AN INTRODUCTION TO THE INFINITESIMAL CALCULUS

H. S. CARSLAW, M.A.

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INFINITESIMAL CALCULUS

NOTES FOR THE USE OF SCIENCE AND ENGINEERING STUDENTS

BY

H. S. CARSLAW, M.A., D.Sc., F.R.S.E.

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF SYDNEY
FORMERLY FELLOW OF EMMANUEL COLLEGE, CAMBRIDGE
AND LECTURER IN MATHEMATICS AT THE UNIVERSITY OF GLASGOW

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PREFACE

These introductory chapters in the Infinitesimal Calculus were lithographed and issued to the students of the First Year in Science and Engineering of the University of Sydney at the beginning of last session. They form an outline of, and were meant to be used in conjunction with, the course on *The Elements of Analytical Geometry and the Infinitesimal Calculus*, which leads up to a term's work on Elementary Dynamics.

The standard text-books amply suffice for the detailed study of this subject in the second year, but the absence of any discussion of the elements and first principles suitable for the first year work, was found to be a serious hindrance to the work of the class. For such students a separate course on Analytical Geometry, without the aid of the Calculus, is not necessary, and the exclusion of the methods of the Calculus from the analytical study of the Conic Sections is quite opposed to the present unanimous opinion on the education of the engineer. It has been our object to present the fundamental ideas of the Calculus in a simple manner and to illustrate them by practical examples, and thus to enable these students to use its methods intelligently and readily in their Geometrical, Dynamical, and Physical work early in their University course. This little book is not meant to take the place of the standard treatises on the subject, and, for that reason, no attempt is made to do more than give the lines of the proof of some of the later theorems. As an introduction to these works, and as a special text-book for such a "short course" as is found necessary in the engineering schools of the Universities and in the Technical Colleges, it is hoped that it may be of some value.

In the preparation of these pages I have examined most of the standard treatises on the subject. To Nernst and Schönflies' *Lehrbuch der Differential- und Integral-Rechnung*, to Vivanti's *Complementi di Matematica ad uso dei Chemici e dei Naturalisti*, to Lamb's *Infinitesimal Calculus*, and to Gibson's *Elementary Treatise on the Calculus*, I am conscious of deep obligations. I should also add that from the two last-named books, and from those of Lodge, Mellor, and Murray, many of the examples have been obtained.

In conclusion, I desire to tender my thanks to my Colleagues in the University of Sydney, Mr. A. Newham and Mr. E. M. Moors, for assistance in reading the proof-sheets; to my students, Mr. D. R. Barry and Mr. R. J. Lyons, for the verification of the examples; also to my old teacher, Professor Jack of the University of Glasgow, and to Mr. D. K. Picken and Mr. R. J. T. Bell of the Mathematical Department of that University, by whom the final proofs have been revised.

H. S. CARSLAW.

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CHAPTER I

THE ANALYTICAL GEOMETRY OF THE STRAIGHT LINE*

§ 1. Cartesian Co-ordinates.

The position of a point on a plane may be fixed in different ways. In particular it is determined if its distances from two fixed perpendicular lines in the plane are known, the usual conventions with regard to sign being adopted. These two lines Ox and Oy are called the axes of x and y; and the lengths OM and ON, which the perpendiculars from the point P cut off from the axes, are called the co-ordinates of the point P and denoted by x and y. OM and ON are taken positive or negative according as they are measured along Ox and Oy, or in the opposite directions.

Ex. 1. Mark on a piece of squared paper the position of the points $(\pm 2, \pm 3)$.

2. Prove that the distance between the points (2, 3), and (−2, −3) is $2\sqrt{13}$.

3. Prove that the distance d between the points (x_1, y_1), (x_2, y_2) is given by
$$d^2=(x_1-x_2)^2+(y_1-y_2)^2.$$

4. Prove that the co-ordinates of any point (x, y) upon the circle whose centre is at the point (a, b) and whose radius is r, satisfy the equation
$$(x-a)^2+(y-b)^2=r^2.$$

§ 2. The Co-ordinates of a Point dividing the Line joining two given Points in a given Ratio l : m.

Let P_1 and P_2 be the two given points (x_1, y_1), (x_2, y_2); and let $P(x, y)$ divide P_1P_2 in the ratio $l : m$ (see Fig. 1).

* The student is recommended to read pp. 1-25 of Hall's *Introduction to Graphical Algebra* (2nd ed.) before commencing this work.

Draw P_1M_1, PM, and P_2M_2 perpendicular to Ox; P_1HK and PL parallel to Ox, meeting PM and P_2M_2 in H, K, and L.

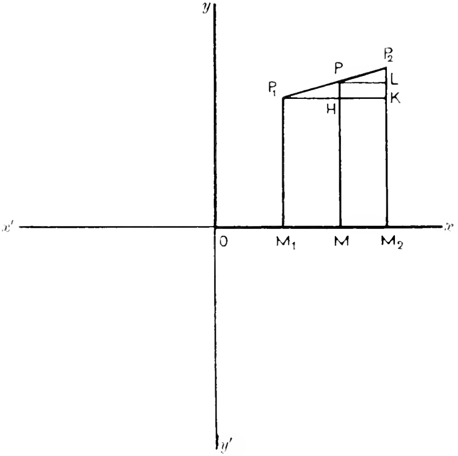

FIG. 1.

Since
$$\frac{P_1H}{PL} = \frac{P_1P}{PP_2} = \frac{l}{m},$$

$$\therefore \quad \frac{x - x_1}{x_2 - x} = \frac{l}{m},$$

$$\therefore \quad x(l + m) = lx_2 + mx_1,$$

$$\therefore \quad x = \frac{lx_2 + mx_1}{l + m}.$$

Similarly
$$y = \frac{ly_2 + my_1}{l + m}.$$

These are the co-ordinates of the internal point of section. Those of the external point may be found in the same way to be

$$x = \frac{lx_2 - mx_1}{l - m}$$

and $$y = \frac{ly_2 - my_1}{l - m}.$$

Ex. 1. Prove that the co-ordinates of the middle point of the line which cuts off unit length from Ox and Oy are $\frac{1}{2}$ and $\frac{1}{2}$.

2. Find the co-ordinates of the points of trisection of this line, and also of the points which divide it externally in the ratio 1 : 2.

3. Prove that the C.G. of the triangle whose angular points are (2, 1), (4, 3), (2, 5) is the point $\left(\frac{8}{3}, 3\right)$; and give the general theorem.

§ 3. **The Equation of the First Degree represents a Straight Line.**

If the point P move along a curve the co-ordinates of the point are not independent of each other. In mathematical language "y is a function of x," and we speak of $y=f(x)$ as the equation of the curve, meaning that this equation is satisfied by the co-ordinates (x, y) of any point upon the curve. For example, the equation of the circle whose centre is at the origin and whose radius is a is $x^2 + y^2 = a^2$. The properties of curves may often be obtained by discussing their equations.

The simplest equation is that of the first degree, $ax + by + c = 0$, a, b, and c being constants.

For example, take the equation

$$x + 2y = 4.$$

By assigning any value to x and solving the equation for y we obtain, as in the accompanying table, the co-ordinates of any number of points upon the locus. Plotting these points on the diagram we see that all lie upon a straight line.

x	y
− 3	3·5
− 2	3
− 1	2·5
0	2
1	1·5
2	1
3	·5

We proceed to prove that this is true in general; in other words, that *all the points whose co-ordinates satisfy the equation*

$$ax + by + c = 0$$

lie upon a straight line.

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points upon the locus.

Then we have $$ax_1 + by_1 + c = 0 \quad . \quad . \quad . \quad (1)$$
$$ax_2 + by_2 + c = 0 \quad . \quad . \quad . \quad (2)$$

Multiplying (1) by m and (2) by l, and adding, we obtain

$$a(lx_2 + mx_1) + b(ly_2 + my_1) + c(l + m) = 0,$$

$$\therefore \quad a\left(\frac{lx_2 + mx_1}{l + m}\right) + b\left(\frac{ly_2 + my_1}{l + m}\right) + c = 0.$$

But $\frac{lx_2 + mx_1}{l + m}$, $\frac{ly_2 + my_1}{l + m}$ are the co-ordinates of the point dividing P_1P_2 in the ratio $l : m$, and l, m may be chosen at random. It follows that if P_1, P_2 are two fixed points on the locus given by

$$ax + by + c = 0,$$

any other point on the unlimited straight line P_1P_2 is also upon the locus; and it can easily be shown that no point *off* this line lies upon the curve.

Therefore the equation

$$ax + by + c = 0$$

represents a straight line.

Ex. Prove this theorem by showing that if PQR are any three points whose co-ordinates satisfy the given equation, the triangle PQR has zero area.

§ 4. In the last article we have shown that the equation of the first degree represents a straight line. It is not then necessary in plotting the locus given by such an equation to proceed as we did above in the example $x + 2y = 4$. Two points fix a straight line. Therefore we have only to find two points whose co-ordinates satisfy the equation. The most convenient points are those where the line cuts the axes, and these are found by putting $x = 0$ and $y = 0$, respectively, in the equation.

Ex. 1. Draw the lines (i.) $x=0, \quad x=1, \quad x=-1$

(ii.) $y=0, \quad y=2, \quad y=-2$

(iii.) $x+y=0, \quad x+y=1$

(iv.) $y=2x, \quad y=2x+3$

(v.) $\frac{x}{4}+\frac{y}{3}=1, \quad \frac{x}{4}-\frac{y}{3}=1.$

2. Determine whether the point (2, 3) is on the line

$$4x+3y=15.$$

3. What is the condition that the point (a, b) should lie upon the line

$$ax+by=2ab?$$

§ 5. **The Gradient of a Line.**

When we speak of "the gradient" of a road being 1 in 200 we usually mean that the ascent is 1 foot vertical for 200 feet horizontal. This might also be called the slope of the road. The same expression is used with regard to the straight line. The "gradient" or the "slope" of a straight line is its rise per unit horizontal distance; or the ratio of the increase in y to the increase in x as we move along the line. This is evidently the same at all points of the straight line, and is equal to the tangent of the angle the line makes with the axis of x measured in the positive direction.

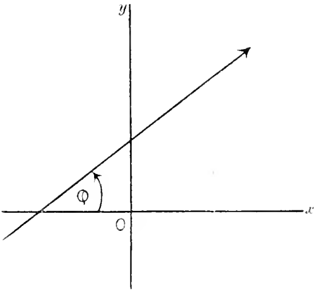

FIG. 2.

To save ambiguity it is well to fix upon the angle to be chosen, and in these pages it will be convenient to consider the line as always drawn upward in the direction of the arrow (Fig. 2), and thus to restrict the angle ϕ to lie between 0° and 180°.

When $0 < \phi < \frac{\pi}{2}$ the gradient is positive.

When $\frac{\pi}{2} < \phi < \pi$ the gradient is negative.

EX. 1. Write down the values of ϕ for the lines in § 4 (i.).

2. Prove that the gradient of the line $y = mx + c$ is m, and interpret the constant c.

§ 6. **Different forms of the Equation of the Straight Line.**

In the preceding articles we have shown that the equation

$$ax + by + c = 0$$

represents a straight line, and we have seen how the line may be drawn when its equation is given. We have now to show *how to obtain the equation of the line when its position is given.*

(A) *The equation of the line through two given points.*

Let (x_1, y_1), (x_2, y_2) be the two given points. Let (x, y) be the co-ordinates of any point upon the line. Then it is clear (cf. Fig. 1) that

$$\frac{y - y_1}{x - x_1} = \text{the gradient of the line,}$$

and that

$$\frac{y_2 - y_1}{x_2 - x_1} = \quad " \quad " \quad .$$

Thus we have the equation

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

between the co-ordinates (x, y) of the representative point and the co-ordinates (x_1, y_1) (x_2, y_2) of the fixed points. This is the equation of the straight line through these points. It is more conveniently written

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.$$

It follows that

(B) *The equation of the line through (x_1, y_1), making an angle ϕ with the axis of x, is*

$$\frac{y - y_1}{x - x_1} = \tan \phi ;$$

and that

(C) *The equation of the line which cuts off a length c from the axis of y, and is inclined at an angle whose tangent is m to the axis of x, is*

$$y = mx + c,$$

and that

(D) *The equation of the line which cuts off intercepts a and b from the axis of x and y is*

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Ex. 1. Write down the equations of the lines through the following pairs of points: (1, 1) (1, −1); (1, 2) (−1, −2); (3, 4) (5, 6); (a, b) $(a, -b)$.

2. Find the equations of the lines through the point (3, 4) with gradient ±5, and draw the lines.

3. The lines $y = x$ and $y = 2x$ form two adjacent sides of a parallelogram, the opposite angular point being (4, 5). Find the equations of the other two sides; and of the diagonals.

4. Write down the equations of the lines making angles 30°, 45°, 60°, 120°, 135°, and 150° with the axis of x, which cut this axis at unit distance from the origin in the negative direction.

§ 7. The "Perpendicular" Form of the Equation of the Straight Line.

A straight line is determined when the length of the perpendicular upon it from the origin, and the direction of this perpendicular are given.

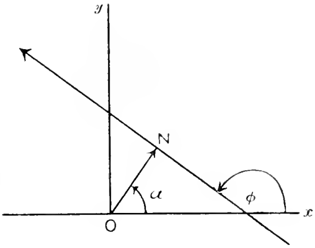

FIG. 3.

Let ON be the perpendicular, p, upon the line.

Let the angle between ON and Ox be α, this angle lying between 0 and 2π (cf. Fig. 3).

Then N is the point $(p\cos\alpha, p\sin\alpha)$.

Using the form (B) of § 6 the equation of the line becomes

$$\frac{y - p\sin\alpha}{x - p\cos\alpha} = \tan\phi = \tan\left(\alpha + \frac{\pi}{2}\right) = -\frac{\cos\alpha}{\sin\alpha}.$$

This reduces to

(E) $$x\cos\alpha + y\sin\alpha = p.$$

N.B.—The quantity p is to be taken always positive, and the angle α is the angle between $\overrightarrow{Ox}$ and $\overrightarrow{ON}$.

§ 8. The Point of Intersection of Two Straight Lines.

Since the point of intersection of the two lines

$$ax + by + c = 0$$

$$a'x + b'y + c' = 0$$

lies on both lines, its co-ordinates x, y satisfy both equations.

Solving the equations we have

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{1}{ab' - a'b}.$$

It is clear that if

$$ab' - a'b = 0,$$

and neither of the other two denominators vanish, the co-ordinates x, y are infinite, and the lines are parallel.

If in addition

$$ca' - c'a = 0$$

we have

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'},$$

and the third denominator $bc' - b'c$ also vanishes.

In this case the two equations are not independent, and they really represent the same straight line.

Ex. 1. Find the co-ordinates of the point of intersection of the lines

$$2x + y = 4.$$
$$x + 2y = 6.$$

Illustrate your result by a diagram.

2. Find the equations of the lines through (2, 3) parallel to

$$3x \pm 4y = 5.$$

3. Find the co-ordinates of the angular points of the triangle whose sides are given by

$$x + y = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$3x - 2y = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$
$$4x + 3y = 24 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Also find the equations of the medians of this triangle and the co-ordinates of its C.G.

§ 9. The Angle between Two Straight Lines whose Equations are given.

The equations of the lines may always be reduced to the forms

$$(1) \quad y = mx + c,$$
$$(2) \quad y = m'x + c',$$

and in this case the angles they make with the axis of x are ϕ and ϕ' where

$$\tan \phi = m, \qquad \text{(cf. Fig. 4)}$$
$$\tan \phi' = m'.$$

Hence

$$\tan (\phi - \phi') = \frac{m - m'}{1 + mm'}, = \tan \theta,$$

and the angle θ between the lines is $\tan^{-1}\left(\frac{m - m'}{1 + mm'}\right)$.

Unless care is shown in taking for the line (1) that with the greater slope, we would obtain a negative value for the tangent of the angle between the lines. The reason for this is obvious.

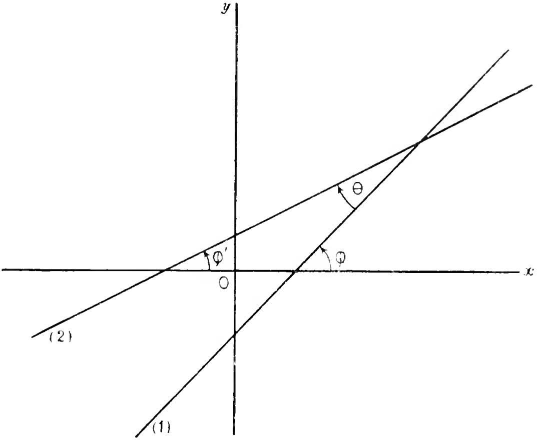

Fig. 4.

It follows that

(i.) *The lines are parallel if* $m = m'$;

(ii.) *The lines are perpendicular if* $mm' + 1 = 0$.

When the equations are

$$ax + by + c = 0$$
$$a'x + b'y + c' = 0$$

(i.) *The lines are parallel if* $\frac{a}{a'} = \frac{b}{b'}$;

(ii.) *The lines are perpendicular if* $aa' + bb' = 0$.

Ex. 1. Write down the equation of the straight line through (1, 2) perpendicular to $x - y = 0$.

2. Find the angles between the lines

$$\left.\begin{aligned} x - 2y + 1 &= 0 \\ x + 3y + 2 &= 0 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} 4x + 3y &= 12 \\ 3x + 4y &= 12 \end{aligned}\right\}$$

and draw the lines.

3. Write down the equation of the straight line through (a, b) perpendicular to $bx - ay = a^2 + b^2$.

4. Write down the equation of the line bisecting the line joining (1, 2) (3, 4) at right angles, and the equations of the perpendiculars upon both lines from the origin.

5. Prove that $l(x-a)+m(y-b)=0$ is a line through (a, b) parallel to $lx+my=0$; and write down the equation of the line through (a, b) perpendicular to $lx+my=0$.

6. Write down the equations of the lines through the C.G. of the triangle whose angular points are at (4, – 5) (5, – 6) (3, 1) parallel and perpendicular to the sides.

§ 10. **The Length of the Perpendicular from a Point (x_0, y_0) upon a Straight Line whose Equation is given.**

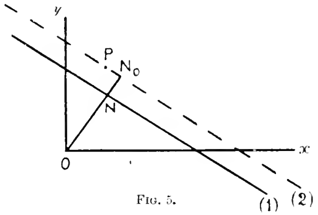

Fig. 5.

(i.) If the equation of the straight line is given in the "perpendicular" form

$$x \cos \alpha + y \sin \alpha = p \qquad (1),$$

the line through P (x_0, y_0) parallel to it is given by

$$(x - x_0) \cos \alpha + (y - y_0) \sin \alpha = 0 \qquad (2),$$

or
$$x \cos \alpha + y \sin \alpha = x_0 \cos \alpha + y_0 \sin \alpha.$$

But if p_0 is the perpendicular ON_0 from O upon the line (2), and if N, N_0 are on the same side of O, the equation of PN_0 may be written

$$x \cos \alpha + y \sin \alpha = p_0.$$

Therefore

$$x_0 \cos \alpha + y_0 \sin \alpha = p_0.$$

Also the perpendicular from P upon the line (1) is

$$ON_0 - ON, \qquad \text{(cf. Fig. 5)}$$

i.e.
$$p_0 - p,$$

i.e.
$$x_0 \cos \alpha + y_0 \sin \alpha - p.$$

In the case when N_0 lies between O and N we have to take

$$p - p_0;$$

and when N, N_0 lie on opposite sides of O, ON_0 makes angle $(a + \pi)$ with Ox, and we have to take

$$p + p_0.$$

In both these cases the length of the perpendicular is given by

$$-x_0 \cos a - y_0 \sin a + p.$$

(ii.) If the equation of the line is given as

$$ax + by = c \quad (c > 0) \quad . \quad . \quad . \quad (1),$$

we have first to throw this into the "perpendicular" form.

Suppose it becomes

$$x \cos a + y \sin a = p \quad . \quad . \quad . \quad (2).$$

Then, by equating the values we find from these two equations for the intercepts upon the axes, we obtain

$$\frac{\cos a}{a} = \frac{\sin a}{b} = \frac{p}{c}.$$

Therefore $$c \cos a = ap,$$

$$c \sin a = bp,$$

and $$c^2 = (a^2 + b^2)p^2;$$

$$\therefore \quad c = \sqrt{a^2 + b^2}\, p,$$

where there is no ambiguity in the square root, as both p and c are positive.

Hence $$\cos a = \frac{a}{\sqrt{a^2 + b^2}},$$

$$\sin a = \frac{b}{\sqrt{a^2 + b^2}},$$

and $$p = \frac{c}{\sqrt{a^2 + b^2}},$$

and the "perpendicular" form of the line

$$ax + by = c \quad (c > 0)$$

is $$\frac{ax}{\sqrt{a^2 + b^2}} + \frac{by}{\sqrt{a^2 + b^2}} = \frac{c}{\sqrt{a^2 + b^2}}.$$

Hence the length of the perpendicular from (x_0, y_0) *upon*

$$ax + by - c = 0$$

is $$\pm \left(\frac{ax_0 + by_0 - c}{\sqrt{a^2 + b^2}} \right).$$

And the positive sign is taken when (x_0, y_0) is upon the opposite side of the line from the origin, the negative sign when it is on the same side of the line as the origin.*

This result holds for the equation of the straight line, in whatever form it is given. The reason for the change of sign in the expression for the length of the perpendicular is that the equation of the first degree $lx + my + n = 0$ divides the plane of xy into two parts, in one of which $lx + my + n$ is positive; and in the other it is negative. Upon the line the expression vanishes.

Ex. 1. Transform the equations

(i.) $3x \pm 4y = 5$ (ii.) $3x \pm 4y = -5$

into the perpendicular form, and from your tables write down the value of a for each.

2. Write down the length of the perpendicular from the origin upon the line joining (2, 3) (6, 7).

3. Write down the length of the perpendicular from the point (2, 3) upon the lines

$$4x + 3y = 7, \quad 5x + 12y = 20, \quad 3x + 4y = 8.$$

4. Find the inscribed and escribed centres of the triangle whose sides are

$$3x + 4y = 0, \quad 5x - 12y = 0, \quad y = 15,$$

and the equations of the internal and external bisectors of the angles of this triangle, distinguishing the different lines.

[The student is referred for a fuller discussion of the subject matter of this chapter to (i.) Briggs and Bryan's *Elements of Co-ordinate Geometry*, Part I. chapters i.-x.; (ii.) Loney's *Co-ordinate Geometry*, chapters i.-vi.; and (iii.) to C. Smith's *Elementary Treatise on Conic Sections*, chapters i. and ii.

In all these books a large number of examples will be found illustrating the points we have discussed.]

EXAMPLES ON CHAPTER I

1. Find the equation of the locus of the point P which moves so that

$$\text{(i.)}\ AP^2 + PB^2 = c^2$$
$$\text{(ii.)}\ AP^2 - PB^2 = c^2$$
$$\text{(iii.)}\ AP \cdot PB = c^2,$$

A and B being the points $(-a, 0)$, $(a, 0)$.

---

* *Rule.*—To find the length of the perpendicular from a given point (x_0, y_0) upon a given straight line

$$lx + my + n = 0,$$

insert the values (x_0, y_0) in place of (x, y) in the linear expression and divide by the square root of the sum of the squares of the coefficients of x and y in this expression.

