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INTRODUCTORY

MODERN GEOMETRY

OF

POINT, RAY, AND CIRCLE

PART I

BY

W. B. SMITH, PH.D.

1432.

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OF

POINT, RAY, AND CIRCLE

PART I

BY

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Aeltestes bewahrt mit Treue,
Freundlich aufgefaßtes Neue.
— *Goethe.*

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GEOMETRY.

INTRODUCTION.

1. **Geometry is the Doctrine of Space.**

What is Space? On opening our eyes we see objects around us in endless number and variety: the book here, the table there, the tree yonder. This vision of a world outside of us is quite involuntary — we cannot prevent it, nor modify it in any way; it is called the *Intuition* (or Perception or Envisagement) *of Space.* Two objects precisely alike, as two copies of this book, so as to be indistinguishable in every other respect, yet are not the same, because they differ in place, in their positions in Space: the one is here, the other is not here, but there. In between and all about these objects that thus differ in place, there lies before us an apparently unoccupied region, where it seems that nothing *is*, but where anything *might be.* We may imagine or suppose all these objects to vanish or to fade away, but we cannot imagine this region, either where they were or where they were not, to vanish or to change in any way. This region, whether occupied or unoccupied, where all these objects are and where countless others might be, is called **Space.**

2. There are certain *elementary facts*, that is, facts that cannot be resolved into any simpler facts, about this Space, and these deserve special notice.

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A. Space is fixed, permanent, unchangeable. The objects in Space, called bodies, change place, or may be imagined to change place, in all sorts of ways, without in the least affecting Space itself. Animals move, that is, change their places, hither and thither; clouds form and transform themselves, drifting before the wind, or dissolve, disappearing altogether; the stars circle eternally about the pole of the heavens; sun, moon, and planets wander round among the stars; but the blue dome of the sky,* the immeasurable expanse in which all these motions go on, remains unmoved and immovable, as a whole and in all its parts, absolutely the same yesterday, to-day, and forever.

B. Space is **homœoidal**; *i.e.* it is precisely alike throughout its whole extent. Any body may just as well be here, there, or yonder, so far as Space is concerned. A mere change of place in nowise affects the Space in which the change, or motion, occurs.

C. Space is **boundless.** It has no beginning and no end. We may imagine a piece of Space cut out and colored (to distinguish it from the rest of Space); the piece will be bounded, but Space itself will remain unbounded.

N. B. When we say that Space is unbounded, we do *not* mean that it is *infinite.* Suppose an earthquake to sink all the land beneath the level of the sea, and suppose this latter at rest; then its outside would be unbounded, without beginning and without end, — a fish might swim about on it in any way forever, without stop or stay of any kind. But it would *not* be infinite; there would be exactly so many square feet of it, a finite number, neither more nor less. Likewise, the fact that bodies may and do move about in space every way without let or hindrance of any kind implies

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* Appearing blue because of the refraction of light in the air.

that Space is boundless, but by no means that it is infinite. For all we know there may be just so many cubic feet of Space; it may be just so many times as large as the sun, neither more nor less. This distinction between unbounded and infinite, first clearly drawn by Riemann, is fundamental.

D. Space is **continuous.** There are no gaps nor holes in it, where it would be impossible for a body to be. A body may move about in Space anywhere and everywhere, ever so much or ever so little. Space is itself simply where a body may be, and a body may be anywhere.

E. Space is **triply extended,** or has **three dimensions.** This important fact needs careful explication.

In telling the size of a box or a beam we find it necessary and sufficient to tell three things about it: its length, its breadth, and its thickness. These are called its dimensions; knowing them, we know the size completely. But to tell the size of a ball it is enough to tell one thing about it, namely, its diameter; while to tell the size of a chair we should have to tell many things about it, and we should be puzzled to say what was its length, or breadth, or thickness. Nevertheless, it remains true that Space and all bodies in Space have just three dimensions, but in the sense now to be made clear.

We learn in Geography that, in order to tell accurately where a place is on the outside of the earth, which may conveniently be thought as a level sheet of water, it is necessary and sufficient to tell *two* things about it; namely, its latitude and its longitude. Many places have the same latitude, and many the same longitude; but no two have the same latitude *and* the same longitude. It is not sufficient, however, if we wish to tell exactly where a thing is in Space, to tell two things about it. Thus, at this moment the bright star Jupiter is shining exactly in the south; we also know its altitude, how high it is above the horizon (this altitude is measured *angularly* — a term to be explained hereafter, but with which we have no present concern). But the knowledge of these two facts merely enables us to *point towards* Jupiter; they do not fix his place definitely, they do not say how far away he is: we should point towards him the same way whether he were a mile or a million of miles distant. Accordingly, a third thing must be known about him, in order to know precisely where he is; namely, his distance from us. But when this third thing is known, no further knowledge about his place is either necessary or possible. Once more, here is the point of a pin. Where is it in this room? It is five feet above the floor. This is not enough, however, for there are many places five feet above the floor. It is also ten feet from the south wall, but there are yet many positions five feet from the floor and ten feet from the south wall, as we may see by slipping a cane five feet long sharpened to a point, upright on the floor, keeping the point always ten feet from the south wall. But as it is thus slipped along, the point of the cane will come to the point of the pin and then will be exactly twelve feet from the west wall. If it now move ever so little either way east or west, it will no longer be at the pin-point and no longer twelve feet from the west wall. So there is one, and only one, point that is five feet from the floor, ten feet from the south wall, and twelve feet from the west wall. Hence it is seen that these three facts fix the position of the pin-point exactly. A fourth statement, as that the point is nine from the ceiling, will either be superfluous, if the ceiling is fourteen feet high, being implied in what is already said, or else incorrect, if the ceiling is not fourteen feet high, contradicting what is already said. In general, with respect to any position in Space it is *necessary* to know three independent facts (or data), and it is *impossible* to know any more. All other knowledge about the position is involved in this knowledge, which is necessary and sufficient to enable us to answer any rational question that can be put with respect to the position. Accordingly, since any position in Space is known completely when, and only when, *three* independent data are known about it, we say that Space is *triply* or *three-fold extended*, or has *three dimensions*. The dimensions are any three independent things that it is necessary and sufficient to know about any position in Space, as of the pin-point or of Jupiter, in order to know exactly where it is.

3. But with respect to the outside of the earth, viewed as a level sheet of water, we have seen that only two data, as of latitude and longitude, are necessary and sufficient to fix any position on it; neither are more than two independent data possible; all other knowledge about the position is involved in the knowledge of these two data about it. Accordingly we say of such outside of the earth that it is *doubly* or *two-fold extended*, is bi-dimensional, or has two dimensions; and we name every such *outside*, every such bi-dimensional region, a **surface.** Such is the top of the table: to know where a spot is on it we need know two, and only two, independent facts about it, as how far it is from the one edge and how far from the other. (Which other? and why?)

We see at once that a surface is no part of Space, but is only a *border* (doubly extended) between two parts of Space. Thus, the whole earth-surface is no part either of the earth-space or of the air-space around the earth, but is the boundary between them. A soap-bubble floating in the air is not a surface; though exceedingly thin, it has some thickness and occupies a part of space; the outside of the film is a surface, and so is the inside, and these are kept apart by the film itself. If the film had no thickness, the outside and the inside would fall together, and the film would be a surface; namely, the outside of the Space within and the inside of the Space without.

4. Consider now once more this earth-surface, still viewed as a smooth level sheet of water. From Geography we learn that there are two extreme positions on this surface that are called *poles* and that do not move at all as the earth spins round on her axis. We also learn that there is a certain region of positions just midway between these poles and called the **Equator.** This Equator is no part of the surface; it is only a border or boundary between two parts of the globe-surface, which are called *hemispheres.* To know where any position is on this border, it is necessary and sufficient to know *one* thing about it, namely, its longitude; neither is any other independent knowledge about the position possible; all other knowledge is involved in this one knowledge. Accordingly we say of this border, the Equator, that it is *simply extended*, or has one dimension only. Every such one-dimensional border is called a **line,** and its one dimension is named *length.* A line, then, has length, but neither breadth nor thickness.

5. Lastly, consider a part of a line, as of the Equator, say between longitudes 40° and 50°. The ends of this part bound it off from the rest of the equator, but they themselves form no part of the Equator. They are called **points**; they have position merely, but no extent of any kind, neither length nor breadth nor thickness, — they are wholly *non-dimensional.*

*6. It is noteworthy that extents, or regions, are bounded by extents of fewer dimensions, and themselves bound extents of more dimensions. Thus, lines are bounded by points, and themselves bound surfaces; surfaces are bounded by lines, and themselves bound spaces; spaces are bounded by surfaces, and themselves bound — what? If anything at all, it must be some extent of still higher order, of *four* dimensions. But here it is that our intuition fails us; our vision of the world knows nothing of any fourth dimension, but is confined to three dimensions. If there be any such fourth dimension, we can know nothing of it by intuition: we cannot imagine it. In music, however, we do recognize four dimensions: in order to know a note completely, to distinguish it from every other note, we must know four things about it: its pitch, its intensity, its length, its timbre, — how high it is, how loud it is, how long it is, how rich it is. While, then, extents of higher dimensions may be unimaginable, they are not at all unreasonable.

This doctrine of dimensions is of prime importance, but rather subtile; let not the student be disheartened, if at first he fail to master it.

6. We may see and handle bodies, which occupy portions of Space; but not so surfaces, lines, points, which occupy no Space, but are merely regions in Space. Here we must invoke the help of the logical process called **abstraction**, *i.e.* withdrawing attention from certain matters, disregarding them, while regarding others. A sheet of paper is not a surface, but a body occupying Space. However thin, it yet has some thickness. But in thinking about it we may leave its thickness out of our thoughts, disregard its thickness altogether; so it becomes for our thought, though not for our senses or imagination, a *surface*. The like may be

said of the film of the soap-bubble. Again, consider the pointer. It is a body or solid, not only long, but wide and thick; it occupies Space. It is neither line nor surface. But we may, and do often, disregard wholly two of its dimensions, and attend solely to the fact that it is *long*. Thus it becomes for our thought a *line*, though not for our senses or imagination. So the mark made with chalk or ink or pencil is a body, triply extended; but we disregard all but its length, and it becomes for our reason a *line*. Lastly, we make a dot with pen or chalk or pencil; it is a body, tri-dimensional, occupying Space. But we may disregard all its dimensions, and attend solely to the fact that it has position, that it is here, and not there. So it becomes in our thought a *point*. By such abstraction the earth, the sun, the stars, the planets, may all be treated as points.

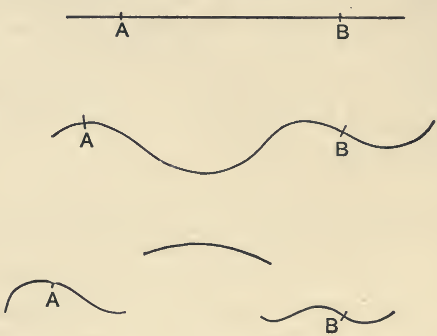

FIG. 1.

7. Inasmuch as Space is continuous, there may also be continuous surfaces and lines; and the only surfaces and lines treated in this book are continuous, without holes, gaps, rents, breaks, or interruptions of any kind in their extent.

It is important to note that in passing from any position A

to another B on a continuous line, a moving point P must pass through a complete series of intermediate positions; *i.e.* there is no position on the line between A and B that the point P would not assume in going from A to B. (Fig. 1.)

*8. Starting from the notion of Space, we have attained the notions of surface, line, and point, in two ways: by treating them as borders, and by the process of abstraction. But we may reverse this order and attain the notions of line, surface, and solid or space from the notion of point, with the help of the notion of *motion*, thus: Let a *point* be defined as having *position without parts or magnitude* of any kind. Let it move continuously through Space from the position A to the position B. To know where it is at any stage of its motion along any definite path, it is necessary

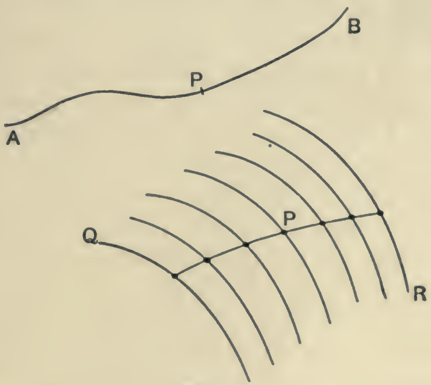

FIG. 2.

and sufficient to know one thing; namely, how far it is from A. Hence its path is a *one*-dimensional extent, or what we call a *line*.

Now let a line move in any definite way from any position Q to any other position R. To know the position of any point of its path, it is necessary and sufficient to know two things; namely, the position of the point on the moving line and the position of the moving line itself: hence the path of the line is a *two*-dimensional extent, which we have already named a *surface*. (Fig. 2.)

Now let a surface move in any definite way from any position U to any other position V. To know the position of any point on its path it is necessary and sufficient to know three things about it; namely, its position on the moving surface (which, we know, counts as two things) and the position of the moving surface itself. Hence the path of the surface is a *three*-dimensional extent, which we have already named a *solid* or a part of *space*.

Now, if we let a solid move, what will its path be? Naturally we should expect it to be a *four*-dimensional extent, but no such extent is yielded in our experience by any motion of a solid — the path of a solid is nothing but a solid. The explanation of the apparent inconsistency is very simple, to-wit: A piece of a line traces out a surface only when it moves out from the line itself, — if one part were to slip round on another part of the same line, it would trace out no surface at all as its path; likewise, a piece of a surface traces out a solid as its path only by moving out from the surface itself, — if one part were to slip round on another part of the surface, it would trace out no solid at all as its path. So, if a piece of our space could move out from space itself, it would trace out a four-fold extent as its path; in fact, however, no part of space can move out from space; on the contrary, it can only slip along *in* space, from one part of space to another, and hence does not trace out any four-fold extended path.

9. Space, we have seen, is homœoidal, everywhere alike. We naturally inquire: Is there any homœoidal surface? In general, surfaces are certainly *not* homœoidal. Consider an egg-shell, and by abstraction treat it as a surface. It is not alike throughout; the ends are not like each other, and neither is like the middle region. Suppose a piece cut out anywhere; if slipped about over the rest of the shell, this piece will *not fit.* But now consider a smooth round ball covered with a thin rigid film, and treat this film as a surface, by disregarding its thickness. Suppose a piece of the film cut out and slipped round over the rest of the film: the piece will fit everywhere perfectly, the surface is homœoidal; it is called a **sphere**-surface.

N.B. The precise definition of this surface is that *all its points are equidistant from a point within, called the centre.* Suppose a rigid bar of any shape, pointed at both ends, and movable about one end fixed at a point; then the other end will move always on a sphere-surface, which is *the whole region where the moving end may be.* Since Space is homœoidal around the fixed point, the surface everywhere equidistant from the point is also homœoidal.

Now turn over the piece cut out of this spherical film and slip it about the film: it no longer fits anywhere at all — the surface is homœoidal, but *not reversible.*

10. But now consider a fine mirror covered with a delicate film, which by abstraction we treat as a surface. Suppose a piece cut out of the film and slipped about over it: the piece fits everywhere; turn it over, re-apply it, and slip it about: it still fits everywhere — the surface is both *homœoidal* and *reversible;* it is called a **plane**-surface.

* N.B. A precise definition of this surface is the following: Take two points A and B and suppose two equal spherical

bubbles formed about A and B as centres. Let them expand, always equal to each other, until they meet, and still keep on expanding. The line where the equal (Fig. 3) spherical bubbles, regarded as surfaces, meet, has all its points just as far from A as from B. As the bubbles still expand, this line, with all its points equidistant from A and B, itself expands and traces out a *plane* as its path through Space.

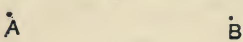

Fig. 3.

Hence we may define the *plane* as the region (or surface) where a point may be that is equidistant from two fixed points. Instead of region it is common to say **locus**, *i.e.* place. Briefly, then, a *plane is the locus of a point equidistant from two fixed points*. It is evident that the plane, as thus defined, is reversible; for since the bubbles about A and B are all the time precisely equal, to exchange A and B, or to exchange the sides of the plane, will make no difference whatever. Thus the plane cuts the Space evenly half in two; and since Space itself is homœoidal, so also is this section or surface that halves it exactly. The superiority of this definition consists in its not only telling what surface the plane is, but also making clear that there actually *is* such a surface.

11. The mirror is the nearest approach that we can make to a perfect plane surface; the blackboard is not plane, it is rough and warped; but we shall disregard all its unevenness and treat it as a plane extended through Space without end. Any surface may be dealt with as a plane by abstraction, being thought as *homœoidal* and *reversible*.

12. On this board, regarded as a plane, we draw a chalk-mark, abstract from all its dimensions but its length, and treat it as a line. This line is plainly not alike throughout; a piece cut out and slipped along it will not fit (Fig. 4).

FIG. 4.

But here is a line homœoidal, alike in all its parts; it is drawn with a pair of compasses and is called a **circle** (Fig. 5). One point of the compasses is held fast at the *centre O,* while the other traces out the circle as its path in

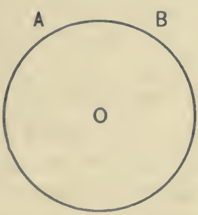

FIG. 5.

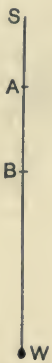

FIG. 6.

the plane. The circle is *the locus of a point in the plane equidistant from a fixed point.* Since the plane is homœoidal, so too is this circle (see Art. 10); a piece, called an **arc,** cut out and slipped round will everywhere fit on the circle. But turn it over and slip it round, — it fits nowhere; the circle is *not* reversible. It divides the plane into two parts, not halves, that are not alike along the dividing line. But now suppose a perfectly flexible string fastened at S and stretched by a weight W. Its length only being regarded, it is a line homœoidal, alike throughout, and also reversible; any part AB will not only fit perfectly anywhere on it, but will also fit when reversed, turned end for end. Such a line is called *right*, or *straight*, or *direct*, or a **ray**. Extended indefinitely, it cuts the whole plane into two halves precisely alike along the ray itself.

*N.B. The common line where the two spherical bubbles of Art. 10 meet is a circle, for it is plainly precisely alike all around; it is homœoidal, being the intersection of two homœoidal surfaces, namely, the two equal sphere-surfaces; it is also in a plane, and in fact traces out the plane by its expansion as the bubbles expand.

To get accurately the notion of the **ray** or *straight line*, we need another point C, and a third expanding bubble always equal to those about A and B. The circular intersection of the bubbles about A and B will trace out one plane; of those about B and C will trace out another plane; of those about C and A will trace a third plane. All the points where the first two planes intersect will be equidistant from A and B and C, and no other points will be; the same may be said of all points where the second and third planes meet, and of all points where the third and first meet; hence all three of the planes meet together, and they meet only together. Also, the line where they meet has every one of its points equidistant from all the three points, A, B, C; hence it is the **locus** *of a point equidistant from three fixed points*. Moreover, it is *homœoidal* and *reversible*, since it is the intersection of two planes, which are homœoidal and reversible; hence it is what we call a straight line, or right line, or *ray*.

13. We may now define:

A **sphere**-surface is the locus of a point at a fixed distance from a fixed point. It is homœoidal, but not reversible.

A **plane** is the locus of a point equidistant from two fixed points. It is both homœoidal and reversible.

A **ray** is the locus of a point equidistant from three fixed points. It is both homœoidal and reversible; it is also the intersection of two planes.

A **circle** is the locus (or path) of a point in a plane at a fixed distance from a fixed point. It is homœoidal, but not reversible. It is also the locus (or path) of a point in space at a fixed distance from two points; it is also the intersection of two equal sphere-surfaces.*

14. It is only with the foregoing figures and combinations of them that we have to deal in this book. Circles and rays may be drawn with exceeding accuracy, but any lines, however roughly drawn, may answer our logical purposes as well as the most accurately drawn; we have only, by abstraction, to treat them as having the character of the lines in question.

Circles and sphere-surfaces are unbounded, without beginning or end, but both are finite: we shall learn how to measure them.

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* In the foregoing free use has been made of the notion of *equidistance* without formal definition, because of its familiarity. We may, however, say precisely: If A and B be two points, the ends of a rigid bar of any shape, and if A be held fast, then all the points on which B can fall are equidistant from A, and no other points are equidistant with them. They all lie on a closed surface, called a sphere-surface. All points within this surface are said to be *less* distant, and all points without are said to be *more* distant, from A than B is. Herewith, then, we tell exactly what we mean by equidistant, less distant, and more distant, but we make no attempt to define *distance* in general, which is difficult and unnecessary to our purpose.

15. Any geometric element or combination of geometric elements, as points, lines, surfaces, is called a geometric **figure.** It is a fundamental assumption, justified by experience, that space is homœoidal, that figures or bodies are not affected in size or shape by change of place. Two figures that may be fitted exactly on each other, or may be thought so fitted, are called **congruent.** Any two points, lines, or parts of the two figures, that fall upon each other in this superposition are said to **correspond.** It is manifest that all planes are congruent and all rays are congruent. Rays and planes are unbounded, but whether or not they are finite is a question that we are unable to answer.

16. Any part of a circle or ray, as AB, is bounded by two end-points, A and B, and is finite; the one is named an **arc** (Fig. 5), the other a **tract**, *sect*, or *line-segment.* Each is denoted by the two letters denoting the ends, as the tract AB, the arc AB. Sometimes it is important to distinguish these end-points as beginning and end proper; we do this by writing the letter at the beginning first.

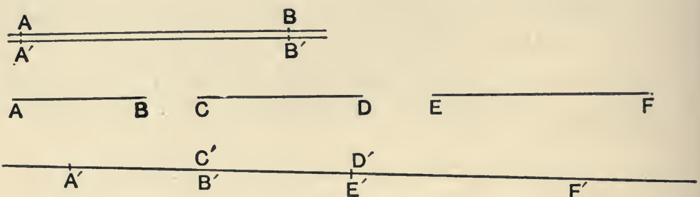

FIG. 7.

17. Two tracts, AB and $A'B'$, are called equal when the end-points of the one may be (Fig. 7) simultaneously fitted on the end-points of the other.

If we have a number of tracts, AB, CD, EF, etc., and we lay off successively on a ray tracts $A'B'$, $C'D'$, $E'F'$, etc., respectively equal to AB, CD, EF, etc., the end of the first being the beginning of the second, and so on, while no part of one falls on any part of another, we are said to *add* or **sum** the tracts AB, etc. Each is called an *addend* or **summand,** and the whole tract from first beginning to last end is called the **sum.**

Equality is denoted by the bars $(=)$ between the equals, as $AB = CD$.

18. If, when the beginning A is placed on the beginning C, the end B does not fall on the end D, the tracts are unequal, and we write $AB \neq CD$. If B falls between C and D, then AB is called less than CD, $AB < CD$; but if D falls between A and B, then AB is called greater than CD, $AB > CD$. In either case, the tract BD or DB, between the two ends of the tracts, whose beginnings coincide, is called the **difference** of the two tracts, and we are said to **subtract** the one from the other. Ordinarily we mention the greater tract first in speaking of difference.

19. The symbols of addition and subtraction are $+$ and $-$ (plus and minus), thus:

$$AB + CD = AD \text{ and } AB - CD = BD.$$

It is important to note here the order of the letters. In summing a number of tracts, as AB, CD, EF, etc., to KL,

FIG. 8.

we have $AB + CD + EF \cdots + KL = AL$ (Fig. 8). The order of the summands is *indifferent*, and this important fact is called the **Commutative Law** of Addition. Thus

$$AB + CD + EF = AB + EF + CD = EF + AB + CD, \text{ etc.}$$

20. When beginning and end of a tract or of any magnitude are exchanged, the tract or magnitude is said to be **reversed**, and the reverse is denoted by the sign $-$. Thus the reverse of AB is BA, or $AB=-BA$. If we add a magnitude and its reverse, the sum is 0, or

$$AB+(-AB)=AB+BA=0.$$

The same result 0 is obtained by subtracting, from a magnitude, itself or an equal magnitude; and, in general, it is plain that to subtract CD yields the same result as to add (Fig. 9) the reverse DC. The reverse of a magnitude is

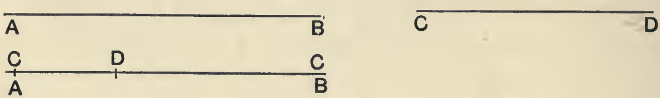

FIG. 9.

often called its *negative*, the magnitude itself being called its *positive*.

Similar rules hold for adding and subtracting arcs of a circle or of equal circles.

ANGLES.

21. The indefinite extent of a ray on one side of a point O, as OA, is called a **half-ray**: it has a beginning O, but no end. Two half-rays, OA and OA', which together make up a whole ray, are called *opposite* or **counter** (Fig. 10).

Now let two half-rays, OA and OB, have the same beginning O; the opening or spread between them is a magnitude: it may be greater or less. Suppose OA and OB to be two very fine needles pivoted at O; then OB may fall exactly on OA, or it may be turned round from OA; and the amount of turning from OA to OB, or the spread between the half-rays, is called the **angle** between them. We may denote it by a Greek letter, as α, written in it; or by a large Roman letter, as O, at its vertex (where the half-rays meet); or by three such letters, as AOB, the middle

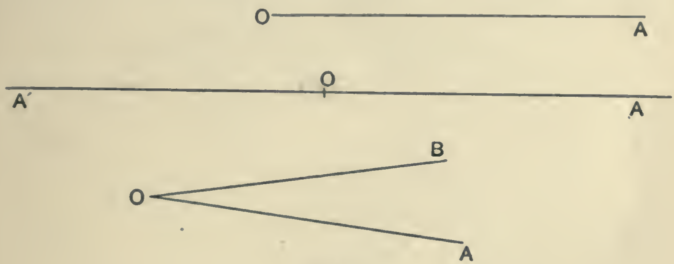

FIG. 10.

one being at the vertex; the other two anywhere on the half-rays. The symbol for angle is $\measuredangle$.

