INFINITESIMALS AND LIMITS.

BY

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CHAPTER I

INFINITESIMALS

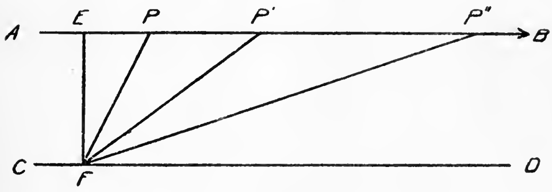

Fig. 1.

1. In Fig. 1 let AB and CD be two parallel straight lines and draw EF ⊥ CD. Let a point P start from E and move along AB to the right so as at the end of the first second to be at P, at the end of the second second to be at P′, at the end of the third second to be at P″, ...

As the point P changes its position,

∠ EFP	takes the different values	EFP, EFP′, EFP″, etc.
∠ PFD	" "	PFD, P′FD, P″FD, etc.
EP	" "	EP, EP′, EP″, etc.
FP	" "	FP, FP′, FP″, etc.

but ∠ EFD does not change its value,
and EF " "

That is, among the quantities which enter into the discussion of the given figure as the point P changes its position, there are some, ∠ EFD and EF, which retain the same values, and which are accordingly called constants.

Others, as EP, ∠ EFP, ∠ PFD and FP, which assume different values, are called variables. .

Fig. 2.

Again, suppose a man starts from A to go to B, going over half the distance between himself and B every hour. At the end of the first hour he will reach M ; at the end of the second he will reach M′ ; at the end of the third at M″ ; etc.

In this illustration we see that AM, the distance the man travels, and MB, the distance from the man to B, change continually ; but no matter how long the man travels, AB always retains the same value. Hence we call AM and MB variables and AB a constant.

In the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \text{etc.},$$

let n be the number of terms taken beginning with the first and let s be the sum of those terms.

Now let n be made greater and greater continually. Then when

$$\begin{aligned}
n&=2, & s&=1 + \tfrac{1}{2} &&= 1\tfrac{1}{2}\\
n&=3, & s&=1 + \tfrac{1}{2} + \tfrac{1}{4} &&= 1\tfrac{3}{4}\\
n&=4, & s&=1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &&= 1\tfrac{7}{8}\\
n&=5, & s&=1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} &&= 1\tfrac{15}{16}
\end{aligned}$$

We see from these equations that n and s change their values continually but that each term of the series always retains the same value.

Hence we call n and s variables and each term a constant.

Thus in any problem there may appear quantities which, under the assumptions made in the problem, always retain the same values. There may also appear others which under the same assumptions take different values. The former are called constants, the latter variables.

2. **A variable** is a quantity which changes its value under the assumptions made in the problem into which it enters.

3. **A constant** is a quantity which does not change its value under the assumptions made in the problem into which it enters.

4. **The absolute value** of a number is simply the number of units it contains, no regard being paid to its sign.

Thus the absolute value — 5 and of 5 is 5.

5. **An infinitesimal** is a variable which approaches zero in such a way that its absolute value may be made to become and remain less than any positive number ϵ that may be named, however small the ϵ may be taken.

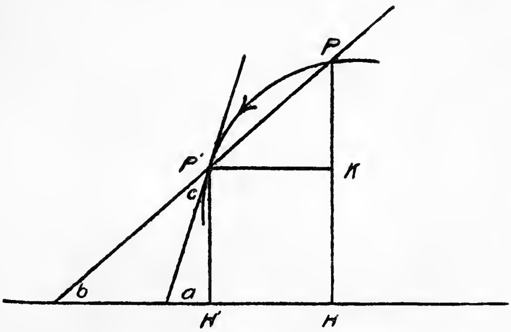

Fig. 3.

In Fig. 3 let the point P′ remain fixed, let P move continually along the curve till it reaches P′, and let it stop there.

Then the $\angle c$ continually approaches zero in such a way that its absolute value may be made to become and remain less than any positive number ϵ however small ϵ may be taken.

For let $$\epsilon = 1^\circ.$$
Then by moving P far enough we can make
$$c < 1^\circ.$$
Now let $$\epsilon = 1'.$$
By moving P farther on we can make
$$c < 1'.$$
Let $$\epsilon = 1''.$$
Again by moving P farther on we can make
$$c < 1''.$$
Similarly no matter how small ϵ is taken we can make
$$c < \epsilon$$
by moving P on far enough.