2. Find the equation of the straight line through (−1, 3), (3, 2), and show that it passes through (11, 0).

3. Show that the lines

$$\begin{aligned} 3x - 2y + 7 &= 0 \\ 4x + y + 3 &= 0 \\ 19x + 13y &= 0 \end{aligned}$$

all pass through one point, and find its co-ordinates.

4. Find the equations of the lines through the origin parallel and perpendicular to the lines of Ex. 3; also those through the point (2, 2).

5. Find the equation of the line joining the feet of the perpendiculars from the origin upon the lines

$$\begin{aligned} 4x + y &= 17 \\ x + 2y &= 5. \end{aligned}$$

6. Draw the lines

$$\begin{aligned} 4y + 3x &= 12 \\ 3y + 4x &= 24. \end{aligned}$$

Find the equations of the bisectors of the angles between them, distinguishing the two lines.

7. The sides of a triangle are

$$\begin{aligned} x - y + 1 &= 0 \\ x - 4y + 7 &= 0 \\ x + 2y - 11 &= 0. \end{aligned}$$

Find
(i.) the co-ordinates of its angular points,
(ii.) the tangents of its angles,
(iii.) the equations of the internal and external bisectors of these angles.

8. The angular points of a triangle are at (0, 0) (2, 4) (−6, 8). Find
(i.) the equations of the sides,
(ii.) the tangents of the angles,
(iii.) the equations of the medians,
(iv.) the equations and lengths of the perpendiculars from the angular points on the opposite sides,
(v.) the equations of the lines through the angular points parallel to the opposite sides,
(vi.) the co-ordinates of the C.G.,
(vii.) the co-ordinates of the centres of the inscribed, circumscribed, and nine-points circles.

CHAPTER II

THE MEANING OF DIFFERENTIATION

§ 11. **The Idea of a Function.**

If two variable quantities are related to one another in such a way that to each value of the one corresponds a definite value of the other, the one is said to be a function of the other. The variables being x and y, we express this by the equation $y=f(x)$; in which case x and y are called the independent and dependent variables respectively. Analytical Geometry furnishes us with a representation of such functions of great use in the experimental sciences. The variables are taken as the coordinates of a point, and the curve, whose equation is

$$y=f(x),$$

gives us a picture of the way in which the variables change.

So far as we are concerned in these chapters the equation $y=f(x)$ may be assumed to give us a curve. There are, however, some peculiar functions which cannot thus be represented.

§ 12. **Examples from Physics and Dynamics.**

If a quantity of a perfect gas is contained in a cylinder closed by a piston the volume of the gas will alter with the pressure upon the piston. Boyle's Law expresses the relationship between the pressure p upon unit area of the piston, and the volume v of the gas, when the temperature remains unaltered. This law is given by the equation

$$pv=p_0v_0,$$

where p_0, v_0 are two corresponding values of the pressure and the volume. When the volume v for unit pressure is unity, this equation becomes

$$pv = 1,$$

and the rectangular hyperbola, whose equation is

$$xy = 1,$$

will show more clearly than any table of numerical values of p and v the way in which these quantities change.

When the pressure is increased past a certain point Boyle's Law ceases to hold, and the relation between p and v in such a case is given by van der Waals's equation :—

$$\left(p + \frac{a}{v^2}\right)(v - b) = 1,$$

a and b being certain positive quantities which have been determined by experiment for different gases. Inserting the values of a and b for the gas under consideration, and drawing the curve

$$\left(x + \frac{a}{y^2}\right)(y - b) = 1,$$

with suitable scales for x and y, the way in which p and v vary is made evident.

Such illustrations could be indefinitely multiplied. We add only two, taken from the case of the motion of a particle in a straight line.

When the *velocity is constant*, the distance s from a fixed point in the line to the position of the particle at time t is given by

$$s = vt + s_0,$$

where s_0 is the distance to the initial position of the particle, and v is the constant velocity.

The straight line

$$y = vx + s_0$$

then represents the relation between s and t.

When the *acceleration is constant*, the corresponding equation is

$$s = \tfrac{1}{2}ft^2 + v_0t + s_0,$$

where f = the acceleration,
v_0 = the initial velocity,
s_0 = the distance to the initial position.

In this case we have the parabola

$$y = \tfrac{1}{2}fx^2 + r_0 x + s_0.$$

Also in both these cases we might obtain an approximate value of s for a given value of t, or the value of t for a given value of s, by simple measurements in the figures representing the respective curves.

§ 13. **The Fundamental Problem of the Differential Calculus.**

The aim of the Differential Calculus is the investigation of the rate at which one variable quantity changes with regard to another, when the change in the one depends upon the change in the other, and the magnitudes vary in a continuous manner. The element of time does not necessarily enter into the idea of a rate, and we may be concerned with the rate at which the pressure of a gas changes with the volume, or the length of a metal rod with the temperature, or the temperature of a conducting wire with the strength of the electric current along it, or the boiling point of a liquid with the barometric pressure, or the velocity of a wave with the density of the medium, etc. etc. The simplest cases of rates of change are, however, those in which time does enter, and we shall begin our consideration of the subject with such examples.

§ 14. **Rectilinear Motion.**

In elementary dynamics the velocity of a point, which is moving uniformly, is defined as its rate of change of position, and this is equal to the quotient obtained by dividing the distance traversed in any period by the duration of the period, the distance being expressed in terms of a unit of length, and the period in terms of some unit of time.

When equal distances are covered in equal times this fraction is a perfectly definite one and does not depend upon the time, but when the rate of change of position is gradually altering, as, for instance, in the case of a body falling under gravity, the value of such a fraction alters with the length of the time considered. If, however, we note the distance travelled in different intervals measured from the time t, such intervals being taken smaller and smaller, we find that the values we obtain for what we might call the *average velocity* in these intervals are getting nearer and nearer to a definite quantity.

For example, in the case of the body falling from rest we have

$$s = \tfrac{1}{2}gt^2.$$

Let $(s + \delta s)$ be the distance which corresponds to the time $(t + \delta t)$.

These quantities δs and δt added to s and t are called the "increments" of these variables.

Then $$s + \delta s = \tfrac{1}{2}g(t + \delta t)^2 = \tfrac{1}{2}gt^2 + gt \,.\, \delta t + \tfrac{1}{2}g(\delta t)^2,$$

and $\therefore$ $$\frac{\delta s}{\delta t} = gt + \tfrac{1}{2}g\delta t.$$

It is clear that as δt gets smaller and smaller, the "average velocity" in the interval δt approaches nearer and nearer to the value gt. This value towards which the average velocity tends as the interval diminishes is called the *velocity at the instant t*, on the understanding that we can get an "average velocity" as near this as we please by taking the interval sufficiently small. The actual motion with these average velocities in the successive intervals would be a closer and closer approximation to the continually changing motion in proportion to the minuteness of the subdivisions of the time. The advantage of the method of the Differential Calculus is that it gives us a means of getting these "instantaneous velocities," or rates of change, at the time considered, and that, when the mathematical formula connecting the quantities is given, we can state what the rate of change of the one is with regard to the other, without being dependent upon an approximation obtained by a set of observations in gradually diminishing intervals.

§ 15. **Limits. Differential Coefficient.**

If a variable which changes according to some law can be made to approach some fixed constant value as nearly as we please, but can never become exactly equal to it, the constant is called the limit of the variable under these circumstances. Now if this variable is x, and the limiting value of x is a, the dependent variable y (where $y = f(x)$) may become more and more nearly equal to some fixed constant value b as x tends to its limit a, and we may be able to make y differ from b by as little as we please, by making x get nearer and nearer to a. In this case b is called the limit of the function as x approaches its limit a, or more shortly, the limit of the function for $x=a$, and this is written $Lt_{x=a}(y)=b$.

E.g. (i.) If
$$y=\frac{\sin x}{x},$$
$$Lt_{x=0}(y)=1.$$

(ii.) If
$$y=\frac{1}{x},$$
$$Lt_{x=0}(y)=\infty,$$

or, more correctly, y has no limit for $x=0$.*

In this last example the function increases without limit as x approaches its limit. We might have the corresponding case of x increasing without limit and the function having a definite limit: *e.g.* if

$$y=a^x \text{ where } 0<a<1,$$
$$Lt_{x=\infty}(y)=0.$$

This idea of a limit has already (§ 14) been employed, and when $s=\frac{1}{2}gt^2$, the velocity at the time t of the moving point is what we here define as

$$Lt_{\delta t=0}\left(\frac{\delta s}{\delta t}\right).$$

In the general case of motion when the relation between s and t is $s=f(t)$, we take the distance at the time $(t+\delta t)$ as $(s+\delta s)$, and we have

$$s+\delta s=f(t+\delta t),$$

or
$$\frac{\delta s}{\delta t}=\frac{f(t+\delta t)-f(t)}{\delta t}.$$

Hence the velocity at the time t is given by

$$v=Lt_{\delta t=0}\left(\frac{\delta s}{\delta t}\right)=Lt_{\delta t=0}\left\{\frac{f(t+\delta t)-f(t)}{\delta t}\right\}.$$

* For a full discussion of the idea of *limit*, see Gibson's *Calculus*, chapter iv.

This limiting value of the ratio of the increment of s to the increment of t as the increment of t approaches zero is called the differential coefficient of s with regard to t. Instead of writing $Lt_{\delta t=0}\left(\frac{\delta s}{\delta t}\right)$, *we use the symbol* $\frac{ds}{dt}$ *for this limiting value. It must, however, be carefully noticed that in this symbol ds and dt cannot, so far as we are here concerned, be taken separately, and that* $\frac{ds}{dt}$ *stands for the result of a definite mathematical operation, viz. the evaluation of the limiting value of the ratio of the corresponding increments of s and t, as the increment of t gets smaller and smaller.*

We shall see later in § 38 that there is another notation in which ds and dt are spoken of as separate quantities, but until that section is reached, it will be well always to think of the differential coefficient as the result of the operation we have just described. It is clear that if δt is very small, the corresponding increment of s, namely δs, will be very approximately given by $\frac{ds}{dt}\cdot\delta t$. Still it is not a true statement, but only an approximation, to say that in this case

$$\delta s = \frac{ds}{dt}\cdot\delta t.$$

This approximation may, however, be employed in finding the change in the dependent variable due to a *small* change in the independent variable, or the error in the evaluation of a function due to a *small* error in the determination of the variable, provided we know the differential coefficient of the function.

We add some examples in which the differential coefficients are to be obtained from the above definition, viz.—

If $$s = f(t), \quad \frac{ds}{dt} = Lt_{\delta t=0}\left\{\frac{f(t+\delta t)-f(t)}{\delta t}\right\}.$$

Ex. 1. If $s = at + b$, $\frac{ds}{dt} = a$.

2. If $s = at^2 + 2bt + c$, $\frac{ds}{dt} = 2(at + b)$.

3. If $\theta = \omega t$, $\frac{d\theta}{dt} = \omega$.

4. If $x = a \sin \omega t$, $\frac{dx}{dt} = a\omega \cos \omega t$.

§ 16. Geometrical Illustration of the Meaning of a Differential Coefficient.

In the last sections we have been led to the idea of a limiting value by the consideration of a moving particle, and have thus been brought to define the differential coefficient of s with regard to t.

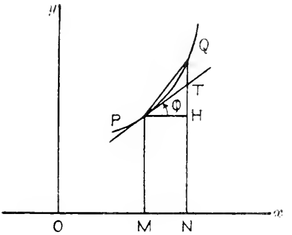

Fig. 6.

We have another illustration of the meaning of the differential coefficient in the consideration of the gradient, or slope, of the curve

$$y = f(x).$$

Let P be the point (x, y) and Q the point $(x + \delta x, y + \delta y)$, and let the tangent at P make an angle ϕ with Ox.

Then in Fig. 6

$$\left.\begin{aligned} OM &= x \\ ON &= x + \delta x \\ MN &= \delta x \end{aligned}\right\} \text{ and } \left.\begin{aligned} MP &= y = f(x) \\ MQ &= (y + \delta y) = f(x + \delta x) \\ HQ &= \delta y \quad\;\; = f(x + \delta x) - f(x) \end{aligned}\right\}$$

Thus the slope of the secant PQ

$$= \tan HPQ$$

$$= \frac{\delta y}{\delta x}$$

$$= \frac{f(x + \delta x) - f(x)}{\delta x}.$$

Now if we keep P fixed and let Q approach P, the secant PQ gets nearer and nearer the tangent at P, and the limiting value of the fraction $\frac{\delta y}{\delta x}$ as δx gets smaller and smaller is $\tan \phi$.

Thus, with the same notation as before

$$\frac{dy}{dx} = Lt_{\delta x = 0}\left(\frac{\delta y}{\delta x}\right) = Lt_{\delta x = 0}\left(\frac{f(x + \delta x) - f(x)}{\delta x}\right) = \tan \phi.$$

Since the slope of the tangent is known when $\frac{dy}{dx}$ is found, we can at once proceed to write down the equation of the tangent at a point on the curve $y = f(x)$, when the value of $\frac{dy}{dx}$ at that point is known.

Ex. 1. If $f(x) = e^{2x}$, write down $f(x \pm h)$; and show that

$$Lt_{h=0}\left(\frac{f(x+h)-f(x)}{h}\right) = Lt_{h=0}\left(\frac{f(x)-f(x-h)}{h}\right).$$

Interpret this result geometrically.

2. Find the value of $\frac{dy}{dx}$ at the point (2, 1) on the curve $4y = x^2$, and show that the equation of the tangent at that point to this parabola is

$$x - y = 1.$$

§ 17. Approximate Graphical Determination of the Differential Coefficient.

When the equation connecting x and y is such that the curve

$$y = f(x)$$

may be easily drawn, the slopes of the various positions of the secants PQ, as Q is made to move nearer and nearer to P, will give a series of values more and more nearly approximating to the value of $\frac{dy}{dx}$ at that point. An instructive example is the case of the curve

$$y = x^2,$$

in which the following table of values of δx, δy and $\frac{\delta y}{\delta x}$ can readily be obtained, and the way in which $\frac{\delta y}{\delta x}$ approaches its limiting value 2 at the point where $x = 1$ be made evident.

δx	1	·9	·8	·7	·6	·5	·4	·3	·2	·1	·09	·08	·07	·06	·05	·04	·03	·02	·01
δy	3	2·61	2·24	1·89	1·56	1·25	·96	·69	·44	·21	·1881	·1664	·1449	·1236	·1025	·0816	·0609	·0404	·0201
$\frac{\delta y}{\delta x}$	3	2·9	2·8	2·7	2·6	2·5	2·4	2·3	2·2	2·1	2·09	2·08	2·07	2·06	2·05	2·04	2·03	2·02	2·01

§ 18. In the chapters which immediately follow we shall show how to obtain the differential coefficients of the most important functions. This process of obtaining the differential coefficient is called differentiating the function. We shall see that in very many cases there is little difficulty in differentiation, and that the knowledge of the differential coefficients is of great value not only in geometry, but in the application of mathematics to physics.

From Fig. 7 it is obvious that when $\frac{dy}{dx}$ is positive the tangent is inclined at an acute angle to the axis of x, and y

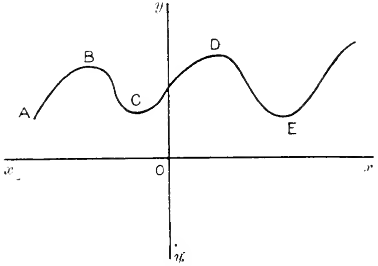

Fig. 7.

increases there with an increase of x, or decreases with a decrease in x. When $\frac{dy}{dx}$ is negative, the tangent is inclined at an obtuse angle to the axis of x, and y decreases as x increases, or *vice versa*. When $\frac{dy}{dx} = 0$, the tangent is parallel to this axis. Let us imagine the curve ABC to be a road, and that a traveller is marching along it in the positive direction of the axis of x, which is horizontal. When the traveller ascends, $\frac{dy}{dx}$ is positive; when he descends, $\frac{dy}{dx}$ is negative; and if the road is properly rounded off and no sharp corners occur, when he passes from ascending to descending, or the reverse, $\frac{dy}{dx}$ changes sign by passing through zero.

The acceleration of a moving point is defined in Dynamics as the rate of change of its velocity. Therefore, if we write v for the velocity at time t, the acceleration at that instant is $\frac{dv}{dt}$. If the position of the point at time t is given by $s=f(t)$,

then the velocity v $=\frac{ds}{dt}$

and the acceleration $=\frac{dv}{dt}=\frac{d}{dt}\cdot\left(\frac{ds}{dt}\right).$

The differential coefficient of the differential coefficient is called the second differential coefficient, and in the case of $s=f(t)$, is written $\frac{d^2s}{dt^2}$.

E.g. If

$$s=\tfrac{1}{2}gt^2,$$

$$\frac{ds}{dt}=gt,$$

and

$$\frac{d^2s}{dt^2}=g.$$

EXAMPLES ON CHAPTER II

The differential coefficients required in the examples on this chapter are to be obtained from the definition.

1. Plot the curves (i.) $y=x+x^2$
(ii.) $y=x^3$,
and show that they have the same gradient when $x=1$.

2. By considering the area of a square and the volume of a cube, show that the differential coefficients of x^2 and x^3 are $2x$ and $3x^2$ respectively.

3. Show that the curves $y=x^2$ and $y=x^4$ intersect at the origin and the points $(1, 1)$ $(-1, 1)$, and that at each of the two latter points the angle between the tangents is $\tan^{-1}\frac{2}{9}$.

4. Show that the gradient of the curve $y=x^3-3x$ at the point where $x=2$ is 9. Find the equation of the tangent there and trace the curve.

5. Find where the ordinate of the curve $y=3x-4x^2$ increases at the same rate as the abscissa, and where it decreases five times as fast as the abscissa increases.

6. If $s=ut-\frac{1}{2}gt^2$, find the values of the velocity and acceleration at the time t.

7. A cylinder has a height h ins. and a radius r ins.; there is a possible small error δr ins. in r. Find an approximate value of the possible error in the computed volume.

8. Find approximately the error made in the volume of a sphere by making a small error δr in the radius r. The radius is said to be 20 ins.; give approximate values of the errors made in the computed surface and volume if there be an error of ·1 in. in the length assigned to the radius. Also calculate the ratio of the errors in the radius, the surface, and the volume.

9. The area of a circular plate is expanding by heat. When the radius passes through the value 2 ins. it is increasing at the rate of ·01 in. per sec. Show that the area is increasing at the rate of $\cdot 04\pi$ sq. in. per sec. at that time.

10. The length of a bar at temperature 0° is unity. At temperature $t°$ its length l is given by the equation

$$l = 1 + at + bt^2,$$

find the rate at which the bar increases in length at temperature $t°$, and give an approximation to the increase in length due to a small rise in temperature.

11. If the diameter of a spherical soap-bubble increases uniformly at the rate of ·1 centimetre per second, show that the volume is increasing at the rate of $\cdot 2\pi$ cub. cent. per second when the diameter becomes 2 centimetres.

12. A ladder 24 feet long is leaning against a vertical wall. The foot of the ladder is moved away from the wall, along the horizontal surface of the ground and in a direction at right angles to the wall, at a uniform rate of 1 foot per second. Find the rate at which the top of the ladder is descending on the wall, when the foot is 12 feet from the wall.

CHAPTER III

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS; AND SOME GENERAL THEOREMS ON DIFFERENTIATION

§ 19. The Differential Coefficient of x^n.

Let $$y = x^n.$$

Then $$y + \delta y = (x + \delta x)^n$$

$$= x^n\left(1 + \frac{\delta x}{x}\right)^n,$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{x^n\left(1 + \frac{dx}{x}\right)^n - x^n}{\delta x}.$$

But by the Binomial Theorem, when $h < 1$,

$$(1 + h)^n = 1 + nh + \frac{n.n-1}{1.2}h^2 + \ldots$$

Therefore

$$\frac{\delta y}{\delta x} = \frac{x^n\left(1 + \frac{n}{x}\delta x + \frac{n.n-1}{1.2}\frac{(\delta x)^2}{x^2} + \ldots\right) - x^n}{\delta x},$$

$$\therefore \quad \frac{\delta y}{\delta x} = nx^{n-1} + \frac{n.n-1}{1.2}x^{n-2}\delta x + \ldots *$$

* The fact that we have an infinite series on the right hand sometimes causes difficulty to the student, as he imagines that what he calls the summing of the infinite number of small terms involving δx, $(\delta x)^2$, etc. . . . may give rise to a finite sum. The answer to this difficulty in general is to be found in a true view of the meaning of a convergent infinite series, but in the particular case of the Binomial Series we are able to say what the possible error by stopping after a

provided that δx is so small that

$$-1 < \frac{\delta x}{x} < +1.$$

Hence $$Lt_{\delta x=0}\left(\frac{\delta y}{\delta x}\right) = nx^{n-1},$$

and the differential coefficient of x^n is nx^{n-1}.

This is true whatever value n may have, provided it is independent of x.

Thus $$\frac{d(x^4)}{dx} = 4x^3,$$

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) = -\frac{4}{x^5},$$

$$\frac{d}{dx}\sqrt[3]{x^2} = \frac{2}{3\sqrt[3]{x}}.$$

§ 20. **General Theorems on Differentiation.**

Before proceeding to obtain the differential coefficients of other functions, it will be useful to show that many complicated expressions can be differentiated by means of this result, with the help of the following general theorems :—

PROPOSITION I. *Differentiation of a Constant.*

It is clear that, if $y = a$, the slope of the line is zero, and $\frac{dy}{dx} = 0$. In other words, it is obvious that if a magnitude remains the same its rate of change is zero.

Thus *the differential coefficient of a constant is zero.*

PROPOSITION II. *Differentiation of the Product of a Constant and a Function of x.*

Let $y = au$, where a is a constant, and u is a function of x.

When x becomes $x + \delta x$, let u become $u + \delta u$, and y become $y + \delta y$.

---

certain number of terms can be, and thus exclude the infinite series from our argument.

It is, however, worthy of note that the formula for the differential coefficient of x^n can be obtained without this series, by taking first of all n a positive integer and then using § 20 Prop. VI.

Then $$y + \delta y = a(u + \delta u),$$

and $$\frac{\delta y}{\delta x} = a\frac{\delta u}{\delta x}.$$

Therefore $$Lt_{\delta x=0}\left(\frac{\delta y}{\delta x}\right) = a\,Lt_{\delta x=0}\left(\frac{\delta u}{\delta x}\right),$$

or $$\frac{dy}{dx} = a\frac{du}{dx}.$$

∴ *The differential coefficient of the product of a constant and a function is equal to the product of the constant and the differential coefficient of the function.*

The geometrical meaning of this theorem is that if all the ordinates of a curve are increased in the same ratio, the slope of the curve is increased in the same ratio.

PROPOSITION III. *Differentiation of a Sum.*

Let $$y = u + v.$$

Then, as before, $$y + \delta y = (u + \delta u) + (v + \delta v),$$

and $$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}.$$

Proceeding to the limit,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

The same argument applies to the sum (or difference) of several functions, and we see that the *differential coefficient of such a sum is the sum of the several differential coefficients.*

Ex. Differentiate the following functions:—

(i.) $x(2+x)^2$

(ii.) $(a+bx+cx^2)\sqrt{x}$

(iii.) $\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+\frac{x}{1}+1-\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{3x^3}$

(iv.) $\frac{2+2x+3x^2}{\sqrt{x}}$.

