JACOBIAN ELLIPTIC FUNCTIONS

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BY

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PREFACE

At one time the study of elliptic functions began with the inversion of Legendre's integral. Every young mathematician was familiar with $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$, and algebraic identities between these functions figured in every examination. But a growing realization that the inversion of a complex integral raises issues which are not all elementary brought about a change. To-day, many a good teacher says nothing of the Jacobian functions until he can utilize theta functions, and many a good student learns nothing of them at all. Moreover, a theory in which the definition of the fundamental function takes the form

$$\frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(u/\vartheta_3^2)}{\vartheta_4(u/\vartheta_3^2)}$$

starts with a handicap of artificiality from which the older treatment, whatever its faults, was free.

This book is an attempt to restore the Jacobian functions to the elementary curriculum by exhibiting them as functions constructed on a lattice. In the course of the general theory of doubly periodic functions, we find that the lowest order possible for such a function is the second, and that therefore the simplest functions have either one double pole or two simple poles in a primitive parallelogram. The investigation of the first possibility is the invariable method of introducing the Weierstrassian function $\wp z$. It is seldom—the first edition of *Modern Analysis* was an honourable exception—that the investigation of the alternative is recognized as the natural sequel. This is our starting-point. We associate with an arbitrary Weierstrassian function a symmetrical group of functions of the second kind, and this group becomes a Jacobian system by an appropriate specialization of one of the parameters fundamental in the theory. So found, the Jacobian functions are known in advance to be doubly periodic, no parameters are restricted to be real, and simple functional proofs of addition theorems and of the transformations of Jacobi and Landen replace the algebraical proofs demanded by the inverted integral.

For a moment we are tempted to think that the problem of inverting an integral need not be faced. The classical functions have come easily into analysis, they display a multitude of fascinating properties, and their relations to their derivatives imply that they can be used for the evaluation of integrals of the forms with which they are traditionally associated. Let them be studied, and they will be available when wanted. But will they? If b and c are certain critical constants in the theory of a known system of elliptic functions, the integral

$$\int_w^\infty \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}}$$

is identifiable as an inverse function. But if it is b and c that are given, as is almost always the case, in both pure and applied mathematics, when the integral turns up first, have we any reason to suppose that a system exists in which the given b and c do play the essential parts? We are back at the fundamental question. Every elliptic function is the inverse of an elliptic integral; is it true also that the inverse of every elliptic integral is an elliptic function?

There is no logical objection to postponing the consideration of this question. Even if we ignore the problem altogether, our theory is no less satisfactory than the elementary theory of $\wp z$, where precisely the same problem is ignored: we find that if $w = \wp z$, then

$$z = \int_w^\infty \frac{dw}{\sqrt{\{4(w-e_1)(w-e_2)(w-e_3)\}}},$$

but we do not discuss whether for arbitrary values of e_1, e_2, e_3 subject to the condition $e_1+e_2+e_3 = 0$ a Weierstrassian function must exist. For practical purposes, only the answer* to the question is required, and there is no difficulty in explaining the answer. Nevertheless, even in an unambitious course something more than a simple question and

* Together naturally with the solution of the more elementary problem, also described sometimes as the problem of inversion: assuming that the system exists, to determine from the constants in the integral either the elliptic function itself or the lattice on which it hangs. To find the only possible lattice is, as will be seen in the text, a simple matter; the difficulty is to prove that the functions on this lattice do provide the assigned constants. Merely to construct an inverse function by direct operations does not solve the theoretical problem. For example, Hancock's exceedingly thorough account of inversion on a Riemann surface (*Theory of Elliptic Functions*, I) is beyond criticism as a solution of the practical problem, but begs the whole theoretical question in the one sentence (p. 163): Instead of the variable u we may introduce any variable quantity, say

$$u(z, s) = \int_{z_0, s_0}^{z, s} \frac{dz}{s}.$$

No reason is adduced for supposing that $u(z, s)$, so defined, can take an arbitrary value, but u presently becomes the independent variable. If a complete solution of the inversion problem along Riemannian lines is wanted, Hancock's treatise needs a supplement equivalent to the excellent ninth chapter of Neumann's *Riemann's Theorie der Abel'schen Integrale* (Die Umkehrung des elliptischen Integrales), where the problem of ubiquity is stated very clearly.

a straightforward answer seems wanted. We dare not say that we *understand* the relation between the function and the integral unless we see how the double periodicity of the function is implicit in the integral form of the relationship, and in the discovery of double periodicity from this end the origin of the constants is not relevant. Since also definition by inversion of an integral is equivalent to definition by a simple form of differential equation and is not in itself a suspicious process, a mystery remains for the student unless we put a finger for him on the ultimate difficulty. In point of fact, the more precisely the problem of inversion is analysed, the narrower the crucial gap becomes and the less formidable the task of bridging it appears.

The design of this treatise will now be intelligible. There are three divisions of the subject, first the direct theory of functions with simple poles derived from a Weierstrassian function whose periods are arbitrary, then the theory of the inverted integral and the solution of the problem of inversion, and lastly the fertile theory of the classical system. To the writer the order of exposition is almost inevitable, but the reader impatient to make the acquaintance of Jacobi's functions can pass to Chapter X from Chapter IV or even from Chapter III, and he can return at any time to read Chapter VI, on the connexion between integration and periodicity, as an independent chapter and not necessarily as a stage in the inversion argument.

Far from being new to analysis, the three 'primitive' functions defined in Chapter I have often been studied. Jordan in his *Cours d'Analyse* and Tannery and Molk in their *Fonctions Elliptiques* allow a few pages to them and define the classical functions in terms of them; in papers on Poncelet's poristic polygons, Chaundy and Baker* use the same three functions, instead of relying explicitly, as does Halphen in the account of this problem in the second volume of his treatise, on the Weierstrassian functions $\wp z$, σz. The point to be emphasized is the deliberate construction of the functions as functions with simple poles. As algebraic functions of $\wp z$, important in the development of the theory of $\wp z$ itself, the functions go back to Weierstrass.

The primitive functions belong to a group of twelve, and it is this group which is the subject of Chapters II-IV. My notation for the functions is new, and is designed to reflect both the structure of the

* Chaundy, *Proc. London Math. Soc.* (2), 22 (1924) p. 104 and 25 (1926) p. 17; Baker, *Proc. Cambridge Phil. Soc.*, 23 (1926) p. 92. Chaundy takes a knowledge of the functions for granted, Baker derives an addition theorem for them from a differential equation.

functions and their relation to the Jacobian system. If I rewrote the book, I should perhaps develop the theory of these functions at much greater length, but at least I have avoided the extremes of presenting the theory merely as an elaboration on that of $\wp z$ and merely as a preparation for that of $\operatorname{sn} u$.

Chapters V–VIII are devoted to a standard elliptic integral and its inversion. In Chapter V we see precisely what relation between an elliptic function and an elliptic integral is established in the direct theory of the elliptic function. Chapter VI deals, as I have said, with the periodicity of the inverted integral. In Chapter VII two proofs are given of the existence theorem which has been shown to be crucial for the inversion problem. The first of these is an application, new in principle as far as I know, of the theory of aggregates; the second is essentially Goursat's, with adapted notation. Whether the first proof or the second is the 'simpler' depends entirely on the reader's equipment. Given the requisite knowledge of the theory of aggregates, the first proof is brief and straightforward: the line of argument, once indicated, is obvious, and the details are easily filled in from an examination of the integral to be inverted. Goursat uses only the familiar processes of analysis, but economical presentation of his proof calls for considerable algebraical ingenuity; the formulae required belong to the theory of the function with which the inverted integral is to be identified, and are not suggested by mere inspection of the integrand. Incidentally, this proof shows that as a problem in analysis the inversion problem is not as deep as the better-known solution by means of a modular equation inclines us to believe. Chapter VIII brings together the main threads from Chapters VI–VII and completes the solution of the fundamental problem; to read this chapter profitably, it is necessary to accept the conclusion of the principal theorem in Chapter VII, but it is not necessary to have mastered a demonstration of this theorem.

The essence of Chapter X, which introduces the classical functions, is that the functions are regarded as functions constructed on a canonical lattice. The condition which a 'Jacobian' lattice is to satisfy is laid down after a comparison of integrals; this presents the condition as a natural condition, while ensuring that the functions will be the classical functions. An arbitrary lattice is rendered Jacobian on multiplication by a 'normalizing factor', which is found as the value at a particular point of the lattice of a definite elliptic function attached to the lattice; whatever the lattice, the normalizing factor exists and is unique. That is to say, a Jacobian lattice may have any shape, but, for a given shape, is determinate in size and orientation.

Since it is the lattice rather than the system of functions attached to it that is standardized, the theory of the Jacobian functions tends in its opening stages to repeat the theory developed in Chapters I–IV. The repetition, however, is slight, for the utter lack of symmetry in the Jacobian system introduces a new element: formulae may be discovered in a typical form, but if they are to be readily available they must be tabulated in detail.

In one respect the influence of the earlier theory permeates the later chapters. We see* the subject of investigation not as a set of three functions but as a group of twelve; in a variety of senses this group is complete, it stratifies naturally into four triads of copolar functions, and since the four triads are closely interrelated, any attempt to express all formulae in terms of the members of one triad is a false economy in the language. Jacobi's original functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ constitute one of the four triads, but the poles of these functions are congruent with iK', not with the origin, and from the functional point of view a treatment in which this triad plays the leading part is strictly analogous to a version of the Weierstrassian theory which should be written round the function $\wp(z-\omega_2)$ instead of round the function $\wp z$. It is only in deference to tradition and for the sake of readers who will expect this book to prepare them for the general literature of the subject that I have frequently given the same prominence to formulae relating specifically to the classical triad as to the corresponding formulae relating to the triad† $\operatorname{cs} u$, $\operatorname{ns} u$, $\operatorname{ds} u$.

Although few details of mathematical notation are accepted with the same unanimity as the use of K and iK' for Jacobian quarter-periods, I usually write instead K_c and K_n. For this iconoclasm I offer in advance three reasons. First, using K_d for $-(K_c+K_n)$, with K_s as

* That this view was not long ago universal is one of the minor mysteries of mathematics, or perhaps one of the major examples of mistaken subservience. It was in 1882 that Glaisher recognized that the group should be completed, and devised the perfect notation, but outside England, from Bobek in 1884 to Tricomi in 1937, Glaisher's nine functions have been completely ignored, in spite of the suggestive table on p. 30 of the Weierstrass-Schwarz *Formeln und Lehrsätze*. The strangest case is that of Tannery and Molk, since they have an explicit notation for the twelve functions on an arbitrary lattice. But Cayley could speak of 'the elliptic functions properly so called, the functions sn, cn, dn', and dismissed the other nine functions with a curt 'These are not required'.

† It is significant that when M. Roberts in his *Tract on the Addition of Elliptic and Hyper-elliptic Integrals* (1871) applies a general theorem of Jacobi's to the elliptic integrals, it is the formulae for this triad that he finds first (p. 10), although his work is wholly in the real domain.

an alternative symbol for the origin, we promote Glaisher's notation from a mere algebraical mnemonic to a structural notation, for pq u is a function with a zero at K_p and a pole at K_q. Secondly, a typical symbol for a cardinal point opens the possibility of typical formulae, and this, in a subject threatened with suffocation by the sheer multitude of individual formulae, is no light relief. Thirdly, when the modulus of the system is arbitrarily complex, the two quarterperiods are alike complex, and the insertion of a factor i in the second of them is for most purposes inconvenient if not misleading. But details of notation are to be judged pragmatically, not logically, and I can only ask the reader to postpone criticism. There is of course work for which the classical notation is wanted, and my intention has been to revert without hesitation whenever the occasion invites.

The fundamental transformations, the subject of Chapter XIII, are found by a comparison of patterns of poles and zeros. In every case the functional relations are obvious, and the ratio of one variable to another is simply a normalizing factor by which a lattice is made Jacobian. In this treatment of the transformations, rather than in any more abstract considerations, is the most powerful argument for an innovation of which a hint has already been let fall. To say that the ratio of iK' to K is arbitrary implies that the customary convention that $\mathrm{Im}(iK'/K)$ is intrinsically positive has been abandoned. To replace this convention I introduce into the formulae a constant υ which is $+i$ or $-i$ according as $\mathrm{Im}(iK'/K)$ is positive or negative. The device sounds childish, and I did not incorporate it without misgiving, but I hope it will commend itself by its effects.

The heading of the next chapter will mislead; the subject is not the reduction of algebraic integrals, but the integration of Jacobi-Glaisher functions and their products. Only one new transcendent is necessary, but surely it is anomalous to welcome the increase from Jacobi's three functions to Glaisher's twelve as an advance but to insist that at all costs twelve corresponding integrals are to be expressed in terms of one of their number. For each of the twelve functions pq u I denote the integral of $\mathrm{pq}^2 u$, with the natural constant of integration, by Pq u. A table, XIV 2, gives the integrating function Pq u in terms of the classical function $E(u)$, which is Dn u, and another, XIV 3, gives Pq u in terms of Dc u, a function which on theoretical grounds has the same standing as Dn u.

Chapter XV deals with dependence on the modulus. Hermite's forgotten method of writing down the derivatives of the Jacobian functions with respect to the parameter c immediately in terms of integrating functions is revived. The results lead naturally to a discussion of the quarter-periods as functions of c, and the linear differential equations satisfied by K and K' and by $E-c'K$ and $E'-cK'$ are solved completely.

Theta functions are the subject of Chapter XVI. In accordance with the general outlook they are introduced as integral functions with specified lattices of zeros. Partitions of the four fundamental functions lead by logarithmic differentiation to series for the twelve Jacobian functions; except for an anomalous first term in six cases, these series are Fourier series. The reader must be warned that much of the notation in this chapter, though so natural as to seem inevitable in the context, is new; in particular, the functions $\vartheta_s(u)$, $\vartheta_c(u)$, $\vartheta_n(u)$, $\vartheta_d(u)$ are constant multiples of Jordan's $\theta(u)$, $\theta_1(u)$, $\theta_2(u)$, $\theta_3(u)$.

The book is an essay in the theory of functions of a complex variable. but the nature of the functions and integrals as real functions of a real variable, when the parameters involved are real, is considered in Chapter IX for the functions of the opening chapters and in Chapter XVII for the Jacobian functions. In this last chapter dissections of $\mathrm{pq}(u+iv)$ and $\mathrm{Pq}(u+iv)$ are tabulated for application to conformal representation. A few pages touch on numerical evaluation, first by Legendre's original process, which uses a succession of Landen transformations, and lastly by direct use of q-series. The type of convergence of a Landen chain is superior in the long run to that of a q-series, but initially the chain and the series are about equally efficient. It is to be remembered that the Landen process comes to an end when the square of a modulus is negligible, and that if $k > k'$ the transformation can be operated in the direction* in which k' tends to zero. On the other hand, whereas the Landen transformation, valid always in theory, is of no practical value unless k and k' are real, q-series can be used when k and u are complex.

For the reader already acquainted with the general theory of doubly periodic functions and with the theory of the Weierstrassian functions the book begins on p. 50, but I have been persuaded to prefix a summary of the elements of these theories and of the theory of lattices rather than to take for granted or to prove incidentally the results I happened to need. The sole purpose of this introduction is to carry the work back logically to Cauchy's theorem.

* Cayley, *Elliptic Functions*, takes as an example $k = \sin 75°$ and reduces k to $0{\cdot}0^2 28260$ in three steps and to $0{\cdot}0^5 20$ in four, but in the other direction k' is reduced to $0{\cdot}0^4 751$ in two steps.

A collection of annotated exercises at the end of the volume provides an informal outlet of which I have been glad to avail myself. Some of the exercises lead to other proofs of theorems in the text: addition theorems turn up more than once, and Fourier series are found by contour integration. Numerical examples demonstrate that the processes recommended in the text for reducing and inverting an integral are eminently practical. Some standard transformations illustrate the importance of elliptic functions in the field of conformal representation. The theory of the functions defined in the first two chapters is carried a little way forward by means of a number of formulae extracted for the most part from Tannery and Molk. Also there are short excursions beyond the range of this treatise; readers to whom the developments are not new may still be interested to see the results under a changed perspective or with a structure exposed by a systematic notation.

Manifestly this treatise makes no pretence to be in any sense complete or impartial, but there is one omission which does call for explanation. As surely as a lattice is the proper background for an elliptic function, a Riemann surface is the proper background for an elliptic integral, but Riemann surfaces are not even mentioned. Several distinctions must however be borne in mind. The lattice is indispensable to our conception of the subject, but to introduce the surface would be to improve the language rather than to modify the arguments. The rudiments of lattice theory are simple and are extensively applied, and every mathematician must acquire them sooner or later. Even the most slovenly description of Riemann surfaces can not be brief, a student who is not particularly interested in algebraic functions and their integration need never know what a Riemann surface is, and a theory of elliptic functions dependent on an understanding of Riemann surfaces is relegated to the category of specialized studies even more fatally than a theory dependent on a knowledge of theta functions. The incidental uses of the theory of aggregates in Chapter VII and of symbolical solutions of differential equations in Chapter XV are not dangerous in the same way. In the first case, it is the result that matters, not this particular proof; also another proof is given. In the second case, a reader to whom the method is strange can verify the conclusions for himself.

Designed to present the subject from one point of view, the book is almost without references. It would not be hard to asterisk the formulae which occur explicitly in *Fundamenta Nova*, and to find others in

Glaisher's writings and in examination papers of the last half of the nineteenth century, but this would not be to trace the evolution of ideas.

To illness in 1940 I owe six months' uninterrupted leisure, and a long-projected work, without introduction or exercises, was completed early in 1941; in accepting the book at the most depressing moment of the war, the Delegates of the Clarendon Press paid me a compliment which I appreciate at its high value. Production has been slow and correction difficult. I am not one of those fortunate—or maybe unfortunate—writers to whom print never reveals defects unnoticed in manuscript, and I am grateful to the compositors for their patience in very trying circumstances.

In preparing the volume I have had the best assistance I could have wished. To enlist my old pupil and friend Mr. W. J. Langford gave me peculiar satisfaction, since it was for his benefit, so to speak, that I devised long ago all that is original in my presentation of the subject; I am proud that he was eager to labour for me, and that his enthusiasm has not dwindled. He undertook the specific task of verifying formulae and cross-references, but he was marvellously alert to every detail of phrasing and printing, and I am confident that few minor blemishes can have evaded his scrutiny.

My last word belongs to Professor T. A. A. Broadbent, formerly my colleague. From the roughest of manuscript notes to the printed page, every sentence and every symbol has come under his eye, and we have argued about grammar as well as about mathematics. If I say that he has checked the Tables and verified the Exercises, that the treatment of the elliptic integral in Chapter VI is the result of his dissatisfaction with my first draft, and that it was he who insisted that a chapter on theta functions must be inserted, it is not that these items exhaust the account but only that they are easy to enumerate. From first to last, making use in every possible way of his craftsmanship, his knowledge, and his wisdom, I have exploited gladly and shamelessly the friendship that has put his help uncalculated and incalculably at my disposal.

READING, E. H. N.

August 1943

CONTENTS

The two-decimal and the three-decimal reference numbers form independent sequences inside each section, the former being used for the more important results; thus in Ch. XV, § 15·4, the formula ·430 is incidental to the proof of the theorem 15·47 and comes later than 15·46. The integral part, signifying the chapter, is not used except in a reference from one chapter to another, and for purposes of reference, sections, theorems, and formulae in the Introduction are given the integral part 0.

LIST OF TABLES

INTRODUCTION: PROLEGOMENA

(i) Lattices

0·1. It is a fundamental principle in the theory of functions of a complex variable that in the absence of a barrier of singularities a function is determined intrinsically over the whole plane by the distribution of its values near any one point; more precisely, a single Laurent series, which may or may not be a Taylor series, determines a function. But this is not to say that if we know one series we have immediate knowledge of significant properties of the function. The series $-1-z-z^2-\ldots$ belongs, so to speak, to the function $1/(z-1)$, not the function to the series, and there is nothing in the series $1-z^2/2!+z^4/4!-\ldots$ to indicate that the function which it represents is a periodic function whose only zeros are real, or in the series

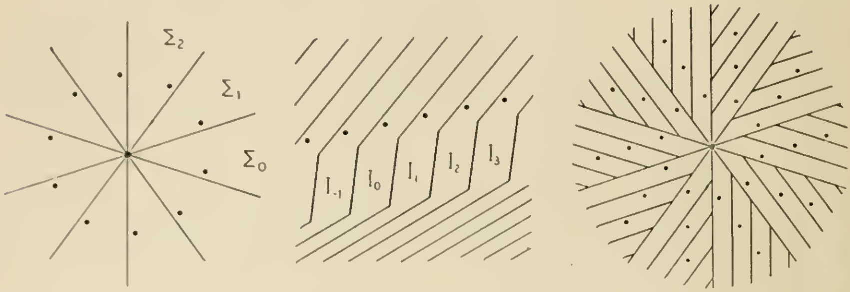
$$1+\frac{1}{2}z+\frac{1.3}{2.4}z^2+\frac{1.3.5}{2.4.6}z^3+\ldots$$

to suggest a branchpoint at $z = 1$. We know a function when we can describe its behaviour, not when we can somehow specify it.

A simple relation between the values of a function in one region and the values in another can be regarded in two ways: we may be content to say that the relation enables us to evade the direct examination of the function in one of the regions, or we may insist that the relation is itself a significant property of the function. The aspects are not distinct; the simpler the relation, both geometrically and analytically, the more fundamental the property. For example, the relation characteristic of an odd function is $f(-z) = -f(z)$, and whatever knowledge we possess of an odd function for values of z whose real part is positive is extended immediately to values of z whose real part is negative. The condition $f(1/z) = f(z)$ concentrates attention on the interior and circumference of the unit circle, leaving properties outside the circle to be inferred, and in particular substituting the neighbourhood of the origin for the distant regions of the plane.

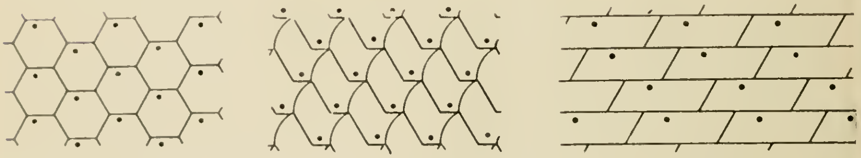
Of all the conditions to which a function may be subject, by far the most effective is a condition which consists geometrically of congruence in the elementary sense and analytically of sheer equality. If two regions have congruent boundaries, then to any point of the one there corresponds a point occupying a congruent position in the other, and if a function has the same values at congruent points, then for that function one region is a copy of the other. If the whole plane is dissected into congruent regions, and if the functional equality holds between every pair of these regions, then one region represents the whole plane.

There are numerous ways of dissecting the plane into congruent regions. Examples, which need not be described in words, are indicated in Figures 1–2.



1_1 1_2 1_3

FIG. 1.



2_1 2_2 2_3

FIG. 2.

In Figure 1_1, suppose that the number of the sectors is n, denote the angle $2\pi/n$ of a sector by 2α, and denote the region

$$(2r-1)\alpha \leqslant \theta < (2r+1)\alpha$$

by Σ_r; the region includes one of the bounding radii of a sector, but not the other. Except that the origin occurs in each region, the n congruent regions $\Sigma_0, \Sigma_1, \ldots, \Sigma_{n-1}$ together just fill the plane; Σ_r may be derived from Σ_0 by a simple rotation through the angle $2r\alpha$, and if z occupies any position in Σ_0, then $\omega^r z$, where $\omega = e^{2i\alpha}$, occupies the congruent position in Σ_r. The function $f(z)$ has the same distribution of values in every sector if $f(\omega^r z) = f(z)$ identically, for all values of z and for $r = 1, 2, \ldots, n-1$. This condition can be simplified to $f(\omega z) = f(z)$, since the more general condition follows by iteration from the simpler form.

In Figure 1_2 the number of regions is infinite, but this circumstance does not complicate either the geometry or the analysis. If z_0, z_1 are corresponding points in adjacent strips I_0, I_1, the difference z_1-z_0 is a number ω independent not only of the position of z_0 in I_0 but also of the choice of I_0, though the difference is replaced by its negative if I_1 is replaced by the strip on the other side of I_0. Assigning the symbol I_0 arbitrarily to one of the strips and I_1 to one of the two neighbours of I_0, we can correlate the strips with the series of symbols $\ldots, I_{-2}, I_{-1}, I_0, I_1, I_2, \ldots$, endless in each direction, and ω, regarded as a vector, defines a displacement which converts I_r into I_{r+1} simultaneously for all values of r. The points congruent with a point z constitute geometrically a paling, which will be said to have ω for a basis; analytically the numbers $z+r\omega$, for all integral values† of r, compose a congruence of which ω is a modulus. If χ and ω are bases of the same paling, the aggregates $r\omega$, $s\chi$ coincide; since χ is a member of the second aggregate there is an integer r_χ such that $\chi = r_\chi \omega$, and since ω is a member of the first aggregate there is an integer s_ω such that $\omega = s_\omega \chi$; since $r_\chi s_\omega = 1$, either $r_\chi = s_\omega = 1$ or $r_\chi = s_\omega = -1$: the only alternative basis to ω is $-\omega$.

The functional relation appropriate to Figure 1_2 is $f(z+\Omega) = f(z)$, to be satisfied if Ω is any step in the characteristic paling, that is, if Ω is any multiple of a basis ω. This relation is secured by the relation

·101 $$f(z+\omega) = f(z):$$

the function $f(z)$ has ω for a period.

The two examples which have been considered illustrate the control which the geometrical form of the congruence exercises over the functional relation. The existence of a functional relation exercises an equally strict control over the geometry, for if $f(z)$ is an analytic function of z, a function like $f(\omega z)-f(z)$ or $f(z+\omega)-f(z)$ can not be zero throughout a restricted region of the plane and different from zero elsewhere; in other words, a relation such as $f(\omega z) = f(z)$ or $f(z+\omega) = f(z)$ can not hold throughout one division of the plane and not be universal. Hence, for example, there can be no functional relation corresponding to the dissection in Figure 1_3, for although a rotation round the origin which carries one of the halfstrips into another carries every component of the pattern into another component, an oblique translation which carries one of the halfstrips into another changes the pattern completely except in one or two sectors.

The sectors of Figure 1_1 and the strips of Figure 1_2 extend to infinity.

† Negative, zero, and positive; integral, unqualified, is usually to be taken in this general sense.

The elements of the patterns in Figure 2 are bounded, and functions whose distributions of values are repeated from cell to cell of such patterns as these run their whole gamut right under our eyes. These functions are peculiarly accessible and possess a multitude of fascinating properties.

From the point of view of a functional relation there is far less difference between Figure 2_1 and Figure 2_2 than a casual glance suggests. We are concerned ultimately not with the shapes of the regions into which the plane is dissected but with the pattern formed by a set of congruent points, and if the geometrical congruence is taken in its simplest form, that is, without reflection or rotation although the cell in Figure 2_1 has the symmetry which admits both these operations, the configuration of corresponding points is of the same kind in the two diagrams. This configuration is known as a lattice. Figure 2_3, as the foundation of a functional relation, presents difficulties, but we can see at once that in this dissection corresponding points compose a pair of congruent lattices.

A lattice is perhaps described most easily as the set of points of intersection of two families of equidistant parallel lines. But we must recognize that the lines, however convenient, are not fundamental. It is not merely that our concern is with the points themselves; we can, as indicated in Figure 3, change the lines completely without changing the aggregate of points.

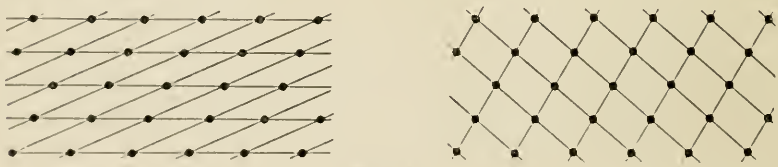


FIG. 3.

For analytical purposes a lattice is best specified by an origin and two vectors, for in the plane of the complex variable points and vectors alike are identified† by complex numbers. The origin is any point O of the lattice. If the lattice is determined by two families of parallel lines, one member a of one family and one member b of the other family pass through O; let A be one of the two points adjacent to O on a, and let B be one of the two points adjacent to O on b. Then if α, β are

† No attempt is made to maintain a consistent distinction between the language of geometry and the language of analysis; 'number', 'vector', and 'point in the complex plane' are interchangeable terms.

the vectors of the steps OA, OB, *the steps from* O *to the lattice points are those whose vectors have the form* $m\alpha+n\beta$, where m, n are independent integers.

We call the pair of vectors α, β, or the pair of complex numbers z_A-z_O, z_B-z_O for which the same symbols may be used, a basis of the lattice. A basis at one origin is a basis at any other origin. The basis is of less significance in the theory of functions than the lattice itself, for a change of basis does not necessarily affect the lattice and may therefore have no effect on functions which are being studied. A basis is none the less essential to the development of analysis.

The pairs of vectors α, β and γ, δ are bases of the same lattice if the aggregates of vectors $m\alpha+n\beta$ and $p\gamma+q\delta$ coincide, the coefficients in each case being integers. Since γ and δ are members of the second aggregate, there are integral coefficients such that

·102 $$\gamma = m_\gamma \alpha + n_\gamma \beta, \qquad \delta = m_\delta \alpha + n_\delta \beta;$$

since α and β are members of the first aggregate, there are integral coefficients such that

·103 $$\alpha = p_\alpha \gamma + q_\alpha \delta, \qquad \beta = p_\beta \gamma + q_\beta \delta.$$

Substituting from one pair of formulae in the other, we have the matrix relation

·104 $$\begin{pmatrix} m_\gamma, & n_\gamma \\ m_\delta, & n_\delta \end{pmatrix} \begin{pmatrix} p_\alpha, & q_\alpha \\ p_\beta, & q_\beta \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},$$

whence

·105 $$\begin{vmatrix} m_\gamma, & n_\gamma \\ m_\delta, & n_\delta \end{vmatrix} \begin{vmatrix} p_\alpha, & q_\alpha \\ p_\beta, & q_\beta \end{vmatrix} = 1,$$

and since the elements of the two determinants are integers,

·106 $$m_\gamma n_\delta - n_\gamma m_\delta = p_\alpha q_\beta - q_\alpha p_\beta = \pm 1.$$

The condition

·107 $$m_\gamma n_\delta - n_\gamma m_\delta = \pm 1$$

is sufficient as well as necessary to secure that γ and δ, defined by ·102, together form a basis of the lattice built on α, β, for with this condition we have from ·102,

·108 $$\pm\alpha = n_\delta \gamma - n_\gamma \delta, \qquad \pm\beta = -m_\delta \gamma + m_\gamma \delta.$$

From ·108, every vector of the form $m\alpha+n\beta$ is of the form $p\gamma+q\delta$; from ·102, every vector of the form $p\gamma+q\delta$ is of the form $m\alpha+n\beta$: the aggregates $m\alpha+n\beta$, $p\gamma+q\delta$ are identical.

Interchange of α and β or of γ and δ reverses the sign of $m_\gamma n_\delta - n_\gamma m_\delta$. It follows that if we are to attach significance to this sign we must regard the basis as an *ordered* pair of vectors. If $\alpha\beta$ and $\gamma\delta$ are ordered pairs, the function $m_\gamma n_\delta - n_\gamma m_\delta$ is known as the discriminant of the transformation of $\alpha\beta$ into $\gamma\delta$. Obviously the alternative of sign presented in ·107 divides the possible bases into two classes, but this division is in the first place a division in relation to $\alpha\beta$. Let a basis $\epsilon\zeta$ be derived from $\gamma\delta$ by the pair of formulae

$$\epsilon = p_\epsilon \gamma + q_\epsilon \delta, \qquad \zeta = p_\zeta \gamma + q_\zeta \delta$$

and also from $\alpha\beta$ by the pair of formulae

$$\epsilon = m_\epsilon \alpha + n_\epsilon \beta, \qquad \zeta = m_\zeta \alpha + n_\zeta \beta.$$

Then
$$\begin{vmatrix} m_\epsilon, & n_\epsilon \\ m_\zeta, & n_\zeta \end{vmatrix} = \begin{vmatrix} p_\epsilon, & q_\epsilon \\ p_\zeta, & q_\zeta \end{vmatrix} \begin{vmatrix} m_\gamma, & n_\gamma \\ m_\delta, & n_\delta \end{vmatrix}.$$

It follows that if two bases $\epsilon'\zeta'$, $\epsilon''\zeta''$ have the same discriminant in relation to $\alpha\beta$, then they have the same discriminant in relation to $\gamma\delta$. The division of the bases into two classes by the sign of the discriminant is therefore absolute, not relative to $\alpha\beta$. Thus

0·11. *The bases of a lattice fall into two classes such that the discriminant of* $\gamma\delta$ *with respect to* $\alpha\beta$ *is* $+1$ *if* $\alpha\beta$ *and* $\gamma\delta$ *are in the same class and is* -1 *if* $\alpha\beta$ *is in one class and* $\gamma\delta$ *is in the other.*

To find the geometrical meaning of the basal condition, let OA, OB as before be steps with the vectors α, β, and let OC, OD be steps with vectors γ, δ given by

·109$_{1\cdot 2}$
$$\gamma = m_\gamma \alpha + n_\gamma \beta, \qquad \delta = m_\delta \alpha + n_\delta \beta.$$

For the moment we make no assumption about the coefficients except that they are real numbers, and we write $m_\gamma n_\delta - n_\gamma m_\delta = J$. Then the areal product of the vectors γ, δ is J times the areal product of the vectors α, β. Hence *the area of the parallelogram determined by* OC, OD *is* J *times the area of the parallelogram determined by* OA, OB. This relation is algebraical, and can be broken into two parts: The numerical value of the ratio of the area of the parallelogram of which OC, OD are sides to the area of the parallelogram of which OA, OB are sides is $|J|$, and minimum rotation† from OC to OD is in the same direction as minimum rotation from OA to OB or in the reverse direction according as J is positive or negative.

A more elementary investigation shows well how the sign of the area and the sign of J are connected with the direction of rotation. Assuming that $n_\delta \neq 0$, the line through C parallel to OD cuts OA in a definite

† That is, rotation through an angle numerically as small as possible.

point C', and the line through D parallel to OA cuts OB in a definite point D'; the parallelograms $(O; CD)$, $(O; C'D)$, $(O; C'D')$ have the same area. Also the relation $J\alpha = n_\delta \gamma - n_\gamma \delta$, written in the form

$$\gamma = (J/n_\delta)\alpha + (n_\gamma/n_\delta)\delta,$$

shows that the vector of OC' is $(J/n_\delta)\alpha$, and $\cdot 109_2$ implies that the vector of OD' is $n_\delta \beta$.

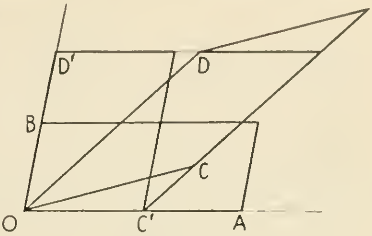


Fig. 4.

Returning to the lattice, and remarking that if the coefficients in $\cdot 109_{1\text{-}2}$ are integers then J is necessarily an integer, we see that if the vectors of OA, OB constitute a basis, and if C, D are any two points of the lattice, the area of the triangle OCD is an integral multiple of the area of the triangle OAB.

0·12. *The vectors of OC, OD constitute a basis if and only if the area of OCD is numerically equal to the area of OAB,*

and further,

0·13. *An undegenerate triangle whose vertices belong to a lattice can not be smaller than a basal triangle.*

When vectors are replaced by complex numbers, the concept of a direction of minimum rotation must be replaced by a definite analytical concept. In a sense we know what this concept must be. An angle of rotation from α to β is an angle of β/α, the complex number which multiplies α to produce β, and minimum rotation from α to β is therefore positive or negative according as β/α is on the positive or the negative side of the real axis, that is, according as $\mathrm{Im}(\beta/\alpha)$ is positive or negative. Since however this account of the concept belongs to the intuitional formulation of the theory of the complex number, it may be supplemented. Let α, β be two complex numbers such that β/α is not real, and let γ, δ be derived from α, β by the pair of formulae

$$\cdot 110 \qquad \gamma = m_\gamma \alpha + n_\gamma \beta, \qquad \delta = m_\delta \alpha + n_\delta \beta,$$

in which the coefficients are real; assuming γ not to be zero there is no loss of generality in assuming $n_\gamma \neq 0$. We have now

$$\frac{\delta}{\gamma} = \frac{m_\delta + n_\delta(\beta/\alpha)}{m_\gamma + n_\gamma(\beta/\alpha)},$$

that is, $$n_\gamma(\beta/\alpha)(\delta/\gamma) + m_\gamma(\delta/\gamma) - n_\delta(\beta/\alpha) = m_\delta,$$

and this relation may be written in the form

$$\{n_\gamma(\beta/\alpha)+m_\gamma\}\{n_\gamma(\delta/\gamma)-n_\delta\} = -J,$$

where $J = m_\gamma n_\delta - n_\gamma m_\delta$. But J is real, and the product of two complex numbers is not real unless one of them is a real multiple of the conjugate of the other; also the product of two conjugate numbers is essentially positive. Hence $n_\gamma(\delta/\gamma)-n_\delta$ is the product of the conjugate of $n_\gamma(\beta/\alpha)+m_\gamma$ by a real number k, and the sign of k is opposite to the sign of J. But since the coefficients are real,

$$\mathrm{Im}\{n_\gamma(\delta/\gamma)-n_\delta\} = n_\gamma \mathrm{Im}(\delta/\gamma), \qquad \mathrm{Im}\{n_\gamma(\beta/\alpha)+m_\gamma\} = n_\gamma \mathrm{Im}(\beta/\alpha),$$

and since $n_\gamma \neq 0$, the condition

$$\mathrm{Im}\{n_\gamma(\delta/\gamma)-n_\delta\} = -k\,\mathrm{Im}\{n_\gamma(\beta/\alpha)+m_\gamma\}$$

is equivalent to $\qquad \mathrm{Im}(\delta/\gamma) = -k\,\mathrm{Im}(\beta/\alpha)$:

0·14. *The imaginary parts of β/α and δ/γ have the same sign or opposite signs according as the discriminant of the transformation from $\alpha\beta$ to $\gamma\delta$ with real coefficients is positive or negative.*

Thus when the basis $\alpha\beta$ of a lattice is regarded as a pair of complex numbers, the two classes described in ·11 are composed simply of those bases for which $\mathrm{Im}(\beta/\alpha)$ is positive and those bases for which $\mathrm{Im}(\beta/\alpha)$ is negative.

It follows from ·13 that if OCD is a basal triangle, there can not be a lattice point between O and C. To investigate the converse of this result, let C be any lattice point such that there is no lattice point between O and C. Then the integers m_γ, n_γ are prime to each other, for if these integers had a common factor d, the vector $\gamma/|d|$ would lead to a lattice point. But if m_γ, n_γ are integers prime to each other, there exist integers x, y satisfying the equation

$$m_\gamma y - n_\gamma x = 1,$$

and if $x = m_\delta$, $y = n_\delta$ is any solution of this equation, and OD is the step from O with vector $m_\delta\alpha+n_\delta\beta$, then OCD is a basal triangle. Hence

0·15. *Two lattice points can serve as vertices of a basal triangle if and only if there is no lattice point between them on the line joining them.*

If the ratio of δ to γ is not real, the ratio of $p\gamma+q\delta$ to γ is not real unless $q = 0$, and therefore the only members of the aggregate $p\gamma+q\delta$ which are real multiples of γ compose the aggregate $p\gamma$. But if O, P are any two points of a lattice, there can be only a finite number of lattice points between O and P, and therefore there is a point C in OP, which may coincide with P, such that there is no lattice point between O and C. It follows that the steps from O to lattice points in the line OP are the integral multiples of the one step OC. In other words,

0·16. *If a line contains more than one point of a lattice, the lattice points which it contains constitute a single paling.*

If OCD is one basal triangle, ·12 implies that the other basal triangles with O and C for two of their vertices have their third vertices either on the line through D parallel to OC or on the parallel line at the same distance on the other side of OC; any lattice point on either of these lines will serve, and the possible positions of the third vertex therefore constitute two palings. This is in agreement with the algebraic solution of the equations

$$m_\gamma y - n_\gamma x = 1, \qquad m_\gamma y - n_\gamma x = -1;$$

if $x = m_\delta$, $y = n_\delta$ is one solution of the first of these equations, the general solution of the first equation is $x = m_\delta + rm_\gamma$, $y = n_\delta + rn_\gamma$, and the general solution of the second equation is $x = -m_\delta + rm_\gamma$, $y = -n_\delta + rn_\gamma$, where r in each case is an arbitrary integer.

0·2. If, as in Figures 2_1 and 2_2, the points geometrically congruent in a dissection of the plane compose a lattice, the distribution of values of the function $f(z)$ is the same in every cell of the pattern if

·201 $$f(z+\Omega) = f(z),$$

for every value of z and for every value of Ω which is a step in the lattice. The functions to be studied in this book are functions subject to a condition of this form.

We say that a function $f(z)$ which satisfies ·201 belongs to the lattice Ω. The fundamental condition is sometimes expressed differently. If $z_1 - z_2$ is a lattice step, the two values z_1, z_2 are said to be congruent, to modulus Ω, and we write $z_1 \equiv z_2$, or, if necessary, $z_1 \equiv z_2$, mod Ω; the condition ·201 is then: The congruence $z_1 \equiv z_2$ implies the equality $f(z_1) = f(z_2)$.

If $\alpha\beta$ is a basis of the lattice, the congruence $z_1 \equiv z_2$, mod $\alpha\beta$, asserts the existence of integers m, n such that $z_2 = z_1 + m\alpha + n\beta$, and the functional relation ·201 becomes

·202 $$f(z+m\alpha+n\beta) = f(z),$$

to be satisfied for all integral values of m and n. In the form ·202 the relation is an immediate consequence of the two simpler relations

·203 $$f(z+\alpha) = f(z), \qquad f(z+\beta) = f(z),$$

of which the first expresses that $f(z)$ has the period α, the second that $f(z)$ has the period β. That is,

0·21. *A function which satisfies a relation $f(z+\Omega) = f(z)$ in which Ω is the typical step in a lattice is a function which has two periods whose ratio is not real.*

Since the number of independent periods is two, such a function is known as a doubly periodic function. It need hardly be said that a doubly periodic function possesses an infinity of distinct periods; every number of the form $m\alpha+n\beta$, including zero, is *a* step in the lattice and is *a* period of the function. No two periods are *the* periods in any more important sense than that they are the periods we happen to be using; this being understood, we may speak of *the* periods as freely as we speak of *the* coordinates of a point.

Two questions now present themselves. (1) Can a function possess two periods whose ratio is real? (2) Can a function possess more than two periods? These questions are bound up with two of a more elementary kind. (1) What is the nature of the aggregate $m\alpha+n\beta$ if α, β are fixed complex numbers whose ratio is real? (2) What is the nature of the aggregate $m\alpha+n\beta+p\gamma$ if α, β, γ are fixed complex numbers?

Let $\beta = u\alpha$, where u is real. We can suppose u positive, for the aggregate $m\alpha+n\beta$ is identical with the aggregate $m(-\alpha)+n\beta$. With each value of the integer l associate the integer p_l such that

·204 $$p_l \leqslant lu < p_l+1,$$

and the point E_l for which the step OE_l is $l\beta-p_l\alpha$; the point E_l either coincides with O or lies between O and A on the line OA. If two points E_r, E_s coincide, then $r\beta-p_r\alpha = s\beta-p_s\alpha$, and therefore u has the rational value $(p_s-p_r)/(s-r)$. Conversely, if $u = h/k$, where h, k are positive integers, the inequalities ·204 are equivalent to

·205 $$p_l+h \leqslant (l+k)u < p_l+h+1,$$

and therefore

·206 $$p_{l+k} = p_l+h, \qquad (l+k)\beta-p_{l+k}\alpha = l\beta-p_l\alpha.$$

Thus the sets of values $p_0, p_1, \ldots, p_{k-1}$ and of positions $E_0, E_1, \ldots, E_{k-1}$ recur.

0·22. *If β/α is real, the number of distinct points in the set $\ldots, E_{-2}, E_{-1}, E_0, E_1, E_2, \ldots$ is finite or infinite according as β/α is rational or irrational.*

If P, Q are any two aggregate-points on the line, the step PQ is of the form $m\alpha+n\beta$, and an equal step from any aggregate-point leads again to an aggregate-point. It follows that if the number of aggregate-points between O and A is finite, the distance between adjacent points is everywhere the same. If E_t is the nearest to O of those of the points $E_1, E_2, \ldots, E_{k-1}$ which are distinct from O, the step OE_t is a number θ, given as $t\beta - p_t\alpha$, which is such that α is an integral multiple $a\theta$ of θ and every number of the form $l\beta - p_l\alpha$ is an integral multiple $g_l\theta$ of θ. Since in particular the number $\beta - p_1\alpha$ is expressible as $g_1\theta$, we have $\beta = b\theta$, where $b = p_1 a + g_1$. Since $\theta = t\beta - p_t\alpha = (tb - p_t a)\theta$, we have $tb - p_t a = 1$, and a and b have no common factor: the ratio b/a is the ratio β/α, known to be rational, expressed in its lowest terms. Since α and β are multiples of θ, every number of the form $m\alpha + n\beta$ is a multiple of θ; since θ is given as $t\beta - p_t\alpha$, every multiple of θ is of the form $m\alpha + n\beta$: the aggregate $m\alpha + n\beta$ is identical with the aggregate of multiples of θ.

0·23. *To say that a function has two periods whose ratio is rational implies no more than that the function has one period of which these two are integral multiples.*

Consider now the case in which β/α is irrational. If N is any whole number, the $N+1$ points $E_0, E_1, E_2, \ldots, E_N$ are all distinct, and if we divide the interval OA into N equal parts, at least one of these parts includes as many as two of the points; also if λ is the length of OA, the distance between two points in the same division is not greater than λ/N. Since the step from one aggregate-point to another is a number of the form $m\alpha + n\beta$, it follows that whatever the value of N, there is a number μ_N of the aggregate $m\alpha + n\beta$ such that $0 < |\mu_N| < \lambda/N$. Now let z_0 be any point in the plane, and let ρ be the radius of any circle with centre z_0. Take a value of N greater than λ/ρ, and with this value of N choose μ_N. Then the point $z_0 + \mu_N$ lies inside the circle. Hence if $f(z)$ is a function satisfying the condition $f(z + m\alpha + n\beta) = f(z)$, an arbitrary circle with z_0 for centre contains a point z_1 distinct from z_0 such that $f(z_1) = f(z_0)$. It follows that if $f'(z_0)$ exists, the value of $f'(z_0)$ is zero. Thus if $f(z)$ is an analytic function, the derivative $f'(z)$ is zero at every point at which it exists:

0·24. *An analytic function with two periods whose ratio is an irrational number is an absolute constant.*

With ·23 and ·24 the question of functions with two periods whose ratio is real is answered completely, and we proceed to the question of functions with three periods, α, β, γ. We can assume at once that no two of the periods have a real ratio, for a rational ratio would reduce the number of periods to two at most, and an irrational ratio would reduce the function to a constant. If the ratio of α to β is not real, any third number γ can be expressed as $u\alpha+v\beta$, where u, v are real; this is only to say that any point in the plane can be identified by coordinates referred to any two axes. If the ratio of u to v is rational, we have $u = wh$, $v = wk$ where h, k are integers; then $\gamma = w(h\alpha+k\beta)$, and since $h\alpha+k\beta$ is a period, this relation reduces the periods to two or the function to a constant according as w is rational or irrational. Similarly if u has a rational value h/k, the relation $kv\beta = k\gamma-h\alpha$ reduces the periods or trivializes the function according to the character of v, and if v has a rational value, the same result follows according to the character of u. Thus the only case that remains for examination is that in which u, v, and the ratio of u to v, are all irrational.

We can suppose u and v positive, for we can replace α by $-\alpha$ or β by $-\beta$ if necessary, and we repeat, with little modification, the construction and the argument leading to ·24. With each value of the integer l we associate the integers p_l, q_l which are such that

·207 $$p_l \leqslant lu < p_l+1, \qquad q_l \leqslant lv < q_l+1;$$

if $l \neq 0$, equality is impossible in either case. The integer l now determines a number $l\gamma-p_l\alpha-q_l\beta$, and a point E_l such that this number represents the step OE_l. The point E_0 is the origin O, and for all other values of l, positive and negative, E_l is inside the parallelogram $(O; AB)$. No two of the points ..., E_{-2}, E_{-1}, E_0, E_1, E_2,... coincide, and any step from one to another of these points is represented by a number in the aggregate $m\alpha+n\beta+p\gamma$. If N is any whole number, the parallelogram $(O; AB)$ can be divided into N^2 equal compartments by means of $N-1$ lines parallel to OA and $N-1$ lines parallel to OB, and if λ is the greatest distance from one point to another of the parallelogram $(O; AB)$, that is, the length of the longer diagonal of this parallelogram, the distance between two points in the same compartment is not greater than λ/N. The N^2+1 points E_0, E_1, E_2,..., E_{N^2} can not all be accommodated in different compartments, and therefore at least one compartment contains as many as two points. Hence the aggregate $m\alpha+n\beta+p\gamma$ includes a member μ_N such that $0 < |\mu_N| \leqslant \lambda/N$, and it follows as before that if $f(z)$ satisfies the condition

$$f(z+m\alpha+n\beta+p\gamma) = f(z),$$

he derivative $f'(z)$ is zero wherever it exists. Thus

0·25. *To say that a singlevalued analytic function has three periods implies no more than that the function is doubly periodic.*

To sum up, the restriction to two periods and the condition that the ratio of one of these periods to the other is not real are not arbitrary limitations but limitations inherent in the subject.

The investigation just completed is not superfluous to our main subject, for it enables us to deal with such dissections as the one in Figure 2_3. If the parallelograms in this diagram have sides α and β, as now indicated, there are displacements with vectors α and 2β, and there is also a displacement with a vector γ which is of the form $u\alpha+\beta$. A function $f(z)$ can not satisfy the condition $f(z+\gamma) = f(z)$ for all positions of z in one parallelogram without satisfying this condition everywhere, that is, without having γ for a period, and then the function is trivial and the pattern ineffective unless u is rational. If 2β and $u\alpha+\beta$ are periods, so also is $2u\alpha$, and if $2u$, in its lowest terms, is h/k, the periods α, $2u\alpha$ are multiples $k\theta$, $h\theta$ of a single period θ. We have now the three periods θ, 2β, $\frac{1}{2}h\theta+\beta$, and we distinguish two cases. If h is even, θ and β are periods. If h is odd, $\frac{1}{2}\theta+\beta$ is a period ϕ, and we have the three periods θ, $2\phi-\theta$, ϕ, of which the second is a direct combination of the other two. The two cases are illustrated in Figures 6_{1-2}, and we see that the pattern in terms of the smaller parallelograms is of the same simple form as the patterns in Figures 2_{1-2}.

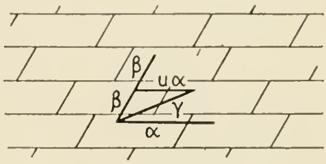
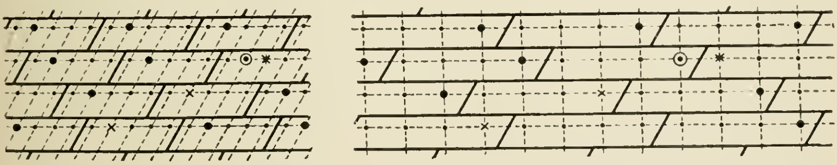


FIG. 5.



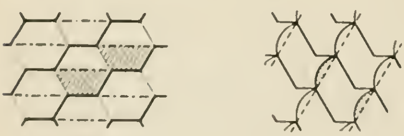
6_1 6_2

FIG. 6.

To put differently the point just illustrated, the periodicity of any particular function we construct may turn out to be better than we anticipated. The functions $\sin z$ and $\cos z$ have the common period 2π, and any rational function of these two has this period, but $\tan z$, defined as $\sin z/\cos z$, is found to have the smaller period π. To say that $f(z)$ belongs to the lattice Ω means only that the identity $f(z+\Omega) = f(z)$ is satisfied; $f(z)$ may in fact possess a period ω which does not belong to the aggregate Ω. What we have shown is that in this case $f(z)$, unless trivial, belongs to a lattice Υ of finer mesh than Ω, and that† the points of Ω are among the points of Υ. But the determination of the minimum lattice is not necessarily the first problem to be attacked when a function is introduced, and if several functions occur in the same investigation, it is a lattice large enough for them all to belong to it that we need, whether or not finer lattices for the individual functions are known.

By a primitive region for a lattice or for a function which belongs to the lattice we mean a region which just represents the whole plane; no two points of the region are congruent, but every point of the plane is congruent with one point of the region. In other words, if Λ is a primitive region, and if Λ_Ω is the region to which Λ is moved by a displacement Ω which is a step in the lattice, every point in the plane belongs to one and only one of the regions Λ_Ω. In terms of the dissection of the plane, with which our discussion began, a primitive region is one of the congruent regions into which the plane is dissected, but unless our definition is formal we have difficulty in dealing with the boundary of a region; a point on the common boundary between two regions, or a point where more than two regions meet, must not be assigned to more than one of the regions, and in consequence only part of the boundary of a region belongs to that region.

In no sense is there a unique or fundamental primitive region. We have only to substitute for any part Δ of a primitive region Λ a region Δ_Ω congruent with Δ, and the combination of $\Lambda-\Delta$ and Δ_Ω is another primitive region. In practice this change usually takes the form of a change of contour of Λ, part of Λ being transferred to adjacent regions and the loss being made good by a corresponding transfer on the other side. For example, in Figure 7_1 the lower halves of the hexagons are all congruent, and by uniting to the upper half of one hexagon the lower half of one of its neighbours we can form a primitive region which is a parallelogram. In Figure 7_2, joining the two ends of each circular arc and replacing the segment in each region by the opposite segment which originally belongs to an adjacent region, we have a primitive region bounded by six straight lines, and this can be further transformed into a parallelogram. In these examples the purpose of the change is to simplify the shape of the region. We can use the

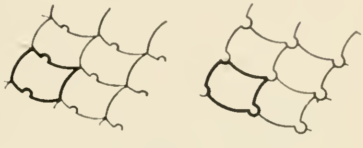


7_1 7_2

Fig. 7.

† The lattice with the finer mesh is a *multiple* of the lattice with the coarser mesh. This is the fundamental notion in the theory of ideals.

change also to avoid particular points on the contour. If a function to be integrated has a pole at a point Q on the contour of Λ, we can replace the contour near Q by part of a small circumference which brings Q inside the region; the congruent changes necessarily remove from the actual boundaries all points congruent with Q, and we have one pole definitely inside the new primitive region and the congruent poles definitely outside. As Figure 8_2 illustrates, the inclusion of one pole may involve the exclusion of more poles than one; that is why we operate by inclusion, not by exclusion.



8_1 8_2

FIG. 8.

If $\alpha\beta$ is a basis of the lattice characteristic of the pattern, any parallelogram $TUWV$ in which the adjacent sides TU, TV have vectors α, β is a primitive region. Only one of the four corners is to be included. Opposite sides are congruent, and if we include a point P on one side we must exclude the corresponding point on the other side. As a rule we include the whole of the sides TU, TV, except the points U, V, and exclude therefore the sides VW, UW. The name† of *cell* is sometimes reserved for a primitive region so constructed. If the vertices of the parallelogram belong to the actual aggregate $m\alpha+n\beta$, the parallelogram is called a period parallelogram or a *mesh*. If T is the origin O, the mesh is said to be fundamental. The fundamental mesh $\alpha\beta$ consists therefore of the interior of the parallelogram whose vertices are the four points 0, α, $\alpha+\beta$, β, together with the point 0, the points between 0 and α, and the points between 0 and β.

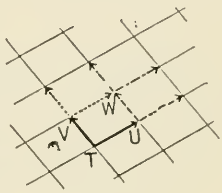


FIG. 9.

There is a distinction to be borne in mind between a parallelogram which is a primitive region and a period parallelogram. If in Figure 9, for example, P is any point on the line VW and Q is the point such that WQ is congruent with VP, the triangles TVP and UWQ are congruent, and the parallelogram $TUQP$ is a primitive region, but this parallelogram is not a period parallelogram unless P is a lattice point.

It is to be noticed also that a region is primitive with regard to a lattice, not with regard to any particular function $f(z)$, which belongs to the lattice. There may be a repetition of values of $f(z)$ inside a region

† Although precise definitions have been laid down, language is free and seldom misleading. Often any primitive region is called a cell, and any parallelogram z_0, z_1, $z_1+z_2-z_0$, z_2 in which z_1-z_0, z_2-z_0 is a primitive pair of periods is called a period parallelogram.

which is primitive for the lattice. A cell of the lattice may be divisible, with reference to $f(z)$, into a number of compartments in each of which $f(z)$ takes an assigned value not more than once, but there is no reason to suppose that even when this is possible different compartments in one cell are congruent geometrically.

(ii) Elliptic Functions in General

0·3. By a theorem known as Liouville's, a singlevalued function of the complex variable, unless a sheer constant, must tend somewhere to infinity. The function may be, like a polynomial or the exponential function, bounded in any finite region of the plane, but in that case the limits as $z \to \infty$ are not all finite. Since a doubly periodic function, if bounded throughout a primitive region, is bounded throughout the whole plane, and can not tend to infinity with z,

0·31. *A singlevalued doubly periodic function which is not a constant has at least one singularity in each cell,*

or in other words,

·301. *If $f(z)$ is a singlevalued doubly periodic function which is not a constant, there is at least one lattice whose points are singularities of $f(z)$.*

Every lattice extends to infinity, and a limiting point of singularities is an essential singularity, even if the individual singularities are poles or branchpoints; hence

0·32. *A doubly periodic function which is not a constant has the point at infinity for an essential singularity.*

From ·31 we learn that the most elementary doubly periodic functions which we can hope to construct are singlevalued doubly periodic functions whose only accessible singularities are poles. These are the functions which, for historical reasons with which we need not concern ourselves, are called elliptic functions. We demonstrate the existence of elliptic functions by particular constructions, but first we prove a few general theorems.

The number of poles of an elliptic function in any bounded region is finite, since otherwise the region would include a limiting point of poles, and this would be an essential singularity of the function. Furthermore, if $f(z)$ is a function not identically zero whose only accessible singularities are poles, then with any finite value of a is associated an expansion

·302 $$f(z) = (z-a)^n\{c_0+c_1(z-a)+c_2(z-a)^2+\dots\},$$

with n an integer and c_0 not zero, valid throughout some neighbourhood of a. The point a is a zero of order n, a neutral point, or a pole of order $-n$, according as n is positive, zero, or negative; c_0 is *the leading coefficient of* $f(z)$ *at* a. For sufficiently small values of $z-a$,

$$|(z-a)\{c_1+c_2(z-a)+...\}| < |c_0|,$$

and therefore within this range $f(z) \neq 0$, except at a itself if a is a zero; that is to say, whether or not a is a zero, a is not a limiting point of zeros, and therefore in any bounded region the number of zeros is finite. Hence

0·33. *The number of poles and the number of zeros of an elliptic function in any cell are finite.*

In other words,

·303. *The poles of an elliptic function constitute a finite number of lattices, and so do the zeros of the function unless the function is identically zero.*

A set of poles or zeros which includes one and only one member of each pole-lattice or zero-lattice is called an irreducible set; the pole or zero is of course given the appropriate multiplicity.

If the only accessible singularities of $f(z)$ are poles, the only accessible singularities of $1/f(z)$ arise from the zeros of $f(z)$; a zero of $f(z)$ of order n implies a pole of $1/f(z)$ of the same order, and if $f(z)$ is not identically zero the zeros of $f(z)$ have no accessible limiting points and can not introduce accessible essential singularities into $1/f(z)$. Alternatively we may say that if $c_0 \neq 0$, then

$$1/\{c_0+c_1(z-a)+c_2(z-a)^2+...\} = d_0+d_1(z-a)+d_2(z-a)^2+...,$$

where $d_0 \neq 0$ and the radius of convergence of the series on the right is not zero; hence the existence of the expansion ·302 for $f(z)$ implies the existence of the expansion

·304 $$1/f(z) = (z-a)^{-n}\{d_0+d_1(z-a)+d_2(z-a)^2+...\},$$

and since a is arbitrary in ·302, a is arbitrary in ·304 also. Thus

·305. *If $f(z)$ is a function not identically zero whose only accessible singularities are poles, then $1/f(z)$ is a function whose only accessible singularities are poles.*

If $f(z)$, not identically zero, is periodic, $1/f(z)$ has the periods of $f(z)$. Hence

0·34. *If $f(z)$ is an elliptic function not identically zero, then $1/f(z)$ is an elliptic function belonging to the same lattice as $f(z)$,*

which, taken with ·31, implies that

0·35. *An elliptic function which is not a constant has at least one zero in each cell.*

If $f(z)$, $g(z)$,... are elliptic functions, finite in number, with a common lattice, any polynomials $P\{f(z), g(z),...\}$, $Q\{f(z), g(z),...\}$ in these functions are elliptic functions with this lattice, and it follows from ·34 that $1/Q\{f(z), g(z),...\}$ also is an elliptic function unless $Q\{f(z), g(z),...\}$ is identically zero; hence $P\{f(z), g(z),...\}/Q\{f(z), g(z),...\}$ is an elliptic function:

0·36. *If a finite number of elliptic functions have a common lattice, any rational function of these functions that is not infinite everywhere is an elliptic function with that lattice.*

The common lattice is not necessarily the fundamental lattice for any of the individual functions. For example, if Ω is a typical period of $f(z)$, then $\frac{1}{2}\Omega$ is a typical period of $f(2z)$ and $\frac{1}{3}\Omega$ is a typical period of $f(3z)$, but the typical period of $f(2z)+f(3z)$ is Ω.

If $f(z)$ is an analytic function, the singularities of the derivative $f'(z)$ are located at the singularities of $f(z)$, and a pole of order n of $f(z)$ gives rise to a pole of order $n+1$ of $f'(z)$. Also the relation $f(z+\Omega) = f(z)$ implies the relation $f'(z+\Omega) = f'(z)$. Hence

0·37. *The successive derivatives of an elliptic function $f(z)$ are elliptic functions with the same lattice as $f(z)$.*

Integration introduces questions of detail. The relation $f(z+\Omega) = f(z)$ implies of course

$$\int_{z_0}^{z} f(z+\Omega)\,dz = \int_{z_0}^{z} f(z)\,dz,$$

provided that the path of integration is the same in the two integrals. But it is only if the residues of $f(z)$ are all zero that the integrals are independent of the path and that we can define a singlevalued function $F(z)$ by the formula

$$F(z) = \int_{z_0}^{z} f(z)\,dz. \qquad \cdot 306$$

Moreover, when this definition is possible, $F(z+\Omega)$ is not

$$\int_{z_0}^{z} f(z+\Omega)\,dz,$$

which can be identified with $F(z)$, but

$$\int_{z_0}^{z+\Omega} f(z)\,dz.$$

We have, still on the assumption that the paths are irrelevant,

$$F(z+\Omega) = \int_{z_0}^{z_0+\Omega} f(z)\,dz + \int_{z_0+\Omega}^{z+\Omega} f(z)\,dz,$$

that is,

·307 $$F(z+\Omega) = F(z)+F(z_0+\Omega),$$

and it is only if $F(z_0+\Omega) = 0$ for every period Ω that $F(z)$ is an elliptic function. If Ω' and Ω'' are any two periods, we have, on substituting $z_0+\Omega'$ for z and Ω'' for Ω in ·307,

·308 $$F(z_0+\Omega'+\Omega'') = F(z_0+\Omega')+F(z_0+\Omega'').$$

Hence if $\alpha\beta$ is a basis of the lattice, and if $\Omega = m\alpha+n\beta$, then

·309 $$F(z_0+\Omega) = m\mathrm{A}+n\mathrm{B},$$

where

·310 $$\mathrm{A} = F(z_0+\alpha), \qquad \mathrm{B} = F(z_0+\beta).$$

Substituting from ·309 in ·307, we have the most general theorem to be expected:

0·38. *If $f(z)$ is an elliptic function whose residues are all zero, belonging to a lattice of which $\alpha\beta$ is a basis, the singlevalued function $F(z)$ defined by the formula*

$$F(z) = \int_{z_0}^{z} f(z)\,dz$$

satisfies the relation

$$F(z+m\alpha+n\beta) = F(z)+m\mathrm{A}+n\mathrm{B},$$

where $$\mathrm{A} = \int_{z_0}^{z_0+\alpha} f(z)\,dz, \qquad \mathrm{B} = \int_{z_0}^{z_0+\beta} f(z)\,dz.$$

A change in z_0 adds a constant to $F(z)$ and is without effect on the functional relation or on the values of A and B, but we must leave z_0 arbitrary, since any particular point we might choose for z_0, such as the origin, might sometimes be a pole of $f(z)$ and would then be unsuitable.

If the basis is changed from $\alpha\beta$ to $\gamma\delta$ by the pair of formulae

·311 $$\gamma = m_\gamma\alpha+n_\gamma\beta, \qquad \delta = m_\delta\alpha+n_\delta\beta,$$

the corresponding constants Γ, Δ are derived from A, B by the same transformation:

·312 $$\Gamma = m_\gamma\mathrm{A}+n_\gamma\mathrm{B}, \qquad \Delta = m_\delta\mathrm{A}+n_\delta\mathrm{B}.$$

In general we can say that the constant $F(z+\Omega)-F(z)$ is the typical member of a lattice which is correlated with the period lattice, but we have to remember that the one lattice may degenerate when the other does not.

A function $G(z)$ which satisfies a relation

$$\cdot 313 \qquad G(z+\alpha) = G(z)+\mathrm{A}$$

is said to possess pseudoperiodicity of the first or additive kind, with A for modulus. The integral in ·38 is a doubly pseudoperiodic function of the first kind, with α, β for periods and A, B for corresponding moduli. Since the derivative of a pseudoperiodic function of the additive kind is a periodic function, the converse of ·38 is true:

0·39. *A doubly pseudoperiodic function of additive type which has no accessible singularities except poles is the integral of an elliptic function whose residues are all zero.*

It is to be noticed that the function z itself is additively pseudoperiodic; any period may be assigned to this function, and the corresponding modulus is equal to the period.

0·4. Let $\alpha\beta$ be a basis of the lattice to which the elliptic function $f(z)$ belongs, and let $O'A'C'B'$ be a parallelogram in which $O'A'$, $O'B'$ have the vectors α, β. Since the number of poles of $f(z)$ in the parallelogram or on its boundary is finite, it is possible to draw a line a parallel to $O'A'$, between $O'A'$ and $B'C'$, which does not pass through any of these poles, and from the periodicity of $f(z)$ in α it follows that a does not pass through any poles of $f(z)$; similarly it is possible to draw a line b parallel to $O'B'$, between $O'B'$ and $A'C'$, which does not pass through any poles of $f(z)$. The lines a, b intersect in a point O, and the parallelogram $OACB$ for which OA, OB have the vectors α, β has no poles of $f(z)$ on OA or OB, and therefore has no poles of $f(z)$ on BC or AC: this parallelogram is a period parallelogram with a pole-free contour.

If $OACB$ is a period parallelogram with a pole-free contour, the function $f(z)$ can be integrated round the contour. If z_0 is the value of z at O, we have

$$\int_{BC} f(z)\,dz = \int_{z_0+\beta}^{z_0+\beta+\alpha} f(z)\,dz = \int_{z_0}^{z_0+\alpha} f(z+\beta)\,dz = \int_{OA} f(z)\,dz,$$

and similarly
$$\int_{AC} f(z)\,dz = \int_{OB} f(z)\,dz:$$

·401. *If a period parallelogram of an elliptic function has a pole-free contour, the integral of the function round the contour is zero.*

Applying Cauchy's theorem, we see that

0·41. *The sum of the residues of an elliptic function at the poles in a primitive region is zero.*

From ·37 and ·36, the logarithmic derivative $f'(z)/f(z)$ is an elliptic function belonging to the same lattice as $f(z)$. In the neighbourhood of a point a, from the expansion ·302,

$$\frac{f'(z)}{f(z)} = \frac{n}{z-a} + \frac{c_1+2c_2(z-a)+3c_3(z-a)^2+\ldots}{c_0+c_1(z-a)+c_2(z-a)^2+\ldots},$$

and since $c_0 \neq 0$, the point a is a neutral point of $f'(z)/f(z)$ if $n = 0$, a simple pole with residue n if $n \neq 0$; that is, the poles of $f'(z)/f(z)$, all simple, are the poles and the zeros of $f(z)$, and the residue of $f'(z)/f(z)$ is m where $f(z)$ has a zero of order m and is $-n$ where $f(z)$ has a pole of order n. Applying ·41 to $f'(z)/f(z)$ and interpreting the result in terms of $f(z)$,

0·42. *The sum of the orders of the zeros of an elliptic function in a primitive region is equal to the sum of the orders of the poles.*

Replacing $f(z)$ by $f(z)-c$, an elliptic function with the same poles as $f(z)$, we have

$0{\cdot}43_1$. *If $f(z)$ is any elliptic function, the sum of the orders of an irreducible set of roots of the equation $f(z) = c$ is independent of the value of c and is equal to the sum of the orders of an irreducible set of poles of $f(z)$.*

The number whose importance is shown by this theorem is called the order of the elliptic function; the order of the function is the sum of the orders of incongruent poles. A multiple root of the equation $f(z) = c$ is a root of the equation $f'(z) = 0$; this equation has a finite number of incongruent roots, $z_1, z_2, \ldots, z_k$, and unless c has one of the k values $f(z_1), f(z_2), \ldots, f(z_k)$, the roots of the equation $f(z) = c$ are all simple. Hence

$0{\cdot}43_2$. *The order of the elliptic function $f(z)$ is the number of incongruent roots of the equation $f(z) = c$ for an arbitrary value of c.*

A function of order 1 would be a function with one simple pole and no others in a cell, and by ·41 the residue at that pole would be zero:

0·44. *There are no elliptic functions of the first order.*

But for every value of n from 2 onward there are elliptic functions corresponding to every partition of n, from the one extreme of functions with a single pole of order n to the other extreme of functions with n distinct simple poles; this is established in due course by the construction of the functions. Since a pole of order p in $f(z)$ implies a pole of order $p+1$ in $f'(z)$, the derivative of a function of order n may have any order from $n+1$ to $2n$.

Let $G(z)$ be any analytic function which has no singularities on the pole-free contour $OACB$ and no singularities except poles inside this contour. Using the same transformation as in the proof of ·401 we have the integral of the product $f(z)G(z)$ round the contour expressed as

$$\int_{OB} f(z)\{G(z+\alpha)-G(z)\}\,dz - \int_{OA} f(z)\{G(z+\beta)-G(z)\}\,dz.$$

If then $G(z)$ is doubly pseudoperiodic, with A, B for moduli corresponding to the periods α, β, this integral reduces to

$$\mathrm{A}\int_{OB} f(z)\,dz - \mathrm{B}\int_{OA} f(z)\,dz.$$

On the other hand, integration round the contour $OACBO$ is in the positive direction or the negative direction, in the sense required for the application of Cauchy's theorem, according as the direction of minimum rotation from OA to OB is positive or negative; that is to say, the sum of the residues of $f(z)G(z)$ must be multiplied by $2\pi i$ or $-2\pi i$ according as the basis $\alpha\beta$ is positive or negative. We introduce υ to denote i or $-i$ as the case may be, and we call υ the *signature* of the basis.

0·45. *Let $\alpha\beta$ be a basis of the elliptic function $f(z)$, and let $G(z)$ be a doubly pseudoperiodic function belonging to the same lattice as $f(z)$, with* A, B *for moduli corresponding to the periods α, β. Then if OA, OB are steps, with vectors α, β, on which neither $f(z)$ nor $G(z)$ has any singularities, and if $G(z)$ has no singularities except poles inside the parallelogram $(O; AB)$, the value of*

$$\mathrm{A}\int_{OB} f(z)\,dz - \mathrm{B}\int_{OA} f(z)\,dz$$

is $2\pi\upsilon$ times the sum of the residues of the product $f(z)G(z)$ at poles inside the parallelogram, υ being the signature of the basis $\alpha\beta$.

In general the residue of a product is the sum of a number of terms, but if the pole under consideration is a simple pole of one factor and a neutral point or a zero of the other, the residue of the product is the product of the residue of the one function and the value of the other.

The cases of ·45 which are of immediate importance are two in which one or other of the functions $f(z)$, $G(z)$ is in a sense trivial.

Taking $f(z)$ as constant, we have:

0·46. *If $G(z)$ is a doubly pseudoperiodic function of additive type with moduli* A, B *corresponding to the periods α, β, whose accessible singularities* *are all poles, then* $\mathrm{A}\beta-\mathrm{B}\alpha$ *is* $2\pi v$ *times the sum of the residues of* $G(z)$ *in any cell of the* $\alpha\beta$ *lattice,* v *being the signature of the basis.*

The distribution of finite values of $G(z)$ differs from cell to cell, but the poles occupy congruent positions in the different cells and the residues of congruent poles are everywhere the same. To this theorem we shall presently return.

Next we take $G(z)$ in ·45 as z, and we replace $f(z)$ by a logarithmic derivative $f'(z)/f(z)$. The factor z has no poles, and the poles of $f'(z)/f(z)$ are simple; if a_r is a pole of $f(z)$, of order p_r, the residue of $zf'(z)/f(z)$ is $-p_r a_r$; if b_s is a zero of $f(z)$, of order q_s, the residue of $zf'(z)/f(z)$ is $q_s b_s$. The sum of the residues is therefore

$$\sum_s q_s b_s - \sum_r p_r a_r,$$

extending to all the zeros and poles in the parallelogram. On the other hand, since $f(z)$ has the same value at B as at O, and the same value at A as at O, each of the integrals

$$\int_{OB} \frac{f'(z)\,dz}{f(z)}, \qquad \int_{OA} \frac{f'(z)\,dz}{f(z)}$$

is the difference between two values of the logarithm of the same number $f(z_O)$, and is therefore an integral multiple of $2\pi i$. Giving A, B their values α, β, we can say that the integral round the contour is of the form $2\pi i(m\alpha+n\beta)$, where m, n are integers, and since we are not attempting to identify these integers, the sign of v is irrelevant and the factors $2\pi i$, $2\pi v$ can be removed:

$0{\cdot}47_1$. *If the poles of an elliptic function in any cell are* $a_1, a_2,\ldots$ *with multiplicities* $p_1, p_2,\ldots$ *and the zeros of the function are* $b_1, b_2,\ldots$ *with multiplicities* $q_1, q_2,\ldots,$ *then the sum* $q_1 b_1+q_2 b_2+\ldots$ *differs from the sum* $p_1 a_1+p_2 a_2+\ldots$ *by a number which is a step in the lattice to which the function belongs.*

We may allow repetition in the enumeration of poles and zeros to replace the explicit use of multiplicities:

$0{\cdot}47_2$. *If* $a_1, a_2,\ldots, a_n$ *is an irreducible set of poles and* $b_1, b_2,\ldots, b_n$ *is an irreducible set of zeros of an elliptic function, each pole and each zero being repeated according to its multiplicity, then* $\sum a_r \equiv \sum b_s$.

If we say that $\sum a_r - \sum b_s$ is a period, we must remember that zero is being admitted as a possibility.

When repetition is allowed in enumeration, a slight extension of vocabulary is convenient. If the point a is in fact p-fold, an irreducible set must include p points congruent with a, but there is no reason to suppose that these points are identical. With this extension we may, for example, secure an equality $\sum a_r = \sum b_s$ to replace the congruence $\sum a_r \equiv \sum b_s$, for if with the sets as originally assigned $\sum a_r - \sum b_s = \Omega_l$, we have only to replace a_n by $a_n - \Omega_l$. If the pole at a_n is simple, the change is possible on any convention, but if the pole at a_n is of multiplicity p, this pole is now being enumerated $p-1$ times at a_n and once at $a_n - \Omega_l$.

Replacing the function $f(z)$ by $f(z)-c$, where c is arbitrary, we have a corollary to $\cdot 47_2$:

0·47_3. *If $f(z)$ is an elliptic function and $z_1, z_2, \ldots, z_n$ is an irreducible set of roots of the equation $f(z) = c$, the congruence to which the sum $z_1+z_2+\ldots+z_n$ belongs is independent of c, being the congruence of which the sum of any irreducible set of poles of $f(z)$ is a member.*

From the simplest cases of ·36 we derive, following Liouville, two theorems which give analytical effect to the consideration that an elliptic function can be identified by its behaviour in one cell.

Let $f(z)$, $g(z)$ be two functions with a common pole a, and let the functions have the same principal part at a: the finite series of negative powers of $z-a$ in the Laurent series representing the functions in the neighbourhood of a are identical for the two functions. Then the difference $f(z)-g(z)$ is represented in the neighbourhood of a by a convergent series of positive powers of $z-a$, beginning as a rule with a constant term, and a is not a pole of $f(z)-g(z)$. If then $f(z)$, $g(z)$ are elliptic functions with a common lattice and with the same poles, and if at every pole in one cell the principal parts of the two functions are identical, the difference $f(z)-g(z)$ is an elliptic function with no singularity in the cell, and is therefore, by ·31, a constant. We may replace any pole by a congruent pole for examination, and the result can be enunciated as follows:

0·48. *If two elliptic functions have a common lattice and the same poles, and if at every point of an irreducible set of poles the principal parts of the two functions are identical, then the difference between the two functions is a constant.*

The poles of the quotient $f(z)/g(z)$ are among the poles of $f(z)$ and the zeros of $g(z)$. If $c_0 \neq 0$, $d_0 \neq 0$, and if each of the series

$$c_0+c_1(z-a)+c_2(z-a)^2+\ldots, \qquad d_0+d_1(z-a)+d_2(z-a)^2+\ldots$$

has a radius of convergence that is not zero, the quotient

$$\{c_0+c_1(z-a)+c_2(z-a)^2+\ldots\}/\{d_0+d_1(z-a)+d_2(z-a)^2+\ldots\}$$

is expressible as a power series in which neither the constant term nor the radius of convergence is zero. It follows that a pole of $f(z)$ of order p is not a pole of $f(z)/g(z)$ if it is also a pole of $g(z)$ of order not less than p, and that a zero of $g(z)$ of order q is not a pole of $f(z)/g(z)$ if it is also a zero of $f(z)$ of order not less than q. If $f(z)$, $g(z)$ are elliptic functions with a common lattice, then $f(z)/g(z)$ is an elliptic function:

$0{\cdot}49_1$. *Let $f(z)$, $g(z)$ be elliptic functions with a common lattice; let $a_1, a_2, \ldots, a_m$ be an irreducible set of poles of $f(z)$, of orders $p_1, p_2, \ldots, p_m$, and let $b_1, b_2, \ldots, b_n$ be an irreducible set of zeros of $g(z)$, of orders $q_1, q_2, \ldots, q_n$. Then if each pole a_r is also a pole of $g(z)$, of order not less than p_r, and if each zero b_s is also a zero of $f(z)$, of order not less than q_s, the function $f(z)$ is a constant multiple of the function $g(z)$.*

It follows from the conclusion of this theorem that the two functions have all their poles and all their zeros the same, in order as well as in position; that is, the order of a_r as a pole of $g(z)$ is exactly p_r and $g(z)$ has no poles incongruent with the set $a_1, a_2, \ldots, a_m$, and the order of b_s as a zero of $f(z)$ is exactly q_s and $f(z)$ has no zeros incongruent with the set $b_1, b_2, \ldots, b_n$. These results follow at once from $\cdot 42$; the order of $g(z)$ is not less than the sum of the orders of $a_1, a_2, \ldots, a_m$ as poles of $g(z)$, and is therefore by hypothesis not less than the sum of the orders of these points as poles of $f(z)$, that is, not less than the order of $f(z)$; on the other hand, the order of $f(z)$ is not less than the sum of the orders of $b_1, b_2, \ldots, b_n$ as zeros of $f(z)$, and therefore not less than the sum of the orders of these points as zeros of $g(z)$, which is the order of $g(z)$. Hence the two functions have the same order, and there is no margin for inequality in the orders at any pole or at any zero, or for additional poles of $g(z)$ or zeros of $f(z)$. We may therefore logically break the theorem $\cdot 49_1$ into two:

$0{\cdot}49_2$. *If two elliptic functions $f(z)$, $g(z)$ have a common lattice, and if every pole of $f(z)$ is a pole of at least as high an order of $g(z)$ and every zero of $g(z)$ is a zero of at least as high an order of $f(z)$, then the two functions have the same poles and the same zeros, to the same multiplicity in every case;*

$0{\cdot}49_3$. *If two elliptic functions with a common lattice have the same poles and the same zeros, to the same multiplicity in every case, one function is a constant multiple of the other.*

The latter of these theorems is the vivid form of the result. We speak of the distribution of poles and zeros as the structure of the function, and we say that an elliptic function is determined, except for a constant multiplier, by its structure. But $\cdot 49_1$ remains the form in which the theorem is used: we seldom investigate a pole or a zero, as $\cdot 49_3$ would require, to verify that its order is not higher for one function than for the other.

(iii) The Weierstrassian Functions

0·5. The construction of specific elliptic functions, to which we proceed, is rendered easy by the observation that if Ω is the typical step in a lattice, any function that is symmetrical in the whole aggregate of arguments $z-\Omega$ satisfies the fundamental condition

$$\cdot 501 \qquad f(z+\Omega) = f(z).$$

For example, the distance of z from the nearest lattice point is such a function. To be analytic in z, and actually to involve the infinity of arguments Ω, the function must be a limit, and we have to find a convergent sequence.

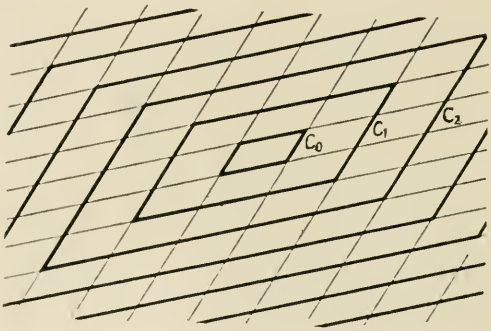


Fig. 10.

Let a lattice be determined by two families of parallel lines, and let P be any point other than a lattice point in or on the boundary C_0 of a cell B_0. The cells which immediately surround B_0 form with B_0 a block B_1 of 9 cells, the next ring of cells forms with B_1 a block B_2 of 25 cells, and so on. The boundary C_r of B_r contains $4(2r+1)$ lattice points, and if ρ is the shorter of the perpendicular distances between opposite sides of a cell, the distance of P from any point on the boundary C_r is greater than or equal to $r\rho$. The series $\sum(2r+1)r^{-k}$ is convergent if k has any real value greater than 2. It follows that, if K is the sum of this series, if σ is any number smaller than the distance of P from the nearest corner of C_0, and if λ is the distance of P from a typical lattice point, then for $k > 2$,

$$\cdot 502 \qquad \sum \lambda^{-k} < 4(\sigma^{-k} + K\rho^{-k}).$$

the summation being extended to any selection whatever of lattice points. Now $\lambda = |z-\Omega|$, and $(z-\Omega)^{-k}$ is singlevalued if k is a whole number. Hence

·503. *If k is any whole number not smaller than 3, the series $\sum (z-\Omega)^{-k}$ extended to all the points of a lattice is absolutely convergent at every point z which is not a lattice point and is uniformly convergent throughout any closed region which does not include any lattice points.*

Thus

0·51. *For any positive integral value of k not smaller than 3, the series $\sum (z-\Omega)^{-k}$ defines a function $\zeta_k z$ of z which is analytic for all finite values of z except the lattice values Ω.*

From its construction,

0·52. *The function $\zeta_k z$ defined for $k \geqslant 3$ by the summation*

$$\zeta_k z = \sum (z-\Omega)^{-k}$$

extended to all the points of a lattice is an elliptic function whose only poles are the lattice points themselves; these are poles of order k.

Although there is an arbitrary whole number in this theorem, only a single function is really being introduced into analysis, for

·504 $$d\zeta_k z/dz = -k\zeta_{k+1} z,$$

and the convergence becomes stronger on each differentiation. Once $\zeta_3 z$ has been defined as $\sum (z-\Omega)^{-3}$, the other functions would follow without independent definition:

·505 $$d^m \zeta_3 z/dz^m = (-)^m \tfrac{1}{2}(m+2)!\, \zeta_{m+3} z.$$

The condition $k > 2$, which is essential for the convergence of the series $\sum (z-\Omega)^{-k}$, raises an urgent question. The function $\zeta_3 z$ is of the third order, and we know that there are no elliptic functions of the first order. Are there functions of the second order?

Consider the passage not from $\zeta_k z$ to $\zeta_{k+1} z$ by differentiation but from $\zeta_k z$ to $\zeta_{k-1} z$ by integration. We have

·506 $$\int_{z_1}^{z_2} \zeta_k z\, dz = -\frac{1}{k-1} \sum \{(z_2-\Omega)^{-(k-1)} - (z_1-\Omega)^{-(k-1)}\},$$

and if $k-1 \geqslant 3$ the two series $\sum (z_1-\Omega)^{-(k-1)}$, $\sum (z_2-\Omega)^{-(k-1)}$ are separately absolutely convergent, and we can write

·507 $$\int_{z_1}^{z_2} \zeta_k z\, dz = -\frac{1}{k-1}\{\sum (z_2-\Omega)^{-(k-1)} - \sum (z_1-\Omega)^{-(k-1)}\},$$

that is,

·508 $$\int_{z_1}^{z_2} \zeta_k z\, dz = -\frac{1}{k-1}(\zeta_{k-1} z_2 - \zeta_{k-1} z_1).$$

The formula ·506 remains true if $k = 3$:

·509 $$\int_{z_1}^{z_2} \zeta_3 z\, dz = -\tfrac{1}{2} \sum \{(z_2-\Omega)^{-2}-(z_1-\Omega)^{-2}\}.$$

But now, although the series on the right is convergent for any two values of z_1 and z_2, the separate series $\sum (z_1-\Omega)^{-2}$, $\sum (z_2-\Omega)^{-2}$ are not convergent. With an arbitrary value of z_1 we may introduce the function

$$-2\int_{z_1}^{z} \zeta_3 z\, dz$$

and identify this function with

·510 $$\sum \{(z-\Omega)^{-2}-(z_1-\Omega)^{-2}\},$$

but whatever value of z_1 we choose we can not avoid the composite form of the typical term in the sum.

To define a standard function from the series ·510, we take $z_1 = 0$. This choice, although almost inevitable, involves us in a difficulty on each side of the equation ·509, because the origin is a lattice point: near the origin, $\zeta_3 z \sim z^{-3}$, and therefore 0 can not be used as a limit of the integral; the series includes a term in which $\Omega = 0$, and in this term we can not put $z_1 = 0$. To meet the difficulty, we segregate the term in which $\Omega = 0$. We have

$$\zeta_3 z = z^{-3} + \sum{}' (z-\Omega)^{-3},$$

where the prime attached to the symbol of summation indicates that the term in which $\Omega = 0$ is omitted, a convention that is maintained, with products as well as with sums, throughout this subject. Since

$$\int^{z} \frac{dz}{z^3} = -\frac{1}{2z^2},$$

where the integral is indefinite, or more strictly has ∞ for lower limit, and

$$\int_0^z \sum{}' \frac{1}{(z-\Omega)^3}\, dz = -\frac{1}{2} \sum{}' \left\{\frac{1}{(z-\Omega)^2} - \frac{1}{\Omega^2}\right\},$$

the function $\zeta_3 z$ can be integrated by means of the singlevalued analytic function $\wp z$ defined by the formula

0·53 $$\wp z = \frac{1}{z^2} + \sum{}' \left\{\frac{1}{(z-\Omega)^2} - \frac{1}{\Omega^2}\right\}.$$

We have

0·54 $$\wp' z = -2\zeta_3 z,$$

·511 $$\int_{z_1}^{z_2} \zeta_3 z\, dz = -\tfrac{1}{2}(\wp z_2 - \wp z_1),$$

·512 $$\int_0^z (\zeta_3 z - z^{-3})\, dz = -\tfrac{1}{2}(\wp z - z^{-2});$$

we can define $\wp z$ from its derivative $-2\zeta_3 z$ if we add the condition

·513 $$\wp z - z^{-2} \to 0,$$

which is implied by ·512; the weaker condition $\wp z \sim z^{-2}$ is often useful, but it is already implied in ·54 and does not distinguish $\wp z$ from any other integral of $\wp' z$.

Being the integral of an elliptic function without simple poles, $\wp z$ is known in advance to have pseudoperiodicity: if $\alpha\beta$ is a basis, the equations

$$\wp'(z+\alpha) - \wp' z = 0, \qquad \wp'(z+\beta) - \wp' z = 0$$

imply $$\wp(z+\alpha) - \wp z = \mathrm{A}, \qquad \wp(z+\beta) - \wp z = \mathrm{B},$$

where A, B are constants. But if Ω is a lattice step, so also is $-\Omega$, and therefore

0·55. *The function $\wp z$ is an even function.*

Also $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$ are not lattice points and are therefore not poles of $\wp z$. Substituting $z = -\frac{1}{2}\alpha$ in the formula for A and $z = -\frac{1}{2}\beta$ in the formula for B we have

$$\mathrm{A} = \wp(\tfrac{1}{2}\alpha) - \wp(-\tfrac{1}{2}\alpha) = 0, \qquad \mathrm{B} = \wp(\tfrac{1}{2}\beta) - \wp(-\tfrac{1}{2}\beta) = 0,$$

whence $$\wp(z+\alpha) = \wp z, \qquad \wp(z+\beta) = \wp z:$$

0·56. *The function $\wp z$ is an elliptic function with the lattice points for double poles.*

Thus $\wp z$ is *an elliptic function of the second order* with coincident poles; it is *the Weierstrassian elliptic function*. The residue of the function at its pole is zero, as required by ·41. We can in fact convert the expansion ·53 into the Laurent expansion for $\wp z$ in the neighbourhood of the origin. If $|z| < |\Omega|$, then

$$\frac{1}{(z-\Omega)^2} = \frac{1}{\Omega^2} + \frac{2z}{\Omega^3} + \frac{3z^2}{\Omega^4} + \dots .$$

For any odd value of r, $\sum' \Omega^{-r} = 0$, and if we write, for $r > 1$,

·514 $$s_r = \sum{}' \Omega^{-2r},$$

we have

0·57 $$\wp z = z^{-2}+3s_2z^2+5s_3z^4+7s_4z^6+\dots,$$

valid inside the circle whose centre is the origin and whose circumference passes through the nearest of the other lattice points. The descriptive formula

0·58 $$\wp z = z^{-2}+O(z^2)$$

adds to ·513 only as much as can be inferred from ·55, but presents the result in the form which is usually the most convenient to use.

If the lattice is referred to a basis $\alpha\beta$, every step Ω is of the form $m\alpha+n\beta$, and the powers Ω^{-k}, $(z-\Omega)^{-k}$ are homogeneous functions of degree $-k$, the former in the pair of variables α, β, the latter in the set of three variables z, α, β. This homogeneity, and its degree, are independent of the choice of basis, and we may say simply that the functions are of degree $-k$ in Ω, or in z and Ω:

0·59$_1$. *The elliptic functions $\zeta_k z$, $\wp z$ are homogeneous functions, of degrees $-k$, -2, in z and Ω.*

The homogeneity of $\wp z$, in the neighbourhood of the origin, is apparent also in the expansion ·57; the sum s_r is homogeneous of degree $-2r$ in Ω, and therefore the sum of terms of the form $s_r z^{2r-2}$ is homogeneous of degree -2 in z and Ω.

If we indicate the dependence of the functions $\zeta_k z$, $\wp z$ on the lattice by writing them in the form $\zeta_k(z|\Omega)$, $\wp(z|\Omega)$, we can express ·59$_1$ symbolically:

0·59$_{2-3}$ $$\zeta_k(\lambda z|\lambda\Omega) = \lambda^{-k}\zeta_k(z|\Omega); \qquad \wp(\lambda z|\lambda\Omega) = \lambda^{-2}\wp(z|\Omega).$$

If the homogeneity is known, its degree is given immediately by the forms of the functions near the origin.

We can arrive at the homogeneity of the elliptic functions somewhat differently. Whatever the complex number λ, the lattice $\lambda\Omega$ which has $\lambda\alpha$, $\lambda\beta$ for a basis is geometrically similar to the lattice Ω which has the basis $\alpha\beta$; the former lattice is derived from the latter by rotation through the angle of λ and magnification by the factor $|\lambda|$. Let $w = \lambda z$, and regard the function $\wp z$ as a function $f(w)$ of w. Addition of Ω to z is equivalent to addition of $\lambda\Omega$ to w; hence $f(w+\lambda\Omega) = f(w)$, and $f(w)$ is a doubly periodic function belonging to the lattice $\lambda\Omega$. A singularity of $f(w)$ arises only from a singularity of $\wp z$; near $z = 0$,

$$f(w) = z^{-2}+O(z^2) = \lambda^2 w^{-2}+O(w^2).$$

Hence $\lambda^{-2}f(w)$ is a function doubly periodic on the lattice $\lambda\Omega$, with the lattice points for double poles and with no other accessible singularities, and such that near $w = 0$, $\lambda^{-2}f(w) = w^{-2}+O(w^2)$. These properties are sufficient, by ·48, to identify $\lambda^{-2}f(w)$ with $\wp(w|\lambda\Omega)$, and replacing w by λz we have $\wp(\lambda z|\lambda\Omega) = \lambda^{-2}\wp(z|\Omega)$, as in $\cdot 59_3$. To adapt this argument to $\zeta_k z$ we must take into account the value of the limit of $\zeta_k z - z^{-k}$ as $z \to 0$; alternatively, $\cdot 59_2$ follows from $\cdot 59_3$ by differentiation.

The homogeneity of these elliptic functions can be expressed geometrically. The two lattices $\lambda\Omega$, Ω are similar, in the elementary geometrical sense, and the point λz occupies in the one lattice the position similar to that occupied by the point z in the other lattice. Apart from constant factors, homogeneous functions are functions of position relative to the lattice, rather than of absolute position in the plane. If z_1, z_2 are associated with the lattice Ω, then λz_1, λz_2 are associated similarly with the lattice $\lambda\Omega$, and the ratios

$$\wp(\lambda z_2|\lambda\Omega)/\wp(\lambda z_1|\lambda\Omega), \qquad \wp(z_2|\Omega)/\wp(z_1|\Omega)$$

are identical. This is only to say that $\wp(\lambda z|\lambda\Omega) = \kappa\wp(z|\Omega)$, where κ is expressible as $\wp(\lambda z_2|\lambda\Omega)/\wp(z_2|\Omega)$ and is independent of z.

0·6. Since the residues of $\wp z$ are zero, to repeat the process of integration does not introduce a manyvalued function. We have

$$\int_0^z \left\{\frac{1}{(z-\Omega)^2}-\frac{1}{\Omega^2}\right\} dz = -\left\{\frac{1}{z-\Omega}+\frac{1}{\Omega}+\frac{z}{\Omega^2}\right\}$$

and we therefore define a function ζz by the formula

0·61 $$\zeta z = \frac{1}{z}+\sum{}' \left\{\frac{1}{z-\Omega}+\frac{1}{\Omega}+\frac{z}{\Omega^2}\right\}.$$

With this definition

0·62 $$\zeta' z = -\wp z,$$

·601 $$\int_0^z (\wp z - z^{-2})\, dz = -(\zeta z - z^{-1}),$$

·602 $$\zeta z - z^{-1} \to 0,$$

and from ·57 or ·61 the Laurent expansion is

$$\zeta z = z^{-1} - s_2 z^3 - s_3 z^5 - s_4 z^7 - \dots .$$

Although the condition $k \geqslant 3$ is indispensable to ·51, and $\wp z$ and ζz can not fit into the sequence $\zeta_k z$, the formulae ·54, ·62 extend the sequence ·504, and we can replace ·505 by

·603 $$d^m \zeta z/dz^m = (-)^m m!\, \zeta_{m+1} z, \qquad m \geqslant 2.$$

Since every residue of ζz is 1, the sum of a number of residues can not be zero and the function is not an elliptic function. Hence, being the integral of an elliptic function,

$0{\cdot}63_1$. *The function* ζz *is pseudoperiodic on the lattice* Ω.

In repeating the argument by which the periodicity of $\wp z$ was demonstrated, we take the basis as $(2\omega_1, 2\omega_2)$. This form of basis, in which the explicit symbols are for halfperiods of the Weierstrassian function attached to the lattice, not for periods of these functions, proves to be incomparably the most economical throughout the theory, and is now to be adopted as the standard form. We have, since ζz is an odd function,

$\cdot 604_{1-2}$ $$\zeta(z+2\omega_1)-\zeta z = 2\eta_1, \qquad \zeta(z+2\omega_2)-\zeta z = 2\eta_2,$$

where

$\cdot 605_{1-2}$ $$\eta_1 = \zeta\omega_1, \qquad \eta_2 = \zeta\omega_2,$$

and for the effect of addition of a general period,

$0{\cdot}63_2$ $$\zeta(z+2m\omega_1+2n\omega_2) = \zeta z+2m\eta_1+2n\eta_2.$$

To the function ζz we can apply the result of ·46, or we may repeat the argument of that theorem, taking for the cell the parallelogram whose corners are the four points $\pm\omega_1 \pm\omega_2$. The function has only one pole in the cell, and the residue there is 1. Hence $4\eta_1\omega_2-4\eta_2\omega_1 = 2\pi v$, that is,

$0{\cdot}64$ $$\eta_1\omega_2-\eta_2\omega_1 = \tfrac{1}{2}\pi v,$$

where v is the signature of the basis $2\omega_1$, $2\omega_2$. The presence of the signature in this formula is easily understood: if the basis is changed from $2\omega_1$, $2\omega_2$ to $2\omega'$, $2\omega''$ by the pair of formulae

$$\omega' = m'\omega_1+n'\omega_2, \qquad \omega'' = m''\omega_1+n''\omega_2,$$

the moduli η', η'' are given by the pair of formulae

$$\eta' = m'\eta_1+n'\eta_2, \qquad \eta'' = m''\eta_1+n''\eta_2$$

with the same coefficients, and therefore

$$\begin{vmatrix} \eta', & \eta'' \\ \omega', & \omega'' \end{vmatrix} = \begin{vmatrix} m', & m'' \\ n', & n'' \end{vmatrix} \begin{vmatrix} \eta_1, & \eta_2 \\ \omega_1, & \omega_2 \end{vmatrix}.$$

Since the function ζz has residues which are not zero, integration of ζz produces a manyvalued function, but since the principal part of ζz near $z = \Omega$ is $1/(z-\Omega)$, the multiplicity of the integral is the multiplicity of the logarithm of a singlevalued function. In other words, we can regard ζz not as a derivative but as a logarithmic derivative, and the function σz defined by the formula

$$\log(\sigma z) = \log z + \int_0^z \sum{}' \left\{\frac{1}{z-\Omega}+\frac{1}{\Omega}+\frac{z}{\Omega^2}\right\} dz$$

is singlevalued. Performing the integration, we have

$$\log(\sigma z) = \log z + \sum{}' \left\{\log\left(1-\frac{z}{\Omega}\right)+\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}\right\},$$

whence

0·65 $$\sigma z = z \prod{}' \left\{\left(1-\frac{z}{\Omega}\right)e^{z/\Omega+z^2/2\Omega^2}\right\}.$$

The definition of σz is equivalent to definition by the relation

0·66 $$\frac{\sigma' z}{\sigma z} = \zeta z$$

coupled with the condition

·606 $$\frac{\sigma z}{z} \to 1.$$

Otherwise expressed,

·607 $$\sigma z = z \exp\left\{\int_0^z (\zeta z - z^{-1})\, dz\right\}.$$

It follows from the uniform convergence of the series for $\zeta_3 z$ that the series for $\wp z$ and ζz also are uniformly convergent, and therefore that σz has no accessible poles and no zeros except those which are immediately in evidence:

0·67. *The function σz is an integral function which has the lattice points for simple zeros.*

The effect on σz of addition of a period to z is to be found from ·66 and ·604. We have

$$\frac{\sigma'(z+2\omega_1)}{\sigma(z+2\omega_1)} - \frac{\sigma' z}{\sigma z} = 2\eta_1,$$

and therefore

·608 $$\frac{\sigma(z+2\omega_1)}{\sigma z} = C_1 e^{2\eta_1 z},$$

where C_1 is a constant to be determined. But since $\zeta z - z^{-1}$ is an odd function,

$$\exp\left\{\int_0^z (\zeta z - z^{-1})\, dz\right\}$$

is an even function, and σz is an odd function, and putting $z = -\omega_1$ in ·608 we have

$$\cdot 609 \qquad C_1 = -e^{2\eta_1\omega_1}.$$

Thus

$$0{\cdot}68_1 \qquad \sigma(z+2\omega_1) = -e^{2\eta_1(z+\omega_1)}\sigma z,$$

and similarly

$$0{\cdot}68_2 \qquad \sigma(z+2\omega_2) = -e^{2\eta_2(z+\omega_2)}\sigma z.$$

These formulae are often conveniently taken in the form

$$0{\cdot}68_{3\text{-}4} \quad \sigma(z+\omega_1) = -e^{2\eta_1 z}\sigma(z-\omega_1), \qquad \sigma(z+\omega_2) = -e^{2\eta_2 z}\sigma(z-\omega_2).$$

If we substitute $z+2\omega_2$ for z in $\cdot 68_1$ and $z+2\omega_1$ for z in $\cdot 68_2$ and compare the results, we find $e^{4\eta_1\omega_2} = e^{4\eta_2\omega_1}$, that is,

$$e^{4(\eta_1\omega_2-\eta_2\omega_1)} = 1,$$

whence $\eta_1\omega_2-\eta_2\omega_1$ is a multiple of $\frac{1}{2}\pi i$, in agreement with ·64, but this argument does not lead to the former precise result.

