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Interaction Between Autocorrelation and Conditional Heteroskedasticity: A Random Coefficient Approach

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ABSTRACT: We consider a linear regression model with random coefficient autoregressive disturbances which provides a convenient framework to analyze autocorrelation and autoregressive conditional heteroskedasticity (*ARCH*) simultaneously. Under our framework, the necessary and sufficient conditions for the process to be stationary are easily derived, and these conditions further reveals the interaction between *ARCH* and autocorrelation. Next we discuss tests for *ARCH* in the presence of autocorrelation and vice versa. A joint test for autocorrelation and *ARCH* is also suggested. An empirical example is provided to illustrate the usefulness of our analysis.

KEY WORDS: ARCH; Random Coefficient Model; Autocorrelation; Stationarity condition; Lagrange multiplier test; Price Expectations; Unbiasedness hypothesis.

1. INTRODUCTION

Recently, the autoregressive conditional heteroskedastic ($ARCH$) model introduced by Engle (1982) and its different generalizations have become popular in modeling the behavior of monetary and financial variables [see, for example, the references cited in Engle and Bollerslev (1986), Bera et. al (1988) and Bollerslev et. al (1988)]. Earlier researchers used to consider only unconditional heteroskedasticity, that too as an arbitrary function of some exogenous variables. In contrast, $ARCH$ models provides a systematic way to specify conditional heteroskedasticity.

$ARCH$ models are applicable exclusively for time series data. For a long time, econometric tradition has been to incorporate autocorrelation in the model to capture the time series dynamics [see Hendry and Mizon (1978)]. Nevertheless, curiously enough, none of the above applied papers considers explicitly serial correlation and $ARCH$ simultaneously. In some cases, the absence of serial correlation was reported using Durbin-Watson, Durbin-h and other tests separately prior to testing for $ARCH$ which was found to be significant. In the presence of $ARCH$, the usual tests for autocorrelation are not valid. In a recent paper, Diebold (1986) demonstrated that the presence of $ARCH$ invalidates the asymptotic distributions of the sample autocorrelation and the Box-Pierce and Box-Ljung test statistics for serial correlation. Furthermore, as indicated in Engle et al. (1985, p.75), the presence of autocorrelation could be mistaken for $ARCH$ even when there is no conditional heteroskedasticity. In light of increasing use of $ARCH$ type models and current econometric practice, there is a need to study the interrelationship between autocorrelation and $ARCH$. The purpose of this paper is to investigate this interrelationship in terms of stationarity of the time series process, testing and estimation.

The presence of $ARCH$ could be interpreted in a number of ways such as non-normality of the disturbance term [Engle (1982, p.992)] and nonlinearity of the process [Higgins and Bera (1988, 1989)]. In a recent paper, Bera and Lee (1988) established a close link between autocorrelation and $ARCH$ through parameter heterogeneity. They applied the White's (1982) information matrix (IM) test to the standard linear regression model with autocorrelated errors and found that the Engel's (1982) Lagrange multiplier (LM) test for $ARCH$ is a special case of one component of the IM test. Given Chesher's (1984) interpretation of the IM test as a test for parameter variation, it can be said that the presence of $ARCH$ is equivalent to random variation in the autocorrelation coefficients [see also Tsay (1987)]. This leads us to consider the regression model with random autocorrelation parameters. In section 3, we show how this formulation helps to study stationarity of the resulting process, after discussing the model in section 2. Next in section 4, we discuss how the tests for $ARCH$ are affected by the presence of autocorrelation and vice versa. An empirical example of testing the unbiasedness of experts' expectation of inflation using the Livingston survey data is provided in section 5. Finally, some concluding remarks are offered in the last section.

2. THE MODEL

To analyze the autocorrelation and the $ARCH$ simultaneously, we need to specify a general framework which encompasses, as special cases, models which are encountered in practice. For this purpose, we consider a linear regression model with random coefficient autoregressive disturbances,

$$
\begin{aligned}
y_t &= x_t'\beta + \varepsilon_t \\
\varepsilon_t &= \sum_{j=1}^{p} \phi_{jt}\varepsilon_{t-j} + u_t \\
&= \sum_{j=1}^{p} (\phi_j + \eta_{jt})\varepsilon_{t-j} + u_t
\end{aligned}
\tag{2.1}
$$

where y_t is the t-th observation on the dependent variable, x_t is a $k \times 1$ vector representing the t-th observation on k fixed regressors with a $k \times 1$ vector of unknown constant coefficients β, ε_t is the disturbances which follow a stochastic p-th order autoregressive process with ϕ_j constant and η_{jt} stochastic for all j. For the model (2.1), a number of fairly general assumptions are made.

ASSUMPTION 1. : *Let* $\{\eta_t = (\eta_{1t}, \ldots, \eta_{pt})', t = 1, 2, \cdots, N\}$ *be a sequence of identically and independently distributed random vectors with* $E(\eta_t) = O_{p\times 1}$ *and* $E(\eta_t \eta_t') = \Sigma$, *where* Σ *is a* $p \times p$ *positive semidefinite matrix.*

ASSUMPTION 2. : $\{u_t, t = 1, 2, \cdots, N\}$ *is a sequence of identically and independently distributed random variable with* $E(u_t) = 0$ *and* $E(u_t^2) = \sigma_u^2$.

ASSUMPTION 3. : $\{\eta_t, t = 1, 2, \cdots, N\}$ *and* $\{u_t, t = 1, 2, \cdots, N\}$ *are mutually independent.*

A wide range of models is encompassed as special cases of the model (2.1). Let Ψ_{t-1} be the information set available at time t. There are many ways to summarize the available information at time t. In particular, a series of the past innovations $\{\varepsilon_{t-i} \mid i = 1, 2, \cdots\}$ is assumed to belong to Ψ_{t-1}. Then the conditional mean and variance of the disturbance ε_t under Assumptions 1–3 can be represented as

$$\begin{aligned} \mu_t \equiv E(\varepsilon_t \mid \Psi_{t-1}) &= E\Big[\sum_{j=1}^{p}(\phi_j + \eta_{jt})\varepsilon_{t-j} + u_t \mid \Psi_{t-1}\Big] \\ &= \sum_{j=1}^{p} \phi_j \varepsilon_{t-j} \\ &= \phi' \underline{\varepsilon_t} \end{aligned} \tag{2.2}$$

where $\phi = (\phi_1, \cdots, \phi_p)'$ and $\underline{\varepsilon_t} = (\varepsilon_{t-1}, \cdots, \varepsilon_{t-p})'$ and

$$\begin{aligned} h_t \equiv Var(\varepsilon_t \mid \Psi_{t-1}) &= E\Big[\Big(\sum_{j=1}^{p} \eta_{jt}\varepsilon_{t-j} + u_t\Big)^2 \mid \Psi_{t-1}\Big] \\ &= E\Big[\underline{\varepsilon_t}'\eta_t\eta_t'\underline{\varepsilon_t} + 2u_t\underline{\varepsilon_t}'\eta_t + u_t^2 \mid \Psi_{t-1}\Big] \\ &= \underline{\varepsilon_t}'\Sigma\underline{\varepsilon_t} + \sigma_u^2 \end{aligned} \tag{2.3}$$

Interestingly, we can observe the two kinds of conditional heteroskedasticity process from (2.3). The diagonality of Σ specifies the linear *ARCH* process proposed by Engle (1982), while the non-diagonality of Σ specifies another $ARCH$ process with additional cross-product terms of the past innovations. We call the latter the augmented autoregressive conditional heteroskedasticity ($AARCH$) to differentiate it from the simple $ARCH$.

