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Ioffe's Normal Cone and the Foundations of Welfare Economics: The Infinite Dimensional Theory

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by

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Abstract. We establish the relevance of Ioffe's normal cone for basic theorems of welfare economics in the context of a commodity space formalized as an ordered topological vector space and endowed with a locally convex topology.

Key Words. Strong Pareto optimal allocations, Ioffe's normal cone, Clarke's normal cone, epi-Lipschitzian sets, public goods.

AMS (MOS) Subject Classifications (1979): Primary 90A14, 90C48. Secondary 49B34.

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1. Introduction

In [18], Khan-Vohra have provided a version of the second fundamental theorem of welfare economics that applies to economies with non-convex preferences and technologies, public goods and with an ordered locally convex space of commodities. In particular, they showed that in economies without public goods and with preferences and technologies formalized as epi-Lipschitzian sets, the Clarke normal cones to the production sets and the "no-worse-than" sets at the respective Pareto optimal production and consumption plans have a non-empty and non-zero intersection. In the presence of public goods, they showed that this statement has to be extended to say that the sum of the relevant normal cones for the consumers has a non-empty and non-zero intersection with those of each of the producers, when these cones are projected into the space of public goods. These results are generalizations of the classical statements that at any Pareto optimal allocation of resources, the marginal rates of substitution in consumption and in production are equated across all agents if the commodities are private goods, and the sum of the marginal rates in consumption are equated to those in production if public goods are involved; see, for example, Lange [19] and Samuelson [24]. These results can also be seen as generalizations of more recent, but also classical, statements that, in the presence of convexity, every Pareto optimal allocation of resources can be sustained through expenditure minimization by consumers and profit maximization by producers at suitably chosen prices; see Arrow [1], Debreu [7, 8, 9] and Foley [10].

In a recent paper [15] limited to a finite dimensional commodity space, the author has presented a reformulation of these basic theorems of welfare economics in terms of the Ioffe normal cone rather than that of Clarke. Since the Ioffe normal cone is based on the Bouligand-Severi contingent cone and since this offers a better local approximation to a set than the Clarke normal cone, our reformulation is in keeping with a more intuitive notion of a marginal rate of substitution, especially in economies whose technologies do not exhibit "free disposal." Moreover, since the Ioffe normal cone is, in general, strictly contained in the Clarke normal cone, this reformulation furnishes sharper results. However, it is natural to ask if the theory presented in [15] can be generalized from an Euclidean space setting to that of an ordered locally convex space of commodities. We offer such a generalization here.

Our generalization is based on Ioffe's [14] recent extension of his approximate subdifferential to locally convex spaces. Such a mathematical object has all the properties that we require for the formulation and proofs of our results provided we limit ourselves to Rockafellar's [21, 22] epi-Lipschitzian sets. Since this was already assumed in [18], a satisfactory generalization of the finite-dimensional theory is obtained. However, it is worth emphasizing that our results neither imply nor are implied by those of Khan-Vohra [18]. The reason for the former implication is twofold. Firstly, the locally convex hypothesis on the topology seems to be essential here in contrast to the situation in [18]; see Remarks 4.1 and 4.2 in that paper. Secondly, we either work under a closedness hypothesis on the "better-than" sets that was not required in [18]; or limit ourselves to a subset of Pareto optimal allocations that is precisely defined below. We do not know if either of these limitations can be removed; the second plays a role in the proofs of our theorems because of the non-convexity of the Ioffe normal cone.

The proofs of the results in [17, 18], and in [2], are essentially based on the Hahn-Banach theorem and revolve around a separation argument. The difference from proofs of corresponding results for convex economies, as in Arrow [1] and Debreu [8], lies in the fact that we now separate the tangent cones to the sets at the Pareto optimal plans rather than the sets themselves. One has only to check whether the intersection principle is satisfied whereby disjoint sets have disjoint tangent cones. Since the Clarke tangent cones satisfy this principle, see [29], and since they are always convex, we can apply the separation theorem provided one of the sets to be separated has a non-empty interior. This is guaranteed by the epi-Lipschitzian hypothesis. Unfortunately, this line of argumentation no longer works in the set-up here for the simple and obvious reason that the Ioffe normal cone, being generally non-convex, cannot arise as a polar to any set and hence a corresponding tangent cone is not a well-defined object. A way around this difficulty is to appeal to the theory developed in Ioffe [14]; in particular, we especially rely on two results. The first of these states that the Ioffe normal cone to a point on the boundary of an epi-Lipschitzian set contains a non-zero element. The second states that, under suitable hypotheses, the Ioffe normal cone to an intersection of sets is contained in the sum of the normal cones. It is of interest that Ioffe uses the Hahn-Banach theorem in the proof of both of these results.

Two final introductory remarks. First, Ioffe [14] makes it clear why the definition of the Ioffe normal cone in a finite dimensional setting, as in [13], does not work in the more general context of a locally convex space. However, one may have success for a more limited class of infinite-dimensional spaces such as Hilbert spaces; see Ward [28, Chapter IV]. Second, in [2], Bonnisseau-Cornet consider an economy without public goods and show that the hypotheses of Theorem 1 in Khan-Vohra [18] can be weakened to the requirement that one, rather than all, of the production and "no-worse-than" sets be epi-Lipschitzian. It is natural to ask if the same can be accomplished here. The problem with this question is that once the epi-Lipschitzian hypothesis is dropped, we are no longer guaranteed that the Ioffe normal cone is strictly contained in the Clarke normal cone; see the example of Treiman [27] and, following him, that of Khan [16]. As such, a generalization along this line may have a mathematical interest, but it does not lead to a better result in terms of the economics.

The remainder of this paper is organized as follows. In Section 2 we present some preliminary material relating to the Ioffe cone--most of it presumably well known but for which we could find no direct reference. Section 3 presents the model and results and Section 4 the proofs.