22. The angle is perfectly definite in size, it has two *ends* or boundaries; namely, the two half-rays, sometimes called **arms**. When we would distinguish these arms as beginning and end, we mention the letter on the beginning-arm first, and the letter on the end-arm last; thus, AOB; here OA is the *beginning* and OB the *end* of the angle.

Exchanging beginning and end reverses the angle; thus, $BOA = -AOB$.

23. Two angles whose ends or arms may be made to fit on each other simultaneously are named **equal**; they are also *congruent*. Two angles whose arms will not fit on each other simultaneously are *unequal;* and that is the less angle whose end-arm falls *within* the other angle when their beginnings coincide; the other is the greater; thus, $AOB > AOC$ (Fig. 11).

24. We sum angles precisely as we sum tracts; we lay off α, β, etc., around O, making the end of each the beginning of the next: the angle from first beginning to last end

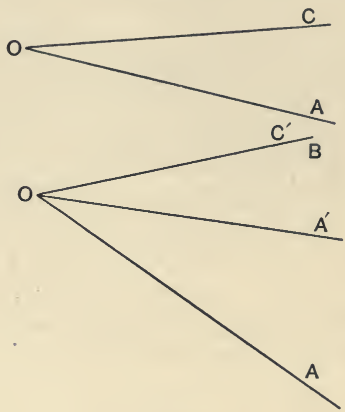

FIG. 11.

is the *sum*. So, too, in order to subtract β from α, lay off β from the beginning towards the end of α; the angle from the end of β to the end of α is the difference, $\alpha - \beta$. Or we may add to α the reverse (or negative) of β: the sum will be $\alpha + (-\beta)$ or $\alpha - \beta$ (Fig. 12).[1]

[1] It is important to note the close correspondence of tract and angle: the former is related to points as the latter is to rays (or half-rays). The tract is the simplest magnitude that lies between points, that distinguishes them and keeps them apart; likewise the angle is the simplest magnitude that lies between rays (in a plane), that distinguishes them and keeps them apart. So, too, we define equality and inequality among tracts and among angles, quite similarly, and without being compelled beforehand to form the notion of the size either of a tract or of an angle. We may now define the *distance* between two points to be the *tract* between them, and the *distance* between two (half-)rays to be the *angle* between them, leaving for future decision which tract and which angle if there should prove to be several.

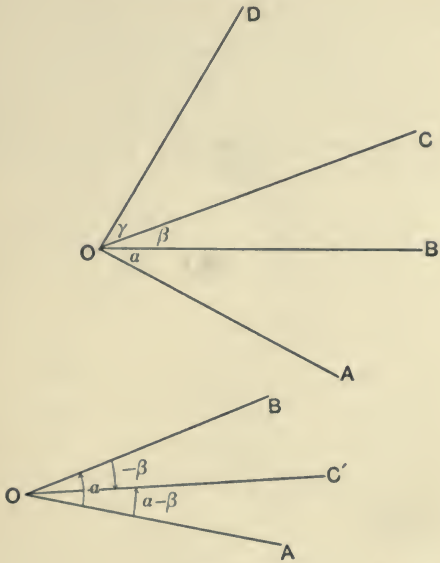

FIG. 12.

AXIOMS.

25. At this stage we must recognize and use certain dictates or irresoluble facts of experience, called **axioms.** ('Αξιωμα means *something worthy*, like the Latin *dignitas;* in fact, older writers use *dignity* in the sense of *axiom*. But Euclid's phrase is κοιναι ἐννοιαι = *common notions*.) Some have no special reference to Geometry, but pervade all of our thinking about magnitudes; such are

(1) Things equal to the same thing are equal to each other.

(2) If equals be added to, subtracted from, multiplied by, or divided by, equals, the results will be equal.

(3) If equals be added to or subtracted from unequals, the latter will remain unequal as before.

(4) The whole equals, or is the sum of, all its distinct parts, and is greater than any of its parts.

(5) If a necessary consequence of any supposition is false, the supposition itself is false.

Others concern Geometry especially, as:

(6) All planes are congruent.

(7) Two rays can meet in only one point.

The extremely important axiom (7) may be stated in other equivalent ways, thus: Two rays cannot meet in two or more points; or, Two rays cannot have two or more points in common; or, Only one ray can go through two fixed points; or, A ray is fixed by two points.

26. A statement or declaration in words is called a *proposition*. The propositions with which we have to deal state geometric facts and are also called **Theorems** (*θεωρημα*, from *θεωρειν*, *to look at*, means the *product of mental contemplation*). Propositions are often incorrect; theorems, never. Subordinate facts, special cases of general facts, and facts immediately evidenced from some preceding facts, are called **Corollaries** or Porisms (*πορισμα* = *deduction*).

We may now proceed to investigate lines and angles, and find out what we can about them. The first and simplest things we can learn concern

CONGRUENCE.

27. **Theorem I.** — *All rays are congruent.*

Proof. Let L and L' be any two rays (Fig. 13). On L take any two points, A and B; on L' take any two points, A' and B', so that the tract AB shall equal the tract $A'B'$. Think of L and L' as extremely fine rigid spider-threads, and in thought place the ends of the tract AB on the ends of the tract $A'B'$, A on A', and B on B'.

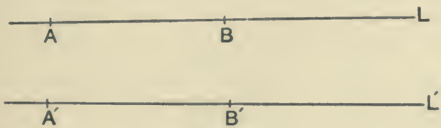

FIG. 13.

Then A and A' become one and the same point, and B and B' become one and the same point; through these two points only one ray can pass (by Axiom 7): hence L and L', which go through these two points, now become one and the same ray; that is, they fit precisely, they are congruent. *Quod erat demonstrandum* = *which was to be proved* = ὅπερ ἔδει δεῖξαι, — the solemn Greek formula; whereas the Hindu, appealing directly to intuition, merely said *Paçya* — Behold!

28. In the foregoing proof we assumed that on any ray we could lay off a tract equal to a given tract, or that on any ray we could find two points, A and B, as far apart as two other points, A' and B'. This assumption that something can be done, is called a **Postulate** (*αἴτημα*), *i.e.* a *demand*, which must be granted before we can proceed further. Actually to carry out the construction, we need a pair of compasses.

29. Theorem II. — *If two points of a ray lie in a certain plane, all points of the ray lie in that plane.*

Proof. Regard the surface of paper or of the blackboard as a plane, and suppose it covered with a fine rigid film, itself a plane. Let L be any ray having two points, A and B, in this plane. Through these two points suppose a second plane drawn or passed; by definition (Art. 13) it will intersect our first plane, or film, along a ray I; this ray I goes through the two points, A and B, and lies wholly (with all its points) in the first plane; also the ray L goes through A and B, and only one ray can go through the same two points, A and B, by Axiom 7; hence L and I are the same ray; but I has all its points in the first plane; hence L has all its points in the first plane. Q. E. D.

Query: What *postulate* is assumed in this proof?

Corollary. If a ray turn about a fixed point P, and glide along a fixed ray L, it will trace out a plane (Fig. 14).

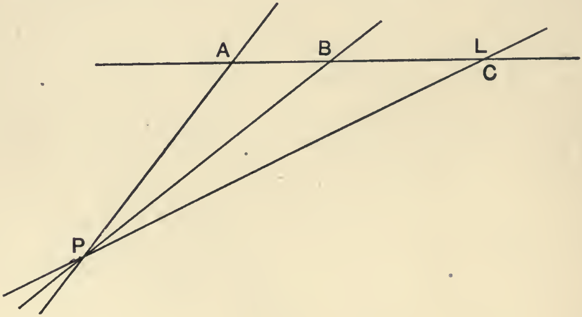

FIG. 14.

For it will always have two points — namely, the fixed point and a point on the fixed ray — in the plane drawn through the fixed point and the fixed ray.

Query: What postulate is here implied? — Henceforth it is understood that all our points, lines, etc., are **complanar**, *i.e.* lie in one and the same plane.

30. In the foregoing Theorem and Corollary we observe clauses introduced by the word **if**. Such a clause is called an **Hypothesis**, *i.e.* a supposition. The result reached by reasoning from the hypothesis and stated immediately after the hypothesis, is called the **Conclusion**.

31. All logical processes consist in one or both of two things: the formation of concepts, as of lines, surfaces, angles, etc., and the combination of these concepts into propositions. Geometric concepts are remarkable for their perfect clearness and precision — we know exactly what we mean by them; this cannot be said of many other concepts, about which diverse opinions prevail, as in Political Economy. Hence it is that Geometry offers an unequalled gymnasium for the reason or logical faculty. We shall now generate some new concepts. Let the student note their definiteness as well as the mode of their formation.

32. Let OA and OB be any two co-initial half-rays, forming the angle AOB. Think of OA as held fast and of OB as turning about the pivot O, starting from the position OA. As it turns (counter-clockwise), the (Fig. 15) angle

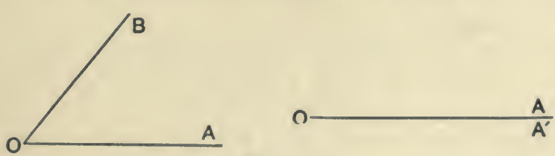

FIG. 15.

AOB increases. Finally, let it return to its original position, OA; then the whole amount of turning from the upper side of OA back to the under side of OA, or the full spread around the point O, is called a *full* angle (or **round** angle, or *circum*-angle, or perigon). Think of a fan opened until the first rib falls on the last. — Note that the upper and under sides of OA are exactly the same in position, and are distinguished only in thought. (Think of a circular piece of paper slit straight through from the edge to the centre.) The like may be said of the two sides of any line or surface. We can now prove

33. **Theorem III.** — *All round angles are congruent.*

Proof. Let AOB and $A'O'B'$ (Fig. 16) be any two round angles. Slip the half-ray OA down, and turn it till OA falls on $O'A'$; they will fit perfectly (why?); the

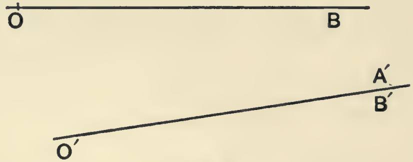

Fig. 16.

whole round angle about O will fit perfectly on the whole round angle about O' (why?); hence the two full angles are congruent. Q. E. D.

N.B. In this slipping of figures about in the plane, it is well to imagine the plane to consist of two very thin, perfectly rigid, smooth and transparent films; also, to imagine one figure drawn in the lower film and one in the upper; and to imagine the upper slipped about at will over the lower.

Query: On what cardinal property of the plane do these considerations hinge?

34. From O draw any half-ray OA; then any second half-ray from O, as OB, will (Fig. 17) cut the round angle AOA into two angles, AOB and BOA. The end OB of the first falls on the beginning, OB, of the second; while

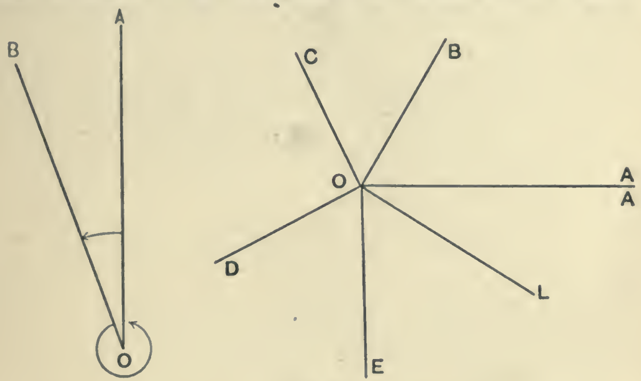

FIG. 17.

the end, OA, of the second falls on the beginning, OA, of the first. Hence the round angle AOA is their sum, by Art. 24.

If we draw any number of half-rays, OB, OC, etc., $\cdots OL$, the round angle will still be the sum of the consecutive angles AOB, BOC, etc., $\cdots LOA$; hence we discover and enounce this

Theorem IV. — *The sum of the consecutive angles about a point in a plane is a round angle.*

N.B. We cannot apply Axiom 1 immediately, because we do not know, except by Art. 24, what is meant by a *sum* of angles.

35. In the foregoing article we have exemplified the *erotetic*, questioning, investigative method, in which the result

is not announced until it is actually discovered and established. In Theorems I., II., III., on the other hand, the *dogmatic* procedure was illustrated, the fact or proposition being announced beforehand, while the demonstration followed after. Each method has its merits, and we shall employ both.

36. As OB turns round from the upper to the under side of OA, the angle AOB begins by being less than BOA and ends by (Fig. 19) being greater than BOA. The plane

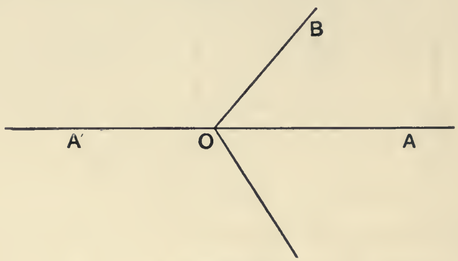

Fig. 19.

is continuous, the turning is continuous, the change in size is continuous; hence, in passing from the stage of being *less* to the stage of being *greater*, the angle has passed through the intermediate stage of being *equal;* let OA' be the position of the rotating half-ray at this stage of equality, then $AOA' = A'OA$. Two equal parts making up a whole are called **halves**; hence AOA' and $A'OA$ are halves of the full angle AOA; they are named **straight** (or flat) angles.

37. Now, — Halves of equals are equal;
All straight angles are halves of equals (namely, equal round angles);

Hence

Theorem V. — *All straight angles are equal.*

This argument here given *in extenso* is a specimen of a syllogism ($\sigma\upsilon\lambda\lambda o\gamma\iota\sigma\mu o\varsigma$ = computation = thinking together). The first two propositions are called **premisses**, the third and last, in which the other two are thought together, is called **conclusion**. All reasoning may be syllogized, but this is rarely done, as being too formal and tedious.

38. **Theorem VI.** — *Two counter half-rays bound a straight angle.*

FIG. 20.

For, let OA and OA' be two such counter half-rays (Fig. 20) forming the whole ray AA'. Turn the upper half of the plane film round O as pivot until the upper OA' falls on the lower OA; then, since the ray is reversible, the ray AA' will fit exactly on the ray $A'A$; *i.e.* the two angles AOA' and $A'OA$ are congruent and equal; and the two compose the round angle AOA; hence each is half of AOA; *i.e.* each is a straight angle. Q. E. D.

39. **Theorem VII.** — Conversely, *The half-rays bounding a straight angle are counter.*

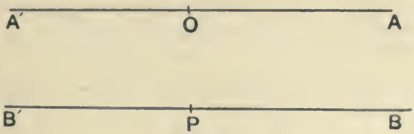

FIG. 21.

Let OA and OA' bound a straight angle (Fig. 21) AOA'; also let PB and PB' be two counter half-rays; then they bound a straight angle BPB', by Theorem VI. Since all straight angles are congruent, we may fit these two on each other; *i.e.* we may fit OA and OA' on PB and PB'; but BB' is a ray; so then is AA'; *i.e.* OA and OA' are counter. Q. E. D.

40. We may *define* a straight angle as *an angle bounded by counter half-rays.* Then we may prove Theorem V. thus:

The ends of all straight angles are pairs of counter half-rays (or form whole rays);

But all such pairs (or whole rays) are congruent (by Theorem I.);

Therefore, all ends of straight angles are simultaneously congruent.

But when the ends of angles are (simultaneously) congruent, so are the angles themselves.

Hence all straight angles are congruent. Q. E. D.

Here the first conclusion, introduced by "therefore," is deduced from two premisses; but the second, introduced by "hence," is apparently deduced from only one. Only apparently, however; for one premiss was understood but not expressed; namely, *all straight angles are angles whose ends are congruent.* Without some such implied additional premiss, it would be impossible to draw the conclusion. Such a maimed syllogism, with only one expressed premiss, is called an **enthymeme.** The great body of our reasoning is enthymematic. We shall frequently call for the suppressed premiss or *reason* by a parenthetic question (Why?).

41. Now draw two rays, LL' and MM', meeting at O. Each divides the round angle about O into two equal straight angles, and together they (Fig. 22) form four angles α, β, α', β'. Two angles, as α and β, that have a common arm, are called **adjacent.** Accordingly we see at once:

Theorem VIII. — *Where two rays intersect, the sum of two adjacent angles is a straight angle.*

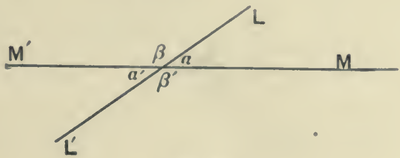

FIG. 22.

Two angles whose sum is a straight angle are called **supplemental**; two angles whose sum is a round angle we may call **explemental.** Two angles as α and α', the arms of the one being counter to the arms of the other, are called *opposite*, or *vertical*, or counter.

Theorem IX. — *When two rays meet, the opposite angles formed are equal.*

For $\alpha+\beta=S$ (a straight angle) (why?); and $\alpha'+\beta=S$ (why?).

Hence $\alpha+\beta=\alpha'+\beta$ (why?); therefore $\alpha=\alpha'$. Similarly let the student show that $\beta=\beta'$. Q. E. D.

An important special case is when the *adjacents*, α and β, are *equal.* Each then is *half* of a straight angle, and therefore *one fourth* of a round angle; and each is called a **right** angle. Now let the student show that if $\alpha=\beta$, then $\alpha'=\beta$ and $\alpha=\beta'$, or

Corollary. When two intersecting rays make two equal adjacent angles, they make all four of the angles equal (Fig. 23).

Def. Rays that make right angles with one another are called **normal** (or perpendicular) to each other. N.B. The normal relation is *mutual.* How?

Def. Two angles whose sum is a right angle are called **complemental.**

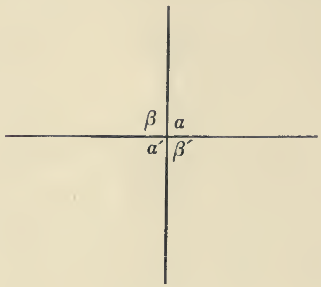

FIG. 23.

42. Are we sure that through any point on a ray we can draw a normal to the ray? Let O be any point on the ray LL' (Fig 24). Let any half-ray, pivoted at O, start

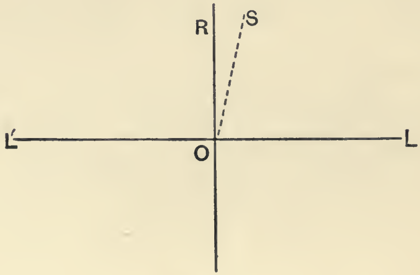

FIG. 24.

from the position OL and turn counter-clockwise into the position OL'. At first the angle on the right is *less* than the angle on the left, at last it is *greater;* the plane, the turning, and the angle are all continuous; hence in passing from the stage of being less to the stage of being greater, it passes through the stage of equality. Let OR be its position in this stage; then $\measuredangle LOR = \measuredangle ROL'$; *i.e.* OR is normal to LL'. Moreover, in no other position, as OS, is the ray normal to LL'; for LOS is not $= LOR$ unless OS falls on OR, but is less than LOR when OS falls within the angle LOR, while SOL' is greater than LOR; hence LOS and SOL' are not equal; *i.e.* OS is not normal to LL' when OS falls not on OR. Similarly, when LOS is greater than LOR. Hence

Theorem X.—*Through a point on a ray one, and only one, ray can be drawn normal to the ray.*

43. *Def.* A ray through the vertex of an angle, and forming equal angles with the arms of the angle, is called the *inner* Bisector or **mid-ray** of the angle. The inner bisector of an adjacent supplemental angle is called the *outer* bisector of the angle itself. Thus OI bisects *innerly* and OE bisects *outerly* the angle AOB (Fig. 25).

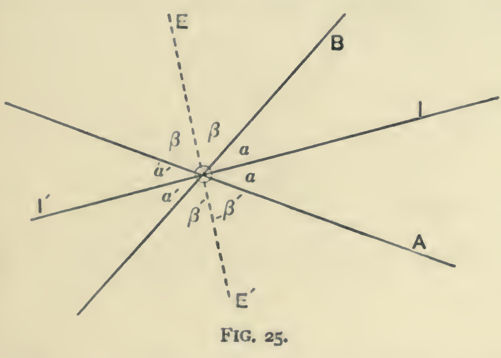

FIG. 25.

Exercise. Prove that there is one and only one such inner mid-ray.

44. Theorem XI. — *The inner Bisector of an angle bisects also its explement innerly.*

Proof. Let OI bisect $\measuredangle AOB$ innerly; then $\measuredangle AOI = \measuredangle IOB$; call each α; then $\alpha + BOI' = \alpha + AOI'$ (why?); take away α; then $BOI' = AOI'$ (why?); *i.e.* the ray II' bisects innerly the angle BOA, the explement of AOB. Show that the angles marked α' are equal.

45. Theorem XII. — *The inner and outer Bisectors of an angle are normal to each other.*

Proof. Let OI and OE bisect (Fig. 25) innerly and outerly the angle AOB. Then, by definition, the angles marked α are equal, and the angles marked β are equal; also the sum of $+\alpha+\alpha+\beta+\beta = S$; hence $\alpha+\beta = \frac{1}{2}S$; or, $IOE =$ a right angle. Q. E. D.

TRIANGLES.

46. Thus far we have treated only of rays intersecting in a single point. But, in general, three rays L, M, N (Fig.

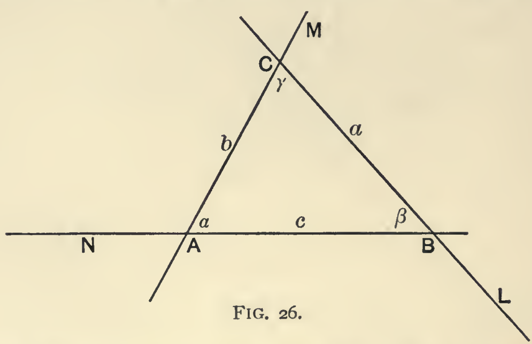

FIG. 26.

26) will meet in three points, since each pair will meet in one point, and there are three pairs: (MN), (NL), (LM).

Denote these points by A, B, C. Then the figure formed by these three rays is called a **triangle**, trigon, or three-side. A, B, C are its **vertices**; α, β, γ its *inner* **angles**; BC, CA, AB, its *inner* **sides**, or simply its **sides**. Its angles and sides are called its *parts*. It is the simplest closed rectilinear figure, and most important. If instead of taking three rays we take three points A, B, C, then we may join them in pairs by rays; and since there are three pairs, BC, CA, AB, then there are three rays, which we may name L, M, N. Thus we see that three points determine three rays, just as three rays determine three points. This equivalent determination of the figure by the same number of points as of rays makes the figure unique and especially important. We denote it by the symbol $\triangle$. We now ask, When are two triangles congruent?

47. **Theorem XIII.** — *Two* $\triangle$ *having two sides and the included angle of the one equal respectively to two sides and the included angle of the other are congruent.*

The **data** are: Two $\triangle$, ABC and $A'B'C'$, having the three equalities, $AB = A'B'$, $AC = A'C'$, $\alpha = \alpha'$ (Fig. 27).

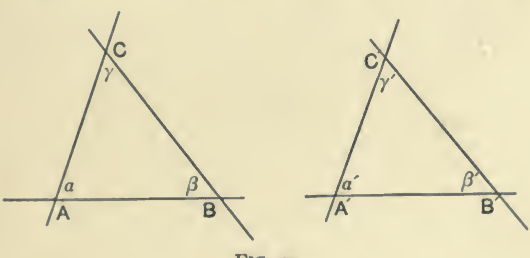

FIG. 27.

Proof. Fit the angle α on the angle α'; this is possible, because the angles are equal and congruent. Then A falls on A'; also the point B falls on B' (why? Because $AB = A'B'$), and C falls on C' (why?). Hence the three vertices of the two ⊿ coincide in pairs; therefore the three sides of the two ⊿ coincide in pairs (why? Because through two points, as A (A') and B (B'), only one ray can pass). Q. E. D.

Corollary 1. The other parts of the two ⊿ are equal or congruent in pairs of correspondents: $\beta = \beta'$, $\gamma = \gamma'$, $BC = B'C'$.

Corollary 2. Pairs of equal parts lie opposite to pairs of equal parts.

48. **Theorem XIV.** — *Two ⊿ having two angles and the included side of the one equal respectively to two angles and the included side of the other are congruent* (Fig. 28).

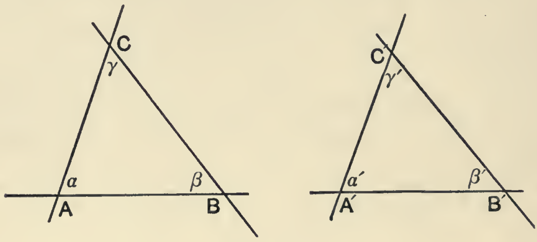

FIG. 28.

