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Wage-Risk Premiums and Workmen's Compensation: A Refinement of Estimates of Compensating Wage-Differential

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Inventory Investment and the Theory of the Firm

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Abstract

This paper presents a model of the joint pricing and product inventory decisions of a monopolistic firm in a continuous time framework. The incentive to hold inventory comes from firm operation in a region of declining average production cost. It is shown that even when firm demand and production cost schedules are stationary, the optimal production and pricing schedule is cyclic. A welfare analysis is presented where it is shown that consumer surplus for this firm may be either smaller or larger than for the firm which doesn't store inventory. In the latter case, the monopolist may be producing in excess of the competitive level.

Inventory Investment and the Theory of the Firm[1]

by

Lanny Arvan and Leon N. Moses

This paper presents a model of investment in inventories of finished products. We believe the approach taken differs in significant ways from existing models. The dynamic behavior it depicts may also provide new insights into aspects of welfare economics, and some of the impacts of monetary policy on certain kinds of firms. The approach is basically a very simple extension of one of the most familiar models in static economics, the model of a firm with a downward sloping demand curve and an average cost function that declines over some range of output. It is shown that this model is at its heart dynamic. The above two conditions provide the firm with a powerful incentive to accumulate inventories even if its demand and cost functions are known with certainty and are completely unchanging over time. Some condition outside the firm, such as randomness or temporal change in demands or costs, provides the essential starting point for every other production-inventory model of which we are aware. This is not true of the present model, though it is fully capable of analyzing situations in which there is randomness or temporal change.

The paper is divided into two parts. The model is introduced and its general workings are discussed in qualitative terms in Part One.

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The relationship of the model to other work in the area is explored. This part of the paper also discusses some of the weaknesses of that work, the fact that the vast majority of the models do nothing with prices and are even ambiguous on the question of whether the firms they study are monopolists or perfect competitors. The basic position taken here is that while the inventory literature is immensely rich and varied, it departs too far from the main constructs and traditions of the theory of the firm.

The model is presented in more quantitative terms in Part Two. The techniques developed in this presentation are then used to investigate two policy issues: the effect of variations in the interest rate on the output cycle of a firm that is behaving dynamically and the prices it charges; the implications of such dynamic behavior for consumer welfare.

Part One: A Qualitative View

In order to make clear the basic workings of the model, let us assume a monopolistic firm with stationary demand and cost functions. Whether due to scale economics in the long run or variable proportions in the short run, the firm's average cost curve declines over some range of output. We are to determine the optimal path over time of the price this firm charges for its product, and the quantities it produces, sells, and carries in inventory.

Our approach is to construct an intertemporal profit function. Future sales are related to current production through the introduction of an inventory capability. The profit function is maximized using standard techniques from optimal control theory. We show that there are realistic circumstances under which a monopolistic firm's ability to store the

product allows it to choose a path that is more profitable than the one in which instantaneous marginal cost and marginal revenue are continually equated. For convenience we will refer to the path in which the firm continually maximizes instantaneous profit as the static strategy, it being understood that production and sales occur through time, not just at points of time.

Instead of following the static strategy the firm adopts a dynamic strategy of producing in excess of current sales and accumulating inventory over a certain time interval. During this time interval, the instantaneous profits that are earned are smaller than they would be with the static strategy. In fact, the firm may have losses. At some point the plant is shut down. This marks the beginning of a second time interval in which sales are made from accumulated inventory. The firm earns profits in this time interval that more than compensate it for the earlier expenses associated with inventory accumulation. Production is resumed when inventory is depleted and a second cycle is begun. The resumption of production after a down period may entail start-up costs. As part of its dynamic strategy the firm may begin each cycle with a certain price and raise it continuously over the cycle.

Figures 1 and 2 illustrate the workings of the model when the firm is free to vary price in the above way and the initial rate of output is where marginal costs of production is at a minimum. We will shortly make clear that this output rate is not the only possible starting point. In the Figures, $y(0)$ is sales at time zero; T is the time when inventory is exhausted and $y(T)$ is sales at that time; $q(0)$ is the starting rate of output; $q(T^*)$ is the rate of output at the instant when production ceases and inventory decumulation begins.

It should be noted that the dynamic strategy can yield greater overall profits than the static even if oligopolistic conditions, fear of entry, regulation, or certain aspects of consumer behavior force the firm to adhere to a policy of a single price over each cycle. The consumer behavior we refer to involves learning. Consumers come to understand the firm's dynamic pricing strategy. They may then counteract it in several ways. The most obvious way is to buy in excess of current use early in the cycle when price is low, store the product and consume from stock when price is high. This will occur if the prices the firm charges over time differ by more than the sum of storage and interest charges consumers incur in stocking the commodity themselves. When the above conditions are met, consumers may also counteract the firm's pricing policy by selling part of their stock when price is high. In effect they become competitors of the firm and further limit its ability to vary prices. Instead of, or in addition to investing in the good, consumers can carry wealth in more general forms, saving more when price is high and buying more later when price is low.[2]

The pricing aspect of the firm's dynamic strategy involves complications that are related to the welfare economics issues considered in Part Two. At this point we simply make the point that consumer reactions to the variable price policy raise questions about the meaning of a stationary demand function and the nature of the dynamic demand sur-

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[2] The firm could counteract these consumer strategies by putting some randomness into its price cycle. Investigation of this kind of gamesmanship between the firm and its customers is beyond the scope of this paper.