2. The Ioffe Normal Cone

In this section we define and develop the basic properties of the Ioffe normal cone. We shall work in a locally convex linear topological space E with E*, its topological dual, endowed with $\sigma(E^*,E)$-topology. For any $x \in E$, $\mathcal{S}(x)$ is the collection of neighborhoods of x. For any $x \in E$ and any $f \in E^*$, we denote the evaluation by $\langle f,x\rangle$. For any positive integer k, E^k denotes the k-fold product of E endowed with the product topology. R will always refer to the set of real numbers.

For any $X \subset E$, the polar cone X^+ of E is given by $\{p \in E^*: \langle p,x\rangle \leq 0 \text{ for all } x \in E\}$.

For any extended real-valued function f on E, we set

$$\text{epi } f = \{(\alpha,x) \in R \times E: \ \alpha \geq f(x)\},$$

$$\text{dom } f = \{x \in E: \ |f(x)| < \infty\},$$

$$f_S(x) = \begin{cases} f(x) & \text{if } x \in S, \\ \infty & \text{if } x \notin S. \end{cases}$$

We denote the indicator function of S by

$$\chi_S(x) = \begin{cases} 0 & \text{if } x \in S, \\ \infty & \text{if } x \notin S. \end{cases}$$

We shall use the symbol $u \to_f x$ to mean that $u \to x$ and $f(u) \to f(x)$. If $\{Q_\alpha\}_{\alpha\varepsilon I}$ is a set of sets, then $\limsup Q_\alpha$ is the collection of limits of converging subnets of nets $\{x_\alpha\}$, $x_\alpha \in Q_\alpha$ for all $\alpha \in I$.

We now develop the notions of the <u>lower Dini directional derivative</u> and the <u>Dini subdifferential</u>. For $x \in \text{dom } f$, let

$$d^-f(x;h) = \liminf_{\substack{u \to h \\ t \to +0}} t^{-1}(f(x+tu)-f(x));$$

$$\partial^- f(x) = \{x^* \in E^*: \langle x^*,h\rangle \leq d^-f(x;h) \text{ for all } h \in E\}.$$

If $x \notin \text{dom } f$, we set $\partial^- f(x) = \emptyset$.

We can now present a concept originally due to Bouligand [3] and Severi [26].

<u>Definition 2.1.</u> For any $X \subset E$ and any $x \in X$, the <u>contingent normal cone</u> to X at x, $N_K(X,x)$, is the set $\partial^- \chi_X(x)$.

The following lemma is well-known and we state it without proof.

<u>Lemma 2.1.</u> For any $X \subset E$ and any $x \in X$, $N_K(X,x)$, is given by the polar of the set $T_K(X,x)$ where

$$T_K(X,x) = \{y \in E: \exists \text{ a net } \{t^\nu,y^\nu\} \text{ in } R \times E$$

$$t^\nu \downarrow 0,\ y^\nu \to y \text{ with } (x+t^\nu y^\nu) \in X\}.$$

The following definitions are taken from Ioffe [14]; also see [12].

<u>Definition 2.2.</u> Let $\mathcal{F}$ denote the collection of all finite dimensional subspaces of E. The set

$$\partial_A f(x) = \bigcap_{L \in \mathcal{F}} \limsup_{u \to_f x} \partial^- f_{u+L}(u)$$

will be called the <u>A-subdifferential</u> of f at x.

Definition 2.3. For any $X \subset E$, and any $x \in X$, the Ioffe normal cone to X at x, $N_A(X,x)$, is the set $\partial_A \chi_X(x)$.

Using Lemma 2.1 we can now present an alternative characterization of $N_A(X,x)$.

Lemma 2.2. Let $\mathcal{F}$ denote the collection of all finite dimensional subspaces of E. For any $X \subseteq E$ and any $x \in X$,

$$N_A(X,x) = \bigcap_{L \in \mathcal{F}} \{y: \exists \text{ a net } \{x^\nu, y^\nu\} \text{ in } E \times E^* \text{ with } x^\nu \to x,\ x^\nu \in X,\ y^\nu \in N_K(X \cap (x^\nu + L), x^\nu) \text{ and } y^\nu \to y\}.$$

Proof. See Proposition 2.1 in [14]. ||

The following two properties are easy to prove and useful for the results to follow.

Lemma 2.3. (i) For any $x \in E$, $N_A(\{x\},x) = E^*$. (ii) For any $X \subseteq E$ and $x \in \text{Int } X$, $N_A(X,x) = \{0\}$.

Proof. For (i), observe that for any finite dimensional subspace L of E

$$d^- \chi_{\{x\} \cap (x+L)}(x;h) = \infty \text{ for all } h \in E.$$

Hence $\partial^- \chi_{\{x\} \cap (x+L)}(x) = E^*$ and therefore $\partial_A \chi_{\{x\}}(x) = E^*$.

For (ii), observe that for any finite dimensional subspace L of E and $x^k \in \text{Int } X$,

$$d^- \chi_{X \cap (x^k+L)}(x^k;h) = \begin{cases} 0 & \text{for all } h \in L \\ \infty & \text{for all } h \notin L. \end{cases}$$

This implies

$$\partial^{-}\chi_{X\cap(x^k+L)}(x^k) = \{x^* \in E^*: \langle x^*,h\rangle = 0 \text{ for all } h \in L\} \equiv L^{\perp}.$$

Hence $N_A(X,x) = \bigcap_{L\in\mathcal{F}} L^{\perp} = \{0\}$ where $\mathcal{F}$ is the family of all finite dimensional subspaces of E. ||

Lemma 2.4. Let $x = (x^1,\ldots,x^k) \in \Pi_{i=1}^k X^i \subset E^k$. If X^i are closed for each i,

$$N_A(\Pi_{i=1}^k X^i, x) = \Pi_{i=1}^k N_A(X^i,x^i).$$

Proof. We shall prove the result only for the case k = 2; the general result then follows easly by induction.