Data: Two ⊿ ABC, $A'B'C'$, having $\alpha = \alpha'$, $\beta = \beta'$, $AB = A'B'$.

Proof. Fit AB on $A'B'$; this is possible (why?). Then α will fit on α' (why?), and β on β' (why?); *i.e.* the ray AC will fit on $A'C'$, and the ray BC on $B'C'$. Then the point C will fall on C' (why? Because two rays meet in only one point); *i.e.* the two ⊿ fit exactly. Q. E. D.

49. We may now use the *conditions of congruence* thus far established to generate new notions that may be used in establishing other Theorems.

Def. The ray normal to a tract at its mid-point is called the **mid-normal** of the tract.

Theorem XV. — *Any point on the mid-normal of a tract is equidistant from its ends* (Fig. 29).

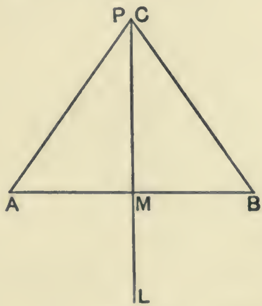

FIG. 29.

Data: AB a tract, M its mid-point, L the mid-normal, P any point on it.

Proof. Compare the $\triangle$ APM and BPM. We have $AM = BM$ (why?). $PM = PM$, $\measuredangle\ AMP = \measuredangle\ BMP$ (why?); hence the $\triangle$ are congruent (why?); and $PA = PB$. Q. E. D.

Def. A $\triangle$ with two equal sides, like APB, is called **isosceles**; the third side is called the *base*, and its opposite angle the *vertical* angle.

50. **Theorem XVI.** — *The angles at the base of an isosceles $\triangle$ are equal;* and *conversely.*

Data: ABC an isosceles $\triangle$, AB its base, AC and BC its equal sides (Fig. 30).

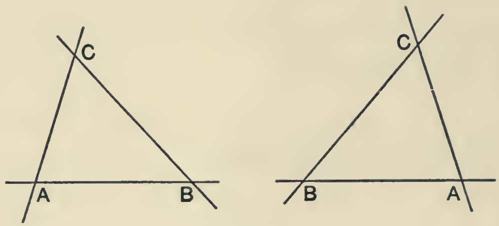

FIG. 30.

Proof. Take up the $\triangle$ ABC, turn it over, and replace it in the position BCA. Then the two $\triangle$s ACB and BCA have the equal vertical angles, C and C, also the side $AC = BC$ (why?) and $BC = AC$ (why?); hence they are congruent (why?), and the $\measuredangle A = \measuredangle B$. Q. E. D.

Conversely, *A $\triangle$ whose basal angles are equal is isosceles.* Let the student conduct a proof quite similar to the foregoing.

Def. The ray through a vertex and the mid-point of the opposite side is called the **medial** of that side.

Corollary 1. In an isosceles $\triangle$ the medial of the base is normal to it, and is the mid-ray of the vertical angle.

Corollary 2. When the medial of a side of a $\triangle$ is normal to the side, the $\triangle$ is isosceles. Prove it.

Corollary 3. When the medial of a side bisects the opposite angle, the $\triangle$ is isosceles. Can you prove it?

LOGICAL DIGRESSION.

51. When the subject and predicate of a proposition are merely exchanged, the proposition is said to be *converted*, and the new proposition is called the *converse*. Thus X is Y; *conversely*, Y is X. In general, converses of true propositions are not true, but false. Thus, *The horse is an animal* is always correct, but *The animal is a horse* is generally false. A proposition remains true after simple conversion only when subject and predicate are properly quantified, thus: *All horses are some animals;* conversely, *Some animals are all horses.* Both propositions are correct and mean the same thing. But they are awkward in expression, and such forms are rarely or never used. When the quantifying word is *all* or its equivalent, the term is said to be taken *universally;* when it is *some* or its equivalent, the term is said to be taken *particularly*. Thus in the foregoing example horse is taken universally, but animal particularly. The only useful conversions are of propositions in which *both* subject and predicate are *universal.* In the great body of propositions only the subject is quantified universally, the quantifier is omitted from the predicate, but a particular one is understood. To show that a universal quantifier is admissible requires in general a distinct proof.

52. In order to convert an hypothetic proposition, we *exchange* hypothesis and conclusion. Thus, if X is Y, U is V; the converse is, if U is V, X is Y. All such hypothetic propositions may be stated *categorically*, thus: All cases of X being Y are cases of U being V; conversely, All cases of U being V are cases of X being Y. This converse is plainly false except when the quantifier *all* is admissible in the first predicate.

53. But while the converse of a true hypothetic proposition is generally false, the **contrapositive** is always true. This latter is formed by *exchanging* hypothesis and conclusion and *denying both*. Thus: If X is Y, then U is V; *contrapositive*, If U is not V, then X is not Y. Or, if a point is on the mid-normal of a tract, then it is equidistant from the ends of the tract; contrapositive, If a point is not equidistant from the ends of a tract, then it is not on the mid-normal of the tract.

---

54. **Theorem XVII.** — *An outer angle of a* $\triangle$ *is greater than either inner non-adjacent angle.*

Data: Let ABC be any $\triangle$, α' an outer angle, β' a non-adjacent inner one (Fig. 31).

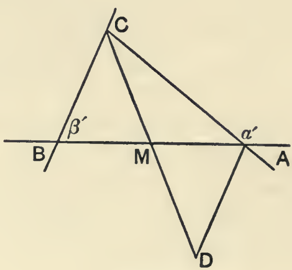

Fig. 31.

Proof. Draw the medial CM and lay off $MD = MC$; also draw AD. Then in the $\triangle$s AMD and BMC we have $AM = BM$ (why?), $MD = MC$ (why?), and $\measuredangle AMD = \measuredangle BMC$ (why?); hence the $\triangle$s are congruent (why?), and $\measuredangle MBC = \measuredangle MAD$ (why?). But $\measuredangle MAD$ is only part of the $\measuredangle \alpha'$; hence $\alpha' > \measuredangle MAD$ (why?); *i.e.* $\alpha' > \beta'$. Q. E. D.

Similarly, prove that $\alpha' > \gamma$.

55. **Theorem XVIII.** — *If two sides of a △ are unequal, then the opposite angles are unequal in the same sense* (*i.e.* the greater angle opposite the greater side) (Fig. 32).

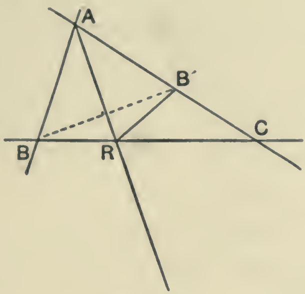

Fig. 32.

Data: ABC a △, $AC > AB$, AR the mid-ray of the angle at A, AB' laid off $= AB$.

Proof. ABR and $AB'R$ are congruent (why?); hence $\measuredangle ABR = \measuredangle AB'R$ (why?); but $\measuredangle AB'R > C$ (why?); *i.e.* $\measuredangle ABC > \measuredangle ACB$. Q. E. D.

Conversely, *If two angles of a △ are unequal, the opposite sides are unequal in the same sense.*

Proof. The opposite sides are not equal; for when the sides are equal, the opposite angles are equal (Theorem XVI.), and *contrapositively*, when the angles are unequal, the opposite sides are unequal. Then, by the preceding Theorem, the greater angle lies opposite the greater side.

56. Join BB'; then AR is the mid-normal of BB' (why?), and hence angle $CBB' =$ angle $BB'R$ (why?). Hence angle $BB'C > B'BC$ (why?); hence $BC > B'C$ (why?). But $B'C = AC - AB$; hence $BC > AC - AB$; *i.e.*

Theorem XIX. — *Any side of a $\triangle$ is greater than the difference of the other two.*

Add AB to both sides of this inequality and there results $AB + BC > AC$; *i.e.*

Theorem XX. — *Any side of a $\triangle$ is less than the sum of the other two.*

This fundamental Theorem is here proved on the supposition that $AB < AC$; if AB were $= AC$ or $> AC$, it would need no formal proof.

57. **Theorem XXI.** — *A point not on the mid-normal of a tract is not equidistant from the ends of the tract.*

Data: AB the tract, MN the mid-normal, Q any point not on MN (Fig. 33).

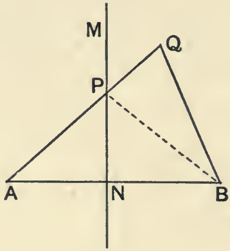

Fig. 33.

Proof. Draw QA and QB; one of them, as QA, must cut MN at some point, as P. Then $QB < QP + PB$ (why?), and $PB = PA$ (why?); hence $QB < QP + PA$; *i.e.* $QB < QA$. Q. E. D.

Of what Theorem is this the converse?

If now we seek for a point equidistant from A and B, we can find it on the mid-normal of AB and *only* there; hence *the locus of a point equidistant from the ends of a tract is the mid-normal of the tract.*

58. Theorem XXII.—*Two* △ *with the three sides of the one equal respectively to the three sides of the other are congruent.*

Data: ABC and $A'B'C'$ the two △, and $AB = A'B'$, $BC = B'C'$, $CA = C'A'$ (Fig. 34).

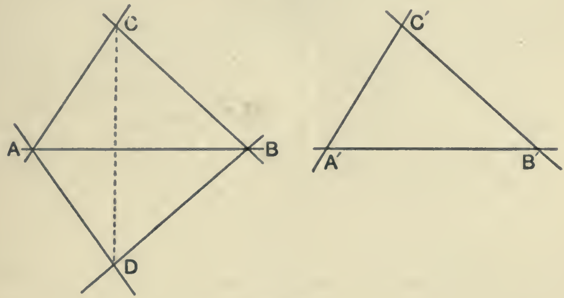

FIG. 34.

Proof. Turn the $\triangle\ A'B'C'$ over and fit $A'B'$ on AB so that C' shall fall (say) at D. Draw CD. Then A and B are on the mid-normal of CD (why?); hence the ray AB is the mid-normal of CD (why?); hence the angle $CAB =$ angle DAB, and angle $CBA =$ angle DBA (why?). Hence the △ are congruent (why?). Q. E. D.

N.B. As to when the △ must be turned round and when turned over, see Art. 94.

59. Theorem XXIII. — A. *From any point outside of a ray one normal may be drawn to the ray.*

Data: P the point, LL' the ray (Fig. 35).

Proof. From P draw a ray far to the left, as PA, making the angle $PAL >$ angle PAL'. Now let the ray turn about P as a pivot into some position far to the right, making angle $PA'L < PA'L'$. The plane, the angle, the motion, all being continuous, in passing from the stage of being unequal in one sense to the stage of being unequal in the opposite sense, the angles made by the moving ray with the fixed ray must have passed through the stage of equality. Let PN be

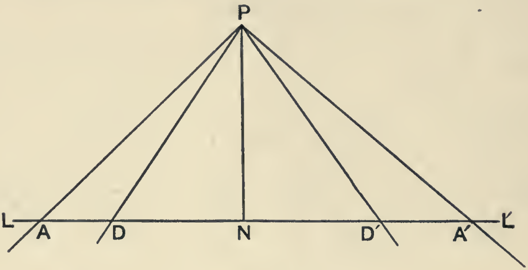

FIG. 35.

the ray in this position so that angle $PNL =$ angle PNL'; then each is a right angle by Definition, and PN is normal to LL'. Q. E. D.

B. *There is only one ray through a fixed point and normal to a fixed ray.*

Proof. Any other ray than PN, as PD, is not normal to LL'; for the outer angle PDL is $>$ the right angle PND (why?). Q. E. D.

C. *The normal tract PN is shorter than any other tract from P to the ray LL'.*

Proof. For the right angle at N is $>$ angle PDN (why?); hence $PN < PD$ (why?). Q. E. D.

D, E. *Equal tracts from point to ray meet the ray at equal distances from the foot of the normal;* and conversely.

Proof. For, if DPD' be isosceles, then the normal PN is the medial of the base (why?).

F. *Two, and only two, tracts of given length can be drawn from a point to a ray.*

Proof. For two, and only two, points are on the ray at a given distance from the foot of the normal.

G. *Of tracts drawn to points unequally distant from the foot of the normal, the one drawn to the remotest is the longest.*

Proof. In the $\triangle$ PDA, angle $PDA > PAD$ (why?); hence $PA > PD$ (why?). Q. E. D.

Similarly, $PA' > PD$.

H. *Equal tracts from the point to the ray make equal angles with the normal from the point to the ray and also equal angles with the ray itself;* and conversely.

I. *Of unequal tracts from the point to the ray, the longest makes the greatest angle with the normal and the least with the ray.*

Let the student conduct the proof of H and I.

60. **Theorem XXIV.** — *Two* ⨺ *having two angles and an opposite side of one equal respectively to two angles and an opposite side of the other are congruent.*

Data: ABC and $A'B'C'$ two ⨺ having $AB = A'B'$, angle $\alpha =$ angle α', angle $\gamma =$ angle γ' (Fig. 36).

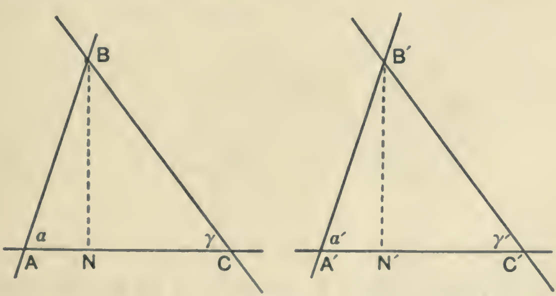

FIG. 36.

Proof. Fit α' on α; then B' falls on B (why?), and $A'C'$ falls along AC. Draw the normal BN. Then BC and BC' make the same angle, $\gamma = \gamma'$, with the ray AN; hence they are = and meet the ray in the same point (why?); *i.e.* C' falls on C; *i.e.* the ⚠ are congruent. Q. E. D.

61. We now come to the so-called **ambiguous** case, of two ⚠ with two sides and an opposite angle in one equal to the two sides and the corresponding opposite angle in the

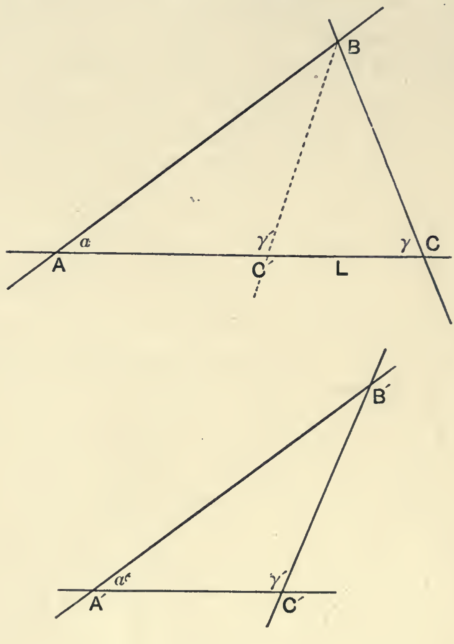

FIG. 37.

other. Let ABC and $A'B'C'$ (Fig. 37) be the two ⚠, with $AB = A'B'$, $BC = B'C'$, and angle α = angle α'. Fit α' on

α; then $A'B'$ falls on AB, B' on B; but since from a point B (B') we may draw two equal tracts to the ray AL, the side $B'C'$ may be either of these equals and *may or may not* fall on BC. In general, then, we cannot prove congruence in this case. But if BC be $> AB$, then angle $\alpha >$ angle γ (why?), and there is *only one* tract on the right of AB drawn from B to the ray AC and equal to BC; the *other* tract equal to BC must be drawn outside of AB and to the left. Hence in this case, when the angle lies opposite the greater side, the ⚠ are congruent. Hence

Theorem XXV. — *Two ⚠ having two sides and an angle opposite the greater in one equal to two sides and an angle opposite the greater side in the other are congruent.*

Corollary. Two right ⚠ having a side and any other part of one equal to a side and the corresponding part of the other are congruent.

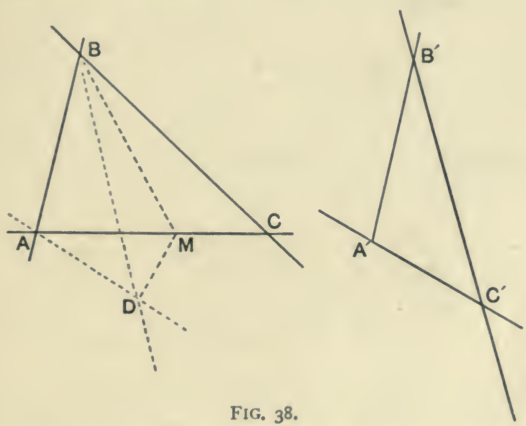

FIG. 38.

62. We have seen (Art. 47) that when two ⚠ have two sides and included angle in one equal to two sides and included angle in the other, they are congruent. But what if the included angles are not equal? Let ABC and $A'B'C'$ be the two △, having $AB = A'B'$, $BC = B'C'$, but $\beta > \beta'$. Slip the upper film of the plane along until $A'B'$ fits on AB and let C' fall on D. Draw the mid-ray BM of the angle CBD, let it cut AC at M, and draw DM. Then the △ CBM and DBM are congruent (why?); hence $AM + MD = AC$ (why?), and $AC > AD$, or $AC > A'C'$. Hence

Theorem XXVI. — *Two △ having two sides of one equal to two sides of the other, but the included angles unequal, have also the third sides unequal, the greater side lying opposite the greater angle.*

Conversely, *Two △ having two sides in one equal to two sides in the other, but the third sides unequal, have the included angles also unequal, the greater angle being opposite the greater side.*

Proof. The included angles are *not equal;* for if they were equal, the △ would be congruent (why?) and the three sides would be equal. Hence the included angles are unequal, and the relation just established holds; namely, the greater angle lies opposite a greater side. Q. E. D.

63. **Theorem XXVII.** — *Every point on a mid-ray of an angle is equidistant from its sides.*

Data: O the angle, MM' the mid-ray, P any point on it.

Proof. From P draw the normals PC and PD; they are (Fig. 39) the distances of P from the ends of the angle. Then the △ POC and POD are congruent (why?); hence $PD = PC$. Q. E. D.

Conversely, *A point equidistant from the ends of an angle is on a mid-ray of the angle* (Fig. 39).

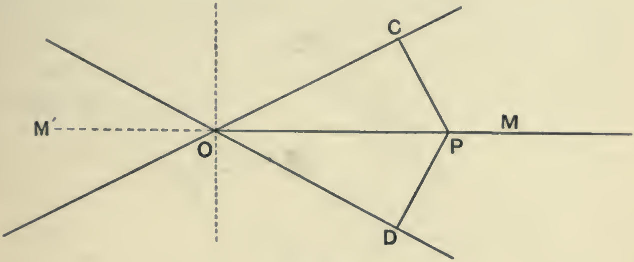

FIG. 39.

Proof. If $PC = PD$, then the $\triangle$ POC and POD are congruent (why?) ; hence angle $POD =$ angle POC. Q. E. D.

Accordingly we say that the *mid-rays of an angle are the locus of a point equidistant from its ends.*

*64. It is just at this stage in the development of the doctrine of the Triangle that we are compelled to halt and introduce a new concept before we can proceed any further. The necessity of this step will appear from what follows (which may, however, be omitted on first reading, at the option of teacher or student).

Def. Two $\triangle$ not congruent are called **equivalent** when they may be cut up into parts that are congruent in pairs.

Theorem XXVIII. — *Any $\triangle$ is equivalent to another $\triangle$ having the sum of two of its angles equal to the smallest angle of the given $\triangle$.*

Data: ABC the $\triangle$, α the least angle (Fig. 40).

Proof. Through M, the mid-point of BC, draw AM and make $MD = MA$. Then the $\triangle$s ACM and DBM are congruent (why?), the part AMB is common to ABC and ABD, and the sum of the angles ADB and $BAD =$ angle BAC. Q. E. D.

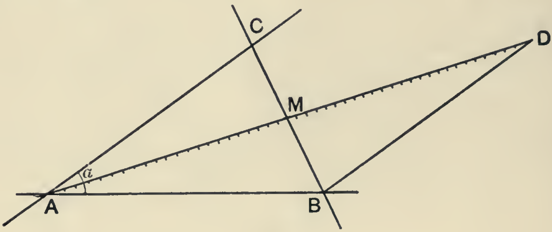

Fig. 40.

Corollary. The sum of the angles in the new $\triangle$ is equal to the sum of the angles in the old $\triangle$.

* **65.** We may now repeat this process, applying it to the smallest angle, as A, of the $\triangle$ ABD. In the new $\triangle$ ABE the smallest angle, as A, cannot be greater than $\frac{1}{4}$ of the original angle α in ABC; after n repetitions of this process we obtain a $\triangle$, as ALB, in which the sum of the angles A and L cannot be $> \frac{1}{2^n}$ of the original angle α in the $\triangle$ ABC. By making n as large as we please, we make $\frac{1}{2^n}$ as small as we please, and so we make $\frac{1}{2^n}$ of angle α smaller than any assigned magnitude no matter how small. Meantime the other angle B has indeed grown larger and larger, but has remained $<$ a straight angle. Hence the sum of the angles in the $\triangle$ ALB cannot exceed a straight angle by any amount however small; but the sum of the angles in $ALB =$ sum of the angles in ABC; hence

Theorem XXIX. — *The sum of the angles in any $\triangle$ cannot exceed a straight angle by any finite amount.*

Corollary 1. The outer angle of a $\triangle$ is *not less* than the sum of the inner non-adjacent angles.

Corollary 2. From any point outside of a ray there may be drawn a ray making with the given ray an angle small at will.

Proof. From P draw any ray PA, and lay off $AB = PA$ (Fig. 41). Then the angle PBA is not greater than

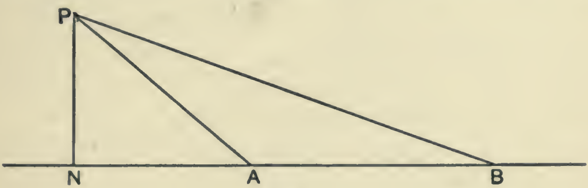

FIG. 41.

$\frac{1}{2}$ PAN (why?); now lay off $BC = PB$ (why?); then angle PCB is not $> \frac{1}{2}$ angle PBA (why?); proceeding this way, we obtain after n constructions an angle PLN not $> \frac{1}{2^n}$ of the angle PAN, and by making n large enough we may make this $\frac{1}{2^n}$ as small as we please. Q. E. D.

*66. **Theorem XXX.** — *If the sum of the angles in any $\triangle$ equals a straight angle, then it equals a straight angle in every $\triangle$* (Fig. 42).

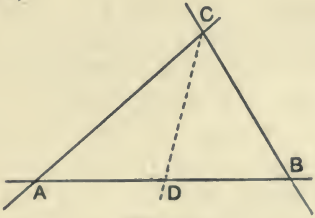

FIG. 42.

Hypothesis: ABC a $\triangle$ with the sum of its angles $A+B+C=S$.

Proof. (1) Draw any ray through C, as CD. Then if the sum of the angles in the $\triangle$s ACD and BCD be $S-x$ and $S-y$, x and y being any definite magnitudes however small, then on adding these sums we get $2S-(x+y)$; and on subtracting the sum, S, of the supplemental angles at D we get $S-(x+y)$ for the sum of the angles of the $\triangle$ ABC. Now if this sum be S, then x and y must each be O; *i.e.* the sum of the angles in each of the $\triangle$s ACD and BCD is S. Now draw DE and DF; in each of the four small $\triangle$s the sum of the angles is still $=S$. (2) We may now make a $\triangle$ as large as we please and of any shape whatever, but the sum of the angles will remain $=S$. For, take the same $\triangle$ ABC, and draw CD normal to AB. Then the sum of the (Fig. 43) angles in the $\triangle$ ACD is S, as has

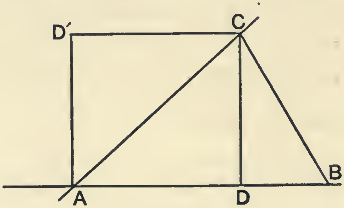

FIG. 43.

been shown above; also angle D is a right angle; hence the angles A and ACD are complementary. Now along AC fit another $\triangle$ ACD' congruent with ACD; then all the angles of the quadrilateral $ADCD'$ are right, and the figure is called a **rectangle.** Now we can place horizontally side by side as many of these rectangles, all congruent, as we please, say p of them; we can also place as many of them vertically, one upon another, as we please, say q of them;

and we can then fill up the whole figure into a new rectangle, as large as we please. About each inner junction-point of the sides of the rectangles there will be four right angles plainly. Now connect the two opposite vertices, as A and Z, of this rectangle. So we get two congruent right $\triangle$s, in each of which the sum of the angles is S. Then any $\triangle$ that we cut off from this right $\triangle$ will, by the foregoing, have the sum of its angles equal to S. Since p and q are entirely in our power, we may make in this way any desired right $\triangle$ and from it cut off any desired oblique $\triangle$, with the sum of its angles $= S$. Q. E. D.

Hence *either no $\triangle$ has the sum of its angles* $= S$, or **every $\triangle$ has the sum of its angles** $= S$.

67. A logical choice between these alternatives is impossible, but the matter may be cleared up by the following considerations:

Across any ray LM draw a transversal T, cutting LM at O, and making the angles α, β, γ, δ. Through any point, as O', of T draw a ray (Fig. 44) $L'M'$ making angle $\alpha' = \alpha$.