The functions ζz, σz, like the elliptic functions $\wp z$, $\zeta_k z$, are homogeneous in z and Ω. As in some other respects, ζz is in sequence with the elliptic functions, and ζz is of degree -1; the homogeneity of σz, and the degree of this function, are most evident in the explicit formula ·65, which shows σz as the product by z of a function of degree 0:

$$0{\cdot}69_{1\text{-}2} \qquad \zeta(\lambda z|\lambda\Omega) = \lambda^{-1}\zeta(z|\Omega), \qquad \sigma(\lambda z|\lambda\Omega) = \lambda\,\sigma(z|\Omega).$$

Elliptic functions in general, and the Weierstrassian functions in particular, depend fundamentally on the shape of the lattice to which they belong, and only to a trivial extent on its size and orientation, for the distribution of values of a function attached to a lattice Ω can be deduced immediately from the distribution of values of the function attached in the same way to any lattice geometrically similar to Ω.

0·7. A zero of the derivative $\wp' z$ is a value of b for which the equation $\wp z - \wp b = 0$ has a multiple root. Since $\wp z$ is of the second order, no root of this equation can be of higher multiplicity than two, and therefore the zeros of $\wp' z$ are necessarily simple. Since the only poles of $\wp' z$ are the triple poles at the lattice points, $\wp' z$ is of the third order. Hence $\wp' z$ has three simple zeros. To locate these zeros, return to the equation $\wp z - \wp b = 0$, taking the equation in the form $\wp z = \wp b$. One root of this equation is $z = b$, and since the function is even, another root is $z = -b$; in general these two roots are incongruent, and every root is congruent with one or other of them. Congruent roots can not coalesce, and therefore if b is a double root, $b \equiv -b$, that is, $2b \equiv 0$; conversely, this condition is sufficient, provided that b is not a pole:

·701. *The zeros of $\wp' z$ are the points other than the lattice points which satisfy the congruence $2z \equiv 0$.*

The points given by $2z \equiv 0$ are the midpoints of steps from the origin to the lattice points. Since the congruence can be expressed also as $2(z-\Omega_t) \equiv 0$, the same points are also the midpoints of steps to the lattice points from any other lattice point. *The points given by $2z \equiv 0$ are the midpoints of steps from one lattice point to another.* If the lattice is referred to a basis $(2\omega_1, 2\omega_2)$, the condition $2z \equiv 0$ becomes

$$2z = 2m\omega_1 + 2n\omega_2,$$

that is, $z = m\omega_1 + n\omega_2$, and can be decomposed according to the parity of m and n.

FIG. 11.

If m and n are both even, the aggregate $m\omega_1 + n\omega_2$ is the original lattice; if m is odd and n even, the aggregate is the congruent lattice which includes the point ω_1; if m is even and n odd, the aggregate is the congruent lattice which includes the point ω_2; if m and n are both odd, the aggregate is the congruent lattice which includes the point $\omega_1 + \omega_2$:

·702. *The midpoints of steps in a lattice compose the lattice itself and three lattices congruent with it.*

We usually apply the name of midpoint lattice only to the three lattices which are distinct from the original lattice.

If $OACB$ is a cell in the lattice, the midpoints of OA, OB, and OC are three points of which no two are congruent, and the three midpoint lattices can be identified as the three which include these points. But it is to be emphasized that the midpoint lattices depend only on the original lattice, not on a particular cell or basis.

If we express ·701 in the form that

0·71. *The zeros of $\wp' z$ constitute the three midpoint lattices of the lattice to which $\wp z$ belongs*

we foresee that the set of three lattices plays a leading part in the theory of the functions. Two of the midpoint lattices are associated with the halfperiods ω_1, ω_2, and since there is no intrinsic difference between the three lattices we associate with these halfperiods a halfperiod belonging to the third lattice, that is, a halfperiod ω_3 congruent with $\omega_1+\omega_2$. To take ω_3 as $\omega_1+\omega_2$ involves a lack of symmetry which ultimately becomes extremely tiresome; we define ω_3 instead by the symmetrical relation

·703 $$\omega_1+\omega_2+\omega_3 = 0.$$

We can use $(2\omega_2, 2\omega_3)$ or $(2\omega_3, 2\omega_1)$ as a basis for the lattice instead of $(2\omega_1, 2\omega_2)$, and it is sometimes useful to exhibit the lattice as the set of points of intersection of three families of parallel lines.

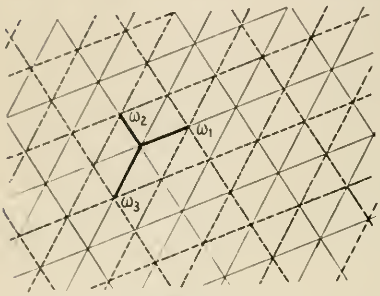


Fig. 12.

The signature υ is the same for the bases $(2\omega_2, 2\omega_3)$ and $(2\omega_3, 2\omega_1)$ as for the basis $(2\omega_1, 2\omega_2)$, and may be regarded as the signature of the triplet of halfperiods. We may admit the result intuitively, but it is evident analytically from ·703, which, written in the form

$$1+(\omega_2/\omega_1)+(\omega_3/\omega_1) = 0,$$

implies that $\mathrm{Im}(\omega_3/\omega_1)$ has the opposite sign to $\mathrm{Im}(\omega_2/\omega_1)$, that is, that $\mathrm{Im}(\omega_1/\omega_3)$ has the same sign as $\mathrm{Im}(\omega_2/\omega_1)$. The result follows also from ·64: writing $\eta_3 = \zeta\omega_3$, we have from ·703 and ·63_2

·704 $$\eta_1+\eta_2+\eta_3 = 0,$$

which with ·703 implies

·705 $$\eta_1\omega_2-\eta_2\omega_1 = \eta_2\omega_3-\eta_3\omega_2 = \eta_3\omega_1-\eta_1\omega_3.$$

It is sometimes worth while to replace ·63_2 by

0·72 $$\zeta(z+2m\omega_1+2n\omega_2+2p\omega_3) = \zeta z+2m\eta_1+2n\eta_2+2p\eta_3.$$

To ·68 we may add

·$706_{1\text{-}2}$ $$\sigma(z+2\omega_3) = -e^{2\eta_3(z+\omega_3)}\sigma z, \qquad \sigma(z+\omega_3) = -e^{2\eta_3 z}\sigma(z-\omega_3).$$

These formulae can be generalized immediately; we have, by ·72,

$$\frac{\sigma'(z+m\omega_1+n\omega_2+p\omega_3)}{\sigma(z+m\omega_1+n\omega_2+p\omega_3)}-\frac{\sigma'(z-m\omega_1-n\omega_2-p\omega_3)}{\sigma(z-m\omega_1-n\omega_2-p\omega_3)} = 2m\eta_1+2n\eta_2+2p\eta_3,$$

and from this, since

$$\sigma(m\omega_1+n\omega_2+p\omega_3) = -\sigma(-m\omega_1-n\omega_2-p\omega_3),$$

integration implies

0·73 $$\sigma(z+m\omega_1+n\omega_2+p\omega_3) = -e^{(2m\eta_1+2n\eta_2+2p\eta_3)z}\sigma(z-m\omega_1-n\omega_2-p\omega_3),$$

for all integral values of m, n, p. Hence also

·707 $$\sigma(z+2m\omega_1+2n\omega_2+2p\omega_3) = -e^{(2m\eta_1+2n\eta_2+2p\eta_3)(z+m\omega_1+n\omega_2+p\omega_3)}\sigma z.$$

We have seen that $\wp z - B$ has a double zero if and only if B has one of the three values $\wp\omega_1$, $\wp\omega_2$, $\wp\omega_3$; these three values, the values of $\wp z$ on the three midpoint lattices, are denoted by e_1, e_2, e_3. Since $\wp z - e_r$ has a double pole at the origin and a double zero at ω_r, the product $(\wp z - e_1)(\wp z - e_2)(\wp z - e_3)$ has a sextuple pole at the origin and double zeros at ω_1, ω_2, ω_3. On the other hand, $\wp' z$ has a triple pole at the origin and simple zeros at ω_1, ω_2, ω_3. That is to say, $\wp'^2 z$ has the same structure as $(\wp z - e_1)(\wp z - e_2)(\wp z - e_3)$, and by Liouville's theorem, $\cdot 49_3$, one function is a constant multiple of the other. Near the origin, $\wp z \sim 1/z^2$, $\wp' z \sim -2/z^3$. Hence

0·74₁ $$\wp'^2 z = 4(\wp z - e_1)(\wp z - e_2)(\wp z - e_3).$$

This fundamental relation between $\wp z$ and $\wp' z$ can be expressed in another form by means of Liouville's other identification theorem, ·48. From the Laurent expansion ·57,

$$\wp z = z^{-2} + 3s_2 z^2 + 5s_3 z^4 + O(z^6),$$

we have

$$\wp^3 z = z^{-6} + 3z^{-2}(3s_2 + 5s_3 z^2) + O(z^2)$$

$$= z^{-6} + 9s_2 z^{-2} + 15s_3 + O(z^2),$$

and also

$$\wp' z = -2z^{-3} + 6s_2 z + 20s_3 z^3 + O(z^5),$$

$$\wp'^2 z = 4z^{-6} - 24s_2 z^{-2} - 80s_3 + O(z^2).$$

Hence

·708 $$\wp'^2 z = 4\wp^3 z - 60s_2 z^{-2} - 140s_3 + O(z^2)$$

$$= 4\wp^3 z - g_2 \wp z - g_3 + O(z^2),$$

where $$g_2 = 60s_2, \qquad g_3 = 140s_3,$$

that is,

·709₁₋₂ $$g_2 = 60 \sum{}' \Omega^{-4}, \qquad g_3 = 140 \sum{}' \Omega^{-6}.$$

But $\wp'^2 z - (4\wp^3 z - g_2 \wp z - g_3)$ is an elliptic function with no possible poles except the lattice points; the formula ·708 proves that the origin is not a pole but a zero of this function, and it follows that the function has the constant value 0:

0·74₂ $$\wp'^2 z = 4\wp^3 z - g_2 \wp z - g_3.$$

Comparing the two formulae for $\wp'^2 z$ we deduce that

0·75₁. *The three midpoint constants* e_1, e_2, e_3 *are the roots of the equation*

$$4t^3 - g_2 t - g_3 = 0.$$

In other words, the midpoint values e_1, e_2, e_3 satisfy the relation

0·75₂ $$e_1 + e_2 + e_3 = 0,$$

and the constants g_2, g_3, which are called the invariants of the lattice, are given in terms of e_1, e_2, e_3 by

0·75$_{3-4}$ $$g_2 = -e_2 e_3 - e_3 e_1 - e_1 e_2, \qquad g_3 = e_1 e_2 e_3.$$

Differentiating ·74$_2$ we have

·710 $$\wp'' z = 6\wp^2 z - \tfrac{1}{2} g_2,$$

whence, substituting the complete Laurent expansions,

$$\begin{aligned} 3z^{-4} + 1.1.3\, s_2 + 2.3.5\, s_3 z^2 + 3.5.7\, s_4 z^4 + 4.7.9\, s_5 z^6 + \ldots \\ = 3z^{-4} + 6(3s_2 + 5s_3 z^2 + 7s_4 z^4 + 9s_5 z^6 + \ldots) \\ + 3z^4 (3s_2 + 5s_3 z^2 + \ldots)^2 - \tfrac{1}{4} g_2, \end{aligned}$$

implying $g_2 = 60 s_2$ as before, and

·711 $$\begin{aligned} 1.7.9\, s_4 + 2.9.11\, s_5 z^2 + 3.11.13\, s_6 z^4 + \ldots \\ = 3(3s_2 + 5s_3 z^2 + 7s_4 z^4 + \ldots)^2, \end{aligned}$$

identically, whence

·712. *The sums* s_4, s_5,... *are polynomials in* s_2 *and* s_3, *with rational coefficients independent of the lattice.*

It follows that while a basis is needed for the evaluation of the invariants of the lattice, the later sums can be deduced from the invariants without further reference to the basis.

When the invariants are known, ·74$_2$ becomes a differential equation,

·713 $$(dw/dz)^2 = 4w^3 - g_2 w - g_3,$$

from which $\wp z$ can be determined as the one solution which satisfies the condition $w \sim 1/z^2$ near $z = 0$.

An alternative argument, leading to simple general theorems which we shall find useful, shows very clearly why the zeros of $\wp' z$, but not those of $\wp z$, can be identified in the lattice. For any lattice step Ω, an elliptic function $f(z)$ of which $\tfrac{1}{2}\Omega$ is not a pole satisfies the condition

·714 $$f(-\tfrac{1}{2}\Omega) = f(\tfrac{1}{2}\Omega).$$

If $f(z)$ is odd, it satisfies also the condition

·715 $$f(-\tfrac{1}{2}\Omega) = -f(\tfrac{1}{2}\Omega),$$

and we have therefore $\quad f(\tfrac{1}{2}\Omega) = -f(\tfrac{1}{2}\Omega)$,

implying that $\tfrac{1}{2}\Omega$, not being a pole, is a zero. Further, if the order of this zero is n, the derivative $f^{(n)}(z)$, which is an even function or an odd function according as n is odd or even, is an elliptic function of which $\tfrac{1}{2}\Omega$ is neither pole nor zero, and therefore is not an odd function: that is, n is odd. Lastly, if $f(z)$ is an odd function which has $\tfrac{1}{2}\Omega$ for a pole of order m, then $1/f(z)$ is an odd function which has $\frac{1}{2}\Omega$ for a zero of order m, and therefore m is odd:

0·76. *Every odd elliptic function has every lattice point and every midpoint for either a pole of odd order or a zero of odd order.*

One corollary, since the derivative of an even function is an odd function, is

·716. *If an even elliptic function has a lattice point or a midpoint for a pole or a zero, the order of this pole or zero is even,*

and another, which leads immediately to ·71, is

·717. *If an odd elliptic function of the third order has one of the four points* $0, \omega_1, \omega_2, \omega_3$ *for a triple pole, it has the other three points for simple zeros.*

Also

·718. *Every odd elliptic function of the second order has two of the four points* $0, \omega_1, \omega_2, \omega_3$ *for simple poles and the other two for simple zeros.*

If $f(z)$ is any function of which $\frac{1}{2}\Omega$ is not a singularity, $f(z)-f(\frac{1}{2}\Omega)$ has $\frac{1}{2}\Omega$ for a zero. It follows from ·716 that

0·77. *If* $f(z)$ *is an even elliptic function,* $f(z)-f(\frac{1}{2}\Omega)$ *has* $\frac{1}{2}\Omega$ *either for a pole of even order or for a zero of even order.*

The theorem with which this section began, which can be enunciated in the form

0·78. *The function* $\wp z - e_r$ *has the midpoint* ω_r *for a double zero,*

is a particular case of this general result, but it is a case of fundamental importance in the sequel.

Since $\wp z - e_1$ has the origin for a double pole and has the point ω_1 for a double zero, $\wp(z+\omega_1)-e_1$ has the origin for a double zero and has $-\omega_1$, and therefore ω_1, for a double pole. Neither function has any other poles or zeros†, and therefore their product, which has the periods of $\wp z$, has no poles, and by ·31 is a constant; but, when $z = \omega_2$,

$$\wp(z+\omega_1) = \wp(-\omega_3) = \wp\omega_3;$$

hence

0·79 $$\{\wp z - e_1\}\{\wp(z+\omega_1)-e_1\} = (e_2-e_1)(e_3-e_1),$$

a formula which shows more clearly than an explicit formula for $\wp(z+\omega_1)$ the effect of the addition of a halfperiod to the argument of the function.

† That is, any incongruent with these. This is a laxity of expression which can do no harm.

0·8. Unless $C = 0$, the function $A+B\wp z+C\wp' z$, in which A, B, C are constants, has a triple pole at the origin and no other poles; this function has therefore three zeros, and their sum is congruent with 0. To determine $A : B : C$ by the equations

$$A+B\wp x+C\wp' x = 0, \qquad A+B\wp y+C\wp' y = 0,$$

where x, y are given complex numbers, is to take the function in the form

$$\begin{vmatrix} 1, & \wp x, & \wp' x \\ 1, & \wp y, & \wp' y \\ 1, & \wp z, & \wp' z \end{vmatrix}$$

in which the two zeros $z \equiv x$, $z \equiv y$ are already obvious. Hence

0·81. *If* $x+y+z \equiv 0$, *then*

$$\begin{vmatrix} 1, & \wp x, & \wp' x \\ 1, & \wp y, & \wp' y \\ 1, & \wp z, & \wp' z \end{vmatrix} = 0.$$

A more complete enunciation is

0·82. *If* x, y *are given, the equation*

$$\begin{vmatrix} 1, & \wp x, & \wp' x \\ 1, & \wp y, & \wp' y \\ 1, & \wp z, & \wp' z \end{vmatrix} = 0$$

is satisfied if z *is congruent with* x, *with* y, *or with* $-(x+y)$, *but not otherwise.*

It follows from ·82 that the equation

$$(\wp x-\wp y)^2\wp'^2 z = \begin{vmatrix} 1, & \wp x, & \wp' x \\ 1, & \wp y, & \wp' y \\ 1, & \wp z, & 0 \end{vmatrix}^2,$$

that is, the equation

$$(\wp x-\wp y)^2(4\wp^3 z-g_2\wp z-g_3) = \begin{vmatrix} 1, & \wp x, & \wp' x \\ 1, & \wp y, & \wp' y \\ 1, & \wp z, & 0 \end{vmatrix}^2,$$

is satisfied if z is congruent with $\pm x$, with $\pm y$, or with $\mp(x+y)$, that is, if $\wp z$ is equal to $\wp x$, to $\wp y$, or to $\wp(x+y)$, but not otherwise: $\wp x$, $\wp y$, $\wp(x+y)$ are the roots of the cubic equation

$$(\wp x-\wp y)^2(4t^3-g_2 t-g_3) = \begin{vmatrix} 1, & \wp x, & \wp' x \\ 1, & \wp y, & \wp' y \\ 1, & t, & 0 \end{vmatrix}^2.$$

Any relation between the roots and the coefficients of this equation is a relation between $\wp(x+y)$ and functions of the separate variables x, y. In particular, expressing the sum of the roots, we have one addition theorem which does not involve the invariants:

0·83. *For any two values of the variable,*

$$\wp(x+y) = \tfrac{1}{4}\{(\wp'x-\wp'y)/(\wp x-\wp y)\}^2-\wp x-\wp y.$$

0·9. The function $\wp z$ having been constructed, a function $f(z)$ can, we proceed to show, be built on the same lattice to an assigned specification. Since $\wp z$ is an even function, any rational function of $\wp z$ is even also, and we suppose first that $f(z)$ is even. Then if b is a zero of $f(z)$, of order q, so also is $-b$, to the same multiplicity, and if $2b \not\equiv 0$, these two zeros are incongruent. Under the same condition, the zeros of $\wp z-\wp b$ are simple zeros at the points congruent with b or $-b$, and the zeros of $(\wp z-\wp b)^q$ are zeros of order q at these points. If $2b \equiv 0$, there are two cases to distinguish. If $b \equiv 0$, no function of the form $\wp z-\wp b$ is available; this case is left aside for the moment. If $b \equiv \omega_r$, the function $\wp z-\wp b$ becomes $\wp z-e_r$ and has a double zero; the zeros of any integral power of $\wp z-e_r$ are of even order. But we have seen that ω_r, if a zero of the even function $f(z)$, is a zero of even order, and if this order is $2q'$, then $(\wp z-e_r)^{q'}$ has zeros equivalent to those of $f(z)$ at the points congruent with ω_r. Thus if $f(z)$ is an even function, an irreducible set of zeros of $f(z)$, excluding a zero congruent with the origin, can be taken to be $\pm b_1, \pm b_2, \ldots, \pm b_n$ with orders $q_1, q_2, \ldots, q_n$, and $\omega_1, \omega_2, \omega_3$ with orders $2q', 2q'', 2q'''$, the last three orders not being necessarily different from zero, and if we write

·901 $$Z(z) = (\wp z-e_1)^{q'}(\wp z-e_2)^{q''}(\wp z-e_3)^{q'''}\prod_s (\wp z-\wp b_s)^{q_s},$$

then $Z(z)$ has, except possibly for the lattice points, which may be zeros of $f(z)$ but are poles of $Z(z)$, the same zero-structure as $f(z)$, and has no poles except at the lattice points. Similarly an irreducible set of poles of $f(z)$, excluding possibly a pole congruent with the origin, can be taken to be $\pm a_1, \pm a_2, \ldots, \pm a_m$ with orders $p_1, p_2, \ldots, p_m$, and $\omega_1, \omega_2, \omega_3$ with orders $2p', 2p'', 2p'''$, and if $P(z)$ is defined by

·902 $$P(z) = (\wp z-e_1)^{p'}(\wp z-e_2)^{p''}(\wp z-e_3)^{p'''}\prod_r (\wp z-\wp a_r)^{p_r},$$

the function $1/P(z)$ has, except possibly for the lattice points, the same pole-structure as $f(z)$, and has no zeros except at the lattice points. It follows that $Z(z)/P(z)$ has, except possibly at the lattice points, the same pole-structure and the same zero-structure as $f(z)$, and therefore the quotient of $f(z)$ by $Z(z)/P(z)$ is an elliptic function with no poles and no zeros incongruent with the origin. But, from ·31 and ·35, an elliptic function which is not a constant has both poles and zeros, and the origin, which could serve in one capacity, can not serve in both capacities. Hence the quotient of $f(z)$ by $Z(z)/P(z)$ is a constant, that is,

0·91 $$f(z) = f_0\, Z(z)/P(z)$$

where, since both $Z(z)$ and $P(z)$ have unity for leading coefficient at the origin, f_0 can be identified with the leading coefficient there of the function $f(z)$ itself.

The argument just used, which is due to Jordan, can be amplified. Let

$$q = q'+q''+q'''+\sum q_s, \qquad p = p'+p''+p'''+\sum p_r.$$

Then if $f(z)$ has the origin for a zero of order q_0, the sum of the orders of the zeros of $f(z)$ is q_0+2q and the sum of the orders of the poles is $2p$; these sums are equal and therefore $p > q$, $q_0 = 2(p-q)$. If $f(z)$ has the origin for a neutral point, the sum of the orders of the zeros of $f(z)$ is $2q$ and the sum of the orders of the poles is $2p$, and therefore $p = q$. If $f(z)$ has the origin for a pole of order p_0, the sum of the orders of the zeros of $f(z)$ is $2q$ and the sum of the orders of the poles is p_0+2p, and therefore $p < q$, $p_0 = 2(q-p)$. Thus the origin is a zero of order $2(p-q)$, a neutral point, or a pole of order $2(q-p)$, according as p is greater than, equal to, or less than, q. On the other hand, near the origin $Z(z)$ is dominated by $\wp^q z$ and therefore by $1/z^{2q}$, and $P(z)$ is dominated by $\wp^p z$ and therefore by $1/z^{2p}$. Hence $Z(z)/P(z)$ also has the origin for a zero of order $2(p-q)$, for a neutral point, or for a pole of order $2(q-p)$, according as p is greater than, equal to, or less than, q. That is to say, $Z(z)/P(z)$, constructed to have the same structure as $f(z)$ except possibly at the lattice points, acquires automatically the character of $f(z)$ at the lattice points themselves, and Liouville's structural identification theorem $\cdot 49_3$ is applicable without modification.

The formula ·91 gives us the descriptive theorem

$0{\cdot}92_1$. *Any even elliptic function belonging to the same lattice as $\wp z$ is a rational function of $\wp z$.*

If $f(z)$ is odd, then $f(z)/\wp' z$ is even, and therefore

$0{\cdot}92_2$. *Any odd elliptic function belonging to the same lattice as $\wp z$ is the product by $\wp' z$ of a rational function of $\wp z$.*

Lastly, any elliptic function $f(z)$ can be expressed as the sum of the two functions $\frac{1}{2}\{f(z)+f(-z)\}$, $\frac{1}{2}\{f(z)-f(-z)\}$, of which the first is even and the second odd. Applying $\cdot 92_1$ and $\cdot 92_2$, 0·92$_3$. *Any elliptic function $f(z)$ can be expressed in the form*

$$R(\wp z)+\wp' z\, S(\wp z),$$

where $\wp z$ belongs to the same lattice as $f(z)$, and $R(\wp z)$, $S(\wp z)$ are rational functions of $\wp z$.

If $f(z)$, $g(z)$ are two elliptic functions belonging to the same lattice, we have

$$f = R_f(\wp)+\wp' S_f(\wp), \qquad g = R_g(\wp)+\wp' S_g(\wp),$$

where $R_f(\wp)$, $S_f(\wp)$, $R_g(\wp)$, $S_g(\wp)$ are rational functions of $\wp$. Between these equations and the relation

$$\wp'^2 = 4\wp^3-g_2\wp-g_3$$

the two auxiliary functions $\wp$, $\wp'$ can be eliminated algebraically, and therefore

0·93. *Any two elliptic functions with a common lattice are connected by an algebraic equation with constant coefficients.*

If m, n are the orders of $f(z)$, $g(z)$, an arbitrary value of $f(z)$ implies not more than† m possible values of $g(z)$ and an arbitrary value of $g(z)$ implies not more than† n possible values of $f(z)$. It follows that if f, g are connected by an irreducible algebraic equation $\phi(f, g) = 0$, the degree of ϕ in g is not greater than m and the degree of ϕ in f is not greater than n.

It is possible to establish ·93 by general functional and algebraical reasoning without the use of the special function $\wp z$, but the argument is delicate.

From ·93 we have the corollary

0·94. *Every elliptic function is connected with its derivative by an algebraic equation.*

In other words, if w is an elliptic function $f(z)$ of z, there is a differential equation $\phi(w', w) = 0$ satisfied by w. The function ϕ is polynomial in w as well as in w' and does not involve z explicitly. Usually, if the order of the elliptic function $f(z)$ is n, the degree of ϕ in w' is n, and the degree of ϕ in w is the order of the elliptic function $f'(z)$, and may have any value from $n+1$ to $2n$.

A second corollary to ·93 comes from taking $f(y+z)$ as a function of z, where y is independent of z. We infer the existence of an equation $\phi\{f(y+z), f(z)\} = 0$, polynomial in the two functions $f(y+z)$, $f(z)$, with coefficients dependent on y; let us write the equation in the form

·903$_1$ $$\Phi\{f(y+z), f(z); y\} = 0.$$

† Usually the numbers are exactly m and n, but we have only to suppose $g(z)$ defined as $\{f(z)\}^2$ to see that reductions in these numbers are possible.

Since this equation is satisfied for all values of y and z, we have also

$$\cdot 903_2 \qquad \Phi\{f(y+z), f(y); z\} = 0.$$

The two equations $\cdot 903_1$, $\cdot 903_2$ are identical, since otherwise we could eliminate $f(y+z)$ algebraically and obtain a relation satisfied identically by y and z, contradicting the assumption that y is independent of z. Hence the function $\Phi\{f(y+z), f(z); y\}$ is a polynomial in $f(y)$ as well as in $f(z)$:

$0{\cdot}95_1$. *If $f(z)$ is any elliptic function, there is an algebraic equation $\Psi\{f(y+z), f(y), f(z)\} = 0$, with coefficients independent of y and z, connecting $f(y+z)$ with $f(y)$ and $f(z)$.*

This result is expressed briefly in the form

$0{\cdot}95_2$. *Every elliptic function possesses an algebraic addition theorem.*

Theoretical interest is focused rather on the converse of this theorem, which, with the obvious exceptions, was established by Weierstrass: the only singlevalued functions to possess algebraic addition theorems are rational functions, functions which in a wide sense may be called circular, and elliptic functions.

The function Ψ of $\cdot 95_1$ is symmetrical in $f(y)$ and $f(z)$. Also the relation $\Psi\{f(-y+z), f(-y), f(z)\} = 0$ is identical with the relation $\Psi\{f(z), f(y), f(z-y)\} = 0$, but this identity does not express symmetry. If however we apply the argument leading to $\cdot 95_1$ to the function $f(-y-z)$ instead of to the function $f(y+z)$, we have succinctly

$\cdot 904$. *If $f(z)$ is any elliptic function, there is a polynomial $F(X, Y, Z)$, symmetrical in the three arguments X, Y, Z, such that the relation $x+y+z = 0$ implies the relation $F\{f(x), f(y), f(z)\} = 0$.*

If the order of the function $f(z)$ is n, the degree of the equation $\cdot 902$ in $f(y+z)$ is usually n, and therefore as a rule the degree of the polynomial Ψ in $\cdot 95_1$ in each of its arguments is n, and so also is the degree of the polynomial F in $\cdot 904$. Algebraically this result is somewhat surprising. For example, it is evident that if $\wp' y$ and $\wp' z$ are removed from the relation

$$(\wp y - \wp z)^2\{\wp(y+z) + \wp y + \wp z\} = \tfrac{1}{4}(\wp' y - \wp' z)^2$$

by means of the relations

$$\wp'^2 y = 4\wp^3 y - g_2 \wp y - g_3, \qquad \wp'^2 z = 4\wp^3 z - g_2 \wp z - g_3,$$

the resulting equation is of degree two in $\wp(y+z)$; it is by no means evident that the coefficients of this equation are not of higher degree in $\wp y$ or $\wp z$. Or to put the matter differently, the eliminant of X', Y', Z' from the four equations

$$\cdot 905_{1\text{-}4} \qquad \begin{vmatrix} 1, & X, & X' \\ 1, & Y, & Y' \\ 1, & Z, & Z' \end{vmatrix} = 0,$$

$$X'^2 = 4X^3 - g_2X - g_3, \qquad Y'^2 = 4Y^3 - g_2Y - g_3, \qquad Z'^2 = 4Z^3 - g_2Z - g_3$$

is obviously symmetrical in X, Y, Z; it is not obviously the product of $\{(Y-Z)(Z-X)(X-Y)\}^2$ by a function quadratic in each separate variable. To find algebraically the significant factor of the eliminant, we remark that $\cdot 905_1$ implies that there exist numbers λ, μ such that

$$X' = \lambda X + \mu, \qquad Y' = \lambda Y + \mu, \qquad Z' = \lambda Z + \mu,$$

and that therefore, from $\cdot 905_{2\text{-}4}$, X, Y, Z are the roots of an equation of the form

$$4t^3 - g_2 t - g_3 = (\lambda t + \mu)^2,$$

whence, relating the coefficients to the roots and eliminating λ and μ, we have

$$(YZ + ZX + XY + \tfrac{1}{4}g_2)^2 = 4(X+Y+Z)(XYZ - \tfrac{1}{4}g_3),$$

a condition of the fourth degree in the set of variables X, Y, Z but quadratic as required in X, Y, Z separately. Thus

$\cdot 906$. *If* $x+y+z = 0$, *then*

$$(\wp y \wp z + \wp z \wp x + \wp x \wp y + \tfrac{1}{4}g_2)^2 = 4(\wp x + \wp y + \wp z)(\wp x \wp y \wp z - \tfrac{1}{4}g_3).$$

The importance of $\cdot 92_3$ is for general theorems rather than for particular applications, for whereas the determination of the functions $Z(z)$, $P(z)$ in the formula $\cdot 91$ depends directly on the structure of $f(z)$, the same can not be said of the rational functions in $\cdot 92_3$; the poles of $\tfrac{1}{2}\{f(z)+f(-z)\}$ and $\tfrac{1}{2}\{f(z)-f(-z)\}$ are among the poles of $f(z)$ and the poles of $f(-z)$ and can be identified, but the zeros of these functions, that is to say, the roots of the equations $f(z) = -f(-z)$ and $f(z) = f(-z)$, are not necessarily discoverable in practice. For example, taking $f(z)$ as $\wp(y+z)$, we can solve the equation $\wp(y+z) = \wp(y-z)$ but we have no means of solving the equation $\wp(y+z) = -\wp(y-z)$; we can therefore express $\wp(y+z) - \wp(y-z)$ in terms of $\wp z$ and $\wp' z$, but we can not proceed to obtain a formula for $\wp(y+z)$. The details of the evaluation of $\wp(y+z) - \wp(y-z)$ are simple and instructive. The function $\wp(y+z)$ has a double pole at $z = -y$; the function $\wp(y-z)$ has a double pole at $z = y$. Hence $\wp(y+z) - \wp(y-z)$ is a function of the fourth order, its poles are the zeros of $(\wp z - \wp y)^2$, and one irreducible set of poles has the sum zero. Because the function is an odd function, three of its zeros are the halfperiods ω_1, ω_2, ω_3 whose sum is zero; hence a fourth zero is the origin, and $\{\wp(y+z)-\wp(y-z)\}/\wp'z$ has no zeros incongruent with the origin. We have therefore

$$\wp(y+z)-\wp(y-z) = A\wp'z/(\wp z-\wp y)^2,$$

where A is independent of z. As $z \to 0$,

$$\{\wp(y+z)-\wp(y-z)\}/2z \to \wp'y,$$

that is, $$\wp(y+z)-\wp(y-z) \sim 2z\wp'y;$$

on the other hand, $$\wp'z/(\wp z-\wp y)^2 \sim -2z.$$

Hence $A = -\wp'y$, and finally

·907 $$\wp(y-z)-\wp(y+z) = \wp'y\wp'z/(\wp y-\wp z)^2,$$

in agreement with ·83.

There are developments of a function in terms of ζz and of σz to which the criticism directed against $\cdot 92_3$ does not apply, and with these developments we conclude our introduction. First let a_r be a pole of the elliptic function $f(z)$, of order m_r, and let the principal part of $f(z)$ in the neighbourhood of a_r be

$$\frac{A_{m_r}^{(r)}}{(z-a_r)^{m_r}}+\frac{A_{m_r-1}^{(r)}}{(z-a_r)^{m_r-1}}+\ldots+\frac{A_3^{(r)}}{(z-a_r)^3}+\frac{A_2^{(r)}}{(z-a_r)^2}+\frac{A_1^{(r)}}{z-a_r}.$$

With the function $\zeta_k z$ defined as in ·52, the principal part of each of the functions $\zeta_{m_r}(z-a_r)$, $\zeta_{m_r-1}(z-a_r),\ldots$, $\zeta_3(z-a_r)$, $\wp(z-a_r)$, $\zeta(z-a_r)$ near a_r consists of a single term whose numerator is unity, and the principal part of the sum

$$A_{m_r}^{(r)}\zeta_{m_r}(z-a_r)+A_{m_r-1}^{(r)}\zeta_{m_r-1}(z-a_r)+\ldots$$
$$+A_3^{(r)}\zeta_3(z-a_r)+A_2^{(r)}\wp(z-a_r)+A_1^{(r)}\zeta(z-a_r)$$

is identical with the principal part of $f(z)$. Denote this sum by $Z_r(z-a_r)$. In $Z_r(z-a_r)$ every term except $A_1^{(r)}\zeta(z-a_r)$ is an elliptic function. Hence $\sum Z_r(z-a_r)$, where the summation extends to all the members of an irreducible set of poles of $f(z)$, is the sum of an elliptic function and the function $\phi(z)$ defined by

$$\phi(z) = \sum_r A_1^{(r)}\zeta(z-a_r).$$

Now if Ω is any lattice step, $\zeta(z+\Omega) = \zeta z+\eta$, where η is independent of z. Hence

$$\zeta(z+\Omega-a_r) = \zeta(z-a_r)+\eta,$$
$$\phi(z+\Omega) = \phi(z)+\eta\sum_r A_1^{(r)}$$
$$= \phi(z),$$

for $\sum A_1^{(r)}$, the sum of the residues of $f(z)$ at an irreducible set of poles, is zero. That is to say, although the individual functions $\zeta(z-a_1)$, $\zeta(z-a_2)$,... and the individual functions $Z_1(z-a_1)$, $Z_2(z-a_2)$,... are not elliptic functions, the particular combinations $\phi(z)$ and $\sum Z_r(z-a_r)$ are elliptic functions. The second of these combinations is an elliptic function whose principal part at every pole in an irreducible set is identical with the principal part of $f(z)$. Hence, by ·48,

0·96$_1$ $$f(z) = c+\sum Z_r(z-a_r),$$

where c is a constant. Conversely, whatever the constants on the right of ·96$_1$, subject to the condition $\sum A_1^{(r)} = 0$, this expression defines an elliptic function:

0·96$_2$. *An irreducible set of poles of an elliptic function may be assigned arbitrarily, together with the principal part of the function at each pole, subject only to the condition that the sum of the assigned residues is zero.*

As a first example, take $f(z) = 1/(\wp y-\wp z)$. Near $z = y$,

$$\wp y-\wp z \sim -(z-y)\wp' y, \qquad \wp' y f(z) \sim -(z-y)^{-1},$$

and near $z = -y$,

$$\wp y-\wp z \sim (z+y)\wp' y, \qquad \wp' y f(z) \sim (z+y)^{-1}.$$

Hence, since $f(z)$ tends to zero with z,

·908$_1$ $$\wp' y/(\wp y-\wp z) = \zeta(y-z)+\zeta(y+z)-2\zeta y,$$

from which ·907 follows by differentiation. Interchanging y and z in ·908$_1$, we have

·908$_2$ $$\wp' z/(\wp y-\wp z) = \zeta(y-z)-\zeta(y+z)+2\zeta z,$$

which gives, in combination with ·908$_1$

·909 $$\zeta(y+z)-\zeta y-\zeta z = \tfrac{1}{2}(\wp' y-\wp' z)/(\wp y-\wp z),$$

and by differentiation

·910 $$\wp(y-z)+\wp(y+z) = \frac{d}{dz}\left(\frac{\wp' z}{\wp y-\wp z}\right)+2\wp z,$$

an unsymmetrical correlative of ·907.

From ·83 and ·909 we have

·911 $$\{\zeta(y+z)-\zeta y-\zeta z\}^2 = \wp(y+z)+\wp y+\wp z,$$

or in a more symmetrical form

·912. *If* $x+y+z = 0$, *then*

$$(\zeta x+\zeta y+\zeta z)^2 = \wp x+\wp y+\wp z.$$

To develop $1/(\wp y-\wp z)^2$ we have to take into account another term in each Taylor series, and we have

$$\wp'^2 y/(\wp y-\wp z)^2 \sim (y-z)^{-2}+(\wp'' y/\wp' y)(y-z)^{-1}$$

near $z = y$, and

$$\wp'^2 y/(\wp y-\wp z)^2 \sim (y+z)^{-2}+(\wp'' y/\wp' y)(y+z)^{-1}$$

near $z = -y$. Hence

$$\wp'^2 y/(\wp y-\wp z)^2 = \wp(y-z)+\wp(y+z)-2\wp y+ \\ +(\wp'' y/\wp' y)\{\zeta(y-z)+\zeta(y+z)-2\zeta y\},$$

which on substitution from $\cdot 908_1$ becomes

·913 $\quad \wp(y-z)+\wp(y+z) = \wp'^2 y/(\wp y-\wp z)^2-\wp'' y/(\wp y-\wp z)+2\wp y,$

that is to say, ·910 with y and z interchanged. Since $\wp'' z = 6\wp^2 z-\frac{1}{2}g_2$, we have

$$(\wp'' y-\wp'' z)/(\wp y-\wp z) = 6\wp y+6\wp z,$$

and therefore

·914 $\quad \wp(y-z)+\wp(y+z) = \frac{1}{2}\{(\wp'^2 y+\wp'^2 z)/(\wp y-\wp z)^2\}-2\wp y-2\wp z,$

a formula which combines with ·907 to reproduce the addition theorem in the form given in ·83.

If the poles and zeros of an elliptic function $f(z)$ are assigned, the properties of the function σz are utilized for the construction of $f(z)$. Let $a_1, a_2,..., a_n$ and $b_1, b_2,..., b_n$ be irreducible sets of poles and zeros of $f(z)$, subject to the equality $\sum b_s = \sum a_r$, a condition which, as we have seen on p. 24, imposes no restriction on the function $f(z)$ itself. Then since $\sigma(z-a_r)$ has the points congruent with a_r for simple zeros, and $\sigma(z-b_s)$ has the points congruent with b_s for simple zeros, and neither function has any other zeros or any accessible poles, the quotient $B(z)/A(z)$, where

$\cdot 915_{1-2}$ $\qquad A(z) = \prod_r \sigma(z-a_r), \qquad B(z) = \prod_s \sigma(z-b_s),$

is a function with precisely the poles and the zeros of $f(z)$. If $2\omega_1$, $2\omega_2$ is a basis of the lattice, we have from $\cdot 68_{3-4}$, for each value of ω and the corresponding value of η,

$$A(z+\omega) = (-)^n e^{2\eta\Sigma(z-a_r)} \prod \sigma(z-\omega-a_r) = (-)^n e^{2\eta\Sigma(z-a_r)} A(z-\omega),$$

and similarly

·916 $\qquad B(z+\omega) = (-)^n e^{2\eta\Sigma(z-b_s)} B(z-\omega).$

But $\qquad \sum (z-a_r) = nz-\sum a_r = nz-\sum b_s = \sum (z-b_s).$

Hence

·917 $\qquad B(z+\omega)/A(z+\omega) = B(z-\omega)/A(z-\omega),$

for all values of z, and therefore the quotient $B(z)/A(z)$ has the period 2ω. Thus $B(z)/A(z)$ is an elliptic function with the structure of $f(z)$, and from Liouville's second identification theorem $\cdot 49_3$,

0·97₁ $$f(z) = gB(z)/A(z),$$

where g is a constant. Conversely, if only the constants implicit in the definitions of $A(z)$ and $B(z)$ by $\cdot 915_{1-2}$ satisfy the relation

$$\sum b_s = \sum a_r,$$

the expression on the right of $\cdot 97_1$ defines an elliptic function, and therefore

$0{\cdot}97_2$. *The poles and the zeros of an elliptic function may be located at arbitrary points and have arbitrary multiplicities, subject only to the conditions that, multiplicity being taken into account, the number of poles is the same as the number of zeros and the sum of the poles is congruent with the sum of the zeros.*

As simple examples of $\cdot 97_1$ we have

0·98 $$\wp y - \wp z = -\sigma(y-z)\sigma(y+z)/\sigma^2 y\,\sigma^2 z,$$

from which we can recover $\cdot 908_1$, and

0·99 $$\wp' z = 2\sigma(z-\omega_1)\sigma(z-\omega_2)\sigma(z-\omega_3)/\sigma\omega_1\,\sigma\omega_2\,\sigma\omega_3\,\sigma^3 z,$$

which, so to speak, extracts the square root in $\cdot 74_1$.

I

THE THREE PRIMITIVE FUNCTIONS

1·1. The simplest elliptic functions are of the second order, and of these there are two kinds, functions with one double pole in each cell, and functions with two simple poles in each cell. The Weierstrassian function $\wp z$, of which a brief account has been given in the introductory essay, is the standard function of the first kind. This book is a study of standard functions of the second kind.

The existence of an elliptic function with one double pole is demonstrated by the actual construction of $\wp z$. The existence of a function with two simple poles is established in the course of the development of the theory of the Weierstrassian function. By the general theorem $0{\cdot}96_2$, the function

·101 $$c+A_1\,\zeta(z-a_1)+A_2\,\zeta(z-a_2)$$

is an elliptic function if $A_1+A_2 = 0$; it is a function of the second order with simple poles at a_1 and a_2 and an assigned residue at one of these poles, and it includes an additive constant c. Similarly by the general theorem $0{\cdot}97_2$, the function

·102 $$f_0\,\sigma(z-b_1)\sigma(z-b_2)/\sigma(z-a_1)\sigma(z-a_2)$$

is an elliptic function if $a_1+a_2 = b_1+b_2$; it is a function of the second order with simple poles at a_1 and a_2 and simple zeros at b_1 and b_2, and it includes a constant factor f_0. In each form the function involves two arbitrary constants in addition to the numbers a_1, a_2 which locate the poles.

We might obtain standard functions with simple poles by choosing constants in ·101 or ·102. But appropriate constants are not easily recognized in advance. Also it is one thing to use the functions ζz and σz for evidence of existence, but to rely on these functions for the definitions and for the most elementary properties of the functions which are to be fundamental is another matter. We approach the problem of construction in a more direct fashion.

1·2. Given a function whose poles are all double, we have only to take a square root to obtain a function whose poles are all simple, but this function is doublevalued unless the zeros as well as the poles of the original function are of even order. The zeros of $\wp z$ are simple, and the branches of $(\wp z)^{\frac{1}{2}}$ can not be separated. But $\wp z-B$, or $\wp z-\wp b$, where $b \not\equiv 0$, has the same poles as $\wp z$; it is a function of the second order, whatever the value of b, and as we have seen in 0·7 its zeros are double if b is congruent with one of the midpoints ω_1, ω_2, ω_3. Thus for $r = 1, 2, 3$ the function $\wp z - e_r$ has all its poles and all its zeros of precisely the second order, and

·201. *The function* $(\wp z - e_r)^{\frac{1}{2}}$ *has no branchpoints.*

It follows that the two values of $(\wp z - e_r)^{\frac{1}{2}}$ are not branches of one function but, like the two square roots of z^2, are separate singlevalued functions. We can discriminate between the two functions by their behaviour in the neighbourhood of $z = 0$; here $\wp z$ resembles $1/z^2$, and therefore one square root of $\wp z - e_r$ resembles $1/z$ and the other resembles $-1/z$. It is with the first of these square roots that our study begins. This function Jordan denotes by $f_r(z)$ and Tannery and Molk denote by $\xi_{r0}(z)$, but to avoid having a suffix as part of the functional symbol we denote the functions that correspond to the three half-periods ω_1, ω_2, ω_3 by $\mathrm{fj}\,z$, $\mathrm{gj}\,z$, $\mathrm{hj}\,z$; then we replace ω_1, ω_2, ω_3 and e_1, e_2, e_3 by ω_f, ω_g, ω_h and e_f, e_g, e_h, a departure from current practice which is trivial in itself but far-reaching in its effect on our notation.

The three functions which we call the primitive functions and denote by $\mathrm{fj}\,z$, $\mathrm{gj}\,z$, $\mathrm{hj}\,z$ are thus three singlevalued functions definable in terms of $\wp z$ by the formulae

1·21$_{1-3}$ $$\mathrm{fj}^2 z = \wp z - e_f, \qquad \mathrm{gj}^2 z = \wp z - e_g, \qquad \mathrm{hj}^2 z = \wp z - e_h,$$

or in another form

1·21$_{4-6}$ $$\mathrm{fj}^2 z + e_f = \mathrm{gj}^2 z + e_g = \mathrm{hj}^2 z + e_h = \wp z,$$

coupled with the conditions that, as $z \to 0$,

1·22$_{1-3}$ $$z\,\mathrm{fj}\,z \to 1, \qquad z\,\mathrm{gj}\,z \to 1, \qquad z\,\mathrm{hj}\,z \to 1,$$

which also have an alternative form

1·22$_{4-6}$ $$\mathrm{fj}\,z \sim 1/z, \qquad \mathrm{gj}\,z \sim 1/z, \qquad \mathrm{hj}\,z \sim 1/z.$$

The notation allows us to speak of a typical function $\mathrm{pj}\,z$, defined by the formula

1·23$_1$ $$\mathrm{pj}^2 z = \wp z - e_p$$

with the condition

1·23$_2$ $$\mathrm{pj}\,z \sim 1/z.$$

The definitions of the primitive functions can be expressed somewhat differently, in terms of the Weierstrassian function σz, which has no accessible poles and has simple zeros at all the lattice points of $\wp z$. The quotient $\sigma(z - \omega_p)/\sigma z$ has the zeros and poles of $\mathrm{pj}\,z$, and therefore

·202 $$\mathrm{pj}\,z = e^{\Delta_p(z)} \sigma(z - \omega_p)/\sigma z,$$

where $\Delta_p(z)$ is an integral function, a function without accessible poles. To obtain $\Delta_p(z)$ and to see this formula in relation to the definition of $\mathrm{pj}\,z$, consider the factorization theorem 0·98, which we may rewrite in the form

·203 $$\wp z - \wp b = -\frac{\sigma(z-b)\sigma(z+b)}{\sigma^2 b\, \sigma^2 z}.$$

The basis of this theorem is that the roots of the equation $\wp z = \wp b$ fall into two classes, the roots congruent with b, which are the zeros of $\sigma(z-b)$, and the roots congruent with $-b$, which are the zeros of $\sigma(z+b)$. In choosing a value of b so that every root of the equation $\wp z = \wp b$ is double, we are choosing b so that the functions $\sigma(z+b)$, $\sigma(z-b)$ have the same zeros. This is easily verified; for all values of z, as we have seen in $0{\cdot}68_{3\text{-}4}$,

$$\sigma(z+\omega_p) = -e^{2\eta_p z}\sigma(z-\omega_p),$$

and substituting for $\sigma(z+\omega_p)$ in ·203 we have

·204 $$\wp z - e_p = \frac{e^{2\eta_p z}\sigma^2(z-\omega_p)}{\sigma^2\omega_p\, \sigma^2 z},$$

whence, since $\sigma(-\omega_p) = -\sigma\omega_p$ and $\sigma z \sim z$ as $z \to 0$,

1·24 $$\mathrm{pj}\,z = -\frac{e^{\eta_p z}\sigma(z-\omega_p)}{\sigma\omega_p\, \sigma z}.$$

This is the required formula of the form ·202, with $\Delta_p(z)$ identified as the linear function $-\log(-\sigma\omega_p)+\eta_p z$, the selection of the branch of the logarithm being irrelevant.

We make very little use of the explicit formula ·24; the distribution of poles and zeros is shown clearly, but otherwise the fundamental properties of the function $\mathrm{pj}\,z$ are not in evidence, and two constants η_p, $\sigma\omega_p$ are involved. It is only in the light of the deduction from $\cdot 23_1$ that the function seems well chosen, and we can almost always base our arguments immediately on the more fundamental definition.

From $\cdot 21_{4\text{-}6}$ we see that we can express the square of one primitive function in terms of the square of another. For brevity we write $e_g - e_f$ as e_{fg}, and so on. Then we have

$1{\cdot}25_{1\text{-}2}$ $$\mathrm{gj}^2 z = \mathrm{fj}^2 z - e_{fg}, \qquad \mathrm{hj}^2 z = \mathrm{fj}^2 z - e_{fh},$$

and also identically

·205 $$e_{gh}\,\mathrm{fj}^2 z + e_{hf}\,\mathrm{gj}^2 z + e_{fg}\,\mathrm{hj}^2 z = 0.$$

1·3. Since $\wp(-z) = \wp z$, identically,

$$\{\mathrm{fj}(-z) - \mathrm{fj}\,z\}\{\mathrm{fj}(-z) + \mathrm{fj}\,z\} = 0,$$

and one of the two functions $\mathrm{fj}(-z)-\mathrm{fj}\,z$, $\mathrm{fj}(-z)+\mathrm{fj}\,z$ is zero for all values of z. As $z \to 0$, $\mathrm{fj}(-z)/\mathrm{fj}\,z \to -1$. Hence this ratio is not identically 1, that is, $\mathrm{fj}(-z)$ is not equal to $\mathrm{fj}\,z$ in the neighbourhood of the origin, and therefore the equality that is valid everywhere is

·301 $$\mathrm{fj}(-z) = -\mathrm{fj}\,z:$$

1·31. *The three primitive functions are odd functions.*

Since $\mathrm{fj}\,z$ is odd, so also is $\mathrm{fj}\,z-z^{-1}$, and the value at the origin of this function, which is regular in that neighbourhood, is zero: near the origin

·302 $$\mathrm{fj}\,z = z^{-1}+O(z).$$

An improvement on this result is derivable immediately from the relation of $\mathrm{fj}^2 z$ to $\wp z$. Since $\mathrm{fj}\,z$ is odd, we may assume

$$\mathrm{fj}\,z = z^{-1}+\alpha z+O(z^3),$$

implying

$$\mathrm{fj}^2 z-z^{-2} = \{\alpha+O(z^2)\}\{2+O(z^2)\} = 2\alpha+O(z^2),$$

and since $$\wp z-z^{-2} = O(z^2),$$

we have from ·21$_1$, $2\alpha = -e_f$, that is,

·303 $$\mathrm{fj}\,z = z^{-1}-\tfrac{1}{2}e_f z+O(z^3).$$

The poles of each primitive function are the poles of $\wp z$. Within a parallelogram that is primitive for $\wp z$, each of the primitive functions has only one pole, and that a simple one; we know therefore that if the functions are doubly periodic, their periods must differ from those of the Weierstrassian function from which they are formed.

To discover the effect of adding one of the Weierstrassian periods, we repeat the argument leading to ·301. The identity $\wp(z+2\omega_k) = \wp z$ implies that either

·304 $$\mathrm{fj}(z+2\omega_k) = \mathrm{fj}\,z$$

everywhere, or

·305 $$\mathrm{fj}(z+2\omega_k) = -\mathrm{fj}\,z$$

everywhere. If in ·304 we substitute $-\omega_k$ for z, we have on the one side $\mathrm{fj}\,\omega_k$, and on the other side, since $\mathrm{fj}\,z$ is an odd function, $-\mathrm{fj}\,\omega_k$. But $\mathrm{fj}\,\omega_k$ and $-\mathrm{fj}\,\omega_k$ can not be equal if ω_k is neither a zero nor a pole of $\mathrm{fj}\,z$. Hence ·304 is not an identity if ω_k is ω_g or ω_h, and the alternative to ·304 being ·305 we have

1·32$_{1-2}$ $$\mathrm{fj}(z+2\omega_g) = -\mathrm{fj}\,z, \qquad \mathrm{fj}(z+2\omega_h) = -\mathrm{fj}\,z,$$

whence further

$$\mathrm{fj}(z-2\omega_g-2\omega_h) = -\mathrm{fj}(z-2\omega_g) = \mathrm{fj}\,z,$$

that is,

1·33 $$\mathrm{fj}(z+2\omega_f) = \mathrm{fj}\,z.$$

Also from ·32,

$$\mathrm{fj}(z+4\omega_g) = \mathrm{fj}\,z, \qquad \mathrm{fj}(z+4\omega_h) = \mathrm{fj}\,z.$$

Thus

1·34. *The function* fj z *is doubly periodic, and* $2\omega_f$, $4\omega_g$, $4\omega_h$ *are three of its periods.*

A parallelogram with sides $2\omega_f$, $4\omega_g$ contains only two poles of fj z, namely, 0 and $2\omega_g$, and these are simple; hence a primitive parallelogram for the function can not be smaller than this, and we infer that $2\omega_f$, $4\omega_g$ is a primitive pair of periods for this function. The pair $2\omega_f$, $4\omega_h$ also is primitive, but the pair $4\omega_g$, $4\omega_h$ is not.

With $2\omega_f$, $4\omega_g$ as a primitive pair of periods, the midpoints of the primitive period parallelogram are ω_f, $2\omega_g$, $\omega_f+2\omega_g$. To describe fj z as an odd function with the two points 0, $2\omega_g$ for poles and the two points ω_f, $\omega_f+2\omega_g$ for zeros is therefore to illustrate the general theorem 0·76; each of the four points 0, ω_f, $2\omega_g$, $\omega_f+2\omega_g$ must be either a pole or a zero, and therefore, since the function is of the second order, two of them must be simple poles and the other two must be simple zeros.

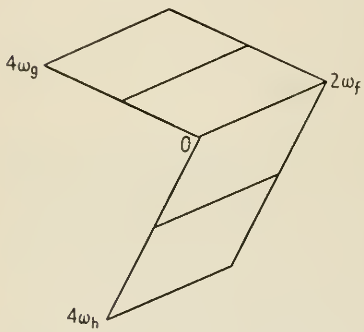


Fig. 13.

Since $4\omega_f$, $4\omega_g$, $4\omega_h$ are periods of all three of the primitive functions, while $2\omega_f$ is not a period of gj z or hj z, we call ω_f, ω_g, ω_h quarterperiods of the set of functions, not forgetting that they are halfperiods of $\wp z$ and that each of them is a halfperiod of one primitive function. We continue to call the poles, which are common to all the functions, the lattice points of the theory.

A doubly periodic function whose poles are all simple is determinate, save to an additive constant, by the poles and the residues attached to them. It is important to be familiar with the patterns formed by the residues of the primitive functions. These three patterns are attached to the same lattice, and there is no qualitative difference between one and another; each pattern consists of alternate rows of positive and negative units. But we must recognize the arrangements of the three patterns relative to one another, and relative to the primitive triad of quarterperiods ω_f, ω_g, ω_h.

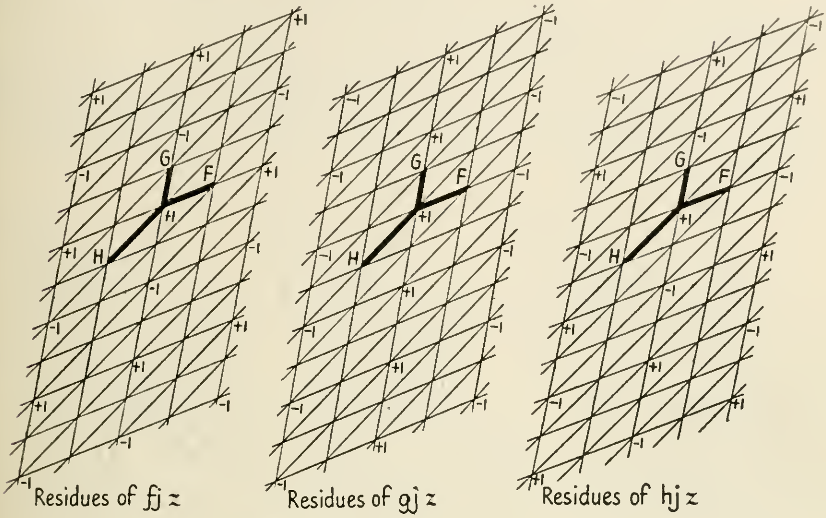


FIG. 14.

The point $2l\omega_f+2m\omega_g+2n\omega_h$ is a pole of fj$z$ for all integral values of l, m, n; if this point is Ω, the principal part of fj$z$ in its neighbourhood is $1/(z-\Omega)$ or $-1/(z-\Omega)$ according as $m+n$ is even or odd, and Ω may be called in the one case a positive pole, in the other case a negative pole. Since fj$z$ is an odd function of $z-\Omega$, to subtract the principal part is to obtain a function, fj$z-1/(z-\Omega)$ or fj$z+1/(z-\Omega)$, in which the pole Ω is not merely removed but replaced by a zero; the function is of course no longer periodic.

1·4. Since the poles of fj$z$ in the primitive parallelogram $2\omega_f$, $4\omega_g$ are simple poles at 0 and $2\omega_g$, there are two values of z at which fj$z$ takes an assigned value B and the sum of these values is congruent with $2\omega_g$. Hence if b is any point in the parallelogram, the only other point in the parallelogram for which fj$z$ has the same value as at b is the point congruent with $2\omega_g-b$, which is one of the four points $2\omega_g-b$, $6\omega_g-b$, $2\omega_f+2\omega_g-b$, $2\omega_f+6\omega_g-b$.

1·41. *The solutions of the equation* fj$z$ = fj$b$ *fall into two sets,*

$$z = 2l\omega_f+2m\omega_g+2n\omega_h+b \quad \textit{with } m+n \textit{ even},$$

$$z = 2l\omega_f+2m\omega_g+2n\omega_h-b \quad \textit{with } m+n \textit{ odd}.$$

Since ω_f is one zero of $\mathrm{fj}\,z$, another is $\omega_f+2\omega_g$, and every zero is congruent with one of these, mod $2\omega_f$, $4\omega_g$, that is to say, is congruent with ω_f, mod $2\omega_f$, $2\omega_g$. The zeros of the primitive function $\mathrm{pj}\,z$ compose a lattice geometrically congruent with the lattice of poles, but with the point ω_p for one of its points.

A doubly periodic function being determinate, save to a constant factor, by the distribution of its poles and zeros, we can identify each primitive function, but for such a factor, by a characteristic pattern.

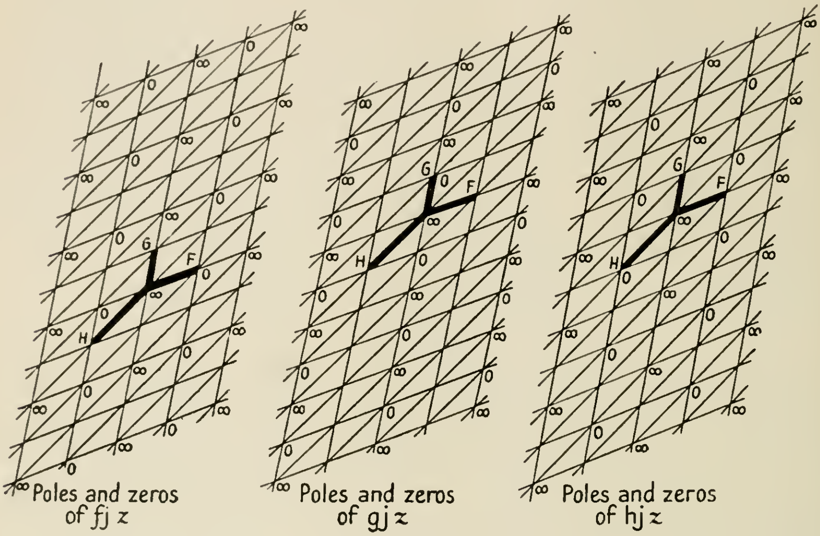


FIG. 15.

The poles have the same positions in the three patterns, but the location of zeros serves instead of the distribution of residues as a means of discrimination, and we do not now distinguish between positive poles and negative poles in the diagrams. In the pattern for $\mathrm{fj}\,z$, lines through F parallel to OG and OH accommodate ranks of zeros; on the line OF, poles and zeros alternate.

1·5. Neither ω_g nor ω_h is a pole or a zero of $\mathrm{fj}\,z$; the values $\mathrm{fj}\,\omega_g$, $\mathrm{fj}\,\omega_h$ of the function at these points are fundamental parameters in the theory. We denote them by f_g, f_h and call them the critical values of $\mathrm{fj}\,z$; the critical values are finite constants, different from zero. If $b=\omega_g$, the two points b, $2\omega_g-b$ coincide; that is, ω_g is a double root of the equation $\mathrm{fj}\,z=\mathrm{fj}\,\omega_g$:

1·51. *The roots of the equation* $\mathrm{fj}\,z = f_g$ *are double roots at the points congruent with* ω_g, *mod* $2\omega_f$, $4\omega_g$,

from which it follows that the roots of $\mathrm{fj}\,z = -f_g$ are double roots congruent with $-\omega_g$. Similarly the roots of $\mathrm{fj}\,z = f_h$ and of $\mathrm{fj}\,z = -f_h$ are double roots congruent with ω_h and with $-\omega_h$. Incidentally we notice that f_g can not be equal to f_h or $-f_h$:

1·52. *The squares of the critical values* f_g, f_h *are unequal.*

The last two theorems are evident algebraically from the identities

·501 $$\mathrm{fj}^2z+e_f = \mathrm{gj}^2z+e_g = \mathrm{hj}^2z+e_h.$$

When $\mathrm{gj}\,z = 0$, $\mathrm{fj}^2z = e_g-e_f$. That is,

·502–·503 $$f_g^2 = e_{fg}, \qquad f_h^2 = e_{fh},$$

and since e_f, e_g, e_h are all different, f_g^2, f_h^2 are different from zero and from each other.

From ·501 we have alternatively

$$g_f^2 = e_{gf}, \qquad h_f^2 = e_{hf},$$

whence

·504 $$f_h^2 = -h_f^2, \qquad g_f^2 = -f_g^2, \qquad h_g^2 = -g_h^2.$$

Although not expressible rationally in terms of e_f, e_g, e_h, the six constants of the form f_g are intrinsically determinate, for they are the values of definite singlevalued functions at specified points. We have in fact from ·24,

1·53 $$f_g = -\frac{e^{-\eta_f\omega_g}\sigma\omega_h}{\sigma\omega_f\,\sigma\omega_g}.$$

We shall return in a moment to an examination of relations between the six constants.

The converse of the set of results typified by ·51 is true also. If b is a double root of the equation $\mathrm{fj}\,z = \mathrm{fj}\,b$, then $2b \equiv 2\omega_g$, mod $2\omega_f$, $4\omega_g$, and b is congruent with one of the four points ω_g, $3\omega_g$, $\omega_f+\omega_g$, $\omega_f+3\omega_g$, that is, with one of the four points ω_g, $-\omega_g+4\omega_g$, $-\omega_h$, $\omega_h+2\omega_f+4\omega_g$:

1·54. *The root* b *of the equation* $\mathrm{fj}\,z = \mathrm{fj}\,b$ *is a double root if and only if* b *is congruent with one of the four points* $\pm\omega_g$. $\pm\omega_h$ *and* $\mathrm{fj}\,b$ *has one of the four values* $\pm f_g$, $\pm f_h$.

1·6. It follows from ·54 that the zeros of the derivative $\mathrm{fj}'z$ are the points congruent with $\pm\omega_g$ or $\pm\omega_h$. This derivative is an elliptic function with the same periodicities as $\mathrm{fj}\,z$ and with the poles of $\mathrm{fj}\,z$ for double poles. It is therefore of order four, and each of the four distinct zeros must be simple. Thus $\mathrm{fj}'z$ has the zeros of each of the functions gj z, hj z for simple zeros, and has the poles which are common to these functions for double poles. In other words, the derivative fj$'z$ and the product gj z hj z have the same zeros and the same poles, and since the functions have the same periodicity, one is a constant multiple of the other. Near the origin, $\mathrm{fj}'z \sim -z^{-2}$, $\mathrm{gj}\,z\,\mathrm{hj}\,z \sim z^{-2}$. Hence†

1·61 $$\mathrm{fj}'z = -\mathrm{gj}\,z\,\mathrm{hj}\,z.$$

This result may be derived directly from the relation of the functions to $\wp z$. The fundamental formula 0·74$_1$ for $\wp'^2 z$ is equivalent to

·601 $$\wp'^2 z = 4\,\mathrm{fj}^2 z\,\mathrm{gj}^2 z\,\mathrm{hj}^2 z,$$

and since $\wp' z \sim -2z^{-3}$, this implies

·602 $$\wp' z = -2\,\mathrm{fj}\,z\,\mathrm{gj}\,z\,\mathrm{hj}\,z,$$

whence, from ·21,

·603 $$\mathrm{fj}\,z\,\mathrm{fj}'z = \mathrm{gj}\,z\,\mathrm{gj}'z = \mathrm{hj}\,z\,\mathrm{hj}'z = -\mathrm{fj}\,z\,\mathrm{gj}\,z\,\mathrm{hj}\,z.$$

The zero ω_f of fj z is simple, and near this point fj z resembles $\mathrm{fj}'\omega_f.(z-\omega_f)$; that is, from ·61,

1·62 $$\mathrm{fj}\,z \sim -g_f h_f (z-\omega_f).$$

Since $2\omega_f$ is a period of the function, the form is the same near $-\omega_f$ as near ω_f. Generally, for all integral values of l, m, n, the point $(2l+1)\omega_f + 2m\omega_g + 2n\omega_h$ is a zero of fj z, and if Υ denotes this point, *the function resembles* $-g_f h_f(z-\Upsilon)$ *or* $g_f h_f(z-\Upsilon)$ *in the neighbourhood of* Υ *according as* $m+n$ *is even or odd.*

Since the step ω_f is a step from any zero of fj z to a pole and from any pole to a zero, the product $\mathrm{fj}\,z\,\mathrm{fj}(z+\omega_f)$ is a doubly periodic function without poles, and is therefore a constant. We can calculate this constant in two ways. Firstly, putting $z = \omega_g$, we have

1·63 $$\mathrm{fj}\,z\,\mathrm{fj}(z+\omega_f) = -f_g f_h.$$

Alternatively, as $z \to 0$,

$$\mathrm{fj}\,z \sim z^{-1}, \qquad \mathrm{fj}(\omega_f+z) \sim z\,\mathrm{fj}'\omega_f,$$

and therefore

1·64 $$\mathrm{fj}\,z\,\mathrm{fj}(z+\omega_f) = \mathrm{fj}'\omega_f.$$

From ·61,

·604 $$\mathrm{fj}'\omega_f = -g_f h_f,$$

† By choosing as a standard function the square root of $\wp z - e_r$ which resembles $-1/z$ we could remove the negative sign from this fundamental formula. Tradition apart, there seems little to recommend one choice rather than the other.

and comparing ·63 and ·64 we have the identity

1·65 $$f_g f_h = g_f h_f,$$

implying

·605 $$\frac{f_h}{h_f} = \frac{g_f}{f_g} = \frac{h_g}{g_h},$$

since the third fraction can be added by symmetry. We have seen already in ·504 that the square of each of the fractions in ·605 is -1, but the equality of the fractions themselves is a much less trivial theorem. Each fraction is the same square root of -1, and we write

·606 $$\frac{f_h}{h_f} = \frac{g_f}{f_g} = \frac{h_g}{g_h} = v,$$

where $v^2 = -1$. To interchange the symbols f and g is to replace v by $1/v$, that is, by $-v$. There is therefore no question of replacing v by i, for unless we impose some condition on the sets of quarterperiods to be used, v is i for some sets, $-i$ for the others.

The significance of v, both geometrically and analytically, can be deduced from ·53. The two formulae

$$f_g = -\frac{e^{-\eta_f \omega_g} \sigma\omega_h}{\sigma\omega_f \sigma\omega_g}, \qquad g_f = -\frac{e^{-\eta_g \omega_f} \sigma\omega_h}{\sigma\omega_f \sigma\omega_g}$$

give

·607 $$v = e^{\eta_f \omega_g - \eta_g \omega_f},$$

and we have seen in 0·64 that $\eta_f \omega_g - \eta_g \omega_f$ is $\frac{1}{2}\pi i$ or $-\frac{1}{2}\pi i$. It follows from ·607 that v is i in the one case, $-i$ in the other, and therefore v, as defined by ·606, is the signature of the basis $2\omega_f$, $2\omega_g$, as defined in the course of the proof of 0·45. The equalities ·605 might have been inferred from the equalities 0·705, and the signature can be described, as on p. 36, as the signature of the triplet $\omega_f \omega_g \omega_h$; the signature is i or $-i$ according as minimum rotation $\omega_f \to \omega_g \to \omega_h$ is positive or negative, or in analytical terms according as $\operatorname{Im}(\omega_g/\omega_f)$, $\operatorname{Im}(\omega_h/\omega_g)$, $\operatorname{Im}(\omega_f/\omega_h)$ are positive or negative.

Since v can be identified without reference to the elliptic functions, ·606 can be regarded as a set of relations

$1·66_{1-3}$ $$f_h = v h_f, \qquad g_f = v f_g, \qquad h_g = v g_h$$

giving three of the critical values in terms of the other three. Identically, $e_{fh} + e_{gf} + e_{hg} = 0$, and to $·66_{1-3}$ we can add by ·503 the relation

$1·67_1$ $$f_h^2 + g_f^2 + h_g^2 = 0,$$

or in the alternative form

$1·67_2$ $$f_g^2 + g_h^2 + h_f^2 = 0.$$

Of the six critical values only two can be independent, but we retain symbols for them all, since any elimination destroys the symmetry of the analysis. We may note however that if we suppose f_g, g_h, h_f, connected by ·67$_2$, to be given, we have not only the other three critical values from ·66, but also, by solving the set of equations

$$e_g - e_f = f_g^2, \qquad e_h - e_f = -h_f^2, \qquad e_f + e_g + e_h = 0,$$

the Weierstrassian constants:

·608 $$e_f = \tfrac{1}{3}(h_f^2 - f_g^2), \qquad e_g = \tfrac{1}{3}(f_g^2 - g_h^2), \qquad e_h = \tfrac{1}{3}(g_h^2 - h_f^2).$$

But a more symmetrical form of the last set of formulae is

·609 $$e_f = \tfrac{1}{3}(g_f^2 + h_f^2), \qquad e_g = \tfrac{1}{3}(h_g^2 + f_g^2), \qquad e_h = \tfrac{1}{3}(f_h^2 + g_h^2).$$

II

THE SET OF ELEMENTARY FUNCTIONS

2·1. As we have seen, the functions fj$z$, gj$z$, hj$z$ have common poles at the lattice points of $\wp z$, and have zeros at the points congruent with ω_f, ω_g, ω_h. Subtraction of ω_f from z interchanges the lattice points with the points congruent with ω_f, and interchanges the points congruent with ω_g with the points congruent with ω_h; also this subtraction brings the particular point ω_f to the origin. Hence the functions fj$(z-\omega_f)$, gj$(z-\omega_f)$, hj$(z-\omega_f)$ have common poles at the points congruent with ω_f, and have zeros at the points congruent with 0, ω_h, ω_g; for each function the principal part near ω_f is $1/(z-\omega_f)$.

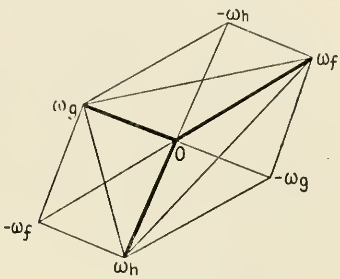


Fig. 16.

To secure a comprehensive notation, we introduce ω_j as an alternative symbol for the origin. We are then able to say that fj$z$, gj$z$, hj$z$ have a positive pole at ω_j and zeros at ω_f, ω_g, ω_h, and that fj$(z-\omega_f)$, gj$(z-\omega_f)$, hj$(z-\omega_f)$ have a positive pole at ω_f and zeros at ω_j, ω_h, ω_g. To perfect the analogy, we may write the functions fj$z$, gj$z$, hj$z$ as fj$(z-\omega_j)$, gj$(z-\omega_j)$, hj$(z-\omega_j)$. The functions fj$(z-\omega_g)$, gj$(z-\omega_g)$, hj$(z-\omega_g)$ have a positive pole at ω_g and zeros at ω_h, ω_j, ω_f, and the functions fj$(z-\omega_h)$, gj$(z-\omega_h)$, hj$(z-\omega_h)$ have a positive pole at ω_h and zeros at ω_g, ω_f, ω_j.

By associating with the three primitive functions the functions obtained by subtracting a quarterperiod from the independent variable, we have therefore a set of twelve functions each of which has simple poles congruent with one of the four points ω_j, ω_f, ω_g, ω_h and simple zeros congruent with another of these points. Since the pole and the zero can be selected in only twelve ways, the set regarded from this point of view is complete. We call the twelve functions the elementary elliptic functions, distinguishing still the three which have a pole at the origin as the primitive functions. We denote the elementary function which has a zero at ω_p and a pole at ω_q by pq$z$, a notation which exposes the structure of the function and is consistent with the notation for the primitive functions. Thus

2·11$_{1-3}$ fj$(z-\omega_f)=$ jf$z$, gj$(z-\omega_f)=$ hf$z$, hj$(z-\omega_f)=$ gf$z$.

2·2. Like any other elliptic function of assigned periodicities, the function pq z is determined, but for a constant factor, by its distribution of poles and zeros, or, as we may say, by its morphology. The constant factor is fixed in the first instance at the pole ω_q, but we need to be able to make comparisons at any fundamental point. We must therefore record the leading coefficient of pq z at each of the four points ω_j, ω_f, ω_g, ω_h, that is, the coefficient of the first significant term in the expansion of pq(ω_k+t) in powers of t, for the four positions of ω_k. If ω_k is the pole ω_q, the expansion is a Laurent series, the first term is the dominating term $1/t$, and the coefficient is 1. If ω_k is not ω_q, the expansion is a Taylor series; if ω_k is not the zero ω_p, the first term is the constant pq ω_k, which is not zero, and this is the leading coefficient; if ω_k is ω_p, the first term is t pq$'\omega_p$, and since the zero is simple, pq$'\omega_p$ does not vanish and the leading coefficient is now this value of the derivative. The coefficients are to be expressed in terms of the six critical constants.

For the primitive function fj z, the values of fj ω_g and fg ω_h define the constants f_g and f_h, and the value of fj$'\omega_f$ is given by 1·61 as $-g_f h_f$. Addition of $2\omega_f$ to ω_k leaves the leading coefficient of fj z unaltered, but addition of $2\omega_g$ or $2\omega_h$ replaces† the coefficient by its negative. The leading coefficients of fj$(z-\omega_f)$ at ω_j, ω_f, ω_g, ω_h are the leading coefficients of fj z at ω_f, $2\omega_f$, $\omega_f+\omega_g$, $\omega_f+\omega_h$, that is, at ω_f, $2\omega_f$, $-\omega_h$, $-\omega_g$, or again at ω_f, $\omega_j+2\omega_f$, $\omega_h-2\omega_h$, $\omega_g-2\omega_g$. The following table gives the fundamental leading terms of the three functions fj$(z-\omega_f)$, gj$(z-\omega_f)$, hj$(z-\omega_f)$, now denoted by jf z, hf z, gf z.

TABLE II I

	Near ω_j	*Near* ω_f	*Near* ω_g	*Near* ω_h
jf z	$-g_f h_f \times z$	$1\div(z-\omega_f)$	$-f_h$	$-f_g$
hf z	$-g_f$	$1\div(z-\omega_f)$	g_h	$h_g f_g \times (z-\omega_h)$
gf z	$-h_f$	$1\div(z-\omega_f)$	$f_h g_h \times (z-\omega_g)$	h_g

Instead of arguing analytically we may read these results from a diagram. Figure 17 shows the manner of determining the entries in the second row. The origin is now marked J, and F, G, H mark the quarter-periods. Inserted are the coefficients of gj z for the lattice points. Since hf z is defined as gj$(z-\omega_f)$, the value of hf z at the point marked Z is the value of gj z at the point marked U, the fourth corner of the parallelogram $JFZU$, and we have only to note the position of U when Z comes to one of the cardinal points J, F, G, H.

† Loosely, 'changes the sign of the coefficient', but there is no real justification for this phrase; a complex number has no sign, but 'the negative of z' is a perfectly good function of z. However, we need not always lapse into pedantry, and the elementary phrase must often be interpreted conventionally.

There is no purpose to be served by adding tables corresponding to Table II 1 for the sets of functions with poles at ω_g and ω_h, for only transliterations are involved, but it is useful to have a table showing

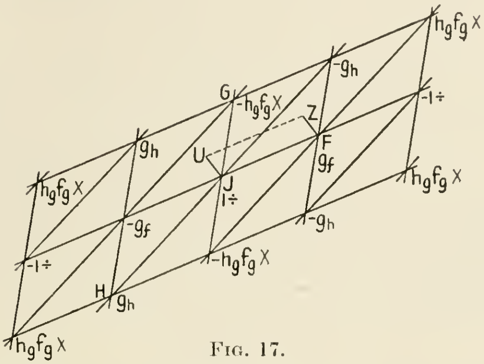


FIG. 17.

the leading coefficients of the twelve functions at the origin. Functional factors in the dominant term are not given.

TABLE II 2

fj z	1÷	jf z	$-g_f h_f\times$	hg z	$-f_g$	gh z	$-f_h$
gj z	1÷	hf z	$-g_f$	jg z	$-h_g f_g\times$	fh z	$-g_h$
hj z	1÷	gf z	$-h_f$	fg z	$-h_g$	jh z	$-f_h g_h\times$

The arrangement of the twelve functions in this table follows a natural cross-classification: functions in the same column have the same poles, and functions in the same row have the same periods.

2·3. The twelve elementary functions have similar morphology; the function pq z has simple zeros and simple poles alternating at equal intervals along the line through ω_p and ω_q, and has this same alternation repeated indefinitely along a succession of parallel lines. Addition of a quarterperiod transfers the zeros and poles of one function to zeros and poles of another, and since every residue is either 1 or -1, no constant factor except 1 or -1 can be introduced: pq$(z+\omega_k)$ is either one of the elementary functions or the negative of one of them.

For the functional change due to the addition of a quarterperiod a rule can be framed: addition of ω_f is an interchange of f with j, and is necessarily accompanied by an interchange of g with h. Thus fg$(z+\omega_f)$ is a multiple of jh z, and hg$(z+\omega_f)$ is a multiple of gh z. To determine whether the factor is 1 or -1, we may return to the primitive functions, using such deductions as

$$\mathrm{fg}(z+\omega_f) = \mathrm{hj}(z-\omega_g+\omega_f) = -\mathrm{hj}(z+\omega_g+\omega_f) = -\mathrm{hj}(z-\omega_h) = -\mathrm{jh}\,z,$$
$$\mathrm{hg}(z+\omega_f) = \mathrm{fj}(z-\omega_g+\omega_f) = -\mathrm{fj}(z+\omega_g+\omega_f) = -\mathrm{fj}(z-\omega_h) = -\mathrm{gh}\,z,$$

or we may discover from a figure whether the translation moves a positive pole of one function to a positive pole or to a negative pole of the other. A complete set of results for the addition of ω_f is recorded.

TABLE II 3

$\mathrm{fj}(z+\omega_f) = \mathrm{jf}\,z$	$\mathrm{jf}(z+\omega_f) = \mathrm{fj}\,z$	$\mathrm{hg}(z+\omega_f) = -\mathrm{gh}\,z$	$\mathrm{gh}(z+\omega_f) = -\mathrm{hg}\,z$
$\mathrm{gj}(z+\omega_f) = -\mathrm{hf}\,z$	$\mathrm{hf}(z+\omega_f) = \mathrm{gj}\,z$	$\mathrm{jg}(z+\omega_f) = \mathrm{fh}\,z$	$\mathrm{fh}(z+\omega_f) = -\mathrm{jg}\,z$
$\mathrm{hj}(z+\omega_f) = -\mathrm{gf}\,z$	$\mathrm{gf}(z+\omega_f) = \mathrm{hj}\,z$	$\mathrm{fg}(z+\omega_f) = -\mathrm{jh}\,z$	$\mathrm{jh}(z+\omega_f) = \mathrm{fg}\,z$

2·4. Since the poles of the elementary functions are all simple, the product $\mathrm{pq}\,z\,\mathrm{qp}\,z$ is an elliptic function without poles, and is therefore a constant. If ω_r is one of the two cardinal points distinct from ω_p and ω_q, the constant is given immediately as $\mathrm{pq}\,\omega_r\,\mathrm{qp}\,\omega_r$. Alternatively, near ω_p we have $\mathrm{qp}\,z \sim 1/(z-\omega_p)$, $\mathrm{pq}\,z \sim (z-\omega_p)\mathrm{pq}'\omega_p$. Hence

2·41 $$\mathrm{pq}\,z\,\mathrm{qp}\,z = \mathrm{pq}'\omega_p.$$

Incidentally,

·401 $$\mathrm{pq}'\omega_p = \mathrm{qp}'\omega_q.$$

In particular,

2·42 $$\mathrm{jf}\,z = -g_f h_f/\mathrm{fj}\,z,$$

·402 $$\mathrm{jf}'0 = -g_f h_f.$$

If p and r are different, the product $\mathrm{pq}\,z\,\mathrm{qr}\,z$ is a function with the zeros of $\mathrm{pq}\,z$ and the poles of $\mathrm{qr}\,z$, and is therefore a constant multiple of $\mathrm{pr}\,z$. As $z \to \omega_r$, $\mathrm{qr}\,z/\mathrm{pr}\,z \to 1$; hence

2·43 $$\mathrm{pq}\,z\,\mathrm{qr}\,z = \mathrm{pq}\,\omega_r\,\mathrm{pr}\,z.$$

For example, $\mathrm{gh}\,z$ is a multiple of $\mathrm{gj}\,z/\mathrm{hj}\,z$, and since by definition $\mathrm{gh}\,0 = \mathrm{fj}(-\omega_h) = -f_h$, we have

2·44 $$\mathrm{gh}\,z = -f_h\,\mathrm{gj}\,z/\mathrm{hj}\,z.$$

By referring to Table II 2 we can avoid direct determination of the constant factors in such formulae as ·42 and ·44. When we know† that $\mathrm{jf}\,z$ is a constant multiple of $1/\mathrm{fj}\,z$ and that $\mathrm{gh}\,z$ is a multiple of $\mathrm{gj}\,z/\mathrm{hj}\,z$, we have only to compare the leading terms at the origin to infer the exact relationships.

We have defined the functions $\mathrm{fj}\,z$, $\mathrm{gj}\,z$, $\mathrm{hj}\,z$ directly, and completed the set of elementary functions from these three, but the set could be completed equally well from other triads. For example if $\mathrm{jf}\,z$, $\mathrm{hf}\,z$, $\mathrm{gf}\,z$,

† If we regard the group of twelve functions as completed algebraically from the three primitive functions by the use of reciprocals and quotients, we are in effect using a modification of Glaisher's device for simplifying the notation of Jacobian elliptic functions. See 10·4 below.

sharing a common pole ω_f, are regarded as fundamental, $\mathrm{fj}\,z$, $\mathrm{gj}\,z$, $\mathrm{hj}\,z$ are definable as $\mathrm{jf}(z+\omega_f)$, $\mathrm{hf}(z+\omega_f)$, $\mathrm{gf}(z+\omega_f)$ and are seen immediately to be multiples of $1/\mathrm{jf}\,z$, $\mathrm{gf}\,z/\mathrm{jf}\,z$, $\mathrm{hf}\,z/\mathrm{jf}\,z$. Or we may use a triad with a common zero: in terms of $\mathrm{jf}\,z$, $\mathrm{jg}\,z$, $\mathrm{jh}\,z$, we can recover the primitive functions as $\mathrm{jf}(z+\omega_f)$, $\mathrm{jg}(z+\omega_g)$, $\mathrm{jh}(z+\omega_h)$ or as $-\mathrm{jf}\,\omega_g\,\mathrm{jf}\,\omega_h/\mathrm{jf}\,z$, $-\mathrm{jg}\,\omega_h\,\mathrm{jg}\,\omega_f/\mathrm{jg}\,z$, $-\mathrm{jh}\,\omega_f\,\mathrm{jh}\,\omega_g/\mathrm{jh}\,z$. It is in transformations from one system to another that these considerations become important; to express a transformation completely, we need not find the primitive functions if some other triad is more convenient.

2·5. Like the primitive function $\mathrm{pj}\,z$, and for the same reason, the function $\mathrm{pq}\,z$ can be expressed in terms of the function σz, with an exponential factor. Writing for convenience, in agreement with the formulae 0·604,

·501 $$\eta_j = \tfrac{1}{2}\{\zeta(z+2\omega_j)-\zeta z\} = 0,$$

identically, we infer from 1·24 that $\mathrm{pq}\,z$ is a constant multiple of

$$\frac{e^{\eta_p z}\sigma(z-\omega_p)}{e^{\eta_q z}\sigma(z-\omega_q)}.$$

Near ω_q, $\mathrm{pq}\,z \sim 1/\sigma(z-\omega_q)$; hence, writing $\omega_{pq} = \omega_q-\omega_p$, $\eta_{pq} = \eta_q-\eta_p$, we have the general formula

2·51 $$\mathrm{pq}\,z = \frac{e^{\eta_{qp}(z-\omega_q)}\sigma(z-\omega_p)}{\sigma\omega_{pq}\,\sigma(z-\omega_q)}.$$

It is not to be expected that the constant factor in this expression, namely

$$\frac{e^{\eta_{pq}\omega_q}}{\sigma\omega_{pq}},$$

can be put into the same symmetrical fractional form as the functional part, for the condition which determines this factor is entirely unsymmetrical as between ω_p and ω_q.

2·6. Ultimately the functions which we are studying depend no less on the periods than on the argument z, and as functions of the four variables z, ω_f, ω_g, ω_h they are all, like the Weierstrassian function $\wp z$, homogeneous. Exposing the dependence on the periods by writing $\mathrm{pq}(z;\omega_f,\omega_g,\omega_h)$ or less explicitly $\mathrm{pq}(z,\omega)$ instead of $\mathrm{pq}\,z$, we can assert that for any value of λ, $\mathrm{pq}(\lambda z,\lambda\omega) = \Lambda\,\mathrm{pq}(z,\omega)$, where Λ is independent of z and the ω, and since $\mathrm{pq}(\lambda z,\lambda\omega) \sim (\lambda z-\lambda\omega_q)^{-1}$ near ω_q and $\mathrm{pq}(z,\omega) \sim (z-\omega_q)^{-1}$ near the same point, $\Lambda = \lambda^{-1}$ and we have

2·61 $$\mathrm{pq}(\lambda z,\lambda\omega) = \lambda^{-1}\,\mathrm{pq}(z,\omega).$$

The assertion of homogeneity which we have made can be justified in two ways. Since $\wp(z,\omega)$ is homogeneous, so in turn are $\wp(z,\omega)-\wp(\omega_k,\omega)$, which is $\mathrm{kj}^2(z,\omega)$, and $\mathrm{kj}(z-\omega_q,\omega)$, which is the function with the periods of $\mathrm{kj}(z,\omega)$ and a positive pole at ω_q. Alternatively, since the periods, the poles, and the zeros, of $\mathrm{pq}(z,\omega)$ are all of the form $l\omega_f+m\omega_g+n\omega_h$, the function $\mathrm{pq}(\lambda z,\lambda\omega)$ has the same periods and the same structure as the function $\mathrm{pq}(z,\omega)$, and the quotient of one by the other is a constant Λ.

The homogeneity of the functions may be expressed and utilized in many ways. We have been considering the functions as dependent on the three quarterperiods ω_f, ω_g, ω_h connected by the relation $\omega_f+\omega_g+\omega_h = 0$. We see now that at the cost of symmetry but at no effective cost of generality we can assign one of the quarterperiods arbitrarily; a second quarterperiod remains as an independent variable, and the third in this manner of treatment becomes a mere function of the second. For example,

·601 $$\alpha\,\mathrm{pq}(z;\alpha,\beta,\gamma) = \mathrm{pq}\left(\frac{z}{\alpha};1,\frac{\beta}{\alpha},-1-\frac{\beta}{\alpha}\right),$$

and the function on the right is explicitly a function of the two variables z/α, β/α. If we are in search of trigonometrical analogies, we may replace ω_f by $\frac{1}{2}\pi$ and use the identity

·602 $$\mathrm{pq}(z;\alpha,\beta,\gamma) = \frac{\pi}{2\alpha}\,\mathrm{pq}\left(\frac{\pi z}{2\alpha};\frac{\pi}{2},\frac{\pi\beta}{2\alpha},-\frac{\pi(\alpha+\beta)}{2\alpha}\right).$$

More generally, the factor λ in ·61 is entirely at our disposal, and if we agree on a normalizing factor λ and write u for λz, we express $\mathrm{pq}(z,\omega)$ as the product of a canonical function $\mathrm{pq}(u,\lambda\omega)$ by a factor independent of the variable u. This process gives rise to the classical elliptic functions associated with the name of Jacobi, with which later chapters are to deal; the choice of the factor is discussed in Chapter X.

Geometrically, the homogeneity in $z:\omega_f:\omega_g:\omega_h$ means that the dependence of the functions on the size and orientation of the period parallelogram is trivial. Any alteration in the shape of the parallelogram disturbs the distribution of the numerical values of the functions, but if the lattice is merely rotated or enlarged, the subsequent value of any of the functions at any point is deducible immediately. Thus a ratio such as $\mathrm{fj}(\frac{1}{2}\omega_g)/\mathrm{hg}\,\omega_f$ is dependent only on shape, or if we divide all our functions by any such constant as f_g we shall have, again at the cost of symmetry, functions of the two variables z/ω_f, ω_g/ω_f.

III

PROPERTIES OF THE ELEMENTARY FUNCTIONS

3·1. The relations between the squares of the primitive functions are unaltered by a change in the argument z, and there is therefore a linear relation between the squares of any two copolar functions. Substituting $z-\omega_f$ for z in 1·501 we have

3·11 $$\mathrm{jf}^2 z+e_f = \mathrm{hf}^2 z+e_g = \mathrm{gf}^2 z+e_h,$$

or in terms of critical values

3·12 $$\mathrm{jf}^2 z = \mathrm{hf}^2 z+f_g^2 = \mathrm{gf}^2 z+f_h^2.$$

The square of any elementary function can be expressed rationally in terms of the square of any other. If the functions are copolar the typical formulae are included in ·12; a general formula is

3·13 $$\mathrm{pq}^2 z = \mathrm{rq}^2 z-\mathrm{rq}^2\omega_p.$$

If $\mathrm{pq}\,z$, $\mathrm{rs}\,z$ are not copolar, then

3·14 $$\mathrm{pq}^2 z = \mathrm{pq}^2\omega_s \frac{\mathrm{rs}^2 z-\mathrm{rs}^2\omega_p}{\mathrm{rs}^2 z-\mathrm{rs}^2\omega_q},$$

since the form of the relation is implied by ·13 and identity at the three points ω_p, ω_q, ω_s determines the constants.

The squares of the elementary functions are all even. Each of the functions is therefore either an odd function or an even function, and since an odd function has the origin either for a zero or for an infinity,

3·15$_1$. *The six elementary functions of which the origin is neither a zero nor a pole are even functions.*

Since the primitive functions are odd, so also are their reciprocals:

3·15$_2$. *The six elementary functions of which the origin is either a zero or a pole are odd functions.*

3·2. The derivative $\mathrm{fj}'(z-\omega_q)$ is given in terms of the two functions $\mathrm{gj}(z-\omega_q)$, $\mathrm{hj}(z-\omega_q)$ copolar with $\mathrm{fj}(z-\omega_q)$ by the formula 1·61 which gives $\mathrm{fj}'z$ in terms of $\mathrm{gj}\,z$ and $\mathrm{hj}\,z$; that is, if $\mathrm{rq}\,z$, $\mathrm{sq}\,z$ are the two functions copolar with $\mathrm{pq}\,z$,

3·21 $$\mathrm{pq}'z = -\mathrm{rq}\,z\,\mathrm{sq}\,z.$$

A formula for integration is evident from ·21: we have

$$\mathrm{rq}'z = -\mathrm{pq}\,z\,\mathrm{sq}\,z, \qquad \mathrm{sq}'z = -\mathrm{pq}\,z\,\mathrm{rq}\,z,$$

whence

·201 $$\mathrm{pq}\,z = -\frac{\mathrm{rq}'z+\mathrm{sq}'z}{\mathrm{rq}\,z+\mathrm{sq}\,z},$$

and therefore

3·22 $$\int_{z_2}^{z_1} \mathrm{pq}\,z\,dz = \log\frac{\mathrm{rq}\,z_1+\mathrm{sq}\,z_1}{\mathrm{rq}\,z_2+\mathrm{sq}\,z_2},$$

the integral having the same multiplicity as the logarithm.

The step ω_{pq} from a zero to a pole of $\mathrm{pq}\,z$ is a halfperiod of the function and is also a step from a pole to a zero. Hence the product $\mathrm{pq}\,z\,\mathrm{pq}(z+\omega_{pq})$ is a constant, and the value of this constant is the limit of the product as $z \to \omega_p$, which is $\mathrm{pq}'\omega_p$. Alternatively, if $z = \omega_r$, then $z+\omega_{pq} = \omega_r-\omega_p+\omega_q$; but if $\mathrm{kj}\,z$ is the primitive function coperiodic with $\mathrm{pq}\,z$, then

·202 $$\mathrm{pq}(z+\omega_q) = \mathrm{kj}\,z = -\mathrm{kj}(-z) = -\mathrm{pq}(\omega_q-z);$$

hence

$$\mathrm{pq}(\omega_r+\omega_{pq}) = -\mathrm{pq}(\omega_p+\omega_q-\omega_r) = -\mathrm{pq}(\omega_s+2\omega_p+2\omega_q),$$

and since $2\omega_p+2\omega_q$, being expressible as $4\omega_p+2\omega_{pq}$, is a period of $\mathrm{pq}\,z$, we have

$$\mathrm{pq}(\omega_r+\omega_{pq}) = -\mathrm{pq}\,\omega_s,$$

and therefore

3·23 $$\mathrm{pq}'\omega_p = -\mathrm{pq}\,\omega_r\,\mathrm{pq}\,\omega_s,$$

whence from ·21,

3·24 $$\mathrm{pq}\,\omega_r\,\mathrm{pq}\,\omega_s = \mathrm{rq}\,\omega_p\,\mathrm{sq}\,\omega_p.$$

From $1{\cdot}21_1$ we have at once

3·25 $$\int_{z_1}^{z_2} \mathrm{fj}^2 z\,dz = (\zeta z_1+e_f z_1)-(\zeta z_2+e_f z_2),$$

where $\zeta'z = -\wp z$ and e_f may be expressed as $-\frac{1}{3}(f_g^2+f_h^2)$. Integration of higher powers of $\mathrm{fj}\,z$ depends on a recurrence. From the formula

·203 $$\mathrm{fj}'^2 z = (\mathrm{fj}^2 z-f_g^2)(\mathrm{fj}^2 z-f_h^2)$$

we have

·204 $$\mathrm{fj}''z = 2\,\mathrm{fj}^3 z-(f_g^2+f_h^2)\mathrm{fj}\,z$$

and therefore

3·26 $$\frac{d}{dz}(\mathrm{fj}^{m-1}z\,\mathrm{fj}'z) = (m+1)\mathrm{fj}^{m+2}z-m(f_g^2+f_h^2)\mathrm{fj}^m z+(m-1)f_g^2 f_h^2\,\mathrm{fj}^{m-2}z.$$

If m is a positive even number we can therefore determine for $\int \mathrm{fj}^m z\,dz$ an expression of the form

$3{\cdot}27_1$ $$(a_{m-3}\mathrm{fj}^{m-3}z+a_{m-5}\mathrm{fj}^{m-5}z+\ldots+a_1\,\mathrm{fj}\,z)\mathrm{fj}'z+a'\zeta z+a''z,$$

with constant coefficients. If m is odd, the recurrence brings us to the integrals of $\mathrm{fj}^3 z$ and $\mathrm{fj}\, z$, and the former of these is expressed in terms of the latter by means of ·204, which is ·26 for the case $m = 1$; thus for an odd index the general form of $\int \mathrm{fj}^m z\, dz$ is

3·27₂ $$(a_{m-3}\mathrm{fj}^{m-3}z + a_{m-5}\mathrm{fj}^{m-5}z + \ldots + a_2\mathrm{fj}^2 z + a_0)\mathrm{fj}'z + a\log(\mathrm{gj}\, z + \mathrm{hj}\, z).$$

Since in ·203 we can substitute $z-\omega_k$ for z, we can replace $\mathrm{fj}\, z$ in ·204 and ·26 by any one of the coperiodic functions $\mathrm{jf}\, z$, $\mathrm{hg}\, z$, $\mathrm{gh}\, z$, and therefore we can express the integral of any odd power of one of these functions by means of the integral of the function itself, as given by ·22, and the integral of any even power by means of the integral of the square of the function. If in ·26 we take $m = 0$ we have

·205 $$\frac{d}{dz}\frac{\mathrm{fj}'z}{\mathrm{fj}\, z} = \mathrm{fj}^2 z - \mathrm{jf}^2 z,$$

whence, $\mathrm{jf}\, z\; \mathrm{fj}\, z$ being constant,

3·28 $$\int \mathrm{jf}^2 z\, dz = \frac{\mathrm{jf}'z}{\mathrm{jf}\, z} - \zeta z - e_f z,$$

while

3·29 $$\int \mathrm{hg}^2 z\, dz = \int (\mathrm{jg}^2 z - g_f^2)\, dz = \frac{\mathrm{jg}'z}{\mathrm{jg}\, z} - \zeta z - e_f z.$$

Thus all positive integral powers, odd or even, of an elementary elliptic function, can be integrated, and since negative powers of one function are positive powers of another, the problem of the integration of integral powers, positive or negative, is completely solved.

3·3. The derivative $\mathrm{fj}'z$ has simple zeros congruent with ω_g and ω_h, and double poles congruent with ω_j. The structure of the logarithmic derivative $\mathrm{fj}'z/\mathrm{fj}\, z$ is simpler, for this quotient has the two points ω_g, ω_h for simple zeros and the two points ω_j, ω_f for simple poles. Again, while $\mathrm{fj}'z$ has the periodicity of $\mathrm{fj}\, z$, addition of $2\omega_g$ or $2\omega_h$ replaces $\mathrm{fj}\, z$ and $\mathrm{fj}'z$ by their negatives, and therefore the quotient has the periods of the original function $\wp z$. More generally,

3·31. *The logarithmic derivative* $\mathrm{pq}'z/\mathrm{pq}\, z$ *has simple poles at* ω_p *and* ω_q *and simple zeros at the other two cardinal points, and has* $2\omega_f$, $2\omega_g$, $2\omega_h$ *for periods.*

Since one of the two functions $\mathrm{pq}\, z$, $\mathrm{pq}'z$ is even and the other is odd, the logarithmic derivative is odd, and referring to 0·718 we recognize that the logarithmic derivatives are, but for arbitrary constant factors, the only odd functions of the second order with the Weierstrassian periods.

