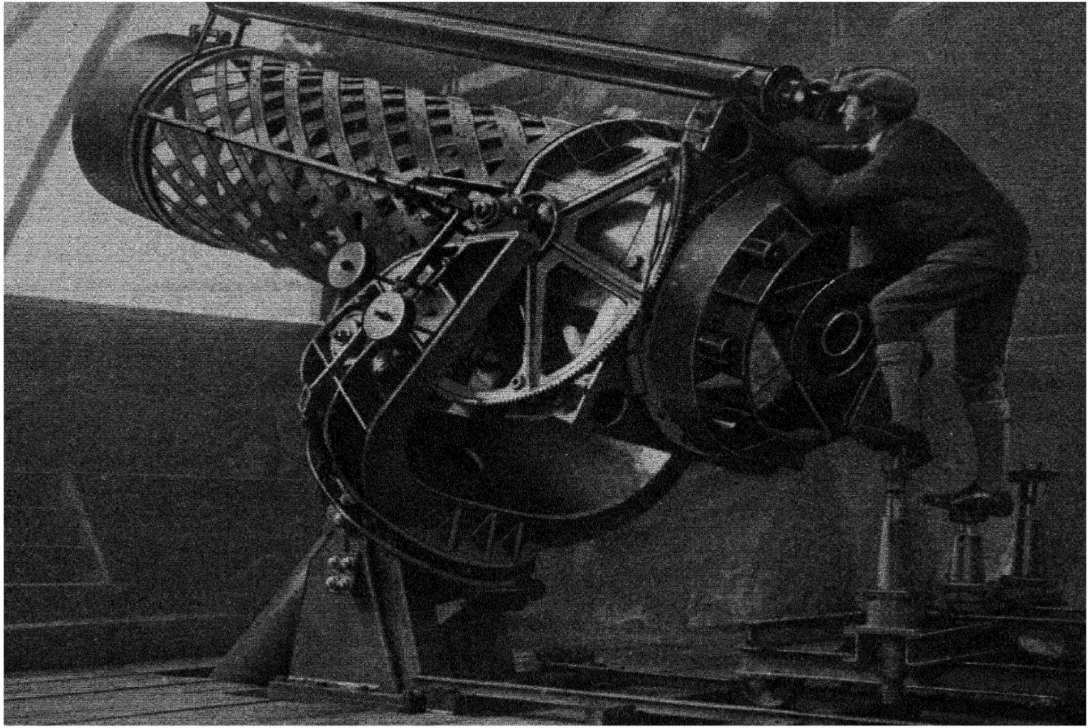
TIGHT BINDING BOOK

AN INTRODUCTION TO

APPLIED OPTICS



Frontispiece *(Sir Howard Grubb, Parsons & Co, Newcastle-on-Tyne)*

ONE-METRE REFLECTOR FOR SIMEIS OBSERVATORY, SOUTH RUSSIA

strument is being erected at the works, and the engineer is adjusting the auxiliary-finder telescope.

AN INTRODUCTION TO
APPLIED OPTICS

BY

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VOLUME II

THEORY AND CONSTRUCTION OF INSTRUMENTS

LONDON

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PREFACE

THE author has been encouraged by the kind reception given to Volume I of this *Introduction to Applied Optics* to proceed with the preparation of Volume II in which the fundamental optical instruments are described. There are, of course, many instruments, of utility and interest, other than those to which attention is given in these pages, notably various scientific instruments, such as spectroscopes, refractometers, interferometers, and the like, but these have generally been dealt with quite adequately in other books, whereas the treatment usually given to the commoner optical instruments, even in optical treatises, is of the slightest description. Even where details of construction have been furnished, the material has sometimes been more appropriate for an optician's catalogue than a text-book. It may be that a need will be felt at some future time for a book dealing with the optical instruments used in physical measurements, but it seemed wiser at present not to re-traverse ground which has been fairly well surveyed by others.

There are some instruments, like the photographic lens, for which no adequate theory can be presented. The refinements of their construction can only be understood by those who have been successful in the design of such systems. Even so, the result of the design is only partially under control. In other cases there are many interesting avenues which have been very inadequately explored, and into which the writer has ventured some short excursions, as in the theory of the microscope. Such is the complexity of even this limited subject, that one person's experience cannot adequately cover the whole range of topics, and previous authority has perforce been followed in many respects. In spite of the numerous shortcomings of the book, it is hoped that the reader may be led to take a more critical interest in his instruments and thus obtain from them the best performance of which they are capable. He is urged especially not to take facile and second-hand opinions about the productions of this maker or that, but to give instruments an intelligent trial where selection has to be made.

As in Volume I, the writer is greatly indebted to the work of Professor A. E. Conrady, whose book on *Applied Optics and Optical Design* may be commended to those who would go more deeply into the subject. He has also received great help from Dr. W. D. Wright, who has been good enough to read the proof sheets of this volume.

One word may be added both as a tribute to optical instrument makers and a warning to those who use their products. Treat your instruments, and especially their optical surfaces, with the greatest respect. Three or four months of intricate calculation may have preceded the production of your oil immersion microscope objective. Its making involves no less important painstaking and delicate work, by operatives of the greatest skill; its surfaces are polished true within a fraction of a wave-length of light. A good lens is as a jewel to its owner.

Acknowledgments and thanks are due to the following for the loan of blocks and permission to reproduce illustrations: Messrs. C. Baker (Figs. 8 and 124), Messrs. R. and J. Beck, Ltd. (Figs. 89 and 125), the School of Optics, Ltd. (Fig. 103), Messrs. Reichert of Vienna (Fig. 129), and Messrs. Carl Zeiss (London), Ltd. (Figs. 64, 88, 121, and 161). Fig. 196 has been reproduced by the kind permission of Dr. Felix Jentsch. The frontispiece is due to Sir Howard Grubb, Parsons, and Co., and the plate in the chapter on "The Microscope" to Messrs. Blackie & Son, Ltd., being reproduced from *Practical Microscopy*, by Martin & Johnson.

L. C. MARTIN.

IMPERIAL COLLEGE OF SCIENCE
AND TECHNOLOGY,
1932.

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INSET

AN INTRODUCTION TO

APPLIED OPTICS

CHAPTER I

THE MAGNIFICATION PRODUCED BY LENSES—THE SIMPLE MICROSCOPE

IN the first volume of this work, the term *magnification* has been restricted to mean the ratio of a linear dimension of the image to the corresponding size of the object; thus the formulae

$$\frac{h'}{h} = -\frac{f}{x} = -\frac{x'}{f'} = m'$$

express the *lateral magnification* measured perpendicular to the axis, while the equation

$$\frac{dx'}{dx} = \left(\frac{h'}{h}\right)^2 \frac{n'}{n}$$

gives the longitudinal magnification, or the ratio of image to object size measured in the axial direction, close to a particular pair of points where the lateral magnification is $\frac{h'}{h}$; it is assumed, of course, that the image exists in three dimensions.

The Lagrange relation permits the derivation of an expression for magnification in terms of the distances of object and image from *any* pair of conjugate points for which the magnification is known. Thus in Fig. 1 we might take B and B', P and P' to be any two pairs of conjugate points. Let PB $= l$ and P'B' $= l'$. Let $BB_1 = h_1$ and $B'B_1' = h_1'$ where BB_1 and $B'B_1'$ are a small object and image both perpendicular to the axis. We will apply the Lagrange relation to the imagery at the points P and P', considering the ray B_1P and its emergent path $P'B_1'$. The ordinary formula is

$$nh\omega = n'h'\omega'$$

h and h' are now the sizes of a small object and image at P and P' respectively, while ω and ω' are the angles with the axis made by the rays B_1P and $P'B_1'$; but $\omega = -\frac{h_1}{l}$ and $\omega' = -\frac{h_1'}{l'}$. Hence

$$nh\left(-\frac{h_1}{l}\right) = n'h'\left(-\frac{h_1'}{l'}\right)$$

or

$$\frac{h_1'}{h_1} = \frac{nhl'}{n'h'l}$$

In a particular case of importance, P and P' may represent the axial points of the entrance and exit pupils; the ray passing through them then becomes the "principal ray" of the bundle entering the instrument. In this case the ratio $\frac{h}{h'}$ above may be written $\frac{p}{p'}$, where p and p' are the radii of the pupils; and l and l' can be written q and q', using these symbols to denote the distances of object

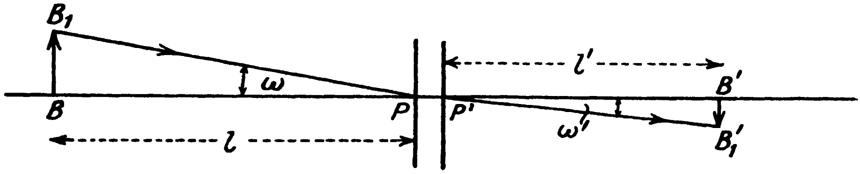


FIG. 1

and image from entrance and exit pupils respectively. The formula then becomes

$$\frac{h_1'}{h_1} = \frac{npq'}{n'p'q} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The great majority of optical instruments are, however, used as direct aids to vision, and the conception of the "magnifying power" of an instrument is then—

Magnifying power

$$= \frac{\text{Size of retinal image obtained with the aid of the instrument}}{\text{Size of image obtained with the unaided eye}}$$

As explained in Vol. I, it has been found that the accommodation of the eye is largely effected by the variation in curvature of the lens. In consequence of this the distance of the principal and nodal points from the retina varies very little, in fact less than half a millimetre in changing the accommodation from distance vision to near vision. The "stop" which limits the bundle of rays arriving at the retina may sometimes be the pupil of the eye itself, and sometimes it may be the exit pupil of some instrument projected into the same approximate position. In most cases the action of the accommodation will secure the sharpness of the image, but, if not, the image position will be assumed to be defined by the intersection of the principal ray with the retina. Further, since the eye entrance pupil and first principal point are only separated by about one or two millimetres, the principal ray may be regarded as that one passing through the principal points.

Refer again to Fig. 1, and let P and P′ now represent the principal points of the eye; the object BB_1 subtends an angle ω at the first principal point P. The ray makes an angle ω' with the axis after refraction. The Lagrange relation applied to the principal points where the magnification is unity gives

$$n\omega = n'\omega'$$

if the discussion is limited to paraxial conditions.

As mentioned above, the size of the image perpendicular to the axis may be defined by the intersection of the principal ray with the image plane. If this plane is at a distance l' from the second principal point, we may then write for the reasons given above,

$$h_1' = -l'\omega' = -l'\frac{n}{n'}\omega$$

the size of the image depending mainly on the angle subtended by the object at the first principal point, even allowing for variations in accommodation. While the angles are small we can deal with their angular measure, but when the angles are large, and the images are measured on flat screens, we shall have to deal with their "tangents." Hence, the above equation for magnifying power can be written—for small angles, at least—

Magnifying power

$$= \frac{\text{Angular subtense of image obtained with the instrument}}{\text{Angular subtense of object seen with the unaided eye}}$$

Magnifying Power of an Optical Instrument. It is always necessary to consider the state of accommodation or "refraction" of the eye when dealing with the magnification produced by an optical instrument. The condition is conveniently specified by giving the position of the accommodation point on which the eye is focused. Call this point M (conjugate to the macula M′), and let its distance from the first principal point of the eye be k. The condition of affairs might be as suggested in Fig. 2, in which "*a*" represents the optical system, and "*b*" is the system of the eye.

The object is situated at some point B, and the lens projects an image of height h' into the accommodation point of the eye.

The angle ω subtended by the image at the first principal point of the eye is

$$\omega = -\frac{h'}{k}$$

In order to investigate the dependence of ω on the size of the object, this equation can be written

$$\omega = -h\left(\frac{h'}{h}\right)\frac{1}{k}$$

$$= h\left(\frac{x'}{f'_a}\right)\frac{1}{k} \text{ (from equation (13), Vol. I)}$$

where x' is the distance F'_aM.

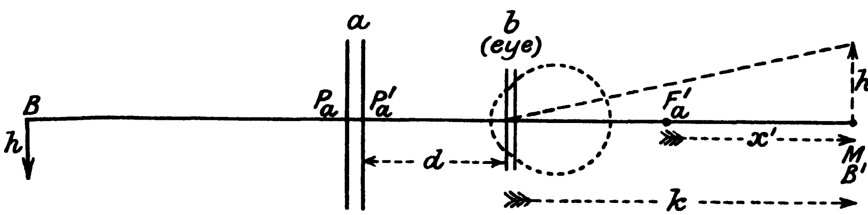


FIG. 2

Since from Fig. 2,

$$x' = d + k - f'_a$$

$$\omega = h\left(\frac{d + k - f'_a}{f'_a k}\right)$$

$$= h\left(\frac{1}{f'_a} - \frac{1}{k} + \frac{d}{f'_a k}\right)$$

$$= h(\mathcal{D}_a - \mathcal{K} + \mathcal{D}_a d\mathcal{K})$$

$$= h\{\mathcal{D}_a(1 + d\mathcal{K}) - \mathcal{K}\} \quad . \quad . \quad . \quad (2)$$

In order to see the object distinctly with the unaided eye in the same state of accommodation, it will be necessary to place it in the point of accommodation, M. This would, of course, be impossible for a real object with the case shown in Fig. 2, but it would be possible in all cases where k is numerically negative.

When placed at M, the angle subtended at the first principal point of the eye by the object is given by

$$\omega_0 = -\frac{h}{k} = -h\mathcal{K}$$

Hence magnification

$$= \frac{\omega}{\omega_o} = \frac{h(\mathcal{F}_a - \mathcal{F} + d.\mathcal{F}_a\mathcal{F})}{-h\mathcal{F}}$$

$$= 1 - \frac{\mathcal{F}_a}{\mathcal{F}} - d.\mathcal{F}_a$$

$$= 1 - \mathcal{F}_a(k + d) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If l' is the distance of the accommodation point from the second principal point of the lens system, then $l' = k + d$, and the formula takes the form in which it is often written—

$$\text{Magnification} = 1 - \frac{l'}{f'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the case of a small lens of high power, it will usually be held close to the eye, and the distance of the point of accommodation will usually be the "least distance of distinct vision." As has been seen from the table in Chapter VIII, Vol. I, this distance varies. A mean value for adult vision may be taken as -250 mm. (or about -10 in.). The magnification then becomes (f' in millimetres)

$$1 + \frac{250}{f'}$$

or sufficiently nearly, if f' is small,

$$\text{Magnification} = 0{\cdot}25.\mathcal{F} = \frac{\mathcal{F}}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $\mathcal{F}$ is the power of the lens in dioptres.

Spectacle Magnification. The user of a pair of spectacles will, however, judge "magnification" rather differently in that he will compare the apparent size of the more or less indistinct image seen with the unaided eye, and the size of the image seen with his spectacles, the object remaining fixed.

From above, the angular subtense of the image will be

$$\omega = h\,\{\mathcal{F}_a(1 + d\mathcal{F}) - \mathcal{F}\}$$

We now calculate the position of the real object corresponding to the image position at M, where it must be formed to be seen distinctly by the eye.

The usual formula,

$$\frac{1}{l'_a} - \frac{1}{l_a} = \frac{1}{f'_a}$$

gives $l_a = \dfrac{f'_a l'_a}{f'_a - l'_a} = \dfrac{l'_a}{1 - l'_a\mathcal{F}}$, omitting the suffix from $\mathcal{F}$

Let the interval $P_aP'_a = t$, then the distance of the object from the *rst* principal point of the eye is given by

$$l_b = l_a - (d + t)$$

$$= \frac{l'_a}{1 - l'_a . A} - (d + t)$$

If the thickness t be neglected, as it is usually small in spectacles, and remembering that $l'_a = d + k$, we have

$$l_b = \frac{l'_a - d + l'_a . A d}{1 - l'_a . A} = \frac{k + (d + k)\, d . A}{1 - (d + k) . A}$$

And the angular subtense of the object is therefore

$$-\frac{h}{l_b} = -h \left\{ \frac{1 - (d + k) . A}{k + (d + k) d . A} \right\}$$

i.e.

$$-\frac{h}{l_b} = -h \left\{ \frac{F - (dF + 1) . A}{1 + (dF + 1)\, d . A} \right\}$$

Hence magnification $= \dfrac{\text{angular subtense of image}}{\text{angular subtense of object}}$

$$= \frac{h\{ . A(1 + dF) - F\}}{-h \left\{ \dfrac{F - (dF + 1) . A}{1 + (dF + 1)\, d . A} \right\}}$$

$$= 1 + (1 + dF)\, d . A \qquad . \quad . \quad . \qquad (6)$$

This expression is quite general for any thin lens and any state of accommodation, provided that a sharp image is obtained with the aid of the spectacle lens. The distance of the lens from the eye is not restricted.

EXAMPLE I.

Take $F = -4$D, $. A = 6$D, and $d = 1{\cdot}0$ (metres)

Then magnification $= 1 + (-3)\, 6$

$= -17$

This shows that the lens is forming an inverted image between the lens and the eye; and the angular subtense of this inverted image is considerably greater than that of the object seen directly. The distance of the object from the lens proves to be $-21{\cdot}4$ cm., while the distance of the inverted image is 75 cm. from the lens, and -25 cm. from the eye. Thus $\dfrac{h'}{h}$ for the lens $= -3{\cdot}5$. From

ERRATA

Page 35, line 6. Equation to read: $\overline{\mathscr{L}}_1' = \overline{\mathscr{L}}_1 + \mathscr{K}_1$.

Page 47, line 21. Π^k_0 should read Π^k_1

Page 49, line 1. (Second equation) denominator to be $1 - \bar{d}\,\overline{\mathscr{F}}_1'$.

Page 49, line 16. $1/\overline{\mathscr{F}}$ should be $1/\overline{\mathscr{F}}'$.

Page 54, line 10. Read $y_1 - \frac{d}{n'} . n'a_1'$.

Page 63, paragraph on "Gaussian Constants," line 3 of paragraph. Replace C by $1/B$.

Page 63, paragraph on "Gaussian Constants," line 4 of paragraph. Read: $h'/h = C$ and $D = -lC$.

Page 94, line 13. $\frac{\mathrm{T}}{t}$ should read $\frac{t}{\mathrm{T}}$.

Page 102, Inscription to Fig. 64 should be as follows:

FIG. 64. ENERGY DISTRIBUTION IN THE SPECTRA OF VARIOUS ILLUMINANTS

A. Blue sky
B. High sun (data from Smithsonian Instn.)
C. Ives suggested standard (black body at 5000°)
D. Low sun (Smithsonian)
E. Gas filled tungsten lamp
F. Acetylene flame

Page 114, line 17. $\frac{AP^2}{l'}$ should read $\frac{AP^2}{l}$.

Page 147, line 21. $\mathscr{K}_i$ should read $\mathscr{K}_t$.

Page 184, equation at foot $+\frac{\lambda}{2}$ should read $-\frac{\lambda}{2}$

Page 230, line 5. $(n_D - n_A)$ should read $(n_D - n_C)$.

Page 305, line 6. *spherical aberration* should read *astigmatism*.

Page 316, equation (f). Missing letter in denominator is $\mathscr{F}$.

these figures the above value for the magnification is easily checked. The student should make a sketch of the arrangement.

EXAMPLE II.

Take $\mathcal{H} = -4$D, and $\mathcal{J} = 6$D (as above), but let d now be equal to 0·01 metre. Then magnification = 1·06, by the formula.

The image is now virtual and erect, and the lens is being used close to the eye as an ordinary magnifier. The low value for the "magnification" is due to the fact that we are comparing the angular subtense of the image seen sharply with the lens, and that of the diffuse image which is seen without it. These angles are not greatly different in magnitude.

EXAMPLE III.

Take $\mathcal{H} = +0{\cdot}5$D, $\mathcal{J} = +0{\cdot}2$D, $d = 3{\cdot}0$ (metres).
In this case the magnification works out to + 2·50. We therefore should obtain an erect image of moderate magnification.

The system is made use of in the so-called "window telescope," made by some opticians, which consists of a large single lens of low power made to hang on the window frame. Most persons of normal vision can manage to relax the accommodation far enough to bring the refraction of the eye to a small positive value, and the combination of lens and eye then produces a system having a focal length considerably longer than that of the eye alone. An erect, magnified image is the result.

Magnification given by Spectacles giving Distance Correction. The above expression for magnification takes a simple form in the case where the "glass" employed is suitable for distance correction. In that case we have the simple relations—

$$\mathcal{H} = \frac{\mathcal{J}}{1 - d.\mathcal{J}}, \text{ and } \mathcal{J} = \frac{\mathcal{H}}{1 + d\mathcal{H}} \quad \text{(See Vol. I, p. 265)}$$

By a simple substitution the expression for the magnification can then be put into the simple forms—

$$\text{Magnification} = \frac{1}{1 - d.\mathcal{J}} = 1 + d\mathcal{H} \quad . \quad . \quad . \quad . \quad (7)$$

We see at once that a positive correcting lens produces a magnification greater than unity, while a negative lens as used in myopia produces a magnification less than unity—sometimes called a "minification."

Single Lenses as Magnifiers. The use of single lenses as magnifiers has been known since very early times. The Greeks were familiar with the effects of refraction of light at curved surfaces, such as might be studied with solid balls of glass or rock crystal, and the perfection of certain antique handwork seems to call for the explanation that it was produced with the aid of a magnifier.

Antony van Leeuwenhoek, born at Delft in 1632, made great advances in grinding and polishing small lenses for magnifiers, and obtained magnifications up to 160. He observed Infusoria and Bacteria for the first time. In 1702 J. Wilson produced a pocket microscope with which, it is said, magnifications up to 400 could be obtained. The convenient mechanical construction of this instrument ensured its popularity for quite 100 years.

We may distinguish two main functions of single-lens magnifiers. First a lens may be used, as in one side of a stereoscope, to obtain a general view of a large object or picture at a suitable angle. Secondly, a lens may be employed to obtain an enlarged image of a very small portion of an object. Naturally the optical arrangements differ in the two cases.

Lenses for Viewing Pictures. The first case above brings us to considerations very similar to those encountered in the discussion of spectacle lenses. The eye turns in its socket in order to view the different parts of the object, and the pencils of light reaching the retina under different angles have an eccentric passage through the lens. The conditions are optically similar to the case in which a small stop is situated in the centre of rotation of the eye (Fig. 3). The conditions have already been discussed in connection with high power spectacle lenses. It was shown in that connection that one of the chief troubles arising was the astigmatism of the oblique pencils. This was shown to be diminished by the choice of a lens of suitable figure, generally of meniscus type with the concavity towards the eye, but correction could not be given beyond a power of about 10D and higher without the employment of aspherical surfaces. The ellipse of Fig. 188, Vol. I, will provide approximate data for the radii required for various powers, but the spectacle lens was intended to form images of *distant* objects, and the results do not hold exactly for near objects. Calculations for near vision have been made by Whitwell.[1]

If it is sought to view a near flat object with such a lens, several additional defects in the image are at once noticed.

1. *Roundness of the Field.* The exterior parts of the object are farther away from the lens than the centre. Even if the lens has been freed from "oblique astigmatism," the object field will not be seen in focus as the eye is moved unless it has the curvature of the Petzval surface, of radius nf' for a single thin lens. With an ordinary

double convex lens, the presence of astigmatism makes matters very much worse. If the object is flat we can first bring radial, then tangential, lines to a focus in the outer parts of the field by bringing the object closer to the lens.

2. *Chromatic Aberration.* On tracing a principal ray through the system of Fig. 3, it will at once be noticed that considerable lateral chromatic aberration must arise through the passage of the pencils of light through the outer parts of the single lens. If the object consists of a number of small bright points on a dark ground (pin-holes in a thin card illuminated from behind) each object point in

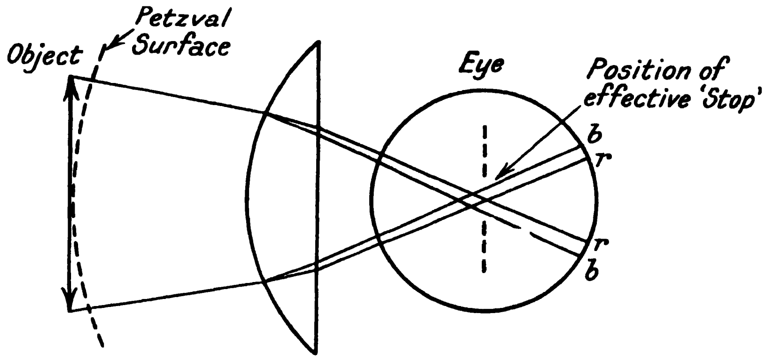


FIG. 3. USE OF A SIMPLE MAGNIFIER WITH ROTATING EYE

the outer part of the field is imaged as a short radial "spectrum," the red being innermost.

In the case of lenses required to give a general view of objects or pictures, they are not often required to be of shorter focal length than about 5 cm., and, as will be shown below, the spherical aberration of the pencils which are transmitted by the pupil of the eye (which will probably not exceed 4 mm. in diameter during the day) does not produce any appreciable deterioration of the image.

3. *Distortion,* of the pin-cushion variety, is noticeable with a single lens. The bending of the rays increases too rapidly in the outer parts of the field, and the magnification thus increases with the distance from the centre.

In Vol. I, Chapter I, the question of the proper presentation of a perspective projection was discussed, and it was shown that a proper judgment of the space values of the picture can only be obtained if the latter is seen under the proper angle.

The majority of pictures obtained by small cameras are made with lenses of considerably shorter focal length than the least distance of distinct vision; hence, if the print is to be seen with the unaided eye, it must be held at too great a distance to attain the proper angle. Matters could be corrected, actually, by placing the projection (or print) back in the camera in which the photograph was made, illuminating it in some way, and then looking at it through the camera lens.

This, however, would be awkward because the camera lens may be provided with a stop through which the eye must look. It would be necessary to move the head about. This type of "keyhole observation" is not convenient as we could not get a good view of the picture as a whole. Hence, although the perspective conditions

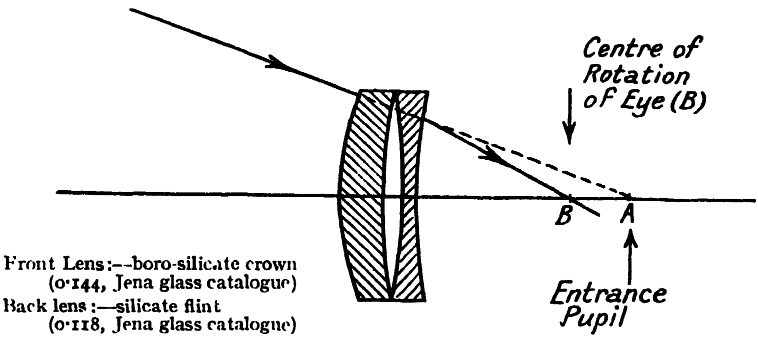


FIG. 4. THE "VERANT" LENS

require that the viewing lens shall have the same focal length as the camera lens, it is desirable that the former shall be corrected for use with an (imaginary) stop situated in the centre of rotation of the eye when properly positioned with respect to the system.

The best-known lens of this type is the "Verant," designed by M. von Rohr and made by Zeiss.[2] It corrects chromatic aberration of the above type, and distortion on the lines laid down by Gullstrand, and it is illustrated in Fig. 4. There is some residual astigmatism at certain angles of obliquity, but it is not large. The curvature of field can be allowed for by variation of the accommodation of the eye.

A lens of somewhat similar character but different construction, intended for use with a stereoscope, has been designed by Albada.[3]

The Magnifiers of Wollaston, Brewster, and Coddington. Where small lenses were required to obtain a fairly great magnification, and yet to give a general view of an object comparable in size with the diameter of the lens itself, a device introduced by Wollaston in 1812 proved a great improvement on the hitherto prevalent use of a small complete sphere of glass; see Fig. 5 (*a*). Wollaston employed two hemispheres of glass mounted together with a small stop between them. This was improved by Brewster who cut a saddle-shaped groove in a complete sphere, thus obtaining a limitation of the pencils by a "stop" effectively situated at the centre of the sphere. The Coddington lens is similar except in the manner of cutting the sphere, which is illustrated in Fig. 5 (*c*).

The idea in all cases remains the same. It is evident from Fig. 6 that pencils of light refracted through the centre from any point on a spherical object surface concentric with the sphere must have similar optical treatment by the system. In every case the prin-

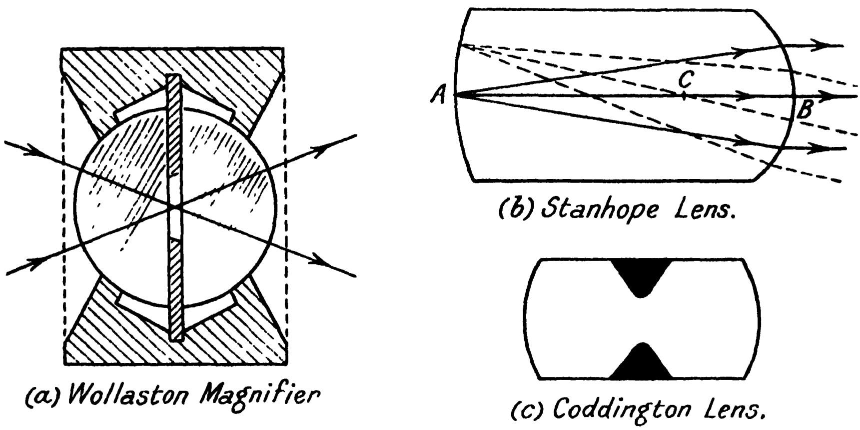


FIG. 5

cipal ray through the centre of the stop suffers no deviation, and the image is therefore free from chromatic aberration of the above type, coma, astigmatism, and distortion.

In cases where the object itself is fairly small, i.e. of the order of a few millimetres in diameter, then the use of a magnifier of the above type, which should be held close to the eye, would produce a retinal image of a part of such a curved object, the part being comparable in size to the pupil, but naturally the object itself will not, in general, be contained in such a curved surface as shown in Fig. 6. This means that either the object should be moved relatively to the lens or a variation of accommodation must come into play. The "roundness of field" is naturally extremely marked, and severely limits the region of the object which is seen sharply in focus at the same time. In the Stanhope lens, Fig. 5 (*b*), the object is intended to be placed on the front curved surface, the central point A lying at the principal focus of the back surface. For glass of refractive index 1·5, this calls for a radius of one-third of the thickness of the lens. The front surface through A is struck about the same centre C, and complete freedom from chromatic difference of magnification, coma, astigmatism, and distortion is therefore secured as far as the lens itself is concerned. These lenses were, at one time, frequently mounted with very small photographs in pencils and the like. The front surface was often made flat, although some advantages were lost. The system still has value for examining small organisms, etc., which can be placed in contact with the front surface.

Nature of the Image to be Presented to the Eye. In spite of the fact that the human eye is admittedly subject to chromatic aberration in the sense of "undercorrection" characteristic of an ordinary lens, and also that it suffers from zonal spherical aberration, it has been shown in the chapter on the eye, Vol. I, that the acuity of normal human vision for the small pupilary diameters characteristic of day-time is little lower than the limits set by the wave-nature of light, even for a perfect optical system.

The vision of actual objects, either at a great distance, or at the near point, manifests no trace of coloured fringes or haziness due to spherical aberration. It is, indeed, just conceivable that the mental receiving apparatus has some means of automatic compensation, which causes the brain to interpret a particular distribution of light (in regard to colour) as characteristic of an elementary point. However that may be, it has always been judged best to design any optical instrument so that the *image* shall be as perfect as possible, in the physical sense, as it is presented to the eye. The defects of vision can then have no worse effect on the appearance of this image than on the appearance of a real object.

A few attempts have, indeed, been made to design systems which should compensate the chromatic aberration of the eye, and secure a greater concentration on the retina. It is possible that systematic research on such lines might yield results of interest, but so far nothing has been done which has led to any departure from the general rule of making the image as physically perfect as possible.

The Simple Microscope. In contra-distinction from the lens required for giving a general view of an object, it is frequently required to obtain a greatly enlarged view of a small area—small, perhaps, in comparison with the diameter of the lens itself.

It will be shown in the chapter on the microscope that the power of the system to yield very sharp images of small objects depends upon the angular divergence of the cone of rays derived from the object, and brought without appreciable aberration to the retinal focus. In Fig. 6, the cone has a small angular diameter w, and this could be enlarged by increasing the size of the central stop. On the other hand, this will rapidly increase the spherical aberration arising through refraction at each surface, and the effects of chromatic aberration will also become serious. The chromatic aberration here referred to is the difference in the focusing position of different colours measured along the axis of symmetry of a bundle; thus if the green is focused on the retina, the other spectral components will be represented by blur patches of lessened concentration. It is to be noted that this chromatic aberration will be in

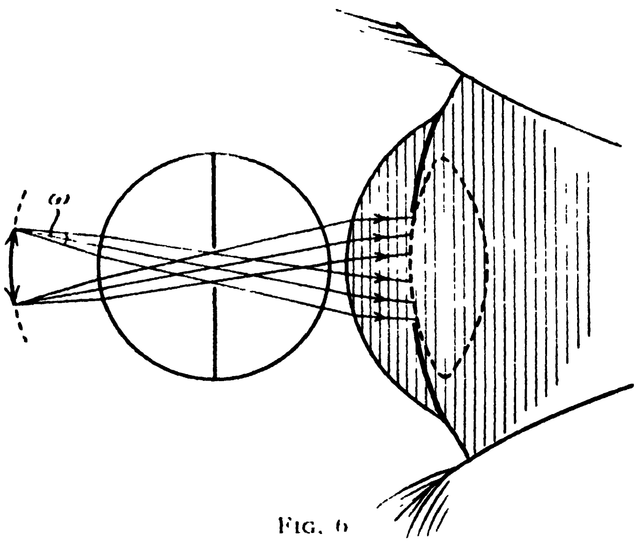


FIG. 6

the same sense as that of the eye itself, the shorter wave-lengths being focused nearer the lens.

In order to obtain a perfect image, as projected by the lens, it may be required that rays diverging from the object point shall all be rendered parallel after leaving the lens, i.e. that the wave-fronts shall be plane on entering the eye. The case is then just the reverse of that when a lens is to form a sharp image of a very distant object. The numerical amounts of the optical path differences are therefore the same.

When we considered the case of primary spherical aberration, the maximum residual optical path differences arising at the best focus position (midway between marginal and paraxial foci) were shown to be one-quarter of the optical path difference between marginal and paraxial rays arising at the paraxial focus. The formula (Vol. I, Chapter IV) for this latter was

$$\Delta p = \frac{y^4}{8}A$$

so that the residual differences at the best focus of a lens exhibiting this primary spherical aberration will be

$$\Delta p_1 = \frac{y^4}{32}\mathrm{A}$$

The coefficient A was found to be

$$\mathrm{A} = -\left\{ j^3 \left(\frac{n}{n-1}\right)^2 + j^2 v_1 \left(\frac{3n+1}{n-1}\right) - j^2 c_1 \left(\frac{2n+1}{n-1}\right) \right.$$
$$\left. + j v_1^2 \left(\frac{3n+2}{n}\right) - j c_1 v_1 \left(\frac{4n+4}{n}\right) + j c_1^2 \left(\frac{n+2}{n}\right) \right\}$$

In the case where a lens receives parallel light, $v_1 = 0$. Suppose, for simplicity, that the lens is plano-convex, with the curved surface turned towards the incident parallel light. Then $c_2 = 0$, and

$$j = (n-1)\, c_1$$

Hence A (the sign need not concern us here),

$$= j^3 \left(\frac{n}{n-1}\right)^2 - j^2 \cdot \frac{j}{n-1}\left(\frac{2n+1}{n-1}\right) + j \cdot \frac{j^2}{(n-1)^2}\left(\frac{n+2}{n}\right)$$
$$= j^3 \left\{ \left(\frac{n}{n-1}\right)^2 - \frac{2(n+1)}{n(n-1)} \right\}$$

As was mentioned above, the path residuals arising in such a case will be the same as those which arise when a small object is situated at the "best focus" of the lens on the plano side, and gives approximately parallel light to a viewing eye.

Hence, the expression for the optical path residuals becomes

$$\mathrm{OPD} = \frac{y^4}{32} j^3 \left\{ \left(\frac{n}{n-1}\right)^2 - \frac{2(n+1)}{n(n-1)} \right\}$$
$$= \frac{1}{32}\frac{y^4}{f'^3} \left\{ \left(\frac{n}{n-1}\right)^2 - \frac{2(n+1)}{n(n-1)} \right\}$$

Take the allowable OPD $= \frac{\lambda}{4} = \cdot 00013$ mm., and $y = 2$ mm. (say), allowing for an average pupilary diameter of 4·0 mm. Then a quick calculation made with an assumed value of $n = 1\cdot 5$ gives

$$f' = 16\cdot 1 \text{ mm.}$$

This is the shortest focal length allowable for a good image on the above criterion. The magnification $= \frac{250}{16\cdot 1} = 15$ approximately.

If the lens is reversed so that the curved side is towards the object, the conditions are less favourable as far as spherical aberration is concerned, and the focal length has then to be increased to about double the above value if the aberrations are to be kept sufficiently small; the allowable magnification therefore sinks to about eight times.

In practice, however, the above limits may be somewhat exceeded before a really marked deterioration of the image begins to be manifest.

In order to test the above conclusions it is interesting to take a plano convex lens, of about 1 cm. radius, and to use it as a magnifier held as close as possible to the eye.

When the plane face is turned towards the eye, the field is of fair extent, but the images of bright spots have haloes. When the lens is reversed so that the plane face is towards the object, the field is more restricted, but the contrast at the centre is perceptibly improved, the haloes being reduced. The test is best made on a number of pinholes in a card held up to the light.

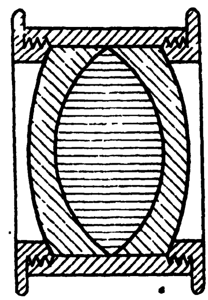


FIG 7 STEINHEIL TYPE APLANATIC MAGNIFIER

The triple lens form most frequently employed in modern pocket magnifiers for magnifications up to × 20

Theoretical or practical tests soon show that a lens of 15 mm. focal length will have fairly pronounced chromatic aberration on the axis; this results in the presence of coloured haloes in the image of a very small source of light, such as a pinhole, and the colour is independent as to whether the lens is held with the curved or the flat side towards the object.

The "Steinheil" Magnifier. Spherical and chromatic aberration can be eliminated by using a biconvex crown-glass lens between two menisci of flint. The lens is designed after Steinheil's "aplanatic" magnifier or "lupe."

The symmetrical shape of the lens would tend to give it freedom from the aberrations of oblique pencils with regard to principal rays passing through the centre, somewhat as in the Coddington lens. When, however, the aberrations are reckoned with respect to a stop outside the lens represented by the pupil of the eye, or an imaginary stop at the centre of rotation of the eye-ball, they may not appear in such a good light. The lens is shown in Fig. 7. The magnification is about × 6.

This lens has the great advantage of symmetry, so that if it is mounted in a pocket-holder it can be used either way round with equal advantage.

Symmetry is a very real advantage in a small pocket magnifier; in cases where it is necessary to obtain a compromise between the opposing claims of freedom from spherical aberration, and freedom from undue astigmatism of the oblique pencils, the double convex form of the lens may be frequently selected.

Small Measuring Magnifiers. It is often very convenient to mount a small scale, engraved or photographed on a thin disc of glass, in the focal plane of a simple magnifier. The instrument can

(C Baker, London)

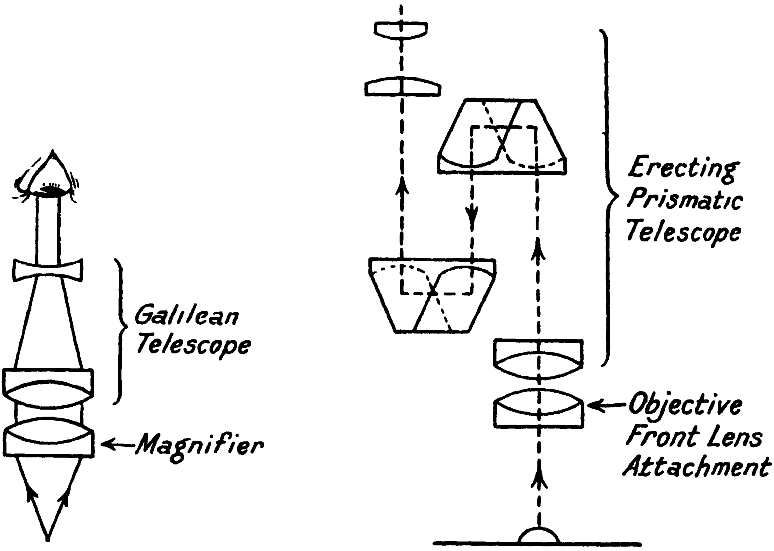
FIG. 8. DISSECTING STAND

then often be placed so that the scale is in contact with the object to be measured. The illumination is secured by a thin plate of glass, placed between the lens and scale, by which light can be reflected down on to the object.

Dissecting Stands. The majority of simple magnifiers are for hand use. They are preferably held quite close to the eye in order to secure the widest possible field in this "keyhole" type of observation. For dissecting purposes, however, "aplanatic" magnifiers of somewhat longer focus are mounted in simple mechanical stands, one of which is shown in Fig. 8.

In order to overcome the difficulty of the short working distance with the higher powers, *Chevalier* proposed, in 1839, to place a concave achromatic lens above the magnifying glass. A convenient form of the arrangement is shown in Fig. 9 (*a*); this is known as the

Brücke lens, but is not often encountered; it is essentially a combination of a Galilean telescope with a long focus microscope objective. A more modern arrangement is to employ a prism erecting telescope system, similar to one member of an ordinary field glass, in combination with a microscope objective as "front lens attachment." Such systems are very convenient for naturalists and



(a) Brücke Lens (b) Telescope Magnifier Combination

FIG. 9

others. The use of the prism erecting system allows of a larger field than is possible with the Brücke system. The theory of the telescope systems will be dealt with in the following chapter.

Such telescopic magnifiers are adaptable for binocular vision, sometimes with the aid of suitable achromatic prism systems.

REFERENCES

1. A. Whitwell: Reference must be made to a series of papers published in *The Optician* from about 1916 onwards. See also Emsley and Swaine, *Ophthalmic Lenses* (Walton Press), p. 243.
2. British Patent 24009 (1903).
3. Albada: *Trans. Opt. Soc.*, XXV (1923–24), 249.

CHAPTER II

THE TELESCOPE

Historical. Roger Bacon (1216–1294) was familiar with convex lenses, which were about this time beginning to be employed for spectacles. He states in his writings that—

> ". . . we can give such figures to transparent bodies, and dispose them in such order with respect to the eye and the objects, that the rays shall be refracted and bent towards any place we please, so that we shall see the object near at hand, or at any distance under any angle we please. And thus from an incredible distance we may read the smallest letters."

The above words may be interpreted as a veiled allusion to a telescope. Undoubtedly, they contain the germ of the scientific idea of the telescope, which is usually an instrument to project an image, of a distant object, which can be viewed under a greater angle than is possible with the unaided eye. Medieval philosophers were not in the habit of giving very explicit descriptions of what discoveries they made, for the ever present danger of being suspected for witchcraft or necromancy was associated with the exhibition of any unfamiliar phenomena.

Somewhat similar hints and allusions appear in the writings of Robert Recorde (1551), Battista Porta (1558), and others. The earliest circumstantial account of the actual construction of a telescope dates from 1590, when Zacharias Jansen, the son of a spectacle maker, Hans Jansen of Middelburg in Holland, is supposed to have invented the instrument. This testimony is due to the son of Zacharias.

There was also, however, another spectacle maker, Hans Lippershey, in the same town. It appears certain that he was in possession of the invention in the year 1608. A popular story ascribes the invention to the children of these two spectacle makers; they mounted a pair of lenses on a piece of wood in play, accidentally securing the correct distance between the lenses. However that may be, Lippershey undoubtedly pushed forward the invention and actually constructed binocular telescopes.

In June, 1609, Galileo heard of the invention without knowing any details of the construction. He returned to his laboratory at

Padua, and in one day had made his first erecting telescope by mounting a plano-convex and a plano-concave lens in a short leaden tube. Subsequently, he made a number of telescopes of greater length and increased magnification of which examples are preserved in the Museo di Fisica, Florence.

Further historical points will best be given in the development of the theory of the instrument.

Elementary Theory. In its simpler form the telescope consists of two lenses mounted coaxially, but the single lenses may be replaced by systems of greater complexity for various reasons, as will be seen. We will, however, treat the telescope as consisting essentially of two main parts: the objective (a), and the eyepiece (b).

Imagine an object perpendicular to the axis which subtends a small angle α at the anterior principal focus F_a of the objective. Having given this angle we know that, wherever the image formed by the objective may be, its height will be given by

$$h'_a = f_a\alpha$$

where f_a is the first focal length of the objective.

The eyepiece acts as a simple magnifier. Turning to equation (2),

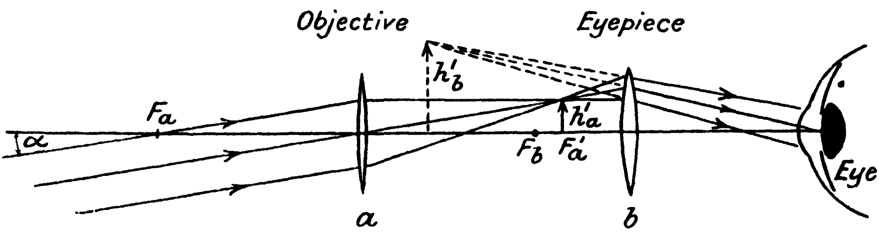


FIG. 10

Chapter I, we find that the angle under which the above image will be viewed (having given the power $\mathcal{P}_b$ of the magnifier, the refraction $\mathcal{A}$ of the eye, and the distance d between the adjacent principal points of magnifier and eye) will be

$$\omega = f_a\alpha\{\mathcal{P}_b(1 + d\mathcal{A}) - \mathcal{A}\}$$

Since, as before,

$$\text{Magnifying power} = \frac{\text{Angular subtense of image}}{\text{Angular subtense of object}}$$

we may write (if α_0 is the angular subtense of the *object* at the eye)

$$\text{Magnifying power of telescope} = \frac{f_a\alpha\{\mathcal{P}_b(1 + d\mathcal{A}) - \mathcal{A}\}}{\alpha_0}$$

This somewhat complex-looking expression becomes greatly simplified if it is assumed that: (1) The object is sufficiently distant to make a and a_0 sensibly equal, as is the case in the great majority of applications of the telescope. The expression then becomes

$$\text{Magnifying power} = f_a\{\mathcal{P}_b(1 + d\mathcal{P}_e) - \mathcal{P}_e\} \qquad (8)$$

The distance l' of the image from the second principal point of the eyepiece or magnifier is $k + d$; hence the corresponding distance l of the conjugate point for the magnifier system is given by

$$\frac{1}{k + d} - \frac{1}{l} = \mathcal{P}_b$$

This gives the position of the intermediate image formed by the objective and viewed by the eyepiece. We take the case when it coincides with F'_a. Then the distance

$$F'_aF_b = g = -(l - f_b)$$

In the usual case $\quad f'_b = -f_b$, so that

$$g = -(l + f'_b)$$

The above equation gives

$$\frac{1}{l} = \frac{1}{k + d} - \mathcal{P}_b = \frac{1 - \mathcal{P}_b(k + d)}{k + d}$$

$$g = -\left\{\frac{k + d}{1 - \mathcal{P}_b(k + d)} + \frac{1}{\mathcal{P}_b}\right\}$$

$$= -\left[\frac{1}{\mathcal{P}_b\{1 - \mathcal{P}_b(k + d)\}}\right]$$

$$= -\left[\frac{\mathcal{P}_e}{\mathcal{P}_b\{\mathcal{P}_e - \mathcal{P}_b(1 + d\mathcal{P}_e)\}}\right]$$

But the Gaussian "power" of the instrument is equal to

$$\mathcal{P} = -\mathcal{P}_a\mathcal{P}_b g = \frac{\mathcal{P}_a\mathcal{P}_e}{\mathcal{P}_e - \mathcal{P}_b(1 + d\mathcal{P}_e)}$$

Hence from equation (8) above,

$$\text{Magnifying power} = \frac{\mathcal{P}_e}{\mathcal{P}} = \frac{\text{Principal point refraction of eye}}{\text{Power of the system}}$$

The "Infinity Adjustment." Further, an important case arises in which the instrument is adjusted to present a sharp image to the normal unaccommodated eye. In this case $\mathcal{P}_e = 0$. Hence

$$\text{Magnifying power} = f_a\mathcal{P}_b$$

Almost invariably, the objective is in air, so that $f'_a = -f_a$. Hence

$$\text{Magnifying power} = -\frac{f'_a}{f'_b}$$

The above formula is easily derived in the simplest case from inspection of a Gaussian diagram.

Fig. 10 shows the course of the rays in the telescope when a virtual image at a finite distance is presented to the eye. If, however, the refraction of the viewing eye is to be zero, it must receive parallel light. This case is shown in Fig. 11, in which F'_a and F_b are coincident. A parallel bundle of rays enters the objective, is focused in the common focal plane, and diverges to the eyepiece from whence it emerges once more parallel.

Tracing one ray through F_a it is seen to pass through F'_b; it is

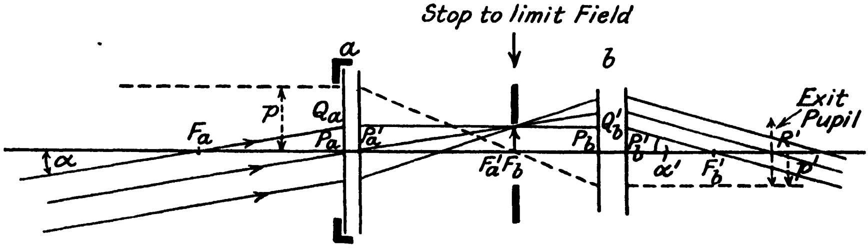


FIG. 11. ASTRONOMICAL TELESCOPE
(Diagrammatic)

evidently parallel to the axis between objective and eye. The angle between the axis and the incident bundle of rays is

$$\tan \alpha = \frac{P_aQ_a}{P_aF_a} \text{ (numerically negative in the figure)}$$

The angle between the axis and the emergent bundle is

$$\tan \alpha' = \frac{P'_bQ'_b}{P'_bF'_b} \text{ (numerically positive in the figure)}$$

The ratio of the tangents of the angles becomes equal to the ratio of the angles themselves when they are small, and clearly gives an equally valid measure of the magnification; it is as good an approximation to relate the impression of "sizes" of the retinal image to the tangent of the angle of subtense as to the angular measure; neither is strictly accurate in the general sense; it is ultimately a matter of verbal definition.

$$\frac{\tan \alpha'}{\tan \alpha} = \frac{P_aF_a}{P'_bF'_b} = -\frac{f'_a}{f'_b}, \text{ in the usual case.}$$

The ratio of the two focal lengths is therefore a measure of the magnifying power. It follows that—

1. To obtain high magnification the focal length of the objective must be great in comparison with that of the eyepiece.

2. If the second focal length or the dioptric "powers" of the objective and eyepiece in such a simple instrument are of the same

sign, the image will be inverted. This is the case in the astronomical telescope.

3. If the second focal lengths or the dioptric powers of the eyepiece and objective have opposite signs the image will be erect. This is the form of telescope invented by Galileo, and also, presumably, in Holland, though the details of the earliest Dutch instruments are not certainly known.

The Galilean Telescope. The action of the Galilean glass is illustrated in Fig. 12. The eyepiece *b* is now a negative or diverging lens, so that the common focal plane containing F'_a and F_b is shown behind the eyepiece. A parallel bundle of rays is focused towards the image point B′ in the common focal plane, but is intercepted

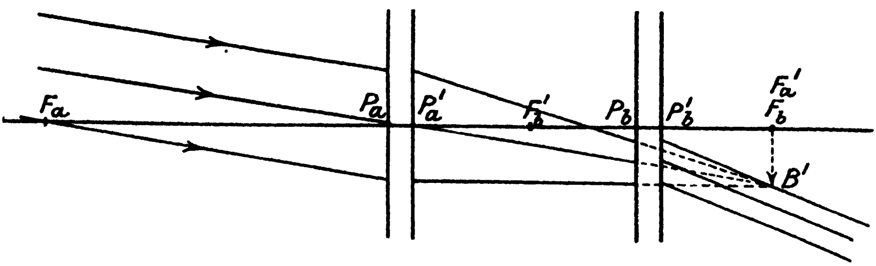


FIG. 12. GALILEAN TELESCOPE
(Diagrammatic)

by the eyepiece before reaching the focus and rendered parallel once again. The image will evidently be erect. If the separation of the lenses be diminished, the virtual image seen by the aid of the instrument can be formed at a finite distance. The formulae for magnification hold good in the present case.

The Pupils of the Telescope System. The entrance pupil of the system is usually considered with reference to an infinitely distant object point. Referring back to Fig. 11, the objective "*a*" is shown diagrammatically as limited in diameter. It is usually the case that a ray parallel to the axis passing, as shown, through the extremity of the objective, is transmitted unhindered by the eyepiece. If, therefore, we follow the usual plan for finding the entrance pupil by determining the images of the various diaphragms and lens rims, etc., formed by the parts of the instrument lying to the left of each such diaphragm, it will be found that all these images have a greater diameter than the boundary of the objective if the above condition regarding the ray is fulfilled.

The entrance pupil is therefore usually represented by the boundary of the objective itself, and the exit pupil is the image of this rim formed by all parts of the system lying to the right of it. The axial position of the exit pupil will be found by tracking through the system a ray from the centre of the entrance pupil. Such a ray is exemplified in the diagram by that through P_a. It cuts the axis in R′ beyond the eyepiece. The radius of the exit pupil is found by tracing a ray from the boundary of the entrance pupil through the system, and finding the axial distance of its intersection of the plane through R′ perpendicular to the axis.

Let p and p' be the radii of the entrance and exit pupils respectively, then inspection of Fig. 11 shows that (for systems in air)

$$\frac{p}{p'} = -\frac{f'_a}{f'_b}$$

in the case of the ordinary adjustment when F'_a coincides with F_b.

But we found above that

$$\text{Magnifying power} = -\frac{f'_a}{f'_b}$$

thus we get also the new result (using diameters now instead of the radii of the pupils)

$$\text{Magnifying power (numerical)} = \frac{\text{diameter of entrance pupil}}{\text{diameter of exit pupil}}$$

For purposes of actual measurement, it may be noted that a real object of height p_1 anywhere in the object space, measured perpendicular to the axis of the telescope, must have an image of height p_1', where

$$-p_1' = p_1\left(\frac{f'_b}{f'_a}\right)$$

which is seen at once by tracing a ray through the top of the object parallel to the axis in the object space. Hence the magnifying power is $\frac{p_1}{p_1'}$. In one very useful practical method of measuring the magnifying power, the real object may be conveniently represented by the points of a pair of dividers opened out to a convenient extent (p_1), and placed immediately in front of the objective. The images of the points will be found just within the position of the exit pupil, and their separation (p_1') can be measured with the aid of a suitable scale mounted with a magnifier, or with a small travelling microscope. This is a very reliable and convenient method of measuring the magnifying power of any small telescope.

In another method the whole objective is illuminated by light diffused from a sheet of paper which more than covers its aperture, when the ex·· pupil can be observed as a uniformly illuminated disc behind the eyepiece, and its diameter can be measured (as above) with a small scale observed by a magnifier.

Object at Finite Distance. It may be the case that the instrument is being used with an object at a finite distance, and that the image is formed in the accommodation point at a distance k from the exit pupil or eye-ring *where the observing eye is situated.* This condition is represented in Fig. 13.

Trace a ray from the axial point of the object through the boundary of the entrance pupil. It must, therefore, pass through the boundary of the exit pupil, and its axial intersection point (real or apparent)

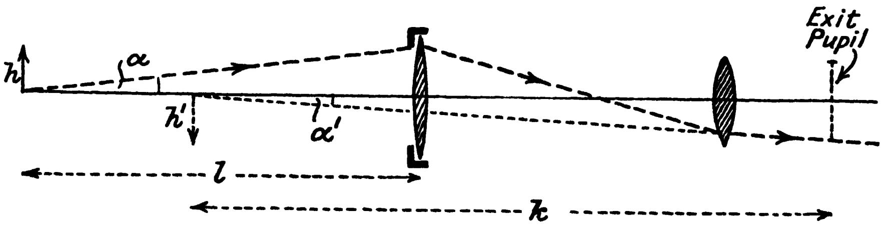


FIG. 13

must be in the final image plane at a distance k. Let h and h' be the perpendicular dimensions of object and image respectively.

The Lagrange relation gives

$$nha = n'h'a'$$

where a and a' are the angles made with the axis by the above incident and emergent rays through the axial points of object and image. If p and p' are the radii of entrance and exit pupils respectively, and l is the distance of the object from the entrance pupil,

$$a = \frac{p}{l} \text{(very nearly)}$$

$$\text{and } a' = \frac{p'}{k}$$

Hence (since n and n', the refractive indices, are almost invariably unity, and will now be assumed to be so)

$$\frac{hp}{l} = \frac{h'p'}{k}$$

$$\text{or } \frac{p}{p'} = \frac{h'/k}{h/l}$$

$$= \frac{\text{Angular subtense of image at the eye}}{\text{Angular subtense of the object at the entrance pupil of the instrument}}$$

We may, therefore, look upon the ratio of the diameters of the entrance and exit pupils as representing the "magnifying power" in the above sense, even though the telescope may not be in the adjustment for infinitely distant object and image.

Position of Principal and Focal Points of the System. The formulae of Chapter II, Vol. I, enable us directly to calculate the position of the principal and focal points of a combination of two optical systems.

If a telescope is in the adjustment when F'_a coincides with F_b, then the intercept "g" between the adjacent foci is zero. The formulae

$$f = \frac{f_a f_b}{g} \qquad P_a P = \frac{f_a d}{g} \qquad P_a F = \frac{f_a}{g}(d + f_b)$$

$$f' = -\frac{f'_a f'_b}{g} \qquad P_b' P' = \frac{f'_b d}{g} \qquad P'_b F' = \frac{f'_b}{g}(d - f'_a)$$

show that the focal lengths of the system are infinite, and also that the principal and focal points are at an infinite distance. The combination is said to be in *afocal adjustment.* Hence the general methods of discussing the optical performance of the system (position of principal planes, focal length, etc.) find very little application.

Telescope Objectives. The early work [1] of *Chester Moor Hall* (1733) and *Dollond* (1758) on the achromatic object glass, and the development of the telescope up to the time of Fraunhofer, can only be mentioned here. Fraunhofer's re-discovery of the dark lines of the solar spectrum and their use for exact measurements of refractive index first placed the matter of achromatism on a definite basis. Fraunhofer achromatized his telescope objectives in the following way. He divided the solar spectrum into regions bounded by the lines A, B, C, D, E, F, and G, and then measured the ratio of the flint dispersion to the crown dispersion for each region. He adopted a mean value in which the ratios were weighted according to the amount of light in each region; thus this work involved one of the earliest essays in heterochromatic photometry. The mean ratio gave a measure of the ratios of the total curvatures ($R_1 + R_2$) for each component. This condition resulted in lenses which had their minimum foci somewhat towards the blue end of the spectrum, and Fraunhofer found empirically that a better result was obtained by a slight change in the ratio which resulted in a minimum focus nearer the red. The condition of bringing together the foci for C and F (introduced somewhat later) ensures that the minimum focal length of an ordinary objective falls in the apple-green region of the spectrum, and produces the maximum bunching together of the radiations which are brightest to the eye. Subsequent systematic work[2] reveals little to be desired in this provision for visual observation.

The type of objective manufactured by Fraunhofer is shown in Fig. 14. It consists of a double-convex crown and a nearly plano-concave flint. Primary chromatic aberration is corrected in the above manner, and spherical aberration is corrected for one zone. The lens is practically free from coma, although it must be doubted whether Fraunhofer knew of the "sine condition." It is the leading type of all small telescope objectives.

If such lenses are designed with the same radius on the adjacent faces, the contact may be cemented. This helps cleanliness and avoids internal reflections. Chromatic and spherical aberration can still be removed, but freedom from coma can only be secured by the choice of suitable glasses.[3]

The absolute elimination of coma is not always considered essential by manufacturers in objectives for small theodolites where

Crown lens $n_D = 1{\cdot}531$
Flint „ „ $= 1{\cdot}616$
$r_1 = +340, r_2 = -139$
$r_3 = -136, r_4 = -620$

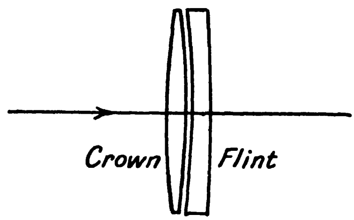


FIG. 14. FRAUNHOFER OBJECTIVE

good central definition suffices. If the axis of the lens happens, however, to be slightly out of its proper alignment, troublesome coma may appear in the middle of the field. It is not usual to cement lenses with diameters greater than 2 in.

Variations of the above general design are sometimes used for particular purposes. The "Steinheil" form of objective is shown in Fig. 15(*a*). The flint lens faces the object, and somewhat steeper curvatures are necessary, but the freedom in removing aberrations is much the same as with the Fraunhofer type.

The removal of spherical aberration for two wave-lengths is possible by bending both components of the achromatic combination, as suggested by Gauss. The profile of such an objective is shown in Fig. 15(*b*). Such a condition is, generally, secured only by the loss of freedom from coma. Such lenses have been used in large theodolites.

Herschel proposed objectives calculated for the removal of spherical aberration for two object distances, but this provision is not often deemed necessary.

Large and Small Objectives. The modern manufacture of small telescope objectives of apertures up to 2 in. is now largely a matter of mass production. The data for the system will be computed before manufacture begins, and the optical performance will be expected to agree with calculation.

The manufacture of large objectives involves much greater uncertainty, and figuring by local rubbing is usually necessary to correct (*a*) zonal spherical aberration, (*b*) errors in the regularity of the surfaces, (*c*) effects of lack of homogeneity in the glass.

The residual secondary spectrum of an astronomical objective is prejudicial in exacting observations. Considerations of cost and the

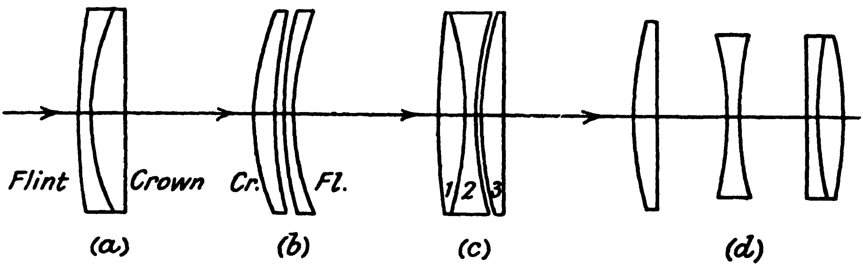


FIG. 15. TYPICAL TELESCOPE OBJECTIVES

(*a*) Steinheil objective
(*b*) Gauss ,,
(*c*) Cooke ,,
(*d*) Astrographic objective Tessar type (Zeiss)

Glasses in Cooke objective—
1 Baryta light flint (O, 543)
2. Borosilicate flint (O, 658)
3. Crown (O, 374)
Jena glass numbers

chemical stability of glass usually call for a doublet of crown and flint for the largest refractors of aperture 10 in. or over, and the secondary spectrum must perforce be tolerated, but the attempts to reduce this in smaller lenses must be noted.

Reduction of Secondary Spectrum. The attempt to produce glasses from which a pair could be selected for the similarity of their run of partial dispersions and so eliminate the residual secondary spectrum of a doublet has already been discussed (Vol. I, p. 231). Failing the requisite chemical stability of suitable glasses, interest is given to the possibilities involved in separating the crown and flint components of a doublet so as to have a space between them. The partial dispersions of a typical "hard crown" and "dense flint" are—

	V	a	β	γ
Hard crown . . .	60·2	·643	·703	·566
Dense flint . . .	36·2	·605	·714	·609

Notice that the dispersion of the flint is relatively too low in the red, but too high in the violet. Separation of the crown and flint components of a telescope objective (Fig. 16) would allow of the use of a smaller flint lens of shorter focal length. At the same time the violet ray is more deviated than the red in passing the crown lens, so that the violet meets the "diverging lens" nearer the axis than the red. This clearly acts in the sense of reduction of the secondary spectrum, but the correction so attainable is not sufficient with a single lens as the rear member without making the curvatures of the lenses so great as to negative the advantage gained. A proposal by Rogers to use a doublet correcting lens which is of zero power for mean wave-lengths, and yet acts in a "divergent" sense as between the red and blue, solves the problem of the secondary spectrum only to introduce troublesome spherical aberration and chromatic differences of magnification. Lenses of similar type have been produced by Plössl and others under the name "dialyte objectives," but they have not come into very common use.

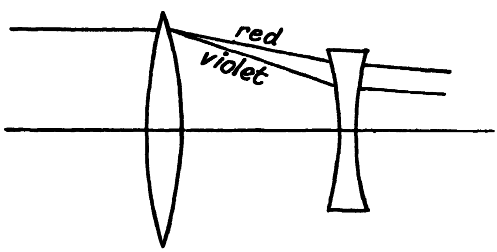


FIG. 16. ACTION OF SEPARATED COMPONENTS

There are certain advantages attainable by a *modest* separation of the lenses of a doublet objective, in spite of the failure appreciably to reduce the secondary spectrum. The objective follows the changes of external temperature with greater facility, and a suitably adjusted lens can be freed from spherical aberration for two colours. As will be understood from the formula quoted below, the shorter focal length of the flint lens tends in the direction of a flatter image field.

The most successful mode of securing the practical elimination of the secondary spectrum is by the use of a three-component lens in which we may have, for example, two positive components which together represent one lens of a glass having a run of partial dispersions similar to that of the negative component. Fig. 15(*c*) shows the section of a Cooke-Taylor photo-visual objective.

The relative partial dispersions of the glasses used for the two positive components of the Cooke photo-visual objective are given in the following table. The mean of their partial dispersions is shown in the third line, while the fourth line gives figures for the boro-silicate flint used in the negative lens.

Glass	n_D	V	Dispersion	Relative Partial Dispersions				
			C–F	A C	D–F	E–F	F–G′	F–H
0·543	1·564	50·7	·01115	·3354	·7085	·3309	·5830	1·1857
0·374	1·511	60·8	·00844	·3507	·7026	·3247	·5675	1·1564
Mean				·3420	·7059	·3282	·5763	1·1730
0·658	1·546	50·1	·01090	·3425	·7052	·3278	·5767	1·1745

The powers of the two positive components are very nearly equal, so that the effect of these lenses combined has a run of partial dispersion exceedingly close to that of the flint negative component.

The limits of size of astronomical refractive objectives are set in practice by the difficulties attendant on the production of discs of optical glass of large diameter. The Yerkes objective has a diameter of 1·02 metres and a focal length of 18·9 metres, and the Lick refractor a diameter of 0·91 metres and focal length 17·6 metres; these are large doublets. Generally speaking, the aperture ratio (aperture : focal length) decreases with the diameter. With very small lenses 1 : 4 may be reached, but 1 : 18 is as much as can be allowed with large refractors on account of the prominence of the secondary spectrum.

When the triplet lenses are used the necessary curvatures are rather large, and it is not easy to obtain a satisfactory performance with an aperture ratio better than 1 : 15. The Gauss condition (spherical correction for two wave-lengths) can be secured by separating two of the components. Such lenses have been made up to 12½ in. clear aperture.

The image given by a triplet apochromatic objective presents practically no trace of colour, and such a lens, if used for astronomical photography, may be focussed visually.

Astrographic Objectives. If the correction is not to be apochromatic as in the photo-visual objectives mentioned above, the lens will be best adapted for astronomical photography if the "F" line focus is united with that for Hδ (wave-length $= 0{\cdot}4101\mu$). This ensures a better bunching of the most "actinic" regions of the spectrum, using the term to refer to those wave-lengths which most affect an ordinary photographic plate. The use of such plates confines the effective light to a limited spectral band, and tends to secure better definition on that account. Modern photographic work in astronomy includes, however, work both with ultra-violet and infra-red radiations, so that the exact requirements for specialized work may be very varied. The optical design of astrographic objectives aims naturally at securing the largest possible *flat field* together with the indispensable high definition required for astronomical photography, which calls for freedom from spherical aberration, coma, and astigmatism. These requirements are met by lenses of which the design resembles photographic lenses, except in so far as they work at a very much smaller aperture ratio. While they imitate the flat field of the photographic lens, they allow of much better central definition. Lenses of the type of the Taylor triplet, and the Tessar (Carl Zeiss) (Fig. 15D) are in use for this purpose. It is frequently the case, however, that astrographic work is done with doublet lenses very carefully designed, and made to be as free as possible from spherical aberration and coma.

The Image Field and its Curvature. The diagrams so far used show the ordinary flat image field of the elementary theory. In practice the phenomena of curvature of field and the accompanying degree of astigmatism are of the greatest significance. Little or nothing can be done to remove the astigmatism of an ordinary doublet telescope objective, and the curvature of the field and astigmatism with those objectives constructed of the usual types of glass, such as hard crown and dense flint, are finite and comparable with those of a single lens of the same focal length.

Thus for thin lenses in air the radius of the Petzval surface is

$$\frac{1}{R} = \Sigma\left(-\frac{1}{nf'}\right)$$

where n is the refractive index of a lens and f' the focal length. E. W. Taylor records an achromatic objective having—

Positive lens: Medium barium crown, $n_D = 1{\cdot}5736$, focal length $= 3{\cdot}543$
Negative lens: Light flint, $n_D = 1{\cdot}6039$, focal length $= 5{\cdot}391$

The focal length of the combination is 10·06; and the radii which can be found by calculation and checked by experiment are—

Tangential image field	2·85
Sagittal image field	6·05
Petzval surface	15·6

In accordance with theory, the tangential surface is three times the distance of the sagittal surface from the Petzval.*

Such a curvature will not be of much significance in the small diameter intercepted by an eyepiece, but the curvature of the image field for the eyepiece has also to be taken into consideration.

* Vol. I, p. 136.

The cheaper forms (see below) are constructed of two separated lenses made out of the same glass. Take, for example, an Huygenian form in which the focal length of the field lens may be $3q$, say, while that of the eye-lens is q. Applying the formula above and assuming $n = 1{\cdot}5$ approximately

$$\frac{1}{R} = -\frac{1}{4{\cdot}5q} - \frac{1}{1{\cdot}5q}$$

The focal length for the Huygenian combination (page 43) at a distance of $2q$ proves to be $\frac{3q}{2}$. This shows that $R = 0{\cdot}75f$, so that

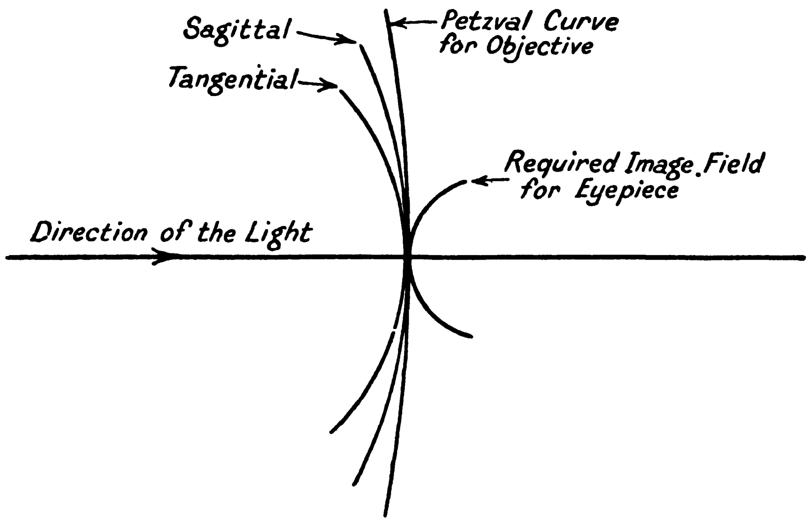


FIG. 17

the Petzval curvature for an eyepiece of focal length 1 in. would be about three-quarters of an inch.

Object points on a surface of such curvature (assuming astigmatism absent) would give the parallel rays required to emerge from the eyepiece. In Fig. 17 is shown in full size the section of the image surfaces for the objective discussed above, assuming a focal length of 10 cm. The right-hand curve is that on which the image should lie if the field of the instrument is all to be "at infinity" when viewed with an eyepiece of focal length 1·3 cm. approximately (if we may assume the above theory, and also neglect the astigmatism of the eyepiece).* The gap between the two increases very rapidly with distance from the axis, but the physical depth of focus

* Actually the eyepiece will be designed to have some over-corrected astigmatism which will flatten the field to some extent.

allows of a certain limited region of the field being apparently sharp. If the outer parts of the field are to be viewed in sharp focus, the eyepiece may be pushed inwards. The central parts of the field may then be viewed clearly by the exertion of accommodation. In attempting to observe this with actual telescopes, allowance will have to be made for the residual astigmatism which will be present. In eyepieces like the Huygenian, where the light has to pass through a larger field lens and then a smaller eye lens, it is

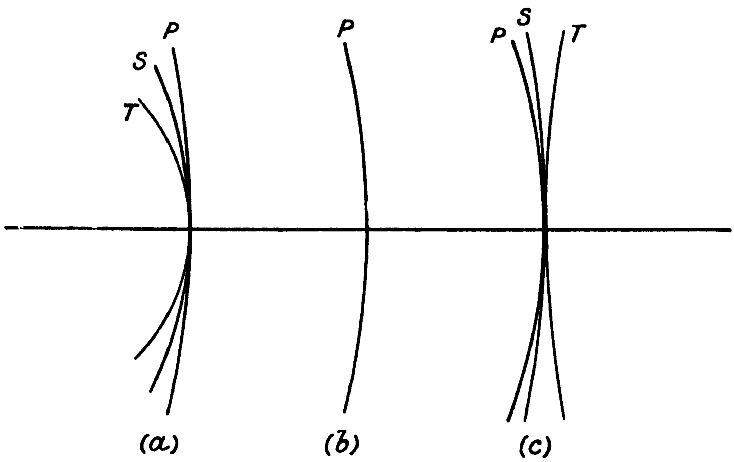


FIG. 18

(*a*) Under-corrected astigmatism
(*b*) Correction; images on Petzval surface
(*c*) Over-corrected astigmatism

N.B. The lens system is imagined to lie to the left of these figures.

possible to modify the amount and direction of the astigmatism by suitable curvatures of the lenses, and to produce over-correction.

Take Fig. 18(*a*) to represent an ordinary case of curvature of the field and "under-corrected" astigmatism (as of the type which arises with a simple lens). If we change the curvatures or separations of the component lenses of a complex system without altering their focal lengths or refractive indices, we shall sometimes be able to alter the amount of the astigmatism, but not the Petzval curvature. In Fig. 18(*b*) the astigmatism is eliminated, and the field has now the curvature of the Petzval surface. In Fig. 18(*c*), the astigmatism is over-corrected in such a way as to make the mean curvature of tangential and sagittal surfaces zero, so that the field is "flattened" in that sense. The tangential field remains at three times the distance of the sagittal field from the Petzval surface. The possibilities mentioned in this paragraph do *not*, however, apply to ordinary close doublet telescope objectives. The subject will be again discussed in connection with photographic lenses.

Design of a Doublet Telescope Objective. The rough design will be worked out for a doublet which is to satisfy (1) the ordinary condition of achromatism, (2) freedom from spherical aberration for some wave-length of the spectrum. If the lens is to be cemented the degrees of freedom will then be exhausted, but if the components may have different radii for the adjacent surfaces, the condition of freedom from coma may be added. Choosing for trial the glasses Hard Crown and Dense Flint selected for the example of Vol. I, p. 229, we had for the powers $\mathcal{K}_a$ and $\mathcal{K}_b$ of the components of a doublet—

$$\mathcal{K}_a = 2{\cdot}68 \qquad \mathcal{K}_b = -1{\cdot}68$$

$$n_D = 1{\cdot}5186 \qquad n_D = 1{\cdot}6041$$

$$V = 60{\cdot}3 \qquad V = 37{\cdot}8$$

The formula for the spherical aberration coefficient of a thin lens is

$$A = \mathcal{K}^3\left(\frac{n}{n-1}\right)^2 + \mathcal{K}^2\mathcal{V}_1\left(\frac{3n+1}{n-1}\right) - \mathcal{K}^2\mathcal{R}_1\left(\frac{2n+1}{n-1}\right)$$

$$+ \mathcal{K}\mathcal{V}_1^2\left(\frac{3n+2}{n}\right) - \mathcal{K}\mathcal{R}_1\mathcal{V}_1\left(\frac{4n+4}{n}\right) + \mathcal{K}\mathcal{R}_1^2\left(\frac{n+2}{n}\right)$$

In the case of the first lens we take $\mathcal{V}_1 = 0$, and hence

$$A_a = \mathcal{K}^3_a\left(\frac{n}{n-1}\right)^2 - \mathcal{K}^2_a\mathcal{R}_1\left(\frac{2n+1}{n-1}\right) + \mathcal{K}_a\mathcal{R}_1^2\left(\frac{n+2}{n}\right)$$

Calculating the requisite numerical coefficients (by four-figure logs.)

$$A_a = 165{\cdot}0 - 55{\cdot}91\ \mathcal{R}_1 + 6{\cdot}21\ \mathcal{R}_1^2$$

It is more convenient, however, to express the relation in terms of $\mathcal{R}_2$ where

$$\mathcal{K} = (n_a - 1)(\mathcal{R}_1 - \mathcal{R}_2)$$

and $$\mathcal{R}_1 = \mathcal{R}_2 + 5{\cdot}168$$

We then obtain

$$A_a = 42{\cdot}0 + 8{\cdot}27\ \mathcal{R}_2 + 6{\cdot}21\ \mathcal{R}_2^2 \qquad . \quad . \quad (a)$$

For the second lens $\mathcal{V}_1 = \mathcal{K}_a = 2{\cdot}68$, and $\mathcal{K}_b = -1{\cdot}68$. The equation, when simplified, reduces to

$$A_b = -11{\cdot}91 + 9{\cdot}58\mathcal{R}_3 - 3{\cdot}77\mathcal{R}_3^2 \qquad . \quad . \quad (b)$$

In order that the lens may be cemented it will be necessary for $\mathcal{R}_2$ to be equal to $\mathcal{R}_3$. Hence if the sum of the aberrations is to be zero

$$A_a + A_b = 0 = 30{\cdot}09 + 17{\cdot}85\,\mathcal{R} + 2{\cdot}44\,\mathcal{R}^2$$

a simple quadratic equation in $\mathcal{R}$ having roots $-4{\cdot}67$ and $-2{\cdot}64$. This determines the forms of two possible lenses for cementing, although one (for reasons given below) is much preferable to the other.

