DTNSRDC-77-0112
Dec 1977

"Introduction to Wiener-Hopf methods in acoustics and vibration."

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(18) DTNSRDC

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER (19) 77-Ø112	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) INTRODUCTION TO WIENER-HOPF METHODS IN ACOUSTICS AND VIBRATION.		5. TYPE OF REPORT & PERIOD COVERED (9) Formal rept.,
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) David G. Crighton		8. CONTRACT OR GRANT NUMBER(s) (15) NØØ167-76-M-8415 New
9. PERFORMING ORGANIZATION NAME AND ADDRESS David W. Taylor Naval Ship R&D Center Bethesda, Maryland 20084		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS J.O. 4-19000-001-32
11. ~~CONTROLLING OFFICE NAME AND ADDRESS~~ Catholic Univ Dept of Mechanical Eng		12. REPORT DATE (11) December 1977
		13. NUMBER OF PAGES 94
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 92p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Wiener-Hopf methods
Structural acoustics
Half-plane flow problems

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Wiener-Hopf technique is [illegible] firmly established as a powerful tool for research in certain types of boundary value problem arising in acoustics. Typical problems which may be solved exactly or asymptotically with this technique concern the sound and vibration levels generated by finite or semi-infinite planar or cylindrical surfaces, of local or extended reaction, immersed in a compressible fluid and subject to acoustic or mechanical forcing. However, even the simplest of these problems involves complications which are

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20. ABSTRACT (Continued)

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ABSTRACT

The Wiener-Hopf technique is now firmly established as a powerful tool for research in certain types of boundary value problem arising in acoustics. Typical problems which may be solved exactly or asymptotically with this technique concern the sound and vibration levels generated by finite or semi-infinite planar or cylindrical surfaces, of local or extended reaction, immersed in a compressible fluid and subject to acoustic or mechanical forcing. However, even the simplest of these problems involves complications which are irrelevant to an understanding of the Wiener-Hopf method itself and its various extensions. Accordingly, this report was written in an attempt to display the operation of the technique in an even simpler physical and mathematical context, and thereby to encourage its more widespread use. The report deals with the application of Wiener-Hopf methods to one-dimensional wave motions on strings and beams, and in particular with the reflection and transmission from discontinuities in the mechanical properties of a string. Also included is a section illustrating how a generalized Wiener-Hopf problem can be set up for a three-part problem involving a string of finite length. Two-dimensional wave problems are then exemplified in a discussion of the acoustic field generated by a vibrating half-plane, and the effect of uniform mean flow over the half-plane is included to show how different types of "edge condition" may be accommodated. The final section sets out in detail the properties of certain functions arising very frequently in application of Wiener-Hopf methods to acoustic problems.

ADMINISTRATIVE INFORMATION

This work was performed under DTNSRDC Contract No. N00167-76-M-8415, financed under DTNSRDC Job Order 4-1900-001-32. At the time, the author, whose permanent address is Department of Applied Mathematical Studies, University of Leeds, England, was a visiting professor in the Department of Mechanical Engineering, Catholic University of America, Washington, D.C.

1. INTRODUCTION

The Wiener-Hopf technique was devised in 1931 [1] to deal with an integral equation arising in neutron transport theory, though its origins—and indeed its essentials—come from Russian work in the 1870's on singular integral equations [2,3]. During the war the technique was extensively applied by Schwinger and his colleagues [4] to problems in electromagnetic wave propagation, and much of the subsequent development of the method has taken place in applications to wave diffraction processes. Standard books on the method are those by Noble [5], Weinstein [6], while several books (e.g., Carrier, Krook and Pearson [7], Morse and Feshbach [8]) have chapters which attempt to introduce the method. In such introductions, two-dimensional boundary value problems involving a partial differential equation for some field variable are invariably used as the simplest demonstration problems (the Sommerfeld problem of plane wave diffraction by a semi-infinite rigid screen being the best known). Such problems, however, bring in at once a number of issues which are irrelevant to the exposition of the W-H method; the introduction of branch cuts in the complex wavenumber plane is one such issue which—while it is an important one, and one which must be understood by anyone wishing to deal with wave diffraction problems—causes great difficulties for most students.

Accordingly, an attempt will be made in these course notes to illustrate the W-H method in a much simpler context than usual. We shall study one-dimensional time-harmonic waves on strings and bars, and in particular we will study the reflection and transmission properties of changes in properties of the medium, abrupt changes of density giving

rise to various kinds of standard and generalized forms of W-H equations. Not only is the derivation of the W-H functional equation much simpler than usual for these problems, but its solution is also much easier, and the final inversion of a Fourier transform integral can be readily performed and the results seen to correspond with familiar undergraduate ideas.

The W-H technique is a method of solving certain types of boundary value problem in which, typically, we have information about the pressure, say, on the half-plane $x < 0$ and about the velocity on the half-plane $x > 0$, and we cannot solve for the radiated acoustic field until we know the pressure all over the whole boundary, $-\infty < x < +\infty$, or about the velocity there. In the W-H method, a single equation is derived, relating the Fourier transforms of the unknown pressure (for $x > 0$), of the unknown velocity (for $x < 0$) and of the given forcing field arising from some prescribed force or source or incident field. The transforms of the two unknown distributions are known (partly because of information supplied by the anticipated physical behavior of the system under study) to have certain analyticity properties as functions of the <u>wavenumber regarded as a complex variable</u>, and a certain crucial step (the W-H method) and the use of some fundamental theorems of the calculus of functions of a complex variable together enable one equation to be solved for two unknown functions. Then the field everywhere can be found in terms of an inverse Fourier integral which in many instances can be estimated by stationary phase or steepest descent/saddle point techniques [e.g., 9,10] or by generalized function methods [11,12]. In the problems to be discussed here such elaborate methods are not needed, and an exact inversion of the Fourier integrals

can be obtained with the aid of elementary residue calculus (see, e.g., [7]).

It is usual, and no doubt more logical, to start an introduction to a technique of this kind by first summarizing all that will be required in the way of results from the theory of functions of a complex variable. As our problems are extremely simple, both from the conceptual and from the manipulative point of view, it seems unnecessary to start here with such a digression, and we shall introduce the various ideas and theorems as they arise in the course of the problems. For a rigorous statement of the theorems the reader is referred to standard references (e.g., Titchmarsh [13]). We should warn the reader, however, that the degree of rigor which is usually adopted by workers in wave theory is not altogether a matter of mathematical pedantry. Certain problems, involving coupled wave-bearing media in particular, are extremely delicate, and require much more attention to mathematical rigor than do the simple problems at hand here.

2. REFLECTION OF WAVES FROM DISCONTINUITIES ON A STRING

Consider a uniform string of line density ρ_0 lying along the portion $-\infty < x < 0$ of the x-axis under tension T. Consider this string to be joined at $x = 0$ by a massless connection to a string of density ρ_1 lying along $0 < x < +\infty$, the tension there also being T. Transverse waves of arbitrary profile propagate at speed $C_0 = (T/\rho_0)^{\frac{1}{2}}$ on the left-hand string, and at speed $C_1 = (T/\rho_1)^{\frac{1}{2}}$ on the right-hand. Alternatively, with time dependence $\exp(-i\omega t)$, $\omega > 0$, understood throughout, waves in $x < 0$ have wavenumber $k_0 = \omega/C_0$ while those in $x > 0$ have wavenumber $k_1 = \omega/C_1$. We wish to determine the reflected and transmitted waves in $x < 0$, $x > 0$ respectively when a progressive wave with displacement $\exp(ik_0 x)$ is incident upon the junction from $x < 0$. [See Figure 1.]

This is a trivial problem to which the solution can be found by elementary methods and which can be generalized to cover a variety of different conditions at the junction. It and its generalizations are also suitable for introducing the W-H method very simply. We start by writing the total displacement as

$$\left.\begin{array}{ll} y + \exp(ik_0 x) & \text{in } x < 0 \\ \text{and as} \quad y & \text{in } x > 0 \end{array}\right\} \qquad (2.1)$$

The reason for this is that then y (which might be called the scattered field) must take the form of an outgoing wave as $x \to +\infty$ and as $x \to -\infty$, and this enables us to state something about the Fourier transform of y. We have first, however, to make the wavenumbers k_0 and k_1 slightly

complex

$$\left.\begin{aligned} k_0 &\rightarrow k_{or} + ik_{oi} \\ k_1 &\rightarrow k_{1r} + ik_{1i} \end{aligned}\right\} \tag{2.2}$$

where real and imaginary parts of both wavenumbers are both <u>positive</u> and where the imaginary parts are small (and in the end vanishingly small compared with the real parts.

The rationale for this is as follows. We start by assuming a time factor $\exp(-i\omega t)$ with $\omega > 0$. (We could just as well take $\exp(+i\omega t)$, but the choice of $\exp(-i\omega t)$ is helpful for reasons that should emerge later.) Then as $x \rightarrow +\infty$ the phase factor of an outgoing wave must be $\exp(+ik_1 x)$ where $k_1 = \omega/C_1$. Giving k_1 a small positive imaginary part therefore makes an outgoing wave <u>decay</u> as $x \rightarrow +\infty$, like $\exp(-k_{1i}x)$, corresponding to the presence of small internal dissipation in the string. Similarly, as $x \rightarrow -\infty$ an outgoing wave will have the phase factor $\exp(-ik_0 x)$, and then will also be exponentially damped if we give k_0 a small positive imaginary part.

We therefore now know that for $x > 0$, $y(x)$ is a continuous function which decays like $\exp(-k_{1i}x)$ as $x \rightarrow \infty$. Its HALF-RANGE FOURIER TRANSFORM then has certain properties as a function of complex wavenumber s (that being a useful symbol for the wavenumber which does not have any particular significance, the symbol k for example often being associated with a particular wavenumber). Define

$$Y_+(s) = \int_0^\infty y(x) \exp isx \, dx. \tag{2.3}$$

Then $Y_+(s)$ exists as an analytic function of s at all points in the complex s-plane for which the integral converges. The integral over any finite range of x certainly converges (and y(x) is certainly integrable near x=0) so that the convergence is dictated by the behavior of y(x) as $x \to +\infty$. There

$$y(x) \exp isx \sim \exp[-(s_i + k_{1i})x] \qquad (2.4)$$

where $s = s_r + is_i$, and the integral up to infinity therefore converges if $s_i + k_{1i} > 0$. It may also converge for some other values of s, but what can be guaranteed on the basis of the anticipated behavior of y(x) as $x \to +\infty$ is that

$Y_+(s)$ is analytic in an upper half-plane

$$R_+ : \operatorname{Im} s > -k_{1i} \qquad (2.5)$$

Moreover, $Y_+(s)$ has non-growing algebraic behavior as $|s| \to \infty$ everywhere within the domain R_+ if y(x) is finite at x = 0+ or has an integrable singularity at x = 0+, so that $|Y_+(s)| = 0(|s|^{-\lambda})$ say for some $\lambda \geq 0$ as $|s| \to \infty$ along any radius in the domain R_+. If, as may occur in some problems, y(x) has a non-integrable singularity at x = 0+ (that is, a singularity at least as strong as x^{-1}) then y(x) has to be regarded as a generalized function whose generalized half-range transform $Y_+(s)$ is still analytic in an upper half-plane R_+, but now can have algebraic growth as $|s| \to \infty$.

By analytic we mean that $Y_+(s)$ is single-valued and has a unique derivative

$$Y'_+(s) = \lim_{h \to o} \frac{Y_+(s+h) - Y_+(s)}{h} \tag{2.6}$$

as the point s+h approaches s along <u>any</u> path in the plane.

In a precisely similar way, the half-range transform of the scattered displacement y(x) for x < 0

$$Y_-(s) = \int_{-\infty}^{0} y(x) \exp isx \, dx \tag{2.7}$$

is an analytic function of s in a lower half of the s-plane,

$$R_- : \quad \mathrm{Im}\, s < + k_{oi} \tag{2.8}$$

and for a function y(x) integrable at x = 0− $Y_-(s)$ has non-growing algebraic behavior at infinity in R_-, so that $|Y_-(s)| = 0(|s|^{-\mu})$ for some $\mu \geq 0$ as $|s| \to \infty$ along any radius in R_-. (Algebraic growth of $Y_-(s)$ is permitted if y(x) has a non-integrable singularity as x → 0−.)

Since k_{oi} and k_{1i} are strictly positive, the FULL-RANGE FOURIER TRANSFORM

$$Y(s) = Y_+(s) + Y_-(s) = \int_{-\infty}^{+\infty} y(x) \exp isx \, dx \tag{2.9}$$

exists as an analytic function of s in a <u>strip</u> D,

$$D = R_+ \cap R_- : -k_{1i} < \mathrm{Im}\, s < + k_{oi} \tag{2.10}$$

formed by the intersection of the overlapping upper and lower half-planes R_+ and R_-. [See Figure 2.]

The notation and conventions for the Fourier transforms are such that ⊕ functions arise from half-range transforms over the positive x-axis, ⊖ functions from half-range transforms over the negative x-axis.

Alternatively, $Y_+(s)$ can be regarded as the full-range transform of the generalized function $y(x)\ H(x)$,

$$Y_+(s) = \int_{-\infty}^{+\infty} y(x)\ H(x) \exp isx\ dx \tag{2.11}$$

and correspondingly

$$Y_-(s) = \int_{-\infty}^{+\infty} y(x)\ H(-x) \exp isx\ dx\ , \tag{2.12}$$

$H(x)$ denoting the Heaviside unit function, $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x < 0$.

The FOURIER INVERSION THEOREMS run as follows:- if C_+ is any contour from $-\infty$ to $+\infty$ lying within the domain R_+ and such that $Y_+(s) \exp(-isx)$ is integrable over this contour, then

$$\begin{aligned} \frac{1}{2\pi} \int_{C_+} Y_+(s) \exp(-isx)\ ds &= y(x) \quad (x > 0) \\ &= 0 \qquad (x < 0) \end{aligned}$$

both of which are contained in

$$y(x)\ H(x) = \frac{1}{2\pi} \int_{C_+} Y_+(s) \exp(-isx)\ ds \tag{2.13}$$

The fact that the integral vanishes for $x < 0$ follows from Cauchy's theorem applied to a closed contour Γ consisting of the contour C_+ with its ends joined by a large circular arc in R_+. Since the integrand is analytic everywhere in R_+

$$\int_\Gamma Y_+(s) \exp(-isx)\ ds = 0$$

and the integral along the circular arc vanishes if $x < 0$

because $\exp(-isx)$ is then exponentially small at infinity in R_+ and $Y_+(s)$ is at least algebraically small there. (This result should really be proven carefully using Jordan's lemma; see [7,13].)

Correspondingly, if C_- is any contour from $-\infty$ to $+\infty$ lying within R_- and such that $Y_-(s)\exp(-isx)$ is integrable over it, then

$$y(x)\,H(-x) = \frac{1}{2\pi}\int_{C_-} Y_-(s)\exp(-isx)\,ds \tag{2.14}$$

Now if C is any contour from $-\infty$ to $+\infty$ lying everywhere within the strip of D of overlap between R_+ and R_-, then we can identify C with C_+ for (2.13), C with C_- for (2.14) and obtain by addition of (2.13) and (2.14) the inversion theorem

$$y(x) = \frac{1}{2\pi}\int_C Y(s)\exp(-isx)\,ds \tag{2.15}$$

We remark here that our convention $\exp(-i\omega t)$ for the time factor is consistent with the formulas (2.9) and (2.15), in the sense that we are really taking F.T.'s in space and time with the definitions

$$Y(s,\omega) = \int\int y(x,t)\exp(isx + i\omega t)\,dx\,dt$$

$$y(x,t) = \frac{1}{(2\pi)^2}\int\int Y(s,\omega)\exp(-isx - i\omega t)\,ds\,d\omega$$

Note that if a time factor $\exp(+i\omega t)$ were taken, and k_o, k_1 were defined as $k_o = \omega/C_c$, $k_1 = \omega/C_1$, then we would have to give k_o, k_1 small <u>negative</u> imaginary parts in order to secure a strip of overlap in which to take F.T.'s in x.

The point of using half-range transforms is that conditions are different according as $x \gtrless 0$, so that a two sided-transform cannot immediately be applied. For $x > 0$ we have the equation of motion

$$\frac{d^2y}{dx^2} + k_1^2 \ y = 0 . \tag{2.16}$$

Multiply by exp(isx) and integrate from 0 to ∞. Then the last term produces

$$k_1^2 \ Y_+(s)$$

provided $s\varepsilon R_+$, while the first term can be integrated by parts to give

$$\int_0^\infty \frac{d^2y}{dx^2} e^{isx} \, dx = \left[\frac{dy}{dx} e^{isx}\right]_0^\infty - is \ [ye^{isx}]_0^\infty - s^2Y_+(s)$$

From the anticipated behavior, $y \sim \exp(\cdot ik_1 x)$ as $x \to +\infty$ we see that $\frac{dy}{dx} e^{isx}$ and ye^{isx} both vanish as $x \to +\infty$ provided $s\varepsilon R_+$. If we write

$$y(0+) = \lim_{x \to 0+} y(x), \ y'(0+) = \lim_{x \to 0+} y'(x) \tag{2.17}$$

then we have

$$(s^2 - k_1^2) \ Y_+(s) = -y'(0+) + isy(0+) \tag{2.18}$$

as a statement, for $s\varepsilon R_+$, of the equation of motion in $x > 0$.