Figure 1

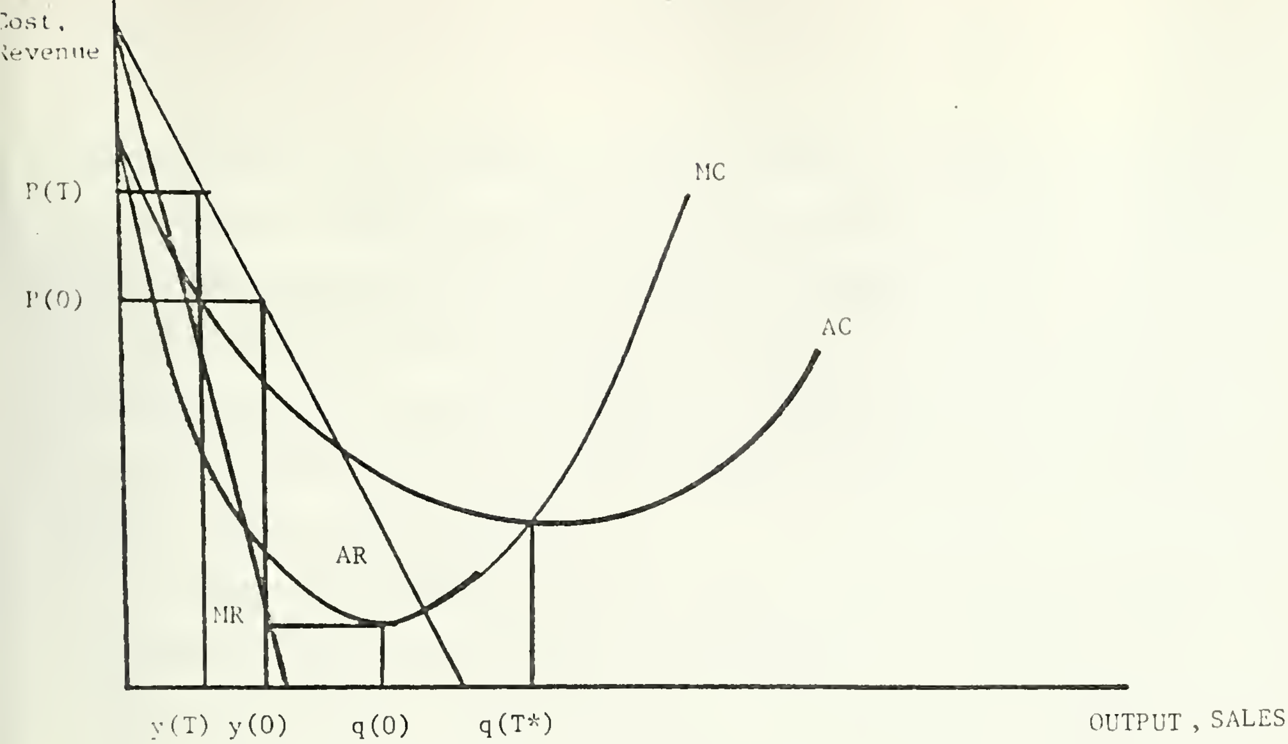
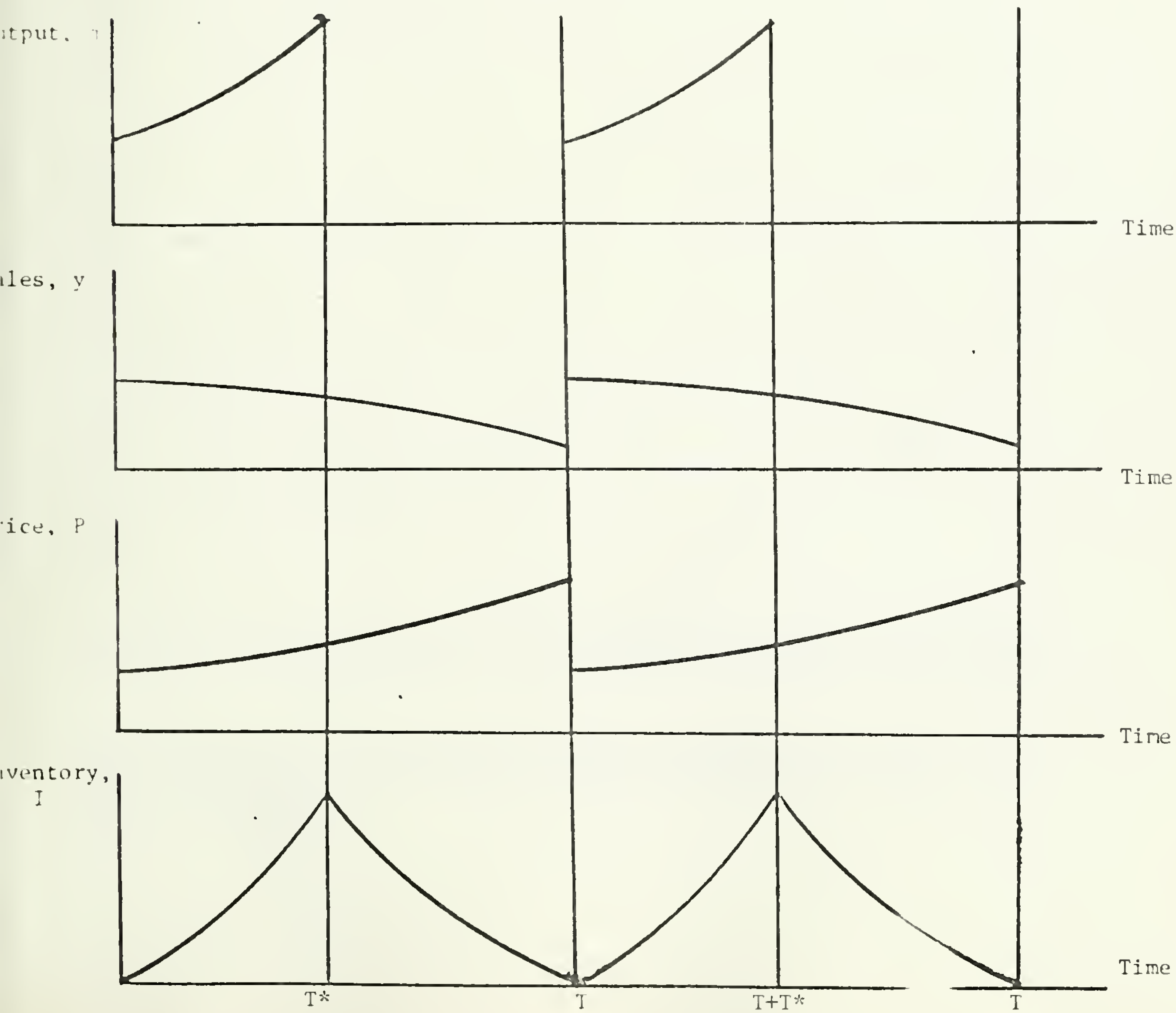


