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The Iterative Step in the Linear Programming Algorithm of N. Karmarkar

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The Iterative Step in the Linear Programming Algorithm of N. Karmarkar

by Charles Blair

Abstract

We simplify and strengthen the analysis of the improvement obtained in one step of Karmarkar's algorithm.

The recently published [1] algorithm of N. Karmarkar uses the following step:

Suppose $x = (a_1, \ldots a_n) > 0$ is a feasible solution to the LP:

$$\begin{array}{l} \text{minimize } cx \\ \text{subject to } Ax=0,\ x \geq 0,\ \Sigma x_i = 1 \end{array} \qquad (1)$$

We will assume the optimal solution to (1) has objective function value ≤ 0, and that $ca > 0$. We refer to [1, section 6] for proofs that a method of solving this type of problem yields a method of solving any LP.

Let $x = (z_1, \ldots z_n)$ be the optimal solution to

$$\begin{array}{l} \min c(a_1 x_1, \ldots a_n x_n) \\ \text{subject to } A(a_1 x_1, \ldots, a_n x_n) = 0,\ \Sigma x_i = 1,\ \| x - (\frac{1}{n}, \ldots \frac{1}{n}) \| \leq \alpha (n(n-1))^{-\frac{1}{2}} \end{array} \qquad (2)$$

where $\alpha < 1$ is a parameter to be specified. [1, Theorem 5] shows that (2), which is a minimization of a linear objective function on a sphere, can be carried out using $O(n^3)$ operations.

The next feasible solution to (1) generated by the algorithm is $w = \gamma(a_1 z_1, \ldots, a_n z_n)$ where the scalar γ is chosen so that $\Sigma w_i = 1$.

Let $f(x) = (cx)^n / \prod_1^n x_i$ (this is the same as the potential function f in [1], except we do not use logarithms). To show that the new solution w is "better" than the previous solution, [1, Theorem 2] shows

Theorem 1: For some $k < 1$ (dependent on α) $f(w) \leq kf(a)$.

Since $\Sigma x_i = 1$ and $x_i \geq 0$ implies $\Pi x_i \leq n^{-n}$, Theorem 1 implies that, if the optimal solution to (1) has objective function value zero and v is obtained from a after t iterations $(cv)^n \leq k^t n^n f(a)$. As indicated in [1], this property yields a polynomial-time algorithm.

In this note, we give a new proof of Theorem 1, which gives a slightly better value of k and is more elementary in that logarithms are not used.

Lemma 2*: $\Sigma c_i a_i z_i \leq n^{-1}(1-\alpha/(n-1))\Sigma c_i a_i$.

Proof: Since the optimal solution to (1) is assumed to have value ≤ 0, there is a $u \geq 0$ satisfying $A(a_1 u_1, \ldots, a_n u_n) = 0$, $\Sigma u_i = 1$, and $\Sigma c_i a_i u_i \leq 0$. Since $\| u - (\frac{1}{n}, \ldots \frac{1}{n}) \|^2 \leq (1 - \frac{1}{n})^2 + (n-1)n^{-2} = (n-1)n^{-1}$, and z is the optimal solution to (2), $\Sigma c_i a_i z_i$ must be

$\leq (1-\lambda)(\Sigma c_i a_i n^{-1}) + \lambda(\Sigma c_i a_i u_i) \leq (1-\lambda)n^{-1}(\Sigma c_i a_i)$, where

$\lambda = (\alpha(n(n-1))^{-\frac{1}{2}})/((n-1)n^{-1})^{\frac{1}{2}} = \alpha/n-1.$ Q.E.D.

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*This is the same as [1, Theorem 3].

<u>Lemma 3</u>: If $Q > R > S > 0$, then there exist $\varepsilon, \delta > 0$ such that (i) $(Q-\varepsilon)^2 + (R+\varepsilon+\delta)^2 + (S-\delta)^2 = Q^2 + R^2 + S^2$ and (ii) $(Q-\varepsilon)(R+\varepsilon+\delta)(S-\delta) < QRS$.