As summarized in Table 1 below, a variety of linear regression models are obtained as special cases of the model (2.1) according to the specification of ϕ and Σ. Furthermore, our unified framework is quite flexible since the order of autocorrelation can be selected differently from that of $ARCH$ without any major adjustment to the model. This can be done by specifying ϕ and Σ so that irrelevant elements should be placed by zeros. For example, the mth-order autocorrelation combined with the kth-order $ARCH$ will be obtained by formulating $\phi = (\phi_1, \phi_2, \cdots, \phi_m, 0, \cdots, 0)'$ and $\Sigma =$diag $(\gamma_1, \gamma_2, \cdots, \gamma_k, 0, \cdots, 0)$. For analytical convenience, the same orders for both AR and $ARCH$ processes will be maintained throughout.

The model (2.1) could be generalized by making the regression parameters β also stochastic. It will introduce the unconditional heteroskedasticity indirectly [see Breush and Pagan (1979)]. However, for the purpose of considering autocorrelation and $ARCH$ simultaneously, the model (2.1) is sufficient. We should also mention that there are various ways to introduce autocorrelation and $ARCH$ jointly. For example, another way would be to take

$$\varepsilon_t = (u_t + \phi'\underline{\varepsilon_t})(1 + \underline{\varepsilon_t}'\Sigma\underline{\varepsilon_t})^{\frac{1}{2}}$$

so that $E(\varepsilon_t \mid \Psi_{t-1}) = \phi'\underline{\varepsilon_t}(1+\underline{\varepsilon_t}'\Sigma\underline{\varepsilon_t})^{\frac{1}{2}}$ and $Var(\varepsilon_t \mid \Psi_{t-1}) = \sigma_u^2(1+\underline{\varepsilon_t}'\Sigma\underline{\varepsilon_t})$ However, the advantage of the model (2.1) is that it allows us to handle the verification of stationarity conditions, testing for different restrictions and estimation much easier. Moreover, note that our model could be generalized further to broaden the information set Ψ_{t-1} [see Tsay (1987)].

3. CONDITIONS FOR STATIONARITY

In this section we examine the necessary and sufficient conditions for the process of random coefficient autoregressive disturbances to be covariance stationary. Then we show that the stationary conditions for the pth-order linear ARCH process discussed in detail in Engle (1982) can be easily derived as a special case.

In statistics literature, the stationarity conditions for the autoregressive series with random parameters has been extensively studied. Andel (1976) derived the necessary and sufficient conditions for a univariate autoregressive series with stochastic coefficients and showed that the covariance function of a stationary autoregressive series with random parameters satisfies the same Yule-Walker equations as in the usual autoregressive model with fixed parameters. Nicholls and Quinn (1982) extended the results of Andel to the multivariate autoregressive model with random coefficients. Based on Andel (1976) and Nicholls and Quinn (1982), Nicholls (1986) provided simple conditions that are easy to check for second-order stationarity of the univariate autoregressive series with random coefficients [see also Ray (1983)]. Following Nicholls (1986), we now state the stationarity conditions for the error process of the model (2.1) as Proposition 1.

PROPOSITION 1. : *In addition to Assumptions 1–3, let* Ξ_t *be the* σ*-field generated by* $\{u_s, \eta_s; s \leq t\}$ *and*

$$M = \begin{bmatrix} 0_{(p-1)\times 1} & I_{p-1} \\ \phi_p & \phi'_{-p} \end{bmatrix}$$

where $0_{(p-1)\times 1}$ *is the* $(p-1) \times 1$ *null vector,* I_{p-1} *is the* $(p-1) \times (p-1)$ *identity matrix and* $\phi_{-p} = (\phi_{p-1}, \cdots, \phi_1)'$. *Then the pth-order autoregressive disturbances process with random coefficients* ($\varepsilon_t = \sum_{j=1}^{p} (\phi_j + \eta_{jt})\varepsilon_{t-j} + u_t$, *where* u_t *is a white noise) has a unique* Ξ_t*-measurable second-order stationary solution if and only if (1)* M *has all its eigenvalues within the unit circle and (2)* $(vec(\Sigma))'\mathrm{a} < 1$, *where* a *is the last column of the matrix* $(I - M \otimes M)^{-1}$.

PROOF: see Theorem 2 and Lemma 3 of Andel (1976) and Corollary 2.3.2 of Nicholls and Quinn (1982).

This result shows that the stationarity conditions for a particular disturbance process discussed in section 2 can be derived simply by imposing some restrictions on ϕ and Σ. It is easily seen that the eigenvalues of the matrix M are the same as the roots of the polynomial $z^p - \phi_1 z^{p-1} - \phi_2 z^{p-2} - \cdots - \phi_p = 0$. This implies that condition (1) of Proposition 1 describes the stationarity condition for the pth-order autoregressive ($AR(p)$) process. If there is no autocorrelation, all the eigenvalues of the matrix M are zeros and condition (1) is automatically satisfied. Condition (2) of Proposition 1 can be viewed as the stationarity conditions for conditionally heteroskedastic process in the presence of autocorrelation. If there is no autocorrelation and Σ is a diagonal matrix, this condition turns out to be exactly identical to the stationarity condition for the simple $ARCH$ as discussed in Theorem 2 of Engle (1982) and Theorem 1 of Milhoj (1985). If there is no conditional heteroskedasticity, Σ becomes the null matrix and condition (2) is automatically satisfied.

To investigate the validity of the stationarity conditions for the Engle's $ARCH$ process in the presence of autocorrelation, we first consider the pth-order linear $ARCH$ process specified as $h_t = \sigma_u^2 + \gamma_1 \varepsilon_{t-1}^2 + \cdots + \gamma_p \varepsilon_{t-p}^2$. In this case, $\Sigma = diag(\gamma_1, \gamma_2, \cdots, \gamma_p)$ and a is a $p^2 \times 1$ vector with one as elements corresponding to $\gamma_j, j = 1, 2, \cdots, p$. Given regularity conditions $\gamma_j \geq 0, j = 1, 2, \cdots, p$, the required stationarity condition is summarized as $\sum_{j=1}^{p} \gamma_j < 1$. However, Proposition 1 can be easily used to show that this condition is not sufficient to ensure stationarity in the presence of autocorrelation. We now consider the pth-order linear $ARCH$ combined with $AR(p)$ process. For this special case, after some simplications we can state the two conditions for stationarity as $\omega(\phi) \sum_{j=1}^{p} \gamma_j < 1$, where $\omega(\cdot)$ is a function of $(\phi_1, \phi_2, \cdots, \phi_p)$, and the roots of the polinomial $z^p - \phi_1 z^{p-1} - \phi_2 z^{p-2} - \cdots - \phi_p = 0$ lie inside the unit circle. It is clear that the influence of autocorrelation which is exercised through $\omega(\phi)$ may invalidate the stationarity conditions for the simple $ARCH$ process. To see it clearly, let us consider an $AR(2)$ process with a second-order $ARCH$. For this case,

$$M = \begin{bmatrix} 0 & 1 \\ \phi_2 & \phi_1 \end{bmatrix}$$

and it can be shown that the stationarity condition (2) is $\omega(\phi)(\gamma_1 + \gamma_2) < 1$, where

$$\omega(\phi) = \frac{1 - \phi_1^2}{(1 + \phi_2^2)(1 - \phi_2 - 2\phi_1^2 + \phi_1^4 - \phi_1^2 \phi_2^2)}.$$

To simplify it further, take the case of first- order $ARCH$ with an $AR(1)$ process, so that $\phi_2 = \gamma_2 = 0$. Then the stationarity condition reduces to $\gamma_1/(1-\phi_1^2) < 1$. In the absence of autocorrelation the condition is only $\gamma_1 < 1$. Figure 1 illustrates this case. Given ϕ_1^*, the range of γ_1 for stationarity is given by $[0, \gamma_1^*)$. This range of γ_1 is a subset of the interval $[0,1)$ which is the stationarity region in the absence of autocorrelation. This example makes it clear that the presence of aucorrelation can make a stationary $ARCH$ process nonstationary and as a consequence close attention should be paid to the interaction between AR and $ARCH$ processes. In the next section, we will investigate how the tests for $AARCH$ or $ARCH$ are affected by the presence of autocorrelation and vice versa.