Figure 2: Time Paths of Production, Sales, Price and Inventory.



face with which we are working. However, it must be remembered that the stationarity assumption is only being made in this introductory section as a way of emphasising the source of the model's dynamic behavior. Also, it is worth repeating that so long as the firm is not a perfect competitor, the production and inventory accumulation - decumulation aspects of the model stand even if it is only free to choose one price to which it must adhere over an entire cycle.

There are either two or three necessary conditions for the dynamic strategy to emerge as the optimal policy, depending on the form of the firm's cost function. First, obviously, the product must be capable of being stored. Second, the firm's average cost curve must decline over some range of output. If the firm's cost function eventually turns up and is U-shaped, either because of diseconomies of scale in the long run or decreasing average productivity of variable factors in the short run, there is a third necessary condition.[3,4] Instantaneous marginal cost must equal instantaneous marginal revenue at an output rate that is smaller than the one at which average cost is a minimum. If instantaneous marginal cost and marginal revenue intersect above minimum average cost, profits are maximised by adopting the static strategy of continually equating marginal cost and marginal revenue and always satisfying current demand from current production.

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[3] There is obviously a question of whether the model is best suited for short or long run analysis. This issue is considered at the end of this part of the paper.

[4] If the average cost curve is declining throughout there are better policies than the static policy but there is no optimal solution. The firm would like to begin production near its efficient scale which in this case is infinite.

If these costs are too high, either individually or in combination, then the static strategy will be optimal. Concerning start-up costs, it should be mentioned that they are a kind of fixed cost. They act to lengthen the period of production and inventory accumulation and reduce the number of shut-downs over time.

If the dynamic strategy is optimal, how are the rates of output and sales selected? The answer depends on the relationship between instantaneous marginal cost and marginal revenue. If they intersect on the rising portion of the marginal cost curve, the initial rate of output is the rate at which this intersection occurs and the price selected is the market price. Suppose the intersection of instantaneous marginal cost and marginal revenue occurs where marginal cost is falling. The initial rate of output is then the rate at which marginal cost is a minimum. A smaller quantity is sold so that inventory accumulation begins immediately in this case. The quantity sold is determined by the condition that the marginal revenue it entails is equal to minimum marginal cost. The initial price is dictated by the demand curve. Again, price is increased through the cycle.

An interesting situation arises if there is no point of intersection of marginal cost and marginal revenue. Indeed, let us make the case even stronger than the one normally used in static theory to explain when a commodity is not produced. That is, suppose average revenue is everywhere below not just average cost but also below marginal cost. A dynamic

strategy can still yield profits if there are any quantities that can be sold at a price that exceeds minimum average cost, i.e., if the price intercept of the demand curve is above minimum average cost. Whether or not profits can be earned depends on storage and interest charges. If production is profitable, the initial rate of output is where marginal cost is a minimum. The initial sales rate is that quantity at which marginal revenue is equal to minimum marginal cost. The initial price is of course the price at which that quantity can be sold. This case is related to some broader issues in welfare economics that are taken up in Part Two. Let us now consider how the approach described above relates to the production - inventory literature.

The modern analytical approach to inventory behavior began with the work of Arrow [1,2,3], Dvoretzky [5,6,7], Hadley [8], Holt [10], Mills [15], Modigliani [16], Wagner [21], and Whitin [22]. These scholars were experts in the theory of the firm. Yet, in the inventory area their work departed from the main constructs and traditions of that theory, in some instances on the demand side, in some on the cost side, and in some on both. As to demand, the models published through the mid 1950's all assumed that the firm sold a fixed amount of finished product in each period, though that quantity could vary deterministically over time, reflect the workings of a random distribution, or be uncertain. The vast majority of the models also imposed a variety of costs on the firm if it failed to meet the fixed demands. Frequently these were stockout costs, penalties the firm experienced that went beyond losses in current revenue. Failure to

meet demand was seen as possibly causing a loss of customer goodwill and a reduction in the quantity of product the firm might sell in the future, though the models did not actually make the distribution of future demand a function of previous stockouts. Some models permitted unfilled orders to be backordered, sometimes at penalty costs of production because overtime labor and emergency operations might be required. Costs of the above kinds indicate clearly that the firms being dealt with in the models were not perfect competitors. Yet they ignored price as a choice variable! They did not have firms raise price as a way of reducing demand and stockouts; neither did the models have firms cut price in order to sell more and reduce excess inventory.

