and GB. Draw a sect from B parallel to CJ and meeting JH at Q, and from A draw a sect parallel to CF meeting BQ at K, and from G draw one parallel to HJ meeting BQ at L. Then trace out the relations called for below.

Q. and E. upon preceding article.

615.—What kind of figure is AIGB and what relation does it bear to BA?

616.—What kind of figure is AJQK and what relation does it bear to JQ?—to BC?

617.—What kind of figure is QHGL and what relation does it bear to HG?—to CA?

618.—What relations do the trigons AJI, IHG, GFB, GBL, and AKB bear to the trigon ABC?

619.—What relation does the sum of the two tetragons AKBC and GFBL bear to the sum of the four trigons AJI, IHG, GFB, and BCA?

620.—If from the whole figure there be subtracted the four trigons just mentioned, there remains the tetragon AIGB, and if from the whole figure there be subtracted the two tetragons mentioned in q. 619, there remains a figure which is the sum of the two tetragons AJQK and QHGL. What relation exists between the tetragon AIGB and the sum of the tetragons AJQK and QHGL?

621.—What relation exists between the square upon the hypotenuse of a right trigon and the sum of the squares upon the two legs of the trigon?

622.—If the lengths of two of the sides of a right trigon be known, how may the length of the third side be computed,—(i) two legs given,—(ii) the hypotenuse and one leg given?

623.—What relation exists between the sum of the squares of two sides of a trigon and the square of the third side, if the first two sides include an acute angle?—a right angle?—an obtuse angle? See q. 164, page 46.

624.—If the lengths of the sides of a trigon are known, how may it be determined without drawing the trigon, whether it is acute or right or obtuse?

625.—Determine the character (with respect to angles) of the trigon whose sides are,—6, 7, 9; 5, 12, 13; 2, 3, 4; 5, 6, 8; 12, 12, 23; 6, 8, 10; 1, 2, 2½; 1½, 2, 2½; 16, 22, and 23.

192.—By means of the relation connecting the lengths of the legs of a right trigon and that of the hypotenuse, the student may easily deduce a formula for the length of the altitude upon any side of a trigon in terms of the three sides, and another for the area of the trigon in terms of the three sides. Thus, let a, b, and c be the sides, c being taken as base, and suppose the altitude h_c drawn. If $a >$ or $< b$, suppose $a > b$. The projection of a upon c call p, and that of b upon c call q. Now, if α is acute, $p+q = c$; if α is right, $p = c$ and $q = 0$, so that $p+q$ still $= c$; if α is obtuse, $p-q = c$. In all cases $p^2+h_c^2 = a^2$, and $q^2+h_c^2 = b^2$, so that $p^2-q^2 = a^2-b^2$.

When $p+q = c$, $p-q = \dfrac{a^2-b^2}{c}$, for $p-q = \dfrac{p^2-q^2}{p+q}$;

then $$(p+q)+(p-q) = 2p = c+\frac{a^2-b^2}{c} = \frac{c^2+a^2-b^2}{c},$$

or $$p = \frac{c^2+a^2-b^2}{2c}.$$

When $p-q = c$, $p+q = \dfrac{a^2-b^2}{c}$, and $2p = c+\dfrac{a^2-b^2}{c}$

$$= \frac{c^2+a^2-b^2}{c}, \text{ so that } p = \frac{c^2+a^2-b^2}{2c} \text{ as before.}$$

Now $h_c^2 = a^2 - p^2 = (a+p)\,(a-p)$

$$= \left(a+\frac{c^2+a^2-b^2}{2c}\right)\left(a-\frac{c^2+a^2-b^2}{2c}\right)$$

$$= \frac{2ac+c^2+a^2-b^2}{2c} \times \frac{2ac-c^2-a^2+b^2}{2c}$$

$$= \frac{(a+c)^2-b^2}{2c} \times \frac{b^2-(a-c)^2}{2c}$$

$$= \frac{(a+c+b)(a+c-b)}{2c} \times \frac{(b+a-c)(b-a+c)}{2c}$$

$$= \frac{(a+b+c)\,(a+b+c-2b)\,(a+b+c-2c)\,(a+b+c-2a)}{4c^2}.$$

Substitute s for the semi-perimeter, *i. e.*, put

$$s = \frac{a+b+c}{2}, \text{ or } 2s = a+b+c.$$

$$h_c^2 = \frac{(2s)(2s-2b)(2s-2c)(2s-2a)}{4c^2}$$

$$= \frac{16s(s-a)\,(s-b)\,(s-c)}{4c^2}$$

$$= \frac{4}{c^2}\Big[s(s-a)(s-b)(s-c)\Big], \qquad \text{whence}$$

$$h_c = \frac{2}{c}\sqrt{s(s-a)\,(s-b)\,(s-c)}. \tag{1}$$

Putting A for the area of the trigon, we have, since

$A = \frac{c}{2}h_c$, $A = \frac{c}{2} \times \frac{2}{c}\sqrt{s(s-a)\,(s-b)\,(s-c)}$, or,

combining, $A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$. (2)

On account of their great practical importance, formulæ (1) and (2) should be carefully remembered.