First lens: $\mathcal{R}_1 = 0{\cdot}498$ $\qquad$ $\mathcal{R}_1 = 2{\cdot}52$

$\mathcal{R}_2 = \mathcal{R}_3 = -4{\cdot}67$ $\qquad$ $\mathcal{R}_2 = \mathcal{R}_3 = -2{\cdot}64$

$\mathcal{R}_4 = -1{\cdot}885$ $\qquad$ $\mathcal{R}_4 = 0{\cdot}145$

It may be deemed that the contact should be left uncemented in order to satisfy the condition for the absence of coma by a possible difference of curvature of the adjacent faces. The most instructive method to proceed is to plot the values of the aberration found from equations (*a*) and (*b*) above in a graphical diagram. The following table gives points for plotting—

R_2 or R_3	-5	-4	-3	-2	0	$+2$	$+4$
A_a . .	155·75	108·22	73·08	50·30	42·0	83·38	174·38
A_b . .	$-154{\cdot}01$	110·55	$-74{\cdot}58$	$-45{\cdot}15$	$-11{\cdot}91$	$-6{\cdot}83$	$-33{\cdot}91$

As is clearly seen from the equations, we obtain two parabolas (Fig. 19) with vertical axes. It is convenient to plot A_b values with reversed sign. The parabolas then intersect in the two points with abscissae $-4{\cdot}67$ and $-2{\cdot}64$ corresponding to the roots of the equation above.

Coma. By an extension of the method given in Chapter IV, Vol. I, we can investigate the value of the coefficient a_2 in equation (50) of that chapter, i.e. the "coma" coefficient. Professor Conrady has shown that it is proportional to

$$\frac{N+1}{N}\mathcal{K}\mathcal{R}_1 - \frac{2N+1}{N}\mathcal{K}\mathcal{V}_1 - \frac{N.\mathcal{K}^2}{N-1}$$

for the case of a thin lens. We insert the necessary numerical quantities, and find for the above coma coefficients of lenses *a* and *b*.

$$C_a = 4{\cdot}445\,\mathcal{R}_2 + 1{\cdot}94$$

$$C_b = -2{\cdot}73\,\mathcal{R}_3 + 4{\cdot}32$$

These lines are drawn in the diagram—again plotting C_b values with reversed sign; it will thus be seen that the difference between C_a and C_b for the abscissa of the lower intersection point of the two

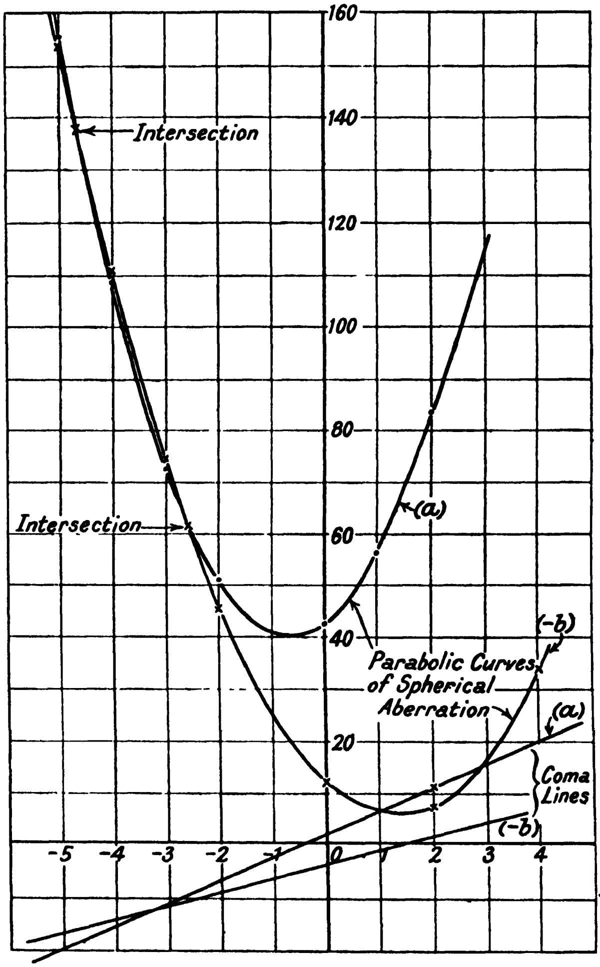


FIG. 19. GRAPHICAL PRESENTATION OF SPHERICAL ABERRATION AND COMA FOR A DOUBLET OBJECTIVE

parabolas is about the same as with that for the higher intersection point. The lower intersection point with the shallower curve for the contact would, however, be preferable. (We can calculate the intersection point of the coma lines as $-3{\cdot}66$.)

If an uncemented doublet is allowable, we can choose a pair of lenses to be free from both spherical aberration and coma.

If $C_a + C_b = 0$, we get by adding the equations

$$4{\cdot}445\ \mathcal{R}_2 - 2{\cdot}73\ \mathcal{R}_3 + 6{\cdot}26 = 0$$

$$\mathcal{R}_3 = 1{\cdot}63\ \mathcal{R}_2 + 2{\cdot}29$$

This relation between $\mathcal{R}_2$ and $\mathcal{R}_3$ will ensure absence of coma.

Substituting this value for $\mathcal{R}_3$ in the expression for A_b above, we obtain

$$A_b = -10{\cdot}02\ \mathcal{R}_2^2 - 12{\cdot}52\ \mathcal{R}_2 - 9{\cdot}73$$

We had

$$A_a = 6{\cdot}21\ \mathcal{R}_2^2 + 8{\cdot}27\ \mathcal{R}_2 + 42{\cdot}0$$

if the sum of the aberrations is to be zero, then we get on adding

$$0 = 3{\cdot}81\ \mathcal{R}_2^2 + 4{\cdot}25\ \mathcal{R}_2 - 32{\cdot}27$$

This quadratic equation yields two solutions, i.e. $\mathcal{R}_2 = -3{\cdot}52$ or $+2{\cdot}405$. The first of these is the one of practical interest, since the second would give a pronounced meniscus form to the lens as a whole. The figures therefore become

$$\mathcal{R}_1 = 1{\cdot}65$$
$$\mathcal{R}_2 = -3{\cdot}52$$
$$\mathcal{R}_3 = -3{\cdot}45$$
$$\mathcal{R}_4 = -0{\cdot}67$$

Suitable thicknesses have now to be assigned, and the system perfected by trigonometrical trials.

The foregoing account explains the approximate method of designing a telescope objective which is to be corrected in itself. If the objective is to be used for visual observation, it is better to correct the spherical aberration for the brightest light in the spectrum ($\lambda = 0{\cdot}55\mu$).

When an objective is intended for use with a definite eyepiece, the axial chromatic and spherical aberration of the latter can be found (as the eyepiece design is usually completed first) by tracing some rays of an axial parallel beam backward through the system. The objective is then designed so as to compensate the axial aberrations of the eyepiece, and will need to be slightly over-corrected if intended for use with an ordinary Huygenian or Ramsden eyepiece. The condition for chromatic correction can be adjusted to allow for this, even in getting the rough design.

The approximate solution is turned into a trial formula by finding the radii, and assigning a suitable thickness and diameter necessary to obtain the required aperture ratio. The trigonometric formulae are then employed in tracing a group of parallel rays through the objective, a paraxial ray, a marginal ray, and one or more rays in mid-zones, preferably using the wave-length for brightest light. "Red" and "blue" rays can also be traced through the mid-zone. The last radius can be adjusted to produce the exact chromatic correction required, and the state of correction as regards spherical aberration and coma can be examined. Chapter IV of Vol. I contains an explanation of the methods by which the phase differences of disturbances arriving in the image can be deduced from a knowledge of the geometrical aberrations of ray paths. The state of correction as regards coma is examined by the so-called "offence against the sine condition."

The optical sine relation

$$nh \sin a = n'h' \sin a'$$

and the corresponding paraxial form

$$nha_0 = n'h'a'_0$$

give two values of the magnification deduced from the marginal and paraxial ray paths, viz.

$$m_m = \left(\frac{h'}{h}\right)_{marginal} = \frac{n \sin a}{n' \sin a'}$$

and

$$m_p = \left(\frac{h'}{h}\right)_{paraxial} = \frac{na_0}{n'a'_0}$$

The "sine" condition for freedom from coma (valid in the absence of spherical correction) is that m_m and m_p shall be identical.* The offence against the sine condition is given by $\left(\frac{m_m}{m_p} - 1\right)$. The computing schedule (Vol. I, p. 19) will usually begin with the same numerical value for $\sin a$ and a_0. Hence

$$\text{Offence against the sine condition} = \frac{m_m}{m_p} - 1 = \frac{a'_0}{\sin a'} - 1$$

The numerical result thus obtained is clearly a measure of the coma, taken as the radial distance between the focussing points of paraxial and marginal zones divided by the radial distance of the

* Note that the magnification for a marginal ray is

$$m_m = \frac{n \sin a}{n' \sin a'} = \frac{n\left(\frac{y_m}{l}\right)}{n' \sin a'}$$

where y_m is the incidence height of the ray and l is the distance of the object when this distance is large. Hence when the incident light is parallel to the axis, the sine condition may be expressed as the necessary constancy of $\frac{y_m}{\sin a'}$.

image point from the centre of the field. It is the general experience with ordinary telescope and microscope objectives that it must not be allowed to rise above one part in 400, or 0·0025.

It is not within the scope of the present book to describe in detail the systematic trials by which the design of an objective is finally completed. The approximate thin lens method saves much time in the early stages. Modern advances in design have arisen through the ability to interpret the aberrations obtained from the numerical work in terms of optical path differences at the focus of the lens, and through an exact knowledge of the tolerances allowable. An introductory account of the general principles involved is given in Vol. I, Chapter IV, but the reader must be referred for fuller details to Prof. Conrady's work[4] on *Applied Optics and Optical Design.*

Resolving Power of a Telescope. In Chapter IV, Vol. I, it was explained that the closest approach of two elementary "star" images which still permits of the recognition of the double nature of the concentration is approximately such that the centre of one Airy disc falls on the first dark ring of the neighbouring image.

The angle subtended by the radius of the Airy disc at the second nodal point of a telescope objective therefore represents the angular resolving power. We have the approximate formula for an image in air

$$\rho = \text{radius of Airy disc} = \frac{0{\cdot}61\lambda f'}{y} \quad \text{(see Vol. I, p. 93.)}$$

The angle w' subtended by this radius at the second nodal point is given by

$$w' = \frac{\rho}{f'} = \frac{1{\cdot}22\lambda}{a}$$

where a is the diameter of the objective.

This is the same angle as that subtended by the objects at the first nodal point. If this is put into English units we easily find—

$$\text{Angular resolving power in seconds} = \frac{5{\cdot}5}{\text{aperture in inches}}$$

It has been made clear that the theoretical basis for this limit is only approximate; the exact figure depends upon the distribution of light in the actual "Airy Disc," and this may be modified by spherical aberration and other causes in an actual lens. Dawes's Rule derived from experiment gives—

$$\text{Angular resolving power in seconds} = \frac{4{\cdot}5}{\text{aperture in inches}}$$

In order that it may be possible for the eye to perceive the doubling of the image of a close double star, the separation of the images must clearly subtend an angle at least as great as the *minimum separabile* for the eye, which Hooke found to be one minute of arc.

As viewed by the eyepiece of focal length f'_l, the angle under which the radius of the Airy disc is seen is

$$\frac{\rho}{f'_b} = \frac{1\cdot22\lambda f'_a}{af'_b}$$

$$= w'\,M. \quad . \quad . \quad \text{(from above)}$$

Note that this is, of course, "angular resolving power" multiplied by the "magnification." If the angle of view is to be 1′ of arc (= 0·00029 radians), then

$$0\cdot00029 = \frac{1\cdot22\lambda}{a}\,.\,M$$

It is easy to get an approximate figure if we take $\lambda = 0\cdot00058$ mm., and thus

$$M = \frac{a\ (\text{mm.})}{2\cdot44} = 10 \times (\text{aperture in inches}) \quad . \quad . \quad . \quad \text{approximately.}$$

At such a magnification, however, the resolvable detail, even if just visible, would be unendurably small, and even though no fresh detail can be rendered it will make for freedom from visual strain to increase the magnification to three or four times the above figures. Further enlargement serves little or no useful purpose; it is "empty magnification."

The above equations give (for 1′ visual angle)

$$f'_b = \frac{1\cdot22\lambda}{0\cdot00029}\left(\frac{f'_a}{a}\right)$$

$$f'_b\ (\text{mm.}) = 2\cdot44 \times (\text{aperture ratio number})$$

As indicated, however, it will be better to take a focal length only one-third or one-quarter of the above value. Thus for an astronomical objective with an aperture ratio number $\frac{f'}{a} = 15$, the minimum value of f'_b is 37 mm., but we shall be able to use with visual advantage eyepieces with focal lengths as short a~ 12 mm. or 9 mm.

The equation which gives the minimum magnification for the telescope

$$\text{Min.} = \frac{a\ (\text{mm.})}{2\cdot44}$$

is useful when we put

$$\text{Min.} = \frac{a\ (\text{mm.})}{\text{diameter of exit pupil in mm.}} = \frac{a}{2{\cdot}44}$$

It will be seen that in order to do justice to the resolving power of the objective the magnification should be *at least* sufficient to reduce the diameter of the exit pupil to 2·4 mm., and we can with visual advantage use exit pupils down to 0·6 mm. If a handy test of the

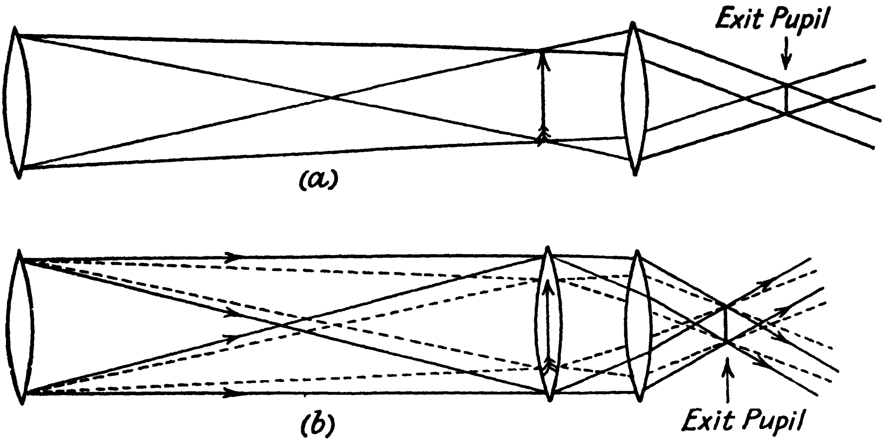


FIG. 20

(*a*) Telescope without field lens
(*b*) Telescope with field lens; note enlargement of field and shift of exit pupil

effective state of magnification is required, the measurement of the diameter of the exit pupil gives the information at once. It will be understood that the majority of binoculars and instruments with large exit pupils come nowhere near making the fullest use of the resolving power of their objectives, although the large exit pupil is important in maintaining the illumination of the image.

The Eyepiece. The early astronomical telescope as employed by Kepler about 1611 consisted of a convex object glass and a convex eye lens. Reference to Fig. 20(*a*) shows that all the rays passing through the extremities of the image are just intercepted by the eye lens, but that if the image were larger, part of the rays would fall outside the eye lens. The illumination of the peripheral parts of the image would therefore be poor, and the optical performance of the instrument would suffer otherwise. It is not known with certainty to whom the suggestion of the field lens (Fig. 20(*b*)) should be credited. It will be seen that the use of a convex lens in the plane of the image bends the rays towards the axis, although it can make no difference to the size of the image. Hence the peripheral parts of the image are now viewed by full pencils, the apparent field of view being controlled by the margin of the field lens in the case shown.

It is worthy of note that the exit pupil is now moved considerably closer to the eye lens, although it remains of the same size. Most modern eyepieces, however, do not place the field lens in the focal plane, because any specks of dirt on the glass are then seen in focus in the field of view.

The addition of yet a third convex lens, as an erector, transformed this eyepiece into a terrestrial erecting eyepiece. This will be dealt with below, but we must first notice two important forms of two-lens eyepiece due to Huygens (1703) and Ramsden (1783). Their forms are illustrated in Figs. 21 and 22 respectively.

In dealing with the simple magnifier, it was mentioned that one serious difficulty arises in the variation of magnification with the wave-length (or with the colour) of the light. This may be regarded as arising from a variation of focal length with wave-length. Consider, however, the expression for the power of a combination of two thin coaxial lenses of the same glass, of refractive index *n* and separated by a distance *d*.

$$\mathcal{P} = \mathcal{P}_a + \mathcal{P}_b - d\,\mathcal{P}_a\mathcal{P}_b$$

This becomes

$$\mathcal{P} = (n-1)\,\mathcal{R}_a + (n-1)\,\mathcal{R}_b - d(n-1)^2\,\mathcal{R}_a\mathcal{R}_b$$

Let the refractive index of the glass change for some given change of wave-length and become $n + \delta n$; then

$$\mathcal{P}_1 = (n+\delta n-1)\,\mathcal{R}_a + (n+\delta n-1)\,\mathcal{R}_b - d(n+\delta n-1)^2\,\mathcal{R}_a\,\mathcal{R}_b$$

Subtracting

$$\mathcal{P}_1 - \mathcal{P} = \delta n\,.\,\mathcal{R}_a + \delta n\,\mathcal{R}_b - d\{2\delta n(n-1) + \delta n^2\}\,\mathcal{R}_a\,\mathcal{R}_b$$

The condition that the focal length may be unchanged, if δn is small enough for its square to be neglected, is thus

$$0 = \mathcal{R}_a + \mathcal{R}_b - 2d(n-1)\,\mathcal{R}_a\,\mathcal{R}_b$$

or

$$d = \tfrac{1}{2}\left\{\frac{1}{(n-1)(\mathcal{R}_a)} + \frac{1}{(n-1)(\mathcal{R}_b)}\right\}$$

$$= \tfrac{1}{2}(f'_a + f'_b)$$

This simple formula furnishes a rough guide to the separation of two lenses of the same glass necessary in order to secure "achromatism of the focal length" of the magnifier, and thus freedom from the most objectionable radial colour effects in the field of view.

The Huygenian Eyepiece. In the Huygenian eyepiece the design varies amongst different makers, being adjusted to suit the thickness of the lenses and the objective distance. In a common form the focal length of the "field" lens is about twice that of the "eye" lens. The separation called for by the elementary theory is about

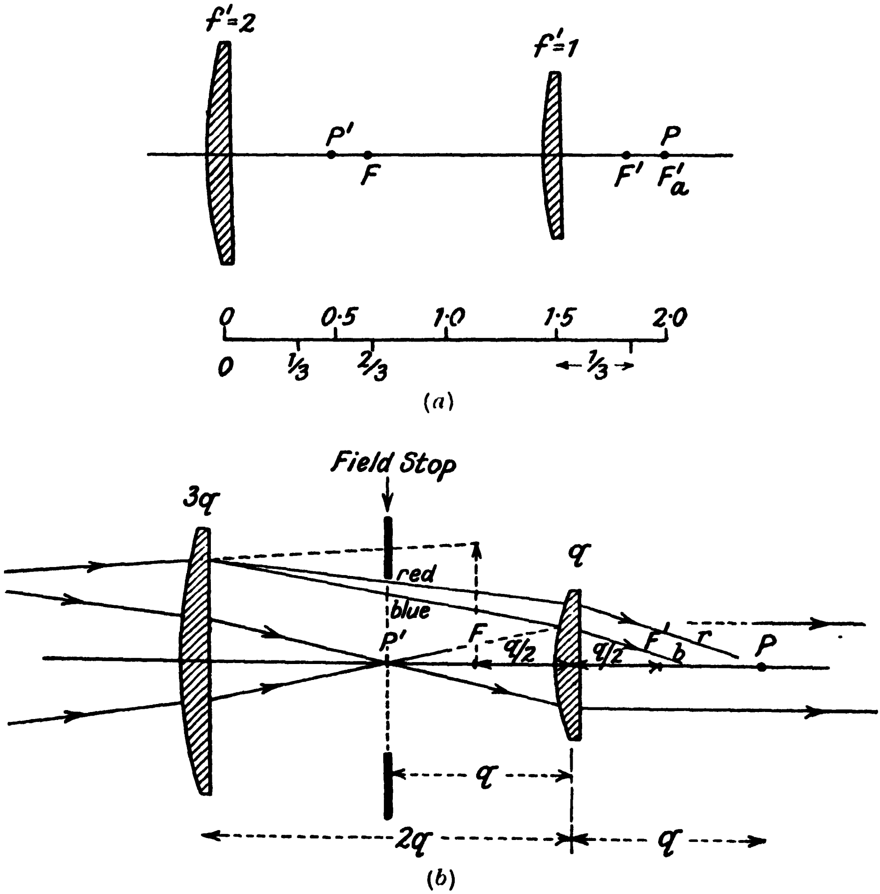


FIG. 21. TWO FORMS OF HUYGENIAN EYEPIECE

(*a*) Illustrates the position of the principal and focal points
(*b*) To illustrate the achromatism

one and a half times the focal length of the eye lens. Working out (by the thin lens theory) the position of the principal and focal points of the combination, they are found as in Fig. 21 (*a*). The principal points are so situated that the first is found behind the system and the second within it. The focal length of the combination is $\frac{4}{3}$ times that of the eye lens, and the power is positive, although it is clearly impossible to put a real object into the first principal focus and obtain an erect magnified image.

The design usually quoted in textbooks is shown in Fig. 21 (*b*); although not of much importance, it will serve to explain some features of the arrangement; in this case the field lens has a focal length of $3q$, say, an eye lens of focal length q, and a separation of $2q$; where q is some suitable unit. The application of the usual formulae gives $P_aP = 3q$; $P'_bP' = -q$; $f = -\frac{3}{2}q$; $f' = \frac{3}{2}q$. The first focal point F is $\frac{3q}{2}$ behind the front lens; the second, F′, is $\frac{q}{2}$ behind the eye lens.

In the ordinary use of the Huygenian eyepiece it will be placed so that the image-forming rays from the objective converge towards

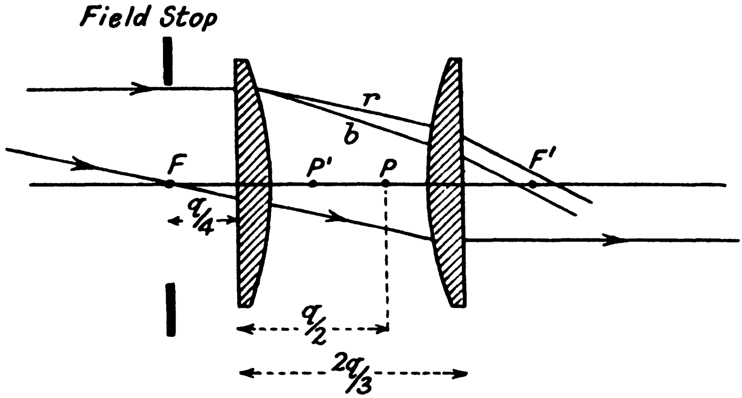


FIG. 22. RAMSDEN EYEPIECE
Showing the position of the principal and focal points

a point in the first focal plane through F. They reach the field lens and form a real image between the two components in the plane of the anterior principal focus of the eye lens *b*, which is the plane of the stop limiting the field of view. The eye lens renders the rays parallel after refraction. By use of the plano-convex form of each component it is possible to produce "over-corrected" astigmatism, and thus to flatten the field to some extent. Distortion may also be reduced. A field of 40° may be attained.

The achromatism of the system arises as follows. A ray directed towards the image point B_1 is shown in the diagram. (It may be considered to be the principal ray from the centre of the objective.) It is refracted by the field lens, and the dispersion causes a greater deviation for the blue than for the red. The separation of the field lens and eye lens is, however, such as to make the blue ray intersect the eye lens nearer the axis than the red, and the greater deviation now produced in the red now renders both of them parallel on final emergence. Hence both the blue and red images will be seen under the same angle by the eye.

The Ramsden Eyepiece. Invented by Ramsden for the observation of the micrometer webs in reading microscopes, this eyepiece has two lenses of equal focal length disposed as shown in Fig. 22. The required theoretical separation for achromatism is equal to the focal length of either, but with a telescope of any practical length this brings the exit pupil too close to the eye lens; for this reason the separation is reduced and the residual chromatic differences

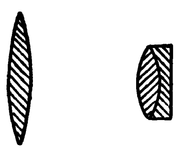


FIG. 23A. KELLNER EYEPIECE

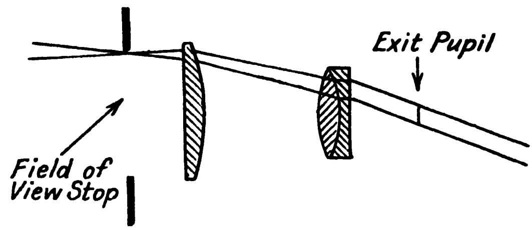


FIG. 23B. ACHROMATIZED RAMSDEN EYEPIECE

are tolerated. In the case shown, the focal length of each lens is q, and the separation is $2q/3$; the focal and principal points are then disposed symmetrically as shown in the figure, the combined focal length being $3q/4$.

The Ramsden eyepiece as actually used does not thus strictly secure achromatism of magnification, but is extremely useful in "measuring instruments" and "optical sights," because it is possible to place cross threads or stadia lines in the common focal plane of the eyepiece and objective, and thus to measure directly the comparatively undistorted image produced by the latter. The angular field represented by the separation of two stadia lines is independent of the eyepiece, and we may thus change from one magnification to another, if required, by the choice of a new eyepiece.

Kellner (1849) invented an eyepiece (Fig. 23A) with a double convex crown front lens and a cemented eye lens of the ordinary "crown and flint" type, which he called "orthoscopic," and for which he claimed a greatly improved colour correction in the outer regions of the field. In this case, however, the first focal plane lies in or very close to the front lens, so that any dust on the surface is seen in focus with the field of view. In a later type (Fig. 23B), introduced by Zeiss the advantages of the Ramsden type were retained, and a still better achromatism was effected by the use of barium silicate crown and silicate flint for the members of the cemented eye lens. This type is usually known as the "Achromatized Ramsden" form.

In the case of providing for a wide field and sufficient magnification, while maintaining sufficient clearance between the eyepiece and the eye ring or exit pupil positions, closely spaced and cemented lens combinations are of service. Fig. 24 shows the Abbe orthoscopic eyepiece.

It is not difficult with such eyepieces to control the chromatic difference of magnification fairly exactly, and "compensating eye-

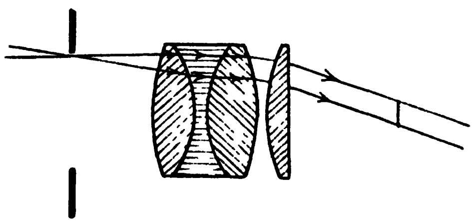


Fig. 24. Abbe Orthoscopic Eyepiece

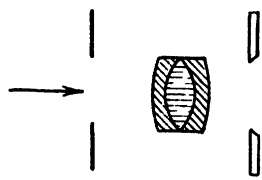


Fig. 25. Triple Cemented Lens Used as Eyepiece

pieces" may be produced to control the defects of the objective, although this is usually only necessary for microscope systems. (See page 87.)

The "Monocentric" eyepieces of Steinheil are not infrequently met with. All the surfaces are struck with one centre and therefore they have the characteristics discussed in connection with the Coddington lens, but they are not so suitable for use in a case where the eye needs to move about a centre which remains steady in relation to the eyepiece. For observation where it is necessary to have the greatest possible freedom from stray light due to back reflections, such eyepieces will be found to give good results up to a field of about 20°.

Fig. 25 illustrates a simple astronomical eyepiece made from a triple cemented lens.

Erecting Systems. *The Terrestrial Eyepiece.* The possibility of erecting the image in his simple astronomical telescope by introducing an erecting lens between objective and eyepiece was known to Kepler. Rheita obtained improved definition by using two intermediate lenses. The essential form of the four-lens terrestrial eyepiece as made by Dollond, Ramsden, Fraunhofer and many others, down to the present day, is the outcome of many empirical trials. A typical construction is shown in Fig. 26, and the diagram will be sufficient explanation of the general mode of action; the erecting lens system is followed by a Huygenian eyepiece of usual form. The stop in the erector may be made to coincide with an intermediate image of the entrance pupil, thus removing any stray light such as that reflected from the interior of the tubes. A shift of the

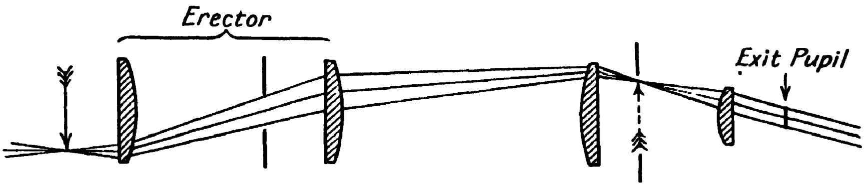


FIG. 26. ERECTING, OR TERRESTRIAL EYEPIECE

erector stop, however, cuts down the aperture of some of the oblique pencils, and some opticians use this method of improving the definition in the outer parts of the field. Improved erecting eyepieces have been devised in which the erecting lens is a cemented triplet. In Fig. 27 the field lens is employed to secure the necessary convergence of the principal rays.

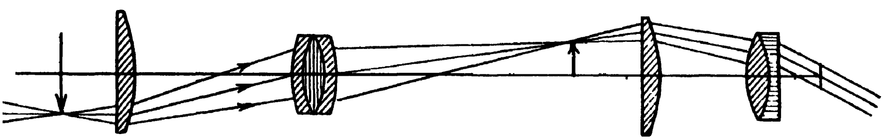


FIG. 27 TERRESTRIAL EYEPIECE WITH TRIPLE CEMENTED ERECTING LENS

It is particularly important with these eyepieces to design the objective to suit them; a certain amount of spherical and chromatic over-correction is necessary.

Pancratic, or Variable-power Telescopes. The possibility of varying the magnification of a telescope with erecting lenses was realized quite early, and some telescopes were fitted with the erecting system in an independent draw-tube so that object and image distances for the erecting lens could be varied.

The best known forms of modern variable power telescope are provided with mechanical means whereby the instrument remains in focus during the change of magnification. In the "Ross" type, the general lay-out of the erector is similar to that shown in Fig. 27 above. The erecting lens is triple and cemented. When it is desired to increase the magnification, the erector is moved nearer the first image, and the enlarged second image moves in the direction of the eyepiece. The eyepiece system (of the achromatized Ramsden type) is mechanically withdrawn in order to preserve the focus.

The eyepiece may also be given an independent movement to adapt the focus for the vision of different observers.

The mechanical means of varying the power consists of three tubes fitting one inside the other, the outermost X (Fig. 29) carries

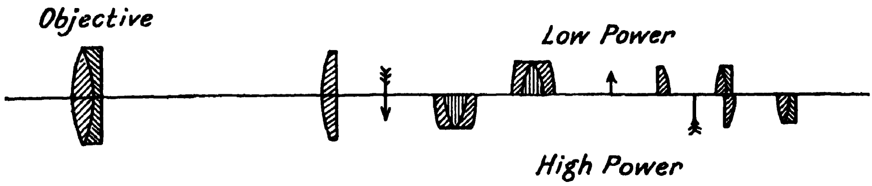


FIG. 28. VARIABLE POWER TELESCOPE (ROSS)

a milled portion which enables it to be turned, carrying with it an inner tube Y having two helical slots cut in it. The third tube Z is screwed to the main tube of the telescope so that it cannot revolve; it has two straight slots cut in it. Sliding in tube Z are two

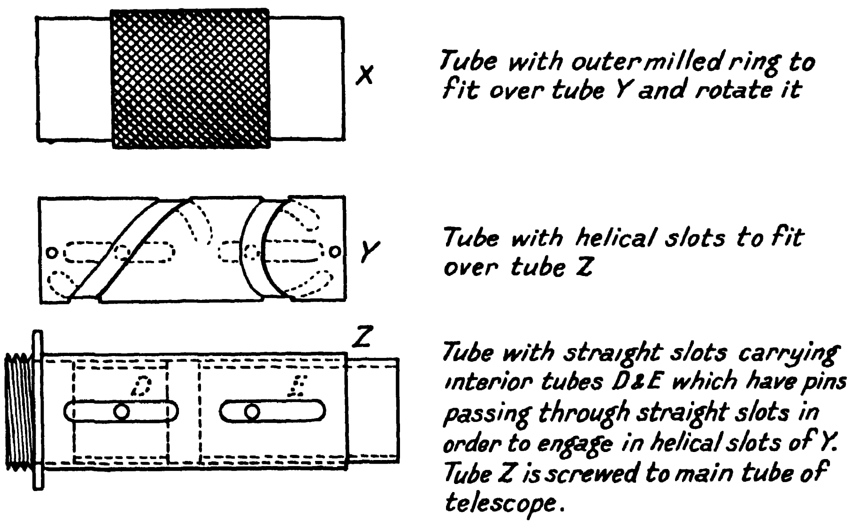


FIG. 29. TYPICAL SYSTEM FOR MECHANICAL VARIATION OF THE SEPARATION OF LENSES CARRIED IN TUBES D AND E

independent tubes in which the erecting lens and the eyepiece respectively are mounted. These tubes each carry a pin which projects through a straight slot of the fixed tube Z into the corresponding helical slot of tube Y. Thus any rotation of the latter causes translatory movements of the erector and eyepiece.

Assuming the usual notation, the distance k between object and image for the erector system is given by

$$k = l' + d - l$$

where d is the distance between the principal points of the erector system.

$$k = d - \frac{l^2}{f' + l}$$

Differentiating, $$\frac{dk}{dl} = -\frac{l^2 + 2lf'}{(l + f')^2}$$

This is clearly zero (and the distance between object and image is a minimum) when $l = -2f'$, the condition when the magnification due to the erector only is -1. The differential coefficient gives the ratio of the pitches of the helical slots controlling the movement of the eyepiece and erector respectively. The diagram shows that the

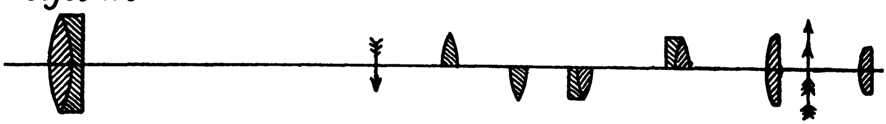


FIG. 30. VARIABLE POWER TELESCOPE (OTTWAY)

pitch of the right-hand (eyepiece) slot is zero at one point, and the eyepiece thus moves very little for comparatively large changes of magnification when it is nearest the objective.

In the Ottway type of variable power telescope, the eyepiece remains stationary, the effective focal length and position of the erector system both being varied. This is accomplished by using a two-lens erector, as shown in Fig. 30, one lens being a cemented doublet. The power of the erecting system is varied by the variation of the distance between the components, each member being given the displacement required to keep the image stationary.

Alternative Powers. An alternative arrangement to the provision of continuous variable power in a telescope is to give two or three magnifications by separate eyepieces mounted on a swivel, so that the change from one to another can be made very quickly. The *apparent* angular field of view in various powers does not vary greatly, so that the real field is smallest with high powers and greatest with low. The great advantage of a variable power is that an object may be "picked up" easily with a large field, and then observed in detail by the use of a higher power. The magnifying powers usually obtainable with variable power systems range from about five to twenty.

The Field of View of a Telescope. In all telescopes we can distinguish a diaphragm which limits the angular divergence of the rays passing through the centre of the exit pupil. Referring to Fig. 11, the stop limiting the field is seen in the common focal plane. If it reaches the size when the full bundle of rays passing through a point in the margin cannot be transmitted by the eyepiece, the illumination of the boundary of the field will suffer. In most eyepieces intended for ordinary observation, the diameter and position of the stop is specified in the complete design. Take the case of the Huygenian eyepiece shown in Fig. 31, for which

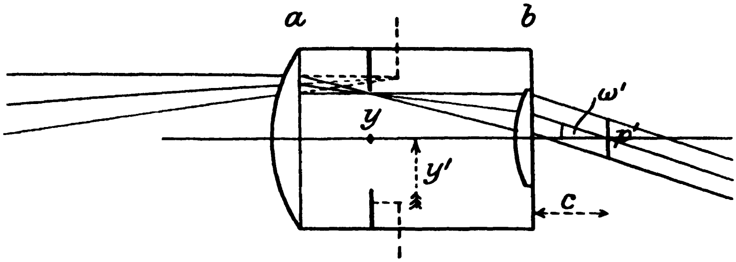


FIG. 31

$f'_a = 2f'_b$ and $d = \frac{3f'_b}{2}$. The stop is situated in the first focal plane of lens b. If its radius is y, the apparent angular field is clearly $2\left(\tan^{-1}\frac{y}{f'_b}\right)$. The entrance pupil of the eyepiece is evidently formed at a distance l from the field lens for which the conjugate $l' = \frac{f'_b}{2}$ and

$$\frac{1}{(f'_b/2)} - \frac{1}{l} = \frac{1}{2f'_b}$$

and thus $l = \frac{2f'_b}{3}$. The radius of the effective entrance pupil is therefore

$$\frac{yl}{l'} = \frac{4y}{3}$$

Hence the real field of view will be $2\tan^{-1}\left(\frac{4y}{3f'_o}\right)$, where f'_o is the focal length of the objective. If f'_e is the focal length of the eyepiece as a whole, i.e. $4f'_b/3$, the apparent angular field is $2\tan^{-1}\left(\frac{4y}{3f'_e}\right)$. If the angles are small, the ratio of apparent field to real field will therefore reduce to $\frac{f'_o}{f'_e}$, the ratio of the focal lengths of objective and eyepiece which is the angular magnification of the telescope. More generally, we calculate the radius y' of the effective stop situated in the first focal plane of the eyepiece system, and find the real field as $2\omega = 2\tan^{-1}\left(\frac{y'}{f'_o}\right)$, and the apparent field as $2\omega' = 2\tan^{-1}\left(\frac{y'}{f'_e}\right)$. The distance of the exit pupil from the eye lens can be calculated. Let it be c, say. The radius p' of the exit pupil can also be calculated from $\frac{p}{m}$ where p is the radius of the entrance pupil and m the magnification of the telescope. Then the required radius of the eye lens will be $p' + c\tan\omega' = p' + \frac{c\,.\,y'}{f'_e}$; this will allow the full oblique pencil to be transmitted.

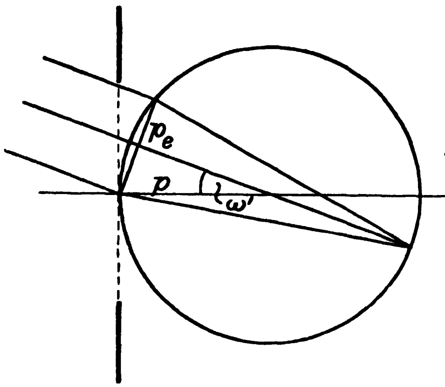


FIG. 32. EYE PUPIL FALLS IN A LARGER EXIT PUPIL

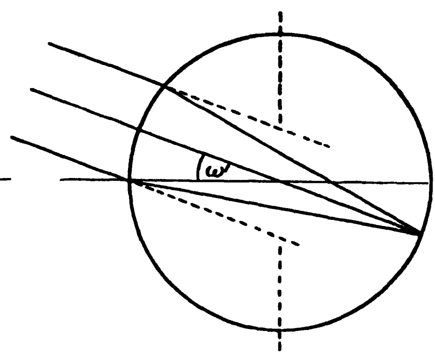


FIG. 33. EXIT PUPIL FORMED IN THE CENTRE OF ROTATION OF THE EYE

Observation with Rolling Eye. In the case when the pupil of the eye is brought into coincidence with the exit pupil of the instrument, and the latter is smaller than the eye pupil, then the whole field of the instrument will be projected on the retina. In using hand instruments, small telescopes or binoculars, the head and instrument can easily be moved together in order to bring different images into the central region of distinct vision. If, on the other hand, the instrument is held on a stand and not easily moved, the eye can observe different parts of the field by turning the head so as to observe as it were through the small window of the exit pupil, or by holding the head stationary and moving the eye in its socket.

The eye, observing through the eye lens, sees the image of the stop (more or less well defined) which limits the field. This image forms the "exit window," and in the case of telescopes with positive eyepieces it is usually seen "at infinity" and sharply defined.

If the exit pupil coincides with the eye pupil, let the point of rotation be situated at a distance ρ behind the pupil, and let p_e be the radius of the pupil of the eye. When the eye is turned so as to bring the margin of the field to the fovea, the required radius of the exit pupil is seen from Fig. 32 to be

$$p' = \rho \tan \omega' + p_e \sec \omega'$$

The eye will then lose no light on rotation.

It may be possible to design the instrument so that the exit pupil falls into the centre of rotation of the eye in ordinary conditions of vision. This case is illustrated in Fig. 33, and it will be seen that the radius of the actual exit pupil need only be $p_e \sec \omega'$, in order that the eye pupil may be kept completely filled with light.

On the other hand, it will often be the case that the exit pupil of the instrument is smaller than the eye pupil, and the limit to

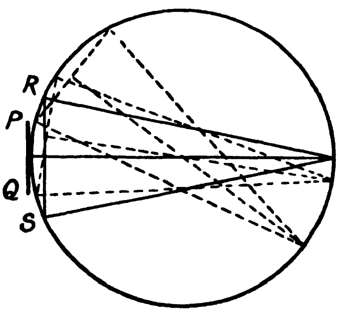


Fig. 34. Eye Pupil Falls Together with a Smaller Exit Pupil

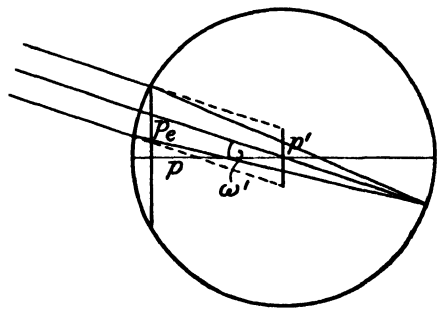


Fig. 35 Observation with Stationary Eye

which the eye can move without experiencing some curtailment of the light will be restricted. In Fig. 34, PQ represents the exit pupil of the instrument, and RS the pupil of the eye. If the eye rotates, the iris will begin to cut off some light from the image when S reaches Q. No illumination of the image will be possible when S reaches P. Hence

Total rotation possible while retaining full illumination

$$= \frac{p_e - p'}{\rho} \text{ (approx. angular measure)}$$

Total rotation possible while retaining partial illumination

$$= \frac{p_e + p'}{\rho}$$

When the instrument exit pupil coincides with the centre of rotation of the eye, no light will be cut off in this way for observation with the rolling eye until the margin of the field falls on the fovea, provided that the optical system gives full parallel bundles through the exit pupil up to the limit of the angular field. On the other hand, it will be noticed that the stationary eye at rest in the symmetrical position (Fig. 35) will not secure full illumination of the whole field simultaneously unless the radius of the eye pupil is at least

$$p_e = \rho \tan \omega' + p'$$

where ρ is the distance of the exit pupil behind the eye pupil.

If it should be smaller than the required amount, we can see that the limit of the angular field for full illumination will be

$$\tan^{-1}\left(\frac{p_e - p'}{\rho}\right)$$

while the limit of the angular field for partial illumination will be

$$\tan^{-1}\left(\frac{p_e + p'}{\rho}\right)$$

Visible Field in Relation to the position of the eye. In observing from the axial point of the exit pupil of an instrument, there

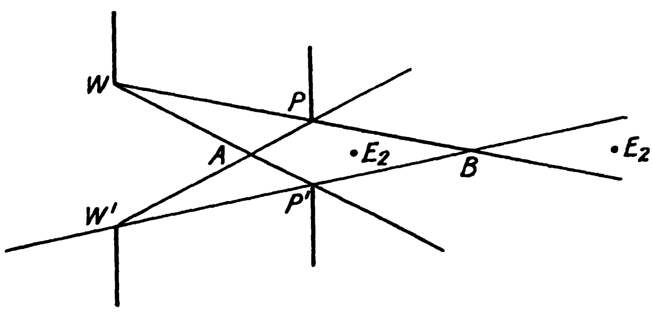


FIG. 36. EXIT PUPIL AND EXIT WINDOW

will be some lens mount or diaphragm which limits the angular extent of the field of view; the image thus seen will be called the exit "window."

Let WW′ (Fig. 36) be the exit window and PP′ the exit pupil of a telescope. If the eye is placed anywhere within the quadrilateral APBP′ it will be clear that the limits of the visible field of view will be controlled by the margin of the "window" WW′. On the other hand, if the eye is placed outside the quadrilateral beyond B, as at E_2, the angular subtense of the visible field will be controlled by the exit pupil PP′ which now takes on the role of the exit window. The term "position of the eye" used above may refer either to the nodal point of a fixed eye, or to the centre of rotation of a rolling eye.

The case just mentioned, when the exit pupil limits the angular field, is typical of the Galilean telescope, and it is important enough to merit a closer discussion.

The Field of View of the Galilean Telescope. The "entrance pupil" of the Galilean telescope will be assumed to be the diaphragm which limits the objective, of which the radius is p. The radius of the exit pupil is $\frac{p}{m}$ where m is the magnification of the telescope.

The distance of the entrance pupil from the first principal point of the eyepiece is

$$l = -(f'_a + f'_b)$$

Hence l', the distance of the exit pupil from the second principal point, is given by

$$\frac{1}{l'} + \frac{1}{f'_a + f'_b} = \frac{1}{f'_b}$$

Then

$$\frac{1}{l'} = \frac{1}{f'_b} - \frac{1}{f'_a + f'_b}$$

$$l' = \frac{f'_b(f'_a + f'_b)}{f'_a}$$

$$= f'_b\left(1 + \frac{f'_b}{f'_a}\right) = f'_b\left(1 - \frac{1}{m}\right).$$

Let the distance of the nodal point of the eye from the second principal point of the eyepiece be d, then its distance ρ from the exit pupil is given by

$$\rho = d - f'_b\left(1 - \frac{1}{m}\right)$$

Fig. 37(a) shows the formation of the exit pupil PP′, and a bundle of parallel rays originally derived from an infinitely distant image point. Let the pupil of the eye be situated at EE′ (shown separately in lower figure). A bundle of parallel rays from the exit pupil must completely fill the eye pupil, provided that the inclination of the rays to the axis is not greater than that of the line (a) in the diagram; the angle with the axis is thus given by

Maximum angle of full illumination

$$= \tan^{-1} \frac{\frac{p}{m} - p_e}{d - f'_b\left(1 - \frac{1}{m}\right)} = \tan^{-1}\left(\frac{p - mp_e}{md - f'_b(m-1)}\right)$$

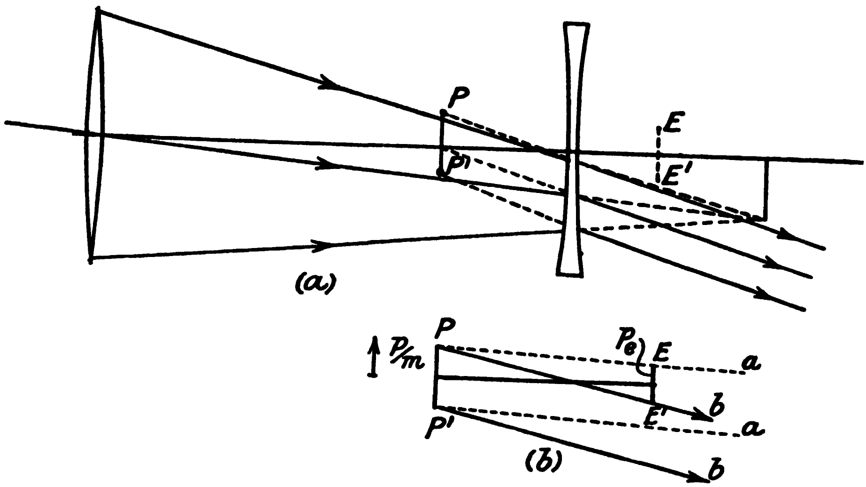


FIG. 37. TO EXPLAIN THE FIELD OF VIEW OF A GALILEAN TELESCOPE

When the inclination of the bundle has increased to that of the ray (*b*) in the figure, no light can reach the retina.

Maximum angle of partial illumination

$$= \tan^{-1} \frac{\frac{p}{m} + p_e}{d - f'_b\left(1 - \frac{1}{m}\right)} = \tan^{-1}\left(\frac{p + mp_e}{md - f'_b(m-1)}\right)$$

The visible field is therefore bounded by a ring of decreasing illumination which may be looked upon as the out-of-focus image of the exit pupil. We may clearly enlarge the field of view by increasing p, and diminishing d as far as possible. The quantity f'_b will be numerically negative so that by decreasing it (other things being equal) we shall increase the field of view, but this would call for a shorter focal length for the object glass to preserve the same magnification. The limit to progress in this connection is the difficulty of correcting the aberration of the negative eye lens when the focal length is reduced below 1 in. or thereabout. The objective radius

p may be increased in a binocular until the mounts of the two objectives practically touch each other (if the interocular distance is to be adjustable, this condition must be reached at the lowest separation), and d is decreased as far as possible by bringing the eyes as close as possible to the eye lens.

The so-called "mean field" is given by

$$\tan^{-1} \frac{p}{md - f'_b (m - 1)}$$

and it will be seen that the effect of increasing m while other things remain the same will be to decrease the field. (The magnification would be increased by a longer focal length of the objective.)

Reflecting Erecting Systems. Perhaps the most important means of erecting the telescope image is now by the use of erecting prisms;

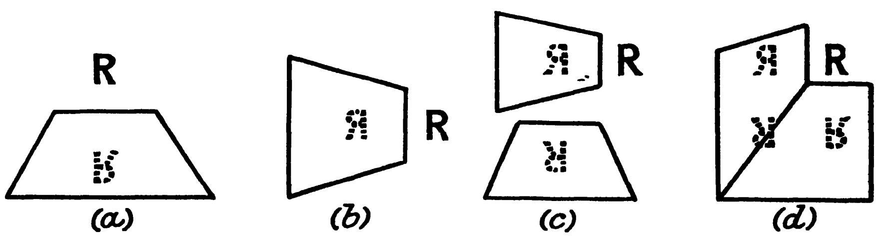


FIG. 38. MIRROR REFLECTIONS

the suggestion was originally made by Porro, and put into practical use by Abbe.

The image of an object formed by reflection in a horizontal mirror is seen inverted (Fig. 38 (*a*)); the image formed by a vertical mirror is reversed, left to right (Fig. 38 (*b*)). The reversed image of the vertical mirror may be viewed in the horizontal mirror (Fig. 38 (*c*)). The result is a complete reversion and inversion such as is found in the real image formed by a telescope objective. If, therefore, the light from an objective suffers two such reflections between the objective and the image plane, it will be seen erect and correctly disposed when examined by the eyepiece. The light must not fall on the mirrors at too great an angle of obliquity, so that in general the direction of the principal ray will be more or less deviated after the two reflections.

If the two mirrors are placed together so that their planes intersect at right angles, reflections may take place at either one first, and in addition to the two images formed by one reflection in each mirror, there will be an inverted and reversed image to which reflection in both mirrors has contributed (Fig. 38 (*d*)).

In Fig. 39 (*a*), a ray shown in broken lines suffers reflection at two mirrors shown by full lines. It is easy to show that if the angle a between the mirrors is reckoned as shown in the diagram, then the deviation of the ray is $2a$. Fig. 39 (*b*) is a "polar diagram" in which

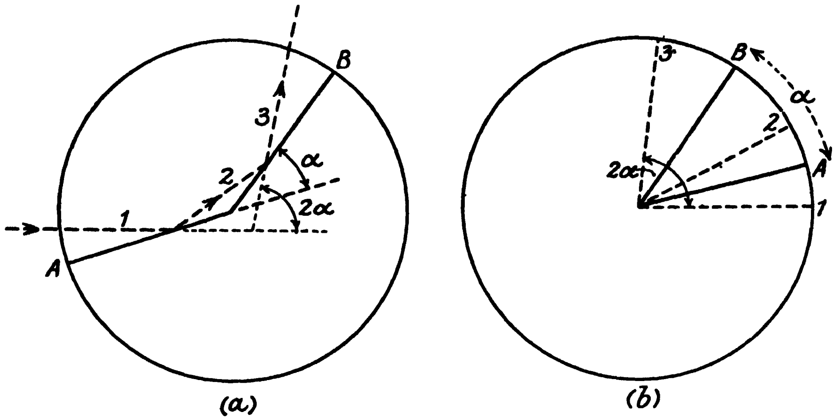


FIG. 39

Reflection at two inclined mirrors (*a*) and the corresponding polar diagram (*b*). (In (*b*) the mirror direction is such that the reflecting side is met when going clockwise round the circle)

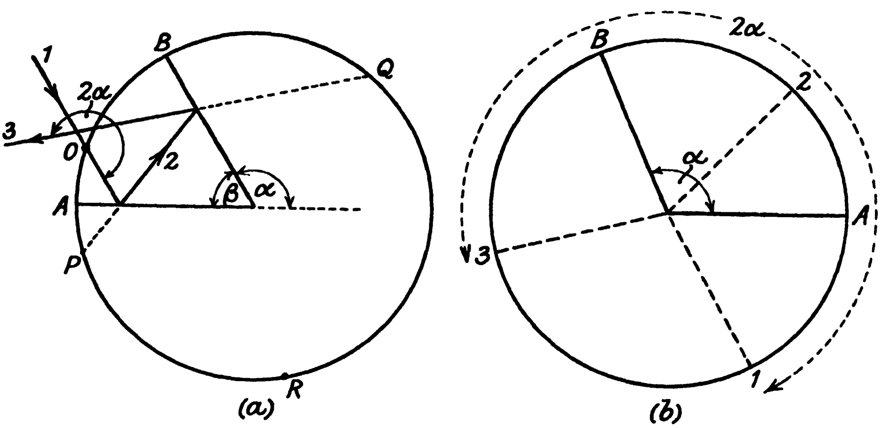


FIG. 40

(*a*) Reflection at two inclined mirrors
(*b*) Polar diagram

the directions and senses of the mirrors A and B are shown by the *outward* directions of the corresponding radii. The convention is that the reflecting surface represented in the polar diagram shall be met by going clockwise round the circle. In these and subsequent polar figures the ray directions are also shown by the outward directions of the corresponding radii. In Fig. 40 (*a*) the reflecting surfaces of the mirrors include an acute angle, and Fig. 40 (*b*) is the corresponding polar diagram. Let ray 1 pass through an object-point O between the reflecting surfaces; the path after reflection in mirror A is in a line passing through P, the mirror image of O. After reflection in B, the ray 3 appears to diverge from the point Q; it is easily proved that the angle subtended at the centre by the arc OBQ is equal to 2β, where β is the angle included between the reflecting surfaces of the mirrors. We might, however, draw a ray through O which strikes first the mirror B, then A. This ray would, after the double reflection, appear to originate from the point R, where the arc OAR also subtends an angle 2β at the centre. The points Q and R will evidently coincide only when $\beta = 90°$, and a

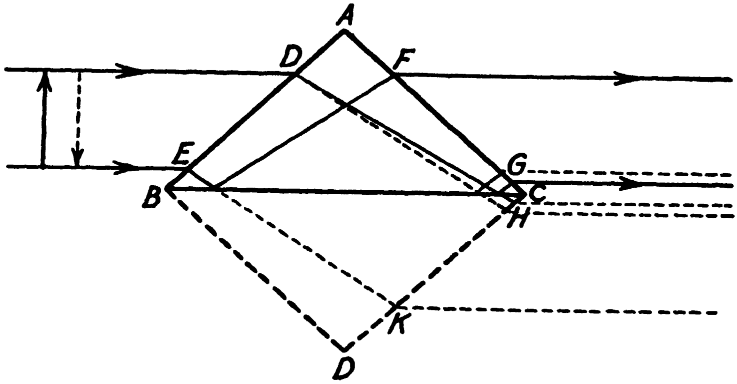


FIG. 41. EQUIVALENCE OF AN ERECTING PRISM AND A PARALLEL PLATE OF GLASS

single image will be seen. If β is not 90°, the angular separation of Q and R will be $360° - 4\beta$.

A combination of two reflecting surfaces mounted so that reflection can first take place at either of them forms a "roof" reflector. If used in an instrument, the doubling of the image produced by any slight difference of β from 90° must not be perceptible to the eye. The tolerance will depend on the optical arrangements, but an accuracy within one second of arc may sometimes be called for. The eye would scarcely detect the doubling of the image if the separation of the components subtends an angle of less than 30 sec. in the visual field of view, but this tolerance is still somewhat large in regard to the full contour acuity of the normal eye. (Vol. I, Chapter V.)

Prism Faces as Reflectors. The equilateral inverting prism ABC (Fig. 41) partly overcomes the difficulty, illustrated in Fig. 38 (*a*), that the observer has to look into a new direction in order to see the image. The optical effect is conveniently studied by "developing" the prism and drawing the mirror image of the refracted ray paths, and we therefore find the effect as that of a thick inclined plate of glass in the path of the light. The effect of dispersion clearly separates a red and blue ray, but they emerge parallel. The lateral displacement has no effect if the object or image is at a distance so great that the confusion of the images is not perceptible, but it will be serious if the image is close to the prism. Parallel rays from a distant object will all be subjected to the same action by a parallel plate, so that no aberration can arise, but if the image is close to the prism surface, so that rays passing through any

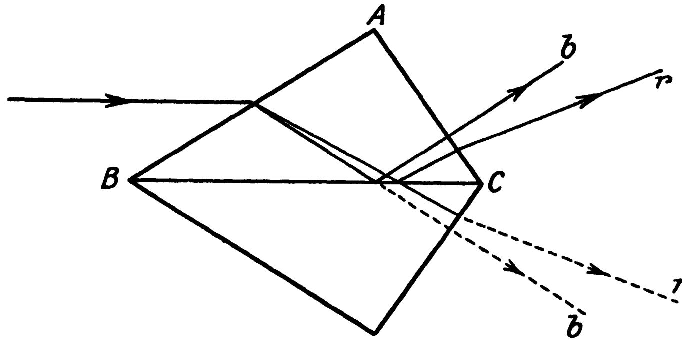


FIG. 42. EFFECT OF ANGLE ERROR

point of it to the prism surface are incident at various angles, then serious aberrations arise by refraction at the plane surfaces, which will clearly be especially objectionable if the axial ray to the central point of the field is obliquely incident. Hence, if any refracting prism surface must be placed in the path of a divergent or convergent beam, the surface should be normal to the axial ray in order that the central part of the field, where the corresponding angles of incidence have small obliquity, may be subject to the least possible disturbance. Fig. 42 clearly shows that if the base angles are not equal, the inverting prism will have a residual dispersive action which would (for example) draw the image of a star into a short spectrum.

Such a prism as that of Fig. 41 can, however, only produce inversion in the "up and down" sense while its base is horizontal. If it is placed in front of the objective of an inverting telescope the image will be erect, but reversed "left to right."

We can, however, obtain a prism which also gives reversion, by substituting the two faces of a roof reflector for the horizontal reflecting face; each of the roof faces will be inclined at 45° to the horizontal plane, and the roof edge will be a horizontal line. Fig. 43 shows the end view and side elevation of such a prism. There is no need to retain the vertical sides of the simple reflecting prism (indicated by the dotted lines in the end view). The top faces are, there-

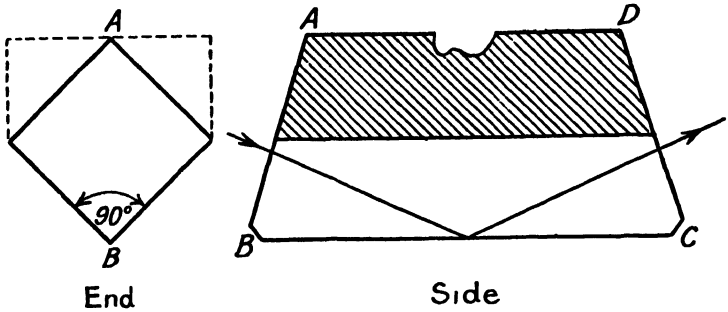


FIG. 43. SIMPLE "ROOF" PRISM

fore, also cut at an angle of 45°, but only the end faces and the two roof faces are polished; the top faces are left rough-ground.

We can follow the action of such a prism by the aid of a three-dimensioned polar diagram as suggested by Instructor Captain T. Y. Baker.[5] It must first be understood that when reflection takes

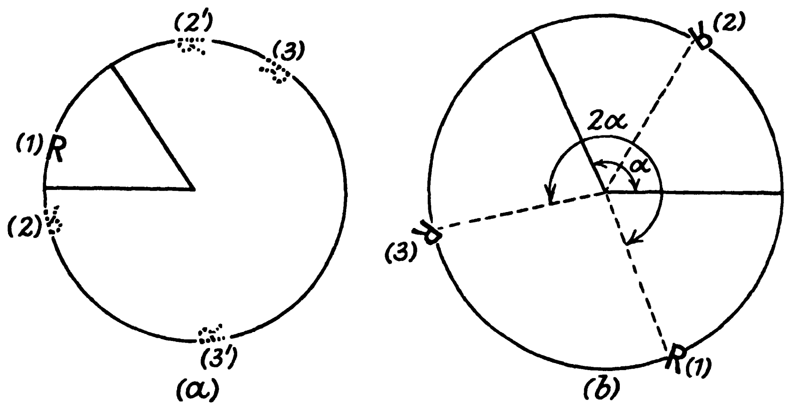


FIG. 44. ANGULAR SWING OF THE IMAGE

place at two mirrors in turn (Fig. 44), *the image may be geometrically regarded as the result of swinging the object around the axis marked by the intersection of the reflecting surfaces.* If R is a real object lying in the plane of the diagram, images 2 and 2′ are formed after one reflection, 3 and 3′ after two reflections. The angle of swing is double the angle included between the reflecting surfaces. Fig. 44 (*b*) is a two-dimensional polar diagram of the same case, having significance only with regard to the orientation of the images and ray directions. The letter R in positions 1, 2, and 3 is parallel to those in the left-hand diagram. The final image orientation at 3 is the result of a swing $2a$ round the axis, and is, of course, independent of the exact position of the reflecting surfaces, provided their line of intersection remains the same and that they include an angle a.

We are now ready for the three-dimensional diagram. Fig. 45 represents a sphere with centre O, and the radius vector O1 to the point 1 on the circumference represents a downward directed ray corresponding to an inverted image suggested by the letter R. The point 1 is in the middle of the vertical stroke. Point 2 gives by the outward radial direction the required direction corresponding to an erect image. If both vertical strokes of the letters R are in a vertical great circle, it will be seen that rotation through 180° about a radius of the sphere, marked OP in the diagram, will be sufficient to effect the required change. This line OP, therefore, is the required direction of the intersection of the two reflecting surfaces required, and the angle between them must be 90°. Their directions can be shown (in the sense of the polar diagrams) by a pair of lines drawn on the surface at 90° through P. This represents the simple case discussed above when we use a roof prism with its roof edge horizontal. The roof prism is allowable since the rotation is to be exactly 180°. If some other angle, say 179°, of rotation were required, it would not be permissible to use a roof prism, because reflection must now take place first at *one* surface and then at the other for all rays if overlapping images are to be avoided.

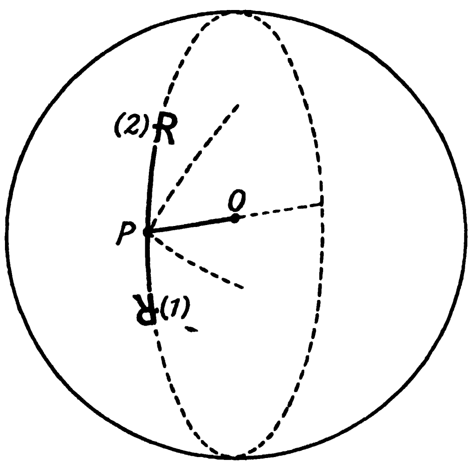


FIG. 45. POLAR DIAGRAM TO ILLUSTRATE ACTION OF "ROOF" REFLECTOR

In general it can be shown that the displacement of a rigid body from one position to any other can be effected by (1) a movement of translation along a certain axis, and (2) a movement of rotation about that axis.

An image can be given a shift of pure translation in any direction and of any amount by reflection in two suitably-disposed parallel mirrors (one reflection produces a reversal and a second is necessary to annul the reversal). Any required movement of rotation can be produced by successive reflection in two surfaces for which the roof edge represents the rotation axis. It follows that reflection from four surfaces will suffice to effect any required change in the position and disposition of an image, so as (for example) to bring it into the field of view of an eyepiece placed in any particular position. The vast majority of prism erecting systems employ four reflections.

The polar diagrams are helpful because they permit attention to be given in the first place to the required change of direction of the ray and orientation of the image. The lateral shift can be dealt with afterwards. General considerations are illustrated by Fig. 46. The outward radius to A represents the direction and sense of the axial ray before entering a prism system, while the point X represents the axial ray on leaving the system. The orientation of the object-space field is represented by the short perpendicular lines AB and AC which might represent the 12 o'clock and 3 o'clock directions respectively. The corresponding lines XY and XZ represent these respective directions in the image field.

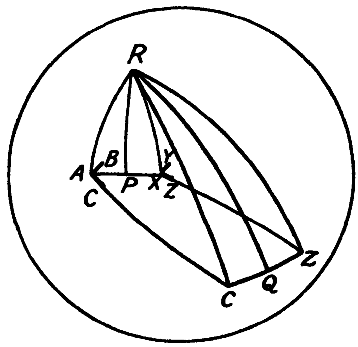


FIG. 46. GENERAL CASE

The pole of the great circle containing AB is the point c, while the point z is the pole of the great circle containing XY. If we are to find some axis of swing, rotation about which will bring AB into the direction XY, this swing will bring the pole c to the point z. If we bisect AX at right angles by the line PR, the axis must lie on this line; equally, it must lie on the line QR which bisects cz at right angles. These bisectors intersect in R, the radius to which point marks the axis of swing. The angle of swing is clearly given by $\widehat{ARX}$. The necessary angle between the reflecting surfaces, considered in the sense of the polar diagrams, will be half $\widehat{ARX}$. Captain Baker's original paper shows how to use the constructions in the graphical design of prism systems.

"Prism Binocular" System. In prism binoculars it is usually required for the final ray to be parallel to the direction of the incident ray. Fig. 47 (*a*) illustrates the case in which the incident ray O travels "south—east—north—vertical—south." This is accomplished by the aid of (1) a pair of reflectors at 90° with roof edge vertical, (2) a second pair with roof horizontal (see Fig. 48). Referring to the polar diagram, in Fig. 47 (*b*), we see that the effect of (1) will be to swing the ꓤ round the vertical axis through 180°. The

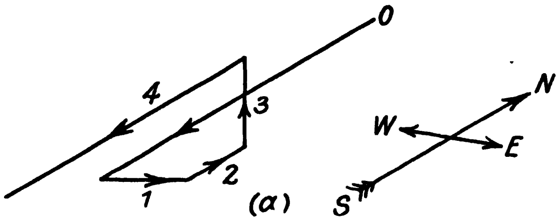


FIG. 47 (*a*). PATH OF RAY IN PORRO PRISM SYSTEM

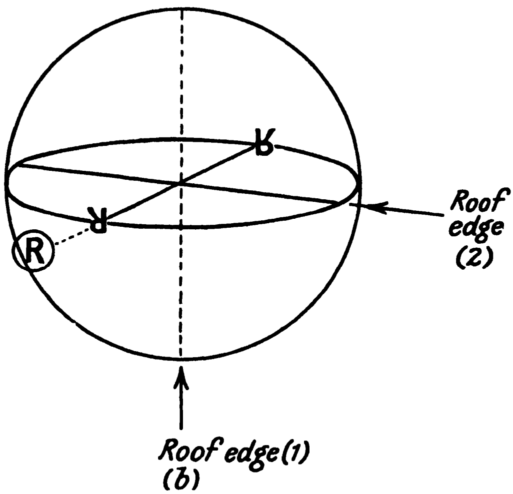


FIG. 47 (*b*). POLAR DIAGRAM FOR PORRO PRISM SYSTEM

effect of (2) swings the image through 180° round the horizontal roof edge. The final result is R—distinguished with a ring in the diagram. Figs. 48 and 49 illustrate the arrangements of the prisms in practice. It will, of course, be clear that the system could be rotated as a whole round the incident ray (O) as an axis; the "erecting" effect would remain precisely the same, although the emergent ray would have moved with the system while remaining parallel to its original direction. The reflecting faces can be grouped in pairs for the purpose of the above theory irrespective of the order in which they are encountered by the light.

Such a system may, if necessary, be divided as shown in Fig. 49 (*a*), where part of the farther prism has been lifted vertically. In certain forms of periscopic telescope, the upper part of the system may be placed just over the objective, while the lower part of the system is immediately followed by the eyepiece. Fig. 49 (*b*) shows a different arrangement; the reflections take place in a different order. All

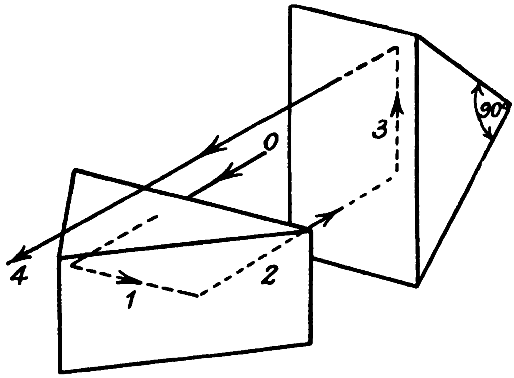


Fig. 48. Prism Binocular System

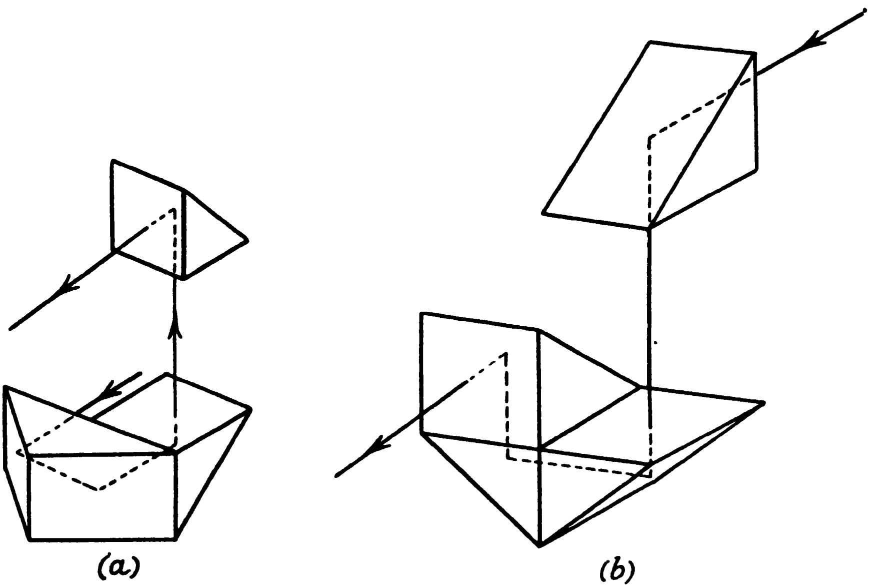


Fig. 49. Variations of an Erecting System

these cases involve a lateral displacement between the incident and emergent principal rays; reflection takes place at the various surfaces consecutively, and the extreme accuracy required for roof edge angles can be avoided. The 90° angles must be accurate to ± 3′ of arc, and the prisms should also be free from pyramidal error, i.e. non-parallelism of the 90° edge to the opposite face.

"Direct Vision Erecting Prisms." In many cases the loss of alignment between the axes of objective and eyepiece is undesirable, and the Abbe prism shown in Fig. 50 (*a*) illustrates an arrangement

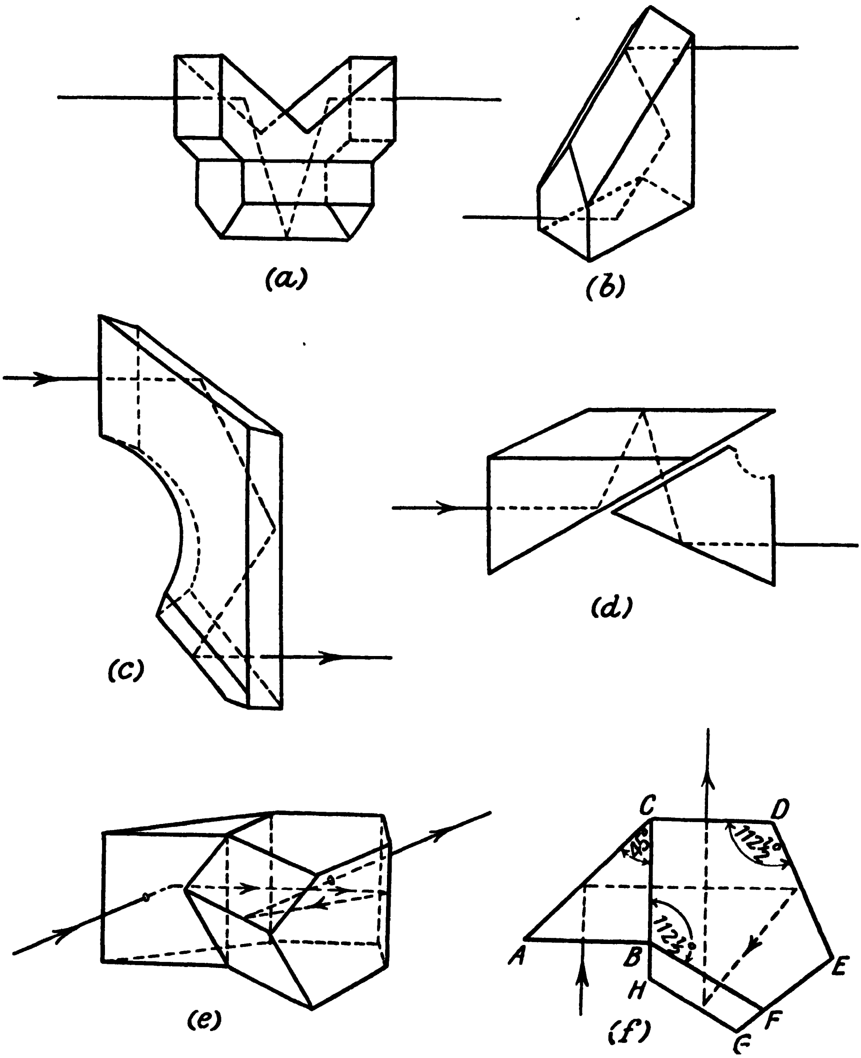


FIG. 50. A GROUP OF ERECTING PRISMS

(*a*) Abbe direct vision erecting prism
(*b*) Leman prism
(*c*) Modified Leman prism
(*d*) König prism
(*e*) Daubresse prism
(*f*) ,, ,, (plan)

by which this is avoided. The disadvantage of this arrangement is the somewhat awkward size of the prism, which calls for a considerable enlargement of the telescope tube. An arrangement due to A. König, Fig. 50 (*d*), is somewhat similar, but allows of the use of a smaller prism; it will be seen that the division of the prism into two parts brings an air film into use, at which total reflection of the incident ray can take place at any point. When the ray returns from the roof edge it is incident at a very small angle, and passes through the air film with but slight loss.