Since by hypothesis P continually approaches P′ and does not pass it, c continually approaches 0 but cannot become less than 0. Therefore when the absolute value of c once becomes less than ϵ it must remain so.

Hence by the definition c is an infinitesimal.

Again in Fig. 3 the line PK continually approaches zero in such a way that its absolute value may be made to become and remain less than any positive number ϵ, however small the ϵ may be made.

Let $$\epsilon = .1.$$
Then by moving P towards P′ we can make
$$PK < .1.$$
Now let $$\epsilon = .01.$$
Then by moving P further on we can make
$$PK < .01.$$

Similarly, no matter how small ϵ may be taken we can make

$$PK < \epsilon$$

by moving P far enough.

Since by hypothesis P continually approaches P′ and cannot pass it the absolute value of PK continually approaches o and cannot become less than o. Therefore when the absolute value of PK once becomes less than ϵ it must remain so.

Hence by the definition PK is an infinitesimal.

Similarly it may be shown that the Δ PKP′ is an infinitesimal.

It is obvious that each of these infinitesimals c, PK and Δ PKP′ will become zero when P reaches P′.

In Fig. 4 let AB and CD be two parallel straight lines. Draw P″E $\perp$ CD.

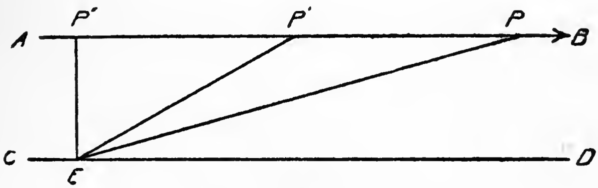

Fig. 4.

Now let E remain fixed and let P move continually along AB towards the right.

The $\angle$ PED continually approaches zero and its absolute value may be made to become and remain less than any positive number ϵ, however small ϵ may be made.

Hence by the definition $\angle$ PED is an infinitesimal.

It is obvious that $\angle$ PED can never be made equal to zero.

Let $s \equiv$ the sum of n terms of the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \text{etc.}$$

Let $$R = 2 - s$$

Now let n increase continually. Then s will increase continually and can be made as nearly equal to 2 as we please. Hence R will continually approach zero and may be made less than a positive number ϵ, however small ϵ may be taken.

Since all the terms are positive, R must be positive; hence when once R becomes less than ϵ, it must remain so.

Hence by the definition R is an infinitesimal.

It is obvious that R can never be made equal to zero.

6. There are therefore two classes of infinitesimals, namely:

(1) Those which may finally become zero.

(2) Those which can never become zero.

CHAPTER II

INFINITES

7. An infinite.—An infinite is a variable whose absolute value can be made to become and remain larger than any positive number that may be assigned.

In Fig. 4 the line P″P is an infinite; for by moving P farther and farther to the right we can make the absolute value of P″P greater than any positive nnmber that may be taken. Also since the absolute value of P″P is always increasing, when once it becomes greater than any positive number, it will remain greater.

Again let

[1] $$s = 1 + 2 + 4 + \text{etc. to } n \text{ terms.}$$

Then s is an infinite.

For the series is a Geometrical Progression and we know by arithmetic that

[2] $$s = \frac{ar - a}{r - 1}$$

[3] $$\therefore \quad s = \frac{2^n - 1}{2 - 1} = 2^n - 1.$$

Now we see from the right-hand side of [3] that by making n larger and larger we can make s larger than any positive number that may be taken. Also since s is always increasing when once it becomes larger than any positive number, it will remain larger.

The symbol for an infinite is ∞.

8. A finite.—Any quantity which is neither an infinitesimal nor an infinite is called a finite.

CHAPTER III

PROPERTIES OF INFINITESIMALS

9. In Fig. 5 draw PQ, RS, etc., $\perp$ xx' and at equal distances from each other. Also draw RT, etc., parallel to xx'.

We will in this way construct the triangular figure PRT bounded by PT, RT and the curve; also the triangular figures a, b, c, and d; and the rectangles q, p, and o.

Now let PQ move continually towards P'Q', the other perpendiculars moving so as always to preserve the same relative position. Then it is obvious that PP′Q′Q. q, p, o, a, b, c, d are all infinitesimals. by § 5.