Observe that

$$\chi_{(X^1,X^2)}(x^1,x^2) = \chi_{X^1}(x^1) + \chi_{X^2}(x^2).$$

Since X^i are closed, the indicator functions are lower semicontinuous. We can now apply Proposition 4.4 of [14] to assert that

$$\partial_A\chi_{(X^1,X^2)}(x^1,x^2) = \partial_A\chi_{X^1}(x^1) \times \partial_A\chi_{X^2}(x^2).$$

But then the result is proved. ||

Lemma 2.5. For any convex closed set X E, and $x \in X$,

$$N_A(X,x) = \{f \in E^*: \langle f,x\rangle \leq \langle f,y\rangle \text{ for all } y \in X\}.$$

Proof. Since X is convex, $\chi_X(\cdot)$ is a convex function. Either X = E or X is a strict subset which is closed by hypothesis. In either case, there exists a point at which $\chi_X(\cdot)$ is continuous. We can now appeal to Proposition 3.2 in [14] to assert that $\partial\chi_X(x) = \partial_A\chi_X(x)$, where $\partial f(x)$ is the subdifferential of f at x the sense of convex analysis. Since $\partial\chi_X(x) = \{f \in E^*: \langle f,x\rangle \leq \langle f,y\rangle \text{ for all } y \in X\}$, the proof is complete. ||

Our next set of results involve epi-Lipschitzian sets introduced by Rockafellar [20]. We first recall the following

Definition 2.4. A function $f: U \to R$, $U \subset E$, U open, is said to be a Lipschitz function if there is a continuous seminorm s(u) on E such that

$$f(u) - f(w) \leq s(u-w) \text{ for all } u,w \in U.$$

We can now present

Definition 2.5. A set $X \subset E$ is said to be epi-Lipschitzian at $x \in X$ if either $x \in \text{int } X$ or, locally near x, X is linearly homeomorphic to the epigraph of a Lipschitz function.

The following characterization is due to Rockafellar [20] for R^n.

Theorem 2.1. A closed set $X \subset E$ is epi-Lipschitzian at $x \in X$ iff there exist $y \in E$, $U_y \in \mathcal{N}(y)$, $U_x \in \mathcal{N}(x)$, $\lambda > 0$ such that

$$(x'+\mu y') \in X \quad \text{for all } x' \in X \cap U_x,$$
$$\text{for all } y' \in U_y, \text{ for all } \mu \in (0,\lambda).$$

Proof. A proof for $E = R^n$ is given in [19]. The difficult part of the proof is to construct a Lipschitz function given the condition of the theorem. This is based on decomposing R^n into a direct sum of two closed subspaces, one of which is one-dimensional. It is well known that this fact is true for a topological vector space; see [6, Theorem 1.4.3] and [11, p. 120]. ‖

Lemma 2.6. If $E = E_1 \times E_2$, $x_i \in X_i \subset E_i$ and X_i are epi-Lipschitzian at x_i $(i = 1,2)$, then $X_1 \times X_2$ is epi-Lipschitzian at (x_1,x_2).

Proof. The proof is a simple consequence of Theorem 2.1 above. ‖

For any $X \subset E$ and $x \in X$, let $H(X,x) = \{y \in E:$ there exist $U_y \in \mathcal{U}(y)$, $U_x \in \mathcal{U}(x)$, $\lambda > 0$ such that $(X \cap U_x) + \mu U_y \subset X$ for all $\mu \in (0,\lambda)\}$. It is clear that $H(X,x) \neq \emptyset$ iff X is epi-Lipschitzian at x.

We can now present

Lemma 2.7. Let $x \in X_i \subset E$ for $i = 1,\ldots,n$. Then

$$\bigcap_i H(X_i,x) \subset H(\bigcap_i X_i,x).$$

Thus, if $\bigcap_i H(X_i,x) \neq \emptyset$, $\bigcap_i X_i$ is epi-Lipschitzian at x.

Proof. Suppose $y \in \bigcap_i H(X_i,x)$. Since $y \in H(X_i,x)$, there exist $U_x^i \in \mathcal{U}(x)$, $U_y^i \in \mathcal{U}(y)$ and $\lambda^i > 0$ such that for each $\mu^i \in (0,\lambda^i)$

$$(X_i \cap U_x^i) + \mu^i U_y^i \subset X_i \qquad i = 1,\ldots,n.$$

Let $U_x = \bigcap_i U_x^i$, $U_y = \bigcap_i U_y^i$ and $\lambda = \operatorname{Min}_i \lambda^i$. Then for each $\mu \in (0,\lambda)$

$$(X_i \cap U_x) + \mu U_y \subset X_i \qquad i = 1,\ldots,n.$$

This implies

$$(\bigcap_i X_i \cap U_x) + \mu U_y \subset \bigcap_i X_i.$$

Since $U_x \in \mathcal{U}(x)$, $U_y \in \mathcal{U}(y)$ and $\lambda > 0$, $y \in H(\bigcap_i X_i, x)$ and the proof of the first statement is complete. The proof of the second statement is obvious. ||

We now recall the definitions of the Clarke tangent and normal cones as presented in [22].

<u>Definition 2.6.</u> For any $X \subset E$ and any $x \in X$, the <u>Clarke tangent cone</u> to X at x, $T_C(X,x)$, is given by

$$\{y \in E\text{: For any net } \{t^\nu, x^\nu\} \text{ in } R \times E \text{ with } t^\nu \downarrow 0,\ x^\nu \in X,$$
$$x^\nu \to x, \text{ there exists } y^\nu \in E \text{ with } (x^\nu + t^\nu y^\nu) \in X\}.$$

We shall denote the polar of $T_C(X,x)$ by $N_C(X,x)$ and refer to it as the <u>Clarke normal cone</u> to X at x.

<u>Lemma 2.8.</u> Let $X \subset E$ by epi-Lipschitzian at $x \in X$. Then $H(X,x) = \text{Int } T_C(X,x) \neq \emptyset$.

<u>Proof.</u> See [22]. ||

We can now present the following results of Ioffe [14].