If we compare the diagram of poles and zeros for the function $\mathrm{fj}'z/\mathrm{fj}\,z$ with the corresponding diagram for $\mathrm{gj}\,z$, which also has simple zeros congruent with ω_g and simple poles congruent with ω_j, we see at once that the difference is a difference of scale in the direction from the origin to ω_f. To be precise, if in $\mathrm{fj}\,z$ the quarterperiods ω_f, ω_g have

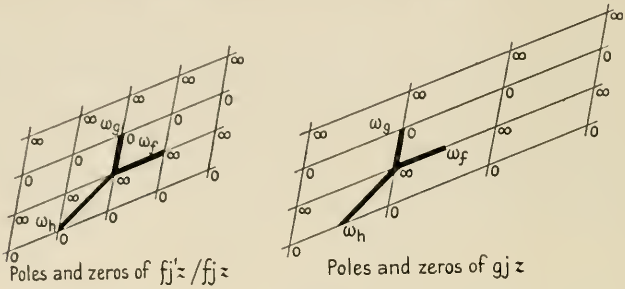


Fig. 18.

the values 2α, 2β, we obtain a pattern identical with that of the poles and zeros of $\mathrm{fj}'z/\mathrm{fj}\,z$ by constructing $\mathrm{gj}\,z$ from a Weierstrassian function with periods 2α, 4β, instead of with periods 4α, 4β. Exhibiting the dependence of each function on its quarterperiods, we can say that the function $\mathrm{gj}(z;\alpha,2\beta,\gamma-\beta)$ has the same morphology as the function $\mathrm{fj}'(z;2\alpha,2\beta,2\gamma)/\mathrm{fj}(z;2\alpha,2\beta,2\gamma)$. Moreover, 4α and 4β compose a primitive pair of periods for both functions, and the leading terms at the origin differ only in sign. Hence

3·32$_{1\text{-}2}$
$$\mathrm{fj}'(z;2\alpha,2\beta,2\gamma)/\mathrm{fj}(z;2\alpha,2\beta,2\gamma)$$
$$= -\mathrm{gj}(z;\alpha,2\beta,\gamma-\beta) = -\mathrm{hj}(z;\alpha,\beta-\gamma,2\gamma),$$

the argument with regard to the third function being the same†.

To replace the logarithmic derivative $\mathrm{fj}'z/\mathrm{fj}z$ by $\mathrm{jf}'z/\mathrm{jf}z$ is only to change the sign, since $\mathrm{jf}z$ is a constant multiple of $1/\mathrm{fj}z$. If we substitute $z-2\gamma$ for z, we have

$$\mathrm{fj}(z-2\gamma;2\alpha,2\beta,2\gamma) = \mathrm{gh}(z;2\alpha,2\beta,2\gamma),$$
$$\mathrm{gj}(z-2\gamma;\alpha,2\beta,\gamma-\beta) = -\mathrm{gj}(z-2\beta;\alpha,2\beta,\gamma-\beta) = -\mathrm{jg}(z;\alpha,2\beta,\gamma-\beta),$$
$$\mathrm{hj}(z-2\gamma;\alpha,\beta-\gamma,2\gamma) = \mathrm{jh}(z;\alpha,\beta-\gamma,2\gamma),$$

and therefore

3·33$_{1\text{-}2}$
$$\mathrm{gh}'(z;2\alpha,2\beta,2\gamma)/\mathrm{gh}(z;2\alpha,2\beta,2\gamma)$$
$$= \mathrm{jg}(z;\alpha,2\beta,\gamma-\beta) = -\mathrm{jh}(z;\alpha,\beta-\gamma,2\gamma).$$

† There is a temptation to write ·32$_1$ in the form

$$\mathrm{fj}'(z;\omega_f,\omega_g,\omega_h)/\mathrm{fj}(z;\omega_f,\omega_g,\omega_h) = -\mathrm{gj}(z;\tfrac{1}{2}\omega_f,\omega_g,\tfrac{1}{2}\omega_{gh}),$$

but the notation in the function $\mathrm{gj}\,z$ is logically indefensible. In Table XIII below, the economy is considerable and the fault perhaps venial.

The connexion between the logarithmic derivatives of $\mathrm{fj}\,z$ and $\mathrm{gh}\,z$ is brought to the surface by ·32 and ·33; the poles of one of these functions are the zeros of the other, and the product of the functions is constant.

3·4. In $\cdot 32_1$ we have evidently a second means of integrating a primitive function. With a change of notation we can write $\cdot 32_1$ as

$$\mathrm{fj}(z;\alpha,\beta,\gamma) = -\mathrm{gj}'(z;\alpha,2\beta,\gamma-\beta)/\mathrm{gj}(z;\alpha,2\beta,\gamma-\beta),$$

and we have therefore

3·41 $$\int_{z_1}^{z_2} \mathrm{fj}(z;\alpha,\beta,\gamma)\,dz = \log\frac{\mathrm{gj}(z_1;\alpha,2\beta,\gamma-\beta)}{\mathrm{gj}(z_2;\alpha,2\beta,\gamma-\beta)}.$$

Comparing this formula with the formula included in ·22 for the same integral, namely

$$\int_{z_1}^{z_2} \mathrm{fj}(z;\alpha,\beta,\gamma)\,dz = \log\frac{\mathrm{gj}(z_1;\alpha,\beta,\gamma)+\mathrm{hj}(z_1;\alpha,\beta,\gamma)}{\mathrm{gj}(z_2;\alpha,\beta,\gamma)+\mathrm{hj}(z_2;\alpha,\beta,\gamma)},$$

we see that $\mathrm{gj}(z;\alpha,\beta,\gamma)+\mathrm{hj}(z;\alpha,\beta,\gamma)$ is a constant multiple of

$$\mathrm{gj}(z;\alpha,2\beta,\gamma-\beta),$$

and identifying the factor from the form near $z = 0$, we have

3·42 $$\mathrm{gj}(z;\alpha,\beta,\gamma)+\mathrm{hj}(z;\alpha,\beta,\gamma) = 2\,\mathrm{gj}(z;\alpha,2\beta,\gamma-\beta).$$

The source of this relation is easily detected. The primitive functions $\mathrm{gj}\,z$, $\mathrm{hj}\,z$ have the same poles, which fall into three groups. Some, like the origin, are positive for both functions, some, like $2\omega_f$, are negative for both functions, and some, like $2\omega_g$, are positive for one and negative for the other; but for the existence of poles of the last kind, the functions would be identical. If we add the functions, the unlike poles disappear, and halving the sum we have a function with a positive pole at the origin and a negative pole at $2\omega_f$. The function has periods $4\omega_f$, $4\omega_g$, $4\omega_h$, and since it has no poles except 0 and $2\omega_f$ in the parallelogram $4\omega_f$, $4\omega_g$, this parallelogram is primitive and the periods $4\omega_f$, $4\omega_g$, $4\omega_h$ form a primitive set. Further, the removal of the principal part $\pm 1/(z-\Omega)$ near any pole Ω of a primitive function leaves a function which is not merely finite but zero at Ω; hence the poles of $\mathrm{gj}\,z$ and $\mathrm{hj}\,z$ which disappear when the functions are added are replaced by zeros. That is, $2\omega_g$ and $2\omega_h$ are zeros of $\mathrm{gj}\,z+\mathrm{hj}\,z$, and since this sum is of only the second order, these zeros are simple and every zero is congruent with one or other of them. To sum up, $\mathrm{gj}\,z+\mathrm{hj}\,z$ is an elliptic function with periods $4\omega_f$, $4\omega_g$, $4\omega_h$, and with simple poles at the origin and $2\omega_f$ and simple zeros at $2\omega_g$ and $2\omega_h$; on the other hand, $\mathrm{fj}'z/\mathrm{fj}\,z$ is an elliptic function with periods $2\omega_f$, $2\omega_g$, $2\omega_h$, and with simple poles at the origin and ω_f and simple zeros at ω_g and ω_h. The similarity is perfect, and comparing the forms near the origin we have

3·43 $\quad \mathrm{gj}(z;\alpha,\beta,\gamma)+\mathrm{hj}(z;\alpha,\beta,\gamma) = -2\,\mathrm{fj}'(z;2\alpha,2\beta,2\gamma)/\mathrm{fj}(z;2\alpha,2\beta,2\gamma),$

a symmetrical relation which combines with ·32$_1$ to give ·42.

Subtraction of $\mathrm{hj}\,z$ from $\mathrm{gj}\,z$ replaces the poles at 0 and $2\omega_f$ by zeros, leaving the poles at $2\omega_g$ and $2\omega_h$ effective. In fact, since the product $(\mathrm{gj}\,z+\mathrm{hj}\,z)(\mathrm{gj}\,z-\mathrm{hj}\,z)$ is a constant, the poles of one factor are the zeros of the other. We have now, comparing residues at 2β,

3·44 $\quad \mathrm{gj}(z;\alpha,\beta,\gamma)-\mathrm{hj}(z;\alpha,\beta,\gamma) = 2\,\mathrm{hg}'(z;2\alpha,2\beta,2\gamma)/\mathrm{hg}(z;2\alpha,2\beta,2\gamma),$

in agreement with the relation which we have already noticed between the logarithmic derivatives of $\mathrm{fj}\,z$ and $\mathrm{gh}\,z$. For the same difference we have also

3·45 $\quad \mathrm{gj}(z;\alpha,\beta,\gamma)-\mathrm{hj}(z;\alpha,\beta,\gamma) = 2\,\mathrm{jg}(z;\alpha,2\beta,\gamma-\beta),$

and we can combine ·42 and ·45. The formulae

3·46$_{1\text{-}2}$
$$\begin{aligned}\mathrm{gj}(z;\alpha,\beta,\gamma) &= \mathrm{gj}(z;\alpha,2\beta,\gamma-\beta)+\mathrm{jg}(z;\alpha,2\beta,\gamma-\beta)\\ &= \mathrm{hj}(z;\alpha,\beta-\gamma,2\gamma)-\mathrm{jh}(z;\alpha,\beta-\gamma,2\gamma)\end{aligned}$$

are typical of a group.

Since α plays the part of ω_f in all the functions, substitution of $z-\alpha$ for z gives

3·46$_{3\text{-}4}$
$$\begin{aligned}\mathrm{hf}(z;\alpha,\beta,\gamma) &= \mathrm{hf}(z;\alpha,2\beta,\gamma-\beta)-\mathrm{fh}(z;\alpha,2\beta,\gamma-\beta)\\ &= \mathrm{gf}(z;\alpha,\beta-\gamma,2\gamma)+\mathrm{fg}(z;\alpha,\beta-\gamma,2\gamma),\end{aligned}$$

formulae which may be found by direct combination of the copolar functions $\mathrm{gf}\,z$, $\mathrm{hf}\,z$ with a view to the removal of one set of poles. There are no results of this kind to be found by combining $\mathrm{jf}\,z$ with one of the functions $\mathrm{gf}\,z$, $\mathrm{hf}\,z$, for although we can find a function with periods $4\omega_f$, $4\omega_g$ and with two simple poles, this function has no symmetry with respect to the origin; it is neither even nor odd, and therefore it can not be a multiple of an elementary function, however the primitive pair of periods is selected.

3·5. It is interesting to discover the integrating formula ·41 by investigating the functional character of the integral

$$\int_{z_1}^{z} \mathrm{fj}\,z\,dz.$$

Since the poles of fj z are simple, the singularities of the integral are logarithmic, and since every residue of fj z is a whole number, the function $F(z)$ defined by

·501 $$F(z) = \exp \int_{z_1}^{z} \mathrm{fj}\, z\, dz$$

is singlevalued. Since every residue of fj z is numerically unity, positive poles of fj z give rise to simple zeros of $F(z)$, and negative poles of fj z to simple poles of $F(z)$. Near the origin, $F(z) \sim Az$, where A is a constant dependent on z_1, and this condition, with the relation

·502 $$F'(z)/F(z) = \mathrm{fj}\, z,$$

leads at once to

$$F(-z) = -F(z), \qquad F(z+2\omega_f) = -F(z),$$
$$F(z+2\omega_g) = B/F(z), \qquad F(z+2\omega_h) = C/F(z),$$

where B, C are constants of integration. The identification of $F(z)$ with a multiple of a function jg z follows immediately, and since $F(z_1) = 1$, we have

·503 $$F(z) = \frac{\mathrm{jg}(z; \alpha, \beta, \gamma)}{\mathrm{jg}(z_1; \alpha, \beta, \gamma)},$$

where, in terms of the quarterperiods of fj z, $\alpha = \omega_f$, $\beta = 2\omega_g$, and therefore $\gamma = \omega_{gh}$.

IV

ADDITION THEOREMS FOR THE ELEMENTARY FUNCTIONS

4·1. The process by which, in the last introductory section, an arbitrary elliptic function is expressed in terms of $\wp z$ and $\wp' z$, is a special case of a process devised by Liouville for expressing an elliptic function $f(z)$ in terms of any coperiodic function $\phi(z)$ of the second order.

If the pole-sum of $\phi(z)$ is 2γ, the sum of incongruent zeros of $\phi(z)-C$, for any value of C, is congruent with 2γ. That is to say, if $\phi(c) = C$, then also $\phi(2\gamma-c) = C$; in other words,

·101 $$\phi(2\gamma-z) = \phi(z),$$

for all values of z. It follows that $f(z)$ can not be a rational function of $\phi(z)$ unless $f(z)$ satisfies the same condition, $f(2\gamma-z) = f(z)$, and since this condition is not implied by the periodicity of $f(z)$, we consider first a function $g(z)$, coperiodic with $\phi(z)$, for which the condition does hold.

If c is a zero or a pole of $g(z)$, of any order, and if

·102 $$g(2\gamma-z) = g(z),$$

then $2\gamma-c$ is a zero or a pole of the same order. As in 0·9, we must examine separately the case in which c is a pole of $\phi(z)$, and the case in which the two points c, $2\gamma-c$ are congruent, that is, the case in which $2c \equiv 2\gamma$. These cases left aside, the zeros of $g(z)$ are the zeros of a product $\prod\{\phi(z)-\phi(b_s)\}^{q_s}$ and the poles of $g(z)$ are the zeros of a product $\prod\{\phi(z)-\phi(a_r)\}^{p_r}$.

The points given by the congruence $2c \equiv 2\gamma$ are the four points $\gamma+\omega_k$, where ω_k is zero or a halfperiod† of $\phi(z)$, and with regard to these points we have a number of theorems analogous to 0·76 and 0·77. Differentiating ·102, we have, for any value of n,

·103 $$(-)^n g^{(n)}(2\gamma-z) = g^{(n)}(z),$$

from which it follows that if n is odd, every point which satisfies the condition $z \equiv 2\gamma-z$ is either a pole or a zero of $g^{(n)}(z)$. Hence, if a point $\gamma+\omega_k$ is a zero of $g(z)$, it is a zero of even order, and since $1/g(z)$ is an elliptic function which also satisfies ·102, it follows that if $\gamma+\omega_k$ is a pole of $g(z)$, it is a pole of even order:

† Not the Weierstrassian halfperiod; as yet the function $\phi(z)$ is not specialized, and when we take $\phi(z)$ as one of the elementary functions, two of the Weierstrassian halfperiods are only quarterperiods of $\phi(z)$.

·104. *If $g(z)$ satisfies the condition $g(2\gamma-z) = g(z)$, the point $\gamma+\omega_k$, if not neutral for $g(z)$, is of even order, whether as a zero or as a pole;*

·105. *If $g(z)$ satisfies the condition $g(2\gamma-z) = g(z)$, the point $\gamma+\omega_k$ is either a pole of even order of $g(z)$ or a zero of even order of $g(z)-g(\gamma+\omega_k)$.* In particular,

·106. *The point $\gamma+\omega_k$ is either a double pole of $\phi(z)$ or a double zero of $\phi(z)-\phi(\gamma+\omega_k)$,*

and therefore if $\gamma+\omega_k$ is not a pole of $\phi(z)$, we can allow for a zero of $g(z)$ located there, of even order $2p$, by including with the zeros a factor $\{\phi(z)-\phi(\gamma+\omega_k)\}^p$, or for a pole of $g(z)$ located there, of even order $2q$, by including with the poles a factor $\{\phi(z)-\phi(\gamma+\omega_k)\}^q$. Thus we construct functions $Z(z)$, $P(z)$, polynomials in $\phi(z)$, such that the zeros of $Z(z)$ are those zeros of $g(z)$ which are not poles of $\phi(z)$ and the zeros of $P(z)$ are those poles of $g(z)$ which are not poles of $\phi(z)$. Let $F(z)$ denote the function $g(z)P(z)/Z(z)$; then $F(z)$ is a function coperiodic with $\phi(z)$, and the only points which can serve either as poles or as zeros of $F(z)$ are the poles of $\phi(z)$. But $F(z)$ satisfies the relation $F(2\gamma-z) = F(z)$; if one pole of $\phi(z)$ is a pole of $F(z)$, so is the other pole of $\phi(z)$, and if one pole of $\phi(z)$ is a zero of $F(z)$, so is the other pole of $\phi(z)$. Hence there can not be both poles and zeros of $F(z)$ among the poles of $\phi(z)$, and $F(z)$ either lacks poles or lacks zeros, from which it follows that $F(z)$ is a constant.

4·11. *If $\phi(z)$ is an elliptic function of the second order whose pole-sum is 2γ, and if $g(z)$ is any function coperiodic with $\phi(z)$ which satisfies the condition $g(2\gamma-z) = g(z)$, then $g(z)$ is a rational function of $\phi(z)$.*

It must not be thought that $g(z)$ can not have the poles of $\phi(z)$ for zeros or poles; the character at these points is determined automatically if the character at all other points is determined deliberately. The factor $\phi(z)$ may appear in the explicit formula for $g(z)$, but this will be because the zeros of $\phi(z)$ are zeros of $g(z)$ and have introduced $\phi(z)$ into $Z(z)$, or because the zeros of $\phi(z)$ are poles of $g(z)$ and have introduced $\phi(z)$ into $P(z)$. Also the poles of $\phi(z)$ are zeros or poles of $g(z)$ if $Z(z)$ and $P(z)$ are not of the same degree in $\phi(z)$.

Whatever the function $f(z)$, the half-sum $\frac{1}{2}\{f(z)+f(2\gamma-z)\}$ satisfies the condition ·102 and is therefore a rational function of $\phi(z)$. To complete the representation of $f(z)$ we must deal with the half-difference $\frac{1}{2}\{f(z)-f(2\gamma-z)\}$, and this is a function $h(z)$ which satisfies the condition

·107 $$h(2\gamma-z) = -h(z).$$

This condition, like ·102, implies that the zeros and poles of a function subject to it, other than any that may be located at one of the four points $\gamma+\omega_k$, fall into pairs. Also, by arguments which need not be repeated,

·108. *Any function $h(z)$ which satisfies the condition $h(2\gamma-z) = -h(z)$ has each of the four points $\gamma+\omega_k$ for a pole of odd order or for a zero of odd order.*

Allowing for this peculiarity, to discuss the analysis of a function satisfying ·107 is only to repeat in substance the arguments leading to ·11, but we can take a short cut. From ·101,

·109 $$\phi'(2\gamma-z) = -\phi'(z),$$

and therefore $$\frac{h(2\gamma-z)}{\phi'(2\gamma-z)} = \frac{h(z)}{\phi'(z)}.$$

That is to say, the quotient $h(z)/\phi'(z)$ is a function to which ·11 applies:

4·12. *If $\phi(z)$ is an elliptic function of the second order whose pole-sum is 2γ, and if $h(z)$ is any function coperiodic with $\phi(z)$ which satisfies the condition $h(2\gamma-z) = -h(z)$, then $h(z)$ is the product of the derivative $\phi'(z)$ by a rational function of $\phi(z)$.*

Combining ·11 and ·12 we have the general theorem of Liouville's of which $0{\cdot}92_3$ is a special case:

4·13. *If $\phi(z)$ is an elliptic function of the second order, any elliptic function coperiodic with $\phi(z)$ is expressible in the form*

$$R\{\phi(z)\} + \phi'(z)\, S\{\phi(z)\},$$

where $R\{\phi(z)\}$, $S\{\phi(z)\}$ are rational functions of $\phi(z)$.

4·2. If $\mathrm{pq}\,z$, $\mathrm{rs}\,z$ are coperiodic elementary functions, ·13 asserts a relation between them, but this relation is in every case evident enough when attention has been called to the form of relation required. We have for example

4·21 $$\mathrm{gh}\,z = -\frac{f_h\,\mathrm{fj}'z}{\mathrm{hj}^2 z} = \frac{f_h\,\mathrm{fj}'z}{f_h^2-\mathrm{fj}^2 z}.$$

It is different when we change the argument of one of the functions from z to† $y+z$; the function $\mathrm{rs}(y+z)$, as a function of z with y playing a parametric part, has the same periods as $\mathrm{rs}\,z$, and this function also can therefore be expressed in terms of $\mathrm{pq}\,z$ and $\mathrm{pq}'z$, with coefficients

† Throughout this chapter, as in 0·8 and 0·9, x and y denote independent complex numbers, not real numbers related to z.

dependent on y. When the functions $\mathrm{rs}\,z$, $\mathrm{pq}\,z$ are identical, the analysis of $\mathrm{rs}(y+z)$ is the discovery of addition theorems for $\mathrm{pq}\,z$.

The application of Liouville's process to $\mathrm{rs}(y+z)$ requires the determination of zeros of functions of the form $\mathrm{rs}(y+z)\pm\mathrm{rs}(y+2\gamma-z)$, that is, the solution of equations of the form $\mathrm{rs}\,u = \mp\mathrm{rs}\,v$. Now not only can we solve the equation $\mathrm{rs}\,u = \mathrm{rs}\,v$, but since $-\mathrm{rs}\,v$ can be expressed as $\mathrm{rs}(v+2\omega_l)$ we can solve the equation $\mathrm{rs}\,u = -\mathrm{rs}\,v$ also. For this reason processes which fail to lead to an addition theorem for $\wp z$ are effective when applied to the elementary functions.

Being coperiodic, the functions $\mathrm{pq}\,z$, $\mathrm{rs}\,z$ are derivable from the same primitive function, and if this primitive function is $\mathrm{kj}\,z$, we have

·201–·202 $$\mathrm{kj}\,z = \mathrm{pq}(z+\omega_q) = \mathrm{rs}(z+\omega_s).$$

Since $\mathrm{kj}\,z$ is odd, $\mathrm{pq}(z+\omega_q) = -\mathrm{pq}(\omega_q-z)$, as in 3·202, and we can take this result in the form

·203 $$-\mathrm{pq}\,z = \mathrm{pq}(2\omega_q-z).$$

A fundamental parallelogram can be formed with $2\omega_k$ for one side; if the other side is $4\omega_l$, then

·204 $$\mathrm{pq}(z+2\omega_l) = -\mathrm{pq}\,z.$$

The function $\mathrm{rs}\,z$ also satisfies the same two conditions:

·205–·206 $$-\mathrm{rs}\,z = \mathrm{rs}(2\omega_q-z), \qquad \mathrm{rs}(z+2\omega_l) = -\mathrm{rs}\,z.$$

The two poles of $\mathrm{pq}\,z$ are ω_q and $\omega_q+2\omega_l$. Hence the pole-sum of this function is $2\omega_q+2\omega_l$, and to analyse the function $\mathrm{rs}(y+z)$ we write $\mathrm{rs}(y+z)$ as $g(z)+h(z)$, where

·207 $$2g(z) = \mathrm{rs}(y+z)+\mathrm{rs}(2\omega_q+2\omega_l+y-z) = \mathrm{rs}(y+z)-\mathrm{rs}(2\omega_q+y-z),$$

·208 $$2h(z) = \mathrm{rs}(y+z)-\mathrm{rs}(2\omega_q+2\omega_l+y-z) = \mathrm{rs}(y+z)+\mathrm{rs}(2\omega_q+y-z),$$

y being regarded as a constant.

The functions $g(z)$, $h(z)$ have the same poles, namely, the poles of $\mathrm{rs}(y+z)$, which are $-y+\omega_s$ and $-y+\omega_s+2\omega_l$, and the poles of $\mathrm{rs}(2\omega_q+y-z)$, which are $y+2\omega_q-\omega_s$ and $y+2\omega_q-\omega_s+2\omega_l$. Except for special values of y, which need not now be considered, these four poles are distinct and simple, and the functions are of the fourth order. The two points $-y+\omega_s$, $y-\omega_s+2\omega_q+2\omega_l$, whose sum is the pole-sum of $\mathrm{pq}\,z$, are zeros of $\mathrm{pq}\,z-\mathrm{pq}(\omega_s-y)$, and the two points $-y+\omega_s+2\omega_l$, $y-\omega_s+2\omega_q$ are for the same reason zeros of $\mathrm{pq}\,z-\mathrm{pq}(\omega_s-y+2\omega_l)$, that is, of $\mathrm{pq}\,z+\mathrm{pq}(\omega_s-y)$. Hence each of the functions $g(z)$, $h(z)$ has for its poles the zeros of the function $\mathrm{pq}^2z-\mathrm{pq}^2(\omega_s-y)$.

Since $\mathrm{rs}\,z$ is of the second order and has the two poles ω_s, $\omega_s+2\omega_l$, the equality $\operatorname{rs} u = \operatorname{rs} v$ implies either $u \equiv v$ or $u+v \equiv 2\omega_s + 2\omega_t$. Hence the roots of the equation $g(z) = 0$ are the solutions of the congruence

$$\cdot 209 \qquad y+z \equiv 2\omega_q + y - z,$$

and the roots of the equation $h(z) = 0$ are the solutions of the congruence

$$\cdot 210 \qquad y+z \equiv 2\omega_q + 2\omega_t + y - z,$$

for in each case the alternative congruence does not involve z.

Since ·209 is simply $z \equiv 2\omega_q - z$, it follows from ·203 that the zeros of $g(z)$ are the zeros and the poles of $\operatorname{pq} z$, and therefore, since the poles of $\operatorname{pq} z$ must be omitted in the construction of $g(z)$ in terms of $\operatorname{pq} z$,

$$\cdot 211 \qquad g(z) = \frac{A(y)\operatorname{pq} z}{\operatorname{pq}^2 z - \operatorname{pq}^2(\omega_s - y)},$$

the unknown factor being a function of y; the poles of $\operatorname{pq} z$ enter as zeros of $g(z)$ because the degree of the denominator is higher than that of the numerator. Near ω_q, $\operatorname{pq} z \sim 1/(z-\omega_q)$; hence

$$A(y) = \lim_{z\to\omega_q} \frac{g(z)}{z-\omega_q} = \lim_{z\to\omega_q} \frac{\operatorname{rs}'(y+z) + \operatorname{rs}'(2\omega_q + y - z)}{2} = \operatorname{rs}'(\omega_q + y)$$
$$= \operatorname{pq}'(2\omega_q + y - \omega_s) = \operatorname{pq}'(\omega_s - y),$$

from ·202 and ·203.

The congruence ·210 is equivalent to $z \equiv 2\omega_q - 2\omega_t - z$, since $4\omega_t$ is a period. From ·203 and ·204 we have

$$\cdot 212 \qquad \operatorname{pq}(2\omega_q - 2\omega_t - z) = \operatorname{pq} z,$$

whence

$$\cdot 213 \qquad \operatorname{pq}'(2\omega_q - 2\omega_t - z) = -\operatorname{pq}' z,$$

and it follows, since the poles of $\operatorname{pq}' z$ are the poles of $\operatorname{pq} z$ and satisfy the congruence $z \equiv 2\omega_q - z$ which is incompatible with $z \equiv 2\omega_q - 2\omega_t - z$, that *the zeros of* $h(z)$ *are the zeros of* $\operatorname{pq}' z$. This could have been predicted from the general discussion in the last section, for $h(z)$ and $\operatorname{pq}' z$ are both of the fourth order, and therefore $h(z)$ can have no zeros in addition to those of $\operatorname{pq}' z$. We have now

$$\cdot 214 \qquad h(z) = \frac{B(y)\operatorname{pq}' z}{\operatorname{pq}^2 z - \operatorname{pq}^2(\omega_s - y)},$$

and since $\operatorname{pq}' z/\operatorname{pq}^2 z \to -1$ as $z \to \omega_q$,

$$B(y) = -h(\omega_q) = -\operatorname{rs}(\omega_q + y) = -\operatorname{pq}(2\omega_q + y - \omega_s) = \operatorname{pq}(\omega_s - y).$$

Replacing z by x to emphasize that it is only for the purposes of the proof that y has been subordinated, we have

4·22 $\quad \mathrm{rs}(x+y)-\mathrm{rs}(2\omega_q-x+y) = 2\,\mathrm{pq}\,x\,\mathrm{pq}'(\omega_s-y)/\{\mathrm{pq}^2x-\mathrm{pq}^2(\omega_s-y)\},$

4·23 $\quad \mathrm{rs}(x+y)+\mathrm{rs}(2\omega_q-x+y) = 2\,\mathrm{pq}'\,x\,\mathrm{pq}(\omega_s-y)/\{\mathrm{pq}^2x-\mathrm{pq}^2(\omega_s-y)\},$

and finally the one general formula

4·24 $$\mathrm{rs}(x+y) = \frac{\mathrm{pq}\,x\,\mathrm{pq}'(\omega_s-y)+\mathrm{pq}'x\,\mathrm{pq}(\omega_s-y)}{\mathrm{pq}^2x-\mathrm{pq}^2(\omega_s-y)}.$$

4·3. Before elaborating this result, we will investigate an equivalent theorem for the elementary functions by a modification of the method used in 0·8 in the discussion of the Weierstrassian function $\wp z$, which is due in essence to Abel.

Squaring the fundamental expression $-\mathrm{gj}\,z\,\mathrm{hj}\,z$ for $\mathrm{fj}'z$ and substituting for gj^2z and hj^2z in terms of fj^2z, we have

·301 $$(\mathrm{fj}'z)^2 = (\mathrm{fj}^2z-e_{fg})(\mathrm{fj}^2z-e_{fh}),$$

and since this equality is unaltered if a quarterperiod is subtracted from z, we can say that if $\mathrm{j}\,z$ is any one of the twelve elementary elliptic functions,

·302 $$(\mathrm{j}'z)^2 = (\mathrm{j}^2z-A)(\mathrm{j}^2z-B),$$

where A, B are constants of the form e_{rs}. If ϕz is the function j^2z, then

·303 $$(\phi'z)^2 = 4\phi z(\phi z-A)(\phi z-B).$$

Since the addition of one of the halfperiods $2\omega_f$, $2\omega_g$, $2\omega_h$ to z either leaves $\mathrm{j}\,z$ unchanged or changes $\mathrm{j}\,z$ to $-\mathrm{j}\,z$, this addition leaves ϕz unchanged; that is to say, ϕz has $2\omega_f$, $2\omega_g$, $2\omega_h$ for periods. Within a parallelogram that is primitive for these periods, $\mathrm{j}\,z$ has only one pole, and this is a simple positive pole ω; hence within such a parallelogram ϕz has only one pole, which is double, and the only pole of $\phi'z$ is a triple pole at the same point, ω. Further, if a, b, c are any three constants, $a+b\phi z+c\phi'z$ is an elliptic function whose only pole in the fundamental parallelogram is a triple pole at ω, and therefore $a+b\phi z+c\phi'z$ is a function $F(z)$ with three zeros whose sum is congruent with 3ω, that is, since 2ω is either zero or a period, is congruent with ω.

If $a+b\phi+c\phi' = 0$, then

·304 $$(a+b\phi)^2 = 4c^2\phi(\phi-A)(\phi-B),$$

where A, B are the constants in ·302. Hence if x, y, t are the three values of z in the fundamental parallelogram which satisfy the equation

·305 $$F(z) = 0,$$

then ϕx, ϕy, ϕt are three values of ϕ which satisfy the equation

·306 $$(a+b\phi)^2 = 4c^2\phi(\phi-A)(\phi-B),$$

and since this is a cubic equation in ϕ, these three values are simply the three roots of the equation. Thus while t is determined from x and y, save for multiples of the halfperiods $2\omega_f$, $2\omega_g$, $2\omega_h$, by the congruence

·307 $$x+y+t \equiv \omega,$$

ϕt is determined from ϕx and ϕy by any formula which gives one root of a cubic equation when two roots are already known; in the present case the simplest formula to apply is that for the product of the roots, since this does not involve A or B, and we have

·308 $$\phi x\, \phi y\, \phi t = a^2/4c^2,$$

whence

·309 $$\mathrm{j}\,x\,\mathrm{j}\,y\,\mathrm{j}\,t = \pm a/2c.$$

But if x and y satisfy the equation

$$a+b\phi z+c\phi' z = 0,$$

the ratios $a:b:c$ are determined by the pair of equations

$$a+b\phi x+c\phi' x = 0, \qquad a+b\phi y+c\phi' y = 0,$$

and therefore we have

·310 $$2\,\mathrm{j}\,x\,\mathrm{j}\,y\,\mathrm{j}\,t = \pm\frac{\phi x\,\phi' y-\phi y\,\phi' x}{\phi x-\phi y},$$

or in another form, since $\mathrm{j}\,t$ is either $\mathrm{j}(-t)$ or $-\mathrm{j}(-t)$,

·311 $$\mathrm{j}(x+y-\omega) = \pm\frac{\mathrm{j}\,x\,\mathrm{j}'y-\mathrm{j}\,y\,\mathrm{j}'x}{\mathrm{j}^2x-\mathrm{j}^2y}.$$

We can remove the ambiguity of sign from this last equation; near the positive pole ω, $\mathrm{j}'z \sim -1/(z-\omega)^2$, and therefore as $y \to \omega$, the fraction on the right tends to $\mathrm{j}\,x$. Hence the positive sign must be taken, and we have definitely

4·31 $$\mathrm{j}(x+y-\omega) = \frac{\mathrm{j}\,x\,\mathrm{j}'y-\mathrm{j}\,y\,\mathrm{j}'x}{\mathrm{j}^2x-\mathrm{j}^2y}.$$

To see that this formula is identical, except in notation, with ·24, we have only to substitute $y+\omega_s-2\omega_q$ for y in the latter; on the right, $\mathrm{pq}(\omega_s-y)$, $\mathrm{pq}'(\omega_s-y)$ become $\mathrm{pq}(2\omega_q-y)$, $\mathrm{pq}'(2\omega_q-y)$, that is, $-\mathrm{pq}\,y$, $\mathrm{pq}'y$, and on the left, $\mathrm{rs}(x+y)$ becomes $\mathrm{rs}(x+y-2\omega_q+2\omega_s)$, that is, $\mathrm{pq}(x+y-\omega_q)$.

It follows from ·31 that in ·310 we must take the positive or the negative sign according as the function $\mathrm{j}\,z$ is even or odd; the simpler plan is to regard the formula with a positive sign as giving $\mathrm{j}(-t)$. We can express the result differently. The condition $x+y+z=\omega$ is symmetrical in x, y, z; so also is the product $\mathrm{j}x\,\mathrm{j}y\,\mathrm{j}(-z)$, which whether $\mathrm{j}z$ is odd or even may be written as $\mathrm{j}(-x)\,\mathrm{j}(-y)\,\mathrm{j}(-z)$. Hence we may replace x and y by y and z or by z and x in the fraction to which $2\,\mathrm{j}x\,\mathrm{j}y\,\mathrm{j}(-z)$ is equated.

4·32. *If* $\mathrm{j}z$ *is any one of the twelve elementary elliptic functions, and if the sum of three arguments* x, y, z *is congruent with a positive pole of* $\mathrm{j}z$, *then*

$$\frac{\phi x\,\phi' y-\phi y\,\phi' x}{\phi x-\phi y}=\frac{\phi y\,\phi' z-\phi z\,\phi' y}{\phi y-\phi z}=\frac{\phi z\,\phi' x-\phi x\,\phi' z}{\phi z-\phi x}=2\,\mathrm{j}(-x)\,\mathrm{j}(-y)\,\mathrm{j}(-z),$$

where ϕz *denotes* $\mathrm{j}^2 z$.

But the equalities of the fractions in ·32 do not really contain additional results, for each of the equalities is equivalent, but for a factor ϕx, ϕy, or ϕz, to

$$\begin{vmatrix} 1 & \phi x & \phi' x \\ 1 & \phi y & \phi' y \\ 1 & \phi z & \phi' z \end{vmatrix}=0,$$

and

$$\begin{vmatrix} 1 & \phi x & \phi' x \\ 1 & \phi y & \phi' y \\ 1 & \phi z & \phi' z \end{vmatrix}$$

is the simplest linear function of ϕz and $\phi' z$ that is zero when z is x or y.

4·4. If ω is a pole of $\mathrm{j}z$, the origin is a pole of $\mathrm{j}(z-\omega)$; thus the function for which ·31 provides a direct addition formula is not $\mathrm{j}z$ itself but the primitive function coperiodic with $\mathrm{j}z$. In other words, ·31 gives in the first place four formulae for each of the three primitive functions, not one formula for each of the twelve elementary functions. In ·24 no restriction is imposed, but the functions $\mathrm{pq}\,z$, $\mathrm{pq}(\omega_s-z)$ are effectively different functions unless ω_s is zero, that is, unless $\mathrm{rs}\,z$ is primitive. The explicit formula for $\mathrm{fj}(x+y)$ involves $\mathrm{pq}(-y)$ and $\mathrm{pq}'(-y)$, and the ultimate simplification depends on whether $\mathrm{pq}\,z$ is one of the two odd functions $\mathrm{fj}\,z$, $\mathrm{jf}\,z$ or one of the two even functions $\mathrm{gh}\,z$, $\mathrm{hg}\,z$. We can avoid the complication by taking as the standard form

4·41 $$\mathrm{fj}(x-y)=\frac{\mathrm{pq}\,x\,\mathrm{pq}'y+\mathrm{pq}\,y\,\mathrm{pq}'x}{\mathrm{pq}^2x-\mathrm{pq}^2y},$$

which holds if $\mathrm{pq}\,z$ is any one of the four elementary functions coperiodic with $\mathrm{fj}\,z$. The generality of this formula when the argument is taken as a difference is trivial, for the result is only the particular formula

4·42₁
$$\mathrm{fj}(x-y) = \frac{\mathrm{fj}\,x\,\mathrm{fj}'y+\mathrm{fj}\,y\,\mathrm{fj}'x}{\mathrm{fj}^2x-\mathrm{fj}^2y}$$

with $x-\omega_q$, $y-\omega_q$ substituted for x, y.

The fundamental addition theorem for $\mathrm{fj}\,z$ is this last formula with the sign of y restored, when we have†

4·42₂
$$\mathrm{fj}(x+y) = \frac{\mathrm{fj}\,x\,\mathrm{fj}'y-\mathrm{fj}\,y\,\mathrm{fj}'x}{\mathrm{fj}^2x-\mathrm{fj}^2y}.$$

As we have already had occasion to remark, if $\mathrm{pq}\,z$ is an elementary function coperiodic with $\mathrm{fj}\,z$, then

·401
$$\mathrm{pq}'^2z = (\mathrm{pq}^2z-e_{fg})(\mathrm{pq}^2z-e_{fh}),$$

whence for any two arguments,

·402
$$\mathrm{pq}^2x\,\mathrm{pq}'^2y-\mathrm{pq}^2y\,\mathrm{pq}'^2x = (\mathrm{pq}^2x-\mathrm{pq}^2y)(e_{fg}\,e_{fh}-\mathrm{pq}^2x\,\mathrm{pq}^2y).$$

We may therefore, so to speak, rationalize the numerator in ·41, and we have an alternative formula:

4·43. *If* $\mathrm{pq}\,z$ *is coperiodic with* $\mathrm{fj}\,z$, *then*

$$\mathrm{fj}(x-y) = \frac{e_{fg}\,e_{fh}-\mathrm{pq}^2x\,\mathrm{pq}^2y}{\mathrm{pq}\,x\,\mathrm{pq}'y-\mathrm{pq}\,y\,\mathrm{pq}'x}.$$

In particular, we have

4·44
$$\mathrm{fj}(x+y) = \frac{e_{fg}\,e_{fh}-\mathrm{fj}^2x\,\mathrm{fj}^2y}{\mathrm{fj}\,x\,\mathrm{fj}'y+\mathrm{fj}\,y\,\mathrm{fj}'x},$$

the addition formula given for these functions by Jordan‡, whose proof is that verification of poles and zeros which is of so little value to the average student if no hint of a process for discovering the result is provided.

4·5. Addition formulae for the elementary functions whose poles are not congruent with the origin may be inferred in two ways. Since $\mathrm{jf}\,z$ is $-f_g f_h/\mathrm{fj}\,z$, a formula for $\mathrm{fj}(x-y)$ gives us at once a formula for $\mathrm{jf}(x-y)$; similarly, since $\mathrm{gh}\,z$ is $-f_h\,\mathrm{gj}\,z/\mathrm{hj}\,z$, we can write down formulae for $\mathrm{gh}(x-y)$ from those for $\mathrm{gj}(x-y)$ and $\mathrm{hj}(x-y)$. Alternatively, by regarding $\mathrm{jf}\,z$ as $\mathrm{fj}(z-\omega_f)$ and $\mathrm{gh}\,z$ as $\mathrm{fj}(z-\omega_h)$, we can express $\mathrm{jf}(x-y)$ in terms of functions of x and functions of $y+\omega_f$ and $\mathrm{gh}(x-y)$ in terms of functions of x and functions of $y+\omega_h$, and we can then replace the functions of $y+\omega_f$ and $y+\omega_h$ by elementary functions of y; in effect,

† This is the formula used by Chaundy and by Baker in the papers cited in the Preface.

‡ *Cours d'Analyse*, 2 (3 éd. 1913), p. 458.

this is to use the general formula for $\mathrm{rs}(x+y)$ in terms of $\mathrm{pq}\,x$ and $\mathrm{pq}(\omega_s-y)$ which we found by Liouville's process.

By the first method we have at once for $\mathrm{jf}(x-y)$ the general formulae

$4{\cdot}51_{1-2}$

$$\mathrm{jf}(x-y) = -\frac{f_g f_h(\mathrm{pq}^2x-\mathrm{pq}^2y)}{\mathrm{pq}\,x\,\mathrm{pq}'y+\mathrm{pq}\,y\,\mathrm{pq}'x} = -\frac{f_g f_h(\mathrm{pq}\,x\,\mathrm{pq}'y-\mathrm{pq}\,y\,\mathrm{pq}'x)}{e_{fg}e_{fh}-\mathrm{pq}^2x\,\mathrm{pq}^2y},$$

where $\mathrm{pq}\,z$ is any one of the four functions $\mathrm{fj}\,z$, $\mathrm{jf}\,z$, $\mathrm{gh}\,z$, $\mathrm{hg}\,z$. The addition theorem, in the strictest sense, is

$4{\cdot}52_{1-2}$ $$\mathrm{jf}(x+y) = -\frac{f_g f_h(\mathrm{jf}^2x-\mathrm{jf}^2y)}{\mathrm{jf}\,x\,\mathrm{jf}'y-\mathrm{jf}\,y\,\mathrm{jf}'x} = -\frac{f_g f_h(\mathrm{jf}\,x\,\mathrm{jf}'y+\mathrm{jf}\,y\,\mathrm{jf}'x)}{e_{fg}e_{fh}-\mathrm{jf}^2x\,\mathrm{jf}^2y}.$$

Since the functions in terms of which $\mathrm{gj}(x-y)$ is expressible are coperiodic with $\mathrm{gj}\,z$ and those in terms of which $\mathrm{hj}(x-y)$ is expressible are coperiodic with $\mathrm{hj}\,z$, we can not express $\mathrm{gj}(x-y)$ and $\mathrm{hj}(x-y)$ in terms of the same function. Nevertheless we can choose expressions for $\mathrm{gj}(x-y)$ and $\mathrm{hj}(x-y)$ with a common denominator, for the relations

$$\mathrm{fj}^2z+e_f = \mathrm{gj}^2z+e_g = \mathrm{hj}^2z+e_h$$

remain true if $z-\omega_q$ is substituted for z and imply that if $\mathrm{rq}\,z$, $\mathrm{sq}\,z$ are copolar, then for any two arguments x, y,

$\cdot 501$ $$\mathrm{rq}^2x-\mathrm{rq}^2y = \mathrm{sq}^2x-\mathrm{sq}^2y.$$

We take then one of the four cardinal points, ω_q, and we use $\cdot 41$ to express $\mathrm{gj}(x-y)$ by means of the function $\mathrm{rq}\,z$ which has a pole at ω_q and is coperiodic with $\mathrm{gj}\,z$, and $\mathrm{hj}(x-y)$ by means of the function $\mathrm{sq}\,z$ which has a pole at ω_q and is coperiodic with $\mathrm{hj}\,z$; thus we have

$4{\cdot}53$ $$\mathrm{gh}(x-y) = -\frac{f_h(\mathrm{rq}\,x\,\mathrm{rq}'y+\mathrm{rq}\,y\,\mathrm{rq}'x)}{\mathrm{sq}\,x\,\mathrm{sq}'y+\mathrm{sq}\,y\,\mathrm{sq}'x},$$

a formula which in spite of its simplicity does not exhibit well the structure of $\mathrm{gh}(x-y)$, since neither of the functions $\mathrm{rq}\,z$, $\mathrm{sq}\,z$ is coperiodic with $\mathrm{gh}\,z$. To modify the formula, we rationalize the denominator or the numerator. The derivative $\mathrm{rq}'z$ is the negative of the product of the two functions different from $\mathrm{rq}\,z$ but copolar with $\mathrm{rq}\,z$; one of these is $\mathrm{sq}\,z$, and the other, which we will denote by $\mathrm{pq}\,z$, is the function which has a pole at ω_q and is coperiodic with $\mathrm{fj}\,z$, and therefore with $\mathrm{gh}\,z$. The derivative $\mathrm{sq}'z$ is the negative of the product of $\mathrm{rq}\,z$ and this same function $\mathrm{pq}\,z$. Hence

$\cdot 502$ $$(\mathrm{rq}\,x\,\mathrm{rq}'y+\mathrm{rq}\,y\,\mathrm{rq}'x)(\mathrm{sq}\,x\,\mathrm{sq}'y-\mathrm{sq}\,y\,\mathrm{sq}'x)$$
$$= (\mathrm{rq}\,x\,\mathrm{pq}\,y\,\mathrm{sq}\,y+\mathrm{rq}\,y\,\mathrm{pq}\,x\,\mathrm{sq}\,x)(\mathrm{sq}\,x\,\mathrm{pq}\,y\,\mathrm{rq}\,y-\mathrm{sq}\,y\,\mathrm{pq}\,x\,\mathrm{rq}\,x)$$
$$= (\mathrm{pq}^2y-\mathrm{pq}^2x)\mathrm{rq}\,x\,\mathrm{sq}\,x\,\mathrm{rq}\,y\,\mathrm{sq}\,y+(\mathrm{rq}^2y\,\mathrm{sq}^2x-\mathrm{rq}^2x\,\mathrm{sq}^2y)\mathrm{pq}\,x\,\mathrm{pq}\,y.$$

But the product $\mathrm{rq}\,z\,\mathrm{sq}\,z$ is $-\mathrm{pq}'z$, and since $\mathrm{pq}\,z$, $\mathrm{rq}\,z$, $\mathrm{sq}\,z$ are the functions $\mathrm{fj}(z-\omega_q)$, $\mathrm{gj}(z-\omega_q)$, $\mathrm{hj}(z-\omega_q)$, we have

$$\mathrm{rq}^2z = \mathrm{pq}^2z-e_{fg}, \qquad \mathrm{sq}^2z = \mathrm{pq}^2z-e_{fh},$$

·503 $$\mathrm{rq}^2y\,\mathrm{sq}^2x-\mathrm{rq}^2x\,\mathrm{sq}^2y = (e_{fg}-e_{fh})(\mathrm{pq}^2y-\mathrm{pq}^2x) = -e_{gh}(\mathrm{pq}^2y-\mathrm{pq}^2x).$$

On the other hand,

$$\mathrm{sq}'^2z = (\mathrm{sq}^2z+e_{fh})(\mathrm{sq}^2z+e_{gh}),$$

whence

·504 $$\mathrm{sq}^2x\,\mathrm{sq}'^2y-\mathrm{sq}^2y\,\mathrm{sq}'^2x = (\mathrm{sq}^2y-\mathrm{sq}^2x)(\mathrm{sq}^2x\,\mathrm{sq}^2y-e_{fh}\,e_{gh}) = (\mathrm{pq}^2y-\mathrm{pq}^2x)(\mathrm{sq}^2x\,\mathrm{sq}^2y-e_{fh}\,e_{gh}),$$

and removing the common factor throughout we have the required simplification. The steps involved in rationalizing the numerator are the same, but for the interchange of $\mathrm{rq}\,z$ and $\mathrm{sq}\,z$, and they are also the steps involved in rationalizing the denominator of $\mathrm{hg}(x-y)$. Thus

4·54. *If* $\mathrm{pq}\,z$ *is any one of the four elementary functions coperiodic with* $\mathrm{gh}\,z$, *if* $\mathrm{sq}\,z$ *is the function copolar with* $\mathrm{pq}\,z$ *and coperiodic with* $\mathrm{hj}\,z$, *and if* $\mathrm{rq}\,z$ *is the function copolar with* $\mathrm{pq}\,z$ *and coperiodic with* $\mathrm{gj}\,z$, *then*

·54₁₋₂ $$\mathrm{gh}(x-y) = -\frac{f_h(e_{gh}\,\mathrm{pq}\,x\,\mathrm{pq}\,y-\mathrm{pq}'x\,\mathrm{pq}'y)}{e_{fh}\,e_{gh}-\mathrm{sq}^2x\,\mathrm{sq}^2y} = -\frac{f_h(\mathrm{rq}^2x\,\mathrm{rq}^2y+e_{fg}\,e_{gh})}{e_{gh}\,\mathrm{pq}\,x\,\mathrm{pq}\,y+\mathrm{pq}'x\,\mathrm{pq}'y}.$$

In terms of the one function $\mathrm{pq}\,z$ and its derivative,

4·55₁₋₂ $$\mathrm{gh}(x-y) = \frac{f_h(e_{gh}\,\mathrm{pq}\,x\,\mathrm{pq}\,y-\mathrm{pq}'x\,\mathrm{pq}'y)}{\mathrm{pq}^2x\,\mathrm{pq}^2y-e_{fh}(\mathrm{pq}^2x+\mathrm{pq}^2y)+e_{fg}\,e_{fh}} = -\frac{f_h\{\mathrm{pq}^2x\,\mathrm{pq}^2y-e_{fg}(\mathrm{pq}^2x+\mathrm{pq}^2y)+e_{fg}\,e_{fh}\}}{e_{gh}\,\mathrm{pq}\,x\,\mathrm{pq}\,y+\mathrm{pq}'x\,\mathrm{pq}'y}.$$

The addition theorem for $\mathrm{gh}\,z$ is explicitly

4·56₁₋₂ $$\mathrm{gh}(x+y) = -\frac{f_h(e_{gh}\,\mathrm{gh}\,x\,\mathrm{gh}\,y+\mathrm{gh}'x\,\mathrm{gh}'y)}{e_{fh}\,e_{gh}-\mathrm{jh}^2x\,\mathrm{jh}^2y} = -\frac{f_h(\mathrm{fh}^2x\,\mathrm{fh}^2y+e_{fg}\,e_{gh})}{e_{gh}\,\mathrm{gh}\,x\,\mathrm{gh}\,y-\mathrm{gh}'x\,\mathrm{gh}'y},$$

where if we wish to have no other function than $\mathrm{gh}\,z$ we must substitute

$$\mathrm{jh}^2x = \mathrm{gh}^2x-e_{fh}, \qquad \mathrm{jh}^2y = \mathrm{gh}^2y-e_{fh}$$

in the first fraction and

$$\mathrm{fh}^2x = \mathrm{gh}^2x-e_{fg}, \qquad \mathrm{fh}^2y = \mathrm{gh}^2y-e_{fg}$$

in the second.

4·6. If we derive addition formulae for the functions other than the primitive functions directly from ·24, the expressions which we find are unsymmetrical in appearance, since the functions pq z and pq$(\omega_s - z)$ are essentially different if ω_s is not zero. If rs z is the same function as pq z, then pq$(z-\omega_s)$ becomes the primitive function coperiodic with pq z. We have for example

4·61 $$\mathrm{jf}(x+y) = (\mathrm{jf}\,x\,\mathrm{fj}'y - \mathrm{fj}\,y\,\mathrm{jf}'x)/(\mathrm{jf}^2x - \mathrm{fj}^2y),$$

4·62 $$\mathrm{hg}(x+y) = (\mathrm{hg}\,x\,\mathrm{fj}'y + \mathrm{fj}\,y\,\mathrm{hg}'x)/(\mathrm{hg}^2x - \mathrm{fj}^2y).$$

These formulae have the advantage of involving no constants, and they are easily converted into more symmetrical forms by direct algebra.

In this connexion we may notice the result of trying to avoid one of the two steps in Liouville's process for obtaining ·24. We shall be able to infer rs$(x+y)$ directly from ·22 if the interchange of x and y converts rs$(2\omega_q - x + y)$ into its negative; since

$$\mathrm{rs}(2\omega_q + x - y) = \mathrm{rs}\{4\omega_q - (2\omega_q - x + y)\}$$

and $4\omega_q$ is either zero or a whole period, the requisite condition is that rs z must be an odd function. In this case we have immediately

4·63 $$\mathrm{rs}(x+y) = \frac{\mathrm{pq}\,x\,\mathrm{pq}'(\omega_s - y)}{\mathrm{pq}^2x - \mathrm{pq}^2(\omega_s - y)} = \frac{\mathrm{pq}\,y\,\mathrm{pq}'(\omega_s - x)}{\mathrm{pq}^2(\omega_s - x) - \mathrm{pq}^2y},$$

but algebraical manipulation is necessary to provide a common denominator unless ω_s is zero, that is, unless we are dealing with the primitive functions, which from this point of view also are seen as the simplest of the group.

4·7. A peculiarly terse form of the addition theorem, derivable immediately from $·42_2$, is

$4·71_1$ $$\mathrm{gj}(x+y) + \mathrm{hj}(x+y) = \frac{\mathrm{gj}\,x\,\mathrm{hj}\,y + \mathrm{gj}\,y\,\mathrm{hj}\,x}{\mathrm{fj}\,x + \mathrm{fj}\,y}.$$

Addition of $2\omega_g$ to y gives

$4·71_2$ $$\mathrm{gj}(x+y) - \mathrm{hj}(x+y) = \frac{\mathrm{hj}\,x\,\mathrm{gj}\,y - \mathrm{hj}\,y\,\mathrm{gj}\,x}{\mathrm{fj}\,x - \mathrm{fj}\,y},$$

and therefore the whole mass of addition formulae is recoverable from $·71_1$ alone.

Results away from the origin are again more complicated:

4·72 $$\mathrm{gf}(x+y) + \mathrm{hf}(x+y) = \frac{g_f\,\mathrm{gf}\,x\,\mathrm{gf}\,y + h_f\,\mathrm{hf}\,x\,\mathrm{hf}\,y}{\mathrm{jf}\,x\,\mathrm{jf}\,y - g_f h_f},$$

4·73 $$\mathrm{jf}(x+y) + \mathrm{gf}(x+y) = \frac{h_f(\mathrm{jf}\,x\,\mathrm{hf}\,y - g_f\,\mathrm{gf}\,x)(\mathrm{jf}\,y\,\mathrm{hf}\,x - g_f\,\mathrm{gf}\,y)}{\mathrm{jf}^2x\,\mathrm{jf}^2y - g^2h^2}.$$

V

THE NATURE OF THE PROBLEM OF INVERSION

5·1. In a form which we have already found useful, the relation between the function fj z and its derivative is

·101 $$(\mathrm{fj}'z)^2 = (\mathrm{fj}^2z - f_g^2)(\mathrm{fj}^2z - f_h^2).$$

In other words, if $w = \mathrm{fj}\,z$, then

5·11 $$(dw/dz)^2 = R_f(w),$$

where, in a notation which we shall retain,

·102 $$R_f(w) = (w^2 - f_g^2)(w^2 - f_h^2).$$

Written as

·103 $$\frac{dz}{dw} = \pm\frac{1}{\sqrt{R_f(w)}},$$

·11 can be integrated immediately, and we have

·104 $$z = \pm\int_{\infty}^{w}\frac{dw}{\sqrt{R_f(w)}},$$

where the path of integration in the w plane is determined by the transformation $w = \mathrm{fj}\,z$ from some path in the z plane from the origin to the point z. Also, near $z = 0$ we have $w \sim 1/z$, $dw/dz \sim -1/z^2$, and therefore $dz/dw \sim -1/w^2$; that is, if we make the radical in the integrand precise by requiring it to resemble w^2 towards infinity along the path of integration, and to be continuous along that path, we must prefix the minus sign or interchange the limits. Thus with no ambiguity,

5·12. *If $w = \mathrm{fj}\,z$, there is a path of integration in the w plane such that along this path*

$$\int_{w}^{\infty}\frac{dw}{\sqrt{R_f(w)}} = z.$$

For a given value of w, the relation $w = \mathrm{fj}\,z$ is satisfied as we know by an infinity of values of z, and it follows from ·12 that if the path of integration is arbitrary, the integral

$$\int_{w}^{\infty}\frac{dw}{\sqrt{R_f(w)}},$$

which, with the understanding that the radical resembles w^2 towards infinity along the path, we shall denote by $I_f(w)$, is susceptible of an infinity of values: *the aggregate of solutions of the equation* $\text{fj}\,z = w$ *is contained in the aggregate of values of the integral* $I_f(w)$, but the identity of the two aggregates is not yet asserted.

Consider now the relation

·105 $$I_f = \int_w^\infty \frac{dw}{\sqrt{R_f(w)}}.$$

where the path of integration is given and w is a current point of that path. This relation implies

·106 $$\frac{dI_f}{dw} = -\frac{1}{\sqrt{R_f(w)}},$$

whence

·107 $$(dw/dI_f)^2 = R_f(w),$$

and by hypothesis dw/dI_f resembles $-w^2$ for large values of w. In other words, if I_f has a given value, the corresponding value of w is the value when $z = I_f$ of a solution $w(z)$ of the differential equation ·11,

$$(dw/dz)^2 = R_f(w),$$

which is such that w is large and dw/dz resembles $-w^2$ for small values of z.

If in ·11 we write for a moment $w = 1/y$, this equation becomes

·108 $$(dy/dz)^2 = (1-f_g^2 y^2)(1-f_h^2 y^2).$$

This equation has two particular solutions for which $y = 0$ when $z = 0$. The initial values of dy/dz for these two solutions are 1 and -1, and therefore the only solutions which vanish at the origin are one that is expansible near the origin by the power series

$$z+a_2 z^2+a_3 z^3+\dots$$

and one† that is expansible near the origin by a power series

$$-z+a_2' z^2+a_3' z^3+\dots.$$

It follows that the only solutions of ·11 which are large at the origin are one that is expressible near the origin in the form

$$1/(z+a_2 z^2+a_3 z^3+\dots),$$

and therefore, since $1/(1+a_2 z+a_3 z^2+\dots)$ can be converted into $1+b_0 z+b_1 z^2+\dots$, in the form

$$z^{-1}+b_0+b_1 z+\dots,$$

† Obviously the second of these solutions is the negative of the first, but this relation is so irrelevant to the argument that it is hardly worth while to use it to shorten the algebra.

and one that by the same argument is expressible near the origin in the form

$$-z^{-1}+b_0'+b_1'z+\dots.$$

For the first of these, but not for the second, dw/dz resembles $-z^{-2}$ and therefore resembles $-w^2$: the equation ·11 possesses one and only one solution which is such that for small values of z, w is large and dw/dz resembles $-w^2$. Since fj z satisfies the equation and has these characteristic properties, the unique solution is identified as the known function fj z, and it follows that if the relation ·105 is satisfied, then $w = \text{fj}\, I_f$:

5·13. *If z is the value of the integral*

$$\int_w^\infty \frac{dw}{\sqrt{R_f(w)}}$$

along any path, then $w = \text{fj}\, z$.

Combining ·12 and ·13 we have a fundamental theorem:

5·14. *When the multiplicity of values due to a possible variation of path is taken into account, the relation*

$$\int_w^\infty \frac{dw}{\sqrt{\{(w^2-f_g^2)(w^2-f_h^2)\}}} = z$$

is equivalent to the relation $w = \text{fj}\, z$, provided that the radical in the integrand resembles w^2 towards infinity along the path of integration.

5·2. As special cases of ·12 we have the three theorems collected in the following enunciation:

5·21. *There are curves in the w plane, from 0, from f_g, and from f_h, to infinity, such that if these are taken for the paths of integration, then*

$$\int_0^\infty \frac{dw}{\sqrt{R_f(w)}} = \omega_f, \qquad \int_{f_g}^\infty \frac{dw}{\sqrt{R_f(w)}} = \omega_g, \qquad \int_{f_h}^\infty \frac{dw}{\sqrt{R_f(w)}} = \omega_h,$$

the radical in the integrand in each case resembling w^2 towards infinity along the path.

From ·14 we have the more complete results:

5·22. *The aggregates of values of the integrals*

$$\int_0^\infty \frac{dw}{\sqrt{R_f(w)}}, \qquad \int_{f_g}^\infty \frac{dw}{\sqrt{R_f(w)}}, \qquad \int_{f_h}^\infty \frac{dw}{\sqrt{R_f(w)}},$$

if the radical in each integral resembles w^2 towards infinity along the path, are respectively

$$(2m+1)\omega_g+(2n+1)\omega_h,$$

$$(2m+1)\omega_g+2n\omega_h \quad \textit{with } m+n \textit{ even},$$

$$2m\omega_g+(2n+1)\omega_h \quad \textit{with } m+n \textit{ even}.$$

The integral of $-1/\sqrt{R_f(w)}$ along any path from w to ∞ is the negative of the integral of $1/\sqrt{R_f(w)}$ along the path from $-w$ to ∞ obtained by reflection in the origin. The aggregate of values of the integral from 0 to ∞ is therefore unaltered if the condition on the radical is reversed; in fact the aggregate $-(2k+1)\omega_g-(2l+1)\omega_h$ is converted into the aggregate $(2m+1)\omega_g+(2n+1)\omega_h$ by the substitution of m, n for $-(k+1)$, $-(l+1)$. More simply, in describing the aggregate of values of the integral from 0 to ∞ we may omit any specification of the radical. But in the integrals from f_g and f_h the specification of the radical is essential; the aggregate $-(2k+1)\omega_g-2l\omega_h$ is expressed in the form $(2m+1)\omega_g+2n\omega_h$ by the substitution of m, n for $-(k+1)$, $-l$, and if $k+l$ is even, $m+n$ is odd. That is, the aggregates $(2m+1)\omega_g+2n\omega_h$, $2m\omega_g+(2n+1)\omega_h$ with $m+n$ odd consist of the values of the integrals from $-f_g$, $-f_h$ to ∞, with the radical subject to the familiar convention.

If p and q are whole numbers, $p\omega_g+q\omega_h$ is an integral of $1/\sqrt{R_f(w)}$ from 0 to ∞ if p and q are both odd, from f_g or $-f_g$ to ∞ if p is odd and q is even, from f_h or $-f_h$ to ∞ if p is even and q is odd. If p and q are both even, $p\omega_g+q\omega_h$ is a typical pole of fj z. Hence

5·23. *The aggregate of values of the integral of $1/\sqrt{R_f(w)}$ along a path which comes from and returns to infinity is $2m\omega_g+2n\omega_h$.*

Reflection in the origin does not alter the nature of the path, and therefore no specification of the radical need be included.

5·3. The relation $w = \text{fj}\, z$ is a particular solution of the differential equation ·11; the general solution is $w = \text{fj}(\delta \pm z)$, and we have an elementary function again as a solution if the constant δ has one of the values ω_f, ω_g, ω_h. If the equation is taken in the form ·103, the general integral becomes

$$z = \pm \int_k^w \frac{dw}{\sqrt{R_f(w)}},$$

where in effect it is the fixed limit of integration that is the constant of integration of the differential equation; the integrand is unaltered.

With $\delta = \omega_f$, the elementary function involved is jf z, and since this function vanishes with z, the fixed limit of integration is zero. The value of $\mathrm{jf}'z$ at the origin is $-f_g f_h$, and the value of the radical to be selected at any point of the path of integration is determined if the initial value is selected. For an integration from zero it seems natural to take the factors of $R_f(w)$ in the form $f_g^2-w^2$, $f_h^2-w^2$. We need not repeat the details of the argument developed at length for the function $w = \mathrm{fj}\,z$; no transformation of the dependent variable in the equation ·11 is now necessary:

5·31. *If the value of the radical involved is $f_g f_h$ at the origin, the relation*

$$\int_0^w \frac{dw}{\sqrt{\{(f_g^2-w^2)(f_h^2-w^2)\}}} = z$$

is equivalent to the relation

$$w = -\mathrm{jf}\,z.$$

As a matter of elementary calculus we can convert the integral in one of the theorems ·14, ·31 into the integral in the other by substituting $f_g f_h/w$ for w:

5·32. *If $w_1 w_2 = f_g f_h$, then*

$$\int_0^{w_1} \frac{dw}{\sqrt{\{(f_g^2-w^2)(f_h^2-w^2)\}}} = \int_{w_2}^{\infty} \frac{dw}{\sqrt{\{(w^2-f_g^2)(w^2-f_h^2)\}}}.$$

In virtue of ·14 and ·31, the functional theorem

$$\mathrm{jf}\,z\,\mathrm{fj}\,z = -f_g f_h$$

is fundamentally this relation between integrals expressed in a different language.

With the function $\mathrm{fj}(\omega_g-z)$, which is $-\mathrm{hg}\,z$, the fixed limit of integration is f_g, and the point f_g in the w plane is a branchpoint of the radical in the integrand. We have therefore no means of specifying the radical by a universal rule applicable to an arbitrary path from f_g, though we must necessarily specify it in some particular way along any proposed path before the integral can have a meaning. With a chosen path in the z plane from 0 to z, and the corresponding path in the w plane determined by the transformation $w = -\mathrm{hg}\,z$, the value of dz/dw along the w path is either $1/\sqrt{R_f(w)}$ or $-1/\sqrt{R_f(w)}$, supposing $\sqrt{R_f(w)}$ to be a value of the radical specified for the path. Hence if J is the value of the integral

$$\int_{f_g}^{w} \frac{dw}{\sqrt{R_f(w)}},$$

z has one of the two values $\pm J$, or in other words, J has one of the two values $\pm z$. Since $\operatorname{hg}(-z) = \operatorname{hg} z$, we have $w = -\operatorname{hg} J$ in either case:

5·33. *If* $w = -\operatorname{hg} z$, *there is a path of integration such that for an appropriate selection of the radical*

$$\int_{f_g}^{w} \frac{dw}{\sqrt{R_f(w)}} = z.$$

At first glance, the deferred selection of the radical is a restriction on the possibility of discovering a path, but this is not really true.

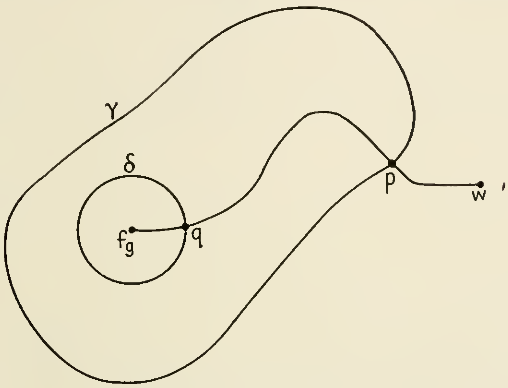


FIG. 19.

With a selected radical, let J be the value of the integral along a path from f_g to w, and let p be a point of the path such that the arc $f_g p$ is simple. Let γ be a circuit through p which surrounds the point f_g and the whole of the arc $f_g p$, but has none of the points $-f_g$, $\pm f_h$ in its interior. Let q be a point of the arc $f_g p$ so near to f_g that the circle δ through q with f_g for centre is entirely inside the region surrounded by γ. Then the arc qp and the circuit γ form with the circle δ and the arc qp the complete boundary of a region throughout which $1/\sqrt{R_f(w)}$ is regular. It follows that the integral of $1/\sqrt{R_f(w)}$ has the same value K along the path $f_g qp + \gamma + pw$ as along the path $f_g q + \delta + qpw$, if the integrand has the same values along the initial arc $f_g q$ in the two cases. On the second path, the value of the radical at q is changed into its negative by the description of the circle δ, and therefore the value of the integrand at every point of the path qpw in this second integral is the negative of the value at that point in the original integral along the path $f_g qpw$: the contribution of the path qpw to the value K is the negative of the contribution of the same path to the value J. Hence

$$\cdot 301 \qquad K - \left(\int_{f_g}^{q} + \int_{\delta}\right)\frac{dw}{\sqrt{R_f(w)}} = -\left\{J - \int_{f_g}^{q}\frac{dw}{\sqrt{R_f(w)}}\right\};$$

that is to say, $K = -J + L$ where

$$\cdot 302 \qquad L = \left(2\int_{f_g}^{q} + \int_{\delta}\right)\frac{dw}{\sqrt{R_f(w)}}.$$

Now K, as defined by means of the circuit γ, is independent of q, as also is J. Hence L is in fact independent of the position of q on the arc $f_g p$, and can be evaluated as the limit when q tends to f_g. The substitution $w - f_g = t^2$ renders the integrand finite while the transformed paths still tend to disappear. Hence $L = 0$ and $K = -J$.

5·34. *If there is a path from f_g to w along which the integral*

$$\int_{f_g}^{w}\frac{dw}{\sqrt{R_f(w)}}$$

has a value z, there is also a path from f_g to w, coincident with the first from f_g to a point p distinct from f_g, along which the integral has the value $-z$, although the radical in the integrand has the same value in the two integrals at any point of the common arc $f_g p$.

Once a second path has been found, it can be deformed out of all obvious relationship to the first.

It is now clear that not only is it impossible to discriminate naturally between the two values of the radical $\sqrt{R_f(w)}$ in the neighbourhood of the point f_g, but no artificial discrimination would restrict the values which the integral from f_g to a variable point w can assume. In associating the function hg z with an integral it is in fact unnecessary to pay any attention to the ambiguity of the radical involved. This difference between hg z and fj z or jf z is seen equally well from the standpoint of the differential equation

$$\cdot 303 \qquad (dw/dz)^2 = (w^2 - f_g^2)(w^2 - f_h^2).$$

As an equation of the first order this equation has only a finite number of solutions with a given initial value of w; the number depends entirely on the number of initial values of dw/dz available, and if ambiguity disappears from the first derivative, it is not in any sense transmitted to a derivative of higher order. We have in fact from ·303

·304 $$d^2w/dz^2 = 2w^3-(f_g^2+f_h^2)w,$$

an equation which has one and only one solution with given initial values of w and dw/dz; this solution of ·304 is a solution of ·303 if and only if the initial values of w and dw/dz satisfy ·303. If the initial value of w is f_g, the initial value of dw/dz, to satisfy ·303, is necessarily zero. Thus there is one and only one solution of ·303 with initial value f_g, and this unique solution we know to be $w = -\mathrm{hg}\, z$:

5·35. *If* $$\int_{f_g}^{w} \frac{dw}{\sqrt{R_f(w)}} = z,$$

then $w = -\mathrm{hg}\, z$, *whichever choice is made of the radical in the integrand.*

Combining ·35 with ·33 we have

5·36. *If no restriction is placed on the path of integration, the relation*

$$\int_{f_g}^{w} \frac{dw}{\sqrt{\{(w^2-f_g^2)(w^2-f_h^2)\}}} = z$$

is equivalent to the relation $w = -\mathrm{hg}\, z$.

Formally this theorem does not include ·34, but ·34 is certainly essential to a real grasp of ·36.

5·4. The theory of elliptic functions had its origin in problems of integration. Legendre made an exhaustive study of integrals involving the square root of a polynomial of the fourth degree, and in particular of the integral

$$\int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$$

and integrals closely allied to it, k being for him a real parameter between 0 and 1. Making in ·31 the substitution $w = f_g x$, we have the direct relation between an elliptic function and an integral of Legendre's standard form:

5·41. *The relation*

$$u = \int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$$

is equivalent to the relation

$$f_g x = -\mathrm{jf}(u/f_h)$$

if $k = f_g/f_h$ *and if the value of the radical in the integrand is* 1 *at the origin.*

Or, since

·401 $$\int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} = \int_{1/x}^{\infty} \frac{dx}{\sqrt{\{(x^2-1)(x^2-k^2)\}}},$$

we may connect Legendre's integral with the function which we have treated as fundamental:

5·42. *The relation*

$$u = \int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$$

is equivalent to the relation

$$f_h/x = \mathrm{fj}(u/f_h)$$

if $k = f_g/f_h$ *and if the value of the radical in the integrand at the origin is* 1.

Historically the elliptic functions were discovered when Legendre's relation

·402 $$u = \int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$$

was taken to express x as a function of u, and it follows from ·14 that the elementary functions which we have studied could in a sense be defined by the inversion of the integrals, whatever the values of f_g and f_h. Whether the fundamental integral is taken in the symmetrical form which appears in ·14 or in the form standardized by Legendre is not a matter of principle.

But it is to be observed that in the relation

·403 $$\int_w^{\infty} \frac{dw}{\sqrt{\{(w^2-f_g^2)(w^2-f_h^2)\}}} = z$$

the constants f_g, f_h are already derived from the function fj$z$: they are the numbers fjω_g, fjω_h. The whole of the theory which identifies the integral relation ·403 with the functional relation $w = \mathrm{fj}\,z$ rests on the particular association of the numbers f_g, f_h with the function fj$z$. It follows that unless the function fj$z$ is already known, the integral in ·403 is itself unrecognizable and definition by means of this integral means nothing. In other words, although we are justified by ·14 in asserting that there exists an integral of the form

$$\int_w^\infty \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}}$$

by means of which the function $\mathrm{fj}\,z$ could be defined, in order to identify the necessary integral we must know the constants $\mathrm{fj}\,\omega_g$, $\mathrm{fj}\,\omega_h$. In fact, if we are to use the relation

·404 $$\int_w^\infty \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}} = z$$

as the fundamental relation between w and z or specifically as a definition of w as a function of z, we must regard the constants b, c as given parameters, leaving the parts which they play in the theory of the function $w(z)$ to be discovered; we can not assign these parts in advance while professing ignorance of the nature of the function. Whether we think of ·402 with Legendre as defining u as a function of x or with Jacobi as defining x as a function of u, we think of k in the first instance as an arbitrary constant, not as a parameter whose value is determined by a part played in a theory already developed. Even if the functions obtained by inversion of the integrals are the elliptic functions with which we are already acquainted, their discovery from the integrals is not just a formal alternative to their definition in terms of a lattice. The problem of the inversion of the elliptic integral requires the determination of the lattice if it exists, not merely a proof of its existence.

However approached, the problem of inversion presents difficulties of a higher order than those of sheer manipulation. Nevertheless, it should be explained, if not solved, in any account of elliptic functions, not so much for its historical interest as for its practical importance. It is the integrals to which the knowledge of the functions was due which operate to bring the functions into many branches of analysis and geometry, to say nothing of applied mathematics, and we cut away from the theory of elliptic functions all its applications if we can not pass from integral to function.

5·5. For the standard form of the integral to be considered we shall use

$$\int_w^\infty \frac{dw}{\sqrt{R(w)}}$$

where $$R(w) = (w^2-b^2)(w^2-c^2)$$

and it is understood that the radical resembles w^2 towards infinity along the path of integration. To emphasize that an integral, either directly or inversely, is the basis of discussion, we shall denote the integral by I. We have then a relation between I and w expressed initially in the form

·501 $$I \equiv I(w) = \int_w^\infty \frac{dw}{\sqrt{R(w)}},$$

and the problem with which we are concerned is the nature of the function $w(I)$ defined by this relation if I is taken as the independent variable.

There are two ways of attempting to identify the function $w(I)$ with a function fj I. We may look for a suitable lattice without considering the function $w(I)$ itself, or we may investigate the properties of the function $w(I)$ with a view to establishing that $w(I)$ must be an elliptic function. In the first method, the argument is to be concluded by an appeal to ·14, a suitable lattice meaning indeed a lattice which renders ·14 applicable to the integral. In the second method, we anticipate that the proof that the function has the essential property of periodicity involves a determination of periods.

The first method involves the solution, which for practical purposes must be explicit, of the pair of equations

·502 $$\mathrm{fj}(\omega_g; -\omega_g-\omega_h, \omega_g, \omega_h) = b, \qquad \mathrm{fj}(\omega_h; -\omega_g-\omega_h, \omega_g, \omega_h) = c,$$

or rather, since it is b^2, c^2 that are given, of the pair of equations

·503 $$\mathrm{fj}^2(\omega_g; -\omega_g-\omega_h, \omega_g, \omega_h) = b^2, \qquad \mathrm{fj}^2(\omega_h; -\omega_g-\omega_h, \omega_g, \omega_h) = c^2,$$

as simultaneous equations in ω_g and ω_h. Supposing ω_g, ω_h to be the pair of halfperiods from which the function is constructed, we have from the definition of $\wp z$,

·504 $$\wp\omega_f = \frac{1}{(\omega_g+\omega_h)^2} + \sum_{m,n}{}' \left\{\frac{1}{\{(2m+1)\omega_g+(2n+1)\omega_h\}^2} - \frac{1}{\{2m\omega_g+2n\omega_h\}^2}\right\},$$

·505 $$\wp\omega_g = \frac{1}{\omega_g^2} + \sum_{m,n}{}' \left\{\frac{1}{\{(2m+1)\omega_g+2n\omega_h\}^2} - \frac{1}{(2m\omega_g+2n\omega_h)^2}\right\},$$

·506 $$\wp\omega_h = \frac{1}{\omega_h^2} + \sum_{m,n}{}' \left\{\frac{1}{\{2m\omega_g+(2n+1)\omega_h\}^2} - \frac{1}{(2m\omega_g+2n\omega_h)^2}\right\},$$

and therefore the equations ·503 are explicitly

$$\cdot 507 \quad \sum_{m,n}\left\{\frac{1}{\{(2m+1)\omega_g+2n\omega_h\}^2}-\frac{1}{\{(2m+1)\omega_g+(2n+1)\omega_h\}^2}\right\}=b^2,$$

$$\cdot 508 \quad \sum_{m,n}\left\{\frac{1}{\{2m\omega_g+(2n+1)\omega_h\}^2}-\frac{1}{\{(2m+1)\omega_g+(2n+1)\omega_h\}^2}\right\}=c^2,$$

where the term for which $m=0$, $n=0$ is now included in the summation.

There is nothing in the form of ·507–·508 to suggest that a solution is always possible. The functions f_g, f_h are homogeneous functions of ω_g and ω_h, and the equation

$$\cdot 509 \qquad \frac{\mathrm{fj}(\omega_g;\,-\omega_g-\omega_h,\omega_g,\omega_h)}{\mathrm{fj}(\omega_h;\,-\omega_g-\omega_h,\omega_g,\omega_h)}=\frac{b}{c}$$

is an equation in the single variable ω_g/ω_h. If ω_g, ω_h are any two values satisfying ·509, and if

$$\lambda=b^{-1}\,\mathrm{fj}(\omega_g;\,-\omega_g-\omega_h,\omega_g,\omega_h),$$

then $\lambda\omega_g$, $\lambda\omega_h$ satisfy $f_g=b$, $f_h=c$. Thus, functionally speaking, the distinction between the pair of equations ·507–·508 and the one equation ·509 is trivial. But again it can not be said that from the form of the two series in ·507–·508 their ratio is obviously susceptible of an arbitrary value; the result is true, but it is in establishing it that the difficulty of this attack on the inversion problem lies.

It may seem at first glance that the solution of the pair of equations $f_g=b$, $f_h=c$ is implicit in ·21 or ·22. The integrals $I(b)$, $I(c)$, that is,

$$\int_b^\infty\frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}},\qquad \int_c^\infty\frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}},$$

determine, not indeed two definite numbers, if the paths of integration are unspecified, but two definite aggregates. Does not the existence of these aggregates prove the existence of a lattice, and is not the detection of a primitive pair of periods a problem likely to demand only some quite elementary technique? On this question the first comment to be made is that we have not proved, except in the case in which b, c are derived from a lattice, that the aggregates of values of the integrals $I(b)$, $I(c)$ are connected in any simple way with a lattice; the direct investigation of the multiplicity of values of these integrals is as much part of the process now proposed for the solution of the problem of inversion as it is part of the process which depends entirely on the study of the integral $I(w)$. But this is no point of principle; the difficulty comes later. Suppose that we have found particular values β, γ of the two integrals $I(b)$, $I(c)$ which we are satisfied form a primitive pair in relation to the aggregates of values. We can form a function $\mathrm{fj}(z; -\beta-\gamma, \beta, \gamma)$ on the lattice determined by β and γ, and this function has determinate values f_β, f_γ for the values β, γ of z. Have we any reason to assert that f_β, f_γ are equal to the constants b, c? We can not hope to answer this question by inserting β, γ as values of ω_g, ω_h into the series in ·507–·508. The alternative is to appeal to the relation between β, γ and f_β, f_γ in the form ·21: *There are paths of integration from f_β, f_γ to ∞ on the one hand and from b, c to ∞ on the other hand such that*

·510 $$\int_{f_\beta}^{\infty} \frac{dw}{\sqrt{\{(w^2-f_\beta^2)(w^2-f_\gamma^2)\}}} = \int_{b}^{\infty} \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}},$$

·511 $$\int_{f_\gamma}^{\infty} \frac{dw}{\sqrt{\{(w^2-f_\beta^2)(w^2-f_\gamma^2)\}}} = \int_{c}^{\infty} \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}}.$$

Unless we can prove that these conditions alone are sufficient to identify f_β, f_γ with b, c, the theorem which directs us to the only lattices in which b, c could play the required parts supplies us with no reason for concluding that b, c actually play these parts.

It is the second method of attacking the inversion problem, namely, the study of the functional character of the relation defined by the integral, that we shall pursue. Although we have proved the identity of the functional relation $w = \mathrm{fj}\, z$ with the integral relation

$$\int_{w}^{\infty} \frac{dw}{\sqrt{R_f(w)}} = z,$$

we have said nothing to explain it, that is, to show how the form of the integrand imposes on the aggregate of values of which the integral becomes susceptible when the path is arbitrary a quality corresponding to double periodicity in the inverse function. Without this explanation, ·14 remains unintelligible, and with it, since the origin of the constants f_g, f_h is irrelevant for the purpose and we can deal throughout with the integral $I(w)$, we are taking one step towards the solution of the larger problem along the proposed lines.

To avoid misapprehension, it should be said as clearly as possible that there is no difficulty intrinsic in the notion of inverting an integral to provide a function. On the contrary, a functional relation between two variables, whatever its formal expression, can not be one-sided. To say that the relation

·512 $$I = \int_{w}^{\infty} \frac{dw}{\sqrt{R(w)}}$$

can be regarded as defining w as a function of I is logically a platitude, if mathematically it was a revolutionary discovery. We can go farther: this relation, from its form, implies the existence of dI/dw, and therefore the existence, except possibly at certain discoverable points in the w plane, of dw/dI; we can safely say that w is, generally speaking, regular in the sense that if w_0 corresponds to I_0, then $w-w_0$ is expressible for sufficiently small values of $I-I_0$ as a power series in $I-I_0$. To put the matter differently, ·512 is equivalent to

$$dw/dI = \sqrt{R(w)},$$

or in rational form to

·513 $$(dw/dI)^2 = R(w),$$

coupled with boundary conditions, and the existence of solutions of differential equations is guaranteed by a mass of general theory.

It is true that integrals in a complex plane require paths of integration for their precise determination, but this is not a potential complication of the function $w(I)$. To suppose the path in ·512 arbitrary is to admit that to an assigned value of w corresponds an aggregate of possible values of I, but this remark, read in the reverse direction, says only that the function $w(I)$ may have a common value for a number of distinct values of the argument I.

After this digression the fundamental difficulty in the study of the inverted integral will not be misunderstood. Although we can define $w(I)$ by ·512, this formula gives us no clue to the range of values of I for which the function $w(I)$ exists. In the construction of the Weierstrassian function $\wp z$ and of the functions which we have defined in terms of $\wp z$, an arbitrary value can be given to z; the functions exist over the whole of the z plane. But there is nothing whatever in the form of the relation ·512 to justify us in taking for granted that if we equate the integral I to an arbitrary complex number, there necessarily exist a limit and a path from which the integral acquires the assigned value; the domain of existence of the function $w(I)$ defined by ·512 may well fall short of the complete I plane, and there is no obvious means of finding the extent of this domain of existence. We are no better off if we replace the integral by a differential equation. The function $w(I)$ is a particular solution of the equation ·513, identified by its character near the origin. All that we learn from the theory of differential equations is that there is some circle round the origin throughout which this function exists, and that if the function can be continued analytically across the circumference of this circle it does not cease to satisfy the equation. If the continuation is held up by a line of singularities, the particular solution with which we are concerned exists only in a restricted domain: there are values of I which can not serve as arguments to the function $w(I)$.

Here is the drawback to the classical use† of the integral as the basis of the theory. We can prove by elementary methods that the function $w(I)$ is regular where it exists and that it is doubly periodic where it exists. But these properties are entirely consistent with the possibility that the domain throughout which the function exists is some complicated pattern of perforated shreds and patches, and to dispose of this possibility is a mathematical problem sufficiently serious to be deferred as long as progress is made without its solution. Only, as we have said, however much we learn about elliptic functions before solving the problem of inversion, we can not learn when and how to use them.

5·6. The course of the next three chapters follows the account we have given of the problem to be investigated. In Chapter VI we examine the dependence of the integral $I(w)$ on the path of integration, and deduce that the function $w(I)$ is doubly periodic. In Chapter VII we prove first that any point near which a branch of the function exists is either an ordinary point or a pole of that branch, and next that there are no finite values of I near which the function does not exist; we infer that $w(I)$ is meromorphic. Of the existence theorem, which as we have explained is at the heart of the problem of inversion, two proofs are given. The first proof derives $w(I)$ from $I(w)$ and depends on propositions in the theory of aggregates; these propositions are assumed to be known. As an appendix to this proof an argument is given in which the

† It must not be thought that the original introduction of the elliptic functions was wildly illogical: Abel and Jacobi were not blind to fallacies that to us are glaring. But at first only real variables were involved; to reverse the functional relation when the limit and the integral are both real requires little more than the determination of ranges throughout which the integral is a monotonic function of the limit, and these ranges, by Rolle's theorem, are bounded by zeros and infinities of the integrand. The difficulty of the inversion problem as well as the beauty of the lattice theory belongs essentially to the domain of the complex variable.

multiplicative axiom is used, for this is the argument which is most easily invented; the axiom may be invalid, but its use supplies the clue to the construction of the proof which dispenses with it. The second proof of the existence theorem takes $w(I)$ as a particular solution of a differential equation and shows that there can be no upper limit to the radius of the circle round the origin within which this solution is a meromorphic function of I. This proof depends on analytical formulae peculiar to the function under consideration; one feels that it is artificial, that no recollection of it is likely to be helpful in any problem except the one for which it was invented, but undeniably it is the easier of the two proofs to understand, if the harder to reconstruct. In Chapter VIII the various threads are gathered together, and the solution of the inversion problem is complete.

VI

THE AGGREGATE OF VALUES OF AN ELLIPTIC INTEGRAL

6·1. The subject of this chapter is the integral

$$I(w) \equiv \int_w^\infty \frac{dw}{\sqrt{R(w)}},$$

where $R(w)$ denotes $(w^2-b^2)(w^2-c^2)$ and the radical is a continuous function asymptotic to w^2 towards the end of the path of integration. For a given value of w the value of the integral depends to some extent on the path of integration, and it is the nature of this dependence that we are to investigate. If $I_1(w_*)$, $I_2(w_*)$ are different values of the integral, corresponding to different paths from the same point w_* to ∞, then when we look at the relation between I and w from the other side, I_1 and I_2 are different arguments for which the function $w(I)$ has the same value w_*.

The integral $I(w)$ is elementary if b or c is zero, or if $b^2 = c^2$; we therefore assume that none of these conditions is satisfied, that is, that in the w plane the four points b, c, $-b$, $-c$ are all distinct.

We find that discussion of the multiplicity of values of $I(w)$ for an arbitrary value of w can be made to depend on evaluation of the integral along a path which comes from and returns to infinity; we first find a canonical shape into which such a path can be deformed without alteration in the value of the integral, and we then evaluate the integral in terms of the necessary constants, which are only two in number. Returning to the integral $I(w)$, we describe the aggregate of values of the integral associated with one and the same lower limit w by means of these constants, which are themselves values of $I(b)$ and $I(c)$ and which we now find to be quarterperiods of the inverse function $w(I)$. In the last section of the chapter we prove that the ratio of a value of $I(b)$ to a value of $I(c)$ can not be real.

The integral $I(w)$ is regular at infinity, and the branchpoints b, c, $-b$, $-c$ of $\sqrt{R(w)}$ are its only singularities. We assume throughout that no path of integration passes through a branchpoint. This assumption is wanted not to keep our integrals finite but to keep them unambiguous. Near b, for example, $R(w)$ resembles a multiple of $w-b$; the integrals

$$\int_{w_1}^{w_2} \frac{dw}{\sqrt{(w-b)}}, \quad \int_{w_1}^{w_2} \frac{dw}{\sqrt{R(w)}}$$

remain significant and finite if one of the limits tends to b, and there is no reason why b should not be inserted as a limit. But if we have a path of integration $w_1 b w_2$, each value of $\sqrt{R(w)}$ tends to zero as w tends to b in any way, and it is not possible to specify the function to be integrated along bw_2 by saying that it is continuous at b with the function integrated along $w_1 b$. If a path passes through branchpoints, the integrand requires a separate specification on each section of the path, and this is a complication seldom worth incurring.

6·2. If p, q are points on a path of integration, the arc between p and q can be replaced by another arc joining these points if the two arcs together form the boundary of a simply connected region which includes no singularities of the integrand. Hence the relation between the integrals $I_1(w)$, $I_2(w)$ from the same point w to ∞ along two different paths W_1, W_2 is bound up with topographical relations between the two paths W_1, W_2 and the four points b, c, $-b$, $-c$.

A preliminary reduction of the analytical problem simplifies the topographical problem. The two paths W_1, W_2 together form a path S which comes from and returns to infinity. Let $\phi(w)$ denote temporarily the function such that $\{\phi(w)\}^2 = R(w)$, that $\phi(w) \sim w^2$ towards infinity along W_1, and that $\phi(w)$ is a continuous function of w along S; since W_1 and W_2 avoid the branchpoints, so also does S, and $\phi(w)$ has a definite value at each point of S. Integrating along S in the same direction as along W_1, we have

$$\cdot 201 \qquad \int_S \frac{dw}{\phi(w)} = \int_{W_1} \frac{dw}{\phi(w)} - \int_{W_2} \frac{dw}{\phi(w)},$$

since the direction of integration along W_2 is opposite to the direction of integration along S. Now the integral along W_1 in ·201 is $I_1(w)$ as already defined. But whether $\phi(w)$, which resembles w^2 towards one end of S, resembles w^2 or resembles $-w^2$ towards the other end of S, depends on topographical relations of S to the branchpoints; in the former case the integral along W_2 in ·201 is $I_2(w)$, but in the latter $I_2(w)$ is the integral along W_2 of $-\phi(w)$ and the last integral in ·201 is $-I_2(w)$. Thus defining J_S as

$$\int_S \frac{dw}{\phi(w)},$$

we have J_S equal in the one case to $I_1(w)-I_2(w)$, in the other case to $I_1(w)+I_2(w)$. In other words

6·21. *If the paths* W_1, W_2 *together form a path* S, *and if the radical* $\sqrt{R(w)}$ *is the same along the part of* S *which coincides with* W_1 *as along* W_1, *then if* J_S *is the integral of* $1/\sqrt{R(w)}$ *along* S, *the integral* $I_2(w)$ *is equal to* $I_1(w)-J_S$ *or to* $J_S-I_1(w)$ *according as variation of* w *along* S *from one end to the other restores or reverses the asymptotic resemblance of* $\sqrt{R(w)}$ *to* w^2.

This theorem reduces the discussion of the multiplicity of values of $I(w)$ at the accessible point w to the discussion of the variation of $\sqrt{R(w)}$ and the integration of $1/\sqrt{R(w)}$ along a path which comes from and returns to infinity, and our next task is to express suitably the topographical relations of such a path to the four branchpoints; as we accomplish this, we can see that the evaluation of the integral J_S will follow naturally.

6·3. To describe the relevant topographical relations between the four fixed points b, c, $-b$, $-c$ and a variable path S which comes from and returns to infinity, we suppose paths B, C, B', C' to be drawn to infinity from the four points; the four fixed paths are subject to the conditions that B', C' are the reflections of B, C in the origin, that

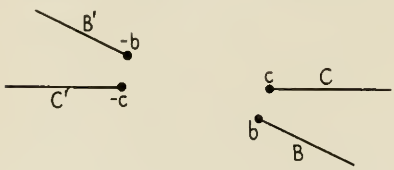


FIG. 20_1.

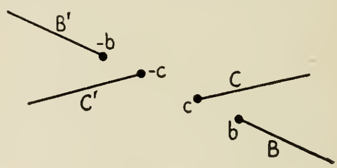


FIG. 20_2.

none of the paths have multiple points, that no two of them have any points in common, and that any sufficiently large circle with its centre at the origin cuts each path in only one point. These paths are drawn once for all, and we call them the critical paths.

There is no difficulty in finding a set of critical paths. If the points b, c are not in line with the origin, that is, if b/c is not real, we may take for B, C the half-lines which prolong beyond b, c the radii to these points from the origin. In the excepted case, if b is the more distant of the points b, c from the origin, we may take B as before; C can not now lie along the line joining the origin to c, but may be any half-line from c which does not lie along this line. Half-line paths are geometrically the simplest, but it is not at all necessary that the critical paths should be of this form, and to stipulate half-line paths is to invest with significance details that are accidental.

The purpose of introducing a circle round the origin is easily seen. If $|w| > \max(|b|, |c|)$, the product of the binomial series representing the values of the two square roots

$$\left(1-\frac{b^2}{w^2}\right)^{-\frac{1}{2}}, \qquad \left(1-\frac{c^2}{w^2}\right)^{-\frac{1}{2}}$$

which tend to 1 as $w \to \infty$ is a convergent series

$$1+\frac{a_1}{w^2}+\frac{a_2}{w^4}+\dots,$$

and at any point the integrand $1/\sqrt{R(w)}$ has one of the two values $A(w)$, $-A(w)$, where

·301 $$A(w) = \frac{1}{w^2}\left(1+\frac{a_1}{w^2}+\frac{a_2}{w^4}+\dots\right).$$

The two functions $A(w)$, $-A(w)$ are distinct functions throughout the region of convergence of the series in the definition of $A(w)$, and the integrand is specified unambiguously if it is identified with one of these two functions. To say that on a path to infinity the radical $\sqrt{R(w)}$ is ultimately to resemble w^2, or to write $\sqrt{R(w)} \sim w^2$, is only a way of expressing that beyond the last intersection of the path with the circle $|w| = \max(|b|, |c|)$ the function denoted by $\sqrt{R(w)}$ is $1/A(w)$. Outside the circle of convergence we can identify $\sqrt{R(w)}$ with one or other of the functions $1/A(w)$, $-1/A(w)$ at an isolated point or along a path which does not extend to infinity; only the assertion of identity can not then be expressed in the form $\sqrt{R(w)} \sim w^2$ or $\sqrt{R(w)} \sim -w^2$ without violence to the strict use of the asymptotic symbol. If a path crosses into the circle $|w| = \max(|b|, |c|)$ from outside, the identification of $1/\sqrt{R(w)}$ with one of the two functions $A(w)$, $-A(w)$ is interrupted, and if the path recrosses and identification with one of the two functions again becomes possible, there is no reason why the same function should serve a second term; it is only if the representation of the integrand with the help of the series is uninterrupted that it is impossible for one of the functions $A(w)$, $-A(w)$ to give way to the other.

If a path pq is entirely outside the circle $|w| = \max(|b|, |c|)$, then

·302 $$\int_p^q A(w)\,dw = G(p)-G(q),$$

where

·303 $$G(w) = \frac{1}{w}\left(1+\frac{a_1}{3w^2}+\frac{a_2}{5w^4}+\dots\right).$$

Two conclusions can be drawn. Firstly, the value of the integral depends only on the endpoints; in other words, any two paths from p to q are reconcilable if neither of them penetrates the circle of convergence. Secondly, if $|p| \geqslant \rho$, $|q| \geqslant \rho$, then

$$\left|\int\limits_p^q A(w)\,dw\right| < \frac{2+\epsilon}{\rho},$$

where $\epsilon \to 0$ as $\rho \to \infty$: if p, q can be taken upon or outside a circle of arbitrarily large radius, the integral from p to q along a path which does not penetrate the circle $|w| = \max(|b|, |c|)$ is then negligible. These conclusions apply also to the integrand $-A(w)$, and apply therefore to the integrand $1/\sqrt{R(w)}$ which necessarily either coincides with $A(w)$ along the whole path or coincides with $-A(w)$ along the whole path:

6·31. *If the path of integration pq lies wholly outside the circle $|w| = \max(|b|, |c|)$, the value of the integral*

$$\int\limits_p^q \frac{dw}{\sqrt{R(w)}}$$

is independent of the path, and if also $|p| \geqslant \rho$, $|q| \geqslant \rho$, then the absolute value of the integral is less than $(2+\epsilon)/\rho$, where $\epsilon \to 0$ as $\rho \to \infty$.

Given a circle Γ whose centre is the origin and whose circumference cuts each of the critical paths in one and only one point, any path S which comes from and returns to infinity can be deformed, without passage across a branchpoint, into a path T such that every multiple point of T and every intersection of T with a critical path is outside Γ. The path T may cross and recross the circle Γ any number of times; if T enters the circle at $p_1, p_2, \dots, p_n$ and leaves it at $q_1, q_2, \dots, q_n$, we have T expressed as

$$\infty p_1 + p_1 q_1 + q_1 p_2 + p_2 q_2 + \dots + q_{n-1} p_n + p_n q_n + q_n \infty,$$

where each of the portions

$$\infty p_1,\ q_1 p_2, \dots,\ q_{n-1} p_n,\ q_n \infty$$

is wholly outside the circle, and each of the portions

$$p_1 q_1,\ p_2 q_2, \dots,\ p_n q_n$$

is wholly inside the circle and is a path without multiple points which joins one point of the circumference to another without cutting a critical path.

Before considering the integration, let us examine a little more closely the forms of path inside the circle Γ. A simple path σ joining two points p, q of the circumference divides the interior into two regions Σ_1, Σ_2, and if σ does not cut any of the critical paths, the part of each critical path inside Γ lies wholly in one of these regions. We say that σ is impeded on the side of a region Σ_1 or Σ_2 by the critical paths which are partly in that region, or, less exactly, by the branchpoints from which those paths are drawn. Three cases are distinguishable: (i) One region contains no branchpoints and the other contains four; the path is unimpeded on one side, impeded on the other side by the four critical paths. (ii) The path is impeded on one side by a single critical path, on the other side by three critical paths. (iii) The path is impeded on each side by two critical paths.

We can recognize these three cases in other ways. The critical paths cut Γ in four points f, g, f', g'; the two points p, q divide the circumference into two circular arcs, and each of these forms with σ the

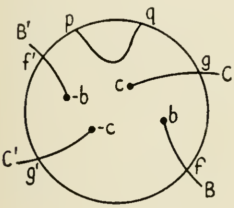


FIG. 21_1.

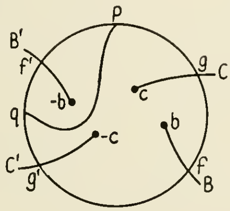


FIG. 21_2.

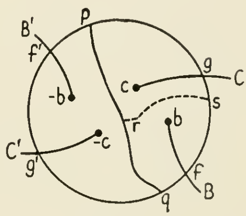


FIG. 21_3.

boundary of one of the regions Σ_1, Σ_2. A region contains part of the critical path B if the circular arc which forms part of the boundary of the region includes the point f. (i) The path σ is unimpeded on one side and impeded on the other side by the four critical paths if one of the two circular arcs pq includes none of the points f, g, f', g' and the other includes them all, that is, if the two points p, q are in the same one of the four circular arcs fg, gf', $f'g'$, $g'f$. (ii) The path σ is impeded on one side by one critical path and on the other side by three critical paths if one of the two circular arcs pq includes one of the points f, g f', g' and the other includes three, that is, if the two points p, q are in adjacent arcs of the set fg, gf', $f'g'$, $g'f$. (iii) The path σ is impeded on each side by two critical paths if each of the circular arcs pq includes two of the four points f, g, f', g', that is, if p and q are on diametrically opposite arcs of the set fg, $f'g$, $f'g'$, $g'f$.

The classification of loops inside the circle Γ is applicable to the arcs inside Γ of the path T by which the original path of integration S has been replaced. (i) If the T-arc $p_m q_m$ is unimpeded on one side, the circular arc $p_m q_m$ on that side of the T-arc forms with the T-arc the boundary of a simply connected region containing none of the branch-points, and the integral has the same value along this circular arc as along the T-arc. To replace the T-arc from p_m to q_m by an arc of Γ joining the same two points is to remove $p_m q_m$ from the second set of T-arcs, and to replace the two arcs $q_{m-1} p_m$, $q_m p_{m+1}$ in the first set by one arc $q_{m-1} p_m q_m p_{m+1}$, that is, $q_{m-1} p_{m+1}$: the form of the two sets is unchanged, and the unimpeded arc is eliminated from the second set. (iii) If the T-arc $p_m q_m$ divides the interior of the circle into two regions each of which contains parts of two critical paths, we can divide one of these regions, by a curve joining a point r in $p_m q_m$ to a point s in the circumference of Γ, into two regions each of which contains a part of one critical path and no part of any other; the arcs of the set fg, gf', $f'g'$, $g'f$ to which p_m and q_m belong are diametrically opposite, and s must be taken on one of the other two arcs. The construction is illustrated in Figure 21_3. Integration along the T-arc $p_m q_m$ is then equivalent to integration along $p_m rs$, srq_m in succession, and on account of the construction each of these paths is impeded by one critical path only. The number of arcs in the second set is increased by one, and for convenience an evanescent arc ss may be added to the first set.