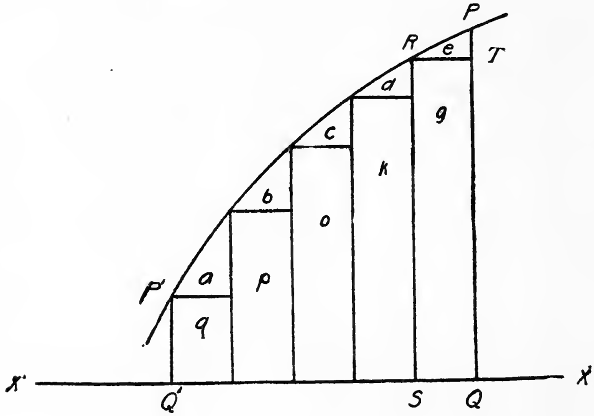

Fig. 5.

We see also that $q < p < o <$ PP′Q′Q.

Hence one infinitesimal may be greater than another.

Also since

$a + b + c + d +$ etc. $+ q + p + o +$ etc. $=$ PP′Q′Q.
it is obvious that the sum of a finite number of infinitesimals may be an infinitesimal.

Finally let there be any $n + 1$ infinitesimals like q, p, o.

Then since $$q < p < o \ldots$$

we get $$q < \frac{p + o + \ldots}{n}$$

Let $$p + o + \ldots = s$$

Then $$q < \frac{s}{n}$$

But q and s are both infinitesimals.

Hence an infinitesimal may be less than another infinitesimal divided by a constant.

It is obvious from this discussion that an infinitesimal can be properly said to be very small only at certain stages of its variation.

When a variable becomes smaller and smaller in such a way that it may be made smaller than any positive number that may be named, it is said to decrease indefinitely.

When a variable becomes larger and larger in such a way that it may be made larger than any positive number that may be named, it is said to increase indefinitely.

PROPOSITION 1

10. *The reciprocal of an infinitesimal is an infinite.*

For let $\alpha \equiv$ any infinitesimal.

Then $\frac{1}{\alpha} \equiv$ its reciprocal.

Now when α decreases indefinitely,

$\frac{1}{\alpha}$ increases indefinitely;

for when the denominator of a fraction decreases indefinitely the value of the fraction increases indefinitely.

Hence $\frac{1}{\alpha}$ is an infinite. by § 7.

Q. E. D.

PROPOSITION 2

11. *The product of a constant and an infinitesimal is itself an infinitesimal.*

Let ϵ and $s \equiv$ two infinitesimals

and $n \equiv$ any constant.

Then we may take $s < \frac{\epsilon}{n}$. by § 5.

Hence $ns < \epsilon$. Q. E. D.

PROPOSITION 3

12. *The sum of any finite number of infinitesimals is also an infinitesimal.*

Let there be n infinitesimals.

$$\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_p, \epsilon_{p+1}, \ldots, \epsilon_n.$$

Let ϵ_p be that one of these infinitesimals which has at any instant the greatest absolute value.

[1] Let $\qquad s = \epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_n.$

[2] Then $\qquad s < n\epsilon_p$

But $n\epsilon_p$ is an infinitesimal. by § 11.

Hence by [2] s is also an infinitesimal. Q. E. D.

PROPOSITION 4

13. *The product of any finite number of infinitesimals is also an infinitesimal.*

Let there be n infinitesimals

$$\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_n.$$

Let x be a variable which can be made to increase indefinitely. We can always find a value of x such that

$$\epsilon_1 < \frac{1}{x}, \qquad \text{by § 5.}$$

$$\epsilon_2 < \frac{1}{x},$$

$$\epsilon_3 < \frac{1}{x},$$

$$\text{etc} < \text{etc.},$$

$$\epsilon_n < \frac{1}{x}.$$

[1] Hence $\qquad \epsilon_1\,\epsilon_2\,\epsilon_3\,\ldots\ldots\,\epsilon_n < \dfrac{1}{x_n}$

Now making x increase indefinitely we can make $\frac{1}{x_n}$ an infinitesimal. Hence by [1] $\epsilon_1\ \epsilon_2\ \epsilon_3\ \ldots\ \epsilon_n$ is an infinitesimal.