<u>Theorem 2.2.</u> If $X \subset E$ is epi-Lipschitzian at $x \in X$, then

$$N_C(X,x) = \text{cl con } N_A(X,x).$$

Moreover, if x is a boundary point of X,

$$N_A(X,x) \neq \{0\}.$$

Proof. See the proof of Proposition 3.3 and Corollary 3.3.2 in [14]. ||

Theorem 2.3. Suppose that $X_1,\ldots,X_n$ are closed subsets of E and all of them, except for at most one, are epi-Lipschitzian at x. Suppose further that the following condition is satisfied:

$$x_i^* \in N_A(X_i,x_i^*) \ (i = 1,\ldots,n) \text{ and } \Sigma_i x_i^* = 0$$
$$\Rightarrow \quad x_i^* = 0 \text{ all } i.$$

Then $N_A(\bigcap_i X_i,x) \subset \Sigma_i N_A(X_i,x_i)$.

Proof. See Corollary 4.1.2 and its proof in [14]. ||

Our final result relates to level sets generated by A-differentiable functions in the sense of Ioffe. As above, let $\mathcal{F}$ denote the family of finite dimensional subspaces of E. For any $L \in \mathcal{F}$, let $\|\cdot\|_L$ be a fixed norm.

Definition 2.7. Let E and F be locally convex, linear topological spaces and ϕ: $E \to F$. ϕ is said to be A-differentiable at $x \in E$ if there exists a continuous linear operator T: $E \to F$ such that for any $L \in \mathcal{F}$, and any $U \in \mathcal{B}_E(0)$ and $V \in \mathcal{B}_F(0)$,

$$(\phi(u+h) - \phi(u) - Th) \in \|h\|_L V \quad \text{for all } u \in x+U, \text{ for all } h \in U \cap L.$$

We shall denote T by $\phi'(x)$.

Remark. Ioffe [14, p. 115] remarks that $\phi'(x)$ is unique where it exists and, in a normed space, any A-derivative is also a Gateaux derivative and any strict Frechet derivative is an A-derivative.

Before we present our next result, we need to recall (see [4, 5, 21] for details)

Definition 2.8. For any Lipschitz $f\colon E \to R$ the Clarke generalized derivative at $x \in E$, $\partial_C f(x)$, is given by

$$\partial_C f(x) = \{y^* \in E^*\colon \ (-1,y^*) \in N_C(\text{epi } f, (f(x),x)\}.$$

We can now present a corollary of a result of Rockafellar [21].

Theorem 2.4. Suppose $f\colon E \to R$ is Lipschitz in a neighborhood of $x^* \in E$ and that f is A-differentiable at x^*. Let $X = \{x \in E\colon \ f(x) \leq f(x^*)\}$. If $\{f'(x^*)\} \neq 0$, then

$$N_A(X,x^*) = \bigcup_{\lambda \geq 0} \lambda\{f'(x^*)\}.$$

Proof. We first claim that $0 \notin \partial_C f(x^*)$. Since f is A-differentiable at x^*, by Proposition 3.1 in [14],

$$\partial_A f(x^*) = f'(x^*).$$

Since f is Lipschitz around x^*, by Proposition 3.3 in [14]

$$\text{cl con } \partial_A f(x^*) = \partial_C f(x^*).$$

But then $0 \in \partial_C f(x^*)$ implies $0 = f'(x^*)$, a contradiction.

Next, we appeal to [21; Corollary 1 to Theorem 5] to state that X is epi-Lipschitzian at x* with

$$(1) \qquad N_C(X,x^*) \subseteq \bigcup_{\lambda \geq 0} \lambda\{\partial_C f(x^*)\}$$

Since X is epi-Lipschitzian and f is Lipschitz in the neighborhood of x*, we can appeal to [14; Proposition 3.3] to rewrite (1) as

$$(2) \qquad \text{cl con } N_A(X,x^*) \subseteq \bigcup_{\lambda \geq 0} \lambda\{\text{cl con } \partial_A f(x^*)\}$$

Next we claim that x* is a boundary point of X. To see this, first note that there exists $\hat{y} \in E$ such that

$$(3) \qquad f^\circ(x^*,\hat{y}) \equiv \limsup_{x \to_f x^*,\, t \downarrow 0} \frac{f(x^*+t\hat{y})-f(x^*)}{t} > 0$$

If not, $f^\circ(x^*,y) \leq 0$ for all $y \in E$. Since $\partial_C f(x^*) = \{z \in E^*: \langle z,y\rangle \leq f^\circ(x^*,y)$ for all $y \in E\}$, as a consequence of [21, Corollary to Proposition 2], we obtain $0 \in \partial_C f(x^*)$, a contradiction. But (3) implies that in any neighborhood of x*, there exists $\hat{x} \in E$ such that $f(\hat{x}) > f(x^*)$. This implies that x* is a boundary point of X.

We now appeal to Theorem 2.2 and to the fact that f is A-differentiable at x* to rewrite (2) as

$$N_A(X,x^*) = \bigcup_{\lambda \geq 0} \lambda\{f'(x^*)\}. \qquad \|$$

For our final result, we assume that E is an ordered topological vector space with a locally convex linear topology; see, for example, [25] for the necessary terminology. Let E_+ be its positive cone and $E_- = -E_+$.

Lemma 2.9. Let $X \subseteq E$ and $x \in X$.

(i) If $E_+ \subseteq X$, then $p \in N_A(X,x)$ implies $p \in E_-^*$.

(ii) If $E_- \subseteq X$, then $p \in N_A(X,x)$ implies $p \in E_+^*$.