Briefly, a path of type (i) can be ignored, and a path of type (iii) can be replaced by two paths of type (ii):

6·32. *Given a circle of sufficiently large radius with the origin for centre, integration of* $1/\sqrt{R(w)}$ *along a path* S *which comes from and returns to infinity is equivalent to integration along a succession of paths*

$$\infty p_1,\ p_1 q_1,\ q_1 p_2,\ p_2 q_2, \dots,\ q_{n-1} p_n,\ p_n q_n,\ q_n \infty$$

in which each of the paths

$$\infty p_1,\ q_1 p_2, \dots\ q_{n-1} p_n,\ q_n \infty$$

is outside the circle, reducing possibly to a single point, and each of the paths

$$p_1 q_1,\ p_2 q_2, \dots\ p_n q_n$$

is a simple loop inside the circle and is impeded by one only of the four critical paths.

6·4. There is now only one type of path inside the circle Γ to be taken into account, and we proceed to investigate the integral along a path of this type. We take a path pq which joins a point p in one of the two arcs fg, fg' to a point q in the other of these two arcs, and is therefore impeded by B alone. The integrand has everywhere one of the values of $1/\sqrt{R(w)}$, and for the sake of definiteness we specify the value at q; we select, in the notation already adopted, the value $A(q)$, the function $A(w)$ being the sum of a series

$$\frac{1}{w^2}+\frac{a_1}{w^4}+\frac{a_2}{w^6}+\dots$$

which is convergent if $|w| > \max(|b|, |c|)$.

The path pq is deformable into a path which begins with the circular arc pf, then follows the critical path B from f to a point t between

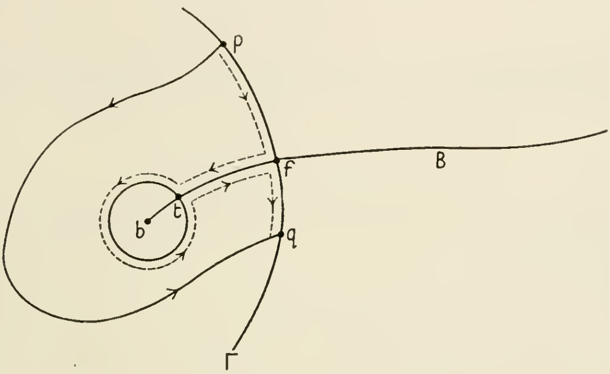


FIG. 22.

The paths of integration, in this figure and in Fig. 23, are the boundary lines themselves, not curves vaguely 'just inside' the boundaries. The dotted lines are merely guides to the actual paths.

f and b, describes a complete circuit γ round b, returns from t to f along B, and finishes with the circular arc fq. As we have said in $\cdot 1$, we may take t as near to b as we wish; near b the dominant part of $1/\sqrt{R(w)}$ is one of the branches of $1/\sqrt{\{2b(b^2-c^2)(w-b)\}}$, and the integral of this function along a path inside a circle of radius τ round b tends to zero with τ, notwithstanding the infinity of the integrand. But however small the circuit γ may be, the passage round this circuit multiplies $\sqrt{(w-b)}$ by -1, and in the return from t to f the integrand $1/\sqrt{R(w)}$ has at each point of B the negative of its value there during the approach from f to t. This change has two consequences.

Firstly, the value of the integrand at f when f is the end of the path tf and the beginning of the path fq is the negative of the value of the integrand at f when f is the beginning of the path ft and the end of the path pf. But on account of the choice at q, the integrand along the concluding arc fq is the function $A(w)$, whose value at f is $A(f)$. Hence the value at f of the function integrated along pf is $-A(f)$, and the function is $-A(w)$:

·401. *Because the value of the integrand at q is $A(q)$, therefore the value of the integrand at p is $-A(p)$.*

Secondly, the multiplication of the integrand by -1 cancels the effect of the change in the direction of integration along B, and the first integral, from f to t, is equal to the second integral, from t to f: we can write

·402 $$\int_p^q = \int_p^f + \int_\gamma + 2\int_t^f + \int_f^q .$$

Since the integral along B is convergent at b, we can replace the integral from t to f by the difference between two integrals from b, and we have

$$\int_p^q = \int_p^f + \left(\int_\gamma - 2\int_b^t\right) + 2\int_b^f + \int_f^q .$$

Since t does not occur, implicitly or explicitly, outside the bracketed terms, the difference

$$\int_\gamma - 2\int_b^t$$

has a value independent of t, and since each term tends to zero with t, this value is zero, whence more simply

·403 $$\int_p^q = \int_p^f + 2\int_b^f + \int_f^q .$$

This formula does not require the circle Γ to be in any sense 'large': for example, if the critical paths are radial, Γ may have any radius greater than $\max(|b|, |c|)$. If however we do anticipate applications in which integrals along paths that are not inside the circle are to be disregarded, we see that the significant part of the integral from p to q comes entirely from the integral along the critical path B. We can go farther. The integral from b to f still involves Γ, but if we replace this integral by the difference between two integrals to ∞, one of these integrals is independent of Γ, and the other has its path outside Γ. We write therefore

·404 $$\int_b^\infty \frac{dw}{\sqrt{R(w)}} = \beta,$$

the path of integration being the critical path B and the radical in the integrand being asymptotic to w^2 towards infinity along the path; β is

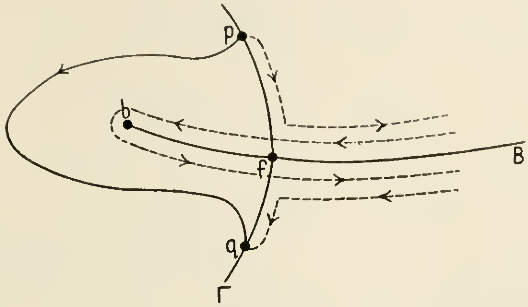


FIG. 23.

a constant, a value of $I(b)$, and is independent altogether of the path pq. We have now, since the integrand at f is $A(f)$,

$$\int_b^f \frac{dw}{\sqrt{R(w)}} = \beta - \int_f^\infty A(w)\,dw,$$

and substituting in ·403,

$$\cdot 405 \quad \int_p^q \frac{dw}{\sqrt{R(w)}} = 2\beta + \int_p^f \{-A(w)\}\,dw - 2\int_f^\infty A(w)\,dw + \int_f^q A(w)\,dw.$$

Expressing this result more symmetrically, in a form which suggests Figure 23, and incorporating ·401, which is vital to the result, we have the fundamental theorem:

6·41. *A circle* Γ *round the origin as centre cuts each of the critical paths in one point only; pq is a loop joining one point of* Γ *to another, lying wholly inside* Γ, *and impeded only by the critical path B. If the value of the integrand* $1/\sqrt{R(w)}$ *at q is* $A(q)$, *then the value of the integrand at p is* $-A(p)$, *and*

$$\int_p^q \frac{dw}{\sqrt{R(w)}} = 2\beta - \left(\int_p^f + \int_f^\infty\right) A(w)\,dw + \left(\int_\infty^f + \int_f^q\right) A(w)\,dw,$$

where
$$\beta = \int_b^\infty \frac{dw}{\sqrt{R(w)}},$$

and f is the point in which the circle Γ *cuts B. The path of integration for* β *is the critical path B, and* $\sqrt{R(w)}$ *resembles* w^2 *towards* ∞ *in this* *integral; the paths of integration pf, fq are arcs of the circle* Γ, *and the path of integration between f and* ∞ *lies along B.*

We can evaluate the integrals along the circumference and outside the circle Γ as in ·302, and we have explicitly

·406
$$\int_p^q \frac{dw}{\sqrt{R(w)}} = 2\beta - G(p) - G(q),$$

where as before $$G(w) = \frac{1}{w} + \frac{a_1}{3w^3} + \frac{a_2}{5w^5} + \dots.$$

Very serviceable is the descriptive theorem, which indeed was foreseen in the proof of ·41:

·407. *If pq is a loop inside* Γ, *impeded only by the critical path B, the value of*

$$\int_p^q \frac{dw}{\sqrt{R(w)}}$$

along the loop differs from 2β *by a sum of integrals along paths wholly outside* Γ, *if the integrand has the value* $A(q)$ *at q.*

To replace the integrand in ·41 at one point by its negative implies replacing it by its negative throughout, and as we do not change the meanings of β and $A(w)$ we have to change signs throughout the formula:

6·42. *If, with the notation of* ·41, *the value of the integrand at q is* $-A(q)$, *then the value of the integrand at p is* $A(p)$, *and*

$$\int_p^q \frac{dw}{\sqrt{R(w)}} = -2\beta + \left(\int_{pf\infty} - \int_{\infty fq} \right) A(w)\, dw.$$

An enunciation similar to that of ·407 is of course possible.

In ·41 the circular arcs on which p and q are situated are not specified more precisely than that one of them is fg and the other is fg'. To reverse the situations is in effect to interchange the allocations of the symbols C, C', and since these allocations do not enter in any way into the argument, the value of the integral is not altered. This does not mean that we have an integration in which the direction in which the path is described is immaterial. In ·41 we can not interchange p and q without altering the *value* of the integrand at any point, for to retain the value $A(q)$ is to alter automatically the *rule* by which the integrand is selected. If we write q', p' for p, q, with the condition that the value of the integrand at p' is $A(p')$, it is ·42 that is relevant if p' is to be the lower limit of integration, and we have

$$\int_{p'}^{q'} \frac{dw}{\sqrt{R(w)}} = -2\beta + \left(\int_{p'f\infty} - \int_{\infty f q'} \right) A(w)\, dw,$$

that is,

$$\int_{q}^{p} \frac{dw}{\sqrt{R(w)}} = -2\beta + \left(\int_{pf\infty} - \int_{\infty f q} \right) A(w)\, dw,$$

in agreement with the formula in ·41, since now the path is reversed and the integrand is unchanged.

To change the critical path by which a loop is impeded is to make only a formal change in ·41, replacing the integrals along B and the point of intersection f by integrals along one of the other critical paths and the point in which Γ cuts that path. We write

·408 $$\int_{c}^{\infty} \frac{dw}{\sqrt{R(w)}} = \gamma,$$

the path of integration being the critical path C and the radical resembling w^2 towards ∞ along the path; γ is a value of $I(c)$. The paths B', C' do not introduce new constants, for if W is any path from a point w to ∞, and W' is the reflection of W in the origin, the integrand $1/\sqrt{R(w)}$ has the same value at corresponding points of W and W' if its asymptotic form is the same on the two paths; since the element of one path is the negative of the element of the other, the integral from $-w$ to ∞ along W' is the negative of the integral from w to ∞ along W. In particular

·409 $$\int_{-b}^{\infty} \frac{dw}{\sqrt{R(w)}} = -\beta, \qquad \int_{-c}^{\infty} \frac{dw}{\sqrt{R(w)}} = -\gamma,$$

if the paths of integration are B', C' and if $\sqrt{R(w)} \sim w^2$ towards ∞ on each path.

6·5. We can now resume the evaluation of J_S, the integral, along a path S which comes from and returns to infinity, of the continuous function $1/\sqrt{R(w)}$ which resembles $1/w^2$ towards the end of S. Having drawn a circle Γ which cuts each of the critical paths once only, we have S deformed into a succession of paths

$$\infty\, p_1,\; p_1 q_1,\; q_1 p_2,\; p_2 q_2, \ldots,\; q_{n-1} p_n,\; p_n q_n,\; q_n \infty.$$

The form of a path outside Γ is irrelevant, and we may suppose each of the paths $q_1p_2, q_2p_3, \ldots, q_{n-1}p_n$ to be an arc of Γ; it is not necessary to lay down a rule by which to make the choice between the two arcs of Γ joining q_{m-1} to p_m, since integrals along these two arcs are in any case equal. Each of the paths $p_1q_1, p_2q_2, \ldots, p_nq_n$ is a simple loop inside Γ impeded by a single critical path.

By hypothesis, the integrand along $q_n\infty$ can be identified with the function $A(w)$: hence the value of the integrand at q_n is $A(q_n)$, and it follows from ·41 that the value of the integrand at p_n is $-A(p_n)$; hence the integrand along $q_{n-1}p_n$ is the function $-A(w)$, the value of the integrand at q_{n-1} is $-A(q_{n-1})$, and by ·42 the value of the integrand at p_{n-1} is $A(p_{n-1})$, whence the integrand along $q_{n-2}p_{n-1}$ is the function $A(w)$:

6·51. *The function integrated along the paths*

$$q_n\infty, q_{n-1}p_n, q_{n-2}p_{n-1}, \ldots, q_1p_2, \infty p_1$$

outside the circle Γ *is alternately* $A(w)$ *and* $-A(w)$; *in particular, the integrand at the beginning of the path resembles* $1/w^2$ *or* $-1/w^2$ *according as the number of loops inside the circle is even or odd.*

Knowing now the terminal values of the integrand on each loop, we can apply ·41 or ·406 or a corresponding theorem with a change of sign. The result to be anticipated is perhaps clear if we first apply the descriptive theorem ·407. If the integral, from the branchpoint to infinity, along the critical path which impedes the loop p_mq_m, has the value λ_m, a constant independent of the path S, it follows from ·51 and ·407 that

·501. *The integral* J_S *differs from*

$$2\lambda_n - 2\lambda_{n-1} + 2\lambda_{n-2} - \ldots \pm 2\lambda_2 \mp 2\lambda_1$$

by a finite number of integrals along paths which do not penetrate the circle Γ.

Since the values of J_S and of $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ do not involve the radius ρ of the circle Γ, and the values of the integrals outside Γ are negligible if ρ is sufficiently large, we are tempted to say that the difference between J_S and $2\lambda_n - 2\lambda_{n-1} + \ldots \mp 2\lambda_1$ is arbitrarily small and is therefore zero. But the argument is not quite as simple as this, for the original deformation of S depends on the choice of the circle Γ, and we have no reason to assert that with a different circle the deformation would have led to the same set of impeding paths arranged in the same order. To rescue the argument we must make a deformation accommodated to the larger circle from the path as we now have it, and we must use the precise results of ·41 and ·42.

We take then a circle Γ' round the origin, with radius ρ' greater than ρ. Let the critical path which impedes the loop $p_m q_m$ cut Γ in f_m and cut Γ' in f'_m; the initial and final arcs $\infty\, p_1$ and $q_n \infty$ of S may be deformed in any manner, subject to the conditions of lying outside Γ, and they may therefore be assumed to cut Γ' only once, in points p'_1 and q'_n. In ·41 we have the difference

$$\int_p^q \frac{dw}{\sqrt{R(w)}} - 2\beta$$

expressed as
$$-\left(\int_{pf\infty} - \int_{\infty fq}\right) A(w)\, dw,$$

or, as we may say for brevity, as $-pf\infty + \infty fq$. In this form, with the integrand $A(w)$ throughout and with the initial sign $-$ or $+$ according as n is even or odd,

$$\begin{aligned} J_S &- (\mp 2\lambda_1 \pm 2\lambda_2 \mp \ldots - 2\lambda_{n-1} + 2\lambda_n) \\ &= \pm\infty\, p_1 \pm (p_1 f_1 \infty - \infty\, f_1 q_1) \mp q_1 p_2 \mp (p_2 f_2 \infty - \infty\, f_2 q_2) \pm q_2 p_3 \pm \cdots \\ &\quad \cdots + q_{n-2} p_{n-1} + (p_{n-1} f_{n-1} \infty - \infty\, f_{n-1} q_{n-1}) - q_{n-1} p_n - \\ &\qquad - (p_n f_n \infty - \infty\, f_n q_n) + q_n \infty \\ &= \pm\infty\, p_1 f_1 \infty \mp \infty\, f_1 f_2 \infty \pm \ldots - \infty\, f_{n-1} f_n \infty + \infty\, f_n q_n \infty \\ &= \pm\infty\, p'_1 p_1 f_1 f'_1 \infty \mp \infty\, f'_1 f_1 f_2 f'_2 \infty \pm \ldots \\ &\quad \ldots - \infty\, f'_{n-1} f_{n-1} f_n f'_n \infty + \infty\, f'_n f_n q_n q'_n \infty, \end{aligned}$$

since $p'_1, f'_1, f'_2, \ldots, f'_n, q'_n$ are points in the paths $p_1\infty, f_1\infty, f_2\infty, \ldots, f_n\infty, q_n\infty$. But the paths $p'_1 p_1 f_1 f'_1, f'_1 f_1 f_2 f'_2, \ldots, f'_{n-1} f_{n-1} f_n f'_n, f'_n f_n q_n q'_n$ and the circular arcs $p'_1 f'_1, f'_1 f'_2, \ldots, f'_{n-1} f'_n, f'_n q'_n$ are all outside the circle $|w| = \max(|b|, |c|)$, and therefore the arcs of the circle Γ' may be substituted for the three-sided paths which include arcs of the circle Γ, and we have

$$\cdot 502 \qquad \begin{aligned} J_S &- (\mp 2\lambda_1 \pm 2\lambda_2 \mp \ldots - 2\lambda_{n-1} + 2\lambda_n) \\ &= \pm\infty\, p'_1 f'_1 \infty \mp \infty\, f'_1 f'_2 \infty \pm \ldots - \infty\, f'_{n-1} f'_n \infty + \infty\, f'_n q'_n \infty. \end{aligned}$$

Since the left-hand side does not involve the radius ρ', the value of the right-hand side is independent of ρ', and since the right-hand side consists of not more than $n+1$ integrals each of which tends to zero as $\rho' \to \infty$, the constant value of the right-hand side for all values of ρ' not less than ρ is zero, and the value of the left-hand side is zero:

6·52. *If a circle* Γ *whose centre is the origin cuts each of the four critical paths in one point only, and if the path* S *which comes from and returns to infinity is deformable into a path of which the portions inside* Γ *are a succession of loops* $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n$ *each of which is impeded by one and only one critical path, then* J_S, *the integral of* $1/\sqrt{R(w)}$ *along* S, *is given by*

$$J_S = 2\lambda_n - 2\lambda_{n-1} + \ldots \pm 2\lambda_2 \mp 2\lambda_1,$$

where λ_m *is the integral of* $1/\sqrt{R(w)}$, *from the branchpoint to infinity, along the critical path which impedes* σ_m, *provided that the radical in every integral is asymptotic to* w^2 *towards infinity in the direction of integration.*

Strictly speaking it is superfluous to specify the asymptotic form of the radical in this theorem, for if w^2 is replaced throughout by $-w^2$, each term is replaced by its negative and the formula remains valid.

We should perhaps remark that what we have shown in the course of the proof of ·52 is not that any deformation of S which is adapted to the circle Γ' must resemble closely a deformation which is adapted to the circle Γ, but that there must exist one deformation with the appropriate degree of resemblance. The value of the integral J_S is perfectly definite, but the steps of its evaluation offer infinite variety.

The proof of ·52 suggested by ·407 is instructive, but the result is established much more easily by the actual evaluation, by means of the function $G(w)$, of the integral along each of the paths which together compose the path into which S has been deformed. Allowing for the alternation in integrand, we have, by ·302 and ·406,

$$\int_{q_n}^{\infty} \frac{dw}{\sqrt{R(w)}} = G(q_n), \qquad \int_{p_n}^{q_n} \frac{dw}{\sqrt{R(w)}} = 2\lambda_n - G(p_n) - G(q_n),$$

$$\int_{q_{n-1}}^{p_n} \frac{dw}{\sqrt{R(w)}} = -G(q_{n-1}) + G(p_n),$$

$$\int_{p_{n-1}}^{q_{n-1}} \frac{dw}{\sqrt{R(w)}} = -2\lambda_{n-1} + G(p_{n-1}) + G(q_{n-1}),$$

. . . .

$$\int_{q_1}^{p_2} \frac{dw}{\sqrt{R(w)}} = \mp G(q_1) \pm G(p_2), \qquad \int_{p_1}^{q_1} \frac{dw}{\sqrt{R(w)}} = \mp 2\lambda_1 \pm G(p_1) \pm G(q_1),$$

$$\int_{\infty}^{p_1} \frac{dw}{\sqrt{R(w)}} = \mp G(p_1),$$

and addition recovers at once the formula for J_S.

The simplest case of ·52 is that in which only one loop occurs. The path S comes from infinity to a point on a circle whose radius is greater than $\max(|b|, |c|)$, forms inside this circle a loop impeded by only one of the critical paths, and returns to infinity; the value of the integral along S is then 2λ, where λ is the value of the integral along the complete critical path; the integrand, which resembles $1/w^2$ towards the end of S, resembles $-1/w^2$ towards the beginning of S.

Since each of the terms $2\lambda_n$, $2\lambda_{n-1}, \ldots$, $2\lambda_1$ in ·52 can be recognized as the value of the integral along a simple infinite loop, we may express ·52 by saying that

6·53. *An arbitrary path which comes from and returns to infinity is equivalent to a succession of infinite loops each of which is impeded by one and only one of the critical paths.*

But if we break the geometrical continuity of the path we must add explicitly that the asymptotic form of the integrand is the same towards the beginning of each loop as towards the end of its predecessor, for this relation is no longer secured automatically. Nevertheless the language of ·53 is convenient.

The three cases of ·52 in which only two loops L_1, L_2 are required call for separate comment. Taking the loop L_2 to be impeded by B, the loop L_1 may be impeded by B, by B', or by one of the other two paths.

If both loops are impeded by B, then $\lambda_1 = \lambda_2 = \beta$, and $J_S = 0$. Essentially this result is implicit in the discussion, after ·42 above, of the interchange of the arcs on which the endpoints p, q are situated. If we return along a path without altering the integrand at any point, we naturally annihilate the integral. But if the integral from p to q with the integrand specified as $A(p)$ has substantially the same value as the integral from q to p with the integrand specified as $A(q)$, a repetition of the path from p to q is ultimately equivalent to a return from q to p along the path that has already been followed.

The repeated loop is derived from a continuous path

$$\infty\, p+\sigma+qq'f'p'p+\sigma+q\,\infty,$$

where σ is a loop pq inside the circle Γ, impeded by B, and $q'f'p'$ is an arc of a larger circle Γ' which cuts B in f'. The repeated infinite loop is the limiting form in which p' and q' have tended to infinity along $p\,\infty$ and $q\,\infty$. The path $p'p+\sigma+qq'$ is described twice, once with the integrand whose value at p is $-A(p)$ and once with the integrand whose value at p is $A(p)$. These integrals from p' to q' cancel out, and there remains only the sum

$$\left(\int_{\infty}^{p'} - \int_{q'}^{p'} + \int_{q'}^{\infty}\right) A(w)\, dw,$$

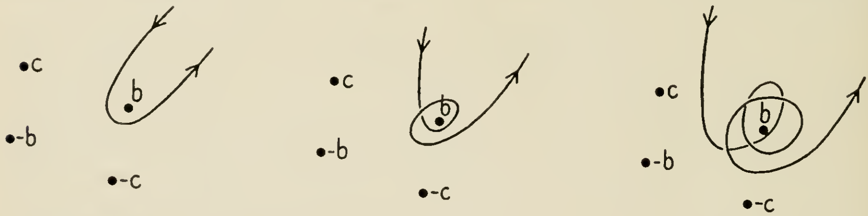
which is identically zero, since the value of $A(w)$ at any point is independent of the path of integration. When we have proved that the integral along the path

$$\infty\, p+\sigma+qfp+\sigma+q\, \infty$$

is zero, the result can be extended by the usual methods to any path into which this can be deformed. For example, instead of elongating the circuit $\sigma+qfp$ into an infinite loop we can shrink it to a coil, as small as we please, round the branchpoint b; if $|b| < |c|$, this coil may be wholly inside the region of divergence of the series which defines $A(w)$:

6·54. *The value of the integral* $\int dw/\sqrt{R(w)}$ *along any path which comes from infinity, describes a complete coil round one of the branchpoints, and returns unimpeded to infinity, is zero.*

This result shows that we do not multiply the value of an integral along a single loop by repeating the loop. In the formation of a succession of simple loops from a path S, the same loop may occur k times

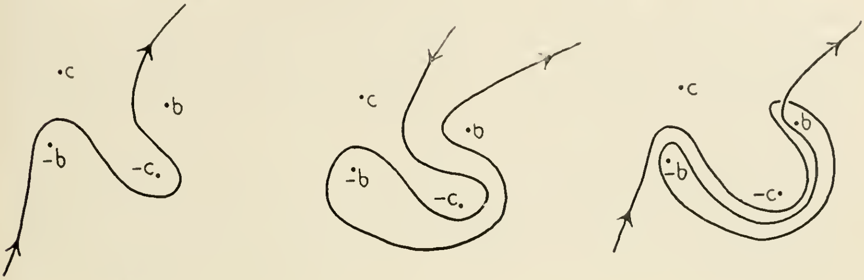


$J_S = 2\beta$ $J_S = 0$ $J_S = 2\beta$

FIG. 24.

consecutively. If k is even, the repeated loop makes no contribution to the value of the integral, and the integrand has the same value at the end of the last loop as at the beginning of the first loop: the set of loops has no effect, direct or indirect, on the integral, and may be ignored altogether in the evaluation. If k is odd, the set of loops makes the same contribution as its first member, both to the value of the integral and to the variation of the integrand. In other words, although the deformation of a path S may lead to a succession of loops $k_1 L_1$, $k_2 L_2, \ldots, k_m L_m$, in which $k_1, k_2, \ldots, k_m$ are any whole numbers, the most general form for evaluation is $L_1, L_2, \ldots, L_l$ in which consecutive loops are not impeded by the same branchpoint, and the value of the integral takes the corresponding form $2\lambda_l - 2\lambda_{l-1} + \ldots \mp 2\lambda_1$ in which consecutive terms are not formally† equal.

If L_1 is impeded by B' and L_2 by B, the path S is equivalent to a path which has the two points b, $-b$ on one side and the two points



$J_S = 4\beta$ $J_S = 6\beta$ $J_S = 8\beta$

FIG. 25.

c, $-c$ on the other side; the value of λ_1 is $-\beta$, and $J_S = 4\beta$. Herein lies the possibility of multiplying to any desired extent the value of the integral. The integrand has the same asymptotic form towards the end of the path as towards the beginning, and if instead of allowing the path to proceed to infinity we cast coils round b and $-b$ alternately, each coil adds 2β to the value of the integral. If k is any whole number, we can express $2k\beta$ as $2\beta - (-2\beta) + 2\beta - (-2\beta) + \ldots$, to k terms, and the casting of coils round the two points alternately translates this identity. In this construction $2k\beta$ is essentially a positive multiple of 2β; to obtain a path along which the integral has the value $-2k\beta$, we cast coils round b and $-b$ alternately as before, but ending with a coil round $-b$.

For the third case of a path equivalent to two loops, let L_1 be impeded by C' and L_2 by B. The value of the integral is $2\beta + 2\gamma$, and we must take L_1 to approach between $-b$ and $-c$ and to recede between $-c$ and b, and L_2 to approach between $-c$ and b and to recede between b and c. The pair of loops is therefore equivalent to a path which comes from infinity between B' and C' and returns to infinity between B and C, that is, which separates b and $-c$ from c and $-b$. The integral in this case can be expressed in another form, for we may take for the path a path passing through the origin and symmetrical with respect to the origin. If we denote the half of this path from the

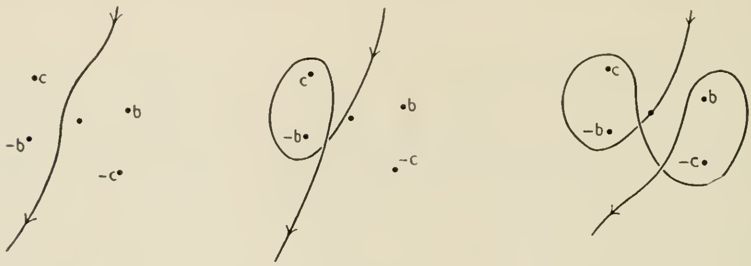
† We have not yet proved that β can not be accidentally equal to γ or to $-\gamma$; this is, however, true, as we shall see in ·8 below.

origin to infinity between B' and C' by A, and the other half by A', and if the value of the integral from 0 to ∞ along A is α, the integrals from ∞ to 0 along A and from 0 to ∞ along A' both have the value $-\alpha$. Hence if the asymptotic form of the integrand is the same along A' as along B, we have $2\beta+2\gamma = -2\alpha$, that is,

·503 $$\alpha+\beta+\gamma = 0.$$

The asymptotic form of the integrand is the same towards each end of the composite path AA'. It follows that if the path is repeated again and again in one direction, the value of the integral is multiplied. We obtain a continuous path equivalent to a repetition of AA' by casting a coil round the two points $-b$, c or round the two points b, $-c$, and we can find in this way a path to give to the integral the value $2k\alpha$ where k is any assigned whole number, positive or negative.

The integral along a path which comes from infinity between B' and



$J_S = 2\alpha$ $J_S = 4\alpha$ $J_S = 6\alpha$

FIG. 26.

C and returns to infinity between B and C', thus separating b and c from $-b$ and $-c$, has the value $2\beta-2\gamma$, and any positive multiple of this value can be obtained by the insertion of coils cast round b and c or round $-b$ and $-c$. For negative multiples of $2\beta-2\gamma$ the direction of integration must be reversed.

Since β, $-\gamma$ are integrals from b, $-c$ to ∞, it is to be expected that $\beta+\gamma$ is an integral from b to $-c$, as well as an integral from 0 to ∞. In fact the substitution
$$\frac{t^2}{c^2} = \frac{w^2-b^2}{w^2-c^2}$$
implies
$$\frac{(dt)^2}{(t^2-b^2)(t^2-c^2)} = \frac{(dw)^2}{(w^2-b^2)(w^2-c^2)},$$
and $t = 0$, ∞ correspond to $w = b, -c$. But to discuss the relations between a path from b to $-c$ and a path from 0 to ∞ would take us a long way from our present subject, since $t = 0$ corresponds to $w = -b$

as well as to $w = b$, and $t = \infty$ corresponds to $w = c$ as well as to $w = -c$.

In the general expression $2\lambda_t - 2\lambda_{t-1} + \ldots \mp 2\lambda_1$ for the value of the integral J_S, each λ has one of the four values $\pm\beta$, $\pm\gamma$. The value of J_S is therefore of the form $2m\beta + 2n\gamma$, where m, n are integers, not necessarily positive. Conversely, if m, n are integers, $2m\beta + 2n\gamma$ can be expressed, in an infinite number of ways, in the form $2\lambda_t - 2\lambda_{t-1} + \ldots \mp 2\lambda_1$, and since any succession of loops inside a circle can be joined by arcs of the circle to form a continuous path, every sum of the form $2m\beta + 2n\gamma$ is the value of the integral J_S along some path or other. Thus

6·55. *The values of the integral* $\int dw/\sqrt{R(w)}$ *along paths which come from and return to infinity are the numbers of the form* $2m\beta + 2n\gamma$.

The aggregate of values is the same whether or not the asymptotic form of the radical is prescribed.

6·6. We can now complete the statement of the multiplicity of values of the integral $I(w)$. The integral J_S of ·21 is the integral evaluated in ·52. The integrand which is $A(w)$ towards the end of S is $A(w)$ or $-A(w)$ towards the beginning of S according as the number of terms in the expression for J_S is even or odd, and this number differs from the number $m+n$ in ·55 by an even number. Hence from ·21 and ·55,

6·61. *If I is one value of the integral $I(w)$ for a given value of w, the aggregate of values of the integral for that value of w consists of all the numbers of the form $2m\beta + 2n\gamma + I$ in which $m+n$ is even and all the numbers of the form $2m\beta + 2n\gamma - I$ in which $m+n$ is odd, m and n being whole numbers, positive zero or negative, and β, γ being the values of the integrals $I(b)$, $I(c)$ along paths subject to certain topographical conditions which are satisfied in particular if the paths are reconcilable with half-lines.*

From the present point of view no special significance attaches to rectilinear paths, but if for any purpose paths must be specified precisely, rectilinear paths are naturally the simplest to use.

6·7. Reversed to express properties of $w(I)$, ·61 combines the two relations

6·71 $$w(2m\beta + 2n\gamma + I) = w(I) \quad \text{if } m+n \text{ is even,}$$

6·72 $$w(2m\beta + 2n\gamma - I) = w(I) \quad \text{if } m+n \text{ is odd,}$$

with the theorem that

6·73. *All the solutions of the equation* $w(J) = w(I)$ *are of one of the two forms*

$$J = 2m\beta+2n\gamma+I \quad \text{with } m+n \text{ even},$$

$$J = 2m\beta+2n\gamma-I \quad \text{with } m+n \text{ odd}.$$

It is from ·71 that periodicity of the function $w(I)$ is to be inferred; for this purpose ·72 is irrelevant. Clearly 4β and 4γ are periods, for $m+n$ is even if m and n are both even: the function $w(I)$ is doubly periodic. But 4β and 4γ can not constitute a primitive pair of periods, for since $m+n$ is even if m and n are equal, ·71 implies that $2\beta+2\gamma$ is a period, and the parallelogram $(2\beta+2\gamma, 4\beta)$ has only half the area of the parallelogram $(4\beta, 4\gamma)$. Since the integral α satisfies

6·74 $$\alpha+\beta+\gamma = 0,$$

we can replace ·71 by

6·75 $$w(2h\alpha+4k\beta+I) = w(I)$$

with no restrictions on the integers h, k, and since from ·73 this formula gives all the values of Ω such that $w(\Omega+I) = w(I)$ for every value of I, it follows that the pair of periods 2α, 4β is primitive. Equally 2α, 4γ is a primitive pair, and, incorporating the definitions of α, β, γ as integrals, we have the theorem that

6·76. *The function* $w(I)$ *is a doubly periodic function, of which values of* $2I(0)$, $4I(b)$, $4I(c)$ *are periods; with suitable restrictions on the paths determining the integrals, the first of these periods constitutes with either of the others a primitive pair.*

If $m+n$ is odd, $m\beta+n\gamma$ has the form $h\alpha+(2k+1)\beta$; we can therefore express ·73 in the alternative form

6·77. *The aggregate of values of* J *satisfying the equation* $w(J) = w(I)$ *consists of the numbers congruent with* I *and the numbers congruent with* $2\beta-I$, *to moduli* 2α *and* 4β.

In general the congruences to which I and $2\beta-I$ belong are distinct, but they coincide if w has either of the four values $\pm b$, $\pm c$, when I is congruent with one of the four corresponding values $\pm\beta$, $\pm\gamma$.

Neither 0 nor ∞, as a value of w, creates an exception to ·77. If w is 0, one value of I is α, and the general value of I is the sum of odd multiples of β and γ. If w is ∞, one value of I is 0, and the general value of I is the sum of even multiples of β and γ; in fact the values of I for which w is ∞ are precisely the values of those integrals whose paths begin and end at infinity with which this chapter has been predominantly occupied.

6·8. A genuine doubly periodic function can not be built on two periods if the ratio of one period to the other is real. We can hardly doubt that for arbitrary complex values of b and c, the ratio of one of the integrals β, γ to the other is in general complex, and on investigation we are able to dispose of the possibility of exceptional cases, within the conditions imposed on b and c.

We take for the paths of integration the prolongations of the radii from the origin to b, c; the case in which b/c is real is therefore reserved for subsequent examination. We make the substitution

·801 $$w^2 = W,$$

and we have

6·81 $$2\beta = \int_{b^2}^{\infty} \frac{dW}{\sqrt{\{W(W-b^2)(W-c^2)\}}}, \quad 2\gamma = \int_{c^2}^{\infty} \frac{dW}{\sqrt{\{W(W-b^2)(W-c^2)\}}},$$

where the paths of integration are again the prolongations of radii from the origin. Since for the present purpose a change of sign throughout is immaterial, we need not attempt to specify the radical, but the integrand is of course continuous along each path.

In general the half-lines in the W plane from the origin through the

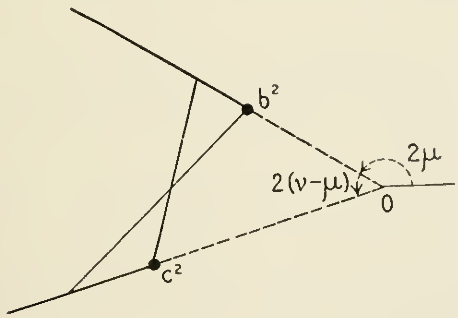


FIG. 27.

points b^2, c^2 are the two arms of a simple angle whose measure is between 0 and π. For definiteness we suppose that rotation through this angle from $0b^2$ to $0c^2$ is positive; interchange of the symbols b^2, c^2 does not affect the conclusion of the argument. We denote by 2μ, 2ν the angles of the complex numbers b^2, c^2 which are positive and less than 2π; then $2(\nu-\mu)$ is the angle of the sector determined, in the most elementary sense, by the half-lines along which the paths of integration lie. In the extreme case in which b^2/c^2 is real and negative, $2(\nu-\mu) = \pi$; this case can be admitted.

On the first path of integration, $dW/\sqrt{\{W(W-b^2)\}}$ is everywhere real, and the possible angles of the element of the integral at any point are the angles of the complex number $1/\sqrt{(W-c^2)}$. At any point on the path, $W-c^2$ has an angle not less than $2\nu-\pi$, the angle of the step from c^2 to the origin, and not greater than 2μ, the angle of the step from c^2 to the point at infinity on the path. Hence $1/\sqrt{(W-c^2)}$, and the element of the integral, can be taken to have an angle not greater than $-(\nu-\frac{1}{2}\pi)$ and not less than $-\mu$.

On the second path of integration, $dW/\sqrt{\{W(W-c^2)\}}$ is everywhere real, $W-b^2$ has an angle not less than 2ν and not greater than $2\mu+\pi$, and $1/\sqrt{(W-b^2)}$, and the element of the integral, can be taken to have an angle not greater than $-\nu$ and not less than $-(\mu+\frac{1}{2}\pi)$.

We appeal now to a simple lemma†:

6·82. *If at every point of a path of integration the element of integral* $f(z)\,dz$ *has its angle within an assigned range, then the angle of the integral* $\int f(z)\,dz$ *also is within that range.*

From this it follows that the angle of one of the complex numbers $\pm\beta$ is in the range from $-\mu$ to $-(\nu-\frac{1}{2}\pi)$ and the angle of one of the complex numbers $\pm\gamma$ is in the range from $-(\mu+\frac{1}{2}\pi)$ to $-\nu$. Since

$$-(\mu+\tfrac{1}{2}\pi) \leqslant -\nu < -\mu \leqslant -(\nu-\tfrac{1}{2}\pi) < -(\mu+\tfrac{1}{2}\pi)+\pi,$$

these ranges do not overlap and they are on the same side of one diameter; in other words, no angle which belongs to one of them either belongs to the other or differs by π from an angle belonging to the other:

6·83. *If the ratio of b to c is not real, and if the critical paths radiate from the origin, the ratio of β to γ is not real.*

If $2(\nu-\mu) = \pi$, that is, if the ratio of b^2 to c^2 is real and negative, one integral has the angle $-\nu$ and the other has the angle $-\mu$, and these, in this case, differ by $\frac{1}{2}\pi$:

6·84. *If the ratio of b to c is purely imaginary, so also is the ratio of β to γ, if the critical paths radiate from the origin.*

If the ratio of b to c is real, let $|b| > |c|$. The prolongation of the

† With the transformation $w = f(z)$, this is merely the theorem that if every tangent to an arc has its direction within a given angular range, the chord of the arc also has a direction within that range, an immediate corollary to the theorem that there is a tangent parallel to the chord.

radius from the origin in the w plane to c passes through b or $-b$ and can not be used as a critical path. But this prolongation indented at the point which must be avoided is reconcilable with a half-line from c to ∞, and in this form it can be used for the evaluation of a quarter-period γ: the indent is needed only to connect the radical on one side of the branchpoint with the radical on the other side. In the W plane, the ratio of b^2 to c^2 is real and positive. The path of integration for 2γ is the prolonged radius from c^2, and $\pm 2\gamma$ is the sum of an integral from c^2 to b^2 and an integral from b^2 to ∞. The second of these integrals is $\pm 2\beta$; to the first, which has therefore one of the four values $\pm 2\gamma \mp 2\beta$, is applicable the argument which establishes ·84, and the ratio of either $\gamma-\beta$ or $\gamma+\beta$ to β is a purely imaginary number which can not vanish:

6·85. *If b/c is a real number numerically greater than unity, and if the critical paths are rectilinear, then γ/β is a complex number whose real part is 1 or -1 and whose imaginary part is not zero.*

The steps $\gamma+\beta$, $\gamma-\beta$ are steps in the lattice built on β and γ, and therefore this lattice does contain steps at right angles to β; the difference between this case and that of ·84 is that γ is not itself one of these steps.

Since ·85 deals with the only case omitted from ·83, the conclusion of ·83 is general with respect to the values of b and c. Also we can remove the restriction on the paths. By ·61, the general values of the integrals $I(b)$, $I(c)$ are given in terms of particular values β, γ by

$$I(b) = (2m_1+1)\beta + 2n_1\gamma, \qquad I(c) = 2m_2\beta + (2n_2+1)\gamma.$$

Since the ratios $(2m_1+1)/2m_2$, $2n_1/(2n_2+1)$ can not be identical, a real ratio of any value of $I(b)$ to any value of $I(c)$ would imply a real ratio of β to γ. Hence

6·86. *If b^2, c^2 are different and neither of them is zero, the ratio of one of the integrals*

$$\int_b^\infty \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}}, \qquad \int_c^\infty \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}}$$

to the other can not be purely real, whatever the paths of integration.

VII

THE UBIQUITY OF THE FUNCTION INVERSE TO AN ELLIPTIC INTEGRAL

7·1. The sense in which the function $w(I)$ has been shown to be doubly periodic is as follows: If w_* is a value of w associated with a value I_* of I, then w_* is associated also with every value of I that is of the form $2h\alpha+4k\beta+I_*$. This does not imply that there are not other values of w associated with the same set of values of I, or that there are not values of I with which no value of w is associated; in other words, the property of double periodicity may belong to a manyvalued function or to a lacunary function, and indeed it must belong to any function that is an algebraic function or a lacunary function of an elliptic function. For example, if $w^3 = \wp z$, the three values of w associated with a value z_0 of z are all associated also with every value of z that is of the form $z_0+2m\omega_1+2n\omega_2$. Similarly, for the function defined by the expansion

·101 $$w = 1+\wp z+\wp^2 z+\wp^4 z+\wp^8 z+\ldots,$$

if the value w_0 is assumed by w when z has the value z_0, the same value w_0 is assumed when z has any value congruent with z_0, but in this example w has no meaning unless $|\wp z| < 1$, and since there are no values which $\wp z$ does not take, it follows that there are values of z for which w does not exist. The function defined by $w^3 = \wp z$ and the function defined by the expansion ·101 are doubly periodic functions of z in precisely the sense in which the function $w(I)$ has been shown to be a doubly periodic function of I, but they are not elliptic functions. An elliptic function is required to be singlevalued, and to have no singularities other than poles except at infinity. These conditions are not mere simplifications adopted in a first approach to the subject only to be abandoned if they become irksome; they are essential to the arguments that depend on integration round the perimeter of a period parallelogram, and the whole general theory presupposes them in one way or another.

To suppose that every value of I is possible as an argument of the function $w(I)$ is to suppose that to a value of I given arbitrarily must correspond at least one path to infinity along which the integral of $1/\sqrt{R(w)}$ has the given value. To suppose that w as a function of I is singlevalued is to suppose that integrals to infinity from different points of the w plane can in no case be equal. We have only to state the assumptions in this explicit language to know that nothing that we have yet said has any bearing on either of them.

Taken literally, the question whether integrals from different points of the w plane can be equal is not a local question: the whole plane, not only the immediate neighbourhood of one point, is involved. We recall however that in constructing the function fj$z$ we did take into account that a singlevalued function was required, and that in that case we secured the result by conditions each of which was purely local: we could see that a branchpoint must be either a zero or an infinity, and by direct inspection of the zeros and infinities of the function we proved that there could be no branchpoints; we concluded that the function was singlevalued. The inference seems immediate, but there is a tacit assumption, which it would have been pedantic to emphasize then, that if a function has branches it has branchpoints.

Let us analyse this assumption. If a function w of z is not a singlevalued function or a formal aggregate of a number of distinct singlevalued functions, there is some simple closed circuit Γ in the z plane with the property that if the point z describes the circuit and w varies continuously, nevertheless the value of w after description of the circuit may be different from the starting value. To reduce this characteristic of the function to the local property that there is some point t such that the function is not singlevalued within a circle drawn round t as centre, however small the circle may be, we may talk naively of contracting the circuit Γ, or we may apply the Heine-Borel theorem† to a reticulation, but however it is conducted the argument breaks down if the region in which it is applied is not simply connected, that is, if the mutating circuit surrounds points at which the function is not defined. The function fj$z$ is defined from the beginning for every finite value of z, and is proved to be singlevalued as soon as it is proved to have no branchpoints. To search the I plane for branchpoints of the

† Briefly, the argument is as follows. If there are circuits that are not conservative for the function, we can associate with a variable point P of the z plane the largest circle with P as centre which does not surround such a circuit. The radius of this circle is a continuous function of the position of P, and in any closed region this function attains its lower bound. If the lower bound is zero, a point where the bound is attained is a point in whose immediate neighbourhood a passage can be made from one branch to some other. If the lower bound is not zero, the closed region can be covered by a finite number of overlapping circles no one of which contains a mutating circuit; if the region is simply connected it then follows that the region as a whole does not contain such a circuit, but this concluding step can not be taken if there is multiple connectivity.

function $w(I)$ is premature until we know whether the function exists throughout the whole plane.

As we shall see, to discuss the relation between w and I near a value of I which the integral is known to take is a simple matter. The difficulty in discussing a value of I which the integral is not known to take is that we do not even know how to connect that value by any expansions with a value which the integral does take. While this is true in general, there must be at least one point in the I plane where the difficulty does not arise. For the points of the I plane which are values of the integral $I(w)$ compose an aggregate Δ. If there are finite values that the integral can not take, the aggregate Δ does not cover the whole plane, and there is at least one accessible boundary point to this aggregate; that is, either there is at least one accessible point which does not belong to Δ but is a limit of members of Δ, or there is at least one accessible member of Δ which is a limit of points that do not belong to Δ. As far as logical classification can tell us, there may be any number of points of each kind, but unless there is one point of one kind or the other there can be no finite values impossible for $I(w)$.

We proceed to investigate the alternatives, and we consider first the neighbourhood of a member of Δ, because results obtained by working outwards from a known centre are needed in the subsequent more difficult discussion.

7·2. Assuming the relation

·201 $$I_* = \int_{w_*}^{\infty} \frac{dw}{\sqrt{R(w)}},$$

we are to consider the neighbourhood of I_*, or rather, to consider together the neighbourhoods of I_* and w_* in their two planes.

The definition of I as an integral implies the existence of dI/dw, and therefore implies the analytic character of the branch of $I(w)$ involved, near any point at which w, I, and the integrand are all finite, and implies also the analytic character of the corresponding branch of w as a function of I unless the integrand dI/dw is zero. In detail, there is a number σ such that if $|w-w_*| < \sigma$ the relevant branch of $1/\sqrt{R(w)}$ is expansible in a series

$$k_0+k_1(w-w_*)+k_2(w-w_*)^2+\dots,$$

and if I is the integral along a path which comes from w to w_* inside the circle $|w-w_*| < \sigma$ and then follows the path of I_*, we have

·202 $$I-I_* = -k_0(w-w_*)-\tfrac{1}{2}k_1(w-w_*)^2-\tfrac{1}{3}k_2(w-w_*)^3-\dots.$$

Unless $k_0 = 0$, this expansion is reversible in the form

$$\cdot 203 \qquad w-w_* = h_1(I-I_*)+h_2(I-I_*)^2+h_3(I-I_*)^3+...,$$

valid for sufficiently small values of $|I-I_*|$.

That is, there is a number ρ, not zero, such that if I is any number satisfying the inequality $|I-I_*| < \rho$, the expansion in ·203 is convergent and determines a value w for which $|w-w_*| < \sigma$; for this value of w, the integral to ∞ has the value I if the path of integration is the path $ww_*\infty$.

7·21. *In general, if $I_* = I(w_*)$, there is a number ρ such that if $|I-I_*| < \rho$, then there exists a point w such that I is a value of $I(w)$.*

We must examine the cases, of zero or infinite integrand, not covered by our proof of ·21 to see if there are any exceptions, to the result.

First, the integrand $1/\sqrt{R(w)}$ is zero at w_* only if w_* is infinite, that is, if I_* is the value of the integral along a path S which comes from and returns to infinity. This is the type of integral to which the greater part of the last chapter was devoted. If w is any point of S, one value of $I(w)$ is the integral from w along S, and one value of $I_*-I(w)$ is the integral to w along S. Since S comes from infinity, there is a point v of S before which every point of S is outside the circle

$$|w| = \max(|b|, |c|),$$

and in the notation of 6·3 the integrand along S from ∞ to v is one of the two functions $\pm A(w)$. Thus for any point w of S before v, we have $I(w) = I$, where

$$\cdot 204 \qquad I_*-I = \pm \int_\infty^w A(w)\,dw = \mp\frac{1}{w}\left(1+\frac{a_1}{3w^2}+\frac{a_2}{5w^4}+\cdots\right).$$

For sufficiently small values of $I-I_*$ this expansion is reversible in the form

$$\cdot 205 \qquad \frac{1}{w} = \pm(I-I_*)\{1+h_1(I-I_*)^2+h_2(I-I_*)^4+\ldots\},$$

which is further equivalent to

$$\cdot 206 \qquad w = \pm\frac{1}{I-I_*}\{1+g_1(I-I_*)^2+g_2(I-I_*)^4+\ldots\}.$$

From this expansion we can argue as from ·203. For any sufficiently small value of $I-I_*$, the expansion gives a definite value of w, and we can secure the condition $|w| > \max(|b|, |c|)$ by a reduction if necessary of the limit imposed on $|I-I_*|$. Then it follows that for this value of w, given by ·206, the integral $I(w)$ has the value I if the path of integration lies along S. That is to say, *the case in which* $w_* = \infty$ *is no exception to* ·21.

Incidentally, since the upper sign or the lower must be taken in ·204 according as passage along S restores or reverses the integrand at infinity, that is, according as the form of I_* is $2h\alpha+4k\beta$ or $2h\alpha+(4k+2)\beta$, we have proved that

7·22. *The infinities of the function $w(I)$ are simple poles; those congruent with the origin have residue* 1 *and those not congruent with the origin have residue* -1.

Next, the integrand $1/\sqrt{R(w)}$ is infinite at w_* if w_* has one of the critical values $\pm b$, $\pm c$. We have already seen that the integral remains finite if the path starts actually from the critical point; the value of the integral is of the form $m\beta+n\gamma$, with m odd and n even at $\pm b$ and with m even and n odd at $\pm c$. Writing

$$w-b = t^2,$$

we have
$$R(w) = t^2(2b+t^2)\{(b^2-c^2)+2bt^2+t^4\},$$

and since $b(b^2-c^2) \neq 0$, $1/\sqrt{R(w)}$ is expansible for sufficiently small values of t in the form

$$\frac{1}{t}(k_0+k_1 t^2+k_2 t^4+\ldots),$$

where $k_0 \neq 0$; more generally, this is the form of $1/\sqrt{R(w)}$ near any critical point w_*, if t denotes either value of $\sqrt{(w-w_*)}$. Since $dw/dt = 2t$, integration gives

$$\int_{w_*}^{w} \frac{dw}{\sqrt{R(w)}} = 2k_0 t+\tfrac{2}{3}k_1 t^3+\tfrac{2}{5}k_2 t^5+\ldots,$$

if the path of integration remains inside the circle $|w-w_*| = \delta^2$, where δ is the radius of convergence of the series $k_0+k_1 t^2+k_2 t^4+\ldots$. If then the path of integration to ∞ from a point w inside this circle consists of a path to w_* inside the circle followed by the path which provides the value I_*, the value I of $I(w)$ is given by

·207 $$I-I_* = -t(2k_0+\tfrac{2}{3}k_1 t^2+\tfrac{2}{5}k_2 t^4+\ldots),$$

implying a reversal

·208 $$t = (I-I_*)\{h_0+h_1(I-I_*)^2+h_2(I-I_*)^4+\ldots\}$$

and therefore an expansion

·209 $$w-w_* = (I-I_*)^2\{g_0+g_1(I-I_*)^2+g_2(I-I_*)^4+\ldots\}.$$

As before, any value of I sufficiently near to I_* is a value for which a point w and a path of integration $ww_*\infty$ exist: *the case in which w_* is a branchpoint is no exception to* ·21.

We have dealt with the only possible cases: there are no exceptions to ·21, and we can assert that

·210. *Any finite value which the integral $I(w)$ assumes is completely embedded in values which it assumes,*

or in other words that

7·23. *A finite value taken by the integral $I(w)$ can not be a limit of values which the integral does not take.*

In the language of ·1, we have disposed of one alternative regarding boundary points of the aggregate Δ.

7·3. From the formulae ·202, ·204, ·207 used to establish ·23, it follows† that

7·31. *If I_* is a value of the integral $I(w_*)$, then I_* is a limit of values of $I(w)$ as w tends to w_*.*

What we have now to prove is the converse theorem, that if J is a limit of values that $I(w)$ can take, then there is some value w_* of w such that J is a value of $I(w_*)$. The proof is simple in principle, but in detail complications arise because $I(w)$ is not a singlevalued function of w, and $w(I)$ must not be assumed to be a singlevalued function of I.

We first show that we can, in effect, treat $I(w)$ as singlevalued: we can surround J by a circle within which there can not be two distinct values of I associated with the same value of w. The evidence is naturally to be drawn from the conclusions in 6·77 regarding the multiplicity of values of I.

If I is at a distance less than ρ from J, then $2m\alpha+4n\beta+I$ is at a distance less than ρ from $2m\alpha+4n\beta+J$, and $2m\alpha+(4n+2)\beta-I$ is at a distance less than ρ from $2m\alpha+(4n+2)\beta-J$. Let us surround each point of the form $2m\alpha+4n\beta+J$ and each point of the form $2m\alpha+(4n+2)\beta-J$ by a circle of radius ρ. Then if one value of I is inside the circle round J, there can not be a second value of I inside the same circle unless this circle is overlapped by one of the other circles, and we ask if ρ can be chosen small enough to make overlapping impossible.

Since, as we have proved in 6·8, the ratio of β to γ is not real, the points $2m\alpha+4n\beta$ form an undegenerate lattice. A distance between

† The reader will notice that ·31 is not a corollary of ·23.

one point of the form $2m\alpha+4n\beta+J$ and another, or between one point of the form $2m\alpha+(4n+2)\beta-J$ and another, is the same as a distance between two points of this lattice, and is at least as large as the smaller of the perpendicular distances between opposite sides of the period parallelogram 2α, 4β; this distance has therefore a minimum value δ_1 which is not zero.

Again, since

$$\{2m_1\alpha+4n_1\beta+J\}-\{2m_2\alpha+(4n_2+2)\beta-J\}$$
$$= 2(m_1-m_2)\alpha+4(n_1-n_2)\beta-2(\beta-J),$$

identically, a distance between a point of the form $2m\alpha+4n\beta+J$ and a point of the form $2m\alpha+(4n+2)\beta-J$ is the same as a distance between the point $2(\beta-J)$ and a lattice point. As we have just seen, the minimum distance between two lattice points is not zero; hence the aggregate of lattice points can not have $2(\beta-J)$ for a limiting point, and either $2(\beta-J)$ is an actual lattice point, or the distances from $2(\beta-J)$ to lattice points, that is, the distances between points of the form $2m\alpha+4n\beta+J$ and points of the form $2m\alpha+(4n+2)\beta-J$, have a minimum value δ_2 which is not zero.

If $2(\beta-J)$ is the lattice point $2m\alpha+4n\beta$, then $J = -m\alpha-(2n-1)\beta$; if m is even, J is congruent either with β or with $-\beta$ and is a value either of $I(b)$ or of $I(-b)$; if m is odd, J is congruent either with γ or with $-\gamma$ and is a value either of $I(c)$ or of $I(-c)$. Thus if $2(\beta-J)$ is a lattice point, J is known already to have the form $I(w_*)$, and further argument to this end is unnecessary.

Setting aside the case in which $2(\beta-J)$ is a lattice point, we have a distance $\min(\delta_1, \delta_2)$, not zero, which is the least distance between two points each of which has one of the two forms $2m\alpha+4n\beta+J$, $2m\alpha+(4n+2)\beta-J$, and therefore if every point of each form is the centre of a circular region of radius $\frac{1}{2}\min(\delta_1, \delta_2)$, no two of these regions overlap, and if, for a given value of w, there is a value of $I(w)$ inside one of these circles, then, for that value of w, there is one and only one value of $I(w)$ inside each circle.

We can now suppose the point J, which is a limit of values taken by $I(w)$ as w varies, to be the centre of a circle within which there is at most one value of $I(w)$ corresponding to any one value of w. The radius μ of this circle is not zero, and in dealing with J as a limiting point we may ignore altogether points outside the circle.

Let ρ be any radius between 0 and μ, and denote by $P(\rho)$ the interior of the circular region with centre J and radius ρ. Inside the region

$P(\rho)$ there is an infinity of values that $I(w)$ can take, since J is a limit of these values; since one value of w can not be the source of more than one value of $I(w)$ inside $P(\rho)$, the values of w which give values of $I(w)$ inside $P(\rho)$ form a set $W(\rho)$ which also contains an infinity of members, and therefore has at least one limiting point. That is, the limiting points of the set $W(\rho)$ compose a set $D(\rho)$ with at least one member, which may be the point at infinity in the w plane. Now if $0 < \sigma < \rho$, the circular region $P(\sigma)$ forms part of the circular region $P(\rho)$, and the sets $W(\sigma)$, $D(\sigma)$ therefore form parts† of the sets $W(\rho)$, $D(\rho)$; also the set $D(\rho)$ is closed, since the w plane is completed by the point at infinity. The collection of sets $D(\rho)$, for all values of ρ in the open interval $0 < \rho < \mu$, determines a set Π composed of the values of w that belong to every member of this collection, and because the individual sets are closed and the collection is a nest, the set Π is not empty‡ but has at least one member. Let w_* be a member of Π and therefore a limiting point of $W(\rho)$. By ·31, any value of $I(w_*)$ is a limit of values of $I(w)$ for values of w belonging to $W(\rho)$, and therefore each circle of radius ρ round a point of one of the forms $2m\alpha+4n\beta+J$, $2m\alpha+(4n+2)\beta-J$ includes, possibly on its circumference, one and only one of the values of $I(w_*)$. That is, if $0 < \rho < \mu$, there is one and only one value of $I(w_*)$ in or upon the circle round J with radius ρ. But w_* and the values of $I(w_*)$ are independent of ρ; the value of $I(w_*)$ belongs therefore to the only point which is common to all the circles, namely, the centre J itself.

7·32. *If J is a finite limit of values which $I(w)$ can take, then J is itself a value which $I(w)$ can take.*

Or, in the form of the enunciation of ·23,

7·33. *A finite value which the integral $I(w)$ does not take can not be a limit of values which the integral does take.*

An alternative method of reaching the conclusion ·33 will seem simpler; logically it is less satisfactory, since the controversial multiplicative axiom is assumed.

Since J is a limit of values that $I(w)$ can take inside the circle round J with radius μ, we can select from these values a sequence I_1, I_2,... of which J is the only limit. These values belong to arguments w_1, w_2,..., and since one value of

† Perhaps the whole, as far as we know at present, but this is immaterial.

‡ If the collection was formed for values of ρ in an interval $\lambda \leqslant \rho < \mu$ closed at the lower end, the set Π_λ of common members would be simply $D(\lambda)$, and the fact that $D(\rho)$ is closed would be irrelevant. But since our whole object is to find a set Π_0 and ρ can not actually be 0, it is essential to have an argument which allows the interval of values of ρ to be open.

w can not account for more than one of the values of I, the sequence $w_1, w_2, \ldots$ consists of an infinite number of distinct terms and has at least one limit, finite or infinite. If w_* is a limit of the sequence $w_1, w_2, \ldots$, there is a subsequence $w_{m_1}, w_{m_2}, \ldots$, where $m_1 < m_2 < \ldots$, of which w_* is the only limit. By ·31, if $I_\dagger$ is a value of $I(w_*)$, then $I_\dagger$ is a limit of values of $I(w_{m_1}), I(w_{m_2}), \ldots$. Since every value of each of the integrals $I(w_{m_1}), I(w_{m_2}), \ldots$ is in some circle with radius μ round a point of one of the forms $2m\alpha + 4n\beta + J$, $2m\alpha + (4n+2)\beta - J$, the limiting point $I_\dagger$ is in or upon one of these circles, and therefore each of the circles contains one and only one value of $I(w_*)$. In particular, there is one value I_* of $I(w_*)$ in or upon the circle round J, and this value is a limit of the values I_{m_1}, $I_{m_2}, \ldots$ of $I(w_{m_1}), I(w_{m_2}), \ldots$ which are inside the same circle. But $I_{m_1}, I_{m_2}, \ldots$ are terms of the sequence $I_1, I_2, \ldots$ which by hypothesis has only the one limit J. Hence I_* coincides with J, that is, J is identified with a value of $I(w_*)$.

Essentially what is done in the earlier proof of ·33 is to transform this argument into a form in which picked sequences are not invoked, without losing the thread which runs so clearly through the unsophisticated version.

7·4. The combination of the two results ·23, ·33 is the theorem that

·401. *There exists no boundary to the aggregate of values taken by the integral $I(w)$ if the range of w is unrestricted.*

This implies that these values cover the whole of the I plane:

7·41. *There is no finite value which the integral $I(w)$ does not take for some value, finite or infinite, of the lower limit w.*

In other words, with the convention that a function exists at a point where its value is unequivocally infinite as well as at a point where it has a finite value,

7·42. *The function $w(I)$ exists for every finite value of I.*

It follows now that the discussion in ·2 of the character of $w(I)$ in the neighbourhood of a point determined by an arbitrary value of w was also a discussion of the function in the neighbourhood of an arbitrary value of I, and from an inspection of the three types of expansion ·203, ·206, ·209, we see that

7·43. *The only accessible singularities of the function $w(I)$ are simple poles with residue* 1 *or* -1.

In particular, $w(I)$ has no branchpoints, and therefore

7·44. *The function $w(I)$ is either a singlevalued function or an aggregate of distinct singlevalued functions.*

The second alternative recognized in this theorem will be understood from the example of such simple relations as $w^2 = 1/z^2$ and $w^2 = \csc^2 z - 1$. In each of these relations w, regarded as a function of z, satisfies the conditions of both ·42 and ·43, and it is only our previous acquaintance with the functions that enables us to see at once that the one function is the combination of the two distinct functions $1/z$, $-1/z$ and the other the combination of the two distinct functions $\cot z$, $-\cot z$.

7·5. For the last step in determining the character of $w(I)$ we replace the integral relation defining $I(w)$ by the differential equation

·501 $$(dw/dI)^2 = (w^2-b^2)(w^2-c^2).$$

The argument is a repetition of that used in 5·1 and 5·3. Substituting $w = 1/y$, we have

·502 $$(dy/dI)^2 = (1-b^2y^2)(1-c^2y^2),$$

whence

·503 $$d^2y/dI^2 = -(b^2+c^2)y+2b^2c^2y^3.$$

In ·502, $y = 0$ implies $dy/dI = 1$ or $dy/dI = -1$, and there is one and only one solution of ·503 for which initially $y = 0$, $dy/dI = 1$, that is, one and only one solution expansible near $I = 0$ in the form

·504 $$y = I(1+b_1 I+b_2 I^2+\ldots).$$

Hence there is one and only one solution of ·501 expansible near $I = 0$ in the form

·505 $$w = \frac{1}{I}+c_0+c_1 I+c_2 I^2+\ldots.$$

That is to say, near the origin $w(I)$ is one singlevalued function, whence from ·44

7·51. *The function $w(I)$ is singlevalued throughout the I plane, except possibly at infinity,*

implying with ·43 the fundamental theorem to which we have been working:

7·52. *The function $w(I)$ obtained by inverting the integral relation*

$$I = \int_w^\infty \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}},$$

in which the radical is asymptotic to w^2 towards infinity along the path of integration, is a meromorphic function.

7·6. In 5·1, the identification of the function $\mathrm{fj}\, z$ with a particular solution of the differential equation

·601 $$(dw/dz)^2 = (w^2-f_g^2)(w^2-f_h^2)$$

over the whole z plane was made without comment, and it is worth while to remark on the difference between ·601 and ·501 in respect of the use that can be made of a particular solution selected at the origin. The solution of ·501 is not necessarily confined to the circle of convergence of the series $c_0+c_1I+c_2I^2+\dots$ which constitutes the regular part of the Laurent expansion ·505. If p is any point inside this circle, the expansion

·602 $$\frac{1}{p+(I-p)}+c_0+c_1\{p+(I-p)\}+c_2\{p+(I-p)\}^2+\dots$$

can be rearranged as a power series in $I-p$, and the function represented by ·505 can be continued analytically from the new series; thus continued, the function nowhere ceases to satisfy the differential equation ·501. The function exists at a point I_* if there is† a curve D joining p to I_* with a finite sequence of overlapping circles such that each point of D lies inside at least one of the circles, that the centre of each circle is inside the preceding circle, and that each circle is the circle of convergence of the Taylor series constructed at its centre from the function defined in its predecessor. Whether such a curve and such a sequence of circles exist for a specified point I_* depends ultimately on the sequence of coefficients c_0, c_1, c_2,..., and in this sense the domain of existence of the function $w(I)$ is undoubtedly determined theoretically by the constants b^2, c^2. From this point of view the extent of the domain is a subject for investigation, and the proper assumption to make is that it is only if the constants b^2, c^2 satisfy some set of conditions at first unknown that the domain of existence of the function as a meromorphic function extends over the whole plane, except perhaps at infinity.

In the case of the equation ·601, we make no effort to continue the solution analytically. The solution being identified with a known function fj z, the expansion obtained by rearrangement from

·603 $$\frac{1}{p+(z-p)}+c_0+c_1\{p+(z-p)\}+c_1\{p+(z-p)\}^2+\dots,$$

where the coefficients depend on f_g, f_h in precisely the same way as the coefficients in ·602 on b, c, is the expansion of fj$(p+h)$ as a power series in h, and any continuation of fj z from the centre p is equally a continuation from the series ·603. Knowing that fj z can be continued to any point that does not belong to its lattice of poles, we infer that whatever conditions are necessary for the continuation of ·603 must in

† This is the classical form of continuation. For a meromorphic function, continuation by means of overlapping Laurent circles is more efficient, and no more difficult to justify theoretically.

fact be satisfied. In other words, *if there are any conditions which b, c must satisfy in order that a solution of the pair of equations $f_g = b$, $f_h = c$ should exist, these conditions certainly include conditions sufficient to ensure that the domain covered by the continuation of the series* ·602 *is the whole plane with the exception of the points forming a single lattice.* This result helps us not at all in the determination of the domain of existence of the continuation of ·505 for arbitrary values of b and c.

There is however a method, altogether different from that of analytic continuation, due in principle to Weierstrass and applied in detail by Goursat†, for dealing with the inversion problem by extending the region of existence of $w(I)$ as a meromorphic solution of the differential equation

·604 $$(dw/dI)^2 = (w^2-b^2)(w^2-c^2).$$

We know that *if* $w(I)$ is an elliptic function, *then* $w(I+J)$ is expressible rationally in terms of $w(I)$, $w(J)$ and their derivatives, and in particular $w(2I)$ is expressible rationally in terms of $w(I)$ and $w'(I)$. A rational function of two meromorphic functions is itself meromorphic. If then we can calculate $w(I)$ as a meromorphic function of I throughout a region $|I| < \rho$, we can calculate $w(2I)$ as a meromorphic function throughout the same region, that is, we can extend the calculation of $w(I)$ to the larger region $|I| < 2\rho$, and it follows that there can be no maximum to ρ. We propose therefore to use the following lemma:

7·61. *If the function $w(I)$ exists as a meromorphic function throughout some circle round the origin, and if $w(2I)$ can be expressed rationally in terms of $w(I)$ and $w'(I)$, then $w(I)$ exists as a meromorphic function throughout the whole plane, the point at infinity perhaps excepted.*

If $w(I)$ is the solution of ·604 for which

·605 $$w(I) \sim 1/I$$

near the origin, the first condition in ·61 is satisfied, as we have proved in ·505. For the second condition, we are not at liberty simply to quote an addition theorem for $\mathrm{fj}\,z$ or a formula for $\mathrm{fj}\,2z$; whatever formulae we need we must establish from the definition of $w(I)$ in terms of the differential equation. We can however say that because, from 4·44,

$$\mathrm{fj}\,2z = (f_g^2 f_h^2 - \mathrm{fj}^4 z)/2\,\mathrm{fj}\,z\,\mathrm{fj}'z,$$

it follows that *if* $w(I)$ is $\mathrm{fj}\,I$, *then* $w(2I)$ must be $\{b^2c^2-w^4(I)\}/2w(I)w'(I)$;

† *Cours d'Analyse Mathématique* (1st ed., 1905) II, 505. The notation and the details of the analysis are adapted to our own treatment. Goursat deals with the Jacobian function $\mathrm{sn}\,u$.

the glance forward† does its work in suggesting a formula for verification.

Suppressing the argument I throughout, we accept for examination the function W defined by the formula

·606 $$W = \frac{b^2c^2 - w^4}{2ww'},$$

where w satisfies the differential equation ·604 and the initial condition ·605. Writing also

·607 $$T = \frac{w'^2}{w^2} = \frac{b^2c^2}{w^2} - (b^2+c^2) + w^2$$

we have

·608 $$T' = -2\left(\frac{b^2c^2}{w^3} - w\right)w' = -4WT.$$

On the other hand, since ·606 can be written

$$\frac{2w'W}{w} = \frac{b^2c^2}{w^2} - w^2,$$

we have from ·607

$$4W^2T = \{T+(b^2+c^2)\}^2 - 4b^2c^2,$$

that is, $$4W^2 = T + 2(b^2+c^2) + \frac{(b^2-c^2)^2}{T},$$

whence

·609 $$4(W^2-b^2)T = \{T-(b^2-c^2)\}^2, \qquad 4(W^2-c^2)T = \{T+(b^2-c^2)\}^2$$

and also, from ·608,

$$2W' = -T\left(1 - \frac{(b^2-c^2)^2}{T^2}\right),$$

that is,

·610 $$2W'T = -\{T^2-(b^2-c^2)^2\}.$$

From ·609, ·610,

$$W'^2 = 4(W^2-b^2)(W^2-c^2),$$

so that if $J = 2I$, then

·611 $$(dW/dJ)^2 = (W^2-b^2)(W^2-c^2).$$

Also, near $I = 0$, from ·605, $W \sim 1/2I$, that is, near $J = 0$,

·612 $$W(J) \sim 1/J.$$

† Historically, addition theorems were discovered in the form of relations between integrals before the integrals were inverted.

Hence W, as a function of J, satisfies precisely the conditions which define w as a function of I. That is to say,

$$W(I) = w(2I),$$

and the second condition in ·61 is satisfied as well as the first:

7·62. *The solution of the equation*

$$(dw/dI)^2 = (w^2-b^2)(w^2-c^2)$$

which resembles $1/I$ *near* $I = 0$ *exists as a meromorphic function throughout the whole* I *plane, except perhaps at infinity.*

It will not be disputed that the deduction from ·61 is simpler than the series of proofs in ·3–·5. On the other hand, while the earlier propositions depend on general principles that can be expected to have applications elsewhere, the later proof depends on the exact form of the differential equation, on a formula that is peculiar to this equation and not to be discovered without some trouble, and on a lemma of which the range of usefulness is necessarily limited, since addition theorems are rare.

VIII

THE SOLUTION OF THE PROBLEM OF INVERSION

8·1. The characterization of the function $w(I)$ inverse to the integral $I(w)$ is now complete. By 7·52 the function is meromorphic except at infinity, and by 6·76 it has two periods whose ratio, by 6·86, is not real. By 7·22 the poles are simple, and by 6·55 and 6·76 there are only two poles in a primitive parallelogram. Hence

8·11. *The function $w(I)$ is an elliptic function of the second order with distinct simple poles.*

Once known to be an elliptic function, $w(I)$ is readily identified by its structure. By 6·55 the poles form the lattice built on 2β and 2γ, and since 2α is a period, α is a step from a pole to a zero, and therefore, the origin being a pole, α and $-\alpha$ are zeros.

8·12. *If b^2 and c^2 are unequal and neither of them is zero, and if*

$$I = \int_w^\infty \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}}$$

where the radical resembles w^2 towards infinity along the path of integration, then

$$w = \mathrm{fj}(I; \alpha, \beta, \gamma),$$

where α, β, γ are appropriate values of $I(0)$, $I(b)$, $I(c)$ connected by the relation $\alpha+\beta+\gamma = 0$.

In particular,

·101–·102 $\quad b = \mathrm{fj}(\beta; -\beta-\gamma, \beta, \gamma), \qquad c = \mathrm{fj}(\gamma; -\beta-\gamma, \beta, \gamma).$

But this pair of formulae is a corollary of ·12, not, as it might be if the formulae were proved independently, the foundation of the theorem.

8·2. The restrictions imposed in 6·3 on the paths B, C from b, c to ∞, the paths by which β, γ are defined, had the sole purpose of clearing the ground for the subsequent discussion. It is easy to see that they do not render the integrals β, γ determinate: without infringing the restrictions, we can change the paths in such a way as to change the integrals also. For example, without altering the path B we may replace C by a path C_1 such that C and C_1 together form a simple loop impeded by B. The integral along C_1, which then replaces γ throughout the discussion, has the value $2\beta-\gamma$, which is necessarily different from

γ, but the resultant function $\mathrm{fj}(I;\gamma-3\beta,\beta,2\beta-\gamma)$ is identical with $\mathrm{fj}(I;-\beta-\gamma,\beta,\gamma)$.

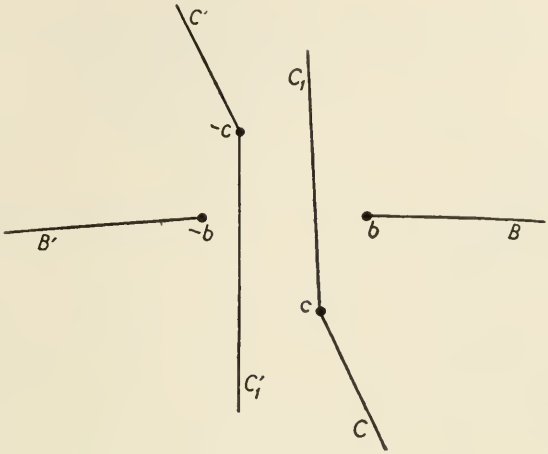


FIG. 28.

If B_*, C_* are any two paths from b, c to ∞, the integrals along these paths are values of $I(b)$, $I(c)$ and are given by

·201 $$\beta_* = (2m_1+1)\beta+2n_1\gamma, \qquad \gamma_* = 2m_2\beta+(2n_2+1)\gamma,$$

where m_1, n_1, m_2, n_2 are whole numbers, not necessarily positive, and m_1+n_1, m_2+n_2 are even. The function $\mathrm{fj}(I;\alpha_*,\beta_*,\gamma_*)$ is identical with $\mathrm{fj}(I;\alpha,\beta,\gamma)$ only if β_*, γ_* is a primitive pair of quarterperiods of the latter function, that is, only if

·202 $$(2m_1+1)(2n_2+1)-4n_1m_2 = \pm 1.$$

It follows that some restrictions on the paths from b, c are essential to the identification of $w(I)$ with $\mathrm{fj}(I;\alpha,\beta,\gamma)$. But the restrictions in 6·3 are of no intrinsic significance.

8·3. The definition of $w(I)$ as a particular solution of the differential equation
$$(dw/dI)^2 = R(w)$$
is almost equivalent to definition as the inverse of $I(w)$. Since the Weierstrass-Goursat proof that the function is meromorphic avoids the topographical problems inseparable from the study of the integral $I(w)$, it is of some interest to see how the double periodicity of the function can be established from the differential equation alone.

Given a function $f(z)$, if we can express $f(z+\omega)$ in terms of $f(z)$, with functions of ω playing a parametric part, we can easily see if any choice of ω reduces $f(z+\omega)$ identically to $f(z)$. The periodicity of a function can therefore be verified immediately if an addition theorem is known.

As we verified in 7·6 that $w(I)$ satisfies the duplication formula of fj$z$, so we can verify that $w(I)$ satisfies the addition formula

·301 $$w(I+J) = \frac{w(I)w'(J)-w(J)w'(I)}{w^2(I)-w^2(J)}.$$

Write temporarily $$W = \frac{kw-hw'}{w^2-h^2},$$

where h, k are constants and the argument I of the functions w, w', W is understood. Since

·302–·303
$$w'^2 = (w^2-b^2)(w^2-c^2), \qquad w'' = 2w^3-(b^2+c^2)w,$$

we have

$$\begin{aligned}(w^2-h^2)^2W' &= (kw'-hw'')(w^2-h^2)-2ww'(kw-hw') \\ &= -kw'(h^2+w^2)+hw\{2(h^2w^2+b^2c^2)-(b^2+c^2)(h^2+w^2)\},\end{aligned}$$

on substitution and reduction. Hence simultaneous interchange of h with w and of k with w' leaves W' unaltered. It follows that *if*

·304 $$h = w(J), \qquad k = w'(J)$$

then

·305 $$\frac{\partial W}{\partial I} = \frac{\partial W}{\partial J},$$

whence, with h, k given by ·304, $W(I, J)$ is a function $\phi(I+J)$ of $I+J$. Since I can be connected continuously with the origin, and since $W \to h$ as $I \to 0$, the function $\phi(J)$ is $w(J)$, that is, $W(I, J) = w(I+J)$, and ·301 is proved.

Attention may be called to the part played by 7·62 in this argument. Without 7·62 we reach the point that the function on the right-hand side of ·301 is some function of $I+J$, but if I and J could belong to domains separated from the neighbourhood of the origin, a condition derived from some point in one of these domains would be necessary before this function of $I+J$ could be identified. It is for this reason that investigation of periodicity by means of the addition theorem could not be included in Chapter VI, where it would seem appropriate.

It is a curious feature of the formal development of the theory from the differential equation that the duplication formula, or some other weakened form of the addition theorem, appears to be essential to the proof of the general theorem.

Since $w'(I)$ is an irrational function of $w(I)$, the expression for $w(I+J)$ can not reduce to $w(I)$ for all values of I unless the term $w(J)w'(I)$ disappears, on account of the value of J, either absolutely or, as we shall see is possible, in comparison with $w(I)w'(J)$. We have to consider the two possibilities,

·306 $$\text{(i)} \quad w(J) = 0 \quad \text{when } J = \omega,$$

·307 $$\text{(ii)} \quad w(J)/w'(J) \to 0 \quad \text{as } J \to \delta.$$

Firstly, if $w(\omega) = 0$, that is, if ω is a value of $I(0)$, then from ·302

$$w'^2(\omega) = b^2c^2 \neq 0,$$

and we have from ·301,

·308 $$w(I+\omega) = w'(\omega)/w(I).$$

Hence ω is not a period, but writing ·308 in the form

$$w(I)\,w(I+\omega) = w'(\omega)$$

and replacing I by $I+\omega$ we have, since $w(I+\omega)$ is not identically zero,

$$w(I+2\omega) = w(I).$$

8·31. *If ω is any value of $I(0)$, that is, is the integral of $1/\sqrt{R(w)}$ along any path from 0 to ∞, the function $w(I)$ is periodic in 2ω but not in ω.*

Secondly, to the condition ·307 we may add $w(\delta) \neq 0$, since we do not need to recapitulate the first case. But with this condition added, ·307 requires

·309 $$w'(J) \to \infty,$$

implying, from the differential equation, $w(J) \to \infty$, and further

·310 $$\frac{w'^2(J)}{w^4(J)} \to 1.$$

Hence from ·301,

·311 $$w(I+\delta) = \mp w(I)$$

according as

·312 $$\lim_{J\to\delta} \frac{w'(J)}{w^2(J)} = \pm 1.$$

Now from the duplication formula, in the form

·313 $$w(J) = \frac{b^2c^2 - w^4(\frac{1}{2}J)}{2w(\frac{1}{2}J)w'(\frac{1}{2}J)},$$

it follows that an infinity of w at δ is associated with three possibilities at $\frac{1}{2}\delta$; we may have there (i) a zero of w, (ii) a zero of w', or (iii) an infinity of w.

Differentiating ·313 logarithmically we have

·314 $$\frac{2w'(J)}{w(J)} = -\frac{4w^3w'}{b^2c^2-w^4}-\frac{w'}{w}-\frac{w''}{w'}$$

and therefore, from ·303,

·315 $$\frac{w'(J)}{w^2(J)} = -\frac{4w^4w'^2}{(b^2c^2-w^4)^2}-\frac{w'^2}{b^2c^2-w^4}-\frac{2w^4-(b^2+c^2)w^2}{b^2c^2-w^4},$$

the unwritten argument on the right of ·314 and ·315 being everywhere $\frac{1}{2}J$, and this formula gives us, in each of the three cases, the value of the limit required for applying ·312 to ·311.

(i) If $w(\frac{1}{2}\delta) = 0$, then $w'^2(\frac{1}{2}\delta) = b^2c^2$, and ·315 gives

$$\frac{w'(J)}{w^2(J)} \to -1,$$

as $J \to \delta$; hence $w(I+\delta) = w(I)$. This is ·31, reached from the other end.

(ii) If $w'(\frac{1}{2}\delta) = 0$, then from ·302

$$2w^4(\tfrac{1}{2}\delta)-(b^2+c^2)w^2(\tfrac{1}{2}\delta) = -\{b^2c^2-w^4(\tfrac{1}{2}\delta)\},$$

and from ·315 $$\frac{w'(J)}{w^2(J)} \to 1,$$

whence $w(I+\delta) = -w(I)$; the condition $w'(\frac{1}{2}\delta) = 0$ implies that $w(\frac{1}{2}\delta)$ is $\pm b$ or $\pm c$:

8·32. *If $\frac{1}{2}\delta$ is any value of $I(b)$ or any value of $I(c)$, then $w(I)$ is periodic in 2δ but not in δ.*

(iii) If $w(\frac{1}{2}J) \to \infty$ as $J \to \delta$, then $w'^2(\frac{1}{2}J)/w^4(\frac{1}{2}J) \to 1$, and therefore from ·315,

$$\frac{w'(J)}{w^2(J)} \to -1,$$

implying $w(I+\delta) = w(I)$. This however is only a deduction from ·31 and ·32 combined, since δ is now of one of the forms $2^nI(0)$, $\pm 2^nI(b)$, $\pm 2^nI(c)$, with $n \geqslant 2$.

The cumulative argument establishes that

8·33. *Every period of the function $w(I)$ is of one of the three forms $2I(0)$, $\pm 4I(b)$, $\pm 4I(c)$,*

but does not indicate how a primitive pair of periods is to be found.

The proof of periodicity from the differential equation possesses the doubtful merit of invoking the minimum of theoretical principles, neither the theory of aggregates nor the topography of paths being used. But the comment made in 5·5 on 5·14 is again apt. When we construct $\wp z$ as a doubly infinite series, we are deliberately constructing a function that will be doubly periodic. When we have investigated the effect of varying the path of the elliptic integral, we understand why the aggregate of values is a pair of doubly infinite congruences. But double periodicity emerges from the addition theorem as an inexplicable accident, and the addition theorem though easy to verify is hard to discover. And is not the use of the duplication formula in the proof of ·32 ingenious enough to be pleasing but too ingenious to be satisfying?

8·4. The introduction of an elliptic function with simple poles has now been effected in two ways, radically different. If the sole purpose is to have such a function to study, there can be no doubt that it is simpler to construct the function by means of doubly infinite series than to invert an integral. But, as we have seen, the construction and the inversion do not really solve the same problem: one process is not an alternative to the other. The direct process discovers a function with assigned quarterperiods; in the inverse process the function is one for which given parameters play in the end the part of the critical values f_g, f_h. In the one case, f_g, f_h are implicitly determined from ω_g, ω_h, in the other case, ω_g, ω_h are implicitly determined from f_g, f_h.

It is important to remark that in each case the primary object is the function $\mathrm{fj}\,z$ itself; any evaluation of parameters associated with the function is incidental. As we have said, the determination of ω_g, ω_h from f_g, f_h is not unique. This is not because the function $\mathrm{fj}\,z$ is not unique: the relation

·401 $$\int_w^\infty \frac{dw}{\sqrt{R_f(w)}} = z$$

determines w as one definite elliptic function, not as one or other of a group of elliptic functions; it is because the lattice to which the function $\mathrm{fj}\,z$ is attached can be constructed from any primitive pair of its periods. The points at which $\mathrm{fj}\,z$ has the values f_g, f_h, or in other words the solutions of the two equations

·402 $$\mathrm{fj}(\Omega_g; -\omega_g-\omega_h, \omega_g, \omega_h) = f_g, \qquad \mathrm{fj}(\Omega_h; -\omega_g-\omega_h, \omega_g, \omega_h) = f_h$$

in Ω_g, Ω_h, are

·403 $$\Omega_g = (2m_1+1)\omega_g+(2m_1+4n_1)\omega_h,$$
$$\Omega_h = (4m_2+2n_2)\omega_g+(2n_2+1)\omega_h.$$

This is therefore the solution of the pair of equations

·404 $$\mathrm{fj}(\Omega_g; -\Omega_g-\Omega_h, \Omega_g, \Omega_h) = f_g, \qquad \mathrm{fj}(\Omega_h; -\Omega_g-\Omega_h, \Omega_g, \Omega_h) = f_h$$

if the lattice built on Ω_g, Ω_h is geometrically identical with the lattice built on ω_g, ω_h, that is, if

·405 $$(2m_1+1)(2n_2+1)-(2m_1+4n_1)(4m_2+2n_2) = \pm 1.$$

The significance of the sign in the last condition was investigated in the first section of the introduction. Applying 0·14 as a criterion to ·403–·405, we see that

8·41. *The pairs of quarterperiods* Ω_g, Ω_h *such that the function* $\mathrm{fj}(z; -\Omega_g-\Omega_h, \Omega_g, \Omega_h)$ *is identical with the function* $\mathrm{fj}(z; -\omega_g-\omega_h, \omega_g, \omega_h)$ *and that rotation from* Ω_g *to* Ω_h *is in the same direction as rotation from* ω_g *to* ω_h *are given by*

·41₁
$$\Omega_g = (2m_1+1)\omega_g+(2m_1+4n_1)\omega_h,$$
$$\Omega_h = (4m_2+2n_2)\omega_g+(2n_2+1)\omega_h,$$

with the condition

·41₂
$$(2m_1+1)(2n_2+1)-(2m_1+4n_1)(4m_2+2n_2) = 1,$$

the pairs such that the functions are identical and that rotation from Ω_g *to* Ω_h *is in the opposite direction to rotation from* ω_g *to* ω_h *are given by the same pair of formulae with the condition*

·41₃
$$(2m_1+1)(2n_2+1)-(2m_1+4n_1)(4m_2+2n_2) = -1.$$

It follows from the way in which the formulae have come into our work, and it can be verified immediately by elementary algebra, that the aggregate of pairs of numbers given by $\cdot 41_1$ with the condition $\cdot 41_2$ can be constructed from any one of its members by precisely the same formulae subject to precisely the same condition. For this reason the aggregate is called automorphic, and because there is no ambiguity of sign in $\cdot 41_2$ the aggregate is said to be definite. The aggregate given by the same formulae with the less restrictive condition ·405 also is automorphic: it too can be reconstructed from any one of its members with the same formulae and the same condition. The aggregate formed with the ambiguous sign is said to be extended from the definite aggregate.

There are many types of automorphic aggregate, but since we are dealing with only one problem we shall not attempt a general definition. If the formulae ·403 are understood, the aggregate can be said to be conditioned by $\cdot 41_2$ or ·405. Alternatively we can speak of the definite aggregate and the extended aggregate generated by ·403.

Every pair of complex numbers belongs to one and only one definite automorphic aggregate with the generating relation $\cdot 41_1$, and to one and only one extended automorphic aggregate with the same generating relation. Each aggregate, though it has an infinity of members, is determined by any one of them.

If the ratio of ω_g to ω_h is real, ω_g and ω_h are real multiples of one complex number ω, and every pair of numbers determined by formulae such as ·403 is a pair of real multiples of ω. Conversely therefore, if there is one member of the aggregate for which the ratio of Ω_g to Ω_h is not real, there are no members for which the ratio is real: either the aggregate degenerates completely, or it has no degenerate members. We are not concerned with degenerate automorphic aggregates.

We must be on guard against supposing that because the condition ·405 is resolved into the exclusive alternatives $\cdot 41_2$, $\cdot 41_3$, the extended automorphic aggregate is the sum of two aggregates of different kinds, a 'positive' aggregate conditioned by $\cdot 41_2$ and a 'negative' aggregate conditioned by $\cdot 41_3$. Geometrically, the mistake is clear enough: the aggregate of pairs of quarterperiods for which rotation from ω_g to ω_h is in one direction is just the same kind of aggregate as that in which rotation is in the reverse direction. Analytically, the fallacy lies in overlooking that whereas ·403 and ·405, or $\cdot 41_1$ and $\cdot 41_2$, define an aggregate in relation to one of its members, describing as we may say the internal structure of the aggregate, $\cdot 41_1$ and $\cdot 41_3$ define an aggregate by a relation of its members to an external term: $\cdot 41_3$ is not satisfied if m_1, n_1, m_2, n_2 are all zero, and ω_g, ω_h do not constitute a member of the alleged 'negative' aggregate. As they stand, $\cdot 41_1$ and $\cdot 41_3$ give us no reason to suspect that the aggregate which they define is automorphic. To determine the internal structure of this aggregate, we must find the relation of the member Ω_g, Ω_h to some member of the aggregate itself. Since $\cdot 41_3$ is satisfied by $m_1 = 0$, $n_1 = 0$, $m_2 = 1$, $n_2 = -1$, the pair ω_g, $2\omega_g - \omega_h$ belongs to the aggregate conditioned by $\cdot 41_3$, and if we write

$$\bar{\omega}_g = \omega_g, \qquad \bar{\omega}_h = 2\omega_g - \omega_h$$

the generating formulae become

$$\Omega_g = (6m_1+8n_1+1)\bar{\omega}_g - (2m_1+4n_1)\bar{\omega}_h,$$

$$\Omega_h = (4m_2+6n_2+2)\bar{\omega}_g - (2n_2+1)\bar{\omega}_h,$$

that is,

$$\Omega_g = (2\bar{m}_1+1)\bar{\omega}_g + (2\bar{m}_1+4\bar{n}_1)\bar{\omega}_h, \qquad \Omega_h = (4\bar{m}_2+2\bar{n}_2)\bar{\omega}_g + (2\bar{n}_2+1)\bar{\omega}_h,$$

where

$$\bar{m}_1 = 3m_1+4n_1, \qquad \bar{n}_1 = -2m_1-3n_1,$$

$$\bar{m}_2 = m_2+2n_2+1, \qquad \bar{n}_2 = -n_2-1.$$

Reciprocally,

$$m_1 = 3\bar{m}_1+4\bar{n}_1, \qquad n_1 = -2\bar{m}_1-3\bar{n}_1,$$

$$m_2 = \bar{m}_2+2\bar{n}_2+1, \qquad n_2 = -\bar{n}_2-1,$$

and therefore the condition that m_1, n_1, m_2, n_2 are integers is equivalent to the condition that $\bar{m}_1$, $\bar{n}_1$, $\bar{m}_2$, $\bar{n}_2$ are integers, and since identically

$$2m_1+1 = (2\bar{m}_1+1)+2(2\bar{m}_1+4\bar{n}_1), \qquad 2m_1+4n_1 = -(2\bar{m}_1+4\bar{n}_1),$$
$$4m_2+2n_2 = (4\bar{m}_2+2\bar{n}_2)+2(2\bar{n}_2+1), \qquad 2n_2+1 = -(2\bar{n}_2+1),$$

the 'negative' condition

$$(2m_1+1)(2n_2+1)-(2m_1+4n_1)(4m_2+2n_2) = -1$$

becomes the 'positive' condition

$$(2\bar{m}_1+1)(2\bar{n}_2+1)-(2\bar{m}_1+4\bar{n}_1)(4\bar{m}_2+2\bar{n}_2) = 1.$$

That is to say, *the* internal *structure of the aggregate determined by* $\cdot 41_1$ *and* $\cdot 41_3$ *is expressed by a condition of the form* $\cdot 41_2$. The aggregate determined by $\cdot 41_1$ and $\cdot 41_3$ is a definite automorphic aggregate, definite in the same sense as the aggregate for which the positive sign is chosen from ·405. The extended aggregate is composed of two mutually exclusive definite aggregates.

The poles and zeros of fj$z$ form in the z plane lattices which exist independently of the notation by which the function is studied, but the notation is governed to some extent by the uses to which it is to be put. To take the simplest example, neither ω_g nor ω_h can be used as a symbol for a zero of fj$z$. If ω_g and ω_h are to be replaced by another pair of quarterperiods without the meanings of f_g and f_h being changed, ·403 and ·405 give the conditions to be observed.

8·42. *If the function* fj$z$ *is given, the pairs of quarterperiods* ω_g, ω_h *with which it can be associated form an extended automorphic aggregate.*

The fundamental existence theorem of the inversion problem can now be put succinctly:

8·43. *If* b^2 *and* c^2 *are unequal and neither of them is zero, the pair of equations*

$\cdot 43_{1-2}$ $\quad \mathrm{fj}(\omega_g; -\omega_g-\omega_h, \omega_g, \omega_h) = b, \qquad \mathrm{fj}(\omega_h; -\omega_g-\omega_h, \omega_g, \omega_h) = c$

is soluble, and the solutions compose a single extended automorphic aggregate.

In ·42 and ·43 we recover in the automorphic aggregate the uniqueness which one pair of quarterperiods can not display. Of course it is always possible to secure verbal uniqueness in the solution of any problem by speaking of the aggregate of solutions rather than of an individual solution, and we need not even assume that the problem is soluble if we remember that logically an aggregate may have no members, but there is much more in ·42 and ·43 than a verbal trick: the aggregate has been shown not to be nul, and its structure has been discovered.

The significance in the analytical theory of the geometrical interpretation of the distinction between the two conditions $\cdot 41_2$, $\cdot 41_3$ was seen on p. 59:

8·44. *If the function* fj$z$ *is given, the pairs of quarterperiods* ω_g, ω_h *for which the signature of* $(-\omega_g-\omega_h, \omega_g, \omega_h)$ *is* $+i$ *compose a definite automorphic aggregate, and the pairs for which the signature is* $-i$ *compose the complementary definite aggregate.*

That is to say, for every member of the first aggregate,

·406 $$f_g = ig_f, \qquad g_h = ih_g, \qquad h_f = if_h,$$

and for every member of the second,

·407 $$f_g = -ig_f, \qquad g_h = -ih_g, \qquad h_f = -if_h.$$

The two definite aggregates of ·44 together form the extended aggregate of ·42.

We have broken the aggregate defined by ·403 and ·405 into two halves by taking a definite sign in ·405. There is another line along which we can divide this aggregate. Let us require the function denoted by gj$z$ as well as the function denoted by fj$z$ to be unaltered. Since g_f and f_g are to have the same value in every specification, their ratio is always the same, and is either i for all pairs of quarterperiods or $-i$ for all pairs of quarterperiods. Hence the aggregate is necessarily definite. Also f_h, g_h, and the ratios h_f/f_h, h_g/g_h, have the same values for all pairs of quarterperiods. Hence h_f, h_g are unaltered, and the third function hj$z$ is unaltered†.

8·45. *If two of the three functions* fj$z$, gj$z$, hj$z$ *are given, the third function is determinate, and the pairs of quarterperiods* ω_g, ω_h *with which the set of functions can be associated compose a definite automorphic aggregate.*

The values of Ω_g which satisfy the equation

$$\text{hj}(\Omega_g; -\omega_g-\omega_h, \omega_g, \omega_h) = h_g$$

are given by $$\Omega_g = (4m+1)\omega_g + 2n\omega_h,$$

and therefore in ·403 the number m_1 is even if hj$z$ is unaltered; because gj$z$ also is unaltered, n_2 is even, and

8·46. *The aggregate of pairs of quarterperiods* ω_g, ω_h *such that the three functions* fj$(z;\Omega)$, gj$(z;\Omega)$, hj$(z;\Omega)$ *are identical with the three functions*

† More simply, if we introduce derivatives, hj $z = -$fj$'z$/gj z.

$\mathrm{fj}(z;\omega)$, $\mathrm{gj}(z;\omega)$, $\mathrm{hj}(z;\omega)$ *is the definite automorphic aggregate generated by the pair of relations*

$$\Omega_g = (4m'+1)\omega_g+4n'\omega_h, \qquad \Omega_h = 4m''\omega_g+(4n''+1)\omega_h$$

with the condition

$$(4m'+1)(4n''+1)-16n'm'' = 1.$$

8·5. To modify ·43 to concern one complex variable, not a pair of variables, we may write

·501–·503 $$\frac{\omega_g}{\omega_h} = \tau, \quad \frac{\mathrm{fj}(\omega_g; -\omega_g-\omega_h, \omega_g, \omega_h)}{\mathrm{fj}(\omega_h; -\omega_g-\omega_h, \omega_g, \omega_h)} = \phi(\tau), \quad \frac{b}{c} = k.$$

The pair of equations $\cdot 43_{1-2}$ is then replaced by the one equation

·504 $$\phi(\tau) = k,$$

and the pair of formulae ·403 by a single formula

·505 $$\mathrm{T} = \frac{(2m_1+1)\tau+(2m_1+4n_1)}{(4m_2+2n_2)\tau+(2n_2+1)}.$$

By an automorphic aggregate we now mean an aggregate of values of one complex number. Examples of automorphic aggregates in one variable are the aggregates generated by ·505 with coefficients subject to one or other of the conditions $\cdot 41_2$, ·405, the aggregate being definite in the one case, extended in the other. If the aggregate is definite, it follows from ·41 that its members lie all on one side of the real axis. The generating relation and condition being given, every complex number belongs to one and only one definite automorphic aggregate, and to one and only one extended automorphic aggregate; an undegenerate automorphic aggregate has no real members.

Translated into terms of one variable, ·45 becomes

8·51. *If k^2 is finite and neither* 0 *nor* 1, *the equation $\phi(\tau) = k$ is soluble and the solutions compose an undegenerate extended automorphic aggregate.*

In other words,

8·52. *If k^2 is finite and neither* 0 *nor* 1, *the equation $\phi(\tau) = k$ is an automorphic equation with one and only one solution.*

It is to be remembered that the function $\phi(\tau)$ is a defined function of τ, involving no parameters whatever.

If the inversion problem is attacked as the problem of satisfying the conditions $f_g = b$, $f_h = c$, the fundamental theorem to establish is ·52. The function $\phi(\tau)$ has the property, easily verifiable, that its value is unaltered by the substitution ·505 if the coefficients are subject to the condition ·405; for this reason the function itself is called automorphic. That the equation $\phi(\tau) = k$ is automorphic is almost trivial; the difficulty lies in proving that it has a solution.

From a slightly different point of view, the theorem ·52 asserts that if the variable τ is subject to no restrictions except that it is not to be purely real, there are no finite values except 0, 1, and -1 which the function $\phi(\tau)$ does not assume. Essentially this is a theorem on the correspondence established between the two complex variables τ, k by the relation $\phi(\tau) = k$. It is a theorem of exactly the same kind as the theorem we have proved in Chapter VII regarding the correspondence established between w and I by the relation

$$\int_w^\infty \frac{dw}{\sqrt{R(w)}} = I.$$

IX

FUNCTIONS AND INTEGRALS WITH REAL CRITICAL VALUES

9·1. If b^2 and c^2 are real, consideration of the possible reality of quarter-periods falls into three cases.

(i) If $b^2 > c^2 > 0$, with $b > 0$, the integral

$$\int_b^\infty \frac{du}{\sqrt{\{(u^2-b^2)(u^2-c^2)\}}}$$

is real, and the integral

$$\int_0^\infty \frac{i\,dv}{\sqrt{\{(v^2+b^2)(v^2+c^2)\}}}$$

is imaginary†; thus, in the notation of 6·4 and 6·5, β has a real value and α an imaginary value.

(ii) If $b^2 > 0 > c^2$, with $c = iq$, $b > 0$, $q > 0$, the integral

$$\int_b^\infty \frac{du}{\sqrt{\{(u^2-b^2)(u^2+q^2)\}}}$$

is real, and the integral

$$\int_q^\infty \frac{i\,dv}{\sqrt{\{(v^2+b^2)(v^2-q^2)\}}}$$

is imaginary; β has a real value and γ an imaginary value.

(iii) If $0 > b^2 > c^2$, with $b = ip$, $c = iq$, $q > p > 0$, the integral

$$\int_0^\infty \frac{du}{\sqrt{\{(u^2+p^2)(u^2+q^2)\}}}$$

is real, and the integral

$$\int_q^\infty \frac{i\,dv}{\sqrt{\{(v^2-p^2)(v^2-q^2)\}}}$$

is imaginary; α has a real value and γ an imaginary value.

† Throughout this chapter, and again in Chapter XVII below, it is important to remember that 'imaginary' is not synonymous with 'complex'; an imaginary number is a complex number whose real part is zero, and to call such a number *purely* imaginary is redundant if emphatic. A complex number can be called *pure* if it is either real or imaginary.

Thus in all cases the system includes one real quarterperiod and, as might have been inferred from 6·84 and 6·85, one that is imaginary.