Q. E. D.

PROPOSITION 5

14. *First.—A power of an infinitesimal is itself an infinitesimal if its exponent is a positive finite other than 0.*

Second.—It is an infinite if its exponent is a negative finite other than 0.

Third.—It is equal to 1 if its exponent is 0.

First.

Let α be any infinitesimal and p a positive finite other than 0.

[1] Then $\alpha^p = \alpha\ \alpha\ \alpha\ \alpha\ \ldots$ to p factors.

[2] But $\alpha\ \alpha\ \alpha\ \ldots$ to p factors is an infinitesimal. by § 13.

[3] Hence $\alpha^p =$ an infinitesimal.

Second.

Let $-p$ be a negative finite other than 0.

[4] Then $\alpha^{-p} = \frac{1}{\alpha^p}$.

But since α is an infinitesimal

[5] $\alpha^p =$ an infinitesimal, by case 1.

[6] and $\frac{1}{\alpha^p} =$ an infinite. by § 10.

Hence by [4] we get

[7] $\alpha^p =$ an infinite.

Third.

Let $p = 0$.

Then $\alpha^p = \alpha^0 = 1$.

Q. E. D.

CHAPTER IV

LIMITS

15. Limit.—The limit of a variable is that constant which differs from the variable by an infinitesimal.

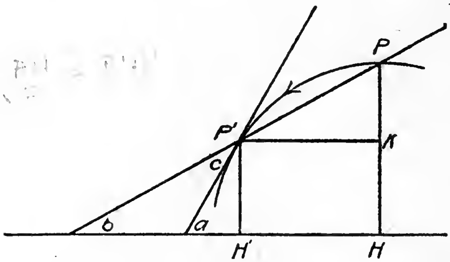

Fig. 6.

In Fig. 6

[1] $\angle a = \angle b + \angle c$ by Geom.

[2] $\therefore$ $\angle a - \angle b = \angle c$

Now let P move along the curve till it reaches P'.

Then $\angle a$ is a variable, $\angle b$ is a constant and $\angle c$ is an infinitesimal.

Hence by [2] $\angle b$ is that constant which differs from the variable $\angle a$ by an infinitesimal $\angle c$.

Therefore by the definition $\angle b$ is the limit of $\angle a$.

Again

[3] PH — P'H' = PK.

Now again let P move along the curve till it reaches P'. Then PH is a variable, P'H' is a constant, and PK is an infinitesimal.

Hence by [3] P'H' is that constant which differs from the variable PH by the infinitesimal PK.

Therefore by the definition P'H' is the limit of PH.

It is obvious that both the variable $\angle a$ and the line PH reach their limits.

In Fig. 4,

[4] $\angle P''ED - P''EP = \angle PED.$

Let P move continually to the right. Then $\angle P''EP$ is a variable, $\angle P''ED$ is a constant and $\angle PED$ is an infinitesimal.

Hence by [4] $\angle P''ED$ is that constant which differs from the variable $\angle P''EP$ by the infinitesimal $\angle PED$.

Therefore by the definition $\angle P''ED$ is the limit of $\angle P''EP$.

Let $s_n =$ the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{4}$ etc.

Let $R_n = 2 - s_n$.

Let n increase indefinitely. Then s is a variable, 2 is a constant, and R_n is an infinitesimal. by § 5.

Hence 2 is that constant which differs from the variable s_n by the infinitesimal R_n.

Therefore by the definition 2 is the limit of s_n.

It is obvious that neither of these last two variables can ever reach its limit.

16. Hence there are two classes of *Limits :*

[1] Those to which the variable finally becomes equal.

[2] Those to which the variable can never become equal.

The first class are called *Attainable Limits.*

The second class are called *Unattainable Limits.*

17. In Fig 4,

let $v \equiv$ the variable $\angle P''EP$,

let $l \equiv$ its limit $\angle P''ED$,

and $\epsilon \equiv$ the infinitesimal $\angle PED$.

[1] Since $\angle P''EP = \angle P''ED - \angle PED$. by Geom.

[2] $v = l - \epsilon$

And in general whatever the variable may be,

[3] $v = l - \epsilon.$

In [3] ϵ may be either positive or negative. For if as in Fig. 4 the variable always increases as it approaches its limit ϵ will be negative, but if in any case the variable always decreases as it approaches its limit ϵ will be positive. There are some cases in which the variable alternately passes from one side of its limit to the other as it approaches it. In these cases ϵ is alternately positive and negative.