A "prism" due to Leman is shown in Fig. 50 (*b*), and a modified form in Fig. 50 (*c*); the latter was widely used during the War in an optical machine-gun sight in which the emergent ray is lowered by about 2 in. in order to secure useful protection to the eye of the

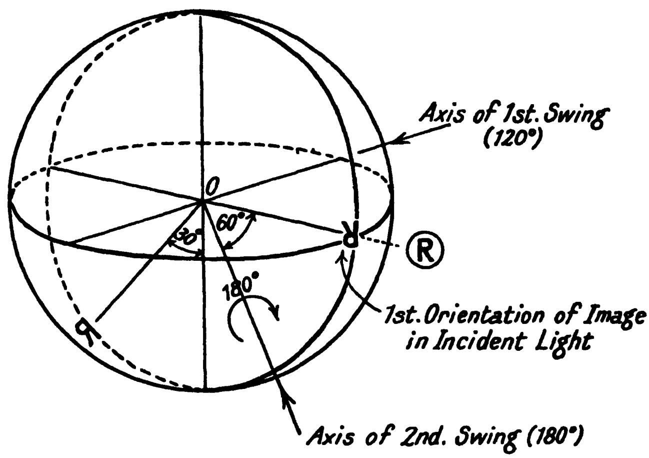


FIG. 51. ACTION OF MODIFIED LEMAN PRISM

gunner. The optical action of the system of Fig. 50 (*c*) is illustrated by the polar diagram of Fig. 51. The first axis of swing is horizontal, and the angle of swing is 120°. The next swing is one of 180° about an axis inclined at 60° to the horizontal (the lower roof edge). These two swings are clearly sufficient to erect the image.

The prism of Fig. 50 (*e*) is due to Daubresse, and can, perhaps, be understood more readily from the plan view (Fig. 50 (*f*)). It consists essentially of a right-angled prism ABC, together with one of the familiar prisms (the so-called "pentag") used in rangefinders (CDEFB). The "pentag" alone produces a deviation of 90° in the path of a beam, when it is important that the ***deviation*** shall not alter with any angular shift of the prism itself. Reflection in the "pentag" takes place at ***silvered*** surfaces corresponding to the lines DE and FB; the directions of these lines are at an angle of 45° with each other. Thus the "pentagonal prism" system consists essentially of two inclined mirrors *plus* a plane parallel block of glass. Rotation of such a system produces no angular deviation of a parallel beam doubly reflected from the mirrors.

In the Daubresse prism, the face FB is replaced by two roof faces which meet at 90° in a horizontal line. The action can now be understood from the polar diagram (Fig. 52). The inverted ꓤ at (1) marks the direction of the incident beam. The first axis of swing is vertical, and the swing is 225° produced by surfaces AC and ED inclined (in the sense of the polar diagram) at 112·5°. This brings the image to the position (2). The next swing must naturally be

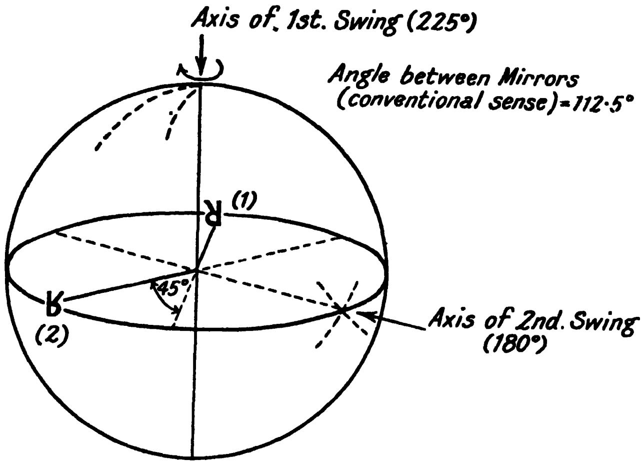


FIG. 52. ACTION OF DAUBRESSE PRISM

one of 180° about an axis symmetrically inclined to both (1) and (2), which makes an angle of 112·5° with the final direction 1. This is the direction of the roof edge HG in Fig. 50 (*f*). It will be clear that the prism ABC could be separated from the remainder of the system if it were desired to obtain a considerable separation of the incident and emergent rays, as, for example, in a periscope.

Reflecting Telescopes. In 1639 *Mersenne* proposed to make a telescope from spherical reflecting surfaces* by substituting a concave reflector for the objective, and a convex reflector for the negative eyepiece of a Galilean telescope. While it is true that reflecting astronomical telescopes are of great importance, the eyepiece in modern instruments is always of the refracting form.

The "parabolic" mirror (of which the surface is a paraboloid of revolution) brings a bundle of rays parallel to the axis to a focus

* Reflecting systems are sometimes spoken of as "catoptric" systems, in distinction from "dioptric" systems of lenses. Combinations of mirrors and lenses may be called "catadioptric" systems.

without spherical aberration. In the first method of using the mirror (Fig. 53), a photographic plate is mounted in the focal plane to register the image, which is not observed visually. The plate obscures a certain proportion of the mirror aperture; the concentration of light in the image "star discs" is slightly diminished; a circular mount will be desirable for the plate in order to avoid radial irregularities in the concentration.

Aberrations of the Image. Consider the image formation at the focus of a parabola AP (Fig. 54(*a*)); while it is free from spherical aberration, it is afflicted with coma. It was shown above that the optical sine condition for freedom from coma is

$$\frac{y_m}{\sin a'} = \text{const.}$$

In our case $\dfrac{y_m}{\sin a'}$ is represented

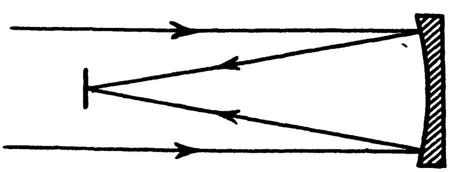


FIG. 53. PLATE IN FOCUS OF ASTRONOMICAL REFLECTOR

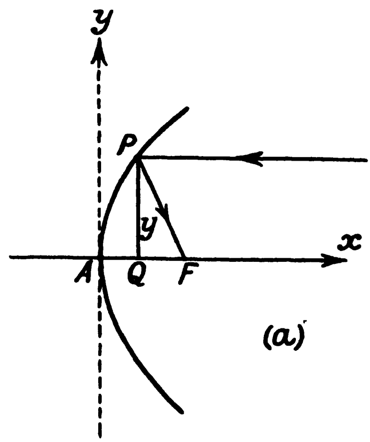


FIG. 54(*a*). REFLECTION AT A PARABOLOIDAL SURFACE

by the length FP for each zone, and is clearly not a constant; let P be the point x,y, A being the origin of co-ordinates, and AF the axis of x. Let AF $= f$, the "focal length."

$$\begin{aligned} FP^2 &= FQ^2 + PQ^2 \\ &= (f-x)^2 + y^2 \end{aligned}$$

The equation to the parabola is $y^2 = 4ax$, and $a = f$, so that

$$FP^2 = \left(f - \frac{y^2}{4f}\right)^2 + y^2$$

i.e.

$$FP^2 = \left(f + \frac{y^2}{4f}\right)^2$$

In the absence of spherical aberration, the magnification, m_m, for a marginal zone may be written, using the sine theorem and putting $\sin a = \dfrac{y}{l}$, where y is the incidence height on the mirror and l is the object distance (assumed to be very great),

$$m_m = \left(\frac{h'}{h}\right)_m = \left(\frac{n}{n'}\right)\frac{\sin\alpha}{\sin\alpha'} = \frac{n}{n'}\,\frac{y}{l\sin\alpha'} = \frac{n}{n'}\,\frac{\mathrm{FP}}{l}$$

For the paraxial zone the magnification is

$$m_p = \frac{n}{n'}\,\frac{\mathrm{AF}}{l}$$

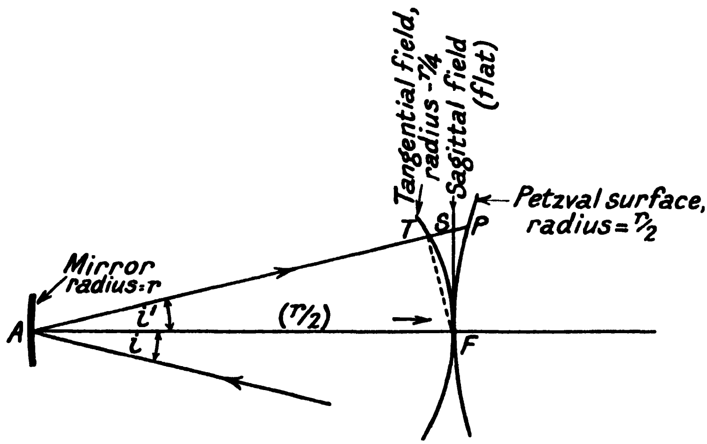
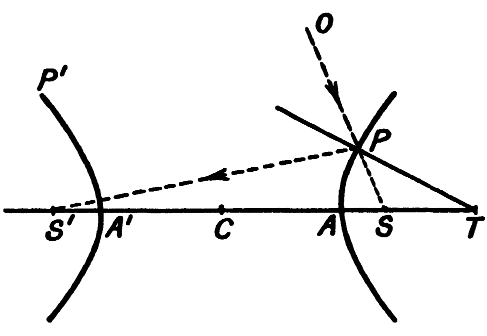
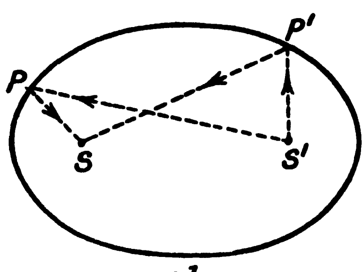


Fig. 54(*b*). Image Fields when Object Field is at Infinity



(*c*)

Fig. 54(*c*). Reflection at a Hyperboloidal Surface



(*d*)

Fig. 54(*d*). Reflection at an Ellipsoidal Surface

Hence the offence against the sine condition is

$$\frac{m_m}{m_p} - 1 = \frac{\mathrm{FP}}{\mathrm{AF}} - 1$$

$$= \frac{y^2}{4f^2}$$

For the common value $\frac{v}{f} = \frac{1}{20}$ for a mirror, the above amount comes to $\frac{1}{1600}$. We mentioned above that general experience shows that for ordinary aperture ratios, the offence against the sine condition should not be allowed to exceed one part in 400; hence for parabolic mirrors of ordinary aperture, the coma is well within the tolerance and compares well with that characteristic of well-designed refractors; the amount speedily grows, however, when relatively large aperture ratios are used.

The Astigmatism of the Oblique pencils will be encountered in precisely the same way and for the same cause as that in a refracting telescope objective passed centrally by the principal ray, i.e. the fore-shortening of the aperture with regard to the tangential rays; the amount will be approximately the same as that calculated on page 147 of Vol. I.

The curvature of the field may be calculated from the ordinary expression for that characteristic of a refracting surface (Vol. I, page 139). We had

$$\frac{1}{n'R'_i} - \frac{1}{nR_i} = -\left(\frac{n'-n}{nn'r}\right)$$

Putting $n' = -n$, the field curvature for an infinite radius in the object field comes out to $\frac{2}{r}$; the radius of the Petzval surface is equal to the focal length. Hence the result is better than with a refracting telescope objective.

Tangential and Sagittal Fields for a Concave Mirror. When the ordinary equations for the conjugate distances in narrow tangential and sagittal fans refracted at a spherical surface (Vol. I, page 296) are adapted to the case of reflection by putting $n' = -n$, we obtain

Formula for tangential fan: $\frac{1}{t'} + \frac{1}{t} = \frac{2}{r \cos. i}$

Formula for sagittal fan: $\frac{1}{s'} + \frac{1}{s} = \frac{2 \cos. i}{r}$

Referring to Fig. 54(*b*) we see that the image distance when t is infinite is given by $t' = \left(\frac{r}{2}\right) \cos. i$, and is therefore found by drawing a perpendicular FT from the focus F to the principal ray AP. The tangential field near the axis is therefore spherical and has a radius

$\frac{r}{4}$, since from elementary geometry: "the angle in a semicircle is a right angle."

Again, $s' = \frac{r}{(2 \cos. i)}$. The sagittal field is evidently flat. If FP is the Petzval surface,

$$\text{the interval TS} = \frac{h^2}{2\left(\frac{r}{4}\right)}, \text{ approx., where } h = \text{FS},$$

and

$$\text{SP} = \frac{h^2}{2\left(\frac{r}{2}\right)}$$

so that TS = 2 SP, and, as we expect from theory, the tangential focal line is three times as far as the sagittal line from the Petzval surface. The above results apply strictly, of course, to very narrow apertures, but the system compares favourably with a refracting telescope.

The most important advantage of the reflector is, however, that the chromatic aberrations, both axial and radial, are absent.

The main difficulty in securing a good performance with a reflector is due to the necessary extreme accuracy of figure of the surface. In order that the disturbance from the object point may meet in the image within the usual tolerance of $\frac{1}{4}\lambda$ path difference, the figure of the surface must be accurate to within $\frac{1}{8}\lambda$. For methods of testing see page 249.

Cassegrain Telescope. In order to appreciate the action of the Cassegrain and Gregorian telescopes, we may introduce a simple theorem in analytical geometry. If AP, A′P′ are two branches of a hyperbola (Fig. 54(*c*)),

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

the normal at the point P $(x'y')$ is

$$\frac{x - x'}{x'/a^2} = \frac{y - y'}{-y'/b^2}$$

which intersects the x axis at the point T given by putting $y = 0$, and obtaining

$$x = x'\left(1 + \frac{b^2}{a^2}\right)$$

But if e is the eccentricity of the hyperbola,

$$\frac{b^2}{a^2} = -(1 - e^2)$$

so that $$x = x'e^2$$

and the distances of the intersection point from the two foci are

$$ST = CT - CS = x'e^2 - ae$$
$$S'T = CT + CS' = x'e^2 + ae$$

Now we have

$$(S'P)^2 = (ae + x')^2 + y'^2$$

and $$y^2 = (a^2 - x^2)(1 - e^2)$$

so that $$(S'P)^2 = (xe + a)^2$$

and similarly $$(SP)^2 = (xe - a)^2$$

Hence $$\frac{SP}{S'P} = \frac{xe - a}{xe + a} = \frac{ST}{S'T} \text{ (from above)}$$

By Euclid, VI, A: "If, in any triangle the segments of the base produced have to one another the same ratio as the remaining sides of the triangle, the straight line drawn from the vertex to the point of section bisects the external angle."

Hence a ray OP proceeding to one focus S of the hyperbola would, if reflected at P, proceed towards the other focus S′. The action is exactly analogous to that of the conjugate foci of an elliptical reflector (Fig. 54(d)).

In the Cassegrain reflector (Fig. 55(a)), the main mirror is pierced by a central aperture. Instead of the Newtonian plain mirror, a convex mirror of hyperbolic form is placed so that its two foci are situated at S′ and S (the main mirror focus and the region of the aperture respectively). In this case an image free from spherical aberration will be formed at S. Sampson recommends an eccentricity of about 3. This determines the position of the auxiliary reflector.

It will be noticed that the use of such a reflector is like that of a negative lens in a telephoto system; it increases the magnification, approximately in the ratio of the distances of the surface from the two foci of the hyperbola. The diameter of the small mirror and that of the aperture in the large one are about equal in the usual case. If the diameter of the small mirror is $\frac{1}{p}$ that of the large one, then the resultant focal length can easily be shown to be $(p - 1)f'$ approximately, where f' is that of the main mirror.

The Gregorian Reflector. In this case the auxiliary reflector is placed beyond the focus of the main mirror, and should therefore be given an elliptical section. The Gregorian form is less common than the Cassegrain.

Herschel's Telescope. Herschel's method of using the reflector is not widely used now; it is shown diagrammatically in Fig. 55(*b*). While it possesses some advantages in simplicity, and consequent conservation of light, over the arrangements described below, it is clear that the minimum angular tilt will be about $\frac{y}{2f}$, or $\frac{y}{r}$. Herschel's mirrors had an approximate aperture ratio of $\frac{1}{10}$, i.e. $\frac{y}{2f} = \frac{1}{40}$. Consequently, the inclination of the principal ray to the axis will be about 1·5°. Therefore the difference of the sagittal and tangential focussing distances will be $f \tan^2 (1{\cdot}5°)$. As König[6] points out, for a mirror of 1·22 metres diameter and 12·2 metres focal length the intercept between the astigmatic foci is therefore 4·2 mm., and if an eyepiece is used which will make the telescope magnification 400, the intercept requires a 4-dioptre difference of accommodation for the two foci. The removal of this defect calls for special figuring of the mirror. In addition to the astigmatic defect, the effects of coma also become objectionable at such a comparatively large distance from the axis.

Newton's Telescope. It will be remembered that Newton saw no hope of making achromatic lenses, and therefore turned his attention to reflecting telescopes (1672). In the Newtonian form (Fig. 55(*c*)), a small auxiliary mirror, at 45° to the principal ray, deflects the convergent beam through 90°, so that the focus is formed approximately in the locus of the surface of the telescope tube where it may be conveniently examined by an eyepiece. If D is the diameter of the objective, the distance of the mirror from the focus is $\frac{D}{2}$. The shape of the mirror must clearly be an elliptical conic section, the minor axis being $\frac{D^2}{2f}$, and the major axis approximately $\frac{D^2}{f\sqrt{2}}$. By this arrangement, the centre of the field lies on the axis, and unsymmetrical aberration is avoided.

General Remarks on Reflectors. It is sometimes the practice to equip a reflecting telescope with alternative methods of securing the image. Thus in the Mount Wilson photographic reflectors, the image can be secured either with a Newtonian reflector, or with a combination of Cassegrain reflector and auxiliary Newtonian reflector, the latter being placed close to the mirror. Practically all modern mirrors are silvered on glass; the use of speculum metal has been very largely discontinued. Many difficulties are encountered when very large mirrors are required, since thick glass discs are neces-

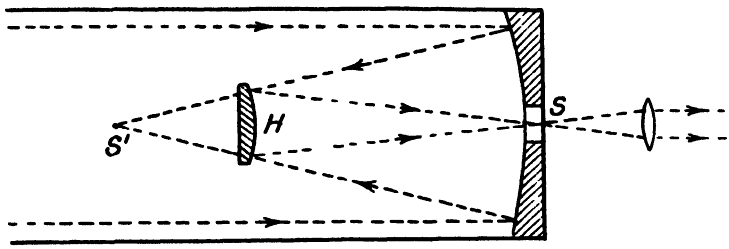


FIG. 55(*a*). CASSEGRAIN REFLECTING TELESCOPE

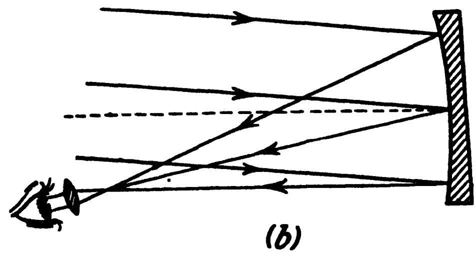


FIG. 55(*b*). HERSCHEL'S REFLECTING TELESCOPE

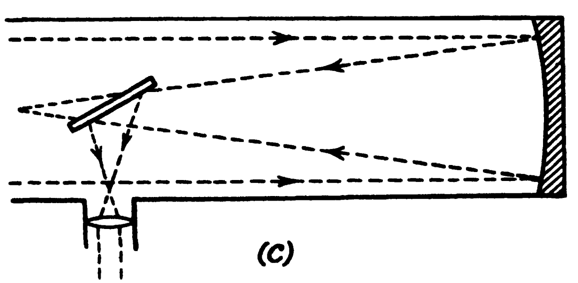


FIG. 55(*c*). NEWTON'S REFLECTING TELESCOPE

sary which are difficult to keep free from the disturbing effects of internal strain and temperature variations; these may distort the surfaces. For this reason it has been proposed to grind the discs in fused quartz, which has a much lower coefficient of expansion than glass.

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2. R. Kingslake: *Trans. Opt. Soc.*, XXVIII (1926–27), No. 4.

3. Smith and Cheshire: *Constructional Data of Small Telescope Objectives* (Harrison, London, 1915).
4. Conrady: *Applied Optics and Optical Design* (Oxford University Press).
5. Baker: *Trans. Opt. Soc.*, XXIX (1927–28), 49.
6. König: *Die Fernrohre und Entfernungsmesser* (Berlin, 1923), p. 70.

CHAPTER III

THE MICROSCOPE

It is possible that compound microscopes were constructed by Giambattista della Porta, but the actual introduction of the instrument must be credited to Zacharias Jansen, and Lippershey, of Middelburg, about the year 1610. The principle of the telescope would naturally suggest that of the microscope, and it is known that Galileo also developed both these instruments.

The practice of microscopy was developed by Hooke, whose *Micrographia* was published in 1665. The researches of Leeuwenhoek (born 1632) with the simple microscope, and Bonnani with his compound instrument (1697), helped on the development, but the early objectives were necessarily made of very small aperture in order to avoid overwhelming chromatic aberration. The discovery (1733–1758) of the achromatic lens finally enabled a great advance to be made in this instrument, but progress was slow until the beginning of the nineteenth century when Marzoli (1808–1811) constructed plano-convex cemented objective lenses, used with the plane side turned towards the object. He was followed in this step by Chevalier (1825). About this time J. J. Lister, in England, and G. B. Amici, in Italy, began their work. In 1830 Lister published his discovery of the two pairs of spherical-aberration-free conjugate points for these plano-convex doublets. Principles of compensation were used in Amici's objectives of 1827. Lister was one of the earliest to realize the value of a wide aperture in the object glass.

The theory of the "Lister" points will be understood from Fig. 56(*a*). The "thin lens" approximate discussion of Chapter II indicates a shape somewhat similar to that of the diagram for the aplanatic crown-flint achromatic combination used for a telescope object glass. If the object point B is brought nearer the lens it is still possible to find a lens corrected for spherical aberration, and approximately for coma, which has a form not greatly different. Reference to the spherical aberration equation shows, however, that having once calculated such a form we could derive an equation giving the aberration of the crown lens in terms of $\mathcal{S}_1$, the vergence $\left(\frac{1}{l_1}\right)$ to the object. The flint lens would furnish another equation in $\mathcal{S}_{1b}$, which would be transformed by putting

$$\mathcal{S}_{1b} = \mathcal{S}_{1a} + \mathcal{F}_a$$

We should therefore obtain a quadratic equation in $\mathcal{S}_1$, for the spherical aberration of the complete lens, and the solution would furnish *two* roots which represent *two* pairs of conjugate points free from spherical aberration for small apertures of the lens. It is found that the residual coma has an opposite sign in the two cases so represented.

Two such pairs are shown in Fig. 56(*a*). One pair, B and B′, are real object and image points, but of the other points, C and C′, the point C represents a virtual object.

Fig. 56(*b*) shows an aplanatic combination of two cemented doublets. The first lens (*a*) works on the conjugate pair represented by

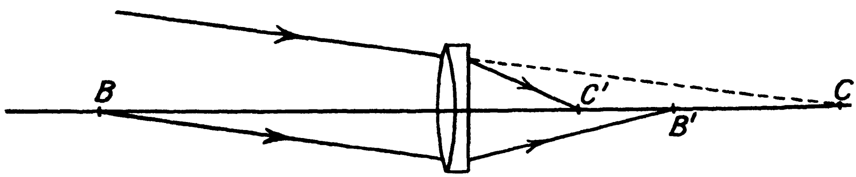


FIG. 56(*a*). THE APLANATIC POINTS OF A DOUBLET LENS

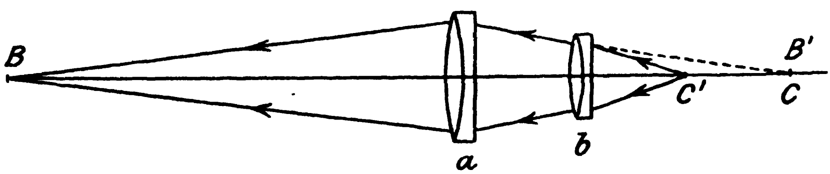


FIG. 56(*b*). LISTER OBJECTIVE

B and B′ in Fig. 56(*a*); but lens (*b*) works on the conjugate pair C and C′. Reverse the direction of the light, treating C′ as the object point, and we have an aplanatic objective of the Lister type. The advantage thus gained is the production of a system of short focal length in which the aberrations are not excessive. If it is sought to attain the same short focal length by using one doublet lens and increasing the curvatures of both components, then the aberrations of the system become much more troublesome, for even if spherical aberration is corrected for one zone, there are very serious amounts of spherical aberration for other zones (zonal aberration), for the restriction of which it is necessary to diminish the *aperture* of the lens very severely.

The importance and exact significance of the "aperture" of the lenses will be dealt with before discussing the further development of the miscroscope objective.

Magnification. The microscope consists of two positive systems, "*a*" and "*b*" (Fig. 57), of which the adjacent principal foci are separated by a distance g; the systems are the objective and eyepiece respectively.

The object of height h has an image of height h' which is projected by the objective into the first principal focus of the eyepiece.

$$h' = -h\frac{g}{f'_a}$$

The image is viewed under the angle ω' where

$$\omega' = \frac{h'}{f'_b} = -\frac{hg}{f'_a f'_b}$$

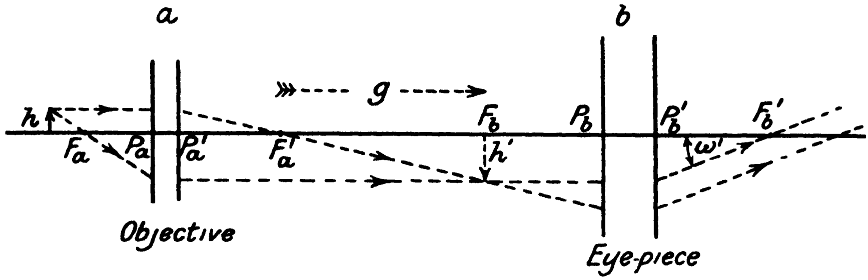


FIG. 57. A GAUSSIAN DIAGRAM OF THE MICROSCOPE

An object viewed by an unaided eye will naturally be placed in the nearest point of accommodation, which is at a distance β, say, from the first principal point of the eye. Hence,

$$\text{Angular subtense of object to unaided eye} = -\frac{h}{\beta}$$

Hence,

$$\text{Magnification} = \frac{\text{Angular subtense of image to eye}}{\text{Angular subtense of object when at near point}}$$

$$= \frac{\beta g}{f'_a f'_b}$$

The expression can be written $\left(\frac{g}{f'_a}\right)\left(\frac{\beta}{f'_b}\right)$, and it will be seen from above that the first bracket represents the "first magnification" of the objective, i.e. ratio of linear size of image to object measured transversely to the axis. The second bracket $\left(\frac{\beta}{f'_b}\right)$ is clearly the "magnification" of the eyepiece used as a simple magnifier.

Modern models of the microscope have a length such that the normal value of g, the "optical tube length" is 180 mm. The older English tube had an optical tube length of 10 in.

Amongst English makers it is usual to specify the eyepieces by the magnification, i.e. (approximately)

$$\frac{10 \text{ in.}}{\text{focal length of eyepiece in inches}}$$

or

$$\frac{250}{\text{focal length of eyepiece in millimetres}}$$

taking 10 in. or 250 mm. as measures of the least distance of distinct vision. In the Continental catalogues, however, the objective magnification is listed as

Objective magnification (Continental)

$$= \frac{250}{\text{focal length of objective in millimetres}}$$

and

Eyepiece magnification (Continental)

$$= \frac{180}{\text{focal length of eyepiece in millimetres}}$$

This is equivalent to writing the microscope magnification as

$$\text{Magnification of microscope} = \left(\frac{\beta}{f'_a}\right)\left(\frac{g}{f'_b}\right)$$

which arose from Abbe's special method[1] of deriving the magnification formula, which need not be reproduced here. We may, however, note that when the image formed by an instrument is projected "to infinity," then

$$\text{Magnification} = -\frac{\beta}{f'}$$

where f' is the focal length of the combined system, given by

$$f' = -\frac{f'_a f'_b}{g}$$

Thus $$\text{Magnification} = \frac{\beta g}{f'_a f'_b}$$

Resolving Power of the Microscope System. It will be assumed in the first instance that the object field consists of an assembly of discrete self-luminous points, although such a condition rarely obtains in the actual use of the microscope, and it will be necessary later to review the conclusions reached by such an assumption.

In Fig. 58(*a*) let R and R′ be the entrance and exit pupils respectively of the whole image-forming system, including the eye if visual observation takes place. Let a and a' be the extreme angles with the axis, made by rays passing through the axial points of object and image respectively. Then

$$\text{Radius of Airy disc in the image} = h' = \frac{\cdot 61\lambda'}{\sin a'},$$

where λ' is the wave-length of light in the image space.

In order to find the dimension h in the object plane, which has an image of size h' (the radius of the Airy disc) in the image field, (assuming the system to fulfil the "sine condition") we use the sine relation

$$nh \sin a = n'h' \sin a'$$

whence

$$h = \frac{n'h' \sin a'}{n \sin a} = \frac{0{\cdot}61\ n'\ \lambda'}{n \sin a} = \frac{0{\cdot}61\ \lambda_0}{n \sin a}$$

where λ_0 is the wave-length of light in the air.

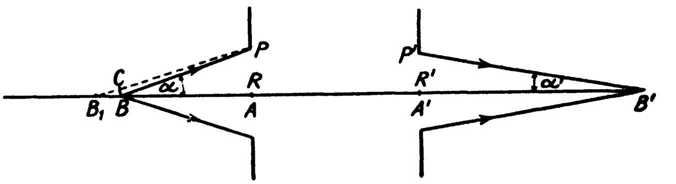


FIG. 58(*a*)

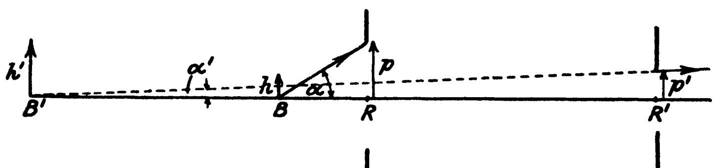


FIG. 58(*b*)

It was explained in Vol. I, page 108, that the closest possible approach for two elementary star images which allows of recognizable resolution is approximately this radius of the Airy disc. Thus the distance of closest approach for two *object* points is inversely proportional to the product of the refractive index of the object space, and the sine of the semi-aperture of the cone of rays entering the entrance pupil of the system from the object point. Abbe called this product the "Numerical Aperture" of the system, and we shall write it NA, thus

$$h = \frac{0{\cdot}61\lambda_0}{NA}$$

It is remarkable that the only necessary assumption about the construction of the optical system is its ability to produce images free from aberrations of optical path.

Visual Resolutions. As in the case of the telescope, the visual resolution of the images requires that they shall be presented to the eye under a sufficient angular magnitude. If ω is the angle subtended by the image at the eye, then

$$\omega = \text{(Angle subtended by the object at the near point)} \times \text{(magnification)}$$

$$= \left(\frac{h}{\beta}\right) m_a, \text{ numerically, where } m_a \text{ is the magnification,}$$

$$= \frac{0{\cdot}61\lambda\, m_a}{\beta NA}$$

if h is the separation necessary to secure that the images are just resolved physically. In order to present the image separation under the very minimum angle for visual resolution (say one minute of arc, which is ·00029 in angular measure), we get from the formula

$$m_a = \frac{\omega NA\beta}{0{\cdot}61\lambda}$$

by taking $\lambda = {\cdot}00058$ mm. and $\beta = 250$ mm.

m_a (the minimum essential magnification) $= 200\, NA$ approx.

It will be found, however, that this critical image distance should, in actual practice, be presented under a much larger angle than one minute. Four or five minutes will not be too large. The formula required is

$$m_a = 200\, NA \;(\omega \text{ in minutes})$$

so that if we are working with $NA = 1{\cdot}2$, and we desire to make $\omega = 3'$ in order to have comfortable observation, a magnification of 720 will be called for. The required eyepiece can then easily be calculated from the formula

$$f'_b = \frac{250 \times 160}{f'_a \times m_a}$$

Take, for example, a case in which $m_a = 750$ and $f'_a = 2$ mm.

Then $$f'_b = \frac{250 \times 160}{2 \times 750} = 27 \text{ mm. approx.}$$

In the English system this represents a magnifying power for the eyepiece of $\frac{250}{27} = \times 9$ approx. The "Continental magnifying power" is $\frac{180}{27} = \times 7$ approx. It is quite usual to work with magnifying powers considerably higher than these when employing a 2 mm. immersion lens. In fact, the ordinary microscope system employs a great deal of "empty" magnification in contrast to the case of small telescopes and prism binoculars, where the magnification is usually not nearly high enough to use the full resolving power of the system.

Size of Exit Pupil and Measurement of Magnification. Let R and R′ (Fig. 58(*b*)) be the axial points of the entrance and exit pupils of the optical system of a microscope, while B and B′ are the axial points of object and image respectively. The extreme ray from B passes through the margin of the exit pupil after refraction by the system, since entrance and exit pupils are conjugates, and it must be directed from B′. If h and h' are the sizes of object and image, then

$$nh \sin a = n'h' \sin a'$$

But $n \sin a = NA$, and $\sin a' = \frac{p'}{\text{R}'\text{B}'}$, within allowable approximation if R′B′ is large. The sine relation becomes

$$h\,NA = \frac{n'h'p'}{\text{R}'\text{B}'}$$

$$= (n'p') \times \text{(angular subtense ot image taken at centre of exit pupil)*}$$

Hence,
$$\frac{h}{\beta} . NA = \left(\frac{n'p'}{\beta}\right) \times \text{(angular subtense of image)}.$$

But $\frac{h}{\beta}$ is the numerical value of the angular subtense of the object held at the near point. Hence since $n' = 1$,

$$\frac{NA\ .\ \beta}{p'} = \frac{\text{Angular subtense of image}}{\text{Angular subtense of object}} = \text{Magnification.}$$

This gives a useful method of finding the magnification of the microscope.

Since we found above a minimum value for the necessary magnification

$$m_a = 200\,NA$$

$$p' = \frac{NA\beta}{200\,NA}$$

* We are only concerned with numerical relations, and shall not need to consider the signs of the angles in this section.

The conventional value for β is 250 mm., so that p' should then be 1·25 mm. The diameter of the exit pupil of the microscope ought to be *at most* 2·5 mm. if all the resolvable detail is to be visible to the eye. It will be an advantage, as shown above, to employ magnification four or five times as great, so that the exit pupil diameter may well be reduced to 0·5 mm.

Depth of Focus in the Object Space. Referring to Fig. 58(*a*), let B and B′ represent conjugate points for which paraxial and marginal optical paths are equal. We may displace the object point to B_1 along the axis, so as to cause a difference of paraxial and marginal optical paths of $\frac{\lambda}{4}$ before the deterioration of the image becomes apparent through the phase differences of the disturbances arriving in the image. See Vol. I, pages 108–112 and page 141. Join B_1P and draw BC perpendicular to B_1P. The difference

Increase of axial path – increase of marginal path

$$= B_1B - B_1C$$

within a small quantity of the second order, giving

Optical path difference $= n \,.\, B_1B\,(1 - \cos\alpha)$

and writing $B_1B = dx$, we obtain

Optical path difference $= 2n \,.\, dx \,.\, \sin^2\frac{\alpha}{2}$

If this may amount to $\pm\frac{\lambda_0}{4}$,

$$dx = \pm\frac{\lambda_0}{8n\sin^2\alpha/2}$$

where λ_0 is the wave-length of light in air. Note that α is $\sin^{-1}\left(\frac{NA}{n}\right)$ where NA is the numerical aperture of the microscope objective, and n is the refractive index of the object space.

Since the step is allowed on either side of the focus, the total range is double the above, and the following approximate table may be calculated for blue-green light.

TOTAL DEPTH OF FOCUS

Numerical Aperture of Objective	Depth of Focus in Air	Depth of Focus in Medium of Refractive Index 1·5
·25	·0079 mm.	·0122 mm.
·50	·0019 ,,	·0030 ,,
·75	·0008 ,,	·0013 ,,
1·00		·0007 ,,
1·25		·0004 ,,

The depth of focus in the image depends on the aperture ratio of the convergent beam. With average objectives it may be of the order of one or two millimetres. When a projection eyepiece is used to form the final image for photomicrography the depth of

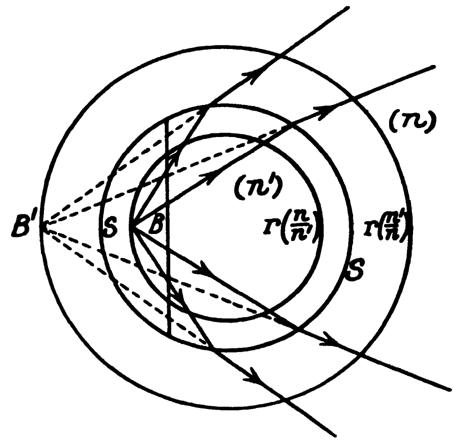


FIG. 59(*a*)

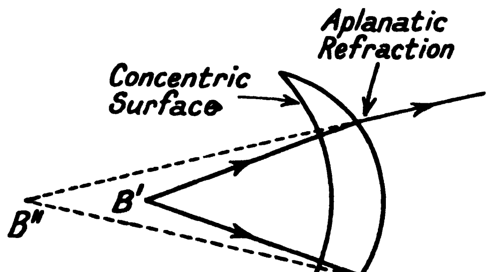


FIG. 59(*b*)

focus in the final image will be very considerably greater still, owing to the very small angular apertures of the convergent pencils; it may be several inches or more. Note that this does *not* refer to focus changes produced in other ways than by moving the final receiving screen.

High Power Microscope Objectives. It will now be appreciated that the resolving power of the objective is dependent on the Numerical Aperture which can be attained, and, hence, the means for increasing the numerical aperture to the widest possible limits must receive attention.

In Vol. I, page 20, an explanation was given of the aplanatic surfaces of a sphere. Referring to Fig. 59A, the sphere S of radius r has a refractive index n', and is immersed in a medium of refractive index n. An object point B within the sphere lies on a concentric circle drawn with radius $\frac{rn}{n'}$; all rays from this point suffer aplanatic refraction at the surface of the sphere. The corresponding virtual image B′ is situated on a corresponding circle of radius $\frac{rn'}{n}$, and is free from spherical aberration and coma. Thus the wide divergence of the fan of rays from B is very considerably reduced.

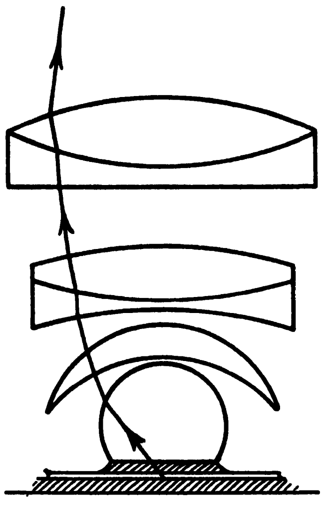


FIG. 60(*a*). SECTION OF IMMERSION ACHROMAT

2 mm., N.A. = 1·25

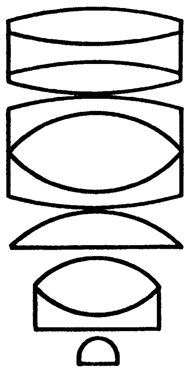


FIG. 60(*b*). SECTION OF TYPICAL APOCHROMATIC OBJECTIVE

2 mm., N.A. = 1·40

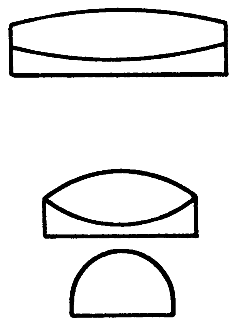


FIG. 60(*c*). SECTION OF TYPICAL DRY LENS

8 mm., N.A. = 0·65

Such a fan, with reduced divergence, may now encounter a spherical refracting surface which has its centre coincident with the common divergence point B′ of all the rays, which thus suffer no deviation and no aberration. This condition is illustrated in Fig. 59(*b*), where the rays enter the first surface of the meniscus lens. It is clear, however, that the back surface of this lens can also be made to produce an aplanatic refraction if B′ is one of the aplanatic points, and the divergence of the rays can be still further reduced. In this stage they may be made to encounter a "Lister" combination or other arrangement by which the divergence from a virtual image is changed into convergence towards a real image, as suggested in Fig. 60(*a*).

The object point cannot be placed, of course, within a spherical lens; hence this front lens is cut off by a plane face, and the continuity of refractive index of the media is ensured by placing a film of liquid, usually cedar oil of refractive index 1·517, between the lens and the cover glass ($n = 1·515$, about) which is practically in contact with the object. There is then little change of refractive index between the front lens and the farther side of the cover glass, as the front lens is usually of glass of fairly low refractive index, not above 1·54. The above principles represent the main basis of the control of spherical aberration and coma in high power immersion lenses.

On the other hand, the two aplanatic refractions involve a large amount of chromatic under-correction which, therefore, has to be compensated by the back lens or lenses of the system, and which is usually only attained by the use of fairly deep curves in the contact faces of these lenses. The result is that although spherical aberration can be compensated for one zone without very great difficulty, it is somewhat difficult to obtain it for all zones, i.e. zonal aberration is likely to be present. The successful balancing of the aberrations calls for considerable skill and resource on the part of the designer.

Apochromatic Objectives. Efforts to reduce the secondary spectrum characteristic of the "achromatic" systems made of glasses of the older types were made by Abbe when the new Jena glasses were introduced, but the success of his "apochromatic" lenses was largely due to the use of "fluorite" as a component of some of the lenses. The "optical constants" of fluorite as compared with two other glasses are (compare Vol. I, page 233)—

	n_D	$n_F - n_C$	V	$n_F - n_D$	$n_{G1} - n_F$	β	γ
Fluorite	1·4338	·00454	95·5	·00321	·00256	·707	·563
Boro-silicate crown	1·5160	·00809	68·3	·00567	·00454	·701	·561
Telescope flint	1·5237	·01003	52·2	·00708	·00577	·706	·575
Dense flint	1·6225	·01729	36·0	·01237	·01052	·715	·608

It will be recalled in the discussion of achromatism (Vol. I, Chapter VII) that the "powers" of opposing lenses in achromatic combinations are lowest when there is a large difference of V values; when the powers of the lenses are relatively low, the curves are relatively shallow, and zonal aberration is less likely to arise. It was also shown that the residual secondary spectrum was reduced by a large difference of V values; on the other hand, the relative partial dispersions of fluorite are quite comparable with those of ordinary crown glasses, and also come fairly close to those of an extra light flint such as "telescope flint." Hence with combinations using fluorite in place of the "crown" lens, and "extra light flint" in place of the ordinary flint lens of an achromatic combination, it is possible not only to reduce the secondary spectrum, but also largely to reduce the zonal aberrations. Fig. 60(*b*) shows an "apochromat." Low powers (dry) are also made.

Apochromatic objectives are found usually to possess noticeable chromatic difference of magnification, which is best compensated by the use of special eyepieces. (See page 87.)

High Power "Dry" Lenses. Such lenses are made with numerical apertures up to 0·95, and, therefore, the extreme rays in the object space may have to make an angle of $\sin^{-1} 0{\cdot}95 = 72°$ with the axis. The arrangement of the majority of dry lenses of intermediate power is similar to that shown in Fig. 60(*c*), which might be regarded as a Lister pair combined with a hemispherical front lens. In this case, however, considerable spherical and chromatic aberration arises at the front face of the "hemispherical" lens, and this has to be corrected by the rear components; not only this, but the aberration arising at the cover glass surface must be considered. If the front lens and the cover glass are of equal refractive index, the cover glass may be considered as an addition to the thickness of the front lens in the course of design, but this means that the system will usually be very sensitive to the thickness of cover glass with which it is used. Referring to Vol. I, page 14, it will be found that refraction at a plane surface transforms a spherical wave-front into an elliptical wave-front, and, if a decrease of refractive index is involved, the *major* axis of the ellipse is the normal to the surface, so that the disturbances travelling along oblique paths suffer relative retardation. Hence, since an infinitely thin cover glass could produce no effect, it will be clear that increasing thickness of cover glass produces more "over-correction" in the spherical aberration sense.

In practice, slight over-correction arising through the use of a too thick cover glass can be compensated by using the objective with a shorter tube length and, consequently, a greater working distance between the front lens and the object point; this produces an opposing tendency to under-correction, mainly in the front lens.

Eyepieces. Little need be added to the discussion of eyepieces given in the preceding chapter. The Huygenian type is perhaps the most widely employed, the design varying with the required magnification; the focal length of the field lens may not exceed 1·5 times that of the eye-lens at the lowest magnifications. Rings are fitted to a set of eyepieces so that they are parfocal, and can be interchanged without large changes of focus; the optical tube-length is thus kept constant. Various eyepieces of the "orthoscopic" and other types are in favour with some microscopists who prefer a very wide angular field.

Ordinary achromatic objectives have little chromatic difference of magnification, and are best used in conjunction with ordinary Huygenian eyepieces. With the $\frac{1}{12}$ in. oil immersion objectives, and with the apochromatic objectives, there is a slightly greater magnification for the "blue" or shorter wave-lengths, as compared with the red, and this is best overcome by the use of so-called "compensating" eyepieces which are designed to give a correspondingly greater magnification to the red than to the blue. If held to the light the border of the field stop appears tinged with orange in a compensating eyepiece, and blue in an ordinary Huygenian. Abbe designed a series of apochromatic lenses for the firm of Zeiss in which all had the same chromatic difference of magnification, and thus could be used to advantage with the same compensating eyepieces, and various other makers have special arrangements in this connection.

The "projection eyepiece" usually has a single field lens with a triple projecting lens. The exit pupil is limited by a small stop in order to cut out stray light.

As mentioned above, the ordinary forms of eyepiece have a negative curvature of field according to the usual formula—

$$\frac{1}{R} = \Sigma\left(\frac{-1}{nf'}\right)$$

where n is the refractive index and f' the focal length of a constituent lens of the system reckoned as for an infinitely thin lens with faces of the same curvature. In 1918, Conrady proposed to introduce as the front lens of an eyepiece system, a carefully computed achromatic lens of negative power which would be sufficient to give a positive curvature to the field of a telescope objective. This might then be followed by a suitable eyepiece.

An anastigmatic flat field telescope on these lines was later independently designed and produced by H. Dennis Taylor.

In microscope objectives, the curvature of field of the primary image is so much greater that an achromatic negative combination can hardly be expected to do more than flatten the field of the primary image without compensating the curvature due to an observing eyepiece.

In 1922, H. Boegehold and A. Köhler, of Messrs. Carl Zeiss, brought out the "Homal," a projection lens of negative power, which flattens the field of the microscope objective for which it is designed. As in the case of the Galilean telescope, the exit pupil lies within the lens; the field of view to an eye placed behind it is very small indeed, and the system is useless for visual work, but the system is quite satisfactory for photography.

Numerical Aperture: General Remarks. Since "numerical aperture" is the product of refractive index and the sine of the angle of obliquity of the extreme ray, a medium of refractive index n cannot transmit a ray of numerical aperture greater than n. Hence, if an object is in air, the numerical aperture of the extreme rays cannot exceed unity. The law of refraction, $n \sin i = n' \sin i'$, shows that the "numerical aperture" characteristic of a ray is independent of any refractions at plane surfaces normal to the axis, such as those of the cover glass, etc.; therefore, if the original object is in air, the objective cannot work at a numerical aperture greater than unity. If the medium is water (refractive index 1·333), the limit of possible numerical aperture is equal to or under 1·333, provided that all the media have at least this refractive index. To take full advantage of an objective having any high *NA* such as 1·6 (such could be constructed), it would be necessary to have the mounting medium, cover glass, and immersion medium of this refractive index at least. The action of dark-ground illuminating systems has to be considered with this point in mind. (See below, page 119.)

Historical Note. As remarked above, Lister's paper on the aplanatic points of an achromatic doublet was published by the Royal Society in 1830. About the same time, G. B. Amici, in Italy, was working somewhat on the same lines, but attempting to find how the spherical aberration of higher orders and the coma could be eliminated by the mutual action of doublet lenses, none of which were necessarily working strictly aplanatically. In the "fifties," Amici introduced the strongly curved plano-convex lens as a front lens, which later developed into the hemispherical and hyper-hemispherical form of modern high power objectives. This allowed of a great increase of numerical aperture, the importance of which in resolution of fine structures had been realized by various workers, and also noticed by Lister in 1830.

Immersion systems were first introduced as a means of overcoming the effects of the thickness of the cover glass, and also its possible variations and irregularities; the liquid employed was almost invariably water. Such systems were employed by Amici and others. The great advantages of immersion systems in high power work were only slowly realized, although advances had been made by Tolles in America, who used glycerine and balsam immersion lenses. Abbe published his diffraction theory in 1877, and explained on this basis the theory of the effect of homogenous immersion in increasing numerical aperture and securing increased resolving power. He also introduced the apochromatic lenses.

The Illumination of the Object. The assumption made above, that the object in the microscope consists of an assembly of discrete self-luminous points, is seldom realized in practice. The

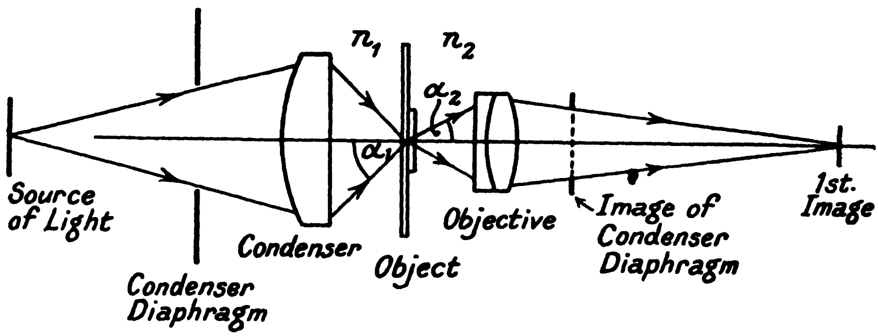


FIG. 61. THE USE OF THE SUB-STAGE CONDENSER (Diagrammatic)

majority of biological specimens are viewed by transmitted light, while opaque objects, such as metallurgical specimens, are illuminated by light which they reflect into the instrument.

Material objects which are non-self-luminous may obstruct light completely or partially. They may also reflect, refract, diffract, or scatter light. By these actions they become "visible," and the interpretation of the "image" formed on the retina usually leads us to a more or less correct knowledge of the physical character of the corresponding object, but often mainly through collected experience. The simplest case is that of a "silhouette" pattern in which the geometrical outline pattern of a flat object is seen by the obstruction of light, partial or complete. If the object exercises selective absorption of light, the geometrical boundaries may be indicated by the fields of various colours.

If such an object is to be viewed away from a microscope, we may hold it in front of a uniformly luminous white surface to obtain the best view, and the "true" colours. In the microscope we *may* project the image of a uniformly illuminated or luminous white surface approximately into the plane of the object by the aid of the substage "condenser" system (Fig. 61), and this affords a close approximation to the best condition for obtaining an image pattern similar to the object itself in a geometrical sense.

The above conditions may not, however, be at all suitable for yielding an image from which the physical characteristics of certain other classes of object can be inferred. As an experiment, take a piece of glass with a pattern moulded in its surface, such as is employed for doorways and the like. Where such a glass is held close to a broad, evenly illuminated source of light, the "pattern" practically disappears. In order to obtain a strongly marked, easily visible appearance of some kind, the glass is held between the eye and a small source of light which is thus giving a more or less directed beam. The "pattern" stands out strongly because certain parts of the glass refract or reflect light into the eye, and other parts do not. The effects of cumulative experience again largely enter into the interpretation, which may or may not be true. Hence, we need to guard against the easy assumption that there is always a possibility of obtaining a "true" picture of the object. The successful microscopist will adapt his illumination to the end in view. He will regard any image pattern not so much as a picture of the object, as a piece of physical evidence, from which the physical characteristics of the object can possibly be established. The different methods of illumination possible with the modern substages should be regarded as additional weapons in the armoury.

Substage Condensers. Theoretical considerations (to be developed more fully below) show that in order to obtain the optimum resolving power, i.e. the distinction of the finest possible detail in many classes of object, (but not necessarily every case), it should be possible, if required, to pass light through the object in such a way that the rays diverging from any one point of it spread out and fill, uniformly, the whole aperture of the objective. The case is illustrated in Fig. 61. If the refractive indices of the media on the two sides of the object slide are n_1 and n_2, and the angular divergence of the extreme ray which can enter the objective from an axial point of the object plane is a_2, then we must have

$$n_1 \sin a_1 \nless n_2 \sin a_2$$

i.e. the numerical aperture of the condenser system must at least be equal to that of the objective if the above condition is to be fulfilled.

In the simplest case, there is no lens system in the substage.

An image of a fairly broad source of light, i.e. a lamp flame or an opal glass electric lamp, is thrown approximately into the object plane by a concave mirror. The divergence of the light from the object plane is sufficient to fill an objective of small numerical aperture—up to about 0·25. It is then advisable to use a small stop immediately below the object to limit the illumination to the part of the object required. Sometimes when a broad source of light is available, the plane mirror may be used to reflect the light through

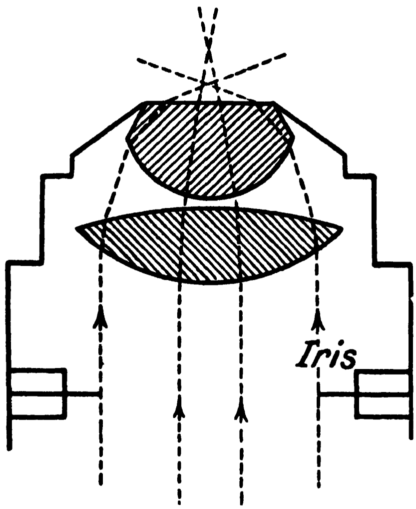


FIG. 62. ABBE TWO-LENS CONDENSER

N.A. = 1·2

the object. The angular divergence of the extreme rays passing through any point of the object plane is then determined by the angular subtense of the source of light.

For larger apertures a lens system in the substage is employed. Fig. 62 shows a type frequently employed, due to Abbe, with which a numerical aperture of 1·2 can be obtained. There is, however, considerable spherical aberration, so that the rays do not all pass through one point, and there is no correction for chromatic aberration. The disadvantage of this aberration will be understood from Fig. 63, in which the condenser S brings the marginal parallel rays from a small distant source to an axial focus at C—intermediate rays to B, and paraxial rays to A. If an object is placed in the plane of the point C, any point of it can only be illuminated by marginal and paraxial rays. A pinhole at C would block the rays from an intermediate zone. In practice, the difficulty is usually overcome by using a broad source of light, and rays from another part of the source may then pass through the intermediate zone and through the axial point of the object. This could be studied by drawing a figure similar to Fig. 63, and bringing a bundle of rays into the condenser at a suitable angle with the axis. The use of a broad source of light is, however, likely to be accompanied by difficulty in controlling the light and restricting the presence of "stray" and unwanted light.

This uniform distribution of light into the objective is most important in practice. It should always be checked by removing the eyepiece of the focused microscope, and observing the image

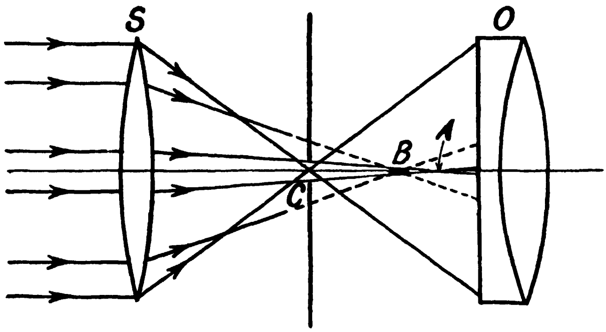


Fig. 63. Effect of Spherical Aberration in a Condenser

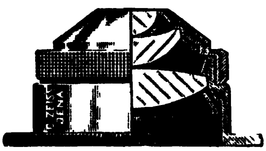
(formed by the objective) of the diaphragm below the condenser, or of whatever stop limits the numerical aperture of the extreme rays passing through the system. If light diverges uniformly to the objective from any point of the object, this image (which is seen just above the objective) will appear as a uniform disc of light. If a small source is employed, and the condenser is subject to spherical aberration, then the disc at the back of the objective will probably be illuminated either at the margin and the centre only, or in an intermediate zone. This must be remedied by the use of a larger source of light, or by the use of an auxiliary condensing lens as described below.

If three or four lenses are employed in the construction of the substage condenser, as shown in Fig. 64, it may be made free from spherical aberration and given an *NA* up to 1·40. It is sometimes useful thus to be able to use quite a small source, such as a glowing thorium pastille heated by a small gas flame, or the glowing tungsten ball of a small Point'olite lamp*, as the illumination of the object

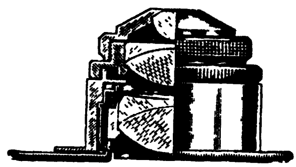
* Manufactured by Messrs. Ediswan Electric Co., Ltd. If a Point'olite lamp is used for visual work the intensity of light will have to be reduced by a suitable light filter or screen.

can then be limited to one small part under examination. This is most helpful in avoiding strong light. In order to make perfectly sure of the above test, it is advisable to make the observation through a small pinhole in a card placed at the end of the tube. This will save any mistake due to glare.

Chromatic aberration is a little troublesome in cases where it is desired to use such small sources, and the best substage condensers are of the achromatic type. They are made on practically the same lines as objectives of equivalent numerical aperture, but the degree of precision required in their manufacture is naturally less, as the sole requirement is that they shall secure this uniform distribution



Three-lens "Aplanatic" Condenser *NA* = 1·40



Achromatic Condenser *NA* = 1·0

FIG. 64. TYPICAL CONDENSERS (*Zeiss*)

of light into the objective, and any small deficiencies are not noticed because a source of light of finite size is invariably employed. The main advantage they confer is the possibility of obtaining the proper conditions with the minimum of trouble when it is desired to limit the illuminated area of the object. A diagram of such an achromatic condenser is shown in Fig. 64. In certain cases it may be useful to use an objective (reversed) as a condenser, especially when using immersion systems of high numerical aperture; in such a case the object will be mounted between two cover glasses.

The diaphragm which limits the aperture of the condenser is an important adjunct. It is usually of the "iris" variety, so that its aperture can be reduced from the full lens diameter down to a few millimetres, and it is preferably mounted so that it can be displaced perpendicular to the axis of the system in order that a narrow cone of very oblique light may be projected through the object. This is useful in cases where evidence of the very finest structures is sought for.

Condensers required to give a greater numerical aperture than 1·0 *must* be used as immersion systems, a spot of cedar-wood oil being placed between the surface of the objective and the slide, otherwise $n \sin \alpha$ can never rise above unity, even if the extreme rays emerge from the condenser into the air practically at right angles with the axis. Even if the condenser is not designed for an

NA greater than 1·0, the use of oil immersion saves much loss of light by reflection at very oblique angles. If the objective in use has an *NA* of 1·2 or more, it is only very seldom, however, that the allowable *NA* of the condenser (as actually stopped down for use in observation) may exceed 1·0, and some observers, if pressed for time, will omit the use of the oil for the condenser on this account. Naturally, an immersion objective must *always* be oiled on.

The working distance of the condenser should be so arranged that if the source of light is placed at about 10 to 12 inches from the mirror, an image of the source will be projected into the plane of the object as mounted on a slide of normal thickness, say,

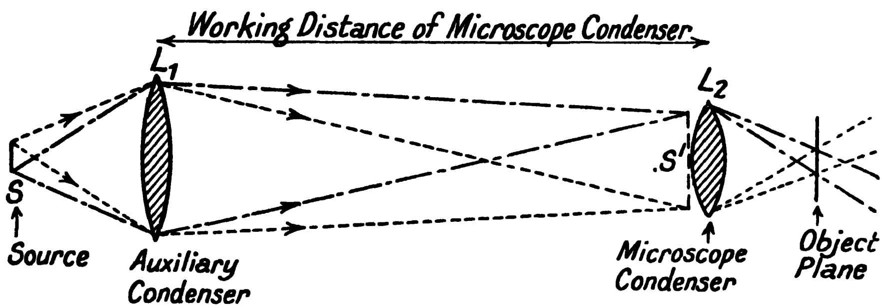


FIG. 65. USE OF AUXILIARY CONDENSER

1·0 mm. The distance between the front surface of the condenser and the slide should then be about 0·2 mm., so that a drop of oil may be squeezed out between the surfaces. The condenser will always be held in a focusing mount, and a good rack and pinion gives ample accuracy for the adjustment. It saves anxiety if a suitable stop prevents the surface of the condenser from actually rising above the plane of the surface of the stage on which the slide has to rest. There can then never be any danger of breaking the slide in that way.

In practice, a number of difficulties may be encountered in securing correct illumination which is evenly distributed over the required region of the object, amongst which are the following—

1. *The Image of the Source is Too Small to Illuminate the Required Area of the Object.* In this case an auxiliary condenser must be used, as shown in Fig. 65. The lens L_1 projects an image of the source S into the lens L_2, which represents the microscope condenser. If L_1 is placed at the working distance for L_2, and further, if L_1 projects the image without spherical aberration, it (L_1) will act as a uniform source, projecting light into the image of S. Clearly, this image

should be large enough to fill the aperture of L_2 sufficiently to attain the required *NA* of the illuminating beam.

Special "aplanatic" auxiliary condensers can be obtained for the above purpose. If only a plano-convex bulls-eye condenser is available, it should be employed with the plane side towards the shorter conjugate distance to minimize the spherical aberration as far as possible.

If the above arrangement proves impossible for any reason, owing, perhaps, to the difficulty of filling the microscope condenser aperture, use may be made of the double condenser system shown in Fig. 66. The first auxiliarly lens L_1 projects an image S′ of the source S into the lens L_2. The lens L_2 now projects an image of

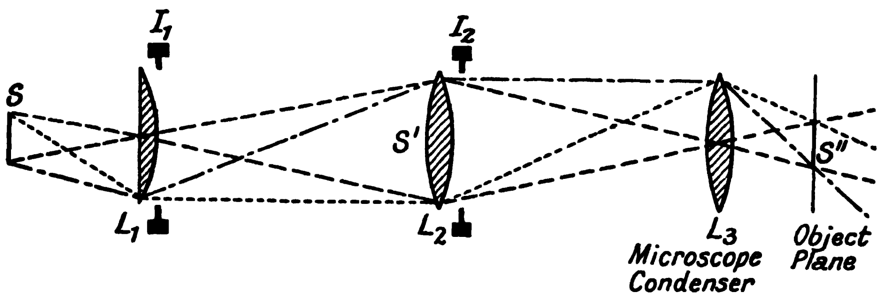


FIG. 66. DOUBLE CONDENSER SYSTEM

L_1 into the aperture of the microscope condenser, and if L_1 is reasonably free from spherical aberration, the illumination of the condenser will be uniform. The microscope condenser projects an image of L_2 into the plane of the object, so that the original source of light is finally imaged there also.

If the lenses L_1 and L_2 are fitted with iris diaphragms, shown in the diagram by I_1 and I_2, the use of I_1 clearly controls the aperture of the microscope condenser which is illuminated and, hence, the numerical aperture of the illumination, while I_2 clearly controls the area of the object plane which is illuminated.

2. *The Working Distance of the Condenser is Too Small or Too Great.* If a very thick slide is used, it may be found that the condenser will not give a satisfactory image of the source or effective source when used at the ordinary working distance, even when the condenser is racked up as far as possible. If this results in uneven distribution of the light into the objective, the source must be brought much nearer to the condenser, or an image of it projected into such a position as may be required.

If, on the other hand, the working distance is too great, the condenser may, of course, be racked down to obtain the required focus, but difficulty may then be found in retaining a film of immersion oil between the condenser and the slide. The space may, however, be partly filled by one or two cover glasses, oiled on both sides, and this usually overcomes the difficulty.

3. *The Source of Light may be Unsuitable.* If an electric lamp is used it should have an opal glass bulb inside a suitable housing with one or two windows, the size of the aperture being regulated by iris or other diaphragms (see Fig. 67).

If the electric lamp has visible filaments, the light should be diffused by a screen of "ground" or "opal" glass, or even by tissue

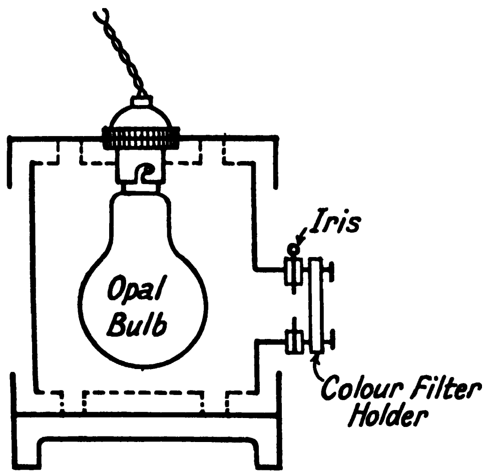


FIG. 67. SIMPLE MICROSCOPE LAMP

paper in emergency. In any case, some scheme should be adopted for visual observation to obtain a small, bright, uniform patch of light about 1 in. diameter; it is helpful to have an iris or other means of reducing this diameter if required. The Point'olite tungsten arc may be adapted for visual observation by one or more of the means described above, but some "light filter" preferably with variable density will usually be required. Suitable "wedges" of neutral glass may be mounted to be moved oppositely so as to secure more or less dimming; an alternative is to vary the thickness of a layer of a suitable absorbing liquid.

Setting up the Instrument for Observation. Perhaps the best approach to the theory of the microscope is by an experiment which can be followed either practically, or from the description below. This description gives, in fact, an outline of the procedure usually adopted in setting up a microscope for careful observations on any object. A microscope is chosen with a well-corrected achromatic condenser fitted with an iris diaphragm. The source of light is a lamp, such as is shown in Fig. 67, placed at, say, 10 in. from the mirror. The objects for the experiment are two diffraction rulings mounted on an ordinary 3 in. × 1 in. microscope slide and covered by a suitable cover glass. The gratings should have bands of 30,000 and 15,000 lines to the inch respectively, and are to be used with a 16 mm. apochromatic objective of first class quality, which has a numerical aperture of 0·30. The eyepiece may be a " × 15 compensating." (Such rulings as mentioned were obtainable, commercially, under the name of Grayson's rulings,* but there is at the time of writing some difficulty in procuring specimens.)

In setting up the instrument, a 1 in. or 2 in. objective is first used in the microscope with a low power eyepiece. The plane side of the mirror is used to reflect the maximum light through the system, and the diaphragm of the condenser is closed to its smallest diameter. By racking the objective up and down, the image of the diaphragm may be found in the field, and it is made central by the use of the centring screws of the condenser. The object slide is clipped on the stage and the image of the object plane is then focused. The condenser is racked up till the image of the diaphragm on the lamp is brought into the object plane. Hence the lamp iris permits of the control of the illuminated area in the object plane.

This illuminated area is reduced to a small portion in the centre of the field. Then the 16 mm. apochromatic lens and the compensating eyepiece are put in position and the object focused. In order to see the rulings it will probably be necessary to close down the aperture of the condenser, and this will easily enable the "15,000" band to be focused.

Now remove the eyepiece and look into the back of the objective. A symmetrically placed circular patch of light should be seen, apparently filling a part of the aperture of the back lens. If the iris of the substage condenser is opened out the patch expands, and *vice versa*. It should appear perfectly uniformly illuminated. If this is not the case, the illumination system must be put into better adjustment, until uniformity is obtained even for the largest substage apertures. Disregard for the moment any dimly coloured lateral patches of light which may be seen.

Having made sure of this point, replace the eyepiece, and, having focused the plane of the object as carefully as possible, open out

* A typical Grayson's ruling may have bands of 5,000, 10,000, 15,000 up to 60,000 lines to the inch in the same slide.

the condenser aperture slowly, watching for the appearance of the 30,000 band. (If it fails to appear, a blue filter placed in front of the source of light may aid matters. If it is desired to use monochromatic blue light, the opal bulb lamp may have to be replaced by a Point'olite with auxiliary condenser in order to obtain sufficient intensity.) Provided the system is adequately corrected, the 30,000 band will probably appear when the condenser has a wide aperture —nearly "filling" the objective; but diminution of the condenser aperture will cause the bands to vanish. Returning for a moment to the theory already developed, it can be ascertained from the formula, i.e. $h = \frac{0{\cdot}61\lambda}{NA}$, that the closest limit of approach for two self-luminous objects which are to be still resolved in the image given by a lens of NA 0·3 is

$$h = \frac{0{\cdot}61\lambda}{0{\cdot}3}$$

Taking $\lambda = 0{\cdot}45 \times 10^{-3}$ mm., and remembering that 1 mm. $= \frac{1}{25{\cdot}4}$ of an inch, we find that we should not expect to resolve a pair of elementary objects spaced much more closely than $\frac{1}{27{,}000}$ of an inch apart if they were self-luminous with blue-violet light. Actually, then, if we are resolving lines in a 30,000 band, the performance is rather better than the "self-luminous" theory would seem at first to indicate, but the formula, it will be remembered, is not of a hard and fast character*, so that great significance must not be attached to the lack of correspondence. The possibility of the resolution of the 30,000 band under the above conditions may, however, be established as an experimental fact.

Now remove the eyepiece and observe that almost the whole of the back of the objective is filled with the direct light from the condenser, i.e. the objective is working at practically full aperture for the direct light. What will occur if the condenser is stopped down, so that the direct light only fills a small fraction of the aperture of the objective? If the resolving power is dependent on the aperture thus filled with the direct light, we may expect that on stopping down the condenser to give, say, one-eighth of the full aperture the distance between the lines would have to be eight

* The distribution of intensity in the neighbourhood of the image of a self-luminous line object has been investigated. The correspondence in the intensity distribution perpendicular to the elementary line image and across the diameter of the elementary "Airy disc" image is close enough to justify the tentative use of the same formula.

times as great to permit of resolution. A little care will, however, show that the 15,000 band can be resolved with quite small condenser apertures; the resolution is therefore diminished to one-half instead of to one-eighth. *It is evident that the above theory is inadequate to deal with this case.*

While the condenser aperture is small, the eyepiece may be removed and the back of the objective inspected. The eye is placed as close as possible to the plane of the image formed by the objective. (A suitable spectacle lens will help to bring the back of the objective into focus.) In this way the direct beam will be seen to be flanked by coloured lateral patches of light which have been

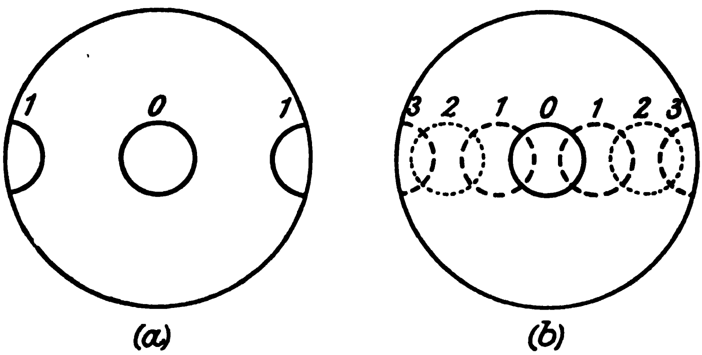


FIG. 68

(*a*) Appearance at back of objective with "15,000 lines band" just resolved, using small aperture condenser
(*b*) Appearance at back of objective with "5,000 lines band"; same condenser and objective

diffracted by the grating, their size and shape corresponding to that of the circular patch of the direct beam. The diffraction colours naturally show violet and blue nearer the centre followed by green and red, but they will not be well seen with the 15,000 band and the 0·3 lens, since only the blue parts of the diffracted images will probably be visible on each side. The succession will be better seen by using a coarser ruling, say, 5,000 to the inch, when not only the first order, but also the second order, diffraction patches will be seen. When using the 30,000 band, the diffracted light cannot enter the objective till the condenser aperture approaches its maximum.

Such experiments as the foregoing form an instructive introduction to the theory of the microscope. Abbe carried out experiments in which special gratings were used as objects; these were so ruled that the elements had definite angular shapes and the paths of all refracted rays could be calculated. It was found, however, that although in certain cases the rulings could not be resolved when the aperture of the objective of the observing microscope was just wide enough to admit all the directly refracted rays, the resolution of the images could be secured by a still greater increase of aperture. This led Abbe to the realization of the part played by the diffracted light, and to the formulation of the famous "Abbe Theory."

The Theory of the Microscope. It was mentioned above that a very suitable illumination for a microscopic object consisting of a plane containing opaque and transparent areas would be the provision of a self-luminous surface immediately behind it. The transparent areas would then function as self-luminous sources, and the theory of the formation of the image would be similar to the simple case of the telescope. Subject to the absence of aberrations and stray light, the resolving power so obtained would depend only on the numerical aperture of the objective.

In practice, lenses suffer from certain imperfections, and the object has to be illuminated with the aid of a condenser. It is an experimental fact that certain very delicate structures cannot be detected unless the illuminating cone is of somewhat smaller angular aperture than is required to fill the whole aperture of the objective. In such cases it is easily shown experimentally that the light entering the objective from the object is partly *diffracted* light, and any theory of image formation *must* take account of the effects of such diffracted light.

Now the theoretical investigation of the diffraction of light by material objects is generally a matter of the most difficult nature. It only becomes simple to calculate the main effects when the "object" is of a very simple character. A straight edge, a circular hole, a rectangular slit, or a row of similar apertures or structures are among the cases where simple methods are possible. This partly accounts for the fact that the question of the formation of the image of a "grating" (or row of slits or similar small structures) is of such theoretical importance. It is true, of course, that very many natural objects show a regular structure, and this makes the corresponding theory of particular interest.

The thorough analysis of the mode of image formation in any condition of illumination is a very difficult and intractable problem, but light may be gained in considering specially simple cases, especially that of the grating.

1. The theory developed by Abbe and others[2] supposes that the light derived from the condenser and transmitted by the object may be resolved into systems of plane waves. The basis for this

assumption has been examined by Stoney.* Thus in Fig. 69, if we concentrate attention on one such system represented in the diagram by a "parallel beam" travelling in a particular direction, we shall have diffraction occurring in the grating in the object plane; the objective will receive the direct and such diffracted beams of which the angular directions fall within its limits of aperture. Such beams will be concentrated by the objective to form sharp maxima near the back focal surface, and these maxima will have related phases, so that their effect in any other surface can be calculated. They will give an interference system, and it

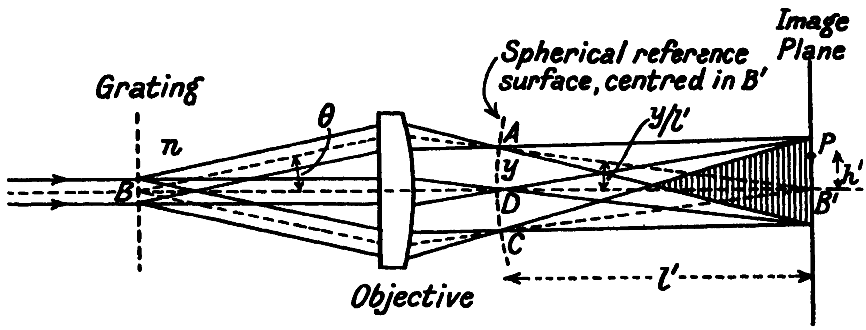


FIG. 69. DIFFRACTION IN THE MICROSCOPE
(Abbe principle)

can be shown that the *frequency* of the maxima in the image plane conjugate to the object plane for the objective corresponds to the enlarged dioptric grating image. In order to deal with the practical problem, however, the superposed effects of *all* the various beams passing in different directions through the object plane have to be worked out.

2. If it is assumed that the source of light is accurately focused in the plane of the object, then it is possible to proceed in the first place by dealing with one elementary point of the source at a time. The distribution of intensity in the plane of the object becomes known, and thus also the amount of light diffracted into different directions. In this way the distribution of light in the final image plane can be arrived at, but even in this case it is convenient first to consider the distribution of intensity near the back focal surface of the objective. This procedure has been worked out by the present writer[3], but only for the simple case of a grating, and assuming an optical system with rectangular apertures. Once more it is

* *Phil. Mag.*, Oct., 1896, p. 335.

necessary to extend the results by considering the superposition of the effects of all points in the source.

The Abbe Principle. On the basis of the experiments which he made with specially ruled gratings, Abbe reached the conclusion that the necessary condition for the resolution of a regular structure was that the directly refracted light, and at least one of the diffracted beams should enter the objective.

In Fig. 70 we have pictured a parallel beam incident on a grating at an angle θ_0 in a medium of refractive index n_0. The angle of transmission is θ_1 in a medium of refractive index n_1. The general

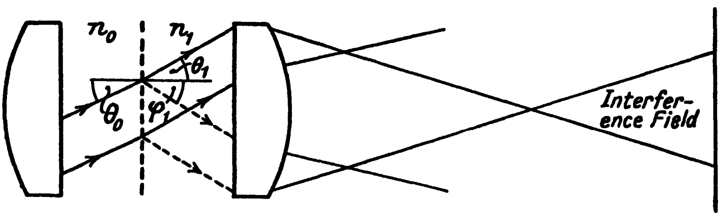


FIG. 70. DIFFRACTION OF AN OBLIQUE BEAM

formula giving the position of a diffraction maximum in the second medium is, writing x for the width of the grating element,

$$x(n_1 \sin \theta_1 - n_0 \sin \theta_0) = p\lambda \text{ (see page 283)}$$

where p is an integer and λ is the wave-length of light in air. If $p = 0$, this reduces to the law of refraction.

Now, if the incident parallel light is incident normally, as in Fig. 69, the angle of the first order diffracted beam is

$$xn_1 \sin \theta_1 = \lambda$$

or

$$x = \frac{\lambda}{n_1 \sin \theta_1}$$

If, therefore, the numerical aperture of the objective enables it to receive rays at a maximum obliquity of θ_1, this equation would give, for the minimum value of x enabling the first order diffraction maximum on each side of the axis to enter the objective,

$$x_{min} = \frac{\lambda}{NA}$$

On the other hand, the light may be incident obliquely as in Fig. 70, and it is clear that the most favourable conditions under which the refracted beam and first diffracted beam can enter the objective is that they should lie symmetrically on each side of the normal.