PROPOSITION IV. *Differentiation of the Product of Two Functions.*

Let $$y = uv,$$

Then, as before, $$y + \delta y = (u + \delta u)(v + \delta v).$$

Thus $$\delta y = v\delta u + u\delta v + \delta u \,.\, \delta v,$$

and $$\frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \delta u \cdot \frac{\delta v}{\delta x}.$$

Proceeding to the limit,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx};$$

since as δx approaches its limiting value zero, δu approaches zero also, and $\frac{\delta u}{\delta x}$, $\frac{\delta v}{\delta x}$ become $\frac{du}{dx}$ and $\frac{dv}{dx}$.

This result may be written

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx},$$

and when $y = uvw$, we would obtain in the same way,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx}. \quad \text{(Cf. § 31.)}$$

In the case of two functions *it is easy to remember that the differential coefficient of the product of two functions is equal to the first function × the differential coefficient of the second + the second function × the differential coefficient of the first.*

Ex. Differentiate the following functions :—

(i.) $(1+x^2)(2x^2-1)$
(ii.) $(2x^2+1)(x+2)^2$
(iii.) $(ax+b)^2(cx+d)^2$
(iv.) $x(x+1)(x+2)$,

and show that the results are the same if the expressions be multiplied out and then differentiated.

PROPOSITION V. *Differentiation of a Quotient.*

Let $$y = u/v.$$

Then $$y + \delta y = \frac{u + \delta u}{v + \delta v},$$

and $$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v} = \frac{v\delta u - u\delta v}{v^2\left(1 + \frac{\delta v}{v}\right)}.$$

Therefore
$$\frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v^2\left(1 + \dfrac{\delta v}{v}\right)}.$$

Proceeding to the limit, it follows that
$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.*$$

In words, to *find the differential coefficient of a quotient, from the product of the denominator and the differential coefficient of the numerator subtract the product of the numerator and the differential coefficient of the denominator, and divide the result by the square of the denominator.*

Ex. Differentiate the following expressions :—

(i.) $\dfrac{x+1}{2-x}$ (ii.) $\dfrac{(x+1)(x+2)}{(x+3)}$ (iii.) $\dfrac{x}{(x+1)(x+2)}$

(iv.) $\dfrac{(x+1)^2}{(x+2)^3}$ (v.) $\dfrac{1+x^2}{1-x^2}$ (vi.) $\dfrac{ax^2+2bx+c}{ax^2-2bx+c}$.

These five formulæ, with the help of the result of § 19, enable us to differentiate a large number of expressions, but they do not apply directly to such cases as $\sqrt{a+x}$, $\sqrt{a^2+x^2}$, $\dfrac{1}{ax+b}$, $\dfrac{1}{(ax+b)^2}$, etc.

Each of the above expressions is a function of a function of x, and we proceed to prove the general theorem :—

PROPOSITION VI. *Differentiation of a Function of a Function.*

Let $y = F(u)$

where $u = f(x)$

$$\left(\begin{array}{ll} e.g. & y = \sqrt{u} \\ \text{where} & u = a^2 + x^2 \end{array}\right)$$

Then when x is changed to $x + \delta x$,

let u become $u + \delta u$,

and y become $y + \delta y$;

* This result may be obtained by writing
$$vy = u,$$
and then differentiating both sides of this equation.

the functions being such that for a small change in x, we have a definite and small change both in u and y.

But
$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x};$$

$\therefore$ proceeding to the limit,

$$Lt_{\delta x=0}\left(\frac{\delta y}{\delta x}\right)=Lt_{\delta u=0}\left(\frac{\delta y}{\delta u}\right)\times Lt_{\delta x=0}\left(\frac{\delta u}{\delta x}\right).$$

Thus
$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

Ex. 1. When $y=\dfrac{1}{(x+a)^2}$, we may put

$$\left.\begin{aligned}y&=\frac{1}{u^2}\\ \text{and } u&=x+a\end{aligned}\right\}$$

$$\therefore\quad \frac{dy}{dx}=\frac{d\left(\frac{1}{u^2}\right)}{du}\cdot\frac{du}{dx},\text{ where } u=x+a,$$

$$=-\frac{2}{u^3}\cdot 1$$

$$=-\frac{2}{(x+a)^3}.$$

2. When $y=(ax+b)^n$, prove $\dfrac{dy}{dx}=na(ax+b)^{n-1}$.

3. When $y=(1-x)\sqrt{1+x^2}$

$$\frac{dy}{dx}=(1-x)\frac{d}{dx}(\sqrt{1+x^2})+\sqrt{1+x^2}\frac{d}{dx}(1-x).$$

But
$$\frac{d}{dx}\sqrt{1+x^2}=\frac{d\sqrt{u}}{du}\cdot\frac{du}{dx},\text{ where } u=1+x^2,$$

$$=\left(\frac{1}{2\sqrt{1+x^2}}\right)(2x)$$

$$=\frac{x}{\sqrt{1+x^2}}$$

$$\therefore\quad\frac{dy}{dx}=\frac{(1-x)x}{\sqrt{1+x^2}}-\sqrt{1+x^2}$$

$$=-\frac{2x^2-x+1}{\sqrt{1+x^2}}.$$

4. When $y=\sqrt{\dfrac{ax+b}{cx+d}}$, prove $\dfrac{dy}{dx}=\dfrac{ad-bc}{2\sqrt{(ax+b)(cx+d)^3}}$.

EXAMPLES ON CHAPTER III

1. Find $\frac{dy}{dx}$ in the following cases:—

(i.) $y=\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^3$

(ii.) $y=\sqrt{2ax-x^2}$

(iii.) $y=\sqrt{(x+1)(x+2)}$

(iv.) $y=(x+a)^p(x+b)^q$

(v.) $y=\sqrt{\frac{1+x}{1-x}}$

(vi.) $y=\frac{(a-x)^p}{(b-x)^q}$

(vii.) $y=\frac{x^n}{a+bx^m}$

(viii.) $y=(1+x^n)^m$

(ix.) $y=\sqrt{x^2+a^2}+\sqrt{x^2-a^2}$

(x.) $y=\frac{1}{\sqrt{x^2+a^2}}+\frac{1}{\sqrt{x^2-a^2}}$

(xi.) $y=\frac{x^3}{\sqrt{(1-x^2)^3}}$

(xii.) $y=\sqrt{\frac{1+x+x^2}{1-x+x^2}}$

2. Find the gradient at the point (x_0, y_0) in the following curves:—

(i.) $y^2=4ax$

(ii.) $x^2+y^2=a^2$

(iii.) $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1$

(iv.) $2xy=c^2$.

3. Prove that the equations of the tangents at (x_0, y_0) to these curves are respectively

(i.) $yy_0=2a(x+x_0)$.

(ii.) $xx_0+yy_0=a^2$.

(iii.) $\frac{xx_0}{a^2}\pm\frac{yy_0}{b^2}=1$.

(iv.) $xy_0+yx_0=c^2$.

4. A boy is running on a horizontal plane in a straight line towards the base of a tower 50 yards high. How fast is he approaching the top, when he is 500 yards from the foot, and he is running at 8 miles per hour?

5. A light is 4 yards above and directly over a straight horizontal path on which a man six feet high is walking, at a speed of 4 miles per hour, away from the light.

Find (i.) The velocity of the end of his shadow;

(ii.) The rate at which his shadow is increasing in length.

6. A man standing on a wharf is drawing in the painter of a boat at the rate of 4 feet per second. If his hands are 6 feet above the bow of the boat, prove that the boat is moving at the rate of 5 feet per second when it is 8 feet from the wharf.

7. A vessel is anchored in 10 fathoms of water, and the cable passes over a sheave in the bowsprit which is 12 feet above the water. If the cable is hauled in at the rate of 1 foot per second, prove that the vessel is moving through the water at a rate of $1\frac{1}{4}$ feet per second when there are 20 fathoms of cable out.

8. If a volume v of a gas, contained in a vessel under pressure p, is compressed or expanded without loss of heat, the law connecting the pressure and volume is given by the formula

$$pv^{\gamma} = \text{constant},$$

where γ is a constant.

Find the rate at which the pressure changes with the volume.

9. In Boyle's Law, where $pv = c^2$, show that $\dfrac{dv}{dp} = -\dfrac{c^2}{p^2}$. What does the negative sign in this expression mean?

10. In van der Waals's equation

$$\left(p + \frac{a}{v^2}\right)(v - b) = \text{constant}.$$

Prove that

$$\frac{dv}{dp} = -\frac{(v-b)}{\left(p - \frac{a}{v^2} + \frac{2ab}{v^3}\right)}.$$

CHAPTER IV

THE DIFFERENTIATION OF THE TRIGONOMETRIC FUNCTIONS

(The angles are supposed to be measured in Radians)

§ 21. The Differential Coefficient of the Sine.

Let $$y = \sin x.$$

Then $$y + \delta y = \sin (x + \delta x),$$

and
$$\delta y = \sin (x + \delta x) - \sin x$$
$$= 2 \cos \left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}.$$

Therefore
$$\frac{\delta y}{\delta x} = \cos \left(x + \frac{\delta x}{2}\right) \left\{\frac{\sin \left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}\right\}$$

Proceeding to the limit, and remembering that

$Lt_{\theta=0}\left(\frac{\sin \theta}{\theta}\right) = 1$, it follows that

$$\frac{dy}{dx} = \cos x.$$

N.B.—When $y = \sin (mx + n)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ where } u = mx + n$$
$$= \frac{d(\sin u)}{du} \cdot \frac{du}{dx}$$
$$= \cos u \,.\, m$$
$$= m \cos (mx + n).$$

Ex. Prove from the definition of $\frac{dy}{dx}$, that when $y = \sin (mx + n)$,

$$\frac{dy}{dx} = m \cos (mx + n).$$

§ 22. The Differential Coefficient of the Cosine.

Let $y = \cos x.$

Then $y + \delta y = \cos (x + \delta x)$

and $\delta y = \cos (x + \delta x) - \cos x$

$$= -2 \sin \left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}.$$

Thus $$\frac{\delta y}{\delta x} = -\sin \left(x + \frac{\delta x}{2}\right)\left(\frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}\right).$$

Proceeding to the limit,

$$\frac{dy}{dx} = -\sin x.$$

N.B.—When $y = \cos (mx + n)$, $\frac{dy}{dx} = -m \sin (mx + n)$.

Ex. Prove from the definition of $\frac{dy}{dx}$, that when $y = \cos (mx + n)$,

$$\frac{dy}{dx} = -m \sin (mx + n).$$

§ 23. The Differential Coefficient of the Tangent.

Let $y = \tan x = \frac{\sin x}{\cos x}$.

Then $$\frac{dy}{dx} = \frac{\cos x \frac{d(\sin x)}{dx} - \sin x \frac{d(\cos x)}{dx}}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

N.B.—When $y = \tan (mx + n)$, $\frac{dy}{dx} = m \sec^2(mx + n)$.

Ex. Prove from the definition that when $y = \tan (mx + n)$,

$$\frac{dy}{dx} = m \sec^2(mx + n).$$

From these three results it is easy to deduce the following:—

$$\frac{d}{dx}\cdot\cot x = -\operatorname{cosec}^2 x\,;\quad \frac{d}{dx}\cdot\cot(mx+n) = -m\operatorname{cosec}^2(mx+n)$$

$$\frac{d}{dx}\cdot\sec x = \frac{\sin x}{\cos^2 x}\,;\quad \frac{d}{dx}\cdot\sec(mx+n) = m\frac{\sin(mx+n)}{\cos^2(mx+n)}$$

$$\frac{d}{dx}\cdot\operatorname{cosec} x = -\frac{\cos x}{\sin^2 x}\,;\quad \frac{d}{dx}\cdot\operatorname{cosec}(mx+n) = -m\frac{\cos(mx+n)}{\sin^2(mx+n)}.$$

§ 24. **Geometrical Proofs of these Theorems.**

All these cases of differentiation may be discussed geometrically. The method will be followed easily from the case of the tangent, which we now examine.

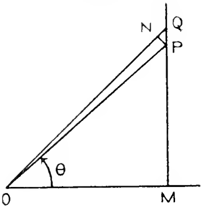

Fig. 8.

Let $\angle MOP$ be the angle θ radians, and let OM be 1 unit in length.

Let $\angle POQ$ be $\delta\theta$, and let QPM be perpendicular to the line OM from which θ is measured.

Let PN be perpendicular to OQ.

Then
$$\begin{aligned}\delta(\tan\theta) &= PQ\\ &= PN\sec\angle NPQ\\ &= PN\sec(\theta+\delta\theta)\\ &= OP\sec(\theta+\delta\theta)\sin\delta\theta\\ &= \sec\theta\sec(\theta+\delta\theta)\sin\delta\theta.\end{aligned}$$

Thus
$$\frac{\delta(\tan\theta)}{\delta\theta} = \sec\theta\sec(\theta+\delta\theta)\left(\frac{\sin\delta\theta}{\delta\theta}\right),$$

and proceeding to the limit,

$$\frac{d(\tan\theta)}{d\theta} = \sec^2\theta.$$

Examples. Find $\frac{dy}{dx}$ in the following cases:—

(i.) $y = 2a\sin(bx+c)\sin(bx-c)$.

(ii.) $y = x^2\cos 2x$.

(iii.) $y = \tan 3x + \cot 3x$.

(iv.) $y = \frac{\sin 2x - \sin x}{\cos x}$.

(v.) $y = x^m\sin^n x$.

(vi.) $y = x^m\sin nx$.

(vii.) $y = \sin^p x\cos^q x$.

(viii.) $y = \sec^2(ax+b) + \operatorname{cosec}^2(cx+d)$.

§ 25. **The Inverse Trigonometrical Functions.**

Since the sine of an angle varies continuously from -1 to $+1$ as the angle passes from $-\frac{\pi}{2}$ through zero to $+\frac{\pi}{2}$, it is convenient to take the inverse sine as lying in these two quadrants. In other words, for

$$y = \sin^{-1}x$$

we take that part of the curve

$$\sin y = x,$$

which lies between $y = -\frac{\pi}{2}$ and $y = +\frac{\pi}{2}$.

In this case, when

$$y = \sin^{-1}x \qquad \left(-\frac{\pi}{2} < y < \frac{\pi}{2}\right)$$

$$\sin y = x,$$

and differentiating, $$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x).$$

Therefore $$\frac{d(\sin y)}{dy} \cdot \frac{dy}{dx} = 1,$$

or $$\cos y \cdot \frac{dy}{dx} = 1$$

But $$\cos y = +\sqrt{1 - x^2}, \text{ since } -\frac{\pi}{2} < y < \frac{\pi}{2},$$

and therefore $$\frac{dy}{dx} = \frac{1}{+\sqrt{1 - x^2}}.$$

Hence $$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{+\sqrt{1 - x^2}}.$$

§ 26. **The Differentiation of the Inverse Cosine.**

In the case of $$y = \cos^{-1}x,$$

it is convenient to take y as lying between 0 and π, and in this case the equation

$$\cos y = x,$$

on differentiation, gives

$$\frac{d}{dy}(\cos y) \cdot \frac{dy}{dx} = 1,$$

or, $$-\sin y \cdot \frac{dy}{dx} = 1;$$

and since $$\sin y = +\sqrt{1-x^2} \qquad (0<y<\pi)$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}},$$

or $$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}.$$

This result may obviously be derived from that for $\sin^{-1}x$,

since $$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.$$

§ 27. **The Differentiation of the Inverse Tangent.**

In the case of the inverse tangent we get a complete set of values by taking y in the interval $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

When $$y = \tan^{-1}x$$

$$\tan y = x,$$

and differentiating,

$$\frac{d(\tan y)}{dy} \cdot \frac{dy}{dx} = 1,$$

or $$\sec^2 y \cdot \frac{dy}{dx} = 1.$$

Hence $$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}.$$

If the student will examine the graphs of the functions $\sin^{-1}x$, $\cos^{-1}x$, etc., he will see that without the above restrictions on the size of the angle there would be an ambiguity in the results for the sine, cosine, secant, and cosecant. For a given value of x, within the possible range of values for x, we have an infinite number of values of y, and at these points on the curve the gradients are equal in magnitude, but may be opposite in sign.

Ex. Write down the values of $\frac{dy}{dx}$ in the following cases :—

(i.) $y=\sin^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{x}{a}\right)$.

(ii.) $y=\sin^{-1}(1-x)$.

(iii.) $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

(iv.) $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

(v.) $y=x^2\sin^{-1}(x^2)$.

(vi.) $y=\tan^{-1}(x^3)$.

EXAMPLES ON CHAPTER IV

1. Differentiate the following functions :—

(i.) $\sin^3 x + \cos^3 x$.

(ii.) $\tan x + \frac{1}{3}\tan^3 x$.

(iii.) $\sec^2 x + \tan^2 x$.

(iv. $\text{cosec}^2 x + \cot^2 x$.

(v.) $\frac{1+\sin x}{1-\sin x}$.

(vi.) $\frac{1-\cos x}{1+\cos x}$.

2. If $y = \frac{\sin x}{1+\tan x}$, prove that $\frac{dy}{dx} = \frac{\cos^3 x - \sin^3 x}{(\cos x + \sin x)^2}$.

3. If $y = \cos(x^3)$, prove that $\frac{dy}{dx} = -3x^2 \sin(x^3)$, and find $\frac{dy}{dx}$ when

(i.) $y = x^m \sin x^n$.

(ii.) $y = x^m \cos x^n$.

(iii.) $y = x^m \tan x^n$.

4. Differentiate the following functions :—

(i.) $(x^2+1)\tan^{-1} x - x$.

(ii.) $x \sin^{-1} x + \sqrt{1-x^2}$.

(iii.) $\tan^{-1}\left(\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}\right)$ . . (Put $\sqrt{x} = \tan\theta$, $\sqrt{a} = \tan\alpha$.)

(iv.) $\tan^{-1}\left(\frac{1+x+x^2}{1-x+x^2}\right)$.

(v.) $\cot^{-1}\left(\frac{1+\sqrt{1+x^2}}{x}\right)$ . . (Put $x = \tan\theta$.)

5. A particle P is revolving with constant angular velocity ω in a circle of radius a. The line PM is drawn from P perpendicular to the line from the centre to the initial position of the particle. Find the velocity and acceleration of M.

6. If the position of a point is given at time t by the equations

$$x = a(\omega t + \sin \omega t),$$
$$y = a(1 - \cos \omega t),$$

where a and ω are constants, find its component velocities and accelerations, and its direction of motion at the time t.

7. Prove that when

$$x < -1, \quad \frac{d}{dx}(\sec^{-1} x) = -\frac{1}{x\sqrt{x^2-1}},$$

and that when

$$x > 1, \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}},$$

and illustrate your results from the graph of the inverse secant.

8. Prove that when

$$x > 1, \ \frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}},$$

and that when

$$x < -1, \ \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{1}{x\sqrt{x^2-1}},$$

and illustrate your results from the graph of the inverse cosecant.

CHAPTER V

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS—PARTIAL DIFFERENTIAL COEFFICIENTS—DIFFERENTIALS

§ 28. In this chapter we assume a knowledge of the properties of the following series :—

$$e^x = 1 + x + \frac{x^2}{2!} + \ . \ . \ .$$

$$a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \ . \ . \ .$$

which hold for all values of x, and

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ . \ . \ .$$

which holds when $-1 < x < +1$, using "$\log x$" for "$\log_e x$."

We shall now show how to differentiate e^x, a^x, $\log x$, and other functions whose differential coefficients may thus be obtained.

§ 29. **To differentiate e^x.**

Let $y = e^x$.

Then $y + \delta y = e^{x+\delta x}$,

$$\therefore \quad \delta y = e^{x+\delta x} - e^x,$$

$$= e^x(e^{\delta x} - 1),$$

$$= e^x\left(1 + \frac{\delta x}{1} + \frac{(\delta x)^2}{2!} + \ . \ . \ . - 1\right).$$

$$\therefore \quad \frac{\delta y}{\delta x} = e^x(1 + \frac{\delta x}{2!} + \frac{(\delta x)^2}{3!} + \ . \ . \ .)$$

Proceeding to the limit,

$$\frac{dy}{dx} = e^x.$$

Thus
$$\frac{d}{dx}(e^x) = e^x.$$

It follows that the equation

$$\frac{dy}{dx} = y$$

is satisfied by $y = ce^x$, where c is any quantity independent of x.

Ex. 1. If $y = ae^{bx}$, prove that $\frac{dy}{dx} = by$.

2. If $y = e^{x^2}$, prove that $\frac{dy}{dx} = 2xy$.

3. If $y = xe^{x^2}$, prove that $x\frac{dy}{dx} = (1 + 2x)y$.

4. Prove, *from the definition of the differential coefficient*, that if $y = ae^{bx}$,

$$\frac{dy}{dx} = abe^{bx}.$$

§ 30. **To differentiate log x.**

Let
$$y = \log x.$$

Then
$$y + \delta y = \log (x + \delta x),$$

and
$$\delta y = \log (x + \delta x) - \log x,$$

$$= \log \left(1 + \frac{\delta x}{x}\right),$$

$$= \frac{\delta x}{x} - \frac{1}{2}\left(\frac{\delta x}{x}\right)^2 + \frac{1}{3}\left(\frac{\delta x}{x}\right)^3 - \ldots \quad \text{if } \left(\frac{dx}{x}\right) < 1.$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{1}{x} - \frac{1}{2x}\left(\frac{\delta x}{x}\right) + \frac{1}{3x}\left(\frac{\delta x}{x}\right)^2 - \ldots$$

Proceeding to the limit,

$$\frac{dy}{dx} = \frac{1}{x}.$$

Ex. 1. If $y = a^x$, $\log y = x \log a$. Hence show that

$$\frac{d}{dx}(a^x) = a^x . \log_e a.$$

2. If $y = \log_a x$, prove that $\frac{dy}{dx} = \frac{\log_a e}{x} = \frac{1}{x . \log_e a}$.

3. If $y = \log (ax + b)$, prove that $\frac{dy}{dx} = \frac{a}{ax + b}$.

4. Prove the result of (3) from the definition of the differential coefficient.

5. If $y=\log\left(\frac{a-x}{b-x}\right)$, prove that $\frac{dy}{dx}=\frac{(a-b)}{(a-x)(b-x)}$.

6. If $y=\log(x+\sqrt{x^2\pm a^2})$, prove that $\frac{dy}{dx}=\frac{1}{\sqrt{x^2\pm a^2}}$.

7. If $y=\log\sqrt{\frac{a-b\cos\theta}{a+b\cos\theta}}$, prove that $\frac{dy}{d\theta}=\frac{ab\sin\theta}{a^2-b^2\cos^2\theta}$.

§ 31. Logarithmic Differentiation.

We have already obtained a general rule for the differentiation of a product or quotient. We are now able to prove another method which often leads more quickly to the result. This method is called *Logarithmic Differentiation.*

Let $$y = uvw.$$

Then $$\log y = \log u + \log v + \log w,$$

$$\therefore \quad \frac{d}{dx}(\log y) = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w),$$

$$\therefore \quad \frac{d}{dy}(\log y)\cdot\frac{dy}{dx} = \frac{d}{du}(\log u)\cdot\frac{du}{dx} + \frac{d}{dv}(\log v)\cdot\frac{dv}{dx} + \frac{d}{dw}(\log w)\cdot\frac{dw}{dx},$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}.$$

In other words, before differentiation of an expression involving the product or quotient or powers of other expressions, take logarithms of both sides of the given equation.