PROPOSITION I

18. *If two variables are always equal, and each approaches a limit, their limits are equal.*

Let v and v' be two variables,
l " l' be their respective limits,
and ϵ " ϵ' be two infinitesimals.

Let $v = v'$

We are to prove that $l = l'$.

[1] $v = l - \epsilon$ by § 17.

[2] and $v' = l' - \epsilon'$. by § 17.

Subtracting [2] from [1] we get

[3] $v - v' = l - l' - \epsilon + \epsilon'$

[4] But by hypothesis $v = v'$, and hence $v - v' = 0$.

Substituting this value of $v - v'$ into [3]

[5] we get $0 = l - l' - \epsilon + \epsilon'$.

[6] Hence $l' - l = \epsilon' - \epsilon$.

If ϵ and ϵ' are not equal to each other suppose that the absolute value of $\epsilon >$ the absolute value of ϵ'.

[7] Then $\epsilon' - \epsilon$ is not greater than 2ϵ,

and from [6] it follows that

[8] $l' - l$ is not greater than 2ϵ.

Since by § 15 l and l' are constants, $l' - l$ is a constant. And, since by § 5 ϵ can be made as nearly zero as we please, 2ϵ can also be made as nearly zero as we please. Then it follows from [8] that $l' - l$ is a constant which can never be greater than 2ϵ. But the only such constant is zero.

[9] Hence $l' - l = 0$.

[10] or $l' = l$. Q. E. D.

PROPOSITION 2

19. *If the sum of any finite number of variables be variable, then the limit of their sum is equal to the sum of their limits.*

Let v and $v' \equiv$ two variables.

l " $l' \equiv$ their respective limits.

and ϵ " $\epsilon' \equiv$ two infinitesimals.

We are to prove that $\lim (v + v') = l + l'$.

[1] $v = l - \epsilon$ by § 17, [3]

[2] and $v' = l' - \epsilon'$ by § 17, [3]

[3] By addition $v + v' = l + l' - \epsilon - \epsilon'$

[4] and limit $(v + v')$ = limit $(l + l' - \epsilon - \epsilon')$. by § 18.

Since by hypothesis ϵ and ϵ' are infinitesimals, $(-\epsilon - \epsilon')$ is an infinitesimal by § 12.

Also since by § 15 l and l' are both constants, $l + l'$ is a constant.

Hence $l + l'$ is a constant which differs from the variable $[l' + l - \epsilon - \epsilon']$ by the infinitesimal $[-\epsilon = \epsilon']$.

[5] Therefore limit $(l + l' - \epsilon - \epsilon') = l + l'$ by § 15.

Substituting into [4] we get

[6] $$\text{limit } (v + v') = l + l'.$$

Q. E. D.

Similarly the theorem may be proved for the sum of any finite number of variables since ($\epsilon - \epsilon' - \epsilon'' \dots$ etc.) is an infinitesimal. by § 12.

PROPOSITION 3

20. *If the product of a finite number of variables be variable, then the limit of their product is the product of their limits.*

Let v and $v' \equiv$ two variables,
l " $l' \equiv$ their limits,
and ϵ " $\epsilon' \equiv$ two infinitesimals.

We are to prove that lim $vv' = ll'$.

[1] $$v = l - \epsilon$$ by § 17, [3]

[2] $$v' = l' = \epsilon$$ " "

Multiplying [1] by [2] we get

[3] $$vv' = ll' - \epsilon l' - \epsilon' l + \epsilon\epsilon'$$

[4] Therefore
$$\text{limit } (vv') = \text{limit } (ll' - \epsilon l' - \epsilon' l + \epsilon\epsilon' l).$$
by § 18.

Now $\epsilon l'$ and $\epsilon' l$ are infinitesimals by § 11.
and $\epsilon\epsilon'$ is an infinitesimal. by § 13.

Hence $- \epsilon l' - \epsilon' l + \epsilon\epsilon'$ is an infinitesimal. by § 12.

But since by § 15 l and l' are constants ll' is a constant.

Now ll' is a constant which differs from the variable $ll' - \epsilon t' - \epsilon' l + \epsilon\epsilon'$ by the infinitesimal $- \epsilon l' - \epsilon' l + \epsilon\epsilon'$.