4. THE TEST STATISTICS

4.1 The Likelihood Function and The Information Matrix

Before proceeding to derive the test statistics, some discussion of the likelihood function and the information matrix is provided because the approach will be based on the Lagrange multiplier (LM) test of Rao (1948) and Silvey (1959). It is assumed that the stationarity conditions for our model are satisfied. Recognizing that Σ is symmetric, let $\gamma = vech(\Sigma)$, where $vech(\Sigma)$, the 'vector-half' of Σ, is a column vector obtained by stacking the elements of the lower triangle of Σ [see Nicholls and Quinn (1982), p.12]. To rewrite $Var(\varepsilon_t \mid \Psi_{t-1})$ in terms of γ instead of Σ, we define a selection matrix K_p such that for a matrix A, $vech(A) = K_p vec(A)$ [see Nicholls and Quinn (1982), pp.12–13]. Rewriting the conditional variance,

$$h_t = Var(\varepsilon_t \mid \Psi_{t-1}) = \gamma' Z_t + \sigma_u^2 \tag{4.1}$$

where $Z_t = K_p vec(\underline{\varepsilon_t}\underline{\varepsilon_t}')$.

Let $L_N(\theta)$ denote the likelihood function conditional on Ψ_{t-1}, where $\theta = (\beta', \phi', \gamma', \sigma_u^2)'$ is an $[k + p + (\frac{p(p+1)}{2}) + 1]$ vector of all parameters in the model and N denotes the total number of observations which are used effectively. The likelihood function $L_N(\theta)$ is assumed to be the true one and to be sufficiently regular to give the familiar asymptotic results. Under the normality assumtion, the log-likelihood function $l_N(\theta)$ is

$$l_N(\theta) = const - \frac{1}{2}\sum_{t=1}^{N} ln(h_t) - \frac{1}{2}\sum_{t=1}^{N}\frac{1}{h_t}(\varepsilon_t - \phi'\underline{\varepsilon_t})^2 \tag{4.2}$$

Note that $v_t = \varepsilon_t - \phi'\underline{\varepsilon_t}$ is a function of β and ϕ. Explicitly, $v_t = (y_t - \underline{y_t}'\phi) - (x_t - \underline{x_t}'\phi)'\beta$, where $\underline{y_t} = (y_{t-1}, \cdots, y_{t-p})'$ and $\underline{x_t} = (x_{t-1}, \cdots, x_{t-p})'$. The first and second partial derivatives of $l_N(\theta)$ with respect to θ can be expressed as follows

$$\frac{\partial l_N(\theta)}{\partial\theta} = -\frac{1}{2}\sum_{t=1}^{N}\frac{1}{h_t}\left(\frac{\partial h_t}{\partial\theta}\right) - \frac{1}{2}\sum_{t=1}^{N}\frac{1}{h_t^2}\left[2v_t\left(\frac{\partial v_t}{\partial\theta}\right)h_t - v_t^2\left(\frac{\partial h_t}{\partial\theta}\right)\right] \tag{4.3}$$

and

$$\begin{aligned}\frac{\partial^2 l_N(\theta)}{\partial\theta\partial\theta'} &= \frac{1}{2}\sum_{t=1}^{N}\frac{1}{h_t^2}\left(\frac{\partial h_t}{\partial\theta}\right)\left(\frac{\partial h_t}{\partial\theta'}\right) - \frac{1}{2}\sum_{t=1}^{N}\frac{1}{h_t}\left(\frac{\partial^2 h_t}{\partial\theta\partial\theta'}\right)\\ &- \frac{1}{2}\sum_{t=1}^{N}\left[\frac{1}{h_t}2v_t\left(\frac{\partial^2 v_t}{\partial\theta\partial\theta'}\right) + \frac{1}{h_t}\left(\frac{\partial v_t}{\partial\theta}\right)\left(\frac{\partial v_t}{\partial\theta'}\right) - \frac{1}{h_t^2}4v_t\left(\frac{\partial v_t}{\partial\theta}\right)\left(\frac{\partial h_t}{\partial\theta'}\right)\right]\\ &+ \frac{1}{2}\sum_{t=1}^{N}\left[\frac{1}{h_t^2}v_t^2\left(\frac{\partial^2 h_t}{\partial\theta\partial\theta'}\right) - \frac{1}{h_t^3}2v_t^2\left(\frac{\partial h_t}{\partial\theta}\right)\left(\frac{\partial h_t}{\partial\theta'}\right)\right]\end{aligned} \tag{4.4}$$

The detailed derivation with respect to each parameter vector is provided in the Appendix A.

Now we define the asymptotic information matrix $I(\theta)$ as $I(\theta) = -\frac{1}{N}E\left(\frac{\partial^2 l_N(\theta)}{\partial\theta\partial\theta'}\right)$. Using the property of iterated expectation on the information set Ψ_{t-1}, we can simplify our expectation procedure greatly. Then the information matrix $I(\theta)$ is found to be block diagonal between θ_1 and θ_2, where $\theta = (\theta_1', \theta_2')'$ with $\theta_1 = (\beta', \phi')'$ and $\theta_2 = (\gamma', \sigma_u^2)'$. Here θ_1 and θ_2 could be regarded as mean parameters and variance parameters, respectively [see equations (2.2) and (4.1)]. The detaiied derivation of the information matrix $I(\theta)$ is given in the Appendix B. In practice, it will be necessary to get a consistent estimate of the information matrix $I(\theta)$, say $\hat{I}$, if we wish to carry out tests of hypotheses. From the results of the Appendix B, it is quite a straightforward matter to construct $\hat{I}$. The submatrices of $\hat{I}$ are given by

$$\hat{I} = \begin{bmatrix} \hat{I}_{11} & 0 & 0 & 0 \\ 0 & \hat{I}_{22} & 0 & 0 \\ 0 & 0 & \hat{I}_{33} & \hat{I}_{34} \\ 0 & 0 & \hat{I}_{43} & \hat{I}_{44} \end{bmatrix} \qquad (4.5)$$

where

$$\hat{I}_{11} = \frac{1}{N}\sum_{t=1}^{N}\frac{1}{\hat{h}_t}(x_t - \underline{x_t}'\hat{\phi})(x_t - \underline{x_t}'\hat{\phi})'$$

$$\hat{I}_{22} = \frac{1}{N}\sum_{t-1}^{N}\frac{1}{\hat{h}_t}\underline{\hat{\varepsilon}_t\hat{\varepsilon}_t}'$$

$$\hat{I}_{33} = \frac{1}{2N}\sum_{t=1}^{N}\frac{1}{\hat{h}_t^2}\hat{Z}_t\hat{Z}_t'$$

$$\hat{I}_{34} = \frac{1}{2N}\sum_{t=1}^{N}\frac{1}{\hat{h}_t^2}\hat{Z}_t'$$

$$\hat{I}_{44} = \frac{1}{2N}\sum_{t=1}^{N}\frac{1}{\hat{h}_t^2}$$

Note that when carrying out the LM tests, quantities with the hat are evaluated at the maximum likelihood estimates (MLE) of θ under the null hypothesis. The LM test is based on the score vector $d = \partial l_N(\theta)/\partial\theta$, which can be partitioned as $d = (d_1', d_2', d_3', d_4)'$ to be conformable with $\theta = (\beta', \phi', \gamma', \sigma_u^2)'$. Let $\hat{d}$ be the score vector evaluated at the restricted MLE of θ. Then the LM statistic is $LM = \frac{1}{N}\hat{d}'\hat{I}^{-1}\hat{d}$ which, under the null hypothesis, has a limiting χ^2 distribution with m degrees of freedom , where m is the number of restrictions imposed by the null hypothesis. From the expression of $\hat{I}_{22}$, it is clear that the tests for autocorrelation crucially depend on the estimate of γ (through h_t) and therefore standard tests for autocorrelation will give misleading results if the presence of $ARCH$ is neglected. In the next sections, we discuss the tests for $AARCH$ or $ARCH$ in the presence of AR and vice versa.

