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Abstract

We simplify and strengthen the analysis of the improvement obtained in one step of Karmarkar's algorithm.

The recently published [1] algorithm of N. Karmarkar uses the following step:

Suppose $x = (a_1,\ldots a_n) > 0$ is a feasible solution to the LP:

$$\begin{aligned}&\text{minimize } cx\\ &\text{subject to } Ax=0,\ x\geq 0,\ \Sigma x_i=1\end{aligned} \tag{1}$$

We will assume the optimal solution to (1) has objective function value ≤ 0, and that $ca > 0$. We refer to [1, section 6] for proofs that a method of solving this type of problem yields a method of solving any LP.

Let $x = (z_1,\ldots z_n)$ be the optimal solution to

$$\begin{aligned}&\min\ c(a_1x_1,\ldots a_nx_n)\\ &\text{subject to } A(a_1x_1,\ldots,a_nx_n)=0,\ \Sigma x_i=1,\ \|x-(\tfrac{1}{n},\ldots\tfrac{1}{n})\| \leq \alpha(n(n-1))^{-\frac{1}{2}}\end{aligned} \tag{2}$$

where $\alpha < 1$ is a parameter to be specified. [1, Theorem 5] shows that (2), which is a minimization of a linear objective function on a sphere, can be carried out using $O(n^3)$ operations.

The next feasible solution to (1) generated by the algorithm is $w = \gamma(a_1z_1,\ldots,a_nz_n)$ where the scalar γ is chosen so that $\Sigma w_i=1$.

Let $f(x) = (cx)^n / \prod_1^n x_i$ (this is the same as the potential function f in [1], except we do not use logarithms). To show that the new solution w is "better" than the previous solution, [1, Theorem 2] shows

Theorem 1: For some $k < 1$ (dependent on α) $f(w) \leq kf(a)$.

Since $\Sigma x_i = 1$ and $x_i \geq 0$ implies $\Pi x_i \leq n^{-n}$, Theorem 1 implies that, if the optimal solution to (1) has objective function value zero and v is obtained from a after t iterations $(cv)^n \leq k^t n^n f(a)$. As indicated in [1], this property yields a polynomial-time algorithm.

In this note, we give a new proof of Theorem 1, which gives a slightly better value of k and is more elementary in that logarithms are not used.

Lemma 2*: $\Sigma c_i a_i z_i \leq n^{-1}(1-\alpha/(n-1))\Sigma c_i a_i$.

Proof: Since the optimal solution to (1) is assumed to have value ≤ 0, there is a $u \geq 0$ satisfying $A(a_1 u_1, \ldots, a_n u_n) = 0$, $\Sigma u_i = 1$, and $\Sigma c_i a_i u_i \leq 0$. Since $\| u - (\frac{1}{n}, \ldots \frac{1}{n}) \|^2 \leq (1 - \frac{1}{n})^2 + (n-1)n^{-2} = (n-1)n^{-1}$, and z is the optimal solution to (2), $\Sigma c_i a_i z_i$ must be $\leq (1-\lambda)(\Sigma c_i a_i n^{-1}) + \lambda(\Sigma c_i a_i u_i) \leq (1-\lambda)n^{-1}(\Sigma c_i a_i)$, where $\lambda = (\alpha(n(n-1))^{-\frac{1}{2}})/((n-1)n^{-1})^{\frac{1}{2}} = \alpha/n-1$. Q.E.D.

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*This is the same as [1, Theorem 3].

Lemma 3: If $Q > R > S > 0$, then there exist $\varepsilon, \delta > 0$ such that (i) $(Q-\varepsilon)^2 + (R+\varepsilon+\delta)^2 + (S-\delta)^2 = Q^2 + R^2 + S^2$ and (ii) $(Q-\varepsilon)(R+\varepsilon+\delta)(S-\delta) < QRS$.

Proof: For δ close to zero, there exists an ε close to zero such that (i) holds. Since $\frac{\varepsilon}{\delta} \to (R-S)/(Q-R)$ as $\delta \to 0$,
$\text{Lim}\ \frac{1}{\delta}(QRS-(Q-\varepsilon)(R+\varepsilon+\delta)(S-\delta)) = (QR-QS) + (S^2-RS) = (R-S)(Q-S) > 0$.