Applying the same procedure, this time for $s\varepsilon R_-$, to the equation

$$\frac{d^2y}{dx^2} + k_o^2 \ y = 0 \tag{2.19}$$

which holds in $x < 0$, produces

$$(s^2 - k_o^2) \ Y_-(s) = + y'(0-) - isy(0-) \tag{2.20}$$

The system (2.18), (2.20) is completed when the boundary condition at $x = 0$ is specified. In the present simplest case, in which there can be no difference of transverse force from $x = 0-$ to $x = 0+$, the boundary condition is that

$$\lim_{x \to 0-} T \frac{d}{dx}(y + e^{ik_o x}) = \lim_{x \to 0+} T \frac{dy}{dx} \quad , \tag{2.21}$$

so that

$$y'(0-) + ik_o = y'(0+) \tag{2.22}$$

while the total displacement of the string must also be continuous, so that

$$\lim_{x \to 0-} (y + e^{ik_o x}) = \lim_{x \to 0+} y \tag{2.23}$$

or

$$y(0-) + 1 = y(0+) \tag{2.24}$$

Eliminate say $y(0-)$ and $y'(0-)$ from (2.20) by using (2.22) and (2.24), giving

$$(s^2 - k_o^2)\, Y_-(s) = y'(0+) - isy(0+) + i(s - k_o) \tag{2.25}$$

and if we now add this is to (2.18), the unknowns $y'(0+)$ and $y(0+)$ disappear and we have

$$(s^2 - k_1^2)\, Y_+(s) + (s^2 - k_o^2)\, Y_-(s) = i(s - k_o) \ .$$

It is convenient to divide through by $(s^2 - k_o^2)$, which we may do because the equation itself is only meaningful in the strip D and the zeros $\pm k_o$ lie outside that strip. Then we get a standard form of Wiener-Hopf functional equation

$$K(s)\, Y_+(s) + Y_-(s) = \frac{1}{s+k_o} \tag{2.26}$$

where the kernel is

$$K(s) = \frac{s^2 - k_1^2}{s^2 - k_o^2} \quad . \tag{2.27}$$

The W-H equation relates a linear combination of the unknown half-range transforms $Y_\pm(s)$ to a function related to the incident wave field $\exp ik_o x$. The equation holds in the strip of overlap D, and the coefficients, $K(s)$ and $(s + k_o)^{-1}$, are analytic and non-zero in the interior of the strip in the simplest cases, though cases in which $K(s)$ has zeros or poles in the interior of D can also be handled straighforwardly.

In the next section we show how the general situation of the W-H equation can be obtained by inspection and how the Fourier transform integrals can be inverted to yield explicit solutions for the transmitted and reflected waves. Following that we look at differences which arise when the boundary conditions at $x = 0$ are changed, and when the strings are replaced by elastic beams.

3. SOLUTION OF THE W-H EQUATION

Assuming that we have obtained an equation (2.26) with the coefficient of one or other of $Y_\pm(s)$ reduced to unity, the first crucial point lies in the W-H FACTORIZATION of $K(s)$. In this we express

$$K(s) = K_+(s)\, K_-(s) \tag{3.1}$$

as the product of two functions of which $K_+(s)$ is analytic and non-zero in R_+ and of at most algebraic growth at infinity in R_+, while $K_-(s)$ is analytic and non-zero in R_- and of at most algebraic growth at infinity

there. (Here we assume that $K(s)$ has no zeros in the strip D, a point we shall examine again later.) It is remarkable that such a factorization exists for any $K(s)$ which is analytic and non-zero in D but which may have any kind of singular behavior outside D; the proof was given by Wiener and Hopf [1]. Here the truth of the theorem is obvious;

$$K_+(s) = \left(\frac{s+k_1}{s+k_o}\right) \quad , \; K_-(s) = \left(\frac{s-k_1}{s-k_o}\right) \tag{3.2}$$

is one factorization with the required properties, and is such that $K_\pm(s) \to 1$ at infinity in $R_\pm$. After we have completed the solution we shall return to the uniqueness or otherwise of this factorization.

Because $K_-(s)$ is free from zeros in R_-, and in particular in D, we can divide (2.26) through by it, to get

$$K_+(s)\; Y_+(s) + \frac{Y_-(s)}{K_-(s)} = \frac{i}{(s+k_o)\; K_-(s)} \tag{3.3}$$

The analyticity properties of the terms on the left here are known, while the function on the right is neither a ⊕ function nor a ⊖ function. Our next object is to write it as the <u>SUM</u> of the two functions analytic in R_+ and R_- respectively, and of algebraic behavior at ∞ in those half-planes. Again, the existence of this ADDITIVE SPLIT is assured by the W-H theorem [1], and here we can again see how to perform the split by inspection (and in this case the method of inspection is widely useful and should be carefully noted).

The function $i/(s+k_o)\, K_-(s)$ is analytic in R_- except for the pole at $s = -k_o$. Near the pole, the function behaves like $i/(s+k_o)\, K_-(-k_o)$, so that we can isolate the pole behavior by adding and subtracting this term, to give

$$\frac{1}{(s + k_o)\, K_-(s)} = \frac{1}{(s + k_o)}\left\{\frac{1}{K_-(s)} - \frac{1}{K_-(-k_o)}\right\} + \frac{1}{(s + k_o)\, K_-(-k_o)} \tag{3.4}$$

The first term no longer has a pole at $s = -k_o$, for near there

$$\frac{1}{K_-(s)} - \frac{1}{K_-(-k_o)} = (s + k_o) \times \text{(function analytic at } -k_o) ,$$

and it is therefore a $\ominus$ function $G_-(s)$ say, while the correction term $1/(s + k_o)\, K_-(-k_o)$ is evidently a $\oplus$ function, $G_+(s)$ say. Note that this additive split is not restricted to any particular form of $K_-(s)$, but turns on the presence of a <u>pole</u> term $(s + k_o)^{-1}$ only.

We now rewrite the equation (3.3) as

$$K_+(s)\, Y_+(s) - G_+(s) = G_-(s) - \frac{Y_-(s)}{K_-(s)} \quad , \tag{3.5}$$

and consider the function E(s) defined by

$$E(s) = K_+(s)\, Y_+(s) - G_+(s) \ . \tag{3.6}$$

This is a function defined and analytic throughout R_+ and of algebraic behavior (algebraic growth or decay) at ∞ in R_+. E(s) is not defined by (3.6) except in R_+. However, within D, E(s) can equally well be defined by

$$E(s) = G_-(s) - \frac{Y_-(s)}{K_-(s)} \tag{3.7}$$

and this definition then CONTINUES ANALYTICALLY the function E(s), defined originally only in R_+ by (3.6), through the strip D of overlap into the lower half-plane R_-, and there E(s) is also analytic and of algebraic growth or decay at infinity.

Thus we have a function $E(s)$, defined by (3.6) in R_+ and (3.7) in R_-, these two definitions being identical throughout D, which is analytic and of at most algebraic growth at infinity in the entire complex plane. According to the extended version of Liouville's theorem, the most general such function is a polynomial $P(s)$, and hence the most general solution to the W-H equation (2.26) is

$$\begin{aligned} Y_+(s) &= \frac{G_+(s) + P(s)}{K_+(s)} , \\ Y_-(s) &= K_-(s)\,[G_-(s) - P(s)] \end{aligned} \tag{3.8}$$

for any polynomial $P(s)$.

In most applications the polynomial is fixed in degree, and sometimes also completely, from considerations of "edge conditions" at $x = 0$. The edge conditions determine the behavior of $Y_\pm(s)$ at infinity in $R_\pm$. This is seen most clearly if we go to infinity along the imaginary axis. Let $s = iu$ with u purely real and positive. Then

$$Y_+(iu) = \int_0^\infty y(x)\,\exp(-ux)\,dx \tag{3.9}$$

which is the one-sided Laplace transform of $y(x)$. When $u \to +\infty$ only small values of x make any contribution to the integral, and in fact WATSON's LEMMA [see, e.g.,9] states that under appropriate conditions, the asymptotic behavior of $Y_+(iu)$ as $u \to +\infty$ is obtained by inserting in (3.9) the asymptotic expansion of $y(x)$ as $x \to 0+$ and integrating term by term. The process can also be used in an inverse fashion to find the behavior of $y(x)$ as $x \to 0+$ by examining the behavior of $Y_+(iu)$ as $u \to +\infty$[14]. Note, however, that the behavior of $y(x)$ as $x \to 0+$ is NOT simply related to the behavior of the full-range transform $Y(s)$

for large values of s. If one knows $Y(s)$, one has then to decompose $Y(s)$ into $Y_+(s) + Y_-(s)$ and then look at $Y_+(s)$ for large values of s, a procedure explained in detail in [14].

In a similar way, the asymptotics of $y(x)$ as $y \to 0-$ are determined by those of $Y_-(-iv)$ as $v \to +\infty$, using Watson's lemma.

Now in our problem we assume that the deflection y in the reflected or transmitted wave is finite as $x \to 0\pm$. Then the leading order term in the expansion of $Y_+(iu)$ is

$$(\text{Const}) \int_0^\infty \exp(-ux)\, dx = (\text{Const})u^{-1} \tag{3.10}$$

and similarly for $Y_-(-iv)$, so that $Y_\pm(x)$ are each $O(s^{-1})$ at infinity in $R_\pm$, respectively. $K_\pm(s)$ each tend to 1 at infinity, while $G_\pm(s)$ are each $O(s^{-1})$. It then follows from (3.6) that $E(s)$, which is the polynomial $P(s)$, is $O(s^{-1})$ at infinity in R_+, and from (3.7) that it is $O(s^{-1})$ at infinity in R_-, and hence, because $R_\pm$ have a common strip, $P(s)$ is a polynomial which vanishes everywhere at infinity like s^{-1}. Therefore $P(s)$ must be identically zero, and the solution subject to the condition of finiteness at the junction $x = 0$ is

$$\left.\begin{aligned} Y_+(s) &= G_+(s)/K_+(s) \\ Y_-(s) &= G_-(s)\, K_-(s) \end{aligned}\right\} \tag{3.11}$$

and in explicit form this gives

$$\left.\begin{aligned} Y_+(s) &= \frac{2ik_o}{(s+k_1)(k_o+k_1)} \quad , \\ Y_-(s) &= \frac{i}{s+k_o} - \frac{2ik_o}{(s+k_o)(k_o+k_1)}\left(\frac{s-k_1}{s-k_o}\right) \end{aligned}\right\} \tag{3.12}$$

Before proceeding to the inversion of the Fourier transform we return to the question of the uniqueness of the $K_{\pm}(s)$. Let one specific factorization be $K_+(s)\ K_-(s)$. Then in any other factorization, the factors can be written

$$[A_+(s)\ K_+(s)][A_-(s)\ K_-(s)]$$

where, since $K_{\pm}(s)$ are analytic and free of zeros in $R_{\pm}$, $A_{\pm}(s)$ must also be analytic and free of zeros in $R_{\pm}$, respectively. Further,

$$A_+(s)\ A_-(s) = 1$$

for $s\epsilon D$, and because $A_-(s)$ has no zeros in R_-

$$A_+(s) = \frac{1}{A_-(s)}$$

again for $s\epsilon D$. Define

$$F(s) = A_+(s) \qquad s\epsilon R_+$$

$$F(s) = \frac{1}{A_-(s)} \qquad s\epsilon R_-$$

Then $F(s)$ is analytic throughout the whole s-plane (an ENTIRE function). Suppose further that the factors are required to have some specified algebraic behavior at infinity in that respective half-plane. Then $A_{\pm}(s)$ each tend to <u>constant</u> values at infinity in $R_{\pm}$, and the entire function $F(s)$ has constant values <u>everywhere</u> at infinity. By Liouville's theorem

the only such function is $F(s)$ = Constant F_o, so that if $K_+(s)$, $K_-(s)$ are one pair of factors, any other pair must be of the form

$$F_o\, K_+(s),\ F_o^{-1}\, K_-(s)\ .$$

In other words, a factorization with prescribed algebraic behavior at infinity is unique up to multiplication of $K_+(s)$ by a constant F_o and division of $K_-(s)$ by the same constant F_o. Note the importance of the restriction to algebraic behavior at infinity. In many applications, part of $K(s)$ can be represented as an infinite product from which the split $K_+(s)\ K_-(s)$ can be effected by inspection. Usually, however, the infinite series of factors which gives $K_+(s)$ or $K_-(s)$ has <u>exponential</u> behavior at infinity in R_+ or R_-, and then it is necessary to divide say $K_+(s)$ by an entire function with the appropriate exponential behavior at infinity in R_+, so that the resulting factor behaves algebraically, at the same time multiplying $K_-(s)$ by the same factor to eliminate exponential behavior at infinity in R_-. Several examples of this are given in the book by Noble [5].

4. INVERSION OF THE FOURIER INTEGRALS

From (3.12), the full-range transform of the scattered field $y(x)$ is given by

$$Y(s) = Y_+(s) + Y_-(s) \; ,$$

and

$$y(x) = \frac{1}{2\pi} \int_C Y(s) \exp(-isx) \, ds$$

where C runs from $-\infty$ to $+\infty$ in the strip D. Since $Y_\pm(s)$ behave algebraically at infinity in $R_\pm$, the convergence of the integral is dictated by the $\exp(-isx)$ factor.

For $x > 0$, close the contour C with a large semi-circle in the lower half-plane. The contribution from this semi-circle vanishes as the radius becomes infinite because $\exp(-isx)$ is exponentially small for $x > 0$ and $\text{Im } s < 0$. (Again, a more careful proof of this should really be given.) Further, inside the closed contour consisting of C and the large semi-circle, $Y_-(s)$ is analytic and by Cauchy's theorem makes zero contribution to the integral. $Y_+(s)$ does have singularities in R_-, but these consist in fact of just a simple pole at $s = -k_1$. Noting that the contour is described clockwise rather than counter-clockwise, we have

$$y(x > 0) = \frac{1}{2\pi} (-2\pi i)(\text{Residue of } Y_+(s) \text{ at } s = -k_1) \quad \exp(ik_1 x)$$

$$= (-i) \frac{2ik_0}{k_0 + k_1} \exp(ik_1 x)$$

$$= \tau \exp(ik_1 x)$$

where the transmission coefficient is

$$\tau = \frac{2k_o}{k_o + k_1} \tag{4.1}$$

For $x < 0$, complete the contour C with a large semi-circle in R_+. There is no contribution from the large semi-circle, and application of Cauchy's theorem now gives

$$y(x < 0) = \frac{1}{2\pi} (+ 2\pi i)(\text{Residue of } Y_-(s) \text{ at } s = + k_o) \quad \exp(- ik_o x)$$

$$= + i \frac{(2ik_o)}{(2k_o)(k_o + k_1)} (+ k_o - k_1) \exp(- ik_o x)$$

$$= R \exp(- ik_o x)$$

with a reflection coefficient

$$R = \left(\frac{k_o - k_1}{k_o + k_1}\right). \tag{4.2}$$

It is a trivial matter to check that these solutions for the reflected and transmitted waves satisfy the conditions (2.23) and (2.24).

In the kinds of problems encountered in acoustics, the function $Y_+(s)$ usually has a branch point singularity in R_- as well as simple poles, and correspondingly $Y_-(s)$ may have one or more branch points and simple poles in R_+. The pole contributions are unaffected (except in certain critical circumstances) and give rise to natural modes of the system, analogous to the reflected and transmitted waves found here. When the dissipation factors are small, some poles will lie close to the real axis and give rise to propagating modes, as here. Others may lie close to the imaginary axis and give non-propagating modes, of the kind found in wave-guide problems and in the motion of plates and beams. In other problems poles may be present in the complex plane

and yet may never be captured in the appropriate deformations of the integration path which will have to be made, and such poles then represent no distinct and identifiable structure of the field. Pole contributions also serve, in acoustics problems, to represent the abrupt changes that would occur according to geometrical optics as one crosses boundaries between illuminated, reflected, and shadow wave zones. These pole contributions have to be supplemented in more complicated problems by integrals around the branch cuts which join branch points. There are many techniques for estimating the contributions from branch cut integrals [9,10,11,12] including cases where various kinds of singularities come close together, and even coincide. Generally branch cut integrals represent forced near-field behavior, which decays algebraically away from a junction or discontinuity of the two-part system usually studied by the W-H technique, leaving only the natural propagating modes at large distances.

5. DIFFERENT CONDITIONS AT $x = 0$

Suppose now that the strings are each stretched to tension T, but are joined by a mass m which is free to slide on a smooth wire perpendicular to the strings. The condition of continuity of displacement of the string remains in force, so that as in (2.24)

$$y(0-) + 1 = y(0+) \tag{5.1}$$

while the equation of motion of the particle is

$$T\frac{dy}{dx}(0+) - T\frac{d}{dx}\left(y + e^{+ik_o x}\right)(0-) = -m\omega^2\, y(0+) \tag{5.2}$$

since $y(0+)e^{-i\omega t}$ is the particle displacement. We now find that it is impossible to eliminate all of $y(0\pm)$, $y'(0\pm)$ from the equations (2.18), (2.20), (5.1), and (5.2), which are statements of the equations of motion and the boundary conditions. The simplest W-H equation one can get, replacing (2.26), turns out to be

$$K(s)\, Y_+(s) + Y_-(s) = \frac{i}{s + k_o} + \frac{m\omega^2}{T}\,\frac{y(0+)}{s^2 - k_o^2} \tag{5.3}$$

containing two unknown functions, $Y_\pm(s)$, and the unknown constant $y(0+)$ also. Supposing $y(0+)$ were known, however, we proceed as before. The additive split of $i/(s + k_o)\, K_-(s) = G_+(s) + G_-(s)$ has been given in (3.4); for the other term in (5.3) we proceed in a similar fashion and get

$$\frac{m\omega^2}{T}\,\frac{y(0+)}{(s^2 - k_o^2)\, K_-(s)} = H_+(s) + H_-(s) \tag{5.4}$$

where

$$H_+(s) = \frac{m\omega^2}{T}\, y(0+)\, \frac{1}{(s+k_o)}\, \frac{1}{(-2k_o)\, K_-(-k_o)}$$

$$H_-(s) = \frac{m\omega^2}{T}\, y(0+)\, \frac{1}{(s+k_o)} \left[\frac{1}{(s-k_o)\, K_-(s)} - \frac{1}{(-2k_o)\, K_-(-k_o)} \right] \tag{5.5}$$

The entire function E(s) is again zero and thus we find

$$K_+(s)\, Y_+(s) = \frac{i}{(s+k_o)\, K_-(-k_o)} - \frac{m\omega^2}{2k_o T}\, y(0+)\, \frac{1}{(s+k_o)\, K_-(-k_o)} \tag{5.6}$$

From this equation we can now determine the value of $y(0+)$, for we recall from §3 that the behavior of $Y_+(s)$ as $s \to \infty$ in R_+ is related to the behavior of $y(x)$ as $x \to 0+$ — i.e., to $y(0+)$. As $s \to \infty$ in R_+ we have

$$\begin{aligned} Y_+(s) &\sim \int_0^\infty y(0+)\, \exp(isx)\, dx + \ldots \\ &= \frac{iy(0+)}{s} + \ldots . \quad (\text{provided } \mathrm{Im}\, s > 0) \end{aligned} \tag{5.7}$$

while $K_+(s) \to 1$ as $s \to \infty$ in R_+. Therefore the leading terms of (5.6) state that

$$\frac{iy(0+)}{s} = \frac{1}{sK_-(-k_o)} - \frac{m\omega^2}{2k_o T}\, \frac{y(0+)}{s}\, \frac{1}{K_-(-k_o)} \tag{5.8}$$

which requires

$$y(0+) \left\{ i + \frac{m\omega^2}{2k_o T\, K_-(-k_o)} \right\} = \frac{i}{K_-(-k_o)} \tag{5.9}$$

The solution can now be completed in precisely the same manner as before. This method—of examining the detailed behavior, possibly to several terms, in the expansion of $Y_+(s)$ as $s \to \infty$ in R_+ —is

frequently used to determine unknown constants arising from boundary conditions. Another common method, which is illustrated in the next section on waves in beams, involves obtaining a solution like (5.6) for general values of the unknown constants, and then arguing that unless the constants have certain special values, a plus function will have a pole somewhere in R_+ or a minus function will have a pole somewhere in R_-. In complicated problems involving coupled elastic plate/acoustic fluid motions the plus or minus function which apparently has the singularity may not be the obvious Y_+ or Y_- function, but some more complicated related function.