4.2 Testing for AARCH (ARCH) in the Presence of Autocorrelation

In our model, to test for the $AARCH$ (or $ARCH$) process is to test whether or not the coefficients of autoregressive disturbances are varying. Obviously, the hypothesis to be tested is that $H_0 : \gamma = vech(\Sigma) = 0$. However, some analytical difficulties appear since under the null hypothesis the values of γ lies on the boundary of the parameter space. In this context, the standard theory associated with tests based on maximum likelihood estimators will not be valid. Large sample properties of the maximum likelihood estimators and associated tests in boundary situations have been examined by Chernoff (1954), Moran (1971), Chant (1974) and Self and Liang (1987). One important general result from their investigation is that the Wald (W) and the likelihood ratio (LR) tests in the boundary situation will not follow their usual asymptotic χ^2 distribution, while the asymptotic properties of the LM test are not altered. As a result, it has been argued that the LM test is paricularly suitable for testing the hypotheses under which parameter values are at the boundary of the parameter space [see Godfrey (1988), p.95].

Following Moran (1971) and Chant (1974)'s findings, a LM statistic for testing $H_0 : \gamma = 0$ in the presence of autocorrelation, denoted by $LM_{AARCH|AR}$, is derived. Under the null hypothesis, our model is reduced to a linear regression model with $AR(p)$ process. The restricted maximum likelihood estimates of β, ϕ and σ_u^2 are easily obtained from the most of computer packages. Then we can define the two kinds of residuals $\hat{\varepsilon}_t$ and $\hat{u}_t$ as $\hat{\varepsilon}_t = y_t - x_t'\hat{\beta}$ and $\hat{u}_t = \hat{\varepsilon}_t - \hat{\phi}'\underline{\hat{\varepsilon}_t}$, respectively. The basic results are formally stated as Proposition 2.

PROPOSITION 2. : *Let* $\underline{\hat{\varepsilon}_t} = (\hat{\varepsilon}_{t-1}, \cdots, \hat{\varepsilon}_{t-p})'$ *and* $\hat{Z}_t = K_p vec(\underline{\hat{\varepsilon}_t}\underline{\hat{\varepsilon}_t}')$ *and* $W_t = (1, \hat{Z}_t')'$. *Then the LM statistic for testing* $H_0 : \gamma = vech(\Sigma) = 0$ *in the presence of autocorrelation* ($LM_{AARCH|AR}$) *can be expressed as one half of the explained sum of squares from the regression of* $f_t = (\frac{\hat{u}_t^2}{\hat{\sigma}_u^2} - 1)$ *on* W_t *and has an asymptotic* χ^2 *distribution with* $\frac{p(p+1)}{2}$ *degrees of freedom when the null hypothesis is true.*

PROOF: Using the block diagonality of the information matrix $I(\theta)$, and hence its consistent estimate $\hat{I}$, discussed in section 4.1 and the inverse matrix formula, $LM_{AARCH|AR}$

can be written as

$$LM_{AARCH|AR} = \frac{1}{N}\hat{d}_3' C^{-1}\hat{d}_3 \tag{4.6}$$

where

$$\hat{d}_3 = \frac{1}{2\hat{\sigma}_u^2}\sum_{t=1}^{N}\left(\frac{\hat{u}_t^2}{\hat{\sigma}_u^2} - 1\right)\hat{Z}_t$$

$$C = (\hat{I}_{33} - \hat{I}_{43}\hat{I}_{44}^{-1}\hat{I}_{43}')$$

$$= \left[\left(\frac{1}{2N}\sum_{t=1}^{N}\frac{1}{\hat{\sigma}_u^4}\hat{Z}_t\hat{Z}_t'\right) - \left(\frac{1}{2N}\sum_{t=1}^{N}\frac{1}{\hat{\sigma}_u^4}\hat{Z}_t\right)\left(\frac{1}{2N\hat{\sigma}_u^4}\right)^{-1}\left(\frac{1}{2N}\sum_{t=1}^{N}\frac{1}{\hat{\sigma}_u^4}\hat{Z}_t'\right)\right]$$

Letting $\bar{Z} = \frac{1}{N}\sum_{t=1}^{N}\hat{Z}_t$, we can rewrite C as

$$C = \left[\frac{1}{2N\hat{\sigma}_u^4}\sum_{t=1}^{N}(\hat{Z}_t - \bar{Z})(\hat{Z}_t - \bar{Z})'\right].$$

Then the statistic can be expressed as

$$LM_{AARCH|AR} = \frac{1}{2}\left(\sum_{t=1}^{N} f_t\hat{Z}_t\right)'\left(\sum_{t=1}^{N}(\hat{Z}_t - \bar{Z})(\hat{Z}_t - \bar{Z})'\right)^{-1}\left(\sum_{t=1}^{N} f_t\hat{Z}_t\right) \tag{4.7}$$

where $f_t = \left(\frac{\hat{u}_t^2}{\hat{\sigma}_u^2} - 1\right)$. To write $LM_{AARCH|AR}$ in matrix form, define $f = (f_1, \cdots, f_N)'$, $\hat{Z} = (\hat{Z}_1, \cdots, \hat{Z}_N)'$ and ι as an $(N \times 1)$ unit vector. Then the statistic takes the form of

$$LM_{AARCH|AR} = \frac{1}{2}f'\hat{Z}\left[(\hat{Z} - \iota\bar{Z}')'(\hat{Z} - \iota\bar{Z}')\right]^{-1}\hat{Z}'f \tag{4.8}$$

Using the fact that $f'\iota = 0$, we have

$$LM_{AARCH|AR} = \frac{1}{2}f'(\hat{Z} - \iota\bar{Z}')\left[(\hat{Z} - \iota\bar{Z}')'(\hat{Z} - \iota\bar{Z}')\right]^{-1}(\hat{Z} - \iota\bar{Z}')'f$$

$$= \frac{1}{2}f'W(W'W)^{-1}W'f \tag{4.9}$$

where $W = (\iota, \hat{Z})$, which is equal to one half of the explained sum of squares from the regression of f on W. By the asymptotic properties of the maximum likelihood estimators, $LM_{AARCH|AR}$ follows an asymptotic χ^2 distribution with $\frac{p(p+1)}{2}$ degrees of freedom. Q.E.D.

A simple form which is asymptotically equivalent to and is also computationally more convenient than $LM_{AARCH|AR}$ can be derived by noting that under the conditional normality assumption, $Var(f_t \mid \Psi_{t-1}) = 2$ and $plim\left(\frac{f'f}{N}\right) = 2$. The alternative statistics is

$$LM^*_{AARCH|AR} = N\frac{f'W(W'W)^{-1}W'f}{f'f} = NR^2 \tag{4.10}$$

where R^2 is the square of the multiple correlation in the regression of f on W. Since the squared multiple correlation of a linear regression is not altered by adding a constant to the dependent variable and by multiplying it by a scalar, R^2 can be easily calculated from the regression of $\hat{u}_t^2$ on $(1, \hat{Z}'_{1t}, \hat{Z}'_{2t})'$ where $\hat{Z}_{1t} = (\hat{\varepsilon}^2_{t-1}, \hat{\varepsilon}^2_{t-2}, \cdots, \hat{\varepsilon}^2_{t-p})'$ and $\hat{Z}_{2t} = (\hat{\varepsilon}_{t-1}\hat{\varepsilon}_{t-2}, \hat{\varepsilon}_{t-1}\hat{\varepsilon}_{t-3}, \cdots, \hat{\varepsilon}_{t-p+1}\hat{\varepsilon}_{t-p})'$. Note that $\hat{Z}_{1t}$ is a vector of p lagged squared residuals and $\hat{Z}_{2t}$ is a $\frac{p(p-1)}{2} \times 1$ vector of distinct cross products among p lagged residuals. Following the line of argument in Koenker (1981) and Koenker and Bassett (1982), it is expected that when the u_t are not normally distributed, $LM^*_{AARCH|AR}$ is still an appropriate statistic while $LM_{AARCH|AR}$ may have asymptotically incorrect size and its power which is extremely sensitive to the kurtosis of the distribution of u_t.