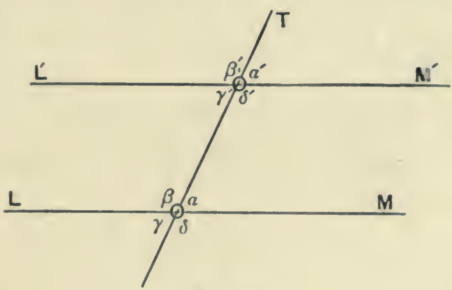

FIG. 44.

This is evidently possible (why?). Then plainly $\beta' = \beta$, $\gamma' = \gamma$, $\delta' = \delta$, $\alpha' = \alpha$; they are called **corresponding** angles;

also α and γ', β and δ' are equal, — they are called **alternate** angles; also α and δ', as well as β and γ', are supplemental, — they are called **interadjacent** angles.

68. Now let P be the mid-point of OO'; on it as a pivot turn the whole right side of the plane round through a straight angle until O falls on O', and O' falls on O. Then, since the angles about O and O' are equal as above stated, the half-ray OL will fall and fit on the half-ray $O'M'$, and the half-ray $O'L'$ on the half-ray OM. Accordingly, *if the rays LM and $L'M'$ meet on one side of the transversal T, they also meet on the other side of T.*

69. Three possibilities here lie open:

(1) The rays LM and $L'M'$ may meet on the left and also on the right of T, in *different* points.

(2) They may meet on the left and also on the right of T, in the *same* point.

(3) They may **not meet at all.**

No logical choice among these three is possible. But *in all regions accessible to our experience* the rays neither converge nor show any tendency to converge. Hence we *assume* as an

Axiom A. *Two rays that make with any third ray a pair of corresponding angles equal, or a pair of alternate angles equal, or a pair of interadjacent angles supplemental, are* **non-intersectors.**

70. But another query now arises. Is it possible to draw another ray through O' so close to L' that it will not meet OL however far both may be produced? Here again it is impossible to answer from pure logic. An appeal to experience is all that is left us. This latter testifies that no ray can be drawn through O' so close to $O'L'$ as not to approach and finally meet the ray OL. Hence we *assume* as another

Axiom B. *Through any point in a plane* **only one non-intersector** *can be drawn for a given straight line.*

This single non-intersector is commonly called the **parallel,** through the point, to the straight line.

71. It cannot be too firmly insisted, nor too distinctly understood, that the existence of any non-intersector at all, and the existence of only one for any given point and given ray, are both assumptions, which cannot be proved to be facts. The best that can be said of them, and that is quite good enough, is that they and all their logical consequences accord completely and perfectly with all our experience as far as our experience has hitherto gone. Even then, if there be any error in our assumptions, we have thus far been utterly unable to find it out.

A geometry that should reject either or both of these assumptions would have just as much logical right to be as the geometry that accepts them, and such geometries lack neither interest nor importance. They may be called **Hyper-Euclidean** in contradistinction from this of ours, which from this point on is Euclidean (so-called from the Greek master, Euclides, who distinctly enunciated the *equivalent* of our Axioms in a Definition and a Postulate).

NOTE. — Observe the relation of Axioms **A** and **B**: *the one is the converse of the other.*

Observe also that the necessity of assuming the first lies in our ignorance of the *indefinitely great,* and the occasion of assuming the other lies in our ignorance of the *indefinitely small.* **See Note, Art. 301.**

72. Accepting our Axioms as at least *exacter than any experiment we can make,* we may now easily settle the question as to the sum of the angles in a △. Let ABC be any △; through the vertex C draw the one parallel to the base AB. Then $\alpha = \alpha'$, $\beta = \beta'$ (why?); also $\alpha' + \gamma + \beta' = S$; hence $\alpha + \gamma + \beta = S$; *i.e.* (Fig. 45)

Theorem XXXI. — *The sum of the angles in a △ is a straight angle.*

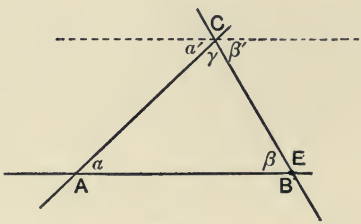

FIG. 45.

Corollary 1. The outer angle E equals the sum of the inner non-adjacent angles α and γ (why?).

Corollary 2. If two angles of a △ be known, the third is also known.

Corollary 3. If two △s have two angles, or the sum of two angles of the one equal to two angles, or the sum of two angles of the other, then the third angles are equal.

Corollary 4. To know the three angles of a △ is not to know the △ completely, for many △s may have the same three angles. Such △s are **similar**, as we shall see, but are not congruent; they are alike in **shape**, but not in **size**.

73. Next to normality, parallelism is the most important relation in which rays can stand to each other, and we must now use the new relation in the generation of new concepts.

Theorem XXXII. — *Parallel Intercepts between parallels are equal.*

Data: L and L', M and M', two pairs of parallels (Fig. 46).

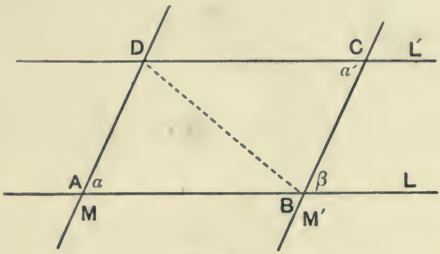

FIG. 46.

Proof. Draw BD. Then the $\triangle$ ABD and CDB are congruent (why?), and $AB = CD$, $BC = DA$. Q. E. D.

Def. The figure $ABCD$ formed by two pairs of parallel sides is called a **parallelogram,** and may be denoted by the symbol $\square$.

A join of opposite vertices, as BD, is called a **diagonal.**

74. **Theorem XXXIII. — Properties of the parallelogram.**

A. *The opposite sides of a parallelogram are equal.*

This has just been proved.

B. *The opposite angles of a parallelogram are equal.*

Proof. $\alpha = \beta$ (why?); $\beta = \alpha'$ (why?); hence $\alpha = \alpha'$. Q. E. D.

Corollary. Adjacent angles of a parallelogram are supplementary.

C. *Each diagonal of a parallelogram cuts it into two congruent* $\triangle$. Prove it.

D. *The diagonals of a parallelogram bisect each other* (Fig. 47).

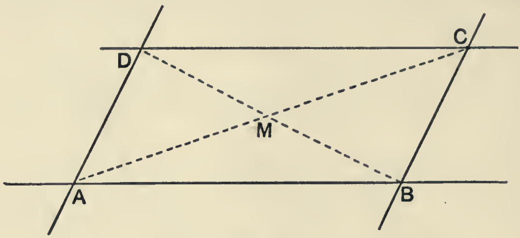

FIG. 47.

Proof. The △ AMB and CMD are congruent (why?); hence $AM = CM$, $BM = DM$. Q. E. D.

75. We may now convert all the foregoing propositions and obtain as many criteria of the parallelogram.

Theorem XXXIV. — A′. *A 4-side with its opposite sides equal is a parallelogram.*

Data: $AB = CD$, $AD = CB$ (Fig 48).

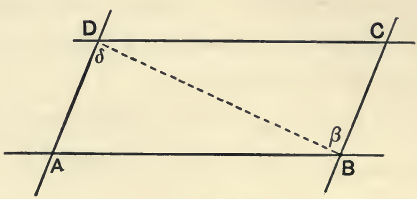

FIG. 48.

Proof. Draw BD. Then ABD and CDB are congruent (why?); hence $\beta = \delta$; and AD and CB are parallel; similarly, AB and CD are parallel; hence $ABCD$ is a parallelogram. Q. E. D.

B$'$. *A 4-side with opposite angles equal is a parallelogram.*

Data: $\alpha = \alpha'$, $\beta + \beta' = \gamma + \gamma'$ (Fig. 49).

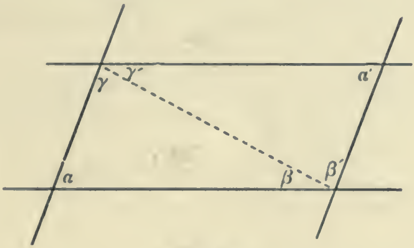

FIG. 49.

Proof. Since $\alpha = \alpha'$, $\beta + \gamma = \beta' + \gamma'$ (why?). Hence $\beta = \gamma'$, $\beta' = \gamma$; *i.e.* opposite sides are parallel, the 4-side is a parallelogram. Q. E. D.

C$'$. *A 4-side that is cut by each diagonal into two congruent* △ *is a parallelogram.*

For the opposite angles must be equal (why?); hence, etc. Q. E. D.

D$'$. *A 4-side whose diagonals bisect each other is a parallelogram.*

For the opposite sides are equal, being opposite equal angles in congruent △; hence, etc. Q. E. D.

E$'$. *A 4-side with one pair of sides equal and parallel is a parallelogram.*

For the other two sides are equal and parallel (why?); hence, etc. Q. E. D.

76. The foregoing properties and criteria of the parallelogram illustrate excellently the nature of a *definition.* This latter defines or *bounds off* by stating something that is true of the thing defined, but of nothing else. Accordingly, the characteristic of every definition or definitive property is that the proposition that states it may be converted simply. Thus:

Every parallelogram is a 4-side with opposite angles equal; and *conversely*, every 4-side with opposite angles equal is a parallelogram.

Not every property is definitive, and hence not every property may be used as test or criterion.

77. Special Parallelograms.

Def. An equilateral parallelogram is called a **rhombus.**

Theorem XXXV. — *The diagonals of a rhombus are normal to each other.*

Let the student conduct the proof suggested by the figure (Fig. 50).

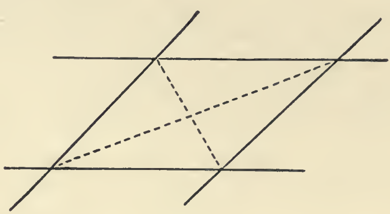

FIG. 50.

Conversely, *A parallelogram whose diagonals are normal to each other is equilateral, or a rhombus.* Let the student supply the proof.

78. *Def.* An equiangular parallelogram is called a **rectangle** (for all the angles are *right* angles).

Theorem XXXVI. — *The diagonals of a rectangle are equal* (Fig. 51).

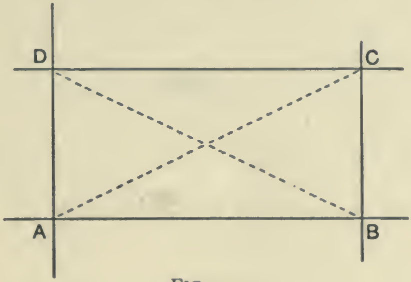

FIG. 51.

For the △ ABC and BAD are congruent (why?) ; hence $AC = BD$. Q. E. D.

Conversely, *A parallelogram with equal diagonals is equiangular, or a rectangle.*

For the △ ABC and BAD are again congruent, though for another reason. What reason ?

79. *Def.* A parallelogram both equilateral and equiangular is called a **square**.

Theorem XXXVII. — *The diagonals of a square are equal and normal to each other.*

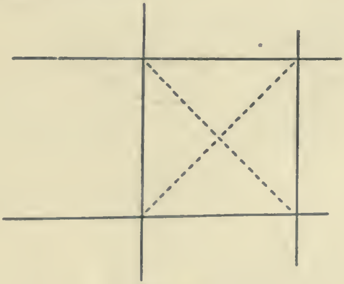

FIG. 52.

For the square, being both rhombus and rectangle, has all the definitive properties of both. Or the student may prove the proposition directly from the figure (Fig. 52), as well as its converse:

A parallelogram with diagonals equal and normal to each other is a square.

80. Can we convert Theorem XXXII. and prove that equal intercepts between parallels are parallel? Manifestly no (Fig. 53), for from the point C we may draw two equal

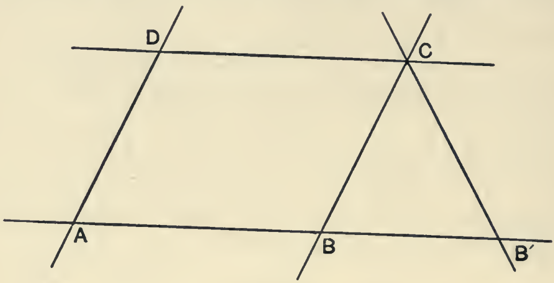

FIG. 53.

tracts to the other parallel, the one CB parallel to AD, the other CB' sloped at the same angle to the parallels but in opposite ways. We may call CB' *anti-parallel* to AD, and the figure $AB'CD$ an **anti-parallelogram.** Since from any point C only two equal tracts, or tracts of given length, may be drawn to the other parallel through A, we have the

Theorem XXXVIII. — *Equal intercepts between parallels are either parallel or anti-parallel.*

Corollary 1. Adjacent angles of an anti-parallelogram are alternately equal or supplemental.

Corollary 2. Anti-parallels prolonged meet at the vertex of an isosceles △.

THE GENERAL QUADRILATERAL OR 4-SIDE.

81. A Quadrilateral is determined by four intersecting rays. These determine six points, the four *inner* vertices, C, D, E, F, and the two *outer* ones, A, B. The cross-rays, CE, DF, AB, are the diagonals, CE and DF *inner*, AB *outer*. Commonly the outer diagonal is little used, and the inner ones are called *the* diagonals. When none of the angles C, D, E, F, of the 4-side is greater than a straight angle, the 4-side is called the **normal**, as $CDEF$. It is the only form ordinarily considered. The other two forms are (2) the *crossed*, $ACBE$, and (3) the *inverse*, $ADBF$ (Fig. 54). For all forms let the student prove

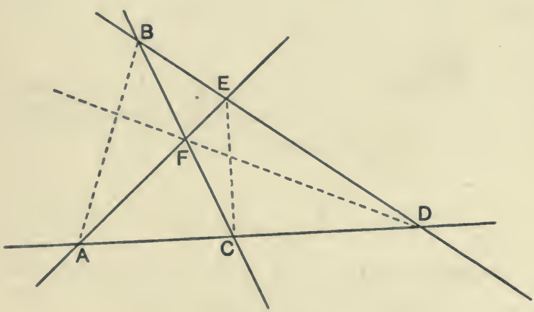

FIG. 54.

Theorem XXXIX. — *The sum of the inner angles of a 4-side is a round angle.*

Corollary. When two angles of a 4-side are supplemental, so are the other two.

82. Theorem XL. — *The angles between two rays equal the angles between two normals to the rays.*

Data: OL and OM any two rays, PA and PB any two normals to them (Fig. 55).

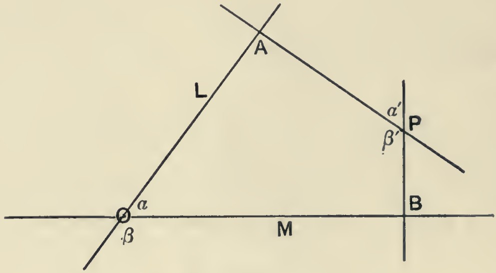

FIG. 55.

Proof. The angles at A and B are right angles and therefore supplemental (why?); hence $\alpha = \alpha'$, and $\beta = \beta'$. Q. E. D.

N.B. The 4-side with its *opposite angles supplemental* is very important and has received the name **encyclic** 4-side, for reasons to be seen later on (Arts. 126–7).

THREE OR MORE PARALLELS.

83. **Theorem XLI.** — *Three parallels that make equal intercepts on one transversal, make equal intercepts on any transversal.*

Data: L, M, N, three parallels, and $AB = BC$, and DEF any transversal (Fig. 56).

Proof. Draw $D'EF'$ parallel to ABC. Then $AB = BC$ (why?), $AB = D'E$ (why?), and $BC = EF'$ (why?); hence $D'E = EF'$ (why?), hence the $\triangle$ DED' and FEF' are congruent (why?); hence $DE = EF$ (why?). Q. E. D.

84. *Def.* A 4-side formed by two parallels and two transversals is called a **trapezoid**. Thus $ACFD$ is a trapezoid. The parallel sides are called the **bases** (major and minor); the parallel through the mid-points of the transverse sides is the *mid-parallel.*

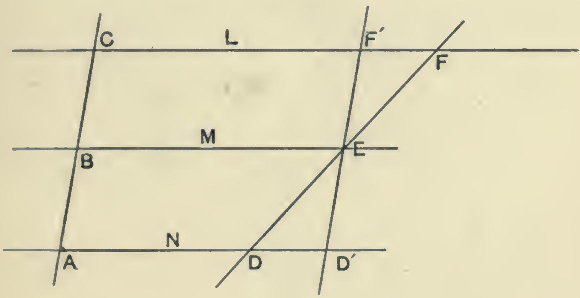

FIG. 56.

Theorem XLII. — *The mid-parallel of a trapezoid equals the half-sum of its bases.*

Let the student elicit the proof from the foregoing figure.

Corollary 1. A parallel to a base of a $\triangle$ bisecting one side bisects also the other. (*Hint.* Let D fall on A.)

Corollary 2. A ray bisecting two sides of a $\triangle$ is parallel to the third.

For only one ray can bisect two sides (why?), and we have just seen (Cor. 1) that a ray parallel to the base does this; hence, Q. E. D.

Corollary 3. The mid-parallel to the base of a $\triangle$ equals half the base.

85. *Def.* Three or more rays that pass through a point are said to **concur** or be concurrent.

Theorem XLIII. — *The medials of a △ concur.*

Data: ABC a △, AP and BQ two medials (Fig. 57).

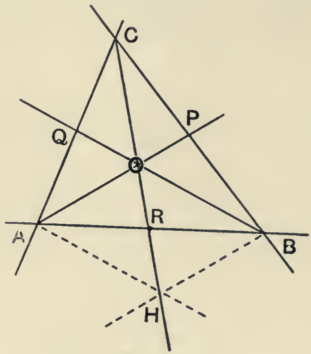

FIG. 57.

Proof. Draw a ray from C through O, the intersection of the two medials, and lay off $OH = CO$. Draw AH and BH; they are parallel to BQ and AP (why?); hence $AOBH$ is a parallelogram (why?); hence $AR = BR$ (why?). Hence COR is the third medial; *i.e.* the three medials pass through O. Q. E. D.

Corollary. Each medial cuts off a third from each of the other two. For $CO = 2\,OR$ (why?).

Def. The point of concurrence of the medials is called the **centroid** of the △. It is two-thirds the length of each medial from the corresponding vertex.

86. **Theorem XLIV**. — *The mid-normals of the sides of a △ concur.*

Data: ABC a △, L and M mid-normals to the sides BC and CA, meeting at S.

Proof. S is equidistant from B and C (why?), and from C and A (why?); hence S is equidistant from A and B (why?), or is on the mid-normal of AB (why?); hence the mid-normals concur (Fig. 58). Q. E. D.

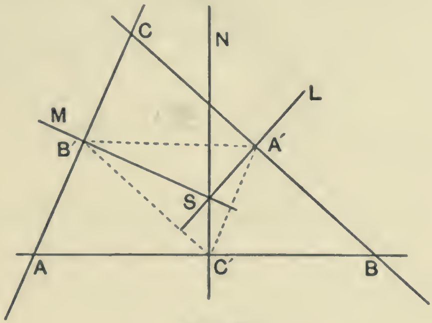

FIG. 58.

Corollary. S is equidistant from A, B, and C, and no other point in the plane is (why?).

Def. The point of concurrence of the mid-normals is called the **circumcentre** of the $\triangle$.

87. *Def.* A tract from a vertex of a $\triangle$ normal to the opposite side is called an **altitude** of the $\triangle$. Sometimes, when length is not considered, the whole ray is called the altitude.

Theorem XLV. — *The altitudes of a $\triangle$ concur.*

Proof. Using the preceding figure, draw the $\triangle$ $A'B'C'$. Its sides are parallel to the sides of ABC (why?); hence its altitudes are the mid-normals L, M, N; and these have just been found to concur. Also, since ABC may be any $\triangle$, $A'B'C'$ may be any $\triangle$; hence the altitudes of any $\triangle$ concur. Q. E. D.

Def. The point of concurrence of altitudes is called the **orthocentre** (or alticentre) of the △.

Def. In a right △ the side opposite the right angle is called the **hypotenuse** (= subtense = under-stretch).

Queries: Where do circumcentre and orthocentre lie: (1) in an acute-angled △? (2) in an obtuse-angled △? (3) in a right △?

88. Theorem XLVI.—*The inner mid-rays of the angles of a △ concur.*

Data: ABC a △, AL, BM, CN the inner mid-rays of its angles (Fig. 59).

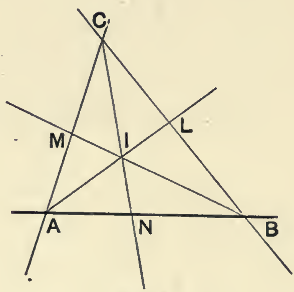

FIG. 59.

Proof. Let AL and BM intersect at I. Then I is equidistant from AB and AC, and from AB and BC (why?); hence I is equidistant from AC and BC; hence I is on the inner mid-ray of the angle C; *i.e.* the three inner mid-rays concur in I. Q. E. D.

Def. The point of concurrence of the inner mid-rays is called the **in-centre** of the △.

89. **Theorem XLVII.** — *The outer mid-rays of two angles and the inner mid-ray of the other angle of a △ concur.*

Let the student conduct the proof (Fig. 60).

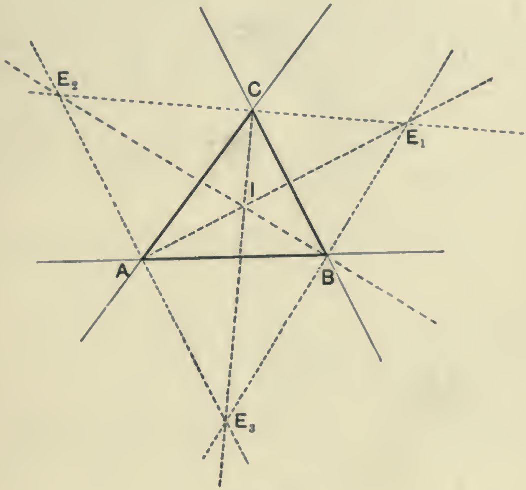

FIG. 60.

Def. The points of concurrence are called **ex-centres** of the △ : there are *three*.

EXERCISES I.

Little by little the student has been left to rely more and more upon his own resources of knowledge and ratiocination in the conduct of the foregoing investigations. He has now possessed himself of a large fund of concepts, and he must test his ability to wield, combine, and manipulate them in forging original proofs of theorems. Let him bear always in mind the fundamental logical principle that *every example* *of a general concept has all the marks of that general concept.* Let him begin his proof by stating precisely the *data*, the given or known facts, let him draw a corresponding diagram in order to have a clearer view of the spatial relations involved, let him note carefully what concepts are present in the proposition, let him draw auxiliary lines and introduce auxiliary concepts at pleasure. But let him exhaust simple means before trying more complicated, let him distinguish, by manner of drawing, the principal from the auxiliary rays, and especially let him be systematic and consistent in the literation of his figures.

1. How many degrees in a straight angle? In a right angle?

HISTORICAL NOTE. — For purposes of computation the round angle is divided into 360 equal parts called **degrees**, each degree into 60 equal **minutes** (partes *minutæ* primæ), each minute into 60 equal **seconds** (partes minutæ *secundæ*), denoted by $^\circ$, $'$, $''$ respectively. This sexagesimal division is cumbrous and unscientific, but is apparently permanently established. It seems to have originated with the Babylonians, who fixed approximately the length of the year at 360 days, in which time the sun completed his circuit of the heavens. A *degree*, then, as is indicated by the name, which means *step* in Latin, Greek, Hebrew (*gradus*, βαθμος (or τμημα), *ma'alah*), was primarily the daily *step* of the sun eastward among the stars. The Chinese, on the other hand, determined the year much more exactly at $365\frac{1}{4}$ days, and accordingly, in defiance of all arithmetic sense, divided the circle into $365\frac{1}{4}$ degrees.

2. The angles of a △ are equal; how many degrees in each?

REMARK. — Such a △ is called *equiangular*, more commonly *equilateral,* but better still **regular**.

3. Show that this regular △ is equilateral.

4. One angle of a △ is a right-angle; the others are equal; how many degrees in each?

5. One angle of a $\triangle$ is twice and the other thrice the third; what are the angles?

6. Two angles of a $\triangle$ are measured and found to be $46^\circ\,37'\,24''$ and $52^\circ\,48'\,39''$; what is the third?

7. One angle of a $\triangle$ is measured to be $61^\circ\,22'\,40''$; the others are computed to be $49^\circ\,34'\,28''$ and $69^\circ\,2'\,43''$; what do you infer?

8. A half-ray turns through two round angles counter-clockwise, then through half a right-angle clockwise, then through a straight angle counter-clockwise, then through $\frac{1}{8}$ of a round angle counter-clockwise, then through $\frac{7}{6}$ of a straight angle clockwise; what angle does it make in its final position with its original position?

9. O is a fixed point (called *origin*) on a ray, A and B are any pair of points, M their mid-point. Show and state in words that $2OM = OA + OB$.

10. A, B, C are three points on a ray, A', B', C' are mid-points of the tracts BC, CA, AB, and O is any point on the ray; show that $OA + OB + OC = OA' + OB' + OC'$.