The three cases are not as distinct as they have been allowed to seem. If f_g^2, f_h^2 have the real values C, $-B$, then g_f^2, h_f^2 have the real values $-C$, B, and since

·101 $$f_g^2+g_h^2+h_f^2 = 0,$$

g_h^2, h_g^2 have the real values A, $-A$, where

·102 $$A+B+C = 0;$$

that is, g_h^2, g_f^2 have the real values A, $-C$, and h_f^2, h_g^2 have the real values B, $-A$. Of the three real numbers A, B, C subject to ·102, the one which is algebraically greatest is necessarily positive, and the one which is algebraically least is necessarily negative. If then $A > B > C$, the pair of numbers A, $-C$ belongs to case (i); if further $B > 0$, the pair of numbers B, $-A$ belongs to case (ii) and the pair of numbers C, $-B$ to case (iii), while if $B < 0$, the pair of numbers C, $-B$ belongs to case (ii) and the pair of numbers B, $-A$ to case (iii). That is to say, the three cases must occur together, one of the three primitive functions fj$z$, gj$z$, hj$z$ satisfying the conditions associated with each case, and it is *the simultaneous occurrence of the three cases in the triplet of inseparable functions* with which we are really dealing. If the triplet possesses this property, then the system has a real and an imaginary quarterperiod.

That the converse is true follows immediately from the definition of $\wp z$. If ω_f has a real value ω and ω_g an imaginary value $i\omega'$,

·103 $$\wp(x+iy) = \frac{1}{(x+iy)^2}+\sum{}'\left\{\frac{1}{\{(x+2m\omega)+i(y+2n\omega')\}^2}-\frac{1}{(2m\omega+2in\omega')^2}\right\}.$$

If $y = 0$, the first term and the terms for which $n = 0$ are real, and the terms for which $n \neq 0$ can be added in conjugate pairs; if $x = 0$, the first term and the terms for which $m = 0$ are real, and the terms for which $m \neq 0$ can be added in conjugate pairs:

9·11. *If $\wp z$ has one real and one imaginary period, then $\wp z$ is real if z is either real or imaginary.*

In particular, $\wp\omega_f$ and $\wp\omega_g$, that is, e_f and e_g, are real, and since $e_f+e_g+e_h = 0$, e_h is real, and so also are the differences which are the squares of the critical values of the primitive functions. Hence

9·12. *A system in which the squares of the critical values of the primitive functions are all real is a system which has one real quarterperiod and one imaginary quarterperiod.*

9·2. In our direct investigation into the inversion of the integral $I(w)$, we have identified the particular integrals α, β, γ with the quarter-periods ω_f, ω_g, ω_h and the function $w(I)$ with $\mathrm{fj}\, I$. To allocate the integrals differently is to permute the symbols α, β, γ, but to permute the symbols ω_f, ω_g, ω_h would be to deny the notation of which these symbols form part. To put the matter differently, the six functions

$$\mathrm{fj}(z;\alpha,\beta,\gamma), \qquad \mathrm{gj}(z;\gamma,\alpha,\beta), \qquad \mathrm{hj}(z;\beta,\gamma,\alpha),$$
$$\mathrm{fj}(z;\alpha,\gamma,\beta), \qquad \mathrm{gj}(z;\beta,\alpha,\gamma), \qquad \mathrm{hj}(z;\gamma,\beta,\alpha)$$

are identically the same function of z, but such a collection of symbols as $\mathrm{gj}(z;\omega_h,\omega_f,\omega_g)$ is literally a contradiction in terms.

If then we are to translate the results of ·1 into results concerning a set of functions in which ω_f is real and ω_g imaginary, it is the functional symbol which is different in the different cases, corresponding in each case to the part played by α: in (i), α is imaginary and coincides with ω_g; in (ii), α is complex and coincides with ω_h; in (iii), α is real and coincides with ω_f.

9·21. *If ω_f is real and ω_g imaginary, then f_g and f_h are both imaginary, g_f and g_h are both real, h_f is real and h_g is imaginary.*

We can render these results almost self-evident by locating more completely the real values of $\wp z$. We suppose as above that ω_f has a real value ω and that ω_g has an imaginary value $i\omega'$; we do not assume that the real numbers ω, ω' are positive, for we have presently to make comparisons in which this restriction would have to be removed.

One period parallelogram for the function $\wp z$ is now a rectangle of which one side extends along the real axis from 0 to 2ω and one along the imaginary axis from 0 to $2i\omega'$. By the midlines of the rectangle we mean the lines $x = \omega$ and $y = \omega'$, which cross at right angles at the midpoint $\omega + i\omega'$, which is $-\omega_h$.

Since e_f, e_g, e_h are real, two formulae typified by 0·79, namely

·201 $$\{\wp z - e_f\}\{\wp(z+\omega_f) - e_f\} = (e_g - e_f)(e_h - e_f),$$

·202 $$\{\wp z - e_g\}\{\wp(z+\omega_g) - e_g\} = (e_f - e_g)(e_h - e_g),$$

imply that if $\wp z$ is real, so also are $\wp(z+\omega)$ and $\wp(z+i\omega')$. Hence

9·22. *If $\wp z$ has two pure periods, then $\wp z$ is real along the midlines of a period rectangle.*

It is convenient to denote the four points 0, ω, $i\omega'$, $\omega+i\omega'$ now by J, F, G, H, and to call the rectangle of which these points are the vertices the fundamental rectangle. The function $\wp z$ is real at every point of the perimeter of the fundamental rectangle, and if z describes this perimeter continuously in the direction $JGHFJ$, the value of $\wp z$ varies continuously from $-\infty$ to $+\infty$, being dominated by $-1/y^2$ on the imaginary axis near the origin and by $1/x^2$ on the real axis near the origin; hence *$\wp z$ assumes every real value at least once on the perimeter.* If z_1, z_2 are two incongruent points where $\wp z$ has the same value, $z_1+z_2 \equiv 0$, and therefore $\frac{1}{2}(z_1+z_2)$ is congruent with 0 or with a halfperiod of $\wp z$; but if z_1, z_2 are two points on the perimeter of the fundamental rectangle, $\frac{1}{2}(z_1+z_2)$ is either inside the rectangle or on the perimeter and can not be zero or a halfperiod unless z_1, z_2 coincide at a corner—in any case $\frac{1}{2}(z_1+z_2)$ is not zero or a halfperiod if z_1, z_2 are distinct, and *$\wp z$ does not assume any value more than once on the perimeter.* Hence

9·23. *As z describes the perimeter $JGHFJ$ of the fundamental rectangle, $\wp z$ increases steadily through all real values from $-\infty$ to $+\infty$.*

Incidentally we have established the inequalities

·203 $$e_g < e_h < e_f,$$

which taken with $e_f+e_g+e_h = 0$ imply that e_f is positive and e_g negative, and therefore that *in the case under consideration $\wp z$ is positive for all real values of z and negative for all imaginary values of z.*

Since $\wp z = \wp z_1$ implies $z \equiv \pm z_1$, we can complete ·22 from ·23:

9·24. *If $\wp z$ has the pure periods 2ω, $2i\omega'$, then $\wp(x+iy)$ is real if x is a multiple of ω or y of ω', but not otherwise.*

To subtract e_f, e_g, or e_h from $\wp z$ does not affect the monotonic property expressed in ·23 but brings the zero to a known point. Hence

9·25. *On the perimeter of the fundamental rectangle, the squares of the functions* $\mathrm{fj}\,z$, $\mathrm{gj}\,z$, $\mathrm{hj}\,z$ *are everywhere real;* $\mathrm{fj}\,z$ *is real along FJ and imaginary along $FHGJ$;* $\mathrm{gj}\,z$ *is real along $GHFJ$ and imaginary along GJ;* $\mathrm{hj}\,z$ *is real along HFJ and imaginary along HGJ.*

On the perimeter, each function has only one zero and only one infinity, and these are the points which divide the real stretch from the imaginary stretch. Hence there is no change of sign along a real stretch or along an imaginary stretch, and the signs which the function has near the origin persist along the two stretches. Near the origin each function is dominated by the term $1/z$, which is positive for positive real values of z and negatively imaginary for positively imaginary values of z. The sign of x along JF is the sign of ω, and the sign of y along JG is the sign of ω'.

9·26. *The real values of* fj z, gj z, hj z *along the perimeter of the fundamental rectangle have the sign of* ω, *the imaginary values are negatively or positively imaginary according as* ω' *is positive or negative.*

The results of ·25 and ·26 can be extended immediately to the whole set of elementary functions. To the function pq z, where p, q are two of the four letters j, f, g, h, there correspond two points P, Q which are two of the four points J, F, G, H.

9·27. *The function* pq z *is real with a definite sign along one of the stretches into which the points* P, Q *divide the perimeter of the fundamental rectangle, imaginary with a definite sign along the other of these two stretches; the function is purely real or imaginary if* x *is a multiple of* ω *or* y *of* ω', *but not otherwise.*

Near the point Q the function resembles $1/(z-z_Q)$ or $-1/(z-z_Q)$, and of the two sides of the rectangle which meet at Q, one is parallel to the real axis and the other is parallel to the imaginary axis. The real stretch for the function pq z is the stretch PQ which includes the former of these sides, the imaginary stretch the stretch PQ which includes the latter. If Q is J, F, or G, the pole is necessarily positive, but H is $-\omega_h$ and is a positive pole of jh z and a negative pole of fh z and gh z. Hence the two functions fh z, gh z are real and have the opposite sign to $x-x_H$ at points near H on the line $y = y_H$; on the line $x = x_H$ these functions are positively imaginary for small positive values of $y-y_H$ and negatively imaginary for small negative values of $y-y_H$. In each of the other ten cases, the function has the same sign as $x-x_Q$ near Q on the line $y = y_Q$ and the opposite sign to $i(y-y_Q)$ near Q on the line $x = x_Q$. So far the determinations are independent of the signs of ω and ω', but the signs of $x-x_Q$ and $y-y_Q$ on the perimeter depend on the signs of ω and ω' as well as on the position of Q. Results are most easily read from a diagram.

9·3. Combining ·25 and ·26 to determine the nature of the six critical values, remembering that the values of fj z and gj z at H are $-f_h$ and $-g_h$, we see that if ω and ω' are positive, f_g is negatively imaginary, f_h is positively imaginary, g_f is real and positive, g_h is real and negative, h_f is real and positive, and h_g is negatively imaginary. Thus g_f/f_g, h_g/g_h, f_h/h_f are all positively imaginary, and since the square of each of these fractions is -1, the fractions have the common value i. If the sign of ω is changed, the three critical values that are real change sign together, and if the sign of ω' is changed, each of the imaginary critical values is replaced by its negative. That is, in agreement with 1·66,

9·31. *According as ω'/ω is positive or negative,*

$$\cdot 31_1 \qquad g_f/f_g = h_g/g_h = f_h/h_f = i$$

or

$$\cdot 31_2 \qquad g_f/f_g = h_g/g_h = f_h/h_f = -i.$$

The relation between the three real constants g_f, g_h, h_f is

$$9\cdot 32 \qquad g_h^2+h_f^2 = g_f^2.$$

To put the results of ·31, ·32 differently, let b, c, d be the positive square roots of the positive real numbers e_f-e_g, e_h-e_g, e_f-e_h; then

$$\cdot 301 \qquad b^2 = c^2+d^2,$$

identically, the three real critical values are given by

$$\cdot 302\text{–}\cdot 304 \qquad g_f = \pm b, \qquad g_h = \mp c, \qquad h_f = \pm d,$$

the upper or the lower signs being taken according as ω is positive or negative, and the three imaginary critical values are given by

$$\cdot 305\text{–}\cdot 307 \qquad f_g = \mp ib, \qquad h_g = \mp ic, \qquad f_h = \pm id,$$

the upper or the lower signs being taken according as ω' is positive or negative.

It will be noticed that ·302–·307 restrict the values possible simultaneously to the critical values. Naturally gj z and hj z must have the same sign at F, if they are real along JF, for they have the same approximate form near J and no change of sign takes place. They are distinct functions, and there is nothing in our previous work with which this result seems to clash. But g_f and g_h are the critical values of the same function gj z, and the qualitative restriction, that these values must have different signs, needs explanation. It is quite possible to define a function with real critical values that are both positive; in fact it is quite possible to arrange for b and c to be the critical values of the very function gj z with which we are dealing. And there is no fallacy in the proof that the function has both a real and an imaginary quarterperiod. But what is implied by ·302–·303 is that a choice of a real period and an imaginary period to constitute a primitive pair for the function is incompatible with a choice of critical values with the same sign: the point $-\omega_g$ is the corner opposite to the origin in the parallelogram ω_f, ω_h, and therefore if ω_g is on the imaginary axis, ω_f and ω_h are on lines equidistant from that axis and the values of gj z on these lines have opposite signs. Without changing the function gj z or the quarterperiod ω we can take $\omega_h = \omega+i\omega'$ and secure $g_f = b$, $g_h = c$, but now $\omega_g = -2\omega-i\omega'$ and neither ω_g nor ω_h is purely imaginary.

If we refer to the conditions imposed in 6·3 and 6·5 on paths from which the integrals α, β, γ are made definite, we see that the first of these paths and its reflection in the origin divide the plane into two distinct parts, and that the paths to ∞ from b, c lie wholly in the same division. If b, c are real, with $b^2 > c^2$, we can satisfy these conditions whether or not b and c have the same sign, but the proof that α can be real depends on using one half or the other of the imaginary axis itself as the path of integration from 0 to ∞, and this particular choice is impossible if the signs of b and c are different. That is to say, if the signs of b and c are different, the transformation in $\cdot 1$ (i) is impossible and we have no reason to expect a primitive pair of quarterperiods of the form ω, $i\omega'$; what we have now learned is that in this case such a pair can not in fact exist.

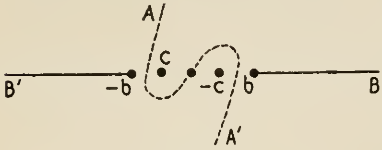


FIG. 29.

We can write down both ω and ω' in terms of b, c, d from any one of the functions fj z, gj z, hj z. The immediate theorems are

9·33$_{1-3}$. *Expressions for* $\pm\omega$ *as integrals are*

$$\int_0^\infty \frac{du}{\sqrt{\{(u^2+b^2)(u^2+d^2)\}}}, \quad \int_b^\infty \frac{du}{\sqrt{\{(u^2-b^2)(u^2-c^2)\}}}, \quad \int_d^\infty \frac{du}{\sqrt{\{(u^2+c^2)(u^2-d^2)\}}};$$

9·33$_{4-6}$. *Expressions for* $\pm\omega'$ *as integrals are*

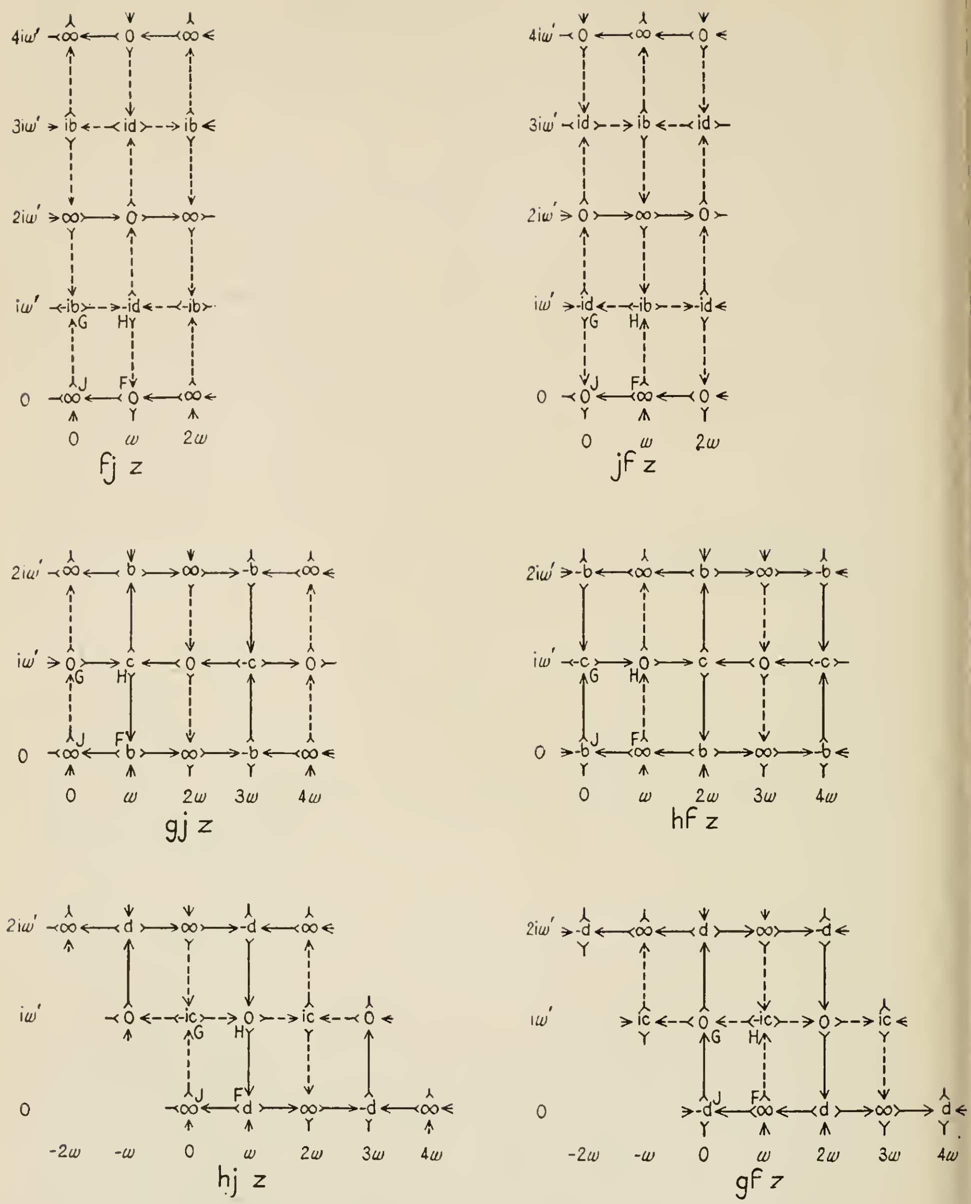
$$\int_b^\infty \frac{dv}{\sqrt{\{(v^2-b^2)(v^2-d^2)\}}}, \quad \int_0^\infty \frac{dv}{\sqrt{\{(v^2+b^2)(v^2+c^2)\}}}, \quad \int_c^\infty \frac{dv}{\sqrt{\{(v^2-c^2)(v^2+d^2)\}}}.$$

The implied equalities between integrals are made obvious by a preliminary substitution $u^2 = U$, $v^2 = V$.

Since the integrals

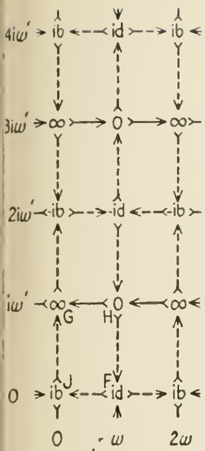
$$\int_0^{ib} \frac{dw}{\sqrt{\{(w^2+b^2)(w^2+d^2)\}}}, \quad \int_0^{b} \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}}, \quad \int_{ic}^{d} \frac{dw}{\sqrt{\{(w^2+c^2)(w^2-d^2)\}}}$$

are combinations of an integral in $\cdot 33_{1-3}$ with an integral in $\cdot 33_{4-6}$, each of them has a value $\pm\omega\pm i\omega'$ for an appropriate path in the complex plane. It follows from the investigation in Chapter VI that a path which does not surround any of the branchpoints is appropriate in this sense, and therefore that we may take the first path along the imaginary axis, the second path along the real axis, and the third path to the origin along one axis and away from the origin along the other axis. The first two paths pass through branchpoints, and although the substitution has the same form along the whole path, the radical is real on one side of the branchpoint and imaginary on the other side; on

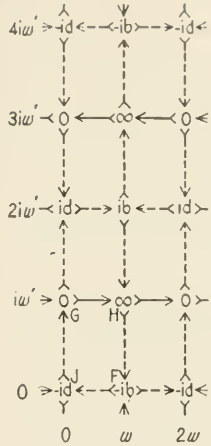


F

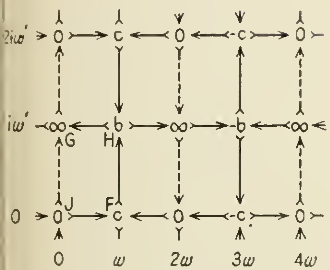
Full lines indicate real values, brok



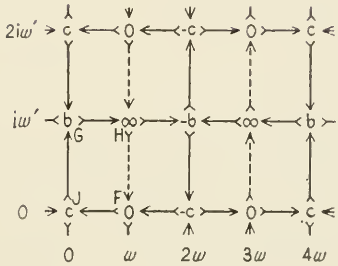
hg z



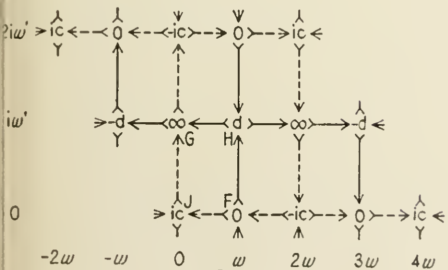
gh z



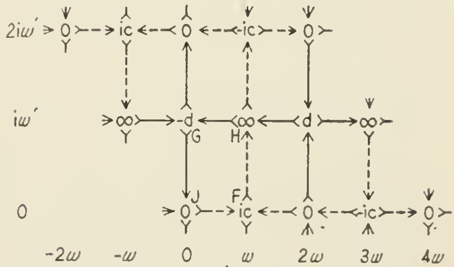
jg z



Fh z



Fg z



jh z

imaginary values, of the function.

the third path the substitution changes form at the origin. Thus we have $\pm\omega\pm i\omega'$ expressed in the three forms

$$i\int_0^d \frac{dv}{\sqrt{\{(b^2-v^2)(d^2-v^2)\}}}+\int_d^b \frac{dv}{\sqrt{\{(b^2-v^2)(v^2-d^2)\}}},$$

$$\int_0^c \frac{du}{\sqrt{\{(b^2-u^2)(c^2-u^2)\}}}+i\int_c^b \frac{du}{\sqrt{\{(b^2-u^2)(u^2-c^2)\}}},$$

$$\int_0^c \frac{dv}{\sqrt{\{(c^2-v^2)(d^2+v^2)\}}}+i\int_0^d \frac{du}{\sqrt{\{(c^2+u^2)(d^2-u^2)\}}},$$

and to $\cdot 33_{1-6}$ we can add

$9{\cdot}33_{7-9}$. *Expressions for* $\pm\omega$ *as integrals are*

$$\int_d^b \frac{du}{\sqrt{\{(b^2-u^2)(u^2-d^2)\}}},\quad \int_0^c \frac{du}{\sqrt{\{(b^2-u^2)(c^2-u^2)\}}},\quad \int_0^c \frac{du}{\sqrt{\{(c^2-u^2)(d^2+u^2)\}}};$$

$9{\cdot}33_{10-12}$. *Expressions for* $\pm\omega'$ *as integrals are*

$$\int_0^d \frac{dv}{\sqrt{\{(b^2-v^2)(d^2-v^2)\}}},\quad \int_c^b \frac{dv}{\sqrt{\{(b^2-v^2)(v^2-c^2)\}}},\quad \int_0^d \frac{dv}{\sqrt{\{(c^2+v^2)(d^2-v^2)\}}}.$$

The formulae $\cdot 33_8$, $\cdot 33_9$ for ω come from $\cdot 33_2$, $\cdot 33_3$ by the substitutions of bc/u, cd/u for u, and the formulae $\cdot 33_{10}$, $\cdot 33_{12}$ for ω' from $\cdot 33_4$, $\cdot 33_6$ by the substitutions of bd/v, cd/v for v.

Interchange of c with d interchanges the sets of formulae for ω with the sets for ω'. This was to be anticipated, for if $\mathrm{pq}\,z$ is a function with quarterperiods ω, $i\omega'$, then $\mathrm{pq}\,iz$ is a function with quarterperiods $i\omega$, ω'. To put ω', $i\omega$ into the parts of ω_f, ω_g involves minor adjustments equivalent to a change of sign of one quarterperiod, but substantially the change is from a system in which the critical values g_f, $-g_h$, h_f are b, c, d to one in which these values are b, d, c.

Detailed descriptions of the behaviour of the twelve elementary functions are better incorporated in diagrams than tabulated or formulated, and a set of diagrams for positive values of ω and ω' constitutes Figure 30. Lines along which the function is real are continuous, lines along which it is imaginary, broken. Zeros and infinities are indicated, and critical values are inserted, with the notation of $\cdot 302$–$\cdot 307$: b, c, d are positive real numbers definable as g_f, $-g_h$, h_f. Barbs show the direction of algebraic increase, and therefore the values along a stretch between a zero and an infinity are positive or positively imaginary if the barbs point from 0 towards ∞, negative or negatively imaginary if the barbs point from ∞ towards 0.

The diagrams are extended to cover a complete period parallelogram of each function. All that is necessary for this extension is to remember that a direction of increase is unchanged at a zero or an infinity, since the zeros and infinities are all simple, but is reversed at a critical point which is neither one nor the other. For the functions $\mathrm{fj}\,z$ and $\mathrm{gj}\,z$ we naturally choose as period parallelograms the rectangles 2ω, $4i\omega'$ and 4ω, $2i\omega'$; for $\mathrm{hj}\,z$ no rectangle is available, but by choosing 4ω, $-2\omega+2i\omega'$ as the primitive pair of periods we secure that the fundamental rectangle $JFHG$ is included in the period parallelogram.

9·4. We can express ·27 as a property of the transformation $w = \mathrm{pq}\,z$ rather than of the function $\mathrm{pq}\,z$:

9·41. *In the transformation* $w = \mathrm{pq}\,z$, *there is a point to point correspondence between the perimeter of the fundamental rectangle in the* z *plane and the pair of half-lines bounding a definite quadrant of the* w *plane.*

This theorem prompts us to ask, before turning to the classification of real integrals, if the transformation associates points inside the rectangle with points belonging to the quadrant.

The lines $x = m\omega$, $y = n\omega'$ dissect the z plane into rectangles each of which is congruent geometrically with the fundamental rectangle; we will call these rectangles cells. Each of the period parallelograms 2ω, $4i\omega'$ and 4ω, $2i\omega'$ is dissected into eight cells. The period parallelogram 4ω, $-2\omega+2i\omega'$ of $\mathrm{hj}\,z$ is dissected into six cells and four triangles, and the triangles have to be associated in pairs to form regions equivalent to two more cells. For our immediate purpose we can however avoid even verbal conventions by recalling the distinction between a period parallelogram and a parallelogram which is a primitive region. The rectangle 4ω, $2i\omega'$ is not a period parallelogram for $\mathrm{hj}\,z$, for $2i\omega'$ is not a period of the function. But the triangle $(0, 2i\omega', -2\omega+2i\omega')$ is congruent with the triangle $(4\omega, 4\omega+2i\omega', 2\omega+2i\omega')$, and therefore the rectangle 4ω, $2i\omega'$, like the period parallelogram 4ω, $-2\omega+2i\omega'$, is† a primitive region for $\mathrm{hj}\,z$: of any set of points $z+m.4\omega+n(-2\omega+2i\omega')$ congruent for the function, this rectangle contains one and only one.

† The reader is invited to consider why both the rectangles 2ω, $4i\omega'$ and 4ω, $2i\omega'$, which are period parallelograms for $\mathrm{fj}\,z$ and $\mathrm{gj}\,z$, are primitive regions for $\mathrm{hj}\,z$, whereas the first is not a primitive region for $\mathrm{gj}\,z$ or the second a primitive region for $\mathrm{fj}\,z$.

Thus we can say that each of the twelve elementary elliptic functions has a primitive region consisting of precisely eight cells.

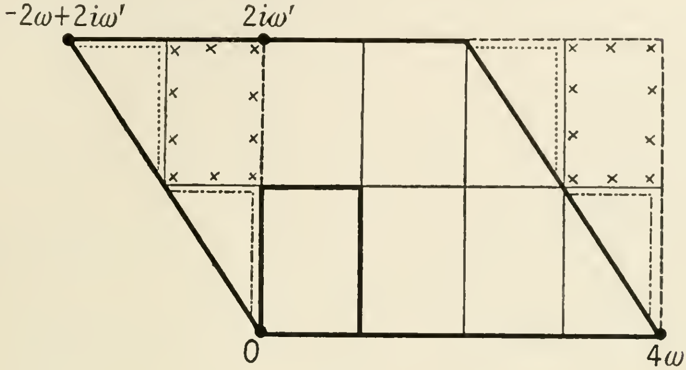


FIG. 31.

We can repeat for the interior of a cell the argument used in ·2 regarding the perimeter of the fundamental rectangle. If z_1, z_2 are inside the same cell, then $\frac{1}{2}(z_1+z_2)$ is inside that cell and is of the form $(h+\xi)\omega+(k+\eta)i\omega'$, where h, k are integers and ξ, η real numbers between 0 and 1. Hence z_1+z_2 is of the form $2(h+\xi)\omega+2(k+\eta)i\omega'$ and is not of the form $2m\omega+2ni\omega'$ with integral values of m, n, and therefore the values of $\mathrm{pq}\, z_1$, $\mathrm{pq}\, z_2$ are not equal:

9·42. *Two points at which* $\mathrm{pq}\, z$ *has the same value can not be inside the same cell.*

Again, from ·27, no points inside any cell give either real or imaginary values to $\mathrm{pq}\, z$; it follows that if z_1, z_2 are inside the same cell, the arc in the w plane corresponding to an arc $z_1 z_2$ which lies wholly in the cell is an arc which joins w_1 to w_2 without crossing either the real axis or the imaginary axis, even at infinity:

9·43. *If z_1, z_2 are in the same cell of the z plane, then w_1, w_2 are in the same quadrant of the w plane.*

Now let w_1, w_2, w_3, w_4 be points one in each quadrant of the w plane, and, for $r = 1, 2, 3, 4$, let z_r', z_r'' be the two points in a primitive region of $\mathrm{pq}\, z$ which satisfy the equation $\mathrm{pq}\, z = w_r$. Of the eight points so defined, ·42 implies that two for which r is the same are not in the same cell, and ·43 implies that two for which r is different are not in the same cell. Hence the eight points are in eight different cells, and since the primitive region consists of eight cells, there is one and only one of the eight points in each cell.

Since w_1, w_2, w_3, w_4, inside their several quadrants, are independent, it follows that

9·44. *Each quadrant of the w plane is associated by the function* $\mathrm{pq}\,z$ *with a pair of cells in a primitive region of the z plane.*

As a corollary,

9·45. *If z_2 is inside the same cell as z_1, the second point in any primitive region at which* $\mathrm{pq}\,z$ *has the value* $\mathrm{pq}\,z_2$ *is inside the same cell as the second point at which* $\mathrm{pq}\,z$ *has the value* $\mathrm{pq}\,z_1$.

In other words, the function $\mathrm{pq}\,z$ couples the eight cells composing a primitive region in four pairs. Different functions with a common primitive region may couple the cells differently.

We can now absorb ·41 into a much more complete theorem:

9·46. *If one quarterperiod is real and one imaginary, the transformation* $w = \mathrm{pq}\,z$ *establishes a point-to-point correspondence between a rectangle and its perimeter in the z plane and a quadrant and its boundary in the w plane.*

On the perimeter of a rectangle there are four exceptional points, the four corners. On the boundary of a quadrant there are only two exceptional points, the origin and the point at infinity. Since the angles are all right angles, the transformation remains conformal at the two corners ω_p, ω_q in the z plane since these correspond to 0 and ∞ in the w plane, but there must still be two points where the correspondence, though definite, is not conformal. In the z plane, these are two corners of the rectangle; in the w plane, they are points on the boundary, with no geometrical peculiarity.

9·47. *The transformation of a rectangle into a quadrant by means of the functional relation* $w = \mathrm{pq}\,z$ *is conformal except at the points* ω_r, ω_t *which are zeros of* $\mathrm{pq}'z$ *in the z plane and the corresponding points* $\mathrm{pq}\,\omega_r$, $\mathrm{pq}\,\omega_t$ *in the w plane.*

We anticipate three types of function: if the singularities in the w plane are both on the real radius of the quadrant, the function is coperiodic with $\mathrm{gj}\,z$, if one is on the real radius and one on the imaginary radius, the function is coperiodic with $\mathrm{hj}\,z$, and if both are on the imaginary radius, the function is coperiodic with $\mathrm{fj}\,z$. This is the classification of ·1, from another point of view.

9·5. In discussing the evaluation of real integrals we suppose that ω and ω', determined from given positive constants b, c, d such that

$b^2 = c^2+d^2$ by any of the equivalent formulae in ·33, are chosen to be positive, and that elliptic functions are constructed with ω for the first quarterperiod and $i\omega'$ for the second. The radicals in the integrals are all taken to be positive.

If x is the value of the real integral

$$\int_t^\infty \frac{dt}{\sqrt{\{(t^2-b^2)(t^2-c^2)\}}},$$

then $t = \mathrm{gj}\,x$. Even with the restriction to real values the functional equation alone does not determine x when t is given, but since the integral is to be real we are supposing $t \geqslant b$. As t decreases from ∞ to b, the value of the integral increases steadily from 0 to ω; that is, $0 \leqslant x \leqslant \omega$, and in this range the function $\mathrm{gj}\,x$ is monotonic and can not assume any value for more than one value of x.

The differential equation

$$(dw/dz)^2 = (w^2-b^2)(w^2-c^2)$$

satisfied by $\mathrm{gj}\,z$ is satisfied by the coperiodic functions $\mathrm{hf}\,z$, $\mathrm{jg}\,z$, $\mathrm{fh}\,z$ also, and we see from Figure 30 that as x increases from 0 to ω, $\mathrm{hf}\,x$ decreases from $-b$ to $-\infty$, $\mathrm{jg}\,x$ increases from 0 to c, and $\mathrm{fh}\,x$ decreases from c to 0.

9·51. *The equations*

$$\mathrm{gj}\,x_1 = t_1, \qquad \mathrm{hf}\,x_2 = -t_2$$

with the condition $0 \leqslant x \leqslant \omega$ *determine* x_1, x_2 *as singlevalued real functions of* t_1, t_2 *for the range* $b \leqslant t$, *and* x_1, x_2 *so determined are the values of the integrals*

$$\int_{t_1}^\infty \frac{dt}{\sqrt{\{(t^2-b^2)(t^2-c^2)\}}}, \qquad \int_b^{t_2} \frac{dt}{\sqrt{\{(t^2-b^2)(t^2-c^2)\}}}.$$

9·52. *The equations*

$$\mathrm{fh}\,x_3 = t_3, \qquad \mathrm{jg}\,x_4 = t_4$$

with the condition $0 \leqslant x \leqslant \omega$ *determine* x_3, x_4 *as singlevalued real functions of* t_3, t_4 *for the range* $0 \leqslant t \leqslant c$, *and* x_3, x_4 *so determined are the values of the integrals*

$$\int_{t_3}^c \frac{dt}{\sqrt{\{(b^2-t^2)(c^2-t^2)\}}}, \qquad \int_0^{t_4} \frac{dt}{\sqrt{\{(b^2-t^2)(c^2-t^2)\}}}.$$

There are four elementary functions associated with the equation

$$(dw/dz)^2 = (w^2+c^2)(w^2-d^2);$$

$\mathrm{hj}\,x$ decreases from ∞ to d and $\mathrm{gf}\,x$ decreases from $-d$ to $-\infty$, as x increases from 0 to ω. Since $\mathrm{fg}\,x$ and $\mathrm{jh}\,x$ are imaginary in this range, $-i\,\mathrm{fg}\,x$ and $-i\,\mathrm{jh}\,x$ are real; in real terms, these functions, of which the first decreases from c to 0 and the second increases from 0 to c, satisfy the equation

$$(dt/dx)^2 = (c^2-t^2)(d^2+t^2).$$

9·53. *The equations*

$$\mathrm{hj}\,x_5 = t_5, \qquad \mathrm{gf}\,x_6 = -t_6$$

with the condition $0 \leqslant x \leqslant \omega$ *determine* x_5, x_6 *as singlevalued real functions of* t_5, t_6 *for the range* $d \leqslant t$, *and* x_5, x_6 *so determined are the values of the integrals*

$$\int_{t_5}^{\infty} \frac{dt}{\sqrt{\{(t^2+c^2)(t^2-d^2)\}}}, \qquad \int_{d}^{t_6} \frac{dt}{\sqrt{\{(t^2+c^2)(t^2-d^2)\}}}.$$

9·54. *The equations*

$$\mathrm{fg}\,x_7 = it_7, \qquad \mathrm{jh}\,x_8 = it_8$$

with the condition $0 \leqslant x \leqslant \omega$ *determine* x_7, x_8 *as singlevalued real functions of* t_7, t_8 *for the range* $0 \leqslant t \leqslant c$, *and* x_7, x_8 *so determined are the values of the integrals*

$$\int_{t_7}^{c} \frac{dt}{\sqrt{\{(c^2-t^2)(d^2+t^2)\}}}, \qquad \int_{0}^{t_8} \frac{dt}{\sqrt{\{(c^2-t^2)(d^2+t^2)\}}}.$$

There remain the functions for which the differential equation is

$$(dw/dz)^2 = (w^2+b^2)(w^2+d^2).$$

As x increases from 0 to ω, $\mathrm{fj}\,x$ decreases from ∞ to 0, and $\mathrm{jf}\,x$ decreases from 0 to $-\infty$. Since $\mathrm{hg}\,x$ and $\mathrm{gh}\,x$ are imaginary, $-i\,\mathrm{hg}\,x$ and $-i\,\mathrm{gh}\,x$ are real; the first decreases from b to d, the second decreases from $-d$ to $-b$, and they both satisfy the equation

$$(dt/dx)^2 = (b^2-t^2)(t^2-d^2).$$

9·55. *The equations*

$$\mathrm{fj}\,x_9 = t_9, \qquad \mathrm{jf}\,x_{10} = -t_{10}$$

with the condition $0 \leqslant x \leqslant \omega$ *determine* x_9, x_{10} *as singlevalued real functions of* t_9, t_{10} *for the range* $0 \leqslant t$, *and* x_9, x_{10} *so determined are the values of the integrals*

$$\int_{t_9}^{\infty} \frac{dt}{\sqrt{\{(t^2+b^2)(t^2+d^2)\}}}, \qquad \int_{0}^{t_{10}} \frac{dt}{\sqrt{\{(t^2+b^2)(t^2+d^2)\}}}.$$

9·56. *The equations*

$$\operatorname{hg} x_{11} = it_{11}, \qquad \operatorname{gh} x_{12} = -it_{12}$$

with the condition $0 \leqslant x \leqslant \omega$ *determine* x_{11}, x_{12} *as singlevalued real functions of* t_{11}, t_{12} *for the range* $d \leqslant t \leqslant b$, *and* x_{11}, x_{12} *so determined are the values of the integrals*

$$\int_{t_{11}}^{b} \frac{dt}{\sqrt{\{(b^2-t^2)(t^2-d^2)\}}}, \qquad \int_{d}^{t_{12}} \frac{dt}{\sqrt{\{(b^2-t^2)(t^2-d^2)\}}}.$$

In real terms, the undegenerate integral

$$\int \frac{dt}{\sqrt{\{\pm(t^2-P)(t^2-Q)\}}},$$

in which we may suppose t positive, takes one of six essentially distinct forms. If P, Q have positive values p^2, q^2, with $0 < q < p$, the character of the integral is different in the three ranges $(0, q)$, (q, p), (p, ∞); if P is negative and Q has the positive value q^2, the ranges $(0, q)$, (q, ∞) need separate consideration; only if P and Q are both negative is there no subdivision. For each of the six forms there are two standard integrals, for the fixed limit of integration may be taken at either end of the range to which t is confined. Thus a set of twelve functions, closely allied analytically, but differing in detail in the real domain, is naturally associated with this problem of integration.

The sum of the two integrals associated with the same range of values of t, if the variable limits coincide, is on the one hand the integral over the whole range of values of t, and on the other hand the difference between the smallest and the largest values of x, which in every case is ω. In this way each of the six pairs of formulae ·51–·56 is bound up with one of the formulae $\cdot 33_{1-3,7-9}$.

It is particularly to be emphasized that the formulae of this section are read, without suppressed algebra, from the diagrams composing Figure 30. An alternative set of formulae is based on the consideration of values along the imaginary axis. On this axis the functions, written as functions of iy, are functions of the real variable y. The formulae are derivable from those already given by the interchange of ω, c, f with ω', d, g and the substitution of iy, $\pm it$ for x, t in the functional equations. For example, the value y_1 of the integral

$$\int_{t_1}^{\infty} \frac{dt}{\sqrt{\{(t^2-b^2)(t^2-d^2)\}}}$$

is determined by

$$\mathrm{fj}\, iy_1 = -it_1, \qquad 0 \leqslant y_1 \leqslant \omega',$$

and the value of the integral

$$\int\limits_{t_7}^{d} \frac{dt}{\sqrt{\{(c^2+t^2)(d^2-t^2)\}}}$$

is determined by

$$\mathrm{gf}\, iy_7 = -t_7, \qquad 0 \leqslant y_7 \leqslant \omega'.$$

Although it is sometimes it and sometimes $-it$ that replaces t, there is no ambiguity in any individual formula, and it is simpler to refer to the diagrams than to a set of rules.

Again the expressions for the quarterperiod can be recovered, but the point that should now be clear is that one formula for ω and one for ω' can be read from each of the diagrams in Figure 30 without subsidiary analysis. The variable in the integral is t or it for ω according as the function with which the diagram deals is real or imaginary along the real axis, and is t or it for ω' according as the function is imaginary or real along the imaginary axis. The limits of integration are marked on the axes. The factors in the radical are w^2-r^2, w^2-s^2, where $\pm r$, $\pm s$ are the critical values, real or imaginary, marked on the diagram, and these factors become t^2-r^2, t^2-s^2 or $-t^2-r^2$, $-t^2-s^2$ according as the variable is t or it. Automatically the radical absorbs if necessary a factor i and is real within the range of integration. In this way the diagram for $\mathrm{hf}\, z$ gives the formulae $\cdot 33_2$ for ω and $\cdot 33_{11}$ for ω'. The whole operation is far simpler to perform than to describe. Each of the six expressions for each quarterperiod is implicit in two diagrams, and two diagrams which give the same formula for one quarterperiod give different formulae for the other.

9·6. A formula expressing one of the elementary functions in terms of another is a substitution reducing one of the standard integrals to another. For example, the relation $\mathrm{jf}\, z\, \mathrm{fj}\, z = -g_f h_f$, that is, $\mathrm{jf}\, z\, \mathrm{fj}\, z = -bd$, expresses that the transformation $t_9 = bd/t_{10}$ converts the first of the integrals in $\cdot 55$ into the second. Similarly,

$$\mathrm{fg}^2 z = \frac{h_g^2(\mathrm{gj}^2 z - g_f^2)}{\mathrm{gj}^2 z},$$

and writing $\mathrm{fg}\, z = it_7$, $\mathrm{gj}\, z = t_1$, we have the transformation

$$t_7^2 = c^2 - \frac{b^2c^2}{t_1^2}$$

which converts the first integral in ·54 into the first integral in ·51. Verification is immediate, but it is from the functional side that the transformations can be foreseen most readily.

In the same way, the equality of alternative expressions for the same integral, by means of a function of x and by means of a function of iy, depends ultimately on the identity

$$\mathrm{pq}(z;\alpha,\beta,\gamma) = \lambda\,\mathrm{pq}(\lambda z;\lambda\alpha,\lambda\beta,\lambda\gamma),$$

which, for $\lambda = i$, gives

$$\mathrm{pq}(z;\omega,i\omega',-\omega-i\omega') = i\,\mathrm{pq}(iz;i\omega,-\omega',-i\omega+\omega').$$

To arrange the periods on the right as $-\omega'$, $i\omega$, $\omega'-i\omega$ involves only an interchange of f with g, if either of these symbols occurs, and the minor adjustments necessary when the sign of ω_f is changed are obvious in each individual case.

Table IX i

Relations between functions of z with quarterperiods ω, $i\omega'$, $-\omega-i\omega'$ and functions of iz with quarterperiods ω', $i\omega$, $-\omega'-i\omega$

$\mathrm{fj}\,z = i\,\mathrm{gj}\,iz$	$\mathrm{jf}\,z = i\,\mathrm{jg}\,iz$	$\mathrm{hg}\,z = -i\,\mathrm{hf}\,iz$	$\mathrm{gh}\,z = -i\,\mathrm{fh}\,iz$
$\mathrm{gj}\,z = i\,\mathrm{fj}\,iz$	$\mathrm{hf}\,z = i\,\mathrm{hg}\,iz$	$\mathrm{jg}\,z = i\,\mathrm{jf}\,iz$	$\mathrm{fh}\,z = i\,\mathrm{gh}\,iz$
$\mathrm{hj}\,z = i\,\mathrm{hj}\,iz$	$\mathrm{gf}\,z = i\,\mathrm{fg}\,iz$	$\mathrm{fg}\,z = -i\,\mathrm{gf}\,iz$	$\mathrm{jh}\,z = -i\,\mathrm{jh}\,iz$

9·7. We have said that the classical inversion of the elliptic integral presented none of the theoretical difficulties which we have found serious, the reason being that the integrals involved were real functions of real variables†. Although we have taken the general solution of the inversion problem for granted in the present chapter, it is interesting to discuss the restricted problem. The difficulty is rather in discovering what has to be proved than in constructing proofs, and explanation tends to be in language too deliberately elementary.

Given two real numbers ω, ω', we can construct from them a system in which the first two quarterperiods are ω and $i\omega'$, and we find as in ·2 and ·3 that the critical values g_f, $-g_h$, h_f are real numbers with the same sign satisfying the condition $g_f^2 = g_h^2+h_f^2$. The question is whether, if b, c, d are given real numbers with the same sign satisfying the condition $b^2 = c^2+d^2$, there necessarily exist two real numbers ω, ω' such that b, c, d play the parts of g_f, $-g_h$, h_f in the system constructed on

† Strictly speaking, functions and variables that are imaginary in the sense in which we are using the word were immediately brought into the analysis; the double periodicity could not otherwise have been discovered. But the freedom of the complex plane was not conferred on the integrals, and it is this freedom, not the formal substitution of it for t in a real integral, that demands a new discipline.

ω, $i\omega'$. There is no loss of generality in supposing b, c, d all positive, for we change their common sign by changing the sign of one of the numbers ω, ω'.

We follow the argument which in 5·5 we could not press to a conclusion. From the given positive real numbers b, c, d subject to the condition

·701 $$b^2 = c^2+d^2$$

we calculate the positive real numbers ω, ω' by the formulae

9·71$_{1\text{-}2}$ $$\omega = \int_0^\infty \frac{dt}{\sqrt{\{(t^2+b^2)(t^2+d^2)\}}}, \qquad \omega' = \int_0^\infty \frac{dt}{\sqrt{\{(t^2+b^2)(t^2+c^2)\}}},$$

chosen from ·33. With the real number ω as ω_f and the imaginary number $i\omega'$ as ω_g we construct a system of primitive functions, and in this system the critical constants g_f, $-g_h$, h_f have definite positive real values which satisfy the condition

·702 $$g_f^2 = g_h^2+h_f^2.$$

Can we identify these values with b, c, d?

Suppose first that b has the fixed value 1; then ·71$_1$ defines a relation between the real variables ω, d, and ·71$_2$ defines a relation between the real variables ω', c. If c, d are subject to the condition

·703 $$c^2+d^2 = 1,$$

they both lie between 0 and 1 and one increases as the other decreases. As d decreases from 1 to 0, ω increases steadily from $\frac{1}{2}\pi$ to ∞; as c increases from 0 to 1, ω' decreases steadily from ∞ to $\frac{1}{2}\pi$. It follows that as c increases and d decreases, the ratio of ω to ω' increases steadily from 0 to ∞, and acquires any given value for one and only one pair of values of c and d; for this pair of values, ω and ω' as well as the ratio of ω to ω' are determinate. In other words, if ω and ω' are given, the conditions

$$\mu^2+\nu^2 = 1,$$

$$\int_0^\infty \frac{dt}{\sqrt{\{(t^2+1)(t^2+\nu^2)\}}} = \lambda\omega, \qquad \int_0^\infty \frac{dt}{\sqrt{\{(t^2+1)(t^2+\mu^2)\}}} = \lambda\omega'$$

are satisfied by one and only one set of positive real values of λ, μ, ν. If now we substitute t/λ for t in the integrals, we find that

9·72. *The equations*

$$\mu^2+\nu^2 = 1$$

$$\int_0^\infty \frac{dt}{\sqrt{\{(t^2+\lambda^2)(t^2+\lambda^2\nu^2)\}}} = \omega, \qquad \int_0^\infty \frac{dt}{\sqrt{\{(t^2+\lambda^2)(t^2+\lambda^2\mu^2)\}}} = \omega'$$

are satisfied by one and only one set of positive real values of λ, μ, ν,

and since these equations are identical with ·701, $\cdot 71_{1\text{-}2}$ with λ, μ, ν written for b, c/b, d/b, it follows that, for given positive real values of ω and ω', *the relations* $\cdot 71_{1\text{-}2}$ *with the condition* ·701 *are satisfied by one and only one set of positive real values of* b, c, d. Since the relations are satisfied on the one hand by the set of values b, c, d from which ω and ω' are calculated, and on the other hand by the set of critical values g_f, $-g_h$, h_f in the system of functions with $\omega_f = \omega$, $\omega_g = i\omega'$, the identification of the original constants with the critical values is complete:

9·73. *Given three positive constants* b, c, d, *none of which is zero, satisfying the condition* $b^2 = c^2+d^2$, *there is one and only one system of elliptic functions in which the quarterperiod* ω_f *has a real value* ω, *the quarterperiod* ω_g *has an imaginary value* $i\omega'$, *and the critical values* g_f, g_h, h_f *are* b, $-c$, d; *the values of* ω *and* ω' *are given by*

$$\omega = \int_0^\infty \frac{dt}{\sqrt{\{(t^2+b^2)(t^2+d^2)\}}}, \qquad \omega' = \int_0^\infty \frac{dt}{\sqrt{\{(t^2+b^2)(t^2+c^2)\}}}.$$

This theorem does not include ·12, for it does not deny the possibility of a system with real critical values but without pure quarterperiods; we could not expect to disprove such a possibility without entering the complex field. But in the majority of applications the restricted theorem is sufficient, without the general theory completed in the last chapter and used to establish ·12, to justify the introduction of elliptic functions when they are required. In particular, the real integrations in ·5 need no deeper foundation.

X

INTRODUCTION OF THE JACOBIAN FUNCTIONS

10·1. In the study of elliptic integrals and functions, standardization, reduction to normal forms, naturally plays a part. From a practical point of view, if a function is to be used in numerical work it is always worth while in the long run to reduce the number of independent parameters if this can be done by trivial transformations: we do not tabulate $\log_a x$ and $\sin ax$ as functions of two variables, although we are prepared to tabulate $\log_{10} x$ as well as $\log_e x$ and $\sin \frac{1}{2}\pi x$ as well as $\sin x$. In theoretical work, when there is a question of functional dependence on parameters, a reduction which makes available the methods of the theory of functions of one variable may be the first step to a solution: we have glanced at an illustration of this use of reduction in connexion with the inversion problem.

By substituting λw for w, we replace the integral

$$\int_w^\infty \frac{dw}{\sqrt{\{(w^2-b^2)(w^2-c^2)\}}}$$

by a constant multiple of

$$\int_{\lambda w}^\infty \frac{dw}{\sqrt{\{(w^2-b_\lambda^2)(w^2-c_\lambda^2)\}}},$$

where $b_\lambda = b/\lambda$, $c_\lambda = c/\lambda$, and λ is arbitrary. In particular, by a trivial modification we can deal with an integral involving only one parameter c/b instead of with an integral involving two that are independent.

From the point of view of the elliptic functions, the change is associated most simply with the periods. The identity

$$\mathrm{pq}(z;\alpha,\beta,\gamma) = \lambda\,\mathrm{pq}(\lambda z;\lambda\alpha,\lambda\beta,\lambda\gamma)$$

implies that the detailed behaviour of an elliptic function depends on the ratios $\omega_f:\omega_g:\omega_h$ rather than on the values of the quarterperiods, or, to put it graphically, on the shape of a period parallelogram rather than on its size and orientation: except for a constant factor, the distribution of values of the function is governed by position relative to the cardinal points. To multiply the three quarterperiods simultaneously by λ is also to divide the six critical values simultaneously by λ.

We can say that among all the triplets α, β, γ subject to the condition $\alpha+\beta+\gamma = 0$ and having the same shape, we are free to select one; we agree upon a normalizing factor λ, and the triplet $\lambda\alpha$, $\lambda\beta$, $\lambda\gamma$ is then the canonical triplet of shape $\alpha:\beta:\gamma$. The factor λ must be homogeneous of degree -1 in α, β, γ, and the choice is otherwise arbitrary: we could take $\lambda = 1/\alpha$ and secure a unit quarterperiod; we could take $\lambda = \pi/2\alpha$ and assimilate the functions to the circular functions by providing a real quarterperiod $\frac{1}{2}\pi$; possibly if the theory had originated on the functional side, one of these selections would have been made.

It was in fact the development of the theory from the side of the integrals which determined the normalizing factor and the canonical functions. The first integral to be inverted was Legendre's integral

$$\int\limits_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

and although the functions associated with this integral can not have the symmetry and the formal simplicity of the functions associated with an integral in which the radical has the more general form $\sqrt{\{(w^2-b^2)(w^2-c^2)\}}$, their importance now is far more than historical. The choice of functions and parameters in current use was determined by the lines along which the subject actually developed, and the choice can not be made to appear in every respect natural when the whole subject is approached in another way. But our object is to exhibit the classical results in a functional setting, and this requires the use of the classical notation. Only it is to be remembered that, as soon as we have found how to fit the notation into our scheme, we are dealing with functions of complex variables, and the parameters we use, whatever their traditional origin, are subject only to such restrictions as prevent functions or integrals from degenerating.

10·2. To reduce the integral

$$\int\limits_0^w \frac{dw}{\sqrt{\{(b^2-w^2)(c^2-w^2)\}}}$$

to the form of Legendre's integral, with $0 < k < 1$ if $0 < c < b$, we substitute $w = cx$, and the relation

·201 $$z = \int\limits_0^w \frac{dw}{\sqrt{\{(b^2-w^2)(c^2-w^2)\}}}$$

becomes

·202 $$u = \int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

with $k = c/b$, $u = bz$. That is to say, since ·201 is equivalent to $w = \mathrm{jg}\, z$ in a system in which $g_f = b$, $g_h = -c$,

10·21. *Legendre's relation is equivalent to*

$$kx = -\mathrm{jg}\, u$$

in a system in which $g_f = 1$, $g_h = -k$.

Since it was by the inversion of Legendre's integral that Jacobi introduced the elliptic functions with which his name is associated, we therefore say that

·203. *A set of quarterperiods* ω_f, ω_g, ω_h *is a Jacobian set if* $g_f = 1$.

In other words, admitting a constant multiplier instead of determining the specific function $\mathrm{gj}\, z$ by its residue,

10·22. *The pair of quarterperiods* ω_f, ω_g *determines a Jacobian lattice if an elliptic function with simple zeros congruent with* ω_g *and simple poles congruent with the origin has its residue at the origin equal to its value at* ω_f.

Or, replacing the function by its reciprocal,

10·23. *The pair of quarterperiods* ω_f, ω_g *determines a Jacobian lattice if an elliptic function with simple zeros congruent with the origin and simple poles congruent with* ω_g *has its derivative at the origin equal to its value at* ω_f.

To verify the form of the condition in ·23, we remark that

$$\mathrm{jg}'0 = \lim_{z\to 0} \mathrm{jg}\, z\, \mathrm{gj}\, z = \mathrm{jg}\, \omega_f\, \mathrm{gj}\, \omega_f,$$

whence the condition $g_f = 1$ is equivalent to $\mathrm{jg}'0 = \mathrm{jg}\, \omega_f$.

If α, β, γ is any set of quarterperiods, then $\mathrm{gj}(\lambda\alpha; \lambda\alpha, \lambda\beta, \lambda\gamma) = 1$ if and only if $\lambda = \mathrm{gj}(\alpha; \alpha, \beta, \gamma)$. That is,

10·24. *There is one and only one Jacobian set of quarterperiods similar to any given set, and the normalizing factor of the set* ω_f, ω_g, ω_h *is the critical value* g_f.

The Jacobian triplet is the unique representative of the class of similar triplets to which it belongs. In general the normalizing factor λ is complex and the Jacobian parallelogram $\lambda\alpha$, $\lambda\beta$ differs in orientation as well as in size from the parallelogram α, β which it represents. For example, if $\alpha = i\omega'$, $\beta = \omega$, where ω, ω' are real, then $\operatorname{gj} iy$ is imaginary, and if $-2\omega' \leqslant y \leqslant 2\omega'$, the sign of $\operatorname{gj} iy$ is opposite to that of y, and $iy \operatorname{gj} iy$ is real and positive in that range; in particular, $g_f \alpha$ is real and positive whether ω' is positive or negative, and the rectangle is turned through a right angle, negatively or positively, as well as brought to the right size. If α is real and β imaginary, $\operatorname{gj} x$ is real with the same sign as x if $-2\alpha \leqslant x \leqslant 2\alpha$, and $x \operatorname{gj} x$ is real and positive in that range; in particular, $g_f \alpha$ is real and positive, implying if α is negative that the rectangle is turned through two right angles. Since every rectangle is congruent with some rectangle whose sides are along the real and imaginary axes, we have proved incidentally that

·204. *If ω_f and ω_g are at right angles, the first member of the corresponding Jacobian set of quarterperiods is real and positive and the second member is positively or negatively imaginary according as rotation from ω_f to ω_g is positive or negative.*

Briefly,

10·25. *If a Jacobian parallelogram is a rectangle, its first side is along the positive half of the real axis.*

10·3. Legendre expresses the fundamental elliptic integral in the form

·301 $$F(\phi) = \int_0^\phi \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}$$

as well as, with $x = \sin\phi$, in the form

·302 $$u = \int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

which we have been using. The angle ϕ is called the amplitude of the integral $F(\phi)$, but in spite of this terminology the functional relationship is not seen as a dependence of ϕ on $F(\phi)$. The crucial step was taken when the relation

·303_1 $$u = F(\phi)$$

was treated as a relation

·303_2 $$\phi = \operatorname{am} u.$$

Circular functions of ϕ, with which Legendre's pages abound, became $\sin \operatorname{am} u$, $\cos \operatorname{am} u$, and so on, and the radical $\sqrt{(1-k^2\sin^2\phi)}$, which is $d\phi/du$, became $d \operatorname{am} u/du$, abbreviated by Jacobi to $\Delta \operatorname{am} u$.

Gudermann introduced a more compact notation, writing $\operatorname{sn} u$, $\operatorname{cn} u$

for $\sin\operatorname{am} u$, $\cos\operatorname{am} u$, and† $\operatorname{dn} u$ for $\Delta\operatorname{am} u$. Any exposition that is to facilitate access to the subject must deal with Jacobi's functions and conform to Gudermann's notation and its accepted extension.

That $\operatorname{sn} u$ is an elliptic function, a multiple of $\operatorname{jg} u$ in the Jacobian lattice, we already know. In symbols, ·21 can be written

·304 $$\operatorname{sn} u = \operatorname{jg} u/\operatorname{jg}'0 = (1/g_h)\operatorname{jg} u;$$

the value of g_f is implicit. It follows that $\operatorname{cn} u$, $\operatorname{dn} u$ are multiples of the copolar functions $\operatorname{fg} u$, $\operatorname{hg} u$, and these also are therefore elliptic functions. To prove this directly is to repeat the arguments of 1·2: the functions are doubly periodic functions with simple poles at the poles of $\operatorname{sn} u$, and it has only to be shown that their zeros are not branchpoints. Alternatively we may utilize the general theory. If $\operatorname{pq} z$ is an elementary elliptic function satisfying the equation

·305 $$(dw/dz)^2 = (w^2-b^2)(w^2-c^2),$$

then $\operatorname{pq}^2 z-b^2$, $\operatorname{pq}^2 z-c^2$ are the squares of the elementary functions copolar with $\operatorname{pq} z$. To replace $\operatorname{pq} z$ by a constant multiple $\mu\operatorname{pq} z$ is to replace ·305 by the equation

$$\mu^2(dx/dz)^2 = (x^2-\mu^2b^2)(x^2-\mu^2c^2),$$

which may appear in the form

$$(dx/dz)^2 = (\kappa x^2-\xi)(\lambda x^2-\eta),$$

where κ, λ are any two constants such that $\kappa\lambda = 1/\mu^2$. But $\xi = \kappa\mu^2b^2$, $\eta = \lambda\mu^2c^2$, and the factors $\kappa x^2-\xi$, $\lambda x^2-\eta$ are necessarily multiples of $\operatorname{pq}^2 z-b^2$, $\operatorname{pq}^2 z-c^2$. It follows from the equation

$$(dx/du)^2 = (1-x^2)(1-k^2x^2),$$

which is implied by the integral relation ·302, that $\operatorname{cn}^2 u$, $\operatorname{dn}^2 u$ are multiples of $\operatorname{fg}^2 u$, $\operatorname{hg}^2 u$, that is, that $\operatorname{cn} u$, $\operatorname{dn} u$ are multiples of $\operatorname{fg} u$, $\operatorname{hg} u$, one way round or the other, and since $\operatorname{cn} u = 0$ when $\operatorname{sn} u = 1$, it is $\operatorname{cn} u$ which is a multiple of $\operatorname{fg} u$ and $\operatorname{dn} u$ which is a multiple of $\operatorname{hg} u$. The constant multipliers are determined by the values at the origin: since $\operatorname{cn} 0 = 1$ and $\operatorname{dn} 0 = 1$,

·306 $$\operatorname{cn} u = \operatorname{fg} u/\operatorname{fg} 0 = -(1/h_g)\operatorname{fg} u,$$

·307 $$\operatorname{dn} u = \operatorname{hg} u/\operatorname{hg} 0 = -(1/f_g)\operatorname{hg} u.$$

If ω_f is real and ω_g imaginary, the functions $\operatorname{fg} u$ and $\operatorname{hg} u$ are real between ω_g and $\omega_f+\omega_g$, imaginary for real values of u and in particular

† Also $\operatorname{tn} u$ for $\tan\operatorname{am} u$, but this symbol is completely superseded by the equivalent symbol $\operatorname{sc} u$ in Glaisher's systematic notation; see ·4 below.

at the origin; $\operatorname{fg} u/\operatorname{fg} 0$ and $\operatorname{hg} u/\operatorname{hg} 0$ are real if u is real, but rather than describe a real function of a real variable in such a manner that an imaginary factor has to be removed, we describe the functions directly in structural terms, and including in one description the original function $\operatorname{sn} u$ and the functions $\operatorname{cn} u$, $\operatorname{dn} u$ we can say that

10·31. *The Jacobian elliptic functions* $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ *are the functions constructed with a Jacobian set of quarterperiods* ω_f, ω_g, ω_h *to have simple poles congruent with* ω_g, *and simple zeros congruent with the origin, with* ω_f, *and with* ω_h, *respectively, and to have unity for leading coefficient at the origin.*

We have followed history in introducing $\operatorname{sn} u$ from Legendre's integral, but whatever the lattice it is not surprising to find $\operatorname{jg} z/\operatorname{jg}' 0$ as a canonical function. For theoretical purposes, a function is dominated by its infinities; hence the choice of $\operatorname{fj} z$, with a pole at the origin, for a primitive function. But for applications, and especially for calculations, infinities are to be avoided in favour of zeros: a canonical integral has 0 rather than ∞ for a fixed limit, the corresponding function has the origin for a zero rather than for a pole. And if the standard is to be set at the origin, we shall concern ourselves not with the residue at the pole which we have been at pains to avoid, but with the coefficient at the zero which we have located. If the origin is a simple pole it is natural to introduce the factor which causes a function to resemble $1/z$; if the function is a simple zero we arrange† for the function to resemble z. To this end we may apply a constant factor either to the function or to the independent variable: if $\psi(z)$ is a function which resembles z/λ near $z = 0$, then $\lambda\psi(z)$ as a function of z resembles z, and $\psi(\lambda u)$ as a function of u resembles u near $u = 0$. We may say that having chosen the canonical function as $\operatorname{jg} z/\operatorname{jg} \omega_f$ in order to secure the value 1 at ω_f, we have still in hand a factor λ to be chosen so that $\operatorname{jg}(\lambda u)/\operatorname{jg}(\lambda\omega_f)$ resembles u; this unique factor is the normalizing factor which produces the Jacobian lattice.

We now introduce an expressive notation‡ for a Jacobian parallelogram, writing the three quarterperiods ω_f, ω_g, ω_h as K_c, K_n, K_d, and using K_s, as we have hitherto used ω_j, as an alternative symbol for zero. In this notation $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ are functions with zeros congruent with K_s, with K_c, and with K_d, respectively, and the three functions have poles congruent with K_n. But whereas in the earlier chapters $\operatorname{pq} z$ is rendered specific, its structure being implicit in the notation, by its form near its own pole ω_q, the Jacobian functions are rendered specific by their forms near the origin. This change is marked by the change of symbol for the independent variable as well as for the quarterperiods. And we have to remember that whereas ω_f, ω_g, ω_h

† As we adopt circular measure to secure the condition $\sin\theta \sim \theta$.

‡ The reader must be warned that this notation is not in the literature of the subject. I would call it new, had I not been using it in lectures since 1925.

are subject to only the one condition $\omega_f+\omega_g+\omega_h = 0$, the Jacobian quarterperiods are subject to the characteristic condition

10·32 $$\text{gj}(K_c; K_c, K_n, K_d) = 1$$

as well as to the condition

·308 $$K_c+K_n+K_d = 0.$$

It is sometimes convenient to write K'_d for K_c+K_n, that is, for $-K_d$, to facilitate comparison with classical formulae.

The three quarterperiods in a Jacobian set play distinct parts. In the characteristic condition ·32, K_n corresponds to the function that occurs, and K_c is the argument. Precisely because the three parts are distinct, there is nothing artificial in ignoring one of the quarterperiods in a specification. If α, β are the values of K_c, K_n, we say that the Jacobian system has the basis α, β, leaving the value of K_d, which is $-\alpha-\beta$, to be inferred.

A set of quarterperiods is Jacobian only with a definite allocation of parts: if the triplet α, β, γ is Jacobian in this order, there is no reason to suppose that it is Jacobian in any other order. In symbols, ·24 becomes

10·33. *The Jacobian triplet similar to* ω_f, ω_g, ω_h *is given by*

$$K_c = g_f\,\omega_f, \qquad K_n = g_f\,\omega_g, \qquad K_d = g_f\,\omega_h.$$

To permute α, β, γ among the parts ω_f, ω_g, ω_h is to bring each of the critical values of one system α, β, γ in turn into the part of g_f. Each permutation has its own normalizing factor as well as its own allocation, and the six permutations give rise to six different Jacobian triplets. The significance of this multiplicity will appear in 13·4.

10·4. Having indicated our right to the classical notation, we now reverse the deductions and treat ·31 not as a theorem but as the definition of the functions to be studied. The advantage of this course is that we can develop the theory of the functions in complete generality, that is, for complex values of the parameter, without assuming the solution of the inversion problem, while the simple theory of inversion for a real parameter, as given in 9·7, will justify in the end the uses to which the functions are commonly put. Logically we could have dispensed with preliminary analysis and laid down our definitions dogmatically, but the set of letters s, c, d, n is a queer one to impose without explanation, and it is better to incur the cost of a little repetition. In repeating ·31 as a definition we incorporate the notation for the quarterperiods.

10·41. *The functions* $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ *are defined as the elliptic functions constructed with a Jacobian set of quarterperiods* K_c, K_n, K_d, *to have simple poles congruent with* K_n, *and simple zeros congruent with the origin, with* K_c, *and with* K_d, *respectively, and to have unity for leading coefficient at the origin.*

Thus

10·42$_{1-3}$ $$\operatorname{sn}' 0 = 1, \qquad \operatorname{cn} 0 = 1, \qquad \operatorname{dn} 0 = 1.$$

Also, because† the lattice is Jacobian,

10·43 $$\operatorname{sn} K_c = 1.$$

The whole theory of the functions is implicit in ·41 and ·43.

Jacobi's three functions are standardized functions with a common pole at K_n and zeros at K_s, K_c, K_d, just as the primitive functions defined in 1·2 are standardized functions with a common pole at ω_j and zeros at ω_f, ω_g, ω_h. And the Jacobian functions, like the primitive functions, are best understood as belonging to a set of twelve functions, each choice of a zero and a pole among the four cardinal points providing one function. The typical function of the complete set has a zero at K_p and a pole at K_q, and the standardizing factor is chosen in every case in relation to the origin, where the leading coefficient is required to be unity; with this condition we denote the function by $\operatorname{pq} u$.

Thus $\operatorname{sc} u$, $\operatorname{sn} u$, $\operatorname{sd} u$ have simple zeros at the origin, and the quotient of each of them by u tends there to 1:

·401 $$\operatorname{sq} u \sim u.$$

The reciprocal functions $\operatorname{cs} u$, $\operatorname{ns} u$, $\operatorname{ds} u$ have simple poles at the origin, and the product of each of these by u tends there to 1:

·402 $$\operatorname{ps} u \sim 1/u;$$

in fact these three functions are the primitive functions of the lattice, in the sense of our earlier chapters. If the origin is neither zero nor pole, then

·403 $$\operatorname{pq} 0 = 1.$$

The utility, if not the importance, of the set of twelve functions was first seen by Glaisher, who introduced‡ the nine functions which complete the set by defining $\operatorname{ns} u$ as $1/\operatorname{sn} u$, $\operatorname{sc} u$ as $\operatorname{sn} u/\operatorname{cn} u$, and so on,

† Logically, the relation ·43 can not be regarded as a characterizing property of a Jacobian lattice, since the notation assumes already that the lattice is Jacobian; the characteristic property must be expressed as $\operatorname{gj} \omega_f = 1$. But as regards the Jacobian functions, nothing more is to be learnt mathematically from one way of expressing the result than from another.

‡ *Messenger of Mathematics*, 11 (1882), p. 85.

regarding the notation purely as mnemonical. That the functions we have defined satisfy the relations

$$\cdot 404\text{–}\cdot 405 \qquad \mathrm{pq}\,u\,\mathrm{qp}\,u = 1, \qquad \mathrm{pq}\,u\,\mathrm{qr}\,u = \mathrm{pr}\,u$$

and are therefore in fact Glaisher's functions, follows immediately from Liouville's theorem. No constant factors other than unity now occur, because the functions are all standardized at the same point: the leading coefficient at the origin is 1 for every function. Glaisher constructed the set from Jacobi's functions, the three functions with a pole at K_n, but the set can be reconstructed by the same rules from any triplet with a common pole or a common zero: if the primitive functions $\mathrm{cs}\,u$, $\mathrm{ns}\,u$, $\mathrm{ds}\,u$ are regarded as fundamental, Jacobi's functions are given by

$$\cdot 406\text{–}\cdot 408 \qquad \mathrm{sn}\,u = 1/\mathrm{ns}\,u, \qquad \mathrm{cn}\,u = \mathrm{cs}\,u/\mathrm{ns}\,u, \qquad \mathrm{dn}\,u = \mathrm{ds}\,u/\mathrm{ns}\,u,$$

or if we begin with the three functions $\mathrm{sc}\,u$, $\mathrm{sn}\,u$, $\mathrm{sd}\,u$ which vanish at the origin, we have

$$\cdot 409\text{–}\cdot 410 \qquad \mathrm{cn}\,u = \mathrm{sn}\,u/\mathrm{sc}\,u, \qquad \mathrm{dn}\,u = \mathrm{sn}\,u/\mathrm{sd}\,u.$$

The set of Glaisher's functions, unlike the set of elementary functions defined in 2·1, is wholly lacking in symmetry. A formula may be typical in its algebraical structure of a group of three or more formulae, but the constants in one formula can seldom be obtained from those in another by mere transliteration.

Each of the elementary functions built on the Jacobian lattice is a constant multiple of the corresponding Glaisher function, and in a sense the factor is known, for it is the leading coefficient of the elementary function, as given in Table II 2. But the coefficients in this table are given in terms of the critical values of the earlier theory. If we propose to translate theorems from Chapters I–IV into theorems on Glaisher's functions, we shall have to relate the parameters as well as the functions to the Jacobian system. It is usually better to apply the methods of the general theory than to translate the results.

XI

PROPERTIES OF THE JACOBIAN FUNCTIONS

11·1. Many of the arguments used in the first part of the book are unaffected by the presence of constant factors in the functions considered, and lead to theorems that are true of the Jacobian† functions. If arguments are repeated, they will be given succinctly.

Since the function $\mathrm{pq}(-u)$ has the same poles and the same zeros as $\mathrm{pq}\,u$, one function is a constant multiple of the other. If the origin is neither a zero nor a pole, the functions have the same value there and everywhere: $\mathrm{pq}\,u$ is an even function. If the origin is a simple pole or a simple zero, $\mathrm{pq}(-u)/\mathrm{pq}\,u \to -1$ as $u \to 0$, and the constant value of the ratio is -1: the function $\mathrm{pq}\,u$ is odd.

11·11. *The three functions* $\mathrm{sc}\,u$, $\mathrm{sn}\,u$, $\mathrm{sd}\,u$ *and their reciprocals are odd functions; the three functions* $\mathrm{cn}\,u$, $\mathrm{dn}\,u$, $\mathrm{cd}\,u$ *and their reciprocals are even functions.*

If K_l is a step from a zero to a pole of the function $\mathrm{pq}\,u$, the product $\mathrm{pq}\,u\,\mathrm{pq}(u+K_l)$ has no poles and is therefore a constant; hence

$$\mathrm{pq}\,u\,\mathrm{pq}(u+K_l) = \mathrm{pq}(u+K_l)\mathrm{pq}(u+2K_l)$$

for all values of u, and therefore

$$\mathrm{pq}(u+2K_l) = \mathrm{pq}\,u:$$

11·12. *Any step from a zero to a pole of the function* $\mathrm{pq}\,u$ *is a halfperiod of the function.*

In particular,

11·13. *The step* K_{pq} *from* K_p *to* K_q *is a halfperiod of* $\mathrm{pq}\,u$.

If K_l is any one of the three numbers K_c, K_n, K_d, the function $\mathrm{pq}(u+2K_l)$ has the same zeros and the same poles as $\mathrm{pq}\,u$, and $\mathrm{pq}(u+2K_l)/\mathrm{pq}\,u$ is a constant which can be equated to $\mathrm{pq}\,K_l/\mathrm{pq}(-K_l)$ if K_l is neither zero nor pole and can in any case be equated to

$$\lim_{u\to 0} \frac{\mathrm{pq}(u+K_l)}{\mathrm{pq}(u-K_l)}.$$

Whether $\mathrm{pq}\,u$ is even or odd, and whether it is a value or a limit which we find, the constant is either -1 or 1, and therefore *each of the numbers*

† Usually I speak of the twelve functions as Jacobian, for to attach Glaisher's name, to the exclusion of Jacobi's, to nine of the twelve would be to exaggerate Glaisher's contribution to the theory of the subject. If it is necessary to discriminate, $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$ may be described as Jacobi's functions.

$2K_c$, $2K_n$, $2K_d$ is a halfperiod if not a period of $\mathrm{pq}\,u$. From ·12 it follows that one of these numbers is a period, and since $\mathrm{pq}\,u$ has only one pole, and that a simple one, in the parallelogram $2K_c$, $2K_n$, it follows that not more than one of the numbers is a period; thus one of the three is a period and two are halfperiods:

11·14. *Of the three numbers K_c, K_n, K_d, the one which is equal to a step from a zero to a pole of* $\mathrm{pq}\,u$ *is a halfperiod of the function, the other two are quarterperiods.*

If K_l is a halfperiod, $\mathrm{pq}\,u\,\mathrm{pq}(u+K_l)$ is constant; if K_l is a quarterperiod, $\mathrm{pq}(u+2K_l) = -\mathrm{pq}\,u$. It is easy to confirm the latter result by a direct examination of the ratio $\mathrm{pq}(u+2K_l)/\mathrm{pq}\,u$ in the different cases.

If the function $\mathrm{pq}\,u$ is odd, one of the symbols p, q is s and the other belongs to a halfperiod; if the function is even, p, q are two of the symbols c, n, d and belong to quarterperiods, while the third of these symbols belongs to a halfperiod.

Since K_q-K_p differs from K_p+K_q by $2K_p$, which is at least a halfperiod if it is not zero,

11·15. *The sum K_p+K_q is a halfperiod of the function* $\mathrm{pq}\,u$.

This form of the result, with the identity

$$K_s+K_c+K_n+K_d = 0, \tag{·101}$$

is the clearest analytical explanation of the grouping of the functions with respect to periodicity: the four terms can be split into pairs in three ways, and each pair is associated with two functions.

The natural classification of the twelve functions is shown in the following scheme:

TABLE XI1

Poles and periods of the twelve Jacobian functions

	Pole K_s	*Pole K_c*	*Pole K_n*	*Pole K_d*
Periods $2K_c$, $4K_n$, $4K_d$	$\mathrm{cs}\,u$	$\mathrm{sc}\,u$	$\mathrm{dn}\,u$	$\mathrm{nd}\,u$
Periods $4K_c$, $2K_n$, $4K_d$	$\mathrm{ns}\,u$	$\mathrm{dc}\,u$	$\mathrm{sn}\,u$	$\mathrm{cd}\,u$
Periods $4K_c$, $4K_n$, $2K_d$	$\mathrm{ds}\,u$	$\mathrm{nc}\,u$	$\mathrm{cn}\,u$	$\mathrm{sd}\,u$

The double stratification was perceived by Glaisher, but it is perhaps fair to say that he did not quite understand it, for he uses the phrase 'groups having the same denominator', taking this denominator to be one of Jacobi's three functions $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$ and attaching no functional significance to his notation. He observes, without offering an explanation, that formulae relating to the group $\mathrm{cs}\,u$, $\mathrm{ns}\,u$, $\mathrm{ds}\,u$ are sometimes simpler in respect of literal coefficients than formulae relating to other groups. The reason is, that in this group alone the factor which renders an elliptic function specific when its poles and zeros are assigned bears the same organic relation to each of the three functions.

The recognition of the congruences of points at which a function $\mathrm{pq}\,u$ has a common value, or in other words the solution of the equation $\mathrm{pq}\,u = \mathrm{pq}\,a$, is implicit in the table of periods and poles. If the function is an even function, a and $-a$ are distinct solutions of the equation, and every solution is congruent with one of these. For example, the general solution of

·102 $$\mathrm{cn}\,u = \mathrm{cn}\,a$$

is

·103 $$u = 4mK_c + 2nK_d \pm a,$$

or in terms of K_c and K_n,

·104 $$u = 2mK_c + 2nK_n \pm a, \quad \text{with } m+n \text{ even};$$

the general solution of

·105 $$\mathrm{dn}\,u = \mathrm{dn}\,a$$

is

·106 $$u = 2mK_c + 4nK_n \pm a.$$

If $\mathrm{pq}\,u$ is an odd function, and $2K_r$ is one of the halfperiods, the sum of the two poles is congruent with $2K_r$, and distinct solutions of the equation are a and $2K_r - a$. Thus the general solution of

·107 $$\mathrm{sn}\,u = \mathrm{sn}\,a$$

is

·108 $$u = 4mK_c + 2nK_n + a \quad \text{or} \quad u = (4m+2)K_c + 2nK_n - a.$$

11·2. We have explained in 10·1 that the effect of standardizing the elliptic integral is that only one parameter remains. The constants in the elliptic integrals are the critical values in the corresponding system of elliptic functions, and we have in effect asserted that the mutual relations between the Jacobian functions depend on a single constant.

Since $\mathrm{pq}(u+2K_l)$ is equal to $\mathrm{pq}\,u$ or to $-\mathrm{pq}\,u$ according as K_l is a halfperiod or a quarterperiod,

·201 $$\mathrm{pq}^2(u+2K_l) = \mathrm{pq}^2 u$$

in either case. That is, $\mathrm{pq}^2 u$ is doubly periodic in $2K_c$ and $2K_n$, and since there is only one pole in a period parallelogram, the principal part of the expansion of $\mathrm{pq}^2 u$ near K_q consists of a single term $A_p/(u-K_q)^2$; a linear term $B_p/(u-K_q)$ which would supply a residue can not occur. It follows that if $\mathrm{pq}\,u$, $\mathrm{rq}\,u$ are copolar functions, the functions $(\mathrm{pq}^2u)/A_p$, $(\mathrm{rq}^2u)/A_r$ have the same principal part $1/(u-K_q)^2$ near the common pole, and their difference, a doubly periodic function everywhere finite, is constant:

11·21. *The squares of any two copolar Jacobian functions are connected by a linear relation with constant coefficients.*

The relation is known if two pairs of corresponding values of the related functions are known; the relation between pq^2u and rq^2u is expressible as

11·22 $$\frac{\mathrm{pq}^2u}{\mathrm{pq}^2K_r}+\frac{\mathrm{rq}^2u}{\mathrm{rq}^2K_p}=1.$$

Since the functions are known at or near the origin, only one other point need in fact be examined, but we can not write down a general formula if we introduce the origin with no regard to its functional relation to the two functions which are to be connected. The relation ·22 is equivalent to

·202 $$\frac{\mathrm{pq}^2u}{A_p}-\frac{\mathrm{rq}^2u}{A_r}=C,$$

and therefore

·203 $$A_p:A_r=-\mathrm{pq}^2K_r:\mathrm{rq}^2K_p.$$

If the pole K_q is not the origin, the value of the constant C in ·202 is $1/A_p-1/A_r$, and ·22 implies similarly that

11·23 $$\mathrm{qp}^2K_r+\mathrm{qr}^2K_p=1,$$

provided that the common zero is not the origin. For a function $\mathrm{ps}\,u$ with a pole at the origin, the principal part of ps^2u there is $1/u^2$; hence for two functions $\mathrm{ps}\,u$, $\mathrm{qs}\,u$, the difference $\mathrm{ps}^2u-\mathrm{qs}^2u$ is constant, and this may be evaluated either at K_p or at K_q:

11·24 $$\mathrm{ps}^2u-\mathrm{qs}^2u=-\mathrm{qs}^2K_p=\mathrm{ps}^2K_q.$$

11·3. The three original Jacobian functions are copolar and the linear relations between their squares are generally regarded as expressing cn^2u and dn^2u in terms of sn^2u. Since simultaneously at the origin

·301 $$\mathrm{sn}\,u=0,\qquad \mathrm{cn}\,u=1,\qquad \mathrm{dn}\,u=1,$$

we have $$\mathrm{cn}^2u=1-b\,\mathrm{sn}^2u,\qquad \mathrm{dn}^2u=1-c\,\mathrm{sn}^2u,$$

where b, c are constants. Since also $\mathrm{cn}\,u=0$, $\mathrm{sn}\,u=1$ simultaneously when $u=K_c$, the constant b is 1, and we have

11·31 $$\mathrm{cn}^2u=1-\mathrm{sn}^2u,$$

11·32 $$\mathrm{dn}^2u=1-c\,\mathrm{sn}^2u,$$

where c remains as the one parameter involved in the algebraic relations between the functions of the system; when we speak of *the* parameter, it is c that we mean. If we put the relations ·31, ·32 into the form of ·22, we have

·302–·303 $$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \qquad c\,\mathrm{sn}^2 u + \mathrm{dn}^2 u = 1,$$

and we recognize that these relations depend on the specific values of $\mathrm{cn}\, K_s$, $\mathrm{dn}\, K_s$, and $\mathrm{sn}\, K_c$, while $1/c$ is identified with $\mathrm{sn}^2 K_d$:

11·33 $$c = \mathrm{ns}^2 K_d.$$

The relation between $\mathrm{cn}^2 u$ and $\mathrm{dn}^2 u$ is

·304 $$\mathrm{dn}^2 u = c' + c\,\mathrm{cn}^2 u,$$

where c' defined by

11·34 $$c' = 1 - c$$

is the *complementary parameter* of the system, identifiable also from ·304:

11·35 $$c' = \mathrm{dn}^2 K_c.$$

From ·304,

·305 $$\mathrm{cn}^2 K_d = -c'/c.$$

We can find the identities similar to ·31, ·32, ·304 for the other pairs of copolar functions without returning to first principles. Dividing ·31, ·32 by $\mathrm{sn}^2 u$ we have

·306–·307 $$\mathrm{cs}^2 u = \mathrm{ns}^2 u - 1, \qquad \mathrm{ds}^2 u = \mathrm{ns}^2 u - c.$$

If we divide ·304 by $\mathrm{sn}^2 u$ we obtain not the relation between $\mathrm{cs}^2 u$ and $\mathrm{ds}^2 u$ but the homogeneous relation

·308 $$\mathrm{ds}^2 u = c'\,\mathrm{ns}^2 u + c\,\mathrm{cs}^2 u,$$

and it is from the homogeneous relation

·309 $$\mathrm{dn}^2 u = c'\,\mathrm{sn}^2 u + \mathrm{cn}^2 u$$

that the relation

·310 $$\mathrm{ds}^2 u = \mathrm{cs}^2 u + c'$$

comes by mere division. We therefore add the homogeneous identity at each pole, and set out the complete scheme of formulae as follows:

TABLE XI 2

$\mathrm{cs}^2 u + 1 = \mathrm{ns}^2 u$	$\mathrm{cs}^2 u + c' = \mathrm{ds}^2 u$	$\mathrm{ds}^2 u + c = \mathrm{ns}^2 u$	$c\,\mathrm{cs}^2 u + c'\,\mathrm{ns}^2 u = \mathrm{ds}^2 u$
$\mathrm{sc}^2 u + 1 = \mathrm{nc}^2 u$	$c'\,\mathrm{sc}^2 u + 1 = \mathrm{dc}^2 u$	$c'\,\mathrm{nc}^2 u + c = \mathrm{dc}^2 u$	$c\,\mathrm{sc}^2 u + \mathrm{dc}^2 u = \mathrm{nc}^2 u$
$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1$	$c\,\mathrm{sn}^2 u + \mathrm{dn}^2 u = 1$	$c\,\mathrm{cn}^2 u + c' = \mathrm{dn}^2 u$	$c'\,\mathrm{sn}^2 u + \mathrm{cn}^2 u = \mathrm{dn}^2 u$
$c'\,\mathrm{sd}^2 u + \mathrm{cd}^2 u = 1$	$c\,\mathrm{sd}^2 u + 1 = \mathrm{nd}^2 u$	$c\,\mathrm{cd}^2 u + c'\,\mathrm{nd}^2 u = 1$	$\mathrm{sd}^2 u + \mathrm{cd}^2 u = \mathrm{nd}^2 u$

To understand the individual formulae in Table XI 2, we must recognize those which are not homogeneous as identities of the form ·22. For this purpose we must be able to determine, otherwise than from the formulae themselves, the squares of the critical values of Glaisher's twelve functions, in terms of the constants of the system. There is no difficulty in writing down the required values of $\mathrm{ns}^2 u$, $\mathrm{nc}^2 u$, $\mathrm{nd}^2 u$, since zero and infinite values do not concern us. But if, for example, we consider $\mathrm{cd}\, u$ as $\mathrm{cn}\, u/\mathrm{dn}\, u$, we can not write down $\mathrm{cd}^2 K_n$, one of the constants wanted in the determination of the relation between $\mathrm{cd}^2 u$ and $\mathrm{nd}^2 u$. Knowing that $\mathrm{cd}^2 u$ and $\mathrm{nd}^2 u$ are simultaneously unity at the origin, we have

·311 $$(\mathrm{nd}^2 K_c - 1)\mathrm{cd}^2 u = \mathrm{nd}^2 K_c - \mathrm{nd}^2 u,$$

and since $\mathrm{nd}^2 K_c = 1/c'$, we can infer $\mathrm{cd}^2 K_n$, but we might as well find the relation between $\mathrm{cd}^2 u$ and $\mathrm{nd}^2 u$ from the relation between $\mathrm{cn}^2 u$ and $\mathrm{dn}^2 u$ as write it down in the form ·311.

There is however another line of argument. We can evaluate $\mathrm{cd}^2 u$ as $\mathrm{cn}^2 u/\mathrm{dn}^2 u$ even at the common pole K_n if we know the principal parts of the functions $\mathrm{cn}^2 u$, $\mathrm{dn}^2 u$ there. In other words, although we can not pass directly from the six critical values of one copolar triad to the six critical values of another, we can pass directly from the twelve leading coefficients of one copolar triad to the twelve leading coefficients of another.

If the principal part of $\mathrm{sn}\, u$ near K_n is $a_s/(u - K_n)$, the leading coefficient of $\mathrm{sn}^2 u$ at K_n is a_s^2. Now the product $\mathrm{sn}\, u\, \mathrm{sn}(u+K_n)$ has no poles, and therefore has a constant value, whence

·312 $$\lim_{u\to 0} \mathrm{sn}\, u\, \mathrm{sn}(u+K_n) = \mathrm{sn}\, K_c\, \mathrm{sn}(K_c+K_n).$$

But, as $u \to 0$, $(\mathrm{sn}\, u)/u \to 1$, $u\, \mathrm{sn}(u+K_n) \to a_s$; hence

$$a_s = \mathrm{sn}\, K_c\, \mathrm{sn}\, K'_d,$$

and *the leading coefficient of* $\mathrm{sn}^2 u$ *at* K_n *is* $1/c$. It follows from ·31, ·32 that the leading coefficients of $\mathrm{cn}^2 u$, $\mathrm{dn}^2 u$ at K_n are $-1/c$, -1.

The leading coefficients of $\mathrm{cn}\, u$ at K_c and of $\mathrm{dn}\, u$ at K_d are values of the derivatives $\mathrm{cn}' u$, $\mathrm{dn}' u$, but without anticipating the discussion of derivatives we can find the squares of these leading coefficients by repeating the argument we have just used. The products

$$\mathrm{cn}\, u\, \mathrm{cn}(u+K_d), \qquad \mathrm{dn}\, u\, \mathrm{dn}(u+K_c)$$

are constants, and therefore

·313 $$\mathrm{cn}(u+K_c)\mathrm{cn}(u-K_n) = \mathrm{cn}\, K_d,$$

·314 $$\mathrm{dn}(u+K_d)\mathrm{dn}(u-K_n) = \mathrm{dn}\, K_c;$$

as we have just seen, as $u \to 0$

$$u^2 \operatorname{cn}^2(u-K_n) \to -1/c, \qquad u^2 \operatorname{dn}^2(u-K_n) \to -1;$$

hence, from ·305 and ·35,

·315–·316 $\quad \operatorname{cn}^2(u+K_c) \sim c'u^2, \qquad \operatorname{dn}^2(u+K_d) \sim -c'u^2.$

Thus the scheme of leading coefficients for the squares of Jacobi's original functions is as follows:

TABLE XI 3

	At K_s	*At* K_c	*At* K_n	*At* K_d
$\operatorname{sn}^2 u$	$1\times$	1	$1/c\div$	$1/c$
$\operatorname{cn}^2 u$	1	$c'\times$	$-1/c\div$	$-c'/c$
$\operatorname{dn}^2 u$	1	c'	$-1\div$	$-c'\times$

In each row, the product of the coefficients at the zero and the pole is equal to the product of the other two coefficients.

With Table XI 3 in front of us, we can write down the corresponding scheme for any other copolar triad. The scheme for the functions with a pole at the origin is remarkably simple:

TABLE XI 4

	At K_s	*At* K_c	*At* K_n	*At* K_d
$\operatorname{cs}^2 u$	$1\div$	$c'\times$	-1	$-c'$
$\operatorname{ns}^2 u$	$1\div$	1	$c\times$	c
$\operatorname{ds}^2 u$	$1\div$	c'	$-c$	$-cc'\times$

In this scheme, for a reason which we shall discover in the next section, the coefficient at a zero is the product of the other coefficients in the same column.

We can read from either of the tables XI 3, XI 4 the square of any critical value $\operatorname{pq} K_t$, and so we can write down any required relation between the squares of two copolar functions immediately in the form of ·22. Further, the square of any one of the twelve Jacobian functions can be expressed rationally in terms of the square of any other:

11·36. *If K_t is neither a pole nor a zero of* $\operatorname{pq} u$, *then*

$$\frac{\operatorname{pq}^2 u}{\operatorname{pq}^2 K_t} = \frac{\operatorname{rt}^2 u - \operatorname{rt}^2 K_p}{\operatorname{rt}^2 u - \operatorname{rt}^2 K_q}.$$

11·4. The square of $\operatorname{pq} u$ is a function $\phi(u)$ of the second order with $2K_c$, $2K_n$, $2K_d$ for periods and with the one pole K_q, which is double; the derivative $\phi'(u)$ is therefore of the third order, with K_q for a triple pole. The points where the function $\phi(u)$ has the same value as at a given point a are the points congruent with a and the points congruent with $-a$; two of these points coincide, that is, a is a double root of the equation $\phi(u) = \phi(a)$ and therefore a root of the equation $\phi'(u) = 0$, only if $2a \equiv 0$, that is, if a is congruent with one of the four numbers K_s, K_c, K_n, K_d. Of the four numbers, K_q is a triple pole of $\phi'(u)$, and therefore each of the other three is a simple zero. Expressing $\phi'(u)$ as $2\,\mathrm{pq}\,u\,\mathrm{pq}'u$ and removing the pole and the zero of $\mathrm{pq}\,u$, we see that

11·41. *The derivative* $\mathrm{pq}'u$ *has double poles congruent with* K_q, *and simple zeros congruent with the two cardinal points other than* K_p *and* K_q.

It follows from ·41 that

11·42. *If* K_r, K_t *are the two cardinal points other than* K_p, K_q, *the derivative* $\mathrm{pq}'u$ *is a constant multiple of* $\mathrm{rq}\,u\,\mathrm{tq}\,u$.

For a function $\mathrm{sq}\,u$ with a zero at the origin, the values of $\mathrm{sq}'u$ and of $\mathrm{rq}\,u$, $\mathrm{tq}\,u$ at the origin are all 1, and the constant factor is 1:

11·43 $$\mathrm{sq}'u = \mathrm{rq}\,u\,\mathrm{tq}\,u.$$

For a function $\mathrm{ps}\,u$ with a pole at the origin, $\mathrm{ps}\,u \sim 1/u$, and therefore $\mathrm{ps}'u \sim -1/u^2$ while $\mathrm{rs}\,u\,\mathrm{ts}\,u \sim 1/u^2$:

11·44 $$\mathrm{ps}'u = -\mathrm{rs}\,u\,\mathrm{ts}\,u.$$

If the origin is neither a pole nor a zero, we have

·401 $$\mathrm{pq}^2u = 1-\mathrm{qs}^2K_p\,\mathrm{sq}^2u,$$

and therefore

11·45 $$\mathrm{pq}'u = -\mathrm{qs}^2K_p\,\mathrm{sq}\,u\,\mathrm{rq}\,u = \mathrm{ps}^2K_q\,\mathrm{sq}\,u\,\mathrm{rq}\,u,$$

where $\mathrm{rq}\,u$ is the third function copolar with $\mathrm{pq}\,u$ and $\mathrm{sq}\,u$; the coefficient in ·45 is supplied by either of the Tables XI 3, XI 4.

We tabulate for reference the coefficients in the twelve derivatives; the functional contribution to the complete formula for $\mathrm{pq}'u$ is supplied by the two functions with which $\mathrm{pq}\,u$ shares a column in the table.

Table XI 5

$\mathrm{cs}'u = -1\times$	$\mathrm{sc}'u = 1\times$	$\mathrm{dn}'u = -c\times$	$\mathrm{nd}'u = c\times$
$\mathrm{ns}'u = -1\times$	$\mathrm{dc}'u = c'\times$	$\mathrm{sn}'u = 1\times$	$\mathrm{cd}'u = -c'\times$
$\mathrm{ds}'u = -1\times$	$\mathrm{nc}'u = 1\times$	$\mathrm{cn}'u = -1\times$	$\mathrm{sd}'u = 1\times$

It must be remembered that this table gives the expression of $\mathrm{pq}'u$ as a function of u, not the form of $\mathrm{pq}\,u$ near the zero K_p. The leading coefficient of $\mathrm{pq}\,u$ at K_p is the product of the entry in XI 5 by $\mathrm{rq}\,K_p\,\mathrm{tq}\,K_p$. If the entry against $\mathrm{pq}'u$ in XI 5 is ± 1, then

$$\mathrm{pq}'^2u = \mathrm{rq}^2u\,\mathrm{tq}^2u,$$

and the leading coefficient of $\mathrm{pq}^2 u$ at K_p is the product of the values there of $\mathrm{rq}^2 u$ and $\mathrm{tq}^2 u$; this is the property of Table XI 4 noticed in the last section, and we see that in Table XI 3 it is possessed by $\mathrm{sn}^2 u$ and $\mathrm{cn}^2 u$ but not by $\mathrm{dn}^2 u$.

The classical formulae in differentiation are in the third column of Table XI 5:

$11{\cdot}46_{1-3}$ $\mathrm{sn}'u = \mathrm{cn}\,u\,\mathrm{dn}\,u, \quad \mathrm{cn}'u = -\mathrm{sn}\,u\,\mathrm{dn}\,u, \quad \mathrm{dn}'u = -c\,\mathrm{sn}\,u\,\mathrm{cn}\,u.$

These formulae can be regarded as a set of simultaneous differential equations which with the set of initial conditions

$$\mathrm{sn}\,0 = 0, \qquad \mathrm{cn}\,0 = 1, \qquad \mathrm{dn}\,0 = 1$$

determines completely the set of functions $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$. From this point of view it is clear that

11·47. *There can not be more than one set of Jacobian functions with a given parameter c.*

To prove however that there is a set of Jacobian functions for an arbitrary value of c, that is, that the set of functions determined from the set of differential equations is necessarily a Jacobian set of which c is the parameter, is to meet all the difficulties of the inversion problem.

11·5. Addition of a quarterperiod transfers the poles and zeros of one Jacobian function to the poles and zeros of another. If $2K_t$ is a period of $\mathrm{pq}\,u$, addition of K_t interchanges poles and zeros, and $\mathrm{pq}(u+K_t)$ is a multiple of $\mathrm{qp}\,u$. This is the theorem by which we established the periodicities of $\mathrm{pq}\,u$ and of which we used particular cases in compiling Table XI 3: $\mathrm{sn}(u+K_n)$, $\mathrm{cn}(u+K_d)$, $\mathrm{dn}(u+K_c)$ are multiples of $\mathrm{ns}\,u$, $\mathrm{nc}\,u$, $\mathrm{nd}\,u$. If $2K_t$ is not a period of $\mathrm{pq}\,u$, the zero K_p+K_t and the pole K_q+K_t of $\mathrm{pq}(u+K_t)$ are congruent with the two cardinal points other than K_p and K_q. For example, $\mathrm{sn}(u+K_c)$, $\mathrm{cn}(u+K_n)$, $\mathrm{dn}(u+K_d)$ are multiples of $\mathrm{cd}\,u$, $\mathrm{ds}\,u$, $\mathrm{sc}\,u$.

The functional change is obvious geometrically in each particular case. Symbolically we may say that in replacing $\mathrm{pq}(u+K_t)$ by $\mathrm{rm}\,u$, that is, by $\mathrm{rm}(u+K_s)$, we interchange t with s and we must interchange at the same time the other two of the four symbols s, c, n, d. But to ascertain the constant factor we must be able to compare the two functions $\mathrm{pq}(u+K_t)$, $\mathrm{rm}\,u$ at some one point. Most simply, if

$$\mathrm{pq}(u+K_t) = \lambda\,\mathrm{rm}\,u,$$

then λ is the leading coefficient of $\mathrm{pq}\,u$ at K_t. The square of any such coefficient is determinable from either of the tables XI 3, XI 4; what we have now to consider is the determination of the coefficient itself.

As before, we can write down the leading coefficients of any one of the twelve functions if we know them for one copolar triad. The leading coefficient of $\mathrm{pq}\,u$ at the zero K_p is the value of $\mathrm{pq}'K_p$, and is given by Table XI 5 in terms of the values $\mathrm{rq}\,K_p$, $\mathrm{tq}\,K_p$ of the two functions copolar with $\mathrm{pq}\,u$. Also the product of the leading coefficients at K_p and K_q is given in terms of values by an argument that is now familiar: this product is

$$\lim_{u\to 0} \mathrm{pq}(u+K_p)\mathrm{pq}(u+K_q),$$

and since K_q-K_p is a step from zero to pole and from pole to zero, the product $\mathrm{pq}(u+K_p)\mathrm{pq}(u+K_q)$ is independent of u and can be evaluated directly. If $\mathrm{pq}\,u$ is an odd function, either K_p or K_q is zero, and since $(K_r+K_q)+(K_t+K_p) = 0$, the product is expressible as $\mathrm{pq}\,K_r\,\mathrm{pq}(-K_t)$, that is, as $-\mathrm{pq}\,K_r\,\mathrm{pq}\,K_t$. If $\mathrm{pq}\,u$ is even, neither K_p nor K_q is zero and we may take $K_r = 0$, $K_t = -(K_p+K_q)$; putting $u = -K_p$ we have, since $2K_q$ is now a halfperiod,

$$\mathrm{pq}(u+K_p)\mathrm{pq}(u+K_q) = \mathrm{pq}\,0\,\mathrm{pq}(2K_q+K_t) = -\mathrm{pq}\,K_r\,\mathrm{pq}\,K_t,$$

and the result is the same as before:

11·51. *The product of the leading coefficients of the Jacobian function* $\mathrm{pq}\,u$ *at the zero* K_p *and the pole* K_q *is the negative of the product of the values of the function at the other two cardinal points.*

It follows that to form a complete set of leading coefficients we require only the values of each of the three members of one triad at the two cardinal points where that function is neither zero nor infinite. Taking the original Jacobian triad $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$, we have by definition

·501–·503 $\quad \mathrm{sn}\,K_c = 1, \qquad \mathrm{cn}\,0 = 1, \qquad \mathrm{dn}\,0 = 1;$

the constants unidentified are $\mathrm{sn}\,K_d$, $\mathrm{cn}\,K_d$, $\mathrm{dn}\,K_c$, of which only the squares are known:

·504–·506 $\quad \mathrm{sn}^2K_d = 1/c, \qquad \mathrm{cn}^2K_d = -c'/c, \qquad \mathrm{dn}^2K_c = c'.$

As we shall see in the next section, the values of $\mathrm{sn}\,K_d$, $\mathrm{cn}\,K_d$, $\mathrm{dn}\,K_c$ are not only unidentified but unidentifiable: without altering the parameters c, c', we can alter the basis and replace any one of these constants by its negative. What we must do therefore is to accept these values, or a set of constants rationally equivalent to them, as fundamental constants in the theory.