Proof. We only prove (i). Towards this end, suppose $p \in N_A(X,x)$ and $p \notin E_-^*$. Then there exists $z \in E_+$ such that

$$\langle p,z\rangle > 0.$$

Let F be the one-dimensional subspace of E generated by z. By Lemma 2.2

$$p \in \{y: \exists \text{ a net } \{x^\nu,y^\nu\} \text{ in } E\times E^* \text{ with } x^\nu \to x,$$
$$x^\nu \in X,\ y^\nu \in N_K(X \cap (F+x^\nu),x^\nu) \text{ and } y^\nu \to y\}.$$

Let $\{x^\nu,p^\nu\}$ be such a net. Observe that

$$z \in T_K(X \cap (F+x^\nu),x^\nu).$$

To see this, take any sequence of positive numbers $\{t^k\}$, $t^k \to 0$ and the constant sequence $\{z^k\}$ equal to z. Then for all k,

$$(x^\nu+t^kz^k) = (x^\nu+t^kz) \in X \cap (F+x^\nu).$$

This proves the claim. Hence, for all ν, $\langle p^\nu,z\rangle \leq 0$. Since p^ν converges in the $\sigma(E^*,E)$-topology to p, $\langle p,z\rangle \leq 0$, a contradiction. ||

3. The Model and Results

From now on we shall assume that E is an ordered topological vector space endowed with a locally convex topology.

An economy consists of a finite number of consumers and a finite number of firms. We shall index consumers by t, $t = 1,\ldots,T$, and shall assume that each has a consumption set $X^t \subset E$ and a reflexive preference relation $\gtrsim_t$. $>_t$ denotes $\gtrsim_t$ and not $\lesssim_t$. Let the "better-than" set for t at x^t be given by $P^t(x^t) = \{y \in X^t | y >_t x^t\}$ and the "no-worse-than" set by $\overline{P}^t(x^t) = \{y \in X^t | y \gtrsim_t x^t\}$. Firms are indexed by j, $j = 1,\ldots,F$, and each has a production set $Y^j \subset E$. The aggregate endowment is denoted by $w \in E_+$. An economy is thus denoted by $\mathcal{E} = ((X^t, \gtrsim_t)_1^T, (Y^j)_1^F, w)$ and we shall need the following concepts for it.

Definition 3.1. $((x^{*t}),(y^{*j}))$ is an allocation of $\mathcal{E}$ if for all $t = 1,\ldots,T$, $x^{*t} \in X^t$, for all $j = 1,\ldots,F$, $y^{*j} \in Y^j$ and $\Sigma_t x^{*t} - \Sigma_j y^{*j} \leq w$.

Definition 3.2. $((x^{*t}),(y^{*j}))$ is a Pareto optimal allocation of $\mathcal{E}$ if there does not exist any other allocation $((x^t),(y^j))$ of $\mathcal{E}$ such that $x^t \in \overline{P}^t(x^{*t})$ for all t and $x^t \in P^t(x^{*t})$ for at least one t.

Definition 3.3. $((x^{*t}),(y^{*j}))$ is a strong Pareto optimal allocation of $\mathcal{E}$ if there does not exist an allocation $((x^t),(y^j))$ of $\mathcal{E}$ with $((x^t),(y^j)) \neq ((x^{*t}),(y^{*j}))$, $x^t \in \overline{P}^t(x^{*t})$ for all t.

Definition 3.4. $((x^{*t}),(y^{*j}))$ is a locally Pareto optimal allocation of $\mathcal{E}$ if there exists a neighborhood $V = ((V^t),(V^j))$ of $((x^{*t}),(y^{*j}))$ and there does not exist any other allocation $((x^t),(y^j))$ of $\mathcal{E}$ such that $x^t \in (\overline{P}^t(x^{*t}) \cap V^t)$ for all t, $x^t \in (P^t(x^{*t}) \cap V^t)$ for at least one t, and $y^j \in (Y^j \cap V^j)$ for all j.

Definition 3.5. $((x^{*t}),(y^{*j}))$ is a strong locally Pareto optimal allocation of $\mathcal{E}$ if there exists a neighborhood $V = ((V^t),(V^j))$ of $((x^{*t}),(y^{*j}))$ and there does not exist an allocation $((x^t),(y^j))$ of $\mathcal{E}$ with $((x^t),(y^j)) \neq ((x^{*t}),(y^{*j}))$, $x^t \in (\overline{P}^t(x^{*t}) \cap V^t)$ for all t, and $y^j \in (Y^j \cap V^j)$ for all j.

It is clear that every Pareto optimal allocation of $\mathcal{E}$ is a locally Pareto optimal allocation of $\mathcal{E}$; and given reflexivity of $\succsim_t$, that every strong locally Pareto optimal allocation of $\mathcal{E}$ is a locally Pareto optimal allocation of $\mathcal{E}$. Figure 1a gives an example of a two agent economy with $E = R^2$ in which a locally Pareto optimal allocation (x^{*1},y^{*1}) is not Pareto optimal but a strong locally Pareto optimal allocation. Figure 1b exhibits a locally Pareto optimal allocation which is not a strong locally Pareto optimal allocation. In either figure, $P^1(x^{*1})$ is the interior of $\overline{P}^1(x^{*1})$.

Since the primary concern of this paper is the derivation of necessary conditions for Pareto optimality, we shall confine our attention to local allocations. We shall also need the following assumption.

(A1) For all t and all $x^t \in X^t$, $E_+ \subset \overline{P}^t(x^t)$. For all j, $Y - E_+ \subset Y^j$.

We can now present

Theorem 3.1. If $((x^{*t}),(y^{*j}))$ is a strong locally Pareto optimal allocation of $\mathcal{E}$, (A1) is satisfied and $\overline{P}^t(x^{*t})$ and Y^j are closed and respectively epi-Lipschitzian at $((x^{*t}),(y^{*j}))$, then there exists $p^* \in E_+^*$, $p^* \neq 0$ such that

(a) $-p^* \in N_A(\overline{P}^t(x^{*t}),x^{*t})$ for all t,

(b) $p^* \in N_A(Y^j,y^{*j})$ for all j.

For a discussion of (A1) and the epi-Lipschitzian hypotheses, the reader is referred to Khan-Vohra [18] and the references therein.