11. A, B, C, D, O are points on a ray; A', B', C' are mid-points of AB, BC, CD; A'', B'', are mid-points of $A'B'$, $B'C'$; M is the mid-point of $A''B''$; prove $8OM = OA + 3OB + 3OC + OD$.

12. What are the conditions of congruence in isosceles $\triangle$s? In right $\triangle$s?

13. In what $\triangle$ does one angle equal the sum of the other two?

Def. A number of tracts joining consecutively any number of points (first with second, second with third, etc.) is called a *broken line*, or *train of tracts*, or **polygon**. Where the last point falls on the first the polygon is **closed**; otherwise it is *open*. Unless otherwise stated, the polygon is supposed to be closed. The points are the **vertices**, the tracts are the **sides** of the polygon. The closed polygon has the same number of vertices and sides, and we may call it an **n-angle** or **n-side.** The angles between the pairs of consecutive sides are the **angles** of the polygon, either inner or outer; unless otherwise stated, *inner* angles are referred to. Inner and outer angles at any vertex are supplemental. When each inner angle is less than a straight angle, the polygon is called **convex**; otherwise, *re-entrant*. Unless otherwise stated, convex polygons are meant. Sides and angles of a polygon may be reckoned either clockwise or counter-clockwise.

14. Prove that the sum of the inner angles of an n-side is $(n-2)$ straight angles. What is the sum of the outer angles?

15. Find the angle in a regular (*i.e.* equiangular and equilateral) 3-side, 4-side, 5-side, 8-side, 12-side. (For proof that there **is** a regular n-side, see Art. 137.)

16. Show that a (convex) polygon cannot have more than three obtuse outer angles, nor more than three acute inner angles.

17. Two angles of a $\triangle$ are α and β; find the angles at the intersection of their mid-rays.

18. If two $\triangle$s have their sides parallel or perpendicular in pairs, then the $\triangle$s are mutually equiangular.

19. The medial to the hypotenuse of a right $\triangle$ cuts the $\triangle$ into two isosceles $\triangle$s.

20. An angle in a $\triangle$ is obtuse, right, or acute, according as the medial to the opposite side is less than, equal to, or greater than, half the opposite side.

21. A medial will be greater than, equal to, or less than, half the side it bisects, according as the opposite angle is acute, right, or obtuse.

22. If P and Q be on the mid-normal of AB, then $\triangle APQ \equiv \triangle BPQ$ ($\equiv$ indicates congruence).

23. AB is the base, C the opposite vertex of an isosceles $\triangle$; show that $ABN \equiv BAM$ (1) when AM and BN are altitudes, (2) when they are medials, (3) when they are mid-rays of angles A and B, (4) when MN is normal to the mid-normal of AB.

24. P is any point within the $\triangle$ ABC; show that $AP + BP < AC + CB$, $AP + PB + CP > \frac{1}{2}(AB + BC + CA)$.

25. $ABC \cdots L$ and $AB'C' \cdots L$ are two convex polygons, not crossing each other, between the same pairs of points, A and L; which is the longer? Give proof.

26. P is a point within $\triangle ABC$; show that angle $APB > ACB$ and sum of angles at $P = 2(A + B + C)$.

27. P is equidistant from A, B, and C; show that angle $APB = 2$(angle ACB).

28. Conversely, if angle $APB = 2$ (angle ACB), angle $BPC = 2$ (angle BAC), and angle $CPA = 2$ (angle CBA), then P is equidistant from A, B, C.

29. The mid-rays of the angles at the ends of the transverse axis of a kite cut the sides in the vertices of an anti-parallelogram (Art. 99).

30. The four joins of the consecutive mid-points of the sides of a 4-side form a parallelogram.

31. The joins of the mid-points of the pairs of opposite sides and of the pairs of diagonals of a 4-side concur, bisecting each other.

32. The mid-parallels to the sides of a △ cut it into 4 congruent △s.

33. What figures are formed by the mid-parallels when the △ is right? isosceles? regular?

34. A parallelogram is a rhombus if a diagonal bisects one of its angles.

35. A parallelogram is a square if its diagonals are equal and one bisects an angle of the parallelogram.

36. From any point in the base of an isosceles △ parallels are drawn to the sides; the parallelogram so formed has a constant *perimeter* (= measure round = sum of sides).

37. The sum of the distances of any point on the base of an isosceles △ from the sides is constant.

38. The sum of the distances of any point within a regular △ from the sides is constant.—What if the point be without the △?

39. P is on a mid-ray of the angle A in the △ ABC; compare the difference of PB and PC: when P is within the △, and when P is without.

40. The inner mid-ray of one angle of a △ and the outer mid-ray of another form an angle that is half the third angle of the △.

41. O is the orthocentre of the △ ABC; express the angles AOB, BOC, COA, through the angles A, B, C.

42. Do the like for the circum-centre S and the in-centre I.

43. The medial to the hypotenuse of a right △ equals one-half of that hypotenuse.

44. The mid-rays of two adjacent angles of a parallelogram are normal to each other.

45. In a 5-pointed star the sum of the angles at the points is a straight angle. What is the sum in a 7-pointed star?

46. Parallels are drawn to the sides of a regular △, trisecting the sides; what figures result?

47. A side of a △ is cut into 8 equal parts, through each section point parallels are drawn to the other sides; how are the other sides cut and what figures result?

48. Two ⨺ are congruent when they have two mid-tracts of two corresponding angles equal, and *besides* have equal

(1) these angles and a pair of the including sides; or

(2) two pairs of corresponding angles; or

(3) one pair of corresponding angles and the corresponding angles of the mid-tract with the opposite side; or

(4) one pair of including sides and the adjacent segment of the opposite side.

49. Two ⨺ are congruent when they have two corresponding sides and their medials equal, and *besides* have equal

(1) another pair of sides; or

(2) the angles of the medial with its side (in pairs); or

(3) a pair of angles of the bisected side with another side, the angles of the medial with this side being both acute or both obtuse; or

(4) a pair of angles of the medial with an including side, the corresponding angles of the medial with its side being both acute or both obtuse.

50. Two ⨺ are congruent when they have a pair of corresponding altitudes equal, and *besides* have equal

(1) the pair of bases and a pair of adjacent angles; or

(2) the pair of bases and another pair of sides; or

(3) the pairs of angles of the altitude with the sides; or

(4) two pairs of corresponding angles; or

(5) the two pairs of sides, when the altitudes lie both between or both not between the sides of the $\triangle$s.

SYMMETRY.

90. We have seen that congruent figures are alike in size and shape, different only in place, and may be made to fit point for point, line for line, angle for angle. The parts that fit one on the other are said to **correspond** or be **correspondent.** Plainly only like can correspond to like, as point to point, etc.

Def. The ray through two points we may call the **join** of those points, and the point on two rays the **join** of the rays.

91. It is now plain that if A corresponds to A' and B to B', then the join of A and B must correspond to the join of A' and B'; for in fitting A on A' and B on B' the ray AB must fit on the ray $A'B'$ (why?). Also if the ray L corresponds to L', and M to M', then the join of L and M must correspond to the join of L' and M' (why?). These facts are very simple but very important.

We shall think of the plane as a thin double film, the one figure drawn in the upper layer, the other in the lower.

92. Two congruent figures may be placed anywhere and any way in the plane, but there are *two* positions especially important: (1) the one in which the one figure may be superimposed on the other by turning the one half of the plane through a straight angle about a ray called an **axis**; (2) the one in which the one figure may be fitted on the other by turning the one half of the plane through a straight angle about a point called a **centre.**

Congruent figures in either of these two positions are called **symmetric**: in the first case **axally**, as to the axis of symmetry; in the second case **centrally**, as to the centre of symmetry.

93. In two symmetrics, corresponding angles, like all other correspondents, are of course congruent; but they are reckoned *oppositely* if the symmetry be axal, *similarly* if

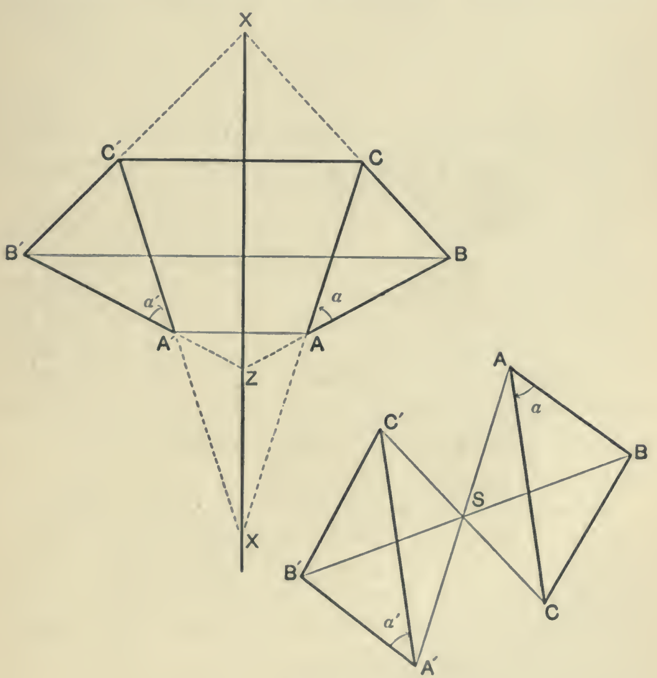

FIG. 61.

it be central. To parallels correspond parallels; to normals, normals; to mid-points, mid-points; to mid-rays, mid-rays; to the axis corresponds the axis, each point to itself; to the centre corresponds the centre itself (Fig. 61).

Elements, whether points or lines, that correspond to themselves may be called *self-correspondent* or **double.**

It is also manifest that centre and axis are the *only* self-correspondents; hence if a point be self-correspondent, it must lie on the axis in axal symmetry, *or* be the centre in central symmetry; and if two counter half-rays be correspondent, they (or the ray) must be normal to the axis in axal symmetry, *or* go through the centre in central symmetry.

94. These facts are all perfectly obvious, but a more vivid exemplification of the nature of these two kinds of symmetry may perhaps be found in the following:

Suppose the axis of symmetry to be a perfect plane mirror; then either half of the plane may be treated as the reflection or exact image of the other, and will be the symmetric of the other as to the mirror-axis. For the image of any point A is the point A' such that the axis is the mid-normal of AA', as we know from Physics; also, on folding over the one half of the plane about the axis upon the other half, the point A falls on A' (why?); hence A' is the symmetric of A as to the axis.

Suppose the centre of symmetry S to be also a reflector; then the reflection or image of any point A will be a point A' such that S is the mid-point of the tract AA', and on rotation through a straight angle about S the point A falls on A', and the half-ray SA fits on the half-ray SA'. Hence either of two centrally symmetric figures is the exact image of the other reflected from the centre of symmetry S.

Note carefully that these two species of symmetry depend upon the two fundamental definitive properties of the plane: central symmetry upon the *homœoidality* of the plane, axal symmetry upon the *reversibility* of the plane. Moreover, axally symmetric figures can *not* be fitted on each other without reversion, folding over; by movement *in* the plane their corresponding parts can at best be *op*posed, but never *super*posed; while on the other hand central symmetrics may be *super*posed, but cannot be *op*posed, along any ray, by motion in the plane. In central symmetrics the corresponding parts follow one another in the same order, but in axal symmetrics they follow in opposite orders.

95. We must now discuss these two symmetries more minutely, and to exhibit a certain remarkable relation holding between them we arrange their properties in parallel columns.

In Axal Symmetry.	In Central Symmetry.
1. The axis corresponds to itself.	1. The centre corresponds to itself.
2. Every point of the axis corresponds to itself.	2. Every ray through the centre corresponds to itself (each half to the other).
3. Every self-correspondent point lies on the axis.	3. Every self-correspondent ray goes through the centre.
4. The join of two correspondent **rays** is on the axis.	4. The join of two correspondent **points** goes through the centre.
(For it is self-correspondent.)	(For it is self-correspondent.)
5. Correspondent **points** are equidistant from every **point** on the axis.	5. Correspondent **rays** are equally inclined (isoclinal) to every **ray** through the centre; hence **they are parallel,** as is otherwise manifest.

6. The axis is a mid-**ray** of every **angle** between correspondent **rays**, and in fact the *inner* mid-ray.	6. The centre is a mid-**point** of every **tract** between correspondent **points**, and in fact the *inner* mid-point.
N.B. The *outer* mid-ray is a **normal to the axis.**	N.B. The *outer* mid-point is a **point at infinity.**
7. The join of two correspondent *points* is a **normal to the axis.**	7. The join of two correspondent *rays* is **at infinity.** (For they are parallel.)
8. Correspondent **tracts** are anti-parallel.	8. Correspondent **angles** are contra-posed (*i.e.* have their arms extended oppositely).
9. Correspondent **points** are equidistant from the axis.	9. Correspondent **rays** are equidistant from the centre.
10. The join of two **rays** and the join of their correspondents themselves correspond.	10. The join of two **points** and the join of their correspondents themselves correspond.

96. On regarding closely these correlated propositions, it becomes clear that the one set differs from the other only in the interchange of certain notions, as *point* and *ray*, *tract* and *angle*, etc. Every property of axal symmetry has its obverse in central symmetry, and *vice versa*. This most profound, important, and interesting fact has received the name of the **Principle of Reciprocity.** We make this notion more precise by the following

Def. Two figures such that to every point of each corresponds a ray of the other, and to every ray of each a point of the other, are called **reciprocal.** For example :

Suppose rays drawn through a point O to any number of points, A, B, C, D, E, . . . on a ray L. Then the point O with its ray through it, and the ray L with its point on it, are two reciprocal figures (Fig. 62). The first is called a (flat) **pencil** of rays, O being the centre; the second is called a **row** (or range) of points, L being the axis. Suppose we have now a second pencil through O' and a second row on L'. These two figures are again reciprocal, and the two pairs of reciprocals together make up another more complex pair of reciprocals. In this latter pair we find our definition fully exemplified. To O and O' correspond L and L'; to the rays through O and O' correspond the points on L and L'; also, to the join (ray) of O and O' corresponds the

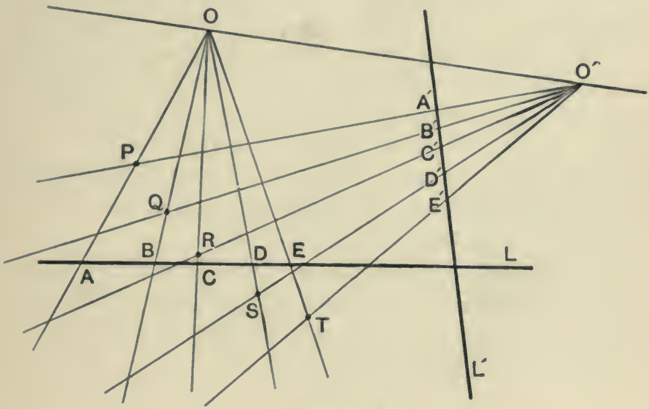

FIG. 62.

join (point) of L and L'; to any point as P, the join of two rays (OA, $O'A'$), corresponds a ray AA', the join of two points (A, A'). So Q, R, S, T are points corresponding to the rays BB', CC', DD', EE'. We may notice further that **angle** and **tract** correspond in the reciprocal figures; thus the angle AOB corresponds to the tract AB, and the angle BOC to the tract BC; while the angle OPO' corresponds to the tract AA' and the tract RS to the angle between the rays corresponding to R and S; namely, between CC' and DD'. Let the student trace out as many correspondences as possible.

97. To three points fixing a triangle in either of two reciprocals must correspond also three rays fixing a triangle in the other reciprocal; hence, in general, **triangle** corresponds to **triangle** in reciprocals. But notice: the *sides* of one correspond to the *vertices* of the other; hence if the sides of one all go through the same point, the vertices of the other all lie on the same ray; that is, **three concurrent rays** in either reciprocal correspond to **three collinear points** in the other.

It now appears that axal and central symmetry are reciprocal to each other; the reciprocal of an axal symmetric is a central symmetric, and the reciprocal of a central symmetric is an axal symmetric; the reciprocal properties of axal symmetry are the properties of central symmetry, and the reciprocal properties of central symmetry are the properties of axal symmetry.

Very often the two symmetric figures may be regarded as the two halves of one figure; this one figure is then said to be **symmetric** as to the axis of symmetry or as to the centre of symmetry, as the case may be.

98. If our figure be two points, A and A', then the mid-normal X of the tract AA' is the axis of symmetry, manifestly. If, now, any double point D on the axis be joined with A and A', there results the isosceles $\triangle$ ADA', whence it appears that (Fig. 63)

The isosceles $\triangle$ is a symmetric $\triangle$.

It is plain that any two points on the ray AA' equidistant from N are symmetric as to X, that all points on the ray, and indeed in the whole plane, may be arranged in symmetric pairs, the members of each pair equidistant from the axis X.

99. Now take two points on the axis, as D and D', or D and D'', and consider the 4-side $DAD'A'$. It is composed of two $\triangle$, ADD' and $A'DD'$, symmetric with each other as to the axis X, and opposed along that axis. Hence the 4-side is itself symmetrical as to X.

Def. Such a 4-side, with an axis of symmetry, is called a **kite.**

If we hold D fast, and let D' glide along X, the 4-side $ADA'D'$ remains a kite. We see that there are two kinds

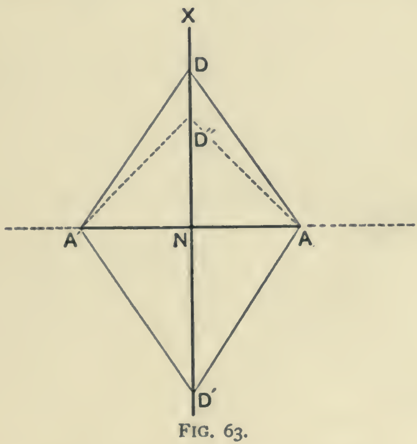

FIG. 63.

of kites, the convex kite, as $ADA'D'$, and the re-entrant, as $ADA'D''$. As the gliding point passes through N the kite changes from one kind to the other, passing through the intermediate form of the symmetrical $\triangle$.

When the gliding point reaches a position D' such that $ND = ND'$, then the four sides of the kite are all equal (why?), and the kite becomes a **rhombus** (why?). In this case D and D' are symmetric as to AA' as an axis of symmetry. Hence *the rhombus has two axes of symmetry;* namely, its *two diagonals.*

In all cases the diagonals, AA' and DD', of the kite are normal to each other (why?).

100. Now consider a pair of points, B and B', symmetric as to the axis X (Fig. 64). Then X is mid-normal of BB'.

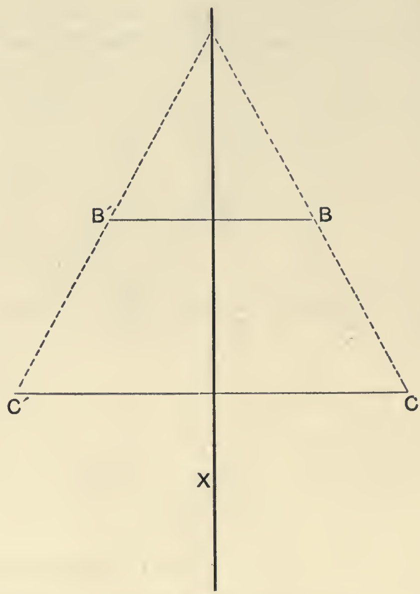

FIG. 64.

If C and C' be any other pair of symmetric points, then X is also mid-normal of CC'; hence BB' and CC' are parallel (why?). Also the tracts BC and $B'C'$ are symmetric as to X (why?), and the 4-side $BB'C'C$ is itself symmetric as to the axis X. Hence the angles at C and C' are equal, also the angles at B and B' are equal (why?) ; hence the angles at B and C and at B' and C' are supplemental (why?), and the 4-side $BB'C'C$ is an **anti-parallelogram** (why?). Hence we see that *another symmetric 4-side is an anti-parallelogram.*

It is plain that every anti-parallelogram is symmetric, for we know that the oblique sides prolonged yield an isosceles △. Let the student complete the proof.

101. There is *only one* kind of symmetric △, the isosceles. For, let ABA' (Fig. 65) be symmetric and A' correspondent

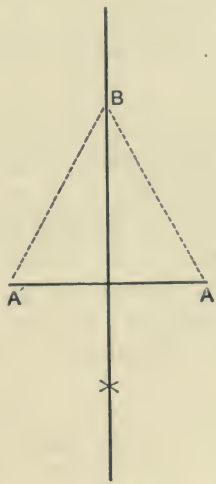

FIG. 65.

to A. Then B must correspond to itself (why?) ; hence B must lie on the axis (why?) ; hence $BA = BA'$ (why?).

Now let the student prove that

(1) *In a symmetric △ the axis of symmetry is a medial;* (2) *it is also a mid-ray;* (3) *it is also a mid-normal.*

Conversely, let him show that

A medial that is a mid-ray, or a mid-normal, is an axis of symmetry.

102. There are *only two* axally symmetric 4-sides; namely, the kite and the anti-parallelogram. For, in a symmetric 4-side a vertex must correspond to a vertex (why?). Also, not all vertices can be on the axis (why?). Also, a vertex on the axis is a double point (why?). Also, the vertices not on the axis must appear in pairs (why?); hence there must be *either two or four* of them. If there be two only, then the other two are on the axis and the 4-side is a kite; if there be four of them, we have just seen that the 4-side is an anti-parallelogram.

103. Now let us turn to the *reciprocals*. The reciprocals of the two *points* A and A' symmetric as to the *axis* X will be two *rays* L, L', symmetric as to the *centre* S. But rays symmetric as to a centre are parallel (why?); hence we have two parallels symmetric as to S, which is midway between them. The rays are symmetric as to any other point S' midway between them (why?). The piece of plane between these parallels is called a *parallel strip*, or **band** (Fig. 66).

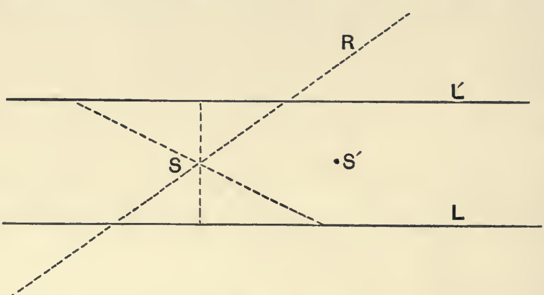

FIG. 66.

But what corresponds to the point D on the axis X? The answer is: a ray R through S (why?). Hence to the symmetric $\triangle$ of the three points A, A', D, there corresponds the figure formed by two parallels L, L', and a transverse R through S, — a so-called *half-strip*. This is truly a *three-side*, but not apparently a $\triangle$ (3-angle), for the parallels do not meet in finity, in regions accessible to our experience. Hence, instead of saying that the reciprocal of a $\triangle$ in axal symmetry is a $\triangle$ (3-angle or 3-point) in central symmetry, we should have said, accurately, that the reciprocal of a $\triangle$ in axal symmetry is a 3-side (or trilateral) in central symmetry, which will always be a $\triangle$ except when sides are parallel or all concur. In higher Geometry it is very convenient to remove this apparent exception by using this form of expression: the parallels meet *not in finity*, but *in infinity*.

104. It is indeed plain that

A $\triangle$ can have no centre of symmetry.

For, since vertex corresponds to vertex, and since correspondents appear in pairs, one vertex must be a double point; hence it would have to be the centre S (why?). But the other two vertices would have to lie on a ray through S, being correspondents; hence the three vertices would be collinear, and the $\triangle$ would be flattened out to a *triply-laid ray*.

105. But there is a centrally symmetrical 4-side; namely, the **parallelogram**. For, consider once more the kite $AXA'X'$ and let us reciprocate it into a centrally symmetric figure (Fig. 67). To the axis XX' will correspond the centre S; to the symmetric pair of rays AX and $A'X$ will correspond a symmetric pair of points P and P'; to the join of those on the axis X will correspond the join of these through the centre (PP'). Similarly, to the symmetric rays AX' and $A'X'$ will correspond the symmetric points Q and Q', and to the join X' will correspond the join QQ'. Also, AX and AX' have a join A while $A'X$ and $A'X'$ have a join A', and these joins are symmetric as to the axis XX'; recipro-

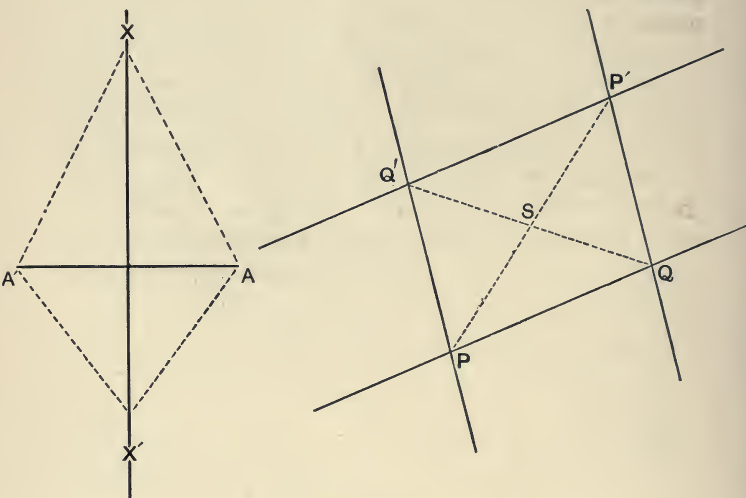

FIG. 67.

cally, P and Q have a join PQ, and P' and Q' have a join $P'Q'$, and these joins are symmetric as to S; that is, they are parallel (why?). Similarly, PQ' and $P'Q$ correspond to B $(AX, A'X')$ and B' $(A'X, AX')$; but B and B' are symmetric as to XX' (why?); hence PQ' and $P'Q$ are symmetric as to S, *i.e.* are parallel. Hence $PQ\,P'Q'$ is symmetric as to S, and is a *parallelogram*. Q. E. D.