193.—By means of the relation among the three sides of a right trigon, a method of constructing a square equivalent to a given rectangle may be developed. Suppose a and b the legs of a right trigon, c the hypotenuse, h the altitude upon c, p the projection of a on c, and q the projection of b on c. Then we have $c = p+q$, and $c^2 = p^2+q^2+2pq$; but $c^2 = a^2+b^2$, and $a^2 = p^2+h^2$ while $b^2 = q^2+h^2$, so that $c^2 = p^2+h^2+q^2+h^2 = p^2+q^2+2h^2$. Then $2h^2 = 2pq$, or $h^2 = pq$; *i. e.*, the square upon the altitude drawn to the hypotenuse of a right trigon is equivalent to the rectangle upon the projections of the legs on the hypotenuse. Thus, to construct a square equivalent to a given rectangle, lay off upon a straight line two adjacent sects respectively equal to two consecutive sides of the given rectangle; at the junction point of these sects erect a perpendicular and upon it take such a point as may serve as the vertex of a right angle whose two sides shall pass through the outermost extremities of the sects before mentioned. This point determines such a portion of the perpendicular erected as will equal a side of the required square. A reference to q. 111, page 40, will suggest a means of determining the desired point upon the perpendicular.

194.—It will be noticed that in the discussion just given, since $pq = h^2$, h is a mean proportional between p and q; for, dividing each member by qh, we have $\frac{p}{h} = \frac{h}{q}$;

i. e., in any right trigon the projection of one leg on the hypotenuse is to the altitude upon the hypotenuse as that altitude is to the projection of the other leg on the hypotenuse. The method of finding the side of a square which shall be equivalent to a given rectangle also enables us then to find the mean proportional between two given sects. In this connection it is to be particularly noticed that if $h^2 = pq$, $\frac{p^2}{h^2} = \frac{p}{q}$; for when $h^2 = pq$, $\frac{p}{h} = \frac{h}{q}$. Multiply both sides of last equation by $\frac{p}{h}$ and there will result $\frac{p^2}{h^2} = \frac{p}{h} \times \frac{h}{q} = \frac{p}{q}$.

Q. and E. 626.—Which has the larger perimeter, a rectangle or its equivalent square?

627.—Reduce a given concave hexagon to an equivalent square.

628.—Devise a method for constructing graphically the square roots of 2, 3, 5, 6, 7, 8, and 10, assuming any convenient sect to represent unity. What relation exists between the lines you get representing the square roots of 2 and 8?—what among unity, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$?

195.—The ratio between the areas of two figures is called their **areal ratio.**

Q. and E. 629.—What relation, if any, exists between the areal ratio of two similar trigons and their ratio of similitude, or linear ratio?

630.—Same for similar polygons.

631.—Supply the missing words (if there are any) in the following propositions:—"The areas of two similar $\left\{\begin{matrix}\text{trigons}\\\text{polygons}\end{matrix}\right\}$ are proportional to .... any two homologous sects."

632.—Devise a method of constructing a polygon similar to a given polygon and bearing a given areal ratio to it,—

633.—similar to one of two given dissimilar polygons and equivalent to the other.

PROBLEMS.

*634.—Devise a method for transforming a given trigon into an equivalent trigon one angle of which shall equal a designated angle of the given trigon, and one side of which shall be adjacent to prescribed angle and shall equal a given sect;—

635.—Which shall have a given angle and a side of given length;—

636.—Which shall have two sides of given lengths.

637.—Devise a method of drawing from a given point in one side of a trigon a line such that it shall divide the trigon into two equivalent parts;—

638.—Two lines such that they shall divide the trigon into three equivalent parts;—three lines such that they shall divide it into four equivalent parts, etc.;—

639.—A line such that it shall divide the trigon into two parts whose areas are proportional to two given sects.

640.—Find a point within a trigon such that if lines be drawn from it to the vertices of the trigon, the trigon will be divided into three equivalent parts;—three parts whose areas shall be proportional to the sides of the original trigon pertaining to them.

641.—Transform a given parallelogram into an equivalent parallelogram,—(i), which shall have one of its angles equal to a given angle; (ii), which shall have one of its angles equal to a given angle and each leg equal to a given sect; (iii), which shall have its leg equal to a given sect and its base equal to another given sect.

642.—Transform the sum of two trigons into an equivalent square.

643.—Find a square equivalent to the sum of two given

* See Art. 190, page 81.

unequal squares,—equivalent to their difference. See q. no. 621, page 84.

644.—Find a square whose area shall be to that of a given square as one of two given sects is to the other.

645.—Find a trigon which shall be similar to two given similar trigons and equivalent to their sum,—to their difference.