6. WAVES ON BEAMS

We now replace the strings of §2 by beams of specific mass ρ_o, ρ_1 and bending stiffness B_o, B_1 in $x < 0$, $x > 0$, respectively. The free wavenumbers will be denoted by k_o, k_1 as before, for time dependence $\exp(-i\omega t)$, where

$$k_o^4 = (\rho_o\omega^2/B_o) \ , \ k_1^4 = (\rho_1\omega^2/B_1). \tag{6.1}$$

For the moment we leave conditions at the junction $x = 0$ unspecified and take a wave with displacement $\exp(ik_o x)$ incident from $x = -\infty$, denoting the <u>total</u> displacements again by $y + \exp(ik_o x)$ in $x < 0$ and by y in $x > 0$. This ensures exponential decay of $y(x)$ as $x \to +\infty$ and as $x \to -\infty$ provided Im k_o, Im k_1 are given small positive values, for we anticipate that in $x > 0$

$$y(x) = A \exp(ik_1 x) + B \exp(-k_1 x)$$

and (6.2)

$$y(x) = C \exp(-ik_o x) + D \exp(k_o x).$$

The near-field terms here, $B \exp(-k_1 x)$ and $D \exp(k_o x)$, decay as $x \to +\infty$ and as $x \to -\infty$, respectively, because k_o and k_1 have positive real parts; they arise because the equations of motion are

$$\frac{d^4y}{dx^4} - k_o^4 \, y = 0 \qquad \text{in } x < 0$$

(6.3)

$$\frac{d^4y}{dx^4} - k_1^4 \, y = 0 \qquad \text{in } x > 0$$

Taking half-range Fourier transforms of (6.3) gives

$$(s^4 - k_1^4)\, Y_+(s) = y'''(0+) - isy''(0+) - s^2y'(0+) + is^3y(0+) \tag{6.4}$$

$$(s^4 - k_o^4)\, Y_-(s) = -y'''(0-) + isy''(0-) + s^2y'(0-) - is^3y(0-) \tag{6.5}$$

the first of these holding in R_+, the second in R_-. Now the poles $s = +k_1$, $s = +ik_1$ lie in R_+ (see Figure 3), so that $Y_+(s)$ will have poles at $+k_1$, $+ ik_1$ unless we choose

$$y'''(0+) - ik_1\, y''(0+) - k_1^2\, y'(0+) + ik_1^3\, y(0+) = 0 \tag{6.6}$$

$$y'''(0+) + k_1\, y''(0+) + k_1^2\, y'(0+) + k_1^3\, y(0+) = 0 \tag{6.7}$$

Similarly, $Y_-(s)$ will have poles at $s = -k_o$, $s = -ik_o$ in R_- unless

$$-y'''(0-) - ik_o\, y''(0-) + k_o^2\, y'(0-) + ik_o^3 y(0-) = 0 \tag{6.8}$$

$$-y'''(0-) + k_o\, y''(0-) - k_o^2\, y'(0-) + k_o^3\, y(0-) = 0 \tag{6.9}$$

Whatever the boundary conditions at $x = 0$, conditions (6.6-6.9) must be satisfied. (Two analogous relations could have been deduced in §2, but it seemed unecessary to emphasize such a point at an early stage.)

Four further conditions may be imposed at $x = 0$, at least one of these being a nonhomogenous condition, so that we shall have a uniquely soluble set of eight equations for the eight unknown boundary constants. We shall not take any particular set of boundary conditions, as these tend only to lead to complicated expressions without any special structure. By addition of (6.4) and (6.5) we get

$$(s^4 - k_1^4)\, Y_+(s) + (s^4 - k_o^4)\, Y_-(s) = Q_1(s) + Q_o(s) \tag{6.10}$$

where

$$\left.\begin{aligned} Q_1(s) &= is^3y(0+) - s^2y'(0+) - isy''(0+) + y'''(0+) \\ Q_o(s) &= -is^3y(0-) + s^2y'(0-) + isy''(0-) - y'''(0-) \end{aligned}\right\} \qquad (6.11)$$

and conditions (6.6-6.9) are satisfied, so that

$$\left.\begin{aligned} Q_1(k_1) &= Q_1(ik_1) = 0 \\ Q_o(-k_o) &= Q_o(-ik_o) = 0 \end{aligned}\right\} \qquad (6.12)$$

Equation (6.10) holds in the strip

$$D: -k_{1i} < \text{Im } s < + k_{oi} \quad .$$

For simplicity we can take $k_{oi} = k_{1i}$, so that the strip D is symmetric about the real axis and the points $+k_o$, $+k_1$ lie on the upper boundary of D, the points $-k_o$, $-k_1$ on its lower boundary. Because k_{oi} is supposed to be very small, the other points of interest, $+ik_o$, $+ik_1$, and $-ik_o$, $-ik_1$, lie well above and well below D, respectively. If therefore we work in the interior of D we can divide (6.10) through by $(s^4 - k_o^4)$ say, and get a W-H equation

$$K(s)\ Y_+(s) + Y_-(s) = \frac{Q_1(s) + Q_o(s)}{(s^4 - k_o^4)} \qquad (6.13)$$

in which the kernel is

$$K(s) = \frac{s^4 - k_1^4}{s^4 - k_o^4} \qquad (6.14)$$

The W-H product split, into factors analytic, non-zero and of algebraic behavior at infinity in $R_\pm$, respectively, is again obvious:

$$K_+(s) = \frac{(s + k_1)(s + ik_1)}{(s + k_o)(s + ik_o)}, \quad K_-(s) = \frac{(s - k_1)(s - ik_1)}{(s - k_o)(s - ik_o)} \tag{6.15}$$

Then division by $K_-(s)$ ($K_-(s) \neq 0$ in R_- and so $\neq 0$ in D) gives

$$K_+(s)\ Y_+(s) + \frac{Y_-(s)}{K_-(s)} = \frac{Q_1(s) + Q_0(s)}{(s + k_o)(s + ik_o)(s - k_1)(s - ik_1)} \tag{6.16}$$

and we have to make an <u>additive</u> split of the right hand side. Now because of (6.12), $Q_o(s)$ must contain the factors $(s + k_o)(s + ik_o)$, while $Q_1(s)$ must contain the factors $(s - k_1)(s - ik_1)$ so that we can write

$$\left.\begin{aligned} Q_o(s) &\equiv (s + k_o)(s + ik_o)(a_o s + b_o) \\ Q_1(s) &\equiv (s - k_1)(s - ik_1)(a_1 s + b_1) \end{aligned}\right\} \tag{6.17}$$

where the coefficients a_o, a_1, b_o, b_1 are known when any particular set of conditions is specified at $x = 0$. Now the right side of (6.16) has the form

$$\frac{a_o s + b_o}{(s - k_1)(s - ik_1)} + \frac{a_1 s + b_1}{(s + k_o)(s + ik_o)}$$

which is already in the desired form $G_-(s) + G_+(s)$. The reason for this is that analyticity arguments have already been used to remove pole singularities where they are not permitted and because pole singularities are the only kinds of singularity which are present in these one-dimensional problems this effectively means that the additive split must already have been carried out.

By the usual arguments we then have the solutions

$$\left.\begin{aligned} K_+(s)\, Y_+(s) - G_+(s) &= P(s) \\ G_-(s) - \frac{Y_-(s)}{K_-(s)} &= P(s) \end{aligned}\right\} \tag{6.18}$$

where $P(s)$ is a polynomial. And because $G_\pm(s) = O(s^{-1})$, $K_\pm(s) = O(1)$ and $Y_\pm(s) = O(s^{-1})$ (because y is finite at $x = 0$) at infinity in their respective half-planes of analyticity, the polynomial $P(s)$ must be identically zero, so that

$$\begin{aligned} Y_+(s) &= \frac{a_1 s + b_1}{(s + k_1)(s + ik_1)} , \\ Y_-(s) &= \frac{a_o s + b_o}{(s - k_o)(s - ik_o)} \end{aligned} \tag{6.19}$$

The inverse transforms can then be performed explicitly, the poles $s = k_o$ and $s = ik_o$ in R_+ gives rise to the reflected wave and a decaying mode in $x < 0$, the poles $s = -k_1$ and $-ik_1$ gives the transmitted wave and a decaying mode in $x > 0$. Specifically we find

$$\begin{aligned} y(x > 0) &= i\,\frac{(a_1 k_1 - b_1)}{k_1(1 + i)} \exp(ik_1 x) \\ &\quad + i\,\frac{(a_1 ik_1 - b_1)}{k_1(1 - i)} \exp(-k_1 x) \\ y(x < 0) &= i\,\frac{(a_o k_o + b_o)}{k_o(1 - i)} \exp(-ik_o x) \\ &\quad - i\,\frac{(a_o ik_o - b_o)}{k_o(1 - i)} \exp(+k_o x) \end{aligned} \tag{6.20}$$

which is of the anticipated form (6.2). For any specified boundary conditions the values of y and its first three derivatives are known at $x = 0_\pm$, so that the coefficients a_o, b_o, a_1, b_1 in (6.20) can be

obtained from a direct comparison of (6.17) with (6.11). But of course, the W-H was never intended to be used for solving such simple problems as these one-dimensional ones. We are using them as the simplest vehicle on which many features of common occurrence in the W-H method can be demonstrated.

7. GENERALIZED W-H EQUATIONS: THREE-PART BOUNDARY VALUE PROBLEMS

The standard W-H equation, (2.26), arises in many, though by no means all, boundary value problems in which different boundary data are prescribed on, say, $x < 0$ and $x > 0$. Many problems of interest in acoustics fall into this category; for instance, the problem of energy conversion from the elastic to the acoustic mode when a surface wave in an elastic plate encounters a junction in the plate across which the plate properties change abruptly can be modeled in terms of two semi-infinite plates $x < 0$ and $x > 0$ for many purposes. The boundaries concerned do not always have to coincide with just the x-axis. For example, the standard W-H equation arises in the diffraction of waves by an open-ended parallel plate waveguide, or by an open ended circular duct, provided these are both semi-infinite. On the other hand, diffraction by three parallel equi-spaced semi-infinite plates is a completely open problem, though diffraction by an infinite cascade of semi-infinite staggered plates is a relatively simple standard W-H problem. If the waveguides referred to above have closed ends (diffraction by a semi-infinite thick rigid plate, or by a semi-infinite solid rod) the standard W-H method does not lead to a closed form solution, but to an infinite set of coupled linear equations, whose solution can only be approximated by the solution of a finite subset of the equations in the low frequency limit.

Thus it is clearly difficult to give any general guidelines as to when the standard W-H method would work except to say that it will not work, without modification, in the case of three-part boundary value problems where data is given on, say, $-\infty < x < 0$; $0 < x < \ell$;

$\ell < x < +\infty$ in three different forms. There are, however, a number of problems which have been successfully tackled—though always only in approximate form—in this category, using modified W-H methods. We quote as examples the problems of diffraction by an infinite rigid strip of finite width in the high frequency limit, of the resonances of a circular tube of finite length open at both ends and of wave motion in an elastic plate set in an infinite rigid surrounding baffle.

Now in many of these problems the approximate solution is achieved on the basis of a "weak interaction" simplification. For example, in diffraction of waves of wavenumber k_o by a rigid strip of width ℓ in the high frequency limit, $(k_o\ell) >> 1$, one can suppose that to first order each edge is unaware of the presence of the other, so that one can start the approximation with two semi-infinite problems of standard W-H type. Then if the incident wave amplitude is $O(1)$, the first interaction of one edge with the other will be through a cylindrical wave emanating from one edge, due to the incident wave, and of amplitude $O((k_o\ell)^{-\frac{1}{2}})$ near the other edge. For the second approximation we therefore solve another two semi-infinite problems, but now with more complicated forcing terms arising from the mutual interaction between the edges. While one can see how to continue the process in that simple case, it is advantageous to derive a modified W-H equation whose approximate solution throws up these successive interaction problems in a natural way. The advantage is that one can see from the generalized W-H equation what to try in more complicated problems where the physical situation is less clear. For example, near <u>resonance</u> the "weak interaction" sort of approximation is quite inappropriate, as the

whole phenomenon depends on strong coupling between the ends of the system.

We shall try here to use our one-dimensional examples to illustrate the possibility of generalizing the W-H technique to deal with three-part problems. For such one-dimensional problems, however, there is not generally any "weak interaction" approximation that one can make, because waves on strings do not decay in the way that two and three dimensional acoustic fields do, so that in a sense one is always confronted with the strong coupling situation. Nonetheless, a number of interesting points arise in the string problems which have direct comparisons in more serious three-part problems.

Consider a uniform string of line density ρ_o in $-\infty < x < 0$ and in $\ell < x < +\infty$, these two semi-infinite strings being joined by a string of line density ρ_1 in $0 < x < \ell$. A wave with displacement $\exp ik_o x$ is incident from $x = -\infty$, and we want to find the displacements everywhere, subject to conditions at $x = 0$ and at $x = \ell$ which we will specify later.

Write the total displacement in $x < 0$ as $y + \exp ik_o x$, so that y represents an outgoing field as $x \to -\infty$. Define

$$Y_-(s) = \int_{-\infty}^{0} y(x) \exp isx \, dx \tag{7.1}$$

so that $Y_-(s)$ is analytic in R_- ($\text{Im } s < k_{oi}$) and of algebraic decrease everywhere at infinity in R_- (because $y(x)$ is <u>finite</u> at $x = 0$). From the differential equation

$$\frac{d^2y}{dx^2} + k_o^2 y = 0 \qquad (-\infty < x < 0)$$

we have

$$(s^2 - k_o^2)\, Y_-(s) = + y'(0-) - isy(0-) \tag{7.2}$$

and in order to avoid having a pole at $s = -k_o \varepsilon R_-$ we must have

$$+ y'(0-) + ik_o\, y(0-) = 0 \tag{7.3}$$

Also as a check, as $|s| \to \infty$ in R_-,

$$Y_-(s) \sim - \frac{i}{s}\, y(0-)$$

which we also get from

$$Y_-(s) \sim \int_{\infty}^{0} y(0-) + \ldots)\, e^{isx}\, dx \tag{7.4}$$

using Watson's lemma.

The same differential equation holds in $\ell < x < \infty$. Define

$$Y_+(s) = \int_{\ell}^{\infty} y(x) \exp isx\, dx \tag{7.5}$$

$y(x)$ being the total displacement in $\ell < x < \infty$. Then $Y_+(s)$ is analytic in the upper half-plane

$$R_+ : \text{Im } s > - k_{oi}$$

because of the exponential decrease, $\exp(-k_{oi}x)$ of $y(x)$ as $x \to +\infty$. However, $Y_+(s)$ does not now have algebraic behavior at infinity in R_+. To find the behavior we write

$$Y_+(s) = \int_0^\infty y(X + \ell)\ e^{is(X+\ell)}\ dX$$

$$= e^{is\ell}\int_0^\infty y(X + \ell)\ e^{isX}\ dX$$

$$\sim e^{is\ell}\int_0^\infty [y(\ell+) + X\ y'(\ell+) + \ldots]\ e^{isX}\ dX$$

$$= +\frac{i}{s}\ e^{is\ell}\ y(\ell+) + 0\left(\frac{e^{is\ell}}{s^2}\right) \tag{7.6}$$

as $s \to \infty$ in R_+. Thus $Y_+(s)$ has exponential behavior, exp $is\ell$, at infinity in R_+. From the differential equation we have

$$(s^2 - k_o^2)\ Y_+(s) = [-y'(\ell+) + is\ y(\ell+)]\ e^{is\ell} \tag{7.7}$$

from which (7.6) is obvious. Again, there is a pole at $s = k_o \varepsilon R_+$ unless

$$-y'(\ell+) + ik_o\ y(\ell+) = 0 \tag{7.8}$$

For the middle portion of the string we define

$$Y_1(s) = \int_0^\ell y(x)\ \exp\ isx\ dx \tag{7.9}$$

Since the integration is over a finite range only and the integrand is bounded and continuous, $Y_1(s)$ is <u>analytic in the entire complex plane</u> (the Fourier Transform of a function with compact support — i.e., vanishing outside some finite range, is an ENTIRE function). As $s \to \infty$ with Im $s > 0$,

$$Y_1(s\ ) \sim \int_0^\infty (y(0+) + xy'(0+) + \ldots)\ e^{isx}\ dx = 0(s^{-1}) \tag{7.10}$$

because when Im $s > 0$ only small values of x contribute to the integral, so that we can expand y(x) about $x = 0+$ and also extend the integration

up to ∞. As $s \to \infty$ with Im $s < 0$ we need to write $x = \ell - z$, to get

$$Y_1(s) = \exp(is\ell) \int_0^{\ell} y(\ell - z) \exp(-isz)\, dz$$

and because of the exponential $\exp(-isz)$ we can expand about $z = 0$ again to get

$$Y_1(s) \sim \exp\, is\ell \int_0^{\infty} [y(\ell-) - zy'(\ell-) + \ldots] e^{-isz}\, dz$$

$$= \exp(is\ell) \left\{ -\frac{i}{s}\, y(\ell-) + 0(\frac{1}{s^2}) \right\} \qquad (7.11)$$

Thus the entire function $Y_1(s)$ is <u>algebraically small</u>, $0(s^{-1})$ or smaller, in Im $s > 0$, but <u>exponentially large</u> like $\exp(is\ell)$, in Im $s < 0$.