Wagner and Whitin [21] appear to have been the first to introduce a downward sloping demand curve into a model that had storage, interest, and other costs usually found in inventory studies. After publication of this paper a number of authors constructed models in which price was determined. By all standards, one of the clearest expositions of the role of demand is that of Mills [15] who puts into clear relief the difference between monopolistic and perfectly competitive firms in their motivations for holding inventories. Karlin and Carr [11] also treat price as a choice variable, make quantity demanded a random variable that depends on price, and examine the effects of uncertainty. Thomas [19] examines the joint production-price decision of a firm that faces a deterministic demand function that is different in each of T periods. The model is interesting in that it implies concave costs, an issue that will be discussed later in relation to the work of Lundin [14] who explicitly assumes

such costs. Kunreuther and Richard [12] consider the price and inventory decision of a retail firm that incurs a fixed cost in ordering goods and constant marginal costs of purchasing and holding them. Since demand is stationary, a single price is determined. A similar problem is taken up in Kunreuther and Schrage [13] but demand is seasonal. It is assumed that the firm wishes to charge a single price, and the optimal price is determined. In his model, Vanthienen [20] also determines a single best price. He has linear holding costs, but assumes convex production costs and also has adjustment costs for rates of increase or decrease in production. He finds that even when the quantity intercept of the demand function and the coefficient on price vary with time, the overall production scheduling problem has to be solved only once to determine the optimal price. Cohen [4] deals with the simultaneous production and price setting problem for a product that decays exponentially. The way in which perishability interacts with optimal price and ordering decisions is examined. Most models are presented in a discrete time framework. However, Pekelman [17] is a continuous time version of Wagner-Whitin. It does not have setup costs. Since convex production costs are assumed, the incentive to hold product inventories is due entirely to variations in demand.

Given that the Wagner-Whitin paper was published in 1957, it is surprising to find that the number of studies in which price is taken into account is still quite small. In this regard it is worth noting that a recent, excellent and very comprehensive volume on inventory theory by Peterson and Silver [18] devotes only two of its approximately eight hundred pages to a review of models in which firms make price decisions.

Almost all production-inventory models preclude scale economies in the long run and the influence of changing factor proportions on the productivity of variable factors in the short run. They assume linear or quadratic total costs of production, with the latter appearing to be the favorite. Those models that deal with uncertain demand must assume a quadratic or a linear cost function in order to achieve certainty equivalence. For the rest, the explanation for the use of the simplest cost functions seems to be an absorption with a class of fixed charges such as setup and ordering costs.

The former are a carryover from old fashioned industrial engineering to modern management science. They are a lumpy cost that is incurred when the machinery of a plant is prepared for a run of a particular product. Since the firm is planning over time, it can choose the time between runs as well as the quantity produced in a run. In this sense setup costs are variable costs. The longer the time between setups, the lower the average setup cost per unit of output, but the higher the inventory carrying cost. The optimal policy depends on the height of storage, interest, and setup costs and the initial level of inventory.

Ordering costs are much the same as set-up costs, except they occur on the input side. The ordering of inputs involves clerical and other costs that are internal to the firm. The amount of such costs may be independent of the quantity being ordered. The firm can choose how frequently it orders as well as quantity ordered. Clearly, setup and ordering costs

are equivalent in discrete time models. One gives rise to inventories of finished products and the other to inventories of inputs.

Lumpy costs cause problems in both discrete and continuous time models. They make computations difficult in the former, though recent progress has reduced the difficulty.[5] The equivalence of setup and ordering costs does not hold in continuous time models. While they cause computational difficulties, ordering costs can be handled. Setup costs are a different matter. They call into question the existence of an optimum solution, which is why they are ignored in Part Two of the paper.

There are two additional points that we would like to make in this part of the paper. First, while our model has clear implications for short run behavior, it is necessary to be careful about long run interpretations because the model ignores costs of adjusting scale of plant. Such costs are a critical feature of most studies that explain how firms invest optimally as demand and cost conditions change over time. On the other hand, these investment models do not consider inventories. As Zabel [23] points out, "The assumption of zero inventories is a key assumption often made, implicitly or explicitly, in studies of capital theory or in the theory of the firm".[6]

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Lundin [14] has two theorems giving conditions when the problem of determining planning horizons for simultaneous price and production-inventory decisions can be decomposed in order to simplify computation. His model is a generalization of Thomas [19] in the sense that he explicitly assumes convex production and holding costs. Lundin does not characterize the paths of production, sales and price. Clearly, if he did they would be somewhat similar to ours. However, since his model is in discrete time, production is never positive when inventory is carried over from the previous period.

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As an example of an otherwise excellent volume in which inventories are ignored, see Hirshleifer [9].

Yet if the firm's product can be stored, the influence of inventory accumulation on capital investment and on optimal scale should be taken into account. Inventory capability can both allow the firm to operate with a smaller scale of plant and reduce the frequency with which it changes fixed plant. The development of a model that considers both kinds of investment would seem to be a worthwhile effort for future research. Second, our model has the plant shut down for a period of time. This may bother some readers. An obvious way out of the unreality of a shut down is to introduce multiple products. Instead of shutting down, the plant would shift over to another product at a certain point of time. This is what actually happens in many lines of industry.

By way of summary, we hope that our research gives scholars interested in inventory models more choice as to approach than they have had in the past. They can, following the suggestions of this paper, choose one based on the traditional economic model of production and costs. This will limit the extent to which demand uncertainty and costs that are lumpy in time can be treated. Alternatively, they can continue in what is the tradition of management science. This approach gives more scope for the introduction of uncertainty and lumpy costs, but limits the researcher to the simplest production cost functions.

Part Two: The Model

Consider a non-stochastic infinite horizon problem in which start-up costs are initially ignored, the interest rate is r and the other variables are:

$y(t)$, sales at time t;

$q(t)$, production at t;

$I(t)$, inventories at t;

$P(y,t)$, demand price at t when quantity sold is y;

$C(q,t)$, production cost at t when quantity produced is q;

$S(I,t)$, storage costs of inventory at t when inventory is I.

We make the following assumptions and note that subscripts mean partial derivatives:

(i) P, C, and S are twice continuously differentiable;

(ii) instantaneous demand curves are downward sloping, $P_y < 0$;

(iii) instantaneous total cost of production increases with output, $C_q > 0$;

(iv) storage cost increases with inventory, $S_q > 0$.

The firm's problem is

$$(1) \quad \underset{y,q}{\text{Maximize}} \quad \int_0^\infty e^{-rt}\left[P\big(y(t),\, t\big)y(t) - C\big(q(t),t\big) - S\big(I(t),t\big)\right]dt$$

subject to

$$(2) \quad \dot{I}(t) = q(t) - y(t).$$

Additionally, I, y, $q \geq 0$ for all t and there are no starting inventories, $I(0) = 0$.