646.—Transform one of two given trigons into an equivalent trigon which shall be similar to the other.

BOOK II.

CURVILINEAR FIGURES IN A SINGLE PLANE.

CHAPTER I.

CURVES IN GENERAL.

196.—Curves have already been defined (see art. 79, page 26,) and frequent use has been made of one especial sort. In this chapter a more systematic examination will be undertaken.

197.—Curves are divided into two principal groups; **graphic curves,** and **mathematical,** or **geometric, curves.** All those curves the relations between whose various points can not be reduced to any mathematical law, or exact statement, are called graphic curves: all others are mathematical, or geometric, curves. Such curves as those drawn freehand are almost always graphic. When we have to treat graphic curves mathematically we are obliged to content ourselves with approximate results, using mathematical curves as nearly like the given curves as we can conveniently get.

198.—As has before been said, the sect joining any two points of a curve is called a chord, and any open curve is called an arc. Sundry other terms are also used in the discussion of curves, and some of them are below defined.

199.—The indefinite straight line through any two separate points of a curve is called a **secant** (*i. e.*, cutting) **line**, or, more briefly, a secant. The points of the curve through which the secant passes are **points of secancy.** (See also art. 201, page 92.)

200.—If one of the two points of secancy be fixed while the other is made to move along the curve, gradually approaching the first and carrying the secant with it, the secant will approach more and more closely to a certain definite position, depending upon the curve and the fixed point of secancy, as the movable point of secancy comes more and more nearly into coincidence with the fixed point. The line occupying this definite position in which the hitherto secant line is left as the movable point of secancy becomes consecutive (see art. 43, page 10,) with the fixed point is said to be **tangent** to the curve at this (fixed) point, and is called a tangent (*i. e.*, touching) line, or, more briefly, a tangent; conversely the curve is said to be tangent to the line when the line is tangent to it. The point at which a line is tangent to a curve is called a **point of tangency,** or **point of contact,** or point of touch.

201.—Any line having any other point than the extremity of a curve in common with the curve, and not tangent at that point, is said to be secant at that point, and to intersect (or to cut) the curve at that point. Obviously the same line may be tangent at one point and secant at another with certain curves; or the same line may be tangent at some points and secant at others.

Q. and E. 647.—Draw any graphic arc, and show its chord, a secant, and a tangent.

648.—Draw a graphic arc such that the same line may be tangent at one point and secant at another:

649.—such that the same line may be tangent at two different points and secant at three others:

650.—such that the same line may be both tangent and secant at the same point. In the curve obtained, how many times would the tracing point pass through the point of secancy and tangency in tracing the curve?

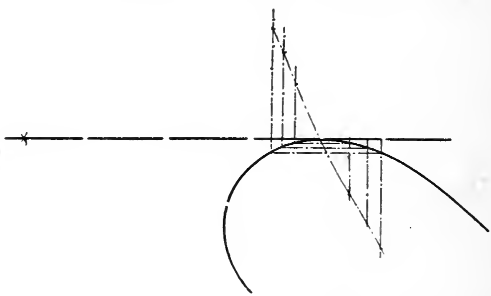
651.—Such that there may be two different tangents at the same point.

202.—**Two curves** are said to be **tangent to each other** at any common point when they both have the same tangent at that point. They are said to be **tangent externally** at any common point when they lie on opposite sides of the common tangent at that point: when both lie on the same side of the common tangent, the one not lying between the other and the tangent is said to be **tangent internally** to that other. In both cases it is to be understood that only those portions extending a very small distance from the point of tangency are to be considered.

203.—**Two curves** are said to **intersect** (or cut) each other at any common point when each has parts lying on opposite sides of the other at the common point.

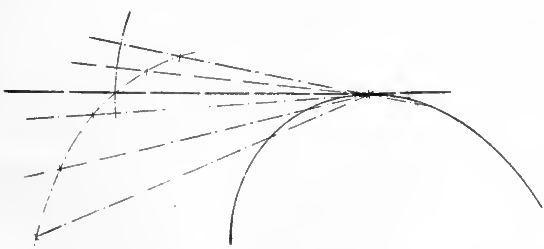
204.—By the **angle between two curves** at a common point is meant the angle between their tangents at the common point.

205.—In mathematical curves, there are exact methods of drawing tangents at given points on the curves, and also from given points outside the curves. With graphic curves, however, the draughtsman is obliged to content himself with approximate methods of performing these things. When the tangent is to be drawn from any point outside the curve, the ruler is made to revolve about that point until, in the draughtsman's judgment, the line drawn will just be tangent. This method is fairly satisfactory, especially when the given point is at a considerable distance from the point of tangency. In



the case of a tangent so drawn, the point of tangency, if required, may be approximately found by this method:—Sundry chords are drawn parallel to the tangent and at as small intervals as are practicable; and through their extremities lines are drawn perpendicular to the tangent. On these lines points are taken at distances from the tangent equal to the lengths of the corresponding chords, the points on the perpendiculars on opposite sides of the point of tangency being taken on opposite sides of the tangent. Through the points so obtained a smooth curve is to be drawn. Where the curve so obtained cuts the tangent is approximately the point of tangency. Should the curve so obtained cut the tangent too obliquely, twice the length of each chord may be laid off on the corresponding perpendicular, or thrice, etc.