The differential equation

$$\frac{d^2y}{dx^2} + k_1^2 y = 0$$

(where y is the total displacement in $0 < x < \ell$) gives

$$(s^2 - k_1^2)\, Y_1(s) = [y'(\ell-)\, e^{is\ell} - y'(0+)]$$

$$- is[y(\ell-)\, e^{is\ell} - y(0+)] \qquad (7.12)$$

and confirms (7.10) and 7.11). Further, $Y_1(s)$ can have no singularities for any finite value of s, so that the right side of (7.12) must vanish for both $s = k_1$ and $s = -k_1$, giving

$$[y'(\ell-)\, e^{ik_1\ell} - y'(0+)] - ik_1[y(\ell-)\, e^{ik_1\ell} - y(0+)] = 0\,, \qquad (7.13)$$

$$[y'(\ell-)\, e^{-ik_1\ell} - y'(0+)] + ik_1[y(\ell-)\, e^{-ik_1\ell} - y(0+)] = 0\,. \qquad (7.14)$$

To make the algebra minimal we choose a definite set of conditions at $x = 0$ and at $x = \ell$, namely the simple junction conditions that there is no change to the total displacement or to the slope at $x = 0$, $x = \ell$, the tension in all three strings being the same. Thus we take

$$\left.\begin{aligned} y(0-) + 1 &= y(0+) \\ y(\ell-) &= y(\ell+) \\ y'(0-) + ik_o &= y'(0+) \\ y'(\ell-) &= y'(\ell+) \end{aligned}\right\} \tag{7.15}$$

If $k_o \neq k_1$ the set of 8 equations (7.3, 7.8, 7.13, 7.14, 7.15) has a finite solution provided

$$\sin k_1\ell \neq 0 \tag{7.16}$$

(excluding <u>resonance</u> of the middle portion) and then

$$\left.\begin{aligned} y(\ell) &= -\frac{2ik_o k_1}{k_o^2-k_1^2}\operatorname{cosec}(k_1\ell) \\ y(0+) &= -\frac{2k_o^2}{k_o^2-k_1^2} - \frac{2ik_o k_1}{k_o^2-k_1^2}\cot k_1\ell \end{aligned}\right\} \tag{7.17}$$

etc. This of course completes the solution for these simple problems, for now that all constants are known, $Y_+(s)$ is known from (7.7), $Y_-(s)$ from (7.2) and $Y_1(s)$ from (7.12).

A method which shows how a generalized W-H equation may be treated ignores the detailed solution (7.17), and instead eliminates the unknown constants from (7.7), (7.2), and (7.12) in just the same way that the corresponding constants were eliminated in §2 to get a standard W-H equation. Thus we write

$$(s^2 - k_1^2)\, Y_1(s) = e^{is\ell}[y'(\ell-) - is\, y(\ell-)]$$

$$- [y'(0+) - is\, y(0+)]$$

$$= - (s^2 - k_o^2)\, Y_+(s)$$

$$- (s^2 - k_o^2)\, Y_-(s)$$

$$+ i(s - k_o)$$

on use of (7.15), (7.2), and (7.7), so that

$$Y_+(s) + Y_-(s) + K(s)\, Y_1(s) = \frac{i}{s + k_o} \tag{7.18}$$

where the kernel is, as in §2,

$$K(s) = \frac{s^2 - k_1^2}{s^2 - k_o^2} \tag{7.19}$$

This is the required generalized W-H equation, a single equation for <u>three</u> unknown functions $Y_+(s)$, $Y_-(s)$ and $Y_1(s)$, given the kernel $K(s)$ and the forcing field $(s + k_o)^{-1}$.

Equations of this kind have been considered by Noble [5 p. 196]. Methods exist for solving such equations approximately in "high frequency" limits in which the finite part of the boundary is many wavelengths long, in some appropriate sense. Generally these methods rely on weak interaction between the ends $x = 0$ and $x = \ell$, though as remarked before, the ends are rather delicately coupled in cases such as the resonance of a finite open-ended waveguide, and methods have also been developed to deal with such cases. Here, because only pole singularities are involved, it is possible to solve (7.18) exactly.

We carry the analyses through in a general form as far as possible, to indicate the procedure which has to be followed in more complicated problems.

First of all we extract a factor $\exp is\ell$ from $Y_+(s)$, writing

$$Y_+(s) = \exp(is\ell)\, Z_+(s) \tag{7.20}$$

so that $Z_+(s) = O(s^{-1})$ at infinity in R_+. The necessity for doing this will be apparent in a moment. Then write $K(s) = K_+(s)\, K_-(s)$ as usual, and divide through by $K_-(s)$ to get

$$\frac{e^{is\ell} Z_+(s)}{K_-(s)} + \frac{Y_-(s)}{K_-(s)} + K_+(s)\, Y_1(s) = \frac{i}{(s + k_o)\, K_-(s)}$$

The second term on the left is analytic in R_- and $O(s^{-1})$ at infinity there, the third is analytic in R_+ and $O(s^{-1})$ at infinity there. The term on the right can be split in the familiar way as

$$\frac{i}{(s + k_o)\, K_-(s)} = G_+(s) + G_-(s) \tag{7.21}$$

and we also make an additive split of the first term in this way,

$$\frac{e^{is\ell}\, Z_+(s)}{K_-(s)} = U_+(s) + U_-(s) \tag{7.22}$$

using a general theorem to be given in a moment. $G_\pm(s)$ are each $O(s^{-1})$ at infinity in $R_\pm$, and we assume that is also true of $U_\pm(s)$. Then we have

$$\begin{aligned} U_+(s) &+ K_+(s)\, Y_1(s) - G_+(s) \\ &= G_-(s) - \frac{Y_-(s)}{K_-(s)} - U_-(s) \end{aligned}$$

and each side is the representation, in R_+ or R_- as the case may be, of a single entire function which behaves like s^{-1} everywhere at infinity, and is therefore identically zero. Hence

$$K_+(s)\ Y_1(s) - G_+(s) + U_+(s) = 0 \tag{7.23a}$$

$$G_-(s) - \frac{Y_-(s)}{K_-(s)} - U_-(s) = 0\ . \tag{7.23b}$$

Now return to the generalized W-H equation, and this time, instead of dividing by $K_-(s)$, we divide by $\exp(is\ell)\ K_+(s)$ to get

$$\frac{Z_+(s)}{K_+(s)} + \frac{e^{-is\ell}\ Y_-(s)}{K_+(s)} + e^{-is\ell}\ K_-(s)\ Y_1(s) = \frac{ie^{-is\ell}}{(s + k_o)\ K_+(s)} \tag{7.24}$$

The first term is analytic in R_+ and $0(s^{-1})$ at infinity there while the third is analytic in R_- and $0(s^{-1})$ there only because of the factor $\exp(-is\ell)$. If we had not divided by $\exp is\ell$ we would have been left with a third term which was analytic in R_-, but exponentially large at infinity in R_- (see Eq. 7.11) and the function theoretic argument would not go through (in particular, the entire function would not be zero, but would be exponentially large in R_-, and there is no general way of constructing functions of this kind).

The function appearing second on the left in (7.24) has mixed properties, so we again try to split it as

$$\frac{e^{-is\ell}\ Y_-(s)}{K_+(s)} = V_+(s) + V_-(s) \tag{7.25}$$

with $V_{\pm}(s) = 0(s^{-1})$ at infinity in $R_{\pm}$. On the other hand, the function occurring on the right of (7.24) is a $+$ function—but it is exponentially large, like $\exp(s_2 \ell)$ in the upper half-plane, and so it is necessary to make a split

$$\frac{ie^{-is\ell}}{(s + k_o)\, K_+(s)} = H_+(s) + H_-(s) \tag{7.26}$$

with $H_{\pm}(s) = 0(s^{-1})$ in $R_{\pm}$ in order to remove this exponential increase.

Now we can split (7.24) into $+$ and $-$ parts, each of which is $0(s^{-1})$ at infinity in $R_{\pm}$ and each of which therefore vanishes identically. Thus we get

$$H_-(s) - V_-(s) - e^{-is\ell}\, K_-(s)\, Y_1(s) = 0\ , \tag{7.27a}$$

$$\frac{Z_+(s)}{K_+(s)} + V_+(s) - H_+(s) = 0\ . \tag{7.27b}$$

The equations (7.23a,b; 7.27a,b), obtained by the W-H argument, are not, in general, solutions to the problem, but they constitute a pair of integral equations which are in a form suitable for approximate solution in the "high frequency limit" (i.e., for $k_o\ell \gg 1$, $k_1\ell \gg 1$ in general). To see this we have to use the general formula (see Section 10) for expressing a function $F(s)$, analytic in the strip D and with suitable behavior at infinity in D, as the sum of functions analytic in $R_{\pm}$ and $0(s^{-1})$ at infinity there. If

$$F(s) = F_+(s) + F_-(s)$$

where the functions are to have the stated properties, then

$$F_+(s) = \frac{1}{2\pi i} \underset{\smile}{\int} \frac{F(t)}{t - s}\, dt \quad , \tag{7.28a}$$

$$F_-(s) = \frac{-1}{2\pi i} \underset{\frown}{\int} \frac{F(t)}{t - s}\, dt \quad , \tag{7.28b}$$

where in (7.28a) the path runs from $-\infty$ to $+\infty$ in the strip D passing <u>below</u> the point $t = s$ while in (7.28b) the path passes above $t = s$.

Applying (7.28) here we have, for (7.23b) and (7.27b) in particular,

$$G_-(s) - \frac{Y_-(s)}{K_-(s)} + \frac{1}{2\pi i} \underset{\smile}{\int} \frac{e^{it\ell}\, Z_+(t)}{K_-(t)(t - s)}\, dt = 0 \tag{7.29}$$

$$-H_+(s) + \frac{Z_+(s)}{K_+(s)} + \frac{1}{2\pi i} \underset{\smile}{\int} \frac{e^{-it\ell}\, Y_-(t)}{K_+(t)(t - s)}\, dt = 0 \; , \tag{7.30}$$

the forcing fields $G_-(s)$ and $H_+(s)$ being known, in principle. Clearly (7.29), (7.30) are a pair of coupled integral equations for the unknown functions $Y_-(s)$ and $Z_+(s)$ which determine the reflected and transmitted fields in $x < 0$, $x > \ell$, respectively. Once these are known, the field $Y_1(s)$ in the middle portion of the string can be found directly from the generalized W-H equation (7.18), for example.

To see the structure of these equations, suppose that the integral term in (7.29) were zero. Then we would have

$$Y_-(s) = K_-(s)\, G_-(s)$$

and noting the definition (7.21) of $G_-(s)$ and comparing with (3.11) we see that this $Y_-(s)$ is precisely the field in $x < 0$ if the wave $\exp(ik_0x)$ were incident upon a <u>semi-infinite</u>, rather than finite, string to the right. Similarly, the situation $Z_+(s) = K_+(s)\, H_+(s)$ is the solution for a semi-infinite problem of reflection at the

junction $x = \ell$, the left hand portion of the string now being regarded as extending to $x = -\infty$. The integral terms represent interactions between the ends $x = 0$, $x = \ell$, giving rise to a sequence of reflections and transmissions. Methods have been devised for dealing approximately with these interactions both when they are weak and when they are strong, as in resonance situations (see Noble [5 (Ch.5]).

Here it is of course possible to solve the coupled integral equations completely. First, however, we show how they may be <u>uncoupled</u> when the kernel $K(s)$ is <u>even</u> in s, as it is here. It is then possible to work in a strip D which is completely symmetrical about the real axis, so that $Y_-(-s)$ is a $\oplus$ function with the same domain R_+ of analyticity as $Z_+(s)$, while $Z_+(-s)$ is a $\ominus$ function. It also follows from Section 10 that $K_+(-s) = K_-(s)$ for $s \in D$.

Consider then (7.29) and (7.30) for $s \in D$, and choose the path $\frown$ above s to be a straight line from $-\infty + ia$ to $+\infty + ia$, $a > 0$. Choose $\smile$ to be the image of this path in the real axis, going from $-\infty - ia$ to $+\infty - ia$. In (7.29) change s to $-s$; the integration path can be chosen to run above both s and $-s$, and so

$$G_-(-s) - \frac{Y_-(-s)}{K_+(s)} + \frac{1}{2\pi i} \int_{\frown} \frac{e^{it\ell} Z_+(t)}{K_-(t)(t+s)} dt = 0 \qquad (7.31)$$

In (7.30) change the integration variable from t to $-t$; $(-t)$ runs from $+ia + \infty$ to $+ia - \infty$ as t runs from $-ia - \infty$ to $-ia + \infty$, so that the path for $(-t)$ is the same as that in (7.31). This gives

$$-H_+(s) + \frac{Z_+(s)}{K_+(s)} - \frac{1}{2\pi i} \int_{\frown} \frac{e^{it\ell} Y_-(-t)\, dt}{K_-(t)(t+s)} = 0\,, \qquad (7.32)$$

Now add (7.31) and (7.32) defining the difference

$$D_+(s) = Z_+(s) - Y_-(-s) \ ,$$

to get

$$[G_-(-s) - H_+(s)] + \frac{D_+(s)}{K_+(s)} + \frac{1}{2\pi i}\int \frac{e^{it\ell}\, D_+(t)\, dt}{K_-(t)(t+s)} = 0 \ , \tag{7.33}$$

while if we subtract (7.31) from (7.32) and define

$$S_+(s) = Z_+(s) + Y_-(-s)$$

we get

$$[-G_-(-s) - H_+(s)] + \frac{S_+(s)}{K_+(s)} - \frac{1}{2\pi i}\int \frac{e^{it\ell}\, S_+(t)\, dt}{K_-(t)(t+s)} = 0 \tag{7.34}$$

so that (7.33) and (7.34) are a pair of similar <u>uncoupled</u> integral equations.

Let us now look at the forcing functions in these equations. The additive split of G(s) in (7.21) is simple, giving

$$\left.\begin{aligned} G_+(s) &= \frac{i}{(s+k_o)\, K_-(-k_o)} \\ G_-(s) &= \frac{i}{s+k_o}\left\{\frac{1}{K_-(s)} - \frac{1}{K_-(-k_o)}\right\} \end{aligned}\right\} \tag{7.35}$$

For the function H(s) in (7.26) we have to use the Cauchy integrals, which give

$$H_+(s) = \frac{1}{2\pi i}\int \frac{ie^{-it\ell}}{(t+k_o)\, K_+(t)(t-s)}\, dt \tag{7.36}$$

We cannot complete the contour with a large semicircle in R_+, because the factor exp(-itℓ) is exponentially large there. Instead we complete the

contour with a large semicircle in R_-, along which the integral vanishes because of exponential smallness of $\exp(-it\ell)$ there. To interpret the meaning of $K_+(t)$ in R_- we write it as $K(t)/K_-(t)$, so that

$$H_+(s) = \frac{1}{2\pi i} \int \frac{ie^{-it\ell} K_-(t)}{(t + k_o) K(t)(t - s)} dt \tag{7.37}$$

The pole at $t = s$ lies outside the contour, $K_-(t)$ is analytic within the contour, and

$$(t + k_o) K(t) = \left(\frac{t - k_1}{t - k_o}\right) (t + k_1)$$

so that the only singularity within the closed contour is at $t = -k_1$, and therefore

$$H_+(s) = \frac{1}{2\pi i} (-2\pi i) \frac{ie^{ik_1\ell} K_-(-k_1)}{\left(\frac{-k_1 - k_1}{-k_1 - k_o}\right) (-k_1 - s)} = \frac{ie^{ik_1\ell}}{(s + k_1)} , \tag{7.38}$$

$$H_-(s) = \frac{ie^{-is\ell}}{(s + k_o) K_+(s)} - \frac{ie^{ik_1\ell}}{(s + k_1)} = \frac{i}{s + k_1}(e^{-is\ell} - e^{ik_1\ell}) . \tag{7.39}$$

This instructive example shows the importance of removing exponential growth at infinity; although the original $H(s)$ is analytic in R_+ it is not algebraically small at infinity there, and use of the Cauchy integrals shows how it can be split into $H_+(s) + H_-(s)$, with $H_\pm(s) = O(s^{-1})$ at infinity in $R_\pm$ as in (7.38) and (7.39).

Coming now to the integral terms in the integral equations (7.33), (7.34), we can this time complete the contour with a large semicircle in R_+, because along that semicircle the factor $\exp(it\ell)$ will be

exponentially small and the integral along the semicircle will contribute nothing. The functions $D_+(t)$, $S_+(t)$ are analytic within the closed contour, and the function $1/K_-(t) = K_+(t)/K(t)$ has a pole at $t = k_1$ only. Therefore (7.33) gives

$$[G_-(-s) - H_+(s)] + \frac{D_+(s)}{K_+(s)} + \frac{(k_1 - k_o)\, e^{ik_1\ell}\, D_+(k_1)}{(s + k_1)} = 0 \tag{7.40}$$

and (7.34) gives

$$[-G_-(-s) - H_+(s)] + \frac{S_+(s)}{K_+(s)} - \frac{(k_1 - k_o)\, e^{ik_1\ell}\, S_+(k_1)}{(s + k_1)} = 0 \tag{7.41}$$

The unknown constants $D_+(k_1)$, $S_+(k_1)$ are found by putting $s = k_1$ in (7.40), (7.41) and then we have completely determined the functions $D_+(s)$, $S_+(s)$, from which $Z_+(s)$, $Y_-(-s)$ can be found, and hence the whole solution is determined. While there are no difficulties of principle, the algebra is very tedious and there is no point in giving it here.