Since $\mathrm{ns}^2K_d = c$, $\mathrm{dn}^2K_c = c'$, we have in effect first to choose definite square roots of c and c'. The choice is governed by the consideration that in the classical case of a positive real first quarterperiod and a positively imaginary second quarterperiod, the functions chosen are to become the positive square roots of the positive real numbers c, c'. In the language of Chapter IX, the real values of an elementary function on the perimeter of the fundamental rectangle all have the same sign. It follows that in the classical case $\operatorname{sn} u$, which resembles u and is therefore positive for sufficiently small positive values of u, is positive when $u = K_c + K_n$, and $\operatorname{dn} u$, which has the positive value 1 at the origin, is positive when $u = K_c$. Accordingly $\operatorname{ns}(K_c + K_n)$ is chosen for one constant, $\operatorname{dn} K_c$ for another, and we write

11·52$_{1-2}$ $$k = \operatorname{ns}(K_c + K_n), \qquad k' = \operatorname{dn} K_c,$$

thus defining the constants known as the *modulus*, k, and the *complementary modulus*, k'. With these definitions,

·507–·509 $$c = k^2, \qquad c' = k'^2, \qquad k^2 + k'^2 = 1.$$

Since the condition $K_c + K_n + K_d = 0$ is essential to ·51, we have to notice specially that

·510 $$\operatorname{ns} K_d = -k.$$

There remains the critical value $\operatorname{cn} K_d$, whose square is now expressible as $-k'^2/k^2$. In the classical case, there are positive real values of $\operatorname{cn} u$ along the line from the zero K_c towards the origin K_s, and therefore the imaginary values along the line from K_c towards $K_c + K_n$, which makes a negative right angle with the line from K_c towards K_s, are negatively imaginary; in particular, since k' and k are positive, $\operatorname{cn}(K_c + K_n) = -ik'/k$, and since $\operatorname{cn} u$ is an even function, $\operatorname{cn} K_d$ has the same value. We write therefore in general

11·53 $$\operatorname{cn} K_d = \operatorname{cn}(K_c + K_n) = -vk'/k,$$

where

11·54 $$v^2 = -1.$$

Always v has one of the two values i, $-i$, but for some bases v has one value, for other bases the other value, and we can not dispense with the symbol.

We can now complete the set of leading coefficients, using ·46$_2$, ·46$_3$, and ·51:

·511–·513 $$\operatorname{sn}'0 = 1, \qquad \operatorname{cn}'K_c = -\operatorname{sn} K_c \operatorname{dn} K_c = -k',$$
$$\operatorname{dn}'K_d = -k^2 \operatorname{sn} K_d \operatorname{cn} K_d = -vk',$$

and therefore, since

$$\operatorname{sn} K_c \operatorname{sn} K_d = -1/k, \qquad \operatorname{cn} K_s \operatorname{cn} K_d = -vk'/k, \qquad \operatorname{dn} K_s \operatorname{dn} K_c = k',$$

·51 implies that near K_n,

·514–·516 $\operatorname{sn} u \sim \dfrac{1/k}{u-K_n}$, $\operatorname{cn} u \sim -\dfrac{v/k}{u-K_n}$, $\operatorname{dn} u \sim -\dfrac{v}{u-K_n}$.

In collecting the leading coefficients of the original Jacobian functions into a table we include a column for the point K'_d, that is, K_c+K_n, since it is to this point more often than to K_d that classical results refer, and since in the case of real moduli this point becomes important as the fourth corner of the fundamental rectangle.

Table XI 6

Leading coefficients of Jacobi's functions

	At K_s	At K_c	At K_n	At K_d	At K'_d
sn u	1×	1	$1/k$÷	$-1/k$	$1/k$
cn u	1	$-k'$×	$-v/k$÷	$-vk'/k$	$-vk'/k$
dn u	1	k'	$-v$÷	$-vk'$×	vk'×

To see more clearly the significance of v, let us look at the table of coefficients for the functions cs u, ns u, ds u, the primitive functions of the Jacobian system; the table is constructed from XI 6:

Table XI 7

Leading coefficients of the primitive Jacobian functions

	At K_s	At K_c	At K_n	At K_d
cs u	1÷	$-k'$×	$-v$	vk'
ns u	1÷	1	k×	$-k$
ds u	1÷	k'	$-vk$	vkk'×

This table includes the six critical values of the primitive functions; in the earlier notation we have

·517–·519 $g_f = \operatorname{ns} K_c = 1$, $h_g = \operatorname{ds} K_n = -vk$, $f_h = \operatorname{cs} K_d = vk'$,

·520–·522 $f_g = \operatorname{cs} K_n = -v$, $g_h = \operatorname{ns} K_d = -k$, $h_f = \operatorname{ds} K_c = k'$.

These values satisfy the relations

$$\frac{g_f}{f_g} = \frac{h_g}{g_h} = \frac{f_h}{h_f} = v.$$

Thus the constant v, which is definable by

11·55 $$\operatorname{sc} K_n = v,$$

is the signature of the set of quarterperiods K_c, K_n, K_d, according to the definition in 1·6; we call it the signature of the basis K_c, K_n.

11·56. *The signature is $+i$ or $-i$ according as minimum rotation $K_c \to K_n \to K_d$ is positive or negative.*

In the theory of the elementary functions constructed from an arbitrary set of quarterperiods, the explicit use of the signature is slight, since the product of one critical value by the signature is expressible as another critical value, and the six critical values though interdependent are all of the same standing. In the Jacobian theory, equality of standing and symmetry are sacrificed at the outset, and the signature becomes one of the insistent constants associated with a basis.

If the leading coefficient of pq u at the pole K_q is a_p, the quotient $(\mathrm{pq}\,u)/a_p$ is the elementary function, in the sense of Chapter II, constructed from the quarterperiods K_c, K_n, K_d. The set of elementary functions is therefore as follows:

TABLE XI 8

$\mathrm{fj}\,u = \mathrm{cs}\,u$	$\mathrm{jf}\,u = -k'\,\mathrm{sc}\,u$	$\mathrm{hg}\,u = v\,\mathrm{dn}\,u$	$\mathrm{gh}\,u = -vk'\,\mathrm{nd}\,u$
$\mathrm{gj}\,u = \mathrm{ns}\,u$	$\mathrm{hf}\,u = -\mathrm{dc}\,u$	$\mathrm{jg}\,u = k\,\mathrm{sn}\,u$	$\mathrm{fh}\,u = k\,\mathrm{cd}\,u$
$\mathrm{hj}\,u = \mathrm{ds}\,u$	$\mathrm{gf}\,u = -k'\,\mathrm{nc}\,u$	$\mathrm{fg}\,u = vk\,\mathrm{cn}\,u$	$\mathrm{jh}\,u = vkk'\,\mathrm{sd}\,u$

This table, which may be otherwise compiled from Table II 2 and the set of critical values ·517–·522, shows the substitutions by which formulae concerning the elementary functions become formulae in the theory of Jacobian functions.

11·6. Since the conditions which render the Jacobian functions specific when their poles and zeros are known are conditions at the origin, the functions themselves depend only on the distributions of poles and zeros, not in any way on the pair of quarterperiods chosen for the basis of the system. Hence the parameter c, which is the constant value of $\mathrm{ns}^2 u - \mathrm{ds}^2 u$, is uniquely determinate. But the constants k, k', v required for the complete scheme of leading coefficients of the functions are in a different category. Their squares c, c', -1 are determinate, but when the constants are defined as $\mathrm{ns}\,K'_d$, $\mathrm{dn}\,K_c$, $\mathrm{sc}\,K_n$ it is with respect to the particular set of quarterperiods in use that they are unambiguous, and their relation to the system of functions is still in question.

Supposing the system of functions to be given, we may attach the symbol K_c to any point at which $\mathrm{sn}\,u = 1$ and the symbol K_n to any pole of $\mathrm{sn}\,u$, provided only that the pair of quarterperiods K_c, K_n is then a primitive pair. That is to say, from ·108,

11·61. *If α, β is one basis of a set of Jacobian functions, the general basis of the same set is given by*

·61$_{1-2}$ $\qquad K_c = (4m_1+1)\alpha+2n_1\beta, \qquad K_n = 2m_2\alpha+(2n_2+1)\beta$

with the one condition

$$\cdot 61_3 \qquad (4m_1+1)(2n_2+1)-4n_1 m_2 = \pm 1.$$

Since 2α is a halfperiod and 2β is a period of $\operatorname{ns} u$,

$$\operatorname{ns}(K_c+K_n) = \pm\operatorname{ns}(\alpha+\beta) \quad \text{according as } m_2 \text{ is even or odd;}$$

since 2α is a period and 2β is a halfperiod of $\operatorname{dn} u$,

$$\operatorname{dn} K_c = \pm\operatorname{dn}\alpha \quad \text{according as } n_1 \text{ is even or odd;}$$

since 2α is a period and 2β is a halfperiod of $\operatorname{sc} u$,

$$\operatorname{sc} K_n = \pm\operatorname{sc}\beta \quad \text{according as } n_2 \text{ is even or odd.}$$

In other words, if with the basis α, β the values of k, k', v are a, a', ι, then when K_c, K_n are given by $\cdot 61_1$, $\cdot 61_2$,

$\cdot 601 \qquad k = \pm a \quad$ according as m_2 is even or odd,

$\cdot 602 \qquad k' = \pm a' \quad$ according as n_1 is even or odd,

$\cdot 603 \qquad v = \pm\iota \quad$ according as n_2 is even or odd.

These alternatives are independent, for the condition $\cdot 61_3$ is satisfied by $n_2 = 0$, $m_1 = n_1 m_2$ and by $n_2 = -1$, $m_1 = -n_1 m_2$, whatever the values of m_2 and n_1. Hence for a given set of functions, the eight possibilities latent in the set of equations

$\cdot 604$–$\cdot 606 \qquad k^2 = c, \qquad k'^2 = c', \qquad v^2 = -1$

are all realized. For a particular choice of K_c and K_n we may ask which square root of c is playing the part of k, which square root of c' is playing the part of k', and whether i or $-i$ is playing the part of v, but the answers depend on the choice of K_c and K_n; we can change the answers by changing the basis.

In $\cdot 61_3$, the sign on the right is positive or negative according as n_2 is even or odd; hence v is ι or $-\iota$ according as the transformation from α, β to K_c, K_n is direct or reverse. This is in agreement with 0·14. We can in fact conclude from the simple arguments of the present section that *for a given set of functions* the sets of quarterperiods for which $\operatorname{sc} K_n$ is i are those for which the rotation $K_c \to K_n \to K_d$ is in one direction and the sets for which $\operatorname{sc} K_n$ is $-i$ are those for which the rotation is in the reverse direction, but we can not tell which direction is associated with $\operatorname{sc} K_n = i$, which with $\operatorname{sc} K_n = -i$. If K_c is real and positive and K_n is imaginary, then between the origin and K_n, $\operatorname{cn} u$ is real and positive and $\operatorname{sn} u$ is positively or negatively imaginary according as K_n is positively or negatively imaginary; thus *in this case* $\operatorname{sc} K_n$ is i or $-i$ according as rotation from K_c to K_n through a right angle is positive or negative, and ·56 is proved for this case without the analysis used in Chapter III. Since i is an absolute constant, the association of the value of $\operatorname{sc} K_n$ with the direction of rotation can not vary from one Jacobian system to another, and we could in fact appeal to continuity and identify the v of 1·606 with the signature retrospectively from ·56.

There is a temptation to remove the signature from the formulae connected with Jacobian functions by including the condition $\operatorname{sc} K_n = i$ in the definition of a Jacobian basis; the pair of conditions

·607–·608 $$\operatorname{sn} K_c = 1, \qquad \operatorname{sc} K_n = i$$

is attractively complete. As we have seen, the second condition is a restriction not on the functions with which we deal, but only on the period systems with which we work. If α, β is a Jacobian basis, so also is $\alpha, -\beta$; the Jacobian functions constructed on the two foundations are identical, and one of the two values $\operatorname{sc}\beta$, $\operatorname{sc}(-\beta)$ is i and the other is $-i$. Since $\operatorname{ns}(\alpha+\beta)$ and $\operatorname{ns}(\alpha-\beta)$ are equal, k has the same value on each basis, and so also has k', which is $\operatorname{dn}\alpha$. Thus to impose the condition $\operatorname{sc} K_n = i$ means only that of the two potential bases α, β and $\alpha, -\beta$, one is accepted and one rejected. If α, β is an acceptable basis, the general basis for the same set of functions is given by the symmetrical pair of formulae

·609 $$K_c = (4m_1+1)\alpha+2n_1\beta, \qquad K_n = 2m_2\alpha+(4n_2+1)\beta$$

with the condition

·610 $$(4m_1+1)(4n_2+1)-4n_1m_2 = 1,$$

which is now definite since the expression on the left can not be equal to -1 for any integral values of m_1, n_1, m_2, n_2.

The question is of course purely one of convenience in vocabulary and notation. If the change is made, the theorem that will be lost is the first part of 10·24: it will no longer be true that every set of quarterperiods $\omega_f, \omega_g, \omega_h$ is represented by a Jacobian set geometrically similar to it. We shall be able to assert only that of the two pairs of numbers $g_f\omega_f, g_f\omega_g$ and $g_f\omega_f, -g_f\omega_g$, one is a Jacobian basis and the other is not; we shall then define the set of quarterperiods $(\alpha, -\beta, -\alpha+\beta)$ as the complement or conjugate of the set $(\alpha, \beta, -\alpha-\beta)$, and instead of saying that $g_f\omega_f, g_f\omega_g$ is necessarily a Jacobian basis and insisting that its signature may be either i or $-i$, we shall have to say that $g_f\omega_f, g_f\omega_g$ is either a Jacobian basis or the conjugate of a Jacobian basis. The duplexity removed from the value of $\operatorname{sc} K_n$ reappears in the procedure of standardization.

The alternative vocabulary might be introduced in another way. The proof that in the system of elliptic functions constructed on the set of quarterperiods ω_f, ω_g, ω_h, the fractions g_f/f_g, h_g/g_h, f_h/h_f have a common value v whose square is -1 is simple. Also if ω_f is kept fixed at a value α and ω_g is changed from β to $-\beta$, then g_f, g_h, h_f are unaltered and f_g, h_g, f_h are replaced by their negatives, and therefore v is replaced by $-v$. It follows that by taking

·611 $$\omega_f = \alpha, \qquad \omega_g = \frac{v(\alpha, \beta)}{i}\beta$$

we have a set of quarterperiods for which the signature is automatically given the value i. Thus we could define the Jacobian basis corresponding to ω_f, ω_g, ω_h by

·612 $$K_c = g_f\,\omega_f, \qquad K_n = \frac{v}{i} g_f\,\omega_g,$$

that is, by

·613 $$K_c = g_f\,\omega_f, \qquad K_n = i f_g\,\omega_g,$$

and secure the definite identity $\operatorname{sc} K_n = i$ without attempting the comparatively difficult interpretation of the alternatives $v = i$, $v = -i$. It will have been established that of the two sets of quarterperiods $(\alpha, \beta, -\alpha-\beta)$, $(\alpha, -\beta, -\alpha+\beta)$ one is geometrically similar to the Jacobian set which represents it and the other is not, and the transformation formulae ·609, ·610 will follow from the condition $\operatorname{sc} K_n = i$ as before.

The obvious criticism of this course is that there is no merit in evading an interpretation; the only question can be whether there are advantages in postponing it. The content of our theorems will not be entirely preserved: we shall not know in advance that rotation in a basis defined by ·612 is necessarily positive. From ·610 we shall learn that rotation in equivalent bases is in the same direction, but we shall have either to appeal to continuity or to develop sooner or later analysis equivalent to that in Chapter III if we are to compare directions of rotation in bases that are not equivalent.

There is no doubt that in practice we need to replace the pair of quarterperiods ω_f, ω_g by the similar pair $g_f\,\omega_f$, $g_f\,\omega_g$ without knowing whether rotation from ω_f to ω_g is positive or negative; if we are to be debarred from the Jacobian notation in the latter case, we shall have to introduce a notation to indicate that the normalizing factor g_f has been used. That is, if K_c, K_n were defined by ·612, we should presently be writing $g_f\,\omega_g = \epsilon K_n$ with ϵ defined as 1 or -1 according as $g_f\,\omega_f$, $g_f\,\omega_g$ was or was not a Jacobian basis in the restricted sense: an adaptable

ϵ would replace an adaptable v, and the notation would be no less and no more complicated than before.

After all, when we have said that the restriction imposed by the condition $\mathrm{sc}\, K_n = i$ would be imposed on the period system but not on the function, is not that an overwhelming argument against imposing the restriction? Should the pair of equations $K_c = \alpha$, $K_n = \beta$ mean more or mean less than that the set of functions constructed on the basis α, β is the Jacobian set for which $\mathrm{sn}\, \alpha = 1$ and β is a pole?

11·7. We have seen at the beginning of ·5 that the effect of the addition of quarterperiods is to be read from a table of leading coefficients. Thus from XI 7 we have an almost equivalent table:

TABLE XI 9

Addition of quarterperiods in the primitive Jacobian functions

$\mathrm{cs}(u+K_c) = -k'\,\mathrm{sc}\, u$	$\mathrm{ns}(u+K_c) = \mathrm{dc}\, u$	$\mathrm{ds}(u+K_c) = k'\,\mathrm{nc}\, u$
$\mathrm{cs}(u+K_n) = -v\,\mathrm{dn}\, u$	$\mathrm{ns}(u+K_n) = k\,\mathrm{sn}\, u$	$\mathrm{ds}(u+K_n) = -vk\,\mathrm{cn}\, u$
$\mathrm{cs}(u+K_d) = vk'\,\mathrm{nd}\, u$	$\mathrm{ns}(u+K_d) = -k\,\mathrm{cd}\, u$	$\mathrm{ds}(u+K_d) = vkk'\,\mathrm{sd}\, u$
$\mathrm{cs}(u+K_d') = -vk'\,\mathrm{nd}\, u$	$\mathrm{ns}(u+K_d') = k\,\mathrm{cd}\, u$	$\mathrm{ds}(u+K_d') = vkk'\,\mathrm{sd}\, u$

The similar table for Jacobi's original functions can be written down either from XI 6 or from XI 9:

TABLE XI 10

Addition of quarterperiods in Jacobi's functions

$\mathrm{sn}(u+K_c) = \mathrm{cd}\, u$	$\mathrm{cn}(u+K_c) = -k'\,\mathrm{sd}\, u$	$\mathrm{dn}(u+K_c) = k'\,\mathrm{nd}\, u$
$\mathrm{sn}(u+K_n) = (1/k)\mathrm{ns}\, u$	$\mathrm{cn}(u+K_n) = -(v/k)\mathrm{ds}\, u$	$\mathrm{dn}(u+K_n) = -v\,\mathrm{cs}\, u$
$\mathrm{sn}(u+K_d) = -(1/k)\mathrm{dc}\, u$	$\mathrm{cn}(u+K_d) = -(vk'/k)\mathrm{nc}\, u$	$\mathrm{dn}(u+K_d) = -vk'\,\mathrm{sc}\, u$
$\mathrm{sn}(u+K_d') = (1/k)\mathrm{dc}\, u$	$\mathrm{cn}(u+K_d') = -(vk'/k)\mathrm{nc}\, u$	$\mathrm{dn}(u+K_d') = vk'\,\mathrm{sc}\, u$

An individual function $\mathrm{pq}(u+K_t)$ is found more readily from XI 9 than from XI 10, since there are no fractional coefficients in the former table, but it is to the formulae in the latter table that historical interest attaches.

11·8. Since $\mathrm{pq}'u$ is a multiple of $\mathrm{rq}\, u\, \mathrm{tq}\, u$, and $\mathrm{rq}^2 u$, $\mathrm{tq}^2 u$ are multiples of $\mathrm{pq}^2 u - \mathrm{pq}^2 K_r$, $\mathrm{pq}^2 u - \mathrm{pq}^2 K_t$, the function $\mathrm{pq}\, u$ satisfies a differential equation

·801 $$(dx/du)^2 = \lambda(x^2-\mu)(x^2-\nu),$$

where λ, μ, ν are constants determinable from Tables XI 5, 2. In particular, $x_1 \equiv \mathrm{cs}\, u$, $x_2 \equiv \mathrm{ns}\, u$, $x_3 \equiv \mathrm{ds}\, u$ satisfy the equations

·802 $$(dx_1/du)^2 = (x_1^2+1)(x_1^2+c'),$$

·803 $$(dx_2/du)^2 = (x_2^2-1)(x_2^2-c),$$

·804 $$(dx_3/du)^2 = (x_3^2+c)(x_3^2-c').$$

Instead of referring again to Tables XI 5, 2, we can utilize XI 9. The function $\mathrm{pq}(u+K_t)$ satisfies the same differential equation of the form ·801 as $\mathrm{pq}\,u$, and if $\mathrm{pq}\,u$ satisfies ·801, the function $(\mathrm{pq}\,u)/\kappa$, where κ is a constant, satisfies the equation

·805 $$\kappa^2(dx/du)^2 = \lambda(\kappa^2x^2-\mu)(\kappa^2x^2-\nu).$$

The complete set of expressions for the squares of the derivatives is as follows:

TABLE XI 11

x	$(dx/du)^2$	x	$(dx/du)^2$
$\mathrm{cs}\,u$	$(x^2+1)(x^2+c')$	$\mathrm{sc}\,u$	$(1+c'x^2)(1+x^2)$
$\mathrm{ns}\,u$	$(x^2-1)(x^2-c)$	$\mathrm{dc}\,u$	$(x^2-1)(x^2-c)$
$\mathrm{ds}\,u$	$(x^2+c)(x^2-c')$	$\mathrm{nc}\,u$	$(c'x^2+c)(x^2-1)$

x	$(dx/du)^2$	x	$(dx/du)^2$
$\mathrm{dn}\,u$	$(1-x^2)(x^2-c')$	$\mathrm{nd}\,u$	$(1-c'x^2)(x^2-1)$
$\mathrm{sn}\,u$	$(1-x^2)(1-cx^2)$	$\mathrm{cd}\,u$	$(1-x^2)(1-cx^2)$
$\mathrm{cn}\,u$	$(1-x^2)(c'+cx^2)$	$\mathrm{sd}\,u$	$(1-c'x^2)(1+cx^2)$

The coefficient λ in ·801 is 1 for a primitive function; hence if $\mathrm{pq}\,u = \{\mathrm{rs}(u+K_q)\}/\kappa$, the coefficient of x^4 in the entry against $\mathrm{pq}\,u$ in this table is κ^2. But if $\mathrm{rs}(u+K_q) = \kappa\,\mathrm{pq}\,u$, then κ^2 is the leading coefficient of $\mathrm{rs}^2 u$ at K_q. Thus the coefficient of x^4 in any entry in XI 11 is the corresponding entry in XI 4. Also the constant term against $\mathrm{pq}\,u$ in XI 11 is the coefficient of x^4 against $\mathrm{qp}\,u$.

From Table XI 11 we derive the details of the fundamental theorems connecting the functions of Jacobi and Glaisher with elliptic integrals. To demonstrate the results is only to repeat the arguments of 5·1 and 5·3 in each case. As before, if c and c' come from a known system of functions, the equivalence of the functional relation with the integral relation is proved without difficulty; it is when c and c' are given first that the problem of inversion is acute.

11·81. *If the radicals resemble x^2 towards infinity on the paths of integration, the relations*

$\cdot 81_{1\text{-}3}$ $$u_1 = \int_{x_1}^{\infty} \frac{dx}{\sqrt{\{(x^2+1)(x^2+c')\}}}, \qquad u_2 = \int_{x_2}^{\infty} \frac{dx}{\sqrt{\{(x^2-1)(x^2-c)\}}},$$

$$u_3 = \int_{x_3}^{\infty} \frac{dx}{\sqrt{\{(x^2+c)(x^2-c')\}}}$$

are equivalent to

$$x_1 = \mathrm{cs}\,u_1, \qquad x_2 = \mathrm{ns}\,u_2, \qquad x_3 = \mathrm{ds}\,u_3.$$

11·82. *If the radicals have the value* 1 *at the origin, the relations*

$$\cdot 82_{1-3} \quad u_4 = \int_0^{x_4} \frac{dx}{\sqrt{\{(1+x^2)(1+c'x^2)\}}}, \qquad u_8 = \int_0^{x_8} \frac{dx}{\sqrt{\{(1-x^2)(1-cx^2)\}}},$$

$$u_{12} = \int_0^{x_{12}} \frac{dx}{\sqrt{\{(1+cx^2)(1-c'x^2)\}}}$$

are equivalent to

$$x_4 = \operatorname{sc} u_4, \qquad x_8 = \operatorname{sn} u_8, \qquad x_{12} = \operatorname{sd} u_{12}.$$

11·83. *The relations*

$$\cdot 83_{1-3} \quad u_5 = \int_1^{x_5} \frac{dx}{\sqrt{\{(x^2-1)(x^2-c)\}}}, \qquad u_6 = \int_1^{x_6} \frac{dx}{\sqrt{\{(c'x^2+c)(x^2-1)\}}},$$

$$u_7 = \int_{x_7}^{1} \frac{dx}{\sqrt{\{(1-x^2)(x^2-c')\}}}$$

are equivalent to

$$x_5 = \operatorname{dc} u_5, \qquad x_6 = \operatorname{nc} u_6, \qquad x_7 = \operatorname{dn} u_7,$$

and the relations

$$\cdot 83_{4-6} \quad u_9 = \int_{x_9}^{1} \frac{dx}{\sqrt{\{(c'+cx^2)(1-x^2)\}}}, \qquad u_{10} = \int_1^{x_{10}} \frac{dx}{\sqrt{\{(x^2-1)(1-c'x^2)\}}},$$

$$u_{11} = \int_{x_{11}}^{1} \frac{dx}{\sqrt{\{(1-x^2)(1-cx^2)\}}}$$

are equivalent to

$$x_9 = \operatorname{cn} u_9, \qquad x_{10} = \operatorname{nd} u_{10}, \qquad x_{11} = \operatorname{cd} u_{11}.$$

If to the variable u in any of the formulae in these three theorems we give the value K_c, we have from Table XI 7 a limit of integration by means of which this quarterperiod is expressible as an integral. Since some of the limits of integration involve k and k', we use k^2, k'^2 instead of c, c' in the integrands also. It is to be remembered that the variables and parameters are complex, and that the paths of integration in ·81, ·82, ·83 are arbitrary.

The set of formulae we obtain for K_c from ·81, ·82, ·83 is both redundant and in a sense incomplete. On the one hand, $\operatorname{dc} u$ gives the same integral as $\operatorname{ns} u$, and $\operatorname{cd} u$ as $\operatorname{sn} u$. On the other hand, the integrals given by $\operatorname{ds} u$ and $\operatorname{nc} u$ are

$$\int_{k'}^{\infty} \frac{dt}{\sqrt{\{(t^2+k^2)(t^2-k'^2)\}}}, \qquad \int_{1}^{\infty} \frac{dt}{\sqrt{\{(k'^2t^2+k^2)(t^2-1)\}}};$$

the distinction between these is trivial, and if they are both to be recorded, the equivalent form

$$\int_{k'/k}^{\infty} \frac{dt}{\sqrt{\{(t^2+1)(k^2t^2-k'^2)\}}}$$

must be added, although this is not provided by any of our twelve formulae. With this extension, the ten distinct formulae for K_c become sixteen; if we allow the possible changes of this kind to be made mentally, the ten become six. Further, the substitution of $1/t$ for t, equivalent to the use of $\operatorname{qp} u$ instead of $\operatorname{pq} u$, is trivial, and if the possibility of this substitution also is borne in mind, four integrals remain:

11·84. *For each of the integrals*

$\cdot 84_{1-4}$
$$\int_{0}^{\infty} \frac{dt}{\sqrt{\{(t^2+1)(t^2+k'^2)\}}}, \qquad \int_{0}^{1} \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}},$$
$$\int_{0}^{1/k'} \frac{dt}{\sqrt{\{(1+k^2t^2)(1-k'^2t^2)\}}}, \qquad \int_{k'}^{1} \frac{dt}{\sqrt{\{(1-t^2)(t^2-k'^2)\}}}$$

there are paths of integration such that the value of the integral is K_c.

If K_n is substituted for u in ·81 or ·82, each limit of integration either has the signature v for an explicit factor or is 0 or ∞. By substituting $\pm vt$ for x we transfer the factor v from the limit to the entire integral. In ·81 the radical has then to resemble $-t^2$ towards infinity and the substitution $x = -vt$ removes the negative sign; in ·82 the determining value of the radical is not affected by the substitution and we put $x = vt$. In ·83 the limits of integration do not involve v, but for the sake of comparison we can introduce v as a factor of the integral by reversing one of the two factors in the radical. These changes having been made, the discussion is exactly parallel to that leading to ·84:

11·85. *For each of the integrals*

$$\cdot 85_{1\text{-}4} \qquad \int_0^\infty \frac{dt}{\sqrt{\{(t^2+1)(t^2+k^2)\}}}, \qquad \int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(1-k'^2t^2)\}}},$$

$$\int_0^{1/k} \frac{dt}{\sqrt{\{(1+k'^2t^2)(1-k^2t^2)\}}}, \qquad \int_k^1 \frac{dt}{\sqrt{\{(1-t^2)(t^2-k^2)\}}}$$

there are paths of integration such that the value of the integral is K_n/v.

The classical integrals giving K_c and K_n are $\cdot 84_2$ and $\cdot 85_2$:

11·86. *If $K_c = K$ and $K_n = vK'$, there are paths of integration such that*

$$\cdot 86_{1\text{-}2} \quad K = \int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}, \qquad K' = \int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(1-k'^2t^2)\}}}.$$

In the simple theory in which k and k' are real and therefore it is possible, as we have really shown in 9·7, to take for K and K' the real values obtained by treating the integrals in ·84 and ·85 as integrals along the real axis, the interpretation of the similarity between the two sets of integrals, and in particular between the two integrals in ·86, is immediate: K' is the same function of k' as K is of k. But in the general theory we are not yet in a position to make this comparison, for we have established no relations between paths of integration.

Among the integral expressions for K_d are two, namely, those given by $\mathrm{nc}\,u$ and $\mathrm{cn}\,u$, in which one limit is 1 and the other has v for a factor. It is impossible to eliminate v from the formal expression of such an integral, and if we replace the integral by the difference between two integrals from 0 or by the difference between two integrals to ∞, we can be doing nothing but putting the integral into the form $A+vB$. The expressions obtainable for K_d in this way are identifiable immediately with forms of $-K_c-K_n$ derivable from ·84 and ·85, and nothing is gained by deriving them from ·81, ·82, ·83. The simplest formal expressions are

$$\int_0^\infty \frac{dt}{\sqrt{\{(t^2+k^2)(t^2-k'^2)\}}}, \qquad \int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(t^2-k'^2)\}}},$$

and if $0 < k' < 1$ the formal simplicity of these integrals is a transparent illusion.

XII

ADDITION THEOREMS FOR THE JACOBIAN FUNCTIONS

12·1. The Jacobian system of functions generates a profusion of addition theorems, and the total lack of symmetry in the system renders general theorems hard to express and tends to deprive general formulae of their utility. For the construction of isolated results it is often better to return to first principles than to attempt a substitution.

For a first example of a general method, let us find a formula for $\operatorname{cn}(u+v)$ by Liouville's process. As functions of u, the functions

$$\operatorname{cn}(u+v)+\operatorname{cn}(u-v), \qquad \operatorname{cn}(u+v)-\operatorname{cn}(u-v)$$

both have four simple poles, the two poles of $\operatorname{cn}(u+v)$, which are $-v+K_n$ and $-v+K_n+2K_c$, and the two poles of $\operatorname{cn}(u-v)$, which are $v+K_n$ and $v+K_n+2K_c$. Of these four poles, $-v+K_n$ and $v+K_n+2K_c$ are zeros of $\operatorname{cn} u-\operatorname{cn}(v-K_n)$, and $v+K_n$ and $-v+K_n+2K_c$ are zeros of $\operatorname{cn} u-\operatorname{cn}(v+K_n)$, that is, $2K_n$ being a halfperiod, of $\operatorname{cn} u+\operatorname{cn}(v-K_n)$. Thus the four poles are the zeros of $\operatorname{cn}^2 u-\operatorname{cn}^2(v-K_n)$. Again, the zeros of $\operatorname{cn}(u+v)+\operatorname{cn}(u-v)$ are the solutions of the equation

$$\operatorname{cn}(u+v) = \operatorname{cn}(u-v+2K_c),$$

and by 11·104, since the difference between the two arguments does not involve u, these are the points for which the sum $2u+2K_c$ is of the form $2mK_c+2nK_n$ with $m+n$ even; the points required are K_c and $-K_c$, which are zeros of $\operatorname{cn} u$, and K_n and K_n+2K_c, which are poles of $\operatorname{cn} u$. Similarly the zeros of $\operatorname{cn}(u+v)-\operatorname{cn}(u-v)$ are the solutions of the equation $\operatorname{cn}(u+v) = \operatorname{cn}(u-v)$ and are identified as zeros of $\operatorname{cn}' u$. Thus

·101 $$\operatorname{cn}(u+v)+\operatorname{cn}(u-v) = \frac{2A \operatorname{cn} u}{\operatorname{cn}^2 u-\operatorname{cn}^2(v-K_n)},$$

·102 $$\operatorname{cn}(u+v)-\operatorname{cn}(u-v) = \frac{2B \operatorname{cn}' u}{\operatorname{cn}^2 u-\operatorname{cn}^2(v-K_n)},$$

where A, B are independent of u. Putting $u = 0$ in ·101 we have

$$A = \operatorname{cn} v \operatorname{sn}^2(v-K_n);$$

letting $u \to 0$ in ·102, we have, since $\operatorname{cn}' u \sim -u$,

$$B = -\operatorname{sn}^2(v-K_n)\lim_{u\to 0}\frac{\operatorname{cn}(v+u)-\operatorname{cn}(v-u)}{2u} = -\operatorname{cn}' v \operatorname{sn}^2(v-K_n).$$

Hence

·103 $$\operatorname{cn}(u+v) = \frac{(\operatorname{cn} u \operatorname{cn} v-\operatorname{cn}' u \operatorname{cn}' v)\operatorname{sn}^2(v-K_n)}{\operatorname{cn}^2 u-\operatorname{cn}^2(v-K_n)}.$$

From Table XI 10,

$$\operatorname{ns}^2(v-K_n) = c\operatorname{sn}^2 v, \qquad \operatorname{cs}^2(v-K_n) = -\operatorname{dn}^2 v,$$

whence

$$\operatorname{cn}^2 u \operatorname{ns}^2(v-K_n)-\operatorname{cs}^2(v-K_n) = 1+c\operatorname{sn}^2 v(\operatorname{cn}^2 u-1) = 1-c\operatorname{sn}^2 u\operatorname{sn}^2 v,$$

and we have a classical formula

12·11 $$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u\operatorname{cn} v-\operatorname{cn}' u\operatorname{cn}' v}{1-c\operatorname{sn}^2 u\operatorname{sn}^2 v}.$$

12·2. To illustrate another process, let $\phi(u)$ denote $\operatorname{dn}^2 u$ and let $F(w)$ denote

$$\begin{vmatrix} 1 & \phi(u) & \phi'(u) \\ 1 & \phi(v) & \phi'(v) \\ 1 & \phi(w) & \phi'(w) \end{vmatrix}.$$

This function $F(w)$ of w has the periods $2K_c$, $2K_n$, and is of the third order with a triple pole at K_n; it has obvious zeros at u and v, and therefore a third zero at $3K_n-u-v$. But

·201 $$\phi'(w) = 2\operatorname{dn} w\operatorname{dn}' w,$$

and therefore

·202 $$\{\phi'(w)\}^2 = \lambda\phi(w)\{\phi(w)-\mu\}\{\phi(w)-\nu\},$$

where λ, μ, ν are constants whose actual values we do not need. Hence the equation $F(w) = 0$ implies that $\phi(w)$ satisfies an equation

·203 $$\begin{vmatrix} 1 & \phi(u) & \phi'(u) \\ 1 & \phi(v) & \phi'(v) \\ 1 & t & 0 \end{vmatrix}^2 = \lambda\{\phi(u)-\phi(v)\}^2 t(t-\mu)(t-\nu),$$

and since this is a cubic equation, its roots are $\phi(u)$, $\phi(v)$, $\phi(3K_n-u-v)$. The product of these three roots is therefore a constant multiple of

$$\frac{\{\phi(u)\phi'(v)-\phi(v)\phi'(u)\}^2}{\{\phi(u)-\phi(v)\}^2},$$

and it follows that $\operatorname{dn}(u+v+K_n)$ is a constant multiple of

$$\frac{\phi(u)\phi'(v)-\phi(v)\phi'(u)}{\{\phi(u)-\phi(v)\}\operatorname{dn} u\operatorname{dn} v},$$

that is, a constant multiple of

·204 $$\frac{\operatorname{dn} u\operatorname{dn}' v-\operatorname{dn} v\operatorname{dn}' u}{\operatorname{dn}^2 u-\operatorname{dn}^2 v}.$$

Since $\operatorname{dn}(u+K_n)$ is a constant multiple of $\operatorname{cs} u$, the simplest interpretation of this result is that $\operatorname{cs}(u+v)$ is a constant multiple of the fraction ·204; when $v = 0$ the fraction becomes $\mathrm{dn}'u/(1-\mathrm{dn}^2u)$, which is $\mathrm{cs}\,u$, and therefore the constant factor is unity:

12·21 $$\mathrm{cs}(u+v) = \frac{\mathrm{dn}\,u\,\mathrm{dn}'v-\mathrm{dn}\,v\,\mathrm{dn}'u}{\mathrm{dn}^2u-\mathrm{dn}^2v}.$$

If it is a formula for $\mathrm{dn}(u+v)$ that we require, we replace v by $v-K_n$. Since $\mathrm{dn}(v-K_n)$ is a constant multiple of $\mathrm{cs}\,v$, the numerator in ·204 then becomes a multiple of

$$\mathrm{dn}\,u\,\mathrm{cs}'v-\mathrm{cs}\,v\,\mathrm{dn}'u,$$

that is, of $$-\mathrm{dn}\,u\,\mathrm{ns}\,v\,\mathrm{ds}\,v-\mathrm{dn}'u\,\mathrm{cs}\,v$$

and therefore of $$(\mathrm{dn}\,u\,\mathrm{dn}\,v+\mathrm{dn}'u\,\mathrm{sn}\,v\,\mathrm{cn}\,v)\mathrm{ns}^2v.$$

Thus $\mathrm{dn}(u+v)$ is a constant multiple of

$$\frac{\mathrm{dn}\,u\,\mathrm{dn}\,v-(1/c)\mathrm{dn}'u\,\mathrm{dn}'v}{\{\mathrm{dn}^2u-\mathrm{dn}^2(v-K_n)\}\mathrm{sn}^2v}.$$

Substituting $-\mathrm{cs}^2v$ for $\mathrm{dn}^2(v-K_n)$, we have the denominator

$$\mathrm{dn}^2u\,\mathrm{sn}^2v+\mathrm{cn}^2v,$$

that is, $1-c\,\mathrm{sn}^2u\,\mathrm{sn}^2v$, and since the fraction becomes 1 when u and v are zero, the constant factor is again unity, and we have

12·22 $$\mathrm{dn}(u+v) = \frac{\mathrm{dn}\,u\,\mathrm{dn}\,v-(1/c)\mathrm{dn}'u\,\mathrm{dn}'v}{1-c\,\mathrm{sn}^2u\,\mathrm{sn}^2v},$$

another classical result.

12·3. Up to a point the argument of the last paragraph is perfectly general, for the values of the coefficients in the cubic equation ·203 are not used. If $\mathrm{pq}\,u$ is any one of Glaisher's twelve functions, and if $\phi(w) = \mathrm{pq}^2w$, then $\phi'(w)$ has a triple pole at K_q, and $4K_q$ is either zero or a period of $\phi(w)$. Hence $\mathrm{pq}(u+v+K_q)$ is a constant multiple of

·301 $$\frac{\mathrm{pq}\,u\,\mathrm{pq}'v-\mathrm{pq}\,v\,\mathrm{pq}'u}{\mathrm{pq}^2u-\mathrm{pq}^2v}.$$

The function $\mathrm{pq}(u+K_q)$ is infinite at the origin, and is therefore a multiple of the primitive function which is coperiodic with $\mathrm{pq}\,u$; if then this primitive function is $\mathrm{rs}\,u$, it follows that $\mathrm{rs}(u+v)$ is a constant multiple of the fraction ·301. Suppose that for small values of u,

·302 $$\mathrm{pq}(u+K_q) \sim a_p/u.$$

Then

·303 $$\lim_{v\to 0}\frac{\mathrm{pq}(u+K_q)\mathrm{pq}'(v+K_q)-\mathrm{pq}(v+K_q)\mathrm{pq}'(u+K_q)}{\mathrm{pq}^2(u+K_q)-\mathrm{pq}^2(v+K_q)} = \frac{\mathrm{pq}(u+K_q)}{a_p}$$
$$\sim \frac{1}{u}.$$

On the other hand,

·304 $$\lim_{v\to 0} \operatorname{rs}(u+v+2K_q) = \operatorname{rs}(u+2K_q) \sim \pm\frac{1}{u},$$

the negative sign occurring if $2K_q$ is a halfperiod of $\operatorname{rs} u$, that is, if K_q is not one of the two points K_r, K_s:

12·31. *If* $\operatorname{pq} u$ *is any one of the four functions coperiodic with the primitive function* $\operatorname{rs} u$, *then*

$$\operatorname{rs}(u+v) = \pm\frac{\operatorname{pq} u \operatorname{pq}' v - \operatorname{pq} v \operatorname{pq}' u}{\operatorname{pq}^2 u - \operatorname{pq}^2 v},$$

the sign being positive if $\operatorname{pq} u$ *is* $\operatorname{rs} u$ *or* $\operatorname{sr} u$, *negative if* $\operatorname{pq} u$ *is one of the other two coperiodic functions.*

This theorem, which gives four formulae for each of the functions $\operatorname{cs}(u+v)$, $\operatorname{ns}(u+v)$, $\operatorname{ds}(u+v)$, and therefore for each of their reciprocals $\operatorname{sc}(u+v)$, $\operatorname{sn}(u+v)$, $\operatorname{sd}(u+v)$, is the simplest comprehensive addition theorem for Jacobian functions. As soon as algebraical combinations are formed and simplified, the constants c, c' enter and repetitions are rare.

The expression $\operatorname{pq} u \operatorname{pq}' v - \operatorname{pq} v \operatorname{pq}' u$ is an awkward denominator. If

·305 $$\operatorname{pq}'^2 u = \lambda \operatorname{pq}^4 u + \mu \operatorname{pq}^2 u + \nu,$$

then

·306 $$\operatorname{pq}^2 u \operatorname{pq}'^2 v - \operatorname{pq}^2 v \operatorname{pq}'^2 u = (\operatorname{pq}^2 u - \operatorname{pq}^2 v)(\nu - \lambda \operatorname{pq}^2 u \operatorname{pq}^2 v),$$

and we replace the reciprocal of the fraction in ·31 by

$$\frac{\operatorname{pq} u \operatorname{pq}' v + \operatorname{pq} v \operatorname{pq}' u}{\nu - \lambda \operatorname{pq}^2 u \operatorname{pq}^2 v},$$

the coefficients ν, λ being taken from Table XI 11. Since identically

·307 $$\frac{\operatorname{qp} u \operatorname{qp}' v + \operatorname{qp} v \operatorname{qp}' u}{\lambda - \nu \operatorname{qp}^2 u \operatorname{qp}^2 v} = \frac{\operatorname{pq} u \operatorname{pq}' v + \operatorname{pq} v \operatorname{pq}' u}{\nu - \lambda \operatorname{pq}^2 u \operatorname{pq}^2 v}$$

we record only one of a pair of formulae related in this way.

12·32. *Addition theorems for the functions with zeros at the origin are as follows*:

·32₁₋₂ $$\operatorname{sc}(u+v) = \frac{\operatorname{sc} u \operatorname{sc}' v + \operatorname{sc} v \operatorname{sc}' u}{1 - c' \operatorname{sc}^2 u \operatorname{sc}^2 v} = -\frac{\operatorname{dn} u \operatorname{dn}' v + \operatorname{dn} v \operatorname{dn}' u}{\operatorname{dn}^2 u \operatorname{dn}^2 v - c'},$$

·32₃₋₄ $$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{sn}' v + \operatorname{sn} v \operatorname{sn}' u}{1 - c \operatorname{sn}^2 u \operatorname{sn}^2 v} = -\frac{\operatorname{cd} u \operatorname{cd}' v + \operatorname{cd} v \operatorname{cd}' u}{1 - c \operatorname{cd}^2 u \operatorname{cd}^2 v},$$

·32₅₋₆ $$\operatorname{sd}(u+v) = \frac{\operatorname{sd} u \operatorname{sd}' v + \operatorname{sd} v \operatorname{sd}' u}{1 + cc' \operatorname{sd}^2 u \operatorname{sd}^2 v} = -\frac{\operatorname{cn} u \operatorname{cn}' v + \operatorname{cn} v \operatorname{cn}' u}{c' + c \operatorname{cn}^2 u \operatorname{cn}^2 v}.$$

We can recognize the coefficients in the denominator of the formula giving $\mathrm{sq}(u+v)$ in terms of $\mathrm{sq}\,u$ and $\mathrm{sq}\,v$ and their derivatives. If

·308 $$\mathrm{sq}'^2 u = \lambda\,\mathrm{sq}^4 u + \mu\,\mathrm{sq}^2 u + \nu,$$

then putting $u = 0$ we have $\nu = 1$; also

·309 $$\lambda = \lim_{u\to K_q}\left(\frac{\mathrm{sq}'u}{\mathrm{sq}^2 u}\right)^2 = \mathrm{qs}'^2 K_q:$$

12·33. *For a Jacobian function* $\mathrm{sq}\,u$ *which has a zero at the origin,*

$$\mathrm{sq}(u+v) = \frac{\mathrm{sq}\,u\,\mathrm{sq}'v + \mathrm{sq}\,v\,\mathrm{sq}'u}{1 - \mathrm{qs}'^2 K_q\,\mathrm{sq}^2 u\,\mathrm{sq}^2 v}.$$

If K_p, K_q, K_r are the three cardinal points distinct from K_s, then

·310–·311 $$\mathrm{ps}'u = -\mathrm{qs}\,u\,\mathrm{rs}\,u, \qquad \mathrm{sq}'u = \mathrm{pq}\,u\,\mathrm{rq}\,u.$$

We can therefore replace the derivatives in ·31 by products without complicating the formulae if the functions differentiated have poles or zeros at the origin:

12·34$_{1\text{-}2}$ $\mathrm{ps}(u+v)$

$$= \frac{\mathrm{qs}\,u\,\mathrm{rs}\,u\,\mathrm{ps}\,v - \mathrm{qs}\,v\,\mathrm{rs}\,v\,\mathrm{ps}\,u}{\mathrm{ps}^2 u - \mathrm{ps}^2 v} = \frac{\mathrm{sp}\,u\,\mathrm{qp}\,v\,\mathrm{rp}\,v - \mathrm{sp}\,v\,\mathrm{qp}\,u\,\mathrm{rp}\,u}{\mathrm{sp}^2 u - \mathrm{sp}^2 v}.$$

Similarly from ·33

12·35 $$\mathrm{sq}(u+v) = \frac{\mathrm{sq}\,u\,\mathrm{pq}\,v\,\mathrm{rq}\,v + \mathrm{sq}\,v\,\mathrm{pq}\,u\,\mathrm{rq}\,u}{1 - \mathrm{qs}'^2 K_q\,\mathrm{sq}^2 u\,\mathrm{sq}^2 v},$$

and in detail from ·32$_{1,3,5}$

12·36$_1$ $$\mathrm{sc}(u+v) = \frac{\mathrm{sc}\,u\,\mathrm{nc}\,v\,\mathrm{dc}\,v + \mathrm{sc}\,v\,\mathrm{nc}\,u\,\mathrm{dc}\,u}{1 - c'\,\mathrm{sc}^2 u\,\mathrm{sc}^2 v},$$

12·36$_2$ $$\mathrm{sn}(u+v) = \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v + \mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1 - c\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v},$$

12·36$_3$ $$\mathrm{sd}(u+v) = \frac{\mathrm{sd}\,u\,\mathrm{cd}\,v\,\mathrm{nd}\,v + \mathrm{sd}\,v\,\mathrm{cd}\,u\,\mathrm{nd}\,u}{1 + cc'\,\mathrm{sd}^2 u\,\mathrm{sd}^2 v}.$$

The classical formula in this set, and indeed the most celebrated addition formula in the whole subject, is ·36$_2$, the addition theorem for Jacobi's first function $\mathrm{sn}\,u$.

12·4. If neither the zero K_p nor the pole K_q is at the origin, then

$$\mathrm{pq}\,u = \mathrm{sr}\,K_q\,\mathrm{rs}(u+K_q),$$

where K_r is the one of the three points K_c, K_n, K_d that is distinct from K_p and K_q. We have therefore

$$\cdot 401 \qquad \mathrm{pq}(u+v) = \frac{\mathrm{rs}\,u\,\mathrm{pq}'v - \mathrm{pq}\,v\,\mathrm{rs}'u}{\mathrm{rs}^2u - \mathrm{rs}^2K_q\,\mathrm{pq}^2v}.$$

But
$$\mathrm{rs}'u = -\mathrm{ps}\,u\,\mathrm{qs}\,u = -\mathrm{pq}\,u\,\mathrm{qs}^2u,$$
$$\mathrm{rs}^2u - \mathrm{rs}^2K_q\,\mathrm{pq}^2v = \mathrm{rs}^2u - \mathrm{rs}^2K_q(1+\mathrm{ps}^2K_q\,\mathrm{sq}^2v) = \mathrm{qs}^2u - \mathrm{qs}'^2K_q\,\mathrm{sq}^2v.$$

Hence

$$\cdot 402 \qquad \mathrm{pq}(u+v) = \frac{\mathrm{pq}\,u\,\mathrm{pq}\,v + \mathrm{rq}\,u\,\mathrm{sq}\,u\,\mathrm{pq}'v}{1 - \mathrm{qs}'^2K_q\,\mathrm{sq}^2u\,\mathrm{sq}^2v}.$$

Expressing $\mathrm{pq}'v$ as $-\mathrm{qs}^2K_p\,\mathrm{rq}\,v\,\mathrm{sq}\,v$, we have the comprehensive theorem

12·41. *If* $\mathrm{pq}\,u$ *is a Jacobian function for which the origin is neither a pole nor a zero, then*

$$\mathrm{pq}(u+v) = \frac{\mathrm{pq}\,u\,\mathrm{pq}\,v - \mathrm{qs}^2K_p\,\mathrm{sq}\,u\,\mathrm{rq}\,u\,\mathrm{sq}\,v\,\mathrm{rq}\,v}{1 - \mathrm{qs}'^2K_q\,\mathrm{sq}^2u\,\mathrm{sq}^2v},$$

which includes the classical formulae

$$12\cdot 42_1 \qquad \mathrm{cn}(u+v) = \frac{\mathrm{cn}\,u\,\mathrm{cn}\,v - \mathrm{sn}\,u\,\mathrm{dn}\,u\,\mathrm{sn}\,v\,\mathrm{dn}\,v}{1 - c\,\mathrm{sn}^2u\,\mathrm{sn}^2v},$$

$$12\cdot 42_2 \qquad \mathrm{dn}(u+v) = \frac{\mathrm{dn}\,u\,\mathrm{dn}\,v - c\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{sn}\,v\,\mathrm{cn}\,v}{1 - c\,\mathrm{sn}^2u\,\mathrm{sn}^2v}.$$

Alternatively, replacing $\mathrm{rq}\,u\,\mathrm{sq}\,u$ in ·402 by $-\mathrm{sq}^2K_q\,\mathrm{pq}'u$, we have the numerator expressed purely in terms of one function and its derivative:

12·43. *If* $\mathrm{pq}\,u$ *is a Jacobian function for which the origin is neither a pole nor a zero, then*

$$\mathrm{pq}(u+v) = \frac{\mathrm{pq}\,u\,\mathrm{pq}\,v - \mathrm{sq}^2K_p\,\mathrm{pq}'u\,\mathrm{pq}'v}{1 - \mathrm{qs}'^2K_q\,\mathrm{sq}^2u\,\mathrm{sq}^2v}.$$

This is the theorem of which ·11 and ·22 are cases; the denominator can be written

$$1 - \mathrm{sq}^2K_p\,\mathrm{qs}^2K_r(1-\mathrm{pq}^2u)(1-\mathrm{pq}^2v).$$

By adding a quarterperiod K_l simultaneously to u and v we obtain from ·43 other expressions for $\mathrm{pq}(u+v)$ in terms of coperiodic functions of u and v. Two constant factors are involved, one for the functions in the numerator and the other for those in the denominator, but it is simpler to adjust a factor to the whole fraction by putting u and v zero than to attend to the first of these factors. The complete set of explicit formulae follows. The classical formulae for $\mathrm{cn}(u+v)$ and $\mathrm{dn}(u+v)$ reappear.

TABLE XII·1

$$\mathrm{nc}(u+v) = \frac{\mathrm{nc}\,u\,\mathrm{nc}\,v+\mathrm{nc}'u\,\mathrm{nc}'v}{1-c'\,\mathrm{sc}^2u\,\mathrm{sc}^2v} = \frac{\mathrm{cn}\,u\,\mathrm{cn}\,v+\mathrm{cn}'u\,\mathrm{cn}'v}{(\mathrm{dn}^2u\,\mathrm{dn}^2v-c')/c} = \frac{\mathrm{ds}'u\,\mathrm{ds}'v+\mathrm{ds}\,u\,\mathrm{ds}\,v}{\mathrm{cs}^2u\,\mathrm{cs}^2v-c'} = \frac{\mathrm{sd}'u\,\mathrm{sd}'v+\mathrm{sd}\,u\,\mathrm{sd}\,v}{(1-c'\,\mathrm{nd}^2u\,\mathrm{nd}^2v)/c}$$

$$\mathrm{dc}(u+v) = \frac{\mathrm{dc}\,u\,\mathrm{dc}\,v+(1/c')\mathrm{dc}'u\,\mathrm{dc}'v}{1-c'\,\mathrm{sc}^2u\,\mathrm{sc}^2v} = \frac{\mathrm{sn}'u\,\mathrm{sn}'v+c'\,\mathrm{sn}\,u\,\mathrm{sn}\,v}{(\mathrm{dn}^2u\,\mathrm{dn}^2v-c')/c} = \frac{\mathrm{ns}'u\,\mathrm{ns}'v+c'\,\mathrm{ns}\,u\,\mathrm{ns}\,v}{\mathrm{cs}^2u\,\mathrm{cs}^2v-c'} = \frac{\mathrm{cd}\,u\,\mathrm{cd}\,v+(1/c')\mathrm{cd}'u\,\mathrm{cd}'v}{(1-c'\,\mathrm{nd}^2u\,\mathrm{nd}^2v)/c}$$

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}\,u\,\mathrm{dn}\,v-(1/c)\mathrm{dn}'u\,\mathrm{dn}'v}{1-c\,\mathrm{sn}^2u\,\mathrm{sn}^2v} = \frac{\mathrm{nd}\,u\,\mathrm{nd}\,v-(1/c)\mathrm{nd}'u\,\mathrm{nd}'v}{(1-c\,\mathrm{cd}^2u\,\mathrm{cd}^2v)/c'} = \frac{\mathrm{cs}'u\,\mathrm{cs}'v-c\,\mathrm{cs}\,u\,\mathrm{cs}\,v}{\mathrm{ns}^2u\,\mathrm{ns}^2v-c} = \frac{\mathrm{sc}'u\,\mathrm{sc}'v-c\,\mathrm{sc}\,u\,\mathrm{sc}\,v}{(\mathrm{dc}^2u\,\mathrm{dc}^2v-c)/c'}$$

$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}\,u\,\mathrm{cn}\,v-\mathrm{cn}'u\,\mathrm{cn}'v}{1-c\,\mathrm{sn}^2u\,\mathrm{sn}^2v} = \frac{\mathrm{sd}'u\,\mathrm{sd}'v-\mathrm{sd}\,u\,\mathrm{sd}\,v}{(1-c\,\mathrm{cd}^2u\,\mathrm{cd}^2v)/c'} = \frac{\mathrm{ds}'u\,\mathrm{ds}'v-\mathrm{ds}\,u\,\mathrm{ds}\,v}{\mathrm{ns}^2u\,\mathrm{ns}^2v-c} = \frac{\mathrm{nc}\,u\,\mathrm{nc}\,v-\mathrm{nc}'u\,\mathrm{nc}'v}{(\mathrm{dc}^2u\,\mathrm{dc}^2v-c)/c'}$$

$$\mathrm{cd}(u+v) = \frac{\mathrm{cd}\,u\,\mathrm{cd}\,v-(1/c')\mathrm{cd}'u\,\mathrm{cd}'v}{1+cc'\,\mathrm{sd}^2u\,\mathrm{sd}^2v} = \frac{\mathrm{dc}\,u\,\mathrm{dc}\,v-(1/c')\mathrm{dc}'u\,\mathrm{dc}'v}{c+c'\,\mathrm{nc}^2u\,\mathrm{nc}^2v} = \frac{\mathrm{ns}'u\,\mathrm{ns}'v-c'\,\mathrm{ns}\,u\,\mathrm{ns}\,v}{\mathrm{ds}^2u\,\mathrm{ds}^2v+cc'} = \frac{\mathrm{sn}'u\,\mathrm{sn}'v-c'\,\mathrm{sn}\,u\,\mathrm{sn}\,v}{c\,\mathrm{cn}^2u\,\mathrm{cn}^2v+c'}$$

$$\mathrm{nd}(u+v) = \frac{\mathrm{nd}\,u\,\mathrm{nd}\,v+(1/c)\mathrm{nd}'u\,\mathrm{nd}'v}{1+cc'\,\mathrm{sd}^2u\,\mathrm{sd}^2v} = \frac{\mathrm{sc}'u\,\mathrm{sc}'v+c\,\mathrm{sc}\,u\,\mathrm{sc}\,v}{c+c'\,\mathrm{nc}^2u\,\mathrm{nc}^2v} = \frac{\mathrm{cs}'u\,\mathrm{cs}'v+c\,\mathrm{cs}\,u\,\mathrm{cs}\,v}{\mathrm{ds}^2u\,\mathrm{ds}^2v+cc'} = \frac{\mathrm{dn}\,u\,\mathrm{dn}\,v+(1/c)\mathrm{dn}'u\,\mathrm{dn}'v}{c\,\mathrm{cn}^2u\,\mathrm{cn}^2v+c'}$$

If in any formula for $\mathrm{pq}(u+v)$ we add a quarterperiod to one variable and not to the other, we obtain an addition formula in which u and v are arguments of different functions. There is a very large number of these mixed formulae, a few of which we have already used incidentally, but although their origin is simple they are of no obvious intrinsic interest. An example is

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}'u\,\mathrm{cs}\,v-\mathrm{dn}\,u\,\mathrm{cs}'v}{\mathrm{dn}^2u+\mathrm{cs}^2v},$$

which was virtually the link between ·21 and ·22.

Another type of formula for the function $\mathrm{pq}(u+v)$ in which neither a pole nor a zero is at the origin comes from ·31 by simple division. The denominators of $\mathrm{ps}(u+v)$ and $\mathrm{qs}(u+v)$ are effectively the same if $\mathrm{ps}(u+v)$ and $\mathrm{qs}(u+v)$ are expressed in terms of copolar functions, as can be done in four ways, the choice of pole determining the functions that must be used. We have, if K_r is the fourth cardinal point,

$$\mathrm{qs}^2 u-\mathrm{qs}^2 v = \mathrm{ps}^2 u-\mathrm{ps}^2 v, \qquad \mathrm{rp}^2 u-\mathrm{rp}^2 v = -\mathrm{ps}^2 K_r(\mathrm{sp}^2 u-\mathrm{sp}^2 v),$$
$$\mathrm{sq}^2 u-\mathrm{sq}^2 v = -\mathrm{sq}^2 K_r(\mathrm{rq}^2 u-\mathrm{rq}^2 v),$$
$$\mathrm{pr}^2 u-\mathrm{pr}^2 v = \mathrm{pq}^2 K_r(\mathrm{qr}^2 u-\mathrm{qr}^2 v),$$

and noting that a negative sign is introduced if $\mathrm{ps}(u+v)$ is expressed in terms of $\mathrm{qr}\,u$ or $\mathrm{rq}\,u$, or if $\mathrm{qs}(u+v)$ is expressed in terms of $\mathrm{pr}\,u$ or $\mathrm{rp}\,u$, we have the general theorem:

12·44. *If K_p, K_q are two of the three points K_c, K_n, K_d, and K_r is the third of these points, then $\mathrm{pq}(u+v)$ is expressible in the four forms*

·44$_{1-2}$ $$\frac{\mathrm{ps}\,u\,\mathrm{ps}'v-\mathrm{ps}\,v\,\mathrm{ps}'u}{\mathrm{qs}\,u\,\mathrm{qs}'v-\mathrm{qs}\,v\,\mathrm{qs}'u}, \qquad \mathrm{ps}^2K_r.\frac{\mathrm{sp}\,u\,\mathrm{sp}'v-\mathrm{sp}\,v\,\mathrm{sp}'u}{\mathrm{rp}\,u\,\mathrm{rp}'v-\mathrm{rp}\,v\,\mathrm{rp}'u},$$

·44$_{3-4}$ $$\mathrm{sq}^2K_r.\frac{\mathrm{rq}\,u\,\mathrm{rq}'v-\mathrm{rq}\,v\,\mathrm{rq}'u}{\mathrm{sq}\,u\,\mathrm{sq}'v-\mathrm{sq}\,v\,\mathrm{sq}'u}, \qquad \mathrm{pq}^2K_r.\frac{\mathrm{qr}\,u\,\mathrm{qr}'v-\mathrm{qr}\,v\,\mathrm{qr}'u}{\mathrm{pr}\,u\,\mathrm{pr}'v-\mathrm{pr}\,v\,\mathrm{pr}'u}.$$

There are six functions to which this theorem is applicable, but since the fractions, unlike those in ·31 and ·32, retain their structure if denominator and numerator are interchanged, there are only twelve distinct formulae in a complete explicit set.

12·5. From ·31, substituting for the derivatives, we have

12·51 $$\mathrm{ps}(u+v)+\mathrm{qs}(u+v) = \frac{\mathrm{ps}\,u\,\mathrm{qs}\,v+\mathrm{ps}\,v\,\mathrm{qs}\,u}{\mathrm{rs}\,u+\mathrm{rs}\,v},$$

or in an elegant form, due for Jacobi's functions to J. J. Thomson,

12·52 $$\frac{\mathrm{pr}(u+v)+\mathrm{qr}(u+v)}{\mathrm{sr}(u+v)} = \frac{\mathrm{pr}\,u\,\mathrm{qr}\,v+\mathrm{pr}\,v\,\mathrm{qr}\,u}{\mathrm{sr}\,u+\mathrm{sr}\,v}.$$

Addition of K_r to v gives on reduction

12·53 $$\mathrm{sp}\,K_r\,\mathrm{pr}(u+v)+\mathrm{sq}\,K_r\,\mathrm{qr}(u+v) = \frac{\mathrm{sp}\,K_r\,\mathrm{pr}\,u\,\mathrm{pr}\,v+\mathrm{sq}\,K_r\,\mathrm{qr}\,u\,\mathrm{qr}\,v}{1-\mathrm{rs}\,K_p\,\mathrm{rs}\,K_q\,\mathrm{sr}\,u\,\mathrm{sr}\,v}.$$

XIII

THE JACOBI AND LANDEN TRANSFORMATIONS

13·1. One row of poles and zeros, regularly spaced along a line, is very like another, and a system of parallel rows of this kind forming a latticework can always be compared in general terms with a system associated with a particular Jacobian function. The differences, for example, between the pattern formed by the poles and zeros of $\operatorname{sc} u$ and the pattern formed by the poles and zeros of $\operatorname{sn} u$ are quantitative, not qualitative. We have to remember however that in the Jacobian theory shape can not be divorced from size. The normalizing factor is the key to every problem of fitting a Jacobian function into a given frame.

Suppose that we do wish to interchange, while retaining geometrical similarity, the parts played by the first two quarterperiods. We have a system in which $K_c = \alpha$, $K_n = \beta$. We can not postulate a system in which $K_c = \beta$, $K_n = \alpha$, for we have no reason to think that such a system exists. But we may legitimately postulate a system in which $K_c : K_n = \beta : \alpha$, or in other words postulate a factor μ such that $(\mu\beta, \mu\alpha)$ is a Jacobian basis, and we can investigate the relation of the system in which $K_c = \mu\beta$, $K_n = \mu\alpha$ to the system in which $K_c = \alpha$, $K_n = \beta$.

It is convenient to introduce a comprehensive notation to be used in the various transformations which we are about to study. We write v for the new variable μu, and H_c, H_n, H_d for the quarterperiods of the functions of v, with H_s as an alternative symbol for the origin; we use b, b' for the parameters, h, h' for the moduli, and ι for the signature, of the Jacobian system with basis H_c, H_n. In each of our problems we take a relation between the basis H_c, H_n and the basis K_c, K_n, and we infer relations between b, b' and c, c', or if possible between h, h', ι and k, k', v, and also between functions constructed on the one basis and functions constructed on the other; we find also the ratio of v to u, that is, the normalizing factor μ.

Throughout this work the table of leading coefficients, XI 7, is invaluable.

13·2. Our first problem is defined by the pair of formulae

·201–·202 $$H_c = \mu K_n, \qquad H_n = \mu K_c,$$

implying at once

·203 $$H_d = \mu K_d,$$

and, since the direction of rotation is reversed,

·204 $$\iota = -\upsilon.$$

The relation between the two systems is symmetrical.

Functions for which the origin is neither a zero nor a pole can be identified by their structure, since the value at the origin is unity in each system. Thus

·205–·206 $$\operatorname{cn} v = \operatorname{nc} u, \qquad \operatorname{dn} v = \operatorname{dc} u.$$

But $\operatorname{sn} v$, which has the zeros and poles of $\operatorname{sc} u$, is given by

·207 $$\operatorname{sn} v = \mu \operatorname{sc} u,$$

since the relations $\operatorname{sn} v \sim v$, $\operatorname{sc} u \sim u$ must be consistent with $v = \mu u$.

We have now only to take u and v at cardinal points to obtain from ·205, ·206, ·207 relations between the constants of the systems. Explicitly, since $\operatorname{sn} H_c = 1$, and $u = K_n$ corresponds to $v = H_c$, we have from ·207

13·21 $$\mu = \operatorname{cs} K_n = -\upsilon.$$

From ·206,

13·22 $$h' = \operatorname{dn} H_c = \operatorname{dc} K_n = k,$$

and it follows that reciprocally

·208 $$h = k',$$

a relation which we can verify in the form

·209 $$h = -\operatorname{ns} H_d = -\upsilon \operatorname{cs} K_d.$$

Since $\operatorname{sn} v$ is an odd function, the relation

$$\operatorname{sn}(\mu u; h) = \mu \operatorname{sc}(u; k),$$

with $\mu^2 = -1$, is equivalent to

·210 $$\operatorname{sn}(iu; h) = i \operatorname{sc}(u; k)$$

whether μ is i or $-i$, and since $\operatorname{cn} v$ and $\operatorname{dn} v$ are even functions, the relations ·205, ·206 are equally independent of the signature. The relation between the parameters is

13·23 $$b = c', \qquad b' = c,$$

and since it is the parameters rather than the moduli which characterize a system, we express the conclusion in terms of parameters:

13·24. *If* $b = c'$, *then*

·24_{1-3} $$\operatorname{sn}(iu, b) = i \operatorname{sc}(u, c), \qquad \operatorname{cn}(iu, b) = \operatorname{nc}(u, c),$$
$$\operatorname{dn}(iu, b) = \operatorname{dc}(u, c).$$

This theorem describes *Jacobi's imaginary transformation* in the form that is customary, but the nature of the transformation as a sheer interchange is made more evident if attention is focused on the set of functions with the origin for a zero:

13·25$_{1-3}$ $$\operatorname{sc}(iu,b) = i\operatorname{sn}(u,c), \qquad \operatorname{sn}(iu,b) = i\operatorname{sc}(u,c),$$ $$\operatorname{sd}(iu,b) = i\operatorname{sd}(u,c).$$

It need hardly be said that although we use the accepted name for the transformation, we do not think of u as a real variable and iu as an imaginary variable.

If we write

·211–·214 $$K_c = K, \qquad K_n = \upsilon K', \qquad H_c = H, \qquad H_n = \iota H',$$

thus defining K', H' in terms of the signatures of the bases to which they belong, the initial conditions ·201, ·202 become, on account of the value of μ,

·215–·216 $$H = K', \qquad H' = K.$$

This is the theorem foreshadowed on p. 199:

13·26. *If α, $\upsilon\delta$ is a basis with signature υ in the Jacobian system in which the parameter and its complement are a, a', then δ, $\iota\alpha$ is a basis with signature ι in the Jacobian system in which the parameter and its complement are a', a.*

Instead of reversing the signature we may take the initial conditions in the form $H_c = \mu K_n$, $H_n = -\mu K_c$; ultimately the same functions are found, for α, $-\beta$ is always an alternative basis to α, β, but we have now $H_d = \mu(K_d + 2K_c)$, and since cardinal points in the one system no longer correspond to cardinal points in the other system, the comparison of relevant values of the functions is much more troublesome.

13·3. We consider next the transformation in which the first and third elements change parts. Again the signature is reversed, and the initial conditions are

·301–·303 $$H_c = \mu K_d, \qquad H_d = \mu K_c, \qquad H_n = \mu K_n,$$

·304 $$\iota = -\upsilon.$$

The functional relations can be taken as

·305–·307 $$\operatorname{cn} v = \operatorname{dn} u, \qquad \operatorname{dn} v = \operatorname{cn} u, \qquad \operatorname{sn} v = \mu \operatorname{sn} u,$$

implying

13·31 $$\mu = \operatorname{ns} K_d = -k,$$

13·32 $$h = -\operatorname{ns} H_d = -(1/\mu)\operatorname{ns} K_c = 1/k,$$

·308 $$h' = \operatorname{dn} H_c = \operatorname{cn} K_d = -\nu k'/k,$$

13·33$_{1-2}$ $$b = 1/c, \qquad b' = -c'/c.$$

As before, the sign of μ is eliminated in the end, and the result takes the form:

13·34. *If $b = 1/c$ and $k^2 = c$, then*

·34$_{1-3}$ $$\operatorname{sn}(ku, b) = k \operatorname{sn}(u, c), \qquad \operatorname{cn}(ku, b) = \operatorname{dn}(u, c),$$
$$\operatorname{dn}(ku, b) = \operatorname{cn}(u, c).$$

With an alternative triplet,

$$\operatorname{sc}(ku, b) = k \operatorname{sd}(u, c), \qquad \operatorname{sn}(ku, b) = k \operatorname{sn}(u, c), \qquad \operatorname{sd}(ku, b) = k \operatorname{sc}(u, c).$$

The transformation described in ·34 is known as *Jacobi's real transformation.* Its importance in the restricted theory is that by connecting a modulus greater than unity with a modulus less than unity it enables all investigations in which the modulus is real to be conducted with the useful limitation $0 < k < 1$.

13·4. The transformations considered in the last two sections can be combined and repeated, and they generate a group of transformations. To understand this group, we have only to think of the Jacobian functions as derived by means of a normalizing factor from the elementary functions constructed on an arbitrary set of quarterperiods ω_f, ω_g, ω_h. The normalizing factor and the parts played by the individual elementary functions in the unsymmetrical Jacobian scheme depend on the assignment of parts among the quarterperiods, and there are six Jacobian sets which differ only in factor and notation. The possible Jacobian bases are

·401–·402 $$K_c^1 = g_f\,\omega_f,\ K_n^1 = g_f\,\omega_g; \qquad K_c^2 = f_g\,\omega_g,\ K_n^2 = f_g\,\omega_f;$$

·403–·404 $$K_c^3 = h_g\,\omega_g,\ K_n^3 = h_g\,\omega_h; \qquad K_c^4 = g_h\,\omega_h,\ K_n^4 = g_h\,\omega_g;$$

·405–·406 $$K_c^5 = f_h\,\omega_h,\ K_n^5 = f_h\,\omega_f; \qquad K_c^6 = h_f\,\omega_f,\ K_n^6 = h_f\,\omega_h.$$

After each standardization, the twelve Jacobian functions are constant multiples of the twelve elementary functions constructed on ω_f, ω_g, ω_h, and therefore the Jacobian functions in one set are constant multiples of the Jacobian functions in any other set.

A transformation determined by a condition

$$H_p : H_q = K_r : K_t$$

is a transformation which connects two Jacobian sets that are derivable from one and the same elementary set. Hence no combination of such transformations can take us outside the group of six sets with a common origin, and the totality of these transformations is a group in the mathematical sense. Symbolically, let $\mathscr{J}$ denote the transformation of ·2 and $\mathscr{K}$ the transformation of ·3, and denote the six Jacobian sets for the moment by the affixes in the scheme ·401–·406. The transformations $\mathscr{J}, \mathscr{K}$ are symmetrical; $\mathscr{J}$ effects a passage between 1 and 2, between 3 and 4, and between 5 and 6; $\mathscr{K}$ effects a passage between 1 and 4, between 2 and 5, and between 3 and 6. Writing $\mathscr{I}$ for identity, we can express the dependence of the six sets on set 1 by the formulae

$$13{\cdot}41_{1\text{–}6} \quad 1 = \mathscr{I}1, \quad 2 = \mathscr{J}1, \quad 3 = \mathscr{J}\mathscr{K}1, \quad 4 = \mathscr{K}1, \quad 5 = \mathscr{K}\mathscr{J}1,$$
$$6 = \mathscr{J}\mathscr{K}\mathscr{J}1 = \mathscr{K}\mathscr{J}\mathscr{K}1.$$

The symmetrical or involutionary character of the two Jacobian transformations is expressed by the formulae

$$\cdot 407\text{–}\cdot 408 \qquad \mathscr{J}^2 = \mathscr{I}, \qquad \mathscr{K}^2 = \mathscr{I}.$$

If we reverse the formulae $\cdot 41_{1\text{–}6}$ we have

$$13{\cdot}41_{7\text{–}12} \quad 1 = \mathscr{I}1 = \mathscr{J}2 = \mathscr{K}\mathscr{J}3 = \mathscr{K}4 = \mathscr{J}\mathscr{K}5 = \mathscr{J}\mathscr{K}\mathscr{J}6 = \mathscr{K}\mathscr{J}\mathscr{K}6.$$

To find the dependence of the set m on the set n, in terms of the transformations $\mathscr{J}, \mathscr{K}$, we have only to substitute the expression for 1 in terms of n as given in $\cdot 41_{7\text{–}12}$ into the expression for m in terms of 1 as given in $\cdot 41_{1\text{–}6}$ and to reduce by suppression of $\mathscr{J}^2$ and $\mathscr{K}^2$.

The factor by which one Jacobian set is transformable into another is a ratio of the normalizing factors by which the two sets are derivable from a common origin. These normalizing factors are given in the first place as critical values in the elementary set, but if the problem is the transformation of a Jacobian set, the ratios of the normalizing factors must be found in terms of the constants of the set to be transformed, or, to put the determination differently, the elementary set must be identified temporarily with that Jacobian set. Thus if the first set is to be transformed, the transforming factors

$$\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6$$

can be regarded either as the quotients by g_f of the six normalizing factors

$$g_f, f_g, h_g, g_h, f_h, h_f,$$

or as the values in the first set itself of the constants

$$\mathrm{ns}\, K_c,\ \mathrm{cs}\, K_n,\ \mathrm{ds}\, K_n,\ \mathrm{ns}\, K_d,\ \mathrm{cs}\, K_d,\ \mathrm{ds}\, K_c,$$

and we have

$$13{\cdot}42_{1\text{–}6} \qquad \mu_m = 1,\ -v,\ -vk,\ -k,\ vk',\ k'.$$

We recognize the values of μ_2 and μ_4 found already in ·21 and ·31.

The signature v_m of set m is the constant $\operatorname{sc} K_n^m$; the six values are therefore

$$\mu_1 \operatorname{sc} K_n,\ \mu_2 \operatorname{sn} K_c,\ \mu_3 \operatorname{sn} K_d,\ \mu_4 \operatorname{sd} K_n,\ \mu_5 \operatorname{sd} K_c,\ \mu_6 \operatorname{sc} K_d;$$

thus

13·43$_{1-6}$ $$v_m = v,\ -v,\ v,\ -v,\ v,\ -v,$$

as is directly obvious. The modulus k_m is the value of $-\operatorname{ns} K_d^m$; the six values are therefore

$$-\frac{\operatorname{ns} K_d}{\mu_1},\ -\frac{\operatorname{cs} K_d}{\mu_2},\ -\frac{\operatorname{ds} K_c}{\mu_3},\ -\frac{\operatorname{ns} K_c}{\mu_4},\ -\frac{\operatorname{cs} K_n}{\mu_5},\ -\frac{\operatorname{ds} K_n}{\mu_6},$$

giving

13·44$_{1-6}$ $$k_m = k,\ k',\ -vk'/k,\ 1/k,\ 1/k',\ vk/k'.$$

The complementary modulus k'_m is $\operatorname{dn} K_c^m$, to which the transforming factor is irrelevant, and the six values are

$$\operatorname{dn} K_c,\ \operatorname{dc} K_n,\ \operatorname{cd} K_n,\ \operatorname{cn} K_d,\ \operatorname{nc} K_d,\ \operatorname{nd} K_c;$$

that is,

13·45$_{1-6}$ $$k'_m = k',\ k,\ 1/k,\ -vk'/k,\ vk/k',\ 1/k'.$$

Since the transformation $\mathscr{J}$ applied to the basis K_c, K_n replaces the signature v by $-v$, we may write $\mathscr{J} v = -v$; similarly $\mathscr{K} v = -v$. The set of six signatures corresponding to the six Jacobian bases is generated from any one signature by the pair of operators $\mathscr{J}, \mathscr{K}$ regarded as operating on v itself.

There is no similar generation of the modulus k, for while $\mathscr{K} k = 1/k$, the complementary modulus k' is not determined uniquely by k, and $\mathscr{J} k$ is ambiguous. If we treat the operations as performed simultaneously on k and k', then $\mathscr{J}(k, k') = (k', k)$, but $\mathscr{K} k'$ is $-vk'/k$ and there is still an ambiguity. If however we adjoin the signature, and operate on the set of parameters (k, k', v), we have the rational transformations

·409–·410 $$\mathscr{J}(k, k', v) = (k', k, -v), \qquad \mathscr{K}(k, k', v) = (1/k, -vk'/k, -v),$$

and now the group of six sets of parameters is again derivable rationally from a single member by the two generators.

Much simpler is the case of the parameter c. We have

·411–·412 $$\mathscr{J} c = 1-c, \qquad \mathscr{K} c = 1/c,$$

and these two operations generate the complete set

13·46$_{1-6}$ $$c_m = c,\ 1-c,\ -(1-c)/c,\ 1/c,\ -c/(1-c),\ 1/(1-c).$$

Now if α, β, γ, δ are any four numbers, the anharmonic ratio $(\alpha\beta, \gamma\delta)$ is

$$\frac{\alpha-\gamma}{\gamma-\beta}\Big/\frac{\alpha-\delta}{\delta-\beta},$$

a number depending on the order in which α, β, γ, δ occur. There are twenty-four permutations of α, β, γ, δ, but since identically

$$(\alpha\beta, \gamma\delta) = (\beta\alpha, \delta\gamma) = (\gamma\delta, \alpha\beta) = (\delta\gamma, \beta\alpha),$$

not more than six of the ratios can be distinct. Also

$$(\alpha\gamma, \beta\delta) = 1-(\alpha\beta, \gamma\delta), \qquad (\alpha\beta, \delta\gamma) = 1/(\alpha\beta, \gamma\delta).$$

Hence if c is the value of one anharmonic ratio of α, β, γ, δ, then $\mathscr{J}c$, $\mathscr{K}c$ are values of other anharmonic ratios of the same four numbers, and every number generated from c by combinations and repetitions of the two operators $\mathscr{J}, \mathscr{K}$ is an anharmonic ratio of α, β, γ, δ. Since the group of operators generates from c a set of six numbers which are in general all different, this set is precisely the anharmonic set to which c belongs, and regarded as operators on a single variable, $\mathscr{J}, \mathscr{K}$ generate the anharmonic group.

13·47. *Each set of Jacobian elliptic functions belongs to an anharmonic group of six sets. The group is generated from any one of its members by combinations and repetitions of Jacobi's two transformations, and the parameters of the six sets are the members of an anharmonic set of numbers. The six Jacobian sets are derivable from one and the same set of elementary elliptic functions by the use in turn of each of the critical values as a normalizing factor.*

The complete set of transformations is given explicitly in the following table, where each column consists of the same function in its six different forms, and each row contains the three primitive functions belonging to the same Jacobian set.

Table XIII 1

The anharmonic group of sets of primitive Jacobian functions

$\mathrm{cs}(u, c)$	$\mathrm{ns}(u, c)$	$\mathrm{ds}(u, c)$
$i\,\mathrm{ns}(iu, c')$	$i\,\mathrm{cs}(iu, c')$	$i\,\mathrm{ds}(iu, c')$
$ik\,\mathrm{ds}(iku, -c'/c)$	$ik\,\mathrm{cs}(iku, -c'/c)$	$ik\,\mathrm{ns}(iku, -c'/c)$
$k\,\mathrm{ds}(ku, 1/c)$	$k\,\mathrm{ns}(ku, 1/c)$	$k\,\mathrm{cs}(ku, 1/c)$
$ik'\,\mathrm{ns}(ik'u, 1/c')$	$ik'\,\mathrm{ds}(ik'u, 1/c')$	$ik'\,\mathrm{cs}(ik'u, 1/c')$
$k'\,\mathrm{cs}(k'u, -c/c')$	$k'\,\mathrm{ds}(k'u, -c/c')$	$k'\,\mathrm{ns}(k'u, -c/c')$

If one member of an anharmonic set of numbers is real, the six members are all real, and in general one and only one of them satisfies the condition $0 \leqslant c \leqslant \frac{1}{2}$. The only cases in any sense exceptional are those in which the inequality takes one of its limiting forms $c = 0$, $c = \frac{1}{2}$; then two members of the set coincide, but the value which satisfies the condition is still unique. Omitting the case $c = 0$ which implies a degenerate set of functions, we can say that Jacobi's transformations can be used to reduce any set for which c is real to dependence on a set for which $0 < c \leqslant \frac{1}{2}$, and that the conditioned set is unique.

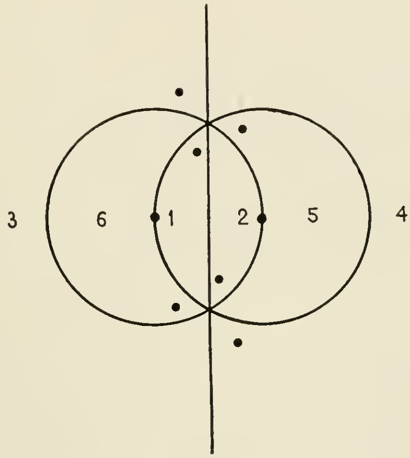


Fig. 32

When the variable c is complex, the two points c, $1-c$ lie on opposite sides of the line through $c = \frac{1}{2}$ parallel to the imaginary axis, that is, the line $|c| = |c'|$, and of the two points c, $1/c$, one is inside and one outside the circle $|c| = 1$. The circle $|c-1| = 1$, that is, $|c'| = 1$, is at once the locus derived from the circle $|c| = 1$ by the substitution of $1-c$ for c, and the locus derived from the line $|c| = |c'|$ by the substitution of $1/c$ for c. The two circles $|c| = 1$, $|c'| = 1$ and the line $|c| = |c'|$ divide the c plane into six regions such that if c is in one of these regions, the other five points in the anharmonic set to which c belongs are one in each of the other five regions; this is the anharmonic dissection of the c plane. To impose the two conditions

·413–·414 $$|c| \leqslant 1, \qquad |c'| \leqslant 1$$

is to confine c to two of the six regions, and to add the condition

·415 $$|c| \leqslant |c'|$$

is to confine c to a single region, namely, the segment of the circle

$|c'| = 1$ which lies on the same side of the line $|c| = |c'|$ as the origin.

In other words, if c_m belongs to the same anharmonic group as c, the conditions

$$\cdot 416\text{–}\cdot 418 \qquad |c_m| \leqslant 1, \qquad |c'_m| \leqslant 1, \qquad |c_m| \leqslant |c'_m|$$

provide c with a representative in a fundamental region. In general the conditions ·416–·418 determine c_m uniquely, but if c lies on one of the boundaries of the anharmonic dissection, that is, satisfies one of the equalities

$$\cdot 419\text{–}\cdot 421 \qquad |c| = 1, \qquad |c'| = 1, \qquad |c| = |c'|,$$

the anharmonic group consists of three conjugate pairs and the conditions ·416–·418 are satisfied by both members of one of these pairs; the two members coincide only in the case already noticed, when c has one of the real values -1, 2, $\frac{1}{2}$ and c_m has the value $\frac{1}{2}$. The equalities ·419–·421 are all satisfied simultaneously at the points where the circles and the line intersect; the anharmonic group consists then of the two complex cube roots of -1 each taken thrice, and c has one of these values, which both satisfy the conditions imposed on c_m: the case of triple coincidence is not an exception to the exception.

13·48. *The anharmonic group of numbers to which the parameter c of a Jacobian system belongs includes one member c_m which satisfies the conditions*

$$|c_m| \leqslant |c'_m| \leqslant 1.$$

In general this member is unique, but if c satisfies one of the conditions $|c| = 1$, $|c'| = 1$, $|c| = |c'|$, then c_m may have one of two conjugate complex values, unless c has one of the three real values -1, 2, $\frac{1}{2}$, when c_m must have the value $\frac{1}{2}$.

By describing c as a Jacobian parameter we both indicate the relevance of this theorem to our subject and avoid specific mention of the degenerate values 0, 1, ∞. We can render c_m unique in all cases if we stipulate that in the boundary cases the imaginary part of c_m is to be positive, thus allocating to the fundamental region that part of its boundary which lies on the positive side of the real axis, but this stipulation has little functional significance.

The anharmonic group can be studied from the integral side. If in the relation

$$\cdot 422 \qquad \int_x^\infty \frac{dx}{\sqrt{\{(x^2-1)(x^2-c)\}}} = u,$$

which is equivalent to $x = \operatorname{ns} u$, we substitute y for x^2, we see that

13·49. *The relation*

·49_1 $$\int_y^\infty \frac{dy}{\sqrt{\{y(y-1)(y-c)\}}} = 2u$$

is equivalent to

·49_{2-4} $$y = \operatorname{ns}^2 u, \qquad y-1 = \operatorname{cs}^2 u, \qquad y-c = \operatorname{ds}^2 u.$$

A transformation in which $\operatorname{ns}^2 v$, $\operatorname{cs}^2 v$, $\operatorname{ds}^2 v$ are multiples, in some order, of $\operatorname{ns}^2 u$, $\operatorname{cs}^2 u$, $\operatorname{ds}^2 u$ is a linear substitution $y = \kappa z + \lambda$ replacing ·49_1 by a relation

·423 $$\int_z^\infty \frac{dz}{\sqrt{\{z(z-1)(z-b)\}}} = 2v.$$

In this substitution the values 0, 1, b of z correspond in some order to the values 0, 1, c of y, and ∞ corresponds to ∞. But b is the value of the anharmonic ratio $(\infty 0, 1b)$, and therefore, since anharmonic values are unchanged by a linear substitution, is the value of one of the anharmonic ratios of the four numbers ∞, 0, 1, c, while c is the value of the particular anharmonic ratio $(\infty 0, 1c)$ of the same four numbers. Hence b, the parameter of the functions of v, belongs to the anharmonic group which includes c, the parameter of the functions of u.

It is a simple matter to connect each value of b with the appropriate linear substitution and with the appropriate relation between $H_c : H_n : H_d$ and $K_c : K_n : K_d$, but since only the squares of Jacobian functions are identifiable from ·49_{2-4}, we can not expect to discover unambiguous relations between the functions themselves. Rather, the reason why the integral relation ·49_1 is the simplest foundation for a theorem concerning a group of values of the parameter is precisely that the irrelevant distinctions between different bases for the same system are not explicit in this relation.

13·5. In the Jacobian transformations, the patterns of poles and zeros are those of the Jacobian functions themselves, modified only by a kind of rechristening. We turn now to some transformations in which the Jacobian patterns are first modified by combinations which have the effect of deleting some of the poles and zeros.

The function $\operatorname{ds} u$ has poles with residue 1 at 0 and $2K_c$, and poles with residue -1 at $2K_n$ and $2K_d$; the function $\operatorname{cs} u$ has poles with residue 1 at 0 and $2K_d$, and poles with residue -1 at $2K_n$ and $2K_c$. Hence the sum $\operatorname{ds} u + \operatorname{cs} u$ has a pole with residue 2 at 0 and a pole with residue

-2 at $2K_n$, and the difference $\operatorname{ds} u - \operatorname{cs} u$ has a pole with residue 2 at $2K_c$ and a pole with residue -2 at $2K_d$. Since the product $\operatorname{ds}^2 u - \operatorname{cs}^2 u$ is the constant c', the poles of one factor are the zeros of the other, and $\operatorname{ds} u + \operatorname{cs} u$, which has periods $4K_c$, $4K_n$, has poles at 0 and $2K_n$ and zeros at $2K_c$ and $2K_n + 2K_c$.

A function with a precisely similar pattern of poles and zeros is the logarithmic derivative $\operatorname{sn}' u / \operatorname{sn} u$, which has periods $2K_c$, $2K_n$, poles at 0 and K_n, and zeros at K_c and K_d. The factor which converts the pattern of the latter function into the pattern of the former is 2, and therefore $\operatorname{ds} 2u + \operatorname{cs} 2u$, which resembles $1/u$ near the origin, is identical with $\operatorname{sn}' u / \operatorname{sn} u$.

This result is easily confirmed from duplication formulae. From $12{\cdot}42_2$, $12{\cdot}42_1$, $12{\cdot}36_2$ we have, putting $v = u$,

·501–·502 $$\operatorname{ds} 2u = \frac{\operatorname{dn}^2 u - c \operatorname{sn}^2 u \operatorname{cn}^2 u}{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}, \qquad \operatorname{cs} 2u = \frac{\operatorname{cn}^2 u - \operatorname{sn}^2 u \operatorname{dn}^2 u}{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u},$$

and therefore

·503–·504 $$\operatorname{ds} 2u + \operatorname{cs} 2u = \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}, \qquad \operatorname{ds} 2u - \operatorname{cs} 2u = \frac{c' \operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u},$$

as required.

Save that the residues at the poles are different, the arguments applied to the pair of functions $\operatorname{ds} u$, $\operatorname{cs} u$ apply also to the pair $\operatorname{dn} u$, $\operatorname{cn} u$; the first of these has poles with residue $-v$ at K_n and $K_n + 2K_c$, and poles with residue v at $3K_n$ and $3K_n + 2K_c$, the second has poles with residue $-v/k$ at K_n and $3K_n + 2K_c$, and poles with residue v/k at $3K_n$ and $K_n + 2K_c$. Hence $\operatorname{dn} u + k \operatorname{cn} u$ has a pole with residue $-2v$ at K_n and a pole with residue $2v$ at $3K_n$, and $\operatorname{dn} u - k \operatorname{cn} u$ has a pole with residue $-2v$ at $K_n + 2K_c$ and a pole with residue $2v$ at $3K_n + 2K_c$. Also $\operatorname{dn}^2 u - k^2 \operatorname{cn}^2 u$ has the constant value c'. Hence $\operatorname{dn} u + k \operatorname{cn} u$ has periods $4K_c$, $4K_n$, poles at K_n and $3K_n$, and zeros at $K_n + 2K_c$ and $3K_n + 2K_c$, while for $\operatorname{dn} u - k \operatorname{cn} u$ poles and zeros are interchanged.

We now recognize that the patterns of poles and zeros for the four functions

$$\operatorname{ds} u + \operatorname{cs} u, \qquad \operatorname{ds} u - \operatorname{cs} u, \qquad \operatorname{dn} u + k \operatorname{cn} u, \qquad \operatorname{dn} u - k \operatorname{cn} u$$

are geometrically similar to the patterns for the four functions

$$\operatorname{cs} v, \qquad \operatorname{sc} v, \qquad \operatorname{dn} v, \qquad \operatorname{nd} v$$

if the quarterperiods in the two systems satisfy the relation

·505 $$H_c : H_n = 2K_c : K_n.$$

If

·506–·507 $$H_c = 2\mu K_c, \qquad H_n = \mu K_n,$$

the transformation

·508 $$v = \mu u$$

renders the functions of u constant multiples of the functions of v. The constant factors are given by comparison at suitable points: K_n, H_n correspond, $\operatorname{ds} K_n = -\nu k$, $\operatorname{cs} K_n = -\nu$, and the two systems have the same signature; also the origins correspond, and $\operatorname{dn} 0 = 1$, $\operatorname{cn} 0 = 1$. Thus we have the first four of the formulae set out in ·51 below.

If we write the relation ·505 in the form

·509 $$K_c : K_n = H_c : 2H_n,$$

we see that another set of similarities is implied: firstly, $\operatorname{ns} v + \operatorname{ds} v$ has poles at 0 and $2H_c$ and zeros at $2H_n$ and $2H_c + 2H_n$, forming a pattern similar to that associated with $\operatorname{ns} u$; secondly, $\operatorname{dc} v + h' \operatorname{nc} v$ has poles at H_c and $3H_c$ and zeros at $H_c + 2H_n$ and $3H_c + 2H_n$, forming a pattern similar to that associated with $\operatorname{dc} u$. That is, the four functions

$$\operatorname{ns} v + \operatorname{ds} v, \qquad \operatorname{ns} v - \operatorname{ds} v, \qquad \operatorname{dc} v + h' \operatorname{nc} v, \qquad \operatorname{dc} v - h' \operatorname{nc} v$$

are multiples of $\quad \operatorname{ns} u, \quad \operatorname{sn} u, \quad \operatorname{dc} u, \quad \operatorname{cd} u$

if $u = \nu v$, where $K_c = \nu H_c$, $K_n = 2\nu H_n$. We are dealing with the same pair of quarterperiods H_c, H_n as before, since otherwise there would be two distinct sets of functions with the same ratio for $H_c : H_n$. Hence

$$2\mu\nu = 1,$$

and the transformations $v = \mu u$, $u = \nu v$ are not the same. Writing for a moment w instead of v in the second transformation and retaining v in the first, we have $v = \mu u = \mu\nu w$, and therefore $w = 2v$. Thus, since $u = K_c$ implies $2v = H_c$, and $\operatorname{ns} H_c = 1$, $\operatorname{ds} H_c = h'$, and since $\operatorname{dc} 0 = 1$, $\operatorname{nc} 0 = 1$, we have a second set of formulae, completing the following theorem:

13·51. *In the transformation $v = \mu u$ which implies the quarterperiod relations*

$$H_c = 2\mu K_c, \qquad H_n = \mu K_n,$$

the following functional relations hold:

$\cdot 51_{1-2}$ $$\operatorname{ds} u + \operatorname{cs} u = (1+k)\operatorname{cs} v \qquad \operatorname{ds} u - \operatorname{cs} u = (1-k)\operatorname{sc} v$$

$\cdot 51_{3-4}$ $$\operatorname{dn} u + k \operatorname{cn} u = (1+k)\operatorname{dn} v \qquad \operatorname{dn} u - k \operatorname{cn} u = (1-k)\operatorname{nd} v$$

$\cdot 51_{5-6}$ $$\operatorname{ns} 2v + \operatorname{ds} 2v = (1+h')\operatorname{ns} u \qquad \operatorname{ns} 2v - \operatorname{ds} 2v = (1-h')\operatorname{sn} u$$

$\cdot 51_{7-8}$ $$\operatorname{dc} 2v + h' \operatorname{nc} 2v = (1+h')\operatorname{dc} u \qquad \operatorname{dc} 2v - h' \operatorname{nc} 2v = (1-h')\operatorname{cd} u.$$

We have determined the factors in $\cdot 51_{1-2}$ and $\cdot 51_{5-6}$ without reference to the origin, since comparison there involves the factor μ. Making the comparison in $\cdot 51_1$ and $\cdot 51_5$ we have $2\mu = (1+k)$, $\mu(1+h') = 1$:

13·52. *In the transformation* $v = \mu u$ *with the quarterperiod relations* $H_c = 2\mu K_c$, $H_n = \mu K_n$, *the factor* μ *is* $\frac{1}{2}(1+k)$, *and moduli in the two systems are connected by the relation*

$$(1+h')(1+k) = 2.$$

Other relations between constants can be obtained algebraically from ·52 or functionally by substitutions in the formulae of ·51. We have

·510–·512 $$h' = \frac{1-k}{1+k} = \left(\frac{1-k}{k'}\right)^2, \qquad h^2 = \frac{4k}{(1+k)^2},$$

and in the other direction

·513–·515 $$k = \frac{1-h'}{1+h'} = \left(\frac{1-h'}{h}\right)^2, \qquad k'^2 = \frac{4h'}{(1+h')^2}.$$

If in $\cdot 51_5$ we substitute $u = \frac{1}{2}K_n$, $v = \frac{1}{2}H_n$, we have

·516 $$h = \frac{2v}{1+k} \operatorname{ns} \tfrac{1}{2}K_n.$$

It is easily shown that

·517 $$\operatorname{ns}^2 \tfrac{1}{2}K_n = -k,$$

and ·516, necessarily consistent with ·512 and ·514, is an instance of an unambiguous relation between three square roots which can not be extracted severally. Similarly,

·518–·519 $$\operatorname{cs}^2 \tfrac{1}{2}H_c = h', \qquad k' = \frac{2}{1+h'} \operatorname{cs} \tfrac{1}{2}H_c.$$

The set of functional relations in ·51 is in a sense complete, for if the required periodicities are to be preserved, poles can not be removed by additions and subtractions except in the combinations given in this enunciation. This is one reason for giving the full tale of eight relations. A second reason is that, although the relations are interdependent, the explicit dependence of individual functions in one system on functions in the other system is by no means obvious unless the eight relations are all in view. The relations are interdependent, but they are not deducible algebraically from any one of them without irrationalities, that is, without ambiguities that have to be removed by functional considerations.

The transformation described in ·51 is equivalent to a transformation of elliptic integrals discovered by Landen, and it is known by his name. The usefulness of the transformation in the elementary theory in which attention is concentrated on real values of the variables will be seen in our concluding chapter. In the original view of the functional relationships which are now absorbed into the theory of elliptic functions, the function $F(\phi; k)$ with amplitude ϕ and modulus k is defined as

$$F(\phi; k) = \int_0^\phi \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}},$$

and a transformation is a relation between integrals corresponding to relations between amplitudes and moduli. From the later point of view, if $u = F(\phi; k)$, $v = F(\chi; h)$, the amplitudes ϕ, χ are regarded as functions $\mathrm{am}(u; k)$, $\mathrm{am}(v; h)$, but the transformation expresses the same correspondence of relations. In practice the relation between amplitudes takes a trigonometrical form, and therefore becomes implicitly if not explicitly a relation between Jacóbian functions, since $\sin\phi$, $\cos\phi$, $d\phi/du$ are identical with $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$.

The integral

$$\int_0^\phi \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}$$

is not of the form of the integrals whose inversion has been studied, but the relation $x = \sin\phi$ which converts this integral into Legendre's form

$$\int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}$$

is a familiar relation between complex variables x, ϕ, and the use of the relation $u = F(\phi; k)$ as a definition of ϕ as a function of u, with k parametric, is entirely justified by the investigation in Chapters V–VIII. Alternatively, we may define the function $\mathrm{am}\,u$ by the pair of equations

$$\sin(\mathrm{am}\,u) = \mathrm{sn}\,u, \qquad \cos(\mathrm{am}\,u) = \mathrm{cn}\,u.$$

The amplitude is indeterminate, by an arbitrary multiple of 2π, but a trigonometrical relation which does not involve submultiples of an amplitude is not ambiguous in any respect.