206.—When the point of tangency is given and we are required to construct the tangent we may proceed thus:—Draw through the given point various secants at slight angular intervals, and so distributed that some of the second points of secancy shall be on one side of



the given point and some on the other. About the given point as center with any convenient radius strike a circular arc, crossing such region as it is judged the tangent will pass through. On each secant at a distance from the second point of secancy equal to the radius used, take a point, and through the points so obtained draw a smooth curve. Where this curve crosses the circular arc before mentioned is a second point on an approximate tangent.

These are called "graphic methods" of construction,* and give tangents exactly or approximately with all curves. The exact, or "geometric," methods will be discussed in connection with the curves to which they pertain.

Q. and E. 652.—Draw any graphic curve, draw a tangent from some point outside, and find the point of tangency.

653.—Draw any graphic curve, choose a point of tangency, and draw the tangent, using the method discussed in art. 205. How may this method be modified so that the auxiliary circle and the other auxiliary curve will intersect less obliquely?

207.—A $\left\{\begin{matrix}\text{polygon}\\ \text{chain}\end{matrix}\right\}$ is said to be **inscribed in a curve** when all its vertices are points of the curve. It is said to be **circumscribed about a curve** when all its $\left\{\begin{matrix}\text{sides}\\ \text{links}\end{matrix}\right\}$ are tangent to the curve. When a rectilinear figure is inscribed in any curve, the curve is circumscribed about the rectilinear figure, and conversely.

208.—As has before been said, the unit of length is the length of a certain sect chosen arbitrarily. We have no direct means, then, of finding the length of a curve, for we can not divide the curve into parts each of which

* See "Traite de Geometrie Descriptive"---De la Gournerie; premiere partie, arts. 100 and 101.

shall be equal to a sub-multiple of our unit sect, since no appreciable part of a curve is straight. We are thus reduced to an indirect method. We find the limit (see art. 12, page 3,) of the length of an inscribed chain (or of the perimeter of an inscribed polygon) as the links (or sides) are taken shorter and shorter, the number of them being, of course, correspondingly increased, and this limit we take to be the length of the curve; for as the links (or sides) become shorter and shorter the inscribed chain (or polygon) approaches closer and closer to coincidence both in position and extent with the curve, and becomes indistinguishable from it (except in thought) when the links (or sides) become inappreciable.

209.—The lengths of mathematical curves are capable of computation; the lengths of graphic curves must be found graphically. In practical drawing we usually content ourselves with the approximation got by taking the length of an inscribed chain each of whose links is about a tenth of an inch in length. When the curve leaves the tangent very rapidly as in the case of a circle whose radius is an eighth of an inch or less, it is necessary of course to use a much shorter link; but it is found in practice that with ordinary curves unavoidable errors in the work more than counter-balance the additional accuracy which might be expected from using a link shorter than a tenth of an inch. It will be noticed that as the curve is more and more nearly straight, the links of the auxiliary chain used may be taken longer and longer, without introducing any considerable error.

210.—Finding the length of a curve is called **"rectifying"** it, for it is essentially the same as finding an equivalent "right" line, or sect. For rectifying curves graphically, a pair of light steel-spring dividers, the distance between whose points may be regulated very exactly by means of a thumb-screw, is very convenient. Such dividers are called "stepping-dividers."

Q. and E. 654.—Draw an arc which shall have a chord of at least six inches and a point at least two inches away from the chord. Find the length of an inscribed chain all of whose links but the last shall be half an inch long, the last being not more than half an inch in length; do the same with links a quarter of an inch long,—an eighth of an inch,—a sixteenth of an inch. What relation exists among the various lengths obtained?

211.—The area of the plane surface bounded by any closed plane curve is found by taking the limit of the area of any inscribed polygon as its sides are made shorter and shorter indefinitely. By the expression "the area of a closed plane curve" will be meant the area of the plane surface bounded by it.

212.—When the area of any closed plane curve is to be obtained graphically, by means of the methods of elementary geometry, especially when it need be found only with rough approximation, we frequently get a polygon whose size is about that of the given figure by taking the vertices at such points outside the curve as will cause the parts cut off by the sides of the polygon to balance the parts added, as nearly as can be estimated by the eye. A fine black silk thread, and a collection of fine cambric needles to insert at the vertices of the polygon, will be found useful in this connection, the thread being stretched around the inserted needles to represent the polygon. Instruments called planimeters (*i. e.*, plane measurers) are frequently used in engineers' offices for finding the areas of plane figures, but they are too complicated to be discussed here.