Ex. 1. If $y=\frac{(ax+b)^p(cx+d)^q}{(ex+f)^s}$, $\frac{1}{y}\cdot\frac{dy}{dx}=\frac{ap}{ax+b}+\frac{qc}{cx+d}-\frac{se}{ex+f}$.

2. If $y=\sqrt{\frac{1+x^2}{1-x^2}}$, $\frac{dy}{dx}=\frac{2x}{\sqrt{(1+x^2)(1-x^2)^3}}$.

3. If $y=\sqrt{\frac{a+2bx+cx^2}{a-2bx+cx^2}}$, $\frac{dy}{dx}=\frac{b(a-cx^2)}{(a-2bx+cx^2)^{\frac{3}{2}}(a+2bx+cx^2)^{\frac{1}{2}}}$.

§ 32. Important Example.

If $$y=e^{-ax}\sin bx,$$

$$\frac{dy}{dx} = \sin bx\,\frac{d}{dx}(e^{-ax}) + e^{-ax}\cdot\frac{d}{dx}(\sin bx)$$

$$= e^{-ax}(-a\sin bx + b\cos bx).$$

Now if $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$, a and b being positive,

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}},$$

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}},$$

and $$\frac{dy}{dx} = -\sqrt{a^2 + b^2} \, . \, e^{-ax} (\sin bx \cos \alpha - \cos bx \, . \, \sin \alpha),$$

$$= -\sqrt{a^2 + b^2} \, . \, e^{-ax} \sin (bx - \alpha).$$

Thus the tangent to $y = e^{-ax} \sin bx$ is parallel to the axis of x

when $$bx = n\pi + \alpha,$$

and the equation defines an oscillating curve with continually diminishing amplitude in the waves as we proceed along Ox.

It is easy to show that when

$$y = e^{ax} \sin (bx + c),$$

$$\frac{dy}{dx} = \sqrt{a^2 + b^2} \, e^{ax} \sin (bx + c + \alpha),$$

and that here the waves increase in amplitude; and corresponding results hold for the case of the cosine.

§ 33. **Maxima and Minima Values of a Function of one Variable.**

The student is already familiar with the graphical and algebraical discussion of the maxima and minima of certain simple algebraical expressions. The methods of the Differential Calculus are well adapted to the solution of such problems, since, if the graph of the function is supposed drawn, the *turning-points*, or places where the ordinate changes from increasing to decreasing, or *vice versa*, can only occur where the tangent is parallel to the axis of x, as in the points A_1, A_2 . . . of Fig 9, or where it is parallel to the axis of y as in the points B_1, B_2 . . ., except in such cases as the points C_1, C_2 . . ., where, although the curve is continuous, the gradient suddenly changes sign, without passing through the value zero or becoming infinitely great.

In case (A): $\frac{dy}{dx}$ is zero at the turning-point; and if this point is one at which the curve ceases to ascend and begins to descend, $\frac{dy}{dx}$ changes from being positive just before that point to being

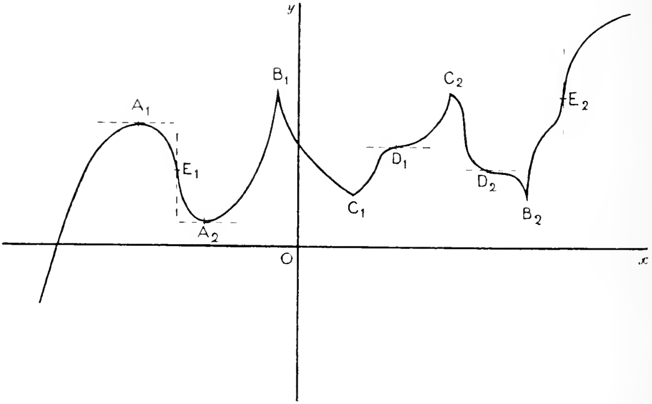

FIG. 9.

negative just after. At such a point the function is said to have a *maximum* value. In the other case, where the curve ceases to descend and begins to ascend, $\frac{dy}{dx}$ changes from negative to positive, and we have a *minimum*. In Fig. 9, at A_1 there is a *maximum*; at A_2 there is a *minimum*.

In case B: $\frac{dy}{dx}$ is infinitely great at the turning-point, and at B_1, where there is a *maximum*, it changes from positive to negative, while at B_2, where there is a *minimum*, it changes from negative to positive.

The other turning-points C_1, C_2 in Fig. 9 correspond to discontinuities in $\frac{dy}{dx}$, but it can be shown that these will not occur in the functions with which we are dealing.

§ 34. **Points of Inflection.**

Although the vanishing of $\frac{dy}{dx}$ is a *necessary* condition for a maximum or minimum, it is not a *sufficient* condition, since the gradient of the curve may become zero without changing its sign as we pass through that point. Examples of such points are to be found in D_1, D_2 of Fig. 9. In the case of D_1 the gradient is positive before and after the zero value; in the case of D_2 it is negative. At these points the curve *crosses* its *tangent*, and when this occurs, whether the tangent is horizontal or not, the point is called a *point of inflection.*

We cannot here discuss the analytical conditions for such a point in general; but in the cases of horizontal tangent (case D) we see that $\frac{dy}{dx}$ vanishes and *does not change sign*; and in the case of vertical tangent (case E), $\frac{dy}{dx}$ is infinitely great at the point, and *does not change sign* as we pass through it.

Ex. 1. Show that $y=ax^2+2bx+c$ has always *one* turning-point; and point out when it is a maximum and when it is a minimum.

2. Find the maximum and minimum ordinates of the curve $y=x^3-6x^2+12$, and also find the points of maximum gradient.

3. Find the turning-points of the curve $y=(x+1)^3(x-2)^4$, and show that $(-1, 0)$ is a point of inflection.

4. Find the turning-points of $y=\frac{(x-1)}{(x-2)}$.

§ 35. **Partial Differentiation.**

So far we have been considering functions of only one independent variable, $y=f(x)$. Cases occur in Geometry and in all the applications of the Calculus where the quantities which vary depend upon more than one variable. For instance, in Geometry the co-ordinates of any point (x, y, z) upon the sphere of radius a, whose centre is at the origin, satisfy the relation

$$x^2+y^2+z^2=a^2.$$

Hence we have

$$z^2=a^2-x^2-y^2,$$

and if we cut the sphere by a plane parallel to the yz plane, along the circle where this plane cuts the sphere x is constant and the change in z is due to a change in y only. In the section by a plane parallel to the zx plane, the change in z would be due to a change in x only. Similar results hold for other surfaces.

Again, the area of a rectangle whose sides are x in. and y in. is xy sq. in., and we may imagine the sides x and y to change in length independently of each other; while the volume of a rectangular box whose edges are x, y, and z in. is xyz cub. in., and x, y, z may be supposed to change here independently.

The ordinary gas equation

$$\frac{pv}{\mathrm{T}} = \text{constant}$$

is another example of the same sort of relation, and it would be easy to multiply these instances indefinitely.

§ 36. Let the equation

$$z = f(x, y)$$

express such a relation between two independent variables x and y, and a dependent variable z.

Let us suppose that the independent variable y is kept constant, and that x changes.

Then the rate at which z changes with regard to x, when y is kept constant, will be given by

$$Lt_{\delta x=0}\left\{\frac{f(x+\delta x, y) - f(x, y)}{\delta x}\right\}.$$

In the second case let x be kept constant and let y change. Then the rate at which z changes in this case is

$$Lt_{\delta y=0}\left\{\frac{f(x, y+\delta y) - f(x, y)}{\delta y}\right\}.$$

These two differential coefficients are called the Partial Differential Coefficients of z with regard to x and y respectively, and are written $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively.*

* It is hardly necessary to point that this symbol $\frac{\partial z}{\partial x}$ stands for an operation, and that ∂z, ∂y are not to be considered separately; also that this is a different notation from the δx of our earlier work.

Ex. 1. When $z=xy$, prove from the definition that $\frac{\partial z}{\partial x}=y$, and $\frac{\partial z}{\partial y}=x$.

2. When $2az=x^2+y^2$, prove from the definition that $\frac{\partial z}{\partial x}=\frac{x}{a}$, and $\frac{\partial z}{\partial y}=\frac{y}{a}$.

3. If $u=xyz$, prove from the definition that $\frac{\partial u}{\partial x}=yz$.

§ 37. Total Differentiation.

When the variables x and y in the above examples both depend upon a third variable t, say, z will vary in value as x and y change with t.

In the case
$$z = xy$$
$$z + \delta z = (x + \delta x)(y + \delta y)$$

and
$$\frac{\delta z}{\delta t} = y\frac{\delta x}{\delta t} + x\frac{\delta y}{\delta t} + \frac{\delta x}{\delta t} \cdot \delta y,$$

so that, proceeding to the limit,
$$\frac{dz}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}.$$

But
$$y = \frac{\partial z}{\partial x} \text{ and } x = \frac{\partial z}{\partial y}, \text{ when } z = xy,$$

therefore, in this case
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

In the second example,
$$2az = x^2 + y^2,$$

we find
$$2a\delta z = 2x\delta x + 2y\delta y + (\delta x)^2 + (\delta y)^2$$

and
$$a\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt},$$

so that again
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

It can be shown that this holds in general, but the proof of the theorem cannot be taken at this stage of our work.

The differential coefficient $\frac{dz}{dt}$ is called the Total Differential Coefficient in such cases, as compared with the Partial Differential Coefficient defined above.

As a special case, when $z = f(x, y)$ and y is a function of x, we obtain

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx},$$

and the left-hand side is called the Total Differential Coefficient of z with regard to x.

Also the result that when $z = f(x, y)$ and x, y are functions of t,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

may be used to obtain an approximation to the small change δz in z due to the small changes δx and δy in x and y, when t becomes $t + \delta t$.

For, as we have seen already (p. 19),

δx will be approximately $\frac{dx}{dt} \cdot \delta t$,

δy ,, ,, $\frac{dy}{dt} \cdot \delta t$,

δz ,, ,, $\frac{dz}{dt} \cdot \delta t$,

and we thus have, on multiplying the above equation by δt,

$$\delta z = \frac{\partial z}{\partial x} \cdot \delta x + \frac{\partial z}{\partial y} \cdot \delta y.$$

§ 38. **Differentials.***

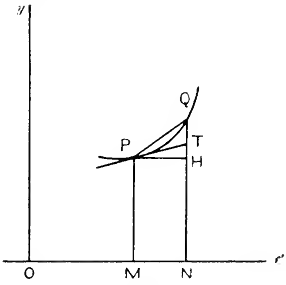

Fig. 10.

In the case of the curve $y = f(x)$ the increment δy of y which corresponds to the increment δx of x is given in Fig. 10 by HQ.

Also

$$HQ = HT + TQ = \delta x \cdot \frac{dy}{dx} + TQ,$$

$$\therefore\ \delta y = \delta x \cdot \frac{dy}{dx} + TQ.$$

As δx gets smaller and smaller, TQ gets smaller and smaller, at least in the neigh-

* § 38 may be omitted on first reading.

bourhood of P; and the "small quantity" TQ is a smaller "small quantity" than δx, since

$$\frac{\delta y}{\delta x}=\frac{dy}{dx}+\frac{TQ}{\delta x},$$

and in the limit $\frac{\delta y}{\delta x}$ is equal to $\frac{dy}{dx}$, so that $\frac{TQ}{\delta x}$ must disappear in the limit. In mathematical language, if δx is an infinitesimal (or small quantity) of the first order, TQ will be at least an infinitesimal of the second order.

It is convenient to have a name and symbol for this quantity $\frac{dy}{dx}\cdot\delta x$. The name adopted is the "differential of y," and the symbol is "dy."

Hence with this definition of the term "differential,"

$$dy=\left(\frac{dy}{dx}\right)\cdot\delta x,$$

where we have enclosed $\frac{dy}{dx}$ in brackets on the right-hand side so that it may be clear that this stands for the differential coefficient obtained by the processes we have been developing in the preceding pages.

By the above definition

$$d(f(x))=f'(x)\,.\,\delta x,\text{ where }f'(x)=\frac{dy}{dx};$$

and $$dx=\delta x.$$

So that $$dy=f'(x)\,.\,dx,\text{ when }y=f(x).$$

Hence we may restate our definition as follows :—

The differential of the independent variable is the actual increment of that variable.

The differential of a function is the differential coefficient of the function multiplied by the differential of the independent variable.

In this definition it is not necessary to assume that the differentials are small quantities or infinitesimals, but in all the applications of this notation this assumption is made. In that case the equation

$$dy=f'(x)dx$$

will give the increment of y, if small quantities of the second order be neglected.

Such an equation as

$$dy = f'(x)dx,$$

a differential equation as it is called, may be used in this way to give the approximate change in the dependent variable, and from this point of view it saves the trouble of writing down the equation between the increments, and then cutting out the terms whose smallness is such that they may be neglected.

Ex. 1. Write down a table of differentials corresponding to the standard differential coefficients.

$$e.g.\ d(x^n) = nx^{n-1}dx.$$

2. If $x = a\cos\theta$, $y = a\sin\theta$, prove by differentials that $\dfrac{dy}{dx} = -\cot\theta$.

3. If $x = a(\omega t + \sin\omega t)$, $y = a(1 - \cos\omega t)$, prove that $\dfrac{dy}{dx} = \dfrac{\sin\omega t}{1 + \cos\omega t}$.

4. If $z = xy$, prove that $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$.

EXAMPLES ON CHAPTER V

1. Find the differential coefficients of

$$\text{(i.) } xe^x,\quad \text{(ii.) } x^m e^{nx},\quad \text{(iii.) } (ax+b)e^{cx+d},\quad \text{(iv.) } e^{x\sin^{-1}x}.$$

2. Find the differential coefficients of

$$\text{(i.) } e^{1+x^2},\quad \text{(ii.) } x^2e^{ax^2},\quad \text{(iii.) } x^m e^{ax^n},\quad \text{(iv.) } x^m a^{x^n}.$$

3. Find the differential coefficients of

$$\text{(i.) } x^n\log x,\quad \text{(ii.) } \log\left(\frac{x+1}{x+2}\right),\quad \text{(iii.) } \log(\sqrt{x+1}+\sqrt{x-1}),\quad \text{(iv.) } \log\left(\frac{1-x^2}{4-x^2}\right);$$

$$\text{(v.) } \log\left(\frac{x}{\sqrt{x^2+1}-x}\right),\quad \text{(vi.) } \log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right).$$

4. Differentiate the following expressions logarithmically :—

$$\text{(i.) } \sqrt{(2x+1)(x-2)},\quad \text{(ii.) } \frac{x}{\sqrt{a^2+x^2}},\quad \text{(iii.) } \frac{1}{x^2(x-1)^3},\quad \text{(iv.) } x^x,$$

$$\text{(v.) } \frac{\sin^n mx}{\cos^m nx},\quad \text{(vi.) } \left(1+\frac{1}{x}\right)^x;$$

and point out why we cannot apply our formula for the differential coefficient of x^n to the case of x^x.

5. If $y = \dfrac{1}{\sqrt{ac-b^2}}\tan^{-1}\left(\dfrac{ax+b}{\sqrt{ac-b^2}}\right)$, prove that $\dfrac{dy}{dx} = \dfrac{1}{ax^2+2bx+c}$. $(ac > b^2.)$

6. If $y = \dfrac{1}{6}\log\dfrac{(x+1)^3}{(x^3+1)} + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right)$, prove that $\dfrac{dy}{dx} = \dfrac{1}{x^3+1}$.

7. If $y = 2\cos^{-1}\sqrt{\dfrac{\alpha - x}{\alpha - \beta}}$, prove that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(\alpha - x)(x - \beta)}}$. $(\alpha > x > \beta.)$

8. If $y = \dfrac{2}{\sqrt{\beta - \alpha}}\cos^{-1}\sqrt{\dfrac{\alpha - x}{\beta - x}}$, prove that $\dfrac{dy}{dx} = \dfrac{1}{(\beta - x)\sqrt{\alpha - x}}$. $(x < \alpha < \beta.)$

9. If $y = \log\left(\dfrac{b + a\cos x + \sqrt{b^2 - a^2}\sin x}{a + b\cos x}\right)$, prove that $\dfrac{dy}{dx} = \dfrac{\sqrt{b^2 - a^2}}{a + b\cos x}$.

$(b^2 > a^2.)$

10. If $r = \dfrac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left\{\sqrt{\dfrac{a - b}{a + b}}\tan\dfrac{\theta}{2}\right\}$, prove that $\dfrac{dr}{d\theta} = \dfrac{1}{a + b\cos\theta}$.

$(a^2 > b^2.)$

11. In the curves whose equations in polar co-ordinates are (i.) $r = ae^{\theta\cot\alpha}$, (ii.) $r^n = a^n\sin n\theta$, (iii.) $r^n = a^n\cos n\theta$, (iv.) $r^n = a^n\sec n\theta$, (v.) $r^n = a^n\operatorname{cosec} n\theta$, find $r\dfrac{d\theta}{dr}$. Can you give any geometrical meaning to this expression?

12. If $y = e^{-2x}\sin(2x + 1)$, prove that $\dfrac{dy}{dx} = 2\sqrt{2}\cdot e^{-2x}\cos\left(2x + 1 + \dfrac{\pi}{4}\right)$.

13. Find the value of $\dfrac{dy}{dx}$ in the following curves; discuss the way in which it changes as x passes along the axis; and find the turning-points, if there are any, of each curve:—

(i.) $y = x(x - 1)^2$.

(ii.) $y = x^2(x - 1)^3$.

(iii.) $y = (x - 1)^2(x - 2)^2$.

(iv.) $y = \dfrac{x^2 + x + 1}{x}$.

(v.) $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$.

(vi.) $y = \dfrac{(x - 1)(x - 2)}{x^2 + x + 1}$.

(vii.) $y = \dfrac{x^2 + x + 1}{(x - 1)(x - 2)}$.

(viii.) $y = \dfrac{(x - 1)(x - 3)}{(x - 2)(x - 4)}$.

(ix.) $y = \dfrac{(x - 1)(x - 2)}{(x - 4)}$.

(x.) $y = \dfrac{x^3 + 1}{x^2}$.

[These curves are discussed algebraically and drawn to scale in Chrystal's *Introduction to Algebra*, pp. 391-404. The student is recommended to compare his results with those to be deduced from these figures.]

14. If $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$.

15. If $z = \tan^{-1}\left(\dfrac{2xy}{x^2 + y^2}\right)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.

16. The formula for the index of refraction μ of a gas at temperature θ° and pressure p is

$$\mu - 1 = \frac{\mu_0 - 1}{1 + a\theta} \cdot \frac{p}{760},$$

where μ_0 = the index of refraction at 0°,
a = the coefficient of expansion of the gas.

Prove that the effect of small variations $\delta\theta$ and δp of the temperature and pressure on the index of refraction is to cause it to vary by an amount

$$\delta\mu = \frac{\mu_0 - 1}{760}\left(\frac{\delta p}{1 + a\theta} - \frac{pa\delta\theta}{(1 + a\theta)^2}\right).$$

17. If $pv = R\theta$ is the ordinary gas equation, where $\theta = 1 + at$, write down the values of

(i.) $\frac{\partial p}{\partial v}$,

(ii.) $\frac{\partial p}{\partial \theta}$,

(iii.) The approximate increase in the pressure due to a small decrease in the volume, the temperature being unchanged,

(iv.) The approximate increase in the volume due to a small increase in the temperature, the pressure remaining the same,

(v.) The approximate increase in the pressure due to a small increase in both temperature and volume.

18. Assuming that the H.P. required to propel a steamer of a given design varies as the square of the length and the cube of the speed, prove that a 2% increase in length, with a 7% increase in H.P., will result in a 1% increase in speed.

19. The area of a triangle is calculated from measurements of two sides and their included angle. Determine the error in the area arising from small errors in these measurements.

20. Assuming that the area of an ellipse whose semiaxes are a and b inches is πab sq. in., and that an elliptical metal plate is expanded by heat or pressure, so that when the semiaxes are 4 and 6 inches, each is increasing at the rate ·1 in. per second, prove that the area of the plate is increasing at the rate of π sq. in. per second.

CHAPTER VI

THE CONIC SECTIONS *

§ 39. In this chapter we shall very briefly examine the properties of the Conic Sections, or the curves in which a plane cuts a Right Circular Cone. It is shown in the Geometry of Conics that these curves are the loci of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line. The fixed point S is called the focus; the fixed line, the directrix; and the constant ratio, e, the eccentricity.

When $e < 1$, the curve is called an Ellipse;
when $e = 1$, the curve is called a Parabola;
when $e > 1$, the curve is called a Hyperbola;
and the circle is a special case of the ellipse, the eccentricity being zero, and the directrix at infinity.

§ 40. **The Parabola** ($e = 1$).

(i.) *To find its equation.*

Let the focus S be the point $(a, 0)$, and the directrix the line $x + a = 0$ (Fig. 11).

Let P be the point (x, y).

Then since $SP^2 = PM^2$,

$$(x - a)^2 + y^2 = (x + a)^2,$$

$$\therefore \qquad y^2 = 4ax.$$

* The student is referred for a fuller discussion of the properties of the conic sections to the books mentioned on p. 12.

This is the equation of the parabola with the origin at the point where the curve cuts the perpendicular from S on the directrix. This point is called the vertex of the curve; the

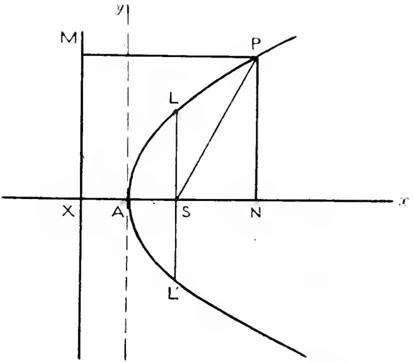

Fig. 11.

axis of x is called the axis of the curve; and the ordinate L′SL through the focus is called the Latus Rectum.

(ii.) *The shape of the curve.*

From the form of the equation of the curve we see that the curve lies wholly to the right of the axis of y, and that it is symmetrical with regard to the axis of x.

Also since
$$y\frac{dy}{dx} = 2a,$$

$$\frac{dy}{dx} = \frac{2a}{y} = \sqrt{\frac{a}{x}} \text{ when } y > 0.$$

It follows that the tangent at the vertex coincides with the axis of y, and that as we move along the curve in the direction of x increasing, the curve continually ascends, the slope getting less and less the greater x becomes.

(iii.) *The equations of the tangent and normal at* (x_0, y_0).

Since the value of $\frac{dy}{dx}$ at (x_0, y_0) is $\frac{2a}{y_0}$, the equation of the tangent there is

$$\frac{y - y_0}{x - x_0} = \frac{2a}{y_0},$$

or

$$y_0(y - y_0) = 2a(x - x_0),$$

which becomes $yy_0 = 2a(x + x_0)$, since $y_0^2 = 4ax_0$.