Our next result extends Theorem 3.1 to economies with public goods. Recall from Samuelson that a public good is a commodity whose consumption is identical across individuals and such that each individual's consumption is equal to aggregate supply, see [24] and [10]. Let E_π refer to the commodity space for private goods and E_g to that for public goods. We shall assume that both E_π and E_g are real vector lattices each endowed with a locally convex linear topology in which the positive cone is closed. Let $E = E_\pi \times E_g$ where E is endowed with the product topology and the induced ordering.

An economy with public goods $\mathcal{E}^G = ((X^t, \succsim_t)_1^T, (Y^j)_1^F, w)$ is such that for all t, $X^t = (X_\pi^t, X_g^t)$ where $X_\pi^t \subset E_\pi$, $X_g^t \subset E_g$ are its projections onto the space of private and public goods respectively. We assume that $X_g^t = X_g$ for all t; that $Y^j \subseteq E$ for all j and that $w \in E$, $w = (w_\pi, 0)$, $w_\pi \in E_{\pi+}$. Let x_π^t and x_g^t refer to the consumption of the private and public goods respectively. $((x^{*t}),(y^{*j}))$ is an <u>allocation</u> for $\mathcal{E}^G$ if for all j, $y^{*j} \in Y^j$, $x^{*t} \in X^t$, $x_g^{*t} = x_g^*$ for all t, and $(\Sigma_t x_\pi^{*t}, x_g^*) - \Sigma_j y^{*j} \leq w_\pi$. The definitions of a <u>Pareto optimal allocation</u> and their local and strong variants are then identical to the ones given in Definitions 3.2 to 3.5.

We can now present our second result.

<u>Theorem 3.2</u>. If $((x_\pi^{*t}, x_g^{*t}),(y^{*j}))$ is a strong Pareto optimal allocation of $\mathcal{E}^G$, (A1) is satisfied, and $\overline{P}^t(x^{*t})$ and Y^j are closed and respectively epi-Lipschitzian at $((x^{*t}),(y^{*j}))$, then there exist $p_\pi^* \in E_{\pi+}^*$, $p_g^* \in E_{g+}^*$, $(p_\pi^*, p_g^*) \neq 0$, $p_g^{*t} \in E_{g+}^*$ such that

(a) $\Sigma_t p_g^{*t} = p_g^*$,

(b) $-(p_\pi^*, p_g^{*t}) \in N_A(\overline{P}^t(x^{*t}), x^{*t})$ for all t,

(c) $p^* \in N_A(Y^j, y^{*j})$ for all j.

So far we have confined our attention to strong Pareto optimal allocations. We can also present

<u>Theorem 3.3</u>. Theorems 3.1 and 3.2 are valid with the term "strong" deleted and with $P^t(x^{*t}) \cup \{x^{*t}\}$ substituted for $\overline{P}^t(x^{*t})$.

Our final result relates to the special case when the preferences and technologies are generated by differentiable functions as, for example, in Lange [19], and Samuelson [23].

Theorem 3.4. Let $((x^{*t}),(y^{*j}))$ be a strong locally Pareto optimal allocation; $\overline{P}^t(x^{*t}) = \{x \in X^t: U_t(x) \geq U_t(x^{*t})\}$ and $Y^j = \{y \in E: F_j(y) \leq 0\}$ where

(i) for all t, U_t: $X^t \to R$ is Lipschitz in a neighborhood of x^{*t}, A-differentiable at x^{*t}, and $x' \in X^t$, $x \geq x'$ implies $U_t(x) \geq U_t(x')$,

(ii) for all j, F_j: $E \to R$ is Lipschitz in a neighborhood of y^{*j}, A-differentiable at y^{*j}, and $F(y') \leq 0$, $y \leq y'$ implies $F(y^j) \leq F(y')$.

If for any t, $U_t'(x^{*t}) \neq 0$, or for any j, $F_j'(y^{*j}) \neq 0$, there exists $p \in E^*$, $p \neq 0$ such that

$$p = U_t'(x^*) = F_j'(y^{*j}) \quad \text{for all t, for all j}$$

4. Proofs of Results in Section 3

We begin with an elementary lemma.

Lemma 4.1. Let E^{*k} be the k-fold Cartesian product of E^*. Then E^{*k} can be identified with the dual of E^k such that for any $x = (x^i) \in E^k$, $x^* = (x^{*i}) \in E^{*k}$, the canonical bilinear form is given by

$$\langle x^*, x\rangle = \sum_{i=1}^{k} \langle x^{*i}, x^i\rangle.$$

Proof. See [11, p. 266]. ‖

Proof of Theorem 3.1

Let $v^* = ((x^{*t}),(y^{*j})) \in E^k$ be the strong locally Pareto optimal allocation and $V = ((V^t),(V^j))$ the corresponding closed neighborhood of v^*. Define the following sets

$$\overline{P}_v^t(x^{*t}) = \overline{P}^t(x^{*t}) \cap V^t \qquad t = 1,\ldots,T$$

$$Y_v^j = Y^j \cap V^j \qquad j = 1,\ldots,F$$

$$V(x^*) = \Pi_t \overline{P}_v^t(x^{*t}) \times \Pi_j Y_v^j$$

$$W = \{v \in E^k: \ \Sigma_t x^t \leq \Sigma_j y^j + w\}.$$

The closedness hypotheses of the theorem guarantee that $V(x^*)$ is a closed set. Since the Clarke tangent cone to a set at an interior point is the whole space, we can appeal to Lemma 2.8 to assert that $H(V^t, x^{*t}) = H(V^j, y^{*j}) = E$. Lemma 2.7 then ensures that $\overline{P}_v^t(x^{*t})$ and Y_v^j are epi-Lipschitzian at x^{*t} and y^{*j} respectively. Hence by Lemma 2.6, $V(x^*)$ is epi-Lipschitzian at v^*. Since E is an ordered locally convex space, W is also closed.