213.—**Two curves** are **similar** when for every polygon or chain which may be inscribed in one a similar polygon or chain may be inscribed in the other, and their ratio of similitude is the ratio of similitude of any two similar inscribed polygons or chains. Homologous points, sects, etc., are determined in the same way as in the case of similar rectilinear figures.

Having given a brief discussion of some of the more important properties pertaining to curves in general, the simplest geometric curve, the circle, will now be more fully considered.

CHAPTER II.

THE CIRCLE.

214.—The simplest and the most important of all geometric curves is the circle, which has already been defined. (See art. 82, page 27.) The word **circle** is **applied indiscriminately to** the curve and to the plane surface bounded by it; but the context will denote with sufficient clearness which of these two meanings is intended. When it is to be especially indicated that the curve alone is denoted, it is called a **circumference**.

215.—In connection with the circle, we speak of center, radius, and diameter, all of which have been defined (arts. 83 to 87.) Besides these and arcs, tangents, secants, and chords, whose general meanings have been shown, we speak also of the things below defined.

216.—A **semi-circle** (*i. e.*, half-circle) is an arc which is half of the whole circle or it is the plane surface bounded by such an arc and its chord. When the former meaning is to be especially denoted, the arc is called a **semi-circumference**.

217.—Half of a semi-circle is a **quadrant.**

218.—Any circular arc larger than a semi-circumference is called a **major arc**; one smaller than a semi-circumference is called a **minor arc.** When an arc is named by means of its two extremities, the minor arc is to be understood unless the contrary is stated or clearly indicated.

219.—Any portion of a circle bounded by an arc of that circle and the chord of the arc is called a **segment** of that circle, **major** if the arc is major, **minor** if the arc is minor. The chord is the **base** of the segment, its ends the two **vertices**; the distance of the farthest point of the arc from the chord is the **altitude** of the segment. Such a segment as has just been described is a segment of one base. The portion of a circle bounded by two parallel chords of the circle and the arcs intercepted by them is called a **segment of two bases**. The ends of the bases are the vertices of the segment. The distance between the bases is the altitude. Unless the contrary is stated or implied, a segment is to be understood to have but one base.

220.—Any portion of a circle bounded by an arc of that circle and the radii drawn to its extremities is called a **sector** of the circle, or a circular sector, or more briefly and usually, a sector. It is **major** if its arc is major, **minor** if its arc is minor. The extremities of the arc and the center of it (see art. 83, page 27, for meaning of "center of an arc") are the **vertices** of the sector. When we speak of *the vertex* of a sector *the one at the* *center* of the arc is meant. The **angle of the sector** is the angle at the vertex.

221.—By the **centric angle** of any **arc** is meant the one at the vertex of the sector of which the arc forms the curvilinear boundary. The centric angle of any arc is said to intercept the arc, and to be **subtended** (*i. e.*, stretched across or under) by the arc, also by the chord of the arc. We say also that an arc is subtended by its chord.

222.—An **arc** is **produced** by being extended so that the portion thus obtained shall be an arc about the same center as the first. Any **arc** is **added** to another of equal radius by producing that other until the produced part equals the first. The resulting arc is the **sum** of the two given arcs.

223.—Two arcs whose sum is a $\left\{\begin{array}{c}\text{quadrant}\\ \text{semi-circle}\\ \text{circle}\end{array}\right\}$ are $\left\{\begin{array}{c}\text{complementary}\\ \text{supplementary}\\ \text{explementary}\end{array}\right\}$ and each is the $\left\{\begin{array}{c}\text{complement}\\ \text{supplement}\\ \text{explement}\end{array}\right\}$ of the other.

224.—Circumferences are divided into 360 equal parts, each of which is called a **degree of arc** or an arc degree. Each degree of arc is divided into 60 equal parts each of which is called a **minute** of arc; each minute of arc, into 60 equal parts each of which is called a **second** of arc. The symbols for degrees, minutes, and seconds of arc are the same as in the case of angles. Evidently the size of a degree of arc depends upon the size of the radius of the arc.

225.—If the vertex of an angle is some point (except an extremity) of the arc of a segment, and its sides pass through the vertices of the segment, the **convex angle** between the two sides is said to be **inscribed in** the **segment.** An inscribed angle is said to intercept the arc included between its sides and explementary to the arc of the segment in which the angle is inscribed.

226.—Any **convex angle** is **inscribed** in a **circle** if its vertex is a point of the circumference and its sides are chords or chords produced. The centric angle intercepting the same arc as an inscribed angle is said to be the **corresponding centric angle.**

Q. and E. 655.—Is the circle sym-centric?—sym-axic? If so, how many sym-centers has any one circle?—sym-axes?

656.—What relation exists among the various sym-axes of a circle, if it has more than one?

657.—What relation exists between a sym-axis and a diameter?

658.—Same three questions for segments of one base,—of two bases.

659.—Same for sectors.

660.—How many radii has a circle?