In this case if the angle of the diffracted light is φ_1 as shown in the figure, the above equation becomes

$$x\,(n_1 \sin \varphi_1 + n_o \sin \theta_o) = \lambda$$

Note the "plus" because φ_1 is measured on the other side of the normal. But if θ_1 is the angle of refraction,

$$n_1 \sin \theta_1 = n_o \sin \theta_o$$

so that the diffraction equation is

$$x\,(n_1 \sin \varphi_1 + n_1 \sin \theta_1) = \lambda$$

and thus if $\theta_1 = \varphi_1$

$$x = \frac{\lambda}{2n_1 \sin \theta_1}$$

And if the objective can receive rays as oblique as θ_1

$$x_{min} = \frac{\lambda}{2NA} \text{ (for oblique light)*}$$

Now in the case of a full cone of illumination there will be some components of the light incident obliquely, and, if these secure resolution, the resolving power of the microscope will be given by the last formula, which closely agrees with that determined for a self-luminous object structure. On the other hand, if the illuminating beam is cut down to an extremely narrow angle, then, according to Abbe's principle, the resolving limit would be $x = \frac{\lambda}{NA}$, which agrees with experiment.

Investigation of the Intensity Distribution in the Image. We must now go more thoroughly into this mode of analysis of the image formation. Referring to Fig. 69, we have the case of a normally incident beam which is diffracted at an angle within the limits of numerical aperture of the objective which thus forms three maxima, one (D) due to the directly transmitted light, and one on each side of the centre (A and C) due to the first order diffraction. The grating is symmetrical with regard to the point B.

In Vol. I it was shown how to find the resultant of two vibrations of equal amplitude, but different phases. If one phase angle is $+\,a$ and the other is $-\,a$, the phase angle of the resultant will be zero. Arguing on this simple basis we can see that if the grating is symmetrical with regard to the axial point B, and symmetrically illuminated, the phases of the disturbances sent in any chosen

* If the numerical aperture of the condenser beam cannot exceed a value NA_c, which is less than NA_o the numerical aperture of the objective, the optimum condition then is $x_{min} = \frac{\lambda}{(NA_c + NA_o)}$.

direction from grating elements on one side of the axis will have positive phase angles, while those from the other side will be numerically equal but negative. Hence, *the phase of the resultant disturbance in any direction will agree with that from* B.

Further, we know that if the objective is free from spherical aberrations, any disturbances originating from B will meet without phase differences in the conjugate image point B′, and they will, therefore, be in the same phase in a spherical surface ADC struck with B′ as centre. Therefore the diffraction maxima must have the same phase in such a spherical reference surface.*

Let AD = DC = y, and let the distance from the point D to the image point B′ be l'. Let P be a point (near the axis) in a plane through B′ perpendicular to the axis, and let PB′ = h'. Then if AC is small in comparison with l', the disturbance from A arriving at the point P will *lead* in phase by

$$\frac{2\pi}{\lambda}\left(\frac{yh'}{l'}\right)$$

while that from C will *lag* by an equal amount. Let us first suppose that the grating apertures are all indefinitely small; then the direct and diffracted maxima all tend to have the same amplitude which we will write, a.

Hence the resultant of A and C in the image plane will be an amplitude of

$$2a\cos\left(\frac{2\pi yh'}{\lambda l'}\right)$$

and adding the effect of D (same phase) we get the result

$$S = a\left\{1 + 2\cos\left(\frac{2\pi yh'}{\lambda l'}\right)\right\} \quad . \quad . \quad . \quad . \quad (a)$$

This expression represents a curve (see Fig. 71) with positive maxima of value $3a$ for the values of $\left(\frac{\pi yh'}{\lambda l'}\right)$ equal to 0, π, 2π, etc., and negative maxima $= a$ for $\frac{\pi}{2}$, $3\frac{\pi}{2}$, etc. The intensities will, therefore, be proportional to $9a^2$ and a^2 respectively.

* Although their amplitudes may have positive or negative signs. In the case where there is a grating aperture on the axis of symmetry, and the apertures are all very small, the amplitudes of the maxima will have the same sign; if the grating is moved so that a "bar" is on the axis the first lateral maximum will be opposite in sign to the central one.

It can easily be shown that the separation of the main maxima given by

$$\frac{\pi y h'}{\lambda l'} = \pi$$

or

$$h' = \frac{\lambda l'}{y}$$

corresponds to the separation of the dioptric images of the grating apertures which would be formed if the latter were self-luminous. The angle of diffraction for the first order diffracted beam is given by

$$n_1 x \sin \theta_1 = \lambda$$

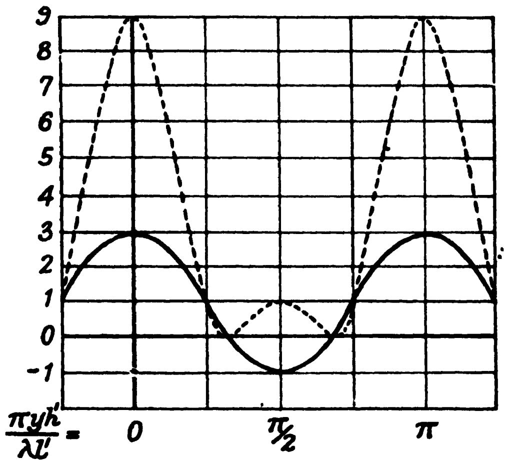


FIG. 71. AMPLITUDE AND INTENSITY
Dotted curve: Relative intensity
Full curve: $1 + 2\cos(2\pi\, yh'/\lambda l')$ i.e. Amplitude

where x = the grating element separation and θ_1 is the angle of diffraction. Now referring to Fig. 69 we see that the angle made with the axis by the extreme ray reaching the axial point in the image plane will be $\left(\frac{y}{l'}\right)$. Hence, if h'_x is the image element corresponding to the dimension x in the object space, the optical sine relation gives

$$n_1 x \sin \theta_1 = \frac{h'_x y}{l'},$$

the image being in air. And hence $y = \frac{\lambda l}{h'_x}$, which gives $h' = h'_x$. It will be noticed that if further maxima are taken up by the objective, the resultant becomes of the type

$$S = a\,(1 + 2\cos\varphi + 2\cos 2\varphi + 2\cos 3\varphi + \text{etc.}) \quad . \quad (\beta)$$

the number of terms depending on the number of maxima. This series approximates the more closely to a discontinuous succession of equal isolated very narrow maxima as the number of terms increases. The object was a grating of indefinitely narrow spaces. Hence we realize the importance of a wide angular aperture for the objective if the "image" is to bear the closest possible resemblance to the object. Fig. 72 shows the resultant of the first three terms

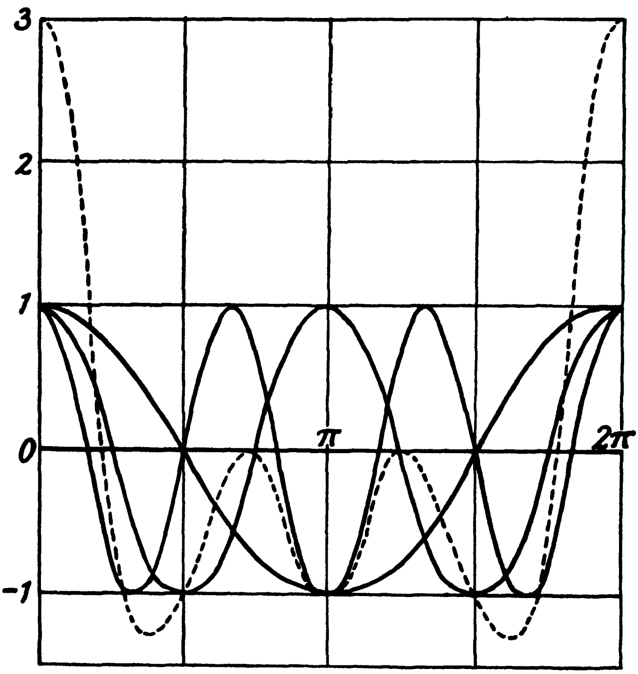


FIG. 72. FIRST THREE TERMS OF SERIES
$\cos\varphi + \cos 2\varphi + \cos 3\varphi +$ etc.
Resultant: dotted curve

of a cosine series which already has large values for $\varphi = 0, 2\pi, 4\pi$, etc., and much smaller values elsewhere; the addition of further terms will add to the effect.

The above discussion relates to a mere succession of indefinitely small apertures for the grating in the object plane, but if the spaces are widened it is found that the phases and relative intensities of the lateral maxima may change, and thus produce a distribution of light more closely representing the relative widths of dark and light spaces as the numerical aperture of the objective is increased and more diffraction maxima are allowed to contribute their part to the image. This action has been discussed by Conrady.[4]

Notice also that the blocking out of the central maximum in the above case (first order only admitted) would have brought the expression into the form

$$S = 2a \cos\left(\frac{2\pi yh'}{\lambda l'}\right)$$

in which the maxima are all of the same brightness, and have twice the frequency of the main maxima in the case of equation (a) above. Therefore if one observes the image plane while the central maximum is screened, the spacing of the bright maxima appears to be halved. A similar effect would be noticed if the first and second order images were acting at first, and then the first order is screened off.

Abbe himself performed many experiments of this type, and thereby somewhat alarmed microscopists, who felt that the conditions employed by him, viz. the use of a very narrow-angled illuminating cone derived from a small aperture in the condenser diaphragm might easily lead to artificial and erroneous results.

While there is nothing invalid in the theory so far as it goes, the real difficulty in applying it to practical cases is in integrating the effects for the beams at varying obliquities which must be assumed present in illumination by condensers of finite aperture. Closer discussion shows that while interferences from one set of related maxima could be observed in any plane where the beams overlap, the interferences from various sets will only agree in the focal plane. Thus the use of illumination of increasing angular aperture gives increasingly the characteristics of an ordinary image formation.

New Method of Analysis. We may now consider the second mode of analysis in which the source of light is focused in the grating, assumed to be a row of indefinitely narrow apertures as before.

Fig. 73 gives a diagrammatic representation of the conditions. The source of light is considered to be a point source in the first place; it will ultimately be necessary to integrate the results to find the effect of a continuous source. Assuming the condenser to be a well corrected optical system of not too great aperture, it may be assumed that the distribution of light in the focus of the condenser is of the "Airy Disc" type in the case of a circular aperture; the case of a circular aperture is, however, too difficult mathematically, and we are forced to take the simpler case of a rectangular aperture, in which the distribution of the light in the image follows a simpler law. By assuming this aperture very small in one direction, the discussion can be limited, in a well-understood manner, to two dimensions, neglecting the spreading in three dimensions which must occur in practice. In this case, the illumination in the object plane due to the condenser will be represented by an expression of the form (see Fig. 74 and Appendix, page 278)

$$\text{Amplitude} = \frac{\sin U}{U}$$

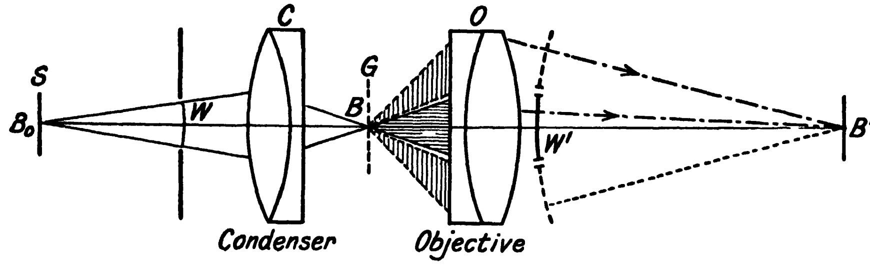


FIG. 73. ILLUMINATION BY A SOURCE OF LIGHT FOCUSED IN THE OBJECT PLANE

It must be carefully remembered that these vibrations in the object plane G are practically in the same phase,[5] and that although the amplitudes in the successive maxima alternate in sign, there

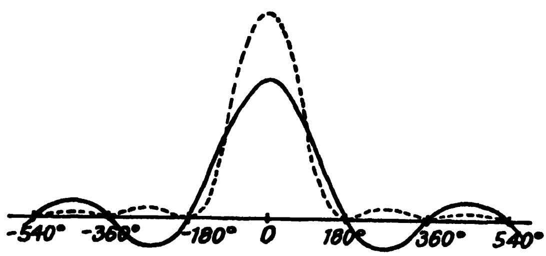


FIG. 74

Full line: amplitude $\frac{\sin u}{u}$ relative

Dotted line: intensity $\frac{\sin^2 u}{u^2}$ relative

is no continuous change of the phase from that of one maximum to the next. It is shown in the Appendix that the phase of the resultant vibration in the diffraction disc must be identical with that of the disturbance derived from the mid-element of the aperture, and therefore the actual phase change in the plane of G is negligible near the axis, in fact, over the range of many maxima and minima of the diffraction pattern.

Note that the first lateral zero value of the amplitude occurs when the difference of phase between the disturbances arriving in the "illumination image" from the extremities of the condenser aperture is equal to 2π, and the linear distance of this from the central maximum will be given by

$$h = \frac{0{\cdot}5\lambda}{NA_c}$$

where NA_c is the numerical aperture of the condenser.

Let the row of apertures forming the "object," and situated in the plane G, be situated symmetrically about the central maximum of illumination. The relative amplitudes of the vibrations occurring in successive apertures may then be given by such terms as

$$\frac{\sin u}{u}, \frac{\sin 3u}{3u}, \frac{\sin 5u}{5u}, \text{ etc.}$$

The final resultant of the light diffracted into any given direction will be found by

$$A^2 = [\Sigma\{a \sin \delta + a \sin (-\delta)\}]^2 + [\Sigma\{a \cos \delta + a \cos (-\delta)\}]^2$$

since there will be equal lag and lead in the phases of elements on each side of the centre. Hence

$$A = \Sigma\, 2a \cos \delta$$

Let x be the grating interval, then the first pair of apertures contribute an amplitude, in the direction making an angle θ with the axis, given by

$$2a_1 \cos \delta = 2\left(\frac{\sin u}{u}\right) \cos\left(\frac{\pi}{\lambda} x \sin \theta\right)$$

The second pair give

$$2\left(\frac{\sin 3u}{3u}\right) \cos\left(\frac{\pi}{\lambda} 3x \sin \theta\right)$$

The whole effect of the grating is thus

$$\text{Amplitude} = 2\left\{\frac{\sin u}{u} \cos\left(\frac{\pi}{\lambda} x \sin \theta\right) + \frac{\sin 3u}{3u} \cos\left(\frac{3\pi}{\lambda} x \sin \theta\right) + \text{etc.}\right\}$$

We have implicitly adopted the conception usual in the discussion of Fraunhofer diffraction phenomena, i.e. that the effects are realized at infinity, or in the focal plane of some lens which brings the disturbances together without further relative changes of phase. Referring again to Fig. 73, notice that the phase of the resultant of each pair of apertures, symmetrically situated about the centre, will be identical with that of an imaginary disturbance *starting* from the centre; all vibrations being in the same phase in the object plane.

But the phases of disturbances starting from the centre point B will be identical at B′, and therefore in any surface concentric with B′. Hence the phases of the resultants will be identical in such a surface as W′ of Fig. 73, and the contributions of the various pairs may be directly added together. Note that the reference surface must lie close to the principal focal surface of the lens O, in order that the above summation may be considered to be effective.

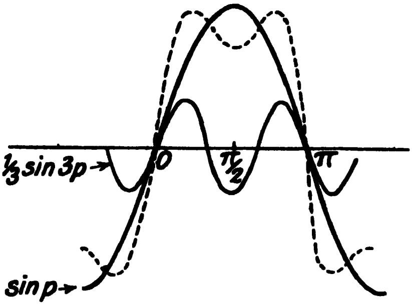


FIG. 75(*a*). SHOWS SUPERPOSITION OF FIRST TWO TERMS IN THE SERIES $\sin p + \frac{1}{3}\sin 3p + \frac{1}{5}\sin 5p + \text{etc.}$

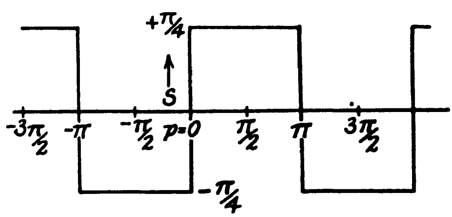


FIG. 75(*b*). FOURIER SINE SERIES RESULTANT

The number of terms to be included clearly depends only on the number of apertures in the grating free to transmit light.

Writing $v = \left(\frac{\pi}{\lambda}\right)x \sin \theta$ we obtain the series in the form,

Amplitude

$$= \frac{-}{u}\left\{\sin u \cos v + \tfrac{1}{3}\sin 3u \cos 3v + \tfrac{1}{5}\sin 5u \cos 5v + \text{etc.}\right\}$$

$$= \frac{1}{u}\left\{(\sin \overline{u+v} + \tfrac{1}{3}\sin 3\,\overline{u+v} + \text{etc.}) + (\sin \overline{u-v} + \tfrac{1}{3}\sin 3\,\overline{u-v} + \text{etc.})\right\}$$

These series represent well-known "Fourier expansions." If we plot graphically a series of terms such as

$$\sin p + \tfrac{1}{3}\sin 3p + \tfrac{1}{5}\sin 5p + \tfrac{1}{7}\sin 7p + \text{etc. to infinity}$$

we obtain a result suggested in Fig. 75(*a*) and Fig. 75(*b*). It will be seen that the tendency of the addition of successive terms is to give a value which is stationary $\left(\text{actually it is } \frac{\pi}{4}\right)$ from $p = 0$ to $p = \pi$, and then again stationary with negative value from $p = \pi$ to $p = 2\pi$, and so on. In Fig. 75(*a*) the effect of the addition of the first two terms is seen; the student should draw the figure himself, and investigate the addition of the term $\frac{1}{5}\sin 5p$.

The effect of the *two* series in the equation above may then be easily found graphically in particular cases. Take, for example, the case when $u = \frac{\pi}{4}$. We then have the two series as represented in Fig. 76. The sum of the two is seen to represent regions of amplitude 2, alternatively positive and negative, separated by regions of no disturbance.

The central band of amplitude 2 extends from $v = -\frac{\pi}{4}$ to $+\frac{\pi}{4}$, but $v = \left(\frac{\pi}{\lambda}\right)x \sin\theta$ where λ is the wave-length of light in the object space. The equation becomes more general by introducing n, the refractive index of the object space, and λ_o, the wave-length of light in air. The equation thus modified is

$$v = \frac{\pi x\, n \sin\theta}{\lambda_o} = \frac{\pi x}{\lambda_o}(NA)_\theta$$

where $(NA)_\theta$ is the "numerical aperture" of the ray direction given by θ. Now it is easily shown that having given a condenser of numerical aperture NA_c, the difference of optical path with which disturbances from the extremes of the condenser meet in the focus of the condenser system at a lateral distance h from the central maximum is

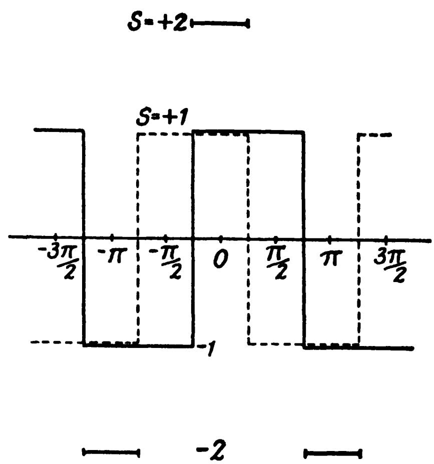


FIG. 76. EFFECT OF TWO SERIES ($u = \pi/4$)

$$\frac{2\pi}{\lambda_o} \cdot 2h \cdot NA_c = 2U$$

Hence using λ for the wave-length of light in air from this point onwards,

$$h = \frac{\lambda \cdot U}{2\pi\, NA_c}.$$

When the value of h is $\frac{x}{2}$, the distance of the innermost grating aperture from the central point, the corresponding value of U is u (say). Hence

$$\frac{x}{2} = \frac{\lambda \cdot u}{2\pi NA_c}$$

or

$$x = \frac{\lambda \cdot u}{\pi\, NA_c}$$

Hence the above expression for v becomes

$$v = \frac{u\,(NA)_\theta}{NA_c}$$

We saw that when $u = \frac{\pi}{4}$, we found v for the central maximum to extend to $\pm \frac{\pi}{4}$, showing then that $(NA)_\theta = \pm NA_c$, and that in this case, at any rate, the central maximum corresponds to the aperture of the objective which is "filled" by the condenser. This can be justified in the general case; it can also be shown that a similar distribution of the maxima arises, no matter what the relative positions of the grating apertures in regard to the illumination, and also even if the grating has finite apertures.

The calculation of the resultant effects in the final image plane of the microscope involves now, *first*, the summation of the effects of the central maximum and all the lateral diffraction maxima considered as derived from one elementary point of the source of light, and, *second*, the addition of such effects for all the elementary points in the source of light.

It will not be possible in the limited treatment necessary here to do more than indicate the procedure. In the case above, the intensity distribution in the reference surface showed a central maximum of amplitude + 2, and lateral maxima with amplitudes alternatively − 2 and + 2. Let us suppose that the aperture of the objective is only wide enough to focus the central and two lateral maxima. We can then find the resultant effect in our case (which case is limited to the elementary axial source of light).

The central maximum forms its "iso-phasal" image in a plane perpendicular to the axis, and the distribution of amplitude is represented by

$$\frac{\sin W}{W}$$

where 2W is the difference of phase of the disturbances arriving, at the point in the image plane, from the extremities of the central maximum. If Y is the diameter of the central maximum, f' is the focusing distance, and h' the distance of the point in the image plane from the axis, then the path difference δ corresponding to 2W is

$$\delta = \frac{Yh'}{f'}$$

and the phase difference 2W is

$$2W = \frac{2\pi}{\lambda} \cdot \frac{Yh'}{f'}$$

i.e.

$$W = \frac{\pi}{\lambda} \frac{Yh'}{f'}$$

Consider now one of the lateral maxima in the above case $\left(u = \frac{\pi}{4}\right)$. Its centre lies at a distance from the axis equal to 2Y, where Y is the diameter of the central maximum. It may be regarded as producing a distribution of the above type in a plane whose normal makes an angle $y = \frac{2Y}{f'}$ with the axis (see Fig. 77).

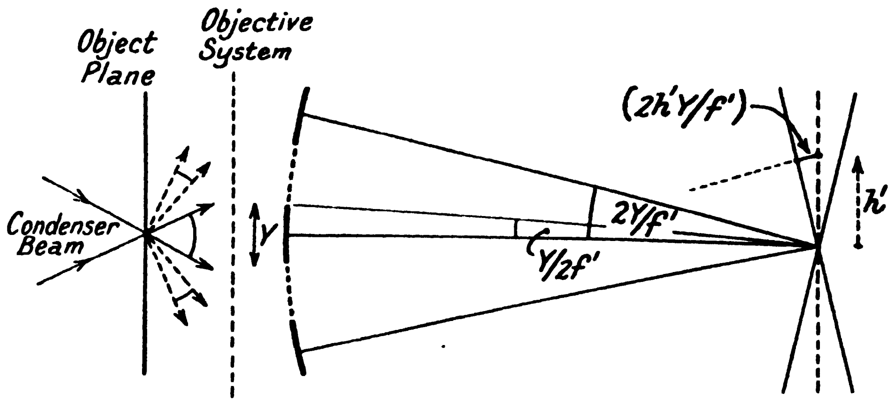


FIG. 77

If the aperture is small there will be considerable depth of focus, so that we may consider the lateral maxima to produce a $\frac{(\sin W)}{W}$ amplitude distribution in the normal plane, but with a progressive relative phase change. This phase change φ at a distance h' from the axis will be

$$\varphi = \frac{2\pi}{\lambda} h' \cdot \frac{2Y}{f'}$$

Taking account of the first order maxima on the two sides of the central one, and calling the amplitude at any chosen point due to the central maximum a_0, while the amplitude due to the lateral maximum is a_1, the resultant will be

$$A^2 = \{a_1 \sin \varphi + a_1 \sin(-\varphi)\}^2 + \{a_0 + a_1 \cos \varphi + a_1 \cos(-\varphi)\}^2$$

i.e. $$A = a_0 + 2a_1 \cos \varphi$$

Hence we find the expression for the amplitude to be

Amplitude

$$= \frac{\sin W}{W} - 2\frac{\sin W}{W}\cos\varphi = \frac{\sin W}{W}\left\{1 - 2\cos\left(\frac{2\pi}{\lambda} \cdot h' \cdot \frac{2Y}{f'}\right)\right\}$$

Putting in the value of h', i.e.

$$h' = \frac{W\lambda f'}{\pi Y}$$

the amplitude is found to be

$$\frac{\sin W}{W}(1 - 2\cos 4W)$$

This function is plotted in Fig. 78, the dotted curve shows the variation of $\frac{\sin W}{W}$. It will be seen that the separation of the main intensity maxima is given by a difference of W values of 90° or $\frac{\pi}{2}$. The corresponding h' value is therefore

$$h' = \frac{\left(\frac{\pi}{2}\right)\lambda f'}{\pi Y} = \frac{\lambda f'}{2Y}$$

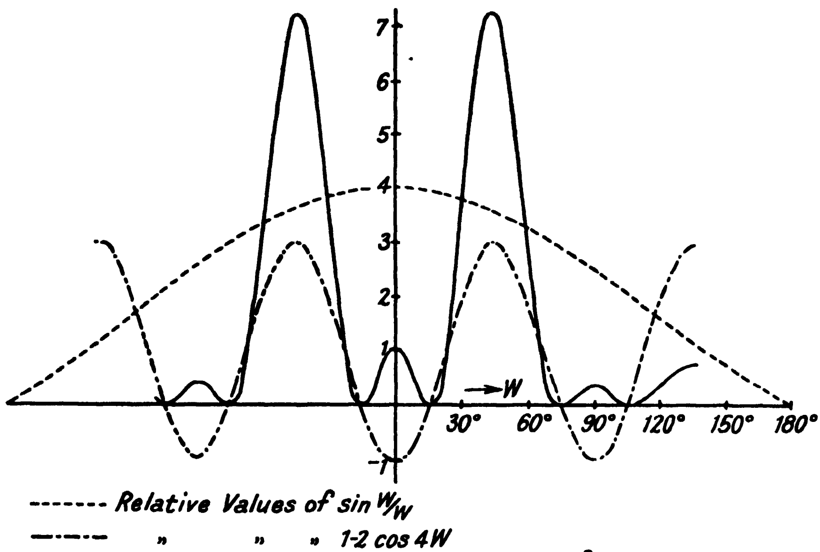


FIG. 78

We shall see that this corresponds to the separation of the spaces in the object plane, for we had the separation of the apertures x given by

$$x = \frac{\lambda u}{\pi NA_c} = \frac{\lambda\left(\frac{\pi}{4}\right)}{\pi NA_c} = \frac{\lambda}{4NA_c}$$

in our case.

Now we may use the optical sine relation to derive the magnification. The "numerical aperture" of the extreme ray from the condenser, NA_c, corresponds to that of the corresponding ray imagined to pass from the boundary of the central diffraction maximum in the reference surface to the focal point. The sine of the angular inclination of this latter ray is $\frac{Y}{2f'}$, where Y is the width of the maximum (see Fig. 77). Hence the sine relation gives

$$x\,NA_c = \frac{h'Y}{2f'}$$

Since from above

$$x\,NA_c = \frac{\lambda}{4}$$

we get

$$h' = \frac{\lambda f'}{2Y}$$

Hence the main intensity maxima in the figure represent the "images" of the apertures in the grating so far as they can be rendered by a very small source of light. If, now, the aperture of the objective could be extended we should obtain the effects of higher order diffraction "spectra," and, provided that the above treatment can be taken as more or less valid, the amplitude would become

$$\frac{\sin W}{W}(1 - 2\cos 4W + 2\cos 8W - \text{etc.})$$

the number of terms in the bracket depending on the number of spectra admitted by the objective. The above series can be written

$$\frac{\text{Sin } W}{W}\left\{1 + 2\cos 4\left(W + \frac{\pi}{4}\right) + 2\cos 8\left(W + \frac{\pi}{4}\right)\right.$$
$$\left. + 2\cos 12\left(W + \frac{\pi}{4}\right) + \text{etc.}\right.$$

It was pointed out above that if we take a series of the general form

$$\cos\theta + \cos 2\theta + \cos 3\theta + \text{etc.}$$

and plot the cosine curves and add the amplitudes, we find that by taking a great number of terms the ordinates of the resulting curve get very large at the values of 0, 2π, 4π, etc., for θ. Hence, in the above series, we shall get finite values for

$$4\left(W + \frac{\pi}{4}\right) = 0,\ 2\pi,\ 4\pi,\ \text{etc.},$$

i.e.

$$W = -\frac{\pi}{4},\ +\frac{\pi}{4},\ +\frac{3\pi}{4},\ \text{etc.}$$

Hence the maxima will represent the grating apertures both in spacing and relative position, but the maxima will only be illuminated in proportion to their illumination by the "illumination image" formed in the object plane from the elementary point source of light.

Such a discussion has to be completed by extending the argument to take account of the case of an assembly of such point sources of illumination; this has been done in a paper[3] by the present writer. It appears that the effective result will be to illuminate more and more of the maxima in the final image, but the intensity maxima in the reference surface will retain the same locations and relative intensities.

Summary and Conclusions. In the Abbe method, the final image is regarded as due to the superposition of an indefinite number of "secondary" interference systems derived from sets of related maxima. No account is taken of any phase relations between such sets, which are found at the so-called "homologous points" of the observed diffraction maxima of finite extent.

At first sight this view appears to contradict the other discussion in which the broad maxima appear as regions of uniform phase, but this, it will be noted, was considering only one elementary point in the source of illumination. These apparent differences arise only in the method of analysis, and have nothing to do with the physical nature of the action. The great mistake very many theorists have made was in attributing an undue physical significance to the details of the theory. Properly understood, however, each method of analysis has something to teach us, and we may proceed to draw a few conclusions.

1. *The objective must be well corrected.* The necessity of equal optical paths for various routes between corresponding object and image points is clearly brought out by the second mode of analysis. If the paths are unequal, the phases of the maxima in the spherical reference surface for any image point will vary, and the concentration of light in the image will deteriorate. If, however, a poorly corrected lens is used, the "resolution" obtained with narrow aperture pencils may not seem markedly inferior to that of a good lens; but such resolution largely depends on secondary interference phenomena, such as were discussed in connection with the Abbe principle.

2. *Where the object consists mainly of opaque and transparent portions the fullest aperture illuminating cones must be employed.* The point to point correspondence of the illumination of object and image plane depends largely on the concentration represented by

the $\frac{\sin W}{W}$ term in the expression above (page 114). This concentration is improved by having the largest possible aperture both for objective and illumination. The limit is set by considerations of the increase of aberration with aperture and the effects of "glare." Unless the objective is of extremely good quality, it will usually be found that the optimum results are secured with the condenser aperture about two-thirds to three-quarters that of the objective.

3. *Where evidence of the finest detectable regular structures in the object is sought for, it is advisable to use an oblique beam of narrow aperture to illuminate the object.*

Under such conditions the appearance in the image plane is largely of the nature of an interference phenomenon, and the one to one correspondence of object and image is spoilt in great measure, only the coarser features of the former being rendered by the limited angular aperture of the illuminating beam. If, however, there are regular structures present in the object, even periodic variations of refractive index, these latter have their best chance to produce an intensity variation of proportionate frequency in the image plane. The location of the interferences in the "picture" will roughly correspond to that of the corresponding structure in the object, but we must regard any element of the interference pattern as due to the whole of that structure, and abandon the idea of point to point correspondence.

Abbe pointed out that under the conditions of oblique lighting by narrow pencils, used with an object consisting of a layer containing media of varying refractive index, the optical path differences arising between disturbances traversing any neighbouring parts of the structure may vary with the inclination of the light. Hence the relative phases of the various diffraction maxima would be likely to show a corresponding variation, and this would be likely to result in a weakening of the interference phenomenon if wide-angled cones were used for illumination. It is a perfectly legitimate conclusion that we are more likely to find evidence of delicate and fine structures when using narrow cones of illumination, but the appearances in such a case are mainly secondary interferences and are not to be interpreted in any other way.

Let us, for example, consider the theoretical aspect of the observation of a typical microscopic object consisting of a regular structure so fine that, when the condenser aperture is very small, the first order diffraction maxima lie entirely outside the limits of the objective aperture. Fig. 79 suggests the appearance within the ring of the objective O. D indicates a direct beam, and D_1 a first order diffracted beam; as we know, the angular apertures of these correspond to the aperture of the condenser. As this aperture is increased, the first order maxima begin to appear, and the homologous points of the Abbe principle may be imagined now to begin to set up their interferences. The conditions should, at first sight, improve till the condenser aperture is equal to that of the objective, but this is dependent on the avoidance of relative phase changes between such pairs of homologous points, a matter clearly dependent on the nature of the object. In the majority of cases the resolution begins again to be lost as the aperture of illumination is yet further increased; this is partly due to glare, and partly due to the loss of the secondary interferences owing to the above reason.

The above case is well illustrated by the familiar diatom, *amphipleure pellucida*, which is "resolved" by a first-class microscope

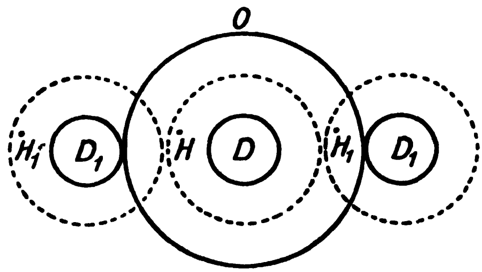


FIG. 79

objective of NA at least 1·2 when working with a condenser giving a symmetrical cone of illumination of NA 0·9 or thereabouts, but not with a smaller aperture of illumination. This resolves the structure of 10,000 lines to the inch, but the (only slightly finer) lines perpendicular to the first can only be "resolved" by the use of oblique light, presumably because the variations of optical path involved in the second structure are much smaller, and the diffraction maxima are much weaker relatively to the direct light. Although the frequency of the structures is thus known, the ultimate form of the siliceous frustules of the diatom is quite unknown.

It seems fair to conclude that whereas the majority of fine-resolution effects with the microscope are obtained with apertures of the illumination considerably smaller than the objective, the finer detail of the image of regular structures must be usually ascribed to "secondary" interference effects. When, however, the conditions are such that the aperture of the condenser can be increased to equality with the objective, and when we are dealing with an object consisting merely of variations of opacity in one plane rather than of refractive index in a thin layer, then we may regard the image formation as equivalent to that of a self-luminous object.

4. *There are thus two ways of regarding and using the microscope. Firstly as a camera, secondly as a kind of interferometer.* In the first case the aim is to obtain a "picture" of the object in the conventional sense, with a "one to one" correspondence between details. A precise meaning can only be attached to the picture when the object consists of a plane containing transparent and, more or less, opaque layers with definite geometrical distribution. When the object is three-dimensional or consists of variations of refractive index between two plane boundaries, the one to one correspondence loses much of its meaning. The effort to interpret the picture in a conventional sense becomes increasingly difficult, and the "image" is best regarded as giving "evidence" of structure (and that only when the illumination has been taken into account) rather than as a picture at all.

The use of these very narrow aperture bundles with a microscope objective suffering from spherical aberration illustrates some of the views expressed above. The presence of a moderate amount of spherical aberration does not preclude the formation of the diffraction maxima and the interference phenomena, but the latter may often be best seen in a different focus from that in which the general features of the object seem to be best revealed.

Dark-ground Illumination. Given a condenser having a greater numerical aperture than the objective of a microscope, a stop may be placed in the condenser diaphragm (Fig. 80), so that the illuminating beam is highly oblique and no "direct" light enters the objective after passing through the object plane, provided that we only have homogeneous media present in the slide, object layer, and cover glass, etc. If, however, the object layer contains bodies which are capable of refracting, scattering, or diffracting the light, a proportion of this deflected light may enter the objective.

The calculation of the amount so deflected is usually very difficult, and the problem only becomes solvable in special cases such as—

(*a*) *Regular structures, or apertures, gratings, etc.* The relative intensities of light diffracted in various directions may be estimated in simple cases.

(*b*) *Spherical globules or other bodies of regular shape present in the object layer.* The laws of reflection and refraction allow the distribution of the emergent light to be calculated if the bodies are large enough.

(*c*) Very small particles or filaments of material having a refractive index differing from the surrounding medium.

In this case the relative amount of light *scattered* into different directions by particles of various sizes can be calculated. A few notes may be helpful in this connection. Referring to Fig. 81, L represents a beam of light, and P a scattering particle. In the theory due to the late Lord Rayleigh,[6] the particle "loads the ether" and

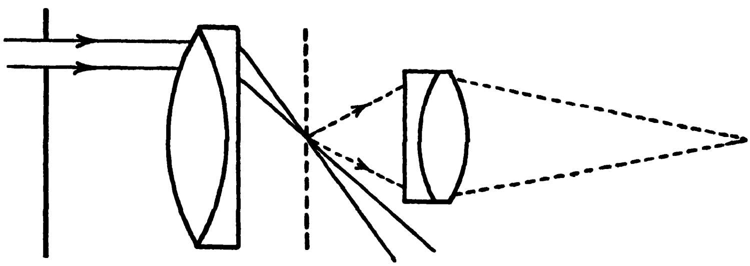


FIG. 80. SIMPLEST FORM OF DARK-GROUND ILLUMINATOR

becomes the seat of disturbances propagated in all surrounding directions. Fig. 81 is a perspective view. The electric forces in the wave-front are perpendicular to the direction of propagation, and may be resolved into horizontal and vertical components H and V.

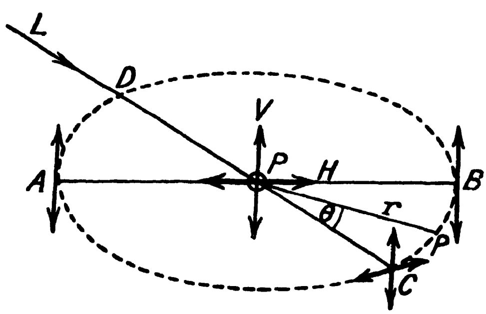


FIG. 81

Both H and V produce equal effects at C in the line of propagation or at D (backwards), but only V can act at A or B in the direction perpendicular thereto.

Rayleigh's formula for the intensity of the scattered light in a direction making an angle θ with the incident ray is, if the incident light is unpolarized,

$$A^2 \frac{(D' - D)^2}{D^2} (1 + \cos^2 \theta)\, m\pi \frac{T^2}{\lambda^4 r^2}$$

where r is the distance of the particle from the point of observation, λ is the wave-length, T is the volume of one particle, m is the number of particles, D′ and D are the densities of the material of the particles and the medium respectively, and A^2 is the intensity of the incident light. Thus the intensity in the forward or backward direction will be double that at right angles to the incident light.

It is easy to understand that the light scattered in a direction perpendicular to the incident beam is plane polarized.

With very small particles, the amount of light scattered is inversely proportional to the fourth power of the wave-length, accounting for

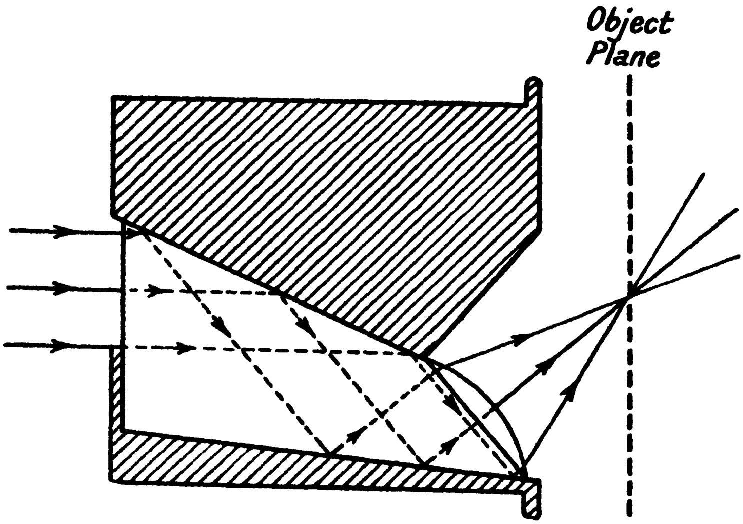


FIG. 82. NACHET'S PRISM

the "blue" of fresh smoke, etc. With coarser particles the light becomes white to ordinary observation. The conditions of transition between the pure scattering effects and the refraction effects characteristic of larger bodies have not been fully explored, and this lack of knowledge causes much difficulty in the interpretation of phenomena observed when using dark-ground illumination.

Optical Systems. One of the earliest inventions for one-sided, dark-ground illumination was Nachet's prism[7]; Fig. 82 will be self-explanatory. Referring now to Fig. 83, it will be seen how the use of a central dark stop in the diaphragm of the condenser produces annular illumination, with a great gain in intensity as compared with the arrangement of Fig. 80. This method is often used with low power objectives.

Another device was, however, described by Wenham[8] in 1850, which was the forerunner of modern reflecting dark-ground illuminators. Fig. 84 shows a section of the paraboloidal reflector; the top of the parabola is absent, so that the rays concentrated through the focus can proceed onwards. Wenham compensated for the spherical aberration arising in the microscope slide by the provision

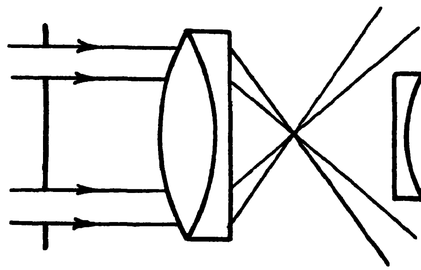


FIG. 83. USE OF ANNULAR STOP

of a meniscus lens as shown in the diagram. In later reflectors a solid truncated paraboloid was made in glass, and the top is oiled on to the slide, thus escaping the spherical aberration* arising by

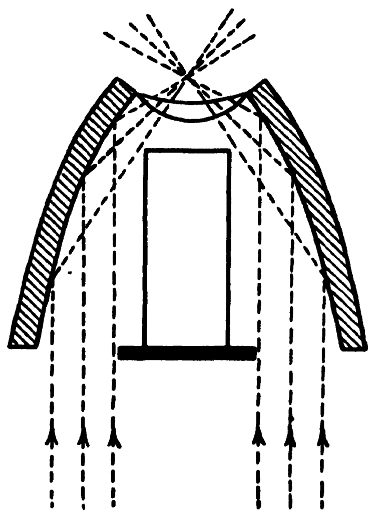


FIG. 84. WENHAM'S PARABOLOID ILLUMINATOR

oblique transmission at a refracting surface, T, Fig. 85. Such paraboloids are still made and used.

The difficulty of manufacturing paraboloidal surfaces of sufficient accuracy led to the invention by Ignatowsky[9] and Siedentopf[10] of double mirror reflecting condensers.

Theory of Cardioid Condenser. To understand their action, we recall the formula expressing the relation between the length of the

* Vol. I of this book, p. 13.

radius vector at any point on a polar curve, and the angle between the radius vector and the normal.

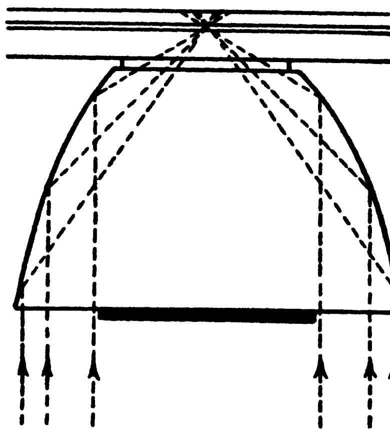


FIG. 85. MODERN PARABOLOID

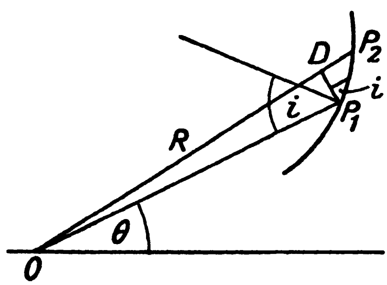


FIG. 86

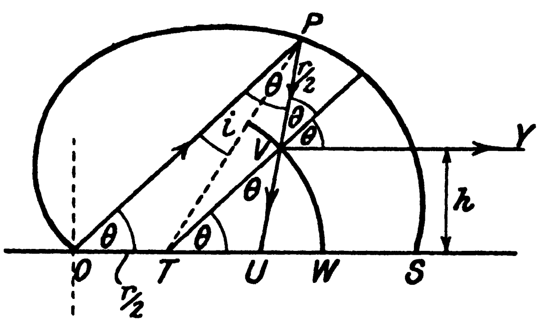


FIG. 87. THE CARDIOID

The equation to the curve is $f(R, \theta) = 0$
and the angle is i.

Joining the origin O to two indefinitely close points P_1P_2 on the curve (Fig. 86) and dropping the perpendicular P_1D to OP_2, we find

$$\tan i = \frac{DP_2}{DP_1} = \frac{\delta R}{R\delta\theta}$$

Hence in the limit,

$$\tan i = \frac{1}{R}\left(\frac{dR}{d\theta}\right)$$

For the cardioid OPS (Fig. 87), whose equation is

$$R = r(1 + \cos \theta),$$

where $r = \frac{OS}{2}$, we obtain

$$\frac{dR}{d\theta} = - r \sin \theta$$

and

$$\tan i = \frac{- r \sin \theta}{r(1 + \cos \theta)} = - \tan\left(\frac{\theta}{2}\right)$$

Hence if a ray started from O and were reflected at the cardioid at P, it would travel towards a point U on the line OS, for which OUP is an isosceles triangle. Take a point T at a distance $\frac{r}{2}$ from O and draw TV cutting PU in V, so that $\widehat{VTU} = \theta$. Then with T as centre and TV as radius, draw the circular arc VW; it is evident that if the ray PV were reflected at this arc, the final direction must be VY parallel to the line OS, since the incident and reflected rays then make equal angles with the normal to the surface. To show that the same spherical *locus* produces the same final direction, parallel to OS, for all rays, it must be shown that TV is independent of θ.

Now $TV = 2TU \cos \theta$, and $TU = \frac{OP}{2 \cos \theta} - \frac{r}{2} = \frac{r(1 + \cos \theta)}{2 \cos \theta} - \frac{r}{2}$

$$= \frac{r}{2 \cos \theta}$$

Hence

$$TV = \frac{2r \cos \theta}{2 \cos \theta} = r$$

If h = distance of ray VY from the axis OS, then evidently

$$\frac{h}{r} = \sin \theta$$

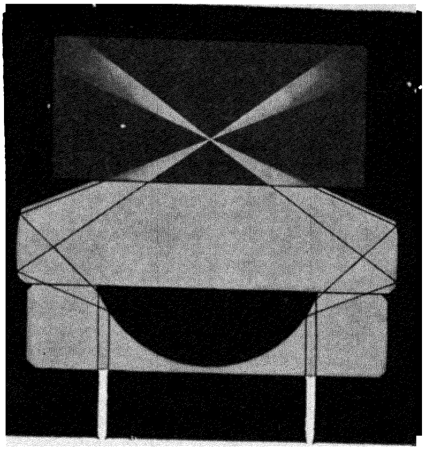
or

$$\frac{h}{\sin \theta} = \text{const.}$$

It, therefore, appears that if we send a parallel beam (from right to left in the diagram) so that reflection takes place first at the sphere and then at the cardioid, we shall obtain an aplanatic refraction free from spherical aberration and coma, so that a distant source of light of reasonably small angular size will be sharply imaged in the plane, perpendicular to the axis, through O.

Fig. 88 shows the course of the rays through a cardioid condenser as made by Zeiss, and a plate of uranium glass placed above it. The rays meet the "spherical" surface first. In practice the second cardioid surface, being only a comparatively narrow one, can be represented sufficiently well in practice by a ring of the approximating spherical surface. The range of numerical aperture in the illuminating cone is about 1·2 to 1·33. If no direct light is to enter the objective the *NA* of the latter must be below about 1·05. Objectives of higher *NA* have to be stopped down.

The *NA* of a ray perpendicular to the axis is equal to n sin 90°, i.e. to n, the refractive index of the medium. Hence if a ray in glass has an *NA* of, say, 1·4, it cannot enter a layer of water perpendicular to the axis, but will be totally reflected at the boundary since the refractive index of water is 1·333. Hence the *NA* of 1·333 is the highest which can be used to illuminate an object in a watery medium.

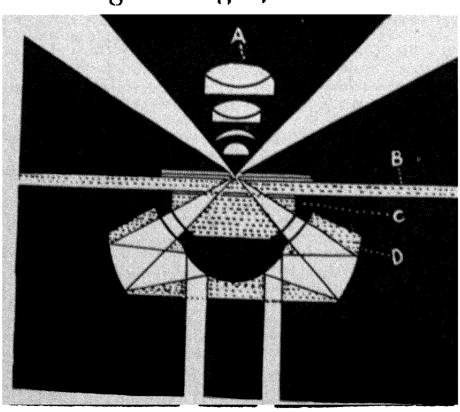


(*Carl Zeiss, Jena*)

FIG. 88. COURSE OF THE RAYS THROUGH A CARDIOID CONDENSER

If the object is immersed in a medium of higher refractive index the *NA* of the illuminating beam can be increased, thus allowing objectives of higher *NA* to be used without stopping down.

When light is totally reflected at a glass-air surface, say, it is well known that a certain amount of light energy does pass into a very narrow layer (of thickness comparable with the wave-length of light) of the air at the surface. Advantage is taken of this in illuminating so-called "smear" preparations of bacilli with such dark-ground illuminators, the organisms adhering to the totally reflecting surface being thus brilliantly illuminated.



(*R. & J. Beck, Ltd.*)

FIG. 89. FOCUSING DARK-GROUND ILLUMINATOR

A number of other types of illuminator have been produced. Fig. 89 shows the focusing dark-ground illuminator of Messrs. R. and J. Beck, Ltd., which allows control of the focusing point. With other types of illuminator it is necessary to choose an exact thickness for the microscope slide (usually about 1·2 mm.). The focusing type can be used with other thicknesses within reasonable limits.

The Slit Ultra-microscope. This arrangement, introduced by Siedentopf and Szigsmondy,[11] has been employed chiefly in the study of colloid particles in various media. The illumination of the medium is concentrated into the region of a sharply limited image of a bright slit, and only such particles as lie within such a region are visible. Fig. 90 illustrates the optical system. The image of the crater of the arc is projected by the lens *f* on the micrometer slit *g*. The second lens *h* then projects an image of *g* into a point at the proper working distance from the objective *i*. Thus we

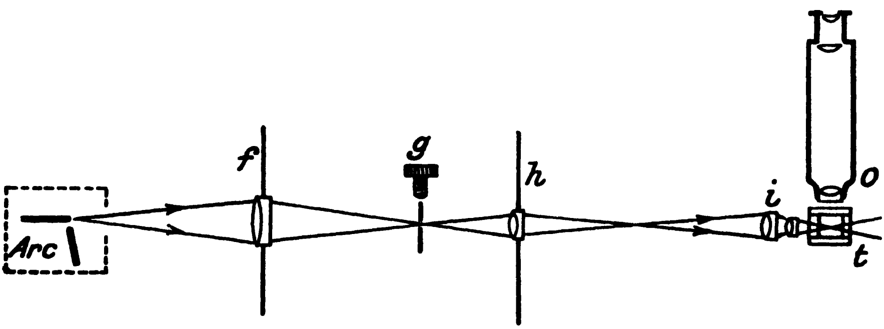


FIG. 90. THE ULTRA-MICROSCOPE
(Diagrammatic)

obtain in the object medium a doubly reduced image of *g*. If the particles to be studied are disposed in a liquid, this is usually contained in a tube of square section *t*. The observing objective O is usually corrected for water immersion. The particles are seen when the region of illumination is brought by adjustment of *i* into the proper position with regard to O. A polarizing prism may be introduced into the illuminating beam for special purposes.

General Remarks on the "Cardioid" and "Slit" Ultra-microscopes. In these ways it is possible to make visible any particles, filaments, or differentiated structures capable of scattering sufficient light into the objective. Small objects thus seen appear bright on a dark background. If the dimensions of such objects are small in comparison with the resolving limit of the objective, the size of the patch of light representing the image will be determined by the considerations of diffraction.

The method is employed for the study of colloidal particles of diameters down to 4×10^{-6} mm. (Siedentopf), and light has been thrown on the phenomena of pedesis (Brownian movements) and the absorption of light by colloids in liquids and glasses, etc., more especially by using the one-sided illumination of Siedentopf and Szigsmondy. The concentric dark-ground illuminator is capable of revealing fine structures, such as flagellae of micro-organisms, delicate spines, etc., which cannot be readily observed in other ways. The inner structure of bacteria is also shown up by dark-ground illumination where it is not detectable by ordinary transmitted light.

While this method has a great advantage in the extreme contrast produced in the image, it suffers from disadvantages in a loss of resolving power under certain conditions.

Resolving Power for Dark-ground Illumination. The first investigation of this subject is due to M. J. Cross.[12] He applies the Abbe principle. Consider the illumination of a grating by an oblique beam of narrow angular aperture from the illuminator of a microscope. Let it be assumed then that in dark-ground illumination none of the "direct light" enters the objective, but that at least two diffraction maxima, say the first order and second order, must be formed at angles within the numerical aperture of the objective if the grating is to be "resolved." If the distribution of the diffracted beams is as represented in Fig. 91, the first order and second order being at angles φ_1 and φ_2, the equation for φ_2 is

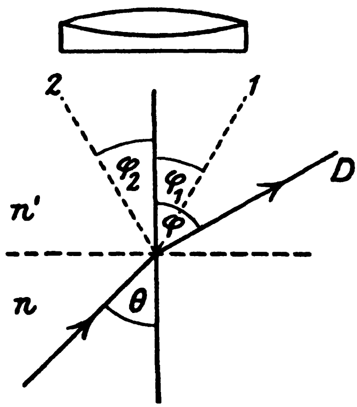


FIG. 91

$$n'x \sin \varphi_2 + nx \sin \theta = 2\lambda$$

where x is the grating space, θ is the angle of the incident beam, and n and n' are the refractive indices on the two sides of the grating. In order to understand the optimum result we may imagine a grating of variable spacing. By broadening the spaces the second order beam will swing towards the direct beam D, and we could alter the angle between beams 1 and 2. By altering the angle of illumination we could then place 1 and 2 at equal angles with both sides of the normal, as shown in the figure, and thus it could be arranged that the first and second order maxima fall just within the objective aperture. This represents the optimum conditions for the objective with dark-ground illumination. Note that if we *increase* or *diminish* the illumination aperture, the maxima may not be focused by the objective.

If a_i is the numerical aperture of the illuminating beam, and a_o is the numerical aperture of the objective, the above condition is

$$x_d\,(a_o + a_i) = 2\lambda$$

If the first order maximum falls just within the objective aperture on the other side, the condition is

$$x_d\,(a_i - a_o) = \lambda$$

Subtracting these equations we obtain

$$2x_d a_o = \lambda$$

or

$$x_d = \frac{\lambda}{2a_o}$$

so that if the optimum conditions are obtainable, the resolving power may become equal to that for bright-ground illumination; but on adding the equations we find that

$$2x_d a_i = 3\lambda$$

so that in order to get these optimum conditions we must have

$$a_i = 3a_o$$

Since with the majority of cases the aperture of the illuminating beam will not greatly exceed that of the objective, and there will be few cases in which the absolute optimum is realized, the theory indicates that x_d must be about double the spacing resolvable with bright ground. This indicates that the resolving power *may* be halved.

Experiments by Siedentopf[13] have shown that this expected result is realized in the case of gratings, but Beck[14] and various other observers working with objects such as diatoms have maintained that the resolution is not impaired. The reasons for the discrepancy in the observations is not yet adequately explained. It may be pointed out that the illumination of special objects is difficult to define very precisely. A considerable amount of light may be reflected back to the top surface of the object from a "total reflection" effect at the top of the cover glass. It should also be remembered that the elementary theory takes no account of the state of polarization of the light, which may be very significant in the diffraction effects.

Owing to the drastic variation of diffraction angles with colour, it is usually the case that "dark-ground" images show brilliant chromatic effects when a lamp giving white light is employed as the source of illumination.

Diffraction Images of Various Objects with Unilateral Dark-ground Illumination. When polarized light is used for unilateral dark-ground illumination, we may consider the intensity of light scattered into different directions by a small particle. The main necessary facts were considered above and are put into graphical form in Fig. 81. If the vibrations in the incident ray are perpendicular to

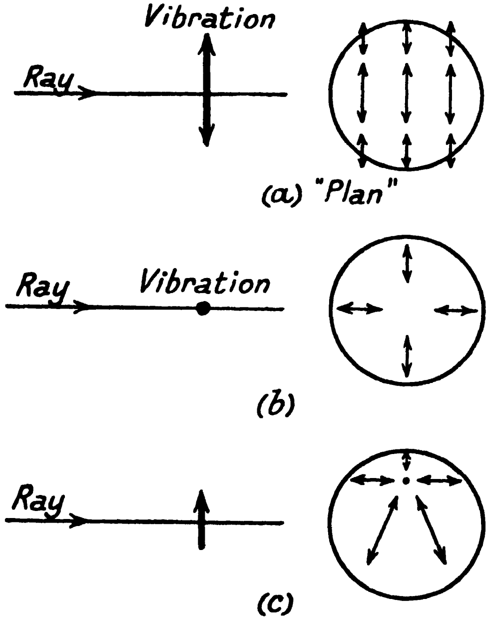


FIG. 92

the plane containing the ray and the microscope axis, the case will be represented by the "plan" diagram of Fig. 92 (*a*). The circle represents the aperture of the objective, and the amplitudes of the vibrations are greatest along one diameter. They are uniform along this line, but vary in any perpendicular direction as suggested by the length of the arrows in the figure.

When the vibrations of the polarized light are parallel to the axis of the microscope, the intensity is zero in the centre of the aperture and rises towards the margin (Fig. 92 (*b*)). When the vibrations are inclined, say, at 25° to the plane containing the ray and the microscope axis, the point of zero illumination is clearly displaced away from the centre of the aperture, and the intensity has an unsymmetrical distribution as suggested in Fig. 92 (*c*).

The resulting image disc in these cases will differ somewhat from the Airy disc. In case (*a*) the disc will be elongated somewhat in the direction of the vibrations. In (*b*) the surrounding rings will be of considerably greater intensity than in the normal Airy disc, and the image distribution has a black centre. In (*c*) there may be an unsymmetrical patch of light with a one-sided dark spot.

When unpolarized light is used for illumination, however, the resultant effect differs little from the usual Airy disc, merely showing a slight "astigmatism" due to the unilateral direction of the illumination. Even this disappears when symmetrical dark-ground illumination is used.

The above particular effects are entirely due to the wave-nature

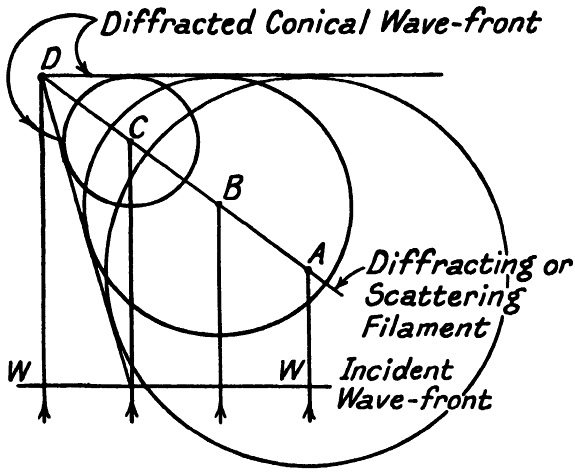


FIG. 93. DIFFRACTION BY A FILAMENT

of light, and must not be interpreted as indicating any structure in the ultra-microscopical particles.

Effect of a Linear Filament. Referring to Fig. 93, let WW be a plane wave-front of an incident parallel bundle of rays falling on a straight filament ABCD capable of scattering light. It is assumed that the elements of the filament act as scattering centres. By the time the wave reaches D, the secondary disturbances scattered from C, B, and A will have spread to proportionate distances. The envelope of the spherical surfaces, which we regard as the diffracted wave-front, is a cone of which the axis is the line DA. The directions of the diffracted *rays* will also be represented by a cone symmetrical round the filament; this cone contains the incident ray, and also that "reflected" ray lying in the plane of the incident ray and the filament. We may look on the incident ray as a generating line. If the filament is perpendicular to the incident ray there will be a cylindrical wave resulting from the diffraction.

Such diffracted wave-fronts will evidently tend to give extremely *astigmatic* images in the microscope; although the concentration of the image will tend to represent a narrow thread, there can be no genuine point to point representation.

Now imagine a straight filament in the "ultra-microscope" inclined at an angle β to the incident ray (see Fig. 94). The diffracted

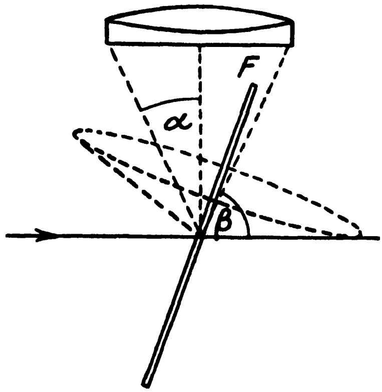


FIG. 94

light is limited in angular direction to the cone of which the incident ray is a generating line. In the case shown, no light will enter the objective of semi-angular aperture α unless $2\beta < (90 + \alpha)$ and

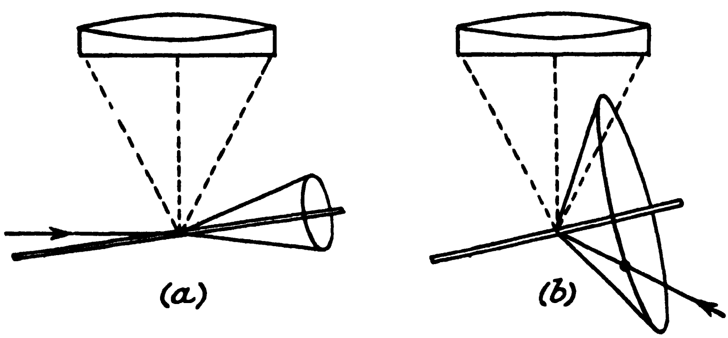


FIG. 95

$2\beta > (90° - \alpha.)$ The filament is assumed to be at the proper working distance for the objective and its size is greatly exaggerated in the figure. If we imagine the illuminating beam rotates round the axis of the microscope, assumed vertical, we may consider the case of a filament inclined at a *small* angle β to the horizontal. In Fig. 95 (a) the cone does not enter the cone of ray directions representing the aperture of the objective, but when the direction of the illuminating ray has been rotated round the vertical Fig. 95 (*b*) (or the filament rotated round a vertical axis) the diffraction cone may enter the objective. Thus there are ranges of filament directions in which no light can reach the objective.

Sometimes the object seen by dark-ground illumination consists of very delicate needle-like crystals which are in motion in the medium, flashing into view as the angles vary. Sometimes the object consists of a spiral thread; only those portions lying within a certain range of angular directions are visible, and the object may appear as a series of detached sections.

Microscopy with Ultra-violet Radiation. The expression for the resolving limit of a microscope objective

$$h = \frac{0{\cdot}5\lambda_0}{NA}$$

shows that when optical design has increased the angular aperture of the objective to the practicable limit, and has employed the practicable immersion media of the highest refractive indices, the only remaining way of increasing resolving power (diminishing h) is to employ radiation of shorter wave-length. The limit of transmission for stable glasses in the ultra-violet region is approximately $0{\cdot}3\mu$ (Vol. I, page 245). Optical media available for regions of shorter wave-length are practically limited to quartz (crystalline and fused), and fluorite. Of these, fused quartz and fluorite are both difficult to obtain in a perfectly homogeneous condition. Crystalline quartz has only limited uses in lens systems owing to its double refraction; it can be employed in eyepiece lenses, or even in the back components of microscope objectives, but not in lenses where a great divergence of angular directions must be allowed for the rays. Both fused quartz and fluorite have rather low refractive indices, and any attempt to make high aperture achromatic combinations from these materials would meet with the greatest difficulties. Present practice is mostly limited to the use of so-called "monochromat" objectives constructed entirely of fused quartz, and using radiation corresponding to a single apparent "line" of the spark spectrum of cadmium, generally $\lambda = 0{\cdot}2749\mu$. The construction of such a lens is shown in Fig. 96 on a greatly enlarged scale. It can be freed from spherical aberration and coma in a very satisfactory way. Projection eyepieces of crystalline quartz are employed, and the image is registered by photography.

Cover glasses are worked in fused quartz; slides usually are of crystalline quartz, but might be made of one of the ultra-violet transmitting glasses.

Homogeneous immersion can be secured by the use of solutions of glycerine or cane sugar which are transparent to this radiation.

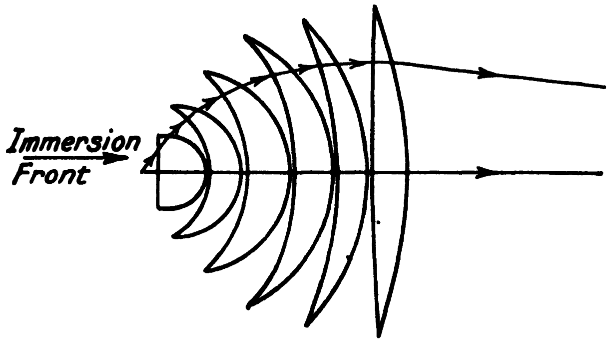
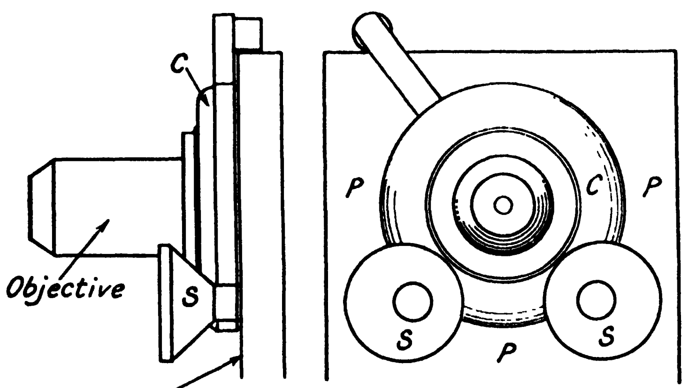


FIG. 96. MONOCHROMAT OBJECTIVE

Suitably adjusted mixtures can be freed from hygroscopic variations of refractive index.

Calculation of the depth of focus of an immersion monochromat of *NA* 1·25 shows it to be about 0·2μ, or eight millionths of an inch. The fine adjustment focusing of the objective must, therefore, be



(*a*) Side view (*b*) Front view

FIG. 97

such that this interval corresponds to a measurable movement of the control. Satisfactory results are obtained from well-made micrometer screws with tangent screw, or by well-regulated elastic displacements.

The image is invisible to the naked eye. A fluorescent finder, which may be likened to an "eye" with a quartz lens, and a "retina" of fluorescing uranium glass viewed from behind by a magnifier, permits the finding and focusing of bright images when the objects have well-marked contrast, but not of the smaller and delicate images unless some opaque material is introduced into the slide. Some workers have used a "carbon pencil" line drawn on the under side of the cover glass. In such cases the method used by Barnard is to employ two objectives in special mounts (Fig. 97), which can thus be interchanged without disturbing either the object or the remainder of the microscope. One is a visual objective and the other a monochromat, and they are made as nearly as possible

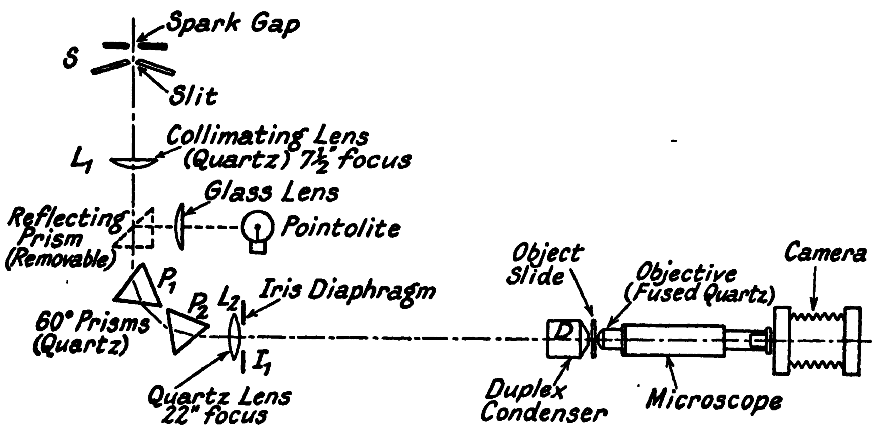


FIG. 98. ILLUMINATING SYSTEM FOR ULTRA-VIOLET MICROSCOPE

"parfocal," so that, when the image has been found and focused by the visual lens, it requires only a small measured motion of the fine adjustment to obtain the ultra-violet focus for the monochromat when the latter is placed in position.

Yet another alternative which is successful in some cases is to use the monochromat (*a*) with "homogeneous" visible light such as that obtained from the green line of the mercury spectrum, (*b*) with the ultra-violet. Although the definition of the visual image obtained in the first place is poor, owing to the spherical aberration of the lens for that wave-length, the difference of the focus in the two cases can be measured with sufficient accuracy to come very near the ultra-violet focus from the visual setting. In all methods, one or two trial "shots" with slight adjustment of the focusing are usually necessary to obtain the sharp image.

The general arrangement of the apparatus for making photomicrographs of transparent objects is shown in Fig. 98, which is largely self-explanatory. The lens L_2 projects the images of the spark gap, i.e. the spectrum, into the plane of the duplex condenser

D; one "line" must be broad enough to fill the aperture of the condenser, of which only the centre part is constructed of quartz. The outer part consists of a dark-ground illuminator of the concentric type as made by Messrs. R. and J. Beck, Ltd. (Fig. 99). The advantage of the arrangement is that objects in a slide can be found very easily with the aid of this illuminator and the visual objectives;

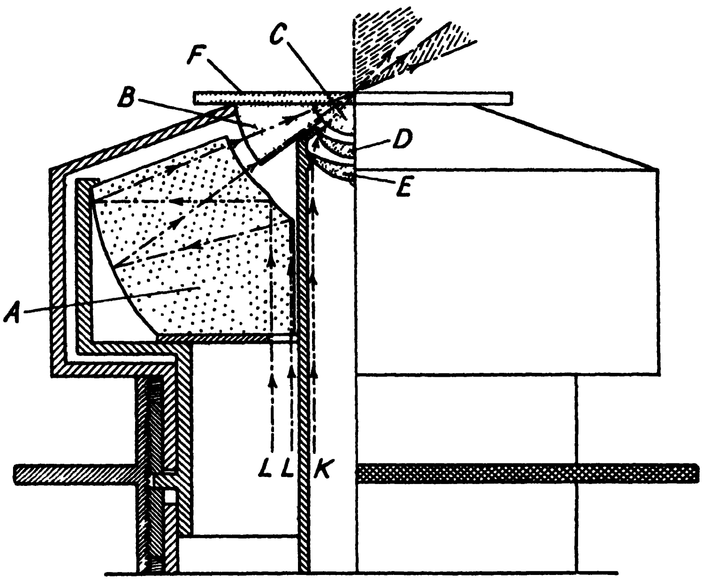


FIG. 99. CONCENTRIC TYPE CONDENSER AND ILLUMINATOR

A = Movable component of dark ground illuminator (glass)
B = Front component of dark ground illuminator (quartz)
C, *D*, *E* = Condenser components in quartz
F Slide (quartz)

they can then be photographed by the ultra-violet radiation without disturbing the slide in any way.

It is convenient to produce the discharge from a small transformer working from an A.C. supply; the brightness is found to be independent of the frequency of the cycles of the current within ordinary limits. The effective brightness of the spark is dependent mainly upon the total energy consumed at the gap. The spark gap with the arrangement of Fig. 98 is about 3 mm., the secondary volts are about 5,000, and there is a condenser of capacity of about ·01 μF in parallel with the gap. The circuit consumes about half a kilowatt.

The earliest steps in ultra-violet microscopy were made by Köhler[15] and von Rohr of the firm of Carl Zeiss prior to 1904, but the method found few users till after the War, when Barnard[16] introduced many alterations in the apparatus and technique in conjunction with the firm of Messrs. R. and J. Beck, Ltd., and used the method with great success in biological work. The experimental methods, particularly in regard to the stabilization of the immersion media, and the use of elastic displacement fine adjustments, have also been explored by Martin and Johnson.[17 18, 19] Johnson has given an experimental demonstration that the expectation of increased resolving power has been almost realized.

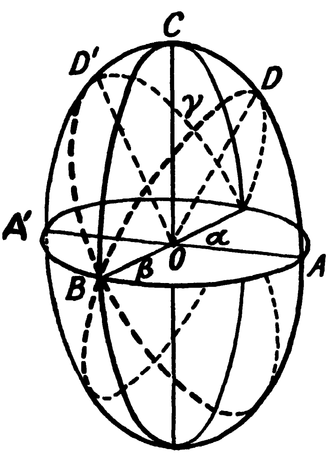


FIG. 100. THE OPTICAL INDICATRIX (FLETCHER)

Polarization Tests—the Optical Indicatrix. Microscopic tests, carried out with the aid of polarized light, are of great importance in the identification of minerals in small crystals, or in thin rock sections. Similar tests are now of increasing importance in biology and in colloid physics.

The section on crystal optics in Vol. I, Chapter VI, may be consulted for introductory study of the phenomena of polarized light. We may add here that in a doubly refracting medium there are in general three chief vibration axes; let them be ***a***, ***b***, and ***c***.

Light travels fastest when its vibrations are parallel to the direction ***a***, and most slowly when parallel to ***c***. The other direction ***b*** is a direction of intermediate case.

The corresponding indices of refraction for rays having vibrations parallel to these directions ***a***, ***b***, ***c***, are α, β, and γ respectively, the velocities being proportional to $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$. Thus it is possible to represent the refractive index for a vibration in any direction by a three-dimensional figure such as shown in Fig. 100. This is the so-called "optical indicatrix," an ellipsoid with semi-axes α, β, and γ. Any section of such an ellipsoid by a plane passing through the centre is, in general, an ellipse with different axes. From the ellipsoid we see, for example, that a disturbance propagated in the direction OB and vibrating parallel to OA has a velocity proportional to $\frac{1}{\alpha}$, while one vibrating parallel to OC has a velocity proportional to $\frac{1}{\gamma}$.

Consider a ray propagated parallel to the direction OC. The vibration, perpendicular to OC, can be resolved into components vibrating parallel to OA and OB respectively, for which the refractive indices are α and β.

Now since $\alpha < \beta < \gamma$, there will be two directions OD, OD′ between OA and OC and between OA′ and OC, for which the semi-diameter of the ellipsoid will be equal to β. The "ellipsoids" through B and D and through B and D′ will, therefore, become circles, and the directions normal to these planes will be ray directions for which there is no difference of velocity for vibrations taking place in various azimuths. These directions of single ray velocity are the "optic axes" of the doubly refracting medium.

In any thin crystal section, or section of other doubly refracting medium, there will be fast and slow vibration directions corresponding to least and highest refractive indices. These may be represented by the "index ellipse" for the section.

The effects of double refraction associated with increasing paths through a crystal plate, the colour effects using white light, and the estimation of the double refraction with the aid of a wedge compensator have been described in Vol. I.

The Polarization Microscope. The polarization microscope is designed in the first place to permit of the application of such tests to very small and thin specimens. It is fitted (Fig. 101) with a sub-stage polarizer (Nicol, Glan-Thompson, etc.), which can be easily swung in and out as desired. This is followed by a diaphragm and by a sub-stage condenser, usually of a two-component type, of which the upper highly converging system can be removed from the path of the light if it is required that the object shall be traversed by comparatively "parallel" pencils of rays. It is thus possible to illuminate the object with plane-polarized light. The objectives and eyepieces are of normal types, but it is considered a great advantage to have the widest possible field of view. An analyser may be inserted in the tube below the eyepiece. It is preferably of the square-ended variety, so that no displacement of the image is produced by its rotation. An auxiliary analyser with a divided circle to measure rotation ("cap analyser") may be fitted over the eyepiece with a swing-out movement.

Slots are usually provided above the objective and sometimes under the eyepiece, so that various compensators, such as a mica-quarter wave plate, quartz wedge, or Babinet compensator, can be inserted into the path of the light.

Petrological tests with the polarization microscope usually aim at the identification of various constituents of a specimen of rock, and for this purpose a thin section is made by grinding, and is cemented between a slip and cover-glass. It is desirable to know the thickness of the specimen, which is usually of the order of one-thousandth of an inch.