To find the trigonometrical relation between the amplitudes ϕ, χ in the Landen transformation we have only to eliminate $\mathrm{ds}\,u$ between $\cdot 51_1$ and $\cdot 51_2$; there results

13·53 $$2\cot\phi = (1+k)\cot\chi-(1-k)\tan\chi.$$

For the determination of ϕ in terms of χ we may modify this formula to

$$(1+k)(\cot\chi-\cot\phi) = (1-k)(\cot\phi+\tan\chi),$$

that is, to

·520 $$\tan(\phi-\chi) = h'\tan\chi.$$

If ϕ is given and χ is required, we write instead

$$\frac{\cos\phi}{\sin\phi} = \frac{\cos 2\chi + k}{\sin 2\chi},$$

implying

·521 $$\sin(2\chi - \phi) = k\sin\phi.$$

Thus we have the two equivalent forms of Landen's theorem:

13·54$_1$. *If the modulus and amplitude of the elliptic integral* $F(\phi;k)$ *are given in terms of the modulus and amplitude of the elliptic integral* $F(\chi;h)$ *by the relations*

$$k = (1-h')/(1+h'), \qquad \tan(\phi-\chi) = h'\tan\chi,$$

then $$F(\phi;k) = (1+h')F(\chi;h).$$

13·54$_2$. *If the modulus and amplitude of* $F(\chi;h)$ *are given in terms of the modulus and amplitude of* $F(\phi;k)$ *by*

$$h' = (1-k)/(1+k), \qquad \sin(2\chi-\phi) = k\sin\phi,$$

then $$F(\chi;h) = \tfrac{1}{2}(1+k)F(\phi;k).$$

13·6. The Landen transformation doubles the ratio of K_c to K_n. It is therefore one of a set of six transformations, which fall into three reciprocal pairs. The transformation which doubles the ratio of K_n to K_c is only the transformation of the last section read in the reverse direction. We do not however obtain a true comparison between the two transformations merely by interchanging u and v in the formulae already found, for if the transformation which implies $H_c = 2\mu K_c$, $H_n = \mu K_n$ is written as $v = \mu u$, the transformation which implies $K_c = \nu H_c$, $K_n = 2\nu H_n$ should be written as $u = \nu v$. Thus if we write $\nu = 1/2\mu$ in order to throw the conditions to be satisfied into the form we require, we must, as we have already noticed, replace $2v$ by v in order to present the transformation itself correctly. This done, we can interchange the two systems throughout:

13·61. *In the transformation* $v = \mu u$ *which implies the quarterperiod relations*

$$H_c = \mu K_c, \qquad H_n = 2\mu K_n,$$

the factor μ *is* $\tfrac{1}{2}(1+k')$, *and moduli in the two systems are connected by the relation*

·61$_1$ $$(1+h)(1+k') = 2.$$

The functional relations are

$\cdot 61_{2\text{-}3}$	$\operatorname{ns} u+\operatorname{ds} u = (1+k')\operatorname{ns} v$	$\operatorname{ns} u-\operatorname{ds} u = (1-k')\operatorname{sn} v$
$\cdot 61_{4\text{-}5}$	$\operatorname{dc} u+k' \operatorname{nc} u = (1+k')\operatorname{dc} v$	$\operatorname{dc} u-k' \operatorname{nc} u = (1-k')\operatorname{cd} v$
$\cdot 61_{6\text{-}7}$	$\operatorname{ds} 2v+\operatorname{cs} 2v = (1+h)\operatorname{cs} u$	$\operatorname{ds} 2v-\operatorname{cs} 2v = (1-h)\operatorname{sc} u$
$\cdot 61_{8\text{-}9}$	$\operatorname{dn} 2v+h \operatorname{cn} 2v = (1+h)\operatorname{dn} u$	$\operatorname{dn} 2v-h \operatorname{cn} 2v = (1-h)\operatorname{nd} u.$

The transformation in this form is sometimes called Landen's second transformation. The trigonometrical form of the transformation, obtained by eliminating $\operatorname{ds} u$ between $\cdot 61_2$ and $\cdot 61_3$, is

·601 $$2 \csc\phi = (1+k')\csc\chi+(1-k')\sin\chi.$$

This relation is not susceptible to modifications corresponding to ·520 and ·521, the reason for the difference between the two transformations in this respect being that the relation of the function $\operatorname{am} u$ to the system is not symmetrical as between the quarterperiods K_c, K_n. To express the second Landen transformation in theorems parallel to $\cdot 54_1$ and $\cdot 54_2$, it is necessary to introduce a *hyperbolic amplitude* θ defined by

·602–·603 $$\sinh\theta = \operatorname{sc} u, \qquad \cosh\theta = \operatorname{nc} u$$

or by

·604 $$u = \int_0^\theta \frac{d\theta}{\sqrt{(1+k'^2\sinh^2\theta)}}.$$

The hyperbolic amplitude θ is connected with the circular amplitude ϕ by the relation

·605 $$\cos\phi\cosh\theta = 1;$$

that is to say, ϕ is the gudermannian of θ.

If ψ is the hyperbolic amplitude of v, then

·606 $$2\coth\theta = (1+k')\coth\psi+(1-k')\tanh\psi,$$

whence we have the two trigonometrical forms of ·61:

$13{\cdot}62_1$. *If* $k' = (1-h)/(1+h)$ *and* $\tanh(\theta-\psi) = h\tanh\psi$, *then*

$$\int_0^\theta \frac{d\theta}{\sqrt{(1+k'^2\sinh^2\theta)}} = (1+h)\int_0^\psi \frac{d\psi}{\sqrt{(1+h'^2\sinh^2\psi)}}.$$

$13{\cdot}62_2$. *If* $h = (1-k')/(1+k')$ *and* $\sinh(2\psi-\theta) = k'\sinh\theta$, *then*

$$\int_0^\psi \frac{d\psi}{\sqrt{(1+h'^2\sinh^2\psi)}} = \tfrac{1}{2}(1+k')\int_0^\theta \frac{d\theta}{\sqrt{(1+k'^2\sinh^2\theta)}}.$$

Another method of deriving ·61 from ·51 and ·52 suggests a simple means of completing the set of transformations. In ·51 let us apply to both the basis K_c, K_n and the basis H_c, H_n Jacobi's imaginary transformation. We have then bases K'_c, K'_n and H'_c, H'_n such that

·607–·608 $$K'_c : K'_n = K_n : K_c, \qquad H'_c : H'_n = H_n : H_c,$$

and the relation

·609 $$H_c : H_n = 2K_c : K_n$$

is equivalent to

·610 $$H'_n : H'_c = 2K'_n : K'_c.$$

But the relation

$$\text{ds}(u;k)+\text{cs}(u;k) = (1+k)\text{cs}(v;h),$$

which is $\cdot 51_1$, becomes

$$\text{ds}(iu;k')+\text{ns}(iu;k') = (1+k)\text{ns}(iv;h'),$$

and to say that, if $(1+h')(1+k) = 2$, then this last relation is satisfied for all values of u, v such that $v = \frac{1}{2}(1+k)u$, asserts the same proposition as that, if $(1+h)(1+k') = 2$, then the relation

$$\text{ds}(u;k)+\text{ns}(u;k) = (1+k')\text{ns}(v;h)$$

is satisfied for all values of u, v such that $v = \frac{1}{2}(1+k')u$; this is $\cdot 61_4$.

Symbolically, if $\mathscr{L}$ is the Landen transformation which doubles the ratio of K_c to K_n, and $\mathscr{J}$ the Jacobi transformation which replaces this ratio by its reciprocal, and if (α, β) denotes the system with the basis $K_c = \alpha$, $K_n = \beta$, we can write

·611–·612 $$\mathscr{J}(\alpha,\beta) = (\beta,\alpha), \qquad \mathscr{L}\mathscr{J}(\alpha,\beta) = (2\beta,\alpha),$$

and therefore

·613 $$\mathscr{J}\mathscr{L}\mathscr{J}(\alpha,\beta) = (\alpha, 2\beta).$$

Fundamentally the last operation is $\mathscr{J}^{-1}\mathscr{L}\mathscr{J}$ rather than $\mathscr{J}\mathscr{L}\mathscr{J}$, since the effect of the first operation of $\mathscr{J}$ has to be reversed, but as the operator $\mathscr{J}$ is involutionary, $\mathscr{J}^{-1}$ and $\mathscr{J}$ are identical.

Because of the differences in detail it is necessary to record in full the functional consequences of the other Landen transformations, but arguments need not be repeated. Poles may be removed, as in ·5, or the first Landen transformation may be combined with the several transformations of the anharmonic group. Actually, knowing the character of the formulae to be found, we avoid almost all algebra by utilizing both processes. From Table XIII 1 we learn the substitutions to make in the formulae of ·51, and from Table XI 7 we have then the coefficients which must be introduced if poles are to disappear. The results are set out in ·63–·66.

13·63. *In the transformation* $v = \mu u$ *which implies the quarterperiod relations*

$$H_n = 2\mu K_n, \qquad H_d = \mu K_d,$$

the factor μ *is* $\frac{1}{2}(k+vk')$, *and moduli are connected by the relation*

·63₁ $$h = (k-vk')^2.$$

The functional relations are

·63₂₋₉

$$\operatorname{ns} u+\operatorname{cs} u = (k+vk')\operatorname{ns} v \qquad \operatorname{ns} u-\operatorname{cs} u = (k-vk')\operatorname{sn} v$$

$$k\operatorname{cd} u+vk'\operatorname{nd} u = (k+vk')\operatorname{cd} v \qquad k\operatorname{cd} u-vk'\operatorname{nd} u = (k-vk')\operatorname{dc} v$$

$$\operatorname{ds} 2v+\operatorname{cs} 2v = (1+h)k^{-1}\operatorname{ds} u \qquad \operatorname{ds} 2v-\operatorname{cs} 2v = (1-h)k\operatorname{sd} u$$

$$\operatorname{dn} 2v+h\operatorname{cn} 2v = (1+h)\operatorname{cn} u \qquad \operatorname{dn} 2v-h\operatorname{cn} 2v = (1-h)\operatorname{nc} u.$$

13·64. *In the transformation* $v = \mu u$ *which implies the quarterperiod relations*

$$H_n = \mu K_n, \qquad H_d = 2\mu K_d,$$

the factor μ *is* $h-vh'$, *and moduli are connected by the relation*

·64₁ $$k = (h-vh')^2.$$

The functional relations are

·64₂₋₉

$$\operatorname{ds} u+\operatorname{cs} u = (1+k)h^{-1}\operatorname{ds} v \qquad \operatorname{ds} u-\operatorname{cs} u = (1-k)h\operatorname{sd} v$$

$$\operatorname{dn} u+k\operatorname{cn} u = (1+k)\operatorname{cn} v \qquad \operatorname{dn} u-k\operatorname{cn} u = (1-k)\operatorname{nc} v$$

$$\operatorname{ns} 2v+\operatorname{cs} 2v = (h+vh')\operatorname{ns} u \qquad \operatorname{ns} 2v-\operatorname{cs} 2v = (h-vh')\operatorname{sn} u$$

$$h\operatorname{cd} 2v+vh'\operatorname{nd} 2v = (h+vh')\operatorname{cd} u \qquad h\operatorname{cd} 2v-vh'\operatorname{nd} 2v = (h-vh')\operatorname{dc} u.$$

13·65. *In the transformation* $v = \mu u$ *which implies the quarterperiod relations*

$$H_d = 2\mu K_d, \qquad H_c = \mu K_c,$$

the factor μ *is* $h'+vh$, *and moduli are connected by the relation*

·65₁ $$k' = (h'+vh)^2.$$

The functional relations are

·65₂₋₉

$$\operatorname{ns} u+\operatorname{ds} u = (1+k')(h')^{-1}\operatorname{ds} v \qquad \operatorname{ns} u-\operatorname{ds} u = (1-k')h'\operatorname{sd} v$$

$$\operatorname{dc} u+k'\operatorname{nc} u = (1+k')\operatorname{nc} v \qquad \operatorname{dc} u-k'\operatorname{nc} u = (1-k')\operatorname{cn} v$$

$$\operatorname{ns} 2v+\operatorname{cs} 2v = (h+vh')\operatorname{cs} u \qquad \operatorname{ns} 2v-\operatorname{cs} 2v = (h-vh')\operatorname{sc} u$$

$$h\operatorname{cd} 2v+vh'\operatorname{nd} 2v = (h+vh')\operatorname{nd} u \qquad h\operatorname{cd} 2v-vh'\operatorname{nd} 2v = (h-vh')\operatorname{dn} u.$$

13·66. *In the transformation* $v = \mu u$ *which implies the quarterperiod relations*

$$H_d = \mu K_d, \qquad H_c = 2\mu K_c,$$

the factor μ *is* $\frac{1}{2}(k'-vk)$, *and moduli are connected by the relation*

·66₁ $$h' = (k'+vk)^2.$$

The functional relations are

$\cdot 66_{2-9}$ $\quad \operatorname{ns} u + \operatorname{cs} u = (k' - vk)\operatorname{cs} v \qquad \operatorname{ns} u - \operatorname{cs} u = (k' + vk)\operatorname{sc} v$

$$k' \operatorname{nd} u - vk \operatorname{cd} u = (k' - vk)\operatorname{nd} v \qquad k' \operatorname{nd} u + vk \operatorname{cd} u = (k' + vk)\operatorname{dn} v$$

$$\operatorname{ns} 2v + \operatorname{ds} 2v = (1+h')(k')^{-1} \operatorname{ds} u \qquad \operatorname{ns} 2v - \operatorname{ds} 2v = (1-h')k' \operatorname{sd} u$$

$$\operatorname{dc} 2v + h' \operatorname{nc} 2v = (1+h')\operatorname{nc} u \qquad \operatorname{dc} 2v - h' \operatorname{nc} 2v = (1-h')\operatorname{cn} u.$$

While the set of six Landen transformations is in one sense complete, it is not mathematically a group, for repetitions and combinations provide an unlimited number of transformations of which no two are identical. If $\mathscr{P}$ is the resultant of any succession of Landen transformations, the inverse transformation $\mathscr{P}^{-1}$ is the resultant of the inverse Landen transformations taken in the reverse order, and the Jacobian system with basis α, β belongs to a chain

$$\ldots \quad \mathscr{P}^{-2}(\alpha,\beta) \quad \mathscr{P}^{-1}(\alpha,\beta) \quad (\alpha,\beta) \quad \mathscr{P}(\alpha,\beta) \quad \mathscr{P}^{2}(\alpha,\beta) \quad \ldots$$

which is endless in both directions. For example, if $\mathscr{L}$ is still the transformation of ·51, and $\mathscr{L}^{-1}$ therefore the inverse transformation of ·61, there is a Landen chain

·614 $\quad \ldots \quad \mathscr{L}^{-2}(\alpha,\beta) \quad \mathscr{L}^{-1}(\alpha,\beta) \quad (\alpha,\beta) \quad \mathscr{L}(\alpha,\beta) \quad \mathscr{L}^{2}(\alpha,\beta) \quad \ldots$

along which the ratio $K_c : K_n$ takes the values

·615 $\quad \ldots \quad \alpha : 2^2\beta \quad \alpha : 2\beta \quad \alpha : \beta \quad 2\alpha : \beta \quad 2^2\alpha : \beta \quad \ldots.$

This chain has, as we shall see, special importance for the evaluation of real integrals and real functions.

13·7. In the practical problem of reducing an integral

$$\int \frac{dz}{\sqrt{\phi(z)}}$$

in which $\phi(z)$ is a polynomial of the fourth degree to a standard elliptic integral, the distinction between real and imaginary is paramount, and this problem belongs to a later chapter, but there are theoretical considerations to which the distinction is irrelevant, by which this problem contributes to the understanding of the transformations of Jacobi and Landen. In the practical problem the coefficient of z^4 in $\phi(z)$ can not be ignored, since the whole character of the result may vary with the sign of this coefficient, just as, in a simpler case, the integrals

$$\int \frac{dt}{\sqrt{(1-t^2)}}, \qquad \int \frac{dt}{\sqrt{(t^2-1)}}$$

are associated with functions between which there is very little resemblance in the real domain. In the practical problem again we must not suppose a polynomial to be decomposed into linear factors unless we are prepared to recombine conjugate complex terms. But in a theoretical investigation a constant factor $\sqrt{a_0}$ is removable by a trivial change in the variable, and we may take the function $\phi(z)$ in the form $(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)$, where, since a repeated factor renders the integral elementary, we may suppose the roots $\alpha, \beta, \gamma, \delta$ to be all distinct.

The two standard forms of the elliptic integral which we shall use are those corresponding to the functions $\operatorname{ns} u$ and $\operatorname{ns}^2 u$, namely,

·701–·702 $$\int_x^\infty \frac{dx}{\sqrt{\{(x^2-1)(x^2-k^2)\}}}, \qquad \int_y^\infty \frac{dy}{\sqrt{\{y(y-1)(y-c)\}}};$$

if u is the value of the first integral, then $x = \operatorname{ns} u$, and with the substitution $x^2 = y$, the value of the second integral is $2u$. The fundamental problem is the reduction of the integrand, and a transformation which affects only the constants of integration is unimportant.

A homographic transformation

·703 $$z' = \lambda\frac{z-\mu}{z-\nu}$$

converts $\int dz/\sqrt{\phi(z)}$ into a multiple of $\int dz'/\sqrt{\psi(z')}$, where the zeros of $\psi(z')$ correspond under the transformation to the zeros of $\phi(z)$. The function $\psi(z')$ is necessarily of the fourth degree unless one of the factors is removed from $\phi(z)$ by the denominator $z-\nu$, that is, unless ν is one of the zeros $\alpha, \beta, \gamma, \delta$; formally, ∞ on the one side then corresponds to a zero on the other side. Thus one transformation which converts $\int dz/\sqrt{\phi(z)}$ into a multiple of the integral in y is

·704 $$y = \frac{\alpha-\gamma}{\gamma-\beta}\cdot\frac{z-\beta}{\alpha-z},$$

where $z = \alpha, \beta, \gamma$ are chosen to correspond to $y = \infty, 0, 1$; since then the factor $y-c$ must be provided by the factor $z-\delta$, the value of c is given by

·705 $$c = \frac{\alpha-\gamma}{\gamma-\beta}\cdot\frac{\delta-\beta}{\alpha-\delta}.$$

That is to say, c is the anharmonic ratio $(\alpha\beta, \gamma\delta)$.

The only arbitrary element in this transformation is the choice among $\alpha, \beta, \gamma, \delta$ of the three zeros to play the definite parts allotted here to α, β, γ. If the zeros are permuted in such a way that the anharmonic ratio is preserved, the same Jacobian system is being used, and the change is no more significant than the use of cosines instead of sines in an elementary integration. Other permutations change the Jacobian system, but from ·705 *the only systems that can be introduced are the six systems composing one anharmonic group.*

It follows from the relation between the integral in x and the integral in y that one transformation for reducing the integral $\int dz/\surd\phi(z)$ to the form ·701 is

13·71 $$x^2 = \frac{\alpha-\gamma}{\gamma-\beta}\cdot\frac{z-\beta}{\alpha-z},$$

and that then

13·72 $$k^2 = (\alpha\beta, \gamma\delta).$$

But the reduction can be effected also by a homographic transformation in which the linear factors $x-1$, $x+1$, $x-k$, $x+k$ correspond to the linear factors $z-\alpha$, $z-\beta$, $z-\gamma$, $z-\delta$. If the correspondence is in this order, the transformation is identified by the first three factors as

13·73 $$\frac{1-k}{k+1}\cdot\frac{x+1}{1-x} = \frac{\alpha-\gamma}{\gamma-\beta}\cdot\frac{z-\beta}{\alpha-z},$$

and since $x = -k$ corresponds to $z = \delta$, the condition to be satisfied by the modulus k is

13·74 $$\left(\frac{1-k}{1+k}\right)^2 = (\alpha\beta, \gamma\delta).$$

That is, if

·706 $$\frac{1-k}{1+k} = l,$$

then l is a modulus of the system with which the integral $\int dz/\surd\phi(z)$ is associated by the transformation ·71. We have seen in ·510 that the ratio $(1-k)/(1+k)$ is also the complementary modulus of the system derived from the system whose modulus is k by the first of the Landen transformations. To replace l by a complementary modulus is only to permute α, β, γ, δ in the transformation ·71; identically,

$$(\alpha\beta, \gamma\delta) = 1-(\alpha\gamma, \beta\delta),$$

and if

13·75 $$x^2 = \frac{\alpha-\beta}{\beta-\gamma}\cdot\frac{z-\gamma}{\alpha-z},$$

then $\int dz/\surd\phi(z)$ is a multiple of $\int dx/\surd\{(x^2-1)(x^2-h^2)\}$, where

13·76 $$h^2 = (\alpha\gamma, \beta\delta), \qquad h'^2 = (\alpha\beta, \gamma\delta).$$

We have now

·707 $$\frac{1-k}{1+k} = \pm h';$$

the relation $$\frac{1-k}{1+k} = -h'$$

is equivalent to $$\frac{1-(1/k)}{1+(1/k)} = h',$$

and to change the modulus from k to $1/k$ is only to permute α, β, γ, δ in the transformation ·73:

13·77. *An integral* $\int dz/\sqrt{\phi(z)}$ *in which* $\phi(z)$ *is a polynomial of the fourth degree is reducible to the standard form* $\int dx/\sqrt{\{(x^2-1)(x^2-k^2)\}}$ *both by a homographic relation between* z *and* x^2 *and by a homographic relation between* z *and* x. *The systems of elliptic functions corresponding to a reduction of the first kind are derivable from the systems corresponding to a reduction of the second kind by Landen's transformation.*

If we take $\phi(z)$ already as $(z^2-1)(z^2-k^2)$, we have Landen's transformation at once in the form

13·78 $$x^2 = \frac{2}{1+k} \cdot \frac{z-k}{z-1},$$

where $x^2 = \infty$, 0, 1 correspond to $z = 1$, k, -1, and if $x^2 = h^2$ corresponds to $z = -k$, then

·708 $$h^2 = \frac{4k}{(1+k)^2}$$

as in ·512. But the origin and the details of the algebraical transformation are clearest if the problem is seen as a special case of a general problem.

XIV

INTEGRATION AND THE INTEGRATING FUNCTIONS

14·1. The product of any number of functions belonging to the same Jacobian system is an elliptic function whose poles and zeros, of arbitrary multiplicities, are situated at cardinal points; such a function we shall call a general Glaisher function.

If we treat a zero as a pole of negative order, or a pole as a zero of negative order, we may say that the typical function of this kind has poles of orders h, k, l, m or zeros of orders $-h$, $-k$, $-l$, $-m$, at K_s, K_c, K_n, K_d, where h, k, l, m are any four whole numbers, positive zero or negative, subject to the condition

·101 $$h+k+l+m = 0.$$

We denote this function by $\mathrm{s}^h\mathrm{c}^k\mathrm{n}^l\mathrm{d}^m u$, or by any variation in which the upper affix is replaced by a lower affix which then defines the order of a zero. One affix may be omitted, since it can be supplied from ·101, and if a cardinal point is known not to be wanted the corresponding letter may be omitted. Thus in this notation $\mathrm{p}_n\mathrm{q}^n u$ can be replaced by $\mathrm{pq}^n u$, whether n is positive or negative, and the function so denoted is the function $(\mathrm{pq}\,u)^n$ already denoted in the same way. It is necessary to agree that $\mathrm{pq}\,u$ is abbreviated from $\mathrm{p}_1\mathrm{q}^1 u$, not from $\mathrm{p}^1\mathrm{q}_1\,u$, and this is a natural convention.

We can express the general function in terms of three functions at whichever of the four cardinal points we wish. For example, in terms of the primitive Jacobian functions,

·102 $$\mathrm{sc}_k\mathrm{n}_l\mathrm{d}_m\,u = \mathrm{cs}^k u\,\mathrm{ns}^l u\,\mathrm{ds}^m u,$$

and in terms of Jacobi's functions,

·103 $$\mathrm{s}_h\mathrm{c}_k\mathrm{d}_m\mathrm{n}\,u = \mathrm{sn}^h u\,\mathrm{cn}^k u\,\mathrm{dn}^m u.$$

The notation is particularly useful for the logarithmic derivatives of Jacobian functions; these are functions with simple poles at two of the cardinal points and simple zeros at the other two, and although they can be expressed as products in the elementary notation, this expression is not unique and compels us to bear in mind that $\mathrm{pq}\,u\,\mathrm{rt}\,u$ is the same function as $\mathrm{pt}\,u\,\mathrm{rq}\,u$. We can now write, omitting one affix,

·104–·105 $$\mathrm{sn}'u = \mathrm{c}_1\mathrm{d}_1\mathrm{n}\,u, \qquad \mathrm{sn}'u/\mathrm{sn}\,u = \mathrm{s}^1\mathrm{c}_1\mathrm{d}_1\mathrm{n}\,u.$$

With positive affixes only, there are six types of function, which, with an arbitrary constant factor included, are

·106 (i) $C\mathrm{p}_h\mathrm{t}^m u,$ $h = m$

·107 (ii) $C\mathrm{p}_h\mathrm{q}_k\mathrm{t}^m u,$ $h+k = m$

·108 (iii) $C\mathrm{p}_h\mathrm{r}^l\mathrm{t}^m u,$ $h = l+m$

·109 (iv) $C\mathrm{p}_h\mathrm{q}_k\mathrm{r}_l\mathrm{t}^m u,$ $h+k+l = m$

·110 (v) $C\mathrm{p}_h\mathrm{q}_k\mathrm{r}^l\mathrm{t}^m u,$ $h+k = l+m$

·111 (vi) $C\mathrm{p}_h\mathrm{q}^k\mathrm{r}^l\mathrm{t}^m u,$ $h = k+l+m.$

A function of type (i) is a multiple of the power $\mathrm{pt}^m u$ of the elementary function $\mathrm{pt}\,u$. If in (ii), $k \geqslant 2$, we can use a relation of the form $\mathrm{qt}^2 u = A + B\,\mathrm{pt}^2 u$ to express the function as a sum of functions of the same type with k diminished by any even number. Hence if k is even, the function is a sum of functions of type (i), and the same is true if h is even. Similarly if two of the three suffixes in (iv) are even, the function is a sum of functions of type (i), and if one is even and two odd, the function is a sum of functions of type (ii) with odd suffixes. But if, in (ii), h and k are odd, the function has $\mathrm{rt}'u$ for a factor, and the quotient involves $\mathrm{pt}\,u$, $\mathrm{qt}\,u$ only in even powers, that is, in such a way that they are expressible in terms of $\mathrm{rt}^2 u$: the function is the sum of terms of the form $C\,\mathrm{rt}^n u\,\mathrm{rt}'u$ with n even. In (iv), with h, k, l all odd, terms of the form $C\,\mathrm{rt}^n u\,\mathrm{rt}'u$ are multiplied by an odd power of $\mathrm{rt}\,u$.

14·11. *A function of type* $\mathrm{p}_h\mathrm{q}_k\mathrm{t}^m u$ *or* $\mathrm{p}_h\mathrm{q}_k\mathrm{r}_l\mathrm{t}^m u$ *is the sum of functions of type* $C\,\mathrm{pt}^m u$ *and functions of type* $C\,\mathrm{pt}^m u\,\mathrm{pt}'u$, *with* $m \geqslant 0$.

As for type (v), by the same argument if h or k is even the function is the sum of functions of type (iii), and if both h and k are odd we can reduce one of them systematically and take provisionally as a standard type

·112 (v′) $C\mathrm{p}_h\mathrm{q}_1\mathrm{r}^l\mathrm{t}^m u,$ $h+1 = l+m$, with h odd.

The reduction of a function with two or more poles and one zero proceeds somewhat differently. The relation $A\,\mathrm{rp}^2 u + B\,\mathrm{tp}^2 u = 1$ between functions copolar at K_p is equivalent to

$$\mathrm{p}_4\mathrm{r}^2\mathrm{t}^2 u = B\,\mathrm{pr}^2 u + A\,\mathrm{pt}^2 u.$$

Hence the function $\mathrm{p}_h\mathrm{r}^l\mathrm{t}^m u$ is the sum of multiples of the functions

$\mathrm{p}_{h-2}\mathrm{r}^{l-2}\mathrm{t}^m u$, $\mathrm{p}_{h-2}\mathrm{r}^l\mathrm{t}^{m-2}u$, and by repetition of the process is the sum of multiples of the functions

·113 $$\mathrm{p}_{h-r}\mathrm{r}^{l-r}\mathrm{t}^m u,\ \mathrm{p}_{h-r-2}\mathrm{r}^{l-r}\mathrm{t}^{m-2}u,\ldots,\ \mathrm{p}_{h-r-s}\mathrm{r}^{l-r}\mathrm{t}^{m-s}u,$$

·114 $$\mathrm{p}_{h-s}\mathrm{r}^l\mathrm{t}^{m-s}u,\ \mathrm{p}_{h-s-2}\mathrm{r}^{l-2}\mathrm{t}^{m-s}u,\ldots,\ \mathrm{p}_{h-r-s}\mathrm{r}^{l-r}\mathrm{t}^{m-s}u,$$

for any even values of r, s. A single even affix does not now reduce the type of a function. If l and m are both even, we can take $r = l$, $s = m$, and the sets of functions ·113, ·114 become

·115 $$\mathrm{p}_m\mathrm{t}^m u,\ \mathrm{p}_{m-2}\mathrm{t}^{m-2}u,\ldots,\ \mathrm{p}_2\mathrm{t}^2 u,\ 1,$$

·116 $$\mathrm{p}_l\mathrm{r}^l u,\ \mathrm{p}_{l-2}\mathrm{r}^{l-2}u,\ldots,\ \mathrm{p}_2\mathrm{r}^2 u,\ 1,$$

composed entirely of functions of type (i). If l is odd and m even, we can not eliminate the point K_r by any choice of r in ·113, but by taking $r = l+1$ we convert this point into a zero; the two sets of functions are

·117 $$\mathrm{p}_{m-1}\mathrm{r}_1\mathrm{t}^m u,\ \mathrm{p}_{m-3}\mathrm{r}_1\mathrm{t}^{m-2}u,\ldots,\ \mathrm{p}_1\mathrm{r}_1\mathrm{t}^2 u,\ \mathrm{p}^1\mathrm{r}_1 u,$$

·118 $$\mathrm{p}_l\mathrm{r}^l u,\ \mathrm{p}_{l-2}\mathrm{r}^{l-2}u,\ldots,\ \mathrm{p}_1\mathrm{r}^1 u,\ \mathrm{p}^1\mathrm{r}_1 u,$$

of which the second consists of functions of type (i), the first of functions of type (ii) together with the one elementary function $\mathrm{rp}\,u$. Similarly, if l and m are both odd, we take $r = l+1$, $s = m+1$ in ·113, ·114, and we have the two sets of functions

·119 $$\mathrm{p}_{m-1}\mathrm{r}_1\mathrm{t}^m u,\ \mathrm{p}_{m-3}\mathrm{r}_1\mathrm{t}^{m-2}u,\ldots,\ \mathrm{p}_2\mathrm{r}_1\mathrm{t}^3 u,\ \mathrm{r}_1\mathrm{t}^1 u,\ \mathrm{p}^2\mathrm{r}_1\mathrm{t}_1 u,$$

·120 $$\mathrm{p}_{l-1}\mathrm{r}^l\mathrm{t}_1 u,\ \mathrm{p}_{l-3}\mathrm{r}^{l-2}\mathrm{t}_1 u,\ldots,\ \mathrm{p}_2\mathrm{r}^3\mathrm{t}_1 u,\ \mathrm{r}^1\mathrm{t}_1 u,\ \mathrm{p}^2\mathrm{r}_1\mathrm{t}_1 u,$$

composed of functions of type (ii) with the elementary functions $\mathrm{rt}\,u$ and $\mathrm{tr}\,u$. Thus in every case a function of type (iii) is the sum of functions of types (i) and (ii), and ·11 is applicable:

14·12. *A function of type* $\mathrm{p}_h\mathrm{r}^l\mathrm{t}^m u$ *is the sum of functions of type* $C\,\mathrm{pt}^m u$ *and functions of type* $C\,\mathrm{pt}^m u\,\mathrm{pt}'u$, *with* $m \geqslant 0$.

Instead of examining the function of type (vi) as it stands, we may regard this function as the product by $\mathrm{p}_k\mathrm{q}^k u$ of the function $\mathrm{p}_{l+m}\mathrm{r}^l\mathrm{t}^m u$ which we have just dissected. In the sets of functions ·115–·120, each non-constant function has only one pole, and if K_r or K_l occurs as a zero, this zero is simple; moreover, $\mathrm{p}^2\mathrm{r}_1\mathrm{t}_1 u$ in ·119 and ·120 is the only function in which the points K_r, K_l both occur as zeros. Hence if we multiply throughout by $\mathrm{p}_k\mathrm{q}^k u$, we obtain, except in this one case, either a function with not more than two poles and with K_p for the only zero, that is, a function of type (i) or (iii), or a function with not more than two poles, with not more than two zeros, and with one of its zeros simple, that is, a function of one of the types (i), (ii), (iii), (v′): the product $\mathrm{p}_k\mathrm{q}^k u\,\mathrm{p}^2\mathrm{r}_1\mathrm{t}_1 u$ may be $\mathrm{p}^1\mathrm{q}^1\mathrm{r}_1\mathrm{t}_1 u$, of type (v′), $\mathrm{q}^2\mathrm{r}_1\mathrm{t}_1 u$, of type (ii), or $\mathrm{p}_{k-2}\mathrm{q}^k\mathrm{r}_1\mathrm{t}_1 u$, of type (iv), but (v′) remains the only type not yet considered.

To deal with (v′), we write the function $\mathrm{p}_h\mathrm{q}_1\mathrm{r}^l\mathrm{t}^m u$ as the product $\mathrm{p}^1\mathrm{q}_1 u\,\mathrm{p}_{h+1}\mathrm{r}^l\mathrm{t}^m u$, and multiply throughout ·115, ·116, ·119, ·120 by the factor $\mathrm{p}^1\mathrm{q}_1 u$. If K_p is already either a pole or a zero, the resulting product has only one pole and is of one of the types (i), (ii), (iv); the functions in which K_p does not figure are the constant in ·115 and ·116, and the functions $\mathrm{r}_1\mathrm{t}^1 u$, $\mathrm{r}^1\mathrm{t}_1 u$ in ·119, ·120, and in these cases the product is either the elementary function $\mathrm{qp}\,u$ or a function, $\mathrm{q}_1\mathrm{r}_1\mathrm{p}^1\mathrm{t}^1 u$ or $\mathrm{q}_1\mathrm{t}_1\mathrm{p}^1\mathrm{r}^1 u$, with two simple poles and two simple zeros, a multiple of a logarithmic derivative. To include the logarithmic derivative in the formula $C\,\mathrm{pt}^m u\,\mathrm{pt}' u$ of ·11 and ·12 we have only to allow m to take the value -1. Replacing $\mathrm{pt}\,u$ by our more familiar $\mathrm{pq}\,u$, we have the result:

14·13. *The general Glaisher function is the sum of a number of terms each of which has one of the two forms* $C\,\mathrm{pq}^m u$, $C\,\mathrm{pq}^{m-1}u\,\mathrm{pq}'u$, *where* C *is a constant and* m *is zero or a positive integer.*

The remarkable features of this theorem are that each term involves only one of the twelve Jacobian functions, and that therefore negative powers are not invoked except in the case of the logarithmic derivative. The theorem is not to be confused in character with Liouville's theorem on the expression of one elliptic function by means of a coperiodic function and its derivative. Liouville's theorem requires rational functions, not merely positive powers, while in ·13 different terms in the sum may involve different elementary functions, and the elementary functions are not all coperiodic.

14·2. From ·13, since $\mathrm{pq}^{m-1}u\,\mathrm{pq}'u$ is immediately integrable, it follows that the problem of integrating the general Glaisher function rests entirely on that of integrating positive integral powers of the twelve Jacobian functions.

For the function $\mathrm{pq}\,u$ there is a relation

·201 $$\mathrm{pq}'^2 u = \lambda\,\mathrm{pq}^4 u + \mu\,\mathrm{pq}^2 u + \nu,$$

given in Table XI II, implying

·202 $$\mathrm{pq}'' u = 2\lambda\,\mathrm{pq}^3 u + \mu\,\mathrm{pq}\,u.$$

We have therefore, for any value of m,

$$\cdot 203 \quad \frac{d}{du}(\mathrm{pq}^{m-1}u\,\mathrm{pq}'u)$$
$$= \{(m-1)(\lambda\,\mathrm{pq}^4u+\mu\,\mathrm{pq}^2u+\nu)+(2\lambda\,\mathrm{pq}^4u+\mu\,\mathrm{pq}^2u)\}\mathrm{pq}^{m-2}u$$
$$= (m+1)\lambda\,\mathrm{pq}^{m+2}u+m\mu\,\mathrm{pq}^m u+(m-1)\nu\,\mathrm{pq}^{m-2}u,$$

from which follows a formula of reduction connecting the integrals of $\mathrm{pq}^{m+2}u$, $\mathrm{pq}^m u$, $\mathrm{pq}^{m-2}u$.

With $m = 1$, ·203 is identical with ·202 and gives a formula for the integral of pq^3u in terms of the integral of $\mathrm{pq}\,u$; we can therefore evaluate the integral of any odd power of $\mathrm{pq}\,u$ in terms of that of $\mathrm{pq}\,u$. With $m = 2$, the constant term ν occurs in ·203, but this term is integrable and we can express the integral of pq^4u, and therefore the integral of any even power of $\mathrm{pq}\,u$, in terms of the integral of pq^2u.

14·21. *The integral of the general Glaisher function is the sum of constant multiples of functions each of which has one of the forms*

$$\mathrm{pq}^m u, \qquad \mathrm{pq}^m u\,\mathrm{pq}'u, \qquad u, \qquad \log\mathrm{pq}\,u, \qquad \int \mathrm{pq}\,u\,du, \qquad \int \mathrm{pq}^2u\,du,$$

where m is zero or a positive integer.

We proceed to consider the integration of $\mathrm{pq}\,u$ and pq^2u.

14·3. The Jacobian function $\mathrm{pq}\,u$ can be integrated by means of the two functions copolar with it, combinations that serve this purpose being evident† from Table XI 5.

TABLE XIV 1

$$\mathrm{cs}\,u = \frac{\mathrm{ns}'u-\mathrm{ds}'u}{\mathrm{ns}\,u-\mathrm{ds}\,u} \qquad \mathrm{ns}\,u = \frac{\mathrm{ds}'u-\mathrm{cs}'u}{\mathrm{ds}\,u-\mathrm{cs}\,u} \qquad \mathrm{ds}\,u = \frac{\mathrm{ns}'u-\mathrm{cs}'u}{\mathrm{ns}\,u-\mathrm{cs}\,u}$$

$$\mathrm{sc}\,u = -\frac{1}{k'}\cdot\frac{\mathrm{dc}'u-k'\,\mathrm{nc}'u}{\mathrm{dc}\,u-k'\,\mathrm{nc}\,u} \qquad \mathrm{dc}\,u = -\frac{\mathrm{nc}'u-\mathrm{sc}'u}{\mathrm{nc}\,u-\mathrm{sc}\,u} \qquad \mathrm{nc}\,u = -\frac{1}{k'}\cdot\frac{\mathrm{dc}'u-k'\,\mathrm{sc}'u}{\mathrm{dc}\,u-k'\,\mathrm{sc}\,u}$$

$$\mathrm{dn}\,u = \frac{1}{v}\cdot\frac{\mathrm{cn}'u+v\,\mathrm{sn}'u}{\mathrm{cn}\,u+v\,\mathrm{sn}\,u}$$

$$\mathrm{sn}\,u = \frac{1}{k}\cdot\frac{\mathrm{dn}'u-k\,\mathrm{cn}'u}{\mathrm{dn}\,u-k\,\mathrm{cn}\,u}$$

$$\mathrm{cn}\,u = \frac{1}{vk}\cdot\frac{\mathrm{dn}'u+vk\,\mathrm{sn}'u}{\mathrm{dn}\,u+vk\,\mathrm{sn}\,u}$$

$$\mathrm{nd}\,u = \frac{v}{k'}\cdot\frac{\mathrm{cd}'u-vk'\,\mathrm{sd}'u}{\mathrm{cd}\,u-vk'\,\mathrm{sd}\,u}$$

$$\mathrm{cd}\,u = \frac{1}{k}\cdot\frac{\mathrm{nd}'u+k\,\mathrm{sd}'u}{\mathrm{nd}\,u+k\,\mathrm{sd}\,u}$$

$$\mathrm{sd}\,u = \frac{1}{vkk'}\cdot\frac{k\,\mathrm{cd}'u+vk'\,\mathrm{nd}'u}{k\,\mathrm{cd}\,u+vk'\,\mathrm{nd}\,u}$$

† See also the argument in 16·6 below.

Signs can be altered in the numerator and denominator of any of these fractions if a negative sign is prefixed to the fraction.

The expression for $\operatorname{dn} u$ as a logarithmic derivative brings us back to the place of this function in Jacobi's work, for if ϕ is such that $\cos\phi = \operatorname{cn} u$, $\sin\phi = \operatorname{sn} u$, then

$$e^{i\phi} = \operatorname{cn} u + i \operatorname{sn} u, \qquad ie^{i\phi}\, d\phi/du = \operatorname{cn}' u + i \operatorname{sn}' u,$$

formulae which together identify $\operatorname{dn} u$ with $d\phi/du$. This alternative suggests that avoidance of radicals and auxiliary functions has perhaps been carried too far. If $\operatorname{pq} u$, $\operatorname{rq} u$, $\operatorname{tq} u$ are copolar, $\operatorname{pq} u \operatorname{rq} u/\operatorname{rq} u$ is of the form $\operatorname{tq}' u/\sqrt{(\lambda \operatorname{tq}^2 u + \mu)}$ and can be integrated in this form, the necessary constants being taken from Tables XI 2 and XI 5. For example, $\operatorname{cs} u = -\operatorname{ns}' u/\operatorname{ds} u$ where $\operatorname{ds}^2 u = \operatorname{ns}^2 u - k^2$, and therefore $\operatorname{cs} u = d\psi/du$ if ψ is defined by $k \cosh\psi = \operatorname{ns} u$, $k \sinh\psi = -\operatorname{ds} u$; the alternative expression for $\operatorname{cs} u$, as $-\operatorname{ds}' u/\operatorname{ns} u$ where $\operatorname{ns}^2 u = \operatorname{ds}^2 u + k^2$, leads to the same substitution. The following table gives substitutions which render the integrations immediate.

TABLE XIV 2

$\begin{cases} k\cosh\psi = \operatorname{ns} u \\ k\sinh\psi = -\operatorname{ds} u \end{cases}$ $\operatorname{cs} u = d\psi/du$	$\begin{cases} k'\cosh\psi = \operatorname{ds} u \\ k'\sinh\psi = -\operatorname{cs} u \end{cases}$ $\operatorname{ns} u = d\psi/du$	$\begin{cases} \cosh\psi = \operatorname{ns} u \\ \sinh\psi = -\operatorname{cs} u \end{cases}$ $\operatorname{ds} u = d\psi/du$
$\begin{cases} k\cosh\psi = \operatorname{dc} u \\ k\sinh\psi = k'\operatorname{nc} u \end{cases}$ $k'\operatorname{sc} u = d\psi/du$	$\begin{cases} \cosh\psi = \operatorname{nc} u \\ \sinh\psi = \operatorname{sc} u \end{cases}$ $\operatorname{dc} u = d\psi/du$	$\begin{cases} \cosh\psi = \operatorname{dc} u \\ \sinh\psi = k'\operatorname{sc} u \end{cases}$ $k'\operatorname{nc} u = d\psi/du$
$\begin{cases} \cos\psi = \operatorname{cn} u \\ \sin\psi = \operatorname{sn} u \end{cases}$ $\operatorname{dn} u = d\psi/du$	$\begin{cases} k'\cosh\psi = \operatorname{dn} u \\ k'\sinh\psi = -k\operatorname{cn} u \end{cases}$ $k\operatorname{sn} u = d\psi/du$	$\begin{cases} \cos\psi = \operatorname{dn} u \\ \sin\psi = k\operatorname{sn} u \end{cases}$ $k\operatorname{cn} u = d\psi/du$
$\begin{cases} \cos\psi = \operatorname{cd} u \\ \sin\psi = k'\operatorname{sd} u \end{cases}$ $k'\operatorname{nd} u = d\psi/du$	$\begin{cases} \cosh\psi = \operatorname{nd} u \\ \sinh\psi = k\operatorname{sd} u \end{cases}$ $k\operatorname{cd} u = d\psi/du$	$\begin{cases} \cos\psi = k'\operatorname{nd} u \\ \sin\psi = -k\operatorname{cd} u \end{cases}$ $kk'\operatorname{sd} u = d\psi/du$

14·4. The integral of $\operatorname{pq}^2 u$ is not expressible in terms of Jacobian functions and more elementary functions, and we have to regard the integral as a function to be investigated. If the origin is not a pole of $\operatorname{pr} u$, we write

·401 $$\operatorname{Pr} u = \int_0^u \operatorname{pr}^2 u\, du.$$

The function $\operatorname{ps}^2 u$ has zero residue at the origin, and $\operatorname{ps}^2 u - 1/u^2$ is regular near the origin: we write

$$\cdot 402 \qquad \operatorname{Ps} u = \int_0^u \left(\operatorname{ps}^2 u - \frac{1}{u^2}\right) du - \frac{1}{u},$$

defining a function such that $\operatorname{Ps}' u = \operatorname{ps}^2 u$ and that $\operatorname{Ps} u + 1/u \to 0$ as $u \to 0$. We call the function $\operatorname{Pq} u$ the integrating function associated with $\operatorname{pq} u$.

The function $\operatorname{Pq} u$ is a function with simple poles at the poles of $\operatorname{pq} u$. If a_p is the residue of $\operatorname{pq} u$ at a pole, the residue of $\operatorname{Pq} u$ there is $-a_p^2$, which has the same value at every pole. Since every residue of $\operatorname{pq}^2 u$ is zero, $\operatorname{Pq} u$ is singlevalued. Since $\operatorname{Pq}'(-u) = \operatorname{Pq}' u$, the sum $\operatorname{Pq} u + \operatorname{Pq}(-u)$ is a constant, which is zero whether† or not the origin is a pole; that is, $\operatorname{Pq} u$ is an odd function. If K_l is any quarterperiod of the Jacobian system, $\operatorname{pq}^2(u+2K_l) - \operatorname{pq}^2 u = 0$, that is,

$$\operatorname{Pq}'(u+2K_l) - \operatorname{Pq}' u \doteq 0,$$

whence $\operatorname{Pq}(u+2K_l) - \operatorname{Pq} u$ has a constant value which is recognizable as $\operatorname{Pq} 2K_l$ if the origin is not a pole, and as $\operatorname{Pq} K_l - \operatorname{Pq}(-K_l)$, that is, as $2\operatorname{Pq} K_l$, if K_l is not a pole; if neither the origin nor K_l is a pole, $\operatorname{Pq} 2K_l = 2\operatorname{Pq} K_l$. Since the two differences

$$\operatorname{Pq}(u+2K_c) - \operatorname{Pq} u, \qquad \operatorname{Pq}(u+2K_n) - \operatorname{Pq} u$$

are constant, the function $\operatorname{Pq} u$ is doubly quasiperiodic. The constants of quasiperiodicity are discussed in the next section.

There are evident relations between the twelve integrating functions. From the relations between the squares of copolar Jacobian functions, given in Table XI 2, we have corresponding formulae.

$14{\cdot}41_{1\text{-}3}$ $\quad \operatorname{Ns} u - \operatorname{Cs} u = u,$

$$\operatorname{Ds} u - \operatorname{Cs} u = c' u,$$

$$\operatorname{Ns} u - \operatorname{Ds} u = c u;$$

$14{\cdot}41_{4\text{-}6}$ $\quad \operatorname{Nc} u - \operatorname{Sc} u = u,$

$$\operatorname{Dc} u - c' \operatorname{Sc} u = u,$$

$$\operatorname{Dc} u - c' \operatorname{Nc} u = c u;$$

$14{\cdot}41_{7\text{-}9}$ $\quad \operatorname{Sn} u + \operatorname{Cn} u = u,$

$$c \operatorname{Sn} u + \operatorname{Dn} u = u,$$

$$\operatorname{Dn} u - c \operatorname{Cn} u = c' u;$$

$14{\cdot}41_{10\text{-}12}$ $\quad c' \operatorname{Sd} u + \operatorname{Cd} u = u,$

$$\operatorname{Nd} u - c \operatorname{Sd} u = u,$$

$$c \operatorname{Cd} u + c' \operatorname{Nd} u = u.$$

† It is to secure this result that $\operatorname{Ps} u$ is defined from the origin, although the integral from K_p would be an easier function to handle.

In addition ·203, with $m = 0$, is a relation between $\mathrm{Pq}\,u$ and $\mathrm{Qp}\,u$; using Table XI ii, we have the six formulae

$$14{\cdot}42_{1\text{-}3} \qquad c'\,\mathrm{Sc}\,u - \mathrm{Cs}\,u = \mathrm{sc}'u/\mathrm{sc}\,u,$$

$$c\,\mathrm{Sn}\,u - \mathrm{Ns}\,u = \mathrm{sn}'u/\mathrm{sn}\,u,$$

$$cc'\,\mathrm{Sd}\,u + \mathrm{Ds}\,u = \mathrm{ds}'u/\mathrm{ds}\,u;$$

$$14{\cdot}42_{4\text{-}6} \qquad c\,\mathrm{Cn}\,u + c'\,\mathrm{Nc}\,u = \mathrm{nc}'u/\mathrm{nc}\,u,$$

$$\mathrm{Dc}\,u - c\,\mathrm{Cd}\,u = \mathrm{dc}'u/\mathrm{dc}\,u,$$

$$\mathrm{Dn}\,u - c'\,\mathrm{Nd}\,u = \mathrm{nd}'u/\mathrm{nd}\,u.$$

The function $\mathrm{pq}'u/\mathrm{pq}\,u$ is not included in ·21 among those required for the integration of the general Glaisher function, and in fact it is not essential if the integrals of pq^2u and qp^2u are both available, but clearly the function is one which we should be ready to use.

To the homogeneous relations between the squares of copolar functions correspond the homogeneous relations

$$14{\cdot}43_{1\text{-}4} \qquad c\,\mathrm{Cs}\,u + c'\,\mathrm{Ns}\,u - \mathrm{Ds}\,u = 0,$$

$$c\,\mathrm{Sc}\,u + \mathrm{Dc}\,u - \mathrm{Nc}\,u = 0,$$

$$\mathrm{Dn}\,u - c'\,\mathrm{Sn}\,u - \mathrm{Cn}\,u = 0,$$

$$\mathrm{Nd}\,u - \mathrm{Cd}\,u - \mathrm{Sd}\,u = 0.$$

For themselves these need hardly be recorded, but if we replace $\mathrm{Pq}\,u$ by $\mathrm{Qp}\,u$ throughout by means of ·42, we have the relations

$$14{\cdot}44_{1\text{-}4} \qquad \mathrm{Sc}\,u + \mathrm{Sn}\,u + \mathrm{Sd}\,u = \mathrm{sc}^1\mathrm{n}^1\mathrm{d}^1u,$$

$$\mathrm{Cs}\,u + c'\,\mathrm{Cd}\,u + \mathrm{Cn}\,u = -\mathrm{cs}^1\mathrm{d}^1\mathrm{n}^1u,$$

$$c\,\mathrm{Nd}\,u - \mathrm{Ns}\,u + \mathrm{Nc}\,u = \mathrm{nd}^1\mathrm{s}^1\mathrm{c}^1u,$$

$$c\,\mathrm{Dn}\,u - c'\,\mathrm{Dc}\,u + \mathrm{Ds}\,u = -\mathrm{dn}^1\mathrm{c}^1\mathrm{s}^1u,$$

which are less obvious in the differentiated form.

14·5. Since we can connect $\mathrm{Pq}\,u$ with $\mathrm{Tq}\,u$ by a formula from ·41, $\mathrm{Tq}\,u$ with $\mathrm{Qt}\,u$ by a formula from ·42, and $\mathrm{Qt}\,u$ with $\mathrm{Rt}\,u$ by a second formula from ·41, we can formulate a direct relation between the two functions $\mathrm{Pq}\,u$, $\mathrm{Rt}\,u$, that is, between any two of the twelve integrating functions. In other words, it is not untrue to say that the integration of even powers of the Jacobian functions requires the introduction of only one integrating function, and the traditional point of view is that to perform an integration is to express a result in terms of one function chosen as canonical. If it is anomalous to recognize that the original Jacobian triad $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$ is not properly understood except as a section of Glaisher's interrelated dozen, and yet to insist on reducing the dozen integrals to the algebraical minimum, again the tradition has determined the notation and permeates the literature.

The integral from which elliptic functions derive their name, the integral giving the length of an elliptic arc, has the form

$$\int_0^\phi \sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)}\,d\phi,$$

that is, but for the factor a,

$$\int_0^\phi \sqrt{(1-k^2\sin^2\phi)}\,d\phi,$$

where $k^2 = (a^2-b^2)/a^2$, and this is Legendre's first elliptic integral $E(\phi)$. When ϕ is regarded as the function $\operatorname{am} u$ of the second integral u, defined by

$$u = \int_0^\phi \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}},$$

we have $$\sqrt{(1-k^2\sin^2\phi)} = d\phi/du = \operatorname{dn} u,$$

and the first integral becomes

$$\int_0^u \operatorname{dn}^2 u\,du.$$

It was therefore almost inevitable, historically, that this integral should become the standard integral of its kind, in spite of the leading position assigned to $\operatorname{sn} u$ in the beginning. The integral is denoted by $E(u)$, and we retain this definition, although of course abandoning any restriction on the parameter k^2. To express the twelve integrating functions in terms of Jacobi's function $E(u)$ is to relate each of them, in the manner already outlined, to the function $\operatorname{Dn} u$. Logarithmic derivatives are expressed in the notation of ·1.

TABLE XIV 3

$$\operatorname{Cs} u = -E(u) - \mathrm{c_1 d_1 s^1 n^1} u$$

$$\operatorname{Ns} u = -E(u) + u - \mathrm{c_1 d_1 s^1 n^1} u$$

$$\operatorname{Ds} u = -E(u) + c'u - \mathrm{c_1 d_1 s^1 n^1} u$$

$$\operatorname{Sc} u = \{-E(u) + \mathrm{s_1 d_1 c^1 n^1} u\}/c'$$

$$\operatorname{Dc} u = -E(u) + u + \mathrm{s_1 d_1 c^1 n^1} u$$

$$\operatorname{Nc} u = \{-E(u) + c'u + \mathrm{s_1 d_1 c^1 n^1} u\}/c'$$

$$\operatorname{Dn} u = E(u)$$

$$\operatorname{Sn} u = \{-E(u) + u\}/c$$

$$\operatorname{Cn} u = \{E(u) - c'u\}/c$$

$$\operatorname{Nd} u = \{E(u) - c\,\mathrm{s_1 c_1 d^1 n^1} u\}/c'$$

$$\operatorname{Cd} u = \{-E(u) + u + c\,\mathrm{s_1 c_1 d^1 n^1} u\}/c$$

$$\operatorname{Sd} u = \{E(u) - c'u - c\,\mathrm{s_1 c_1 d^1 n^1} u\}/cc'$$

Generally speaking, there is no reason for preferring one of the twelve integrating functions to another, and we should use the functions appropriate to any investigation without supposing that a solution is unfinished if it is not stated explicitly in terms of $E(u)$.

The value of the difference $\mathrm{Pq}(u+2K_c)-\mathrm{Pq}\,u$ is evident from Table XIV3 in terms of $E(2K_c)$, a constant which is equal to $2E(K_c)$, since neither the origin nor K_c is a pole of $\mathrm{dn}\,u$. Writing E_c for $E(K_c)$, we have

·501 $$\mathrm{Cs}(u+2K_c)-\mathrm{Cs}\,u = -2E_c,$$

·502 $$\mathrm{Ns}(u+2K_c)-\mathrm{Ns}\,u = 2(K_c-E_c),$$

·503 $$\mathrm{Sn}(u+2K_c)-\mathrm{Sn}\,u = 2(K_c-E_c)/c,$$

and so on.

We must notice a distinction between the last six functions in XIV3 and the first six. In the last six we have

·504–·507 $$\mathrm{Dn}\,2K_c = 2\,\mathrm{Dn}\,K_c = 2E_c, \qquad \mathrm{Sn}\,2K_c = 2\,\mathrm{Sn}\,K_c = 2(K_c-E_c)/c,$$

and so on. In the first six, we have $\mathrm{Cs}\,K_c = -E_c$ but $2K_c$ is a pole, $\mathrm{Dc}\,2K_c = 2(K_c-E_c)$ but K_c is a pole, and so on; in no case are K_c and $2K_c$ both available as arguments. This contrast reappears as a difficulty in the expression of $\mathrm{Pq}(u+2K_n)-\mathrm{Pq}\,u$. Since $2K_n$ is not a pole of $\mathrm{dn}\,u$, $E(2K_n)$ is finite, but $\frac{1}{2}E(2K_n)$ needs identification. The immediate solution is to admit $\mathrm{Dc}\,u$ as a second canonical function $D(u)$. Corresponding to XIV3 we have another table:

Table XIV4

$$\mathrm{Cs}\,u = D(u)-u-\mathrm{n_1d_1s^1c^1}u$$
$$\mathrm{Ns}\,u = D(u)-\mathrm{n_1d_1s^1c^1}u$$
$$\mathrm{Ds}\,u = D(u)-cu-\mathrm{n_1d_1s^1c^1}u$$
$$\mathrm{Sc}\,u = \{D(u)-u\}/c'$$
$$\mathrm{Dc}\,u = D(u)$$
$$\mathrm{Nc}\,u = \{D(u)-cu\}/c'$$
$$\mathrm{Dn}\,u = -D(u)+u+\mathrm{s_1d_1n^1c^1}u$$
$$\mathrm{Sn}\,u = \{D(u)-\mathrm{s_1d_1n^1c^1}u\}/c$$
$$\mathrm{Cn}\,u = \{-D(u)+cu+\mathrm{s_1d_1n^1c^1}u\}/c$$
$$\mathrm{Nd}\,u = \{-D(u)+u+c'\,\mathrm{s_1n_1d^1c^1}u\}/c'$$
$$\mathrm{Cd}\,u = \{D(u)-c'\,\mathrm{s_1n_1d^1c^1}u\}/c$$
$$\mathrm{Sd}\,u = \{-D(u)+cu+c'\,\mathrm{s_1n_1d^1c^1}u\}/cc'$$

Writing D_n for $D(K_n)$, we have now

$$\mathrm{Cs}\,K_n = D_n-K_n, \qquad \mathrm{Dn}\,2K_n = 2(K_n-D_n),$$

while $2K_n$ is a pole of $\mathrm{Cs}\,u$ and K_n is a pole of $\mathrm{Dn}\,u$, but

·508–·509 $$\mathrm{Sc}\,2K_n = 2\,\mathrm{Sc}\,K_n = 2(D_n - K_n)/c',$$

·510–·511 $$\mathrm{Dc}\,2K_n = 2\,\mathrm{Dc}\,K_n = 2D_n$$

and so on.

If the quasiperiodicity of the function $\mathrm{Pq}\,u$ in the Jacobian half-periods $2K_c$, $2K_n$ is expressed by the formula

·512 $$\mathrm{Pq}(u+2lK_c+2mK_n) = \mathrm{Pq}\,u+2l\mathrm{A}+2m\mathrm{B},$$

the values of the constants A, B for the twelve functions are given as follows:

TABLE XIV 5

Moduli of quasiperiodicity of the integrating functions

$\mathrm{Cs}\,u$	$\mathrm{Ns}\,u$	$\mathrm{Ds}\,u$
$-E_c,\ -(K_n-D_n)$	$(K_c-E_c),\ D_n$	$-(E_c-c'K_c),\ (D_n-cK_n)$
$\mathrm{Sc}\,u$	$\mathrm{Dc}\,u$	$\mathrm{Nc}\,u$
$-E_c/c',\ -(K_n-D_n)/c'$	$(K_c-E_c),\ D_n$	$-(E_c-c'K_c)/c',\ (D_n-cK_n)/c'$
$\mathrm{Dn}\,u$	$\mathrm{Sn}\,u$	$\mathrm{Cn}\,u$
$E_c,\ (K_n-D_n)$	$(K_c-E_c)/c,\ D_n/c$	$(E_c-c'K_c)/c,\ -(D_n-cK_n)/c$
$\mathrm{Nd}\,u$	$\mathrm{Cd}\,u$	$\mathrm{Sd}\,u$
$E_c/c',\ (K_n-D_n)/c'$	$(K_c-E_c)/c,\ D_n/c$	$(E_c-c'K_c)/cc',\ -(D_n-cK_n)/cc'$

One relation between the two functions $E(u)$, $D(u)$ is apparent from the two tables XIV 3, 4:

14·51 $$D(u)+E(u) = u+\mathrm{s_1d_1c^1n^1}u.$$

Also we can express the constant which belongs primarily to one function as a limit associated with the other function:

14·52₁ $$K_n-D_n = \lim_{u\to K_n}\left\{E(u)-\frac{1}{u-K_n}\right\},$$

14·52₂ $$K_c-E_c = \lim_{u\to K_c}\left\{D(u)+\frac{1}{u-K_c}\right\},$$

for $$\lim_{u\to K_n}\left\{\mathrm{s_1d_1c^1n^1}u-\frac{1}{u-K_n}\right\} = 0, \qquad \lim_{u\to K_c}\left\{\mathrm{s_1d_1n^1c^1}u+\frac{1}{u-K_c}\right\} = 0.$$

But the fundamental relation between the functions is implied in the interchange of the parts played by K_c and K_n. Jacobi's imaginary transformation replaces one of the functions $\mathrm{dn}\,u$, $\mathrm{dc}\,v$ by the other, and we have, exhibiting the dependence on the parameter,

·513 $$\mathrm{Dn}(u,c) = \nu\,\mathrm{Dc}(v,b)$$

if $$v = \nu u, \qquad b = c', \qquad b' = c,$$

so that we may write

·514 $$D_n(c) = v\,E_c(c').$$

In view of the relation of the primitive functions $\mathrm{cs}^2 u$, $\mathrm{ns}^2 u$, $\mathrm{ds}^2 u$ to the Weierstrassian function $\wp(u; K_c, K_n, K_d)$, there is a third function to which the integrating functions are naturally reducible, namely, the function ζu by which $\wp u$ is integrated. By definition,

·515 $$\zeta u = \frac{1}{u} - \int_0^u \left(\wp u - \frac{1}{u^2}\right) du,$$

and therefore

·516–·518 $$\mathrm{Cs}\,u + u\wp K_c = \mathrm{Ns}\,u + u\wp K_n = \mathrm{Ds}\,u + u\wp K_d = -\zeta u.$$

These formulae however introduce the constants $\wp K_c$, $\wp K_n$, $\wp K_d$ themselves, whereas only differences between these constants are required elsewhere in our work.

14·6. In the parallelogram whose vertices are 0, $2K_c$, $2K_c+2K_n$, $2K_n$, the function $\mathrm{sd}^2 u$ has only one pole, a double pole at K'_d with leading coefficient $-1/cc'$. Hence the only pole of $(u-K'_d)\mathrm{sd}^2 u$ in the parallelogram is a simple pole with residue $-1/cc'$, and the integral of the function round the perimeter is $-2\pi i/cc'$ or $2\pi i/cc'$ according as the description of the perimeter is in the positive or the negative direction, that is, according as the signature of the basis K_c, K_n is i or $-i$; in other words, the value of the integral is $-2\pi v/cc'$. But

$$\left(\int_0^{2K_c} + \int_{2K_c+2K_n}^{2K_n}\right)(u-K'_d)\mathrm{sd}^2 u\,du = -\int_0^{2K_c} 2K_n\,\mathrm{sd}^2 u\,du = -2K_n\,\mathrm{Sd}\,2K_c$$
$$= -4K_n(E_c - c'K_c)/cc',$$

$$\left(\int_{2K_c}^{2K_c+2K_n} + \int_{2K_n}^{0}\right)(u-K'_d)\mathrm{sd}^2 u\,du = \int_0^{2K_n} 2K_c\,\mathrm{sd}^2 u\,du = 2K_c\,\mathrm{Sd}\,2K_n$$
$$= -4K_c(D_n - cK_n)/cc'.$$

Hence

14·61 $$K_c D_n + K_n E_c - K_c K_n = \tfrac{1}{2}\pi v;$$

that is, writing

·601–·604 $$K_c = K, \qquad K_n = vK', \qquad E_c = E, \qquad D_n = vE',$$

we have

14·62 $$KE' + K'E - KK' = \tfrac{1}{2}\pi,$$

a relation discovered by Legendre.

Legendre's relation is unique, for application of the same method to any of the twelve integrating functions leads, with differences of detail in the proof, to the same result. For example, to use $(u-K_n)\mathrm{dn}^2 u$ we take the parallelogram whose vertices are $-K_c, K_c, K_c+2K_n, -K_c+2K_n$; one pair of sides gives the integral $-2K_n \mathrm{Dn}\, 2K_c$, which is $-4K_n E_c$, and the other pair gives $2K_c\{\mathrm{Dn}(K_c+2K_n)-\mathrm{Dn}\, K_c\}$, which is $4K_c(K_n-D_n)$; the residue is -1. Notice that we do not change the integrand to $\mathrm{dn}^2(K_c+u)$ in this argument.

Priority has been given to the quarterperiods K_c, K_n throughout the discussion of constants associated with the integrating functions. There is of course a constant $\mathrm{Pq}(u+2K_d)-\mathrm{Pq}\, u$, but this is only

$$-\{\mathrm{Pq}(u+2K_c)-\mathrm{Pq}\, u\}-\{\mathrm{Pq}(u+2K_n)-\mathrm{Pq}\, u\}$$

and calls for no comment. The forms taken by Legendre's relation if K_d replaces K_c or K_n are only trivially different from ·61, and when K, K', E, E' are introduced ·62 necessarily reappears.

14·7. From the addition formula 12·33 for a function $\mathrm{sq}\, u$ which has a zero at the origin, namely,

·701 $$\mathrm{sq}(u+v) = \frac{\mathrm{sq}\, u\, \mathrm{sq}' v+\mathrm{sq}\, v\, \mathrm{sq}' u}{1-\lambda\, \mathrm{sq}^2 u\, \mathrm{sq}^2 v},$$

where $\lambda = \mathrm{qs}'^2 K_q$, we have

$$\mathrm{sq}^2(u+v)-\mathrm{sq}^2(u-v) = \frac{4\, \mathrm{sq}\, u\, \mathrm{sq}' u\, \mathrm{sq}\, v\, \mathrm{sq}' v}{(1-\lambda\, \mathrm{sq}^2 u\, \mathrm{sq}^2 v)^2}.$$

Integrating with respect to u,

·702 $$\mathrm{Sq}(u+v)-\mathrm{Sq}(u-v)-2\, \mathrm{Sq}\, v = \frac{2\, \mathrm{sq}^2 u\, \mathrm{sq}\, v\, \mathrm{sq}' v}{1-\lambda\, \mathrm{sq}^2 u\, \mathrm{sq}^2 v}.$$

Interchanging u and v and adding the formula so obtained to ·702 we have

$14{\cdot}71_1$ $$\mathrm{Sq}(u+v)-\mathrm{Sq}\, u-\mathrm{Sq}\, v = \mathrm{sq}\, u\, \mathrm{sq}\, v\, \mathrm{sq}(u+v),$$

or in a more symmetrical form,

$14{\cdot}71_2$. *If* $u+v+w = 0$, *then*

$$\mathrm{Sq}\, u+\mathrm{Sq}\, v+\mathrm{Sq}\, w = \mathrm{sq}\, u\, \mathrm{sq}\, v\, \mathrm{sq}\, w.$$

We must not overlook that in this theorem the sum $u+v+w$ must be actually zero; congruence is not enough.

Corresponding results for a function $\mathrm{pq}\, u$ which has neither a zero nor a pole at the origin can be obtained directly from the addition theorem 12·43, but it is simpler to derive them from $\cdot 71_1$ and $\cdot 71_2$ by means of the elementary formula

$$\mathrm{pq}^2 u = 1+\mathrm{ps}^2 K_q\, \mathrm{sq}^2 u,$$

which implies

·703 $$\operatorname{Pq} u = u+\operatorname{ps}^2 K_q \operatorname{Sq} u,$$

and therefore

14·72$_1$ $$\operatorname{Pq}(u+v)-\operatorname{Pq} u-\operatorname{Pq} v = \operatorname{ps}^2 K_q \operatorname{sq} u \operatorname{sq} v \operatorname{sq}(u+v);$$

14·72$_2$. *If* pq u *is a Jacobian function of which the origin is neither a zero nor a pole, and if* $u+v+w = 0$, *then*

$$\operatorname{Pq} u+\operatorname{Pq} v+\operatorname{Pq} w = \operatorname{ps}^2 K_q \operatorname{sq} u \operatorname{sq} v \operatorname{sq} w.$$

Since the differences $\operatorname{cs}^2 u-\operatorname{ns}^2 u$, $\operatorname{ds}^2 u-\operatorname{ns}^2 u$ are constants, we need examine only one of the three functions with a pole at the origin. From the addition formula

·704 $$\operatorname{ns}(u+v) = \frac{\operatorname{ns} u \operatorname{ns}' v-\operatorname{ns} v \operatorname{ns}' u}{\operatorname{ns}^2 u-\operatorname{ns}^2 v}$$

we have $$\operatorname{Ns}(u+v)-\operatorname{Ns}(u-v)-2\operatorname{Ns} v = \frac{2\operatorname{ns} v \operatorname{ns}' v}{\operatorname{ns}^2 u-\operatorname{ns}^2 v},$$

and therefore

·705 $$\operatorname{Ns}(u+v)-\operatorname{Ns} u-\operatorname{Ns} v = \frac{\operatorname{ns} v \operatorname{ns}' v-\operatorname{ns} u \operatorname{ns}' u}{\operatorname{ns}^2 u-\operatorname{ns}^2 v}.$$

The addition formula ·704 can be written

·706 $$\operatorname{ns} u \operatorname{ns} v \operatorname{ns}(u+v) = \frac{\operatorname{ns}^2 u \,.\, \operatorname{ns} v \operatorname{ns}' v-\operatorname{ns}^2 v \,.\, \operatorname{ns} u \operatorname{ns}' u}{\operatorname{ns}^2 u-\operatorname{ns}^2 v},$$

and since $\operatorname{cs} u \operatorname{cs}' u = \operatorname{ns} u \operatorname{ns}' u$, the corresponding formula for $\operatorname{cs}(u+v)$ can be written

·707 $$\operatorname{cs} u \operatorname{cs} v \operatorname{cs}(u+v) = \frac{\operatorname{cs}^2 u \,.\, \operatorname{ns} v \operatorname{ns}' v-\operatorname{cs}^2 v \,.\, \operatorname{ns} u \operatorname{ns}' u}{\operatorname{ns}^2 u-\operatorname{ns}^2 v}.$$

Hence

$$\operatorname{ns} u \operatorname{ns} v \operatorname{ns}(u+v)-\operatorname{cs} u \operatorname{cs} v \operatorname{cs}(u+v) = \frac{\operatorname{ns} v \operatorname{ns}' v-\operatorname{ns} u \operatorname{ns}' u}{\operatorname{ns}^2 u-\operatorname{ns}^2 v},$$

implying, for each of the three functions ps u with a pole at the origin,

14·73$_1$ $$\operatorname{Ps}(u+v)-\operatorname{Ps} u-\operatorname{Ps} v = \operatorname{ns} u \operatorname{ns} v \operatorname{ns}(u+v)-\operatorname{cs} u \operatorname{cs} v \operatorname{cs}(u+v);$$

14·73$_2$. *If* $u+v+w = 0$, *then*

$$\operatorname{Ps} u+\operatorname{Ps} v+\operatorname{Ps} w = \operatorname{ns} u \operatorname{ns} v \operatorname{ns} w-\operatorname{cs} u \operatorname{cs} v \operatorname{cs} w.$$

The formulae ·71$_1$, ·72$_1$, ·73$_1$ are addition theorems for the integrating functions. Each of them can be expressed in terms of one function and its derivative; for example, we have

14·74. *If* $u+v+w = 0$, *then*

$$(\operatorname{Sq} u+\operatorname{Sq} v+\operatorname{Sq} w)^2 = \operatorname{Sq}' u \operatorname{Sq}' v \operatorname{Sq}' w.$$

But we have to remember that there is no algebraic relation between an integrating function and its derivative; the theorems are not *algebraic* addition theorems.

For the classical integrating function $E(u)$ and its companion $D(u)$ we have

14·75. *If* $u+v+w = 0$, *then*

$$E(u)+E(v)+E(w) = -c\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{sn}\,w,$$

$$D(u)+D(v)+D(w) = c'\,\mathrm{sc}\,u\,\mathrm{sc}\,v\,\mathrm{sc}\,w.$$

14·8. We need not appeal to explicit formulae for evidence that the function $\mathrm{Pq}\,u-2\,\mathrm{Pq}\,\frac{1}{2}u$ is doubly periodic, and by direct inspection of periods and poles we have

14·81 $$\mathrm{cs}\,u+\mathrm{ns}\,u+\mathrm{ds}\,u = \mathrm{Ps}\,u-2\,\mathrm{Ps}\,\tfrac{1}{2}u.$$

In a sense this result has no counterpart at the cardinal points K_c, K_n, K_d, for it is the form of the left-hand side that is attractive and $\mathrm{Pq}\,u-2\,\mathrm{Pq}\,\frac{1}{2}u$ has poles congruent with the origin, mod $2K_c$, $2K_n$, whether K_q is at the origin or not. The limitation is apparent otherwise. The differentiated form of ·81 is

14·82 $$\mathrm{ps}^2\,\tfrac{1}{2}u = (\mathrm{ps}\,u+\mathrm{rs}\,u)(\mathrm{ps}\,u+\mathrm{ts}\,u),$$

and the addition of $2K_q$ to u, which alters the function on the left, only rings changes of sign on the right.

Formulae for $\mathrm{Pq}\,u-2\,\mathrm{Pq}\,\frac{1}{2}u$ are obtainable in a variety of ways, of which perhaps the simplest is that just indicated, namely, the transformation and reintegration of ·82. Particular cases are

14·83 $$2E(\tfrac{1}{2}u)-E(u) = (\mathrm{ns}\,u-\mathrm{cs}\,u)(1-\mathrm{dn}\,u),$$

14·84 $$D(u)-2D(\tfrac{1}{2}u) = (\mathrm{ns}\,u-\mathrm{cs}\,u)(\mathrm{dc}\,u-1).$$

The general result can be written

14·85 $$2\,\mathrm{Pq}\,\tfrac{1}{2}u-\mathrm{Pq}\,u = (\mathrm{qs}\,u-\mathrm{rs}\,u)(\mathrm{qs}\,u-\mathrm{ts}\,u)/\lambda\,\mathrm{qs}\,u,$$

where λ is the leading coefficient at K_q of the square of the primitive function coperiodic with $\mathrm{pq}\,u$; this coefficient is given in Table XI 4.

XV

THE DEPENDENCE OF THE JACOBIAN FUNCTIONS AND QUARTERPERIODS ON THE PARAMETER

15·1. It is as doubly periodic functions of u that the Jacobian functions engage our attention, and we have thought of the parameter c as a constant determining a system of functions. In the transformations examined in Chapter XIII we have allowed discrete changes of the parameter, but we are now to recognize that c is in fact a second variable. The 'constants' of a Jacobian system are functions of c, and the 'functions' we have studied are functions not of one variable u but of two independent variables u, c.

The Jacobian functions are differentiable functions of c, and their derivatives can be written down with unexpected ease, by a process discovered by Hermite. If in the relation

·101 $$\int_x^\infty \frac{dx}{\sqrt{\{(x^2+1)(x^2+c')\}}} = u,$$

which is equivalent to $x = \operatorname{cs} u$, u is constant and c varies, then

·102 $$-\frac{\partial x/\partial c}{\sqrt{\{(x^2+1)(x^2+c')\}}} + \frac{1}{2}\int_x^\infty \frac{dx}{(x^2+c')\sqrt{\{(x^2+1)(x^2+c')\}}} = 0,$$

that is,

·103 $$\frac{\partial x}{\partial c} = \tfrac{1}{2}\operatorname{ns} u \operatorname{ds} u \int_0^u \operatorname{sd}^2 u\, du.$$

As in so many problems, isolated formulae are most readily found from first principles, but in compiling a complete set we utilize relations between the functions.

TABLE XV 1

The derivatives of the Jacobian functions with respect to the parameter

$\partial \operatorname{cs} u/\partial c$	$\partial \operatorname{ns} u/\partial c$	$\partial \operatorname{ds} u/\partial c$
$\frac{1}{2}\operatorname{ns} u \operatorname{ds} u \operatorname{Sd} u$	$\frac{1}{2}\operatorname{cs} u \operatorname{ds} u \operatorname{Sd} u$	$-\frac{1}{2}\operatorname{cs} u \operatorname{ns} u(\operatorname{Sc} u + \operatorname{Sn} u)$
$\partial \operatorname{sc} u/\partial c$	$\partial \operatorname{dc} u/\partial c$	$\partial \operatorname{nc} u/\partial c$
$-\frac{1}{2}\operatorname{nc} u \operatorname{dc} u \operatorname{Sd} u$	$-\frac{1}{2}\operatorname{sc} u \operatorname{nc} u \operatorname{Cn} u$	$-\frac{1}{2}\operatorname{sc} u \operatorname{dc} u \operatorname{Sd} u$
$\partial \operatorname{dn} u/\partial c$	$\partial \operatorname{sn} u/\partial c$	$\partial \operatorname{cn} u/\partial c$
$-\frac{1}{2}\operatorname{sn} u \operatorname{cn} u \operatorname{Nc} u$	$-\frac{1}{2}\operatorname{cn} u \operatorname{dn} u \operatorname{Sd} u$	$\frac{1}{2}\operatorname{sn} u \operatorname{dn} u \operatorname{Sd} u$
$\partial \operatorname{nd} u/\partial c$	$\partial \operatorname{cd} u/\partial c$	$\partial \operatorname{sd} u/\partial c$
$\frac{1}{2}\operatorname{sd} u \operatorname{cd} u \operatorname{Nc} u$	$\frac{1}{2}\operatorname{sd} u \operatorname{nd} u \operatorname{Cn} u$	$\frac{1}{2}\operatorname{cd} u \operatorname{nd} u(\operatorname{Sc} u + \operatorname{Sn} u)$

To proceed to higher derivatives of the Jacobian functions, we need the derivatives of the integrating functions, or at least of the five of these functions which occur in the above table. If we can evaluate one derivative, we can evaluate the others by means of the relations in 14·4, but except for the functions with a pole at the origin a direct method is shortest.

Actually the form of the results is clearest if the problem is generalized. Each of the formulae in Table XV I is of the form

·104 $$\frac{\partial\,\mathrm{pq}(u,c)}{\partial c} = \frac{\partial\,\mathrm{pq}(u,c)}{\partial u}\int_0^u f(u,c)\,du,$$

and this formula implies

·105 $$\frac{\partial}{\partial c}\int_0^u \mathrm{pq}^m(u,c)\,du = \mathrm{pq}^m(u,c)\int_0^u f(u,c)\,du - \int_0^u \mathrm{pq}^m(u,c)f(u,c)\,du,$$

provided that as $u \to 0$

·106 $$\mathrm{pq}^m(u,c)\int_0^u f(u,c)\,du \to 0,$$

a condition that is satisfied except for the functions with a pole at the origin.

We have for example

·107 $$\frac{\partial}{\partial c}\int_0^u \mathrm{sn}^m u\,du = -\tfrac{1}{2}\,\mathrm{sn}^m u\,\mathrm{Sd}\,u + \tfrac{1}{2}\int_0^u \mathrm{sn}^m u\,\mathrm{sd}^2 u\,du,$$

whence in particular

·108 $$\frac{\partial\,\mathrm{Sn}\,u}{\partial c} = -\tfrac{1}{2}\,\mathrm{sn}^2 u\,\mathrm{Sd}\,u + \tfrac{1}{2}\int_0^u \mathrm{sn}^2 u\,\mathrm{sd}^2 u\,du.$$

The last integrand is a function of the kind considered in 14·1; since $\mathrm{ns}^2 u - \mathrm{ds}^2 u = c$, we have

$$c\,\mathrm{sn}^2 u\,\mathrm{sd}^2 u = \mathrm{sd}^2 u - \mathrm{sn}^2 u,$$

$$c\int_0^u \mathrm{sn}^2 u\,\mathrm{sd}^2 u\,du = \mathrm{Sd}\,u - \mathrm{Sn}\,u,$$

whence

·109 $$c\frac{\partial\,\mathrm{Sn}\,u}{\partial c} = \tfrac{1}{2}(\mathrm{dn}^2 u\,\mathrm{Sd}\,u - \mathrm{Sn}\,u).$$

Applying ·109 to $14{\cdot}42_2$ we have

$$\frac{\partial\,\mathrm{Ns}\,u}{\partial c} = \tfrac{1}{2}(\mathrm{dn}^2 u\,\mathrm{Sd}\,u + \mathrm{Sn}\,u) - \frac{\partial}{\partial c}(\mathrm{cn}\,u\,\mathrm{ds}\,u)$$
$$= \tfrac{1}{2}\{\mathrm{Sn}\,u + \mathrm{cs}^2 u(\mathrm{Sc}\,u + \mathrm{Sn}\,u)\}$$
$$= \tfrac{1}{2}(\mathrm{cs}^2 u\,\mathrm{Sc}\,u + \mathrm{ns}^2 u\,\mathrm{Sn}\,u).$$

On account of the relations between the integrating functions their c-derivatives may be expressed in a variety of forms. One set of formulae is as follows:

TABLE XV2

The derivatives of the integrating functions with respect to the parameter

$2\partial\,\mathrm{Cs}\,u/\partial c$	$2\,\partial\,\mathrm{Ns}\,u/\partial c$	$2\partial\,\mathrm{Ds}\,u/\partial c$
$\mathrm{cs}^2 u\,\mathrm{Sc}\,u + \mathrm{ns}^2 u\,\mathrm{Sn}\,u$	$\mathrm{cs}^2 u\,\mathrm{Sc}\,u + \mathrm{ns}^2 u\,\mathrm{Sn}\,u$	$\mathrm{cs}^2 u\,\mathrm{Sc}\,u + \mathrm{ns}^2 u\,\mathrm{Sn}\,u - 2u$
$2c'\partial\,\mathrm{Sc}\,u/\partial c$	$2c'\partial\,\mathrm{Dc}\,u/\partial c$	$2c'\partial\,\mathrm{Nc}\,u/\partial c$
$\mathrm{Sc}\,u - \mathrm{dc}^2 u\,\mathrm{Sd}\,u$	$-\mathrm{Sc}\,u - \mathrm{dc}^2 u\,\mathrm{Sd}\,u$	$\mathrm{Sc}\,u - \mathrm{dc}^2 u\,\mathrm{Sd}\,u$
$2c\,\partial\,\mathrm{Dn}\,u/\partial c$	$2c\,\partial\,\mathrm{Sn}\,u/\partial c$	$2c\,\partial\,\mathrm{Cn}\,u/\partial c$
$-\mathrm{Sn}\,u - \mathrm{dn}^2 u\,\mathrm{Sd}\,u$	$-\mathrm{Sn}\,u + \mathrm{dn}^2 u\,\mathrm{Sd}\,u$	$\mathrm{Sn}\,u - \mathrm{dn}^2 u\,\mathrm{Sd}\,u$
$2c'\partial\,\mathrm{Nd}\,u/\partial c$	$2c\,\partial\,\mathrm{Cd}\,u/\partial c$	$2cc'\partial\,\mathrm{Sd}\,u/\partial c$
$\mathrm{Nd}\,u - \mathrm{cd}^2 u\,\mathrm{Nc}\,u$	$-\mathrm{Cd}\,u + \mathrm{nd}^2 u\,\mathrm{Cn}\,u$	$(c-c')\mathrm{Sd}\,u - c\,\mathrm{cd}^2 u\,\mathrm{Sc}\,u + c'\,\mathrm{nd}^2 u\,\mathrm{Sn}\,u$

Other expressions for the derivatives of $\mathrm{Dn}\,u$ and $\mathrm{Dc}\,u$ will presently be useful. Substituting from Table XIV4, we have

$$2cc'\partial\,\mathrm{Dn}\,u/\partial c = (\mathrm{dn}^2 u - c')D(u) - cu\,\mathrm{dn}^2 u,$$

that is,

$15{\cdot}11_1$ $$2c'\partial E(u)/\partial c = -c'u + \mathrm{cn}^2 u\{D(u) - cu\};$$

similarly,

$15{\cdot}11_2$ $$2c\,\partial D(u)/\partial c = cu - \mathrm{nc}^2 u\{E(u) - c'u\}.$$

In a more reciprocal form

$15{\cdot}12_1$ $$2c'\partial\{E(u) - c'u\}/\partial c = c'u + \mathrm{cn}^2 u\{D(u) - cu\},$$

$15{\cdot}12_2$ $$2c\,\partial\{D(u) - cu\}/\partial c = -cu - \mathrm{nc}^2 u\{E(u) - c'u\},$$

or briefly,

$15{\cdot}13_1$ $$2\,\partial(c'\,\mathrm{Nc}\,u)/\partial c = -u - \mathrm{nc}^2 u\,\mathrm{Cn}\,u,$$

$15{\cdot}13_2$ $$2\,\partial(c\,\mathrm{Cn}\,u)/\partial c = u + \mathrm{cn}^2 u\,\mathrm{Nc}\,u.$$

The duality in ·11, ·12, ·13 becomes exact if we replace the differentiations in $\cdot 11_2$, $\cdot 12_2$, $\cdot 13_2$ by differentiations with respect to c', thus changing the signs on the right-hand side.

In terms of Jacobian functions the amplitude $\mathrm{am}\,u$ is definable by the pair of equations

·110–·111 $\sin \mathrm{am}\, u = \mathrm{sn}\, u, \qquad \cos \mathrm{am}\, u = \mathrm{cn}\, u,$

implying $\mathrm{cn}\, u\, \partial \mathrm{am}\, u/\partial c = \partial \mathrm{sn}\, u/\partial c,$

whence

15·14 $\partial \mathrm{am}\, u/\partial c = -\tfrac{1}{2}\mathrm{dn}\, u\, \mathrm{Sd}\, u.$

The other functions introduced as auxiliaries in Table XIV2 can be differentiated with respect to c in the same way.

15·2. Evaluation of derivatives with respect to c reveals the approximate forms of functions near a value of c for which the Jacobian systems degenerate, that is, near $c = 0$ and near $c = 1$.

When $c = 0$, the integral relation equivalent to $x = \mathrm{cs}\, u$ becomes

·201 $$u = \int_x^\infty \frac{dx}{x^2+1}$$

and identifies $\mathrm{cs}\, u$ with $\cot u$; then $\mathrm{ns}\, u$ and $\mathrm{ds}\, u$ both reduce to $\csc u$; each of the functions $\mathrm{dn}\, u$, $\mathrm{nd}\, u$ becomes constant, consistently with having c for a factor of its derivative. The integrating functions are found by elementary integration.

Table XV3

$\mathrm{cs}(u,0) = \cot u$	$\mathrm{ns}(u,0) = \csc u$	$\mathrm{ds}(u,0) = \csc u$
$\mathrm{sc}(u,0) = \tan u$	$\mathrm{dc}(u,0) = \sec u$	$\mathrm{nc}(u,0) = \sec u$
$\mathrm{dn}(u,0) = 1$	$\mathrm{sn}(u,0) = \sin u$	$\mathrm{cn}(u,0) = \cos u$
$\mathrm{nd}(u,0) = 1$	$\mathrm{cd}(u,0) = \cos u$	$\mathrm{sd}(u,0) = \sin u$
$\mathrm{Cs}(u,0) = -\cot u - u$	$\mathrm{Ns}(u,0) = -\cot u$	$\mathrm{Ds}(u,0) = -\cot u$
$\mathrm{Sc}(u,0) = \tan u - u$	$\mathrm{Dc}(u,0) = \tan u$	$\mathrm{Nc}(u,0) = \tan u$
$\mathrm{Dn}(u,0) = u$	$\mathrm{Sn}(u,0) = \tfrac{1}{2}(u - \sin u \cos u)$	$\mathrm{Cn}(u,0) = \tfrac{1}{2}(u + \sin u \cos u)$
$\mathrm{Nd}(u,0) = u$	$\mathrm{Cd}(u,0) = \tfrac{1}{2}(u + \sin u \cos u)$	$\mathrm{Sd}(u,0) = \tfrac{1}{2}(u - \sin u \cos u)$

From this table, with XV1, we have

15·21. *To the first order in c,*

·21₁₋₆ $$\mathrm{cs}(u,c) = \cot u + \tfrac{1}{4}c \csc^2 u(u - \sin u \cos u)$$
$$\mathrm{ns}(u,c) = \csc u + \tfrac{1}{4}c \cot u \csc u(u - \sin u \cos u)$$
$$\mathrm{ds}(u,c) = \csc u - \tfrac{1}{4}c \cot u \csc u(2\tan u - \sin u \cos u - u)$$

$$\mathrm{sc}(u,c) = \tan u - \tfrac{1}{4}c \sec^2 u(u - \sin u \cos u)$$
$$\mathrm{dc}(u,c) = \sec u - \tfrac{1}{4}c \tan u \sec u(u + \sin u \cos u)$$
$$\mathrm{nc}(u,c) = \sec u - \tfrac{1}{4}c \tan u \sec u(u - \sin u \cos u)$$

·21$_{7\text{-}12}$ $\operatorname{dn}(u,c) = 1-\frac{1}{2}c\sin^2 u$

$\operatorname{sn}(u,c) = \sin u-\frac{1}{4}c\cos u(u-\sin u\cos u)$

$\operatorname{cn}(u,c) = \cos u+\frac{1}{4}c\sin u(u-\sin u\cos u)$

$\operatorname{nd}(u,c) = 1+\frac{1}{2}c\sin^2 u$

$\operatorname{cd}(u,c) = \cos u+\frac{1}{4}c\sin u(u+\sin u\cos u)$

$\operatorname{sd}(u,c) = \sin u+\frac{1}{4}c\cos u(2\tan u-\sin u\cos u-u).$

It will be noticed that no two functions which coincide when $c = 0$ remain indistinguishable to the first order in c.

The amplitude $\operatorname{am} u$ is not a singlevalued function of u, but for the branch which reduces to u when $c = 0$ we have to the first order in c, from ·14,

15·22 $$\operatorname{am} u = u-\tfrac{1}{4}c(u-\sin u\cos u).$$

When $c = 0$, the value of K_c is $\frac{1}{2}\pi$, for the relation $\sin u = 1$ is not satisfied when $u = -\frac{1}{2}\pi$; other multiples of $\frac{1}{2}\pi$ are not primitive quarterperiods of the set of circular functions. The value of K_n, and therefore of every primitive quarterperiod except $\frac{1}{2}\pi$, is infinite. The signature plays no part, for it does not enter into the leading coefficients at K_s and K_c, the two cardinal points which remain accessible.

When $c = 1$, the relation equivalent to $x = \operatorname{sn} u$ is

·202 $$u = \int_0^x \frac{dx}{1-x^2},$$

that is, $x = \tanh u$, and $\operatorname{cn} u$ and $\operatorname{dn} u$ both reduce to $\operatorname{sech} u$; the functions which degenerate to constants are $\operatorname{cd} u$ and $\operatorname{dc} u$.

TABLE XV4

$\operatorname{cs}(u,1) = \operatorname{csch} u$	$\operatorname{ns}(u,1) = \coth u$	$\operatorname{ds}(u,1) = \operatorname{csch} u$
$\operatorname{sc}(u,1) = \sinh u$	$\operatorname{dc}(u,1) = 1$	$\operatorname{nc}(u,1) = \cosh u$
$\operatorname{dn}(u,1) = \operatorname{sech} u$	$\operatorname{sn}(u,1) = \tanh u$	$\operatorname{cn}(u,1) = \operatorname{sech} u$
$\operatorname{nd}(u,1) = \cosh u$	$\operatorname{cd}(u,1) = 1$	$\operatorname{sd}(u,1) = \sinh u$
$\operatorname{Cs}(u,1) = -\coth u$	$\operatorname{Ns}(u,1) = -\coth u+u$	$\operatorname{Ds}(u,1) = -\coth u$
$\operatorname{Sc}(u,1) = \frac{1}{2}(\sinh u\cosh u-u)$	$\operatorname{Dc}(u,1) = u$	$\operatorname{Nc}(u,1) = \frac{1}{2}(\sinh u\cosh u+u)$
$\operatorname{Dn}(u,1) = \tanh u$	$\operatorname{Sn}(u,1) = u-\tanh u$	$\operatorname{Cn}(u,1) = \tanh u$
$\operatorname{Nd}(u,1) = \frac{1}{2}(\sinh u\cosh u+u)$	$\operatorname{Cd}(u,1) = u$	$\operatorname{Sd}(u,1) = \frac{1}{2}(\sinh u\cosh u-u)$

Since c' is $1-c$, not $c-1$, or, to put it differently, since derivatives with respect to c' are the negatives of the c-derivatives tabulated in XV1, we have

15·23. *To the first order in c',*

$\cdot 23_{1-12}$

$$\mathrm{cs}(u,c) = \operatorname{csch} u - \tfrac{1}{4}c' \coth u \operatorname{csch} u(\sinh u \cosh u - u)$$
$$\mathrm{ns}(u,c) = \coth u - \tfrac{1}{4}c' \operatorname{csch}^2 u(\sinh u \cosh u - u)$$
$$\mathrm{ds}(u,c) = \operatorname{csch} u + \tfrac{1}{4}c' \coth u \operatorname{csch} u(\sinh u \cosh u + u - 2\tanh u)$$

$$\mathrm{sc}(u,c) = \sinh u + \tfrac{1}{4}c' \cosh u(\sinh u \cosh u - u)$$
$$\mathrm{dc}(u,c) = 1 + \tfrac{1}{2}c' \sinh^2 u$$
$$\mathrm{nc}(u,c) = \cosh u + \tfrac{1}{4}c' \sinh u(\sinh u \cosh u - u)$$

$$\mathrm{dn}(u,c) = \operatorname{sech} u + \tfrac{1}{4}c' \tanh u \operatorname{sech} u(\sinh u \cosh u + u)$$
$$\mathrm{sn}(u,c) = \tanh u + \tfrac{1}{4}c' \operatorname{sech}^2 u(\sinh u \cosh u - u)$$
$$\mathrm{cn}(u,c) = \operatorname{sech} u - \tfrac{1}{4}c' \tanh u \operatorname{sech} u(\sinh u \cosh u - u)$$

$$\mathrm{nd}(u,c) = \cosh u - \tfrac{1}{4}c' \sinh u(\sinh u \cosh u + u)$$
$$\mathrm{cd}(u,c) = 1 - \tfrac{1}{2}c' \sinh^2 u$$
$$\mathrm{sd}(u,c) = \sinh u - \tfrac{1}{4}c' \cosh u(\sinh u \cosh u + u - 2\tanh u).$$

There is no finite value for K_c. The conditions to be satisfied by K_n may be taken as $\mathrm{sc}\,K_n = v$, $\mathrm{dc}\,K_n = k$; since $\mathrm{dc}(u,1)$ is a constant, the second of these is an identity, while the first gives $K_n = \tfrac{1}{2}\pi v$. The signature remains in the formulae, and K_n is ambiguous until the signature is prescribed.

The equations to be satisfied by the amplitude $\mathrm{am}(u,1)$ are

$\cdot 203$–$\cdot 204$ $$\sin \mathrm{am}(u,1) = \tanh u, \qquad \cos \mathrm{am}(u,1) = \operatorname{sech} u,$$

and these are the equations which define the gudermannian function $\mathrm{gd}\,u$, the function which links circular and hyperbolic functions:

15·24 $$\mathrm{am}(u,1) = \mathrm{gd}\,u.$$

To the first order in c',

15·25 $$\mathrm{am}\,u = \mathrm{gd}\,u + \tfrac{1}{4}c' \operatorname{sech} u(\sinh u \cosh u - u).$$

15·3. If u is not independent of c but is a function of c, then

·301 $$\frac{d\,\mathrm{pq}(u,c)}{dc} = \frac{\partial\,\mathrm{pq}(u,c)}{\partial u}\frac{du}{dc} + \frac{\partial\,\mathrm{pq}(u,c)}{\partial c}.$$

The partial derivative with respect to u is the derivative previously denoted by $\mathrm{pq}'u$, that with respect to c is the derivative investigated in ·1. Each partial derivative has the product of the two functions copolar with $\mathrm{pq}\,u$ as a factor, and we can combine Tables XI 5 and XV 1 into a single table showing the function by which this product is multiplied to give the complete derivative $d\,\mathrm{pq}(u,c)/dc$. For brevity du/dc is denoted in this table by $\dot{u}$.

TABLE XV5

cs u	$-\dot{u}+\frac{1}{2}\,\mathrm{Sd}\,u$	ns u	$-\dot{u}+\frac{1}{2}\,\mathrm{Sd}\,u$	ds u	$-\dot{u}-\frac{1}{2}(\mathrm{Sc}\,u+\mathrm{Sn}\,u)$
sc u	$\dot{u}-\frac{1}{2}\,\mathrm{Sd}\,u$	dc u	$c'\dot{u}-\frac{1}{2}\,\mathrm{Cn}\,u$	nc u	$\dot{u}-\frac{1}{2}\,\mathrm{Sd}\,u$
dn u	$-c\dot{u}-\frac{1}{2}\,\mathrm{Nc}\,u$	sn u	$\dot{u}-\frac{1}{2}\,\mathrm{Sd}\,u$	cn u	$-\dot{u}+\frac{1}{2}\,\mathrm{Sd}\,u$
nd u	$c\dot{u}+\frac{1}{2}\,\mathrm{Nc}\,u$	cd u	$-c'\dot{u}+\frac{1}{2}\,\mathrm{Cn}\,u$	sd u	$\dot{u}+\frac{1}{2}(\mathrm{Sc}\,u+\mathrm{Sn}\,u)$

In Table XV5 there appear only four distinct factors, namely,

$$\dot{u}-\tfrac{1}{2}\,\mathrm{Sd}\,u,\qquad c\dot{u}+\tfrac{1}{2}\,\mathrm{Nc}\,u,\qquad c'\dot{u}-\tfrac{1}{2}\,\mathrm{Cn}\,u,\qquad \dot{u}+\tfrac{1}{2}(\mathrm{Sc}\,u+\mathrm{Sn}\,u).$$

An interpretation of the factors shows that even these four are intimately related although they are functionally distinct. The function pq u is zero identically when $u = K_p$, and neither of the copolar functions is zero there. Hence $d\,\mathrm{pq}(K_p, c)/dc$ is zero in virtue of the factor given in the table, and this factor, equated to zero, gives explicitly the value of dK_p/dc. With K_p at the origin, we have a mere identity, since each of the integrating functions vanishes with u. But from the other entries in the table we are to expect three expressions for dK_c/dc, three for dK_n/dc, and three for dK_d/dc. Two expressions for the same derivative must be ultimately equivalent if they are not identical, and the sum of values of the three derivatives is zero. Explicitly

15·31. *The derivatives of the Jacobian quarterperiods K_c, K_n with respect to the parameter c are given by*

$$dK_c/dc = \tfrac{1}{2}\,\mathrm{Sd}\,K_c,\qquad dK_n/dc = \tfrac{1}{2}\,\mathrm{Sd}\,K_n.$$

In terms of the functional values E_c, D_n, we have from Tables XIV3,4

15·32$_{1-2}$ $\quad 2cc'dK_c/dc = E_c - c'K_c,\qquad 2cc'dK_n/dc = -D_n + cK_n.$

Since E_c is by definition $E(K_c, c)$, that is, $\mathrm{Dn}(K_c, c)$, to find dE_c/dc we have only to substitute K_c for u in $\partial E(u)/\partial u$, which is $\mathrm{dn}^2 u$, and in $\partial E(u)/\partial c$, which is given in several forms in ·1. Since K_c is a simple pole of Dc u and a double zero of $\mathrm{cn}^2 u$, we have from ·11$_1$,

$$[\partial E(u)/\partial c]_{u=K_c} = -\tfrac{1}{2}K_c,$$

and therefore, since $\mathrm{dn}^2 K_c = c'$,

·302 $$\frac{dE_c}{dc} = c'\frac{dK_c}{dc} - \tfrac{1}{2}K_c,$$

whence from ·32$_1$

15·33$_1$ $$2c\,dE_c/dc = E_c - K_c.$$

Similarly

15·33$_2$ $$2c'dD_n/dc = -D_n + K_n.$$

But in view of the forms of the expressions for dK_c/dc and dK_n/dc, we may use $\cdot 12_{1-2}$. Since

$$\cdot 303 \qquad \partial\{E(u)-c'u\}/\partial u = \mathrm{dn}^2 u - c' = c\,\mathrm{cn}^2 u,$$

$$\cdot 304 \qquad \partial\{D(u)-cu\}/\partial u = \mathrm{dc}^2 u - c = c'\,\mathrm{nc}^2 u,$$

it follows that

$$\frac{d}{dc}\{E(K_c)-c'K_c\} = \left[\frac{\partial}{\partial c}\{E(u)-c'u\}\right]_{u=K_c}$$

and that
$$\frac{d}{dc}\{D(K_n)-cK_n\} = \left[\frac{\partial}{\partial c}\{D(u)-cu\}\right]_{u=K_n};$$

thus $\cdot 12_{1-2}$ give at once

$$15\cdot 34_{1-2} \qquad 2d(E_c-c'K_c)/dc = K_c, \qquad 2d(D_n-cK_n)/dc = -K_n.$$

We do not alter the form of the relations $\cdot 32_2$, $\cdot 34_2$ if we remove the signature. In the notation of 14·601–14·604,

$$15\cdot 35_{1-2} \qquad 2cc'dK/dc = E-c'K, \qquad 2cc'dK'/dc = -E'+cK',$$

$$15\cdot 36_{1-2} \qquad 2d(E-c'K)/dc = K, \qquad 2d(E'-cK')/dc = -K'.$$

15·4. Combining ·32 and ·34, we see that

15·41. *As functions of the parameter c, the Jacobian quarterperiods K_c, K_n are solutions of the differential equation*

$$\frac{d}{dc}\left\{cc'\frac{dx}{dc}\right\} = \tfrac{1}{4}x.$$

From the form of this equation it is satisfied also by $-K_c-K_n$ and K_n/v, that is, by K_d and K'.