The only remaining point of interest concerns the inversion of the Fourier integral for this three-part problem. We have

$$y(x) = \frac{1}{2\pi}\int [Y_-(s) + Y_1(s) + Y_+(s)] \exp(-isx)\, ds \tag{7.42}$$

with the integral along a path from $-\infty$ to $+\infty$ in D, and we recall that

$$\left.\begin{aligned}
Y_-(s) &= O(s^{-1}) && \text{as } |s| \to \infty \quad \text{in } R_- \\
Y_1(s) &= O(s^{-1}) && \text{as } |s| \to \infty \quad \text{in } R_+ \\
&= O\left(\frac{e^{is\ell}}{s}\right) && \text{as } |s| \to \infty \quad \text{in } R_- \\
Y_+(s) &= O\left(\frac{e^{is\ell}}{s}\right) && \text{as } |s| \to \infty \quad \text{in } R_+
\end{aligned}\right\} \tag{7.43}$$

that $Y_{\pm}(s)$ are analytic in $R_{\pm}$, respectively, and that $Y_1(s)$ is analytic everywhere (an <u>entire</u> function). For $x < 0$, deform the contour into the upper half-plane R_+. The contributions from the large semicircle in R_+ to the Y_1 and Y_+ integrals are zero, because $\exp(-isx)$ is exponentially small and Y_1 and Y_+ are at least as small as s^{-1} at infinity in R_+. Further, Y_1 and Y_+ are analytic within the closed contour , and so their contour integrals vanish. Thus for $x < 0$

$$y(s) = \frac{1}{2\pi} \int Y_-(s) \exp(-isx)\, ds \tag{7.44}$$

which we may <u>try</u> to evaluate by completing the contour in R_+—though note that we know <u>nothing in advance</u> about the behavior of $Y_-(s)$ at infinity in R_+.

If $x > \ell$ we complete the contour with a large semicircle in R_-, along which, although $Y_1(s)$ is exponentially large, the product $Y_1(s) \exp(-isx)$ is exponentially small when $x > \ell$. Similar arguments then give

$$y(x) = \frac{1}{2\pi} \int Y_+(s) \exp(-isx)\, ds \tag{7.45}$$

Finally, if $0 < x < \ell$, close the contour for $Y_-(s)$ in R_-, that for $Y_+(s)$ in R_+, and we find that the integrals of $Y_-(s)$, $Y_+(s)$ vanish, leaving

$$y(x) = \frac{1}{2\pi} \int Y_1(s) \exp(-isx)\, ds \tag{7.46}$$

Because of (7.43) it is only possible to close the contour here in R_-, $Y_1(s) \exp(-isx)$ being exponentially large in R_+ when $0 < x < \ell$.

These arguments apply generally in three-part boundary value problems. Here the integrals (7.44-7.46) can all be evaluated by

residue calculus, since only simple poles are involved, leading again to results which can be confirmed using elementary methods.

The purpose of this section has not been to solve a particular string problem, but to illustrate how the W-H method, applied to three-part problems, leads in general not to a solution, but to a pair of coupled integral equations. For an even kernel K(s) we can decouple the equations, and deal with a pair of similar independent integral equations for the functions $D_+(s)$, $S_+(s)$. In our case the integral terms can actually be evaluated explicitly in terms of unknown constants $D_+(k_1)$, $S_+(k_1)$, which can then be determined by setting $s = k_1$ in the integral equations. A precisely similar situation exists in "near-resonance" problems, such as the scattering of acoustic waves by a long tube, open at both ends, at frequencies near resonance. There it is argued that, although the function $K_-(t)$ in the integral equation (7.33) has branch point singularities, the dominant contribution near resonance comes from a pole term. Well away from resonance it is anticipated that the dominant contribution to the integral comes from a branch point singularity which represents the rather weak acoustic interaction between the ends of the tube. The functions are expanded about the branch point, and the integral term can then again be evaluated (approximately) as the product of $D_+(-k)$ say (where -k is the branch point) and an integral which is expressible in terms of Whittaker functions (which can be further approximated in most cases). Again the constant $D_+(-k)$ can be found by setting $s = -k$ in the integral equation.

8. TWO-DIMENSIONAL HALF-PLANE PROBLEMS

We go on now to look at a simple problem involving a half-plane embedded in an acoustic fluid. In the first place the fluid will have no bulk motion, whereas later we shall allow the fluid to flow over the half-plane with uniform subsonic velocity, leaving a wake behind the plate if the edge is a trailing edge. The issue we want to examine is the following one. Suppose that the plate were infinite in both the positive and the negative x-directions and a wave were forced to propagate along the plate with some prescribed frequency ω and wavenumber q, the wavenumber q being real and greater than the acoustic wavenumber $k_o = \omega/c_o$. Thus the velocity in the positive y-direction is prescribed in the form

$$v(x,t) = v_o \exp(iqx - i\omega t) \ . \tag{8.1}$$

Then the solution for the potential $\phi(x,y)$ in the fluid in $y > 0$ is (dropping the factor $\exp - i\omega t$)

$$\phi(x,y) = -\frac{v_o}{\gamma_q} \exp(iqx - \gamma_q y) \tag{8.2}$$

where

$$\gamma_q = (q^2 - k_o^2)^{\frac{1}{2}} , \tag{8.3}$$

for this makes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_o^2\right) \phi = 0 ,$$

makes $\phi \to 0$ as $y \to +\infty$

and makes $\frac{\partial\phi}{\partial y}(x,0) = v_o \exp(iqx)$.

The field described by (8.2) is that of a <u>subsonic trapped surface wave</u>. No radiation takes place across any plane y = const. because the pressure,

$p = \rho i\omega\phi$, and the velocity $\partial\phi/\partial y$ are 90 degrees out of phase. The energy is locked in a thin layer of thickness $O(\gamma_q^{-1})$ adjacent to the surface, and none escapes as sound.

Suppose now that the surface is semi-infinite, occupying $(-\infty < x < 0,\ y = 0)$, all dependence on the z-coordinate parallel to the surface edge being excluded. Let the surface $-\infty < x < 0$ still be forced to move with the prescribed velocity

$$v(x) = v_o \exp(iqx).$$

Then (8.2) cannot be the solution, because it is easy to see that since $\partial\phi/\partial y$ is prescribed on $y = 0$, for some range of x at least, ϕ must be an odd function of y and since ϕ must be continuous across the extension $(y = 0,\ 0 < x < \infty)$ of the surface, ϕ must be zero there—whereas (8.2) is not zero. Clearly, no single mode like (8.2), nor even any discrete set of modes of this kind, is capable of making $\partial\phi/\partial y$ have the value $v_o \exp iqx$ on $y = 0$, $x < 0$ and of making $\phi = 0$ on $y = 0$, $x > 0$. The solution for ϕ must therefore contain a continuous spectrum of modes like (8.2) with all values of the wavenumber. In particular, it must contain modes with wavenumbers α, say, with $\alpha < k_o$, and for these modes the exponential decay $\exp\{-(q^2 - k_o^2)^{\frac{1}{2}}y\}$ must be replaced by oscillatory behavior $\exp\{i(k_o^2 - \alpha^2)^{\frac{1}{2}}y\}$, the choice of $+\, i(k_o^2 - \alpha^2)^{\frac{1}{2}}y$ rather than $-\, i(k_o^2 - \alpha^2)^{\frac{1}{2}}y$ being dictated by the radiation condition, that the phase factor $\exp\{i(k_o^2 - \alpha^2)^{\frac{1}{2}}y - i\omega t)\}$ be that of an outgoing wave as $y \to +\infty$ (when $y < 0$ we take $-\, i(k_o^2 - \alpha^2)^{\frac{1}{2}}y)$. Energy is radiated across a plane y = const. by such a mode—and we say that the energy which was trapped in the subsonic mode (8.2) on an infinite plate has

been scattered into other subsonic modes and also into supersonic radiating modes by the discontinuity in the surface $y = 0$.

To analyze the process of scattering, or wavenumber conversion, by the plate edge, we define a scattered field ϕ by

$$\phi_{total} = -\frac{v_o}{\gamma_q}\exp(iqx - \gamma_q y) + \phi \tag{8.4}$$

for $y \geq 0$. Because the derivative $\partial\phi_{total}/\partial y$ has the same value on $y = 0_\pm$, ϕ_{total} must be an odd function of y, and so it is enough to consider only $y \geq 0$. The scattered field ϕ is a solution of

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_o^2\right)\phi = 0 \tag{8.5}$$

with

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on } y = 0,\ x < 0 \tag{8.6}$$

and with

$$\phi - \frac{v_o}{\gamma_q}\exp iqx = 0 \quad \text{on } y = 0,\ x > 0. \tag{8.7}$$

This is a typical two-part mixed boundary value problem which we may expect to solve by the W-H technique. Two further conditions are needed, however, to get a unique solution for ϕ. One comes from conditions expected to hold as $|\underline{x}| \to \infty$, and defines the domains $R_\pm$ of analyticity of half-range transforms and the strip D of overlap. The other comes from conditions at the plate edge, $x = y = 0$, and determines the behavior at infinity in the transform s-plane, and hence determines the entire function arising in the W-H method. We shall leave the matter of edge conditions until we need to look at it in detail. For the moment we just assume that all functions with which we deal have at most integrable singularities at $x = y = 0$.

As to conditions as $x \to \pm\infty$, we give k_c and q small positive imaginary parts,

$$k_c = k_1 + ik_2, \quad q = q_1 + iq_2 \tag{8.8}$$

and then (8.7) gives

$$\phi = O(\exp - q_2 x)$$

as $x \to +\infty$, $y = 0$. Away from $y = 0$ we can expect that as $x \to +\infty$ ϕ will take the form of an outgoing cylindrical wave,

$$\phi \sim r^{-\frac{1}{2}} \exp(ik_o r)\ f(\theta) = O(\exp - k_2 x) \tag{8.9}$$

It then follows that all $\oplus$ functions linearly related to ϕ and its y-derivatives will be analytic in

$$R_+ : \text{Im } s > - \min(q_2, k_2) \tag{8.10}$$

As $x \to -\infty$, ϕ_{total} contains an exponentially growing part which we have split off in (8.4), so that ϕ should behave like an outgoing cylindrical wave,

$$\phi = O(\exp - k_2 |x|) \tag{8.11}$$

as $x \to -\infty$. Then all $\ominus$ functions will be analytic (and with algebraic behavior at infinity) in

$$R_- : \text{Im } s < + k_2, \tag{8.12}$$

and the strip D is

$$D: - \min(q_2, k_2) < \text{Im } s < + k_2. \tag{8.13}$$

For $y > 0$ we can apply Fourier transforms to (8.5), and can integrate the $\partial^2\phi/\partial x^2$ exp isx term by parts twice with no contribution from $x = \pm\infty$ provided s lies in D. This gives

$$\left(\frac{\partial^2}{\partial y^2} - \gamma_s^2\right) \Phi(s,y) = 0 \tag{8.14}$$

where

$$\gamma_s = (s^2 - k_o^2)^{\frac{1}{2}}, \tag{8.15}$$

introducing the square root function whose behavior in the complex s-plane holds the key to many aspects of acoustic diffraction and scattering processes.

The function $\gamma_s = (s - k_o)^{\frac{1}{2}}(s + k_o)^{\frac{1}{2}}$ has branch points at $s = \pm k_o$, and branch cuts must emanate from these points to form a barrier which must not be crossed. We can either make a cut from $+ k_o$ to $-k_o$, or we can make a cut from $+ k_o$ to ∞ (in any direction) and a cut from $-k_o$ to ∞ (in any direction). If the values of $(s + k_o)^{\frac{1}{2}}$, $(s - k_o)^{\frac{1}{2}}$ are specified at any point in the s-plane (not necessarily the same point for the two functions) and the branch cuts are fixed, then a unique value of $(s + k_o)^{\frac{1}{2}}$, $(s - k_o)^{\frac{1}{2}}$ is obtained by starting at the given point and moving to any desired point <u>without crossing any branch cut</u>, and insisting that the function change continuously from its initial value. Figure 4 gives various possible choices of branch cuts. In addition to the choice of branch cuts we shall take each of $(s \pm k_o)^{\frac{1}{2}}$ to be the branch which behaves like $+ s^{\frac{1}{2}}$ (rather than $- s^{\frac{1}{2}}$) when s is large and positive.

Now in our problem we know that Φ must be analytic in D, and since the general solution of (8.14) is

$$\Phi(s,y) = A(s)\ e^{-\gamma_s y} + B(s)\ e^{\gamma_s y} \tag{8.16}$$

that will only be possible if the branch cuts from $\pm\, k_o$ do not enter the strip D. Thus the cut from $+k_o$ must go to infinity above the strip, that from $-k_o$ to infinity below the strip. No further specification of the cuts need be made at this stage, because we shall find a solution for s in D, and the values of γ_s are <u>already fixed</u> for s in D by the requirement $(s \pm k_o)^{\frac{1}{2}} \sim + s^{\frac{1}{2}}$ as $s \to +\infty$ and by the <u>general</u> location of the cuts.

We can now see that

$$\begin{aligned} 0 &< \arg(s + k_o) < \pi \\ -\pi &< \arg(s - k_o) < 0 \end{aligned} \tag{8.17}$$

for all s in D, and therefore

$$-\frac{\pi}{2} < \arg(s^2 - k_o^2)^{\frac{1}{2}} < \frac{\pi}{2}$$

or equivalently

$$\text{Re}\ \gamma_s > 0 \qquad \text{for all } s \text{ in } D \tag{8.18}$$

which is the essential property of γ_s. It is possible to choose the branch cuts so that (8.18) holds throughout the entire complex (cut) plane, but there is no need for this since at the moment we are concerned only with values of s in D. Then the only possible form for B(s) in (8.16) is B(s) = 0, otherwise Φ would be infinite as $y \to +\infty$. Thus

$$\Phi(s,y) = A(s)\ \exp(-\gamma_s y) \tag{8.19}$$

and the original radiation condition of outgoing waves is seen to be equivalent, for $k_2 > 0$ and s in D, to the condition that $\Phi(s,y) \to 0$ as $y \to +\infty$.

The boundary condition (8.6) involves $\Phi'_-(s,0)$, where the ' indicates $\partial/\partial y$, and therefore we differentiate (8.19) to get the pair of equations (equivalent to the differential equation plus the radiation condition)

$$\begin{aligned} \Phi(s,0) &= \Phi_+(s,0) + \Phi_-(s,0) = A(s) \\ \Phi'(s,0) &= \Phi'_+(s,0) + \Phi'_-(s,0) = -\gamma_s A(s) \end{aligned} \tag{8.20}$$

where, for example,

$$\Phi_-(s,0) = \int_{-\infty}^{0} \phi(x,0)\, e^{isx}\, dx \quad ,$$

$$\Phi'_+(s,0) = \int_0^{\infty} \frac{\partial\phi}{\partial y}(x,0)\, e^{isx}\, dx, \text{ etc.}$$

In (8.20) two of the functions are known. From (8.6) we have

$$\Phi'_-(s,0) = 0 \qquad (\text{for } s \text{ in } R_-) \tag{8.21}$$

while from (8.7) we have

$$\Phi_+(s,0) = \frac{-v_o}{\gamma_q (s+q)} \tag{8.22}$$

for Im $s > -q_2$, i.e., for $s \epsilon R_+$. Eliminating A(s) between the two equations in (8.20) and using (8.21) and (8.22) gives

$$K(s)\, \Phi'_+(s,0) + \Phi_-(s,0) = \frac{-iv_o}{\gamma_q (s+q)} \tag{8.23}$$

a standard form of W-H equation, with

$$K(s) = \frac{1}{\gamma_s} = (s^2 - k_o^2)^{-\frac{1}{2}} \tag{8.24}$$

The factorization of K into factors analytic and non-zero in R and of algebraic behavior at infinity there is again obvious:

$$K_+(s) = (s + k_o)^{-\frac{1}{2}}, \; K_-(s) = (s - k_o)^{-\frac{1}{2}},$$

and after division of (8.23) by $K_-(s)$ we again have the additive split of the function $1/(s + q)\,K_-(s)$ to make. We thus arrive at the equation

$$K_+(s)\,\Phi_+'(s,0) + \frac{iv_o}{\gamma_q\,K_-(-q)(s+q)} \tag{8.25}$$

$$= -\frac{\Phi_-(s,0)}{K_-(s)} - \frac{iv_o}{\gamma_q(s+q}\left\{\frac{1}{K_-(s)} - \frac{1}{K_-(-q)}\right\}$$

$$= \text{an entire function } E(s).$$

We anticipate that E(s) will be a polynomial, and consider the implications that the degree of the polynomial be N, $E(s) = a_o s^N + a_1 s^N + \ldots + a_N$.

$$\Phi_+'(s,0) = -\frac{iv_o(s+k_o)^{\frac{1}{2}}}{\gamma_q K_-(-q)(s+q)} + (s+k_o)^{\frac{1}{2}}\{a_o s^N + \ldots\}$$

$$\sim O(s^{-\frac{1}{2}}) + O(a_o s^{N+\frac{1}{2}})$$

as $s \to \infty$ in R_+. The first term vanishes algebraically at ∞ and is therefore the half-range transform of a function with at most an integrable singularity at $x = 0+$. In fact it follows from Watson's lemma that the

corresponding value of $\partial\phi/\partial y$ on $y = 0$, $x > 0$ is $O(x^{-\frac{1}{2}})$ as $x \to 0+$. The second term grows as $s \to \infty$, and must arise as the generalized Fourier transform of a function which has a non-integrable singularity at $x = 0$. According to Lighthill [11,p.43]

$$\int_0^\infty x^{-\lambda} \exp isx \, dx = e^{\frac{\pi i}{2}(1-\lambda)\,\mathrm{sgn}\,s} (-\lambda)! \; |s|^{\lambda-1}$$

for real s, and the correct interpretation of this in R_+ is

$$e^{\frac{\pi i}{2}(1-\lambda)} (-\lambda)! \; s^{\lambda-1} \tag{8.26}$$

where the branch cut for the function $s^{\lambda-1}$ is to go from 0 to ∞ in the lower half-plane.