661.—What relation exists among them? Give your reason for your answer.

662.—What relation exists between a diameter and a radius in the same circle?

663.—What is the longest chord that can be drawn in a circle?

664.—What relation exists between the two arcs into which it divides the circumference?—the two segments into which it divides the circle?

665.—How may the shortest line from any given point to a point of a given circle be drawn?—the longest line?

666.—What relation exists between two circles having equal radii?—equal diameters?

667.—Draw six circles of different sizes and obtain the approximate lengths of their circumferences by the method discussed in art. 208, page 96. What relation, if any, exists among the ratios these lengths bear to the lengths of the corresponding radii? Take no circle with a radius of less than two inches.

668.—At what point of a chord is it met by a perpendicular dropped from the center of the circle?

669.—How might this have been known from your conclusion in the case of q. no. 88?

670.—If two chords of the same circle or of equal circles are equi-distant from the center, or centers of the corresponding circles, what relation exists between their lengths?

671.—What, if they are unequally distant from the center, or from the centers of their respective circles?

672.—What relation exists between the distances from the center, of two equal chords in the same circle?—two unequal chords? Show how the conclusions which you come to in this case might have been known by means of those arrived at in response to the two preceding questions.

673.—What is the locus of the middle point of a chord parallel to a given line?

674.—What relation exists between the arcs intercepted by two parallel chords?

675.—If a diameter be perpendicular to a chord what relation exists between the two parts into which it divides each arc subtended by the chord?—the centric angle of the arc subtended by the chord?

676.—If two arcs are equal what relation, if any, exists between their chords?

677.—What, if the arcs be unequal? Always?

678.—If the chords of two arcs are equal, what relation, if any, exists between the arcs? Always?

679.—What, if the chords be unequal? How might this be known through your conclusion in reply to q. 676?

680.—Supply the missing words in the following proposition: —"If arcs of .... radii have equal chords the arcs are ...."

681.—In the case of unequal arcs, are the chords proportional to the arcs?

682.—If so, show why; if not, show why not. If in some cases they are and in other cases they are not, show what other relation exists between the arcs when they are, which does not exist between them when they are not.

682.—If two arcs are equal, what relation, if any, exists between their centric angles?—what, if the arcs are unequal?

683.—What relation exists between two arcs subtending equal centric angles? Always?

684.—What, if the centric angles are unequal? How might this have been known from the conclusion to q. 682?

685.—What relation exists between the centric angle of the sum of two arcs (of equal radii) and the sum of their respective centric angles?—the centric angle of the difference between two arcs (of equal radii) and the difference between their respective centric angles?

686.—Supply the missing words in the following proposition: —"Arcs of .... are .... to their centric angles; and, conversely, centric angles are .... to the arcs subtending them if the arcs have ...."

687.—What relation exists between the number of degrees, minutes, and seconds in any arc and the number of degrees, minutes, and seconds in its centric angle?

688.—If two arcs are $\left\{\begin{array}{c}\text{complementary}\\ \text{supplementary}\\ \text{explementary}\end{array}\right\}$ what relation exists between their respective centric angles?

689.—What relation does the perpendicular to a radius through its outer extremity bear to the circle?

690.—What relation does any oblique line through the outer extremity of a radius bear to the circle?

691.—How many tangents may a circle have at any one point of tangency?

692.—How many tangents can be drawn to a circle from any point outside a circle?—from any point inside? Give reasons for your answers to the last two questions.

693.—What angle does a radius drawn to a point of tangency make with the tangent?

694.—If the various tangents be drawn from a given point outside a circle, what relation, if any, exists among the distances of the points of tangency from the given point?

695.—What relation exists between the sums of sets of alternate sides in any circumscribed polygon of an even number of sides?

696.—What relation do the tangents from any one point bear to the line through the given point and the center of the circle?

697.—What relation do they bear to the chord (or chords) joining the points of tangency?

698.—What kind of line in the circle is that which joins the points of tangency of two parallel tangents to the same circle?

699.—What relation exists between the arcs intercepted by two such tangents? Show the connection between this relation and that developed in reply to the preceding question.

700.—What relation exists between the arcs intercepted between a tangent and a chord parallel to it?

701.—In each of three or four circles of different sizes draw three or four inscribed angles of various sizes, and compare each with its corresponding centric angle. What relation, if any, seems to exist between each inscribed angle and its corresponding centric angle?

702.—What relation, if any, exists among all the angles which it is possible to inscribe in a given segment?

703.—What relation, if any, exists between the centric angle subtended by a chord, and the acute angle between that chord and a tangent through one end of it?

704.—If two chords AB and CF intersect at a point P, what relation does the angle APC bear to the centric angle intercepting the sum of the arcs AC and BF? Trace the connection between this relation and that developed by the preceding two questions in connection with q. 701.

705.—What relation does the angle between two secants drawn from a point without a circle bear to the angle at the center intercepting the difference between the arcs intercepted by the secants? Trace the connection between the relation here developed and those developed just before it.