Also the normal is the line

$$y_0(x - x_0) + 2a(y - y_0) = 0,$$

since this passes through (x_0, y_0) and is perpendicular to the tangent.

EXAMPLES ON THE PARABOLA

1. Show that the curves $x^2 = \pm 4y$ are parabolas, and plot the curves.

2. Show that the equation $y = ax^2 + 2bx + c$ always represents a parabola, and plot the curves

(i.) $y = x^2 + 4x + 3$,
(ii.) $4y = x^2 + 4x - 8$,
(iii.) $x = y^2 + y$.

Find also

(i.) The co-ordinates of their foci;
(ii.) The co-ordinates of their vertices;
(iii.) The equations of their latera recta;
(iv.) The lengths of their latera recta;
(v.) The equations of their axes;
(vi.) The equations of the tangents at their vertices.

3. Find algebraically and graphically the minimum value of the expression $x^2 - 2x - 4$, and the maximum value of $5 + 4x - 2x^2$.

4. The tangent at P meets the axis of the parabola of Fig. 11 in T, and the normal meets the axis in G. Prove the following properties:—

(i.) AN = AT,
(ii.) SP = ST = SG,
(iii.) NG = 2AS,

and show that the tangents at the ends of a focal chord meet at right angles on the directrix.

5. Prove that the line $y = x + 1$ touches the parabola $y^2 = 4x$, and that the line $y = mx + \frac{a}{m}$ touches the parabola $y^2 = 4ax$. Find the point of contact in each case.

6. Find the equations of the tangent and normal at the point where the line $x = 2$ cuts the parabola $x^2 = 4y$.

7. Find the equations of the tangents and normals at the extremities of the latus rectum of the parabola $y^2 = 4ax$, and show that they form a square.

8. Prove that the locus of the middle points of the chords of the parabola $y^2=4ax$, which make an angle θ with the axis of x, is the straight line

$$y=2a \cot \theta.$$

9. The chord PQ meets the axis of the parabola of Fig. 11 in O. PM and QN are the ordinates of P and Q. Prove that AM · AN = AO², by finding the equation of the chord in its simplest form.

10. The position of a moving point is given by the equations

$$x=v \cos \alpha \cdot t,$$
$$y=v \sin \alpha \cdot t-\tfrac{1}{2}gt^2.$$

Interpret the equations, and prove that the point moves on a parabola

whose axis is parallel to the axis of y;

whose vertex is at the point $\left(\frac{v^2 \sin \alpha \cos \alpha}{g}, \frac{v^2 \sin^2 \alpha}{2g}\right)$;

whose directrix is the line $y=\frac{v^2}{2g}$;

and whose latus rectum is of length $\frac{2v^2 \cos^2 \alpha}{g}$.

§ 41. The Ellipse ($e<1$).

(i.) *To find its equation.*

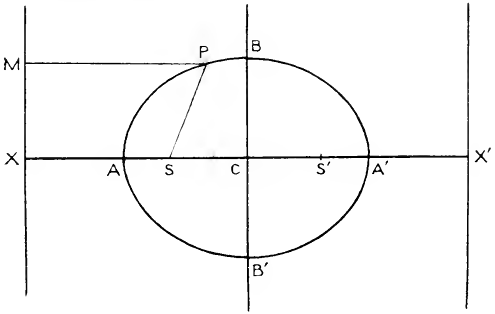

Fig. 12.

Let the axis of x be the axis of the ellipse (*i.e.* the line through the focus perpendicular to the directrix);

and S the point $(d, 0)$;

the axis of y the directrix.

Let P (x, y) be any point upon the curve.

Then $$SP^2 = e^2PM^2.$$

$$\therefore \quad (x-d)^2+y^2=e^2x^2,$$

$$\therefore \quad x^2(1-e^2) - 2xd + y^2 = -d^2,$$

$$\therefore \quad \left(x - \frac{d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{d^2}{(1-e^2)^2} - \frac{d^2}{1-e^2} = \frac{d^2e^2}{(1-e^2)^2}.$$

Now change the origin to the point $\left(\frac{d}{1-e^2}, 0\right)$ keeping the axes parallel to their original directions.

The equation of the ellipse then becomes

$$x^2 + \frac{y^2}{(1-e^2)} = \frac{d^2e^2}{(1-e^2)^2},$$

i.e.

$$\frac{x^2}{\frac{d^2e^2}{(1-e^2)^2}} + \frac{y^2}{\frac{d^2e^2}{(1-e^2)}} = 1.$$

Putting

$$a^2 = \frac{d^2e^2}{(1-e^2)^2}$$

and

$$b^2 = \frac{d^2e^2}{1-e^2},$$

we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1-e^2).$$

In this form the origin C is called the centre of the curve, since it bisects every chord which passes through it. This is clear, since if (x_1, y_1) lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so does $(-x_1, -y_1)$.

Also we notice that $CS = \frac{d}{1-e^2} - d = \frac{de^2}{1-e^2} = ae,$

and that

$$CX = \frac{d}{1-e^2} = \frac{a}{e}.$$

From the symmetry of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

it is clear that there is another focus, namely, the point $(ae, 0)$; and another directrix, the line $x = \frac{a}{e}$, with regard to the axes through the point C.

The axis of x is in this case called the major axis, and the axis of y the minor axis. The one is of length $2a$; the other of length $2b$. If b had been greater than a, the foci would have lain upon the axis of y, and this axis would have been the major axis. When a and b are given the eccentricity e is given by

$$b^2 = a^2(1 - e^2). \quad (a > b.)$$

In the circle $a = b$, and $e = 0$.

(ii.) *The shape of the curve.*

Since the equation involves only the terms x^2 and y^2, the curve is symmetrical about both the axes of x and y.

Also, since $y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$, we see that x must lie between $-a$ and $+a$, and that as x passes from $-a$ to $+a$ the positive value of y gradually increases from zero to b, and then diminishes again to zero.

The curve is thus a closed curve, lying altogether within the rectangle $x = \pm a$, $y = \pm b$.

This is also evident from the property of Ex. 3, p. 59, where it is stated that the curve may be drawn by fixing the two ends of a string of length $2a$ to the points S and S′, and holding the string tight by the point P of the tracing pencil.

(iii.) *The equations of the tangent and normal at* (x_0, y_0).

Since $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0.$$

Therefore the equation of the tangent at (x_0, y_0) is

$$\frac{y - y_0}{x - x_0} = -\frac{b^2 x_0}{a^2 y_0}$$

which becomes

$$(x - x_0)\frac{x_0}{a^2} + (y - y_0)\frac{y_0}{b^2} = 0,$$

or $$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1, \text{ since } \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1.$$

It follows that the equation of the normal is

$$(x - x_0)\frac{y_0}{b^2} - (y - y_0)\frac{x_0}{a^2} = 0,$$

or
$$\frac{xy_0}{b^2} - \frac{yx_0}{a^2} = x_0y_0\left(\frac{a^2 - b^2}{a^2b^2}\right),$$

or
$$\frac{a^2x}{x_0} - \frac{b^2y}{y_0} = a^2 - b^2.$$

EXAMPLES ON THE ELLIPSE

1. Trace the ellipses (i.) $3x^2+4y^2=12$;
(ii.) $3(x-1)^2+4(y-2)^2=12$;
(iii.) $x^2+4y^2=8y$;
(iv.) $4x^2+3y^2=12$;
and find the co-ordinates of the foci and of the extremities of the axes, the length of the latus rectum, and the eccentricity of each.

2. In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that the co-ordinates of any point may be expressed as $x=a\cos\theta$, $y=b\sin\theta$; and interpret the result geometrically.

3. P is the point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that $SP=a+ex_1$, and $S'P=a-ex_1$, and deduce that the curve is the locus of the point which moves so that the sum of its distances from two fixed points is constant.

4. The tangent at P meets the major axis in T, and PN is the ordinate of P, prove that $CN \cdot CT=CA^2$.

5. The normal at P meets the major axis in G. Prove that $SG:SP=e$, and deduce that PG bisects the angle SPS′.

6. Prove that the middle point of the chord $y=x+1$ lies upon $y=-\frac{b^2}{a^2}x$, and that the middle points of chords parallel to $y=mx$ lie upon the chord $y=m'x$, where $mm'+\frac{b^2}{a^2}=0$.

7. If CP bisects chords parallel to CD, prove that CD bisects chords parallel to CP (CP and CD are then said to be *conjugate diameters*); and prove that the tangents at P and D form with CP and CD a parallelogram.

8. If P is the point $(a\cos\theta, b\sin\theta)$, prove that CD is the line $a\sin\theta \cdot y + b\cos\theta \cdot x=0$, and deduce that $CP^2+CD^2=a^2+b^2$.

§ 42. **The Hyperbola** $(e>1)$. (i.) *To find its equation.*

Proceeding as in § 41 (i.) we obtain the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where we have written a^2 for $\frac{d^2e^2}{(e^2-1)^2}$, and $$b^2 \text{ for } \frac{d^2e^2}{e^2-1}, \text{ i.e. for } a^2(e^2-1),$$

and d is the distance from the focus S to the directrix.

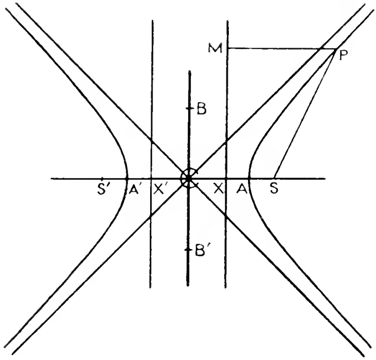

FIG. 13.

It follows that $CS = ae$, $CX = \frac{a}{e}$, and that there are two foci and two directrices.

The line joining the foci S, S′ is called the transverse axis of the hyperbola.

(ii.) *The shape of the curve.*

The form of the equation shows that the curve is symmetrical about both axes, and since $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$ it is clear that x cannot lie between $-a$ and $+a$, while since $x^2 = a^2\left(1 + \frac{y^2}{b^2}\right)$, y can have any value whatsoever.

If we write the equations as

$$\frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2},$$

we see that, when x is numerically very great, $\frac{y^2}{x^2}$ is less than, but very nearly equal to $\frac{b^2}{a^2}$; and that for all points on the curve $\frac{y^2}{x^2}$ is less than $\frac{b^2}{a^2}$.

Also the value of y decreases as x passes from $-\infty$ to $-a$, where it vanishes, and it increases without limit from the value zero at $x = a$ as x passes along the positive axis of x.

The shape of the curve is thus as in the figure. The lines $y = \pm \frac{b}{a} x$ are called the asymptotes, and the curve lies wholly between those lines; while, as the numerical value of x gets greater and greater, it approaches more and more nearly to these lines without ever actually reaching them.

(iii.) *The equations of the tangent and normal at* (x_0, y_0) are easily shown to be

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1,$$

and
$$\frac{y_0}{b^2}(x - x_0) + \frac{x_0}{a^2}(y - y_0) = 0.$$

(iv.) *The product of the perpendiculars from any point on the curve to the asymptotes is constant.*

The asymptotes are the lines $y = \pm \frac{b}{a} x$. Then if PM, PN are the perpendiculars to these lines from the point (x_0, y_0),

$$\text{PM} = \frac{-y_0 + \frac{b}{a} x_0}{\sqrt{1 + \frac{b^2}{a^2}}}, \quad \text{PN} = -\frac{y_0 + \frac{b}{a} x_0}{\sqrt{1 + \frac{b^2}{a^2}}}.$$

Therefore
$$\text{PM} \,.\, \text{PN} = \frac{b^2 x_0^2 - a^2 y_0^2}{a^2 + b^2} = \frac{a^2 b^2}{a^2 + b^2},$$

since
$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1.$$

Hence PM . PN = constant.

Now when $b^2 = a^2$, the asymptotes are at right angles, and the eccentricity is $\sqrt{2}$. In this case, by taking the asymptotes as axes, the equation $x^2 - y^2 = a^2$ is transformed to

$$2xy = a^2.$$

This equation is of the form $xy = c^2$, a relation which is of the greatest importance in Physics. We could obtain an equation of the same form for any hyperbola referred to its asymptotes as oblique axes.

EXAMPLES ON THE HYPERBOLA

1. Trace the hyperbolas :

(i.) $3x^2 - 4y^2 = 12$,
(ii.) $3(x-1)^2 - 4(y-2)^2 = 12$,
(iii.) $x^2 - 4y^2 = 8y$,
(iv.) $4x^2 - 3y^2 = 12$;

and find the co-ordinates of the foci and of the points where each curve cuts its transverse axis, the length of the latus rectum, and the eccentricity of each.

2. Trace the rectangular hyperbolas :

(i.) $xy = \pm 4$,
(ii.) $y = 1 \pm \frac{1}{x}$,

and find the co-ordinates of the foci and of the points where the transverse axis meets each curve.

3. Prove that the tangent at (x_0, y_0) to the hyperbola $xy = c^2$ is $xy_0 + yx_0 = 2c^2$, and that the point of contact bisects the part of the tangent cut off by the asymptotes.

4. P is the point (x_1, y_1) on the hyperbola whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Prove that $SP = ex_1 - a$, and $S'P = ex_1 + a$, and deduce that the curve is the locus of a point which moves so that the difference of its distances from two fixed points is constant.

5. The tangent at P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis in T, and PN is the ordinate of P. Prove that $CN \cdot CT = a^2$.

6. The normal at P meets the major axis in G ; show that $SG = eSP$, and deduce that PG bisects the angle SPS′.

7. Prove that, in the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the middle point of the chord $y = x + 1$ lies upon the line $y = \frac{b^2}{a^2}x$, and that the locus of the middle points of chords parallel to $y = mx$ is the line $y = m'x$, where $mm' = \frac{b^2}{a^2}$.

8. If CP and CD are two conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (*i.e.* if each bisects chords parallel to the other), prove that if P lies upon this curve, CD does not meet the curve, and that if D is the point where CD meets the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$,

$$CP^2 - CD^2 = a^2 - b^2.$$

CHAPTER VII

THE INTEGRAL CALCULUS—INTEGRATION

§ 43. In considering the motion of a point along a straight line, we saw that if

$$s = f(t)$$

is the relation between the distance and the time, the velocity v is given by

$$v = \frac{ds}{dt} = f'(t),$$

and, in general, that the problem of the Differential Calculus is, given the law in obedience to which two related magnitudes vary, to find the rate at which the one changes with regard to the other. The problem of the Integral Calculus is the inverse one: given the rate at which the magnitudes change with regard to each other, to find the law connecting them. In other words, in the Differential Calculus we determine the infinitesimal change in the one magnitude which corresponds to an infinitesimal change in the other, when we know what function the one is of the other. In the Integral Calculus we determine what function the one is of the other when the corresponding infinitesimal changes are known. We have thus to find the function of x, denoted by y, which is such that

$$\frac{dy}{dx} = f(x).$$

The value of y which satisfies this equation is written $\int f(x)dx$ and is called *the integral of $f(x)$ with regard to x.*

E.g. (i.) $\int x dx = \frac{x^2}{2}$, since $\frac{d}{dx}\left(\frac{x^2}{2}\right) = x$.

(ii.) $\int \sec^2 x dx = \tan x$, since $\frac{d}{dx}(\tan x) = \sec^2 x$.

In each of these cases we might have added any constant to the right-hand side, since the differential coefficient of the constant is zero, and the complete result would be

$$\int x dx = \frac{x^2}{2} + C$$

$$\int \sec^2 x dx = \tan x + C,$$

where C is called the constant of integration.

It is thus evident that the equations

$$\frac{d}{dx} F(x) = f(x)$$

and $$F(x) = \int f(x) dx$$

represent the same thing, and that the fuller statement of the second would be

$$F(x) + C = \int f(x) dx.$$

Owing to the presence of the arbitrary constant $\int f(x)dx$ is called the Indefinite Integral of $f(x)$.

The geometrical meaning of the constant of integration is that there is a family of curves all having the same slope as a given curve, or parallel to it; thus the curves

$$y = F(x) + C$$

are all parallel, when C is given different constant values.

§ 44. **Table of Standard Integrals.**

From this point of view of integration, as the process of finding the integral is called, the first requisite is a table of the more important forms. This table is obtained from the corresponding results in differentiation, and any result in integration can always be verified by differentiation. Later we shall see that there are certain general theorems on integration which correspond to the general theorems of differentiation. These will help us to decide upon the most likely ways of finding an answer to the question which the symbol of integration puts to us; namely, What is the function whose differential coefficient is the given expression? To answer this question is in very many cases impossible; but practice soon makes it easy to recognise the cases which can be treated with success.

The following is the table of Standard Forms :—

(i.) $\int x^n dx = \frac{x^{n+1}}{n+1}$, since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \qquad (n \neq -1)$

(ii.) $\int \frac{dx}{x} = \log x$, since $\frac{d}{dx}(\log x) = \frac{1}{x}$

(iii.) $\int e^{ax}dx = \frac{1}{a}e^{ax}$

(iv.) $\int a^x dx = \frac{1}{\log a}a^x$

(v.) $\int \cos x dx = \sin x$

(vi.) $\int \sin x dx = -\cos x$

(vii.) $\int \tan x dx = \log(\sec x)$

(viii.) $\int \operatorname{cosec} x dx = \log\left(\tan\frac{x}{2}\right)$

(ix.) $\int \sec^2 x dx = \tan x$

(x.) $\int \operatorname{cosec}^2 x dx = -\cot x$

(xi.) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$ or $\left(-\cos^{-1}\frac{x}{a}\right) \quad (a^2 > x^2)$ } Radian Measure.

(xii.) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ or $-\frac{1}{a}\cot^{-1}\frac{x}{a}$ } Radian Measure.

(xiii.) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$

(Unless otherwise stated, the logarithms are supposed to be to the base e.)

The student is recommended to draw up a corresponding table for the cases where $mx + n$ takes the place of x in this list.

§ 45. **Two General Theorems.**

$$\text{(i.)} \int (cu)dx = c\int u dx,$$
$$\text{(ii.)} \int (u + v)dx = \int u dx + \int v dx,$$

c being a constant, and u, v functions of x.

To prove these theorems it is sufficient to show that the differential coefficients of the two sides of the equations are the same, since in that case the answers to the questions which the sign of integration puts to us are the same for both sides of the equation.

They may be proved directly as follows:—

(i.) Let $$f(x) = \int u\,dx.$$

Then $$\frac{d}{dx}\cdot f(x) = u,$$

$$\therefore \quad \frac{d}{dx}(cf(x)) = cu,$$

$$\therefore \quad cf(x) = \int cu\,dx,$$ by the definition of the integral, $$\therefore \quad c\int u\,dx = \int cu\,dx.$$

(ii.) Let $$f(x) = \int u\,dx,$$
$$F(x) = \int v\,dx.$$

Then $$\frac{d}{dx}\left\{f(x) + F(x)\right\} = \frac{d}{dx}\cdot f(x) + \frac{d}{dx}\cdot F(x)$$
$$= u + v,$$

$$\therefore \quad f(x) + F(x) = \int(u+v)dx,$$ by the definition,

$$\therefore \quad \int u\,dx + \int v\,dx = \int (u+v)dx.$$

Ex. 1. $$\int(ax^2+2bx+c)dx = a\int x^2dx + 2b\int x\,dx + c\int 1\,dx\ ^*$$
$$= \frac{ax^3}{3} + bx^2 + cx.$$

2. $$\int\frac{2x+1}{2x-1}dx = \int\left(1+\frac{2}{2x-1}\right)dx$$
$$= \int dx + 2\int\frac{dx}{2x-1}$$
$$= x + \log(2x-1).$$

3. $$\int\frac{dx}{x^2-a^2} = \int\frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx\ ^\dagger$$
$$= \frac{1}{2a}\int\frac{dx}{x-a} - \frac{1}{2a}\int\frac{dx}{x+a}$$
$$= \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right),\quad \text{where } x>a.$$

4. $$\int\cos ax\cos bx\,dx = \frac{1}{2}\int[\cos(a+b)x + \cos(a-b)x]dx$$
$$= \frac{1}{2}\int\cos(a+b)x\,dx + \frac{1}{2}\int\cos(a-b)x\,dx$$
$$= \frac{1}{2(a+b)}\sin(a+b)x + \frac{1}{2(a-b)}\sin(a-b)x.$$

§ 46. **Integration by Substitution.**

To prove that $\int f(x)dx = \int f(x)\cdot\frac{dx}{dt}\cdot dt$, *where* $x = \phi(t)$.

* $\int 1\,.\,dx$ is usually written as $\int dx$.

† This is an important example. Cf. (xii.) p. 66.

This important result, which allows us to change an integral with regard to x into an integral in terms of another variable, may be deduced at once from the rule for differentiating a function of a function.

Let $$y = \int f(x)dx, \text{ and } x = \phi(t).$$

From the relation between x and t, y is a function of t.

But $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = f(x) \cdot \frac{dx}{dt}, \quad \text{since } \frac{dy}{dx} = f(x)$$

$$\therefore \quad y = \int f(x) \frac{dx}{dt} \cdot dt,$$ by the definition of an integral.

The expressions under the sign of integration are supposed given in terms of t.

This result may be written

(A.) $$\int f(x)dx = \int f(x) \cdot \frac{dx}{dt} \cdot dt = \int f[\phi(t)] \frac{d}{dt}[\phi(t)]dt.$$

The simple rule for "changing the variable" from x to t is:

Replace dx by $\frac{dx}{dt} \cdot dt$, and by means of the equation connecting x and t, express $f(x)$ as a function of t.

The advantages of this method will be evident from the following examples:—

Ex. (i.) $\int (ax+b)^n dx$. Put $ax+b=u$.

$$\therefore \quad \frac{dx}{du} = \frac{1}{a}$$

and $$\int (ax+b)^n dx = \int u^n \cdot \frac{1}{a} \cdot du = \frac{1}{a}\int u^n du = \frac{u^{n+1}}{a(n+1)} = \frac{1}{a(n+1)}(ax+b)^{n+1}$$

Similarly (ii.) $$\int \sin(ax+b)dx = \frac{1}{a}\int \sin u\,du = -\frac{\cos u}{a} = -\frac{1}{a}\cos(ax+b).$$

(iii.) $$\int \frac{dx}{\sqrt{a^2x^2-b^2}}.$$ Put $ax = u$.

$$\therefore \quad \frac{dx}{du} = \frac{1}{a}$$

and $$\int \frac{dx}{\sqrt{a^2x^2-b^2}} = \int \frac{1}{\sqrt{u^2-b^2}} \cdot \frac{1}{a} \cdot du = \frac{1}{a}\int \frac{du}{\sqrt{u^2-b^2}}$$

$$= \frac{1}{a}\log(u + \sqrt{u^2-b^2})$$

$$= \frac{1}{a}\log(ax + \sqrt{a^2x^2-b^2}).$$

(iv.) $\int \frac{\log x}{x} \cdot dx.$ Put $x = e^u$.

$$\therefore \quad \frac{dx}{du} = e^u$$

and $$\int \frac{\log x}{x} dx = \int \frac{u}{e^u} \cdot e^u \cdot du = \int u du = \frac{1}{2} u^2$$

$$= \frac{1}{2} (\log x)^2.$$

(v.) $\int \frac{dx}{(1-x)\sqrt{1-x^2}}.$ Put $x = \cos\theta$.

$$\therefore \quad \frac{dx}{d\theta} = -\sin\theta,$$

$$\therefore \quad \int \frac{dx}{(1-x)\sqrt{1-x^2}} = \int \frac{1}{(1-\cos\theta)\sin\theta} \cdot (-\sin\theta) \cdot d\theta$$

$$= -\frac{1}{2}\int \frac{d\theta}{\sin^2\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2},$$

$$= \frac{\sqrt{1-x^2}}{1-x}.$$

(vi.) Integrate the following expressions :—

(*a*) $x^{n-1}(ax^n + b)$; $x\sqrt{a^2+x^2}$; $\frac{x^2}{1+x^6}$.