By the same argument, we now can derive the LM test for $ARCH$ in the presence of autocorrelation ($LM_{ARCH|AR}$) simply by assuming that the covariance matrix Σ is diagonal. If Σ is a diagonal matrix, $\hat{Z}_t$ is reduced to $\hat{Z}_{1t}$, a vector of p lagged values of $\hat{\varepsilon}^2_t$. Let $\hat{Z}_1 = (\hat{Z}_{11}, \hat{Z}_{12}, \cdots, \hat{Z}_{1N})'$ and define $W_1 = (\iota, \hat{Z}_1)$. Then the statistic can be written as

$$LM_{ARCH|AR} = \frac{1}{2}f'W_1(W_1'W_1)^{-1}W_1'f \tag{4.11}$$

and a simple statistic which is asymptotically equivalent to $LM_{ARCH|AR}$ is given by

$$LM^*_{ARCH|AR} = N\frac{f'W_1(W_1'W_1)^{-1}W_1'f}{f'f} = NR^2 \tag{4.12}$$

where R^2 is the square of multiple correlation between f and W_1. The R^2 of $LM^*_{ARCH|AR}$ is easily calculated from the regression of $\hat{u}_t^2$ on $(1, \hat{\varepsilon}^2_{t-1}, \cdots, \hat{\varepsilon}^2_{t-p})$. The statistic will follow asymptotically a χ^2 distribution with p degrees of freedom.

$LM_{ARCH|AR}$ and $LM^*_{ARCH|AR}$ can be regarded as the tests for $ARCH$ disturbances which are 'robust' to autocorrelation. However, it should be noted that the validity of these tests again depends on the correct specification of the AR process. The Engle's LM tests for $ARCH$ process, denoted by LM_{ARCH} and LM^*_{ARCH}, are also easily constructed in our framework by assuming no autocorrelation ($\phi = 0_{p\times 1}$) in addition to the diagonality of the covariance matrix Σ. The statistics take the same forms as (4.11) and (4.12) except that $\hat{u}_t$ and $\hat{\sigma}_u^2$ should be replaced by the OLS residual $\hat{\varepsilon}_t$ for the model $y_t = x_t'\beta + \varepsilon_t$ and the estimated residual variance $\hat{\sigma}_\epsilon^2 = \sum_{t=1}^N \hat{\varepsilon}_t^2/N$,respectively, and the elements of W_{1t} are one and p lagged values of the squared OLS residual $\hat{\varepsilon}_t^2$.

From our framework, the $AARCH$ process appears quite naturally as a form of conditional variance. It can be regarded as an extension of the Engle's original $ARCH$. Clearly, in our framework, the LM tests for $AARCH$ process in the absence of autocorrelation (LM_{AARCH} and LM^*_{AARCH}) are similar in form to LM_{ARCH} and LM^*_{ARCH} except that W_1 with the OLS residuals should be replaced by W with the OLS residuals. Then LM^*_{AARCH} can be calculated as NR^2, where R^2 is the square of multiple correlation from the regression of $\hat{\varepsilon}_t^2$ on $(1, \hat{\varepsilon}_{t-1}^2, \cdots, \hat{\varepsilon}_{t-p}^2, \hat{\varepsilon}_{t-1}\hat{\varepsilon}_{t-2}, \cdots, \hat{\varepsilon}_{t-p+1}\hat{\varepsilon}_{t-p})$. The statistic will have an asymptotic χ^2 distribution with $\frac{p(p+1)}{2}$ degrees of freedom.

4.3 Testing for Autocorrelation in the Presence of AARCH (ARCH)

In this section, a simple LM test for autocorrelation which is specifically robust to a specified form of $AARCH$ or $ARCH$ disturbances will be developed. Since a specific form of conditional heteroskedasticity is parameterized, it is obvious that the performance of the test will depend on how accurate the $AARCH$ or $ARCH$ representation of the conditional variance is. The null hypothesis is formulated as $H_0 : \phi = 0$. Under the null hypothesis, our model is reduced to a linear regression model with the $AARCH$ disturbances. Let $\hat{\theta}$ be the maximum likelihood estimators for the $AARCH$ (or $ARCH$ if Σ is diagonal) regression model. Then we can define $\hat{\varepsilon}_t = y_t - x_t'\hat{\beta}$ and $\hat{h}_t = \hat{\gamma}'\hat{Z}_t + \hat{\sigma}_u^2$. The derivation of the tests is very simple since under the null hypothesis $v_t = \varepsilon_t$ and hence $E(v_t \mid \Psi_{t-1}) = 0$

and $E(\underline{\varepsilon_t} \mid \Psi_{t-1}) = 0$, for all t. Furthermore $E(\underline{\varepsilon_t \varepsilon_t}')$ becomes a diagonal matrix which can be consistently estimated by $\hat{\Omega} = diag(\hat{\varepsilon}_{t-1}^2, \cdots, \hat{\varepsilon}_{t-p}^2)$. The basic result is stated in the following proposition.

PROPOSITION 3. : *Let* $\underline{\hat{\varepsilon}_t} = (\hat{\varepsilon}_{t-1}, \cdots, \hat{\varepsilon}_{t-p})'$ *and* $\hat{Z}_t = K_p vec(\underline{\hat{\varepsilon}_t \hat{\varepsilon}_t}')$. *Then the LM statistic for testing* $H_0 : \phi = 0$ *(no autocorrelation) in the presence of AARCH disturbances is*

$$LM_{AR|AARCH} = \left(\sum_{t=1}^{N} \frac{1}{\hat{h}_t} \hat{\varepsilon}_t \underline{\hat{\varepsilon}_t}\right)' \left(\sum_{t=1}^{N} \frac{1}{\hat{h}_t} \hat{\Omega}_t\right)^{-1} \left(\sum_{t=1}^{N} \frac{1}{\hat{h}_t} \hat{\varepsilon}_t \underline{\hat{\varepsilon}_t}\right) \quad (4.13)$$

where $\hat{h}_t = \hat{\gamma}' \hat{Z}_t + \hat{\sigma}_u^2$ *with* $\hat{\gamma}$ = *AARCH estimate of* γ *and* $\hat{\Omega}_t$ =*diag* $(\hat{\varepsilon}_{t-1}^2, \cdots, \hat{\varepsilon}_{t-p}^2)$ *and this has an asymptotic* χ^2 *distribution with p degrees of freedom when the null hypothesis is true.*

PROOF: Under the null hypothesis,

$$LM_{AR|AARCH} = \frac{1}{N} \hat{d}_2' \hat{I}_{22}^{-1} \hat{d}_2$$

where $\hat{d}_2 = \sum_{t=1}^{N} \hat{h}_t^{-1} \hat{\varepsilon}_t \underline{\hat{\varepsilon}_t}$ and $\hat{I}_{22} = \frac{1}{N} \sum_{t=1}^{N} \hat{h}_t^{-1} \hat{\Omega}_t$. Hence the results. Q.E.D.