It is not within the scope of the present book to deal fully with all the numerous tests and criteria which can be applied for the identification of the various materials; we can only mention those which depend on the polarization of light. The doubly refracting materials are immediately recognized by the restoration of light between crossed Nicols. Suppose that an unknown doubly refracting substance is present, the crystallographical axes are likely to have all kinds of inclinations relatively to the section in different parts. Any element traversed by the light has its maximum and minimum refractive indices, represented by the major and minor

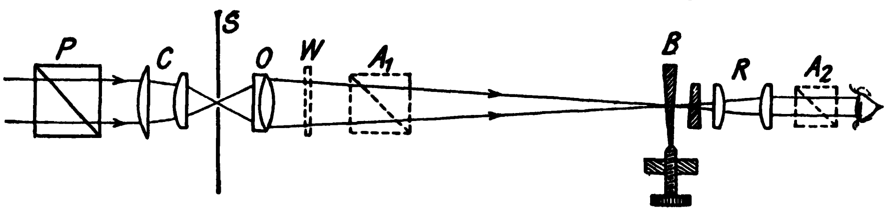


FIG. 101. OPTICAL SYSTEM OF POLARIZING MICROSCOPE FOR ORDINARY OBSERVATION

P = Polarizer
C = Condenser
O = Objective
W = Wedge or retardation plate sometimes introduced
A_1 = Analyser used when compensator B is not in use
B = Babinet compensator. (Other compensators may be introduced here)
R = Ramsden eyepiece
A_2 = Analyser used when compensator is in use

axes of the ellipse representing the section of the indicatrix ellipsoid taken normally to the direction of the light. If the major or minor axis of the ellipse is parallel to the vibration direction of the polarizer, the light is not restored between the Nicols, but if the specimen is rotated through 45° the components of the vibration separated at the crystal surface are equal, and the colour is at maximum intensity. The highest order colour will be found in sections containing OA and OC where the *maximum* difference of refractive index is found; sections normal to OD or OD′ (the axes) would not show double refraction at all. As mentioned in Vol. I, page 211, the Chart of Michel Lévy shows the colours characteristic of different thicknesses of crystal plates of varying bi-refringences, and in this way the maximum or sometimes the average bi-refringence of the mineral can be estimated which, together with other signs, is usually sufficient for identification.

Alternatively, the bi-refringence may be compensated, and thus measured with the aid of the quartz wedge—or more readily with the Babinet compensator.

Extinction Directions. Another important test may concern the angle between a principal extinction direction for the section, and either the cleavage marks or crystal edges which are often recognizable. It is necessary in such a case to have a cross-wire in the field, marking the extinction direction of the analyser, and a means of measuring the rotation of the stage. In order to facilitate the extinction direction setting, various auxiliary devices can be employed.

Examination in Convergent Light. A system for examination of mineral sections in convergent light (the so-called konoscopic

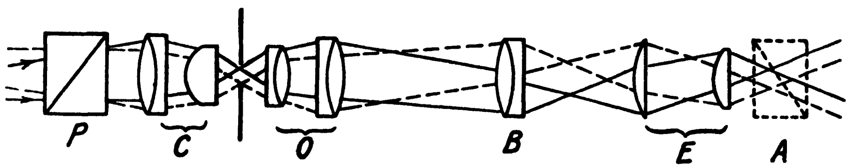


FIG. 102. SYSTEM FOR OBSERVATION OF CRYSTAL SECTIONS IN CONVERGENT LIGHT

P = Polarizer B = Bertrand lens
C = Condenser E = Eyepiece
O = Objective A = Analyser

observation) is obtained in a simple way with the microscope, by using a condenser of power equivalent to that of the objective and removing the eyepiece, so as to study the appearances in the upper focal surface of the objective, where the so-called "stauroscopic" figures are found. A short discussion of the "ring and brush" appearances was given in Vol. I, pages 218–220. The part of the slide illuminated by the condenser should be confined to the area of the specimen under test; this is easily arranged with the eyepiece in position.

The use of a "Bertrand Lens" between the objective and eyepiece facilitates the observation of the upper focal surface of the objective, as shown in Fig. 102, from which it will be seen that the combination of Bertrand Lens and eyepiece forms a low power microscope projecting the image of this focal surface into the focal plane of the eyepiece. In the usual arrangement, this auxiliary lens is so fitted to the microscope that it can be pushed into position or withdrawn easily as required.

The brief mention given to petrological tests in this book by no means represents their great development. Owing to their commercial importance they have received a very great amount of study since Sorby, in 1858, took the initial steps in this branch of microscopy.

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CHAPTER IV

BINOCULAR VISION AND BINOCULAR INSTRUMENTS

Physiology of Binocular Vision. Studies of binocular instruments[1] and binocular vision[2, 3] have formed the subjects of complete treatises, and it will not be possible in the limits of this chapter to do more than give the barest introduction, and to indicate the lines along which the study may be developed.

The optic nerve in each eye leaves the eyeball in the region of the *optic disc*, and passes through the *optic foramen*, an opening in the end of the *orbit* (the conical cavity in the bone which contains the eyeball). The nerve then continues to the *chiasma*, or crossing point of the nerves from each eyeball. It is here that a separation or *decussation* takes place, the nerve fibres belonging to the nasal parts of each fundus passing to the opposite hemisphere of the brain, and those belonging to the temporal parts passing to the corresponding hemisphere of the brain; thus impressions on the right of each fundus are conveyed to the right hemisphere of the brain and *vice versa*. (See Fig. 103.)

In normal health of the visual system, both eyes are fixated on any object under examination, and a single visual *fused* impression results. The fixation is maintained by the motor muscles of the two eyes which are largely *yoked* in their action, so that the fusion of the images is maintained without conscious effort.

There is also an important inherent connection between the relative *convergence* of the visual axes of the two eyes and the *accommodation* of each eye, so that when a healthy pair of eyes views a near object, they converge to exactly the right amount to bring the fixated image to the fovea of each eye. This relation, however, is not of an inseparable character, and the convergence of the visual axes can be varied independently of the accommodation by special means, such as placing a weak prism in front of one eye, or altering the relative positions of pictures in a stereoscope. This, however, is apt to lead to strain and discomfort.

The normal condition of muscular balance in which the visual axes are parallel when the muscles of the eyes are at rest is known as *orthophoria*; the condition of muscular imbalance, in which there is a lack of parallelism with the muscles at rest is known as *heterophoria*. The lack of balance may be in the horizontal or vertical directions; there may be excessive or defective adduction or convergence, or it may be that there is an elevation of one visual axis relatively to that of the other. In such cases, fusion is only maintained by increased innervation of the weaker muscles, and when fusion is prevented by any means, as by the influence of drugs, *diplopia* or double vision will result.

In the case of *strabismus*, or squint, the muscular imbalance is so marked that fusion is impossible, and the visual axes of the two eyes cannot both be directed to one and the same object.

In testing for Heterophoria, use is made of the Maddox Groove,

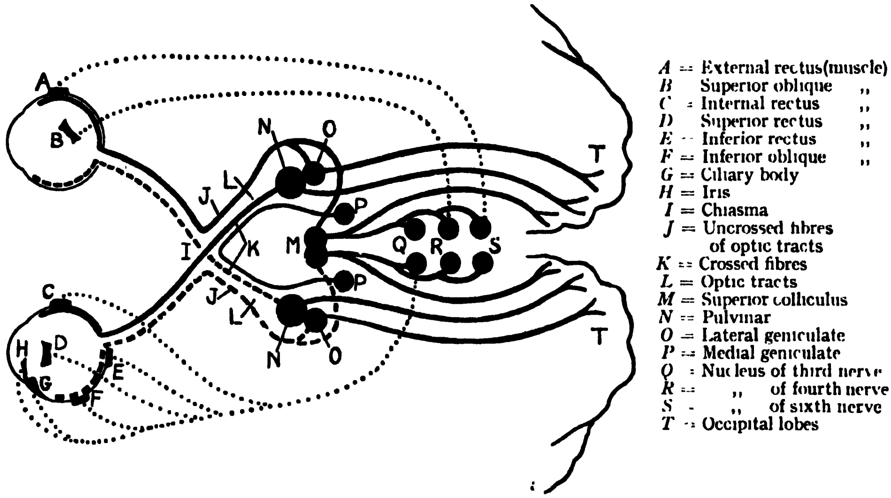


FIG. 103. DIAGRAM OF VISUAL TRACTS AND THE CONNECTIONS OF THE THIRD, FOURTH, AND SIXTH NERVES

consisting of a cylindrical groove ground in a piece of red glass and held before one eye; the glass is blackened with the exception of the groove. This eye then sees a lamp as a bright red streak perpendicular to the axis of the cylinder. The test chart consists essentially of a scale with a small aperture brightly illuminated in the middle, and the test is illustrated in Fig. 104; it is seen that the Maddox groove is held horizontally before the left eye, which then perceives nothing but a vertical streak which cannot possibly be fused with any part of the scale seen by the right eye. There is, therefore, nothing to bring the muscles out of the condition of rest, and the apparent position of the streak on the scale gives an indication of the *Esophoria* (eyes tending to turn inward) or *Exophoria* (eyes tending to turn outwards) which may be present. The vertical test may be made by turning both scale and groove into the vertical direction. In *Hyperphoria* one eye tends to turn upwards;

in *Hypophoria* one eye turns downwards. Combinations of these conditions may be found. The amount is measured in prism diopters. (Vol. I, page 306.)

Diplopia—Inhibition. It was mentioned above that when both eyes fixate the same object a single fused image is perceived. Conditions frequently occur in which different images are presented to the two eyes even in normal vision, while this is always the case in strabismus.

When both eyes are equally good and diplopia is produced, say by slight pressure with the finger on the eyeball at the rim of the orbit, both images are equally strong at first. By slightly increasing the pressure on the eyeball, however, the displaced image is weakened,

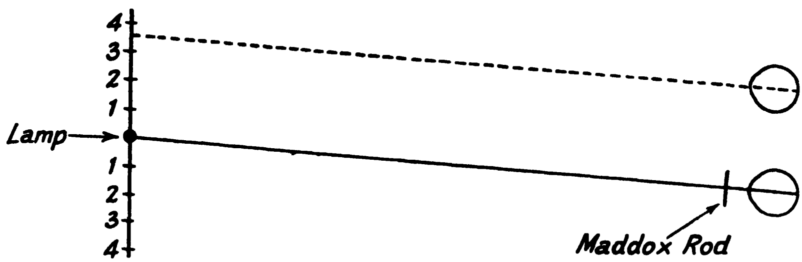


FIG. 104. TEST FOR HETEROPHORIA

and may then practically disappear from consciousness. Sometimes, by mental concentration on one of two almost equally strong images in diplopia, the other can be caused to be hardly noticeable. This effect can well be illustrated by putting two "infusible" patterns, say a circle and a cross respectively, into the two fields of a stereoscope.

Again, if different colours be presented to the two eyes, it is usually the case that there is a "struggle" between the two colours. Either one or the other is perceived almost at will. It is, however, sometimes found that if the colours are distributed in easily fusible patterns, then a binocular colour mixture in the additive sense may occur. Some observers, however, cannot see this effect.

Suppressions of this kind are related to well-known phenomena in the nervous system by which nerve pulses are inhibited, especially at the crossing points of nerve junctions; a strong pulse may, as it were, take up the whole road to the exclusion of any other.

It seems, in fact, as though one eye of the two is generally the one for which the pulse is the master. Let any one try this simple experiment. Make a ring with the thumb and forefinger; then with both eyes open raise the hand quickly so as to view, through the ring, some object on the other side of the room. It will usually occur that the ring comes into the line of sight of the "master eye."

Stereoscopy. Let the two eyes fixate a point represented in Fig. 105 by the point P. This can be represented in practice by a pencil.

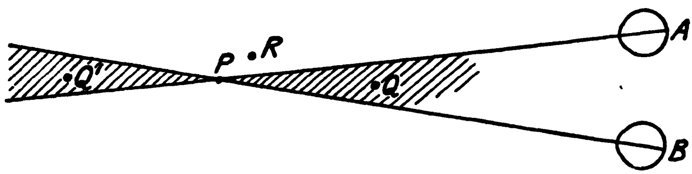


FIG. 105

If another pencil is held at any point Q within the angle formed by the crossing of the visual axes, and either nearer or farther away than P, as at Q′, then the image of Q or Q′ is seen doubled. But if the second pencil is transferred to R, outside this region of diplopia, it no longer appears doubled in consciousness, but shows the stereoscopic effect of a single object situated in space in a recognizable position in respect to P. This effect persists for a certain range of distances of R, both smaller and greater than that of P, but outside such a range the stereoscopic impression breaks down, and a doubled image with no stereoscopic effect is manifest.

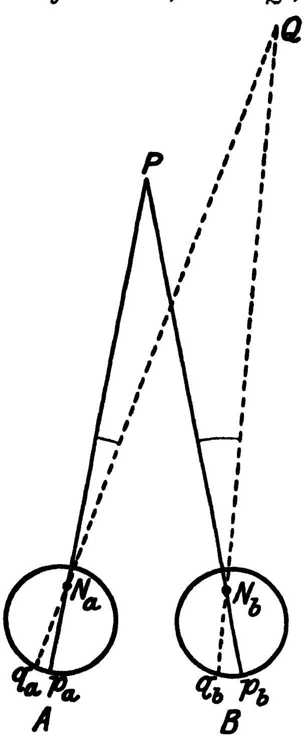


FIG. 106

The Presentation of Space to Consciousness. The visual image presented to consciousness when using either eye is that of a perspective presentation taken with the nodal point of the eye as centre. How this is effected by the retino-cerebral system is unknown. In common with a chick who can start to peck at food on emerging from the egg, we are endowed with this intuitive faculty of recognizing the geometrical relations of the positions of objects.

Let us now consider the binocular vision of two points P and Q at different distances from the observer, whose eyes are A and B (Fig. 106). Now the presentation of these points to eye A is effected through the image points p_a q_a on the retina; similarly, the image points $p_b\, q_b$ determine the presentation to eye B. If both eyes fixate the point P, then p_a and p_b fall on the fovea in each eye respectively, while $q_a\, q_b$ fall at different distances in the two eyes. Let N_a, N_b be the nodal points of the two eyes. It is sufficient to consider one nodal point for each eye as in Vol. I, page 156.

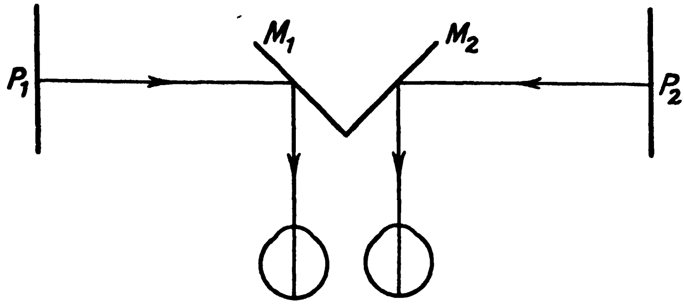


FIG. 107. WHEATSTONE STEREOSCOPE

Let f be the anterior focal length of each eye; then the difference of $p_b q_b$ and $p_a q_a$ is evidently (remember that the distance from the second nodal point to the retina is equal to the *anterior* focal length)

$$p_b q_b - p_a q_a = f\,(\widehat{PN_bQ} - \widehat{PN_aQ})$$

provided the angles are not great. The difference of the two angles is the *binocular parallax.*

The Stereoscope. We therefore ascribe a stereoscopic perception of relative distance to the difference in the perspective images presented to the two eyes. Acting on this reasoning, *Wheatstone*[4] argued that if a suitable perspective picture were presented to each eye simultaneously, then a sensation of a solid object should result; this is the principle of the *stereoscope.* Wheatstone's mirror stereoscope is shown in plan in Fig. 107; M_1 and M_2 are mirrors, reflecting the images of P_1 and P_2, the two perspective pictures.

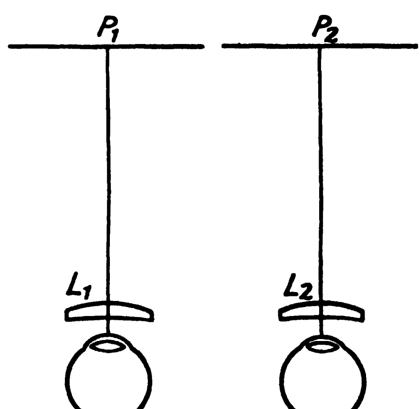


FIG. 108. THE STEREOSCOPE (DIAGRAMMATIC)

The modern stereoscope (Fig. 108) is generally adapted for stereoscopic perspective pairs $P_1\, P_2$ mounted at the interocular distance, and held in the focal plane of viewing lenses L_1 and L_2. If any pair of corresponding points in the pictures are "fused" when the visual axes are parallel, the accommodation which is at "infinity" corresponds perfectly to the convergence for these points, but there will be some deviation from the normal relation when points at different apparent distances are fixated. Wheatstone also used stereoscopes of this type. For the types of lenses used in

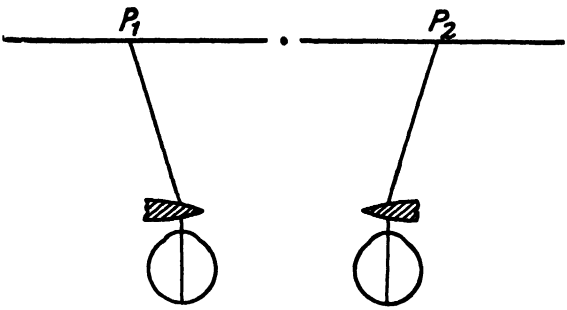


FIG. 109

modern stereoscopes see the Verant, page 10, also Albada's lenses, page 17.

Brewster's stereoscope (Fig. 109), examples of which are still in fairly wide use, was adopted for viewing larger pictures of which the centres were mounted at a greater distance than the ordinary inter-ocular distance; this is effected by the lens-prisms. When such pictures are viewed by a modern stereoscope the inner parts of the viewing lenses are employed, so that the prismatic effect can again be obtained.

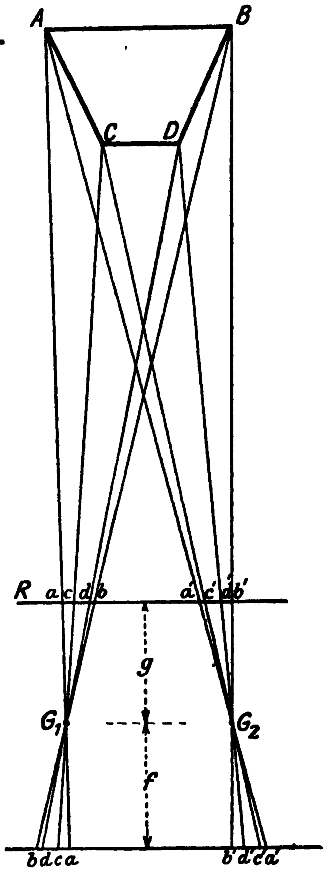


FIG. 110. PERSPECTIVE PROJECTION

Stereoscopic Perspective Pairs. Let G_1 and G_2 (Fig. 110) be two perspective centres situated at the ordinary inter-ocular distance; think of them, for the moment, as the two eyes. We can imagine a projection plane R of glass on which, closing one eye at a time, we mark out the outline of a solid object ABCD; two perspectives are thus obtained of the kind suggested in Fig. 111. Another way of obtaining these perspectives would be to employ two cameras with lenses at G_1 and G_2 having the focusing distance of each lens from the back nodal point to the picture plane exactly equal to g, the distance from perspective centre to projection plane; the lenses are placed more particularly with their front nodal points at G_1 and

G_2. The photographs thus obtained when mounted (duly reversed) in the plane of R will correspond geometrically to the drawings just imagined.

If two such projections are held in the original positions before the eyes, provided that g is great enough, most persons can then "fuse" the pictures without optical aid and obtain a visual sensation of a three-dimensional structure.

Difficulty is, however, experienced by many persons in securing this fusion of the two projections, because, when one of them is viewed, the relation between accommodation and convergence comes into play and causes the visual axes to converge in the projection plane rather than behind it. Success may sometimes be attained by looking downwards into a mirror at the image of a

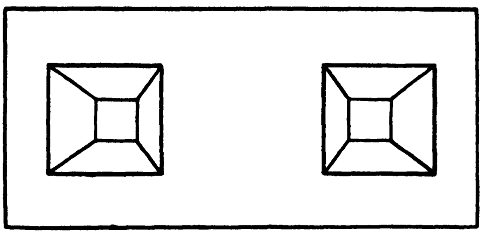


FIG. 111. PERSPECTIVE PAIRS

distant object; the visual axes then become parallel; if now the "stereoscopic picture" is quickly brought between the eyes and the mirror (not too close to the eyes, say about 24 in.) it may be possible to secure the fusion. Another method is to look at the pictures through two pinholes, one before each eye; the accommodation is less definite and a fusion can be effected.

The function of the lenses in the stereoscope will now be more easily understood; the fusion of a pair of corresponding points can be effected without straining the relation between convergence and accommodation. We also secure the important advantage that by using short focus lenses we can deal with wide angle views; with the ordinary parallel-axis stereoscope the pictures have to be placed in the projection plane with their centres, say, 65 mm. apart, and therefore, their effective breadth cannot exceed this amount. It is clear that the correct angular projection of the perspective views cannot be obtained unless the focal lengths of the viewing lenses agree with those of the taking lenses. The importance of viewing a perspective presentation under the correct angle was explained in Vol. I, page 4.

Stereoscopy and Convergence. The following experiment will illustrate some important conclusions. A stereoscopic reconstruction is obtained by fusing two pictures in a stereoscope; the pictures are two projections made in any suitable way, and are mounted separately so that they can be moved independently of one another. When fusion has been obtained they are moved slightly apart. It is found that in the duration of the movement the whole field appears to retreat; if they are made to approach each other the field appears to draw nearer; but such apparent movement from or towards the observer ceases immediately the pictures are still, and the whole field looks to be at the same absolute distance as before. We therefore conclude that the sensation of relative distance in

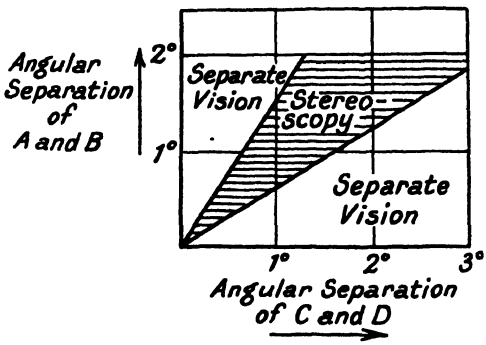


FIG. 112. LIMITS OF BINOCULAR PARALLAX FOR STEREOSCOPY

stereoscopy is entirely independent of the absolute state of convergence or divergence of the visual axes. These axes may, in fact, be made divergent while a perfectly natural-looking stereoscopic view is obtained. If the separation of the picture is increased too far, however, the muscular control breaks down and diplopia results. The only real guide to the apparent absolute distance of the "objects" in a stereoscopic field is their angular size; the stereoscopic sense gives an impression of their relative distances, and this results through the variation of the binocular parallax, but, as mentioned above, this is limited in amount.

According to J. W. French,[5] if we are fusing the binocular images of two points subtending an angle of one degree to one eye, then the limiting binocular parallax is about half a degree, i.e. if two points A and B subtend an angle of 1° to the left eye, and two other points C and D in the same horizontal line as A and B are viewed by the right eye, and if then the images of A and C are fused, it will be impossible to fuse B and D into a stereoscopic image unless the angular separation of C and D to the right eye lies between about 0·5° and 1·5° (French gives 0·6° and 1·6° more exactly); for smaller or greater separations of C and D stereoscopic vision fails, and the images C and D are not fused. The limiting separations depend, according to the same writer, on the angular separation of A and B as shown in Fig. 112. In connection with these results the limited visual acuity even a short distance from the fovea has to be remembered.

Production of Stereoscopic Effects by Projection and Otherwise. The essence of the stereoscopic sensation is the binocular parallax

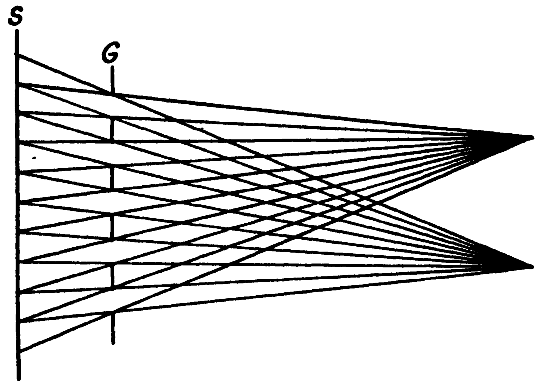


FIG. 113. GRID ARRANGEMENT FOR STEREOSCOPY

arising in the different perspectives presented to the two eyes; it cannot arise when both eyes view one and the same picture; but pictures in different colours may be projected on the same screen, as in red and blue-green. If then the eyes are furnished with colour filters transmitting only these regions of the spectrum, the two pictures (which may be two suitable perspectives) may be combined visually to produce stereoscopic results.

Alternatively, the two pictures may be printed in inks of red and blue-green to be viewed in a similar way.

If a suitable grid is held before a screen, the grid consisting of close equal bars and spaces of equal width, it is possible to arrange matters so that the two eyes view different strips of the same screen (Fig. 113). Thus two perspective projections can be made; one photographed in one set of strips; the other photographed in another set of strips; in this way a stereoscopic effect can be obtained. There seems, however, to be no escape from the necessity of a fairly close relation between the distances of the grid and the observer's head, so that the effect cannot be shown to many people at once.

Yet another possibility is to project the pictures alternatively on the same screen, providing the observer with a rotating sector disc to obscure the eyes alternately, so that one eye views one set of pictures only. Perfectly satisfactory stereoscopic effects can be obtained in this way, but the necessity of the sector disc for each person has prevented the system from being used for entertainment purposes.

Interocular Distance. Binocular instruments must necessarily take account of the variations of interocular distance between different individuals, which ranges in adult men from about 56 mm. up to 72 mm. with a mean of 63 mm.; the mean for women is somewhat smaller, being about 61 mm. Particulars of measurements may be found in a paper by J. W. French.[6] Prismatic binoculars are usually made adjustable between the limits of 57–70 mm., but owing to the much larger eye lenses and exit pupils of Galilean binoculars there is not the same need for the provision of the adjustment, although the necessity of making the glasses usable by a person of small interocular distance limits the diameter of the objective and, therefore, the field of view.

Effect of Duration of Illumination. Dove[7] performed an experiment many years ago (1841) which demonstrated the occurrence of the perception of stereoscopic relief under the very brief illumination given by an electric spark, thus demonstrating that ocular movements or changes of convergence of the visual axes are not an essential element of stereoscopic sensation, for although the sensation lasts much longer than the spark itself, any movements of the eyes would not move the location of the retinal image in either eye. Much more recently, *Langlands*[8] has shown that an improvement in stereoscopic acuity occurs when the duration of the illumination rises above 0·1 sec.; this suggests that ocular movements, when possible, do definitely increase the stereoscopic sense.

Stereoscopic Acuity. Many experiments have been made to determine the lower limit of binocular parallax which is capable of producing the sensation of stereoscopic relief. Under favourable circumstances a parallax of under 5 sec. of arc may be appreciated under conditions of steady observation; but 10 sec. of arc is usually required for instantaneous observations as in the experiments of Dove and Langlands. Such figures are, however, only valid for persons with excellent form vision and adequate training in stereoscopic observations. Some observers may find it difficult to detect parallaxes of whole minutes of arc without experience, and some appear to be lacking in the stereoscopic sense altogether.

Pulfrich[9] has devised interesting test charts for use in a stereoscope; the elements of the chart embody various parallactic displacements, and they afford a ready means of testing the capabilities of various observers.

Pseudoscopic Effects. If the left-hand perspective projection is presented to the right eye, and *vice versa*, the "stereoscopic depth" sensation is reversed and opposes the ordinary interpretation of the

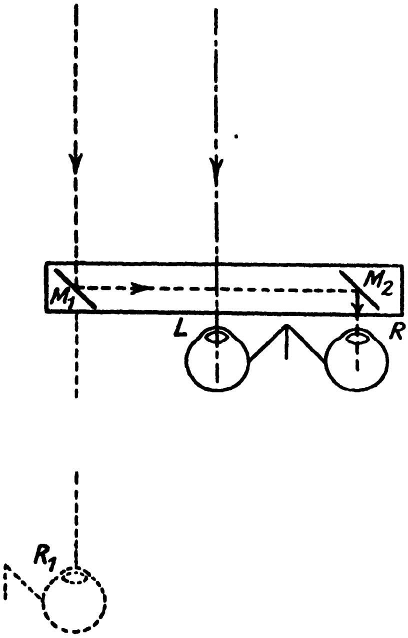


FIG. 114. STRATTON'S PSEUDOSCOPE

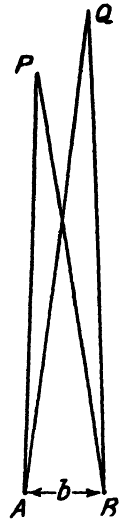


FIG. 115

perspective. The effect may be seen by cutting an ordinary stereoscopic pair and putting the left-hand picture on the right of the stereoscope, the right-hand picture on the left.

A similar effect can be produced by an arrangement of mirrors, as in the so-called "pseudoscope" of Stratton (Fig. 114). Two mirrors M_1 and M_2 are mounted together on a board at 45° to the direction of view of the two eyes L and R, so that rays from any object reach L directly, but must be reflected from M_1 and M_2 before entering R.

The perspective centre for the view presented to the right eye R is evidently at R_1, the image of the nodal point of R formed by the two mirrors. Let us draw the projected image of the right eye; then we may remind ourselves which is the nasal side by the convention in the diagram as suggested by von Rohr.

The experiment itself is a very easy and interesting one to make, and should be carried out by the student.

The Telestereoscope. Referring to Fig. 115, let A and B be two visual perspective centres, and let the separation be b. If the points P and Q are two points in the field situated near the line bisecting AB at right angles, and if their distances are r_p and r_q from the mid-point of AB, then we shall have, sufficiently nearly if b is relatively small

$$\widehat{APB} = \frac{b}{r_p} \text{ and } \widehat{AQB} = \frac{b}{r_q}$$

It was shown above that the parallax for the points P and Q depends on the difference of $\widehat{PAQ}$ and $\widehat{PBQ}$.

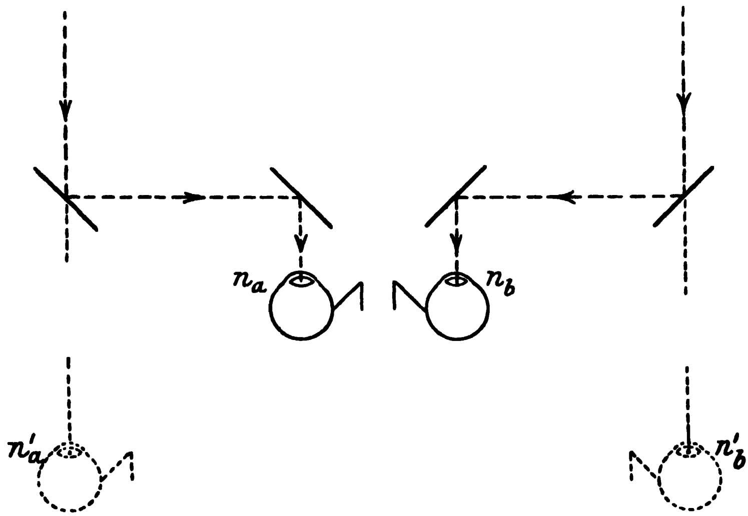


FIG. 116

$$\text{Hence } parallax = \widehat{PBQ} - \widehat{PAQ} = \widehat{APB} - \widehat{AQB}$$
$$= b\left(\frac{1}{r_p} - \frac{1}{r_q}\right)$$
$$= b\left(\frac{r_q - r_p}{r_p r_q}\right)$$

If the difference of distances is small so that we may write it δr, and the product of $r_p r_q$ as r^2, then

$$\text{parallax} = \frac{b\delta r}{r^2}$$

We have made it clear that the stereoscopic phenomenon depends solely on the binocular parallax, and it is in a sense immaterial how the perspectives are presented to the two eyes provided that corresponding points can be fused. By means of optical arrangements such as the pseudoscope above, we may present to the two eyes the perspective obtained from any desired centres, and the binocular parallax valid in vision is now to be calculated from the perspective centres actually in use. Thus in the telestereoscope (Fig. 116), the actual perspective centres are the mirror images of the nodal points of the eyes n'_a, n'_b formed by the double reflection. If the separation is now B the parallax for any pair of points

is
$$\frac{B\delta r}{r^2}$$

The effects of such an arrangement were discussed by Helmholtz.

Stereo-telescopes. If Q is at an infinite distance, and P is a point at a distance r_o which can just be recognized stereoscopically with the ordinary interocular base b, as a nearer point, then the parallax must be the lower limiting value p_o, say, which is recognizable.

Hence since parallax
$$= b\left(\frac{1}{r_p} - \frac{1}{r_q}\right)$$

we have
$$p_o = \frac{b}{r_o}$$

The distance r_o is the "stereoscopic radius."

If the base length is increased to B, the stereoscopic radius R will now be given by
$$p_o = \frac{B}{R}\ .$$

since the stereoscopic radius is increased proportionally to the effective base length.

In the stereo-telescopes we not only have an arrangement of reflectors to increase the base length, but an erecting telescope system is also held in front of each eye. The effect is, therefore, to magnify all visual angles by a factor m, say, which is the magnifying power of the telescope.

The binocular parallax $\frac{p_o}{m}$ valid at the actual perspective centres will now be presented to the eyes magnified m times; it will, therefore, be p_o and thus recognizable. The equation for the stereoscopic radius R_m is now
$$\frac{p_o}{m} = \frac{B}{R_m}\ .$$

The parallax for any given pair of points at an approximate distance r, and difference of distance δr, is now

$$\frac{mB\delta r}{r^2}$$

and if this is equal to or greater than the lower limit of discernible parallax p_o, the stereoscopic separation of the two points will be recognizable; then the limiting condition is

$$p_o = \frac{mB\delta r}{r^2}$$

so that

$$\delta r = \frac{p_o r^2}{mB}$$

Binocular Telescopes. In considering the perspectives presented to the eyes, it is frequently helpful to imagine the image of the eye

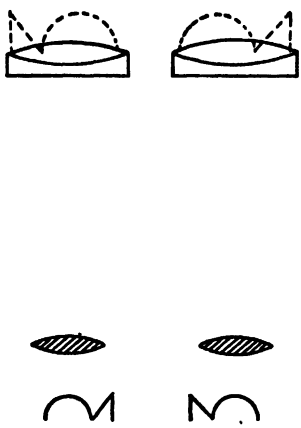


FIG. 117

FIG. 118

centre projected by the optical system into the object space. Fig. 117 is intended to represent a non-erecting binocular telescope. The eyes are diagrammatically represented in a convention suggested by von Rohr, the nasal side being indicated. Since the eye pupil is placed in the exit pupil of the instrument on each side, we consider an inverted image of each eye to be projected by the eyepiece system into the corresponding entrance pupil. It is quite clear that the effective projections will be reversed as well as inverted, and that the space image will be pseudoscopic. The erecting system in ordinary binoculars is therefore necessary for this reason also. We may note that if both eyes use the same optical system (Fig. 118), and if we consider the effective centres, we find that the relation of the two eyes is the same in the projected image in the entrance pupil, and the effect cannot therefore be pseudoscopic, though there will still be inversion and reversion.

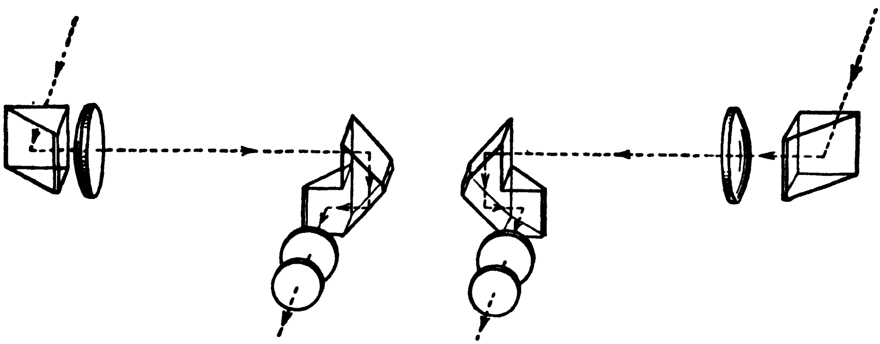


FIG. 119. STEREO TELESCOPE SYSTEM

The chapter on the telescope includes a number of diagrams of prism erecting systems for telescopes in which two parts can be separated; if the ray direction between these parts is at right angles

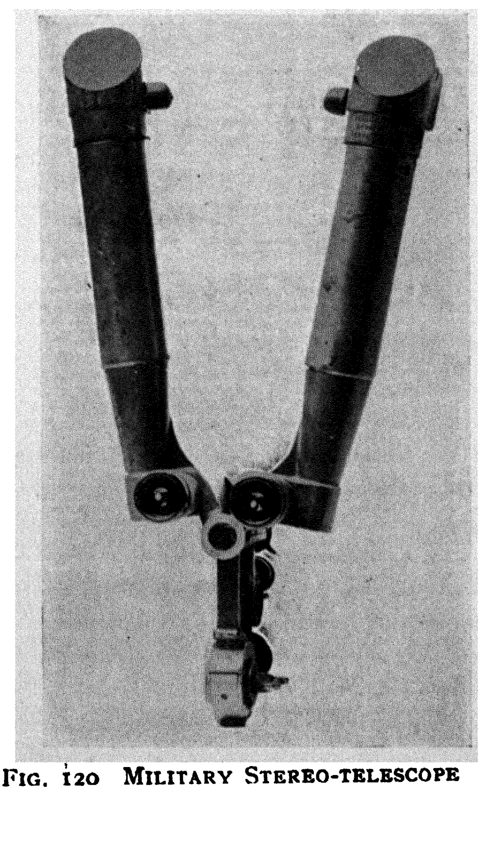
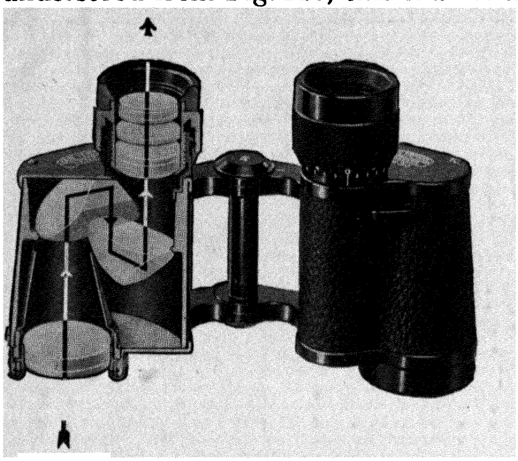


FIG. 120 MILITARY STEREO-TELESCOPE

to the initial and final directions, such a system will be useful in obtaining a binocular system with a long base and enhanced stereoscopic effect. Fig. 119 is a diagram of the arrangement of the optical parts in a stereo-telescope, and Fig. 120 shows a military instrument. The two arms of the apparatus can rotate about an axis parallel to the final directions of the optical axes of the eyepieces, and, as will be understood from Fig. 120, this enables a given inter-



(Carl Zeiss, Jena)

FIG. 121. TRACE OF RAYS THROUGH A FIELD GLASS

ocular distance for the axes to be obtained either for the closed or open positions of the arms. An ordinary "binocular" system is shown in Fig. 121.

Adjustment of the Axes. As mentioned above, the eyes are subject to the habit of a fairly close relation between accommodation and convergence. If in using a binocular telescope the accommodation must be for "infinity," the corresponding normal direction for the visual axes would be parallel. It is not easy to produce or tolerate any considerable divergence of the visual axes; seven to eight minutes of arc may be managed without discomfort. Much greater amounts of convergence can be tolerated with the accommodation still at ∞; 20 to 25 min. causes little or no discomfort. Very little allowance is possible in the variation of the vertical directions of the eyes; the enforced Hyperphoria of one eye should not exceed seven to eight minutes of arc. These limits set a severe demand on the mechanical construction of binocular instruments, which therefore have to be mounted with their axes very closely parallel.

Let P_1 and p_1, P_2 and p_2 (Fig. 122) be the entrance and exit pupils of the two members of a binocular system mounted with the optical axes at an angle α. The systems are presumed to be erecting systems, so that it is clear that if parallel rays enter the centres of the two entrance pupils P_1 and P_2, these rays will emerge from the centres of the exit pupils at a mutual angle of $(M - 1)\alpha$, which must not exceed the limits laid down above if comfort is to be preserved in use.

Thus with a pair of binoculars of magnifying power $\times 6$, the limit of divergence of the optical axes towards the observer (requiring divergence of the visual axes) is, say,

$$\frac{7\frac{1}{2}\text{ min.}}{6 - 1} = 1\frac{1}{2}\text{ min.}$$

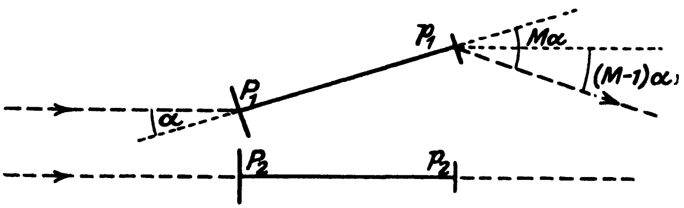


FIG. 122

The same applies to the vertical variation. The limit of convergence of the optical axes is about

$$\frac{22\frac{1}{2}\text{ min.}}{6 - 1} = 4\frac{1}{2}\text{ min.}$$

These are the tolerances generally treated as standard ones.

Correspondence of Magnification. Another important matter in connection with binocular instruments is the necessity of the exact correspondence of the magnification in each part. Any inequality in this respect will produce very unpleasant effects to vision, and will upset the stereoscopic presentation of the binocular space image. Another defect which cannot be tolerated is that of imperfect erection of the images, which will also upset the stereoscopic effect, if present, and prevent the effective fusion of the two parts of the field. Particulars of methods of testing binocular telescopes will be found in a paper by Lt.-Col. Williams.[10]

Space Presentation with Binocular Instruments. As mentioned above, a ready way of obtaining the position of the effective centre when using any instrument before one eye is to find the image of the nodal point of the eye as projected by the instrument. Take, for example, the simple cases represented in Figs. 123 (*a*) and 123 (*b*). A rectangular framework is being viewed with the aid of a lens;

N is the nodal point of the eye, and N′ its image projected by the lens. In the first case the eye is close to the lens (the case of the spectacle lens), and the positions of N and N′ are not far removed from each other; the framework is evidently seen under natural perspective, since the side nearer to the eye is seen under the greater angle. In the second case, the perspective centre is projected

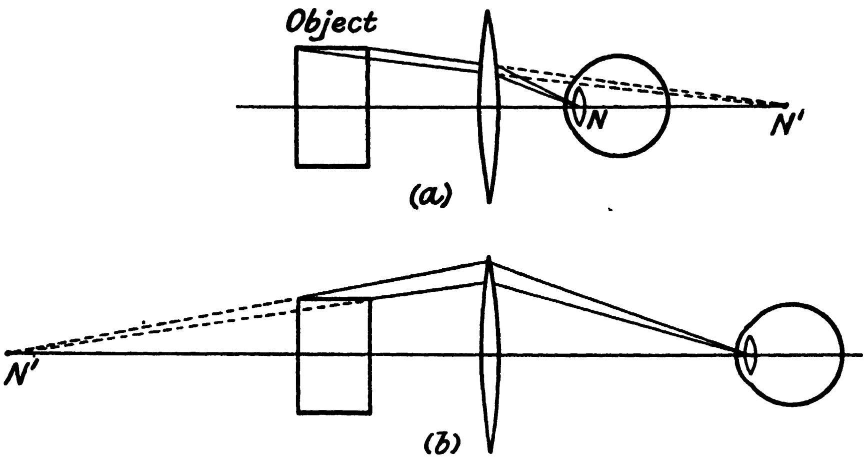


FIG. 123. PERSPECTIVE EFFECTS WITH MAGNIFIER SYSTEM
(a) Effective viewpoint on same side. Natural perspective
(b) Effective viewpoint on far side. Unnatural perspective

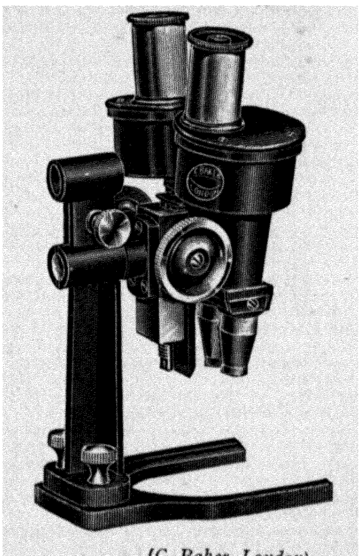
beyond the lens and framework; the latter is seen under unnatural perspective, since the side farther from the eye is seen under the greater angle. The student should draw for himself the diagram of the case when the perspective centre is projected to an infinite distance; objects of equal height then subtend equal angles at the eye. Prof. von Rohr calls these three types of perspective by the names "Entocentric," "Hypercentric," and "Telecentric"; they are all possible with monocular optical systems.

When we consider binocular systems, we may have two general cases of space presentation through the stereoscopic sense, i.e. *orthoscopic,* in which the two perspective centres have their natural relation; *pseudoscopic,* in which the centres are reversed as explained above; and sometimes the system may project the two perspective centres into one point, when the effect is "*synopic,*" and the stereoscopic sense cannot exist since binocular parallax must be absent. Theoretically, as von Rohr points out, the combination of the three kinds of perspective with the three binocular possibilities makes nine modes of space presentation possible with binocular instruments.

Most of the traps in this respect are avoided in the familiar instruments in which Entocentric perspective unites with orthoscopic presentation, but designers of new systems have to bear in mind the possibilities of error.

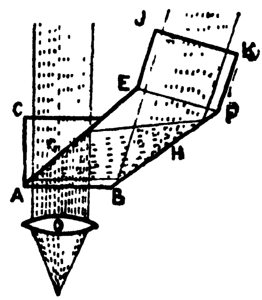
Binocular Systems for Microscopes. Owing to the necessary proximity of the objective and the object, two independent optical systems are only possible with the lowest powers, as in the Greenhough microscope, Fig. 124.

If the argument used above in the case of the binocular telescope is followed, it will be seen that each eye pupil will be projected by



(*C. Baker, London*)

FIG. 124. GREENHOUGH MICROSCOPE



(*R. & J. Beck, Ltd.*)

FIG. 125. BINOCULAR MICROSCOPE SYSTEM

the system roughly into the position of the objective, and that the natural disposition of the nasal and temporal sides will be upset unless each system is erecting. It follows that an erecting device must be included in such a double-barrelled instrument. Use is made of the eccentricity of the eyepiece (when a prism erecting system is used) to obtain interocular distance adjustment, each member being rotatable about the axis of its objective.

When higher powers are to be employed, only a single objective is possible, and the beam from the objective must therefore be divided if a binocular system is used. If the whole aperture of the objective is to remain in action for each image so as to retain the utmost resolving power, then the division must be effected by means of a partly transmitting and partly reflecting surface.

Fig. 125 represents a modern prism system due to Messrs. R. and J. Beck, Ltd., by which this is done with the aid of a half-silvered surface. The final direction of the two principal rays is such as to call for a convergence of the eyes to a point approximately at the

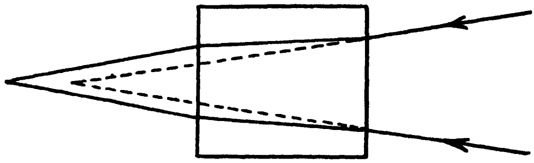


FIG. 126. ALTERATION OF FOCUS POSITION DUE TO BLOCK OF GLASS

near point of the average eye, say 250 mm. distant. This will secure the usual relation between convergence and accommodation if the image is formed at the near point.

The necessity of introducing the double reflection in the prism

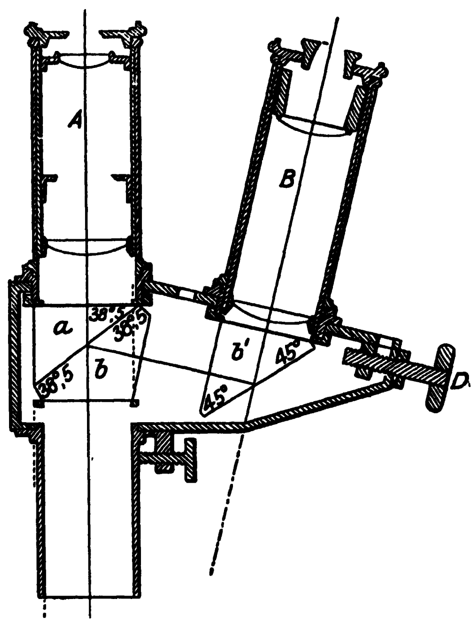


FIG. 127. ABBE STEREOSCOPIC EYEPIECE SYSTEM

elongates the path in the corresponding beam, and this would tend to make it come to a focus at a point unduly below the other eyepiece. This difficulty is countered by adding an additional cubical member to the prism. Fig. 126 will illustrate how such an arrangement may extend the focusing distance.

In this arrangement the interocular distance is varied by the alteration of the tube length in each member. If abnormal adjustment in this respect should be necessary, some deterioration of the image might result, and it would be better to withdraw the prism and make use of the instrument as a monocular.

The division of the field need not be effected *immediately* above the objective; various systems are now in use which can be adapted to the draw-tube of the instrument as binocular eyepieces. The Abbe "stereoscopic" eyepiece is of this kind, and is illustrated in Fig. 127; the separating surface is a thin air film. In order to bring the two exit pupils to the same level, two special eyepieces of differing construction are used. One works as a "Ramsden," the other

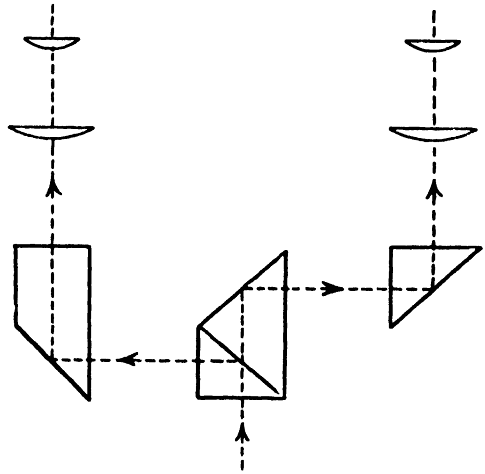


FIG. 128. BINOCULAR EYEPIECE SYSTEM

as a Huygenian eyepiece. In this arrangement the brightness of one image is twice that of the other. Modern arrangements have a more symmetrical design.

Fig. 128 shows the principle of an arrangement used by Leitz and other makers in which the glass path in each arm is equal. A specially built body is usually required for systems of this kind. Other makers have brought out similar systems in which the eyepieces are given a backward inclination, so that a comfortable position for the head may be maintained even when the axis of the objective is vertical.

Fig. 129 shows a construction by Reichert arranged so that monocular or binocular observation can be obtained by turning a knob.

Messrs. R. and J. Beck, Ltd., have recently produced a binocular eyepiece of this kind with converging tubes. Many observers agree that convergent vision is the more natural with the "downward look" position of the head usual when using the microscope, or in viewing objects on the bench.

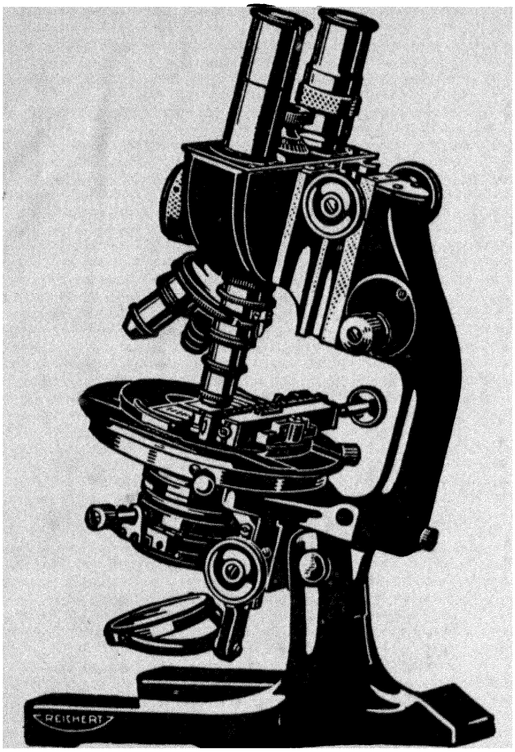


FIG. 129. MODERN BINOCULAR MICROSCOPE (REICHERT)

Production of Stereoscopic Effects. No stereoscopic effect can be obtained while the whole aperture of the objective is in use for each eye. Let us consider the effective perspective centres obtained, say, with the Abbe eyepiece. The essential optical conditions are represented in Fig. 130 (*a*), which shows (conventionally) the two eyes of the observer, R and L. No erecting systems are included (the roughly parallel mirrors in the one path produce no erection), so that the pupils are projected reversed and superposed into the entrance pupil.

Now our discussions so far have not dealt properly with lenses of finite aperture, and we must consider such a case before going further. In Fig. 131, the lens L projects an image of the object points A and B into the plane P. Now the image of A is in focus, while that of B is not in focus, and, consequently, the position of

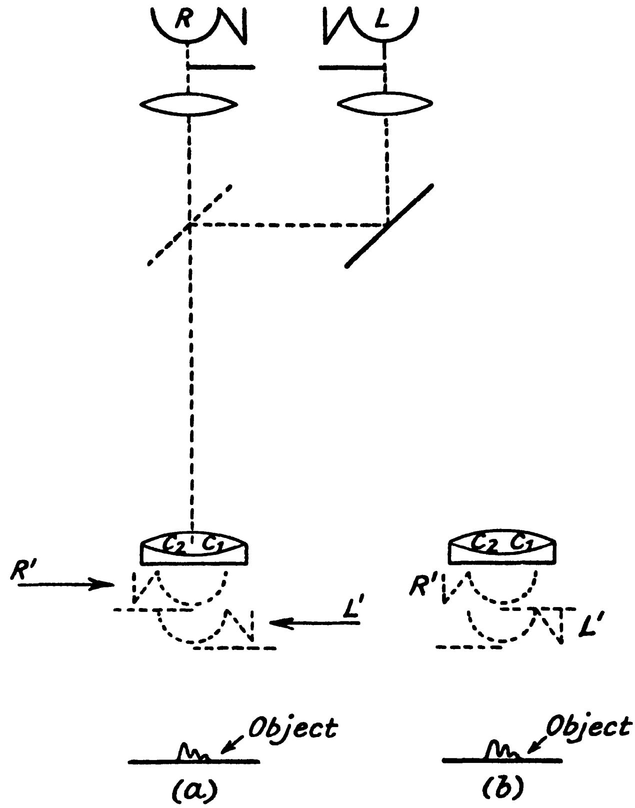


FIG. 130. THEORY OF STEREO EYEPIECE SYSTEM

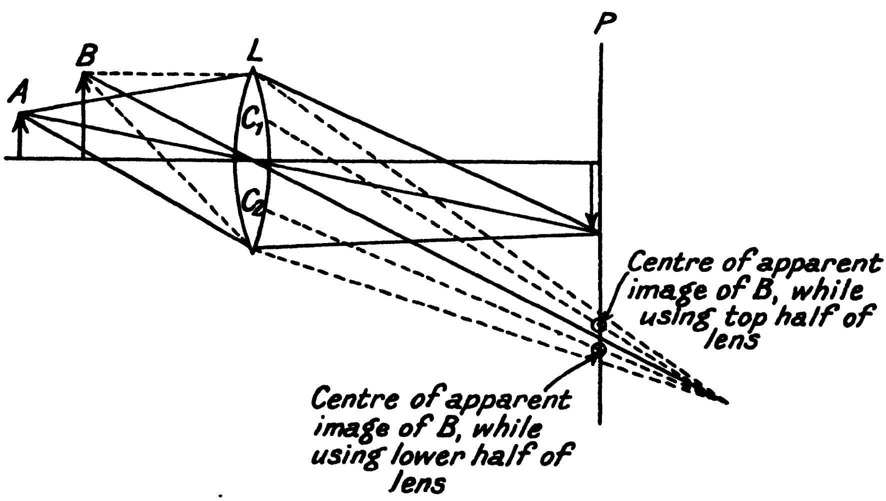


FIG. 131

the centre of the image patch on the focal plane P will differ accordingly as the top or bottom halves of the lens L are used. The two pictures are much as we might obtain from two different lenses with centres at C_1 and C_2, which are really the perspective centres. We can look on the top or bottom half of the lens as two thin lenses plus a prism of small angle; thus it happens that these perspectives are superposed.

Let us now go back to the diagram of the optical arrangements in the microscope; let screens be introduced to cover the nasal halves of each pupil; they are shown imaged in front of the nasal

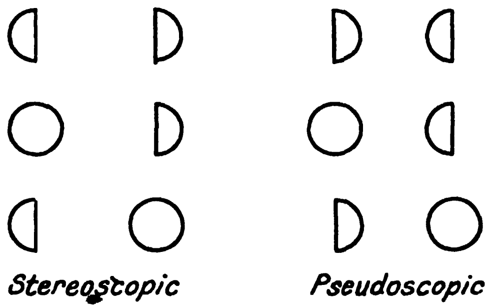


FIG. 132

parts of projected pupils R′ and L′. Consequently, the perspective centre will be on the C_1 side for R′, and on the C_2 side for L′, where C_1 and C_2 are towards the temporal sides of the projected pupils. A true stereoscopic effect will result.

On the other hand, if the screens had covered the temporal sides of each real pupil we should have the conditions suggested in the smaller figure (Fig. 130 (*b*,)); speaking of the projected pupils we now see that the centre for R′ is towards C_2, and that for L′ is towards C_1; it is clear that the effective viewpoints are now interchanged, left for right, and a pseudoscopic effect must result.

Abbe pointed out that it is unnecessary to consider the particular means by which the separation of the centres is made, i.e. whether by a prism dividing the beam from the objective, or whether by screens in the pupils. Nor need the differentiation of the centres be effected except by a screen in *one* pupil, unless the fullest degree of relief is required. If the pupils appear as on the left of Fig. 132, the effect will be stereoscopic; if as on the right, a pseudoscopic effect will be seen.

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CHAPTER V

PHOTOGRAPHIC LENSES

PHOTOGRAPHIC lenses are usually required to project an image on a flat plate or film held perpendicular to the optical axis. As far as *requirements* are concerned, they may be divided into a number of groups of which the first comprises snapshot cameras. These are required to allow "snapshot" exposures of, say, $\frac{1}{25}$th sec. in medium sunlight, or even in the absence of direct sunlight on a bright day. This requires a minimum aperture or *stop number* of "$f/11$," with the photographic materials available at the present time. Such lenses are usually of the "landscape" type, consisting of a single or sometimes an achromatic lens with a suitable stop.

The stop number is the quotient of the focal length of the lens divided by the diameter of the effective stop or diaphragm. See below. The focus is usually non-adjustable, and great depth of focus in the object field is required.

The pictures produced by such landscape lenses are expected to be reasonably sharp to unaided vision, but not necessarily to bear any great enlargement. They are seldom used for architectural subjects, and some distortion in the image can be permitted.

The second group includes lenses which are fitted to more ambitious cameras for amateur work. The main requirements are improved definition, allowing of enlargement; reasonable absence of distortion in the image; focusing by visual setting; maximum relative aperture possible without undue cost. In this group we find lenses of many optical types and relative apertures, ranging from $f/8$ to about $f/4$.

In the third group which comprises lenses for Press and commercial work, the above requirements have to be considered together with the necessity of obtaining the utmost speed (and therefore the greatest relative aperture), more or less regardless of cost. Here we find relative apertures from $f/4$ to $f/2{\cdot}5$ or even $f/2$. Snapshot exposures can be obtained with such lenses in very poor light, and when sunshine is available fast moving objects can be photographed with exposures as short as $\frac{1}{1000}$ sec. or even less.

Kinematograph lenses may be included in this third group. They are required to give the maximum illumination of the image, but, of course, the scale of the picture is in this case a very small one. The definition must be sharp enough to bear very great enlargement.

Lenses are also made to fulfil special requirements such as survey work, where a special freedom from distortion is required; wide angle photography, in which the image must be flat and well defined over an exceptionally wide field; process and copying work, where again special attention has to be paid to freedom from distortion, and (especially in three-colour printing) to the freedom from differences of magnification for different colours; in these cases, however, a large relative aperture may not be called for.

Projection lenses have optical principles similar to those of photographic lenses, although the path of the light is reversed and the object is not self-luminous.

Lastly, the class of telephoto lenses have the function of projecting an image, of a distant object, larger than can be obtained from an ordinary lens with the same working distance. This is done by employing a construction giving an *effective* optical focal length much greater than that of the ordinary camera extension.

The Illumination of the Image. Reference may be made to Chapter VI for the fundamental photometric concepts; it will be convenient to deal with the photometric aspects of photographic lens theory immediately. In photographic lenses the diaphragm is within or close to the lens system; the entrance pupil is usually the image of this diaphragm seen from the front, and the exit pupil is the corresponding image seen from the rear. Let us consider a small elementary object of area σ, and of brightness B perpendicular to the axis in the object space. Fig. 133 will be of help. The corresponding image is of area σ'. If m is the linear magnification, then

$$\sigma' = m^2\sigma$$

Let p be the radius of the entrance pupil, then the amount of light entering the lens from the elementary object area is (if q is the distance of the object from the entrance pupil)

$$\frac{\sigma B \pi p^2}{q^2}$$

This light suffers partial absorption by the lens, and a fraction k reaches the image, where the amount of light falling on unit area (the *illumination*) is

$$\text{Illumination of the image} = \left(\frac{k\sigma B\pi p^2}{q^2}\right) \div m^2\sigma$$
$$= \frac{kB\pi p^2}{m^2 q^2}$$

In accordance with the usual notation, the symbol x denotes the distance of the object from the first focal point of the lens. Then we have

$$q = (f + vf + x), \text{ say,}$$

where vf is the small distance of the first principal point from the entrance pupil; but since $m = \frac{h'}{h} = -\left(\frac{f}{x}\right)$, (see Vol. I, page 40), we get

$$q = f\left\{(1 + v) - \frac{1}{m}\right\}$$

and

$$mq = f\{(1 + v)m - 1\}$$

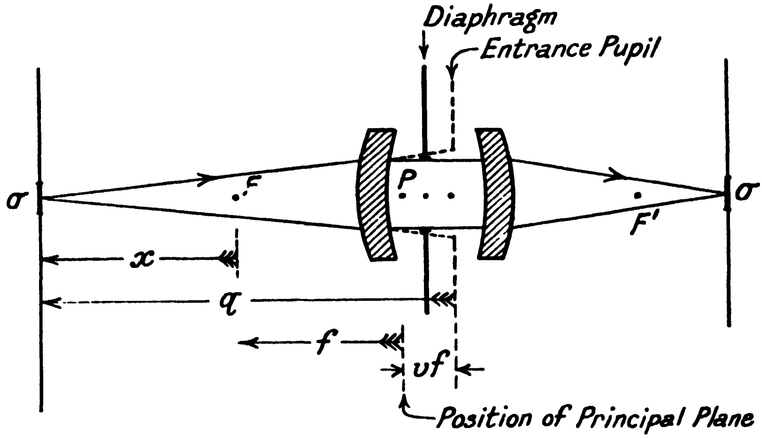


FIG. 133

The expression for illumination is then

$$\text{Illumination} = \frac{kB\pi}{\{(1 + v)\,m - 1\}^2}\left(\frac{p}{f}\right)^2$$

The fraction $\left(\frac{p}{f}\right)$ is half the reciprocal of the stop number, since the stop number was defined above as the quotient of the focal length divided by the diameter of the effective stop or diaphragm. Note that we measure the *entrance pupil,** i.e. the image of the diaphragm seen from the front of the lens, and not the diaphragm seen directly. It is also to be noted that m will be numerically negative for an inverted image, and that when the object is infinitely distant its value will be 0. The value of v will be negligible in many practical cases.

* It may be measured in practice with the help of a reading microscope with a fairly long focus objective.

Illumination of the Image Away from the Axis. Consider the case when the lens projects the image of an extended uniformly diffusing surface of brightness B. A small area ρ (Fig. 134) sends light to the entrance pupil so that the rays make an angle a with the axis of the lens, and with the normal to the surface at ρ. The amount of light dF radiated by the element into a solid angle $d\omega$ in this direction is (page 207)

$$dF = \rho B \cos a \, d\omega$$

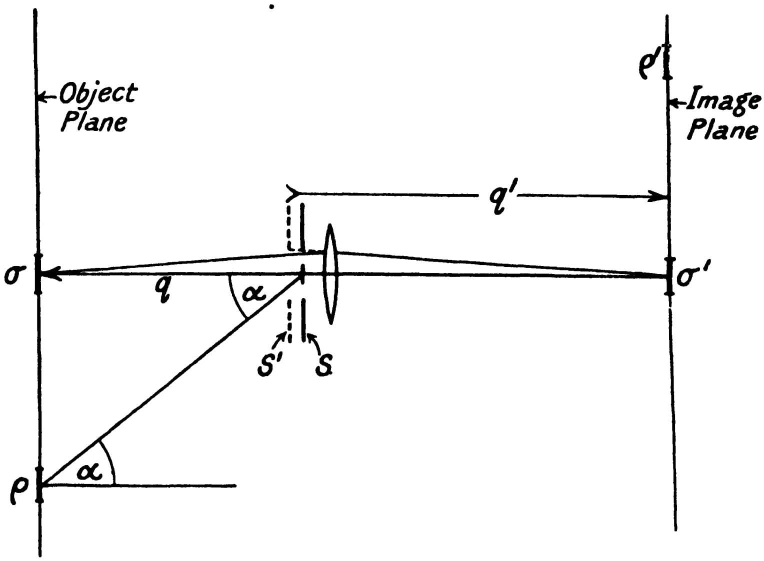


Fig. 134

and the solid angle subtended by the pupil is

$$d\omega = \frac{\pi p^2 \cos a}{(q \sec a)^2}$$

where q is the distance of the entrance pupil from the object plane, and p is the radius of the pupil.

Hence $$dF = \pi B \rho p^2 \cos^4 a$$

In cases where there is no distort'on, the area of the corresponding image patch will be uniform and independent of a. Hence the illumination of the image plane will vary as $\cos^4 a$, even if vignetting by the diaphragm does not occur (see below). It is easily calculated that $\cos^4 a$ becomes 0·56 at 30°, and 0·25 at 45°.

The effects of vignetting were referred to on page 107, Vol. I. It may be illustrated by reference to the symmetrical system shown in Fig. 135, where two similar lenses have a stop between them. If the stop is opened out to full aperture so that an incident bundle of rays parallel to the axis is completely transmitted, we see that the actual separation produces such vignetting that, when a bundle of the same diameter is incident at about 30° with the axis, the part of the back lens transmitting light is limited to the lune-shaped area shown in the side projection. The effect of the vignetting in reducing the *relative* illumination of the outer parts of the image can be avoided by using a sufficiently small stop. Wide-angle lenses are used with small stops for this reason.

Ordinary modern lenses will rarely transmit any rays at all at greater angles with the axis than 45° or 50°. The diagonal of the

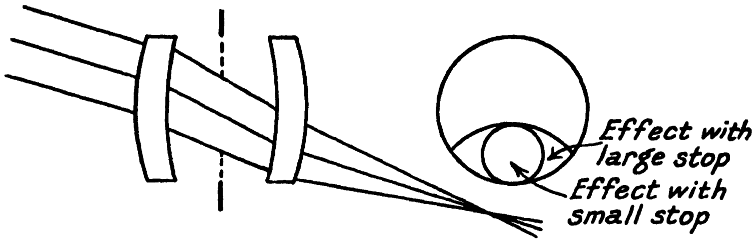


FIG. 135

plate may be expected to subtend an angle of 40° to 50°, so that the most oblique rays will not make angles with the axis much over 25°. The loss of light, as well as the optical aberration, sets a limit to the area which can be usefully covered by the lens.

Requirements for the Formation of Images on a Plane. Fig. 136 shows an optical system which is forming an axial image B′ of an axial object B. The system is free from spherical aberration for these conjugate points; hence all rays from B pass through B′.

If now, in addition, we have another object point B_1, situated in the plane perpendicular to the axis through B, and near to the axis, the condition that the corresponding image B_1' may be sharply defined was worked out in Vol. I, page 110, and is known as the *sine condition.* Let ω, ω'; ω_1, ω_1', be the angular inclinations of corresponding rays passing through B and B′; then the sine condition states that

$$\frac{\sin \omega'}{\sin \omega} = \frac{\sin \omega_1'}{\sin \omega_1} = \text{constant.}$$

This relation must, therefore, be fulfilled if we are to obtain good definition of the images of points surrounding B, wherever these images are situated.

The sine condition is the necessary criterion for freedom from "coma," or differences of magnification for different zones of the lens, but its fulfilment does not, as shown in Vol. I, page 135, secure freedom from oblique astigmatism, which often tends to arise on account of the differential limitation of the perpendicular width of a bundle of rays passing obliquely through a round stop. The foreshortened width in the "tangential" direction is $d \cos a$ (where d is the diameter of the stop, and a is the inclination), as against a width of d in the sagittal direction. If a pencil goes centrally through a thin lens, the same approximate relative retardation is impressed on the central parts, as compared with the marginal parts of each section; since the "tangential" section is the narrower, we can visualize a refracted wave of greater curvature and shorter focal distance than for the sagittal section. In Chapter VIII of Vol. I,

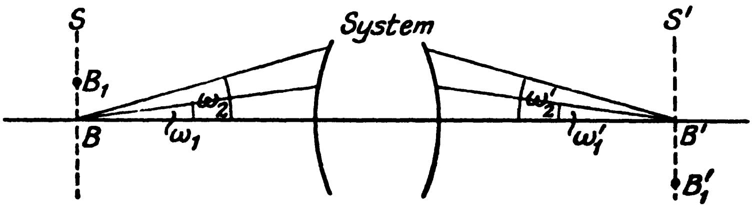


FIG. 136

pages 300 to 305, it was shown, however, that the effects of astigmatism, at least for a very narrow bundle, could be removed by the use of an axial stop at a finite distance from the thin lens when the latter was bent into a suitable meniscus form concave to the stop. We may, therefore, understand that there are similar possibilities in connection with more complex lens systems if a suitably disposed stop or diaphragm is allowed.*

Provided that astigmatism is eliminated, the sharp images are found in cases of simple lens systems to lie on the so-called Petzval surface. From the equation (52) (Vol. I, page 139) it follows that the radius R of this surface (assuming a flat object plane as in our case) is given by the equation

$$\frac{1}{R} = -\frac{1}{n_1 f'_1} - \frac{1}{n_2 f'_2} - \text{etc.}$$

where n_1, n_2, are the refractive indices of the glasses of which the successive lenses are composed, and f_1, f_2, are the focal lengths which would be found for the lenses if calculated from their radii and refractive indices while neglecting the thickness. According to

* Appendix III gives a short discussion of the curvatures of the tangential and sagittal image fields by the simpler "third order" theory. The results are, however, only reliable for rays making small angles with the axis.

the simple first order theory, the sharp image should be flattened, then, by choosing the glasses and radii to make R infinite. In practice, the effect of finite thicknesses in the lenses causes some departure from this provision of the simple theory.

Distortion. Provided then that we satisfy all the above requirements, we may expect to find a reasonably sharp image in the neighbourhood of the axis, which would fall into focus on a flat plate; the definition might be expected, however, to deteriorate when we exceed a certain distance from the axis, where the angular inclination of the rays with the axis invalidates the simpler theory.

We have, however, still to inquire whether the image in the plane S′ will be geometrically similar to that in S. We know that the scale

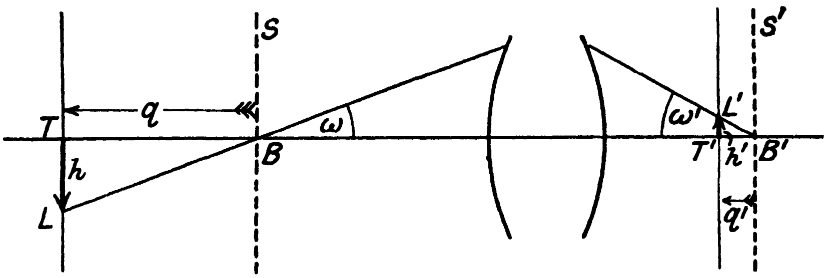


FIG. 137

of the magnification in the plane of B′ is given by $\frac{(\sin \omega)}{(\sin \omega')}$ at least for very small objects, so that there should be geometrical similarity to the object in the close neighbourhood of the axis. Our theory does not allow us to investigate the presence or absence of distortion far from the axial region in these planes S and S′, but as we know that photographic lenses are often required to work at all kinds of relative conjugate distances, it will be of interest to consider a second pair of conjugate planes through T and T′, which we may first assume to fulfil all the requirements of the foregoing paragraphs. These are shown in Fig. 137. Now a ray through B in the object space passes through B′ in the image space, and it passes through the planes through T and T′ in two points L and L′ at distances h and h' from the axis; we will suppose that L′ represents a reasonably sharp image point. Then, the ratio h'/h represents the magnification and is given by

$$\frac{h'}{h} = \frac{q' \tan \omega'}{q \tan \omega}$$

where q and q' are the distances from S and S′ to T and T′ respectively. In order that the magnification ratio may be constant for all sizes of object, we must have relations such as

$$\frac{q' \tan \omega'}{q \tan \omega} = \text{constant for any values of } \omega \text{ and } \omega'.$$

But we found above that

$$\frac{\sin \omega'}{\sin \omega} = \text{constant}$$

was the necessary condition for sharp definition in the planes through S and S′. Since the above conditions are incompatible, we find at once that even if the distortion is completely corrected for planes containing a pair of *truly* aplanatic points, there *must* be definite distortion in any other pair of planes. This conclusion

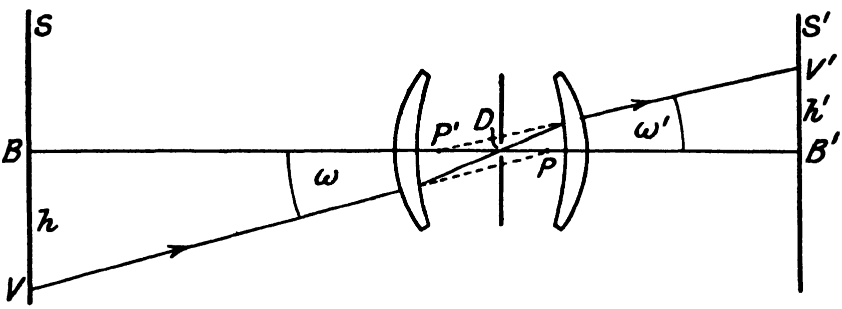


FIG. 138

is justified in practice, but in cases where the extreme angles ω' and ω do not exceed about 4° or 5°, the difference between the sines and tangents will be less than 0·3 in 100, so that the distortion would not necessarily be serious for small aperture systems.

In practice, modern photographic lenses are *not* completely corrected for spherical aberration; this aberration has to be kept below certain limits, but these limits are wide enough to make it much easier to achieve freedom from distortion and other aberrations in a way which is impossible if *strict* aplanatism is called for.

If, then, spherical aberration is to some extent permissible, the position of an image "point" will be fixed by the principal ray through the centre of the diaphragm. We will, therefore, consider a case shown in Fig. 138, where the centre of the diaphragm is situated at the point D. Let us trace a ray through this point in both directions to its intersection with the planes S and S′, which represent the object surface and the photographic plate respectively. The resulting intersection points V and V′ are now object and "image." If the image is subject to spherical aberration, its position is still marked by the intersection of the principal ray with the plane S′.

Looking into the front of the lens, the axial point of the entrance pupil P would be seen as the image of D. From the rear, the exit pupil would be seen at P′. (Vol. I, page 106.)

Now it is likely that the axial positions of the entrance and exit pupils will be to some extent dependent on the inclination of the principal ray through D; these images may be subject to the effects of spherical aberration. Let the Gaussian positions of P and P′, calculated for very small inclinations of the principal ray, be denoted by P_o and P'_o. Let ω and ω' be the axial inclinations of the principal ray in the object and image spaces, while x_o and x'_o are the intervals P_oB and P'_oB' from the Gaussian pupils to the conjugate planes respectively; also let $PP_o = \delta$, and $P'P'_o = \delta'$. Thus PB $= x_o + \delta$ and $P'B' = x'_o + \delta'$. Then object and image heights are related by the equation

$$\frac{B'V'}{BV} = \frac{h'}{h} = \frac{(x'_o + \delta')\tan\omega'}{(x_o + \delta)\tan\omega}.$$

In the case of an indefinitely small object and image let

$$\underset{(h = o)}{\text{limit}}\ \frac{h'}{h} = m_o$$

Then a convenient specification of the amount of the distortion D is

$$D = \frac{1}{m_o}\left\{\frac{(x'_o + \delta')\tan\omega'}{(x_o + \delta)\tan\omega}\right\} - 1$$

This specification is used in Wandersleb's important papers[1] (1907) on distortion, in which the distortion of most of the important photographic lenses of that period is given graphically in terms of D for given angles with the axis.

When the object is at an infinite distance we need to transform the above equation. Remembering from Vol. I, page 40,

$$m_o = \frac{h'}{h} = -\frac{f}{x}$$

$$= \frac{f'}{x_o}, \text{ for a lens in air, sufficiently nearly}$$

so that $$m_o(x_o + \delta) = f' + m_o\delta$$

$$= f'$$

in the limit since m_o is indefinitely small, and δ is also small. Thus

$$D = \left\{\frac{(x'_o + \delta')\tan\omega'}{f'\tan\omega}\right\} - 1$$

Wandersleb uses a value N in his discussion which is the reciprocal of the magnification *m*.