To solve this problem we form the Hamiltonian:

$$(3) \quad H = e^{-rt}[P(\ )y - C(\ ) - S(\ )] + \lambda[q-y] + \eta I + \xi_1 y + \xi_2 q$$

The first order conditions are:

$$(4) \quad \frac{\partial H}{\partial y} = e^{-rt}[P_y y + P] - \lambda + \xi_1 = 0$$

$$(5) \quad \frac{\partial H}{\partial q} = -e^{-rt}C_q + \lambda + \xi_2 = 0$$

$$(6) \quad \frac{\partial H}{\partial \lambda} = \dot{I} = q - y$$

$$(7) \quad \frac{\partial H}{\partial I} = -\dot{\lambda} = -e^{-rt}S_I + \eta$$

There are also complementary slackness conditions.

We begin our examination of optimal policies by looking at what is optimal in a neighborhood of $t = 0$. The Pontryagin principle of optimality will indicate what is optimal at arbitrary times. Three cases are outlined below and examined in greater detail later with the aid of an example.

Case I: The Firm Does Not Exist

There will be no production, sales, and inventory accumulation if the highest price that can be charged for the product is less than the minimum average cost at which it can be produced. It will also be optimal for the firm to withdraw from the market if the current cost of producing each quantity and the inventory cost of transfering it to the future is greater than the discounted price that can be charged.

Case II: The Static Policy

Production and sales are positive and equal at each moment of time. There is no inventory accumulation. In this case, since the Lagrange multipliers for sales and production are necessarily zero, λ can be eliminated from the first order conditions. The result is that instantaneous marginal

cost equals instantaneous marginal revenue and profits over time are maximized by maximizing instantaneous profit at each point of time.

Case III: Dynamic Behavior Is Optimal

Production is greater than sales over certain time intervals and inventory is accumulated. The inventory Lagrange multiplier, η, equals zero in this case. Thus,

$$(8) \qquad \dot{\lambda} = e^{-rt} S_I .$$

Since $\xi_2 = 0$ if we differentiate equation (5), we get:

$$(9) \qquad \dot{C}_q = S_I + rC_q ,$$

meaning that marginal cost grows at an exponential rate to account for interest charges plus a constant rate due to storage costs. Sales are of course continuously positive in this case. Over those time intervals in which production and sales are positive, marginal cost is always equal to marginal revenue. However, the output and sales at which the equality holds are not equal and inventory is accumulated. As long as sales are positive, marginal revenue continues to grow at the rate given in equation (9) even when production falls to zero. This follows from the first order conditions.

Let us examine more closely what optimal policies look like in Case III. Such an examination is facilitated by making the simplifying assumption that the demand and cost functions, and the per unit inventory cost are stationary. Formally, we assume that $P_t \equiv C_t \equiv S_t \equiv 0$, recalling that subscripts refer to partial derivatives. Furthermore, we assume that the second order conditions for our problem are satisfied. Marginal revenue is downward sloping, and marginal cost is upward sloping over the range of output in which our optimal policy occurs. Finally it is assumed that marginal storage costs are non decreasing. These conditions guarantee sufficiency at

the optimum. Thus equations (4) - (9) implicity specify the "solutions" to our problem. It follows from equation (9) above that production increases through time. Since marginal revenue increases sales must fall and inventory is accumulated. Such accumulation can't be optimal forever because inventories are costly, both directly and in the sense of the foregone revenue they represent. At a certain point the firm starts to decumulate and, by the first order conditions, when that occurs production falls to zero. Sales must of course be positive for decumulation to take place. They continue to decline along the same path as they did when production was positive until inventories have been exhausted. By stationarity, the optimal policy when inventories are exhausted is identical to that at time zero. We complete this qualitative description by noting that because demand is downward sloping and sales fall through time, market prices rises through time until the point when inventories are exhausted.

The above verbal description suggests a bang-bang policy, i.e., a policy where our controls are allowed to take discrete jumps at certain points in time rather than be continuous everywhere. To calculate the optimal policy it is necessary to calculate both the value of the controls on each side of the discontinuity and the time at which the discontinuity occurs. The value of the controls after each jump is implied by the first order conditions and stationarity. When inventories begin to be decumulated, production jumps to zero. When inventories have been exhausted and accumulation resumes, sales and production both jump to the same levels as at time zero. The times of these jumps are implied by the conditions that require the Hamiltonian to be continuous everywhere, even at points of discontinuity in the controls. For example, at the time when production stops the contribution that production makes to pro-

its via inventories, λq, must equal the discounted cost of production. Finally note that the Lagrange multiplier λ may also be discontinuous at the time when inventories have been exhausted, i.e. when the firm feels the force of the non-negativity constraint on inventories. More specific consideration of the determination of both jump times and jump values is given in an example that appears later in the paper.

At this stage something should be said about the uniqueness of the optimal policy. To do this we utilize the value function $V(I,t)$, which is the value of the objective from time t to infinity when $I(t) = I$. By stationarity we know that if a plan is optimal at t it is also optimal at all τ such that $I(\tau) = I$. This implies that our policy is not unique because, with two exceptions noted below, each level of inventory is attained twice during a cycle, once during the stage of accumulation and again during decumulation. Since the firm must be indifferent between accumulating or decumulating when inventories are at any given level, how are we to know that it should not be decumulating at a point when we have it accumulating? The explanation lies in two exceptions to this indifference. One of them occurs when inventories are zero, since it is then clearly preferable to begin to accumulate. The other exception is when inventories are at some upper bound and the firm prefers not to produce. The policy we present as "the" solution is the particular optimal path that allows for the longest period of continuous production. It is also the only path for which sales are continuous over a cycle. It is in this sense that we account for start-up costs, even though they are not included in the objective. Given such costs it is clearly preferable to have longer cycles rather than be rapidly switching production on and off.