106. We may indeed see at once that since any two parallels are centrally symmetrical as to any mid-point, a pair of parallels or a parallelogram is symmetric as to the common mid-way point, the intersection of the diagonals. But the foregoing reciprocation is instructive, as illustrating in detail the method to be pursued, and as showing the intimate relation of the different symmetric quadrilaterals; namely, *the* **parallelogram** *is the common reciprocal of both* **kite** *and* **anti-parallelogram**, which are thus seen to be really *one*.

107. Central symmetry does not in general imply anything at all with respect to axal symmetry in a figure. We may draw through any point S any number of rays and lay off on each from S a pair of counter tracts SP and SP', SQ and SQ', etc. No matter how PQ, etc., be chosen, the figure so obtained will be centrally symmetric as to S; but it may have no axal symmetry whatever. Neither does axal symmetry in general imply any central symmetry, but we may establish the following important

Theorem. — *Any figure with two rectangular axes of symmetry has also a centre of symmetry; namely, the intersection of those axes.*

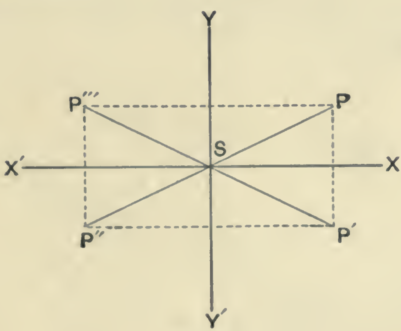

FIG. 68.

Data: XX' and YY' two rectangular axes, P any point of a figure symmetric as to these axes (Fig. 68).

Proof. The point P' symmetric with P as to XX' is a point of the figure (why?); also P'' symmetric with P' as to YY' is a point of the figure (why?); so too is P''' (why?); the figure $PP'P''P'''$ is a rectangle (why?), its diagonals halve each other, and $SP = SP'' = SP' = SP'''$. Hence S is a centre of symmetry. Q. E. D.

THE CIRCLE.

108. We have already discovered the existence of a *homœoidal* plane curve *not reversible* and have named it **circle**.

Defs. A ray cutting a curve is called a **secant**, as L; the part of the secant intercepted by the curve, or the tract between two points of the curve, is called a **chord**, as AB. A finite part of a curve is called an **arc**. A chord and an arc with the same two ends are said to **subtend** each other. Also, the intercept of any line between the ends of an angle is said to *subtend* the angle. Thus BC and DE subtend the angle O (Fig. 69).

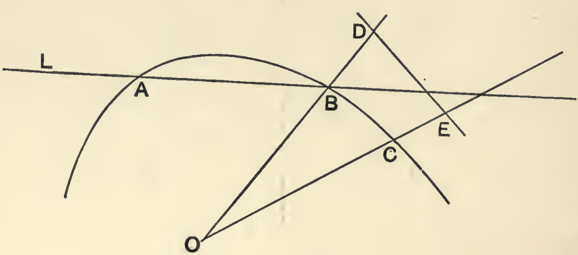

FIG. 69.

109. Theorem XLVIII. — *Congruent arcs subtend congruent chords.*

Proof. Let the arcs AB and CD be congruent; then we may fit A on C and at the same time B on D; then the chords AB and CB fit throughout (why?). Q. E. D.

N.B. We can *not* convert this proposition at once (why?) (Fig. 70).

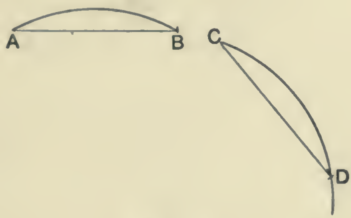

FIG. 70.

110. Theorem XLIX. — *A closed curve is cut by a ray in an even number of points* (Fig. 71).

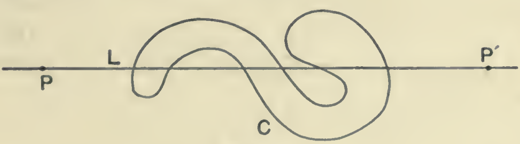

FIG. 71.

Proof. Let L be a ray, C any closed curve. Suppose a point P to trace out the ray L. At first P is without the curve, at last it is also without the curve; hence P has crossed the curve going out as often as it has crossed the curve going in, for every entrance there is an exit; hence the points of intersection appear in pairs, their number is even, as 0, 2, 4, 6, . . . $2n$. Q. E. D.

These preliminary or auxiliary theorems, which prepare the way for a theorem to follow, are sometimes called **lemmas** ($\lambda\eta\mu\mu\alpha$ = assumption, premise, support, prop).

*111. **Theorem L.**—*A circle has an axis of symmetry through every one of its points* (Fig. 72).

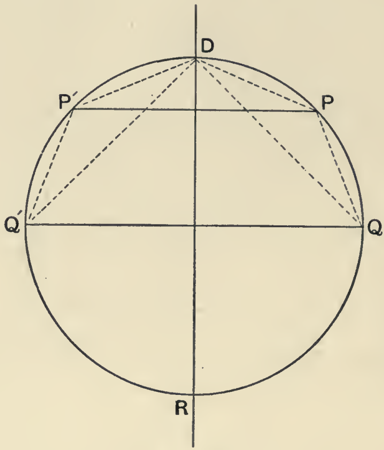

Fig. 72.

Proof. Let D be any point of a circle. Take any arc DP, and slip it round till P falls on D and D on P'; this is possible (why?). Then PDP' is a symmetrical $\triangle$ (why?); and its axis of symmetry DR halves normally the chord PP', and also halves the angle PDP' (why?). Now take any other arc DQ and slip it round till Q falls on D and D on Q', so that DQ and $Q'D$ are congruent. Then the chords DQ and DQ' are congruent (why?). Also, on taking away the congruents DP and DP' we have left PQ and $P'Q'$ as congruent remainders. Hence the chords PQ

and $P'Q'$ are congruent (why?). Hence the $\triangle$ PDQ and $P'DQ'$ are congruent (why?) ; hence the angles PDQ and $P'DQ'$ are equal (why?) ; hence DR halves also the angle QDQ' (why?). But the $\triangle$ QDQ' is symmetric (why?) ; hence DR is also its axis of symmetry, and Q and Q' are symmetric points of the circle ; hence any point of the circle has its symmetric point as to DR ; *i.e.* DR is an axis of symmetry of the circle. Moreover, D was any point of the circle ; hence through any point of the circle passes an axis of symmetry. Q. E. D.

Def. A ray halving a system of parallel chords is called a **diameter** ; the chords and diameter are called **conjugate** to each other.

Corollary 1. In a circle a diameter is normal to its conjugate chords.

Corollary 2. Every mid-normal to a chord in a circle is a diameter and halves the subtended arcs.

*112. **Theorem LI.** — *A circle has a centre of symmetry* (Fig. 72).

For the ray through D must cut the circle in some second point, as R (why?), and as the ray turns round from the position DR to the reversed position RD, through a straight angle, it must pass through some position, QQ', normal to its original position (why?). Hence for any axis of symmetry there is another normal thereto and their intersection is a centre of symmetry (why?). Q. E. D.

N.B. There is *only one* centre of symmetry (why?).

Def. This centre of symmetry is named **centre** *of the circle.* It is often convenient to call the whole ray through the centre a *centre ray or line*, and to restrict the term *diameter* to the centre chord.

Corollary 1. All diameters go through the centre, and halve each other there; *conversely,* chords halving each other are diameters.

Def. Two diameters each halving all the chords parallel to the other are called **conjugate.**

Corollary 2. In the circle two diameters normal to each other are conjugate; and *conversely,* two conjugate diameters are normal to each other.

N.B. Other curves, as Ellipse and Hyperbola, have conjugate diameters *not in general* normal to each other (Fig. 73).

*113. **Theorem LII.** — *All diameters of a circle are equal* (Fig. 74).

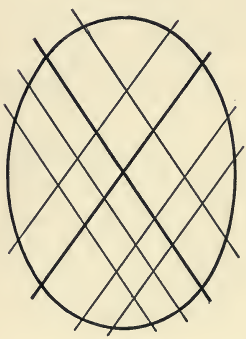

FIG. 73.

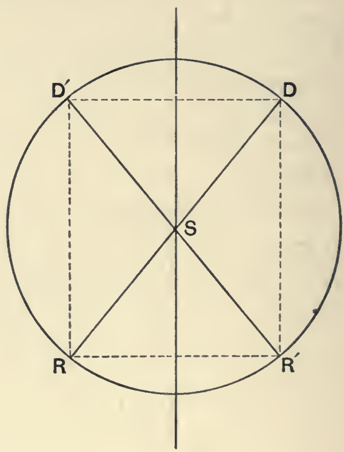

FIG. 74.

Proof. Let DR and $D'R'$ be two diameters. The figure $DD'RR'$ is a parallelogram (why?), and DD' is parallel to RR'; hence the mid-normal of these parallels is a diameter through the centre S; hence SD and SD' are symmetric and equal; hence $DR = D'R'$. Q. E. D.

Def. A half-diameter, from centre to circle, is called **a radius**.

Corollary 1. All radii of a circle are equal; or, all points of a circle are equidistant from the centre.

Corollary 2. Every parallelogram inscribed in a circle is a rectangle.

N.B. By help of this important property the circle is commonly defined as *a plane curve all points of which are equidistant from a point within called the centre.* The common distance of all points of the circle from the centre is often called **the radius.** We have deduced this property from the homœoidality; *conversely*, we may deduce the homœoidality from this property taken as definition. But if there were no such surface as the plane, at least for our intuition, the circle might still exist on the sphere-surface, without centre, but with the body of its properties unimpaired. Hence it seems better to define the circle by its intrinsic homœoidality than by its extrinsic centrality.

Corollary 1. All points within the circle are less, and all points without are more, than the radius distant from the centre.

Defs. The two symmetric halves into which a diameter cuts a circle are called **semicircles.** The part of the plane bounded by an arc and its chord is called a **segment**; the part bounded by an arc and the two radii to its ends is called a **sector.** If the sum of two arcs be a circle, we may call them **explemental,** the one *minor*, the other *major;* every chord belongs equally to each of two explemental arcs, but in general, unless otherwise stated, it is the *minor* that is referred to. Two arcs whose sum is a half-circle are called **supplemental**; two whose sum is a quarter-circle or quadrant are called **complemental.**

Corollary 2. All circles of the same radius are congruent; also, all semicircles of the same radius are congruent, and all quadrants of the same radius are congruent.

Corollary 3. Any circle may be slipped round at will upon itself about its centre as a pivot, like a wheel about its axle, without changing in the least the position of the whole circle.

114. From the foregoing it is clear that if we hold one point of a ray fixed, and turn the ray in the plane about the fixed point, every other point of it will trace out a circle about the fixed point as a centre. An instrument, one point of which may be fixed while the other is movable about in a plane, is called a **compass** or pair of **compasses**, and is both the simplest and the most important of all instruments for drawing.

115. Theorem LIII. — *Through any three points not collinear one, and only one, circle may be drawn.*

Proof. Let A, B, C be the three points not collinear (Fig. 75). We have already seen that the mid-normals to the tracts AB, BC, CA concur in a point S equidistant from A, B, and C; hence a circle about S with radius d passes through A, B, C. Also there is only one point thus equidistant from A, B, C (why?); hence there is only one circle through A, B, C. Q. E. D.

Def. The circle through the vertices A, B, C, of a $\triangle$ is called the **circum-circle** of the $\triangle$.

Corollary 1. A $\triangle$, or a triplet of points, or a triplet of rays, determines one, and only one, circle.

Corollary 2. Through two points, A and B, any number of circles may be drawn. Their centres all lie on the mid-normal of AB.

Corollary 3. As BC turns clockwise about B as a pivot, the intersection S, the centre of the circle through A, B, C, retires upward ever faster and faster along the mid-normal N of AB; when C becomes collinear with A and B, the inter-

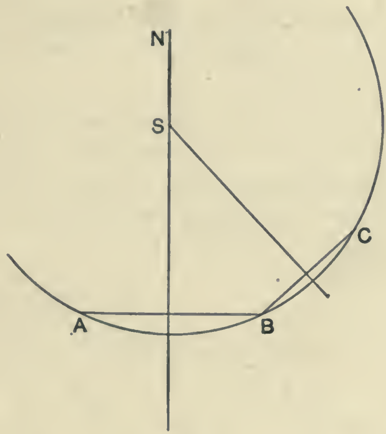

FIG. 75.

section of the mid-normals of AB and BC vanishes from finity, or *retires to infinity*, as the phrase is. As BC keeps on turning, S reappears in finity below and moves slower and slower upward along the mid-normal. Moreover, a circle passes through A, B, and C, no matter how close C may lie to the ray AB, nor on which side of it: only as C falls upon the ray does the centre of the circle vanish into infinity; that is, we may draw a circle that shall fit *as close to the ray AB as we please*, though not upon it, by retiring the centre far enough. Hence a ray may be conceived as a circle with centre retired to infinity; it is strictly the **limit** of a circle whose centre has retired, along a normal to it, *without limit.*

116. Theorem LIV. — *A circle can cut a ray in only two points.*

For there are only two points on a ray at a given distance from a fixed point (why?). Q. E. D.

117. Theorem LV. — *Secants that make equal angles with the centre ray* (or axis) *through their intersection intercept equal arcs on the circle.*

Proof. For both the two semicircles and the two secants are symmetric as to the axis IS (why?); hence, on folding over the one half-plane upon the other, A falls on A', B on B', arc a fits on arc a', and chord c on chord c' (Fig. 76). Q. E. D.

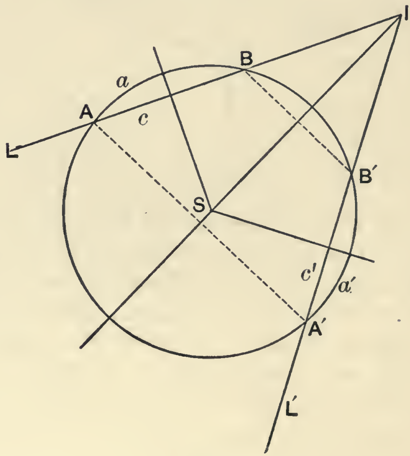

FIG. 76.

Conversely, *Secants that intercept equal arcs make equal angles with the axis through their intersection.*

Proof. Let L and L' intersect equal arcs AB and $A'B'$. Draw the mid-normal of AA'; it is an axis of symmetry (why?). On folding over the left half-plane upon the right half-plane, A falls on A' and B on B' (why?); hence AB and $A'B'$ are symmetric; hence they meet on the axis and make equal angles with it (why?). Q. E. D.

Corollary 1. Equal chords are equidistant from the centre; and *conversely*, Chords equidistant from the centre are equal.

Corollary 2. The greater of two unequal chords is less distant from the centre.

Corollary 3. A diameter is the greatest chord.

Corollary 4. Arcs intercepted by two parallel chords are equal.

Corollary 5. Equal chords or arcs subtend equal *central angles* (angles at the centre), and *conversely*.

Corollary 6. Of two unequal chords or arcs, the greater subtends the greater central angle.

What figure is determined by two parallel chords and the chords of the intercepted arcs? By two secants that intercept equal arcs and the central normals thereto?

118. **Theorem LVI.** — *A central angle subtended by a certain arc* (or chord) *is double the peripheral angle subtended by the same* (or an equal) *arc* (or chord) (Fig. 77).

Proof. Let ASB be a central angle, and APB be a *peripheral* angle (periphery = circumference, the circle itself), subtended by the same arc or chord AB. Draw the diameter PD. Then the $\triangle$ ASP and BSP are isosceles (why?); hence the angle $ASD = 2$ angle APD, and angle $BSD = 2$ angle BPD (why?); hence angle $ASB = 2$ angle APB. Q.E.D.

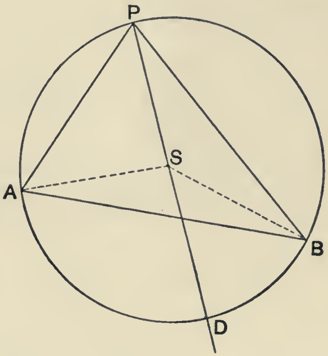

FIG. 77.

Corollary 1. All peripheral angles subtended by (or standing on) the same or equal chords or arcs are equal. Hence, as P moves round from A to B, the angle APB remains unchanged in size.

Def. An angle with its vertex on a certain arc, and its arms passing through the ends of that arc, is said to be **inscribed** in that arc. Hence for an angle to be *inscribed* in a certain arc, and for it to *stand on* the *explemental* arc, are equivalent.

Corollary 2. All angles inscribed in the same or equal arcs of the same or equal circles are equal.

Corollary 3. As the vertex P of a peripheral angle subtended by an arc (or chord) AB, in passing round a circle goes through either end of the arc (or chord), the angle itself leaps in value, changes to its *supplement*.

F. Cajori

119. Theorem LVII. — *The locus of the vertex of a given angle standing on a given tract is two symmetric circular arcs through the ends of the tract* (Fig. 78).

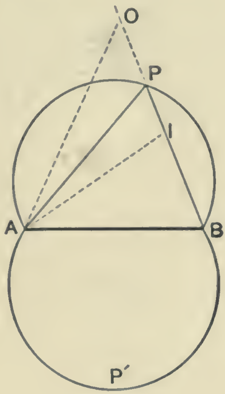

FIG. 78.

Proof. Let P be the vertex of the given angle, in any position, standing on the tract AB. Through A, P, and B draw a circular arc subtended by AB. We have just seen that as long as P stays on this arc, the angle P remains the same in size. Moreover, the point P cannot be without the arc, as at O, because the angle AOB is less than APB (why?) ; neither can it come within the arc, as to I, because the angle AIB is greater than APB (why?) ; hence so long as the angle is constant in size the vertex must remain on the arc APB or on its symmetric arc $AP'B$, of which plainly the same may be said. Q. E. D.

120. Theorem LVIII. — *The angle inscribed in a semicircle* (or standing on a semicircle or diameter) *is a right-angle* (Fig. 79).

Proof. Let ABC be any angle in a semicircle. Then it is half of the central angle ASC (why?), which is a straight angle (why?). Q. E. D.

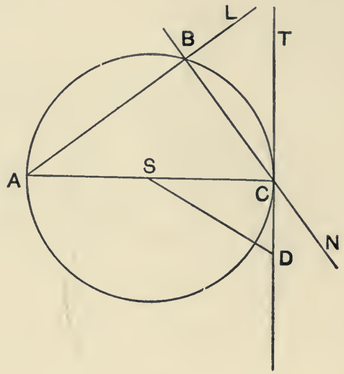

FIG. 79.

Now let the vertex B, the intersection of the rays L and N, move round the circle toward C; the angle ABC remains a right angle, *no matter how close B approaches to C;* moreover, when B passes C, into the lower semicircle, the angle remains a right angle (why?). That is, the angle at B remains a right angle, *no matter from which side nor how close B* approaches to C. Hence it is a right angle *even when B falls on C.* But then the ray L falls on the diameter AC, hence the ray N takes the position T normal to the diameter (or radius) at its end. Such a normal to a radius at its end is called a **tangent** to the circle at the point of *tangence* (or *touch* or *contact*) C.

Def. A ray normal to a tangent to a curve at the point of touch is called **normal** to the curve itself. Hence

Corollary. All radii of a circle are normal to the circle; and *conversely*, all normals to a circle are radii of the circle.

121. Theorem LIX. — *All points on a tangent, except the point of contact, lie outside of the circle.*

Proof. For the point of touch is distant radius from the centre (why?), and all other points, as D, of the tangent are further from the centre (why?) ; hence all other points of the tangent are without the circle (why?). Q. E. D.

122. Theorem LX. — *The point of tangence is a double point.*

Proof. For it is on a diameter, or axis of symmetry, of the circle, and every such point is a double point with respect to that axis.

Independently of this consideration, it is seen that the chord CB becomes the tangent CT *when, and only when,* the points B and C fall together in C.

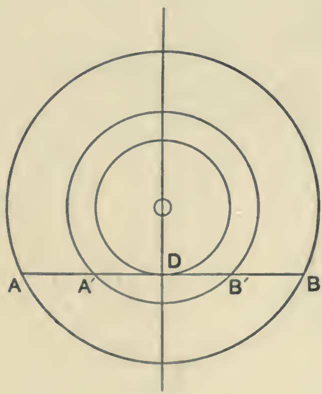

FIG. 80.

Still otherwise, let AB be any chord of a circle about (Fig. 80) O. Draw the mid-normal OD. Now let the circle shrink about the centre O: the points A and B move towards each other, and as D is always mid-way between them they finally fall together in D, and their join is tangent at D to the circle of radius OD.

Def. Two points thus falling together in a double point are called **consecutive** points. Accordingly we may define a tangent to a circle (or to any curve) as *a ray through two consecutive points of the circle* (or curve). Adopting this definition, let the student prove

123. **Theorem LXI.** *Every tangent to a circle is normal to a radius at its end;* conversely, *Every normal to a radius at its end is tangent to the circle.*

124. **Theorem LXII.** *The angle between a tangent and a chord equals the peripheral angle on the same chord, or equals half the angle of the chord* (Fig. 81).

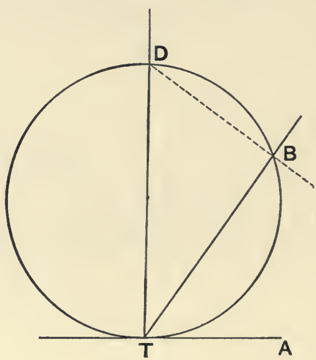

Fig. 81.

Proof. For if DT be a diameter, then the angles BDT and BTA are equal, being complements of the same angle BTD (why?). Or thus: TB is a chord, and TA is also a chord, through the double point T; hence the angle BTA is a peripheral angle standing on the arc TB. Q. E. D.

125. Theorem LXIII. — *The angle between two secants is half the sum or half the difference of the angles of the intercepted arcs, according as the secants intersect within or without the circle.*

Proof. For on drawing AB' the angle I is seen (Fig. 82) to be the sum, and the angle O the difference, of the

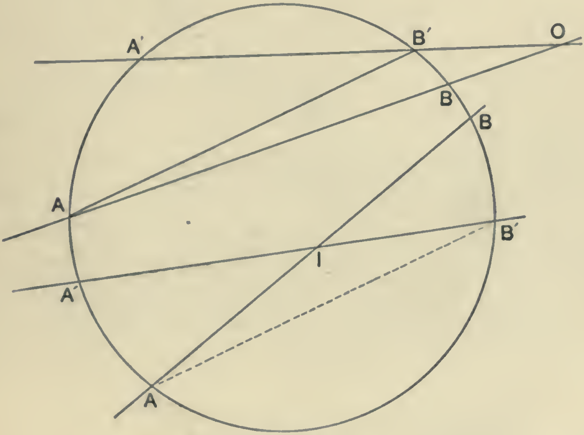

FIG. 82.

angles at A and B' standing on the arcs AA' and BB'. Q. E. D.

126. Theorem LXIV. — *An encyclic quadrangle has its opposite angles supplemental.*

Proof. For the angles B and D are halves of the two central angles ASC and CSA, whose sum is a round angle. Hence the sum of B and D is a straight angle. Q. E. D.

127. Theorem LXV. — Conversely, *A quadrangle with its opposite angles supplemental is encyclic* (Fig. 83).

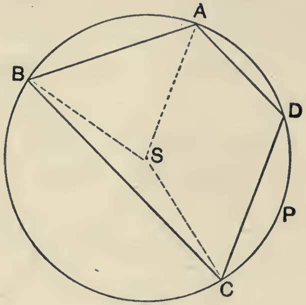

Fig. 83.

Proof. Let $ABCD$ be the quadrangle with the angles A and C, B and D, supplemental. About the $\triangle ABC$ draw a circle. If P be any point on the arc of this circle explemental to ABC, then the angle APC is the supplement of ABC; but if P be not on this arc, then the angle APC is either greater or less than that supplement (why?). Now the angle D is that supplement; hence D is on the arc. Q. E. D.

128. Relations of circles to each other.

Suppose two circles K and K' of radii r and r' to be *concentric, i.e.* to have the same centre O. Then, plainly, the distance between them measured on any half-axis OR is $r - r'$, the difference of the radii. Draw tangents AT, $A'T'$, where OO' cuts the circles. They are parallel (why?). Now let the centre of K' move out on OO' a distance $r - r'$; then A falls on A' and $A'T'$ on AT; the circles have a *common tangent* at A and are said to *touch each other* innerly at A (Fig. 84).

Now let O' move still further along OO'; then the circles will lie partly within, partly without, each other; they will intersect at two points, and only two (why?), symmetric as to OO' (why?), namely P and P'; hence

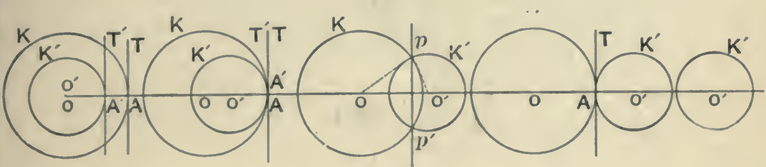

FIG. 84.