If the discussion of Chapter IV, Vol. I, is carefully re-read it will be seen that the expression in equation 50 represents the optical path difference between distances derived from the centre, and from a point in the effective aperture, of the exit pupil *when they arrive at the point given by the intersection of the auxiliary optical axis with the Petzval surface.* Even if the coefficients a_1, a_2, a_3 are zero an optical path difference remains; and thus when the transverse aberration $T'y$ is calculated, a term remains which is proportional to the cube of the image "height," indicating a displacement of the image in a radial direction which does not depend on S and therefore affects the principal ray itself.

Analytical calculations of distortion may be made by the help of formulae quoted in books such as Conrady's *Applied Optics and Optical Design*, but they are not of a simple character, and fail to give useful results when thick lenses and very oblique rays are dealt with, so that they have to be supplemented by empirical calculations. The type of distortion arising in simple cases can often be understood from first principles. No distortion results from thin lenses passed centrally by oblique pencils. If a stop is used with a thin convex lens behind it, the deviation towards the axis of the oblique principal rays tends (for usual forms of lens) to be over-great, and "barrel" distortion, resulting from the lower magnification for the greater image height, results. This is the most common form. A similar thin convex lens in front of a stop produces pin-cushion distortion. The "symmetrical" combination is more likely to be distortion-free, and it is important to retain a symmetrical design in any case where the utmost freedom from distortion is essential, as will be understood when symmetrical systems are discussed below.

Tolerance for Distortion. Distortion may be expressed as above by the percentage

$$100\,D = 100\left(\frac{m}{m_0} - 1\right) = \text{percentage distortion.}$$

In the best symmetrical form lenses the distortion may be kept within 0·1 per cent up to 30° or more with the axis. The value is sometimes greatest for an intermediate inclination of the pencils. With the "Hypergon" (see below) it is only 0·3 per cent at 54° with the axis. It is more difficult to control with the unsymmetrical forms. The curves for an early Cooke lens are given below, but much smaller amounts are found with later lenses. Thus with one form Wandersleb finds the distortion to be zero at 27° with the axis, and reaching a maximum of $\pm$ 0·6 per cent. The tolerance for distortion varies enormously with the object of photography. Several units per cent can be tolerated for landscapes, but the requirements for architectural subjects are much more stringent, say, 0·5 per cent to

1·0 per cent. Photographic mapping requires distortion within 0·1 per cent if possible.

The Landscape Lens. The lenses fitted to inexpensive hand cameras are often of simple meniscus form; they are placed at a suitable distance behind a circular stop. There is no chromatic correction; the focus is fixed once for all, reliance being placed on the fact that the spectral region of actinic activity for *ordinary* photographic emulsions is fairly well defined in the spectrum.

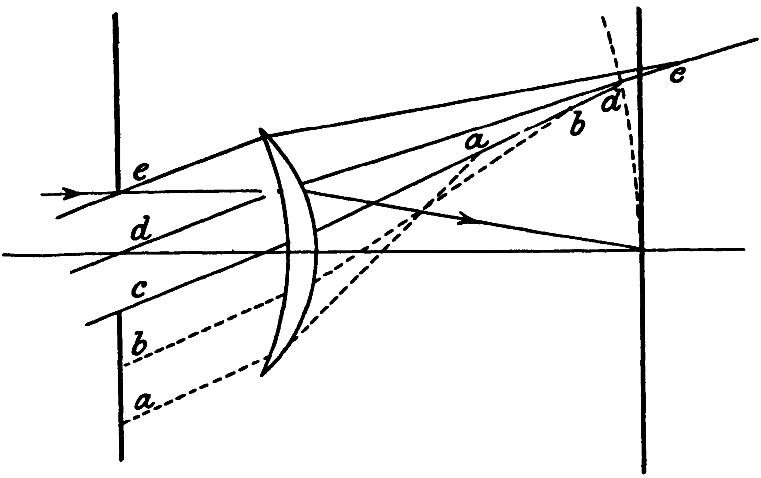


FIG. 139. MENISCUS LANDSCAPE LENS

Fig. 139 shows a diagrammatic section of such a lens, traversed by an oblique bundle of parallel rays. Such a lens suffers in the first place from spherical aberration of the ordinary "under-corrected" type which causes the outer rays to reach the focus closer to the lens than the inner ones. In this case, however, *coma* is also present, the upper part of the lens is traversed more or less symmetrically by the rays in the diagram; the deviation produced by a prism, it will be remembered, is a minimum where there is such a symmetrical disposition. This then tends to lengthen the focus in the upper part, but the rays in the lower part traverse the lens very unsymmetrically, so that there is relatively great deviation in the sense suggested by the figure. Hence, the coma opposes the spherical aberration in the upper part and supports it in the lower part of the lens under the conditions drawn.

For this reason a stop is placed in front of the lens which limits the oblique pencils, as indicated in the diagram; the spherical aberration and coma thus tend to balance their effects in the oblique pencils. Since the spherical aberration must be tolerated, the allowable amount sets a limit to the useful diameter of the stop. $f/11$ can be attained.

Nor is this all; the theory of Chapter VIII, Vol. I, indicated such a disposition of lens and stop to overcome the effects of astigmatism; the problem in designing such a lens consists of striking a suitable balance between the aberrations. It can be shown that a favourable over-correction for astigmatism is obtained when coma and spherical aberration are balanced in the above way, by the disposition of the stop.

Astigmatic Corrections. We will suppose that conditions allow us to abolish astigmatism, or even to over-correct it in a simple lens system.

Fig. 140 shows the cases of under-corrected, corrected, and over-corrected astigmatism. In the first case, the Petzval surface has a

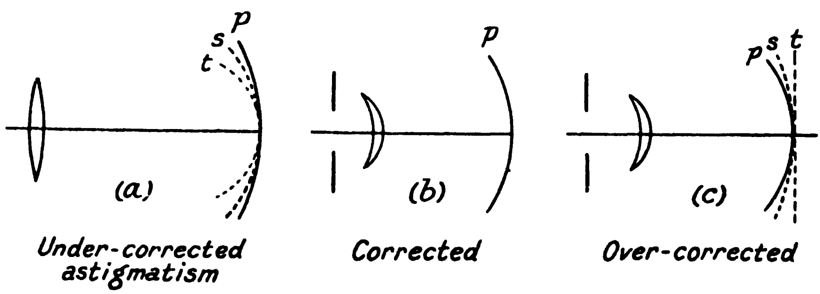


FIG. 140

negative radius of curvature, and the tangential and sagittal image surfaces are similarly disposed; it will be remembered that the distance between the tangential and Petzval surfaces is always three times that between the sagittal and Petzval.

In the second case the astigmatism is corrected, and the sharp image lies on the curved Petzval surface.

In the third case the astigmatism is so far over-corrected that the tangential field is flat while the sagittal one is slightly round; this condition represents a very usual compromise in the direction of getting a flat field for the image. It has been tried in some cases to go still further, so that the tangential and sagittal fields are symmetrically disposed on each side of a plane surface, but this is only at the disadvantage of a greater amount of astigmatism which itself causes a severe loss of definition. The apparently obvious method of improving matters, when more complex lenses are used, is to choose glasses, such that the Petzval curvature will be zero or approximately so: it will be found in practice, however, that the finite thicknesses of the lenses and the effects of the oblique aberrations away from the axis are such as to make the theorem only a very rough guide.

Achromatic Landscape Lenses. In seeking to improve on the single meniscus lens it is natural to seek to produce an achromatic combination which will allow the image to be focused visually. In this case we shall equalize the focus for the lines G′ and D, so that the V values used for the formulae will need to be recalculated. Our new V will be found from

$$V' = \frac{(n_f - 1)}{(n_g' - n_d)}$$

This brings together the brightest visual region of the spectrum and the region which has the greatest action on ordinary photographic emulsions. The formulae now used are of the same form as those of Vol. I, page 228. In practical designs the lens has still an outer meniscus form, with the concavity toward the stop (Fig. 141).

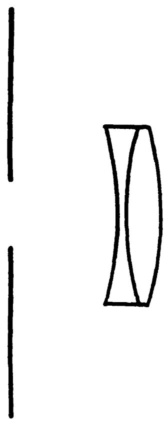


FIG. 141. ACHROMATIC LANDSCAPE LENS

Old and New Achromats. When the glasses used in achromatic combinations have a large difference of V values, it is possible to keep fairly shallow curves for the lenses, and thus to avoid zonal spherical aberration. This is a very important matter in telescope objectives, and keeps the "old achromat" crown and flint combination, with its large difference of V values, of outstanding importance.

Some of the more recently produced glasses have different optical properties. Let us compare the following combinations—

Glass	n_D	V	α	β	γ
1 { Hard crown	1·5186	60·3	·295	·705	·569
1 { Dense flint	1·6041	37·8	·286	·714	·606
2 { Dense barium crown	1·6089	57·4	·294	·706	·572
2 { Light flint	1·5472	45·8	·291	·709	·591

We notice that in the second pair, the glass with the higher refractive index has the higher V value (or lower relative dispersion). Calculating an ordinary achromatic combination from each, we find for unit focal length (*a* and *b* standing for crown and flint respectively)—

Combination 1 $\begin{cases} f'_a = 0·373 \\ f'_b = -0·595 \end{cases}$ (Old achromat)

Combination 2 $\begin{cases} f'_a = 0·2021 \\ f'_b = -0·2533 \end{cases}$ (New achromat)

We evidently have much shorter focal lengths and heavier curves in the units of combination 2, a type of "new achromat." On the other hand, the much closer correspondence between the partial dispersions of the glass used in the new achromat would ensure a

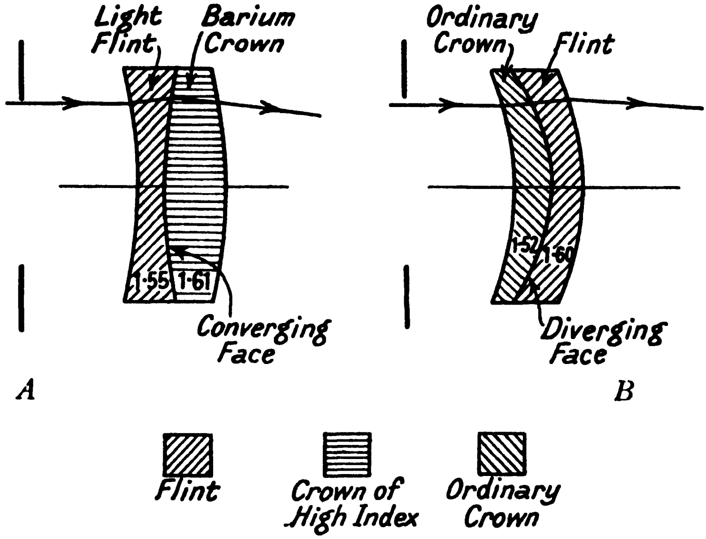


FIG. 142 "NEW" AND "OLD" ACHROMATS

A Achromat of new glasses Contact surface has converging tendency. (Diagrammatic only) Such a landscape lens will cover a 90° field at *f*/16)
B. Achromat of old glasses. Contact surface has diverging tendency. (Diagrammatic only)

much less pronounced secondary spectrum. Calculating the radius of the Petzval surface in each case we find by the usual formula,

$$\frac{1}{R} = \Sigma\left(\frac{-1}{nf'}\right)$$

R = – 1·38 for the old achromat, and R = – 1·94 for the new. Hence the image field, if corrected for astigmatism, will be much flatter with the new achromat than with the old. A diagram (Fig. 142) of possible arrangements of the old and new achromats will help to show that the contact surface in the new has a converging tendency, while in the old the contact has a diverging tendency. On account of the flatter Petzval surface, and the different action of the contact face, it proves to be easier to give satisfactory "photographic" corrections to a new achromat than to one of the older type. A well-designed landscape lens will cover a 90° field at an aperture *f*/16, but the image will be subject to distortion.

Anastigmatic Correction. In Vol. 1, page 302, we obtained the following expression for the astigmatism arising by refraction at a surface—

$$\frac{Q_l^2}{(Q_l - Q_t)^2}\left(\frac{1}{n't'} - \frac{1}{nt}\right)$$

It is clear that the squared terms will always be positive, and therefore the sign of the astigmatic contribution will be given by the second bracket term—

$$\left(\frac{1}{n't'} - \frac{1}{nt}\right)$$

The symbols t' and t may be taken as the conjugate image and object distances respectively for the surface in question. Clearly, in the case where we have a converging surface with the object distance negative and the image distance positive, both terms in the equation would be positive.

For small angles of incidence, the tendency of a convergent face is to give a positive sign to the bracket, while a negative or diverging face would give a negative sign, although contributions given may vary in sign if the incident light is very convergent or divergent. Students should work out typical cases.

Now both with the "old achromat" and the "new achromat" type of landscape lenses it is possible to "flatten the field" in the sense that the usual astigmatism may be over-corrected so that the tangential field is flat or even "hollow" (i.e. turning its convexity towards the lens) but with the ordinary constructions the astigmatic error constantly increases in the same sense towards the margin of the field, as does also the spherical aberration.

If the Petzval surface is "round" as seen above, the flattening of the tangential field can only be achieved if a considerable amount of actual astigmatism is tolerated.

The "concentric" lens designed by Schröder about 1887 and made by the firm of Ross, achieved fairly flat image fields by the choice of suitable glasses. but only at the expense of a large amount of under-corrected spherical aberration.

In seeking to improve on these conditions Steinheil, and also P. Rudolph, evolved (about 1881 and onwards) more or less symmetrical combinations in which thick doublet lenses of outward meniscus shape were mounted on each side of a stop, the concavity of each meniscus being turned towards the stop. With the aid of this construction it was found possible to correct spherical aberration for one zone of the system, chiefly through the opposing action of the inner contact faces, although there was considerable residual zonal aberration. Under these conditions the astigmatism, even if it were of the usual under-corrected type for small angles of the field, and showed at first a tendency to give "round" fields, was now found to be subject to a correcting influence at higher angles so that the fields tended to become hollow. Even with the old glasses it is possible with special constructions to make the tangential and sagittal fields intersect towards the edge of the field, although the residual bending of the image surfaces is large in other regions.

Great improvements were possible by the employment of the new glasses. An "old achromat" used in front of the stop, with a "new achromat" behind, gave the possibility of a flatter Petzval surface,

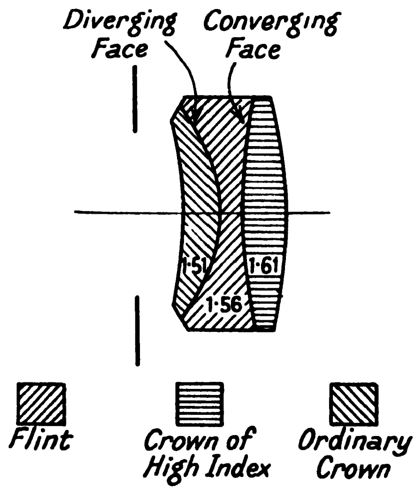


FIG. 143. "TRIPLE CEMENTED TYPE"

and less difficulty in the correction. Lenses of this type were made by the firm of Zeiss, and in such cases it may be considered that the correction is largely secured by the opposing action of the two contact faces, the negative astigmatism of the first with its diverging action acting against the positive astigmatism of the second (converging) contact.

The next step taken by Rudolph, about 1891, was to combine in a triplet lens, three glasses as shown in Fig. 143. The refractive index shows a step upwards at both contact faces, one being diverging and the other converging. It is thus possible to secure anastigmatic correction by the opposing action at these contact faces. Rudolph also showed how it was possible to get a similar effect with other constructions; the middle lens of the three might be "double convex," while the refractive indices are still progressive.

The "Amatar" lenses of Messrs. Zeiss are of the type shown in Fig. 143.

The use of lenses of this type in a symmetrical system was worked out by von Höegh independently, and patented by the firm of Goerz in 1892. (Goerz double anastigmat.)

Still further improvement was found possible by Rudolph in 1894 when the combination now used in the "Protar" lenses of Zeiss was patented. From Fig. 144 it will be seen that a modified "new achromat" is combined with an "old achromat" combination. Although the general construction may be so described, the pairs are not now achromatic in themselves, but achromatism is established for the lens as a whole. There are now three contact faces, and it is possible to secure a very complete anastigmatic correction

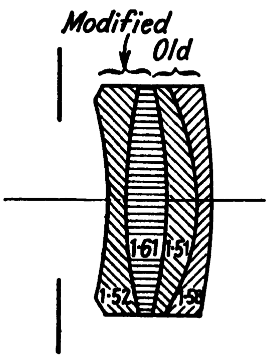


FIG. 144. COMBINATION OF MODIFIED NEW AND OLD ACHROMAT IN THE PROTAR LENS (ZEISS)
(Designed by Rudolph)

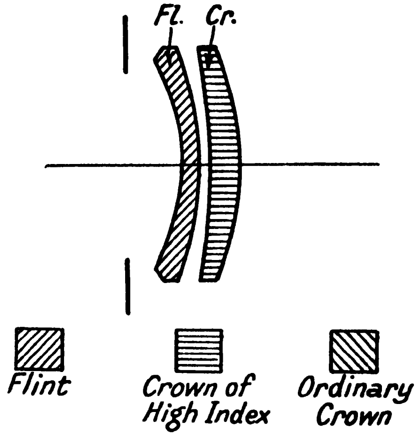


FIG. 145. ANASTIGMATICALLY CORRECTED, SEPARATED DOUBLET

over a field of 30° on each side of the axis at $f/12{\cdot}5$. The lens will cover a 90° field if used with a smaller stop, and since the stop is very close to the system the distortion is small.

Yet another method of securing anastigmat correction is to separate the components of a landscape lens as shown in Fig. 145. The rear faces now have a converging tendency and the front faces a diverging effect, so that opposing spherical and astigmatic contributions may be obtained by the choice of suitable bendings.

Symmetrical Lenses. Important advantages are obtainable from the employment of systems which are built symmetrically with a central stop. In Fig. 146, let X and Y be two lenses of such a system; the central stop is at R. Consider two parallel rays, AB and CD, between the lenses, symmetrically situated with regard to R. If these are traced through the lenses their crossing points B_1

and B_1' in the object and image region must also be symmetrical with regard to the system. Also, there must be some ray through the centre of the stop, which will pass through both B_1 and B_1', Hence the system must be free from coma for one zone at least, when the image size is exactly equal to the object size as must now be the case; further, the exact equality of object and image dimensions must result in the complete elimination of distortion.

Also, since the above reasoning would apply to rays of differing

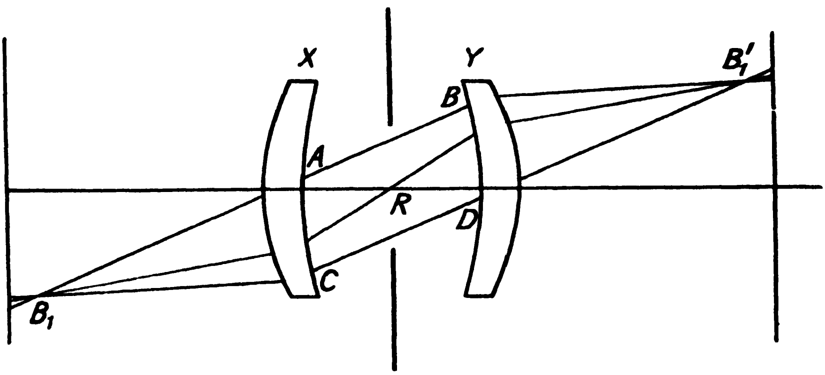


FIG. 146. SYMMETRICAL SYSTEM

wave-length, the image must now be free from chromatic difference of magnification.

The symmetry of the system does nothing to remove the effects of axial chromatic aberration, spherical aberration, astigmatism or curvature of the field, but the freedom from coma, distortion, and chromatic difference of magnification is a very important advantage.

Let us now consider the conditions when the object and image are situated at different distances from the system. In the first place the parts of the principal ray outside the lenses will be parallel to each other, so that $\omega = \omega'$ (see equation above). The absence of distortion then requires $\delta = \delta'$, or a constancy of the positions of the images of the centre point of the stop for all inclinations of the principal ray. A similar argument applies to differences of magnification due to colour; in order to avoid these, the position of the entrance and exit pupil should be independent of wave-length. Further, the argument which established the freedom from coma is no longer valid.

In all these cases, however, the symmetry of the system is a very great advantage. If the separate systems are moderately well corrected in themselves they can be combined in the symmetrical manner without fear cf introducing serious coma, and with the reasonable assurance that the distortion and any chromatic difference of magnification will be made practically negligible. Moreover, the shortening of the focal length due to the combination produces a lens working satisfactorily at double the relative aperture, or more.

The Hypergon lens of *Goerz* (Fig. 147) consists of two deep menisci, made of one glass, of which the outer surfaces very nearly form a sphere. Owing to the absence of chromatic correction it must be used with a small stop, but the symmetry of the system removes the coma, while the astigmatism, as discussed above, page 177, can be removed in such a system by a suitable position of the stop and bending of the lenses. The smallness of the stop produces a sufficient depth of focus to allow satisfactory definition on a flat plate, and the lens has a total field of 135° free from coma, astigmatism, and distortion. It is, therefore, suitable for photographic surveying.

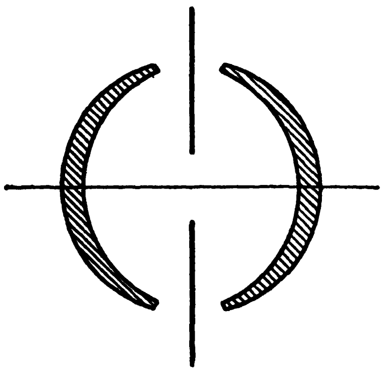


FIG. 147. THE HYPERGON (DIAGRAMMATIC ONLY)

A "Rapid Rectilinear" lens usually consists of a pair of lenses of the cemented landscape type mounted symmetrically with a stop between them. The system is, however, not so important as it used to be, owing to the round field and lack of the anastigmatic correction which can be attained by lenses of simpler construction as will be seen below.

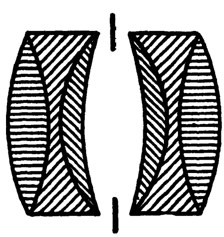


FIG. 148. THE GOERZ DOUBLE ANASTIGMAT

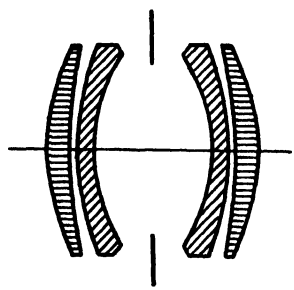


FIG. 149. THE ROSS HOMOCENTRIC LENS

Symmetrical anastigmats are still of importance. The Goerz Double Anastigmat (Fig. 148) will be recognized as the combination of two lenses similar to that of Fig. 144. The Ross Homocentric (Fig. 149) is a combination of two pairs of separated doublets. The

Double Protar of Zeiss is (in one form) a symmetrical combination of two Protar lenses of the type shown in Fig. 143. The components of such systems may be used separately as long focus lenses; thus the Double Protar (*f*/6·3) has a back component which can be used as a long focus lens at an aperture of *f*/12·5. This availability of the back component is of very considerable advantage to a photographer. See also reference to the Taylor Hobson anastigmat below.

Modern anastigmats usually give good definition at full aperture over a plate of which the diagonal is equal to the focal length of the lens; i.e. a field of about 50°. With a small stop the field may be often enlarged to 70° to 80°.

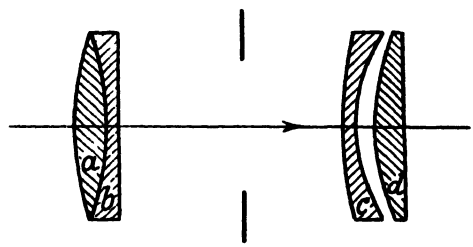


FIG. 150. THE PETZVAL PORTRAIT OBJECTIVE.

a = Crown. n_d = 1·5181 c = Flint 1·5783
b = Flint 1·5783 d = Crown 1·5152

Hemi-symmetrical Systems. Somewhat better results may in some cases be obtained by altering the scale of the two components, while their separation from the stop is adjusted proportionally. The two lenses are still independently corrected; thus the user of such a system has the choice of three focal lengths, i.e. the combination and either the front or the back component.

Asymmetrical Lenses. We can only describe a few lenses of the principal types. The *Petzval* portrait objective (Fig. 150), designed by Petzval as long ago as 1840, secures a large relative aperture, *f*/3, and satisfactory correction of chromatic and spherical aberrations, as well as chromatic difference of magnification. The coma also is small. With an aperture of this magnitude, great pains had to be taken to secure freedom from zonal spherical aberration.

Since the axial region of the image is well corrected, while the marginal points suffer from astigmatism and curvature of field, the lens is mainly suitable for portraiture where good definition is usually only necessary over a limited region to include the features of the sitter, and is even objectionable elsewhere.

Owing to the large separation of the components, there is considerable loss of light away from the axis, due to the restriction of the effective aperture for oblique pencils.

The lens is still in wide use, and is often employed as a projection lens in projection lanterns, and for enlarging.

The Cooke Lens. Perhaps the most famous type of unsymmetrical anastigmats includes the series of "Cooke" lenses, originally designed by Mr. H. Dennis Taylor, who was then optical designer to the firm of T. Cooke and Sons, of York. (The Cooke lenses are now manufactured by Messrs. Taylor, Taylor, and Hobson, Ltd., of Leicester.) The general arrangement of these

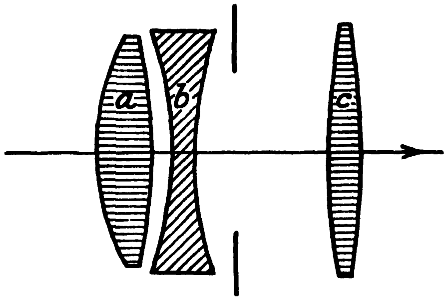


FIG. 151. THE COOKE LENS

a, c = Dense barium crown   b = Light silicate flint

lenses is shown in Fig. 151. The positive outer lenses, in one type of the system, are made of dense barium crown glass, while the negative inner lens is of light silicate flint. It is not within the scope of the present book to give a theoretical account of the optical

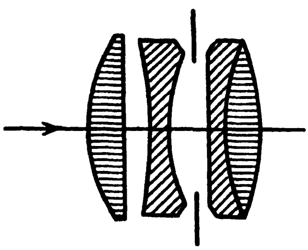


FIG. 152(*a*). THE ZEISS TESSAR (*f*/4·5)

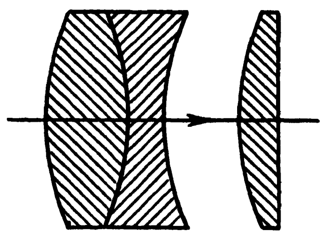


FIG. 152(*b*). THE ALDIS ANASTIGMAT

principles. The system has been modified in many ways, while retaining the same general principle of construction.

In the Zeiss "Tessar," Fig. 152(*a*), the back positive component of a similar system is made into a cemented doublet; in the Taylor Hobson *f*/2·5 anastigmat, Fig. 153(*a*), the back component is separated into two lenses which diminish the spherical aberration component due to the last lens, and allow of a bigger aperture ratio than with the ordinary three-lens system. Compare this with the symmetrical *f*/2 anastigmat of the same firm, Fig. 153(*b*). The Aldis anastigmat, Fig. 152(*b*), is a much modified case of the "Cooke" principle, the front components being cemented together, and the corrections being secured by the last lens of the system.

Graphical Representation of the Aberrations of Photographic Lenses. The multiplicity of types of photographic lenses is somewhat confusing to the would-be user, and comparatively few details of the performance of many modern lenses are available. Von Rohr's treatise[2], *Der Theorie und Geschichte des Photographischen Objectivs*, gave, however, details of the performance of many of the types

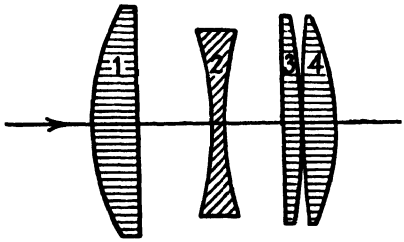


FIG. 153(*a*). THE TAYLOR-HOBSON (*f*/2·5) ANASTIGMAT

	n_d	V		n_d	V
1.	1·613	56·5	3.	1·613	58·5
2.	1·651	33·7	4.	1·613	58·5

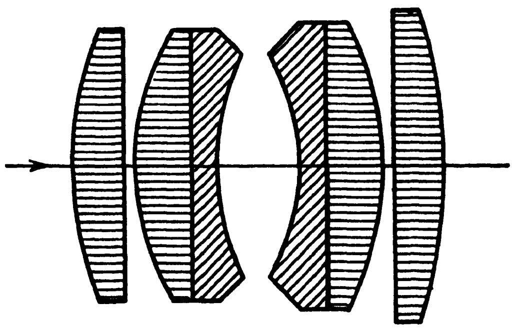


FIG. 153(*b*). THE TAYLOR-HOBSON (*f*/2) ANASTIGMAT

extant up to 1899, and since his method of presenting the facts has frequently been used since that time we may give examples here, viz. for a French landscape lens consisting of an achromatic meniscus behind a stop, next for a Cooke portrait lens, and next for an anastigmat (single) of the Protar type. See Figs. 154, 155, 156. The curves must now be explained.

In order to make the lenses comparable, the results are given in each case for a lens made on a scale which would give it a focal length of 100 mm. for the D line. Then the ordinates of the spherical aberration curves represent the incidence height in millimetres of the incident ray (parallel to the axis); the abscissae of the broken line curves are the longitudinal aberrations of the focal length, and those of the full line curves represent the aberrations of the axial intersection distances of the rays.

In the curves showing the astigmatism, the ordinates represent "Grades"* of the semi-angle of the field, i.e. the angle between the axis and the principal ray in the image space. The abscissae of the full line curves give the distances by which the focusing screen must be removed from the axial focus position, to bring the sagittal bundles into focus; the broken line represents the corresponding distance for the tangential bundles.

The *coma* is not plotted directly in these diagrams, but it will now be shown that a measure of the coma is obtained by finding

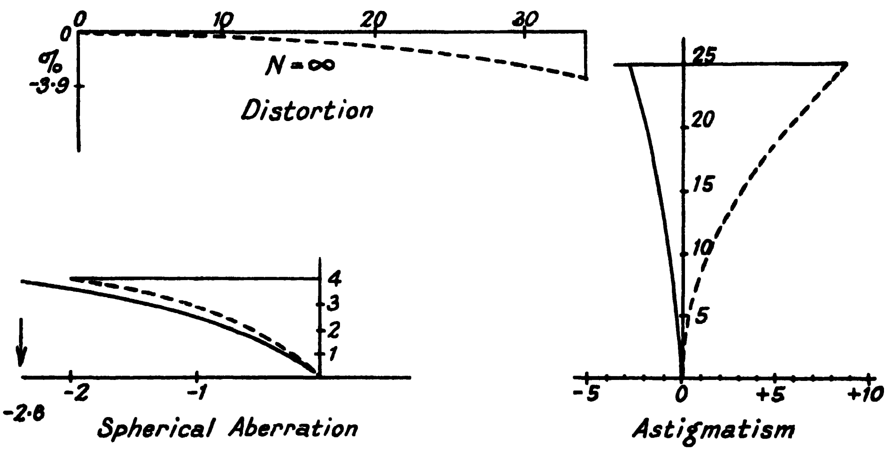


FIG. 154. FRENCH LANDSCAPE LENS ($f/15$)

the horizontal distance between the two curves, dotted and full line, in the spherical aberration diagrams. From the relatively small intercepts in the curves, it will appear that the elimination of coma is regarded as one of the most important conditions to be secured in the design of such lenses.

The Sine Relation and Coma. The investigation of the "optical sine relation," Vol. I, page 110, showed that the dimensions of object and image are governed by the relation

$$nh \sin a = n'h' \sin a'.$$

The formula strictly means that if we have an object point at a distance h from the axis, and we consider the action of a certain zone of the lens system, the corresponding "image" point in which the disturbances from the object point come together in the same phase will be situated at a distance h' from the axis given by the above formula; the angles a and a' are the angles made with the

* The Grade divides the right angle into 100 parts. At one time it was expected that this unit of angular measure would supersede the degree, but it has not come into prominence so far.

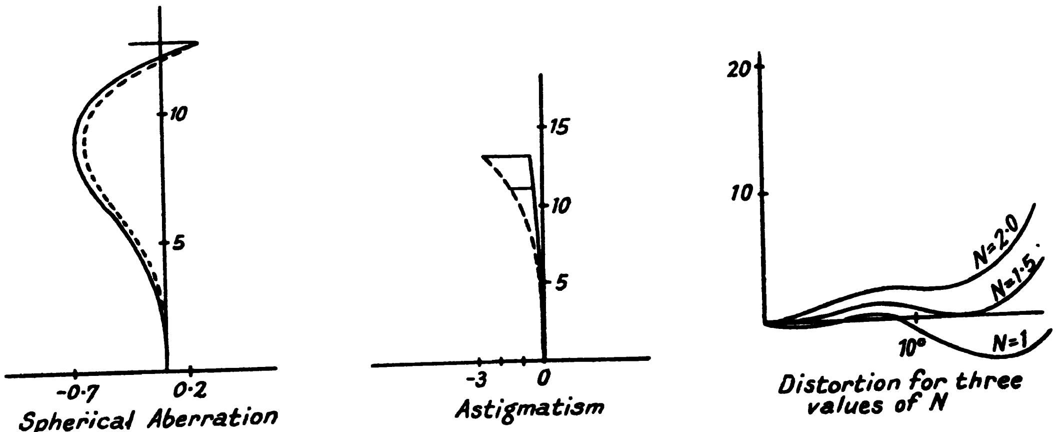


FIG. 155. COOKE PORTRAIT LENS (*f*/4) EARLY FORM

axis by rays starting from the corresponding axial point of the object plane, traversing the above particular zone of the lens system, and coming to a focus at the axial point of the corresponding image surface. In other words, the sine relation determines the physical image point for a particular zone.

Note that if the object is at a great distance, then $\sin \alpha = \frac{y}{l}$ sufficiently nearly, where y = the incidence height of a ray in the

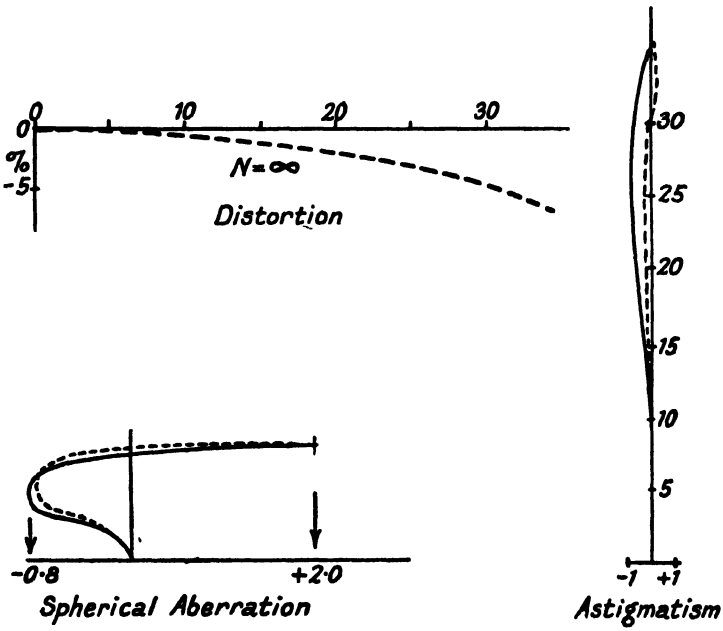


FIG. 156. RUDOLPH ANASTIGMAT, ZEISS PROTAR TYPE ($f/12{\cdot}5$)

first principal plane,* and l = distance of the object from this plane. Hence for a very distant object,

$$h' = \frac{h}{l}\left(\frac{n}{n'}\right)\frac{y}{\sin \alpha'}$$

$$= -\frac{y}{\sin \alpha'} \cdot \frac{n}{n'} \cdot \tan \omega$$

where ω is the angular distance of the object point from the axis.

This shows that for a given value of h, the different zones will

* Which we will here define as the plane perpendicular to the axis, passing through the first principal point. We are not supposing the existence of true "principal planes," as will be seen.

produce an image of the same size if $\frac{y}{\sin \alpha'}$ is constànt. Since by our ordinary conceptions of "focal length" we have

$$h' = f \tan \omega \quad \text{(Vol. I, page 47)}$$

$$= -f' \left(\frac{n}{n'}\right) \tan \omega$$

we may therefore interpret the quantity $\left(\frac{y}{\sin \alpha'}\right)$ as the equivalent focal length of the zone under consideration, and the condition that images formed by successive zones of the system shall be of the same size is that $\left(\frac{y}{\sin \alpha'}\right)$ be constant. Provided the system is

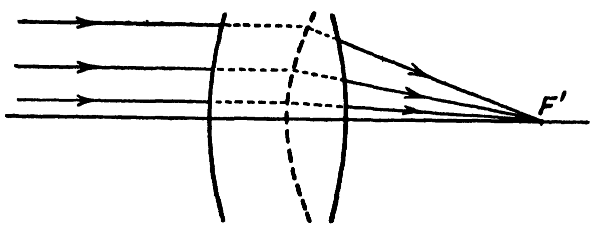


FIG. 157. THE "PRINCIPAL SURFACE" OF A COMA-FREE LENS

free from spherical aberration, this condition is enough to secure the absence of *coma* in the regions near the axis.

If coma is present, let h'_m be the "height" of an image for a marginal zone, and let h' be the corresponding value for a paraxial zone; then the amount of the coma is reckoned by the fraction—

$$\text{Measure of coma} = \frac{h' - h'_m}{h'}$$

$$= 1 - \frac{h'_m}{h'} = 1 - \frac{y_m \sin \alpha'}{y \sin \alpha'_m}$$

$$= 1 - \frac{f'_m}{f'}$$

the suffix m being used to denote "marginal zone" values. This measure of coma is also a measure of the "offence against the sine condition."

Still dealing with very distant object points, and taking a number of incident parallel rays into a system at different distances from the axis (Fig. 157), we see that the constancy of $\frac{y}{\sin \alpha'}$ will require that the incident and emergent rays shall intersect each other in a spherical surface. The second "principal surface" of the system is, therefore, a sphere centred in the principal focal point.

Meaning of the Sine Relation in the Presence of Spherical Aberration. In Fig. 158 we represent a lens system with the apex A of the last surface, the stop, or exit pupil, with centre R, and the marginal and paraxial image points on the axis bv B'_m and B'. respectively. The focal surfaces for a marginal and paraxial zone of the exit pupil will be (sufficiently nearly) planes in the neighbourhood of the axis, and we may represent the corresponding extra-axial image points

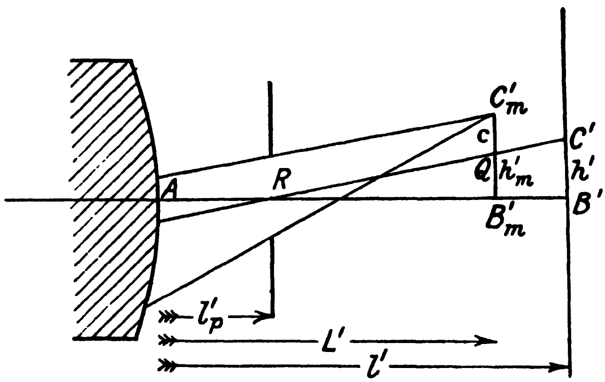


FIG. 158

formed by these marginal and paraxial zones by C'_m and C' respectively, where $B'C' = h'$ and $B'_mC'_m = h'_m$.

Now it is clear the amount of the *coma* under the present condition is the vertical intercept $c = QC'_m$ between the line RC' and the point C'_m. If it is zero, the lateral dis-symmetry of the image patch in any plane will disappear.

Hence in the presence of spherical aberration

$$\text{Measure of coma} = \frac{c}{h'}$$

But

$$c = h'_m - B'_mQ$$

and from the similar triangles RB'_mQ and $RB'C'$, we find

$$B'_mQ = B'C'\left(\frac{RB'_m}{RB'}\right)$$

Let AB' be denoted by l', and AR (the distance of the exit pupil from the apex) by l'_p; also let the distance AB'_m to the marginal focus be L'; then

$$B'_mQ = h'\left(\frac{L' - l'_p}{l' - l'_p}\right)$$

Thus,

$$\text{Measure of Coma} = \frac{c}{h'} = \frac{h'_m}{h'} - \frac{L' - l'_p}{l' - l'_p}$$

Writing $L' - l'_p = (l' - l'_p) - (l' - L')$
we obtain

$$-\frac{c}{h'} = \left(1 - \frac{h'_m}{h'}\right) - \left(\frac{l' - L'}{l' - l'_p}\right)$$

$$= \left(1 - \frac{f'_m}{f'}\right) - \left(\frac{l' - L'}{l' - l'_p}\right)$$

$$= \left(\frac{f' - f'_m}{f'}\right) - \left(\frac{l' - L'}{l' - l'_p}\right).$$

Now if the stop is close to the second principal surface of the lens, then f' will not be very different from $(l' - l'_p)$, when the object is at infinity. Hence very nearly.

$$\text{Measure of Coma} = \frac{(l' - L') - (f' - f'_m)}{f'}$$

$$= \frac{(\text{axial spherical aberration}) - (\text{difference of focal lengths})}{\text{focal length}}$$

Thus the measure of the coma is found as mentioned above, by the horizontal intercepts between the "spherical aberration" curves of von Rohr's diagrams, but only under the limitations stated.

Distortion. Particulars of the *distortion* of many photographic lenses were given graphically by Wandersleb

Put in a simple way, the distortion is given by the ratio between the image dimensions, h', i.e. the actual "image height," and $\mathbf{h}'$, i.e. the corresponding size it would have if free from distortion. In order to get the value of $\mathbf{h}'$ we should multiply h by the magnification ratio found for very small objects and images. Then

$$\text{Distortion} = \frac{h'}{\mathbf{h}'} - 1$$

and the percentage value, i.e. $100\left\{\left(\frac{h'}{\mathbf{h}'}\right) - 1\right\}$ is plotted in the curves.

The distortion curves are also given for different values of N, the reciprocal of the magnification. Thus $N = \infty$ for an infinitely distant object, and $= 1$ for the case when object and image are of the same size. With the old form of the Cooke portrait lens, note how the distortion changes very rapidly with change of magnification.

It is much to be desired that opticians should give specifications of the performances of their lenses on these lines, or by some other simple method. Such information would be of the greatest assistance in determining the suitability of the lenses for various purposes.

Depth of Focus; Photographic Definition. The "depth of focus" in the image may be defined as that total displacement of the plate on each side of the true focus which is possible without producing an appreciable spreading or loss of definition in the image. In Vol. I, page 141, the matter was discussed in terms of optical path differences between marginal and paraxial disturbances, and the equation for the shift on one side of the focus is

$$\delta f = \frac{2\,\delta p}{n'}\left(\frac{f}{y}\right)^2$$

where δp is the allowable difference of optical path. Remembering that the total range will be approximately double the above, and that the stop number is $\left(\frac{f}{2y}\right)$, we get for a medium where $n' = 1$

$$\textit{depth of focus} = 16\,(\textit{allowable path difference})(\textit{stop number})^2$$

The allowable path difference is, however, not easy to specify very exactly; the requirements for photographic recording are usually less severe than for direct visual observation of optical images where the Rayleigh limit of $\frac{\lambda}{4}$ may be necessary. If in photography the limit were taken as $\frac{\lambda}{2}$, say, then with a lens working at $f/8$ and wave-length $= 0{\cdot}5\mu$, the depth of focus would be 256μ, or about a quarter of a millimetre.

The tolerance for the loss of definition of the image is usually such that a photograph held at the least distance of distinct vision should appear reasonably sharp to the eye, and it is usually estimated that the photographic disc or patch representing an image point may subtend an angle of, perhaps, two minutes of arc.

If d is the diameter of the patch in millimetres, the limit would thus be given by the equation

$$\frac{d}{250} = 2 \text{ min. in angular measure}$$

$$d = \frac{250}{1710}$$

so that d will not exceed about one-seventh of a millimetre, or about $\frac{1}{178}$ in. This limit is on the severe side, and a patch of $\frac{1}{100}$ in., or 0·25 mm., may be tolerated in some cases.

If the diameter d of the patch were determined simply by the diameter of a cone of rays passing through a point in the focus, taken at a distance δf therefrom, we should have

$$\frac{\delta f}{d} = \text{stop number}$$

and if d were 0·25 mm., then $\delta f = 2{\cdot}0$ mm. for $f/8$, and the *total depth of focus* would be of the order of 4·0 mm. It is clear that the usual criteria of optical path are far too severe for this case; on

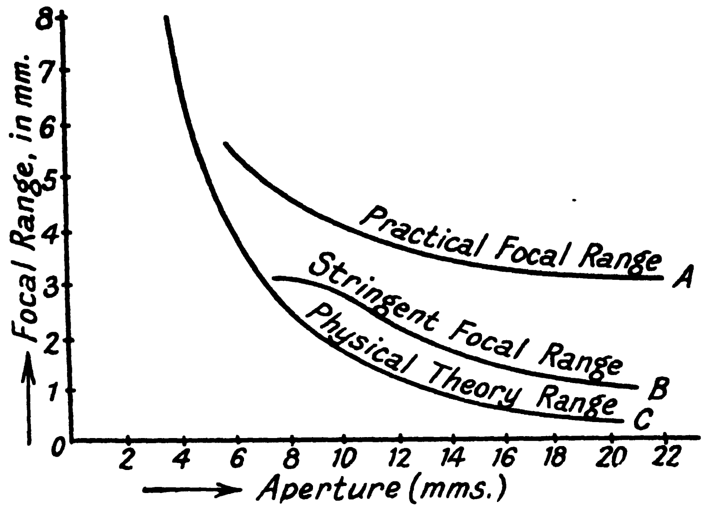


FIG. 159. FOCAL RANGE FOR AN ANASTIGMAT LANDSCAPE LENS

the other hand, the distribution of light is not well represented by the diameter of the supposed "cone of rays," which gives very misleading results near the focus.

It is thus difficult to give a satisfactory theoretical discussion on any simple lines. Considerable light is thrown on the matter by the experimental work of Miss H. G. Conrady,[3] who used an anastigmat landscape lens ($f/7$ approx.) suffering from a residual spherical aberration of known amount, and investigated the *position* of the best focus, and the focal range for various apertures. The results are shown in Fig. 159, in which curve A represents the focal range giving fairly good definition for practical purposes; curve B shows the range over which no loss of definition is at all perceptible; curve C shows the range predicted by physical theory for a perfect lens, using the "Rayleigh limit" of $\frac{\lambda}{4}$ for allowable path differences. It appears that the range is greatly increased by the presence of slight spherical aberration, and that it may be anything from one to eight times the "Rayleigh limit" range, depending on the amount of spherical aberration, which is naturally increasing in the diagram with increasing aperture. On the whole, it is clear that neither the physical theory nor the geometrical discussion of discs of confusion have a precise significance in this problem. Failing more exact knowledge, however, we usually find that it is very rarely with these photographic lenses that the diameter of the effective patch of light exceeds the limits of the calculated geometrical disc of confusion of the rays, and, consequently, it is the general experience that useful tolerances can be obtained by geometrical theory when discussing focal depth in the object space.

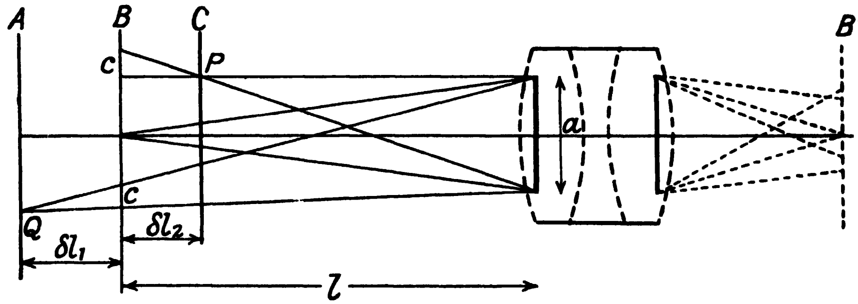


FIG. 160. THEORY OF FOCAL DEPTH IN THE OBJECT SPACE

Focal Depth in the Object Space. With the above limitations in mind, let us consider with the help of Fig. 160 an optical system, having an entrance pupil of diameter a, which is forming, in the plane B′, a sharp image of all points in the plane B. Consider now a point P situated in the plane C. Rays from a limited area (diameter c) of the plane B can reach the entrance pupil by passing through P. Hence we see that the rays from the point P will intersect the image plane B′ in all parts of a disc corresponding to the image of the disc c. If the magnification is m, the diameter of this image $= mc$.

Similarly, the image of a point Q in the plane A will correspond to that of a finite disc in the plane B.

First Criterion. If the criterion for sharpness of definition is merely such that the dimension of mc must not exceed a certain limit, the distances of the planes A and C from B will be limited also. Taking the dimensions shown in the diagram, we obtain from the similar triangles with a common apex in Q the relation

$$\frac{\delta l_1}{c} = \frac{l + \delta l_1}{a}$$

giving $$\delta l_1 = \frac{l}{a}\bigg/\left(\frac{1}{c} - \frac{1}{a}\right) = \frac{lc}{a-c}$$

and from the triangles with their apex in P,

$$\frac{\delta l_2}{c} = \frac{l - \delta l_2}{a}$$

giving $$\delta l_2 = \frac{l}{a}\bigg/\left(\frac{1}{c} + \frac{1}{a}\right) = \frac{lc}{a+c}.$$

These equations give the focal depths on each side of the exact focus. The total focal depth is given by adding the equations. Thus

$$\delta l_1 + \delta l_2 = lc\left(\frac{1}{a-c} + \frac{1}{a+c}\right) = lc\frac{2a}{a^2 - c^2}$$

Total focal depth $$= \frac{2alc}{a^2 - c^2}$$

Second Criterion. On the other hand, it may not be intended that the picture shall be viewed simply at the distance of distinct vision, but, perhaps, under the proper angular magnitude, possibly with the aid of a suitable lens of focal length equal to that of the camera lens. Under these circumstances a disc of confusion will subtend an angle at the eye equal to that subtended by the corresponding disc of confusion in the plane B at the entrance pupil of the camera lens.

But the condition for sharp images is that the angle subtended by the disc of confusion at the entrance pupil shall not exceed a definite limit, say α, i.e.

At the limit, $$c = l\alpha$$

Hence $$\delta l_1 = \frac{l \, . \, l\alpha}{a - l\alpha} = \frac{l\alpha}{\frac{a}{l} - \alpha}$$

and $$\delta l_2 = \frac{l\alpha}{\frac{a}{l} + \alpha}$$

In the position when $\alpha = \frac{a}{l}$, the focal depth will extend outwards to infinity. This gives

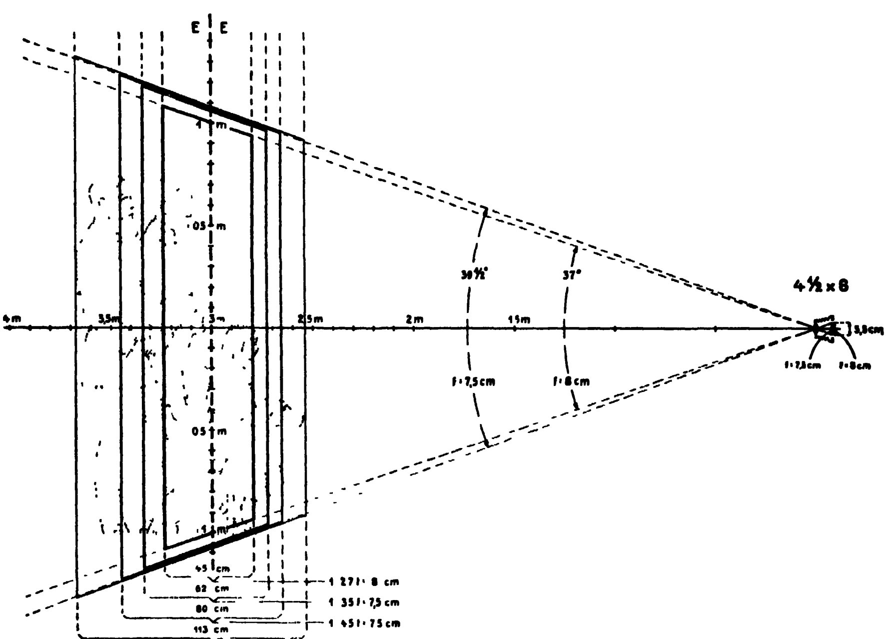
$$\delta l_2 = \frac{a}{2\alpha}$$

and $$l - \delta l_2 = \frac{a}{2\alpha}$$

so that the focal depth of sharp focus will extend from a distance of $\frac{a}{2\alpha}$ to infinity.

We may now note that this depth of focus in the object space only depends on the diameter of the aperture of the entrance pupil. Hence, small snapshot cameras using lenses with big aperture ratios have a greater depth of focus in the object space as compared with larger lenses, and have the advantage that the smaller lenses are much cheaper.

The image diameter corresponding to the angular subtense α above will be $f'\alpha$ where f' is the focal length of the camera lens. If



(*Carl Zeiss, Jena*)

FIG. 161. FOCAL DEPTH FOR A 6 × 4½ CM. CAMERA USING VARIOUS LENSES

the photograph is viewed under the correct conditions it will be held at a distance f' (Vol. I, page 2), so that the value of α must be determined by the resolving power of the eye. A practical value is 2 min., or 0·00058 radians. If, however, f' is shorter than the distance of distinct vision, the picture should be viewed with the aid of a magnifying lens as in a stereoscope (or with such a lens as the Verant), having a focal length equal to that of the camera lens.

Alternatively, the picture may be enlarged photographically, and the scale of the enlargement should be

$$\frac{\text{Distance of viewing of picture}}{\text{Focal length of camera lens}}$$

if the picture is to be seen under correct conditions of perspective. Take, for example, a very small camera with a lens of focal length 5 cm. If the resulting pictures are to be viewed at a distance of, say, 30 cm., the best scale for enlargement would be 6 diameters.

An interesting presentation of the depth of focus effects when using lenses of about 8 cm. focal length covering a plate of about 8 cm. diameter (about $2\frac{1}{4}$ in. by $2\frac{1}{4}$ in. plate) is given by Messrs. Carl Zeiss (Fig. 161). The criterion of focal depth is that the disc of confusion must not subtend an angle greater than $2\frac{1}{2}$ min. The depth of focus is shown for the series of aperture ratios: $\frac{f}{2\cdot7}$, $\frac{f}{3\cdot5}$, $\frac{f}{4\cdot5}$, and $\frac{f}{6\cdot3}$. Since the focal lengths are the same, the actual entrance pupils differ in size.

The first has a focal length of 8 cm., the three latter have focal lengths of 7·5 cm., with a slightly wider angular field. The principal distance, i.e. the distance for best focus, is 3·0 metres; and the diagram is interesting as showing the kind of depth of focus obtainable for "personal" photographs with small pocket cameras.

The Telephoto Lens. Consider a combination of two lenses of positive and negative powers numerically equal. Referring to page 50 of Vol. I, we find the power of a combination given by

$$\mathcal{P} = \mathcal{P}_a + \mathcal{P}_b - d.\mathcal{P}_a.\mathcal{P}_b$$

and the distance of the second principal plane from the second lens is

$$P_bP' = -\frac{\mathcal{P}_a d}{\mathcal{P}}$$

Take, for example, the case where $\mathcal{P}_a = 10\text{D}$, $\mathcal{P}_b = -10\text{D}$, and the separation $d = 4$ cm. ($= \cdot04$ metres), then

$$\mathcal{P} = 10 - 10 - \cdot04\,(10)(-10)$$
$$= 4\cdot0$$

The focal length is therefore 25 cm.

The equation for P_bP' gives this length as $\frac{-10\,(\cdot04)}{4} = -\frac{1}{10}$ metre $= -10$ cm.

Hence the second principal surface lies -10 cm. in front of the second lens.

The effect of the combination is, therefore, that of a lens of power 4·0D placed 6 cm. in front of the first component as suggested in Fig. 162. The back focusing distance of the combination is clearly 15 cm., whereas the actual focal length is 25 cm. Hence the images of distant objects will be larger in the proportion $\frac{25}{15} = \frac{5}{3}$ than those of an ordinary lens with the same back focal distance.

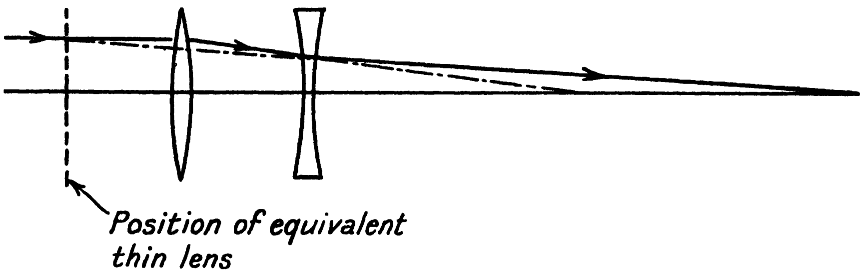


FIG. 162. TELEPHOTO COMBINATION

Modern telephoto lenses employ two separated systems of doublet or triplet type; the first having a positive power, and the second a negative power. Each component must be separately corrected for chromatic aberration owing to the large distance between them,

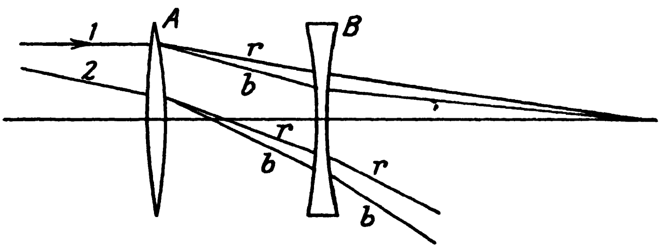


FIG. 163. CONDITION OF ACHROMATISM

for suppose that we consider rays 1 and 2 in Fig. 163; we may imagine that ray 1 is subject to dispersion by lens A, and that the under-correction, by which the blue ray is deviated more than the red ray, is corrected by lens B, the blue now suffering more deviation towards the margin of the negative lens than the red. If correction were thus given for image points very near the axis, consider ray 2 which suffers the same type of aberration in lens A, but meets lens B on the other side of the axis. The aberration is evidently exaggerated. Hence each component must be separately achromatized.

This can also be argued from the expressions for the magnification of *thin* lenses. Let the two thin lenses be *a* and *b*; assuming homogeneous light, let the distances of object and intermediate image from lens *a* be l_a and l'_a; the distances of intermediate image and final image from *b* are l_b and l'_b. Let h_a, h'_a, and h'_b be the sizes of object, intermediate image, and final image. Then

$$\frac{h'_a}{h_a} = \frac{l'_a}{l_a}, \text{ and } \frac{h'_b}{h_a'} = \frac{l'_b}{l_b}$$

so that
$$\frac{h'_b}{h_a} = \frac{l'_a l'_b}{l_a l_b}$$

If now the final magnification is to be independent of the wave-length of the light, and the position of the image is to be constant also, then $\left(\frac{h'_b}{h_a}\right)$, l'_b, and l_a are all constants. Therefore since $\frac{d}{d\lambda}\left(\frac{h'_b}{h_a}\right) = 0$

$$\frac{d(l'_a/l_b)}{d\lambda} = 0$$

and
$$(dl'_a)l_b - l'_a(dl_b) = 0$$

But the separation of the lenses, i.e. $l'_a - l_b$ (remember our sign conventions), is constant, since the separation is prescribed by the equation on page 45, for the achromatism of the focal length. Therefore

$$dl'_a = dl_b$$

and this, substituted in the previous equation, gives

$$dl'_a(l_b - l'_a) = 0$$

But the bracket is equal numerically to the lens separation and cannot be zero. Hence dl'_a must be zero and $dl_b = 0$ also by a similar argument. The position of the intermediate image is independent of the wave-length. Hence each lens must be separately achromatized.

This discussion deals with two separated thin lenses only, and obviously neglects the possible chromatic variation of the principal points with wave-length which may be expected in thick lenses. It often happens in complex lens systems (like the Cooke or Aldis lenses) that we have single uncorrected lenses used, so the meaning of the above discussion must not be pushed too far. It is, however, to be remembered when the use of widely separated and more or less thin lenses is contemplated.

Supposing for a moment that we restrict the further demands to freedom from spherical aberration and coma. Lens A might be a cemented doublet of suitable glass of the telescope objective type. Lens B might also be a cemented doublet with negative lens of crown glass and positive lens of flint.

In the section on microscope objectives, it was explained that a positive cemented doublet has two pairs of conjugate points free from spherical aberration, and we should expect to find similar pairs of points for a negative combination with a strong negative crown lens and a weaker flint lens cemented together. Although it is possible to use more or less aplanatic components, this is only at the expense of marked pincushion distortion, and modern telephoto lenses usually have components which are not separately corrected in themselves, as will be seen below.

The early telephoto lenses mainly used a negative system behind a photographic lens system of some ordinary type; the power could be varied by varying the separation. The "magnifying effect" m of the telephoto attachment is given by

$$m = \frac{\text{Dimensions of image with telephoto attachment}}{\text{Dimensions of image with positive lens only}}$$

But for an infinitely distant object subtending an angle a, the dimension of the image is $f \tan a$; also for a combination of two lens systems placed with their focal points so that $F'_aF_b = g$, we had

$$f = \frac{f_a f_b}{g}$$

Therefore

$$m = \frac{f \tan a}{f_a \tan a} = \frac{f_b}{g}$$

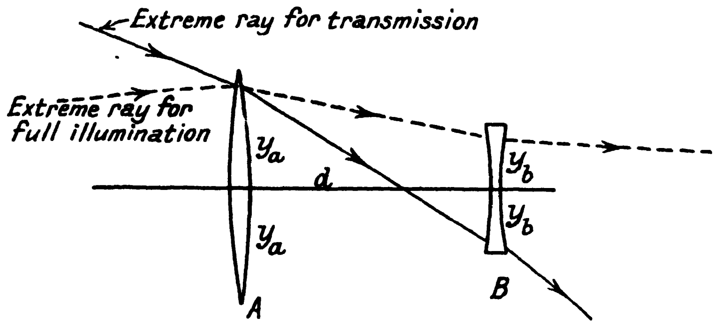


FIG. 164

When opticians supply a negative telephoto attachment with a variable separation, a scale on the mount is usually made to indicate values of m or g. The use of such attachments has, however, largely been abandoned in favour of the modern fixed-focus telephoto lens.

As distinct from the magnifying effect, the expression "telephoto effect" is sometimes used:

$$\text{Telephoto effect} = \frac{\text{Focal length of combination}}{\text{Back focal length}}$$

the "back focal length" being measured from the last surface of the lens system, and representing the approximate focal length of an ordinary lens used at the same camera extension.

It is easily shown that this is equivalent to

$$\text{Telephoto effect} = \frac{f'_a}{(f'_a - d)}$$

where d is the separation of the lenses.

Effective Aperture. Consider a lens system A used first alone, then with a negative attachment B. The entrance pupil will be assumed to be the same as that of the system A; the diameter of B is sufficient to allow of this. Then

$$\text{Stop number} = \frac{\text{Focal length}}{\text{Diameter of entrance pupil}}$$

Hence,
$$\frac{\text{Stop number with telephoto lens}}{\text{Stop number without telephoto}} = \frac{f}{f_a} = m$$

The exposure must be proportional to m^2.

Field of View. In Fig. 164, let A and B be the positive and negative lenses of a thin lens telephoto system, and let y_a and y_b be the radii of the diaphragms which limit them. The tangent of the angle with the axis made by the most oblique ray which can pass between the lenses is clearly

$$\frac{y_a + y_b}{d}$$

If f'_a is the focal length of the thin lens A, the deviation in a ray produced by a transmission through it at a distance y_a from the axis is approximately $\frac{y_a}{f'_a}$. Hence, tracking this most oblique ray backwards through A, we find the inclination to the axis for the most oblique ray which can enter lens A and be transmitted by the system. Provided that we are dealing with angles small enough to take the tangent of an angle as its numerical value, the field of view is given by

$$\frac{y_a + y_b}{d} - \frac{y_a}{f'_a}.$$

This represents the extreme limit of the field. It will be seen that the limit of the fully illuminated field is

$$\frac{y_a - y_b}{d} - \frac{y_a}{f'_a}.$$

These equations are, however, not strictly accurate with the thick lenses encountered in practice.

Modern Telephoto Lenses. The modern telephoto lenses are mostly fixed focus combinations with anastigmatic correction. The telephoto effect is low, being only two to three, but this suffices for a great number of purposes, more especially as several advantages are obtained, viz. high relative aperture (the aperture of some telephoto lenses has been increased to $\frac{f}{3\cdot5}$) anastigmatic correction giving sharp images capable of enlargement, and reasonable economy in size. Fig. 165 shows the Dallon (of Messrs. Dallmeyer,

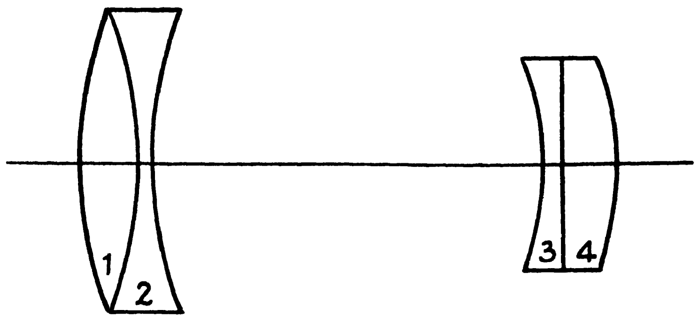


FIG. 165. DALLON LENS (f/5·6)

Ltd.) designed by Mr. L. B. Booth, who was a pioneer in the construction of such systems. Typical glasses in lenses of this kind are—

1. Dense barium crown.
2. Dense flint.
3. Light flint.
4. Medium barium crown.

A general review of the development of modern telephoto objectives has been given by Lee,[4] who was successful in producing a

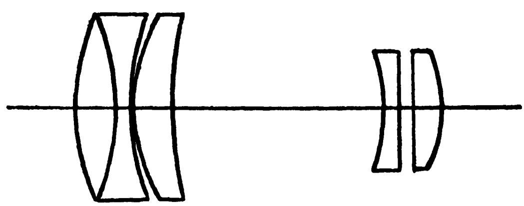


FIG. 166. f/5. DISTORTION-FREE TELEPHOTO LENS DESIGNED BY LEE (Messrs. Taylor, Taylor & Hobson)

distortion-free telephoto lens. He says: "If we consider the construction of the telephoto, a positive lens placed in front of the diaphragm, which will possess pin-cushion distortion, and a negative lens behind the diaphragm which tends to produce the same kind of distortion, it is not surprising that telephotos are afflicted with much pin-cushion distortion, and some designers have considered it inevitable. . . . By separating the components of the negative lens, it was possible to utilize the astigmatism in these surfaces, which is fairly large, to correct the pin-cushion distortion. . . . We are then left with a residuum of under-corrected astigmatism, which is neutralized in the front positive lens by dividing it up into two menisci, one of which is a doublet, for the purpose of achromatism." These words give a brief picture of some of the main stages in the design of a new system. The distortion free lens, B.P. 222, 709, is shown in Fig. 166.

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2. von Rohr: *Der Theorie und Geschichte des Photographischen Objectivs.*
3. H. G. Conrady: *Jour. Roy. Phot. Soc.*, LXVI (1926), 22–25.
4. Lee: *Proc. Opt. Convention*, 1926, p. 869.

CHAPTER VI

THE PHOTOMETRY OF OPTICAL SYSTEMS AND THE PROJECTION OF IMAGES

IN the foregoing discussion of the principles of the telescope and microscope, no attention has been given to the question of the brightness of the image; this aspect of the subject is, however, of the first importance, and must now be considered. A discussion of the sensitiveness of the human eye to light has been given in Vol. I, Chapter V.

For the purposes of elementary discussion, it is assumed that radiant energy would spread out from an elementary source of

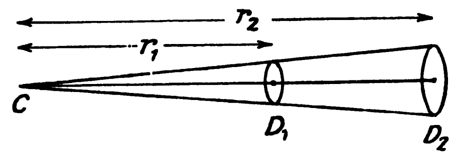

FIG. 167

infinitesimal size along the paths represented by the "rays." The amount of energy per unit time passing any cross-section of a tube whose walls were made up of such rays would, therefore, be constant.

Rays are straight in a homogeneous medium; hence if we consider a conical tube representing a very small solid angle $d\omega$ with its apex in the elementary source C (Fig. 167), the normal cross-sectional areas at distances r_1 and r_2 would be $r_1^2 d\omega$ and $r_2^2 d\omega$. Let the amount of energy passing in unit time be dF; then the energy *per unit area* at these sections will be

$$\frac{1}{r_1^2}\left(\frac{dF}{d\omega}\right) \text{ and } \frac{1}{r_2^2}\left(\frac{dF}{d\omega}\right)$$

and the quantities must be equal in the absence of absorption, i.e. (putting the result into words) the amount of energy per unit area falling on an elementary area held normal to the incident light is inversely proportional to the square of its distance from the source. This is the "inverse square" law; but it only has an exact meaning in regard to an imaginary source of infinitesimal size, and, therefore, may only hold approximately in practical cases.

When the energy is evaluated *according to the luminous sensation* *produced*, the symbol F represents an amount of "light," and $\frac{dF}{d\omega}$ represents, for some particular direction, "the amount of light per unit solid angle," which is the "candle-power" (*intensité lumineuse*) of the source for that direction. The candle-power is usually denoted by J. Thus

Candle-power $$J = \frac{dF}{d\omega}$$

The amount of light falling per unit area of a surface represents the "illumination" (usually denoted by E), so that

Illumination $$E = \frac{dF}{ds}$$

where ds represents an elementary area of the surface.

Lastly, the "brightness" (B) of a surface is defined as the candle-power per unit projected area in the direction under consideration, so that

Brightness $$B = \frac{dJ}{ds}$$

The Cosine Law. Practical observation of self-luminous surfaces (a red-hot poker is a good example), and of many types of diffusely

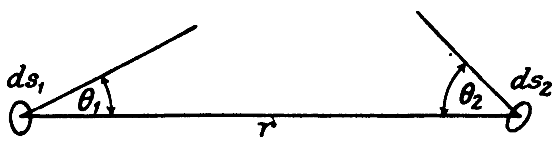

FIG. 168

reflecting surface (such as blotting-paper) shows that their apparent brightness under visual observation is very nearly independent of the direction of observation. It will be shown that this can only be explained by the assumption that the "candle-power" of a small element of such a surface (regarding it for the moment as a source of light) in any direction is proportional to the cosine of the angle between that direction and the normal to the surface.

This relation enables us to calculate the amount of light radiated from one small elementary surface to another, when these surfaces are inclined at any angle to the straight line drawn between them. In Fig. 168 let ds_1 and ds_2 represent the elementary areas, and let the normals to these areas make angles θ_1, θ_2 with the line joining them. Let the normal brightness of ds_1 be B; then the normal candle-power will be Bds_1.

The solid angle subtended by ds_2 at the distance r is given by

$$d\omega = \frac{ds_2 \cos \theta_2}{r^2}$$

Assuming the above cosine law of radiation to hold, the brightness of ds, in a direction inclined at an angle θ_1 to the normal is B $\cos \theta_1$, and therefore the candle-power of the elementary surface in this direction is $Bds_1 \cos \theta_1$. This is the light radiated per unit solid angle in this direction, so that the amount radiated into the solid angle $d\omega$ subtended by ds_2 is

$$B \,.\, ds_1 \,.\, d\omega \,.\, \cos \theta_1 = \frac{B\, ds_1\, ds_2 \cos \theta_1 \cos \theta_2}{r^2}$$

i.e. we multiply the brightness of the source, the area of either one of the elements, and the cosine of its angle, with the solid angle subtended by the other element.

The *illumination* of the element ds_2, the amount of light per unit area, is obtained by dividing the above expression by ds_2, i.e. it is

$$r^{-2} Bds_1 \cos \theta_1 \cos \theta_2$$

and is clearly *proportional* to $\cos \theta_2$; if we hold a small surface in a beam of light, the illumination is proportional to the cosine of the angle turned from the position of normal incidence. This relation is one of a purely geometrical character, it does not depend, as does the "cosine law" of radiation mentioned above, on the physical properties of the surface.

Apparent Brightness of a Radiating Surface. The apparent brightness of a surface is usually directly dependent on the *illumination* of the retinal image. Let the elementary area ds_2 (Fig. 168) represent the entrance pupil of an eye, and let θ_2 be zero, so that the eye "looks" at ds_1. In Vol. I, page 47, it was shown that the size of the image of an object subtending a plane angle ω_p will be

$$h' = f \tan \omega_p$$

Correspondingly, the area of a small retinal image of an object subtending a small solid angle ω_s will be found by squaring the simplified form of the last equation when it has been written

$$h' = f\omega_p$$

thus obtaining

$$h'^2 = f^2\omega^2{}_p$$

or

$$ds' = f^2\omega_s$$

This area ds' is taken to be uniformly illuminated by light reaching the entrance pupil of the eye from the object. The quantity of light dF reaching the pupil is (from above)

$$dF = \frac{B ds_1 \, ds_2 \cos \theta_1}{r^2}$$

Of this, however, only a fraction kdF is transmitted by the media of the eye. But the solid angle subtended by the object at the eye is $\frac{(ds_1 \cos \theta)}{r^2}$, and if the distance from the object to the front focal point is large in comparison with the short distance from the focal point to the pupil, we may rewrite the above expression

$$kdF = kB\omega_s ds_2$$

Hence the illumination of the retinal image is

$$\frac{kdF}{ds'} = \frac{kB\omega_s ds_2}{f^2 \omega_s} = \frac{kBds_2}{f^2}$$

It is clearly independent of the angular position and distance of the object, provided that the accommodation of the eye is unchanged, and that the normal brightness of the surface is constant. Hence, if a surface radiates or reflects in accordance with the cosine law, it will appear to have the same apparent brightness no matter from what distance or under what angle it is seen. This statement is subject to several limitations. In the first place it does not apply when the geometrical image of the object is of a size comparable with, or small in comparison to, the physical concentration of the Airy disc elementary image. Under such circumstances the area of the image is independent of the distance or angular position of the source, and it is too small to give the sensation of any finite extension. The relative apparent "brightness" is, therefore, dependent only on, and directly proportional to, the total amount of light received by the eye, i.e.

$$dF = \frac{B \, ds_1 \, ds_2 \cos \theta_1}{r^2}$$

In order that the light may be perceived by the eye, the quantity must exceed the "threshold value" for the retina. It is found in practice that in very weak illumination, the retina has the power of integrating the light received over a small area subtending about one degree of arc in the visual field, so that if the total light radiated on such an area exceeds a certain amount, the sensation of light will result.

Total Light Radiated by a Self-luminous Surface. Imagine a small element, of area ds, of self-luminous surface at O, Fig. 169;

assuming that radiation takes place in accordance with the cosine law, it is possible to calculate the total light radiated by the element into the space above it. Imagining a hemisphere described above the element, we may calculate the light radiated to a circular strip limited by the angles θ and $\overline{\theta + d\theta}$ between the radii drawn from O and the normal OB. If r is the radius of the hemisphere, the area of the strip is clearly $rd\theta \cdot 2\pi r \sin\theta$, and the solid angle $d\omega$ subtended at O is therefore

$$d\omega = 2\pi \sin\theta d\theta$$

If B is the normal brightness, the candle-power of the element in the direction θ is

$$Bds \cos\theta$$

Hence the light radiated to the annular strip is

$$dF = Bds \cdot 2\pi \sin\theta \cos\theta\, d\theta$$

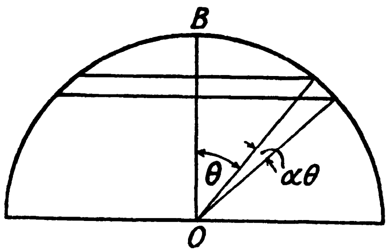

FIG. 169

To find the total light radiated into a symmetrical cone of angle θ we integrate thus

$$F = \int_0^\theta Bds \cdot 2\pi \sin\theta \cos\theta\, d\theta$$

$$= \pi Bds \sin^2\theta$$

When the angle θ is $\frac{\pi}{2}$, so that we consider the whole of the space above the element, the result is πBds.

Brightness of Optical Images. The first case considered is that in which the image is presented directly to the eye, without being projected on a diffusing screen. The second case will be that of the photographic image or projection lantern image where the light falls on a projection plate or screen.

The simplest case to be considered is that of a symmetrical instrument, Fig. 170, of which the centres of the relevant entrance and exit pupils are at R and R′. Let ds be a small element of self-luminous surface with the optical axis as its normal, and let ds' be the corresponding image. The limiting angular divergence of the rays entering the entrance pupil is α, and the corresponding angle made by the extreme rays with the axis in the image is α'.