An Example

Consider a simple example that illustrates the model. Assume that demand is linear and cost is cubic:

(10) $P(y) = A - By;$

(11) $C(q) = \frac{q^3}{3} - Kq^2 + K^2q.$

Marginal and average costs are:

(12) $MC = (q - K)^2$

(13) $AC = \frac{q^2}{3} - Kq + K^2.$

Assume $K^2 < A < 3KB + \frac{K^2}{4}$. $A,B,K > 0$. This guarantees that the intersection of marginal revenue and marginal cost is to the left of minimum average cost. Minimum average cost occurs at $q = \frac{3K}{2}$. Since per unit storage cost, s, is constant, total storage cost is sI.

We solve the differential equation (8) above to get:

(14) $\lambda = -\frac{s}{r}e^{-rt} + c,$

where c is a constant of integration.

With our particular demand, equation (4) above becomes

(15) $e^{-rt}[A - 2By] = \lambda$

we get:

(16) $y = \frac{A + \frac{s}{r} - ce^{rt}}{2B}.$

With the particular cost function assumed, equation (5) above becomes

(17) $e^{-rt}(q - K)^2 = \lambda$

we get:

(18) $q = \sqrt{ce^{rt} - s/r} + K.$

Note that since $\lambda \geq 0$, $c \geq s/r$. Thus, y decreases at an exponential rate while q increases at the square root of this rate.

To complete the solution we must find the value of c and the two times of discontinuity T^*, when production stops, and T, when inventories are exhausted. These three parameters are jointly specified by the following system of equations:

$$\text{(i)} \quad K^2/4 = ce^{rT^*} - s/r$$

$$\text{(ii)} \quad \frac{(A-ce^{rT}+s/r)^2}{4B} = \frac{(A-c+s/r)^2}{4B} - \frac{(K+\sqrt{c-s/r})^2}{3}\left(-K + 2\sqrt{c-s/r}\right)$$

$$\text{(iii)} \quad \int_0^{T^*} \left(\sqrt{ce^{rt}-s/r} + K\right)dt = \int_0^T \left[\frac{A+s/r-ce^{rt}}{2B}\right]dt$$

Equation (i) guarantees continuity of the Hamiltonian at T^*. Equation (ii) guarantees continuity of the Hamiltonian at T, and equation (iii) requires inventories to be zero at T. As a system these equations are somewhat unwieldy. Therefore, we do not derive explicit analytic expressions for c, T^*, and T, though we can demonstrate that (i) by itself requires output to be at its minimum average cost level at the time when production shuts down.

Instead of an analytic approach, we completely characterize the solution in a qualitative sense. Furthermore we use a different method for determining the times of discontinuity. We view our problem as one having a finite time horizon, the time of one cycle of accumulation-decumulation. The firm chooses the time horizon. There is a salvage value which depends on this terminal time. It is the value of discounted profits from all future cycles. The advantage of this approach is that it shows the optimizing decisions the firm must make in calculating its jump times.

Let us begin with our description of parameter values. We claim that c

is the smallest value that is consistent with the first and second order conditions. This means that if marginal revenue intersects marginal costs to the right of minimum marginal cost then at time zero output equals sales, though thereafter production will exceed sales until decumulation begins to take place. If marginal cost intersects marginal revenue to the left of minimum marginal cost, output at time zero is that at which marginal cost is a minimum. Again the quantity sold is that at which marginal revenue equals marginal cost. The proof of this proposition appears in the Appendix where it is also shown that output at T^*, the time when production stops, is less than or equal to the output at which average cost is a minimum.

As to T^*, instead of calculating it directly let us examine the conditions that determine T, the time when inventories are depleted. The rationale for doing this is that T^* is a function of T. Thus, let $\Pi(T)$ be the discounted profits for one cycle of accumulation - decumulation. Total profits are:

$$(19)\qquad \sum_{j=0}^{\infty} e^{-rTj}\Pi(T) = \frac{\Pi(T)}{1-e^{-rT}} = \Pi(T)\frac{e^{rT}}{e^{rT}-1}$$

The optimal T must satisfy the following first order condition:

$$(20)\qquad \frac{\Pi' e^{rT}}{e^{rT}-1} - \frac{\Pi\, re^{rT}}{(e^{rT}-1)^2} = 0$$

or

$$(21)\qquad \frac{\Pi'}{\Pi} = \frac{r}{e^{rT}-1}\ .$$

Since

$$(22)\qquad \Pi(T) = \int_0^{T^*} e^{-rT}[P(y)y - C(q) - sI]dt + \int_{T^*}^{T} e^{-rT}[P(y)y - sI]dt$$

and

$$(23)\qquad I(T) = 0$$

Note that in (22) and (23) y, q, and I denote the optimal values.

we get:

$$\Pi'(T) = -e^{-rT^*} C[q(T^*)] \frac{\partial T^*}{\partial T} + e^{-rT} P[y(T)]y(T). \tag{24}$$

Furthermore, since

$$\frac{\partial T^*}{\partial T} = \frac{y(T)}{q(T^*)} \tag{25}$$

we get

$$\Pi'(T) = [-e^{-rT^*} AC(T^*) + e^{-rT} P(T)]y(T). \tag{26}$$

This derivative has a straightforward interpretation: if the cycle is lengthened by a marginal time unit, an additional y(T) units are sold. These units must be produced. In fact, they are produced at T^* at a per unit cost of $AC(T^*)$, which is the average cost when production stops. These units are sold at T at a price P(T), which is the sales price when inventories are depleted. Our profit function is discounted, though as seen in equation (26), sales price is discounted more heavily than production cost. It is interesting to note that marginal inventory costs in the two integrals cancel each other out.