We claim that $V(x^*) \cap W = \{v^*\}$. If there exists $z \neq v^*$ in this intersection, we contradict the fact that v^* is a strong locally Pareto optimal allocation.

Since W is a closed convex set, we can appeal to Lemma 2.5 to assert that $N_A(W,v^*)$ is identical to the set of normals to W in the sense of convex analysis. Thus

(1) $\quad \rho \in N_A(W,v^*) \Rightarrow \langle \rho, x\rangle \leq \rho, v^* \rangle$ for all $x \in W$.

Next we show that for any $\rho \in N_A(W,v^*)$, $\rho \neq 0$, there exists $p^* \in E_+^*$, $p^* \neq 0$, such that

(2) $\quad \rho = (p^*,\ldots,p^*,-p^*,\ldots,-p^*) \in (E_+^*)^T \times (E_-^*)^F$.

Toward this end, define $m^i(z) \in E^k$ to be the vector of zeros in all coordinates except for $z \in E$ in the i-th and k-th coordinates. Clearly $(m^i(z) + v^*)$ and $(-m^i(z) + v^*)$ are elements of W. Hence from (1) we obtain

(3) $\langle \rho, m^i(z) \rangle = 0 \qquad i = 1,\ldots,T$

By Lemma 4.1, corresponding to any $\rho \in (E^k)^*$, there exist $\rho^i \in E^*$, $i = 1,\ldots,k$, such that for any $v = ((x^t),(y^j)) \in E^k$, $\langle \rho, v \rangle = \Sigma_t \langle \rho^t, x^t \rangle + \Sigma_j \langle \rho^j, y^j \rangle$. Thus from (3) we obtain

(4) $\langle \rho^i, z \rangle + \langle \rho^k, z \rangle = \langle \rho^j, z \rangle + \langle \rho^k, z \rangle \qquad i,j = 1,\ldots,T.$

Since z is arbitrary, (4) implies that there exists $p^* \in E^*$ such that

$$\rho^i = p^* \qquad i = 1,\ldots,T.$$

Similarly, by defining $\bar{m}^i(z) \in E^k$ to be the vector of zeros in all coordinates except for $z \in E$ in the first and (T+i)-th coordinates, and using an identical argument, we can show that there exists $q^* \in E^*$ such that

$$\rho^{T+i} = q^* \qquad i = 1,\ldots,F.$$

Moreover, (3) now implies

$$p^* = -q^*.$$

Finally, since $v^* + (z,0) \in W$, $z \in E_-$, we obtain

$$\langle p^*, z\rangle \geq 0$$

and hence $p^* \in E_+^*$.

Next, given Assumption (A1), we appeal to Lemma 2.9 to assert that

(5) $\quad p \in N_A(\overline{P}^t(x^{*t}), x^{*t}) \Rightarrow p \in E_-^*$ for all t,

(6) $\quad p \in N_A(Y^j, y^{*j}) \Rightarrow p \in E_+^*$ for all j.

Furthermore, by Lemmata 2.4, 2.3 and Theorem 2.3, we obtain

(7) $\quad N_A(V(x^*), v^*) = \Pi_t N_A(\overline{P}^t(x^{*t}), x^{*t}) \times \Pi_j N_A(Y^j, y^{*j}).$

Now suppose there exists $\rho \in N_A(V(x^*), v^*)$, $\rho \neq 0$ such that $-\rho \in N_A(W, v^*)$. Then by combining (2) and (7), the proof of the theorem is complete. Thus we need only consider the case when

(8) $\quad \rho \in N_A(V(x^*), v^*),\ \sigma \in N_A(W, v^*),\ \rho + \sigma = 0 \Rightarrow \rho = \sigma = 0.$

But then we can apply Theorem 2.3 to assert that

(9) $\quad N_A(V(x^*), v^*) + N_A(W, v^*) = N_A(\{v^*\}, v^*)$

By Lemma 2.3, the right-hand side is $(E^k)^*$. By Lemma 2.5, $N_A(W, v^*)$ is a convex cone. If $N_A(W, v^*) = \{0\}$, then $N_A(V(x^*), v^*) = (E^k)^*$ and the proof is complete. If not, pick $\sigma \in N_A(W, v^*)$, $\sigma \neq 0$. Then by (8), $(-\sigma) \notin N_A(V(x^*), v^*)$. Since $(-\sigma) \in (E^k)^*$, by (9) there exist $\alpha \in N_A(V(x^*), v^*)$ and $\beta \in N_A(W, v^*)$ such that

$$(\alpha + \beta) = -\sigma$$

This implies $(\sigma + \beta) = -\alpha$. Since $\sigma \neq 0$, we obtain from (2) and the fact that $N_A(W,v^*)$ is convex, $(\sigma + \beta) \neq 0$ and $(\sigma + \beta) \in N_A(W,v^*)$. But this contradicts (8) and completes the proof of the theorem.

||

Proof of Theorem 3.2

For the strong Pareto optimal allocation $v^* = ((x^{*t}),(y^{*j})) \in E^k$, define the following sets in E^k.

$$V(x^*) = \Pi_t \overline{P}^t(x^{*t}) \times \Pi_j Y^j$$

$$W = \{v \in E^k: \ \Sigma_t x^t_\pi \leq \Sigma_j y^j_\pi + w_\pi; \ x^t_g \leq \Sigma_j y^j_g \text{ for all } t\}.$$

As in the proof of Theorem 3.1, $V(x^*)$ and W are closed sets, and $V(x^*)$ is epi-Lipschitzian. We can also assert that

$$V(x^*) \cap W = \{v^*\}.$$

Suppose there exists $v = ((x^t),(y^j)) \neq v^*$ in $V(x^*) \cap W$. If $x^t_g = \Sigma_j y^j_g$ for all t, we contradict the fact that v^* is a strong Pareto optimal allocation. Suppose therefore that there exist t and $v_t \in E_{g+}$, $v_t \neq 0$ such that $x^t_g + v^t = \Sigma_j y^j$. Denote the set of all such t by M. Then, under assumption (A1), the allocation $((\overline{x}^t),(y^j))$, where

$$\overline{x}^t = x^t \quad \text{for all } t \notin M$$

$$= (x^t_\pi, x^t_g+v^t) \quad \text{for all } t \in M,$$

can be used to contradict the fact that $((x^{*t}),(y^{*j}))$ is a strong Pareto optimal allocation.