Hence $\qquad \Phi_+'(s,o) = O(s^{N+\frac{1}{2}})$

$$\Longleftrightarrow \quad \frac{\partial\phi}{\partial y}(x,0) = O(x^{-N-\frac{3}{2}}) \text{ as } x \to 0+.$$

Therefore the velocity has a singularity at least as bad as $x^{-\frac{3}{2}}$ near $x = 0$, and the kinetic energy in a small region around $x = 0$ will diverge to infinity.

We argue that this singularity is unacceptable, and choose the solution corresponding to

$$E(s) \equiv 0 \; ,$$

thus giving the least singular behavior—like $x^{-\frac{1}{2}}$—in the velocity at $x = 0$. Note that it is impossible to impose a Kutta condition, that the velocity be finite (except by abandoning the radiation condition, or allowing ϕ to be discontinuous, and that cannot be permitted in static fluid). To see what kind of a pressure field exists near $x = 0$

we have

$$\Phi_+(s,0) = \frac{iv_o}{\gamma_q(s+q)}$$

$$\Phi_-(s,0) = \frac{iv_o}{\gamma_q(s+q)}\left\{1 - \frac{1}{(s-k_o)^{\frac{1}{2}}\,K_-(-q)}\right\} .$$

On $y = 0$, $x > 0$ we know from (8.7) what ϕ should be, and this can be confirmed from the expression for Φ_+. On $y = 0$, $x < 0$ we close the inverse Fourier integral path in the upper half-plane. The pole $s = -q$ lies outside the contour and makes no contribution, so that we only need to examine the second contribution to Φ_- as $s \to \infty$ in R_-. For this contribution

$$\Phi_-(s,0) \sim s^{-\frac{3}{2}}$$

and so

$$\phi(x,0) \sim (-x)^{\frac{1}{2}}$$

(times some coefficient) as $x \to 0-$. Thus the pressure and the pressure jump both vanish like $(-x)^{\frac{1}{2}}$ near the plate edge. Note that although the pressure jump <u>does vanish</u> at the plate edge (which would be regarded in aerofoil theory as the satisfaction of a Kutta condition), the velocity is nonetheless infinite at the edge.

In summary, the least singular solution has

$$\left.\begin{aligned} \phi &= O(x^{\frac{1}{2}}) \\ \nabla\phi &= O(x^{-\frac{1}{2}}) \end{aligned}\right\} \tag{8.27}$$

near the edge, and conditions of this kind are often imposed at the outset as <u>edge conditions</u>. It seems preferable not to anticipate the edge behavior in advance, but to follow the W-H method through as far

as (8.25), and then to see in each particular case what behavior must hold near the edge and what freedom exists for minimizing singular behavior (as we shall do in §9). In some cases, in particular in recent work by Rawlins [15] on diffraction of an acoustic wave by a half-plane which is "sound-hard" on one side and "sound-absorbing" on the other, the edge conditions are not at all obvious, being in fact

$$\phi = O(x^{\frac{1}{4}})$$
$$\nabla\phi = O(x^{-\frac{3}{4}})$$

To justify acceptance of a solution with certain edge conditions one has to go beyond the simple linear inviscid wave equation used here, or beyond the simple zero thickness model of the boundary. For example, one can look at linear acoustic propagation with viscous effects included, or one can look at inviscid propagation around a surface which is thin compared with any other relevant length scale but which has a smoothly rounded edge. It can be proven (though the proof has not yet been published) that our solution with conditions (8.27) is the unique one which can be matched to an "inner solution" in which either viscous forces or the continuous curvature of the boundary lead to finite velocities everywhere. That is a rather special kind of proof, however, and we shall refer in §9 to the unavailability of a comparable proof when there is uniform subsonic flow past the radiating half-plane.

This discussion of edge-conditions completes the formal determination of the field as

$$\phi = \frac{1}{2\pi}\int_c \frac{iv_o \exp(-isx - \gamma_s y)}{\gamma_q K_-(-q)\ (s+q)(s-k_o)^{\frac{1}{2}}}\ ds \qquad (8.28)$$

where c runs from $-\infty$ to $+\infty$ in D, i.e., above $s = -q$, above the branch point $s = -k_o$ and below the branch point $s = +k_o$. This holds for $y > 0$, while for $y < 0$ we use $\phi_{total}(-y) = -\phi_{total}(y)$. The integral here can actually be evaluated in closed form, in terms of Fresnel integrals, though the details are complicated, and are nothing to do with the W-H method. We refer the reader to [5,10] for descriptions of the details, and remark simply that the distant radiating acoustic field can be estimated asymptotically from the formula

$$\int_c F(s)\,\exp(-isx - \gamma_s y)\,ds \qquad (8.29)$$

$$\sim \left(\frac{2k_o\pi}{r}\right)^{\frac{1}{2}} \exp(ik_o r - \pi i/4)\,\sin\theta\,F(-k_o\cos\theta)$$

in which $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le \pi$. (This formula may give apparent infinities in particular angular directions, and near those angles it is necessary to use more refined approximations.) To apply it here we have to interpret $(-k_o\cos\theta - k_o)^{\frac{1}{2}}$, which we take as $-ik_o^{\frac{1}{2}}(1 + \cos\theta)^{\frac{1}{2}}$ because the arg of $-k_o\cos\theta - k$ is equal to $-\pi$ for all θ between 0 and π. We also need

$$\frac{1}{K_-(-q)} = (-q - k_o)^{\frac{1}{2}} = -i(q + k_o)^{\frac{1}{2}} ,$$

again because $\arg(-q - k_o) = -\pi$. Then we have

$$\phi \sim \left(\frac{1}{\pi r}\right)^{\frac{1}{2}} \frac{iv_o}{(q - k_o)^{\frac{1}{2}}}\, e^{ik_o r - \pi i/4}\, \frac{\sin\theta/2}{(q - k_o\cos\theta)} \qquad (8.70)$$

which gives the level and directivity of the scattered acoustic field.

In the next section we look at the same problem, but with subsonic mean flow over the surface.

9. HALF-PLANE PROBLEMS WITH MEAN FLOW: WAKES AND THE KUTTA CONDITION

Consider again the half-plane $y = 0$, $x < 0$ with prescribed velocity $v(x) = v_o \exp(iqx)$, but suppose now that there is uniform parallel flow at the same speed U on both sides of the plate, the plate edge being a <u>trailing edge</u>. The potential ϕ_{total} satisfies the convected wave equation

$$\left[\left(-i\omega + U\frac{\partial}{\partial x}\right)^2 - c_o^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\phi_{total} = 0 \qquad (9.1)$$

and the boundary condition on the plate is that of <u>continuity of displacement</u> (not of normal velocity), giving

$$\left(-i\omega + U\frac{\partial}{\partial x}\right)\eta = \frac{\partial\phi_{total}}{\partial y}$$

where $\eta = v/(-i\omega)$ is the surface displacement. Thus

$$-i\omega\frac{\partial\phi_{total}}{\partial y} = (-i\omega + iUq)\, v_o e^{iqx} \quad \text{on } y = 0,\ x < 0. \qquad (9.2)$$

Because of the presence of mean flow there is a new possibility on the extension of the plate. A wake can exist there, across which there may be a discontinuity in tangential velocity $\partial\phi_{total}/\partial x$ as long as the normal displacement (and hence here also the normal velocity) and the pressure are continuous across the wake. Take a single Fourier component $Ae^{i\lambda x}$ of ϕ_{total} for $y = 0+$, $x > 0$; then for $y = 0-$, $x > 0$, below the wake, the corresponding component is $-Ae^{i\lambda x}$, because ϕ_{total} must be an odd function of y. The pressure jump across the wake associated with this particular component of potential is

$$p(x,0+) - p(x,0-) = (-\rho)\left(-i\omega + U\frac{\partial}{\partial x}\right)\left(2Ae^{i\lambda x}\right)$$

and this vanishes either if $A = 0$ (which implies continuity of ϕ_{total} and hence absence of the wake) or, for any value of A, if $\lambda = \omega/U$. Thus the general condition on ϕ_{total} across the wake is that

$$\phi_{total}\ (x,0+) = Ae^{i\omega x/U} \tag{9.3}$$

for some value of A. The tangential velocity is $(i\omega A/u)e^{i\omega x/U}$ above the wake, $(-i\omega A/U)e^{i\omega x/U}$ below it, and hence the wake is an oscillatory vortex sheet of strength $2\omega A/U$, modulated by the phase factor $\exp\left[i\,\frac{\omega}{U}(x - Ut)\right]$, which shows that any element of the vortex sheet propagates downstream at the flow speed U.

We wish to solve the two-part mixed boundary value problem posed by (9.1-9.3), subject here to the restriction to subsonic flow ($M = U/c_o < 1$) and to a radiation condition, but leaving open the issue of edge conditions. We first write

$$\phi_{total} = -\frac{v_o}{\xi(-q)}\,D(-q)\ \exp(iqx - \xi(-q)y) + \phi \tag{9.4}$$

where we shall write generally

$$\xi_s = \left\{s^2 - (k_o + Ms)^2\right\}^{\frac{1}{2}} \tag{9.5}$$

$$\xi(-q) = \xi_{s = -q}$$

$$D_s = 1 + \frac{sU}{\omega} = 1 + \frac{Ms}{k_o}$$

$$D(-q) = 1 - \frac{qU}{\omega} \tag{9.6}$$

The function ξ_s replaces the γ_s in the no-flow case, while D_s is a kind of Doppler factor for wavenumber s. The object of writing ϕ_{total} in the form (9.4) is that the incident field associated with the velocity

$v_o e^{iqx}$ on an infinite plate has been split off; that field is exponentially large as $x \to -\infty$, and the ϕ that remains is assumed to be just an outward propagating wave.

We now have the following problem for ϕ:

$$\left[(-i\omega + U\frac{\partial}{\partial x})^2 - c_o^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\phi = 0 \tag{9.7}$$

$$\frac{\partial\phi}{\partial y} = 0 \qquad (y = 0,\ x < 0) \tag{9.8}$$

$$\phi = A\exp(\frac{i\omega x}{U}) + \frac{v_o D(-q)}{\xi(-q)}\exp(iqx) \qquad (y = 0,\ x > 0) \tag{9.9}$$

and we have to examine conditions as $x \to \pm\infty$. Write $k_o = k_1 + ik_2$, $\omega/U = (k_1 + ik_2)/M$, $q = q_1 + iq_2$. As $x \to -\infty$, ϕ must behave like an outgoing acoustic wave, whose decay will be like that of a plane wave propagating against the flow, so that

$$\begin{aligned}\phi &\sim \exp\left\{-i\omega t - \frac{i\omega x}{c_o - U}\right\}\\ &\sim \exp\left(-\frac{k_2|x|}{1 - M}\right)\end{aligned} \tag{9.10}$$

As $x \to +\infty$, ϕ may behave either like an acoustic wave propagating with the flow

$$\begin{aligned}\phi &\sim \exp\left\{-i\omega t + \frac{i\omega x}{c_o + U}\right\}\\ &\sim \exp\left\{-\frac{k_2 x}{1 + M}\right\}\end{aligned} \tag{9.11}$$

or may behave as it does in the wake,

$$\phi \sim \exp - \left\{\frac{k_2 x}{M}\right\} \tag{9.12}$$

or in the way associated with the forcing field exp iqx,

$$\phi \sim \exp \{- q_2 x\} \tag{9.13}$$

It follows from these that

$$R_+ \text{ is } \operatorname{Im} s > - \min\left(\frac{k_2}{1 + M}, q_2\right)$$

$$R_- \text{ is } \operatorname{Im} s < + \frac{k_2}{1 - M}$$

and that the strip is

$$D: - \min\left(\frac{k_2}{1 + M}, q_2\right) < \operatorname{Im} s < + \frac{k_2}{1 - M} \tag{9.14}$$

provided $M < 1$.

Take Fourier transforms of (9.7) for $y > 0$ to get

$$\Phi(s,y) = A(s) \exp(-\xi_s y) + B(s) \exp(+\xi_s y) \tag{9.15}$$

ξ_s is defined in (9.5). We take a cut from $s = + k_o/(1 - M)$ to infinity above the strip D and one from $s = -k_o/(1 + M)$ to infinity below D, and define $[s - (k_o + Ms)]^{\frac{1}{2}}$ to be the branch which behaves like $+ (1 - M)^{\frac{1}{2}} s^{\frac{1}{2}}$ when s is large, real, and positive, $[s + (k_o + Ms)]^{\frac{1}{2}}$ to be the branch behaving like $+ (1 + M)^{\frac{1}{2}} s^{\frac{1}{2}}$ for large real positive s. With these branch cuts and choices of branch, $\operatorname{Re} \xi_s > 0$ for all s in D, just as for γ_s in §8. Then since (9.15) only holds in D we have to have $B(s) = 0$, and now we can differentiate (9.15) with respect to y, put $y = 0+$ and eliminate the function A(s). Thus

$$\Phi'_+(s,0+) + \Phi'_-(s,0+)$$

$$= -\xi_s[\Phi_+(s,0+) + \Phi_-(s,0+)]$$

From (9.8) we have $\Phi'_-(s,0+) = 0$ while from (9.9) we have

$$\Phi_+(s,0+) = \frac{iA}{(s + k_o/M)} + \frac{iv_o\, D(-q)}{\xi(-q)(s + q)}$$

and hence we get a W-H equation

$$\frac{1}{\xi_s}\,\Phi'_+(s,0+) + \Phi_-(s,0+) + \frac{iA}{(s + k_o/M)} + \frac{iv_o\, D(-q)}{\xi(-q)(s + q)} = 0 \qquad (9.16)$$

We define the factors $\xi_+(s)$ $\xi_-(s)$ by

$$\begin{aligned} \xi_+(s) &= [s + (k_o + Ms)]^{\frac{1}{2}} \\ \xi_-(s) &= [s - (k_o + Ms)]^{\frac{1}{2}} \end{aligned} \qquad (9.17)$$

and by the usual route arrive at

$$\frac{1}{\xi_+(s)}\,\Phi'_+(s,0+) + \frac{iA}{(s + k_o/M)}\,\xi_-(-\frac{k_o}{M}) + \frac{iv_o\, D(-q)}{\xi(-q)(s + q)}\,\xi_-(-q)$$

$$= -\xi_-(s)\;\Phi_-(s,0+) - \frac{iA}{(s + k_o/M)}\left\{\xi_-(s) - \xi_-(-\frac{k_o}{M})\right\}$$

$$- \frac{iv_o\, D(-q)}{\xi(-q)(s + q)}\{\xi_-(s) - \xi_-(-q)\} \qquad (9.18)$$

= an entire function E(s), which must be a polynomial because of the algebraic behavior of all functions involved here. In fact we must have $E(s) \equiv 0$, for if E(s) were even a constant, the $\oplus$ part of (9.18) would give

$$\phi'_+(s,0+) = O(s^{\frac{1}{2}}) \text{ at infinity}$$

which corresponds to the strong singularity $\nabla\phi = O(x^{-\frac{3}{2}})$. With that choice of $E(s)$ the solution is still not unique, because the wake strength A is undetermined. We can argue that conditions are such that there should be no jump in tangential velocity across the extension of the plate, in which case $A = 0$, $\nabla\phi = O(x^{-\frac{1}{2}})$ and $\phi = O(x^{\frac{1}{2}})$ near the plate edge and we have a situation essentially the same as was examined in §8. Because the radiation is emitted into uniformly moving fluid the "stationary phase" formula (8.29) is not immediately applicable, but we shall show in a moment how it can be generalized to the moving fluid case.

We can alternatively argue that a wake will adjust itself to eliminate the high velocities which would otherwise exist at the trailing edge, and that a Kutta condition of finiteness of the velocities at the edge should be imposed whenever possible. The physical basis for such a condition in unsteady trailing edge flow is a matter of controversy at the moment, but that does not concern us here as our interest is merely in seeing if and how such a condition can be applied in this model problem.

Expand the $\oplus$ part of (9.18) as $s \to \infty$ in R_+. We have

$$\phi'_+(s,0) \sim \left(iA\xi_-(-\frac{k_o}{M}) + \frac{iv_o D(-q)\ \xi_-(-q)}{\xi(-q)} \right) (1+M)^{\frac{1}{2}}\, s^{-\frac{1}{2}} + O(s^{-\frac{3}{2}}) \qquad (9.19)$$

the term given explicitly corresponding to $\nabla\phi = O(x^{-\frac{1}{2}})$ the second to $\nabla\phi = O(x^{\frac{1}{2}})$. Thus we can impose a Kutta condition, that the velocities

be <u>finite</u> at the edge (note that the velocities in the part of ϕ_{total} split off in (9.4) are finite, but non-zero, near the edge) by choosing a wake strength

$$A = -\frac{v_o\ D(-q)}{\xi_-\left(-\frac{k_o}{M}\right)\xi_+(-q)} \tag{9.20}$$

It is easy to see that the ⊖ part of (9.18) contains terms like $(s+q)^{-1}$, $(s+k_o/M)^{-1}$ which (as in §8) make no contribution to ϕ for $x < 0$, $y = 0+$ and that with the choice (9.20) the first term in the expansion of $\Phi_-(s,0+)$ as $s \to \infty$ in R_- is $O(s^{-\frac{5}{2}})$. This corresponds to

$$\phi = O(x^{\frac{3}{2}})$$

near the edge, though the pressure is $O(x^{\frac{1}{2}})$ because $p = -\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi$.