Let the normal brightness of the object be B; then the amount of light F entering the instrument is

$$F = \pi Bds \sin^2\alpha$$

The luminous energy flows through the system by various paths which are represented by the ray tracks. If the system fulfils the "sine condition," then when we write the optical sine relation

$$nh \sin \alpha = n'h' \sin \alpha'$$

we know that the ratio of $\sin \alpha$ to $\sin \alpha'$ will be the same for all zones of the system. The energy radiated from the object element into the entrance pupil of the instrument will be subject to some

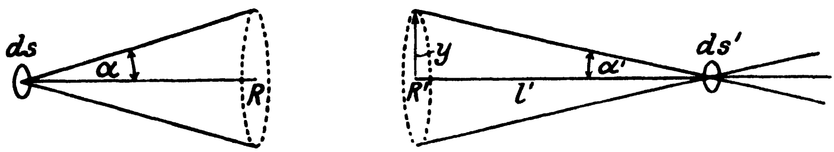

FIG. 170

losses by reflection and absorption. If the transmission factor is k, the amount of light reaching the image will be

$$k\text{F} = k\pi\text{B}ds \,.\, \sin^2 \alpha$$

Assuming now that the corresponding image area, ds', has a normal brightness B′, and that it radiates in accordance with the cosine law, the amount of light passing through the aerial image will also be expressed as

$$k\text{F} = \pi\text{B}'ds' \sin^2 \alpha'$$

i.e.

$$k\pi\text{B}ds \,.\, \sin^2 \alpha = \pi\text{B}'ds' \,.\, \sin^2 \alpha' \quad . \quad . \quad . \quad (a)$$

By squaring the sine relation we obtain

$$n^2 ds \,.\, \sin^2 \alpha = n'^2 ds' \,.\, \sin^2 \alpha'$$

and, dividing into the previous equation

$$\frac{k\text{B}}{n^2} = \frac{\text{B}'}{n'^2}$$

or

$$\text{B}' = k\text{B}\left(\frac{n'}{n}\right)^2$$

Returning to equation (a), let us differentiate it with regard to α and α', obtaining

$$2k\pi\text{B}ds \sin \alpha \cos \alpha \, d\alpha = 2\pi\text{B}'ds' \sin \alpha' \cos \alpha' \, d\alpha'$$

Referring back to the investigation above, page 210, we see that the assumption that the image radiates in accordance with the cosine law is founded on the supposition that the light radiated by the object into an annular cone of angles α and $\alpha + d\alpha$ passes through the image in a corresponding annular cone of angles α' and $\alpha' + d\alpha'$

which can be calculated by the assumption of the sine condition. Differentiating the squared form of the sine relation we obtain

$$2n^2 ds \sin \alpha \cos \alpha \, d\alpha = 2n'^2 ds' \sin \alpha' \cos \alpha' \, d\alpha'$$

which, dividing into the above equation, gives the same result for B′. The investigation is, therefore, in strict accord with the fundamental assumption that the quantity of light remains constant for all cross-sections of a cone whose walls are made up of rays of light. The condition that the image may radiate in accordance with the cosine law is that the transmission factor of the instrument may be the same for all zones. In practice, this is not likely to hold exactly, owing to the different obliquities of the rays at the different surfaces which become much greater towards the margins, thereby greatly increasing the reflection losses.

Before going further, it will be well to re-state the result and draw certain inferences. Few optical theorems have been the subject of so much misunderstanding as this.

Firstly, although the image ds' is radiating in a homogeneous medium, it is unlike the object insomuch as it can only radiate within definite angular limits; the extreme rays cannot exceed an angle α' with the axis.

Provided the eye is so placed that the pupil lies entirely within this angular limit, the image when observed will have the apparent brightness $B' = kB\left(\frac{n'^2}{n^2}\right)$; but if the eye is outside the cone of radiation, the image will not be seen. Now, if the image space medium is air, as is usually the case in visual observation, then $\left(\frac{n'^2}{n^2}\right)$ is unity, and the subjective brightness of the image is that of the object multiplied by a transmission factor k which is always less than unity. The formal proof has only been given for the most simple case, but it may be taken as a general rule that *the apparent brightness of the image of a luminous surface formed by an optical system cannot exceed the apparent brightness of the object surface observed directly*, provided that the apparent size of both object and image is not very small.

As will be seen below, cases arise when the image projected by a system is viewed so that the pupil of the eye is not wholly illuminated, as, for example, when using a microscope. The apparent brightness of the image will be further reduced in the proportion $\left(\frac{p}{p_0}\right)^2$, where p and p_0 are the restricted and full radii of the pupil respectively. This assumes, however, that the restricted pupil is uniformly illuminated. Owing to the reflection and absorption losses in instruments, losses which are usually larger for the more oblique pencils, the assumption is rarely strictly true; it may be used to obtain a first approximation to the answer of several problems, as will be seen.

Transmission Factor of Photographic Lenses and other Instruments. The above theorem indicates a ready means of measuring the transmission factor of an optical instrument such as a photographic lens or a telescope. There are instruments, known as brightness photometers,[1] which allow of the direct measurement of the apparent brightness of a luminous surface. We have, therefore, only to measure (1) the apparent brightness of the surface observed directly; (2) the apparent brightness of the image of this surface as projected by the photographic lens or other system. Then the ratio of image to object brightness gives the transmission factor. It is to be noted that the relative positions and distances of the luminous surfaces are immaterial, provided that a clear view of them can be obtained with the photometer, since the subjective brightness of a surface is, as shown above, independent of its distance provided that the angular subtense does not decrease below a certain value.

A suitable luminous surface is obtained by illuminating a piece of opal glass from behind. A photographic lens may be supported near the surface, and the brightness of the image of the surface is observed, holding the photometer aperture close to the lens. The latter is then removed and the brightness of the surface determined directly. For measurements on telescopes, the surface is held in the exit pupil of the instrument; its image is then observed in the entrance pupil, and the brightness can be determined with the photometer. These methods are much quicker and more accurate than the older methods using the optical bench.

Brightness of the Image in a Telescope. The above theorem regarding the "brightness" of the image projected by an optical system dealt with the brightness as a physical quantity. Given two surfaces of the same physical brightness, and radiating in accordance with the cosine law, they will only *appear* of the same brightness if the eye observes them under the same conditions; for example, with the same diameter of the eye pupil for each observation.

Consider first the case of the telescope; suppose it is used to view a distant luminous surface such as a region of the moon over which the brightness B is constant.

Let the unaided eye regard this surface; it receives an impression of brightness B, using the pupillary area ds_2. Now suppose that the telescope of magnifying power M is held before the eye. Let the area of the object glass be A, and let the transmission factor of the instrument be k. The "brightness" (in the limited physical sense) of the image projected by the telescope is now kB, and the area of the exit pupil of the instrument is $\frac{A}{M^2}$. If this area is larger than that of the eye pupil the impression of the brightness of the telescope image will be kB, since the eye pupil is wholly illuminated; but if the exit pupil is smaller than the eye pupil, the impression of brightness will now be proportional to the available aperture of the eye pupil, i.e.

$$\frac{\text{Apparent brightness with telescope}}{\text{Apparent brightness without}} = \frac{k}{ds_2}\left(\frac{A}{M^2}\right)$$

We may calculate this result in another way which will recall first principles. Let the distance from the surface to the eye, or to the observing telescope, be r; then using the above symbols, the amount of light, from an area ds_1 of the object, radiated to the eye pupil in the case of direct observation must be

$$\frac{B ds_1 \, ds_2}{r^2}$$

Let the area of the retinal image be ds_3, and let the transmission factor of the eye media be t, then the illumination of the retinal image will be

$$\frac{B ds_1 \, ds_2}{r^2} \cdot \frac{t}{ds_3}$$

When using the telescope, provided that the full pencils of light can pass unobstructed into the eye, all the light entering the object glass (less absorption losses, etc.) passes to the retina. The light received by the object glass is

$$\frac{B ds_1 \, A}{r^2}$$

But the retinal image now has an area $M^2 . ds_3$ owing to the telescope magnification. Hence the illumination of the retinal image is

$$\frac{kB ds_1 \, A}{r^2} \cdot \frac{t}{M^2 ds_3}$$

Hence, $$\frac{\text{Illumination of retinal image with telescope}}{\text{Illumination of retinal image without}} = \frac{k}{ds_2}\left(\frac{A}{M^2}\right)$$

$$= k \cdot \left(\frac{\text{Area of exit pupil}}{\text{Area of eye pupil}}\right)$$

It is easy to show that when there is obstruction of the full pencils owing to the exit pupil being larger than the eye pupil, then the effective entrance pupil of the telescope is of area $M^2 ds_2$, so that the ratio of the illumination in the two cases is simply determined by the transmission factor of the telescope.

Case of Star Images. The above considerations do not apply unless the distant luminous surface is of appreciable angular magnitude. In the case of such an object as a star, the area of the retinal image, in the case both of direct and aided observation, will usually be so small that its magnitude will be determined mainly from the optical imperfections of the eye system and the physical spreading of the image.

At any rate, there will not be a great difference in the retinal area illuminated in aided and unaided observation; but whereas in unaided observation we merely have the small area of the pupil to receive light, we can with a telescope capture the light received by the much greater area of a large object glass, and concentrate most of it into a retinal image of much the same extent as that found in unaided observation of a star. Given a large aperture telescope of, say, 1 metre diameter, the area is of the order of fifteen thousand times the eye pupil (8·0 mm.) at night, and the telescope, therefore, has a greater range in detecting distant stars. But since the light received from a star is inversely proportional to the square of the distance, the *range* is only increased by the ratio of the diameter of the object glass to that of the eye pupil, or about 125 to 1 in the above case.

Again, if there is any general luminosity in the "background" of the sky, this will be diminished in the telescope image as calculated above, so that this effect tends to increase the contrast between the brightness of the star and the background.

Night Glasses. It was mentioned above that the eye has the power of integrating the light in feeble stimuli spread over a retinal area corresponding to an angular diameter of about one degree in the field of vision. The "threshold" of perceptible "brightness"* is, therefore, inversely proportional to the square of the angular subtense of the stimulus. The smaller the "threshold," the easier the vision in faint light! When this angle rises above one degree, the threshold is then inversely proportional to the angular subtense itself, and not to the square; this is true up to a subtense of about four or five degrees, after which the threshold tends to become independent of the angular size of the stimulus. We may put these

* Brightness is defined as the candle-power per unit area, in the usual photometric sense.

results into symbols; let t be the threshold brightness, and a the angular subtense, then

$$t \propto \frac{1}{a^2} \quad . \quad . \quad \text{up to one degree}$$

$$t \propto \frac{1}{a} \quad . \quad . \quad \text{from one to four degrees}$$

When using a telescope in faint light we have two cases to consider. In the first the exit pupil is larger than the pupil of the eye; the brightness of the image is kB and independent of the magnification. *While this is true*, we shall obtain continuous advantage by increasing the magnification since the angular size of the image will be proportional to m; this will hold good till the image we wish to observe subtends more than four to five degrees.

In the second case, the magnification has been so far increased that the exit pupil is now smaller than the pupil of the eye. The apparent brightness of the image is now inversely proportional to the square of the magnification, so that to maintain a given apparent brightness of the field the external object brightness B must be proportional to m^2; but if the stimulus is a small one, so that its magnified image subtends less than one degree in the visual field of the eye, the allowable "threshold" varies as $\frac{1}{m^2a^2}$ by reason of the physiological effect (where a is the angular subtense of the stimulus to the unaided eye). Hence

$$t \propto \frac{m^2}{m^2a^2}$$

i.e.

$$t \propto \frac{1}{a^2}$$

so that the threshold for small stimuli is almost independent of the magnification of the telescope. For larger stimuli with an apparent angle, in the instrument, of over one degree, the increase of magnification will obviously cause a disadvantage.

Experiments by the writer[2] seemed to show, however, that with small stimuli the visual threshold for the unaided eye was inversely proportional to a slightly higher power of the angular subtense than the second power up to about one degree of subtense. If this can be accepted then, when using the telescope,

$$t \propto \frac{m^2}{(ma)^{2+x}}$$

where x is a small positive quantity. Confirmatory experiments seemed to indicate that the expected slight advantage with increasing magnification was realized until the stimulus reached an angle of about one degree.

The majority of night glasses are binoculars for hand use, and the magnifying power should then not be greater than about six times, while the greatest efforts are made to secure the largest possible exit pupils and the least possible internal losses of light.

Brightness of the Image in the Microscope. It is seldom that the object is self-luminous; in the majority of cases the condenser projects the image of some luminous surface into the object plane. Let k be the transmission factor of the microscope (supposed uniform for various ray paths), and b the transmission factor of an object element; then, if the eye pupil were wholly filled with light when observing the image, the apparent brightness of the image element would be Bkb. But in the majority of cases the eye pupil is only partially illuminated. It was shown above that the radius p' of the exit pupil of the microscope is

$$p' = \frac{(N.A.)\beta}{m}$$

where $N.A.$ is the numerical aperture, β is the distance of distinct vision, and m is the visual magnification. Hence, if p_0 is the radius of the eye pupil when observing the source directly, we have, to a first approximation,

$$\frac{\text{Apparent brightness of microscope image}}{\text{Apparent brightness of source}} = kb\left(\frac{p'}{p_0}\right)^2$$

$$= kb\beta^2\left(\frac{N.A.}{m}\right)^2$$

Hence, other things being equal, the apparent brightness of the image will be directly proportional to the square of the numerical aperture, and inversely proportional to the square of the magnification.

Note that if the aperture of the objective is not filled by the condenser, then the illuminated area of the exit pupil will be smaller; the "numerical aperture" to be used in the above equation is, therefore, that numerical aperture of the objective which is effectively filled by the condenser.

Exposure in Photomicrography. In photomicrographic work we shall be concerned with the illumination of the image projected on to the screen. The condenser, we will suppose, projects an image of the source or effective source into the object plane, which may, as a first approximation, be supposed to radiate as a perfectly diffusing source into that aperture of the objective which is filled by the condenser. Let the "brightness" be B′, then the light received by the objective from a small area ds will be

$$\pi B' ds \sin^2 \alpha = \frac{\pi B' ds\, (\text{N.A.})^2}{n'^2}$$

where α is the angular divergence of the extreme rays from the condenser, N.A. is the numerical aperture, and n' is the refractive index of the medium. But if B is the original brightness of the source (in air, say), and k_c is the transmission of the condenser, then the light entering the objective will be

$$\pi b k_c B ds\, (\text{N.A.})^2$$

where b is the transmission of the object element as before so that

$$B' = k_c b B n'^2$$

The transmission of the microscope is k_m say, and the size of the corresponding image patch will be $M^2 ds$ where M is the linear magnification. Hence the illumination of the image will be

$$\text{Illumination} = \frac{\pi b k_c k_m B ds\, (\text{N.A.})^2}{M^2 ds}$$

$$= \pi b k_c k_m B \left(\frac{\text{N.A.}}{M}\right)^2$$

In practice the useful part of this result will be that the illumination is proportional directly to the square of the N.A. and inversely to the square of the magnification; a relation which is of service in making photomicrographs of the same object (say) with different objectives. Assuming the reciprocity relation, the *exposure* will be proportional directly to M^2, and inversely to $(\text{N.A.})^2$.

The Projection Lantern. The most widely used types of projection apparatus are those for the projection of transparencies, "lantern slides" and kinematograph film images. The "magic lantern" is said to have been invented by Roger Bacon, but detailed information on the lantern was first given by *della Porta* (1538–1615). One of the chief early difficulties was the lack of suitable light sources, as the early oil lamps without chimneys were not superseded by the Argand burner till the end of the eighteenth century.

It might appear that the only requirements for the projection of a transparency would be the provision of an illuminant and a projection lens; but this is only the case under very restricted conditions. Using an opal bulb lamp O, and a projection lens L, Fig. 171, it is possible to project an image of a small transparency such as might be made from a vest pocket camera picture, but the illumination will not be very strong, and its uniformity will depend on the uniformity of the opal glass. Starting from the boundary of the objective and drawing "rays" intersecting the various points of the object plane, we find that t_1 and t_2 are the extreme points which would be seen projected against the bright background of the bulb if viewed from all parts of the lens L. A point such as t_0 is therefore only partially illuminated, and the projected image will have a boundary in which the illumination fades away. Evidently the region of fading will be the smaller the closer the transparency to the source of light.

In the majority of cases the transparency to be projected will be larger than the source of light. British lantern slides measure $3\frac{1}{4}$ in.

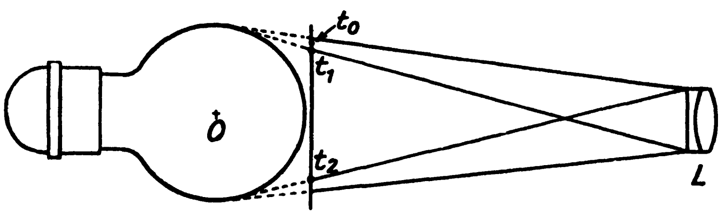

FIG. 171

$\times$ $3\frac{1}{4}$ in. Continental sizes vary, but a common size is 83 mm. $\times$ 100 mm., while larger slides at 12 cm. $\times$ 9 cm. are sometimes used. The American size is $3\frac{1}{4}$ in. $\times$ 4 in. (83 mm. $\times$ 102 mm.). Then in photographic enlargers provision may be required to deal with negative transparencies up to post card size. The most important case is, however, that of the standard kinematograph film picture which is 1 in. wide $\times$ $\frac{3}{4}$ in. high in the direction of the length of the film.

On the other hand, the practical sources of light are mainly of small size. The most important and widely used source is the electric arc. In the common form the crater of the arc is the actual radiant, and this presents a white disc of size varying with the current. Baby arcs taking about 5 amp. give a disc of 3 mm. to 4 mm. diameter, but with 20 to 100 amp. a radiant of much greater diameter can be obtained. According to S. Harcombe, in the *Proc. Opt. Conv.*, 1926, the results of various measurements on the arc crater show that in low current density arcs, where the current density is about 0·15 amp./mm.^{2} of crater area, the intrinsic brilliancy of the crater is about 135 c.p./mm.^{2} In medium current density arcs (·7 amp./mm.^{2}) the intrinsic brilliancy is about 200 c.p./mm.^{2} In high current density arcs, where the current density is 1·0 amp. per sq. mm. of crater, the intrinsic brilliancy is 750 c.p./mm.[2] If d is the diameter of the crater in millimetres, and D that of the positive carbon in millimetres, and I the current in amperes, then $d = K\sqrt{DI}$ where $K = 0{\cdot}344$ for ordinary arcs, and 0·285 for high current density arcs.

Increasing use is being made of these high intensity "searchlight arcs," in which cerium and other salts are introduced into the cores of the carbons; in this way it is possible to maintain stable arcs at much greater current densities, and to reach a still greater intrinsic luminosity in the radiant, since the gases from the core reach a much higher temperature than the carbon itself. With cored carbons taking up to 150 amp., craters up to ½ in. diameter can be obtained, but lower currents are usual in kinema projection, 60 to 70 amp., when the crater will be rather smaller.

The electric arc, although the most satisfactory source of light when high brightness of the picture is required, is somewhat troublesome in operation; even though more or less satisfactory automatic feeding mechanisms are in use, they require a certain amount of skilled attention. Direct current must be employed almost always, especially in kinematograph projection. For small range projection or for photographic enlarging, gasfilled tungsten lamps with coiled filaments are now very widely used. The filament coils are grouped fairly closely together (in a small lamp taking 5 amp. at 100 volts they may be mounted within a projected area 1 cm. square as seen from the centre of the projector lens), but the possibility of short circuiting through thermionic effects sets a limit to the practical closeness of grouping. The higher voltage lamps necessarily have larger filament coils and are less satisfactorily concentrated.

Other lamps of value are the Point'olite tungsten arcs of the Ediswan Electric Company. In the smaller 100 c.p. lamps the chief source of light is a tungsten ball about 2·5 mm. in diameter, but in the larger 500 c.p. lamps the light is mostly derived from a glowing plate 5 mm. square.

The wide-spread availability of electricity is making the older sources of less importance, but the oxy-hydrogen flame used to heat a "lime" cylinder, or (more recently) a thorium pastille, is capable of producing a source of very high intrinsic luminosity distributed in a fairly uniform patch.

Arrangement of Optical Systems for Projection; Screen Brightness. *First Arrangement.* In a very usual optical arrangement, where the source of light is small in comparison with the transparency, it is arranged that the condenser shall project an image of the source into the entrance pupil of the projection lens; the transparency is placed immediately after the condenser. The arrangement is shown diagrammatically in Fig. 172.

Assuming the angular divergence of the beams to be small, we may calculate an approximate expression for the illumination of the screen. Let ds_1 be the area of the source and B its average brightness, and let ds_1' be the size of the corresponding image in the entrance pupil of the condenser projection lens L. Let α' be the angular convergence (relatively to the axis) of the extreme rays from the condenser C to the lens L. If k is the transmission factor

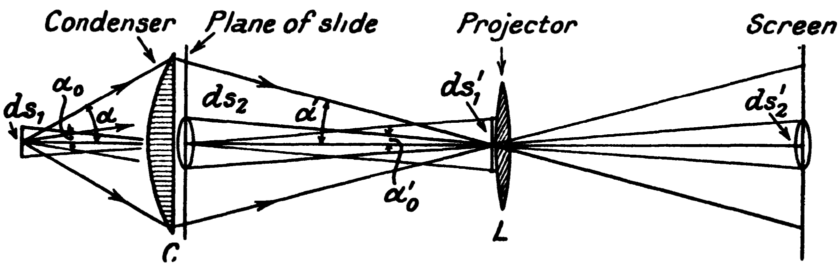

FIG. 172. FIRST ARRANGEMENT FOR PROJECTION (DIAGRAMMATIC ONLY)
The image of the source is formed in the entrance pupil of the projector lens

for the condenser, then the average brightness of the image is kB, and the total amount of light F passing into the projection lens is therefore given by

$$F = \pi k B ds_1' \sin^2 \alpha'$$

The entrance pupil will not lie far from the nodal point; assuming the angular divergence of the rays after passing through the projection lens to be α', the area of the screen illuminated at a distance l' will be (very nearly) $\pi (l' \sin \alpha')^2$; hence the *illumination* of the screen is

$$\frac{\pi k k_1 B\, ds_1' \sin^2 \alpha'}{\pi l'^2 \sin^2 \alpha'} = \frac{k k_1 B\, ds_1'}{l'^2}$$

where k_1 is the transmission factor of the projection lens.

This illumination is evidently proportional to the brightness and area of the source, provided that the projection lens is sufficiently large to transmit the whole of the light.

Further, it is easy to see that our equations imply that the illumination of the transparency is equivalent to illumination by an extended source having the same average brightness as the actual one and placed just behind the slide, but allowing, of course, for the losses in the condenser lens.

Consider a small circular area ds_2 in the plane of the transparency. If B_1 is the effective brightness of this plane, the light sent from this area to the area ds_1' (the image of the source) in the entrance pupil of the projector is

$$\frac{B_1\, ds_2\, ds_1'}{l^2}$$

where l is the distance from transparency to entrance pupil.

Assuming that ds_2 is a small circular area of radius a we obtain

$$\frac{B_1\, ds_2\, ds_1'}{l^2} = \pi B_1\, ds_1' \frac{a^2}{l^2} = \pi B_1\, ds_1' \sin^2 \alpha'_o$$

where the *angle of convergence of the rays* between the axis and the most oblique ray from ds_2 passing through the axial point of the image ds_1' is α'_o.

But we know that the effective brightness of the source image ds_1' must be kB, where k is the transmission of the condenser and B the brightness of the source. Hence the light passing through this image and derived from ds_2 is

$$\pi k B\, ds_1' \sin^2 \alpha'_o$$

Equating the two values for this light from ds_2 we get

$$\pi B_1\, ds_1' \sin^2 \alpha'_o = \pi k B\, ds_1' \sin^2 \alpha'_o$$

so that $$B_1 = kB$$

The effective brightness of the plane of the "object" is the brightness of the source multiplied by the transmission factor k.

Provided that all the light goes unhindered through the projection lens, there is no need for the strict condition that the image of the source shall be formed in the entrance pupil, but we can easily see that if the condenser is free from spherical aberration, so that a sharp image of the source is formed, then the above arrangement will allow us to use the minimum aperture of the projection lens while obtaining the maximum possible light with the given source and condenser, and at the same time illuminating as large a transparency as is possible. If under these conditions the entrance pupil of the projection lens is not completely filled with light, advantage may be obtained by using a larger source, or by pushing the source nearer to the condenser (see below).

Again, there is no need to place the transparency immediately behind the condenser. Provided it is wholly illuminated it may be placed anywhere between the condenser and the projected image of the source, provided that the projector lens can still receive all the light, and that the required projection can be effected.

Second Arrangement (Kinematograph Projector). We noticed above that the illumination of the screen is dependent (granted a sufficient aperture of the projection lens) on the area of the projected image of the source of light. If this source is practically uniform its image may be projected into the plane of the transparency, provided that the picture can be completely covered. In practice, the arrangement is mainly of interest in kinematograph projection, since the small area of the film picture can be covered by the projected image of the arc crater. The "condenser" is usually represented in practice by a mirror, but the theory can be illustrated simply by Fig. 173, in which the condenser is shown diagrammatically by a lens. The symbols *ds*, *ds'*, and *ds"* represent the areas of the source, intermediate image, and final image respectively.

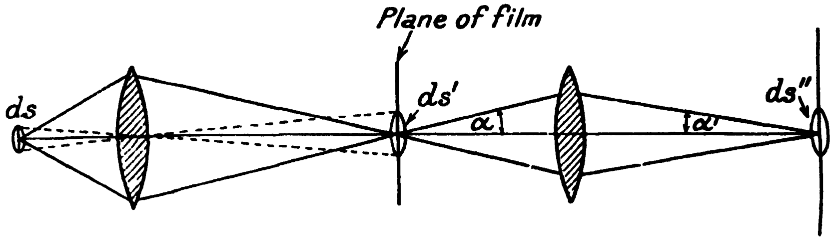

FIG. 173 SECOND ARRANGEMENT FOR PROJECTION APLANATIC CONDENSER UNIFORM SOURCE PROJECTED INTO PLANE OF TRANSPARENCY TO BE PROJECTED

It will be clear that the illumination of the final image will be dependent on the angular aperture of the beam which is transmitted.

If the angles α and α' represent the angles with the axis made by the extreme rays diverging to the projector lens and converging to the final image, we can write, using the symbols in the same sense as above,

Total light received by the projector lens $= \pi k B ds' \sin^2 \alpha$

Hence the illumination of the screen $= \dfrac{\pi k k_1 B ds' \sin^2 \alpha}{ds''}$

But the optical sine relation gives

$$\frac{\sin^2 \alpha}{ds''} = \frac{\sin^2 \alpha'}{ds'}$$

so that the illumination of the screen $= \pi k k_1 B \sin^2 \alpha'$

$$= \frac{k k_1 B \text{ (utilized area of the stop of the projection lens)}}{\text{Square of distance from lens to screen}}$$

With such an arrangement, a projection lens of large aperture (low stop number) can be usefully employed if the condenser gives a beam of equally large angular aperture. The actual illumination on a screen, therefore, depends in each case, first on the illuminated aperture of the projection lens which would be seen to be illuminated when looking through a pinhole in the screen; secondly, on the intrinsic brightness of the source; thirdly, on the transmission of the optical system. With a condenser of large aperture we shall be able to utilize a given area of the stop with a projector of shorter focal length than with a condenser of smaller aperture, and hence to secure a proportionately larger picture of the same brightness. The advantage of the second arrangement is, however, that a mirror can be used as condenser as shown in Fig. 174. The plane of the image of the crater is uniformly illuminated, but other sec-

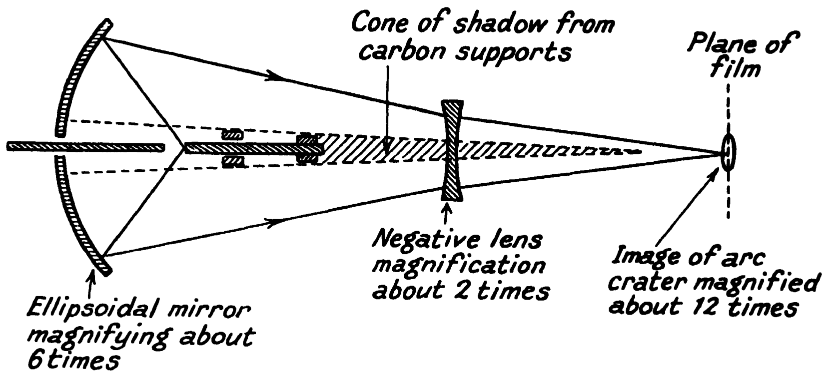

FIG. 174. USE OF MIRROR CONDENSER

tions of the beam will show more or less shadow due to the carbons. Also, when high intensity arcs are employed, the flame of the arc contributes a proportion of the light much greater than in the case of the ordinary arc lamp, in which practically all the light is derived from the crater. The result is that the projection of the image of the source exactly into the plane of the transparency must be very precise, and it is advisable to use some special device to ensure that the negative carbon crater maintains its position very exactly, and that the length of the arc also remains constant.

Selection of Projection Lens. The focal length of the projection lens required under given conditions can be found from the approximate relation—

Focal length in inches

$$= \frac{(\text{Throw from lens to screen in feet}) \times (\text{diameter of transparency in inches})}{(\text{Corresponding diameter of image in feet})}$$

For example, take a standard size slide with a picture 3 in. sq., say, and suppose it is desired to project a picture 8 ft. sq. at a distance of 40 ft. Then

$$\text{Focal length} = \frac{40 \times 3}{8} = 15 \text{ in.}$$

In the case of the kinema projector, the width of the "gate" aperture exposing the film may be about 0·91 in.; to find the focal length of a lens to give a 13 ft. picture at a 50 ft. throw we have

$$\text{Focal length} = \frac{50 \times 0{\cdot}91}{13} = 3\tfrac{1}{2} \text{ in.}$$

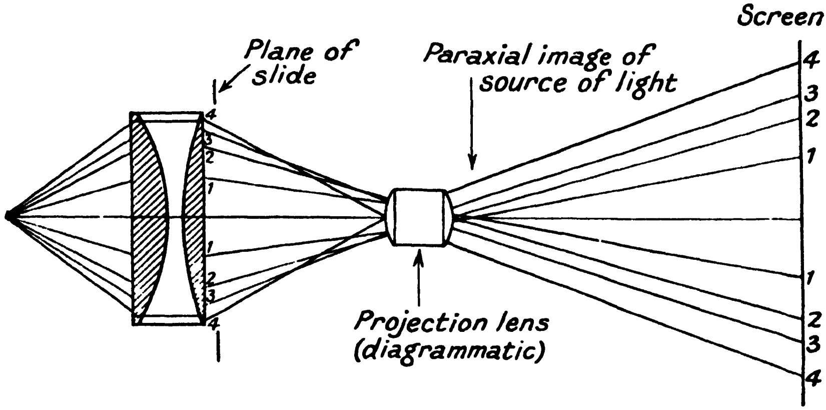

FIG. 175. PROJECTION WITH CONDENSER LENS SUFFERING FROM SPHERICAL ABBERRATION (DIAGRAMMATIC)

If projection lens is moved too far to the right the outer part of field (rays 4) will be cut off, if moved too far to the left an annular zone of light (rays 2 and 3) will be cut off. In the case shown above, the source of light might, if small, be pushed closer to the condenser to obtain better illumination; but the above position allows for a finite size of the image of the source.

Lantern Slide Projection Systems. It commonly occurs that the projection systems in ordinary use employ condensers having large residuals of spherical aberration. A common type of condenser has two "bulls-eye" lenses, each plano-convex, the convexities being turned together. The powers of the components are arranged so that roughly parallel light passes between the two members. The plano-convex lens is cheap to manufacture, and represents an approximation to the crossed lens giving minimum spherical aberration.

The size of the best concentration of light in the beam focused by the condenser is dependent partly on the spherical aberration of the condenser, partly on the chromatic aberration but only to a very small extent, and partly on the size of the source; the latter more especially when the source is of considerable magnitude, as in the case of a gasfilled lamp. The dimensions of the image of the source (neglecting spherical aberration for the moment) will be the greater, the shorter the distance from the source to the condenser; but the total amount of light taken up by the condenser will be the greater if the source is pushed closer to it. It is, therefore, advisable to use a projection lens having a fairly large diameter of the back lens in relation to the focal length; then the source may be pushed in towards the condenser till the cones of light begin to be cut off by the mounts of the projection lens. The best condition will usually be such that the image of the source formed by the paraxial regions of the condenser lies beyond the projector, and since the spreading of the beam will be less with smaller sources,

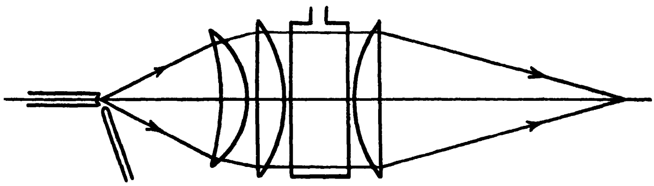

FIG. 176. TRIPLE LENS CONDENSER (WITH DIMINISHED SPHERICAL ABERRATION) INCORPORATING WATER CELL FOR HEAT ABSORPTION

they may be brought closer to the condenser than when using larger ones without loss of light. The condition is illustrated in Fig. 175.

If specially figured or corrected lenses are employed in the condenser system, then the beam will be free from any spreading due to spherical aberration, and this again will allow of the maximum amount of light being taken up by the system consistent with given dimensions of the source. Fig. 176 shows a triple lens condenser which gives considerably diminished aberration. It has a trough containing water for cooling purposes.

The lens type chosen for projection is usually on the lines of the Petzval portrait objective, and may work at a high aperture, $\frac{f}{2{\cdot}5}$ to $\frac{f}{4}$ or thereabouts. The Petzval type has then the advantage of the large diameter of its lenses for a given focal length, and consequent economy of light. These lenses are usually well corrected for spherical aberration and coma, so that the centre of the field is well defined. Owing to the long throw which is usually required, the angular field to be covered is not usually very large, and there is then no need for a more complex and expensive anastigmatic system, although cases do arise in which a high magnification is required for a comparatively short throw, and another type of lens giving a wider field may then be advisable; an anastigmat may then be chosen. The condenser must naturally be capable of giving the angular field required.

Loss of Light in Projection Systems. The average efficiency of ordinary projector systems is very low, only about 5 per cent of the total light from the source reaching the screen in many cases. In kinematography a further loss occurs through the cutting off of light by the revolving shutter.

Even though a mirror may be used behind the source of light, it is difficult to get much more than 10 per cent of the light through the slide carrier with ordinary systems, and reflections at the surfaces of the slide and projection lens cause further serious weakening. Numerical details of the kinematograph projector may be of some interest. It may be reckoned that at least 4 per cent of light is lost by reflection at a glass-air surface, and an absorption of 5 per cent per centimetre thickness of glass is not uncommon in the inferior glass often employed in condenser lenses. A two-lens condenser may therefore transmit only 70 per cent of the incident light, and the loss in a triple condenser would be considerably greater.

The gate at which the film appears is situated either at the focus of the condenser system, or at the "waist" of the converging cone if the system is subject to spherical aberration, and the loss here varies greatly with the source and the optical system employed. It is evident that the area of uniform illumination must overlap the gate widely in order to allow for slight variation in the position of the source of light due to wandering of the arc, or mal-adjustment of the carbons in the focus. Even in favourable circumstances a loss of about half the light seems difficult to avoid if a safe overlap is to be given.

The film itself, even in the most transparent part, removes about 20 to 25 per cent of the direct light, and the objective, usually with six "air-glass" surfaces, may remove a further 40 per cent, even if the absorption in the glass is negligible. We then have to consider the loss due to the kinematograph shutter which masks the image while the film is in motion from one picture to the next. This inevitably cuts off 50 per cent. Hence the transmissions of the various parts are likely to yield the following approximate table—

APPROXIMATE PERCENTAGE TRANSMISSIONS

Condenser	Gate	Film	Projector	Shutter	Combined
70	50	75	60	50	7·8

The arc lamp employed in kinema work may have a candle-power of 12,000, according to the current taken. Assuming that the condenser intercepts a cone of radiation of unit solid angle, which is an approximation to the truth, the total light entering the system will be approximately 12,000 lumens, of which 7·8 per cent reach the screen, i.e. 936. If the screen area is, say, 250 sq. ft.,

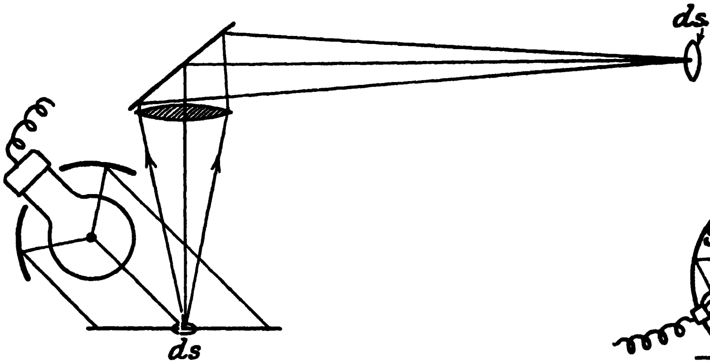

FIG. 177(*a*). SIMPLE EPISCOPE

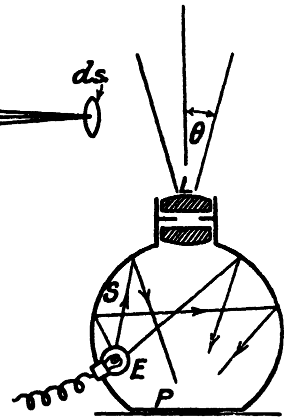

FIG. 177(*b*). SPHERE EPISCOPE

this means an illumination of 3·7 lumens per sq. ft., which is fairly satisfactory in practice.

The condenser losses are considerably reduced with the mirror and negative lens combination. The glass surface, instead of reflecting the light back, in this case sends it onward in the required direction, and the only serious losses are those due to any imperfection of the mirror reflection and reflection at the negative lens, which should not amount to more than 15 or 16 per cent when the surfaces are in good condition.

Episcopes. Episcopes are instruments designed for the projection of images of opaque objects. Such objects are illuminated very intensely with the aid of powerful lamps and reflectors, and their images are projected by anastigmat lenses of very large aperture. Let us assume that an element of perfectly reflecting and diffusing surface, of area ds, is illuminated in the first place by a powerful electric lamp of 2,000 candle-power, say, at 6 in. from the surface, as in Fig. 177(*a*). The lamp may be furnished with a reflector to enhance the illumination, and although the incidence of the light must be oblique, it may be possible to attain an illumination of 8,000 ft. candles on the surface, i.e. 8,000 lumens per sq. ft. must be re-radiated; but the total radiation $= \pi B$ lumens per sq. ft.

Hence, $B = \frac{8000}{\pi}$. The candle-power of the elementary area ds_1 is therefore

$$\frac{(8000\, ds_1)}{\pi}$$

and the lumens radiated into a cone of semi-apical angle θ will be

$$\pi \sin^2 \theta ds_1 \left(\frac{8000}{\pi}\right)$$

If the lens is working at $\frac{f}{2}$, say (the ratio of diameter of entrance pupil to focusing distance from object to pupil is $\frac{1}{2}$, so that $\tan^{-1}\theta = \frac{1}{4}$, and we find $\theta = 14°$), $\sin^2 \theta = \frac{1}{16}$ approximately, so that the amount of light is $500\, ds_1$ lumens. This light, or a proportion of it depending on the transmission of the lens, is to be distributed over the magnified image of ds_1. If the magnification would be sufficient to enlarge a 4 in. picture up to 6 ft., the linear magnification will be 18 times, and the area magnification $= 18^2 = 324$. Hence, if the transmission of the lens is 80 per cent, we shall have $400\, ds_1$ lumens distributed over an area $324\, ds$, so that the lumens per unit area $= \frac{400}{324} = 1{\cdot}23$. This is the illumination of the screen in foot-candles, but allowance must further be made for the reflection factor of any actual object surface.

The principle of the Ulbricht integrating sphere has been applied by Bechstein to the problem of Episcopic projection. Fig. 177(*b*) illustrates the principle. The sphere S has two main openings, one of which is filled with the projection lens L, the other is closed by the picture or surface, the image of which is to be projected. The interior of the spherical surface is coated with a matt white paint. Suitable paints can be secured with very high total reflection coefficients, i.e. up to 98 per cent.

Suppose for a moment that the paint was totally reflecting and absorbed no energy, also that the surface of the diagram P absorbed a negligible amount of energy, then if we introduce a source of light E into the sphere, the aperture of the lens L will be the only path by which the light can escape (perhaps after many internal reflections), and thus clearly *all* the light radiated by E would have to pass through L. It can be shown that the proportion of the light derived from the circular area P to the total light will be represented by $\sin^2 \theta : 1$, where θ is the semi-angular subtense of P (provided that we neglect the light directly escaping from E through L without internal reflection, or assume that a small diffusely reflecting screen is placed to stop such direct escape).

The above performance would be an extremely efficient example of projection, even as compared with the case of transparent objects; but in practice no such efficiency is attainable since the object for projection will absorb much light, the lens L will have a very appreciable aperture and may allow some direct light to escape, and a considerable absorption of light energy will take place at the walls of the sphere. In spite of these drawbacks, a good performance can be obtained.

Epidiascopes. In recent years a number of makers have introduced systems combining an episcope with an ordinary projection lantern; a mirror and condenser can be brought into action when required in order to change from one system to the other; such instruments are known as "epidiascopes." Lantern slides for projection are often called "diapositives" in Continental literature.

The Projection of Light. *Apparent Brightness of Image-forming System (Maxwellian view).* In a previous section we have calculated the apparent brightness of the image projected by an optical system, assumed to be free from spherical aberration and to fulfil the optical sine condition. Referring back to Fig. 170 and the accompanying discussion, the ***illumination*** of the area covered by the image in the image plane (light per unit area) is

$$\pi \mathrm{B}k \sin^2 \alpha'$$

if α' is small, $\sin \alpha' = \frac{y}{l'}$ where y is the radius of the exit pupil of the image-forming system, so that

$$\text{Illumination} = \frac{\mathrm{B}k\pi y^2}{l'^2}$$

$$= \frac{\mathrm{B}k \text{ (area of exit pupil)}}{l'^2}$$

We therefore find that the whole exit pupil of the image-forming system is now radiating as a source of brightness Bk, i.e. the brightness of the source multiplied by the transmission factor of the optical system. The "candle-power" of the radiant area is thus Bk (area of exit pupil).

It is not very easy to give an entirely satisfactory general proof that the same thing holds good whenever the eye views an optical system projecting the image of a uniform source in such a way that any ray from the eye traced back through the system intersects the source; the following treatment, however, may indicate that the principle is wider than might be inferred from the special case above.

Let A be a small area of an object of brightness B, and let RT (Fig. 178) be the trace of a refracting surface having its normal in the plane of the diagram; let the small area be rectangular with one diameter in the plane of refraction and one diameter perpendicular thereto. Let i and i' be the angles of incidence and refraction, then the angular width of the fan of rays (in the plane of the diagram) from the element to a point R of the surface is di, and the corresponding width when refracted is di'. The breadths perpendicular thereto are proportional to $\sin i$ and $\sin i'$ respectively. In order to see this, imagine perpendiculars dropped from each fan to the normal and that the

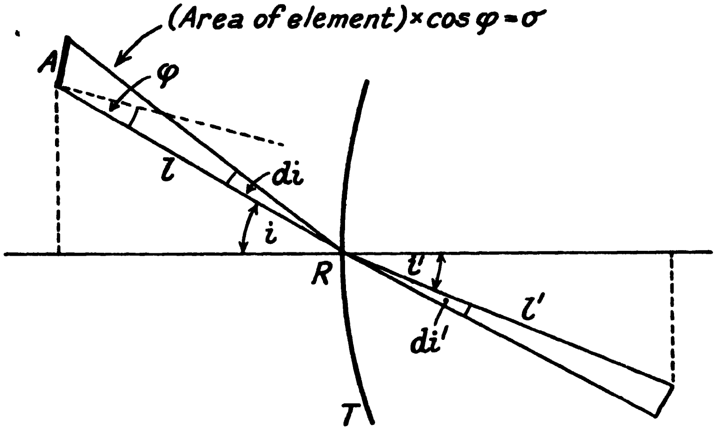

FIG. 178. DIAGRAM

diagram is rotated through a small angle $d\theta$ about the normal. The *locus* of the surface will be constant in the neighbourhood of the point R, and the arcs described by corresponding object space and image space elements at distances l and l' from R will be $l \sin i d\theta$ and $l' \sin i' d\theta$. Let ω_1 and ω_2 be the solid angles of these bundles, then

$$\frac{\omega_1}{\omega_2} = \frac{\sin i\, di}{\sin i'\, di'}$$

But $$n \sin i = n' \sin i'$$

and $$n \cos i\, di = n' \cos i' di'$$

Hence $$\frac{\sin i\, di}{\sin i'\, di'} = \frac{n'^2 \cos i'}{n^2 \cos i}$$

so that $$\frac{\omega_1}{\omega_2} = \frac{n'^2 \cos i'}{n^2 \cos i}$$

Let the projected area of the element A in the direction of R be σ, then the light sent from the radiating element to a small area ds on the surface is $$(B\sigma\, ds \cos i)/l^2 = B \cos i\, ds\, \omega_1$$

and the amount of light it radiates is

$$B_1 \cos i'\, ds\, \omega_2$$

where B_1 is the effective "normal brightness" of the element R, which will agree with its apparent brightness to an observing eye of which the pupil is filled with the light. In order that we may have the amount radiated equal to k times the amount received (k being the transmission factor), we must have

$$B_1 \cos i' \, ds \, \omega_2 = kB \cos i \, ds \, \omega_1$$

so that

$$B_1 = \frac{kB\omega_1 \cos i}{\omega_2 \cos i'} = \frac{n'^2}{n^2} kB$$

Hence the apparent brightness of the optical surface would be equivalent to that of the source if there were no reflection losses; and if the initial and final media were the same, since the equation could be applied to any number of refractions. Thus

$$B_4 = k_1 k_2 k_3 k_4 \left(\frac{n_4'}{n_4} \cdot \frac{n_3'}{n_3} \cdot \frac{n_2'}{n_2} \cdot \frac{n_1'}{n_1}\right)^2 B$$

$$= K\left(\frac{n_4'}{n_1}\right)^2 B$$

Reflection can be looked upon as a particular case of refraction.

This theorem is a very important one; it indicates, *let us repeat*, that when the eye pupil is filled with the beam from a projector of any kind, and when all rays which could be traced from the eye back through the projector intersect the source, then the whole of the projector system has the apparent brightness of the source.

The appliances to be dealt with under this heading comprise searchlights, motor-car headlights, light-house projection systems, signalling lamps, and the like. It will not be possible to do more than to give the briefest outline of the theory and practice, since a very large technical literature exists in regard to all of these.

We showed above that the apparent brightness of the radiating aperture forming an image of a surface of brightness B is kB, where k is the transmission of the system, and this agrees with the apparent brightness of the image when *it* is observed by the eye. If the eye moves to various distances, the apparently illuminated part of the radiating aperture of the system will always have the same apparent brightness kB, provided that the pupil of the eye is filled with light, or lies within the cone of radiation from the optical system; if, however, the pupil is not completely filled with light, the radiating aperture will appear less bright; or if there are possible ray paths between the eye and the optical system which, on being traced backwards, fail to intersect the source of light, then the corresponding parts of the optical system will appear dark.

To make these principles more definite we will refer to Fig. 179. The eye is withdrawn behind the image ds' of a radiating element ds; when the eye pupil was coincident with ds' the whole back surface of the lens appeared to be of brightness kB, since *ds'* was, we will say, slightly larger than the eye pupil. When the eye is withdrawn behind the image *ds'*, the area of the latter acts like a circular stop, and only a limited area in the centre of the lens can

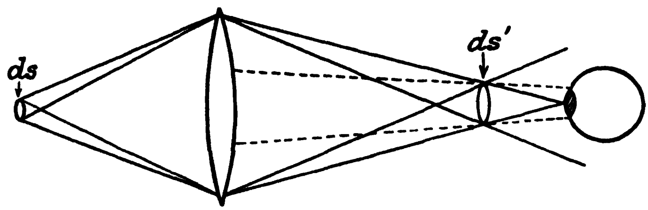

FIG. 179

send light to the whole of the eye pupil; this part, therefore, appears of the full brightness kB, and is surrounded by a penumbral shadow. Again, consider the case of Fig. 180, in which the image of a very small source is projected near the eye by a lens exhibiting strong spherical aberration; the centre of the lens appears filled with light,

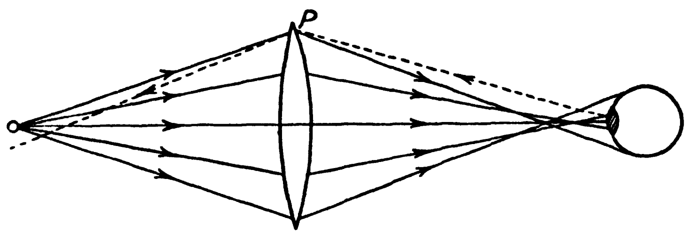

FIG. 180

but the marginal zones are in shadow, since the marginal rays pass outside the pupil. We could infer the same thing if we trace a ray (dotted) from the pupil centre backwards through the margin of the lens system, and find that it passes outside the source; if, again, we take a ray through the marginal point P from the bottom point of the pupil and find that it intersects the source, while a ray from the top of the eye pupil fails to intersect it, we shall infer a partial illumination of the surface of the system near P.

If the aperture appears wholly illuminated to the observing eye, technical parlance speaks of a "complete flash"; otherwise we may have a "partial flash" if the apparent illumination is incomplete.

Effect of non-fulfilment of the Sine Condition. We have seen that in a simple case the apparent brightness of the flash is kB, and this was independent of any fulfilment of the optical sine *condition*, i.e. the constancy of magnification for the different zones of the system, although the optical sine relation expressing the magnification for a zone was used in the simple discussion. If the system does not fulfil the sine condition, the images of the source formed by different zones will have different sizes; hence the projected image will have a diffuse boundary. If the outer zones give the larger image, then an eye observing the lens from a point in the outer region of the image will see the margin of the lens bright while the centre parts are dark, and *vice versa*. Evidently, in order to obtain a sharply-bounded image, the fulfilment of the sine condition will be of importance.

Searchlights and Headlights. Searchlights almost invariably employ mirror reflectors rather than condensing lenses, since mirrors

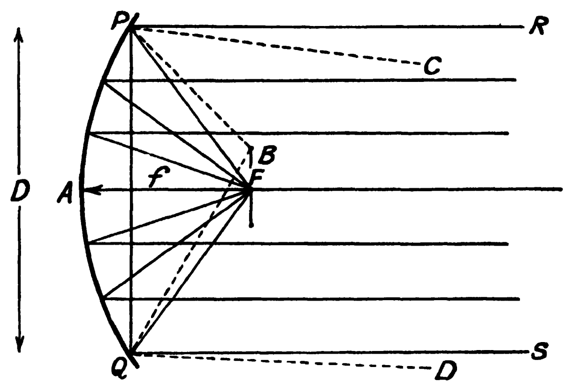

FIG. 181

give freedom from chromatic aberration, and, moreover, involve less loss of light. The geometrical form of the paraboloid of rotation renders rays from the focus strictly parallel on reflection, although the system does not fulfil the optical sine condition. Hence, rays diverging from a point (in the focal plane) away from the axis are not strictly parallel.

The divergence from the sine condition for a zone of diameter D of a parabolic mirror of focal length f is shown by the relation

$$\frac{y}{\sin \alpha'} = f\left\{1 + \frac{1}{16}\left(\frac{D}{f}\right)^2\right\}$$

If the sine condition were fulfilled (page 37), we should have $\frac{y}{\sin \alpha'}$, as a constant. The defects are quite easy to realize geometrically (see Fig. 181); $\frac{y}{\sin \alpha'}$ is the distance PF for any zone; the above equation can easily be calculated from the equation to the parabola (see page 68). The rays FP and FQ are rendered parallel on reflection, but we can see that the angle $\widehat{FPB}$ is greater than the angle $\widehat{FQB}$ if BP and BQ are rays from the extra-axial point B; the paths of these rays after reflection are PC and QD respectively, and hence the angle $\widehat{RPC}$ is greater than $\widehat{SQD}$, so that PC and QD are clearly convergent. Therefore the searchlight beam departs from parallelism, with a finite size of the source, on account of the geometrical properties of the image formation.

In practice, a single reflecting surface is too liable to tarnishing and damage. Hence, reflectors of silvered glass are generally employed, the silver backing being suitably protected by coppering and painting.

One plan is to grind a glass reflector of which both surfaces are congruent paraboloids of rotation;[4] thus the reflected component from the front glass surface should be "parallelized" in addition to that from the back. But, in practice, the finite thickness of the glass introduces a certain amount of spherical aberration into the beam reflected from the back, so that it is advisable to modify[5] the shape of the back surface in order to avoid this defect, and it is no longer a true paraboloid.

In both the above cases, then, the reflected components from both front and back surfaces are more or less parallelized, and this is of importance in long distance projection where the utmost economy of light is required.

In the case of motor headlights, however, the lateral spreading of a certain amount of the light is highly desirable, and it is then possible to use reflectors (as in the Zeiss systems) in which the back reflecting surface is truly spherical, and the front surface is figured to a suitable non-spherical curve, refraction at which corrects the spherical aberration arising from reflection at the back surface.

The Mangin mirror (Fig. 182) (described by A. Mangin in 1876) consists of a glass reflector, silvered on the back, each surface of which is a true sphere. The curvatures may be so chosen that the system is freed from spherical aberration, and also from coma; thus it fulfils the optical sine condition. In consequence of this the projected image of a small source placed at the focus is fairly sharply defined, and the lateral spreading of the light is greatly restricted as compared with the effect of a parabolic reflector. The mirror is, consequently, very useful for signalling lamps where only a very limited region near a receiving station may receive the light.

The form of the glass reflector itself is that of a diverging meniscus lens, and the centre of curvature of the hollow side may nearly coincide with the focus. Since the thickness of the glass rapidly increases towards the margin, this sets a limit to the aperture ratio which can be effectively used; but the mirror can be made to subtend 130° to 140° at the focus.

Though the Mangin mirror is effectively employed for the smaller searchlights up to about 20 in. in diameter, the parabolic reflectors

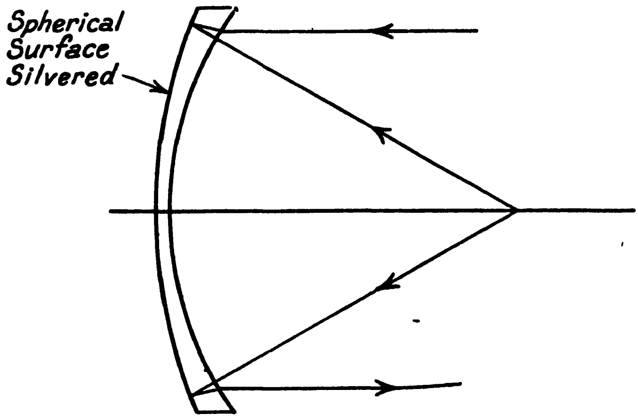

FIG. 182. THE MANGIN MIRROR

(which avoid too great a marginal thickness of glass) are employed for the larger sizes.

Illumination due to Searchlight. Consider a small area in the field of illumination taken near the axis in a plane perpendicular to the axis. It is to be at such a distance from the projector that any ray taken from this area to the exit pupil of the system and traced onwards, will intersect the source of light. Under these conditions an eye placed at the position of our area would observe a "complete flash" in the projector.

According to the theory the illumination will then be

$$\frac{Bk\ (\text{area of projector})}{l'^2}$$

so that the effective "candle-power" of the projector in this area is, as before, $Bk \times$ (area of exit pupil). The coefficient k must, however, include the possible effects of atmospheric absorption.

Large searchlights are made with mirrors up to 5 ft. in diameter, but the difficulty of producing an accurate figure of the surfaces, as compared with the smaller 3 ft. mirrors, causes the results obtained from the larger mirror to fall short of expectations. Assuming a 3 ft. mirror and a carbon arc crater of an intrinsic brightness of 10^5 candles per square inch, a complete flash would give a gross candle-power of

$$\pi \cdot 18^2 \cdot 10^5 = 10^8 \text{ candles (approx.)}$$

But this figure will, in practice, be reduced very considerably by the obstruction of the negative carbon of the arc, the reflection losses in the mirror and in the front "window" which may be fitted to protect the arc from wind; so that the net result is not likely to exceed 60 per cent of the above figure in a clear atmosphere, even with a fairly perfect reflector. The majority of reflectors fail, however, to give a really complete flash owing to optical imperfections.

Effect of Various Sources. In the simple theory above, the source was assumed to be an elementary disc perpendicular to the

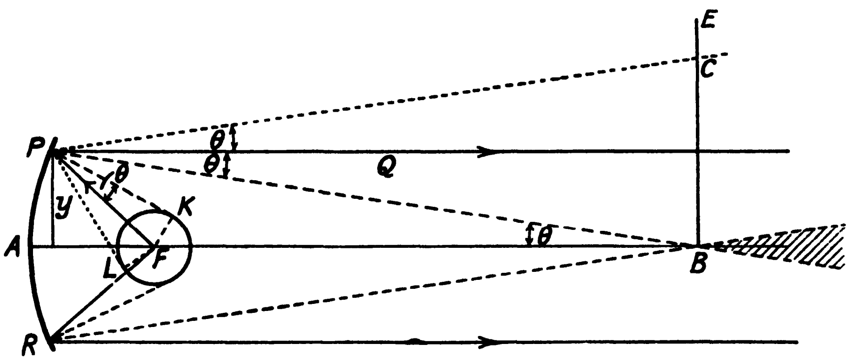

FIG. 183. OPTICAL PROPERTIES OF PARABOLOIDAL REFLECTOR

axis. The nearest approximation to this in practice is the crater of the arc in a searchlight, but smaller projectors may employ other sources, such as filament lamps or acetylene flames. We can usefully consider the illuminated area and the "flashing" of the radiant area under other conditions.

A typical example, Fig. 183, concerns a spherical source of radius FK = r, situated so that its centre falls into the focus. A ray following the path FP would be reflected parallel to the axis, but one derived from the extremity of the source as seen from P (i.e. one following the tangential direction KP) will be reflected through B, where $\widehat{FPK} = \widehat{QPB} = \theta$. Since PQ and AB are parallel, the angle $\widehat{ABP}$ is also θ, and we have

$$\sin\theta = \frac{r}{PF} = \frac{y}{PB} = \frac{y}{AB} \text{ (approx.)}$$

if AB is very large in comparison to y, the radius of the zone marked by P. We have, by the geometry of the parabola,

$$FP^2 = y^2 + \left(f - \frac{y^2}{4f}\right)^2 = \left(f + \frac{y^2}{4f}\right)^2$$

$$\therefore \qquad \frac{r}{\left(f + \frac{y^2}{4f}\right)} = \frac{y}{AB} \text{ very nearly, if } r \text{ is not large}$$

and $$AB = \frac{y}{r}\left(f + \frac{y^2}{4f}\right) \text{ very nearly.}$$

It is clear that a "complete flash" will be obtained at any axial point beyond this distance, and that the illumination at such points will vary (neglecting absorption effects) according to the "inverse square" law. A greater distance is required as the aperture of the mirror is increased.

A ray from the apparent extremity of the source L will be reflected so as to make an angle $\widehat{CPQ}$ with the line PQ. If ECB marks a plane through B perpendicular to the axis, the point P of the mirror will be apparently illuminated to an eye placed anywhere between C and B; on the other hand, a point R on the other extremity of the mirror diameter will be dark, and a complete flash will not be obtained. The flashing of the whole mirror will therefore only be obtained from points within the cone, represented by the shaded area in the figure, the generating line of which is the prolongation of the line PB or RB. At a distance D from the mirror the diameter of the fully illuminated area taken perpendicular to the axis is, writing d for AB,

$$\text{Diameter of fully illuminated area} = 2\,(D - d)\tan\theta$$

$$= \frac{2\,(D - d)\,r}{\left(f + \frac{y^2}{4f}\right)} \text{ very nearly.}$$

The "inverse square" law can only be supposed to hold along those parts of straight lines from the point A which lie within the shaded area.

Lantern and Lighthouse Projection Systems. Lenticular or "dioptric" condensers for the projection of light are of considerable importance in connection with lanterns and lighthouses. In a ship's lantern, concentration is only required in the sense that light should not be wasted in going much above or below the horizontal, but must spread freely in azimuth. The lighthouse beam must often be restricted in both directions. In the lighthouse the "lens" can be made of much larger aperture than would be practicable for any single parabolic reflector of such size, and several projectors can be grouped around a single source. Fig. 184 shows the section of a typical lighthouse "lens"; the whole would be realized by rotating this section about the horizontal axis through the source; a "ship's lantern" system would be obtained by rotating the central elements

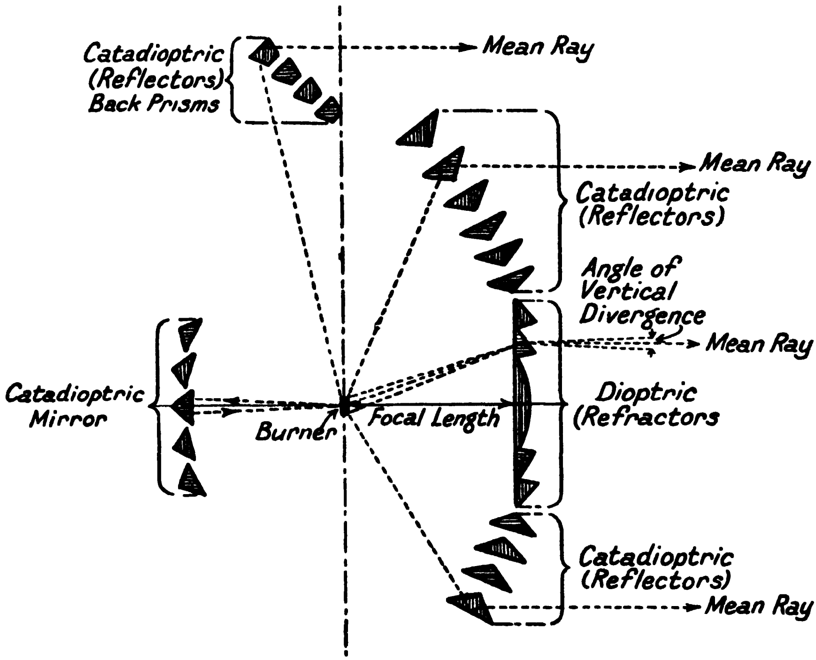

FIG. 184. LIGHTHOUSE PROJECTION SYSTEM

of the section about a *vertical* axis through the source. These stepped lenses were invented by Fresnel in 1822.

The construction of the inner elements is a method of overcoming the great thickness of the lens which would be necessary if the front face were continuous; the discontinuous elements allow a great saving of weight and of absorption of light in the glass. For the outer zones, the elements are reflecting prisms, but one of the refracting surfaces of these, like the surface of the inner prism elements, is not of perfectly straight section; they are given a curvature which should make a ray from a point source at the focus emerge in a horizontal direction. The refraction at the inner prism elements is subject to chromatic aberration, but extreme spherical aberration of the outer refracting zones, which would be characteristic of a single lens with spherical surface, can be avoided by giving the correct form to the outer ring elements.

A theoretical account of the distribution of light in the beams has been given by W. M. Hampton.[6]

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2. Martin: Dept. of Scientific and Industrial Research, *Bulletin No.* 3.
3. German Patents 250314 of 1911, 252920 of 1911, 316050 of 1919.
4. Munker and Schuckert: German Patent 35477 of 18th Aug., 1885.
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6. W. M. Hampton: *Trans. Opt. Soc.*, XXX (1928–29), 185.

CHAPTER VII

THE TESTING OF OPTICAL INSTRUMENTS

NEEDLESS to say, the fundamental test of any instrument or appliance is that of satisfactory performance under ordinary conditions of usage. With optical instruments for visual observation, however, it is not always easy to tell whether or not the performance is really satisfactory without spending much time and trouble. Thus the purchaser of a microscope may obtain what seems a satisfactory image of some object; it appears well defined and free from obvious defects; but a person of keener eyesight, and using a more critical method, may discover deficiencies in the performance. The same is true of a telescope or of projection apparatus. It is, therefore, advisable to adopt methods of testing which are as free as possible from likely errors due to any defective eyesight or prejudices of the observer.

The inspection department of an optical factory will employ sensitive qualitative tests for instruments and their components, as a matter of routine. Occasionally, when a new system is being perfected, the physical *measurement* of any defects has to be undertaken. Similar physical measurements of aberrations are of importance to many who have to use optical instruments for very exact measurements. Thus, astronomers commonly determine numerically the aberrations of their reflectors and refractors, and surveyors using photographic methods may require to measure very accurately the distortion of their lenses. Since there does not seem to be a concise account in English of the chief optical methods, the details being mainly found in scattered handbooks, it is hoped that the following short summary may be of interest.

The "Star" Test. (*Telescope and Microscope Objectives.*) The simplest possible theoretical object is a "point source" of light; since this only exists in imagination, we remember that the image of a very small source of light never decreases in dimensions beyond the theoretical limits set by the aperture ratio of the system and the wave-length of the light in the image region. The reduction of the dimensions of the source beyond certain limits in which its geometrical image becomes small in comparison with the diameter of the "Airy disc," produces no further appreciable change in the diffraction image except a diminution of its brightness.

For telescopes, real stars are ideal test objects; and manufacturers of astronomical telescopes of all sizes naturally consider final tests on real stars as indispensable, but during the course of manufacture "artificial stars" can be of great use.

A small spherical mercury thermometer bulb may form a very small image of the sun, or a nearby source such as a small electric

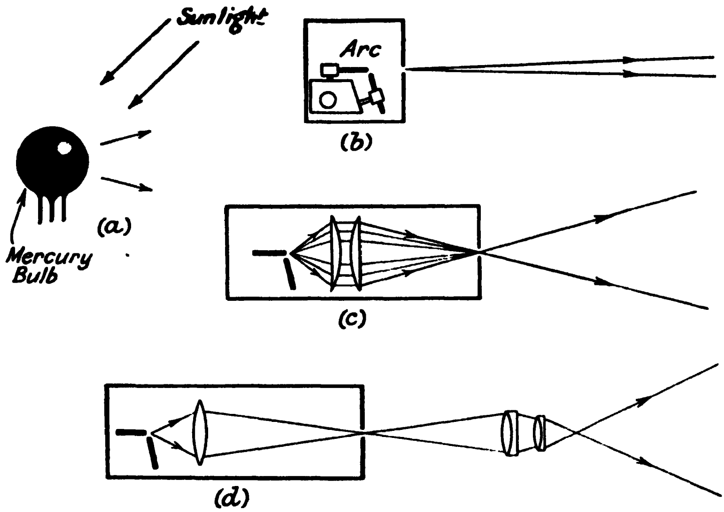

FIG. 185. VARIATIONS OF AN ARTIFICIAL STAR FOR TELESCOPE TESTING

a = Mercury bulb.
b = Source behind pinhole. (This may give narrow beam)
c = Arrangement for wider-angled beam
d = Arrangement where smallest possible "star" is required for closer work

lamp, etc.; when viewed from a distance such a " star " may be very satisfactory. Alternatively, a small pinhole in a sheet of tinfoil may be illuminated from behind by an arc lamp (a carbon arc or sometimes a mercury arc), or perhaps by a condensed electric spark. If the "star" is to radiate light over a wide angle, then the image of the source may be focused on the pinhole by a suitable condenser lens. The pinhole may be too large; it can be used at the proper working distance from a reversed microscope objective, which then projects a diminished image of the pinhole into the plane which would be occupied by the microscopic object in the ordinary use of the lens.

Telescopes intended for use in daylight should be tested by the "sun and thermometer bulb" arrangement, if possible, as the effects of chromatic aberration are more easily recognizable than when artificial light is employed. If it is necessary to use artificial light an "artificial daylight" colour filter can, however, be used. Fig. 185 shows some variations of an artificial " star " for telescope testing.

Artificial "stars" for the microscope may be obtained by silvering the undersides of a number of cover-glasses of varying thickness, without taking too great pains to remove dust and specks from the surface. It will usually be found that a number of small holes result in the film, which are well below the resolution limit of ordinary objectives. The glasses are cemented down on 3 in. × 1 in. slides, silvered faces down, and the "stars" are then illuminated by the microscope condenser, which forms the image of an arc or Point'olite lamp, etc., in the plane of the film.

Another device useful with low powers is to produce, by holding a slide above boiling mercury, a deposit of extremely minute mercury globules which form correspondingly small images of a source of light suitably disposed. The light may be thrown on the globules by the help of a small mirror, or even by a vertical illuminator system in the microscope.

Having secured a suitable "star" source, and made sure that it can send unrestricted light into the whole aperture of the objective, the appearance at the focus is then examined with the eyepiece or eyepieces supplied with the instrument. In many cases, nowadays, the objective and eyepiece systems are designed together and should, therefore, be tested together; but if odd objectives are to be tested, a good idea of their individual performances can still be obtained with the aid of a good set of Huygenian eyepieces of focal lengths ranging from, say, 25 mm. to 7 mm., if objectives of known performance are available for comparison. When using the highest power the Airy disc will be clearly seen in appreciable extension at the best focus with its surrounding rings, that is, with telescope and microscope objectives of usual types.

A complete discussion of the various points in the "star" tests would take too long; space can only be found for brief notes. Fuller details will be found in a booklet by Mr. H. D. Taylor[1] (*The Adjustment and Testing of Telescope Objectives*).

Axial Images; Centring. The first test is for *centring* of the lenses. The star image is brought into the centre of the field and the appearances are observed, both at the best focus and also when the eyepiece has been moved slightly within and without the best position. Complete symmetry of the distribution of light and colour around the axis indicates the correct centring and mounting of the lenses, but if one component lens has its optical axis displaced from the general axis of the system, unsymmetrical colour effects will appear. The eyepiece is assumed to be sufficiently well centred by the tube of the instrument.

Squaring-on. Slight asymmetry of the appearance of the Airy disc rings near the best focus may indicate imperfect "squaring-on" of the objective; the optic axis of the objective does not pass through the centre of the eyepiece. If this is suspected, tests should be made with a special "squaring-on" eyepiece.[2]

Colour Correction. An uncorrected objective, such as may be encountered in old telescopes or microscopes, shows brilliant colour effects with the artificial "star," especially on each side of the best focus. Inside the focus, the disc is fringed with bright red, passing through yellow to blue in the centre; outside, the disc is fringed with blue-violet passing through green and yellow to red in the centre.

An *achromatic objective*, if corrected for visual observation (Vol. I, page 231), has its minimum focus for the apple-green region of the spectrum. Practically no colour is seen at the best focus, but inside the focus a yellowish disc (greenish at the centre) is fringed with an orange-red border, but the colour is very much less marked than with a non-corrected lens; outside the focus there appears an outer fringe of apple-green surrounding a yellowish disc with a faint reddish-violet centre.

The beginner should be warned that some of the colour effects, especially the red fringe inside the focus, must be partly attributed to the eyepiece and the eye, which are not chromatically corrected, and that the amount of colour may vary with different eyepieces, being less marked with the higher powers. It is advisable for him to make a start by observing with lenses known to be well corrected, and to work with different eyepieces so that the eyepiece effect may be allowed for; so, until much experience has been obtained, any unknown lenses should be tested against similar ones of proved performance.

Apochromatic objectives should exhibit practically negligible axial colour (as also will, of course, reflector objectives): any considerable colour effects must be ascribed to the residual errors of eyepiece and eye, unless they are marked enough to be ascribed to faulty construction. Experience is necessary.

Note that the Airy disc with white light is formed of overlapping discs in various colours, and that these have different radii. No correction of the lens can overcome this effect which may be noticed if very careful observation is made.

"Photographic" Colour Correction. It will be remembered that lenses designed for photography are usually achromatized by uniting the foci for the G′ and D lines of the spectrum, thus bringing the minimum focusing distance for the lens into the F line region (blue green). The result of this will naturally be to enhance the red fringe inside the focus, and to make a blue-green instead of an apple-green appearance in the outer fringe outside the focus. Here, again, the performance of any lens should be compared with one of known colour correction—or the actual chromatic variation of the focus can be measured. (See below.)

Spherical Aberration. This defect may affect axial images from telescope and microscope objectives, and may be due to various causes such as incorrect figure, wrong working distance, etc. The

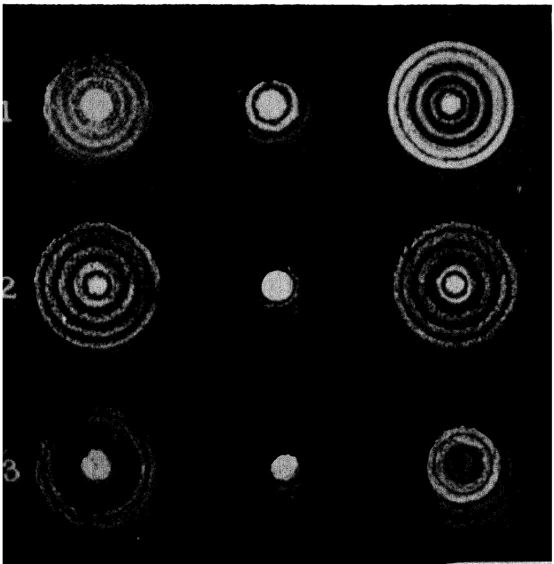

FIG. 186. TYPICAL STAR FIGURES FOR CORRECTED LENS AND FOR SPHERICAL ABERRATION

Row 1. Spherical under-correction. Outside focus; at focus; inside focus
,, 2. Corrected lens: Outside focus; at focus; inside focus
,, 3. Mid zone with short focus: Outside focus; at focus; inside focus
(Zonal spherical aberration)

simple theory was discussed in Vol. I, Chapter IV. In "under-correction," the type of aberration due to a double convex lens, the marginal zones have the shortest focusing distance, and the focal point for any zone is farther from the lens the smaller the radius of the zone. The reverse is the case in "over-correction." In "zonal" spherical aberration, however, the rays from an intermediate zone of the lens may have too small or too great a focusing distance in the cases of zonal under-correction and over-correction respectively. The various typical results must be left to the accompanying diagrams and pictures, which will explain them better than a great deal of verbal description. (See Figs. 186 and 187.)

Defects in the Lens, Striae, Strain, etc. Striae in the glass of the lens may produce a marked "fuzziness" at the best focus and irregularities in the extra-focal appearances. Strain in the glass may be due to bad annealing, or undue pressure by the mount. The resulting distortion of the surfaces is also manifested clearly

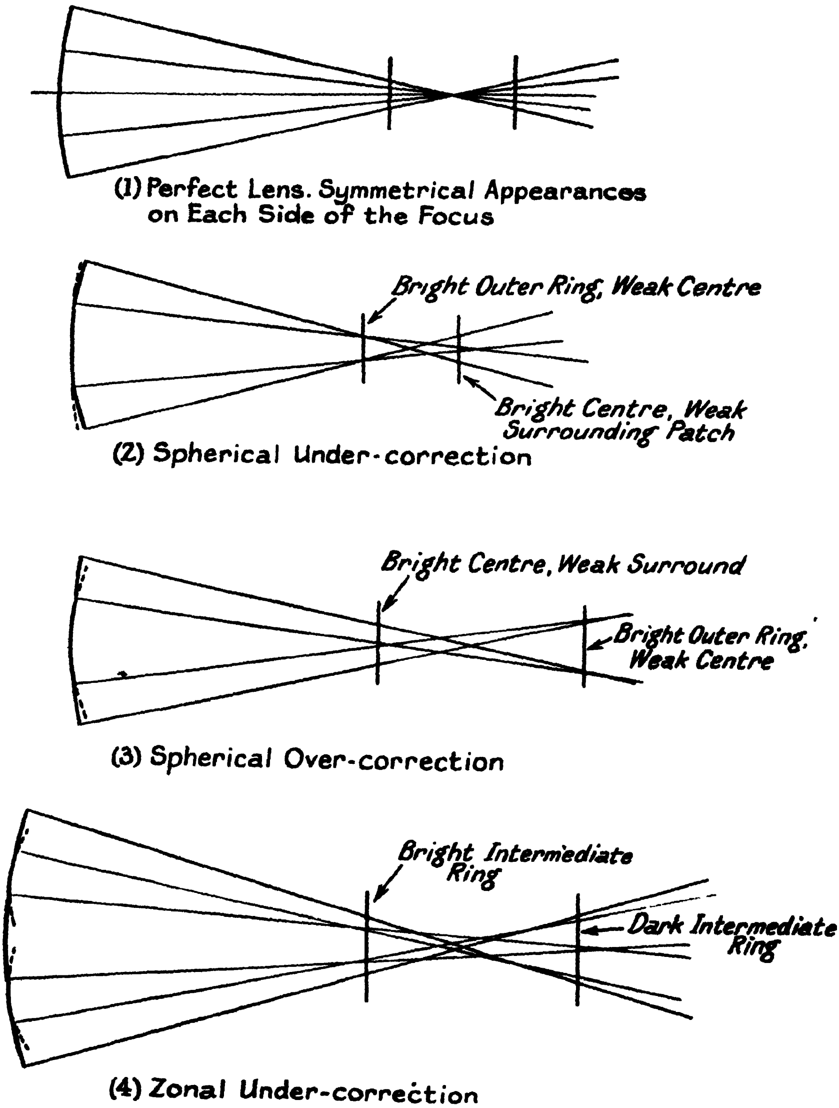

FIG. 187. RAY DIAGRAMS FOR STAR TESTS

in the distortion of the extra-focal rings. Striae and strain are to be tested for independently by the Foucault test and the polariscope respectively. (See Vol. I, pages 257, 208 and below.)

General Note. Unless there is a very marked physical defect in the glasses or surfaces, there will usually be one stage of the focus in which a concentration very closely approximating to the Airy disc appears, but when aberration is present light leaves the centre disc and appears in the surrounding rings.