In summary, y(o) is not sensitive to storage or inventory cost. It is either determined at the intersection of instantaneous marginal cost and marginal revenue, or by the condition that marginal revenue at time zero equals minimum marginal cost. Thus we can write:

$$y(t) = \frac{(A + s/r)(1 - e^{rt})}{2B} + y(o)e^{rt}. \tag{27}$$

The optimal length for one cycle of accumulation - decumulation is determined by the conditions of equation (21), above. Let us now turn to two implications of the model, one having to do with the interest rate and the effect of a tight monetary policy on our firm and the other with welfare economics.

As to the former, it is part of conventional wisdom that an increase in the interest rate can lead to an increase in price because the firm's marginal cost curve shifts up due to a higher user cost of capital. The model presented above demonstrates that there is a second channel through which an increase in the interest rate can have the undesirable effect of increasing prices. This effect is revealed by examining $\partial y/\partial r$ and showing that it is negative. At the outset it should be noted that since marginal revenue is necessarily non-negative, there is an upper bound to sales: $y(o) \leq A/2B$.

Now,

$$(28) \qquad \frac{\partial y}{\partial r} = \frac{-s/r^2 - tce^{rt}}{2B} < 0 \ .$$

This result accords with intuition. Namely, when the firm has settled on its optimal plan as to production, sales, inventory and price, it is indifferent as between selling an additional unit at any given time and selling it in the future. Now, both revenue and storage costs are discounted. Hence, discounted marginal revenue rises through time by the discounted marginal cost of storing an additional unit of output for an additional unit of time. An increase in the interest rate reduces discounted marginal storage cost. However, it reduces future discounted marginal revenue even more, simply because of the time difference between when storage costs are incurred and future revenues are realized. Therefore, non-discounted marginal revenue rises faster with higher interest rates, implying that sales fall faster. Sales are less at any point of time and price is higher.

The above is one part of the effect of an increase in the interest rate. It is also necessary to examine its effect on the length of the cycle. Writing T as T(r) and noting that equation (21) above holds identically, we get:

$$(29)\quad \left[\frac{\Pi\Pi'' - (\Pi')^2}{\Pi^2}\right]\frac{dT}{dr} = \frac{e^{rT} - 1 - rTe^{rT}}{(e^{rT} - 1)^2} - \frac{r^2e^{rT}}{(e^{rT}-1)^2}\frac{dT}{dr}.$$

The first term on the right-hand side is negative. To see this, take the Taylor Series expansion. The second order conditions for the optimal T imply that:

$$(30)\quad \left[\frac{\Pi\Pi'' - (\Pi')^2}{\Pi^2} + \frac{r^2e^{rT}}{(e^{rt}-1)^2}\right] < 0.$$

Therefore, $dT/dr < 0$, meaning that the length of the cycle increases.

Thus, an increase in the interest rate increases average price for two reasons: 1) it increases price at each point of time over what had been the length of time of the cycle before the change in the interest rate; 2) it increases the length of the cycle so that price continues to rise over time intervals during which it would have returned to the low price that prevails at the beginning of each cycle.

Our model has some welfare implications that we now wish to consider. In this regard it should be observed, and was shown by the continuity conditions on H that $q(T^*) = 3K/2$. The former is the output rate at the time that production stops. It is the maximum rate of output produced under the dynamic strategy. 3K/2 is the output at which average cost is a minimum. That the former is equal to the latter implies that no inventories are accumulated when marginal revenue intersects marginal cost to the right of minimum average cost. In this situation, it is always optimal to adopt the static strategy, i.e., Case II above. The incentive to hold inventories, taking advantage of scale economics in the long run or the declining portion of a short run cost function, only exists when this intersection

is to the left of minimum average cost. The below discussion assumes this situation. The behavior of the dynamic monopolist can then be compared with that of the static or "naive monopolist" who always acts as if the behavior of Cases I and II is optimal.

For the naive monopolist the decision to produce is determined by the condition that $P(q) \geq AC(q)$, i.e., the demand curve must cut through or be tangent to the average cost function. The dynamic monopolist finds this condition sufficient but not necessary. As long as there is some t such that $\Pi(t) > 0$, the dynamic monopolist will produce the product even if there are losses at $t = 0$. In those cases when the dynamic strategy leads to a commodity being produced that would not be otherwise produced, there is an obvious welfare gain achieved by introducing a storage capability.

What about the case where both the static and dynamic monopolists find it profitable to produce? Obviously, the latter firm's profits are higher since it has the opportunity to behave statically but chooses not to do so, but what about consumer surplus? A comparison of consumer welfare under the two strategies is more complex than the comparison of profits. Two situations are considered below. Consumers are worse off under the dynamic than under the static regime in the first, and better off in the second.

First, suppose instantaneous marginal revenue and marginal cost intersect to the right of minimum marginal cost. Both strategies call for the same behavior at time zero. After that the dynamic monopolist raises price continuously over the time of each cycle. The consumer is worse off. The producer uses storage capability to convert a part of consumer surplus into profit. In the second situation, marginal revenue intersects marginal cost where the latter is falling.

The output of the dynamic monopolist at time zero is at the minimum of the marginal cost curve, and therefore exceeds that of the static monopolist. Sales are higher and price lower at time zero with dynamic behavior. If sales at T, the time when a cycle ends, are greater than the output at which marginal revenue and marginal cost intersect, then consumers are unambiguously better off with the dynamic monopolist. In fact, it is conceivable that the dynamic strategy leads to an overprovision of the good, judged by the principles of static welfare economics. By overprovision we mean that sales are greater than those where marginal cost and demand intersect.

The above results suggest that the social cost of monopoly is an involved concept that requires more careful treatment than that provided by static consumer surplus triangles. In addition, there is need to rethink the theory of optimal regulation where firms exhibit significant scale economies and a storable product.[7]

Many economists believe that social losses are minor in the relatively free entry situation of competition between firms producing differentiated products. It is argued that profits attract new firms. Demand functions of competitors get flatter with such entry and firms tend to operate at a point very close to the most efficient scale of plant. The findings of this paper offer some support for this conclusion. A dynamic strategy may produce profits and encourage entry even in what has been considered an equilibrium situation in static theory, i.e., the large numbers, tangency solution. It may be that inventory capability brings firms even more closely to the point where individual firm demand functions are perfectly elastic.