[5] Hence lim $(ll' + \epsilon l' - \epsilon' l + \epsilon\epsilon') = ll'$ by § 15.

[6] Hence by [4] lim $(vv') = ll'$ Q. E. D.

Similarly the proposition may be proved for the product of any finite number of variables.

21. *Corollary 1.—The limit of the product of a constant and a variable is the product of the constant and the limit of the variable.*

Let $a \equiv$ any constant
" $v \equiv$ any variable
and $l \equiv$ its limit
Let $\epsilon \equiv$ an infinitesimal.

[1] Then $v = l - \epsilon$ by §17, [3]

[2] $av = al - a\epsilon$

[3] Hence $\lim (av) = \lim (al - a\epsilon)$. by § 18.

Now $a\epsilon$ is an infinitesimal by § 11.
but since by § 15 l is a constant al is a constant.

Hence al is a constant which differs from the variable. $(al - a\epsilon)$ by the infinitesimal $a\epsilon$.

Therefore $\lim (al - a\epsilon) = al$ by § 15.

Substituting into (3) we get

$$\lim (av) = al.$$

Q. E. D.

22. *Corollary 2.—If the product of any finite number of variables be a constant the limit of their product is the same constant.*

Let $v,w,x,$ . . . etc. be variables and let a be a constant.

[1] Let vwx . . . etc. $= a$.

We are to prove that

limit (vwx . . . etc.) $= a$

Let z be another variable

[2] Then by (1) (vwx . . . etc.) $z = az$

[3] lim (vwx . . . etc.) $\lim z = a \lim z$
by § § 18 and 21.

[4] $\therefore$ lim (vwx . . . etc.) $= a$

Q. E. D.

PROPOSITION 4

23. *The limit of any positive integral power of a variable is the same power of the limit of the variable.*

Let $v \equiv$ the variable,
$l \equiv$ its limit,
and $n \equiv$ any positive integral exponent.

We are to prove that $\lim v^n = l^n$

[1] limit ($vvvv$ . . . to n factors) $= l.l.l.$ . . . to n factors. by § 30.

[2] Therefore limit $v^n = l^n$.

Q. E. D.

PROPOSITION 5

24. *The limit of the quotient of two variables is the quotient of their limits, provided that neither of the limits be o.*

Let x and y be the two variables.

We are to prove that $\lim \dfrac{x}{y} = \dfrac{\lim x}{\lim y}$

[1] Let $v = \dfrac{x}{y}$

[2] then $vy = x$

[3] and $\lim (vy'') = \lim x.$ by § 18.

[4] But $\lim (vy) = \lim v \lim y$ by § 20.

[5] hence $\lim v \lim y = \lim x$

[6] and $\lim v = \dfrac{\lim x}{\lim y}$

[7] or by [1] $\lim \dfrac{x}{y} = \dfrac{\lim x}{\lim y}.$

Q. E. D.

25. *Corollary.—The limit of the quotient of a constant by a variable is the constant divided by the limit of the variable.*

Let $a \equiv$ any constant,
" $v \equiv$ " variable,
and $l \equiv$ its limit.

We are to prove that

$$\text{limit } \frac{a}{v} = \frac{a}{\lim v}$$

[1] Let $z = \frac{a}{v}$

[2] Then $zv = a$

[3] Hence $\lim zv = a$ by § 22.

[4] and $\lim z \lim v = a$ by § 20.

[5] hence $\lim z = \frac{a}{\lim v}$

[6] or by [1] $\lim \frac{a}{v} = \frac{a}{\lim v}$

Q. E. D.

PROPOSITION 6

26. *The limit of the nth root of a variable is equal to the nth root of the limit of that variable.*

Let $v \equiv$ any variable
and $l \equiv$ its limit.

[1] $v = \sqrt[n]{v^n}$ and $l = \sqrt[n]{l^n}$,

[2] also limit $v = l$ by hyp.

Substituting in this equation the values of l and v found in [1] we get

[4] $\text{limit } \sqrt[n]{v^n} = \sqrt[n]{l^n}$

Now since v represents any variable whatever, v^n represents any variable whatever and l^n is its limit.

by § 23.

Therefore [4] shows that the limit of the nth root of any variable is equal to the nth root of its limit.

Q. E. D.