(*β*) $\frac{1}{x^2+2x+2}$; $\frac{x+1}{x^2+2x+2}$, putting $x+1=u$.

(*γ*) $\frac{1}{ax^2+2bx+c}$; $\frac{ax+b}{ax^2+2bx+c}$, putting $ax+b=u$. ($ac > b^2$.)

(*δ*) $\frac{1}{\sqrt{x^2+4x+5}}$; $\frac{x+2}{\sqrt{x^2+4x+5}}$, putting $x+2=u$.

(*ε*) $\frac{1}{\sqrt{ax^2+2bx+c}}$; $\frac{ax+b}{\sqrt{ax^2+2bx+c}}$, putting $ax+b=u$. ($ac > b^2$.)

(*ζ*) $\sin^3 x \cos^3 x$; $\frac{\cos x}{a+b\sin x}$; $\cot x$, putting $\sin x = u$.

(*η*) $\frac{1}{a^2\cos^2 x + b^2\sin^2 x}$; $\frac{1}{\cos^2 x \sin^2 x}$, putting $\tan x = u$.

§ 47. Integration by Substitution—*continued*.

Although there are certain general principles that guide us in the choice of a suitable substitution, the second form (B.), p. 70, of the theorem of § 46 will often suggest what the transformation should be. We have seen that

$$\int f[\phi(t)] \frac{d}{dt}[\phi(t)]dt = \int f(x)dx, \text{ where } x = \phi(t),$$

and we may write this result in the form

$$\text{(B.)} \quad \int f[\phi(x)]\frac{d}{dx}[\phi(x)]dx = \int f(u)\cdot du, \text{ where } u = \phi(x),*$$

as the particular symbol we employ is immaterial.

Thus in the case of the examples of last article we obtain our results immediately—

$$e.g. \text{ (i.) } \int(ax+b)^n dx = \frac{1}{a}\int(ax+b)^n\cdot\frac{d}{dx}(ax+b)dx$$

$$= \frac{1}{a}\int u^n du, \text{ where } ax+b=u$$

$$= \frac{1}{n+1}\cdot\frac{u^{n+1}}{a} = \frac{1}{a(n+1)}(ax+b)^{n+1}.$$

$$\text{(ii.) } \int \sin^2 x \cos x dx = \int \sin^2 x \frac{d}{dx}(\sin x)dx$$

$$= \int u^2 du, \text{ where } \sin x = u$$

$$= \frac{1}{3}\sin^3 x.$$

$$\text{(iii.) } \int \frac{x^4}{1+x^5}\cdot dx = \frac{1}{5}\int\frac{1}{1+x^5}\cdot\frac{d}{dx}\cdot(1+x^5)\cdot dx$$

$$= \frac{1}{5}\int\frac{du}{u}, \text{ where } u = 1+x^5$$

$$= \frac{1}{5}\log u$$

$$= \frac{1}{5}\log(1+x^5).$$

$$\text{(iv.) } \int\frac{f'(x)}{f(x)}dx = \log f(x).$$

In this way it is easy to see that

$$\int\frac{ax+b}{ax^2+2bx+c}dx = \frac{1}{2}\log(ax^2+2bx+c),$$

* This can be verified by starting with

$$\int f(u)du,$$

and putting $u = \phi(x)$, as in (A.)

since the integral may be written as

$$\frac{1}{2}\int \frac{1}{ax^2 + 2bx + c} \frac{d}{dx}(ax^2 + 2bx + c)dx,$$

i.e. $$\frac{1}{2}\int \frac{du}{u}, \text{ where } u = ax^2 + 2bx + c.$$

Also $$\int \frac{dx}{ax^2 + 2bx + c} = \int \frac{1}{(ax+b)^2 + ac - b^2} \cdot \frac{d}{dx}(ax+b) \cdot dx.$$

$$= \int \frac{1}{u^2 + ac - b^2} \cdot du, \text{ where } u = ax + b,$$

and this is one of the standard forms.

It follows that any expression of the form

$$\frac{lx + m}{ax^2 + 2bx + c}$$

may be easily integrated, since we can rewrite the numerator as

$$P(ax + b) + Q,$$

where $$P = \frac{l}{a}; \quad Q = \frac{am - lb}{a}.$$

If higher powers of x occur in the numerator, we must first of all divide out by the denominator till we obtain a remainder of the first degree or a constant.*

The expression $\frac{lx + m}{\sqrt{ax^2 + 2bx + c}}$ *may be reduced in a similar way.*

Ex. Integrate the following expressions—

(i.) $\frac{1}{x^2 \pm 4}$; $\frac{1}{a^2x^2 \pm b^2}$; $\frac{1}{4x^2 + 4x \pm 3}$; $\frac{x+1}{4x^2 + 4x \pm 3}$; $\frac{2x+3}{3 + 4x - x^2}$;

$\frac{x^4}{x^2 \pm 1}$; $\frac{x^2 - x + 1}{x^2 + x + 1}$; $\frac{x - 1}{x^2 - 5x + 6}$; $\frac{x^2 + x + 1}{(x-1)(x-2)}$.

(ii.) $\frac{1}{\sqrt{x^2 \pm 4}}$; $\frac{1}{\sqrt{a^2x^2 \pm b^2}}$; $\frac{1}{\sqrt{4x^2 + 4x \pm 3}}$; $\frac{x+1}{\sqrt{4x^2 + 4x \pm 3}}$; $\frac{2x+3}{\sqrt{5 + 4x - x^2}}$.

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* When the factors of the denominator are real, the method of Partial Fractions should be employed.

§ 48. **Integration by Parts.**

The second important method in integration is called integration by parts, and can be used only when the function to be integrated is the product of two functions, one of which can be expressed as a differential coefficient. This method follows at once from the rule for the differentiation of a product.

Since $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx},$$

$$uv = \int\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)dx, \text{ by the definition of integration,}$$

$$= \int u\frac{dv}{dx}\cdot dx + \int v\frac{du}{dx}\cdot dx, \text{ by § 45.}$$

$$\therefore \quad \int u\cdot\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\cdot dx.$$

This result will be of use only if $\int v\frac{du}{dx}dx$ can be more easily evaluated than $\int u\frac{dv}{dx}dx$.

For example—

(i.) $$\int x\cdot\log x\cdot dx = \frac{1}{2}\int\log x\,\frac{d}{dx}(x^2)\cdot dx$$

$$= \frac{1}{2}\left(x^2\log x - \int x^2\frac{d}{dx}(\log x)dx\right)$$

$$= \frac{1}{2}\left(x^2\log x - \int x dx\right)$$

$$= \frac{1}{2}\left(x^2\log x - \frac{x^2}{2}\right)$$

$$= \frac{x^2}{4}\left(2\log x - 1\right).$$

(ii.) $$\int x^2\cdot\cos x\cdot dx = \int x^2\frac{d}{dx}(\sin x)dx$$

$$= x^2\sin x - \int\sin x\cdot\frac{d}{dx}(x^2)\cdot dx$$

$$= x^2\sin x - 2\int\sin x\cdot x\cdot dx$$

$$= x^2\sin x + 2\int x\cdot\frac{d}{dx}(\cos x)dx$$

$$= x^2\sin x + 2\left(x\cos x - \int\cos x\,\frac{d}{dx}(x)\cdot dx\right)$$

$$= x^2 \sin x + 2(x \cos x - \int \cos x dx)$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x.$$

In both of these examples this artifice allows us gradually to reduce the integral to one of a simpler form, and in such cases where powers of x are associated with a trigonometrical, exponential, or logarithmic term, it is of great value.*

An important expression which can be integrated by this method is $\sqrt{x^2 + a^2}$.

We have

$$\int \sqrt{x^2 + a^2} \cdot dx = \int \sqrt{x^2 + a^2} \cdot \frac{d}{dx}(x)\, dx$$

$$= x\sqrt{x^2 + a^2} - \int x \frac{d}{dx} \sqrt{x^2 + a^2} \cdot dx.$$

$$\therefore \int \sqrt{x^2 + a^2} \cdot dx = x\sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2 + a^2}} \cdot dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{x^2 + a^2 - a^2}{\sqrt{x^2 + a^2}}\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}},$$

$$\therefore 2\int \sqrt{x^2 + a^2}\, dx = x\sqrt{x^2 + a^2} + a^2 \log(x + \sqrt{x^2 + a^2})$$

$$\therefore \int \sqrt{x^2 + a^2}\, dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log(x + \sqrt{x^2 + a^2}).$$

Ex. Integrate the following expressions :—

$x^2 \log x$; $x^3 e^x$; $x \tan^{-1} x$; $x^2 \sin ax$; $\sqrt{a^2 - x^2}$; $\sqrt{x^2 - a^2}$.

EXAMPLES ON CHAPTER VII

1. Integrate the following expressions—

(i.) $(x-a)^3$, $\frac{1}{\sqrt{ax+b}}$, $\frac{1+x}{\sqrt{x}}$, $\frac{x+2}{x+3}$.

(ii.) $\frac{1}{x(1-x)}$, $\frac{2x-1}{x^2-3x+2}$, $\frac{x^3}{x^2+x+1}$, $\frac{x^4}{x^2-x+1}$.

(iii.) $\frac{1}{\sqrt{x(1-x)}}$, $\frac{2x-1}{\sqrt{x^2-3x+2}}$, $\frac{x+1}{\sqrt{x^2+x+1}}$.

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* Cf. p. 74 ; Exs. 11, 12, 13, 14, and 15.

2. Integrate the following expressions by parts—

$$\sin^{-1}x, \quad x^2 \tan^{-1}x, \quad x^2 \sin 4x, \quad x^2 \cos 3x, \quad x^m \log x, \quad x^2 e^{-x}.$$

3. Prove that

$$\frac{1}{x^3+1} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)},$$

and hence integrate the expression.

4. Prove that

$$\frac{1}{(x+1)(x-1)^2} = \frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x+1)},$$

and hence integrate the expression.

5. Prove that

$$\frac{x-1}{(x-2)(x-3)} = \frac{2}{x-3} - \frac{1}{x-2},$$

and hence integrate the expression.

6. Integrate the expressions $x\sqrt{1+x}$ and $\dfrac{1}{x\sqrt{1+x}}$ by putting $x+1=u^2$.

7. Prove that

$$\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{\sqrt{1-x^2}}$$

(put $x = \sin\theta$).

8. Integrate the following trigonometrical expressions—

$$\frac{1}{\sin\theta}, \quad \frac{1}{\sin(\theta+\alpha)}, \quad \frac{1}{\sin\theta+\cos\theta}, \quad \frac{1}{\cos^2\theta\sqrt{a^2\tan^2\theta+b^2}}, \quad \frac{\sin x}{\cos^2 x(4\tan^2 x+3)}$$

9. Show that, when $a^2 > b^2$,

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right).$$

$$\left[\text{Put } a+b\cos x \text{ into the form } (a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}\right].$$

Also integrate the expressions

$$\frac{1}{5\pm 4\cos x}, \quad \frac{1}{4\pm 5\cos x}, \quad \frac{1}{3\pm 2\sin x}, \quad \frac{1}{2\pm 3\sin x}.$$

10. Prove, by integration by parts, that

$$\text{(i.)} \int e^{ax}\cos bx\,dx = \frac{b\sin bx + a\cos bx}{a^2+b^2} e^{ax},$$

$$\text{(ii.)} \int e^{ax}\sin bx\,dx = \frac{a\sin bx - b\cos bx}{a^2+b^2} e^{ax}.$$

11. Prove, by integration by parts, that

$$\int \sin^n\theta\,d\theta = -\frac{\cos\theta\sin^{n-1}\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta\,d\theta,$$

and hence show that

$$\int \sin^4\theta\,d\theta = -\frac{\sin^3\theta\cos\theta}{4} - \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta.$$

12. Prove, by integration by parts, that

$$\int \cos^n \theta d\theta = \frac{\sin\theta \cos^{n-1}\theta}{n} + \frac{(n-1)}{n}\int \cos^{n-2}\theta d\theta,$$

and thus obtain the value of $\int \cos^3 \theta d\theta$ and $\int \cos^4 \theta d\theta$.

13. Prove that

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx,$$

and explain how this result may be used in evaluating such integrals as

$$\int x^3 e^{2x} dx, \int x^3 e^{-2x} dx, \text{ etc.}$$

14. Prove that

$$\int x^{n-1}(\log x)^m dx = \int y^m e^{ny} dy,$$

where $x = e^y$, and explain how this result may be used in evaluating integrals such as

$$\int x^2 (\log x)^3 dx, \int x^{-2}(\log x)^3 dx.$$

15. Prove that

$$\int x^n \sin mx dx = -\frac{x^n}{m}\cos mx + \frac{n}{m}\int x^{n-1}\cos mx dx$$

$$= -\frac{x^n}{m}\cos mx + \frac{n}{m^2}x^{n-1}\sin mx - \frac{n.n-1}{m^2}\int x^{n-2}\sin mx dx,$$

and show how this may be used in evaluating such integrals.* Obtain a corresponding result in the case of

$$\int x^n \cos mx dx.$$

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* Examples 3, 4, 5 are cases of the use of the method of *Partial Fractions* in the integration of algebraic functions; 11-15, of the method of *Successive Reduction*. Cf. Lamb's *Infinitesimal Calculus*, §§ 80, 81.

CHAPTER VIII

THE DEFINITE INTEGRAL AND ITS APPLICATIONS

§ 49. In the last chapter we have considered the process of integration as the means of answering the question: What is the function whose differential coefficient is a given function? There is another and a more important way of regarding the subject, in which integration appears as an operation of summation, or of finding the limit of the sum of a number of terms, when these terms increase in number and diminish indefinitely in size. We shall examine integration from this standpoint in the following sections.

§ 50. **Areas of Curves. The Definite Integral as an Expression for the Area.**

Let $y=f(x)$ be the equation of an ordinary continuous curve, and let us consider the area enclosed between the ordinates at $P_0(x_0, y_0)$, and $P(x, y)$, the axis of x and the curve where P_0P is above that axis. This area is obviously a function of x, since to every position of P corresponds a value of the area.

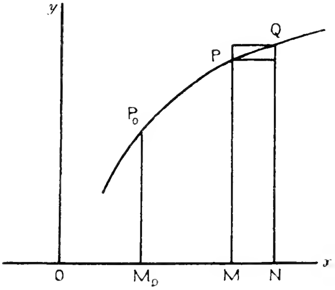

Fig. 14.

Let A stand for the area P_0M_0MP; $A+\delta A$ for the area P_0M_0NQ; and let Q be the point $(x+\delta x, y+\delta y)$. Then if the slope is positive from P to the neighbouring point Q, we see by considering the inner and outer rectangles at P and the element of area there, that

$$y\delta x < \delta A < (y + \delta y)\,\delta x,$$

and if the slope is negative the signs are reversed.

Hence in each case, when we let δx approach its limit zero, we have

$$\frac{dA}{dx} = y = f(x)\,;$$

thus $A = \int f(x)dx$ + const. $= F(x) + C$, say.

Also, since A vanishes when $x = x_0$, $C = -F(x_0)$;

$$\therefore\ A = F(x) - F(x_0).$$

This expression $F(x) - F(x_0)$

is an important one, and the symbol

$$\int_{x_0}^{x} f(x)dx$$

is used to denote it.

$\int_{x_0}^{x} f(x)dx$ *is called the definite integral of* $f(x)$ *with regard to* x *between the limits* x_0 *and* x, *and its value is obtained by subtracting the value of the indefinite integral*—$\int f(x)dx$—*for* $x = x_0$ *from that for* $x = x$.

With this notation the area of the curve $y = f(x)$ included between the ordinates at (x_0, y_0) and (x_1, y_1), the axis of x and the curve is equal to $\int_{x_0}^{x_1} f(x)dx$, and it is clear that if the curve cuts the axis between the limits x_0 and x_1, the definite integral gives the algebraical sum of the areas, those above being taken positive, those below the axis negative.

Ex. 1. To find the area of the part of the circle $x^2 + y^2 = a^2$ cut off by the lines $x = 0$, and $x = x_1$.

The required area $= 2\int_{x_0}^{x_1} \sqrt{a^2 - x^2}\,dx.$

Now it is easy to show that

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \qquad \text{(cf. § 48)};$$

$$\therefore\quad \text{the area} = \left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]_{x_0}^{x_1},$$

where we use these square brackets to denote that we subtract the value for $x=x_0$ from that for $x=x_1$.

If we take $x_0=0$ and $x_1=a$ we find the area of the semicircle as $\frac{\pi}{2}a^2$.

2. To find the area of the part of the parabola $y^2=4ax$ cut off by the lines $x=x_0$ and $x=x_1$.

$$\text{Here the required area} = 2\int_{x_0}^{x_1}\sqrt{4ax}\,dx,$$

$$= 4\sqrt{a}\int_{x_0}^{x_1}\sqrt{x}\,dx,$$

$$= 4\sqrt{a}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_{x_0}^{x_1},$$

$$= \frac{8\sqrt{a}}{3}\left(x_1^{\frac{3}{2}} - x_0^{\frac{3}{2}}\right),$$

and it follows that the area cut off by the latus rectum is $\frac{2}{3}$ of the rectangle upon LL′ as base, with AS for its altitude.

3. Prove that the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.

4. Prove the following:—

(i.) $$\int_0^{\frac{\pi}{4}}\frac{dx}{\cos x} = \log(\sqrt{2}+1) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{\sin x}$$

(ii.) $$\int_0^{\frac{\pi}{2}}\sin^2 x\,.\,dx = \frac{\pi}{4} = \int_0^{\frac{\pi}{2}}\cos^2 x\,dx.$$

(iii.) $$\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\sin^2 x+b^2\cos^2 x} = \frac{\pi}{2ab} = \int_0^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}.$$

(iv.) $$\int_0^1 \sin^{-1}x\,dx = \int_0^{\frac{\pi}{2}}\theta\cos\theta\,d\theta = \frac{\pi}{2}-1.$$

(v.) $$\int_0^a \frac{dx}{\sqrt{a-x}} = 2\sqrt{a}.$$

(vi.) $$\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{3}.$$

5. Prove that when m and n are positive integers

(i.) $$\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\,d\theta = \frac{2n-1\,.\,2n-3\,.\,.\,.\,3\,.\,1}{2n\,.\,\quad 2n-2\,.\,.\,.\,4\,.\,2}\,\frac{\pi}{2} = \int_0^{\frac{\pi}{2}}\cos^{2n}\theta\,d\theta.$$

(ii.) $$\int_0^{\frac{\pi}{2}}\sin^{2n-1}\theta\,d\theta = \frac{2n-2\,.\,2n-4\,.\,.\,.\,4\,.\,2}{2n-1\,.\,2n-3\,.\,.\,.\,5\,.\,3} = \int_0^{\frac{\pi}{2}}\cos^{2n-1}\theta\,d\theta.$$

(iii.) $$\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\,d\theta = \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}}\sin^{m-2}\theta\cos^n\theta\,d\theta.$$

(iv.) $$\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta d\theta = \frac{2}{15}.$$

(v.) $$\int_0^{\frac{\pi}{2}} \sin^6\theta \cos^8\theta d\theta = \frac{5 \,.\, 3 \,.\, 1}{14 \,.\, 12 \,.\, 10} \int_0^{\frac{\pi}{2}} \cos^8\theta d\theta = \frac{5\pi}{212}.$$

In cases where integration is not possible there are various approximate methods of finding the area. The expressions for the area of a trapezium or a portion of a parabola give the trapezoidal and parabolic rules, and we shall see more fully in §§ 51-52 how the inner and outer rectangles may be applied. The value of a definite integral may also be obtained by mechanical means by the use of different instruments, of which the planimeters are perhaps the best known.

Ex. Evaluate the following integrals by the trapezoidal method, *i.e.* find the sum of the inscribed trapeziums instead of the inner or outer rectangles as above :—

(i.) $\int_1^{12} x^2 dx$, dividing the interval into 11 equal parts, and compare with the result of integration.

Answers, $577\frac{1}{2}$; $575\frac{2}{3}$.

(ii.) $\int_{31^\circ}^{32^\circ} \cos dx$, by dividing the interval into 6 equal parts, and compare as above.

Answers, ·0148 ; ·0149.

§ 51. The Definite Integral as the Limit of a Sum.

We have in the last article shown that the symbol $\int_{x_0}^{x_1} f(x)dx$ represents the area between the curve $y = f(x)$, the axis of x, and the bounding ordinates. We shall now obtain an expression for this area as the *limit of a sum*, and thus see in what way the process of integration may be viewed as a summation.

Let P_0P_1 be any portion of the curve on which the slope remains positive.

Divide the interval M_0M_1 into n equal parts δx, so that

$$n\delta x = x_1 - x_0 ;$$

erect the ordinates m_1p_1, m_2p_2, etc., and construct inner and outer rectangles as in Fig. 15.

Then the difference of the sum of these outer rectangles and the sum of the inner rectangles is $(y_1 - y_0)\delta x$, and this may be made as small as we please by increasing the number of intervals and decreasing their size.

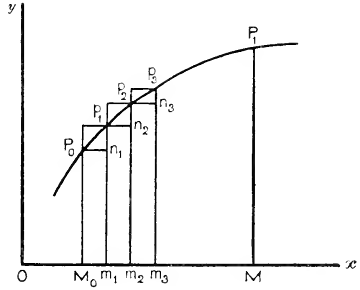

Fig. 15.

Also the area of the curve lies between these two sums, and therefore this area is the limit of either sum as δx approaches zero.

Now the sum of the inner set of rectangles

$$= [f(x_0)\delta x + f(x_0 + \delta x) \,.\, \delta x + \quad . \; . \; . \; . \quad + f(x_0 + \overline{n-1} \,.\, \delta x) \,.\, \delta x]$$

$$= \sum_{r=0}^{r=n-1} f(x_0 + r\delta x)\,\delta x.$$

But the area is $[F(x_1) - F(x_0)]$ where $F(x) = \int f(x)dx$, and we agreed to denote this by $\int_{x_0}^{x_1} f(x)dx$.

$$\therefore \quad \int_{x_0}^{x_1} f(x)dx = Lt_{\substack{\delta x = 0 \\ n\delta x = x_1 - x_0}} \sum_{r=0}^{r=n-1} f(x_0 + r\delta x) \,.\, \delta x,$$

$$= Lt_{\delta x = 0} \sum_{x_0}^{x_1} f(x)\delta x, \text{ written shortly.}$$

It is easy to remove the restriction placed upon $f(x)$ that the slope of the curve should be positive from P_0 to P_1; and to show that this result holds for any ordinary continuous curve whether it ascends or descends, and is above or below the axis in the interval x_0 to x_1.