Theorem LXVI. — *The common axis of two circles is the mid-normal of their common chord.*

When O' is distant $r + r'$ from O, the circles lie without each other, but still have a common tangent (why?) and are said to *touch* **outerly**.

As O' moves still further away from O, the circles cease to touch and henceforth lie entirely without each other.

Thus we find that there are *three critical* positions depending on the distance d between the centres O and O':

$d = o$, when the circles are *concentric.*

$d = r - r'$, when the circles *touch innerly.*

$d = r + r'$, when the circles *touch outerly.*

There are also *three intermediate* positions:

For $o < d < r - r'$ the one circle is *within* the other.

For $r - r' < d < r + r$ the circles *intersect.*

For $r + r < d < \infty$ the circles lie *without* each other.

129. **Theorem LXVII.** — *From any point without a circle two, and only two, tangents may be drawn to the circle* (Fig. 85).

Proof. Let O be the centre of the circle K, and P be the point without. On OP as a diameter draw a circle K'; only one such circle is possible (why?), and it cuts K in two, and only two, points, T and T'. Draw PT and PT': they are tangent to K at T and T' (why?). Moreover, no other ray through P, as PU, is tangent to K, because OUP is not a right angle (why?). Q. E. D.

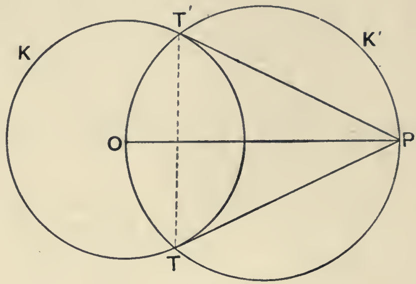

FIG. 85.

Def. The chord TT' through the points of contact of the tangents is called the **chord of contact** for the point P or the **polar** of the **pole** P (see Art.).

The angle between the tangents to two curves at the intersection of the curves is called the **angle between the curves** themselves. When this is a right angle, the curves are said to intersect **orthogonally**.

The distance PT or PT' is called the **tangent-length** from P to the circle.

Corollary 1. Two circles, one having as radius the tangent-length from its centre to the other, intersect orthogonally.

Corollary 2. Two tangents are symmetric as to the axis through their intersection; hence, also, the tangent-lengths are equal.

130. Theorem LXVIII. — *All tangent-lengths to a circle from points on a concentric circle are equal, and intercept equal arcs of the circle* (Fig. 86).

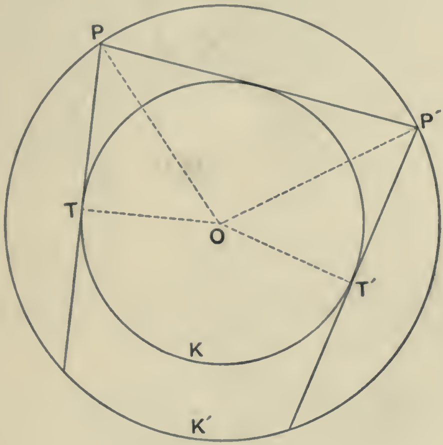

FIG. 86.

Proof. For if P be any point without the circle K', we may turn P round about the centre O on a concentric circle K' without affecting any of the relations obtaining (why?).

Or thus: the right ⚠ TOP and $T'OP'$ are plainly congruent (why?); hence $PT = P'T'$ (why?). Q. E. D.

***131. Theorem LXIX.** — *The intercept between two fixed tangents on a third tangent subtends a constant central angle* (Fig. 87).

Proof. Let PT and PT' be the fixed tangents, VV' the intercept on the variable ray tangent at U. Then TPT' is a constant angle, and VOV' is half of TOT' (why?), and hence is constant. Q. E. D.

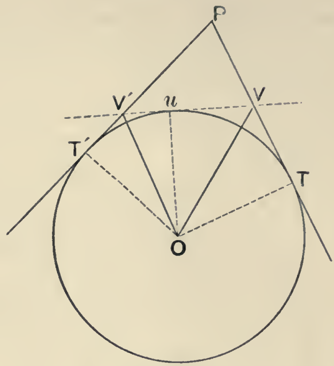

FIG. 87.

132. **Theorem LXX.**—*If the central* (or peripheral) *angles of the common chord of two intersecting circles be equal, the circles are equal.*

Let the student conduct the proof suggested by the figure (Fig. 88), and let him prove the *converse.*

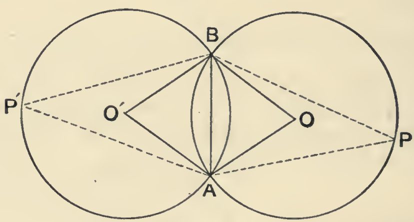

FIG. 88.

*133. **Theorem LXXI.**—*The circumcircle of a* △ *equals the circumcircle of the orthocentre and any two vertices of the* △ (Fig. 89).

Proof. Let K be the circumcircle of the △ ABC, K' the circumcircle of A, B, and O the orthocentre. The angles

C and $B'OA'$ are supplemental (why?); also the angles D and BOA are supplemental (why?); and the angles BOA and $B'OA'$ are equal (why?); hence the angles D and C are equal; hence $K' = K''$ (why?) Q. E. D.

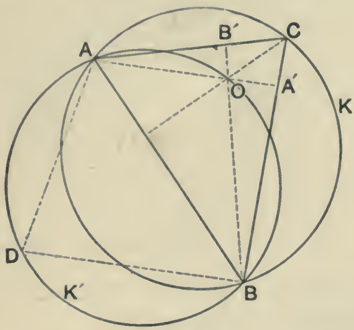

FIG. 89.

*134. **Theorem LXXII.**—*The mid-points of the sides of a* $\triangle$, *the feet of its altitudes, and the mid-points between its orthocentre and vertices, are nine encyclic points.*

Proof. Let a circle through X, Y, Z, the mid-points of the sides, cut the sides in three other points, U, V, W. Then the angle $ZXY =$ angle A (why?), and also $=$ angle ZVY (why?); therefore the $\triangle$ AZV is symmetrical. Hence the $\triangle$ ZVB is also symmetrical, Z is equidistant from A, V, and B, and the angle AVB is a right angle (why?); so also the angles at U and W; *i.e. the circle through the mid-points of the sides goes through the feet of the altitudes* (Fig. 90).

Again, if the circle cuts the altitudes at P, Q, R, then the angle $VPW =$ angle VZW (why?) $= 2$ angle VAW (why?). Moreover, A, V, O, W, are encyclic (why?); hence AO is a diameter of the circle through them (why?); and VAW is a peripheral angle standing on the arc VW; hence the double angle VPW must be the central angle of the same arc; *i.e.* P is the mid-point between a vertex and orthocentre: so, also, are Q and R, similarly. Q. E. D.

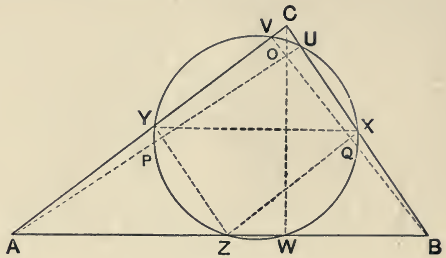

FIG. 90.

Def. This remarkable circle is called the **9-point circle,** or *circle of* **Feuerbach,** of the △ ABC.

Corollary. The radius of the 9-point circle is half the radius of the circumcircle.

135. *Def.* A **Polygon** all of whose sides touch a circle is said to be **circumscribed** about it, and the circle is said to be **inscribed** in the polygon.

Theorem LXXIII. — *A circle may be inscribed in any* △.

Proof. Let ABC be any △ (see Fig. 59). Draw the inner mid-rays of the angles at A, B, C; they concur in the in-centre I of the △, equidistant from the three sides (why?). About this point as centre with this common distance as radius draw a circle; it will touch the three sides of the △ (why and where?). Q. E. D.

N.B. We have seen that the outer mid-rays of the angles concur in pairs with the inner mid-rays of the angles in the three **ex-centres** E_1, E_2, E_3, also equidistant from the sides

(Fig. 60). The circles about these touch only two sides innerly, but the third side outerly, and hence are called **escribed**, or **ex-circles**.

Corollary. Four, and only four, circles touch, each, all the sides of a △.

135 a. Theorem LXXIV. — *In a 4-side circumscribed about a circle the sums of the two pairs of opposite sides are equal* (Fig. 91).

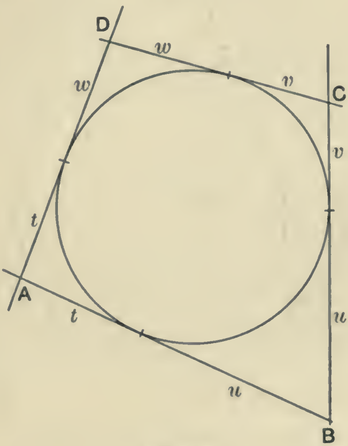

FIG. 91.

Proof. The sum of the four sides is plainly $2t + 2u + 2v + 2w$, and the sum of either pair of opposites is $t + u + v + w$. Q. E. D.

Conversely, *If the sums of two pairs of opposite sides of a 4-side be equal, the 4-side is circumscribed about a circle.*

Proof. Let two counter sides, AB and DC meet in I, and inscribe a circle K in the triangle ADI. Through B

draw a tangent (Fig. 92) to K at U, and let it cut DI at C'. Then since $ABC'D$ is circumscribed about K, we have

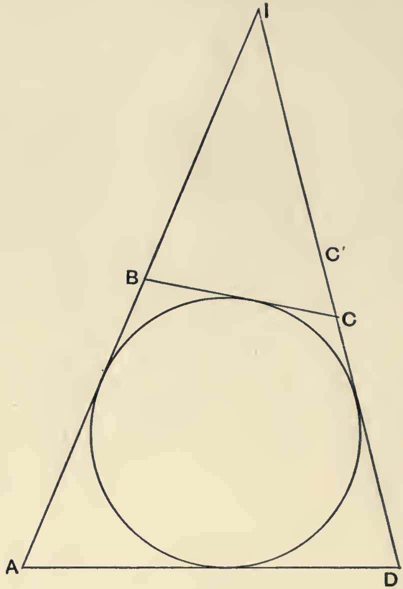

FIG. 92.

$$AB + C'D = BC' + DA.$$

Also $\qquad AB + CD = BC + DA$ (why?).

Whence $\qquad CD - C'D = BC - BC'$,

or $\qquad CC' = BC - BC'$.

Hence C and C' fall together (why? Art. 56). Q. E. D.

136. **Theorem LXXV.** — *The tangent-length from a vertex of a $\triangle$ to the in-circle equals half the perimeter of the $\triangle$ less the opposite side* (Fig. 93).

Proof. For the sum of $CE + CD + BD + BF$ is plainly $2a$ (why?); subtract this from the whole perimeter, $a + b + c$, and there remains $AE + AF = a + b + c - 2a$, or $AE = \frac{b + c - a}{2} = AF$. Q. E. D.

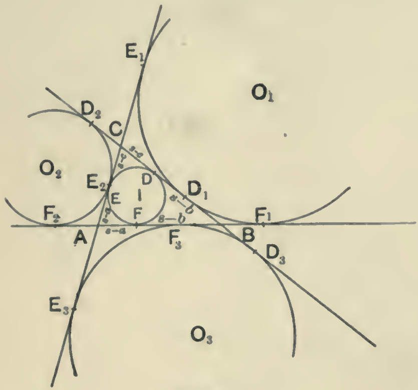

FIG. 93.

It is common and convenient to denote the **perimeter** (Fig. 93) (= measure round = sum of sides) by $2s$; then we see that the tangent-lengths from A, B, C, are $s - a$, $s - b$, $s - c$.

Corollary. The tangent-length from any vertex, A, of a $\triangle$ to the opposite ex-circle and the two adjacent ex-circles are s, $s - b$, $s - c$. Hence $s - a$, s, $s - b$, $s - c$, are the four tangent-lengths from any vertex, A, of a $\triangle$ to the in-circle and the three ex-circles.

These relations are useful and important.

137. **Theorem LXXVI.** — *There is a regular n-side.*

Proof. For the angle is a continuous magnitude (why?); hence there are angles of all sizes from zero to a round angle; hence there is an angle, the $\frac{1}{n}$ part of a round angle, such that, taken n times in addition, the sum will be a round angle. *Suppose* such an angle drawn, *whether or not we can actually draw it*, and suppose n such angles placed consecutively around any point O, so as to make a round angle. In other words, suppose n half-rays drawn cutting the round angle about O into n equal angles. Draw a circle about O, with (Fig. 94) any radius, and draw the n chords

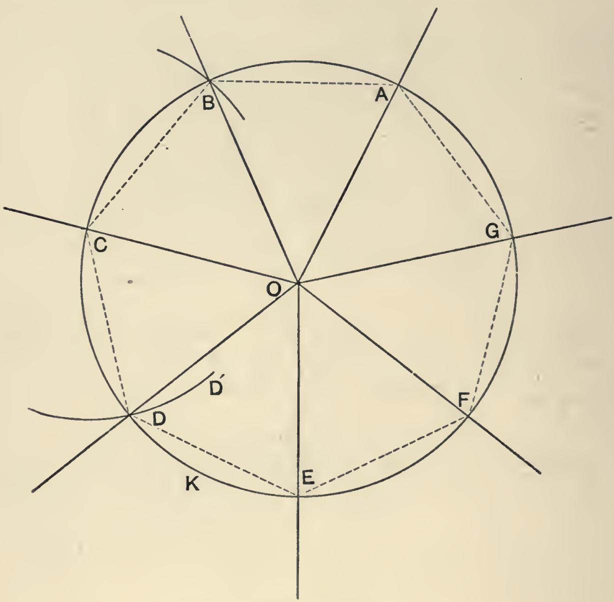

Fig. 94.

subtending the n equal central angles. These chords are all equal (why?), and subtend equal arcs, and they form an n-side. Moreover, the angle between two consecutive sides is constant in size, because it stands on the $\frac{n-2}{n}$ part of the circle. Hence the n-side is both equilateral and equiangular; that is, it is regular. Q. E. D.

Corollary. The inner angle of a regular n-side is the $\left(\frac{n-2}{n}\right)$ part of a straight angle.

Find the value in degrees of the inner angles of the first ten regular n-sides.

N.B. The foregoing demonstration merely settles the question of the existence or *logical possibility* of the regular n-side. The problem of *actually drawing* such a figure is one of the most intricate in all mathematics, and has been solved only for certain very special classes of values of n. But in order to discover the properties of the figure, it is by no means necessary to be able to draw it accurately. It is only since 1864 that we have known how to draw a straight line or ray exactly.

137 a. **Theorem LXXVII.** — *The vertices of a regular n-side are encyclic* (Fig. 94).

Proof. Through any three vertices, as A, B, C, of a regular n-side, draw a circle K; about C with radius CB draw another circle. The fourth vertex D must lie on this circle (why?). If it lie on the circle K, then the angle $BCD =$ angle ABC, as is the case in the regular n-side. Neither *can* it lie off of K, as at D' or D'', because then the angle BCD' or BCD'' would not equal angle BCD (why?), and hence would not equal angle ABC. Hence the next vertex must lie on the same circle K, and so on all around. Q. E. D.

138. **Theorem LXXVIII.** — *The sides of a regular n-side are pericyclic* (that is, they all touch a circle).

Proof. For, on drawing the radii of the circumcircle K (Fig. 95) to the vertices, we get n congruent symmetric △

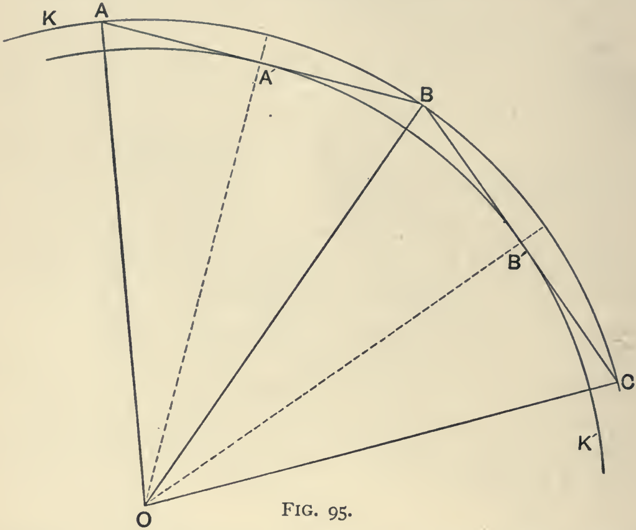

FIG. 95.

(why?). The altitudes of all are the same (why?); with this common altitude as radius draw another circle, K', about the same centre. It will touch each of the sides (why?). Q. E. D.

Corollary. The points of touch of the sides of the regular circumscribed n-side are mid-points of the sides.

139. **Theorem LXXIX.** — *The points of touch of a regular circumscribed n-side are the vertices of a regular inscribed n-side.*

Proof. Connect the points of touch consecutively. Then the △ so formed are all congruent (why?); hence the joining chords are equal; hence the arcs are equal; hence the Theorem. Q. E. D.

THE CIRCLE AS ENVELOPE.

***140a.** Thus far we have regarded the circle from various points of view; from the most familiar it was seen to be the locus of a point in a plane at a fixed distance from a fixed point. An almost equally important conception of the curve treats it not as the *locus* of a point, but as the **envelope** *of a ray*. If the point P moves in the plane always equidistant from O, then its locus is the circle, on which it may always be found; also, if the ray R moves about in the plane always equidistant from O, then its *envelope* is the circle, on which it may always be found, on which it lies, which it continually touches. The point traces the circle, the ray envelops the circle, which is accordingly called the *envelope* (*i.e.* the *enveloped* curve — French *enveloppée*) of the ray. In higher mathematics the notion of the ray, instead of the point, as the determining element in the nature of a curve, attains more and more significance. In this text we are confined to the circle — the envelope of a ray in a plane, at a fixed distance from a fixed point.

***140b.** It is not only rays, however, that may envelop a curve; but circles, and in fact any other curves. Thus, let the student draw a system of equal circles, having their centres on another circle; the envelope will at once be seen to be a pair of concentric circles. Let him also find the envelope of a system of circles equal and with centres on a given ray. In general, let him find the envelope of a circle whose centre moves on any given curve. Lastly, let him draw a large number of circles all of which pass through a fixed point, while their centres all lie on a fixed circle, and let him observe what curve they shadow forth as envelope.

Show that as the pole of a chord (or ray) traces a circle, the chord itself envelops a concentric circle, and conversely.

Show that tangents from two points on a centre ray form a kite, and conversely. Also the chords of contact are parallel, and conversely.

O is the centre of a circle, P any point without it. Show how to find the point of touch of the tangents from P, by drawing a circle about O through P and a tangent where OP cuts the given circle.

CONSTRUCTIONS.

140. Hitherto, in our reasoning about concepts, figures have not been at all necessary, though exceedingly useful in making sharp and precise our imagination of the relations under consideration, in furnishing sensible examples of the highly general notions that we dealt with. The conclusions reached thus far all lie wrapt up in axioms and in our definitions of point, ray, and circle, and our work has been one of explication only; we have merely brought them forth to light. Our demonstrations have not presumed ability to draw accurately, and would remain unshaken if we could not draw at all. Nevertheless, for many practical purposes, it is extremely important and even indispensable that we actually make the constructions and draw the figures that thus far we have merely supposed made and drawn.

141. What is meant by drawing a ray, circle, or any line? Any mark, whether of ink or chalk, though a solid, may be treated as a line by abstraction. Only its length, not its width nor thickness, concerns us. How to make not just any mark, but some particular mark called for, is our **problem** ($\pi\rho o\beta\lambda\eta\mu\alpha$ = anything thrown forward as a task), and its solution consists accordingly of two parts, the logical and the mechanical. The first is accomplished by fixing exactly in thought the position of all the geometric elements (points, rays, circles) in question ; the second, by making marks that by abstraction may be treated as these elements. Now, a point is fixed as the join of two rays, a ray as the join of two points (by what axiom?) ; a circle is fixed or determined by its centre and radius (why?), or by three points on it (why?). Accordingly, when we know two rays through a point, or two points on a ray, or centre and radius, or three points of a circle, we know the point, or ray, or circle completely. The logical part of our work is finished, then, when we determine every point as the join of two known rays, every ray as the join of two known points, every circle as drawn through three known points or about a known centre with a known radius. The mechanical part of the solution requires us to put and keep a point in motion along a circle or a ray. Circular motion is brought about by the compasses already described (Art. 114), of which the shape is arbitrary, the necessary parts being merely a fixed point rigidly connected in any way with a movable point. But in the ruler one edge is supposed *made straight* to begin with, so that a pencil point gliding along it may trace a straight mark. Hence the use of the ruler is really illogical, since it *assumes* the problem of drawing a ray or straight line as already solved in constructing the straight edge. To say that, in order to draw a *straight line*, we must take a *straight edge* and pass a pencil point along it, is no better logically than to say that, in order to draw a circular line, we must take a circular edge and pass a pencil point along it. The question at once arises, How make the edge straight or circular in the first place? It was not until 1864 that Peaucellier won, though he did not at once receive, the Montyon prize from the French Academy by solving the thousand-year-old problem of imparting rectilinear motion to a point without guiding edge of any kind (Page ooo). But, though the ruler is logically valueless, it is practically invaluable, even after the great discovery of Peaucellier. Its edge being assumed as straight and of any desired length, and a pair of compasses of adjustable size being given, we now make the following **Postulates**:

I. *About any point may be drawn a circle of any radius.*

II. *Through any two points may be drawn a ray* (more strictly, a tract of any required length).

Corollary. On any ray from any point on it we may lay off a tract of any required length.

These are the only instruments used or postulates assumed in the constructions of Elementary Geometry.

142. The fundamental relations of rays to each other are two: Normality and Parallelism. Hence

Problem I. — *To draw a ray normal to a given ray.* Since there are many rays normal to a given ray, to make the problem definite we insert the limiting condition, **through a given point.** Two cases then arise:

A. *When the given point is on the given ray.* All we can do is to draw a circle about the point P. It cuts the ray at two points, A and A', symmetric as to P. Hence the midnormal of AA' is the normal sought. Hence any point on this normal lies on two circles of equal radius about A and A'. Hence (Fig. 96)

Solution. From the given point P lay off on the given ray two equal tracts PA, PA'. About A and A' draw two equal circles. Through their points of intersection draw their common chord. It is the normal sought.

Proof. For it is the mid-normal of AA', since it has two of its points equidistant from A and A', and P is the mid-point of AA'.

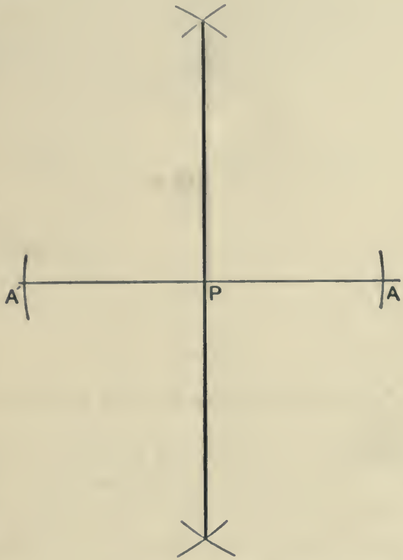

FIG. 96.

Query: What radius shall we take for the circles about A and A'?

B. *When the given point is not on the given ray.* All we can do is to draw a circle about the given point P. Let it cut the ray at A and A'. Then the mid-normal to AA' is the normal required (why?). Hence (Fig. 97)

Solution. Determine the points A, A' on the ray by a circle about the given point P; then proceed as in the first case (A).

Proof. For the mid-normal of AA' goes through P (why?).

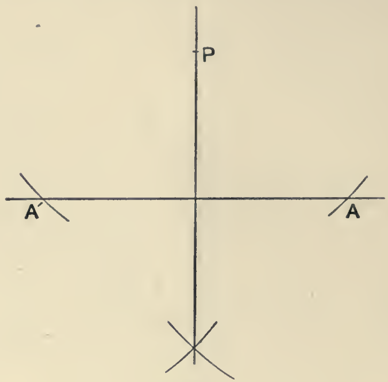

FIG. 97.

Query: What radius shall we take for the circle about P?

143. **Problem II.** — *To draw a parallel to a given ray.* Since there are many parallels to every ray, to make the problem definite we must insert the limiting condition, **through a given point**; then it becomes perfectly definite (why?). Manifestly the point must be not on the ray (why?). We now reflect that a transversal makes equal corresponding angles with parallels, and we have just learned to draw a normal transversal. Hence (Fig. 98)

Solution. Through the point draw a normal to the ray; through the same point draw a normal to this normal. It will be the parallel required.

Proof. For it goes through the point and is parallel to the ray (why?).

These two problems have been discussed at such length as being the hinges on which nearly all others turn. At the end of a problem is sometimes written Q. E. F. = *quod erat faciendum* = *which was to be done*, and translates the Euclidean ὅπερ ἔδει πρᾶξαι.