Extra-axial Aberrations. If chromatic or spherical aberration are present on the axis they will persist over the whole field, but if

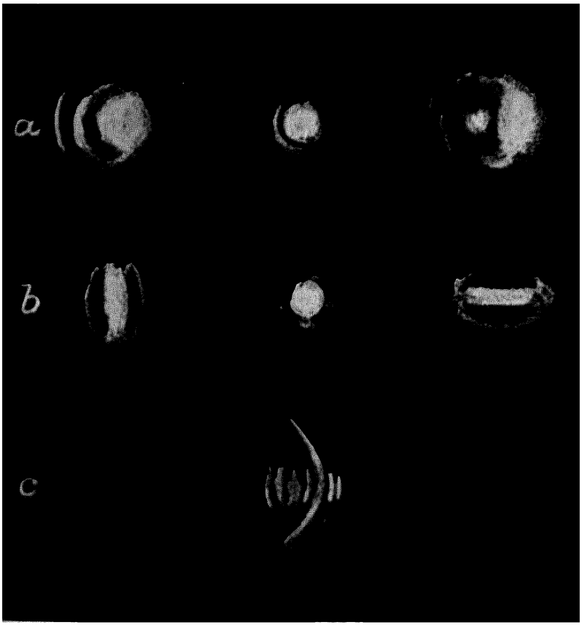

FIG. 188

a = Coma, extra-axial and focal
b = Astigmatism, extra-axial and focal
c = Heavy coma and astigmatism together
The centre image represents the best focus appearance in both (a) and (b)

they are absent on the axis, the aberrations which may afflict the image definition in the outer part of the field are *chromatic difference of magnification*, *coma*, and *astigmatism*. The general nature of these defects have already been explained in Vol. I, Chapter IV. Again, it is important to test an objective with the proper eyepiece, for we cannot easily dissociate the effects of the two. Chromatic difference of magnification causes an extra-axial "star" to appear as a short radial spectrum (with the colours crowded together and mostly overlapping). Coma gives an unsymmetrical side distribution of light (see Fig. 188); while astigmatism gives a radial or tangential line. All ordinary telescope objectives show astigmatism, and if more or less coma happens to be present also, the appearances are very complex; Fig. 188(*c*) shows a typical appearance. This inherent astigmatism does not appear on the axis. If the lens suffers slightly from coma, then the smallest error of squaring-on will cause the central image to show a coma effect.

Other Test Objects. Numerous test objects are employed by practical workers in addition to, or even to the exclusion of, the "star" test, especially in microscopy; reference should be made to the various textbooks. The "star" test for experienced workers yields very satisfactory results, but should be supplemented by a test of another type. The fact that at one focus the "star" image suffering from spherical aberration approximates very closely in size and general appearance to the Airy disc means that any image, even in the presence of this aberration, will show more or less sharp detail. Hence, mere sharpness of detail is not a sufficient indication of a well-corrected system. The supplementary test is one for the contrast of the image, and the object should be one with sharp demarcation between black and white. For the microscope this is represented by Abbe's test plate, consisting of transparent *rulings* in an opaque silver film on the underside of a cover glass.

It is easily possible to interpret the extra-focal appearances on the image so as to inier the defects of the objective, but less simply than with the "star" test.

For the telescope, contrast objects are numerous; the twigs of trees; towers against the sky, and so on, are easily found. For other test objects see Johnson's *Practical Optics*,[3] page 129.

The Measurement of Aberrations. Aberrations may be expressed in terms of the geometrical ray paths, or in terms of phase relations of vibratory disturbances in the image; it is usually possible to calculate the one expression from the other. Likewise, the methods of measurement can be divided into two general classes; one which aims at tracking ray paths by various means, the other which gives a direct indication of the optical phase aberrations in an interferometer pattern. Broadly speaking, the first class involves little apparatus, but much time and trouble; the second involves expensive apparatus, but comparatively little time.

Visual Ray-path Method. The "ray" is a mathematical conception and cannot be physically realized. If we place a diaphragm containing a small aperture of finite size in the path of a convergent wave-front, the maximum concentration of energy will be for practical purposes in the centre of curvature of the element thus exposed. If the aperture has symmetry with respect to some point in its plane, the distribution of light in the convergent beam will show a corresponding symmetry with respect to the line joining this point to the centre of curvature; the centre of symmetry of the diffraction pattern in the beam may thus be conceived as a ray track.

The most direct method of procedure is to arrange a diaphragm behind the lens which forms the image of an artificial "star." The distribution of light in any plane can then be examined with the aid of an eyepiece. (Fig. 189 (*a*) and 189 (*b*).)

A useful method of finding the foci of various zones is suggested by C. Beck, who uses a pair of apertures disposed in the diaphragm, as shown in Fig. 189 (*a*). The position of the focus of the corresponding zone is recognizable by the cruciform symmetry of the

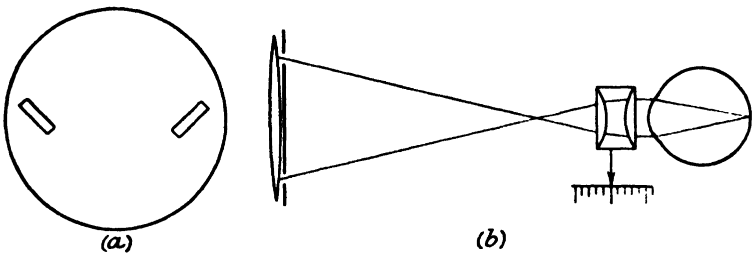

Fig. 189. Direct Observation of Zonal Focus

double diffraction pattern. With the aid of a series of diaphragms, it is thus possible to find the focusing points of a series of zones, or for a series of different wave-lengths of light, and hence to investigate the spherical and chromatic aberrations of the system, although it would be necessary in investigating an objective alone carefully to allow for the corresponding aberrations of the system of the eyepiece and eye, by direct observations on very small "mercury globule" stars supported in the focal plane of the eyepiece.

The difficulty attaching to this and allied methods is, however, very considerable in practice, and seems to arise through the irregularities of the refracting media of the eye. The cornea is traversed, in this test, by two very narrow beams which can easily suffer considerable deflection by a minor irregularity which would make little or no difference to a broad beam. For this reason it is usually difficult to estimate the correct focus of a particular zone; probably some persons would find the experiment much easier than others would.

The Foucault Test. The Foucault test[4] was described in Vol. I, page 257, but some notes will be included here to make the chapter complete; it has the merit that it avoids the criterion of a judgment of position, using rather an estimation of equality of intensity of light; the setting is photometric. Originally applied to test astronomical mirrors, and still mainly employed for that purpose, it can also be applied to "refractors" with useful results.[5]

A small source of light *s*, conveniently a pinhole backed by a flame or opal bulb lamp, sends light to the mirror (Fig. 190), which forms a corresponding image *s'*; the eye is placed immediately behind this image, and sees the mirror under the condition of the Maxwellian view in which it is filled with light, provided that all rays from the mirror enter the pupil unobstructed.

If a knife-edge k_1 is brought upwards into the focus, the whole mirror darkens uniformly if the light from all zones is focused in the same point; but if spherical aberration is present, then characteristic distributions of light and shade appear on the surface of the

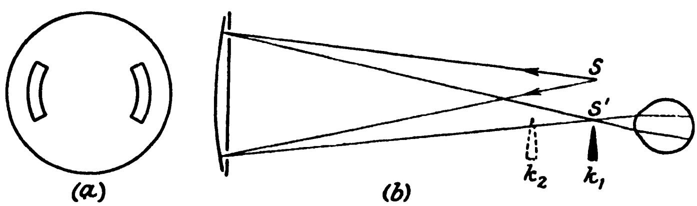

FIG. 190. ZONAL DIAPHRAGM FOR FOUCAULT TEST

mirror. The shadows naturally occur on those regions the rays from which are cut off by the knife-edge. The necessity of conducting the mirror test with image and object point away from the axis makes the characteristic shadow figures unsymmetrical with the reflector, but symmetrical figures are obtained when testing refractors forming really axial images.

Experienced workers recognize the figures characteristic of parabolic, elliptical, hyperbolic, and other forms of reflecting surface, but the test is made quantitative by measuring the focus for successive zones by the use of diaphragms, such as shown in Fig. 190. It is clear from the figure that when the knife-edge intersects the beams either nearer or farther than the focus, the two patches exposed by the diaphragm will not darken simultaneously. Considerable precision is attained by this method if care is taken with the precautions usual in photometric work.

If the pinhole and the knife-edge are moved on the same carrier, the pointer attached thereto may be made to read the relative positions of the centres of curvature of successive zones; but if the knife-edge alone is moved while the pinhole remains stationary, the separation of the images will be clearly double that of the corresponding zonal centres of curvature. Thus, if the radius of the zone is y, and the approximate radius of curvature is r, the centres of curvature of the paraxial and zonal foci are separated by $\frac{y^2}{r}$ for a parabolic mirror.

The Foucault test is not only used for testing the main mirrors, but also the secondary mirrors of Cassegranian and Gregorian reflecting telescopes. According to Hindle (Mon. Notices, *R.A.S*, XCI (1931) 592), it is best carried out for the secondary mirrors independently with the aid of an auxiliary spherical mirror which has itself been tested alone. In this method one focus of the hyperbolic or elliptical mirror will lie at the centre of the spherical mirror, the other focus being at the point of observation.

Of the other tests adapted for visual use, those of the shadow fringe methods developed by Ronchi and Jentsch (see below) may be mentioned, but if actual measurements are desired they may best be made with the help of photographic recording. The same applied to the general methods suggested by Chalmers, which have been revived in a more practical guise by Gardner and Bennett. We shall now proceed to describe several methods involving photographic recording.

The Hartmann Test.[6] The objective under test is directed to a suitable star or artificial star, and a diaphragm placed behind it has circular apertures, h_1, h_2, representing a particular zone of the lens. A photographic plate-holder is arranged so that photographic plates can be exposed when held perpendicular to the axis in two positions, within and without the focus, at a measured distance apart. The light from each aperture forms a small diffraction patch on the plate, the centre of which may be regarded as representing the track of the ray from the centre of the aperture. A suitable diameter for the holes is about $\frac{1}{200}$th of the focal length.

If A_1 and A_2 are the plate-holder scale readings corresponding to two plate positions, and l_1, l_2 are the distance of the centres of the two dots on each plate, then the scale reading A corresponding to the focus of the zone is (see Fig. 191)

$$A = A_1 + \frac{l_1}{l_1 + l_2}(A_2 - A_1)$$

The possibility of cylindrical errors of the surfaces has to be remembered however. It follows from the theory of Vol. I, page 280, the astigmatic *difference* of focus between the diameter of minimum focal length and the diameter at an angle of θ with the first will be given by

$$A_\theta = A_{minimum} + a \sin^2 \theta$$

where a is the "astigmatism." Then

$$A_{\theta + 90^\circ} = A_{min} + a \cos^2 \theta$$

and the mean

$$\frac{A_\theta + A_{\theta + 90^\circ}}{2} = A_{min} + \frac{a}{2}$$

We may, therefore, eliminate irregularities due to astigmatism, when taking results from apertures in different diameters, by always taking observations for one zone in two perpendicular diameters.

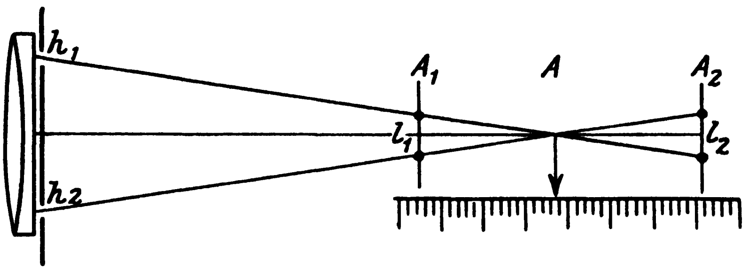

FIG. 191. THE HARTMANN TEST

There is thus no reason to limit the diaphragm to two holes. A specification for the arrangement of apertures in the diaphragm for a typical case is as follows—

Objective 80 mm. Diameter Angles (θ)		1 metre Focal Length Radii of Zones containing Apertures (in mm.)
0°	90°	10, 18, 26, 38
22·5	112·5	30
45	135	6, 14, 22, 34
67·5	157·5	30

The diameter of the holes are 4 mm. in the outer, and 3 mm. in the inner zones.

It is well to add a single hole in the diaphragm, so that the proper orientation of the plates may be recognized after development. The numerical part of the work lies in the systematic measurement of the values of l_1 and l_2 for the various zones and the calculations of the corresponding focusing points.

If the diaphragm itself is not too large it can be measured up on the measuring microscope, and one photograph may thus be dispensed with if the distance from diaphragm to plate is known, and also whether the plate is taken inside or outside the focus.

The main difficulty experienced in this test is the distortion of the dots arising from interference effects between the light from successive dots. This effect sets a limit to the reduction of the distances between the adjacent holes in the diaphragm, and also makes it imperative to take the negatives at a distance from the focus sufficient to avoid marked interferences.

With ordinary aperture ratios it is not difficult in this way to ascertain the relative differences between the foci of successive zones to within a few tenths of a millimetre. The way in which the geometrical aberrations so found can be interpreted in terms of phase differences of disturbances at the focus was shortly explained in Vol. I, pages 119 and 120. A full discussion of the application of the test to a microscope objective will be found in a paper by the writer.[7] Kingslake[8] has pointed out that the assumption made in the above theory that all rays cross the optical axis is not justified, and he prefers to obtain a three-dimensional trace of the rays from a large refractor. The co-ordinates for any ray in any chosen focal plane can thence be calculated.

Hartmann has suggested a criterion for the comparison of astronomical objectives; it is a magnitude expressing the "weighted mean diameter" (in hundred thousandths of the focal length) of the cones of light from the various zones in the plane where the circle of light containing all of the converging pencils is the smallest. Weights are given according to the light gathering power of the zones.

Since the area of a zone is $2\pi r\delta r$, the weight is proportional to the radius and the criterion (denoted by T) is

$$T = \frac{10^5}{F} \frac{\Sigma rd}{\Sigma r}$$

where d is the diameter of the circle of light from a zone of radius r.

Hartmann's original papers gave particulars of similar methods for the measurement of oblique aberrations. The subject has also been developed by Kingslake,[9] who bases the method of the determination of the foci for pairs of rays derived from the extremities of chords of a zone drawn parallel to the axis of tilt of the lens, using formulae developed by Conrady.[10] The method has proved very convenient in practice; reference should be made to the original paper for particulars.

Interference Methods—Fizeau's Experiment.[11] Fizeau placed two parallel slits behind an object glass forming the image of a star. If the distance of the slits is d, Fig. 192, and the distance from slits to focus is f', the appearance found in the focal plane is a series of interference fringes of which the successive maxima are separated by the distance $\frac{\lambda f'}{d}$. The angular subtense, at the back nodal point of the lens, of the distance from a maximum to the neighbouring minimum is therefore $\frac{\lambda}{2d}$. If the telescope is directed towards a double star with components at this angular separation, the maximum of one fringe system will coincide with the minimum of the

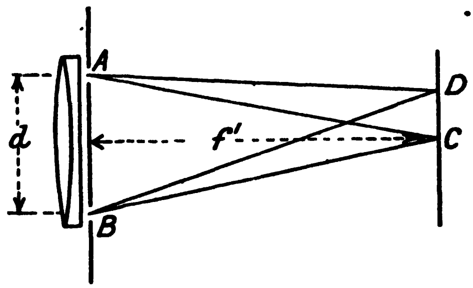

FIG. 192. FIZEAUS' EXPERIMENT

other system, and the pattern will become a uniform band if the two systems are of equal brightness. This is the basis on which was developed Michelson's method of measuring the angular diameters of stars, and a number of methods of measuring the diameters of ultra microscopic particles developed by Gerhardt and others.[12]

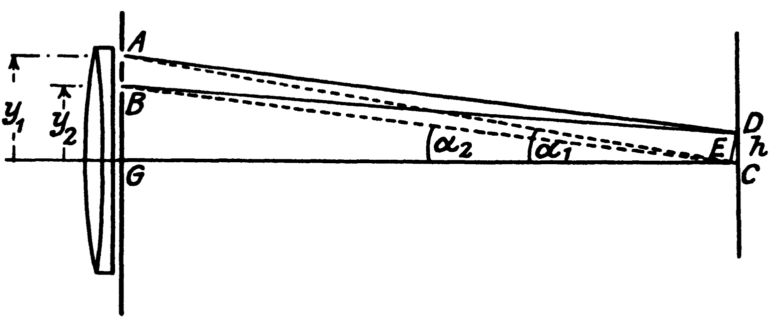

FIG. 193. THEORY OF CHALMERS' TEST

Chalmers' Tests. If the slits in Fizeau's experiment are no longer symmetrically disposed with respect to the axis, the central fringe of the interference system will still fall in the axial focal point if the lens is free from aberration, because the disturbances from all parts of the wave-front will reach this point in the same phase. But if spherical aberration is present, the central fringe will be, in general, displaced away from the axis to some position where the phase agrees. In Fig. 193 let A and B be the two apertures; let C be the axial focal point, and let D be the position of the central "maximum" of zero path difference.

In the actual use of the lens, the image will be formed at the point C, and we are, therefore, concerned to write down the phase difference with which the disturbances from A and B arrive in the point C.

From the point D drop a perpendicular DE on the line AC. The angles $\widehat{ACG}$ and $\widehat{EDC}$ are equal, and will be written a_1. Now

$$AC = AE + EC$$
$$= AD + EC$$

within a small quantity of the second order if AG and CD are small in comparison with GC.

Writing $h = CD$

we thus obtain $AC - AD = h \sin a_1$

and, similarly, $BC - BD = h \sin a_2$

Now (AC - AD) and (BC - BD) represent the net changes of optical path of the disturbances from A and B when meeting at C, rather than at D where we know that they have the same phase. Hence the path difference at C is

Optical path difference

$$= h (\sin a_1 - \sin a_2) = \frac{h}{f'} (y_1 - y_2), \text{ very nearly.}$$

By starting with one aperture on the axis and one at a distance y_1, we may, therefore, find the optical path difference for the first zone with respect to the paraxial zone. Now moving one step outwards, so that the inner aperture falls into the position previously occupied by the first one, and so on, we may find by addition the optical path differences for a number of successive zones with respect to the paraxial zone. Delicate measurements with an eyepiece micrometer are involved in the practical application of this test, which is not particularly easy to carry out quantitatively, although it makes a ready means of testing an objective by visual estimation; for this purpose the slits may be made in a diaphragm held in front of the objective, while the appearances in the focal plane are watched with a high power eyepiece.

Gardner and Bennett's Method.[13] This is essentially an extension of Chalmers' method in which a photographic record of the fringe positions is obtained for a number of zones simultaneously; it is taken away from the focal plane, thus resembling the Hartmann test; in this case, however, the interference effects which spoil the accuracy of the Hartmann record are turned to good use.

In Fig. 194, let A and B be two apertures; C is the focus of the lens for the central zones, and D the displaced position of the central fringe from A and B. If we take a record inside the focal plane, the fringe will be at E′ on the straight line joining D to the mid-point between A and B, whereas it would have been at E on

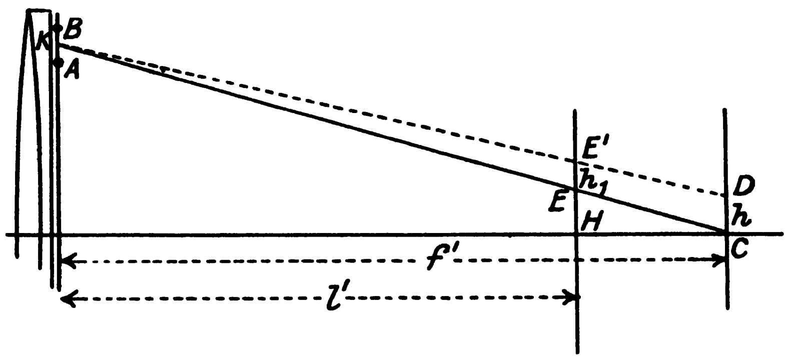

FIG. 194. THEORY OF GARDNER-BENNETT TEST

the corresponding line to C if the aberration were zero. If EE′ is h_1 and the distance from diaphragm to the recording plate is l',

then $$\frac{h_1}{l'} = \frac{h}{f'}$$

If we measure the distance ω between successive maxima in the fringe system due to A and B, the interval will be

$$\omega = \frac{\lambda l'}{y_1 - y_2}$$

But in the theory of Chalmers' method we found that the optical path difference (O.P.D.) for disturbances arriving at C is

$$\text{O.P.D.} = \frac{h}{f'}(y_1 - y_2) = \frac{h_1}{l'}(y_1 - y_2) = \frac{h_1 \lambda}{\omega}$$

or, $$\text{path difference in wave-lengths} = \frac{h_1}{\omega}$$

In practice, a diaphragm is prepared with a number of equidistant holes arranged symmetrically with the axis, and the photograph is taken sufficiently far inside or outside the focus to obtain satisfactory interference fringes in the pattern. The width of the fringes is measured, and also (for each pair of holes) the distance of the centre fringe from the axis, i.e. E′H in the diagram. Then

$$EE' = E'H - \frac{(f' - l')(y_1 + y_2)}{2f'}$$

So that the optical path differences can be obtained for any assumed focus C.

It is to be noticed that the "focus" is not an exactly ascertainable point. We can select a likely point, and then find the optical path differences of the disturbances meeting there; a good criterion for the performance of a well-corrected system is the residual path difference between any two zones, at the focus where these residuals are numerically least. These remarks apply also to the ordinary form of the Hartmann test described above.

The Shadow-fringe Methods. These methods have been developed by Ronchi[14, 15] and Jentsch,[16] who employ gratings of various descriptions, chiefly straight line gratings. Ronchi chiefly recommends rulings with 10–20 lines per mm., while Jentsch uses commercial process screens with 4–8 lines per mm. The gratings are placed in the convergent beam from the test lens which forms an image of an artificial star. If the focus is a perfect one, the projection pattern of the grating has one point, i.e. the focus, as perspective centre, and the shadow pattern in any plane is an undistorted series of dark and light bars; but if the lens suffers from spherical aberration, then the more oblique rays have a different centre, and distortion results in the shadow pattern. Ronchi develops the theory on the basis of the interference of light, which is a more suitable method for dealing with the effects under the conditions he uses. A brief geometrical treatment due to Jentsch may be reproduced here.

Refer to Fig. 195. A ray diverges from the point B at an angle u with the Z axis of co-ordinates, and projects the image of the point P in the grating plane as P′ in the projection plane. The origin of co-ordinates O is the paraxial focus, and the intercept OB $= s$ is the axial spherical aberration associated with the angle u. The co-ordinates of the points P and P′ are x, y, g, and ξ, η, p, respectively.

From the geometry of the figure,

$$\frac{\xi}{x} = \frac{\eta}{y} = \frac{p - s}{g - s}$$

and

$$\tan u = \frac{\sqrt{\xi^2 + \eta^2}}{p - s}$$

In the simplest type of spherical aberration

$$s = A\tan^2 u \quad . \quad . \quad ., \text{ say, sufficiently nearly.}$$

By elimination of s and $\tan u$ in the above equations we obtain

$$A\,(\xi - x)^3(\xi^2 + \eta^2) = (g\xi - px)(p - g)^2\xi^2$$

If the aberration is zero, A = 0, and the equation reduces to

$$(g\xi - px) = 0$$

$$\xi = \frac{px}{g}$$

So that if the grating is rectilinear with parallel bars, then the projection will show a series of parallel bars.

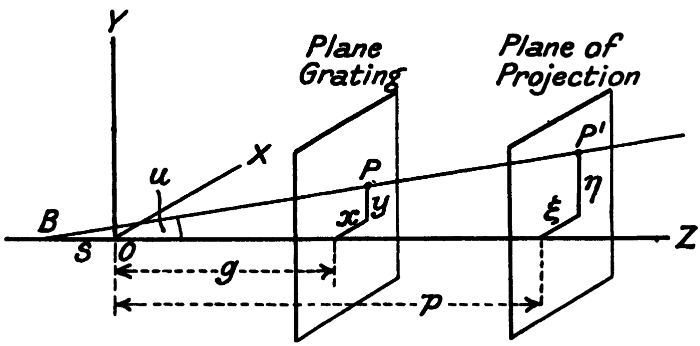

FIG. 195. THEORY OF JENTSCH'S TEST

Again, when A is finite we may write $x = N\delta$, where δ is the grating interval, and N is a constant which may be assigned a succession of positive and negative integral values; thus giving the curves of the various shadows in the projection plane.

For a central bar, $x = 0$, and the equation splits into two parts.

i.e. $\xi = 0$ (a straight line), and

$$\xi^2 + \eta^2 = \frac{g}{A}(p - g)^2$$

this latter curve is a circle, but it can only be real if g and A have the same sign, i.e. it will only appear on one side of the focus. Fig. 196 shows typical shadow fringes inside and outside the focus of a lens suffering from aberration.

For more exact quantitative measurements, Jentsch advises photographic recording of the shadow fringes and measurement of the plates along the X direction; the calculation will be carried through in ways similar to those already described.

Interferometer Tests on Optical Apparatus. The interference of two beams reflected from the surfaces of a thin film was discussed in Vol. I, page 184, *et seq.* It was shown that if the faces of the film are parallel, and the angle of reflection inside the film is i', then the relative optical retardation of the second disturbance is $2n't \cos i'$, where n' is the refractive index of the medium of the film. Remembering the phase change of π consequent on reflection at one face (that which would involve a step from a lower to a higher refractive index for the transmitted light), the condition for destructive interference is

$$2n't \cos i' = m\lambda$$

where m is 0 or an integer.

When the faces of the "film" are truly parallel, interference effects can be seen at separations much greater than those characteristic of the soap bubble, in fact for quite large separations if light of good spectral homogeneity is employed. The limits of optical path difference up to which interference effects can be observed are not certain, but Michelson's experiments showed that they are not less than half a metre.

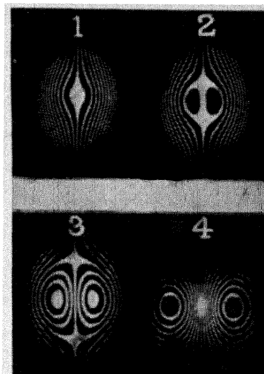

FIG. 196. JENTSCH'S GRID METHOD

1 and 2. Shadow fringes outside the caustic
3 and 4. Shadow fringes inside the caustic
The appearances follow in the progression 1, 2, 3, 4

One very useful method of testing employs the so-called Haidinger's fringes. Referring to Fig. 197, we see that the thin glass plate reflects the green light from a mercury vapour lamp (preferably a low pressure glass lamp) downwards on a piece of glass G with plane parallel faces. The parallel components of the reflected light are brought to a focus on the retina of the eye accommodated for infinity, and the condition for destructive interference is still given by the above equation. The optical path difference for the disturbances following the exactly normal path is $2n't$, and it is clear that $\cos i'$ will be constant for any circle surrounding the normal. The fringes therefore appear as circles with the normal as centre.

If the plate G is moved about and the thickness is slightly irregular, we may find a region in which t increases. Now one definite fringe corresponds to a fixed amount of retardation. It will, therefore, move to a place where $\cos i'$ diminishes in order that $2n't \cos i'$ may remain constant. This means that i' must increase, and hence the fringes move outwards from the centre, fresh fringes developing at the central point and expanding outward; the development of one fringe corresponds to a variation of one wave-length in $2n't$. The reverse effect takes place if the thickness diminishes. Hence by systematic movements of the plate it is possible to examine the variation of thickness over the whole area.

It may be mentioned that the interferences arising in a very similar way at two parallel surfaces are found and examined in the Fabry-Perot interferometer, but in this case two half-silvered glass surfaces held exactly parallel separate a film of air, and the circular fringes are viewed by transmitted light. This system has been used

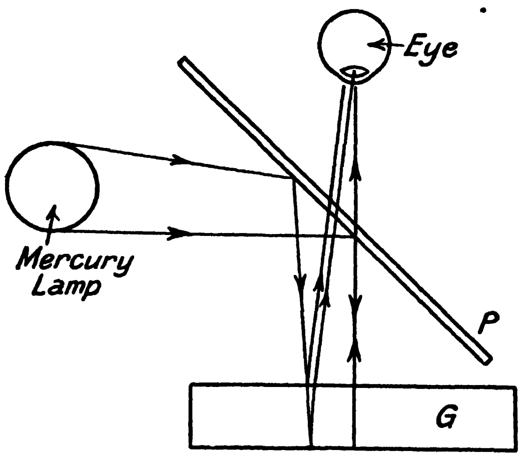

FIG. 197. PRODUCTION OF HAIDINGER'S FRINGES

by Barnard to test the accuracy of the slides of a microscope. One plate is mounted on the stage of the instrument, the other in place of the objective; the fringes remain central while the plates are truly parallel.

Fizeau's Apparatus. Fizeau used a device by which the angle of incidence of the light is constant over the reflecting surfaces, but a lens is then necessary in order to bring the light from the different parts into the eye. The optical parts of an arrangement similar to Fizeau's are shown in Fig. 198, which is adapted for sensitive interferometric tests of an optical surface.

The glass plate S has a truly plane lower surface and is supported by a stand with levelling screws, so that this surface can be brought closely parallel to the upper surface of plate T, which is to be tested for its "figure." The mercury lamp illuminates the pinhole H, which gives a diverging beam reflected downwards by the plate P. The light is then rendered parallel by the lens L. The components reflected from the adjacent glass surfaces of S and T show interferences, the geometrical path difference at any point being twice the thickness of the air film, since the incidence is everywhere normal. The light on its return journey is brought to a focus in the eye, so that a Maxwellian view of the interference field is obtained showing a distribution of fringes which represent, in fact, a contour map of optical thickness.

If the plate S is slightly prismatic, reflected light from its upper face may be avoided; the lower surface of T may be blacked or greased.

In cruder applications of the test the surfaces S and T may be

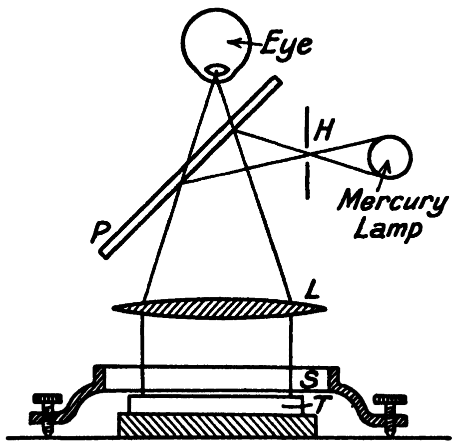

FIG. 198. FIZEAU TYPE INTERFEROSCOPE

carefully cleaned and then slid or wrung together in contact. Any irregularities of the test surface are then again revealed by the variation in thickness of the separating film. The disadvantage in this method lies in the possibility of the deformation of one or both plates.

The Michelson Interferometer. The optical system of the Michelson interferometer is analogous to the arrangement for the production of Haidinger's fringes. It is illustrated in Fig. 199. Light from an extended source meets the mirror M_3, and is half transmitted and half reflected by the lightly silvered back surface. The components proceed to the mirrors M_2 and M_1 respectively, from which they retrace their path, combining again at the surface of M_3. The eye placed in the position E can view the interferences with or without the aid of a telescope; the formation of two interfering components in an oblique direction is very closely analogous to that of Fig. 197 (Haidinger's Fringes), but there is the immense advantage that one surface does not "get in the way" of the other.

The system is equivalent to a plate formed by M_1, and the image of M_2 in M_3.

The interferences are ascribed to the variation of relative retardation with the obliquity of the light, and the fringes are grouped around the normal as before. The component reflected from M_1 has a double transmission through the plate M_3; this is compensated for the M_2 component by the insertion of the plate P equal in thickness to M_3; the fringes are then circular.

Interferences are only visible with white light in very thin films;

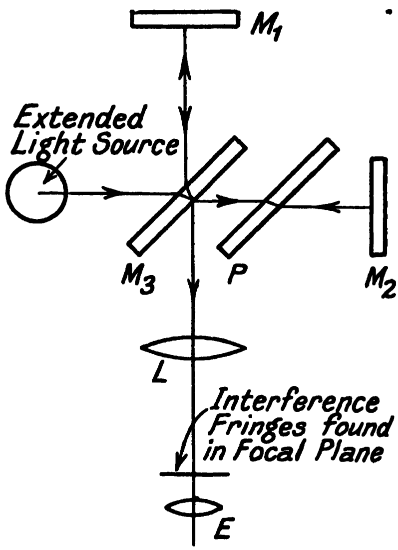

FIG. 199. MICHELSON INTERFEROMETER

the colours in Newton's rings soon fade out as the thickness of the film increases, owing to the superposition of the systems of different sizes in different wave-lengths. Equally, the fringes are only found with white light in the Michelson interferometer when the "equivalent plate" is very thin; i.e. when the effective lengths of the "arms" of the apparatus are very nearly equal, but with spectrally homogeneous light, interference takes place up to very large path differences.

The Prism Testing Interferometer. As the Michelson interferometer is analogous to the Haidinger fringe apparatus, so the lens and prism testing interferometer, the earliest form of which was patented by Twyman and Green in 1916,[17] is analogous to the Fizeau arrangement. A simple form of the apparatus is shown in Fig. 200. As in the Fizeau apparatus, the beams are brought to a focus by a lens which gives a Maxwellian view of the interference field; this is in effect a contour map of the optical thickness of the "equivalent plate" of the system.

We may obtain a simple view of the action by considering a point source and a perfect lens giving two wave-trains of plane waves which unite after passing through the system (Fig. 201). If their directions are inclined at a small angle α and the wave-length is λ, it can easily be seen that a steady interference system of maxima and minima distributed in planes equally inclined to each

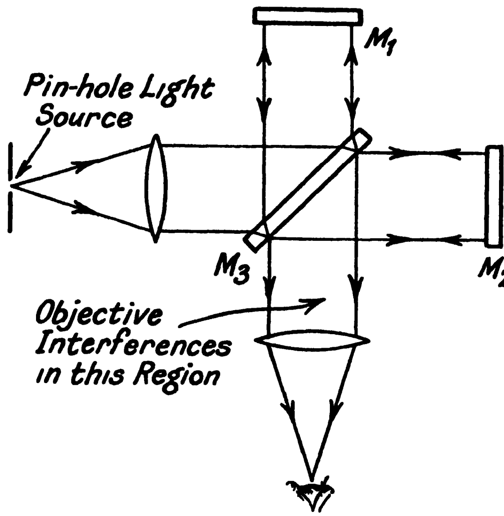

FIG. 200. TWYMAN INTERFEROMETER

group of waves will result. The distance between two "maximum" planes is strictly λ cosec α cos $\left(\frac{\alpha}{2}\right)$, but for all practically interesting cases this becomes $\frac{\lambda}{\alpha}$ since α is always very small. The objective existence of these maxima and minima in the region of superposition can easily be demonstrated with the aid of a piece of ground glass on which the fringes can be found, or by exposing a photographic plate shielded from extraneous light. When the value of α is zero, a single dark or light fringe broadens so as to fill the whole field.

Imagine now a very slightly wedge-shaped piece of glass to be placed in the path of the M_2 beam, the thicknesses at the two ends of the plate being t_1 and t_2 respectively. The increase of optical path owing to the introduction of a thickness t of a medium of refractive index n into a space previously filled with air is $(n - 1)t$.

Hence the relative path difference between disturbances passing through the extremities of the plate is $(n-1)(t_1-t_2)$. The double transmission in the interferometer will make the actual optical path difference $2(n-1)(t_1-t_2)$, and the number of resulting fringes will be $\frac{2(n-1)(t_1-t_2)}{\lambda}$. Note now that the same plate tested by the Fizeau method would show $\frac{2n(t_1-t_2)}{\lambda}$ fringes, so that if $n = 1{\cdot}5$ we shall only find one-third the number of Fizeau fringes when

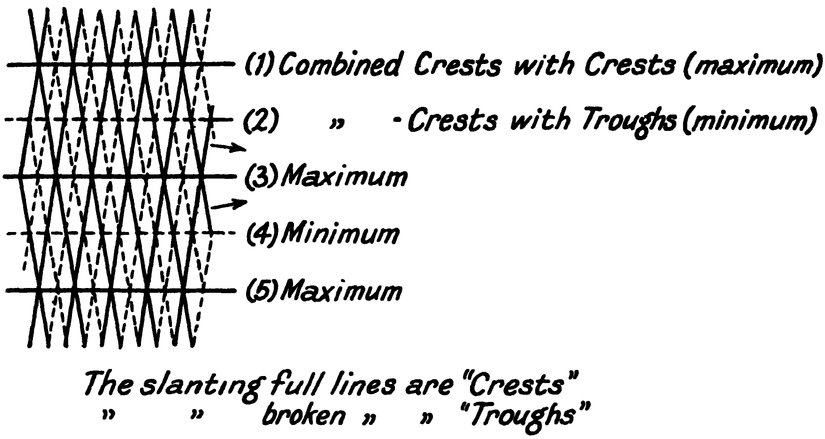

FIG. 201. THE SUPERPOSITION OF PLANE WAVE-TRAINS

using the interferometer; though it is less sensitive for this particular purpose, there are other advantages which more than compensate.

If, for example, a prism of considerable angle is placed in one arm, the mirror can be rotated so that it is normal to the incident light, which therefore retraces its path. The optical path differences along the various ray routes will now be zero if the surfaces of the prism are uniform and the glass is homogeneous; but local irregularities in the surfaces, or variations of refractive index of the specimen, will cause variations which are seen in the contour map of optical thickness.

Slight pressure on the iron bed of the instrument on which the mirrors are mounted can be applied so as to elongate or diminish the path in one arm. The fringes will therefore (moving so as to retain the same retardation) proceed to places of lesser or greater path in the specimen, thus giving a criterion of the sign of the irregularity.

Suppose, for example, that a prism for use in a spectroscope has a local increase of refractive index in one region; this part may be marked out on the surface of the prism by a brush or grease pencil, and subjected to local rubbing by a chamois leather pad charged with rouge. In this way sufficient glass may be removed to compensate for the local variation of index and perfect the interference pattern. In such ways as this it is possible greatly to improve the action of prisms and optical parts, although the compensation will only be exact for one direction of the light, and nothing can wholly compensate for any lack of homogeneity in the glass.

The method of testing lenses for axial aberrations will be understood from Fig. 202. The convex mirror, which must be of reasonably perfect figure (correct to within $\lambda/8$), may be so disposed that its centre of curvature coincides with the focus of the lens. This system now replaces the plane mirror M_2 of Fig. 200. With a perfect lens

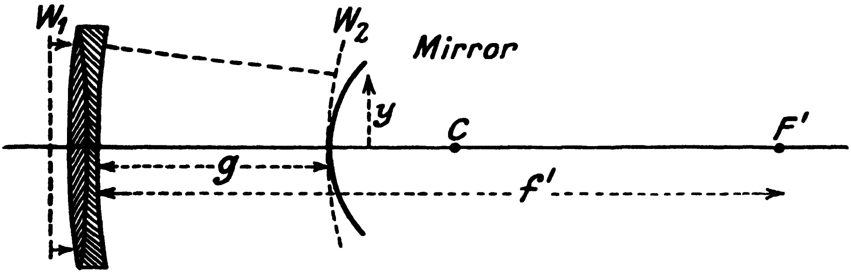

FIG. 202

the "rays" will all fall normally on the surface of the mirror, and will return along their own paths; consequently, no difference of direction can arise for any part of the wave-front, and a uniform interference pattern will be obtained.

Aberration Path Differences with Lens and Mirror System. Referring still to Fig. 202, the lens is shown with principal focus F′, and the mirrors with centre of curvature C. These will not in general coincide at first, although the apparatus will be so constructed that C falls on the axis of the lens, and in cases of practical interest C will be very close to F′, so that rays will return very nearly along their own paths. The wave-front has the radius $(f' - g)$ on reaching the apex of the mirror, f' being the vertex focal length of the lens, and g the separation of the adjacent apices of the lens and mirror.

Assuming as the usual basis of Fermat's theorem (Vol. I, Chapter IV) that we can calculate optical paths along known ray routes, the extra geometrical path for the marginal over the paraxial ray will be represented by double the marginal gap between the circles centred in C and in F respectively; that is, if the lens is free from aberration.

$$\text{Marginal path} - \text{axial path} = 2y^2 \left\{ \frac{1}{2r} - \frac{1}{2(f'-g)} \right\}$$

The optical path difference will be proportional to y^2, and the frequency of the circular fringes will increase from the centre outward, but the central fringe will broaden out to fill the whole field when $r = (f' - g)$, i.e. when the mirror is drawn back so that C and F coincide.

Measurement of Aberration. If the lens suffers from spherical aberration, the optical path difference for marginal and paraxial rays at the paraxial focus can be represented by a series (Vol. I, page 115).

$$\text{Marginal path} - \text{axial path} = c_2 y^4 + c_3 y^6 + \text{etc.}$$

Hence, in general, the optical path difference with lack of coincidence of the centre of the mirror with the paraxial focus, and in the presence of spherical aberration, will be represented by

$$c_1 y^2 + c_2 y^4 + c_3 y^6 + \text{etc.}$$

This equation may be discussed in the manner used in Vol. I, page 121, but it will suffice here to point out that if the focus for any particular local zone of the wave surface coincides with the centre of curvature of the mirror, there must necessarily be a local zonal maximum or minimum of optical path. Therefore, on applying the test above (slightly lengthening one branch) the fringes will gather towards or spread from this zone in both radial directions; a fringe at such a zone will be evanescent if the other fringes crowd towards it.[18] This criterion affords a ready means of setting the radius of curvature of the mirror into coincidence with the foci of successive zones of a lens suffering from spherical aberration, and hence allows the axial aberration to be measured if the mirror is furnished with a suitable micrometer screw. At the same time, the optical fringe system is a "contour map" showing the optical path differences with which the disturbances from particular zones meet in the focus corresponding to the centre of curvature of the mirror.

Sensitiveness. The Rayleigh limit of $\frac{\lambda}{4}$ for the allowable differences of path at the best focus for a telescope objective would, if present in a lens tested on the interferometer, give optical path differences of $\frac{\lambda}{2}$ owing to the double transmission. Taking a case when this arises through "first order" spherical aberration, the fringe system might then be (say) at its brightest for a mid-zone where the phases of the re-united waves agree, and shade off to the minimum both at centre and margin where the path differences would reach half a wave-length. In practice, the difficulties in making the surfaces, especially the convex surface of the mirror, accurate within say $\frac{\lambda}{8}$ are considerable, and the above would represent something like the lower limit of aberration which it is possible to test with any accuracy. The most serious defects of lenses, when the theoretical aberrations are reduced below about $\frac{\lambda}{2}$ or so, are generally faults of regularity, centring, etc., which can readily be detected on the interferometer. For this reason it is not usually possible to "measure" the axial aberration in the way suggested above when the error is small.

Chromatic Aberration. When the centre of curvature of the mirror coincides with the principal focus of the lens for any zone, the combination of lens and mirror is clearly equivalent to a plane reflecting surface, at least in the zone considered. This consideration is the basis of the method of measuring the chromatic aberration on the interferometer; it is, however, advisable to employ a chromatically corrected lens to collimate the light. With the aid of a hydrogen tube and a mercury lamp with a monochromator, or with suitable colour filters of the Wratten series, it is possible to set the mirror to the focus of the lens for a series of wave-lengths throughout the spectrum.[19] With a photographic lens it is possible to study the chromatic variations for one particular zone in observing the evanescence of the fringes, but with more highly corrected lenses it will be necessary to study the pattern with the minimum number of fringes for the lens as a whole.

Measurement of Oblique Aberrations. The earlier simple forms of the lens testing interferometer used for testing telescope objectives have been followed by other more complex arrangements by the aid of which the aberrations of other lenses, such as camera lenses and microscope objectives, may be ascertained. The camera lens interferometer[20] allows of numerical tests of the oblique aberrations, and Kingslake[21] has shown how the numerical coefficients of the Seidel aberrations may be deduced from an interferogram.

Microscope Interferometer. Fig. 203 illustrates the principle of one form of the microscope interferometer. It will be seen that the plane mirror in one arm is replaced by a negative lens followed by the microscope objective. The virtual image of the pinhole source formed by the negative lens must lie at the proper working point fixed by the appropriate tube length of the objective. The spherical mirror in this case is sometimes represented by a globule of mercury. These globules, when sufficiently small, become sufficiently spherical

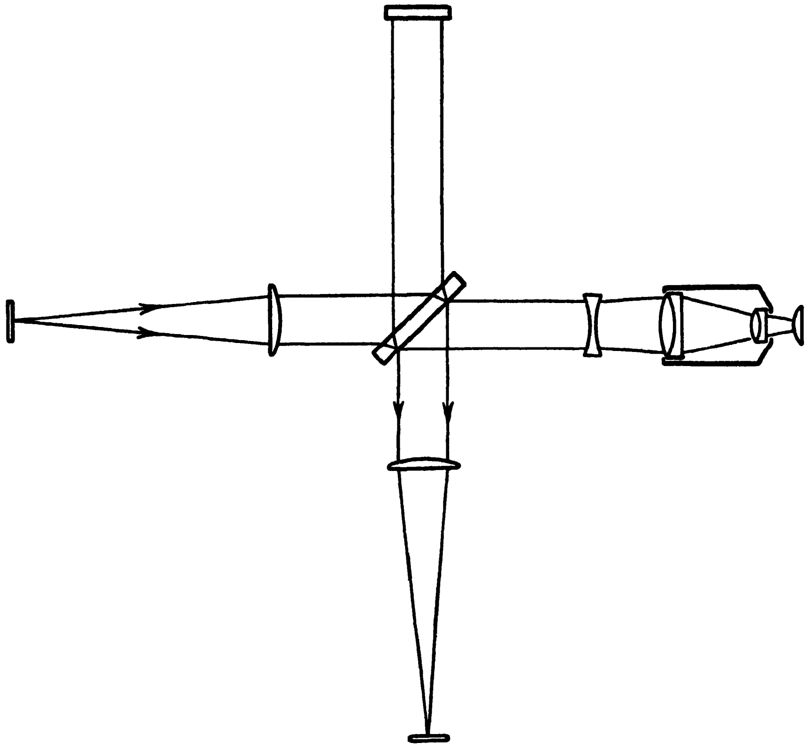

FIG. 203. INTERFEROMETER TEST ON A MICROSCOPE OBJECTIVE

owing to the great surface tension pressure. Details will be found in a paper by Twyman.[22]

General Aspects of the Interferometer Tests. Although they require expensive apparatus, the interferometer tests yield direct results with the least expenditure of time. General points in the theory of the instruments have been discussed by T. Smith.[23] In order to be able definitely to associate the faults of a system with their apparent places in the interference field, we should place the system and mirror as close together as possible, and view or photograph the field with a lens focused on the mirror.

While the interferometer is at the service of the professional optician, the older methods will still be fruitful in the hands of those who may only have occasional necessity for the accurate measurement of aberrations.

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APPENDIX I

OPTICAL CONVENTIONS AND EQUATIONS

It may be a convenience for some readers to have a concise statement of the sign conventions used in the present book, together with some of the elementary equations deduced in the early chapters of Vol. I.

Symbols. Symbols relating to the image space are distinguished from those of the object space by the addition of a dash or accent, thus: l'.

Refractive indices (object and image spaces) . . . n, n'.
Conjugate distances of object and image . . . l, l'
(measured from the principal points)
Conjugate distances of object and image . . . x, x'
(measured from focal points)
Perpendicular heights of object and image . . . h, h'
Angles between a ray and the axis (object and image spaces) . . . . a, a'
The reciprocals of distances are denoted by the cursive form of capitals, thus $1/l$ and $1/l'$. . . . . $\mathscr{L}$ and $\mathscr{L}$
The power of an optical system n'/f' . . . . [illegible]

Sign Convention. Distances measured to the right along the axis are counted positive; those to the left are negative. Similarly, those measured upwards, perpendicular to the axis, are positive, and those measured downwards are negative.

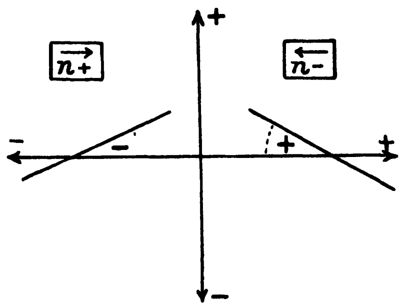

FIG. A1

The angles at which the ray directions meet the axis are counted positive if a clockwise turn will bring a line from the axis direction to the ray direction by the lesser angular movement.

The refractive indices are numerically positive when the direction of the light is from left to right, and numerically negative when the direction of the light is from right to left. The form of the equation remains the same for either direction.

Equations.

(Paraxial equations.)

The law of reflection—

$$i' = -i$$

The law of refraction—

$$n' \sin i' = n \sin i$$

Conjugate distance relation (single refracting surface)—

$$\frac{n'}{l'} - \frac{n}{l} = \frac{n'-n}{r}; \text{ or } \overline{\mathscr{L}}' - \overline{\mathscr{L}} = \mathscr{F}.$$

The above can be converted to the relation for a *single reflecting surface* by putting $\mu' = -\mu$ and simplifying—

Conjugate distance relation (single reflecting surface)

$$\frac{1}{l'} + \frac{1}{l} = \frac{2}{r}.$$

Magnification relation (single refracting surface)—

$$\frac{l'-r}{h'} = \frac{l-r}{h}$$

Smith-Helmholtz-Lagrange relation (for an optical system)—

$$n'h'\alpha' = nh\alpha$$

Conjugate distance relation (thin lens in air)—

$$\frac{1}{l'} - \frac{1}{l} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \text{ or } \mathscr{L}' - \mathscr{L} = (n-1)(\mathscr{R}_1 - \mathscr{R}_2)$$

Magnification relations (optical system)—

$$\frac{h}{h'} = -\frac{x}{f}; \quad \frac{h'}{h} = -\frac{x'}{f'}.$$

The "Newtonian equation"—

$$xx' = ff'$$

Longitudinal magnification—

$$\frac{dx'}{dx} = \left(\frac{h'}{h}\right)^2 \frac{n'}{n}.$$

Focal lengths of an optical system—

$$-\frac{n}{f} = \frac{n'}{f'} = \mathscr{F} \equiv \text{Power}.$$

Power of thick lens; surfaces of power $\mathscr{F}_1$ and $\mathscr{F}_2$
(thickness) ÷ (refractive index) = $\bar{d}$

$$\mathscr{F} = \mathscr{F}_1 + \mathscr{F}_2 - \mathscr{F}_1\mathscr{F}_2\bar{d}.$$

Power of a system of two thin lenses; separation d; and corresponding focal length equation—

$$\mathcal{F} = \mathcal{F}_a + \mathcal{F}_b - d.\mathcal{F}_a.\mathcal{F}_b; \quad \frac{1}{f'} = \frac{1}{f'_a} + \frac{1}{f'_b} - \frac{d}{f_a'f_b'}.$$

Distance from first lens (of system of two thin lenses) to first principal point of system—

$$P_aP = f'd/f'_b,$$

and corresponding distance from second lens to second principal point of system—

$$P_bP' = -f'd/f'_a$$

APPENDIX II

THEORY OF THE DIFFRACTION GRATING

THE simplest approach to the general theory is to consider the grating to consist of a series of parallel rectangular apertures. The investigation first deals then with the diffraction of a plane wave by a single rectangular aperture; the discussion is then extended to the case of a series of apertures.

1. Diffraction of a Plane Wave by a Rectangular Aperture. We may imagine a plane screen of indefinite extent with a rectangular

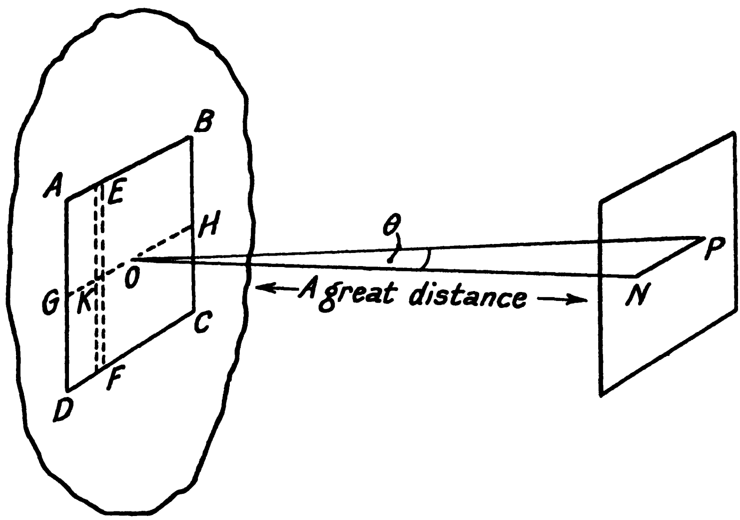

FIG. A2

aperture ABCD (Fig. A2). A wave-train with its wave-fronts parallel to the screen produces vibrations in the plane of the aperture which are all in the same phase. It is now desired to find the relative illumination, due to the aperture, at various points in a second screen at an indefinitely great distance, or a distance so great that if N is a point on the normal through the centre O of the aperture, the difference between the distances AN and ON, or CN and ON, is not more than a negligible fraction of the wave-length of light. In this way the disturbances spreading from all the parts of the aperture and reaching the point N will arrive in sensibly the same phase, and their amplitudes will simply be added together to find the resultant.

Effects of this class were discussed by Fraunhofer, and are usually known as Fraunhofer diffraction phenomena. As will be explained below, the effects on a screen at an "infinite distance" are similar to those found in the focal surface of a lens placed behind the aperture and focusing the light; hence the discussion has more than a theoretical interest.

The second plane is also normal to the line ON; let us consider a point P in this plane such that $\widehat{NOP} = \theta$, say, and NP is parallel to GH, a diameter of the rectangle ABCD. It will be clear that the

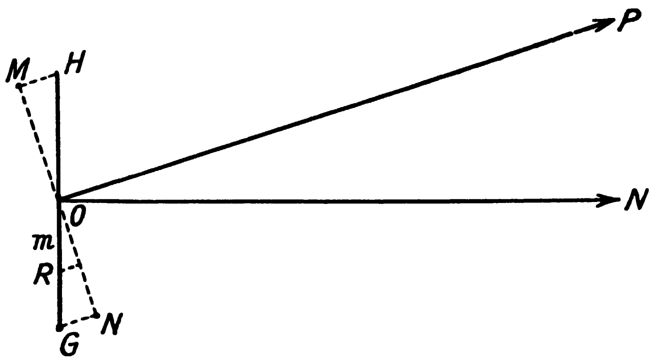

FIG. A3

distances from P of all points on a line, such as AD or BC, perpendicular to GH, will still be sensibly equal, but the distances from P of various points on the line GH will differ considerably if θ is finite. The calculation of the effect due to the whole aperture may therefore be effected by dividing this aperture into a series of strips, such as EF, and imagining the whole effect of the strip as equivalent to a proportionately intense source of disturbance situated at the central point K of the strip.

The problem can now be discussed with a two-dimensional diagram (Fig. A3), as if the rectangular aperture were a line source of light, giving rise to disturbances starting with the same phase.

Let the sides of the rectangle be $m_1 = \text{AB}$, and $n_1 = \text{AD}$ respectively; the area is therefore m_1n_1; let the amplitude at N be written km_1n_1 since all disturbances arrive at N in the same phase. In seeking the effects at a point P_1 the relative phases of the elementary disturbances arriving there must be considered. The dotted line MON in Fig. A3, is perpendicular to OP. All optical paths from points on MON to the point P are considered equal; hence it will be clear that the disturbance derived from the point R at a distance m from the point O will have an extra path to travel as compared with the disturbance derived from O. This extra path is $m \sin \theta$ (where $OR = m$) and the corresponding lag of phase is $(2\pi m \sin \theta)/\lambda$.

In Vol. I, page 81, it was shown that the amplitude A resultant from a number of successive contributions with various phases is given by

$$A^2 = \{\Sigma a \sin \delta\}^2 + \{\Sigma a \cos \delta\}^2$$

where a is the amplitude and δ the phase angle of a single contribution. In our case the area will be divided up into equal strips of width dm so that the contributions of successive elements will have equal amplitudes but differing phases. The area of a single strip is $n_1 dm$, so that the amplitude at P produced by it will be $kn_1 dm$ where k has the same meaning as above.

The summation will therefore be

$$A^2 = \left[\sum_{m=-\frac{m_1}{2}}^{m=\frac{m_1}{2}} (kn_1 dm) \sin \{(2\pi m \sin \theta)/\lambda\}\right]^2$$

$$+ \left[\sum_{m=-\frac{m_1}{2}}^{m=\frac{m_1}{2}} (kn_1 dm) \cos \{(2\pi m \sin \theta)/\lambda\}\right]^2$$

Consider, however, the summation $\Sigma a \sin \delta$ characteristic of the first main term above. For one strip the component will be

$$kn_1 dm \sin \{(2\pi m \sin \theta)/\lambda\},$$

but there will be another strip on the other side of O, at a distance $-m$ from O which will have an equal phase angle, but *negative*, so that the two terms will cancel each other. On the other hand, such terms will *add* numerically in the cosine summation, since the cosine of a negative angle is equal to that of the equal positive angle. We thus see that the first bracket above reduces to zero, and we get

$$A = \sum_{m=-\frac{m_1}{2}}^{m=\frac{m_1}{2}} (kn_1 dm) \cos \{(2\pi m \sin \theta)/\lambda\}$$

$$= kn_1 \int_{m=-\frac{m_1}{2}}^{m=\frac{m_1}{2}} dm \cos \{2\pi m \sin \theta)/\lambda\}$$

$$= kn_1 \left[\frac{\sin \{(2\pi m \sin \theta)/\lambda\}}{(2\pi \sin \theta)/\lambda} \right]_{m = -\frac{m_1}{2}}^{m = \frac{m_1}{2}}$$

$$= \frac{kn_1\lambda}{2\pi \sin \theta} \left[\sin\left(\frac{2\pi}{\lambda} \cdot \frac{m_1}{2} \sin \theta\right) + \sin\left(\frac{2\pi}{\lambda} \cdot \frac{m_1}{2} \sin \theta\right) \right]$$

$$\mathrm{A} = \frac{kn_1\lambda}{\pi \sin \theta} \sin\left(\frac{\pi m_1 \sin \theta}{\lambda}\right)$$

$$= km_1n_1 \frac{\sin\left(\frac{\pi m_1 \sin \theta}{\lambda}\right)}{\left(\frac{\pi m_1 \sin \theta}{\lambda}\right)}$$

$$= \mathrm{K} \frac{\sin \mathrm{U}}{\mathrm{U}},$$

where K is the central intensity, and $\mathrm{U} = (\pi m_1 \sin \theta)/\lambda$. A similar law must also hold for a direction from N taken parallel to the other side of the rectangle; in the case of $\mathrm{U} = (\pi n_1 \sin \theta)/\lambda$.

As regards the phase of the resultant vibration, we can see by the ordinary graphical construction, Vol. I, page 80, that if we have two equal components with phase angles of opposite sign but numerically equal, the resultant phase angle must be zero. In the above case, the resultant disturbance from the whole aperture has the phase of the component derived from the central point O.

Effect of Two Parallel Rectangular Apertures. The result of the investigation for a single aperture must now be extended to the case of a number of apertures. Let us first of all consider the case of two equal parallel rectangular apertures in a screen on which a plane wave-train is incident. The width of each aperture is m_1 as before, and its height n_1. In a very similar manner, we may reduce the effect of each aperture to that of a line source in the plane of the diagram. We found above that the resultant of a single aperture has the phase of a disturbance starting from the mid-point.

Let the distance between the central points of each aperture be x and let θ be the angle of diffraction considered. In Fig. A4, the two apertures are A and B and the mid-point is C. Drawing a dotted line ECD through C perpendicular to the direction of diffraction, it will be clear that the disturbance from B would arrive with a lag of phase (δ say), while that from A would arrive with a numerically equal lead of phase, as compared with an (imaginary) distance derived from C. If A_1 is the amplitude due to a single aperture in the direction θ, then the resultant amplitude due to each of them will be

$$A^2 = \{A_1 \sin \delta + A_1 \sin (-\delta)\} + \{A_1 \cos \delta + A_1 \cos (-\delta)\}^2$$
$$= 4A^2_1 \cos^2 \delta$$

Now δ is evidently given by

$$\frac{2\pi}{\lambda}(BD) = \frac{2\pi}{\lambda}\left(\frac{x}{2}\sin\theta\right) = \frac{\pi x \sin\theta}{\lambda},$$

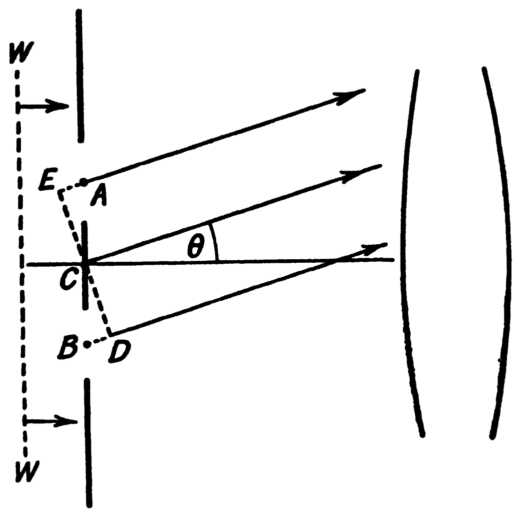

FIG. A4

so that the resultant amplitude is given by

$$A = 2A_1 \cos\left(\frac{\pi x \sin\theta}{\lambda}\right)$$

But from above we had that

$$A_1 = K(\sin U)/U$$

where $U = (\pi m_1 \sin\theta)/\lambda$, so that the complete expression for the amplitude is

$$A = \frac{2K \sin U}{U} \cos\left(\frac{\pi x \sin\theta}{\lambda}\right).$$

Hence the distribution due to a pair of very thin slits, which would be a set of interference fringes in which the amplitude follows the law $A \cos(\pi x \sin\theta/\lambda)$, and the intensity follows the corresponding "$\cos^2$" law, will, in practice, when using slits of finite aperture, be modified by the $(\sin U)/U$ term. Students who are interested in the study of diffraction should make the experiment and plot the curves for particular cases, as they are most instructive. The subject is treated more fully in textbooks of Physical Optics.

Effect of a Series of Parallel Apertures. Let the width of each aperture be m_1 and the length n_1 as above. Exactly as before, we can reduce the effect of each aperture to that of a line source in its small diameter shown in the plane of the diagram (Fig. A5). Let the common distance between the centres of the apertures be x,

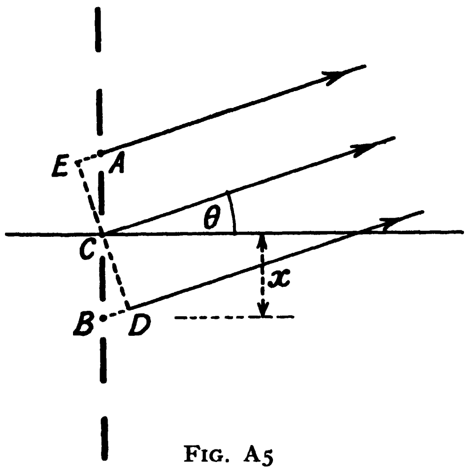

FIG. A5

then also, as before, the disturbances derived from each aperture will have the phase of one derived from the mid-point.

Let C be the central aperture and A and B the first apertures on each side. It will be clear that the disturbance from B will have a lag, and that from A an equal lead in phase (δ say) considering the diffraction angle θ. For the p_{th} aperture away from the centre the phase difference will be $p\delta$. Let A_1 be the amplitude due to a single aperture in the resultant taken for the angle θ, then the effect of all the apertures will be given by

$$A^2 = \left[\sum_{p=1}^{p=(N-1)/2} \{A_1 \sin p\delta + A_1 \sin(-p\delta)\}\right]^2 + \left[A_1 + \sum_{p=1}^{p=(N-1)/2} \{A_1 \cos p\delta + A_1 \cos(-p\delta)\}\right]^2$$

for a total number of apertures N including the central one. It will be clear that the sine terms will mutually cancel each other when N is odd, leaving

$$A = A_1 + 2\sum_{p=1}^{p=(N-1)/2} A_1 \cos p\delta$$

or putting in the value of δ, i.e. $(2\pi \cdot x \sin\theta)/\lambda$

$$A = A_1 + 2\sum_{p=1}^{p=(N-1)/2} A_1 \cos\{(2\pi p x \sin\theta)/\lambda\}$$

The sum of the series may be found as in Vol. I, page 83; it proves to be given by

$$A = A_1 \frac{\sin\{(\pi N x \sin\theta)/\lambda\}}{\sin\{\pi x \sin\theta)/\lambda\}}$$

We can simplify the expression for the case when θ is small in the neighbourhood of the central maximum. It then becomes

$$A = A_1 \frac{\sin\{(\pi N x \sin\theta)/\lambda\}}{(\pi x \sin\theta)/\lambda}$$

$$= NA_1 \frac{\sin\{(\pi N x \sin\theta)/\lambda\}}{(\pi N x \sin\theta)/\lambda}$$

$$= \text{(effect due to whole of apertures) } \{\sin W)/W\}$$

where $W = (\pi N x \sin\theta)/\lambda$.

This last expression will, however, become inaccurate, as θ becomes greater. It is not so easy to see the way in which the amplitude varies from this expression as from the one above, i.e.

$$A = A_1 + 2\sum_{p=1}^{p=(N-1)/2} A_1 \cos p\delta$$

where $\delta = (2\pi x \sin\theta/\lambda)$.

If we plot a succession of curves $y_1 = \cos\delta$, $y_2 = \cos 2\delta$, $y_3 = \cos 3\delta$, etc., and add the ordinates, we shall find that the sum of the ordinates keeps on increasing at $\delta = 0$, $\delta = 2\pi$, $\delta = 4\pi$, etc., but that at intermediate points the contributions of successive terms vary in sign and tend to cancel each other. With two terms

$$A_1 + 2A_1 \cos\delta$$

(which represents three grating apertures) there will be a result shown in Fig. 71 of this volume; the main maxima are separated by one intermediate maximum. With three terms (representing five grating apertures) there will be three intermediate maxima.

The student should draw the amplitude curve, and then the curve given by squaring the amplitudes, thus obtaining an intensity curve with all positive ordinates.

We find that when the number of terms is indefinitely large (i.e. when there are a large number of diffracting apertures, the sum of the amplitudes only has a finite value when δ in the above expression is zero or some positive or negative multiple of 2π, i.e. when

$$x \sin \theta = \pm r\lambda,$$

where r is an integer or zero.

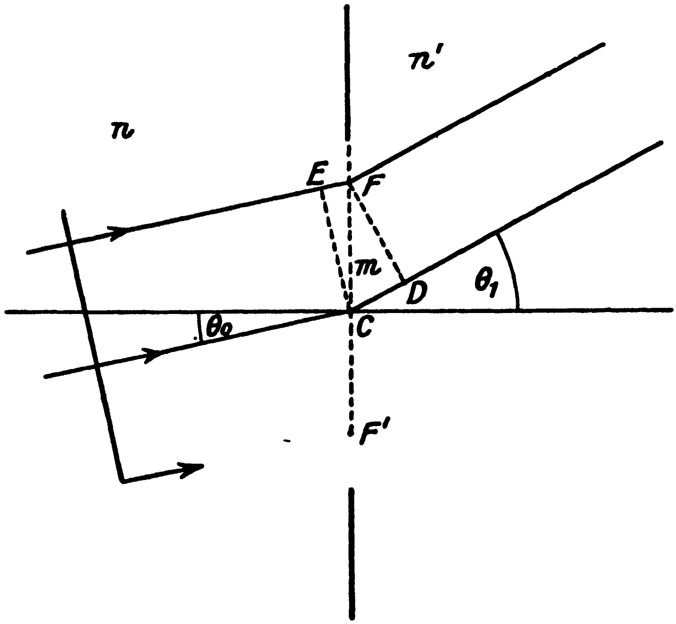

FIG. A6

We may note that in Fig. A4, the amount of $x \sin \theta$ would be EA + BD, or simply the path difference between the disturbances derived from corresponding points of two adjacent apertures. If this is zero, or corresponds to one or more whole wave-lengths, we get a diffraction maximum. The more exact expression for the sum at the foot of page 280 gives

$$A = A_1 \{\sin (N\delta/2\}/\sin \delta/2)$$

We found that A_1, the effect of a single aperture, was given by (K sin U)/U so that the complete expression for the amplitude is

$$A = \text{constant} \times \frac{\sin\left(\dfrac{\pi m_1 \sin \theta}{\lambda}\right)}{\dfrac{\pi m_1 \sin \theta}{\lambda}} \cdot \frac{\sin\left(\dfrac{\pi N x \sin \theta}{\lambda}\right)}{\sin\left(\dfrac{\pi x \sin \theta}{\lambda}\right)}.$$

Wave Incident at Another Angle. We can easily see how the above expressions will be modified to deal with the case in which the plane wave-fronts incident at the apertures are not parallel to the screen, but are inclined with the normal in the plane of the diagram (which is supposed to be perpendicular to the screen) and when there is a change of refractive index at the plane of the screen.

We can show, as before, that the effect of the whole aperture will be equivalent to that of a line source at the middle of the aperture. This centre line can be shown in the diagram (Fig. A6). Let C be its central point, and let F be a point at a distance m from the centre.

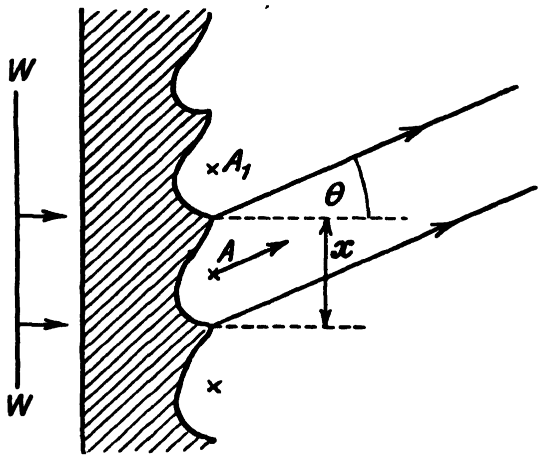

FIG. A7

The plane waves pass up to the aperture at an angle θ_0 with the screen; we desire to find the diffraction effect in a direction making an angle θ_1 with the normal beyond the aperture. Draw the perpendicular CE and FD to the "incident" and "diffracted" paths.

The disturbances passing through F have a shorter optical path than those through C, the difference being

$$n' \cdot CD - n \cdot EF = n'm \sin \theta_1 - nm \sin \theta_0,$$

but for those passing through F′, the symmetrical point on the other side of C, there will be an equal and opposite path difference. Hence the term sin θ in the above expressions will be replaced by $n' \sin \theta_1 - n \sin \theta_0$. In the case of the diffraction grating, for example, the condition for a bright maximum will be

$$x(n' \sin \theta_1 - n \sin \theta_0) = \pm p\lambda$$

where p is zero or an integer.

Actual Gratings. Actual gratings consist in practice of rulings made by a diamond point in the reflecting or transmitting surface.

"Gratings" or test rulings made as objects for testing the resolving power of a microscope are usually on a transparent surface. Diffraction gratings with high dispersion are usually ruled on a reflecting surface, but a cast of the grating can be taken in a thin film of celluloid and mounted on a plate of glass. The general subject may be studied from the accounts given in Baly's *Spectroscopy*, Wood's *Physical Optics*, and other articles.

It must suffice to say here that, although the theory of the grating given above concerns only a series of apertures in a thin opaque screen, the effect of any regular structure producing changes in the optical path of some disturbances comparable with the wave-length of light must be of a similar character. Fig. A7 represents a plane wave-front passing into such a thin film with a regular structure shown in section. The grating element has a spacing x.

In any direction θ with the normal to the face of the grating, any single element will produce a resultant amplitude a_1, say, and having the phase of an elementary disturbance propagated from some point A, say, which will be exactly similarly placed for every grating element. The effect of the grating will therefore be that of a series of sources at A, A_1, A_2, etc., with a constant step of phase if the incident wave is oblique, or co-phasal if the wave-front is incident normally.

APPENDIX III

ASTIGMATISM OF A LENS SYSTEM

In Vol. I, page 302, we obtained the equation for the astigmatism produced by a lens system, in the form

$$(\text{final})\ \frac{I_t I_s}{n'h'^2} - (\text{initial})\ \frac{O_t O_s}{nh^2} = \Sigma \frac{Q_t^2}{(Q_l - Q_t)^2}\left(\frac{1}{n't'} - \frac{1}{nt}\right)$$

where I_t and I_s are the tangential and sagittal image points respectively, both situated on the principal ray through the centre of the stop, and $O_t O_s$ are the corresponding points in the object.

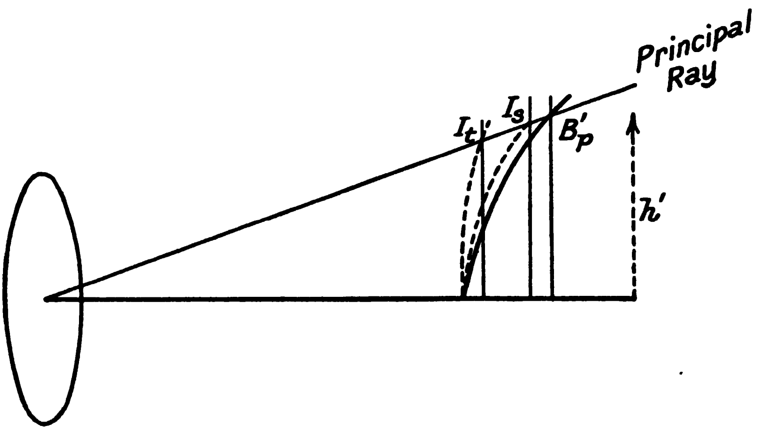

Fig. A8

With the aid of this equation we can obtain a simple expression for the radii of curvature of the tangential and sagittal image fields, supposing them to be represented, near the axis of the system, sufficiently nearly by spherical surfaces. We must also refer to the general discussion of aberrations given on pages 131–139 of Vol. I. The expressions on page 132 give the transverse displacements of a ray from Q in the y and z directions, and it was shown on page 135 that the tangential focus lies at a distance from the Petzval surface which is three times the distance of the sagittal focus from the same surface. In Fig. A8 let I_t,I_s, and $B'p$ represent points on the tangential, sagittal, and Petzval surfaces respectively, of radii R'_t, R'_s, and R'_t. For small angles of slope we may calculate the intercepts by the spherometer formula.

Thus
$$I_tI_s = 2I_sB'_p = 2\left(\frac{h'^2}{2R'_i} - \frac{h'^2}{2R'_s}\right)$$
$$= h'^2\left(\frac{1}{R'_i} - \frac{1}{R'_s}\right)$$

Similarly
$$O_tO_s = h^2\left(\frac{1}{R_i} - \frac{1}{R_s}\right)$$

The above equation therefore takes the form
$$\frac{1}{n'}\left(\frac{1}{R'_i} - \frac{1}{R'_s}\right) - \frac{1}{n}\left(\frac{1}{R_i} - \frac{1}{R_s}\right) = \Sigma\frac{Q_l^2}{(Q_l - Q_t)^2}\cdot\left(\frac{1}{n't'} - \frac{1}{nt}\right)$$

Then
$$\frac{1}{n'R'_s} - \frac{1}{nR_s} = \left(\frac{1}{n'R'_i} - \frac{1}{nR_i}\right) - \Sigma\frac{Q_l^2}{(Q_l - Q_t)^2}\left(\frac{1}{n't'} - \frac{1}{nt}\right)$$
$$= \Sigma\left(\frac{n-n'}{nn'}\cdot\frac{1}{r}\right) - \Sigma\frac{Q_l^2}{(Q_l - Q_t)^2}\left(\frac{1}{n't'} - \frac{1}{nt}\right)$$

from the equation 52 on page 139, Vol. I. This gives the radius of the sagittal image field near the axis.

Since also
$$I_tI_s = \tfrac{2}{3}(I_tB'_p) = \tfrac{1}{3}\left(\frac{h'^2}{R'_i} - \frac{h'^2}{R'_t}\right).$$

we can obtain in a similar way
$$\frac{1}{n'R'_t} - \frac{1}{nR_t} = \Sigma\left(\frac{n-n'}{nn'}\cdot\frac{1}{r}\right) - 3\Sigma\frac{Q_l^2}{(Q_l - Q_t)^2}\left(\frac{1}{n't'} - \frac{1}{nt}\right),$$

giving the radius of the tangential image field. The definitions of Q_l and Q_t were
$$Q_l = n\left(\frac{1}{r} - \frac{1}{l}\right) = n'\left(\frac{1}{r} - \frac{1}{l'}\right)$$

where l and l' refer to the distance of the intermediate entrance and exit pupil for the surface.
$$Q_t = n\left(\frac{1}{r} - \frac{1}{t}\right) = n'\left(\frac{1}{r} - \frac{1}{t'}\right)$$

where t and t' refer to the distance of the intermediate object and image for the surface.

The equations above are not satisfactory for very oblique rays with large values of astigmatic differences of focus, and the curvatures so found only apply to the fields near the axis. It will be well for the student to revise the theory of a thin lens used with a stop, given in Vol. I, pages 302–304. It is clear from the above how the tangential image surface lies at three times the distance of the sagittal surface from the Petzval surface, the radius of the latter being determined by the first sum on the right of each equation.

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