It is only necessary to point out that in the case of such a portion of the curve $y = f(x)$ as is given in Fig. 16, the area of the portion of the curve marked II will appear as a negative area, and if $\int f(x)dx = F(x)$,

$$\int_a^b f(x)dx, \text{ or } [F(b) - F(a)].$$

is equal to

$$(\mathrm{I}) - (\mathrm{II}) + (\mathrm{III}).$$

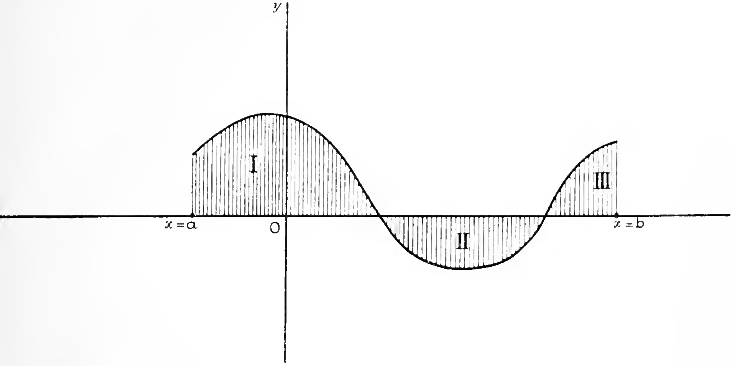

FIG. 16.

The importance of this result lies in the fact that many geometrical and physical quantities (*e.g.* volumes and surfaces of solids, centres of gravity and pressure, total pressure, radius of gyration, etc.) may be expressed in terms of the limits of certain sums. The problem of obtaining these quantities is thus reduced to a question of integration. The symbol of integration $\int$ really stands for the large S of summation, and it was in the attempts to calculate areas bounded by curves that the Infinitesimal Calculus was discovered.

It is also possible to start with the definition of the symbol

$$\int_{x_0}^{x_1} f(x)dx$$

as the limit of a sum, and then obtain its value in terms of the indefinite integral.*

§ 52. **The Evaluation of a Definite Integral from its Definition as the Limit of a Sum.**

It is instructive to see how, by algebraical methods, the values of certain definite integrals may be obtained direct from this summation.

* Cf. Lamb's *Calculus*, §§ 90, 91.

For example, in the case of the parabola

$$y = x^2,$$

we can obtain the area, or the Definite Integral, as follows :—

$$\sum_{r=0}^{r=n-1} \delta x(x_0 + r\delta x)^2$$

$$= \delta x[x_0^2 + (x_0 + \delta x)^2 + (x_0 + 2\delta x)^2 + (x_0 + n - 1 \,.\, \delta x)^2]$$

$$= \delta x\left(nx_0^2 + n \,.\, (n-1) \,.\, x_0\delta x + \frac{(n-1) \,.\, n \,.\, (2n-1)}{6}(\delta x)^2\right)$$

using the results for

$1 + 2 + 3 + \;.\;.\;.\; + (n-1)$, and $1^2 + 2^2 + \;.\;.\;.\; + (n-1)^2$.

Therefore, since $n\delta x = (x_1 - x_0)$,

$$\sum_{r=0}^{r=n-1} \delta x(x_0 + r\delta x)^2 = x_0^2(x_1 - x_0) + x_0(x_1 - x_0)^2\left(1 - \frac{1}{n}\right)$$
$$+ \frac{1}{6}(x_1 - x_0)^3\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right).$$

$$\therefore \quad \underset{\substack{\delta x = 0 \\ n\delta x = x_1 - x_0}}{Lt} \sum_{r=0}^{r=n-1} \delta x(x_0 + r\delta x)^2$$

$$= Lt_{\,n=\infty}\left[x_0^2(x_1 - x_0) + x_0(x_1 - x_0)^2\left(1 - \frac{1}{n}\right)\right.$$
$$\left. + \frac{1}{6}(x - x_0)^3\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$= x_0^2(x_1 - x_0) + x_0(x_1 - x_0)^2 + \frac{1}{3}(x_1 - x_0)^3$$

$$= \frac{x_1^3 - x_0^3}{3}.$$

$$\therefore \quad \int_{x_0}^{x_1} x^2 dx = \frac{x_1^3 - x_0^3}{3}.$$

Ex. Prove in the same way that

$$\int_0^{\frac{\pi}{2m}} \cos mx dx = \frac{1}{m}.$$

$$\int_0^b e^{-ax} dx = \frac{1}{a}(1 - e^{-ab}).$$

§ 53. **Properties of** $\int_{x_0}^{x_1} f(x)dx$.

The following properties of the Definite Integral may be deduced from either of the definitions of this symbol :—

I. $$\int_{x_0}^{x_1} f(x)dx = -\int_{x_1}^{x_0} f(x)dx.$$

II. $$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{\xi} f(x)dx + \int_{\xi}^{x_1} f(x)dx.$$

III. *The integral of an even function between the limits* $-a$ *and* $+a$ = *twice the integral of the function between* 0 *and* a.

E.g. $$\int_{-a}^{a} x^2dx = 2\int_0^a x^2dx = \frac{2}{3}a^3,$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\,.\,d\theta = 2\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta = \frac{\pi}{2}.$$

IV. *The integral of an odd function between the limits* $-a$ *and* $+a$ *is zero.*

E.g. $$\int_{-a}^{+a} x^3dx = 0, \qquad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3\theta d\theta = 0.$$

Similarly $$\int_0^{\pi} \sin^m\theta\cos^{2n+1}\theta d\theta = 0,$$
m, n being positive integers.

V. In applying the method of "change of variable" to the evaluation of definite integrals, we need not express the result in terms of the original variable. We need only give the new variable the values at its limits which correspond to the change from x_0 to x_1 in the variable x, care being taken in the case of a many-valued function that the values we thus allot are those which correspond to the given change in x.

E.g. $$\int_0^a \sqrt{a^2 - x^2}\,.\,dx$$

$$= a^2\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta, \qquad \text{putting } x = a \sin \theta,$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \pi\frac{a^2}{4}.$$

§ 54. **Application to Areas in Polar Co-ordinates.**

When the equation of the curve is given in polar co-ordinates, the area of the sector bounded by $\theta = \theta_0$ and $\theta = \theta_1$ may be shown to be

$$Lt_{\delta\theta=0} \sum_{\theta=\theta_0}^{\theta=\theta_1}\left(\frac{1}{2}r^2\delta\theta\right),$$

with the same notation as before. Hence if the curve is $r = f(\theta)$, the sectorial area is

$$\frac{1}{2}\int_{\theta_0}^{\theta_1} [f(\theta)]^2 d\theta.$$

Polar co-ordinates offer the most convenient method of finding the area of a loop of a curve.

For example, the lemniscate

$$r^2 = a^2 \cos 2\theta$$

has a loop between $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$.

The area of this loop $= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\theta$

$$= \frac{a^2}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta d\theta.$$

$\therefore$ the area of the loop $= a^2\int_0^{\frac{\pi}{4}} \cos 2\theta d\theta$ (Cf. § 53, III.)

$$= a^2\left(\frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{4}}$$

$$= \frac{a^2}{2}$$

Similarly, in the Folium of Descartes, whose equation is

$$x^3 + y^3 = 3axy,$$

there is a loop in the first quadrant; and transferring to polar co-ordinates we find that the area of the loop

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left\{\frac{3a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}\right\}^2 d\theta$$

$$= \frac{9}{2}a^2\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta\sin^2\theta}{(\cos^3\theta + \sin^3\theta)^2} d\theta$$

$$= \frac{9}{2}a^2\int_0^{\infty} \frac{t^2}{(1+t^3)^2}\cdot dt, \text{ putting } \tan\theta = t, \qquad \text{(Cf. § 53, V.)}$$

$$= \frac{3}{2}a^2\int_0^{\infty} \frac{d(t^3)}{(1+t^3)^2}$$

$$= \frac{3}{2}a^2\left(-\frac{1}{1+t^3}\right)_0^{\infty}$$

$$= \frac{3a^2}{2}.$$

Ex.—Prove that the area of the cardioide $r = a(1 - \cos\theta)$ is $\frac{3}{2}\pi a^2$.

§ 55. **Applications to Lengths of Curves.**

The length of an arc P_0P_1 of the curve $y = f(x)$ may be regarded as the limit of the sum of the different chords into which P_0P_1 is divided by the ordinates at m_1, m_2, . . . (cf. Fig. 15).

Hence

$$\text{arc } P_0P_1 = Lt_{\delta x=0} \sum_{x=x_0}^{x=x_1} \sqrt{(\delta x)^2 + (\delta y)^2}$$

$$= Lt_{\delta x=0} \sum_{x=x_0}^{x=x_1} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \, . \, \delta x$$

$$= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, . \, dx = \int_{y_0}^{y_1} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, . \, dy,$$

since $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ will differ from $\sqrt{1 + \left(\frac{\delta y}{dx}\right)^2}$ by a very small quantity when δx is very small, and the sum of these differences multiplied by δx will vanish in the limit.

If polar co-ordinates are used, we obtain in the same way for the curve $r = f(\theta)$ the two expressions

$$\int_{r_0}^{r_1} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \, . \, dr \quad \text{and} \quad \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, . \, d\theta,$$

since the chord is in this case $\sqrt{(\delta r)^2 + (r\delta\theta)^2}$.

Owing to the presence of the radical sign under the sign of integration, the problem of finding the length of the curve has been solved in only a limited number of cases.

Ex. 1. Prove that the length of the arc of the parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum is equal to $a[\sqrt{2} + \log(\sqrt{2}+1)]$

2. Prove that the length of the cardioide $r = a(1 - \cos\theta)$ is $8a$.

§ 56. Volume of Solid, whose Cross-section is given.

If the section of a solid by planes perpendicular to the axis of x is given and denoted by A, the volume of the portion of this solid cut off by two such planes may be obtained by integration, since this volume is readily seen to be *

$$Lt_{\delta x=0} \sum_{x=x_0}^{x=x_1} A \, \delta x,$$

$$\text{or} \quad \int_{x_0}^{x_1} A \, dx.$$

* With the notation of § 49 we have

$$A \, \delta x < \delta V < (A + \delta A)\delta x$$

and

$$\frac{dV}{dx} = A.$$

As a special case, the volume of such a portion of the solid formed by the revolution of the curve $y=f(x)$ about the axis of x is

$$\int_{x_0}^{x_1} \pi\,[f(x)]^2\,dx, \qquad \text{or} \qquad \pi\int_{x_0}^{x_1} y^2 dx,$$

and for revolution about the axis of y, we have in the same way

$$\pi\int_{y_0}^{y_1} x^2 dy.$$

Ex. 1. The portion of the parabola $y^2=4ax$ from the vertex to the point P(x, y) revolves about Ox. Prove that the volume of the cup we thus obtain is $2a\pi x^2$.

2. Obtain the volume of a sphere by considering the rotation of the semicircle $x^2+y^2=a^2$ about Ox.

3. Find the volume (i.) of a right circular cone and (ii.) of a cone in which the base is any plane figure of area A, and the perpendicular from the vertex upon the base is h.

4. Prove that the volume of a spherical cap of height h is $\pi h^2(r-\frac{h}{3})$, where r is the radius of the sphere.

§ 57. **Surface of Solid of Revolution.**

It is easy to show that the surface of a right circular cone whose vertical angle is $2a$ and whose generators are of length l is $\pi l^2 \sin a$, and we can deduce from this that the surface of the slice of a cone obtained by revolving a line PQ about Ox is equal to

$$2\pi\,.\,\text{PQ}\,.\,\text{NR},$$

where NR is the ordinate from the middle point of PQ.

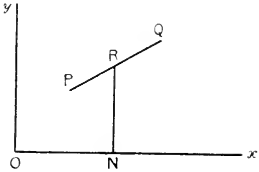

Fig. 17.

Suppose then that an arc P_0P_1 of the curve $y=f(x)$ rotates about Ox, the area of the surface generated by P_0P_1 is the limiting value of the sum of the areas of the surfaces generated by the chords into which we suppose this arc divided. That is, the area of the surface generated by P_0P_1

$$= Lt_{\delta x=0} \sum_{x=x_0}^{x=x_1} 2\pi\left(y+\frac{1}{2}\delta y\right)\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,.\,\delta x$$

$$= 2\pi\int_{x_0}^{x_1} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx, \qquad \text{where } y=f(x).$$

This may be written $2\pi\int_{s_0}^{s_1} y\,ds$, by changing the variable from x to s, where s is the length of the arc from a fixed point to the point (x, y).

When the axis of revolution is the axis of y, we obtain in the same way the expression $2\pi\int_{s_0}^{s_1} x\,ds$.

Ex. 1. Obtain the expression for the surface of a sphere of radius a.

Here we take the curve $y=\sqrt{a^2-x^2}$,

$$\text{and the surface} \quad = 4\pi\int_0^a \sqrt{a^2-x^2}\cdot\sqrt{1+\frac{x^2}{y^2}}\cdot dx$$

$$= 4\pi a\int_0^a dx$$

$$= 4\pi a^2.$$

2. Prove that the area of the portion of a sphere cut off by two parallel planes is equal to the area which they cut off from the circumscribing cylinder whose generators are perpendicular to these planes.

3. Prove that the area of the surface formed by rotating the circle of radius a, whose centre is distant d from the axis of x, about that axis is $4\pi^2 ad$.

§ 58. The Centre of Gravity of a Solid Body.

If a number of particles of masses m_1, m_2, . . . are situated at the points (x_1, y_1, z_1) . . . their C.G. is given by

$$\bar{x}=\frac{\Sigma(m_r x_r)}{\Sigma(m_r)}, \quad \bar{y}=\frac{\Sigma(m_r y_r)}{\Sigma(m_r)}, \quad \bar{z}=\frac{\Sigma(m_r z_r)}{\Sigma(m_r)},$$

and as we may suppose a continuous solid body broken up into small elements of mass δm whose centres are (x, y, z), we may write these results for a solid body in the form

$$\bar{x}=\frac{Lt_{\delta m=0}\ \Sigma x\delta m}{M}, \quad \bar{y}=\frac{Lt_{\delta m=0}\ \Sigma y\delta m}{M}, \quad \bar{z}=\frac{Lt_{\delta m=0}\ \Sigma z\delta m}{M}.$$

In many cases we can transform these expressions into integrals which we can evaluate by the methods already employed, though in general they involve integration with regard to more than one variable, and these cannot be discussed here.

We add some illustrative examples :—

Ex. 1. **The Centre of Gravity of a Semi-circular Plate.**

Take the boundary of the plate along the axis of y, and suppose the semicircle divided by a set of lines parallel to that axis and very near one another. The C.G. of each of these strips PQ′ lies on the axis of x, and therefore the C.G. of the semicircle lies on Ox.

We thus have

$$\bar{x}=\frac{Lt_{\delta m=0}\Sigma x\,.\,\delta m}{M}$$

$$=\frac{2\int_0^a xy\,dx}{\frac{\pi a^2}{2}}$$

$$=\frac{4}{\pi a^2}\int_0^a x\sqrt{a^2-x^2}\,.\,dx$$

$$=\frac{4}{\pi a^2}\left[-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}\right]_0^a$$

$$=\frac{4a}{3\pi}$$

and $\bar{y}=0$.

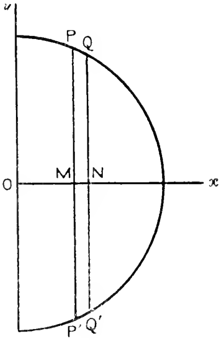

FIG. 18.

2. **The Centre of Gravity of a uniform Solid Hemisphere.**

Let the axis of x be the radius to the pole of the hemisphere, and suppose the solid divided up into thin slices by a set of planes perpendicular to this axis.

Then the C.G. of each of these slices lies on this axis, and therefore the C.G. of the hemisphere does so also.

Then

$$\bar{x}=\frac{\int_0^a xy^2dx}{\frac{2}{3}\pi a^3}$$

$$=\frac{3}{2a^3}\int_0^a x(a^2-x^2)dx$$

$$=\frac{3}{2a^3}\left(a^2\int_0^a x\,dx-\int_0^a x^3dx\right)$$

$$=\frac{3}{2a^3}\left(a^2\left[\frac{x^2}{2}\right]_0^a-\left[\frac{x^4}{4}\right]_0^a\right).$$

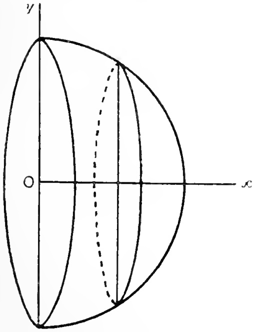

FIG. 19.

$$\therefore\quad \bar{x}=\frac{3}{2}a\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$=\frac{3}{8}a.$$

3. Prove that the C.G. of any cone or pyramid upon a plane base is one fourth of the way up the line from the vertex to the C.G. of the base.

4. Prove that the C.G. of the upper portion of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is at the point $\left(0, \frac{4b}{3\pi}\right)$.

§ 59. Moments of Inertia.

The moment of inertia, I, of a set of particles $m_1, m_2, \ldots$ with respect to an axis from which they are distant r_1, r_2, etc., is the expression

$$m_1r_1^2 + m_2r_2^2 + \ldots$$

and in the case of a continuous solid body we may express this as

$$I = Lt_{\delta m=0} \Sigma r^2\delta m.$$

The radius of gyration k is defined by the equation

$$I = Mk^2.$$

In many cases we may obtain the values of I and k^2 by the use of the methods of integration we have been discussing.

We add some illustrative examples.

Ex. 1. To find the radius of gyration of a thin rod of mass M and length $2l$, about an axis at right angles to the rod and passing through its centre.

Here

$$I = Lt_{\delta m=0} \Sigma x^2 . \delta m$$

$$= \rho\int_{-l}^{l} x^2dx,$$

where $2l\rho = M$

$$= 2\rho\int_0^l x^2dx$$

$$= \frac{2}{3}\rho l^3$$

$$= \frac{Ml^2}{3}.$$

$$\therefore \quad k = \frac{1}{3}l\sqrt{3}.$$

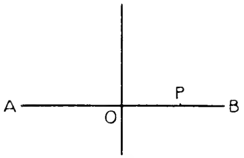

Fig. 20.

2. To find the moment of inertia of a solid circular cylinder about its axis.

Here $\quad I = Lt_{\delta m=0} \Sigma r^2\delta m,$

where $\quad \delta m = \rho h\{\pi(r+\delta r)^2 - \pi r^2\}$

$\quad = \pi\rho h\{2r\delta r + (\delta r)^2\}$, where ρ is the vol. density.

Therefore
$$I=\pi\rho h\int_0^a r^2 . 2rdr$$
$$=2\pi\rho h\int_0^a r^3 dr$$
$$=\frac{\pi}{2}\rho h a^4.$$

But
$$\pi\rho h a^2=M;$$
$$\therefore\quad I=M\frac{a^2}{2}.$$

3. Prove that the radius of gyration of a thin circular plate of radius a about a diameter as axis is $\frac{1}{4}a^2$.

EXAMPLES ON CHAPTER VIII

1. Find the areas bounded by

(i.) $y=\sin 2x$, $x=0$, $x=\frac{\pi}{2}$.

(ii.) $y=e^{-x}\sin 2x$, $x=0$, $x=\frac{\pi}{2}$.

(iii.) The hyperbola $xy=a^2$, $x=x_1$, $x=x_2$.

(iv.) $y=x^3$, $x=0$, $x=4$.

(v.) $y=2x^3$, the axis of y, and the lines $y=2$ and $y=4$.

2. Find the area of the part of the parabola $y=x^2-3x+2$ cut off by the x axis. What does $\int_0^2 ydx$ here represent?

3. Trace the parabola $(y-x-3)^2=x+y$, and find the area of the part of the curve cut off by the lines $x=0$ and $x=4\frac{1}{2}$.

4. Find the areas in polar co-ordinates of

(i.) The part of $r=a\theta$ included between $\theta=0$ and $\theta=2\pi$;

(ii.) A loop of each of the curves $r=a\sin 2\theta$, $a\sin 3\theta$, etc.;

(iii.) A loop of each of the curves $r=a\cos 2\theta$, $a\cos 3\theta$, etc.;

(iv.) The part of the hyperbola $r^2\sin\theta\cos\theta=a^2$ included between $\theta=\theta_1$ and $\theta=\theta_2$;

(v.) A sector of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the centre being the pole.

(vi.) Prove that the area between the two parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16a^2}{3}$.

(vii.) Prove that the area between the two ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ is $4ab\tan^{-1}\frac{b}{a}$.

5. By substituting $x=a\cos\theta$, $y=b\sin\theta$, show that the perimeter of the ellipse of semiaxes a, b is given by $4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\theta}\,.\,d\theta$, and deduce that for an ellipse of small eccentricity the perimeter is approximately $2\pi a\left(1-\frac{e^2}{4}\right)$.

6. Find the lengths of the following curves :—

(i.) The equiangular spiral $r=ae^{\theta\cot a}$ from $\theta=0$ to $\theta=2\pi$;

(ii.) The spiral of Archimedes $r=a\theta$ from $\theta=0$ to $\theta=2\pi$;

(iii.) The catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ from $x=0$ to $x=a$;

(iv.) And show that the length of a complete undulation of the curve

$$y=b\sin\frac{x}{a}$$

is equal to the perimeter of an ellipse whose axes are $2\sqrt{a^2+b^2}$ and $2a$.

7. Find the volumes of the following solids :—

(i.) The solid formed by revolving the part of the line $x+y=1$ cut off by the axes, about the axis of x, and verify your result by finding the volume of the cone in the usual way ;

(ii.) The spheroid formed by rotating the ellipse $9x^2+16y^2=144$ about the axis of x ;

(iii.) The cup formed by the revolution of a quadrant of a circle about the tangent at the end of one of its bounding radii ;

(iv.) The cup of height h formed by the revolution of the curve $a^2y=x^3$ about the axis of y ;

(v.) The ring formed by the revolution of the circle $(x-a)^2+y^2=b^2$ about the axis of y ;

(vi.) The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

And show that if S_0, S_1, S_2 are the areas of three parallel sections of a sphere at equal distances a, the volume included between S_0, S_2 and the spherical boundary is $\frac{a}{3}(S_0+4S_1+S_2)$.

8. The ellipse whose eccentricity is e rotates about its major axis. Prove that the area of the surface of the prolate spheroid thus formed is

$$2\pi b(b+\frac{a}{e}\sin^{-1}e).$$

9. The catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ rotates about the axis of y ; prove that the

area of the surface of the cup formed by the part of the curve from $x=0$ to $x=a$ is $2\pi a^2\left(1-\frac{1}{e}\right)$.

10. The cardioide $r=a(1-\cos\theta)$ revolves about the initial line: prove that the surface of the solid thus formed is $\frac{32}{5}\pi a^2$.

11. Find the C.G. of the following:—

(i.) A thin straight rod of length l in which the density varies as the distance from one end.

(ii.) An arc of a circle of radius a which subtends an angle 2α at the centre.

(iii.) A quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(iv.) A circular sector as in (ii.)

(v.) The segment of the sector of (iv.) bounded by the arc and its chord.

(vi.) A thin hemispherical shell of radius a.

12. Find the moments of inertia of each of the following:—

(i.) A thin straight rod, about an axis through an end, perpendicular to its length.

(ii.) A fine circular wire of radius a, about a diameter.

(iii.) A circular disc of radius a, about an axis through its centre perpendicular to the plane of the disc.