<u>Proof</u>: For δ close to zero, there exists an ε close to zero such that (i) holds. Since $\frac{\varepsilon}{\delta} \to (R-S)/(Q-R)$ as $\delta \to 0$,
$\text{Lim}\ \frac{1}{\delta}\,(QRS-(Q-\varepsilon)(R+\varepsilon+\delta)(S-\delta)) = (QR-QS) + (S^2-RS) = (R-S)(Q-S) > 0.$

Q.E.D.

<u>Lemma 4</u>: If $Q > R > 0$, then there exist $\varepsilon, \delta > 0$ such that (i) $(Q-\varepsilon)^2 + (R+\varepsilon+\delta)^2 + (R-\delta)^2 = Q^2 + 2R^2$ and (ii) $(Q-\varepsilon)(R+\varepsilon+\delta)(R-\delta) < QR^2$.

<u>Proof</u>: For δ close to zero, there exists ε close to zero such that (i) holds. Since $\text{Lim}\ \varepsilon\delta^{-2} = (Q-R)^{-1}$,
$\text{Lim}\ \delta^{-2}(QR^2-(Q-\varepsilon)(R+\varepsilon+\delta)(R-\delta)) = Q - R > 0.$ Q.E.D.

<u>Lemma 5</u>: If $\| x-(\frac{1}{n},\ldots\frac{1}{n})\| = \alpha(n(n-1))^{-\frac{1}{2}}$ and $\Sigma x_i = 1$, then $\Pi x_i \geq n^{-n}(1+\alpha/(n-1))^{n-1}(1-\alpha)$.

<u>Proof</u>: By continuity, there is an x^* which minimizes Πx_i among those x which satisfy the assumptions. $\alpha < 1$ implies $x_i > 0$ for all i, since $(n-1)(n^{-1}-(n-1)^{-1})^2 + n^{-2} = (n(n-1))^{-1}$. By Lemma 3, we cannot have $x_i^* > x_j^* > x_k^*$ for some i,j,k. (Note that $\Sigma x_i = 1$ and $\Sigma x_i^2 = \Sigma(x_i^*)^2$ imply $\| x-(\frac{1}{n},\ldots\frac{1}{n})\| = \| x^* - (\frac{1}{n}, \ldots \frac{1}{n})\|$.) Thus the components of x* have two different values. By Lemma 4, there cannot be more than one component of x* having the smaller value. Thus x* consists of n-1 components with a larger value, and one component with a smaller value. This occurs only if n-1 components of x* are $n^{-1}(1+\alpha/(n-1))$ and one component is $n^{-1}(1-\alpha)$. Q.E.D.

Theorem 6: If a is a feasible solution and w the next solution given by the algorithm

$$f(w) \leq (1-\alpha/n-1)^{n}(1+\alpha/n-1)^{1-n}(1-\alpha)^{-1}f(a) \qquad (3)$$

Proof: Recall that $w = \gamma(a_1z_1,\ldots,a_nz_n)$, hence $f(w) = f(a_1z_1,\ldots,a_nz_n)$. By Lemma 2, $(\Sigma c_i a_i z_i)^n \leq n^{-n}(1-\alpha/(n-1))^n(\Sigma c_i a_i)^n$. Since $(1+\alpha/n-1)^{n-1}(1-\alpha)$ is monotone decreasing as a function of α,

Lemma 5 and $\| z-(\frac{1}{n},\ldots\frac{1}{n})\| \leq \alpha(n(n-1))^{-\frac{1}{2}}$ imply $\Pi a_i z_i = \Pi a_i \Pi z_i \geq (\Pi a_i)n^{-n}(1+\alpha/n-1)^{n-1}(1-\alpha)$. Thus $f(w) \leq (1-\alpha/(n-1))^n(\Sigma c_i a_i)^n/(\Pi a_i)(1+\alpha/n-1)^{n-1}(1-\alpha)$. Q.E.D.

For comparison, [1, Theorem 4] shows that, for $\alpha = \frac{1}{4}$ and n large, $f(w) \leq \exp(-13/96)f(a)$. Theorem 6 yields $f(w) \leq \frac{4}{3}\exp(-\frac{1}{2})f(a)$.

The right-hand-side of (3) is minimized when $\alpha = (n-1)/(2n-3)$. This may be the best single choice of α, if it is to be kept constant through all iterations.

Reference

1. N. Karmarkar, "A New Polynomial-Time Algorithm for Linear Programming," Technical Report, AT&T Bell Laboratories.

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