706.—Same for the angle between two tangents drawn from the same point.

707.—Same for the angle between a tangent and a secant drawn from the same point.

708.—What relation exists between the alternate angles of an inscribed tetragon? *

709.—Are two unequal circles similar figures?—homothetic? If homothetic are they directly homothetic or inversely homothetic?

710.—What relation exists between the ratio of their circumferences and that of the corresponding radii?

711.—What is the locus of a point half-way from a given point to a point upon a given circle?—$\frac{m}{n}$ths of the way?

712.—Is this locus sym-axic?—if so, about what axis?

713.—Is it sym-centric?—if so, about what point?

714.—Does its size depend upon the position of the given point?—upon the size of the given circle?—upon the ratio $\frac{m}{n}$?

715.—Does its position depend upon the position of the given point?—upon the position of the given circle?—upon the ratio $\frac{m}{n}$?

716.—Do all of your conclusions in connection with this locus hold true when $m > n$?

717.—If two chords AB and CF in any circle intersect at the point P, what relation exists between the trigons APC and BPF? —the ratios $\frac{AP}{CP}$ and $\frac{PF}{PB}$?—the products $AP \times PB$ and $CP \times PF$?

718.—If two secants are drawn from any point P without a cir-

* The relations which the student will develop in investigating the cases proposed in q. no. 701--707 are very useful in comparing the different angles of inscribed figures.

cle, one cutting the circle in the two points A and B, and the other in the two points C and F, A and C being nearer P than B and F respectively, what relation exists between the trigons APC and BPF?—between the ratios $\frac{AP}{CP}$ and $\frac{PF}{PB}$?—between the products $AP \times PB$ and $CP \times PF$?

719.—Do these relations still hold when the secant PAB has revolved about P until it has become tangent, so that the points A and B merge into one?

720.—What kinds of trigons may be inscribed in circles?—circumscribed about them?

721.—Same for tetragons. Give reasons for your answers.

722.—What is the locus of the vertex of a trigon whose base is fixed and whose angle at the vertex is constant?

723.—Develop a formula for the area of a circumscribed polygon in terms of its perimeter and the radius of the inscribed circle. See q. 610, page 81.

724.—Develop a formula for the area of a circle in terms of its radius and its circumference. *

725.—Develop a formula for the area of a sector in terms of its radius and its arc.

726.—What relation exists between the areal ratio of similar sectors, similar segments, similar circles, etc., and their ratio of similitude?

727.—What relation exists between the areal ratio of any two similar plane figures and their ratio of similitude? Show why.

PROBLEMS.

728.—Find the locus of one end of a given sect which has the other end on a given circle and is parallel to a given line.

729.—Find the locus of a point on a tangent to a given circle and at a given distance from the point of tangency.

730.—Find the locus of a point at a given distance from a given circle.

* This formula being correctly developed a discussion of the ratio of the circumference to the diameter may be introduced.

731.—Draw a sect of given length which shall be parallel to a given line and have its extremities in two given circles.

Draw the circle which shall

732.—Pass through three given points,—(i), three points independent; (ii), collinear;

733.—Pass through four given points;

734.—Pass through two given points and have a given radius;

735.—Pass through a given point, have a given radius, and be tangent to a given line;

736.—Have a given radius, and be tangent to two given lines;

737.—Be tangent to three given lines,—(i), lines all parallel; (ii), two lines only, parallel; (iii), no two lines parallel; (iv), lines concurrent.

738.—Pass through two given points and be tangent to given line,—(i), when sect joining given points is parallel to given line; (ii), when it is not parallel; (iii), one of given points is the point of tangency. See q. 718, page 107.

739.—Pass through a given point and be tangent to two given lines,—(i), lines parallel; (ii), lines not parallel; (iii), point on one of given lines; (iv), point equidistant from two given lines. In (i) and (ii) make use of fact that the locus of a point equidistant from the two given lines is a sym-axis, and compare with preceding problem.

740.—Pass through a given point, be tangent to a given line and have a given radius—(i), given point on given line;—(ii), not.

741.—Be tangent to given circle and to given line, and have given point of tangency; (i), on given line; (ii), on given circle. Two solutions under each, according as two circles are tangent internally or externally.

742.—Pass through a given point, have a given radius, and be tangent to a given circle,—(i) given point on given circle; (ii) not.

743.—Have a given radius and be tangent to a given line and to a given circle,—(i), the two circles tangent internally; (ii), tangent externally.

744.—Have a given radius and be tangent to two given circles, —(i), tangent externally to both given circles; (ii), tangent internally to one, or conversely, and externally to the other. Two cases under second division. How many solutions in all?

745.—Draw within a given circle three circles equal to each other and each tangent to the given circle and to the other two.

746.—About the vertices of a trigon as centers describe circles so that each pair shall be tangent to each other and to the third, —(i), internally;—(ii), externally.