Q.E.D.

Lemma 4: If $Q > R > 0$, then there exist $\varepsilon, \delta > 0$ such that (i) $(Q-\varepsilon)^2 + (R+\varepsilon+\delta)^2 + (R-\delta)^2 = Q^2 + 2R^2$ and (ii) $(Q-\varepsilon)(R+\varepsilon+\delta)(R-\delta) < QR^2$.

Proof: For δ close to zero, there exists ε close to zero such that (i) holds. Since $\text{Lim}\ \varepsilon\delta^{-2} = (Q-R)^{-1}$,
$\text{Lim}\ \delta^{-2}(QR^2-(Q-\varepsilon)(R+\varepsilon+\delta)(R-\delta)) = Q - R > 0$. Q.E.D.

Lemma 5: If $\|x-(\frac{1}{n},\ldots\frac{1}{n})\| = \alpha(n(n-1))^{-\frac{1}{2}}$ and $\Sigma x_i = 1$, then $\Pi x_i \geq n^{-n}(1+\alpha/(n-1))^{n-1}(1-\alpha)$.

Proof: By continuity, there is an x^* which minimizes Πx_i among those x which satisfy the assumptions. $\alpha < 1$ implies $x_i > 0$ for all i, since $(n-1)(n^{-1}-(n-1)^{-1})^2 + n^{-2} = (n(n-1))^{-1}$. By Lemma 3, we cannot have $x_i^* > x_j^* > x_k^*$ for some i,j,k. (Note that $\Sigma x_i = 1$ and $\Sigma x_i^2 = \Sigma(x_i^*)^2$ imply $\|x-(\frac{1}{n},\ldots\frac{1}{n})\| = \|x^* - (\frac{1}{n}, \ldots \frac{1}{n})\|$.) Thus the components of x^* have two different values. By Lemma 4, there cannot be more than one component of x^* having the smaller value. Thus x^* consists of $n-1$ components with a larger value, and one component with a smaller value. This occurs only if $n-1$ components of x^* are $n^{-1}(1+\alpha/(n-1))$ and one component is $n^{-1}(1-\alpha)$. Q.E.D.

Theorem 6: If a is a feasible solution and w the next solution given by the algorithm

$$f(w) \le (1-\alpha/n-1)^n (1+\alpha/n-1)^{1-n} (1-\alpha)^{-1} f(a) \qquad (3)$$

Proof: Recall that $w = \gamma(a_1 z_1, \dots, a_n z_n)$, hence $f(w) = f(a_1 z_1, \dots, a_n z_n)$. By Lemma 2, $(\Sigma c_i a_i z_i)^n \le n^{-n}(1-\alpha/(n-1))^n (\Sigma c_i a_i)^n$. Since $(1+\alpha/n-1)^{n-1}(1-\alpha)$ is monotone decreasing as a function of α, Lemma 5 and $\| z-(\frac{1}{n}, \dots \frac{1}{n}) \| \le \alpha(n(n-1))^{-\frac{1}{2}}$ imply $\Pi a_i z_i = \Pi a_i \Pi z_i \ge (\Pi a_i) n^{-n} (1+\alpha/n-1)^{n-1}(1-\alpha)$. Thus $f(w) \le (1-\alpha/(n-1))^n (\Sigma c_i a_i)^n / (\Pi a_i)(1+\alpha/n-1)^{n-1}(1-\alpha)$. Q.E.D.

For comparison, [1, Theorem 4] shows that, for $\alpha = \frac{1}{4}$ and n large, $f(w) \le \exp(-13/96) f(a)$. Theorem 6 yields $f(w) \le \frac{4}{3}\exp(-\frac{1}{2}) f(a)$.

The right-hand-side of (3) is minimized when $\alpha = (n-1)/(2n-3)$. This may be the best single choice of α, if it is to be kept constant through all iterations.

Reference

1. N. Karmarkar, "A New Polynomial-Time Algorithm for Linear Programming," Technical Report, AT&T Bell Laboratories.