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[7] Petroleum production and natural gas transmission and distribution would seem to be examples.

The above discussion has some troublesome aspects since it gives a static interpretation to dynamic results. Aside from the dynamics of entry, there is the futher problem that current demands need not be just a function of current prices. In that case a demand surface is required that is more general than the one in our objective function. There is a question about the decision framework that generates the dynamic demand surface. Do consumers know prices in the future when they make decisions on current demand? The dynamic demand surface comes from the maximization of an intertemporal utiltiy function. Future prices of the good can affect current demand via standard substitution effects. If consumers know these prices, a "reverse causality" is introduced into the demand function. From the point of view of the firm the problem is then no longer a variational one. If consumers do not know these prices they try to learn about the firm's pricing policy. The firm in turn tries to learn about consumers' reactions, which in turn motivates consumers again etc. This suggests as a solution a Nash equilibrium of a highly sophisticated dynamic game. The interactive strategies discussed in Part One need not have the Nash property. The formal structure of such a game, showing the existence of equilibrium, and characterizing the equilibrium solution are all extremely difficult problems. How the benefits of a dynamic monopoly strategy are shared between producer and consumer is, therefore, not something about which we can make definite statements.

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We wish to prove that c is the smallest value that is consistent with the first and second order conditions.

<u>Proof</u>: We consider comparison paths which have the property that inventories accumulated at T^* are the same as along the original path and the profits generated by the comparison paths are greater than along the original path. We do this by keeping the same production path and altering the sales path by changing c. We note the restrictions on c.

Thus, let y^* be the sales level where instantaneous MR = MC and output equal sales. Here we assume that $y^* \geq K$, i.e. the intersection of MR and MC is to the right of minimum MC. The case where the intersection of MR and MC is to the left of minimum MC is handled similarly. From the above assumption and since sales at zero are no greater than production at zero:

A1) $$y(o) = \frac{A + s/r - c}{2B} \leq y^* = K - B + \sqrt{B^2 - 2KB + A}$$

or

A2) $$c \geq A + s/r - 2By^* .$$

This restriction is stronger than $c \geq s/r$ since $A/2B \geq y^*$. Since $y(o) \geq 0$, $c \leq A + s/r$. If this second restriction is not binding we can construct a new path with constant $c + \Delta c$. Along this path we know that:

A3) $$y(t) = \frac{A + s/r - (c + \Delta c)e^{rt}}{2B} .$$

The inventories decumulated along this new path until time t are:

A4) $$-I(t) = \frac{(A + s/r)t}{2B} + \frac{(c+\Delta c)(1-e^{rt})}{2Br} .$$

The derivative of discounted revenue from sales is:

$$\text{A5)} \quad \frac{d\left[e^{-rt}\,\frac{(A+s/r\,-\,ce^{rt})}{2B}\,\frac{(A-s/r\,+\,ce^{rt})}{2}\right]}{dc} = \frac{s/r\,-\,ce^{rt}}{2B} < 0\ .$$

The derivative of discounted storage cost from decumulation, which is a positive entry since sales decrease inventories, is:

$$\text{A6)} \quad \frac{d\left[e^{-rt}s\left[\frac{(A+s/r)t}{2B}\right] + sc\left[\frac{e^{-rt}-1}{2Br}\right]\right]}{dc} = \frac{s(e^{-rt}-1)}{2Br} < 0.$$

Thus, at each point in time that there are sales along the comparison path, the contribution to profit is less than along the original path. However, since decumulation takes place more slowly along the comparison path, it must run longer to dispose of the same amount of inventory. Let the terminal time of the comparison path be called $T^*(c+\Delta c)$. If $R(c)$ is discounted total revenue from sales while $S(c)$ is discounted storage cost savings, the difference in profit along the two paths is:

$$\text{A7)} \quad \int_0^{T^*} [R(c+\Delta c)-R(c)+S(c+\Delta c)-S(c)]dt + \int_{T^*}^{T^*(c+\Delta c)} [R(c+\Delta c)+S(c+\Delta c)-\bar{S}]dt$$

where $-\bar{S}$ is the storage cost associated with the constant inventory level $I(T^*)$. We divide the above integral by Δc and let $\Delta c \to 0$. The result is the marginal change in total profits from raising the constant c. Since

$$\text{A8)} \quad \frac{dT^*(c)}{dc} = \frac{e^{rT^*}-1}{2Bry(T^*)} \quad \text{we get:}$$

$$\text{A9)} \quad \frac{d\Pi(T^*)}{dc} = \int_0^{T^*}\left[\frac{s/r-ce^{rt}}{2B} + \frac{s(e^{-rt}-1}{2Br}\right]dt + \left[\frac{e^{rT^*}-1}{2Br}\right]\left[\frac{(A-s/r)e^{-rT^*}+c}{2}\right]$$

$$= \left[\frac{1-e^{-rT^*}}{2r}\right] \quad y(T^*)$$

If there is an internal optimum for c, then $\frac{d\Pi(T^*)}{dc} = 0$. From A9 this would mean that $y(T^*) = 0$, but we know apriori that $y(T^*) > 0$ since there is decumulation after T^*. Consequently there is no internal optimum for c. Thus either $c = A + s/r - 2By^*$ or $c = A + s/r$. In terms of y, $y(o) = y^*$ or $y(o) = 0$. The latter is clearly not optimal when inventories are accumulated. .

Note that while the determination of the boundary between the different cases in terms of the parameters A, B, K, s, and r is quite difficult, we have shown that for a sub-region of the parameter values in which Case 3, above, is optimal, our solution is the correct one.

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