Here, we note that the statistic $LM_{AR|AARCH}$ can be built based on the transformed residuals which are obtained by deviding the residual and the lagged residuals by the estimate of the conditional "standard deviation" for the tth observation. Let $\tilde{\varepsilon}_t = \hat{\varepsilon}_t / \sqrt{\hat{h}_t}$ and let $\tilde{\varepsilon}_{t-j} = \hat{\varepsilon}_{t-j} / \sqrt{\hat{h}_t}$, $j = 1, 2, \cdots, p$. Also define $\underline{\tilde{\varepsilon}_t} = (\tilde{\varepsilon}_{t-1}, \tilde{\varepsilon}_{t-2}, \cdots, \tilde{\varepsilon}_{t-p})'$ and $\tilde{\Omega}_t = diag(\tilde{\varepsilon}_{t-1}^2, \tilde{\varepsilon}_{t-2}^2, \cdots, \tilde{\varepsilon}_{t-p}^2)$. Then $LM_{AR|AARCH}$ can be expressed as

$$LM_{AR|AARCH} = \left(\sum_{t=1}^{N} \tilde{\varepsilon}_t \underline{\tilde{\varepsilon}_t}\right)' \left(\sum_{t=1}^{N} \tilde{\Omega}_t\right)^{-1} \left(\sum_{t=1}^{N} \tilde{\varepsilon}_t \underline{\tilde{\varepsilon}_t}\right)$$

It is clear that $LM_{AR|ARCH}$ will have the same formula as $LM_{AR|AARCH}$ except that Σ is a diagonal matrix and all terms with hat are evaluated at the MLE of parameters for the Engle's $ARCH$ regression model. In the above discussion, we showed that $LM_{AR|AARCH}$ and $LM_{AR|ARCH}$ can be constructed on the basis of the transformed residuals. As a consequence, they are applicable in more general circumstances. In this sense, our results can be constrasted with the $ARCH$-corrected tests for autocorrelation suggested by Diebold (1986) in the time series framework, which are based on the sample autocorrelation.

If we remove the $AARCH$ ($ARCH$) effect, $LM_{AR|AARCH}$ ($LM_{AR|ARCH}$) will be ended up with the familiar LM test for autocorrelation (LM_{AR}) In this case, our model is reduced to a standard linear regression model under the null hypothesis. The $AARCH$ ($ARCH$) residual $\hat{\varepsilon}_t$ and the estimated conditional variance $\hat{h}_t$ should be replaced by the OLS residual $\hat{\varepsilon}_t$ and the OLS estimated variance $\hat{\sigma}_\varepsilon^2 = \frac{1}{N}\sum_{t=1}^{N}\hat{\varepsilon}_t^2$. Let $\hat{r}_k$ be the residual autocorrelation of order k defined by $\hat{r}_k = \sum_{t=1}^{N}\hat{\varepsilon}_t\hat{\varepsilon}_{t-k} / \sum_{t=1}^{N}\hat{\varepsilon}_t^2$. Then after some algebra, we can get

$$\begin{aligned} LM_{AR} &= \frac{1}{\hat{\sigma}_\varepsilon^2}\left(\sum_{t=1}^{N}\hat{\varepsilon}_t\underline{\hat{\varepsilon}_t}\right)'\left(\sum_{t=1}^{N}\hat{\Omega}_t\right)^{-1}\left(\sum_{t=1}^{N}\hat{\varepsilon}_t\underline{\hat{\varepsilon}_t}\right) \\ &= \frac{1}{\hat{\sigma}_\varepsilon^2}\sum_{k=1}^{p}\left[\frac{\left(\sum_{t=1}^{N}\hat{\varepsilon}_t\hat{\varepsilon}_{t-k}\right)^2}{\sum_{t=1}^{N}\hat{\varepsilon}_{t-k}^2}\right] \end{aligned} \tag{4.14}$$

Since for a large N, $\sum_{t=1}^{N}\hat{\varepsilon}_{t-k}^2 = \sum_{t=1}^{N}\hat{\varepsilon}_t^2$, $\quad k = 1, 2, \cdots, p$, we can derive an alternative LM test which is asymptotically equivalent to LM_{AR}. Let us denote it as LM_{AR}^* and define $\hat{r} = (\hat{r}_1, \cdots, \hat{r}_k)'$. Then we get

$$LM_{AR}^* = N\hat{r}'\hat{r} \tag{4.15}$$

which is exactly the conventional LM test for autocorrelation proposed by Breush (1978) and Godfrey (1978). It will have an asymptotic χ^2 distribution with p degrees of freedom when the null hypothesis is true.

4.4 Testing for Autocorrelation and AARCH (ARCH) Jointly

As one way to overcome partially the lack of robustness of one- directional specification test which is constructed to test the validity of one specification at one time, Bera and Jarque (1982) suggest developing joint tests which are capable of testing various specification errors simultaneously. Interestingly, a joint test statistic can be obtained, in certain situations, by adding up the component statistics under the null hypothesis. This is called the additivity property. For our discussion, the formal definition of additivity is provided.

DEFINITION. : *Let the null hypothesis be composed of the two exclusive restrictions* (H_A *and* H_B). *Let* T_{AB} *be a joint test for both restriction,* T_A *the test for* H_A *with the restrictions of* H_B *imposed and* T_B *the test for* H_B *with the restrictions of* H_A *imposed. Then a test based on a particular testing principle is said to be* <u>*additive*</u> *for the null hypothesis if* $T_{AB} = T_A + T_B$.

Recently, Bera and McKenzie (1987) discuss conditions for the additivity of the three testing principles: the LR, W and LM and recommend the LM approach in developing a joint test because of its computational advantage. In particular, they found that the necessary and sufficient condition for the LM test to be additive is the block diagonality of the information matrix among the testing parameters of the component hypotheses.

In this section, we discuss briefly the test statistics for testing autocorrelation and $AARCH$ (or $ARCH$) jointly. The additivity condition holds in our case since the information matrix is block diagonal between the two testing parameter vectors, $\phi = 0$ and $\gamma = 0$. Then the LM test for autocorrelation and $AARCH(ARCH)$ jointly, denoted by $LM_{AR+AARCH}$ ($LM_{AR+ARCH}$), will be constructed as the sum of LM statistics for testing the components of the null hypothesis separately as follows

$$LM_{AR+AARCH} = LM_{AR} + LM_{AARCH} \tag{4.16}$$

$$LM_{AR+ARCH} = LM_{AR} + LM_{ARCH} \tag{4.17}$$

It is straightforward to get the alternative statistics which are computationally simple and asymptotically equivalent to $LM_{AR+AARCH}$ and $LM_{AR+ARCH}$. The test statistics are

$$LM^*_{AR+AARCH} = LM^*_{AR} + LM^*_{AARCH} \tag{4.18}$$

$$LM^*_{AR+ARCH} = LM^*_{AR} + LM^*_{ARCH} \tag{4.19}$$

Note that since our model becomes a linear regression model with no autocorrelation and no conditional heteroskedasticity under the null hypothesis, all relevant quantities are obtained from the OLS estimation procedure. The additivity property of the LM tests derived above allows us to apply the individual tests separately and to combine them in order to form a joint test.

5. AN ILLUSTRATIVE EXAMPLE

To illustrate our methodology, we consider the problem of testing whether forecasts are unbiased estimates of the actual inflation rate using the Livingston biannual survey data from June 1952 to December 1985. Since July 1, 1946, this survey has reported experts' semi-annual predictions of a set of key economic variables including consumer price index. Starting with Turnovsky (1970), the Livingston data have been used extensively in the literature to test different economic hypotheses regarding price expectation [see, for example, Chan- Lee (1980), Brown and Maital (1981) and Figlewski and Wachtel (1981)]. The underlying model for testing unbiasedness of forecasts is

$$y_t = \beta_1 + \beta_2 x_t + \varepsilon_t \tag{5.1}$$

where y_t is the actual rate of inflation and x_t is the expected rate. Forecasts are said to be unbiased if $H_0 : \beta_1 = 0, \beta_2 = 1$ is true. This is essentially a test of weak-form of rationality in which actual and expected inflation rates are the same up to a white noise error term. To test H_0, a proper specification of the distribution of ε_t is crucial, and we will emphasize this point in the following discussion.