The expression for the potential in $y > 0$ is found to be

$$\phi = \frac{1}{2\pi}\int_c F(s)\exp(-isx - \xi_s y)\,ds$$

with

$$F(s) = \frac{1}{\xi_-(s)}\left[\frac{iA}{\left(s+\frac{k_o}{M}\right)}\xi_-\left(-\frac{k_o}{M}\right) + \frac{iv_o\ D(-q)}{\xi_+(-q)(s+q)}\right] \tag{9.21}$$

In the exponential factor we write

$$-isx - [s^2 - (k_o + Ms)^2]^{\frac{1}{2}}\,y$$

$$= -isx - (1-M^2)^{\frac{1}{2}}\,y\quad\left\{\left(s - \frac{Mk_o}{1-M^2}\right)^2 - \frac{k_o^2}{(1-M^2)^2}\right\}^{\frac{1}{2}}$$

so that if we define

$$\Phi'_+(s,0+) = O(s^{\frac{1}{2}}) \text{ at infinity}$$

which corresponds to the strong singularity $\nabla\phi = O(x^{-\frac{3}{2}})$. With that choice of $E(s)$ the solution is still not unique, because the wake strength A is undetermined. We can argue that conditions are such that there should be no jump in tangential velocity across the extension of the plate, in which case $A = 0$, $\nabla\phi = O(x^{-\frac{1}{2}})$ and $\phi = O(x^{\frac{1}{2}})$ near the plate edge and we have a situation essentially the same as was examined in §8. Because the radiation is emitted into uniformly moving fluid the "stationary phase" formula (8.29) is not immediately applicable, but we shall show in a moment how it can be generalized to the moving fluid case.

We can alternatively argue that a wake will adjust itself to eliminate the high velocities which would otherwise exist at the trailing edge, and that a Kutta condition of finiteness of the velocities at the edge should be imposed whenever possible. The physical basis for such a condition in unsteady trailing edge flow is a matter of controversy at the moment, but that does not concern us here as our interest is merely in seeing if and how such a condition can be applied in this model problem.

Expand the $\oplus$ part of (9.18) as $s \to \infty$ in R_+. We have

$$\Phi'_+(s,0) \sim \left(iA\xi_-(-\frac{k_o}{M}) + \frac{iv_o D(-q)\ \xi_-(-q)}{\xi(-q)} \right) (1 + M)^{\frac{1}{2}} s^{-\frac{1}{2}} + O(s^{-\frac{3}{2}}) \tag{9.19}$$

the term given explicitly corresponding to $\nabla\phi = O(x^{-\frac{1}{2}})$ the second to $\nabla\phi = O(x^{\frac{1}{2}})$. Thus we can impose a Kutta condition, that the velocities

$$\sigma = s - \frac{Mk_o}{1-M^2}$$

$$K_o = \frac{k_o}{1-M^2}$$

$$x = R\cos \Theta, \quad (1 - M^2)^{\frac{1}{2}} y = R \sin \Theta$$

then

$$\phi = \frac{\exp\left[-i \frac{Mk_o}{1-M^2} R \cos \Theta\right]}{2\pi} \int_c F \left(\sigma + \frac{Mk_o}{1-M^2}\right) \times$$

$$\exp\left\{-i\sigma R \cos \Theta - (\sigma^2 - K_o^2)^{\frac{1}{2}} R \sin \Theta\right\} d\sigma$$

which is now capable of asymptotic estimation by the formula (8.29) for static fluid.

To make the algebra less complicated, suppose the Mach number M is small so that M^2 can be neglected compared with unity. Then the field associated with the second contribution in (9.21) turns out to be

$$\left(\frac{1}{\pi r}\right)^{\frac{1}{2}} \frac{iv_o(1 - Mq/k_o)}{\left(q - \frac{k_o}{1+M}\right)^{\frac{1}{2}} [q - k_o(\cos\theta - M)]} \sin\theta/2 \exp\{ik_o r - ik_o Mx - \frac{i\pi}{4}\} \quad (9.23)$$

there being no difference between r and κ or between θ and Θ if $M^2 \ll 1$. The field (9.23) is a trivial modification of (8.30).

The distant field associated with the first term in (9.21), with the Kutta condition value (9.20) for A comes out as

$$-\left(\frac{1}{\pi r}\right)^{\frac{1}{2}} \frac{iv_o(1 - Mq/k_o)}{\left(q - \frac{k_o}{1+M}\right)^{\frac{1}{2}} k_o} \frac{M}{(1 - M\cos\theta)} \sin\theta/2 \exp\{ik_o r - ik_o Mx - \frac{i\pi}{4}\}$$

and the ratio of the wake generated field (9.24) to the field (9.23) which would exist in the absence of the wake is (since $q > k_o$) roughly

$$M\left(\frac{q}{k_o}\right) \tag{9.25}$$

This shows that if the plate wave travels at speed ω/q between the flow speed and the sound speed, the wake contribution to the far-field is negligible. If, on the other hand, the plate wave travels more slowly than the flow speed, Mq/k_o exceeds unity and the sound field when the Kutta condition is imposed exceeds that in the absence of any wake.

This is not to be regarded as a general conclusion, for in other problems the wake sound very nearly cancels the primary edge field. That in fact happens here, for when q becomes close to k_o/M it can be seen that the fields (9.23) and (9.24) have small but equal and opposite values. Whether the extra wake field dominates, or mainly cancels the primary edge sound field thus depends very much on details of the basic excitation.

10. CONSTRUCTION OF W-H SPLIT FUNCTIONS

In this section we first outline a general method for effecting either the additive or the multiplicative decomposition of a function analytic in a strip, and then we set down some properties of the functions which arise frequently in acoustics problems. Finally, we record the corresponding properties for strictly incompressible flow problems.

A. Cauchy Integrals

Let $F(s)$ be analytic in some strip D; $\varepsilon_1 < \text{Im } s < \varepsilon_2$ and let R_+ denote the domain $\text{Im } s > \varepsilon_1$, R_- the domain $\text{Im } s < \varepsilon_2$. Suppose also that $|F(s)| \to 0$ uniformly as fast as $|s|^{-\lambda}$ for some $\lambda > 0$ as $|s| \to \infty$ within any closed region within D, i.e., as $|s| \to \infty$ with

$$\varepsilon_1 < \alpha_1 \leq \text{Im } s \leq \alpha_2 < \varepsilon_2$$

Then

$$F(s) = F_+(s) + F_-(s) \tag{10.1}$$

for s in D, where

$$F_+(s) = \frac{1}{2\pi i} \int_{\smile} \frac{F(t)}{t - s}\, dt \tag{10.2}$$

is analytic and bounded in R_+,

$$F_-(s) = \frac{-1}{2\pi i} \int_{\frown} \frac{F(t)}{t - s}\, dt \tag{10.3}$$

is analytic and bounded in R_-.

The path $\smile$ runs from $-\infty$ to $+\infty$ in D below $t = s$, while $\frown$ runs from $-\infty$ to $+\infty$ in D above $t = s$.

Suppose further now that $|F(s)| \to 0$ uniformly as fast as $|s|^{-1-\mu}$ for some $\mu > 0$ as $|s| \to \infty$ in the strip D. Then $\int F(t)dt$ converges absolutely, and

$$F_+(s) \sim -\frac{1}{2\pi i s}\int F(t)\,dt = O(s^{-1}) \tag{10.4}$$

as $|s| \to \infty$ in R_+, while

$$F_-(s) \sim +\frac{1}{2\pi i s}\int F(t)\,dt = O(s^{-1}) \tag{10.5}$$

as $|s| \to \infty$ in R_-.

Without going into fine details, (10.1) is proved by applying Cauchy's theorem

$$F(s) = \frac{1}{2\pi i}\oint \frac{F(t)}{t-s}\,dt$$

to a contour lying within D and enclosing the point s in D ($F(s)$ being defined only for s in D in the first instance). The contour is then deformed to consist of a rectangle with as its lower side as its upper, and with the ends of these sides joined at infinity by short sides parallel to the imaginary axis. In the limit these short sides (of finite length, less than $\varepsilon_2 - \varepsilon_1$) make no contribution to the integral, so that

$$F(s) = \frac{1}{2\pi i}\int \frac{F(t)}{t-s}\,dt - \frac{1}{2\pi i}\int \frac{F(t)}{t-s}\,dt$$

where now both paths of integration run from $-\infty$ to $+\infty$. But then, according to the basic theorem of complex variable analysis, the first term defines an analytic function as s varies without crossing the integration path, i.e., it defines an analytic function in the upper half-plane above the integration path. Similarly, the second term defines a function analytic everywhere below the integration path .

For the behavior at infinity we have

$$F_+(s) - \left\{- \frac{1}{2\pi is} \int F(t)\,dt\right\}$$

$$= \frac{1}{2\pi is} \underbrace{\int} \frac{tF(t)}{t - s}\,dt$$

Without any real loss of generality we can take the path of integration to be the real axis, and $s_2 > 0$ in R_+, so that

$$\left| \underbrace{\int} \frac{tF(t)}{t - s}\,dt \right| \leq \int_{-\infty}^{+\infty} \frac{|t|\ |F(t)|\ dt}{\sqrt{(t - s_1)^2 + s_2^2}}$$

Let $|s| \to \infty$ in R_+ along a ray with $s_1 = Ks_2$, and divide the range of integration at points $t = \pm M$. M is chosen so that $|s| \gg M$, but so that for $|t| \geq M$, $|F(t)| < c|t|^{-\lambda-1}$ for some constant c and some $\lambda > 0$. Then on $(-M,+M)$

$$\int_{-M}^{+M} \frac{|t|\ |F(t)|dt}{\sqrt{(t - Ks_2)^2 + s_2^2}} \sim \frac{1}{|s|} \int_{-M}^{+M} |t|\ |F(t)|\ dt$$

and the integral is finite and independent of s, while on (M,∞) say we put $t = Ks_2 + s_2\tan\theta$ to get

$$\int_M^{\infty} \frac{t|F(t)|dt}{\sqrt{(t - Ks_2)^2 + s_2^2}} \leq \frac{c}{s_2} \int_{\tan^{-1}(-\frac{1}{K})}^{\pi/2} \frac{\sec\theta}{(K + \tan\theta)^{\lambda}}\,d\theta$$

and again the integral is convergent and independent of s. Thus under these conditions

$$F_{\pm}(s) = O(s^{-1}) \text{ at infinity in } R_{\pm}.$$

The above is hardly a proof, but it can be rigorized. In any application the behavior at infinity should be checked out carefully in each case.

The Cauchy integral formulas (10.2, 10.3) enable the product decomposition $K(s) = K_+(s)\, K_-(s)$ to be effected by taking logarithms. Suppose $K(s)$ is analytic in the strip, and that $|K(s)| \to 1$ uniformly as $|s| \to \infty$ in the strip. Suppose further that $K(s) \neq 0$ in the strip. Then $F(s) = \ln K(s)$ is analytic in the strip for any branch of the logarithm, and can be decomposed as in (10.1). Define

$$K_+(s) = \exp F_+(s), \; K_-(s) = \exp F_-(s) \; . \tag{10.6}$$

Then

$$K(s) = K_+(s)\, K_-(s) \tag{10.7}$$

for s in D, and $K_\pm(s)$ are analytic and <u>non-zero</u> throughout $R_\pm$, respectively.

This decomposition is unique up to multiplication of say $K_+(s)$ by a non-zero entire function and division of $K_-(s)$ by the same function. It may be necessary to use this freedom to remove non-algebraic behavior at infinity of $K_\pm(s)$ in certain cases. Noble [5] gives several examples of this minor difficulty.

When $K(s)$ is even, the factors $K_\pm(s)$ <u>as defined by (10.6), (10.2), and (10.3)</u> have the property

$$K_+(-s) = K_-(s) \tag{10.8}$$

If the split is achieved by some way other than use of Cauchy integrals it may be necessary to adjust the functions before (10.8) holds. For example, if $K(s) = (s^2 - k_o^2)^{\frac{1}{2}}$ then the "obvious" split is

$$K_+(s) = (s + k_o)^{\frac{1}{2}}, \; K_-(s) = (s - k_o)^{\frac{1}{2}} \; ,$$

but $K_+(-s) = e^{\pi i/2}\,(s - k_o)^{\frac{1}{2}}$, so that we need to redefine $K_\pm(s)$ as

$$K_+(s) = e^{-\pi i/4}\,(s + k_o)^{\frac{1}{2}},\; K_-(s) = e^{\pi i/4}\,(s - k_o)^{\frac{1}{2}}$$

in order that (10.8) will be satisfied.

B. Decompositions Related to the Square Root Function γ_s

As just noted, the multiplicative decomposition of $\gamma_s = (s^2 - k_o^2)^{\frac{1}{2}}$ into factors analytic, non-zero and algebraic at infinity is

$$\begin{aligned} K_+(s) &= a(s + k_o)^{\frac{1}{2}} \\ K_-(s) &= a^{-1}(s - k_o)^{\frac{1}{2}} \end{aligned} \tag{10.9}$$

for any constant a. If it is useful to require that $K_+(-s) = K_-(s)$ then a should be chosen as $e^{-\pi i/4}$.

For the convected wave equation (with subsonic convection velocities) γ_s is replaced by

$$\xi_s = [s^2 - (k_o + Ms)^2]^{\frac{1}{2}}$$

for which the multiplicative split is

$$\begin{aligned} K_+(s) &= a(1 - M^2)^{\frac{1}{2}}\,(s + \frac{k_o}{1+M})^{\frac{1}{2}} \\ K_-(s) &= a^{-1}(s - \frac{k_o}{1-M})^{\frac{1}{2}} \end{aligned} \tag{10.10}$$

for any constant a. Here of course we cannot make $K_+(-s) = K_-(s)$.

Because the factors $K_+(s)$ $K_-(s)$ for $K(s)$ are analytic and non-zero, the split for $1/K(s)$ is given by $(1/K_+(s)),(1/K_-(s))$.

The additive decomposition of γ_s and functions related to it arises very frequently. Noble [5] gives several ways of calculating the split functions and several representations of those functions. Here we will just verify that if

$$P_+(s) = \frac{\gamma_s}{\pi} \cos^{-1} (s/k_o)$$
$$P_-(s) = \frac{\gamma_s}{\pi} \cos^{-1} (-s/k_o) \tag{10.11}$$

then

$$P_+(s) + P_-(s) = \gamma_s \tag{10.12}$$

for s in D, and $P_\pm(s)$ are analytic in $R_\pm$ with a certain behavior at infinity which will be determined shortly.

Firstly it is necessary to define the function $\cos^{-1} (s/k_o)$ for complex s. If s/k_o is real, $\cos^{-1} (s/k_o)$ is defined as the branch for which $\cos^{-1} (s/k_o) = \pi/2$ when $s/k_o = 0$, i.e., $\cos^{-1} (s/k_o)$ lies between 0 and π when s/k_o is real and between ± 1. Let $\beta = \cos^{-1} (s/k_o)$. Then

$$s/k_o = \frac{e^{i\beta} + e^{-i\beta}}{2}$$

and so

$$i\beta = \ln\left\{\frac{s}{k_o} \pm \frac{(s^2 - k_o^2)^{\frac{1}{2}}}{k_o}\right\}$$
$$= \ln \left(\frac{s \pm \gamma_s}{k_o}\right)$$

where $\gamma_s = (s^2 - k_o^2)^{\frac{1}{2}}$ with the branch cuts as already discussed. The logarithm here is defined to have its principal value, i.e., $\ln z$ is such that

$$-\pi \leq \operatorname{Im} \ln z < +\pi \tag{10.13}$$

with a branch cut along the negative real z-axis. Now take s = 0; the corresponding value of γ is $-ik_o$ and hence $i\beta = \ln\{\pm(-i)\} = \mp \frac{\pi i}{2}$, so that $\beta = \pi/2$ if we choose the lower sign. Hence

$$\beta = -i \ln \left\{\frac{s - \gamma_s}{k_o}\right\} \tag{10.14}$$

which can be rearranged as follows. We have $(s + \gamma_s)(s - \gamma_s) = k_o^2$, hence

$$\beta = -i \ \ell n \left\{\frac{k_o}{s + \gamma_s}\right\}$$

$$= +i \ \ell n \left(\frac{s + \gamma_s}{k_o}\right) \tag{10.15}$$

The definitions (10.14) and (10.15) of the function $\cos^{-1}(s/k_o)$ are not those given in most books on mathematical functions. In those books the square root function is usually understood to have a branch cut from $-k_o$ to $+k_o$, and is quite different from the function γ_s which occurs in wave applications. In particular, if the cut goes from $-k_o$ to $+k_o$, $(s^2 - k_o^2)^{\frac{1}{2}}$ behaves like s both as $s \to +\infty$ and $s \to -\infty$, whereas γ_s behaves like s as $s \to +\infty$ but as $-s$ when $s \to -\infty$.

With the definition

$$\cos^{-1}\left(\frac{s}{k_o}\right) = +i \ \ell n \left(\frac{s + \gamma_s}{k_o}\right) \tag{10.16}$$

consider the functions $P_+(s)$, $P_-(s)$ defined in (10.11). In the first place, no new branch cuts are introduced by the logarithm. A branch cut would be needed only if $s + \gamma_s = 0$ were possible for some value of s, and no such value of s exists. Thus the only singularities are the branch points at $s = \pm k_o$. Consider the function $P_+(s)$ near the point $s = +k_o$, writing $s = k_o + u$ where u is small. We have

$$P_+(k_o + u) = \frac{iu^{\frac{1}{2}}(2k_o + u)^{\frac{1}{2}}}{\pi} \ \ell n \left\{1 + \frac{u}{k_o} + \frac{u^{\frac{1}{2}}(2k_o + u)^{\frac{1}{2}}}{k_o}\right\}$$

$$\sim \frac{iu^{\frac{1}{2}}(2k_o)^{\frac{1}{2}}}{\pi} \ \frac{u^{\frac{1}{2}}(2k_o)^{\frac{1}{2}}}{k_o} + 0(u^2) \tag{10.17}$$

as $u \to 0$, so that $P_+(s)$ is in fact single-valued near $s = k_o$. The branch point singularity of the $\cos^{-1}$ cancels that of γ_s at $s = k_o$, and hence $P_+(s)$ is analytic in R_+. Similarly $P_-(s)$ is analytic in R_-, the branch point singularity of γ_s at $s = -k_o$ being cancelled by that of $\cos^{-1}(-s/k_o)$ there.

To verify (10.12) we have, for s in D,

$$P_+(s) + P_-(s) = \frac{i\gamma_s}{\pi}\left[\ln\left(\frac{s+\gamma_s}{k_o}\right) + \ln\left(\frac{-s+\gamma_s}{k_o}\right)\right]$$

$$= \frac{i\gamma_s}{\pi}\ \ln(-1) = \gamma_s$$

since with (10.13) $\ln(-1) = -i\pi$.