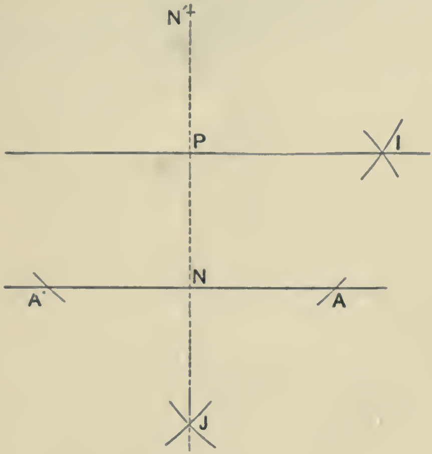

FIG. 98.

144. **Problem III.** — *To bisect a given tract,* or *to draw the mid-normal to a given tract, AB.*

Proceed as in Problem I.

145. **Problem IV.** — *To bisect a given angle.*

Solution. About the vertex draw any circle cutting the arms at A and A', and draw the mid-normal of AA'. It is the mid-ray sought (why?).

Corollary. Show how to bisect any circular arc AB.

146. **Problem V.** — *To bisect the angle between two rays whose join is not given* (Fig. 99).

We reflect that the join AA' of two corresponding points on the rays makes equal angles with the two rays that form the angle. Hence

Solution. From any point P of L draw the normal to it, cutting M at Q. From Q draw the normal to M. Bisect

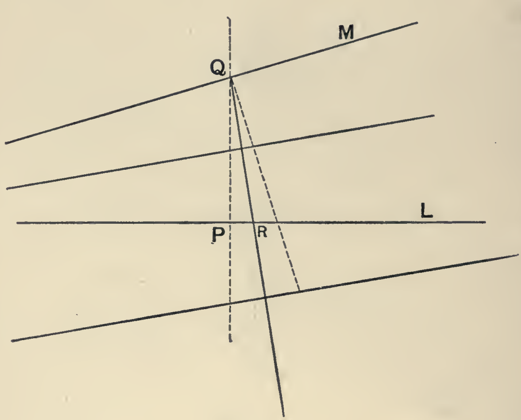

FIG. 99.

the angle at Q between these two normals by the mid-tract QR. Draw the mid-normal of QR. It is the mid-ray sought (why?).

147. **Problem VI.** — *To multisect a given tract AB* (Fig. 100).

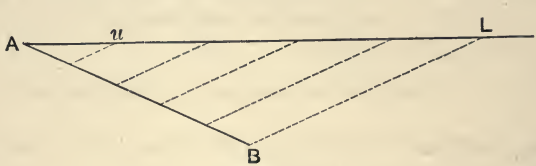

FIG. 100.

Solution. Through either end of the tract, as A, draw any ray, and lay off on it from A successively n equal tracts, L being the end of the last. Draw BL. Through the ends of the equal tracts draw parallels to BL. They cut AB into n equal parts (why?).

148. **Problem VII.** — *To draw an angle of given size, i.e. equal to a given angle* (Fig. 101).

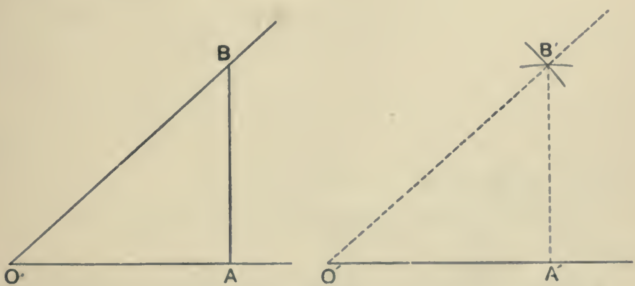

FIG. 101.

Solution. At any point A of either arm of the given angle O erect a normal to OA cutting the other arm at B. From any point O on any other ray lay off $O'A' = OA$, and normal to the ray erect $A'B' = AB$ and draw $O'B'$. Then angle O' = angle O (why?).

When does this construction fail? How proceed then?

149. **Problem VIII.** — *To draw a tract of given length subtending a given angle and parallel to a given ray.*

Data: O the given angle, L the ray, a the length (Fig. 102).

Solution. Through any point P of either arm of the angle draw a parallel to the ray, and lay off on it towards the other arm a tract PA of the given length a. Through A draw a parallel to OP, cutting the other angle arm at Q; through Q draw a parallel to PA meeting OP at R. QR is the subtense sought (why?).

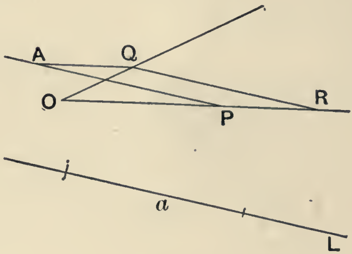

FIG. 102.

150. **Problem IX.** — To construct a △:

A. *When alternate parts* (three sides or three angles) *are given.*

Solution. About the ends of one side AB, with the other sides for radii, draw circles meeting in C. Then ABC is the △ sought (why?) (Fig. 103).

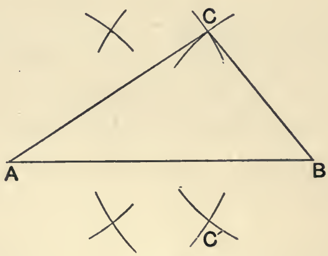

FIG. 103.

How many such △s may be drawn on the same base AB? How are they related? When is the solution impossible?

When the angles are given, apply the construction of Problem VII. How many solutions are possible? What kind of $\triangle$?

B. *When three consecutive parts* (two sides and included angle or two angles and included side) *are given.*

Solution. Apply the construction in Problem VII.

C. *When opposite parts* (two angles and an opposite side or two sides and an opposite angle) *are given.*

Solution. Apply the construction in Problem VII. When is the construction ambiguous?

D. *When two sides and the altitude to the third side are given.*

Solution. Through one end of the altitude draw a normal to it for the base; about the other end C as centre, with the sides as radii, draw circles cutting the base at A and A', B and B'; then ACB or $A'CB'$ is the $\triangle$ required. Why?

E. *When two sides and the medial of the third side are given.*

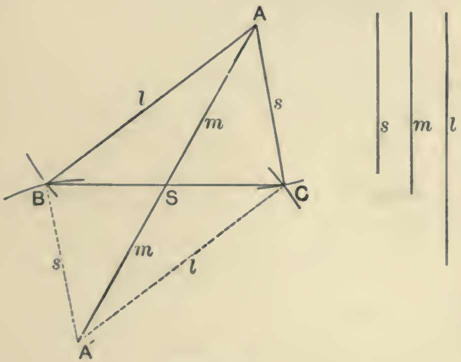

FIG. 104.

If SA be the medial of BC, and SA' be symmetric with (Fig. 104) SA as to S, then $ABA'C$ is a parallelogram (why?); hence

Solution. Take a tract the double of the medial. About its ends as centres with the sides as radii draw circles and then complete the construction. How many △ fulfilling the conditions are possible? How are they related?

F. *When the three medials are given* (Fig. 105).

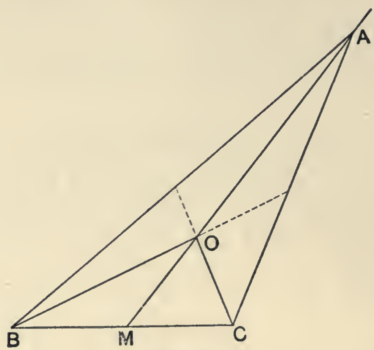

FIG. 105.

Solution. Remember that the medials trisect each other; construct the △ OBC according to (E), and draw OA counter to OM and twice as long.

151. **Problem X.** — *To construct an angle of given size and subtended by a given tract.*

Data: *O the given angle, AB the given tract* (Fig. 106).

Solution. Construct the angle BAD of given size (how?), draw the mid-normal of AB, meeting AD at P; also the normal to AD at A, meeting the mid-normal at S. About

S as a centre with radius SA draw a circle; it touches AD at A (why?). The vertex V of the required angle may be anywhere on the arc AVB or on its symmetric $AV'B$ (why?).

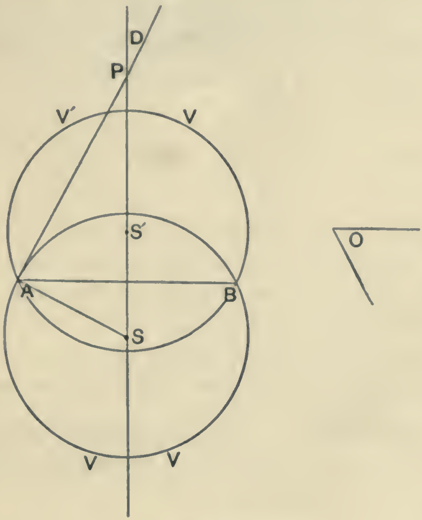

FIG. 106.

152. Problem XI. — *To draw a circle tangent to a given ray.*

Solution. About any point S with a radius equal to the distance of S from the ray, L, draw a circle; it will be a circle required (why?). If the circle must touch the ray L **at a given point** P, then S must be taken on the normal to L through P (why?). If, besides, the circle must go **through a given point** Q, then S must also be on the mid-normal of PQ (why?). Hence the construction.

153. **Problem XII.** — *To draw a circle touching two given rays.*

The centre may be anywhere on either mid-ray (why?). If now the circle is to touch a **third given ray,** the centre must be also on another mid-ray; that is, it must be the intersection of two mid-rays of the three angles of the three rays. There are four such intersections — what are they? Complete the construction. See Fig. 60.

154. **Problem XIII.** — *To draw a circle through two points.*

The centre S may be anywhere on the mid-normal of the tract AB between the points (why?), the radius is — what? If now the circle is to pass **through a third point** C, then S must also be on the mid-normal of BC and CA (why?). There is one, and only one, such point (why?); complete the construction. When is the construction impossible?

155. **Problem XIV.** — *To draw a circle through two given points and tangent to a given ray;* or, *tangent to two given rays and through a given point.*

This double problem is mentioned here because it must naturally present itself to the mind of the student; but the solution involves deeper relations than we have yet explored. See Art. 000.

Several of the foregoing problems were indefinite, admitting any number of solutions: these latter taken all together form a **system** or *family*. Problems concerning parallelograms and other 4-sides may often be solved on cutting the 4-side into two ⚠.

156. **Problem XV.** — *To inscribe a regular 4-side* (square) *in a circle* (Fig. 107).

Solution. Join consecutively the ends of two conjugate diameters. The 4-side formed is inscribed (why?) and is a square (why?).

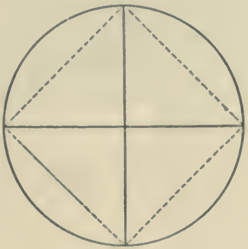

FIG. 107.

157. **Problem XVI.** — *To inscribe a regular 6-side in a circle.*

Solution. Apply the radius six times consecutively as a chord to the circle (Fig. 108). The figure formed will be the regular 6-side (why?).

FIG. 108.

N.B. This seems to have been one of the first geometric problems ever solved. The Babylonians discovered that six radii thus applied would compass the circle, and having already divided the circle into 360 steps, they accordingly divided this number by 6 and thus obtained 60 as the basis of the famous *sexagesimal* notation, which long dominated mathematics and still maintains its authority undiminished in astronomy and chronometry.

In more difficult problems it is often advisable, or in fact necessary, to **suppose the problem solved**, the construction made, and investigate the relations thus brought to light. Then the facts thus discovered may be used regressively in making the construction required. This method is illustrated in the following:

158. **Problem XVII.** — *To draw a square with each of its sides through a given point.*

Let A, B, C, D, be the four given points, and suppose (Fig. 109) $PQRS$ to be the square properly drawn. Draw

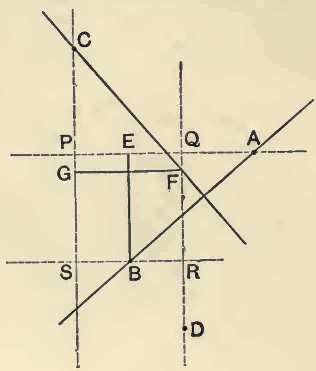

FIG. 109.

AB, cutting a side of the square, and through B draw BE parallel to the side cut. Through a third point C draw a normal to AB, meeting QR in F. Also draw FG parallel to PQ. Then the $\triangle$ ABE and CFG are congruent (why?).

Hence we discover that $CF = AB$. This fact is the key to the

Solution. Join two points A and B; from a third, C, lay off CF equal and normal to AB. The join of D, the fourth point, and F is one side of the square in position (why?). Let the student complete the construction and show that four squares are possible.

159. **Problem XVIII.** — *To trisect a given angle.*

Suppose the problem solved and the ray OT to make $\measuredangle\ TOB = 2\ TOA$ (Fig. 110).

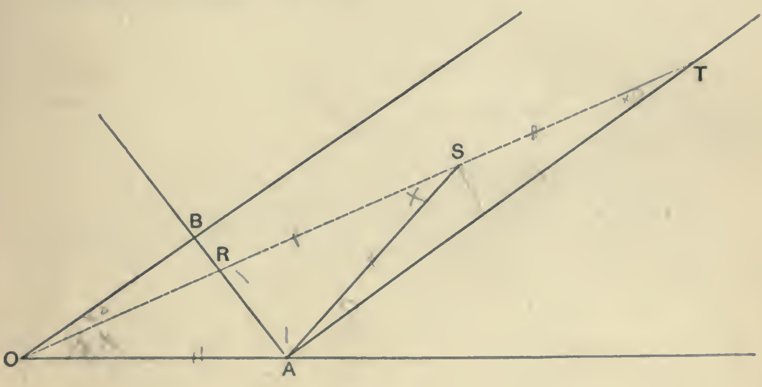

FIG. 110.

From any point A of the one end of the angle draw a parallel and a normal to the other end; also draw to the trisector a tract $AS = OA$. Then the following relations are evident:

$$\measuredangle\ AOS = \measuredangle\ ASO = \measuredangle\ SAT + \measuredangle\ STA;$$

but $\qquad \measuredangle\ AOS = 2 \measuredangle\ TOB = 2\ STA;$

hence $\qquad \measuredangle\ STA = \measuredangle\ SAT$, and $ST = SA$.

Again, $\measuredangle SAR = \measuredangle SRA$, being complements of equal angles; hence $SA = SR$, $\mathbf{TR} = 2\,\mathbf{OA}$. Hence

Solution. From any point A of either arm of the given angle draw a parallel and a normal to the other arm; then, with one point of a straight-edge fixed at the vertex O, *turn the edge until the intercept between the normal and the parallel equals* $2\,\mathbf{OA}$. But to do this we need a *graduated* edge, or a sliding length $2\,OA$ on the edge itself. Accordingly, this construction, while simple, useful, and interesting, is not *elementary geometric* in the sense already defined. To discover such a solution for this famous problem, has up to this time baffled the utmost efforts of mathematicians.

EXERCISES II.

1. State and prove the reciprocals of Exercises 9, 10, 11, page 71.

2. Find a point on a given ray, the sum of whose distances from two fixed points is a minimum.

3. The same as the foregoing, difference supplacing sum.

4. A and A', B and B', C and C', are symmetric as to MN. Show that $\triangle ABC \equiv \triangle A'B'C'$.

5. The inner and outer mid-rays of the basal angles of a symmetric $\triangle$ form a kite.

6. The inner mid-rays of the angles of a trapezium form a kite with two right angles.

7. The joins, of the mid-points of the parallel sides of an anti-parallelogram, with the opposite vertices, form a kite.

8. The mid-rays of the angles at the ends of the transverse axis of a kite cut the sides in the vertices of an anti-parallelogram.

9. How must a billiard ball be struck so as to rebound from the four sides of a table and return through its original place?

10. Trace a ray of light from a focus P, to another given point Q, reflected from a convex polygonal mirror.

11. A ray of light falls on a mirror M, is reflected along S to a second mirror M', is thence reflected along T. Remembering that the angle of incidence equals the angle of reflection, show that the angle between the original ray R and its last reflection T is twice the angle between the mirrors (angle $RT = 2$ angle MM'). On this theorem is grounded the use of the **sextant**.

12. Two mirrors stand on a plane and form an inner angle of 60°; a luminous point P is on the mid-ray of this angle (or anywhere within it); how many images of P are formed? How are they placed? What if the angle of the mirrors be $1/n$ of a round angle?

This theorem is beautifully illustrated in the **kaleidoscope**.

13. A regular n-side has n axes of symmetry concurring in the centre of the n-side, which centre is equidistant from the sides of the n-side.

14. How do these axes lie when n is even? when n is odd? Show that if n be even, the centre is a centre of symmetry.

15. The half-rays from centre to vertices of a regular n-side form a regular pencil of n half-rays, and those from the centre normal to the sides, another regular pencil; also the half-rays of each pencil bisect the angles of the other.

16. In a figure with two rectangular axes of symmetry each point, with three others, determines a rectangle, and each ray, with three others, a rhombus.

17. Find the axes of symmetry of two given tracts.

18. A regular $\triangle$, along with the figure symmetric with it as to its centre, determines a regular 6-angle (6-pointed star).

19. Two congruent squares, the diagonals of one lying on the mid-parallels of the other, form a regular 8-angle; also find the lengths of the intercepts at the corners.

20. The outer angle of a regular n-side is m times the outer angle of a regular mn-side.

EXERCISES III.

1. A circle with its centre on the mid-ray of an angle makes equal intercepts on its arms.

2. Tangents parallel to a chord bisect the subtended arcs, and conversely.

3. Tangents at the end of a diameter are parallel.

4. A and B are ends of a diameter, C and D any other two points of a circle; E is on the diameter, and angle $AED = 2$ angle CAD; find the possible positions of E.

5. From n points are drawn $2n$ equal tangent-lengths to a circle; where do the points lie?

6. In a circumscribed hexagon, or any circumscribed $2n$-side, the sums of the alternate sides are equal.

7. If the vertices of a circumscribed quadrangle, hexagon, or any $2n$-side, be joined with the centre of the circle, the sums of the alternate central angles will be equal.

8. The sums of the alternate angles of an encyclic $2n$-side are equal, namely, each sum is $(n - 1)$ straight angles.

9. The joins of the ends of two parallel chords are symmetric as to the conjugate diameter of the chords.

10. A centre ray is cut by two parallel tangents. Show that the intercepts between tangent and circle are equal.

11. Normals to a chord from the ends of a diameter make, with the circle, equal intercepts on the chord.

12. The joins of the ends of two diameters are parallel in pairs, and form a rectangle, and meet any two parallel tangents in points symmetric in pairs as to the centre.

13. The joins of the ends of two parallel chords meet the tangents normal to the chords in points whose other joins are parallel to the chords.

14. A chord AB is prolonged to C, making $BC =$ radius, and the centre ray CD is drawn; show that one intercepted arc is thrice the other.

15. The intercepts, on a secant, of two concentric circles are equal.

16. A chord through the point of touch of two tangent circles subtends equal central angles in the circles.

17. Two rays through the point of touch of two tangent circles intercept arcs in the circles whose chords are parallel.

18. The transverse joins of the ends of parallel diameters in two tangent circles go through the point of tangence.

19. Four circles touch each other outerly in pairs: 1st and 2d, 2d and 3d, 3d and 4th, 4th and 1st; show that the points of touch are encyclic.

20. Show that three circles drawn on three diameters OA, OB, OC intersect on the sides of the $\triangle ABC$.

21. Find the shortest and the longest chord through a point within a circle.

22. In a convex 4-side the sum of the diagonals is greater than the sum of two opposite sides, less than the sum of all the sides, and greater than half the sum of the sides.

23. Three half-rays trisect the round angle O; on each is taken any point, as A, B, C. Find a point M such that the sum $MA + MB + MC$ is the least possible (a minimum).

24. Two tangents to a circle meet at a point distant twice the radius, from the centre; what angles do they form?

25. The intercept of two circles on a ray through one of their common points subtends a constant angle at the other.

26. What is the envelope of equal chords of a circle?

27. Two movable tangents to a circle intersect under constant angles; find the envelope of the mid-rays of these angles.

28. The vertex V of a revolving right angle is fixed midway between two parallels, and its arms cut the parallels at A and B; find the envelope of AB.

29. From a fixed point P a normal PN is drawn to a movable tangent T of a circle, and through the mid-point M of PN there is drawn a parallel to T; find its envelope.

30. The vertices of a $\triangle$ are V_1, V_2, V_3; the mid-points of its sides are M_1, M_2, M_3; the feet of its altitudes are A_1, A_2, A_3; the inner bisectors of its angles meet the opposite sides at B_1, B_2, B_3; and the outer bisectors at B'_1, B'_2, B'_3; its centroid is C, its in-centre is I, its circumcentre is S; its angles are α_1, α_2, α_3, and their complements are α'_1, α'_2, α'_3. Express through these six angles the angles between: (1) V_1A_1 and V_1V_2; (2) A_1A_2 and V_2V_3; (3) A_1A_2 and V_1A_1; (4) A_1A_2 and A_2A_3; (5) M_1A_2 and V_3V_1; (6) M_1A_2 and V_2V_3; (7) M_1A_2 and M_1A_3; (8) A_1M_2 and A_1M_3; (9) A_1M_2 and A_1A_2; (10) SV_1 and SV_2; (11) SV_1 and V_1V_2; (12) SV_1 and V_2A_2; (13) IV_1 and IV_2; (14) IV_1 and V_2A_2; (15) VB'_1 and $V_2B'_2$.

31. Find the locus of the mid-points of chords through a fixed point upon, within, or without a fixed circle.

32. Find the locus of the mid-points of the intercepts of a secant between a fixed point and a fixed circle.

33. As the ends of a ruler slide along two grooves normal to each other, how does its mid-point move?

34. Two equal hoops move along grooves normal to each other and touch each other; how does the point of touch move?

35. Orthocentre O, centroid C, circum-centre S, and centre F of Feuerbach's (9-point) circle, of a $\triangle$ are collinear, and $OC = 2\,CS$ (Euler), $OF = FS$.

36. Two parallel tangents meet two diameters of a circle at the vertices of a parallelogram concentric with the circle.

37. The inner mid-rays of the angles of a 4-side form an encyclic 4-side.

38. The outer mid-rays of the angles of a 4-side form an encyclic 4-side. How are the 4-sides of 37 and 38 related?

39. The circum-centres of the four $\triangle$s into which a 4-side is cut by its diagonals are the vertices of a parallelogram.

40. The circum-centres of the two pairs of $\triangle$s, into which a 4-side is cut by its diagonals in turn, are how related to each other and to the centres in 39?

EXERCISES IV.

1. Construct a square, knowing

(*a*) Its side; or (*b*) its diagonal.

2. Construct a rectangle, knowing

(*a*) Two sides; or (*b*) a side and a diagonal; or (*c*) either a side or a diagonal and the angle of either with the other; or (*d*) a diagonal and its angles with the other diagonal.

3. Construct a parallelogram, knowing

(*a*) Two sides and one angle; or (*b*) a side, a diagonal, and the included angle; or (*c*) two sides and the opposite diagonal; or (*d*) two sides and the included diagonal; or (*e*) two diagonals and a side; or (*f*) two diagonals and their angles with each other.

4. Construct an anti-parallelogram, knowing

(*a*) Its parallel sides and the distance between them; (*b*) two adjacent sides and their included angle; (*c*) two adjacent sides and the angle between the non-parallel sides; (*d*) a diagonal and two adjacent sides; (*e*) a diagonal, a side, and the included angle.

5. Construct a kite, knowing

(*a*) Two sides and an axis; (*b*) two sides and the included angle; (*c*) a side and the axes.

6. Construct the rays equidistant from three given points.

7. Draw a ray through a given point equally sloped to two given rays.

8. A square has one vertex at a given point, and two others on two given parallel rays; draw it.

9. Hypotenuse and sum of sides of a right $\triangle$ are given; draw it.

10. Construct a regular 2^n-side, and a regular 3.2^n-side.

11. Find the centre of a given circular arc.

12. Trisect a right angle.

13. Two points, A and B, of a ray are given; find any number of points of the ray without drawing it, and without opening the compasses more than AB.

14. Find a point on a given ray or given circle that has a given tangent-length with respect to a given circle.

15. Through a given point draw a secant on which a given circle shall make a given intercept.

16. **Draw four common tangents to two given circles.**

17. Draw a ray touching a given circle and equidistant from two given points.

18. Draw a ray on which two given circles shall make two given intercepts.

19. With three given radii draw three circles, each touching the other two.

20. Draw a circle touching the radii and the arc of a given sector.

21. Draw a circle touching two given equal intersecting circles and their centre ray.

22. On the central intercept of two equal intersecting circles as diameter draw a circle; then draw a circle touching the three circles.

23. Three equal circles touch each other outerly; draw a circle touching the three.

24. Find a point from which two given apposed tracts appear to be equal.

25. Through two given points of a given circle draw a circle that shall cut a third circle orthogonally.

26. Construct a $\triangle$, knowing

(a) The feet of the altitudes; (b) the foot of one altitude and the mid-points of the other two sides; (c) the three ex-centres; (d) two ex-centres and the in-centre.

27. Draw through a given point a ray that shall form with the sides of a given angle a $\triangle$ of given perimeter. Hint: Use the properties of ex-circles.

28. Draw a 5-side, knowing the mid-points of the sides.

29. On a tract AB there is drawn a regular 3-side; draw on it a regular 6-side. Generalize the problem, changing 3 into n, 6 into $2n$, and solve it.

30. Given a regular n-side; draw a regular $2n$-side having the original n vertices for alternate vertices. Do not use the circumcircle.

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