As in the proof of Theorem 3.1, we can show that

(1) $\rho \in N_A(W,v^*) \Rightarrow \langle\rho,x\rangle \leq \langle\rho,v^*\rangle$ for all $x \in W$.

We can now show that for any $\rho \in N_A(W,v^*)$, $\rho \neq 0$, there exist $(p^*_\pi,p^{*t}_g) \in E^*$, $(p^*_\pi,p^{*t}_g) \neq 0$ for all t such that

$$\rho = ((p^*_\pi,p^{*1}_g), \ldots, (p^*_\pi,p^{*T}_g), - (p^*_\pi,\Sigma_t p^{*t}_g), \ldots,$$

$$-(p^*_\pi,\Sigma_t p^{*t}_g)) \in (E^*_+)^T \times (E^*_-)^F.$$

In order to show ρ^i_π is independent of i for i = 1,...,T as well as for i = T+1,...,T+F, and that $\rho^i_\pi = -\rho^j_\pi$, i = 1,...T and j = T+1,...,T+F, we use an argument identical to that in the proof of Theorem 1. The only change is to let $m^i(z)$ $(\overline{m}^i(z))$ to be a vector of zeros in all coordinates except for (z,0) in the i-th ((T+i)-th) and the last (first) coordinates and with $z \in E_\pi$. Thus we need only consider the projections onto the space of public goods. Towards this end, for any $z \in E_g$, define $m^j_g(z)$ to be the vector consisting of $(0,z_g)$ in every coordinate from 1 to T and including T+j, and zero everywhere else. Then $(v^*+m^j_g(z))$ and $(v^*-m^j_g(z))$ are elements of W and we obtain from (1),

$$\Sigma_t\langle\rho^t_g,z\rangle = -\langle\rho^j_g,z\rangle.$$

Since z and the index j are chosen arbitrarily, we obtain

$$\rho^j_g = -\Sigma_t\rho^t_g \qquad j = 1,\ldots,F,$$

which is what we intended to show.

Now the rest of the proof can be completed using arguments identical to those in the proof of Theorem 3.1. ||

Proof of Theorem 3.3

The proof of Theorem 3.1 is valid with $P_v^t(x^{*t}) \cup \{x^{*t}\}$ substituted for $\overline{P}_v^t(x^{*t})$. The former set is closed and epi-Lipschitzian at x^{*t} by hypothesis and we need only check that

$$V(x^*) \cap W = \{v^*\}.$$

This follows as a consequence of the definition of a locally Pareto optimal allocation.

Analogous changes are required in the proof of Theorem 3.2. ||

Proof of Theorem 3.4

The proof is a simple consequence of Theorem 3.1 and Theorem 2.4.

||

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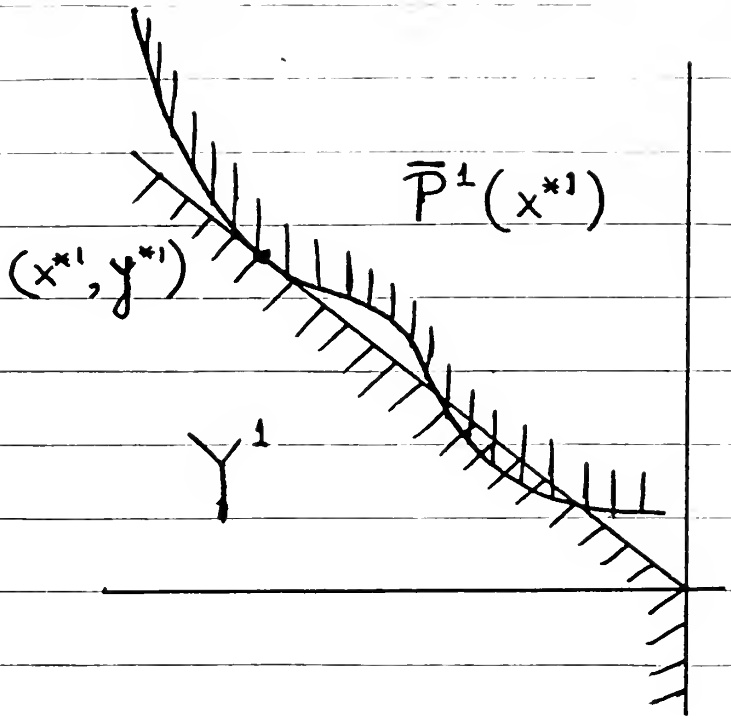

Figure 1a.

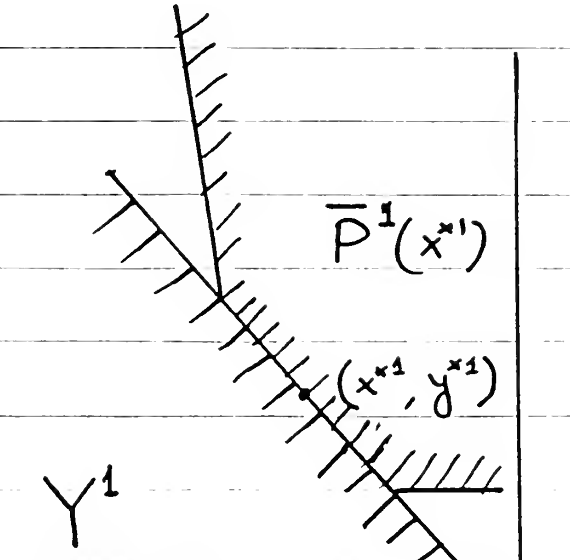

Figure 1b.