The first entry in Table 2 gives the *OLS* estimates. *LM* statistics for testing the presence of autocorrelation and *ARCH* based on the *OLS* residuals are given in the second columns of Table 3 and Table 4. These tests indicate the presence of autocorrelation and *ARCH* up to second order. We can also draw similar conclusion from the results in Table 5 where we present the autocorrelation of *OLS* residuals and their squares. As we noted before, these individual tests for *AR* (*ARCH*) are not valid in the presence of

$ARCH$ (AR). The MLEs with different orders for $ARCH$ and AR are also given in Table 2. These estimates are obtained using the $IMSL$ subroutine $ZXMIN$. First we note that augmented $ARCH$ part is not significant. When a second-order $ARCH$ model is estimated, the first-order $ARCH$ coefficient is also not significant. With only AR model, both first and second-order AR coefficients are found to be significant. However, the first-order AR coefficient becomes insignificant when AR is combined with $ARCH$ as seen from the estimates of Model (7). Looking at the log-likelihood function, we note that there is no improvement where only either AR or $ARCH$ is incorporated. However, when both AR and $ARCH$ are considered simultaneously, there is a substantial improvement from 213.456 to 219.719. Using these estimates we modify the conventional LM statistics (LM_{AR} and LM_{ARCH}). The results are summarized in Table 3 and Table 4. It is interesting to note that LM_{ARCH} is not effected very much in the presence of AR whereas LM_{AR} is effected substantially in the presence of $ARCH$; e.g., LM_{AR} reduces from 10.7732 to 5.4564 when $ARCH(\gamma_2)$ residuals are used, although in both cases the statistics are significant.

Given all the results, it appears that Model (7) in Table 2 may be an acceptable one. Further support for Model (7) comes from the results shown in Table 5 and Table 6. In Table 5, the autocorrelation of the standardized residuals

$$\hat{s}_t = \frac{\hat{\varepsilon}_t - \hat{\phi}_1 \hat{\varepsilon}_{t-1} - \hat{\phi}_2 \hat{\varepsilon}_{t-2}}{\sqrt{\hat{h}_t}},$$

and their squares do not indicate any kind of dependence. Moreover, from Table 6 which summarizes the White (1980)'s test, we see that unlike the OLS residuals these standardized residulas do not suffer from the presence of unconditional heteroskedasticity.

Lastly, coming back to the issue of testing the unbiasedness of experts' expectations, we present the relevant results in Table 7. The numerical magnitudes of the statistics are quite different. Except for $AR(\phi_1, \phi_2)$ model, in all cases we reject the unbiasedness hypothesis. Surprisingly, the OLS model gives similar result to our preferred Model (7). As also seen from Table 2, separately, $ARCH$ and AR have opposite effects on the estimates of β_1 and β_2. Possibly, due to this interaction, the results for OLS regression where both

AR and *ARCH* are ignored are similar to those for a model that incorporates *AR* and *ARCH*.

6. CONCLUDING REMARKS

It is clear that if we misspecify the conditional first moment (mean), inferences on higher order conditional moments will be very misleading. Within our framework, a number of interesting problems could be investigated. First, as we have done in the context of hypothesis testing, it would be interesting to study analytically the effect on estimation if we ignore (or misspecify) the autocorrelation or conditional heteroskedastic structure. Even if we concentrate only on heteroskedasticity, it would be interesting to see whether we will still get consistent estimates if *AARCH* is misspecified as *ARCH*. A recent paper by Pagan and Sabau (1987) sheds some light on this question. Second, it appears that with our random coefficient approach, we can express the *ARCH* type models in Kalman filter framework. This might facilitate the estimation procedure particularly under a multivariate setup.

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APPENDIX A

THE PARTIAL DERIVATIVES OF THE LOG-LIKELIHOOD FUNCTION

For our model the log-likelihood function $(l_N(\theta))$ conditional on the information set Ψ_{t-1} is given in the equation (4.2). Recall that $h_t = \gamma' Z_t + \sigma_u^2$ and $v_t = \varepsilon_t - \phi' \underline{\varepsilon_t} = (y_t - \underline{y_t}'\phi) - (x_t - \underline{x_t}'\phi)'\beta$. Then the first and second partial derivatives with respect to $\theta = (\beta', \phi', \gamma', \sigma_u^2)'$ are easily obtained. The first derivatives are

$$\frac{\partial l_N(\theta)}{\partial \beta} = \sum_{t=1}^{N} \frac{1}{h_t} v_t (x_t - \underline{x_t}'\phi)$$

$$\frac{\partial l_N(\theta)}{\partial \phi} = \sum_{t=1}^{N} \frac{1}{h_t} v_t \underline{\varepsilon_t}$$

$$\frac{\partial l_N(\theta)}{\partial \gamma} = -\frac{1}{2} \sum_{t=1}^{N} \frac{1}{h_t} Z_t + \frac{1}{2} \sum_{t=1}^{N} \frac{1}{h_t^2} v_t^2 Z_t$$

$$\frac{\partial l_N(\theta)}{\partial \sigma_u^2} = -\frac{1}{2} \sum_{t=1}^{N} \frac{1}{h_t} + \frac{1}{2} \sum_{t=1}^{N} \frac{1}{h_t^2} v_t^2$$

and the second derivatives are

$$\frac{\partial^2 l_N(\theta)}{\partial \beta \partial \beta'} = -\sum_{t=1}^{N} \frac{1}{h_t} (x_t - \underline{x_t}'\phi)(x_t - \underline{x_t}'\phi)'$$

$$\frac{\partial^2 l_N(\theta)}{\partial \phi \partial \beta'} = -\sum_{t=1}^{N} \frac{1}{h_t} (x_t - \underline{x_t}'\phi)\underline{\varepsilon_t}'$$

$$\frac{\partial^2 l_N(\theta)}{\partial \phi \partial \phi'} = -\sum_{t=1}^{N} \frac{1}{h_t} \underline{\varepsilon_t \varepsilon_t}'$$

$$\frac{\partial^2 l_N(\theta)}{\partial \gamma \partial \beta'} = -\sum_{t=1}^{N} \frac{1}{h_t^2} v_t (x_t - \underline{x_t}'\phi) Z_t'$$

$$\frac{\partial^2 l_N(\theta)}{\partial \gamma \partial \phi'} = -\sum_{t=1}^{N} \frac{1}{h_t^2} v_t \underline{\varepsilon_t} Z_t'$$

$$\frac{\partial^2 l_N(\theta)}{\partial \gamma \partial \gamma'} = \frac{1}{2} \sum_{t=1}^{N} \frac{1}{h_t^2} Z_t Z_t' - \sum_{t=1}^{N} \frac{1}{h_t^3} v_t^2 Z_t Z_t'$$

$$\frac{\partial^2 l_N(\theta)}{\partial\sigma_u^2\partial\beta'} = -\sum_{t=1}^{N}\frac{1}{h_t^2}v_t(x_t - \underline{x_t}'\phi)'$$

$$\frac{\partial^2 l_N(\theta)}{\partial\sigma_u^2\partial\phi'} = -\sum_{t=1}^{N}\frac{1}{h_t^2}v_t\underline{\varepsilon_t}'$$

$$\frac{\partial^2 l_N(\theta)}{\partial\sigma_u^2\partial\gamma'} = \frac{1}{2}\sum_{t=1}^{N}\frac{1}{h_t^2}Z_t' - \sum_{t=1}^{N}\frac{1}{h_t^3}v_t^2 Z_t'$$

$$\frac{\partial^2 l_N(\theta)}{\partial(\sigma_u^2)^2} = \frac{1}{2}\sum_{t=1}^{N}\frac{1}{h_t^2} - \sum_{t=1}^{N}\frac{1}{h_t^3}v_t^2.$$

APPENDIX B

THE DERIVATION OF THE INFORMATION MATRIX

Let $l_t(\theta)$ be the log-likelihood function for the t-th observation. Then the $I(\theta)$ can be redefined as

$$I(\theta) = -E\left(\frac{\partial^2 l_t(\theta)}{\partial\theta\partial\theta'}\right).$$

We now define

$$I_{\theta_i\theta_j} \equiv -E\left(\frac{\partial^2 l_t(\theta)}{\partial\theta_i\partial\theta_j'}\right)$$

where $\theta_i, \theta_j = \beta, \phi, \gamma, \sigma_u^2$.