(iv.) A hollow circular cylinder of radii a, b and height h, about its axis.

(v.) A sphere of radius a, about a tangent line.

(vi.) (α) A rectangle whose sides are $2a$ and $2b$, about an axis through its centre in its plane perpendicular to the side $2a$;

(β) about an axis through its centre perpendicular to its plane.

(vii.) An ellipse whose axes are $2a$ and $2b$,

(α) about the major axis a;

(β) about the minor axis b;

(γ) about an axis perpendicular to its plane through the centre.

N.B.—The case of the circle follows on putting $a=b$.

(viii.) An ellipsoid, semiaxes a, b, c, about the axis a.

N.B.—For the sphere $a=b=c$.

(ix.) A right solid whose sides are $2a$, $2b$, $2c$, about an axis through its centre perpendicular to the plane containing the sides b and c.

N.B.—Routh's Rule for these last four important cases can be easily remembered:—

$$\left.\begin{array}{r}\textit{Moment of Inertia about an axis}\\ \textit{of symmetry}\end{array}\right\}=\textit{mass}\,\frac{\left(\textit{sum of squares of perpendicular semiaxes}\right)}{3,\ 4,\ \text{or}\ 5}$$

The denominator is to be 3, 4, or 5 according as the body is rectangular, elliptical, or ellipsoidal.

Cf. Routh's *Rigid Dynamics*, vol. i. p. 6.

ANSWERS

CHAPTER I. (p. 12)

1. (i.) $x^2+y^2=\dfrac{c^2-2a^2}{2}$. (ii.) $x=\dfrac{c}{4a}$.

   (iii.) $x^4+2x^2y^2+y^4+2a^2(y^2-x^2)+a^4-c^4=0$.

2. $x+4y-11=0$.

3. $\left(-\dfrac{13}{11}, \dfrac{19}{11}\right)$.

4. The parallel lines through O are

   $$3x-2y=0,\ 4x+y=0,\ 19x+13y=0.$$

   The perpendicular lines through O are

   $$2x+3y=0,\ x-4y=0,\ 13x-19y=0.$$

   The parallels through (2 . 2) are

   $$3x-2y=2,\ 4x+y=10,\ 19x+13y=64.$$

   The perpendiculars through (2 . 2) are

   $$2x+3y=10,\ x-4y+6=0,\ 13x-19y+12=0.$$

5. $x+3y-7=0$.

6. $7x+7y-36=0$ is the bisector of the acute angle.

   $x-y-12=0$ is the bisector of the obtuse angle.

7. (i.) (1 . 2), (3, 4), (5, 3).

   (ii.) $\dfrac{3}{5}$, -3, $\dfrac{6}{7}$.

   (iii.) The internal bisectors are

$$\frac{x-y+1}{\sqrt{2}}=\frac{-x+4y-7}{\sqrt{17}},\quad \frac{x-y+1}{\sqrt{2}}=\frac{-x-2y+11}{\sqrt{5}},\quad \frac{x-4y+7}{\sqrt{17}}=\frac{x+2y-11}{\sqrt{5}}.$$

   The external bisectors are

$$\frac{x-y+1}{\sqrt{2}}=\frac{x-4y+7}{\sqrt{17}},\quad \frac{x-y+1}{\sqrt{2}}=\frac{x+2y-11}{\sqrt{5}},\quad \frac{x-4y+7}{\sqrt{17}}=\frac{-x-2y+11}{\sqrt{5}}.$$

8. If the points (0, 0), (2, 4), (− 6, 8) be called A, B, C respectively, the equation to

   (i.) BC is $x+2y-10=0$,

   to CA is $4x+3y=0$,

   to AB is $2x-y=0$.

   (ii.) $\tan A=2$, $\tan B=\infty$, $\tan C=\dfrac{1}{2}$.

(iii.) Median through A is $y+3x=0$,
Median through B is $y-4=0$,
Median through C is $6x+7y-20=0$.

(iv.) The perpendicular from A on BC is the line AB ; its length is $2\sqrt{5}$.
The perpendicular from B on CA is the line $3x-4y+11=0$; its length is 4.
The perpendicular from C on AB is the line CB ; its length is $4\sqrt{5}$.

(v.) $x+2y=0$,
$4x+3y-20=0$,
$2x-y+20=0$.

(vi.) $x=-\frac{4}{3}$, $y=4$.

(vii.) $\left(\frac{-2(3-\sqrt{5})}{3+\sqrt{5}}\ \frac{2(4+\sqrt{5})}{3+\sqrt{5}}\right)$, $(-3,\ 4)$, $\left(-\frac{1}{2}.\ 4\right)$.

CHAPTER II. (p. 23)

4. $y-9x+16=0$.
5. $x=\frac{1}{4}$, $y=\frac{1}{2}$: $x=1$, $y=-1$.
6. $u-gt$; $-g$.
7. $2\pi rh\delta r$.
8. $\delta V=4\pi r^2\delta r$, 50·27, 502·66.
The proportional errors are $1 : 160\pi : 1600\pi$.
9. $(a+2bt)$ $\delta l=(a+2bt)\delta t$.
12. $\frac{1}{\sqrt{3}}$ feet per second.

CHAPTER III. (p. 31)

1. (i.) $\frac{dy}{dx}=\frac{3(x-1)^2(x+1)}{2x^{\frac{5}{2}}}$.

(ii.) $\frac{dy}{dx}=\frac{a-x}{\sqrt{2ax-x^2}}$.

(iii.) $\frac{dy}{dx}=\frac{2a+3}{2\sqrt{(x+1)(x+2)}}$.

(iv.) $\frac{dy}{dx}=(x+a)^{p-1}(x+b)^{q-1}(p+q)x+qa+pb$.

(v.) $\frac{dy}{dx}=\frac{1}{(1-x)\sqrt{1-x^2}}$.

(vi.) $\frac{(a-x)^{p-1}}{(b-x)^{q+1}}(qa-pb+(p-q)x)$.

(vii.) $x^{n-1}\frac{(b(n-m)x^m+na)}{(bx^m+a)^2}$.

(viii.) $mnx^{n-1}(1+x^n)^{m-1}$.

(ix.) $\frac{x(\sqrt{x^2+a^2}+\sqrt{x^2-a^2})}{\sqrt{x^4-a^4}}$.

(x.) $-x\{(x^2+a^2)^{-\frac{3}{2}}+(x^2-a^2)^{-\frac{3}{2}}\}$.

(xi.) $\dfrac{3x^2}{(1-x^2)^{\frac{5}{2}}}$.

(xii.) $\dfrac{1-x^2}{(1+x+x^2)^{\frac{1}{2}}(1-x+x^2)^{\frac{3}{2}}}$.

2. (i.) $\dfrac{2a}{y_0}$.

(ii.) $-\dfrac{x_0}{y_0}$.

(iii.) $\mp\dfrac{b^2x_0}{a^2y_0}$.

(iv.) $-\dfrac{y_0}{x_0}$.

4. 7·96 miles per hour.
5. 8 miles per hour ; 4 miles per hour.
8. $\dfrac{dp}{dv}=-\dfrac{\gamma p}{v}$.
9. When the pressure decreases, the volume increases, and conversely.

CHAPTER IV. (p. 38)

1. (i.) $3\sin x\cos x(\sin x-\cos x)$.

(ii.) $\sec^4 x$.

(iii.) $\dfrac{4\sin x}{\cos^3 x}$.

(iv.) $\dfrac{-4\cos x}{\sin^3 x}$.

(v.) $\dfrac{2\cos x}{(1-\sin x)^2}$.

(vi.) $\dfrac{2\sin x}{(1+\cos x)^2}$.

3. (i.) $x^{m-1}[m\sin(x^n)+nx^n\cos(x^n)]$.

(ii.) $x^{m-1}[m\cos(x^n)-nx^n\sin(x^n)]$.

(iii.) $x^{m-1}[m\tan(x^n)+nx^n\sec^2(x^n)]$.

4. (i.) $2x\tan^{-1}x$.

(ii.) $\sin^{-1}x$.

(iii.) $\dfrac{1}{2\sqrt{x}(1+x)}$.

(iv.) $\dfrac{1-x^2}{1+3x^2+x^4}$.

(v.) $\dfrac{1}{2(1+x^2)}$.

5. $a\omega\sin\omega t$, $a\omega^2\cos\omega t$.

6. $\dot{x}=2a\omega\cos^2\dfrac{\omega t}{2}$; $\dot{y}=a\omega\sin\omega t$.

$\ddot{x}=-a\omega^2\sin\omega t$; $\ddot{y}=a\omega^2\cos\omega t$.

The direction of motion at time t makes an angle $\dfrac{\omega t}{2}$ with the axis of x.

CHAPTER V. (p. 50)

1. (i.) $e^x(1+x)$. (ii.) $x^{m-1}e^{nx}(m+nx)$. (iii.) $(a+be+cax)e^{cx+d}$.

(iv.) $e^{x\sin^{-1}x}\left(\sin^{-1}x+\dfrac{x}{\sqrt{1-x^2}}\right)$.

2. (i.) $2xe^{1+x^2}$. (ii.) $2xe^{ax^2}(1+ax^2)$. (iii.) $x^{m-1}e^{ax^n}(m+nax^n)$.

(iv.) $x^{m-1}a^{x^n}(m+nx^n\log a)$.

3. (i.) $x^{m-1}(1+m\log x)$. (ii.) $\dfrac{1}{(x+1)(x+2)}$. (iii.) $\dfrac{1}{2\sqrt{x^2-1}}$.

(iv.) $\dfrac{-6x}{(1-x^2)(4-x^2)}$. (v.) $\dfrac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}}$. (vi.) $\dfrac{1}{(1-x)\sqrt{x}}$.

4. (i.) $\dfrac{4x-3}{2\sqrt{(2x+1)(x-2)}}$. (ii.) $\dfrac{a^2}{(a^2\pm x^2)^{\frac{3}{2}}}$. (iii.) $\dfrac{2-5x}{x^3(x-1)^4}$.

(iv.) $x^x(1+\log x)$.

(v.) $\dfrac{mn\cos(m-n)x\sin^{n-1}mx}{\cos^{m+1}nx}$. (vi.) $\left(\log\dfrac{1+x}{x}-\dfrac{1}{x+1}\right)\left(1+\dfrac{1}{x}\right)^x$.

11. (i.) $\tan\alpha$.
(ii.) $\tan n\theta$.
(iii.) $-\cot n\theta$.
(iv.) $\cot n\theta$.
(v.) $-\tan n\theta$.

$r\dfrac{d\theta}{dr}$ is the tangent of the angle between the radius vector to the point (r, θ), and the tangent to the curve at that point.

13. (i.) $\dfrac{dy}{dx}=(3x-1)(x-1)$. Max. at $\left(\dfrac{1}{3}, \dfrac{4}{27}\right)$.
Min. at $(1, 0)$.

(ii.) $\dfrac{dy}{dx}=x(5x-2)(x-1)^2$. Max. at origin.
Min. at $(\cdot 4, \cdot 03456)$.

(iii.) $\dfrac{dy}{dx}=2(x-1)(x-2)(2x-3)$. Min. at $(1, 0)$; $(2, 0)$.
Max. at $\left(\dfrac{3}{2}, \dfrac{1}{16}\right)$.

(iv.) $\dfrac{dy}{dx}=1-\dfrac{1}{x^2}$. Max. at $(-1, -1)$.
Min. at $(1, 3)$.

(v.) $\dfrac{dy}{dx}=2\dfrac{(x^2-1)}{(x^2+x+1)^2}$. Max. at $(-1, 3)$.
Min. at $\left(1, \dfrac{1}{3}\right)$.

(vi.) $\dfrac{dy}{dx}=\dfrac{4x^2-2x-5}{(x^2+x+1)^2}$. Max. at $(-\cdot 9, 6\cdot 1)$ nearly.
Min. at $(1\cdot 4, -\cdot 06)$ nearly.

(vii.) $\frac{dy}{dx} = -\frac{(4x^2-2x-5)}{(x-1)^2(x-2)^2}$. Min. at (– ·9, ·16) nearly.
Max. at (1·4, 18·2) nearly.

(viii.) $\frac{dy}{dx} = -\frac{2(x^2-5x+7)}{(x-2)^2(x-4)^2}$. No turning points.

(ix.) $\frac{dy}{dx} = \frac{x^2-8x+10}{(x-4)^2}$. Max. at (1·5, ·1) nearly.
Min. at (6·45, 9·9) nearly.

(x.) $\frac{dy}{dx} = \frac{x^3-2}{x^3}$. Min. at (1·26, 1·89) nearly.

17. (i.) $-\frac{R\theta}{v^2}$.

(ii.) $\frac{R}{v}$.

(iii.) $\delta p = \frac{R\theta}{v^2}\delta v$.

(iv.) $\delta v = \frac{R\alpha}{p}\delta t$.

(v.) $\delta p = -\frac{R(1+\alpha t)}{v^2}\delta v + \frac{\alpha R}{v}\delta t$.

19. $\frac{\delta\triangle}{\triangle} = \frac{\delta a}{a} + \frac{\delta b}{b} + \cot C\delta C$.

EXAMPLES ON THE PARABOLA (p. 55)

	(1)	(2)	(3)
2. Foci	$\left(-2, -\frac{3}{4}\right)$,	$(-2, -2)$,	$\left(0 - \frac{1}{2}\right)$.
Vertices	$(-2, -1)$,	$(-2, -3)$,	$\left(-\frac{1}{4} - \frac{1}{2}\right)$.
Latera recta	$y = -\frac{3}{4}$	$y = -2$,	$x = 0$.
Lengths of recta	1,	4,	1.
Axes	$x = -2$,	$x = -2$,	$y = -\frac{1}{2}$.
Tangents at vertices	$y = -1$,	$y = -3$,	$x = -\frac{1}{4}$.

3. –5. 7.

5. (1, 2), $\left(\frac{a}{m^2} \cdot \frac{2a}{m}\right)$.

6. $x - y - 1 = 0$, $x + y - 3 = 0$.

7. $x - y + a = 0$, $x + y - 3a = 0$.
$x + y + a = 0$, $x - y - 3a = 0$.

EXAMPLES ON THE ELLIPSE (p. 59)

1. The foci, extremities of the axes, length of latus rectum, and eccentricity are for

(i.) $[\pm 1 . 0]$, $[\pm 2 . 0]$, $[0 \pm \sqrt{3}]$, 3, $\frac{1}{2}$.

(ii.) $[2 . 2]$, $[0 . 2]$, $[3 . 2]$, $[-1, 2]$, $[1, 2+\sqrt{3}]$, $[1, 2-\sqrt{3}]$, 3, $\frac{1}{2}$.

(iii.) $[\pm\sqrt{3}, 1]$, $[\pm 2, 1]$, $[0 . 2]$, $[0 . 0]$, $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$.

(iv.) $[0 \pm 1]$, $[0 \pm 2]$, $[\pm\sqrt{3} . 0]$, 3, $\frac{1}{2}$.

EXAMPLES ON THE HYPERBOLA (p. 62)

1. (i.) $(\pm\sqrt{7}, 0) : (\pm 2, 0) : 3 : \frac{\sqrt{7}}{2}$.

(ii.) $(1 \pm \sqrt{7}, 2) : (3, 2) : (-1, 2) : 3 : \frac{\sqrt{7}}{2}$.

(iii.) $(0, -1 \pm \sqrt{5}) : (0, 0) : (0, -2) : 8 : \sqrt{5}$.

(iv.) $(\pm\sqrt{7}, 0) : (\pm\sqrt{3}, 0) : \frac{4\sqrt{3}}{3} : \frac{\sqrt{21}}{3}$.

2. (i.) $(\pm 2\sqrt{2}, \pm 2\sqrt{2}) : (\pm 2, \pm 2)$.

(ii.) $(\pm 2\sqrt{2}, \mp 2\sqrt{2}) : (\pm 2, \mp 2)$.

CHAPTER VII. (p. 73)

1. (i.) $\frac{(x-a)^4}{4} : \frac{2\sqrt{ax+b}}{a} : \frac{2}{3}\sqrt{x}(3+x) : x - \log(x+3)$.

(ii.) $\log\frac{x}{x-1}$, $\log\frac{(x-2)^3}{(x-1)}$, $\frac{(x-1)^2}{2} + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$,

$$\frac{x^3}{3} + \frac{x^2}{2} - \frac{1}{2}\log(x^2-x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right).$$

(iii.) $\sin^{-1}(2x-1)$; $2\sqrt{x^2-3x+2} + 2\log\left(x - \frac{3}{2} + \sqrt{x^2-3x+2}\right)$;

$$\sqrt{(x^2+x+1)} + \frac{1}{2}\log\left(x + \frac{1}{2} + \sqrt{x^2+x+1}\right).$$

2. $x\sin^{-1}x + \sqrt{1-x^2}$; $\frac{x^3}{3}\tan^{-1}x - \frac{1}{6} . x^2 + \frac{1}{6}\log(1+x^2)$; $\cos 4x\left(\frac{1-8x^2}{32}\right) +$

$$\sin 4x . \frac{x}{8} ;$$

$\left(\frac{9x^2-2}{27}\right)\sin 3x + \frac{2x}{9}\cos 3x$; $\frac{x^{m+1}}{m+1}\log x - \frac{x^{m+1}}{(m+1)^2} - e^{-x}(x^2+2x+2)$.

3. $\frac{1}{6}\log\left(\frac{(1+x)^2}{1-x+x^2}\right)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$

4. $-\frac{1}{2}\frac{1}{x-1}+\frac{1}{4}\log\left(\frac{x+1}{x-1}\right).$

5. $\log\frac{(x-3)^2}{x-2}.$

6. (i.) $2(1+x)^{\frac{3}{2}}\left\{\frac{x}{5}-\frac{2}{15}\right\}.$ (ii.) $\log\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}.$

8. $\log\tan\frac{\theta}{2}.$ $\log\tan\frac{(\theta+\alpha)}{2}.$ $\frac{1}{\sqrt{2}}\log\tan\left(\frac{\theta}{2}+\frac{\pi}{8}\right).$

$\frac{1}{a}\log(a\tan\theta+\sqrt{a^2\tan^2\theta+b^2}).$ $\frac{1}{4}\log\frac{2\sec x-1}{2\sec x+1}.$

11.
$$\int\frac{dx}{5+4\cos x}=\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right).$$

$$\int\frac{dx}{5-4\cos x}=\frac{2}{3}\tan^{-1}\left(3\tan\frac{x}{2}\right).$$

$$\int\frac{dx}{4+5\cos x}=\frac{1}{3}\log\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}.$$

$$\int\frac{dx}{4-5\cos x}=\frac{1}{3}\log\frac{1+3\tan\frac{x}{2}}{1-3\tan\frac{x}{2}}.$$

$$\int\frac{dx}{3+2\sin x}=\frac{2}{\sqrt{5}}\tan^{-1}\sqrt{\frac{1}{5}}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right).$$

$$\int\frac{dx}{3-2\sin x}=\frac{2}{\sqrt{5}}\tan^{-1}\sqrt{5}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right).$$

$$\int\frac{dx}{2+3\sin x}=\frac{2}{\sqrt{5}}\log\frac{\sqrt{5}+\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}{\sqrt{5}-\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}.$$

$$\int\frac{dx}{2-3\sin x}=\frac{2}{\sqrt{5}}\log\frac{1+\sqrt{5}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}{1+\sqrt{5}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}.$$

12. $\frac{1}{3}\cos^2\theta\sin\theta+\frac{2}{3}\sin\theta.$ $\frac{\cos^3\theta\sin\theta}{4}+\frac{3}{4}\cos\theta\sin\theta+\frac{3}{8}\theta.$

15. $\frac{x^n}{m}\sin mx+\frac{n}{m^2}x^{n-1}\cos mx-\frac{n(n-1)}{m^2}\int x^{n-2}\cos mx\,dx.$

CHAPTER VIII. (p. 91)

1. (i.) 1.

(ii.) $\frac{2}{5}\left(e^{-\frac{\pi}{2}}+1\right)$.

(iii.) $a^2 \log \frac{x_2}{x_1}$.

(iv.) 64.

(v.) $3\left(2^{\frac{1}{3}}-\frac{1}{2}\right)$.

2. $\frac{1}{6}$: the difference between the area bounded by the x-axis, the y-axis and the curve, and the area which lies on the negative side of the x-axis.

3. $\frac{343}{12}$.

4. (i.) $\frac{4\pi^3 a^2}{3}$.

(ii.) $\frac{\pi a^2}{8}$, $\frac{\pi a^2}{12}$, $\frac{\pi a^2}{4n}$.

(iii.) $\frac{\pi a^2}{8}$, $\frac{\pi a^2}{12}$, $\frac{\pi a^2}{4n}$.

(iv.) $a^2 \log \frac{\tan \theta_2}{\tan \theta_1}$.

(v.) $\frac{ab}{2} \tan^{-1} \frac{ab(\tan \theta_2 - \tan \theta_1)}{a^2 \tan \theta_1 \tan \theta_2 + b}$.

$\frac{ab}{4} \log \frac{(b+a\tan\theta_2)(b-a\tan\theta_1)}{(b+a\tan\theta_1)(b-a\tan\theta_2)}$.

6. (i.) $a \sec \alpha \, (e^{2\pi \cot \alpha} - 1)$.

(ii.) $\frac{a}{2}\left((2\pi\sqrt{1+4\pi^2}+\log(2\pi+\sqrt{1+4\pi^2})\right)$.

(iii.) $\frac{a}{2}(e-e^{-1})$.

7. (i.) $\frac{\pi}{3}$.

(ii.) 48π.

(iii.) $\frac{5\pi a^3}{3} - \frac{\pi^2 a^3}{2}$.

(iv.) $\frac{3}{5}\pi h^{\frac{5}{3}} a^{\frac{4}{3}}$.

(v.) $2ab^2\pi^2$.

(vi.) $\frac{4}{3}\pi abc$.

11. (i.) $\frac{2}{3}z$ from that end.

(ii.) On the radius bisecting the arc at a distance $\frac{a \sin \alpha}{\alpha}$ from the centre.

(iii.) $\bar{x}=\frac{4b}{\pi}$, $\bar{y}=\frac{4a}{3\pi}$.

(iv.) On the radius bisecting the sector at a distance $\frac{2a}{3}\frac{\sin a}{a}$ from the centre.

(v.) On the bisector of the chord at a distance $\frac{2}{3}a\frac{\sin^3 a}{a-\sin a\cos a}$ from the centre.

(vi.) The middle point of the radius perpendicular to the base.

12. (i.) $\frac{4}{3}Mz^2$. (rod of length $2z$).

(ii.) $\frac{1}{2}Ma^2$.

(iii.) $\frac{1}{2}Ma^2$.

(iv.) $\frac{M}{2}(a^2+b^2)$.

(v.) $\frac{7}{5}Ma^2$.

(vi.) (α) $\frac{Ma^2}{3}$: (β) $M\left(\frac{a^2+b^2}{3}\right)$.

(vii.) (α) $M\frac{b^2}{4}$: (β) $M\frac{a^2}{4}$: (γ) $M\left(\frac{a^2+b^2}{4}\right)$.

(viii.) $M\left(\frac{b^2+c^2}{5}\right)$.

(ix.) $M\left(\frac{b^2+c^2}{3}\right)$.

THE END

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