To find the behavior at infinity we note that as $|s| \to \infty$ in R_+

$$\gamma_s \sim s\left\{1 - \frac{k_o^2}{2s^2} - \frac{1}{8}\,\frac{k_o^4}{s^4} + \ldots\right\} \tag{10.18}$$

if the approach to infinity is below the branch cut from $s = +k_o$,

$$\gamma_s \sim -s\left\{1 - \frac{k_o^2}{2s^2} - \frac{1}{8}\,\frac{k_o^4}{s^4} + \ldots\right\} \tag{10.19}$$

if s goes to infinity above the cut. Since $P_+(s)$ has no singularity at $s = +k_o$ it does not matter which of these is used, and we find

$$P_+(s) \sim \frac{i}{\pi}\left(s - \frac{k_o^2}{2s} + \ldots\right)\left[\ln\left(\frac{2s}{k_o}\right) - \frac{k_o^2}{4s^2} + \ldots\right]$$

$$\sim \frac{i}{\pi}\, s\ \ln\left(\frac{2s}{k_o}\right) + O(s^{-1}\ \ln s) \tag{10.20}$$

These properties of the $P_\pm$ functions arise in a great many applications to wave problems. Corresponding results for the convected wave square

root ξ_s can be derived by making a change of variables to transform ξ_s into γ_s, as was done in §9.

C. Incompressible Flow Problems

Incompressible flow results follow from taking the limit $k_o \to 0$, which has the unfortunate effect of reducing the strip of analyticity D to a line on which the functions are continuous, but not necessarily analytic. The branch cuts from $\pm k_o$ also join up to form a complete barrier along, say, the imaginary axis. To avoid possible difficulties stemming from this it is usual to work with finite k_o and then let $k_o \to 0$. This, however, makes for unnecessary complications in much of the work, and it is useful to be able to tackle the incompressible problem directly. To this end we imagine the branch cuts as starting from $0 + i0$ and going to infinity in $\mathrm{Im}\, s > 0$ and from $0-i0$ to infinity in $\mathrm{Im}\, s < 0$.

Then the limiting form of the function γ_s is real and positive on the whole real s-axis, i.e., it is there the function $|s|$. We shall write $(s^2)^{\frac{1}{2}}$ for this function in the complex plane with the cuts as indicated; it is the continuation to complex s of the function $|s|$ on the real axis, and can also be defined as s(sgn Res) where sgn $x = \pm 1$ for $x \gtrless 0$. Thus

$$\gamma_s \to (s^2)^{\frac{1}{2}} = s\ (\mathrm{sgn}\ \mathrm{Re} s)\ \text{as}\ k_o \to 0 \tag{10.21}$$

The multiplicative split is

$$(s^2)^{\frac{1}{2}} = s_+^{\frac{1}{2}}\, s_-^{\frac{1}{2}} \tag{10.22}$$

where $s_+^{\frac{1}{2}}$ means the branch of $s^{\frac{1}{2}}$ which behaves like $s^{\frac{1}{2}}$ as $s \to +\infty$ with a cut from $0 - 0i$ in the lower half-plane, while $s_-^{\frac{1}{2}}$ behaves like $s^{\frac{1}{2}}$ as $s \to +\infty$ but has the cut from $0+0i$ in the upper half plane.

The additive split of $(s^2)^{\frac{1}{2}}$ can be found by taking the limit of (10.11) as $k_o \to 0$. Define $\ell n_+ s$ to be the branch of $\ell n\ s$ with a cut from 0–0i in the lower half-plane, and with $\ell n_+ s$ real and positive when s is real and greater than 1. Define $\ell n_- s$ similarly except that it has the cut from 0 + 0i in the upper half-plane. Then we can see that

$$\ell n_+ s - \ell n_- s = 0 \quad \text{if } \mathrm{Re}s > 0$$

$$\ell n_+ s - \ell n_- s = 2i\pi \quad \text{if } \mathrm{Re}s < 0 ,$$

and both of these are covered by

$$\ell n_+ s - \ell n_- s = 2i\pi\ H(-\mathrm{Re}s)$$

where H(x) is the Heaviside function equal to 1 or 0 according as $x > 0$ or $x < 0$. Since sgn $x = 2H(x) - 1$, we can write this now as

$$\ell n_+ s - \ell n_- s = i\pi - i\pi \ \mathrm{sgn}\ \mathrm{Re}s$$

and multiplying by s and using (10.21) gives

$$s\ \ell n_+ s - s\ \ell n_- s = i\pi s - i\pi (s^2)^{\frac{1}{2}} \tag{10.23}$$

Now define

$$\left.\begin{aligned} P_+(s) &= \frac{s}{2} + \frac{is}{\pi}\ \ell n_+ s \\ P_-(s) &= \frac{s}{2} - \frac{is}{\pi}\ \ell n_- s \end{aligned}\right\} \tag{10.24}$$

and then it follows from (10.23) that

$$(s^2)^{\frac{1}{2}} = P_+(s) + P_-(s) \tag{10.25}$$

and $P_+(s)$ is analytic in $R_+(\mathrm{Im}\ s > 0)$, $P_-(s)$ is analytic in $R_-(\mathrm{Im}\ s < 0)$,

as required. Further, by careful consideration of the branches it can be shown that these $P_{\pm}(s)$ still have the property

$$P_{+}(-s) = P_{-}(s) \tag{10.26}$$

enjoyed by the function (10.11) for $k_o \neq 0$.

Use of these functions enables incompressible problems to be solved much more elegantly than by taking the limit as $k_o \to \infty$ of the more complicated compressible problems. but we should stress that because there is no strip of overlap all procedures should only be regarded as formal, and the results should be verified by independent checks.

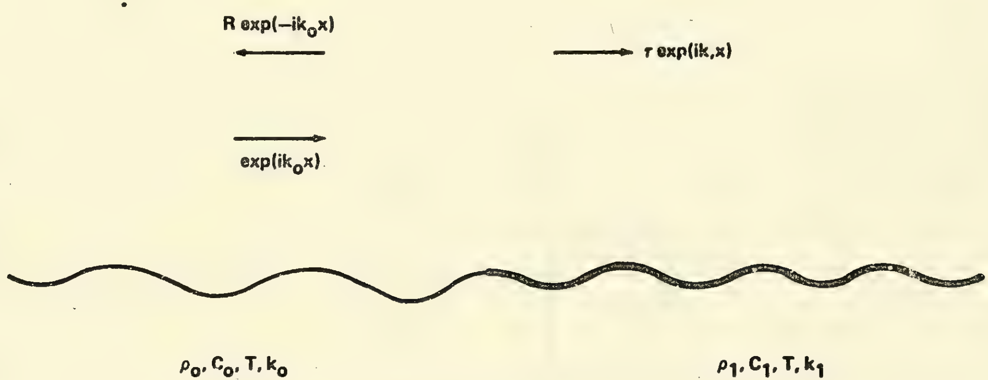


Figure 1 - Reflection and Transmission from a Simple Discontinuity in String Density

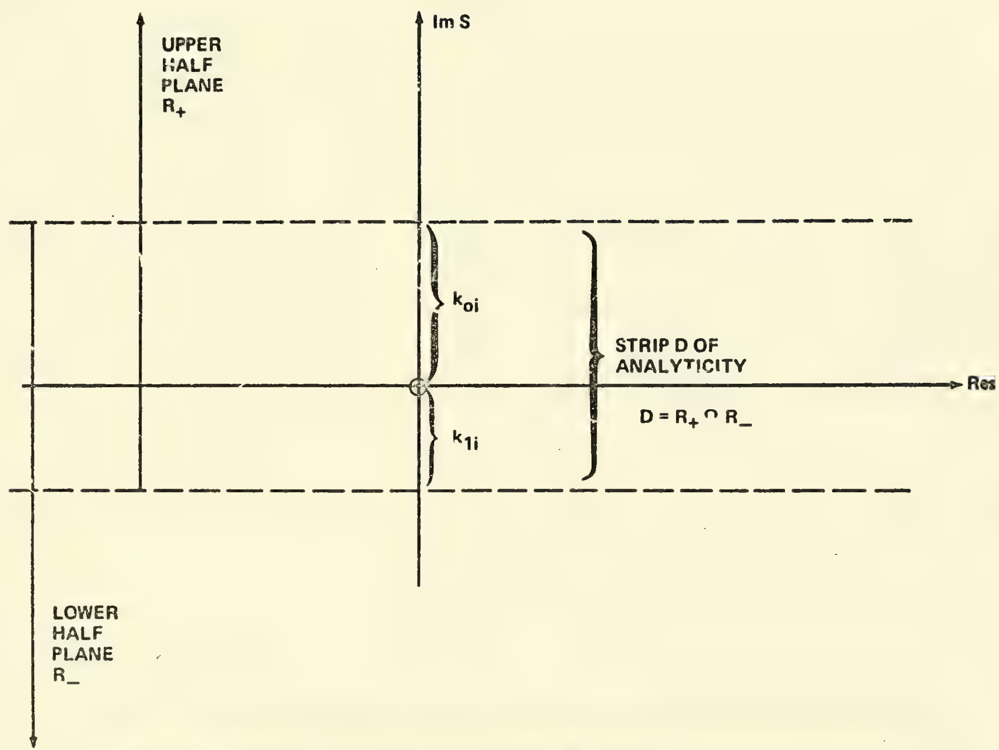


Figure 2 - Complex S-Plane with Overlapping Upper and Lower Half-Planes R_+, R_- and Strip of Analyticity D

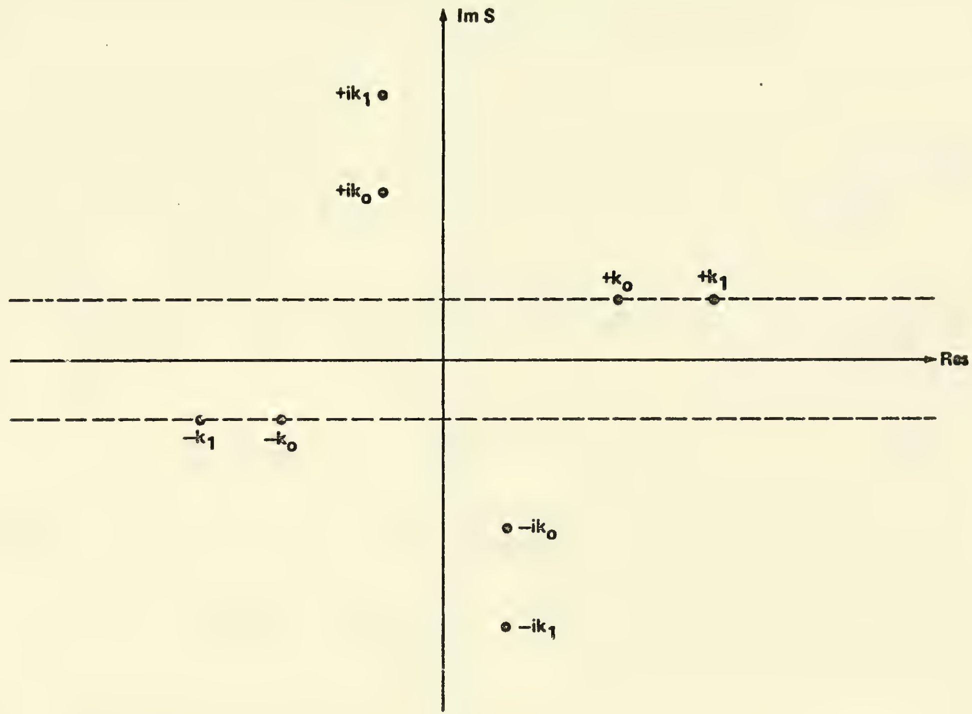


Figure 3 - Singularities Associated with Reflection from Discontinuities on Beams

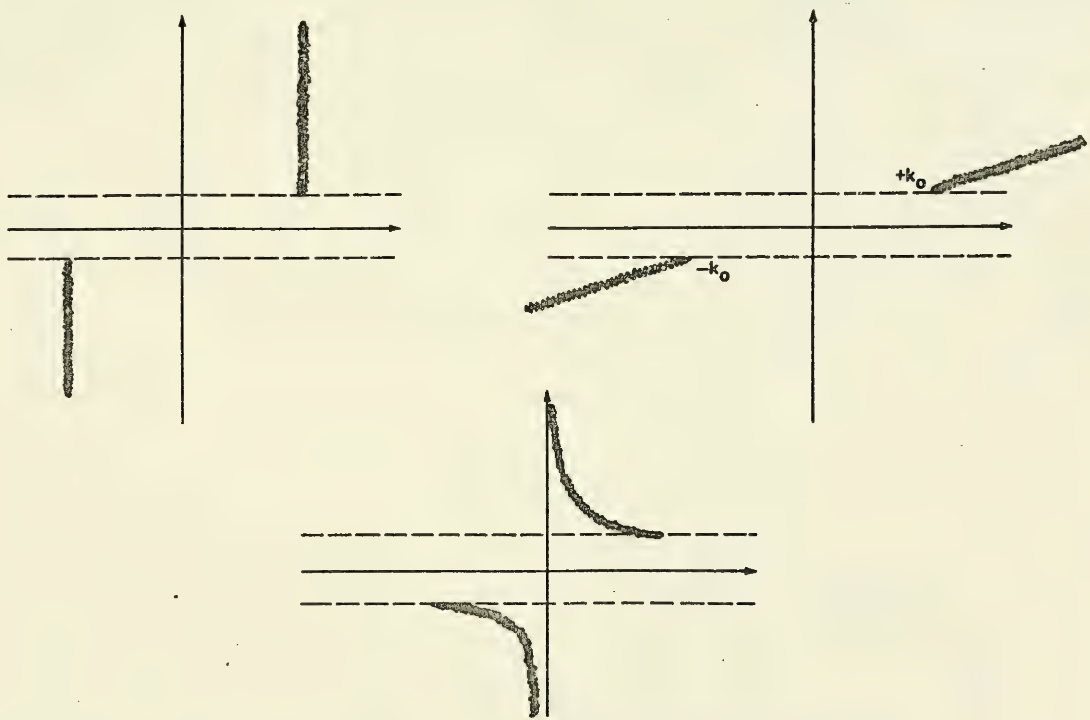
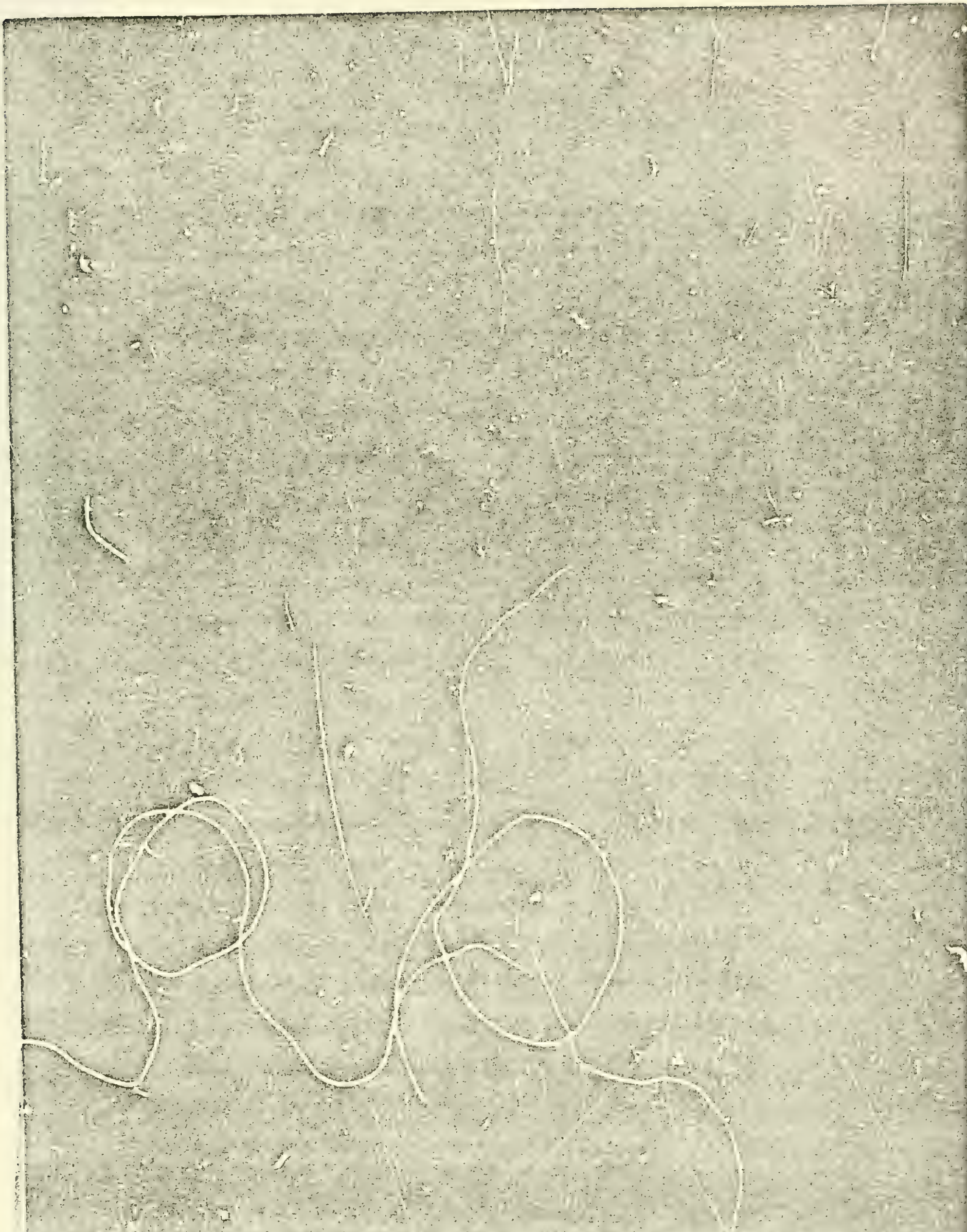


Figure 4 - Common Choices for Location of the Branch Cuts for the Square Root Function γ



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