Using our earlier specifications $E(\varepsilon_t \mid \Psi_{t-1}) = \phi'\underline{\varepsilon_t}$ and $Var(\varepsilon_t \mid \Psi_{t-1}) = h_t$, we have

$$E(v_t \mid \Psi_{t-1}) = E(\varepsilon_t \mid \Psi_{t-1}) - \phi'\underline{\varepsilon_t} = 0$$

$$E(v_t^2 \mid \Psi_{t-1}) = Var(\varepsilon_t \mid \Psi_{t-1}) = h_t$$

where $v_t = \varepsilon_t - \phi'\underline{\varepsilon_t}$. Now, taking expectation conditional on Ψ_{t-1} iteratively, we can easily show that $I_{\beta\gamma}, I_{\phi\gamma}, I_{\beta\sigma_u^2}$ and $I_{\phi\sigma_u^2}$ are null matrices. Further, we note that $I_{\beta\phi}$ is also a null matrix since each element of $\underline{\varepsilon_t}/h_t$ is an odd function of the lagged residuals $\underline{\varepsilon_t}$ which has a symmetric distribution. From these results, it is clear that the information matrix $I(\theta)$ is block diagonal between $\theta_1 = (\beta', \phi')'$ and $\theta_2 = (\gamma', \sigma_u^2)'$, and it is given by

$$I(\theta) = E \begin{bmatrix} \frac{1}{h_t}(x_t - \underline{x_t}'\phi)(x_t - \underline{x_t}'\phi)' & 0 & 0 & 0 \\ 0 & \frac{1}{h_t}\underline{\varepsilon_t \varepsilon_t}' & 0 & 0 \\ 0 & 0 & \frac{1}{2h_t^2} Z_t Z_t' & \frac{1}{2h_t^2} Z_t \\ 0 & 0 & \frac{1}{2h_t^2} Z_t' & \frac{1}{2h_t^2} \end{bmatrix}.$$

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Table 1: Processes of Disturbances

ϕ	Σ	Disturbance Process
$\phi = 0_{p \times 1}$	$\Sigma = 0_{p \times p}$	white noise
$\phi \neq 0_{p \times 1}$	$\Sigma = 0_{p \times p}$	AR
$\phi = 0_{p \times 1}$	Σ: diagonal	ARCH
$\phi = 0_{p \times 1}$	Σ: not diagonal	AARCH
$\phi \neq 0_{p \times 1}$	Σ: diagonal	AR and ARCH
$\phi \neq 0_{p \times 1}$	Σ: not diagonal	AR and AARCH

Table 2: Parameter Estimates of $ARCH$ and AR Models

Model	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\phi}_1$	$\hat{\phi}_2$	$\hat{\sigma}^2$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$	$l_N(\theta)$
(1)OLS	.004146	1.054500							213.456
	(2.178)	(12.036)							
(2)$AARCH$	.003728	1.127542			.000026	.226007	.622228	.214326	215.050
$(\gamma_1, \gamma_2, \gamma_3)$	(3.375)	(14.237)			(2.238)	(1.381)	(1.920)	(.509)	
(3)$ARCH$	.003787	1.117095			.000028	.182881	.702878		214.871
(γ_1, γ_2)	(3.462)	(13.976)			(2.359)	(.989)	(2.354)		
(4)$ARCH(\gamma_2)$	.003728	1.115447			.000038		.777891		213.631
	(.034)	(15.457)			(3.209)		(2.862)		
(5)$AR(\phi_1)$	.005149	.990161	.359050		.000089				213.990
	(.053)	(7.584)	(2.794)		(5.769)				
(6)$AR(\phi_1, \phi_2)$	.005930	.938918	.250781	.340140	.000081				214.143
	(1.537)	(5.685)	(3.460)	(3.190)	(5.677)				
(7)$ARCH(\gamma_2)$	.004504	1.037797	.268719	.332031	.000037		.507311		219.719
$+AR(\phi_1, \phi_2)$	(2.293)	(15.817)	(1.387)	(3.048)	(3.373)		(2.341)		

NOTE: Values in parentheses are t-statistics.

Table 3: LM Statistics for $ARCH$ in the Presence of AR

lag	LM_{ARCH}	$LM_{ARCH\|AR(\phi_1)}$	$LM_{ARCH\|AR(\phi_1,\phi_2)}$
1	3.3264	1.2804	3.6920
2	8.4305	6.9940	8.1705
3	0.6528	1.4464	3.0912
4	3.3705	0.9891	0.7245
5	1.7732	1.2400	1.5810
6	1.1102	0.9638	0.9699

NOTE: Values are the LM statistics for individual lag components which follow an asymptotic χ^2 distribution with 1 degree of freedom. The asymptotic critical values at 5 and 1 % significance levels are 3.841 and 6.635, respectively.

Table 4: LM Statistics for AR in the Presence of $ARCH$

lag	LM_{AR}	$LM_{AR\|ARCH(\gamma_2)}$	$LM_{AR\|ARCH(\gamma_1,\gamma_2)}$
1	7.6836	7.3849	4.4134
2	10.7732	5.4564	5.6407
3	0.0013	0.2788	0.0016
4	0.0595	2.4827	1.0473
5	2.8760	1.6129	2.3710
6	0.5661	1.4537	0.5191

NOTE: Values are the LM statistics for individual lag components which follow an asymptotic χ^2 distribution with 1 degree of freedom. The asymptotic critical values at 5 and 1 % significance levels are 3.841 and 6.635, respectively.

Table 5: Autocorrelation of OLS and Standardized Residuals and Their Squares

lag	$\hat{r}(\hat{\varepsilon}_t)$	$\hat{r}(\hat{s}_t)$	$\hat{r}(\hat{\varepsilon}_t^2)$	$\hat{r}(\hat{s}_t^2)$
1	0.33865	0.07013	0.22379	0.05175
2	0.40402	0.10426	0.35769	-0.09205
3	-0.00449	-0.11381	0.09894	0.06651
4	-0.03048	-0.14551	0.22649	0.22819
5	-0.21365	-0.20876	0.16546	-0.10028
6	-0.09557	-0.05931	0.13142	-0.05300

NOTE: $\hat{\varepsilon}_t$ and $\hat{s}_t$ denote OLS and standardized residuals, respectively. The approximate standard error for these autocorrelations is $\frac{1}{\sqrt{N}} = 0.1213$.

Table 6: Tests for Unconditional Heteroskedasticity

Model: $\hat{\varepsilon}_t^2$ (or $\hat{s}_t^2$)$= a_1 + a_2 x_t + a_3 x_t^2 + v_t$

Residuals	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$	NR^2
OLS	9.56×10^{-8}	6.34×10^{-2}	-9.70^{-3}	17.279
	(0.003)	(1.806)	(0.126)	
Standardized	0.6576	10.673	333.73	2.821
	(1.613)	(0.248)	(0.359)	

NOTE: Values in parentheses are t-statistics. $\hat{\varepsilon}_t$ and $\hat{s}_t$ denote OLS and standardized residuals, respectively. NR^2 follows an asymptotic χ^2 distribution with 2 degrees of freedom and its asymptotic critical values at 5 and 1 % significance levels are 5.991 and 9.210, respectively.

Table 7: Testing Unbiasedness of Forecasts

Model	χ^2-Statistics
(1) OLS	16.906
(3) $ARCH(\gamma_1, \gamma_2)$	30.688
(6) $AR(\phi_1, \phi_2)$	3.380
(7) $ARCH(\gamma_2) + AR(\phi_1, \phi_2)$	13.503

NOTE: Each χ^2-statistics has 2 degrees of freedom. The asymptotic critical values at 5 and 1 % significance levels are 5.991 and 9.210, respectively.

Figure 1: Stationarity Region of ARCH(1) in the Presence of AR(1)