Construct the trigons which shall have given values for

747. a, b, and m_c.

748. a, b, and h_c.

749. a, m_a, and h_b.

750. a, m_a, and α.

751. a, m_b, and h_b.

752. a, m_b, and h_c.

753. a, m_b, and α.

754. a, m_b, and β.

755. a, h_a, and h_b.

756. a, h_a, and α.

757. a, h_b, and h_c.

758. a, h_b, and d_b.

759. a, h_b, and d_c.

760. a, h_b, and α.

761. a, h_b, and β.

762. a, h_b, and p.

763. m_a, m_b, and α.

764. m_a, h_b, and γ.

765. h_a, h_b, and α.

766. h_a, α, and p.

767. a, $b-c$, and h_b.

768. a, $b-c$, and h_c.

769. p, $\alpha+\beta$, and h_c.

770. a, b, and R. *

771. a, m_a, and R.

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* R is radius of circumscribed circle; r of inscribed.

772. a, m_b, and R.
773. a, h_a, and R.
774. a, h_b, and R.
775. a, β, and R.
776. a, β, and r.
777. m_a, h_a, and R.
778. m_a, α, and R.
779. h_a, d_a, and R.
780. h_a, α, and R.
781. h_a, β, and R.
782. d_a, α, and r.
783. α, β, and r.
784. α, β, and R.
785. α, $\beta-\gamma$, and R.
786. $b+c$, β, and R.

Construct the isosceles trigon which shall have given values for

787. b and R.
788. c and R.
789. c and r.

Construct the right trigon which shall have given values for

790. a and r.
791. a and R.
792. c and h_c.
793. c and m_a.
794. c and r.
795. p and h_c.
796. p and r.
797. p and R.
798. p and $\alpha-\beta$.
799. $c-a$ and r.
800. $c-a$ and R.
801. h_c and r.

802. h_c and R.

803. d_c and r.

804. a and r.

805. a and R.

806. $a-\beta$ and r.

807. $a-\beta$ and R.

Construct the tetragon which shall have given values for

808. a, b, g_1, $\angle ab$, and $\angle cd$.

Construct the trapezoid which shall have given values for

809. a, b, c, and d.

810. a, b, d, and m.

811. a, b, p, and m.

812. a, c, g_1, and g_2.

813. a, g_1, g_2, and m.

227.—A **mixtilinear figure**, *i. e.*, one whose outline is made up of curves and right lines, is said to be **inscribed** in a **polygon** when its vertices lie on the sides of the polygon and its curvilinear sides are tangent to those sides of the polygon on which no vertices of the inscribed figure are.

814.—Inscribe a semicircle in a given trigon, so that the vertices of the semicircle shall both rest on the same side of the trigon. How many solutions? Does the character of the trigon have anything to do in determining the number of solutions? If so, what and why?

815.—Same, except that base of semicircle shall be parallel to a given line which is not parallel to any side of the trigon. Same questions as in preceding exercise.

816.—Inscribe in a given trigon a segment similar to a given major segment,—(i), base of segment on one of the sides of trigon; (ii), base of segment not on any side, but parallel to a given line. Same questions as in no. **814**,

817.—Same for minor segment, and same questions as in **814**.

818.—Same for major sector,—(i), one of rectilinear sides on one side of trigon; (ii), not on any side but parallel to a given line. Same questions as in no. 814.

819.—Same for minor sector and same questions as in no. 814.

820 to 825.—Same as from no. 814 to no. 819 except that three indefinite non-concurrent lines, no more than two of which may be parallel, are to be substituted for the sides of the given trigon. Discuss each one fully. Problems from no. 814 to no. 825 inclusive may most easily be solved by making use of the principles of homothesy, using one of the intersections of the given straight lines as center of homothesy.

826.—Construct a circle whose perimeter shall equivale the sum of the perimeters of two given unequal circles,—whose area shall equivale the sum of their areas.

827.—Construct a semicircle whose perimeter shall equivale the difference between the perimeters of two given unequal semicircles,—whose area shall equivale the difference between their areas.

828.—Construct a sector which shall be similar to a given sector and three times as large.

829.—Construct a segment which shall be similar to a given segment of two bases, and half as large.

For problems, no. 826–7–8–9, see art. 191, page 83, and q. 727, page 108.

THE DEPARTMENT OF MATHEMATICS,
THE STATE AGRICULTURAL COLLEGE,
FORT COLLINS, COLORADO.

DEAR SIR:

The accompanying copy of "AN INDUCTIVE MANUAL OF THE STRAIGHT LINE AND THE CIRCLE" is sent you for examination, with view to introduction and use in your school.

Your attention is particularly called to the following features, which, it is believed, distinguish the book from other text books on elementary plane geometry:

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IV. The distinction between the obverse and the reverse of plane figures; see Art. 120.

V. The closeness of relation between regular chains, regular polygons and the circle; see Articles 150–160.

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