The solution of the equation, for sufficiently small values of c, is readily found. The equation can be written

$$\cdot 401 \qquad c'\frac{d}{dc}\left\{c\frac{dx}{dc}\right\} - c\frac{dx}{dc} - \tfrac{1}{4}x = 0;$$

that is, if ϑ denotes the differential operator

$$c\frac{d}{dc},$$

the equation is

$$\cdot 402 \qquad \{(1-c)\vartheta^2 - c\vartheta - \tfrac{1}{4}c\}x = 0,$$

or, since $\vartheta^2 x = 0$ implies $x = A+B\log c$,

$$\cdot 403 \qquad \{\vartheta^2 - c(\vartheta+\tfrac{1}{2})^2\}x = \vartheta^2(A+B\log c),$$

where A, B are constants. Hence

·404 $$x = (1-\Theta)^{-1}(A+B\log c)$$
$$= (1+\Theta+\Theta^2+\Theta^3+...)(A+B\log c),$$

where Θ is the operator defined by

·405 $$\Theta = \frac{1}{\vartheta^2}\{c(\vartheta+\tfrac{1}{2})^2\} = c\left(\frac{\vartheta+\frac{1}{2}}{\vartheta+1}\right)^2;$$

applying repeatedly the fundamental property of ϑ, namely,

$$F(\vartheta)\{cV\} = c\,F(\vartheta+1)V,$$

we see that, for the operator Θ,

$$\Theta^n = c^n\alpha_n(\vartheta),$$

where†

·406 $$\alpha_n(\vartheta) = \left\{\frac{(2\vartheta+2n-1)(2\vartheta+2n-3)...(2\vartheta+1)}{(2\vartheta+2n)(2\vartheta+2n-2)...(2\vartheta+2)}\right\}^2.$$

Substituting

·407 $$\alpha_n(\vartheta).1 = [\alpha_n(\vartheta)]_{\vartheta=0},$$

·408 $$\alpha_n(\vartheta).\log c = [\alpha_n(\vartheta)]_{\vartheta=0}\log c+[d\alpha_n(\vartheta)/d\vartheta]_{\vartheta=0},$$

we have

·409–·410 $$\Theta^n 1 = \alpha_n c^n, \qquad \Theta^n \log c = \alpha_n c^n(\log c+4\beta_n),$$

where

·411 $$\alpha_n = \left(\frac{1.3.....(2n-1)}{2.4.....2n}\right)^2,$$

·412 $$\beta_n = \frac{1}{1.2}+\frac{1}{3.4}+...+\frac{1}{(2n-1)2n},$$

and therefore

15·42 $$x = (A+B\log c)(1+\alpha_1 c+\alpha_2 c^2+...)+4B(\alpha_1\beta_1 c+\alpha_2\beta_2 c^2+...).$$

Both the power series in this solution have radius of convergence unity.

From the explicit solution ·42 it follows that any solution for which $B \neq 0$ has a logarithmic infinity at the origin, and therefore that every solution which is finite at the origin is a multiple of a single solution. Since $K_c = \frac{1}{2}\pi$ when $c = 0$, and K_c is finite unless $c = 1$, we are tempted to infer that

$$K_c = \tfrac{1}{2}\pi(1+\alpha_1 c+\alpha_2 c^2+...),$$

but the result is manifestly absurd, for c determines the lattice, leaving

† From the operational point of view it is misleading to write the factors in ascending order, or even to exhibit $\alpha_n(\vartheta)$ as a square.

the choice of a primitive pair of quarterperiods still open, and no restriction on this choice is implicit in the work leading to the differential equation.

The fallacy is in forgetting that although, as we have seen in ·2, $\frac{1}{2}\pi$ is the only value for K_c when $c = 0$, a value of K_c which is legitimate when $c \neq 0$ need not tend to $\frac{1}{2}\pi$ as $c \to 0$. In fact, as we have seen in 11·61, if α, β is one pair of values of K_c, K_n, the general pair of values is given by

$$K_c = (4m_1+1)\alpha+2n_1\beta, \qquad K_n = 2m_2\alpha+(2n_2+1)\beta$$

with the condition

$$(4m_1+1)(2n_2+1)-4n_1m_2 = \pm 1;$$

if $n_1 = 0$, then $4m_1+1$ is a factor of ± 1 and therefore $m_1 = 0$ and $K_c = \alpha$. Hence if $\alpha \to \frac{1}{2}\pi$ and $\beta \to \infty$, one and only one value of K_c has a finite limit; all the other possible values of K_c tend to infinity and are outside the discussion of the periodicity of the limiting function.

The form of the solution of the differential equation is now intelligible: writing K for K_c,

15·43. *Either* $\qquad K = \frac{1}{2}\pi(1+\alpha_1 c+\alpha_2 c^2+\ldots)$

for $|c| < 1$, *with* α_n *defined by* ·411, *or* K *has a logarithmic infinity at* $c = 0$.

The differential equation in ·41 is unaltered if c and c' are interchanged, and therefore, with the same values of α_n, β_n as before, the general solution is expressible as

15·44 $$x = (A'+B'\log c')(1+\alpha_1 c'+\alpha_2 c'^2+\ldots)+ \\ +4B'(\alpha_1\beta_1 c'+\alpha_2\beta_2 c'^2+\ldots)$$

inside the circle $|c'| = 1$. Also if $K_n = vK'$,

15·45. *Either* $\qquad K' = \frac{1}{2}\pi(1+\alpha_1 c'+\alpha_2 c'^2+\ldots)$

for $|c'| < 1$, *or* K' *has a logarithmic infinity at* $c = 1$.

In any event, K' has a logarithmic singularity at $c = 0$ and K has a logarithmic singularity at $c' = 0$.

Throughout our work one set of quarterperiods has been regarded as essentially equivalent to another. In a discussion of functions with real constants the distinction between real and imaginary leads to a special choice of quarterperiods, and in 6·8 and elsewhere we have used special paths of integration for technical convenience, but the emphasis has been on the view that intrinsically one basis is no different qualitatively from another. We now see that if this is true of K_c, K_n as a basis for a system of Jacobian functions, it is far from true of K_c and K_n as individual functions of c. Within the unit circle round $c = 0$, one value of K_c does behave quite differently from every other value, and within the unit circle round $c' = 0$, the same is true of one value, or rather, since there is an ambiguous sign, of two values, of K_n. We must therefore devote some attention to these special values of K_c and K_n, defined in the first place inside the two circles. We denote the two functions by X_c, X_n, writing

·413 $$X = \tfrac{1}{2}\pi(1+\alpha_1 c+\alpha_2 c^2+\ldots),$$

·414 $$X' = \tfrac{1}{2}\pi(1+\alpha_1 c'+\alpha_2 c'^2+\ldots),$$

·415–·416 $$X_c = X, \qquad X_n = \iota X',$$

where $\iota^2 = -1$ and as in ·411

$$\alpha_n = \left(\frac{1.3.\ldots.(2n-1)}{2.4.\ldots.2n}\right)^2.$$

For any value of c such that $|c| < 1$, the Jacobian system with parameter c has a basis in which the first member is X_c; for any value of c such that $|c'| < 1$, the system has a basis in which the second member is X_n. The two regions $|c| < 1$, $|c'| < 1$ have a common part, in the shape of a lune, but we can not say without investigation whether or not when c is in this lune X_c and X_n can be associated to form a basis of the system.

Let us return to expressions for K_c, K_n as integrals. The theorems 11·84, 11·85 do not specify paths for the integrals given, and therefore do not enable us to recognize the associations of values that are possible. It is through the general theorems of Chapter VI that paths are made precise: for the function obtained by inverting the integral

$$\int_w^\infty \frac{dw}{\sqrt{\{(w^2-B)(w^2-C)\}}},$$

a possible pair of quarterperiods is provided by

·417–·418

$$\beta = \frac{1}{2}\int_B^\infty \frac{dW}{\sqrt{\{W(W-B)(W-C)\}}}, \qquad \gamma = \frac{1}{2}\int_C^\infty \frac{dW}{\sqrt{\{W(W-B)(W-C)\}}},$$

where the paths of integration are any half-lines. To apply this result we use the function $\mathrm{ds}\,u$: since $\mathrm{ds}^2 K_c = c'$ and $\mathrm{ds}^2 K_n = -c$, the Jacobian system with parameter c has a basis defined by

·419–·420 $$K_c = \frac{1}{2}\int_{c'}^{\infty} \frac{dt}{\sqrt{\{t(t-c')(t+c)\}}}, \qquad K_n = \frac{1}{2}\int_{-c}^{\infty} \frac{dt}{\sqrt{\{t(t-c')(t+c)\}}}$$

with rectilinear paths, or, writing $t+c = u$ and then restoring the symbol, by

·421–·422 $$K_c = \frac{1}{2}\int_{1}^{\infty} \frac{dt}{\sqrt{\{t(t-1)(t-c)\}}}, \qquad K_n = \frac{1}{2}\int_{0}^{\infty} \frac{dt}{\sqrt{\{t(t-1)(t-c)\}}},$$

still with rectilinear paths.

Provided that c is not real and greater than 1, we can take for the path in K_c the real axis beyond $t = 1$, and the substitution $t = 1/u$ then replaces this path by the segment of the real axis between 0 and 1:

·423 $$K_c = \frac{1}{2}\int_{0}^{1} \frac{dt}{\sqrt{\{t(1-t)(1-ct)\}}}.$$

But this integral remains finite as $c \to 0$. Hence K_c, defined by ·423 or ·421, is the particular quarterperiod X_c; that is, with the assigned path, and with the appropriate radical,

·424 $$X = \frac{1}{2}\int_{0}^{1} \frac{dt}{\sqrt{\{t(1-t)(1-ct)\}}}.$$

There is no difficulty in verifying this conclusion: the expansion of $1/\sqrt{(1-ct)}$ is a binomial expansion, and the integral

$$\int_{0}^{1} \frac{t^n\,dt}{\sqrt{\{t(1-t)\}}}$$

is elementary.

According to the definitions ·413–·414, X' is the same function of c' as X is of c; it follows, since the path of integration in ·424 is independent of c, that with the same path, and the same determination of the radical,

·425 $$X' = \frac{1}{2}\int_{0}^{1} \frac{dt}{\sqrt{\{t(1-t)(1-c't)\}}}.$$

If the path of integration in K_n, in ·422, is to lie along the real axis, then since the point $t = 1$ must not be in the path, the path is the negative half of the line and c is assumed not to be real and negative.

To replace the path by the segment between 0 and 1 we substitute $(u-1)/u$ for t, and since $t = c$ corresponds to $u = 1/c'$, we have

$$K_n = \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{\{-t(1-t)(1-c't)\}}} = \pm\frac{i}{2}\int_0^1 \frac{dt}{\sqrt{\{t(1-t)(1-c't)\}}},$$

in agreement with ·425, since the integral in K_n remains finite as $c' \to 0$.

We can now use ·424 and ·425 instead of ·413 and ·414 as definitions of the functions X, X'. Two advantages appear at once. Firstly, the range of values of c for which the functions are defined is far less restricted: instead of being confined for each function to the interior of a circle, the point c is subject for that function only to the condition that there is one half-line on which it must not lie; the common domain of existence of the two functions is not the lune common to two circles, but the whole plane except the parts of the real axis outside the segment from 0 to 1 which is the path of integration. Secondly, since the association of the integrals in one basis was assured from the start,

15·46. *If c has any value other than a real value greater than* 1 *or less than* 0, *the functions X, $\pm iX'$ together constitute bases for the Jacobian functions of which c is the parameter; the first of these functions tends to $\frac{1}{2}\pi$ as $c \to 0$, the second to $\pm\frac{1}{2}\pi i$ as $c' \to 0$.*

The bearing of the restrictions imposed on c on the character of the integrals is seen in another transformation of ·421 and ·422. The substitution

$$t-1 = \frac{u-c'}{1-u}$$

replaces a path from $t = 1$ to $t = \infty$ by a path from $u = c'$ to $u = 1$; to $u = \infty$, 0 correspond $t = 0$, c. If t is real, u is on the line through $u = c'$ and $u = 1$. Thus

·426 $$X_c = \frac{1}{2}\int_{c'}^1 \frac{dt}{\sqrt{\{t(1-t)(t-c')\}}},$$

where the path is a rectilinear segment, and this segment must not contain the origin if the integral is to be unambiguous; the radius from 1 to c' does not contain the origin unless c' is real and negative, that is, unless c is real and greater than 1. To express the matter graphically, the radius from 1 to c' sweeps the plane, rotating round the fixed point 1, and this radius is unlimited in every direction except the direction in which it encounters the origin. Similarly, since t is real and negative along the path of the integral in ·422, the substitution

$$-t = \frac{u-c}{1-u}$$

converts the path into the rectilinear segment from c to 1, and since the integral becomes

$$\int_c^1 \frac{dt}{\sqrt{\{-t(1-t)(t-c)\}}},$$

it is now c that must not be real and negative.

The relation between the formulae ·424, ·425 and the formulae ·413, ·414 may be expressed in another way. The series define the functions X_c, X_n only inside certain circles, but these circles are not natural boundaries and the functions can be continued analytically across them. The only singularities of the analytic function of which the series in ·413 is one element are $c = 1$ and $c = \infty$, and if the c plane has a simple cut from one to the other of these points, the continued function is a singlevalued function analytic everywhere in the slit plane; if the cut is along the positive half of the real axis, this is the function given by the integral formula ·424, since the two functions coincide throughout the domain of existence of the series. Similarly, if the plane is cut along the whole of the negative half of the real axis, ·425 represents the continuation of ·414 throughout this slit plane.

From the point of view of continuation, the cuts are arbitrary except for their endpoints. We could for example slit the plane along the positive half of the imaginary axis and continue the series ·414 into the second quadrant across the negative half of the real axis; the function so found would be different in the second quadrant from the function defined by the integral ·425, but it would be equally valid as a standard solution of the differential equation in ·41, finite near $c = 1$. But since the two cuts which are essential if the paths of the integrals are to be rectilinear are also adequate to the purpose of defining the continuations of the series, to utilize these cuts for a double purpose is a natural simplification.

As solutions of the differential equation, the functions X, X' are rendered specific, the one by its form near $c = 0$, the other by its form near $c = 1$. To be in a position to express an arbitrary solution in terms of these two functions, we must, so to speak, reduce the functions to a common origin. We must find the values of the constants A, B

in ·42 if this solution is X', or the values of the constants A', B' in ·44 if this solution is X.

The infinity of the integral

$$\int_c^1 \frac{dt}{\sqrt{\{t(1-t)(t-c)\}}}$$

at $c = 0$ arises from the coalescence of the two factors t, $t-c$ in the radical, and is not modified in character by the presence of the factor $1-t$. Omitting this inoperative factor we have an integral whose infinity must be substantially the same as that of $2X'$, and this integral is elementary:

$$\begin{aligned}\int_c^1 \frac{dt}{\sqrt{\{t(t-c)\}}} &= \left[2\log\{\sqrt{t}+\sqrt{(t-c)}\}\right]_c^1 \\ &= \log\frac{(1+k')^2}{c} \\ &= \log\frac{4}{c}+2\log\left\{1-\frac{c}{2(1+k')}\right\}.\end{aligned}$$

Thus as $c \to 0$,

·427 $$\int_c^1 \frac{dt}{\sqrt{\{t(t-c)\}}}-\log\frac{4}{c} \to 0,$$

where the logarithm, which is singlevalued because the plane is cut along the negative real axis, and is real for real values of c between 0 and 1, has in any case an angle between $-\pi$ and π. Also

$$\begin{aligned}\sqrt{(1-c)}\int_c^1 \frac{dt}{\sqrt{\{t(1-t)(t-c)\}}}-\int_c^1 \frac{dt}{\sqrt{\{t(t-c)\}}} &= \int_c^1 \sqrt{\frac{t-c}{t(1-t)}}\cdot\frac{dt}{\sqrt{(1-c)}+\sqrt{(1-t)}} \\ &\to \int_0^1 \frac{dt}{\sqrt{(1-t)}\{1+\sqrt{(1-t)}\}},\end{aligned}$$

and since the value of the last integral is $\log 4$,

·428 $$X'-\tfrac{1}{2}\log\frac{16}{c} \to 0.$$

Hence ·42 represents X' if

$$A+B\log c \equiv \tfrac{1}{2}\log(16/c),$$

that is, if

·429–·430 $$A = \log 4 \qquad B = -\tfrac{1}{2}.$$

15·47. *If* $|c| < 1$ *and* c *is not real and negative, then*

$$X' = (X/\pi)\log(16/c) - 2(\alpha_1\beta_1 c + \alpha_2\beta_2 c^2 + \ldots),$$

where the logarithm has its principal value and α_n, β_n *have the values given in* ·411, ·412.

The same analysis identifies X near $c = 1$, for the integral involved differs only by the substitution of c' for c:

15·48. *If* $|c'| < 1$ *and* c' *is not real and negative, then*

$$X = (X'/\pi)\log(16/c') - 2(\alpha_1\beta_1 c' + \alpha_2\beta_2 c'^2 + \ldots),$$

where the logarithm has its principal value and α_n, β_n *have the same values as in* ·47.

The relation between ·47, ·48 and the simpler theorems ·43, ·45 may be expressed differently. Without any attention to the source of the differential equation

$$\frac{d}{dc}\left\{cc'\frac{dx}{dc}\right\} = \tfrac{1}{4}x,$$

where $c + c' = 1$, we find that the general solution of the equation is expressible for $|c| < 1$ in the form

$$(A + B\log c)(1 + \alpha_1 c + \alpha_2 c^2 + \ldots) + 4B(\alpha_1\beta_1 c + \alpha_2\beta_2 c^2 + \ldots)$$

and for $|c'| < 1$ in the form

$$(A' + B'\log c')(1 + \alpha_1 c' + \alpha_2 c'^2 + \ldots) + 4B'(\alpha_1\beta_1 c' + \alpha_2\beta_2 c'^2 + \ldots),$$

where α_n, β_n are given by ·411, ·412, and A, B, A', B' are constants of integration. These local investigations give us no means of identifying in one neighbourhood the integral determined by a particular pair of constants in the other neighbourhood; general theory tells us only that there must be coefficients λ_1, μ_1, λ_2, μ_2 such that the integral determined for $|c| < 1$ by A, B coincides with the integral determined for $|c'| < 1$ by A', B' throughout the lune common to the two circles of convergence if and only if

$$A' = \lambda_1 A + \mu_1 B, \qquad B' = \lambda_2 A + \mu_2 B.$$

What we can now do is to evaluate these coefficients: to $A = \tfrac{1}{2}\pi$, $B = 0$ correspond $A' = \log 4$, $B' = -\tfrac{1}{2}$, and to $A' = \tfrac{1}{2}\pi$, $B' = 0$ correspond $A = \log 4$, $B = -\tfrac{1}{2}$. The relation between the pairs of constants A, B and A', B' is symmetrical, and

15·49 $$\lambda_1 = \frac{\log 16}{\pi}, \quad \mu_1 = \frac{(\log 16)^2 - \pi^2}{\pi}, \quad \lambda_2 = -\frac{1}{\pi}, \quad \mu_2 = -\frac{\log 16}{\pi}.$$

The differential equation puts Legendre's relation in a new light. If x_1, x_2 are any two solutions of the equation

$$\frac{d}{dc}\left\{cc'\frac{dx}{dc}\right\} = \tfrac{1}{4}x,$$

then
$$\frac{d}{dc}\left\{x_1 . cc'\frac{dx_2}{dc} - x_2 . cc'\frac{dx_1}{dc}\right\} = 0.$$

Hence, taking X, X' for the solutions,

$$cc'\left(X\frac{dX'}{dc} - X'\frac{dX}{dc}\right)$$

is constant. Now let $c \to 0$. Then $c' \to 1$, $X \to \frac{1}{2}\pi$, and because the infinity of X' is logarithmic, $cX' \to 0$; also dX/dc has no singularity at $c = 0$, and from ·47, $c\,dX'/dc \to -\frac{1}{2}$. Hence

·431
$$cc'\left(X'\frac{dX}{dc} - X\frac{dX'}{dc}\right) = \tfrac{1}{4}\pi.$$

But, for a given value of c, any basis K, $\upsilon K'$ is derivable from X, $\iota X'$ by a pair of formulae

$$K = m_1 X + n_1 \iota X', \qquad \upsilon K' = m_2 X + n_2 \iota X',$$

where $m_1 n_2 - n_1 m_2$ is 1 or -1 according as υ is ι or $-\iota$. Hence

$$\upsilon\left(K'\frac{dK}{dc} - K\frac{dK'}{dc}\right) = \frac{\upsilon}{\iota}\iota\left(X'\frac{dX}{dc} - X\frac{dX'}{dc}\right),$$

implying

·432
$$2cc'\left(K'\frac{dK}{dc} - K\frac{dK'}{dc}\right) = \tfrac{1}{2}\pi,$$

and replacing the derivatives from ·35 we recover Legendre's relation as given in 14·62.

15·5. We derived ·41 from ·32 and ·34 by eliminating $E_c - c'K_c$ and $D_n - cK_n$. Alternatively by eliminating K_c and K_n we have the companion theorem:

15·51. *As functions of c, $E_c - c'K_c$ and $D_n - cK_n$ are solutions of the differential equation*

$$cc'\frac{d^2y}{dc^2} = \tfrac{1}{4}y.$$

Naturally $E' - cK'$ also is a solution.

From the form of this differential equation, every solution which is a regular function of c near $c = 0$ is zero at that point. We have just seen that $c\,dX'/dc \to -\frac{1}{2}$ as $c \to 0$, implying from $·32_2$ that for the basis X_c, X_n, the function $D_n - cK_n$ has the finite value 1 at $c = 0$. Hence the functions $E-c'K$, $E'-cK'$ derived from the basis X_c, X_n are independent solutions of the equation, and we denote these solutions by Y, Y', writing also $Y_c = Y$, $Y_n = \iota Y'$, where ι is the signature of X_c, X_n.

The general solution of the equation in ·51 can be written

·501 $$y = C''Y + D''Y',$$

where C'', D'' are constants. A solution which is zero at $c = 0$ is identifiable by the value of its derivative: Y is the solution which resembles $\frac{1}{4}\pi c$ near $c = 0$, and Y' is the solution which resembles $\frac{1}{4}\pi c'$ near $c = 1$.

Since the equation in ·41 is transformed into the equation in ·51 by the substitution

·502 $$y = 2cc'\,dx/dc,$$

independent investigation of the later equation can be avoided, but the details of interpretation of operators are not without interest, and the coefficients are found immediately in their simplest forms. The equation is

·503 $$\{(1-c)\vartheta(\vartheta-1)-\tfrac{1}{4}c\}y = 0,$$

that is,

·504 $$\{\vartheta(\vartheta-1)-c(\vartheta-\tfrac{1}{2})^2\}y = \vartheta(\vartheta-1)(Cc+D),$$

where C, D are arbitrary constants, and the symbolical solution is

·505 $$y = (1+\Phi+\Phi^2+\ldots)(Cc+D),$$

where

·506 $$\Phi = c\frac{(\vartheta-\frac{1}{2})^2}{(\vartheta+1)\vartheta},$$

and therefore $$\Phi^n = c^n\gamma_n(\vartheta)\frac{1}{\vartheta},$$

if

·507 $$\gamma_1(\vartheta) = \frac{(2\vartheta-1)^2}{(2\vartheta+2)2},$$

·508 $$\gamma_n(\vartheta) = \frac{\{(2\vartheta+2n-3)(2\vartheta+2n-5)\ldots(2\vartheta+1)(2\vartheta-1)\}^2}{(2\vartheta+2n)\{(2\vartheta+2n-2)\ldots(2\vartheta+2)\}^2 2}, \quad n > 1.$$

Interpreting, we have first

·509 $$\Phi^n c = c^n\gamma_n(\vartheta).c = \gamma_n(1)c^{n+1} = \gamma_n\,c^{n+1},$$

where

·510 $$\gamma_1 = \frac{1^2}{2.4},$$

·511 $$\gamma_n = \frac{\{1.3.\ldots.(2n-1)\}^2}{2\{4.6.\ldots.2n\}^2(2n+2)}, \qquad n > 1.$$

Secondly,

·512 $$\Phi^n 1 = c^n\gamma_n(\vartheta).\log c = c^n\{[\gamma_n(\vartheta)]_{\vartheta=0}\log c + [d\gamma_n(\vartheta)/d\vartheta]_{\vartheta=0}\};$$

evaluating, we have

·513–·514 $$\gamma_1(0) = \tfrac{1}{4}, \qquad [d\gamma_1(\vartheta)/d\vartheta]_{\vartheta=0} = -5\gamma_1(0) = -\tfrac{5}{4},$$

and for $n > 1$,

·515 $$\gamma_n(0) = \tfrac{1}{4}\gamma_{n-1}(1) = \tfrac{1}{4}\gamma_{n-1},$$

and, differentiating logarithmically,

·516 $$[d\gamma_n(\vartheta)/d\vartheta]_{\vartheta=0} = -4\gamma_n(0)\delta_{n-1} = -\gamma_{n-1}\delta_{n-1},$$

where

·517 $$\delta_{n-1} = 1 - \left(\frac{1}{1.2} + \frac{1}{3.4} + \ldots + \frac{1}{(2n-3)(2n-2)}\right) + \frac{1}{4.n},$$

and if we take conventionally

·518–·519 $$\gamma_0 = 1, \qquad \delta_0 = 1 + \tfrac{1}{4},$$

then

·520 $$\Phi^n 1 = \gamma_{n-1}c^n(\tfrac{1}{4}\log c - \delta_{n-1}).$$

Hence the solution of ·503 is

15·52 $$y = (C + \tfrac{1}{4}D\log c)(c + \gamma_1 c^2 + \gamma_2 c^3 + \ldots) + \\ + D(1 - \gamma_0\delta_0 c - \gamma_1\delta_1 c^2 - \ldots),$$

a solution which is valid if $|c| < 1$.

To compare this solution with the general solution ·42 of the quarter-period equation, we substitute

·521–·522 $$\gamma_n = \frac{\alpha_n}{n+1}, \qquad \delta_n = 1 - \beta_n + \frac{1}{4(n+1)},$$

relations that hold even for $n = 0$ since we must suppose $\beta_0 = 0$. We have then

$$y = (C - D + \tfrac{1}{4}\log c)(c + \tfrac{1}{2}\alpha_1 c^2 + \tfrac{1}{3}\alpha_2 c^3 + \ldots) + \\ + D\{1 - \tfrac{1}{4}(c + \tfrac{1}{4}\alpha_1 c^2 + \tfrac{1}{9}\alpha_2 c^3 + \ldots) + (\tfrac{1}{2}\alpha_1\beta_1 c^2 + \tfrac{1}{3}\alpha_2\beta_2 c^3 + \ldots)\},$$

implying that

·523 $$dy/dc = \tfrac{1}{2}x$$

if the constants C, D are given in terms of A, B by

·524–·525 $$C = \tfrac{1}{2}A+2B, \qquad D = 2B.$$

If y is found from ·42 by direct integration, there is a constant of integration to be determined. Alternatively, y is obtainable, from the same expansion ·42, as $2cc'\,dx/dc$; the algebra is more substantial, but the determination is complete.

Since $Y/c \to \tfrac{1}{4}\pi$ as $c \to 0$, the constants in ·52 for this solution are $C = \tfrac{1}{4}\pi$, $D = 0$.

15·53. *For* $|c| < 1$,

$$Y = \tfrac{1}{4}\pi(c+\tfrac{1}{2}\alpha_1 c^2+\tfrac{1}{3}\alpha_2 c^3+\ldots),$$

and for $|c'| < 1$,

$$Y' = \tfrac{1}{4}\pi(c'+\tfrac{1}{2}\alpha_1 c'^2+\tfrac{1}{3}\alpha_2 c'^3+\ldots),$$

where $$\alpha_n = \left(\frac{1.3.\ldots.(2n-1)}{2.4.\ldots.2n}\right)^2.$$

Since $Y' = 1$ when $c = 0$, the expression of Y' in the form ·52 requires $D = 1$, but we must not suppose that $C = 0$; on the contrary, ·52 implies, for small values of c,

·526 $$dy/dc = \tfrac{1}{4}D\log c+(C-D)+O(c\log c),$$

and from ·47 we have

·527 $$dY'/dc = -\tfrac{1}{2}X' = -\tfrac{1}{4}\log(16/c)+O(c\log c),$$

confirming the value $D = 1$ and giving also $C-D = -\tfrac{1}{4}\log 16$, that is, $C = 1-\log 2$:

15·54. *For* $|c| < 1$,

$$Y' = \{4-\pi\log(16/c)\}Y+(1-\alpha_0\delta_0 c-\tfrac{1}{2}\alpha_1\delta_1 c^2-\tfrac{1}{3}\alpha_2\delta_2 c^3-\ldots),$$

and for $|c'| < 1$,

$$Y = \{4-\pi\log(16/c')\}Y'+(1-\alpha_0\delta_0 c'-\tfrac{1}{2}\alpha_1\delta_1 c'^2-\tfrac{1}{3}\alpha_2\delta_2 c'^3-\ldots),$$

where the logarithms have their principal values and

$$\alpha_n = \left(\frac{1.3.\ldots.(2n-1)}{2.4.\ldots.2n}\right)^2, \quad \delta_n = 1-\left(\frac{1}{1.2}+\frac{1}{3.4}+\ldots+\frac{1}{(2n-1)2n}\right)+\frac{1}{4(n+1)}$$

with the conventional values $\alpha_0 = 1$, $\delta_0 = 1+\tfrac{1}{4}$.

Taking the general solution near $c = 1$ as

15·55 $$y = (C'+\tfrac{1}{4}D'\log c')(c'+\gamma_1 c'^2+\gamma_2 c'^3+\ldots)+ \\ +D'(1-\gamma_0\delta_0 c'-\gamma_1\delta_1 c'^2-\ldots),$$

we can say that the solution for which $C = \tfrac{1}{4}\pi$, $D = 0$ in ·52 is the solution for which $C' = 1-\log 2$, $D' = 1$ in ·55, and that the solution for which $C' = \frac{1}{4}\pi$, $D' = 0$ in ·55 is the solution for which $C = 1-\log 2$, $D = 1$ in ·52:

15·56. *For the equation*

$$cc'\, d^2y/dc^2 = \tfrac{1}{4}y,$$

the solution near $c = 0$ *given by* ·52 *and the solution near* $c = 1$ *given by* ·55 *coincide throughout the slit plane if*

$$\pi C' = 4(1-\log 2)C - \{4(1-\log 2)^2 - \tfrac{1}{4}\pi^2\}D,$$

$$\pi D' = 4C - 4(1-\log 2)D.$$

The functions Y_c, Y_n of c have been defined from the basis X_c, X_n. They are therefore defined for the whole region throughout which X_c and X_n are defined, that is, for all values of c except real values greater than 1 or less than 0. The expansions in ·53 are particular representations, valid only within restricted domains. We can obtain integral representations of Y and Y' valid throughout the domains of existence of the integrals defining X and X' in ·424 and ·425 by differentiation:

·528–·529 $$Y = cc' \int_0^1 \frac{\sqrt{t}.dt}{\sqrt{\{(1-t)(1-ct)^3\}}}, \quad Y' = -cc' \int_0^1 \frac{\sqrt{t}.dt}{\sqrt{\{(1-t)(1-c't)^3\}}}.$$

The values of the functions $E_c - c'K_c$, $D_n - cK_n$ for an arbitrary basis are implicit in the entries against Dn u and Dc u in Table XIV 5. If

$$K_c = (4m_1+1)X_c + 2n_1 X_n, \qquad K_n = 2m_2 X_c + (4n_2+1)X_n,$$

then since, on the basis X_c, X_n, the values of E_c, D_n are $Y_c + c'X_c$, $Y_n + cX_n$, we have, on the basis K_c, K_n,

·530 $$E_c = (4m_1+1)\{Y_c + c'X_c\} + 2n_1\{X_n - (Y_n + cX_n)\},$$

·531 $$D_n = 2m_2\{X_c - (Y_c + c'X_c)\} + (4n_2+1)\{Y_n + cX_n\},$$

and therefore

·532–·533 $$E_c - c'K_c = (4m_1+1)Y_c - 2n_1 Y_n,$$

$$D_n - cK_n = -2m_2 Y_c + (4n_2+1)Y_n,$$

in agreement with the fact that $E_c - c'K_c$ and $D_n - cK_n$ satisfy a linear differential equation of which Y_c and Y_n are independent solutions.

XVI

THETA FUNCTIONS

16·1. The Jacobian functions are elliptic functions adapted by means of constant factors for use with a standardized lattice. The integrating functions, and more particularly the functions $E(u)$ and $D(u)$, replace the function ζz when the lattice is Jacobian. We anticipate therefore a parallel modification in the function σz, with a representation of the Jacobian function $\mathrm{pq}\,u$ as a quotient of integral functions of u.

We recall that the function σz plays a double part. As a function which facilitates the integration of ζz, this function satisfies the formula and the condition

$$\frac{\sigma' z}{\sigma z} = \zeta z, \qquad \frac{\sigma z}{z} \to 1.$$

As the integral function whose zeros are the lattice points $2m\omega_1 + 2n\omega_2$,

$$\sigma z = z \prod{}' \left\{\left(1 - \frac{z}{\Omega}\right) e^{z/\Omega + z^2/2\Omega^2}\right\},$$

where $\Omega = 2m\omega_1 + 2n\omega_2$ and the product extends over all values of m and n except simultaneous zeros. If we are to introduce a special function into the Jacobian theory and not simply to use a sigma function constructed on the Jacobian lattice, we must verify that the double part is still played.

Since the aggregate of values $2m\pi$ for integral values of m is the aggregate of solutions of the equation $e^{iz} = 1$, the aggregate

·101 $$u = 2mK_c$$

is the aggregate of solutions of the equation

·102 $$e^{2iv} = 1,$$

where†

·103 $$v = (\pi/2K_c)u.$$

In other words, $1 - e^{2iv}$, as a function of u, is a function whose zeros are simple zeros at the points $u = 2mK_c$.

For a fixed value n_p of n, the condition

$$u = 2mK_c + 2nK_n$$

is equivalent to

$$u - 2n_p K_n = 2mK_c,$$

† At the moment the variable $(\pi/K_c)u$ would seem simpler, but $\frac{1}{2}\pi$ corresponds as a *quarterperiod* to K_c, and we shall find that in the long run the insertion of the factor 2 effects a considerable economy.

and therefore if $\rho = \pi K_n/K_c$, then

$$1-e^{2i(v-n_p\rho)}$$

is a function whose zeros are simple zeros at the points

$$u = 2mK_c + 2n_p K_n.$$

Hence for any finite number of values $n_1, n_2, \ldots, n_r$ of n, the product

$$\prod_{p=1}^{r} \{1-e^{2i(v-n_p\rho)}\}$$

is a function with simple zeros at the points $u = 2mK_c+2nK_n$ for all values of m combined with the assigned values of n.

If we are to extend this result to an infinity of values of n, the product must converge and therefore $e^{2i(v-n_r\rho)}$ must tend to zero as r tends to infinity. Now whatever the value of v, $e^{2i(v-n\rho)}$ tends to zero as $n \to +\infty$ only if $\mathrm{Rl}(i\rho)$ is positive, and tends to zero as $n \to -\infty$ only if $\mathrm{Rl}(i\rho)$ is negative. But $\mathrm{Rl}(i\rho)$ is positive or negative according as $\mathrm{Im}\rho$ is negative or positive, that is, according as the signature υ of the basis K_c, K_n is $-i$ or i; in other words, $\mathrm{Rl}(\upsilon\rho)$ is necessarily negative. If we write as before K, $\upsilon K'$ for K_c, K_n, and define σ, q by the formulae

·104–·105 $$\sigma = \pi K'/K, \qquad q = e^{-\sigma},$$

then, by the fundamental property of υ, $\mathrm{Rl}\sigma$ is positive and

·106 $$|q| < 1.$$

And now, $\rho = \upsilon\sigma$; if $\upsilon = i$, then

$$e^{2i(v-n\rho)} = q^{-2n}e^{2iv},$$

tending to 0 if $n \to -\infty$ and to ∞ if $n \to +\infty$, while if $\upsilon = -i$, then

$$e^{2i(v-n\rho)} = q^{2n}e^{2iv},$$

tending to ∞ if $n \to -\infty$ and to 0 if $n \to +\infty$. In either case, n_p must be restricted in one direction or the other if the product

$$\prod_{p=1}^{r} \{1-e^{2i(v-n_p\rho)}\}$$

is to converge.

We need not conclude, however, that a functional product convergent in both directions can not be constructed. The equation

$$e^{-2i(v-n\rho)} = 1$$

has the same roots as the equation

$$e^{2i(v-n\rho)} = 1,$$

and for each value n_p of n we may use the factor $1-e^{-2\upsilon(v-n_p\rho)}$ or the factor $1-e^{2\upsilon(v-n_p\rho)}$, that is, the factor $1-q^{2n_p}e^{-2\upsilon v}$ or the factor

$1-q^{-2n_p}e^{2vv}$, according as n_p is positive or negative: we secure a positive power of q by selecting the exponential function appropriately. Since the two values v, $-v$ are i, $-i$,

16·11. *The aggregate of values*

$$u = 2mK_c+2nK_n$$

consists of the zeros of the two functions

$$1-q^{2n}e^{2iv}, \qquad 1-q^{2n}e^{-2iv}$$

for all positive integral values of n, *together with the zeros of the function*

$$1-e^{2iv}.$$

The function $1-e^{2iv}$ is anomalous in the enunciation of ·11; there is no reason to prefer this function to $1-e^{-2iv}$, but to admit both functions would be to introduce their zeros as double members of the aggregate. We may take the function more symmetrically as $e^{iv}-e^{-iv}$; alternatively, we may give the whole theorem a trigonometrical form:

16·12. *The aggregate* $u = 2mK_c+2nK_n$ *consists of the zeros of* $\sin v$ *and the zeros for all positive integral values of* n *of the function*

$$1-2q^{2n}\cos 2v+q^{4n}.$$

16·2. Since $|q| < 1$, the infinite products†

$$\prod (1-q^{2n}e^{2iv}), \qquad \prod (1-q^{2n}e^{-2iv})$$

are convergent for all values of v, that is, for all values of u:

16·21. *Regarded as a function of* u, *the function*

$$\sin v \prod (1-2q^{2n}\cos 2v+q^{4n}),$$

where $$v = (\pi/2K_c)u, \qquad q = e^{\pi v K_n/K_c},$$

is an integral function with simple zeros at the points

$$u = 2mK_c+2nK_n.$$

Before applying this theorem to the expression of elliptic functions, we consider the transformation of the product into a series. We write temporarily

·201 $$f(t) = \prod (1-q^{2n}t),$$

·202 $$g(t) = (1-t)\prod \{1-q^{2n}(t+t^{-1})+q^{4n}\} = (1-t)f(t)f(t^{-1}).$$

† Throughout this chapter, if no range is indicated, $\Pi\, a_n$ denotes $\prod_1^\infty a_n$; Cayley in his *Elliptic Functions* denotes this infinite product by $[a_n]$, but the notation has not gained currency.

The function $f(t)$ is an integral function of t, expansible in the form

·203 $$f(t) = 1-a_1 t+a_2 t^2-a_3 t^3+\dots,$$

the coefficients being functions of q. For the function $f(t^{-1})$ we have, for all finite values except 0,

·204 $$f(t^{-1}) = 1-a_1 t^{-1}+a_2 t^{-2}-a_3 t^{-3}+\dots.$$

Since identically

·205 $$t^{-1}g(t^2) = -tg(t^{-2}),$$

the expansion of the odd function $t^{-1}g(t^2)$ is of the form

$$-b_1(t-t^{-1})+b_2(t^3-t^{-3})-b_3(t^5-t^{-5})+\dots,$$

and therefore for $g(t)$ there is an expansion

·206 $$g(t) = b_1-(b_1 t+b_2 t^{-1})+(b_2 t^2+b_3 t^{-2})-\dots.$$

If in ·201 we substitute q^2t for t, we lose the first factor; we have therefore

·207 $$(1-q^2t)f(q^2t) = f(t).$$

If we make the same substitution in the product $\prod(1-q^{2n}t^{-1})$, we gain a factor $(1-t^{-1})$; that is,

$$tf(1/q^2t) = -(1-t)f(1/t).$$

Hence

·208 $$tg(q^2t) = -g(t).$$

Substituting the series from ·203 in the functional relation ·207 and comparing coefficients we find

·209 $$q^2(1+a_1) = a_1, \qquad q^4(a_1+a_2) = a_2, \qquad q^6(a_2+a_3) = a_3, \qquad \dots,$$

whence

·210 $$a_1 = q^{1.2}/c_1, \qquad a_2 = q^{2.3}/c_2, \qquad a_3 = q^{3.4}/c_3, \qquad \dots,$$

where

·211 $$c_1 = (1-q^2), \qquad c_2 = (1-q^2)(1-q^4),$$
$$c_3 = (1-q^2)(1-q^4)(1-q^6), \qquad \dots.$$

Similarly, substituting from ·206 in ·208 we find

·212 $$b_2 = q^2b_1, \qquad b_3 = q^4b_2, \qquad b_4 = q^6b_3, \qquad \dots,$$

whence

·213 $$b_2 = q^{1.2}b_1, \qquad b_3 = q^{2.3}b_1, \qquad b_4 = q^{3.4}b_1, \qquad \dots.$$

To determine b_1, we turn to the relation between the two functions $f(t)$, $g(t)$. From ·209,

$$(1-t)f(t) = 1-q^{-2}a_1 t+q^{-4}a_2 t^2-q^{-6}a_3 t^3+\dots,$$

and therefore b_n, the coefficient of $(-)^n t^n$, for positive values of n, in the product of this series and the series in ·204, is given by

$$b_n = q^{-2n}a_n + q^{-2n-2}a_1 a_{n+1} + q^{-2n-4}a_2 a_{n+2} + \dots.$$

When we substitute from ·210 and ·213, we have a series for b_1; the index of q in the numerator of the term containing $c_r c_{n+r}$ is

$$r(r+1)+(n+r)(n+r+1)-2(n+r)-(n-1)n,$$

that is, $2r(n+r)$, and therefore, for all values of n,

·214 $$b_1 = \frac{1}{c_n} + \frac{q^{2(n+1)}}{c_1 c_{n+1}} + \frac{q^{4(n+2)}}{c_2 c_{n+2}} + \frac{q^{6(n+3)}}{c_3 c_{n+3}} + \dots.$$

Now the sequence c_1, c_2, c_3,... converges to a non-zero limit and no terms in this sequence are zero; hence the aggregate of values $|c_1|$, $|c_2|$, $|c_3|$,... has a lower bound μ which is not zero, and $1/|c_r c_{n+r}| \leqslant 1/\mu^2$ for all values of r and n. Also

$$|q^{2(n+1)} + q^{4(n+2)} + q^{6(n+3)} + \dots| < |q|^{2n}/(1-|q^2|).$$

Hence as $n \to \infty$, $$b_1 - \frac{1}{c_n} \to 0;$$

that is,

·215 $$b_1 = 1/\lim c_n.$$

Absorbing the factor $1-q^{2n}$ of c_n into the typical factor of the function $g(t)$, and replacing t by e^{2iv}, we have the fundamental identity

16·22 $$\sin v \prod \{(1-q^{2n})(1-2q^{2n}\cos 2v+q^{4n})\} = \sum (-)^{n-1} q^{(n-1)n} \sin(2n-1)v.$$

16·3. For the standard integral function of u with zeros at the lattice points $2mK_c+2nK_n$ we multiply the function for which ·22 gives two expressions by the constant $2e^{-\sigma/4}$; the series on the right of ·22 is formally a Fourier series, and it is always convenient to have an explicit factor 2 in the coefficients of the sines and cosines in a Fourier series; the exponential factor $e^{-\sigma/4}$, taken in the form $q^{1/4}$, converts the index $(n-1)n$ of q in the typical coefficient into $\{(2n-1)/2\}^2$, thus bringing this coefficient more clearly into relation with the trigonometrical function $\sin(2n-1)v$. With this modification, the function, Jacobi's eta function, is denoted by $\mathrm{H}(u)$:

16·31 $$\mathrm{H}(u) = 2q^{1/4}\sin v - 2q^{9/4}\sin 3v + 2q^{25/4}\sin 5v - \dots,$$

where $v = (\pi/2K_c)u$ and q^m denotes unambiguously $e^{-m\sigma}$, whatever the value of m. From ·22,

16·32 $$\mathrm{H}(u) = 2q^{1/4}\sin v \prod \{(1-q^{2n})(1-2q^{2n}\cos 2v+q^{4n})\}.$$

Addition of $2K_c$ to u is equivalent to addition of π to v; hence

·301–·302 $\quad \mathrm{H}(u+2K_c) = -\mathrm{H}(u), \qquad \mathrm{H}(u+4K_c) = \mathrm{H}(u).$

Addition of $2K_n$ to u is equivalent to addition of $v\sigma$ to v, that is, to multiplication of e^{vv} by q. If $e^{2vv} = t$, the function $\mathrm{H}(u)$ defined by ·32 is a constant multiple of $e^{-vv}g(t)$, where $g(t)$ is the function defined by ·202; hence the functional relation ·208 is equivalent to

·303 $$\mathrm{H}(u+2K_n) = -q^{-1}e^{-2vv}\mathrm{H}(u),$$

whence

·304 $$\mathrm{H}(u+4K_n) = q^{-4}e^{-4vv}\mathrm{H}(u),$$

the exponential factor in $\mathrm{H}(u+2K_n)$ supplying a further factor q^{-2}.

From ·301 and ·303 it follows that if $\Xi(u)$ is defined as the logarithmic derivative $\mathrm{H}'(u)/\mathrm{H}(u)$, then

·305–·306 $\quad \Xi(u+2K_c) = \Xi(u), \qquad \Xi(u+2K_n) = \Xi(u)-\pi v/K_c,$

and therefore $\Xi'(u)$ is doubly periodic in $2K_c$ and $2K_n$. Since $\mathrm{H}(u)$ is an integral function with simple zeros at $2mK_c+2nK_n$, $\Xi(u)$ is a function whose only accessible singularities are simple poles with residue 1 at each of these points, and $\Xi'(u)$ is an elliptic function with principal part $-1/(u-2mK_c-2nK_n)^2$. Hence $\Xi'(u)$ differs by a constant from $-\mathrm{cs}^2u$. But if $\Xi'(u) = A-\mathrm{cs}^2u$, then since $\Xi(u)$ and $\mathrm{Cs}\,u$ are both odd functions,

·307 $$\Xi(u) = Au-\mathrm{Cs}\,u.$$

Since $\Xi(u+2K_c) = \Xi(u)$ and $\mathrm{Cs}(u+2K_c) = \mathrm{Cs}\,u-2E_c$, ·307 implies that $AK_c = -E_c$ and we have

16·33 $$\mathrm{H}'(u)/\mathrm{H}(u) = \Xi(u) = -(E_c/K_c)u-\mathrm{Cs}\,u,$$

·308 $$\Xi'(u) = -(E_c/K_c)-\mathrm{cs}^2u.$$

From the expression of $\mathrm{H}(u)$ as a product,

·309 $$\Xi(u) = \frac{\pi}{2K_c}\left\{\cot v+4\sin 2v\sum\frac{q^{2n}}{1-2q^{2n}\cos 2v+q^{4n}}\right\},$$

·310 $$\Xi'(u) = \left(\frac{\pi}{2K_c}\right)^2\left\{-\csc^2 v+8\cos 2v\sum\frac{q^{2n}}{1-2q^{2n}\cos 2v+q^{4n}}-\right.$$
$$\left.-16\sin^2 2v\sum\frac{q^{4n}}{(1-2q^{2n}\cos 2v+q^{4n})^2}\right\}.$$

As $u\to 0$, $v\to 0$,

$$\mathrm{cs}^2u-\frac{1}{u^2}\to-\frac{1+c'}{3}, \qquad \csc^2v-\frac{1}{v^2}\to\frac{1}{3},$$

and therefore from ·308 and ·310,

·311 $$\frac{E_c}{K_c}-\frac{1+c'}{3}=\left(\frac{\pi}{K_c}\right)^2\left\{\frac{1}{12}-2\sum\frac{q^{2n}}{(1-q^{2n})^2}\right\}.$$

16·4. There is no need to revert to first principles to obtain integral functions with zeros congruent with the cardinal points K_c, K_n, K_d. The function $\mathrm{H}(u+K_c)$ has simple zeros at all the points

$$u=(2m+1)K_c+2nK_n.$$

Addition of K_c to u is equivalent to addition of $\frac{1}{2}\pi$ to v, and we have

16·41$_1$ $$\mathrm{H}(u+K_c)=2q^{1/4}\cos v+2q^{9/4}\cos 3v+2q^{25/4}\cos 5v+\dots,$$

16·41$_2$ $$\mathrm{H}(u+K_c)=2q^{1/4}\cos v\prod\{(1-q^{2n})(1+2q^{2n}\cos 2v+q^{4n})\}.$$

Addition of K_n to u alters more substantially the form of the function. This addition is equivalent to the multiplication of e^{vv} by $q^{1/2}$, and it is convenient here, and occasionally elsewhere, to write r for this parameter. Whether v is i or $-i$, $\sin mv=(e^{vmv}-e^{-vmv})/2v$, and we have from ·31,

$$\begin{aligned} vq^{1/4}\mathrm{H}(u+K_n) &= r(re^{vv}-r^{-1}e^{-vv})-r^5(r^3e^{3vv}-r^{-3}e^{-3vv})+\\ &\quad +r^{13}(r^5e^{5vv}-r^{-5}e^{-5vv})-r^{25}(r^7e^{7vv}-r^{-7}e^{-7vv})+\dots\\ &= -e^{-vv}\{1-r^2(e^{2vv}+e^{-2vv})+r^8(e^{4vv}+e^{-4vv})-\\ &\quad -r^{18}(e^{6vv}+e^{-6vv})+\dots\}, \end{aligned}$$

that is,

16·42 $$\mathrm{H}(u+K_n)=vq^{-1/4}e^{-vv}\Theta(u),$$

where

16·43$_1$ $$\Theta(u)=1-2q\cos 2v+2q^4\cos 4v-2q^9\cos 6v+\dots.$$

Similarly from ·32,

$$\begin{aligned} vq^{1/4}\mathrm{H}(u+K_n) &= r(re^{vv}-r^{-1}e^{-vv})\prod\{(1-q^{2n})(1-q^{2n+1}e^{2vv})(1-q^{2n-1}e^{-2vv})\}\\ &= -e^{-vv}\prod\{(1-q^{2n})(1-q^{2n-1}e^{2vv})(1-q^{2n-1}e^{-2vv})\}, \end{aligned}$$

the bracket outside supplying the missing factor $1-qe^{2vv}$ required to restore the symmetry; thus as a product,

16·43$_2$ $$\Theta(u)=\prod\{(1-q^{2n})(1-2q^{2n-1}\cos 2v+q^{4n-2})\}.$$

The constant factor $vq^{-1/4}$ and the exponential factor e^{-vv} do not affect the zeros of the function, and

16·44. *The function $\Theta(u)$ is an integral function with simple zeros at the points $u=2mK_c+(2n+1)K_n$.*

It is remarkable that $\Theta(u)$, which is connected with the lattice that includes K_n and not with the lattice that includes the origin, is structurally somewhat simpler than $\mathrm{H}(u)$.

Adding π to v we have

·401 $$\Theta(u+2K_c) = \Theta(u),$$

and from ·42 and ·303,

$$\Theta(u+2K_n) = -vq^{1/4}.qe^{vv}\mathrm{H}(u+3K_n),$$
$$\mathrm{H}(u+3K_n) = -q^{-1}.q^{-1}e^{-2vv}\mathrm{H}(u+K_n),$$
$$\mathrm{H}(u+K_n) = vq^{-1/4}e^{-vv}\Theta(u),$$

whence

·402 $$\Theta(u+2K_n) = -q^{-1}e^{-2vv}\Theta(u),$$

a result easily confirmed from $·43_2$. From ·42 and ·303 we have also

$$\Theta(u+K_n) = vq^{1/4}.q^{1/2}e^{vv}.q^{-1}e^{-2vv}\mathrm{H}(u),$$

that is,

·403 $$\Theta(u+K_n) = vq^{-1/4}e^{-vv}\mathrm{H}(u):$$

the relation between the functions $\mathrm{H}(u)$, $\Theta(u)$ is symmetrical.

The logarithmic derivative $\Theta'(u)/\Theta(u)$ is a function $\mathrm{Z}(u)$ such that $\mathrm{Z}'(u)$ is periodic in $2K_c$ and $2K_n$ and has for its only accessible singularities double poles congruent with K_n. Near K_n, $\mathrm{Z}'(u) \sim -1/(u-K_n)^2$, and since this is the form of dn^2u in this neighbourhood,

$$\mathrm{Z}'(u) = \mathrm{dn}^2u-B,$$

where B is a constant. Since $\mathrm{Z}(u)$ and $\mathrm{Dn}\,u$ are odd functions,

·404 $$\mathrm{Z}(u) = \mathrm{Dn}\,u-Bu,$$

and since $\mathrm{Z}(u+2K_c) = \mathrm{Z}(u)$ and $\mathrm{Dn}(u+2K_c) = \mathrm{Dn}\,u+2E_c$, ·404 implies $K_c B = E_c$:

16·45 $$\Theta'(u)/\Theta(u) = \mathrm{Z}(u) = \mathrm{Dn}\,u-(E_c/K_c)u.$$

From $·43_2$,

·405 $$\mathrm{Z}(u) = \frac{2\pi}{K_c}\sin 2v \sum \frac{q^{2n-1}}{1-2q^{2n-1}\cos 2v+q^{4n-2}},$$

·406 $$\mathrm{Z}'(u) = \left(\frac{\pi}{K_c}\right)^2\left\{2\cos 2v \sum \frac{q^{2n-1}}{1-2q^{2n-1}\cos 2v+q^{4n-2}}-\right.$$
$$\left.-4\sin^2 2v \sum \frac{q^{4n-2}}{(1-2q^{2n-1}\cos 2v+q^{4n-2})^2}\right\},$$

and therefore

·407 $$K_c(K_c-E_c) = 2\pi^2 \sum \frac{q^{2n-1}}{(1-q^{2n-1})^2}.$$

In $\Theta(u+K_c)$ we have an integral function whose zeros are at the points $(2m+1)K_c+(2n+1)K_n$; substituting $v+\frac{1}{2}\pi$ for v in $\cdot 43_1$ and $\cdot 43_2$ we have

16·46₁ $$\Theta(u+K_c) = 1+2q\cos 2v+2q^4\cos 4v+2q^9\cos 6v+\ldots,$$

16·46₂ $$\Theta(u+K_c) = \prod \{(1-q^{2n})(1+2q^{2n-1}\cos 2v+q^{4n-2})\}.$$

16·5. On account of the part played by circular functions in their construction, the four functions $\mathrm{H}(u)$, $\mathrm{H}(u+K_c)$, $\Theta(u)$, $\Theta(u+K_c)$ are simply periodic; we have in fact

16·51₁₋₂ $$\mathrm{H}(u+2K_c) = -\mathrm{H}(u), \qquad \Theta(u+2K_c) = \Theta(u),$$

and $u+K_c$ may be substituted for u in these relations. For addition of $2K_n$ we have, since addition of π to v replaces $e^{-\nu v}$ by $-e^{-\nu v}$,

16·52₁ $$\mathrm{H}(u+2K_n) = -q^{-1}e^{-2\nu v}\mathrm{H}(u),$$

16·52₂ $$\mathrm{H}(u+K_c+2K_n) = q^{-1}e^{-2\nu v}\mathrm{H}(u+K_c),$$

16·52₃ $$\Theta(u+2K_n) = -q^{-1}e^{-2\nu v}\Theta(u),$$

16·52₄ $$\Theta(u+K_c+2K_n) = q^{-1}e^{-2\nu v}\Theta(u+K_c).$$

Hence the quotient of one of the four functions by another is a doubly periodic function, and this function is easily identified, save for a constant factor, since its zeros and poles are known.

To express the Jacobian function $\mathrm{pq}\,u$ as a quotient, we replace the four functions $\mathrm{H}(u)$, $\mathrm{H}(u+K_c)$, $\Theta(u)$, $\Theta(u+K_c)$ by functions whose leading coefficients at the origin are 1, writing

16·53₁₋₂ $$\vartheta_s(u) = \frac{\mathrm{H}(u)}{\mathrm{H}'(0)}, \qquad \vartheta_c(u) = \frac{\mathrm{H}(u+K_c)}{\mathrm{H}(K_c)},$$

16·53₃₋₄ $$\vartheta_n(u) = \frac{\Theta(u)}{\Theta(0)}, \qquad \vartheta_d(u) = \frac{\Theta(u+K_c)}{\Theta(K_c)}.$$

We can supply the constant factors piecemeal from the formulae for the functions themselves; in writing down formulae for $\vartheta_s(u)$ we have to remember the factor dv/du, of which the value is $\pi/2K_c$.

Taking the original functions in factors we have

16·54₁ $$\vartheta_s(u) = \frac{2K_c}{\pi}\sin v \prod \frac{1-2q^{2n}\cos 2v+q^{4n}}{(1-q^{2n})^2},$$

16·54₂ $$\vartheta_c(u) = \cos v \prod \frac{1+2q^{2n}\cos 2v+q^{4n}}{(1+q^{2n})^2},$$

16·54₃ $$\vartheta_n(u) = \prod \frac{1-2q^{2n-1}\cos 2v+q^{4n-2}}{(1-q^{2n-1})^2},$$

16·54₄ $$\vartheta_d(u) = \prod \frac{1+2q^{2n-1}\cos 2v+q^{4n-2}}{(1+q^{2n-1})^2}.$$

If the original functions are developed in series, then equally

$$16\cdot55_1 \qquad \vartheta_s(u) = \frac{2K_c}{\pi}\cdot\frac{\sin v - q^{1.2}\sin 3v + q^{2.3}\sin 5v - \dots}{1-3q^{1.2}+5q^{2.3}-\dots},$$

$$16\cdot55_2 \qquad \vartheta_c(u) = \frac{\cos v + q^{1.2}\cos 3v + q^{2.3}\cos 5v + \dots}{1+q^{1.2}+q^{2.3}+\dots},$$

$$16\cdot55_3 \qquad \vartheta_n(u) = \frac{1-2q\cos 2v + 2q^4\cos 4v - 2q^9\cos 6v + \dots}{1-2q+2q^4-2q^9+\dots},$$

$$16\cdot55_4 \qquad \vartheta_d(u) = \frac{1+2q\cos 2v + 2q^4\cos 4v + 2q^9\cos 6v + \dots}{1+2q+2q^4+2q^9+\dots}.$$

These adjustments secure that the quotient $\vartheta_p(u)/\vartheta_q(u)$, which is an elliptic function with the periods the zeros and the poles of $\mathrm{pq}\,u$, also has the same leading coefficient as $\mathrm{pq}\,u$ at the origin:

16·56. *For all values of u, the Jacobian elliptic function $\mathrm{pq}\,u$ is the quotient $\vartheta_p(u)/\vartheta_q(u)$.*

16·6. If in ·56 we express the theta functions as products, we can combine the typical factors of the two functions and express the elliptic function also as a product. We have for example

$$\cdot601 \qquad \mathrm{cs}\,u = \frac{\pi}{2K_c}\cot v \prod \left\{\left(\frac{1-q^{2n}}{1+q^{2n}}\right)^2 \frac{1+2q^{2n}\cos 2v+q^{4n}}{1-2q^{2n}\cos 2v+q^{4n}}\right\}.$$

To find a series for $\mathrm{pq}\,u$, we recall that $\mathrm{pq}\,u$ is expressible as a multiple of a logarithmic derivative; if conditions of convergence are satisfied, a product for $f(z)$ leads immediately to a series for $f'(z)/f(z)$. If $\mathrm{rq}\,u$, $\mathrm{tq}\,u$ are the functions copolar with $\mathrm{pq}\,u$, then $\mathrm{rq}'u$ is a multiple of $\mathrm{tq}\,u\,\mathrm{pq}\,u$ and $\mathrm{tq}'u$ is a multiple of $\mathrm{rq}\,u\,\mathrm{pq}\,u$; hence for an appropriate value λ_p of λ, $\mathrm{rq}'u-\lambda\,\mathrm{tq}'u$ is a multiple of $(\mathrm{rq}\,u-\lambda\,\mathrm{tq}\,u)\mathrm{pq}\,u$, that is, $\mathrm{pq}\,u$ is a multiple† of $(\mathrm{rq}'u-\lambda\,\mathrm{tq}'u)/(\mathrm{rq}\,u-\lambda\,\mathrm{tq}\,u)$. The poles of the logarithmic derivative of a meromorphic function are the zeros and the poles of the function itself; thus the zeros and the poles of $\mathrm{rq}\,u-\lambda_p\,\mathrm{tq}\,u$ constitute a partition of the poles of $\mathrm{pq}\,u$, that is, a partition of the zeros of $\vartheta_q(u)$. These zeros are the common poles of $\mathrm{rq}\,u$ and $\mathrm{tq}\,u$: the constant λ_p has such a value that the combination $\mathrm{rq}\,u-\lambda_p\,\mathrm{tq}\,u$ loses some of the common poles, and these poles are not merely lost as poles but replaced by zeros. If a pole of $\mathrm{tq}\,u$ is a zero of $\mathrm{rq}\,u-\lambda\,\mathrm{tq}\,u$, that is, of $(\mathrm{rt}\,u-\lambda)\mathrm{tq}\,u$, it is a double zero of $\mathrm{rt}\,u-\lambda$,

† This is the form in which $\mathrm{pq}\,u$ is expressed in Table XIV1; of the two combinations available, the one chosen for the table is the one which has K_q for a zero.

and since the only values of $\mathrm{rt}\,u$ at poles of $\mathrm{tq}\,u$ are $\mathrm{rt}\,K_q$ and $-\mathrm{rt}\,K_q$, these are the possible values of λ_p.

We can verify this conclusion. Since $\mathrm{rt}^2u-\mathrm{rt}^2K_q$ is a multiple of qt^2u, every zero of $\mathrm{qt}\,u$ is a zero of one of the two factors $\mathrm{rt}\,u-\mathrm{rt}\,K_q$, $\mathrm{rt}\,u+\mathrm{rt}\,K_q$, and since these functions have no zeros in common, all the zeros of each of them are double. The quotient $(\mathrm{rt}\,u-\mathrm{rt}\,K_q)/(\mathrm{rt}\,u+\mathrm{rt}\,K_q)$ has all its zeros and all its poles precisely double, and is a multiple of $\{(\mathrm{rt}\,u-\mathrm{rt}\,K_q)/\mathrm{qt}\,u\}^2$, that is, of $(\mathrm{rq}\,u-\mathrm{rt}\,K_q\,\mathrm{tq}\,u)^2$:

16·61. *An elliptic function whose logarithmic derivative is a multiple of* $\mathrm{pq}\,u$ *is a multiple of one of the functions* $\mathrm{rq}\,u-\mathrm{rt}\,K_q\,\mathrm{tq}\,u$, $\mathrm{rq}\,u+\mathrm{rt}\,K_q\,\mathrm{tq}\,u$; *the zeros of these two functions constitute a partition of the zeros of* $\vartheta_q(u)$, *and the zeros of one function are the poles of the other.*

Constant multiples being naturally ignored in the enumeration, there are twelve pairs of functions of the form $\mathrm{rq}\,u\mp\mathrm{rt}\,K_q\,\mathrm{tq}\,u$. From the relation to $\mathrm{pq}\,u$ this is evident, and in fact $\mathrm{tq}\,u\mp\mathrm{tr}\,K_q\,\mathrm{rq}\,u$ is simply $\mp\mathrm{tr}\,K_q(\mathrm{rq}\,u\mp\mathrm{rt}\,K_q\,\mathrm{tq}\,u)$. The factorization of qt^2u can be effected either by means of $\mathrm{rt}^2u-\mathrm{rt}^2K_q$ or by means of $\mathrm{pt}^2u-\mathrm{pt}^2K_q$, and leads to two pairs of functions, but the factorization of qr^2u by means of $\mathrm{tr}^2u-\mathrm{tr}^2K_q$ leads to the same pair of functions as the factorization of qt^2u by means of $\mathrm{rt}^2u-\mathrm{rt}^2K_q$.

To express the functions $\mathrm{rq}\,u\mp\mathrm{rt}\,K_q\,\mathrm{tq}\,u$ as infinite products, with a view to expressing $\mathrm{pq}\,u$ as an infinite series, we consider more closely the partitioning of the zeros of $\vartheta_q(u)$. We can describe the zeros of $\vartheta_q(u)$ as the points congruent with K_q to moduli $2K_c$, $2K_n$, $2K_d$. These moduli are not periods of the Jacobian system, but they give rise to the three sets of Jacobian periods

$$2K_c,\ 4K_n,\ 4K_d; \qquad 4K_c,\ 2K_n,\ 4K_d; \qquad 4K_c,\ 4K_n,\ 2K_d.$$

We can therefore partition the zeros of $\vartheta_q(u)$ in three ways into congruences whose moduli are periods of Jacobian functions:

(i) $u\equiv K_q$, mod $2K_c, 4K_n$; $\qquad u\equiv K_q+2K_n$, mod $2K_c, 4K_n$;

(ii) $u\equiv K_q$, mod $4K_c, 2K_n$; $\qquad u\equiv K_q+2K_c$, mod $4K_c, 2K_n$;

(iii) $u\equiv K_q$, mod $4K_c, 2K_c+2K_n$; $\qquad u\equiv K_q+2K_c$, mod $4K_c, 2K_c+2K_n$.

Applying each partition to the four possible values of K_q, we have the twelve pairs of aggregates required for the zeros and poles of functions of the form $\mathrm{rq}\,u\mp\mathrm{rt}\,K_q\,\mathrm{tq}\,u$. Each aggregate can be associated with an integral function whose zeros it provides. We stipulate that the product of two functions whose zeros together comprise the zeros of $\vartheta_q(u)$ is to be the function $\vartheta_q(u)$, that the quotient of one of these functions by the other is to be doubly periodic, and that the leading coefficient at the origin is to be 1 in every case; as we shall see, with these conditions the functions are determinate. We denote the two factors of $\vartheta_q(u)$ each of which has zeros separated by the interval $2K_k$ by $\vartheta_q^{k-}(u)$, $\vartheta_q^{k+}(u)$, using the minus sign for the function which vanishes at K_q.

To determine a function whose zeros are the points $2mK_c+4nK_n$ is to repeat the argument of ·1 and ·2, with the equation

$$e^{2i(v-2n\rho)} = 1$$

replacing the equation $\qquad e^{2i(v-n\rho)} = 1,$

that is, with q^2 replacing q throughout. Hence

·602 $$\vartheta_s^{c-}(u) = Ae^{f(u)}\sin v \prod (1-2q^{4n}\cos 2v+q^{8n}),$$

where A is a constant and $f(u)$ is an integral function which vanishes with u. It follows, since $\vartheta_s^{c-}(u)\vartheta_s^{c+}(u) = \vartheta_s(u)$, that

·603 $$\vartheta_s^{c+}(u) = Be^{-f(u)} \prod (1-2q^{4n-2}\cos 2v+q^{8n-4}).$$

From these formulae

$$\vartheta_s^{c-}(u+4K_c) = e^{f(u+4K_c)-f(u)}\vartheta_s^{c-}(u),$$

$$\vartheta_s^{c+}(u+4K_c) = e^{-\{f(u+4K_c)-f(u)\}}\vartheta_s^{c+}(u),$$

and since $\vartheta_s^{c-}(u)/\vartheta_s^{c+}(u)$ is to be periodic in $4K_c$,

·604 $$2f(u+4K_c) = 2f(u)+2\mu\pi i,$$

where μ is an integer. Again, since by ·303

$$\vartheta_s(u+2K_n) = -q^{-1}e^{-2v v}\vartheta_s(u),$$

we have, replacing K_n by $2K_n$ and $\vartheta_s(u)$ by $A^{-1}e^{-f(u)}\vartheta_s^{c-}(u)$,

·605 $$\vartheta_s^{c-}(u+4K_n) = -q^{-2}e^{-2vv}e^{f(u+4K_n)-f(u)}\vartheta_s^{c-}(u),$$

and therefore, since by ·304

$$\vartheta_s(u+4K_n) = q^{-4}e^{-4vv}\vartheta_s(u),$$

we have

·606 $$\vartheta_s^{c+}(u+4K_n) = -q^{-2}e^{-2vv}e^{-\{f(u+4K_n)-f(u)\}}\vartheta_s^{c+}(u);$$

from ·605 and ·606, since $\vartheta_s^{c-}(u)/\vartheta_s^{c+}(u)$ is periodic in $4K_n$,

·607 $$2f(u+4K_n) = 2f(u)+2\nu\pi i,$$

where ν is an integer. From ·604 and ·607, the derivative $f'(u)$ is a doubly periodic integral function, and is therefore a constant λ, and having inserted the constants A, B in ·602, ·603 in order to postulate a zero value of $f(0)$, we have $f(u) = \lambda u$. But with this form of the function, ·604 and ·607 imply

$$4\lambda K_c = \mu\pi i, \qquad 4\lambda K_n = \nu\pi i,$$

and since the ratio of K_c to K_n is not real, these conditions can not be satisfied unless μ, ν, λ are all zero. Hence $f(u)$ is zero, identically, the factor $e^{f(u)}$ is unity, and

16·62$_1$ $$\vartheta_s^{c-}(u) = \frac{2K_c}{\pi}\sin v \prod \frac{1-2q^{4n}\cos 2v+q^{8n}}{(1-q^{4n})^2},$$

16·62$_2$ $$\vartheta_s^{c+}(u) = \prod \frac{1-2q^{4n-2}\cos 2v+q^{8n-4}}{(1-q^{4n-2})^2}.$$

To infer $\vartheta_s^{n-}(u)$ from $\vartheta_s(u)$, we substitute $\frac{1}{2}v$ for v, and $\frac{1}{2}\sigma$ for σ and therefore q for q^2; again there is no exponential factor, and we find

16·62$_3$ $$\vartheta_s^{n-}(u) = \frac{4K_c}{\pi}\sin \tfrac{1}{2}v \prod \frac{1-2q^{n}\cos v+q^{2n}}{(1-q^{n})^2},$$

16·62$_4$ $$\vartheta_s^{n+}(u) = \cos\tfrac{1}{2}v \prod \frac{1+2q^{n}\cos v+q^{2n}}{(1+q^{n})^2};$$

whereas in ·62$_{1-2}$ the individual factors of $\vartheta_s(u)$, as written in ·54$_1$, are distributed unbroken, some to compose $\vartheta_s^{c-}(u)$ and the others to compose $\vartheta_s^{c+}(u)$, in ·62$_{3-4}$ each factor of $\vartheta_s(u)$ is broken and contributes a component to each function.

The aggregate $4mK_c+2n(K_c+K_n)$ consists of those members of the aggregate $2mK_c+2nK_n$ for which m and n are both even or both odd, and therefore the factors of $\vartheta_s^{d-}(u)$ are the factors of $\vartheta_s^{n-}(u)$ for which n is even and the factors of $\vartheta_s^{n+}(u)$ for which n is odd, together with, possibly, an exponential factor; after verifying that there is no exponential factor, we have

16·62$_5$ $$\vartheta_s^{d-}(u) = \frac{4K_c}{\pi}\sin \tfrac{1}{2}v \prod \frac{1-2(-q)^{n}\cos v+q^{2n}}{\{1-(-q)^{n}\}^2},$$

16·62$_6$ $$\vartheta_s^{d+}(u) = \cos\tfrac{1}{2}v \prod \frac{1+2(-q)^{n}\cos v+q^{2n}}{\{1+(-q)^{n}\}^2}.$$

We derive partitions of $\vartheta_c(u)$ from partitions of $\vartheta_s(u)$ by substituting $v-\frac{1}{2}\pi$ for v. Exponential factors can not become necessary, since the periodicities are unchanged, but the constant factors must be supplied after the substitution; since none of the functional factors vanish when $v = 0$, we have only to divide each separate factor by its value there.

16·63$_1$ $$\vartheta_c^{c-}(u) = \cos v \prod \frac{1+2q^{4n}\cos 2v+q^{8n}}{(1+q^{4n})^2},$$

16·63$_2$ $$\vartheta_c^{c+}(u) = \prod \frac{1+2q^{4n-2}\cos 2v+q^{8n-4}}{(1+q^{4n-2})^2};$$

16·63$_3$ $\vartheta_c^{n-}(u) = \sqrt{2}\sin(\frac{1}{4}\pi-\frac{1}{2}v)\prod\frac{1-2q^n\sin v+q^{2n}}{1+q^{2n}}$,

16·63$_4$ $\vartheta_c^{n+}(u) = \sqrt{2}\cos(\frac{1}{4}\pi-\frac{1}{2}v)\prod\frac{1+2q^n\sin v+q^{2n}}{1+q^{2n}}$;

16·63$_5$ $\vartheta_c^{d-}(u) = \sqrt{2}\sin(\frac{1}{4}\pi-\frac{1}{2}v)\prod\frac{1-2(-q)^n\sin v+q^{2n}}{1+q^{2n}}$,

16·63$_6$ $\vartheta_c^{d+}(u) = \sqrt{2}\cos(\frac{1}{4}\pi-\frac{1}{2}v)\prod\frac{1+2(-q)^n\sin v+q^{2n}}{1+q^{2n}}$.

To substitute $q^{-1}e^{2vv}$ for e^{2vv} in $\vartheta_s^{c-}(u)$ and $\vartheta_s^{c+}(u)$ we must take the numerators of the typical factors in the factorized forms

$$(1-q^{4n}e^{2vv})(1-q^{4n}e^{-2vv}), \qquad (1-q^{4n-2}e^{2vv})(1-q^{4n-2}e^{-2vv}).$$

The factors can not be recombined after the substitution, and the numerator of the typical factor of $\vartheta_n^{c+}(u)$ remains as

$$(1-q^{4n-3}e^{2vv})(1-q^{4n-1}e^{-2vv}).$$

It follows from ·54$_3$ that the numerator of the typical factor of $\vartheta_n^{c-}(u)$ is

$$(1-q^{4n-3}e^{-2vv})(1-q^{4n-1}e^{2vv});$$

we can derive this factor alternatively by absorbing into the first factor deduced from $\vartheta_s^{c-}(u)$ the factor $1-q^{-1}e^{2vv}$ given by the extraneous factor $\sin v$, and representing the factor

$$(1-q^{4n-1}e^{2vv})(1-q^{4n+1}e^{-2vv})$$

as $$(1-q^{4n-1}e^{2vv})(1-q^{4(n+1)-3}e^{-2vv}).$$

We have now

·608 $\vartheta_n^{c-}(u) = Ae^{f(u)}\prod\{(1-q^{4n-3}e^{-2vv})(1-q^{4n-1}e^{2vv})\}$,

·609 $\vartheta_n^{c+}(u) = Be^{-f(u)}\prod\{(1-q^{4n-3}e^{2vv})(1-q^{4n-1}e^{-2vv})\}$,

where A, B are constants and $f(u)$ is an integral function of u which vanishes with u. The infinite products are unaltered if $v+2\pi$ is substituted for v; hence the periodicity of $\vartheta_n^{c-}(u)/\vartheta_n^{c+}(u)$ in $4K_c$ implies

·610 $2f(u+4K_c) = 2f(u)+2\mu\pi i$,

where μ is an integer. Substitution of q^2e^{vv} for e^{vv} reproduces the infinite products, except that in the first the factor $1-q^{-3}e^{-2vv}$ replaces $1-q^3e^{2vv}$ and in the second the factor $1-q^{-1}e^{-2vv}$ replaces $1-qe^{2vv}$. Thus

·611 $\vartheta_n^{c-}(u+4K_n) = -q^{-3}e^{-2vv}e^{f(u+4K_n)-f(u)}\vartheta_n^{c-}(u)$,

·612 $\vartheta_n^{c+}(u+4K_n) = -q^{-1}e^{-2vv}e^{-\{f(u+4Kn)-f(u)\}}\vartheta_n^{c+}(u)$,

and the condition of periodicity of the quotient $\vartheta_n^{c-}(u)/\vartheta_n^{c+}(u)$ is

·613 $2f(u+4K_n) = 2f(u)-2\sigma+2\nu\pi i$, where ν is an integer. As before, $f'(u)$ is a constant, and $f(u)$ is a multiple of u, which we may take to be κv. We have now, from ·610, ·613, since the addition of $2K_c$ to u is equivalent to the addition of π to v and the addition of $2K_n$ to u is equivalent to the subtraction of σ from υv, that is, to the addition of $\upsilon\sigma$ to v,

·614–·615 $$2\kappa\pi = \mu\pi i, \qquad 2\kappa\sigma\upsilon = -\sigma+\nu\pi i.$$

We must again take $\nu = 0$, but the conditions then require $\kappa = \frac{1}{2}\upsilon$, and are satisfied if $\mu = \upsilon/i = \pm 1$. Hence

16·64₁ $$\vartheta_n^{c-}(u) = e^{\frac{1}{2}\upsilon v}\prod \frac{(1-q^{4n-3}e^{-2\upsilon v})(1-q^{4n-1}e^{2\upsilon v})}{(1-q^{4n-3})(1-q^{4n-1})},$$

16·64₂ $$\vartheta_n^{c+}(u) = e^{-\frac{1}{2}\upsilon v}\prod \frac{(1-q^{4n-3}e^{2\upsilon v})(1-q^{4n-1}e^{-2\upsilon v})}{(1-q^{4n-3})(1-q^{4n-1})}.$$

The complete denominator in each function is simply $\prod(1-q^{2n-1})$.

To write down $\vartheta_n^{n-}(u)$ we have only to replace $2K_c$ by $4K_c$ in $\vartheta_n(u)$, that is, to replace q^2 by q and v by $\frac{1}{2}v$. To avoid fractional indices we use, as in ·4, an explicit symbol r for $e^{-\frac{1}{2}\sigma}$:

·616 $$r^2 = q.$$

No exponential factors are required, and we have

16·64₃ $$\vartheta_n^{n-}(u) = \prod \frac{1-2r^{2n-1}\cos v+r^{4n-2}}{(1-r^{2n-1})^2},$$

16·64₄ $$\vartheta_n^{n+}(u) = \prod \frac{1+2r^{2n-1}\cos v+r^{4n-2}}{(1+r^{2n-1})^2}.$$

To find $\vartheta_n^{d-}(u)$ and $\vartheta_n^{d+}(u)$, we replace† $e^{\upsilon v}$ by $r^{-1}e^{\upsilon v}$ in $\cdot62_{5-6}$. Apart from constant factors and an exponential factor $e^{\frac{1}{2}\upsilon v}$, $\cdot62_{5-6}$ give the products

$$(1-e^{-\upsilon v})\{(1+r^2e^{\upsilon v})(1+r^2e^{-\upsilon v})\}\{(1-r^4e^{\upsilon v})(1-r^4e^{-\upsilon v})\}\times$$
$$\times\{(1+r^6e^{\upsilon v})(1+r^6e^{-\upsilon v})\}...,$$
$$(1+e^{-\upsilon v})\{(1-r^2e^{\upsilon v})(1-r^2e^{-\upsilon v})\}\{(1+r^4e^{\upsilon v})(1+r^4e^{-\upsilon v})\}\times$$
$$\times\{(1-r^6e^{\upsilon v})(1-r^6e^{-\upsilon v})\}...,$$

and these are transformed into

$$\{(1-re^{-\upsilon v})(1+re^{\upsilon v})\}\{(1+r^3e^{-\upsilon v})(1-r^3e^{\upsilon v})\}\{(1-r^5e^{-\upsilon v})(1+r^5e^{\upsilon v})\}...,$$
$$\{(1+re^{-\upsilon v})(1-re^{\upsilon v})\}\{(1-r^3e^{-\upsilon v})(1+r^3e^{\upsilon v})\}\{(1+r^5e^{-\upsilon v})(1-r^5e^{\upsilon v})\}....$$

Substitution of $r^4e^{\upsilon v}$ for $e^{\upsilon v}$ multiplies these products by the same factor, $-r^{-4}e^{-2\upsilon v}$. Hence the quotient of one product by the other is

† The reader may consider why we can not apply to $\cdot64_{3-4}$ the process by which $\cdot62_{5-6}$ are derived from $\cdot62_{3-4}$.

periodic in $4K_n$, and since each product is periodic in $4K_c$, no exponential factors are wanted and we have

16·64₅ $$\vartheta_n^{d-}(u) = \prod \frac{1+2(vr)^{2n-1}\sin v - r^{4n-2}}{1-r^{4n-2}},$$

16·64₆ $$\vartheta_n^{d+}(u) = \prod \frac{1-2(vr)^{2n-1}\sin v - r^{4n-2}}{1-r^{4n-2}}.$$

Lastly, the partitions of $\vartheta_d(u)$ are derivable from those of $\vartheta_n(u)$ by the substitution of $v-\frac{1}{2}\pi$ for v:

16·65₁ $$\vartheta_d^{c-}(u) = e^{\frac{1}{4}vv} \prod \frac{(1+q^{4n-3}e^{-2vv})(1+q^{4n-1}e^{2vv})}{(1+q^{4n-3})(1+q^{4n-1})},$$

16·65₂ $$\vartheta_d^{c+}(u) = e^{-\frac{1}{4}vv} \prod \frac{(1+q^{4n-3}e^{2vv})(1+q^{4n-1}e^{-2vv})}{(1+q^{4n-3})(1+q^{4n-1})};$$

16·65₃ $$\vartheta_d^{n-}(u) = \prod \frac{1-2r^{2n-1}\sin v + r^{4n-2}}{1+r^{4n-2}},$$

16·65₄ $$\vartheta_d^{n+}(u) = \prod \frac{1+2r^{2n-1}\sin v + r^{4n-2}}{1+r^{4n-2}};$$

16·65₅ $$\vartheta_d^{d-}(u) = \prod \frac{1-2(vr)^{2n-1}\cos v - r^{4n-2}}{1-r^{4n-2}},$$

16·65₆ $$\vartheta_d^{d+}(u) = \prod \frac{1+2(vr)^{2n-1}\cos v - r^{4n-2}}{1-r^{4n-2}}.$$

To replace the products in ·62–·65 by series is in most cases only to apply, with a change of variables, the identities implicit in the double expressions for the functions $\vartheta_p(u)$ in ·54 and ·55. With the series, the fact that the product of the two functions $\vartheta_q^{k-}(u)$, $\vartheta_q^{k+}(u)$ is the function $\vartheta_q(u)$ is no longer obvious.

16·66₁ $$\vartheta_s^{c-}(u) = \frac{2K_c}{\pi} \cdot \frac{r\sin v - r^9 \sin 3v + r^{25}\sin 5v - \dots}{r-3r^9+5r^{25}-\dots},$$

16·66₂ $$\vartheta_s^{c+}(u) = \frac{1-2r^4\cos 2v + 2r^{16}\cos 4v - 2r^{36}\cos 6v + \dots}{1-2r^4+2r^{16}-2r^{36}+\dots};$$

16·66₃ $$\vartheta_s^{n-}(u) = \frac{4K_c}{\pi} \cdot \frac{\sin \frac{1}{2}v - r^{1.2}\sin\frac{3}{2}v + r^{2.3}\sin\frac{5}{2}v - \dots}{1-3r^{1.2}+5r^{2.3}-\dots},$$

16·66₄ $$\vartheta_s^{n+}(u) = \frac{\cos\frac{1}{2}v + r^{1.2}\cos\frac{3}{2}v + r^{2.3}\cos\frac{5}{2}v + \dots}{1+r^{1.2}+r^{2.3}+\dots};$$

16·66₅ $$\vartheta_s^{d-}(u) = \frac{4K_c}{\pi} \cdot \frac{r\sin\frac{1}{2}v + r^9\sin\frac{3}{2}v - r^{25}\sin\frac{5}{2}v - r^{49}\sin\frac{7}{2}v + \dots}{r+3r^9-5r^{25}-7r^{49}+\dots},$$

16·66₆ $$\vartheta_s^{d+}(u) = \frac{r\cos\frac{1}{2}v - r^9\cos\frac{3}{2}v - r^{25}\cos\frac{5}{2}v + r^{49}\cos\frac{7}{2}v + \dots}{r-r^9-r^{25}+r^{49}+\dots};$$

the signs alternate in pairs in $\cdot 66_{5-6}$, as they do below in $\cdot 67_{3-4}$, $\cdot 68_{1-2}$, $\cdot 68_{5-6}$, and $\cdot 69_{3-4}$.

$16\cdot 67_{1-2}$

$$\vartheta_c^{c-}(u) = \frac{r\cos v + r^9\cos 3v + r^{25}\cos 5v + \ldots}{r + r^9 + r^{25} + \ldots},$$

$$\vartheta_c^{c+}(u) = \frac{1 + 2r^4\cos 2v + 2r^{16}\cos 4v + 2r^{36}\cos 6v + \ldots}{1 + 2r^4 + 2r^{16} + 2r^{36} + \ldots};$$

$16\cdot 67_{3-4}$

$$\vartheta_c^{n-}(u) = \frac{(\cos\tfrac{1}{2}v - \sin\tfrac{1}{2}v) - r^{1.2}(\cos\tfrac{3}{2}v + \sin\tfrac{3}{2}v) - r^{2.3}(\cos\tfrac{5}{2}v - \sin\tfrac{5}{2}v) + \ldots}{1 - r^{1.2} - r^{2.3} + \ldots},$$

$$\vartheta_c^{n+}(u) = \frac{(\cos\tfrac{1}{2}v + \sin\tfrac{1}{2}v) - r^{1.2}(\cos\tfrac{3}{2}v - \sin\tfrac{3}{2}v) - r^{2.3}(\cos\tfrac{5}{2}v + \sin\tfrac{5}{2}v) + \ldots}{1 - r^{1.2} - r^{2.3} + \ldots};$$

$16\cdot 67_{5-6}$

$$\vartheta_c^{d-}(u) = \frac{r(\cos\tfrac{1}{2}v - \sin\tfrac{1}{2}v) + r^9(\cos\tfrac{3}{2}v + \sin\tfrac{3}{2}v) + r^{25}(\cos\tfrac{5}{2}v - \sin\tfrac{5}{2}v) + \ldots}{r + r^9 + r^{25} + \ldots},$$

$$\vartheta_c^{d+}(u) = \frac{r(\cos\tfrac{1}{2}v + \sin\tfrac{1}{2}v) + r^9(\cos\tfrac{3}{2}v - \sin\tfrac{3}{2}v) + r^{25}(\cos\tfrac{5}{2}v + \sin\tfrac{5}{2}v) + \ldots}{r + r^9 + r^{25} + \ldots}.$$

No products like those in $\cdot 64_{1-2}$ and $\cdot 65_{1-2}$ occur in $\cdot 5$, but we can avoid a functional examination by making a substitution directly into $\cdot 66_{1-2}$. Except for constant factors, $\vartheta_n^{c-}(u)$ and $\vartheta_n^{c+}(u)$ are derivable from $\vartheta_s^{c-}(u)$ and $\vartheta_s^{c+}(u)$ by substitution of $q^{-1}e^{2\upsilon v}$ for $e^{2\upsilon v}$, that is, of $r^{-1}e^{\upsilon v}$ for $e^{\upsilon v}$, and multiplication by an exponential factor; the exponential factor is $e^{-\frac{1}{2}\upsilon v}$ in each case, for the factor $1 - e^{-2\upsilon v}$ has been taken from $\sin v$ in the formation of $\cdot 64_1$ from $\cdot 62_1$, and it is a factor $e^{-\frac{1}{2}\upsilon v}$ that must be imported to produce the explicit factor $e^{\frac{1}{2}\upsilon v}$. The relation between the pairs of functions is in no way dependent on the form in which the functions are written, and therefore the series required for $\vartheta_n^{c-}(u)$, $\vartheta_n^{c+}(u)$ are

$$e^{-\frac{1}{2}\upsilon v}\{r(r^{-1}e^{\upsilon v} - re^{-\upsilon v}) - r^9(r^{-3}e^{3\upsilon v} - r^3e^{-3\upsilon v}) + r^{25}(r^{-5}e^{5\upsilon v} - r^5e^{-5\upsilon v})\ldots\},$$

$$e^{-\frac{1}{2}\upsilon v}\{1 - r^4(r^{-2}e^{2\upsilon v} + r^2e^{-2\upsilon v}) + r^{16}(r^{-4}e^{4\upsilon v} + r^4e^{-4\upsilon v}) - \\ - r^{36}(r^{-6}e^{6\upsilon v} + r^6e^{-6\upsilon v}) + \ldots\}.$$

Supplying the denominators, we have

$16\cdot 68_1$
$$\vartheta_n^{c-}(u) = \frac{e^{\frac{1}{2}\upsilon v} - r^{1.2}e^{-\frac{3}{2}\upsilon v} - r^{2.3}e^{\frac{5}{2}\upsilon v} + r^{3.4}e^{-\frac{7}{2}\upsilon v} + \ldots}{1 - r^{1.2} - r^{2.3} + r^{3.4} + \ldots},$$

$16\cdot 68_2$
$$\vartheta_n^{c+}(u) = \frac{e^{-\frac{1}{2}\upsilon v} - r^{1.2}e^{\frac{3}{2}\upsilon v} - r^{2.3}e^{-\frac{5}{2}\upsilon v} + r^{3.4}e^{\frac{7}{2}\upsilon v} + \ldots}{1 - r^{1.2} - r^{2.3} + r^{3.4} + \ldots}.$$

Transformation of $\cdot 64_{3\text{-}6}$ is immediate, since these products are formally identical with the products for $\vartheta_n(u)$ and $\vartheta_d(u)$:

$$16\cdot 68_3 \quad \vartheta_n^{n-}(u) = \frac{1-2r\cos v+2r^4\cos 2v-2r^9\cos 3v+\ldots}{1-2r+2r^4-2r^9+\ldots},$$

$$16\cdot 68_4 \quad \vartheta_n^{n+}(u) = \frac{1+2r\cos v+2r^4\cos 2v+2r^9\cos 3v+\ldots}{1+2r+2r^4+2r^9+\ldots};$$

$$16\cdot 68_5 \quad \vartheta_n^{d-}(u) = \frac{1+2vr\sin v-2r^4\cos 2v-2vr^9\sin 3v+2r^{16}\cos 4v+\ldots}{1-2r^4+2r^{16}-\ldots},$$

$$16\cdot 68_6 \quad \vartheta_n^{d+}(u) = \frac{1-2vr\sin v-2r^4\cos 2v+2vr^9\sin 3v+2r^{16}\cos 4v+\ldots}{1-2r^4+2r^{16}-\ldots}.$$

Lastly, subtracting $\frac{1}{2}\pi$ from v in $\cdot 68_{1\text{-}6}$ we have

$$16\cdot 69_1 \quad \vartheta_d^{c-}(u) = \frac{e^{-\frac{1}{2}vv}+r^{1.2}e^{\frac{3}{2}vv}+r^{2.3}e^{-\frac{5}{2}vv}+r^{3.4}e^{\frac{7}{2}vv}+\ldots}{1+r^{1.2}+r^{2.3}+r^{3.4}+\ldots},$$

$$16\cdot 69_2 \quad \vartheta_d^{c+}(u) = \frac{e^{\frac{1}{2}vv}+r^{1.2}e^{-\frac{3}{2}vv}+r^{2.3}e^{\frac{5}{2}vv}+r^{3.4}e^{-\frac{7}{2}vv}+\ldots}{1+r^{1.2}+r^{2.3}+r^{3.4}+\ldots};$$

$$16\cdot 69_3 \quad \vartheta_d^{n-}(u) = \frac{1-2r\sin v-2r^4\cos 2v+2r^9\sin 3v+2r^{16}\cos 4v-\ldots}{1-2r^4+2r^{16}-\ldots},$$

$$16\cdot 69_4 \quad \vartheta_d^{n+}(u) = \frac{1+2r\sin v-2r^4\cos 2v-2r^9\sin 3v+2r^{16}\cos 4v+\ldots}{1-2r^4+2r^{16}-\ldots};$$

$$16\cdot 69_5 \quad \vartheta_d^{d-}(u) = \frac{1-2vr\cos v+2r^4\cos 2v-2vr^9\cos 3v+2r^{16}\cos 4v-\ldots}{1-2vr+2r^4-2vr^9+2r^{16}-\ldots},$$

$$16\cdot 69_6 \quad \vartheta_d^{d+}(u) = \frac{1+2vr\cos v+2r^4\cos 2v+2vr^9\cos 3v+2r^{16}\cos 4v+\ldots}{1+2vr+2r^4+2vr^9+2r^{16}+\ldots}.$$

16·7. The fundamental connexion between the twenty-four functions which we have now evaluated and the Jacobian elliptic functions was explained in advance.

16·71. *If* ks u *is the primitive function coperiodic with* pq u, *and* rq u, tq u *are the functions copolar with* pq u, *the ratio* $\vartheta_q^{k-}(u)/\vartheta_q^{k+}(u)$ *is equal to a linear combination of* rq u *and* tq u *in which the common pole* K_q *is replaced by a zero, and the logarithmic derivative of this ratio is a constant multiple of* pq u.

It remains to tabulate the results in detail, supplying constant factors from Tables XI 2,7.

TABLE XVI1

$$\frac{2(\operatorname{ns} u-\operatorname{ds} u)}{k^2} = \frac{2\operatorname{sn} u}{1+\operatorname{dn} u} = \frac{\vartheta_s^{c-}(u)}{\vartheta_s^{c+}(u)}$$

$$\frac{2(\operatorname{ds} u-\operatorname{cs} u)}{k'^2} = \frac{2\operatorname{sd} u}{1+\operatorname{cd} u} = \frac{\vartheta_s^{n-}(u)}{\vartheta_s^{n+}(u)}$$

$$2(\operatorname{ns} u-\operatorname{cs} u) = \frac{2\operatorname{sn} u}{1+\operatorname{cn} u} = \frac{\vartheta_s^{d-}(u)}{\vartheta_s^{d+}(u)}$$

$$\frac{\operatorname{dc} u-k'\operatorname{nc} u}{1-k'} = \frac{(1+k')\operatorname{cd} u}{1+k'\operatorname{nd} u} = \frac{\vartheta_c^{c-}(u)}{\vartheta_c^{c+}(u)}$$

$$\operatorname{nc} u-\operatorname{sc} u = \frac{\operatorname{cn} u}{1+\operatorname{sn} u} = \frac{\vartheta_c^{n-}(u)}{\vartheta_c^{n+}(u)}$$

$$\operatorname{dc} u-k'\operatorname{sc} u = \frac{\operatorname{cd} u}{1+k'\operatorname{sd} u} = \frac{\vartheta_c^{d-}(u)}{\vartheta_c^{d+}(u)}$$

$$\operatorname{cn} u+v\operatorname{sn} u = \frac{\operatorname{nc} u}{1+v\operatorname{sc} u} = \frac{\vartheta_n^{c-}(u)}{\vartheta_n^{c+}(u)}$$

$$\frac{\operatorname{dn} u-k\operatorname{cn} u}{1-k} = \frac{(1+k)\operatorname{nd} u}{1+k\operatorname{cd} u} = \frac{\vartheta_n^{n-}(u)}{\vartheta_n^{n+}(u)}$$

$$\operatorname{dn} u+vk\operatorname{sn} u = \frac{\operatorname{nd} u}{1-vk\operatorname{sd} u} = \frac{\vartheta_n^{d-}(u)}{\vartheta_n^{d+}(u)}$$

$$\operatorname{cd} u-vk'\operatorname{sd} u = \frac{\operatorname{dc} u}{1+vk'\operatorname{sc} u} = \frac{\vartheta_d^{c-}(u)}{\vartheta_d^{c+}(u)}$$

$$\operatorname{nd} u+k\operatorname{sd} u = \frac{\operatorname{dn} u}{1-k\operatorname{sn} u} = \frac{\vartheta_d^{n-}(u)}{\vartheta_d^{n+}(u)}$$

$$\frac{k\operatorname{cd} u+vk'\operatorname{nd} u}{k+vk'} = \frac{(k-vk')\operatorname{dc} u}{k-vk'\operatorname{nc} u} = \frac{\vartheta_d^{d-}(u)}{\vartheta_d^{d+}(u)}$$

Particularly interesting is the expression for $\operatorname{cn} u+v\operatorname{sn} u$, for this function is $e^{v\operatorname{am} u}$. Inserting from $\cdot 64_{1\text{-}2}$ the values of $\vartheta_n^{c-}(u)$ and $\vartheta_n^{c+}(u)$ as products, we have explicitly

$$e^{v\operatorname{am} u} = e^{vv}\prod_{n=0}^{\infty}\frac{(1-q^{4n+1}e^{-2vv})(1-q^{4n+3}e^{2vv})}{(1-q^{4n+1}e^{2vv})(1-q^{4n+3}e^{-2vv})}.$$

But
$$\log\prod_{n=0}^{\infty}(1-q^{4n+p}e^{2vv}) = -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}q^{m(4n+p)}e^{2mvv}$$

$$= -\sum_{m=1}^{\infty}\frac{q^{pm}e^{2mvv}}{m(1-q^{4m})},$$

and therefore

$$\operatorname{am} u = v+\sum_{1}^{\infty}\frac{(q^m-q^{3m})(e^{2mvv}-e^{-2mvv})}{vm(1-q^{4m})},$$

that is,

16·72 $$\operatorname{am} u = v + \frac{2q\sin 2v}{(1+q^2)} + \frac{2q^2\sin 4v}{2(1+q^4)} + \frac{2q^3\sin 6v}{3(1+q^6)} + \dots,$$

if the expansion of the logarithms is valid, that is, if $|qe^{2iv}|$ and $|qe^{-2iv}|$ are both less than 1; this condition can be written as $|e^{\pm 2iv-\sigma}| < 1$, that is, as $\mathrm{Rl}\,\sigma \pm 2\,\mathrm{Im}\,v > 0$: the point u is in the strip

$$-\mathrm{Rl}\,\sigma < \mathrm{Im}\{(\pi/K)u\} < \mathrm{Rl}\,\sigma.$$

The rearrangement just applied to the logarithms can be applied to logarithmic derivatives. If λ is any constant such that $|\lambda e^{iv}| < 1$, then

$$\frac{d}{dv}\log(1-\lambda e^{iv}) = -i\sum_{m=1}^{\infty}\lambda^m e^{miv},$$

and therefore, if $|\lambda| < 1$,

$$\frac{d}{dv}\log\prod_{n=1}^{\infty}(1-\lambda^n e^{iv}) = -i\sum_{m=1}^{\infty}\frac{\lambda^m e^{miv}}{1-\lambda^m}.$$

In each of the quotients $\vartheta_p^{n-}(u)/\vartheta_p^{n+}(u)$ the products are of the form suitable for this transformation, and only the restrictions on the range of v have to be supplied. The factor dv/du, that is, $\pi/2K$, enters throughout, and a second factor has to be inserted from Table XIV1, for the logarithmic derivative of $\mathrm{rp}\,u - \mathrm{rq}\,K_p\,\mathrm{qp}\,u$ is not necessarily the function $\mathrm{tp}\,u$ but is a constant multiple of $\mathrm{tp}\,u$. The complete set of formulae is contained in the following pair of theorems.

16·73. *Within the strip* $-\pi\,\mathrm{Rl}(K'/K) < \mathrm{Im}\,v < \pi\,\mathrm{Rl}(K'/K)$,

·73₁ $$\frac{K}{2\pi}\operatorname{cs}\frac{2Kv}{\pi} = \tfrac{1}{4}\cot v - \frac{q^2\sin 2v}{1+q^2} - \frac{q^4\sin 4v}{1+q^4} - \frac{q^6\sin 6v}{1+q^6} - \dots,$$

·73₂ $$\frac{K}{2\pi}\operatorname{ns}\frac{2Kv}{\pi} = \tfrac{1}{4}\csc v + \frac{q\sin v}{1-q} + \frac{q^3\sin 3v}{1-q^3} + \frac{q^5\sin 5v}{1-q^5} + \dots,$$

·73₃ $$\frac{K}{2\pi}\operatorname{ds}\frac{2Kv}{\pi} = \tfrac{1}{4}\csc v - \frac{q\sin v}{1+q} - \frac{q^3\sin 3v}{1+q^3} - \frac{q^5\sin 5v}{1+q^5} - \dots,$$

·73₄ $$\frac{k'K}{2\pi}\operatorname{sc}\frac{2Kv}{\pi} = \tfrac{1}{4}\tan v - \frac{q^2\sin 2v}{1+q^2} + \frac{q^4\sin 4v}{1+q^4} - \frac{q^6\sin 6v}{1+q^6} + \dots,$$

·73₅ $$\frac{K}{2\pi}\operatorname{dc}\frac{2Kv}{\pi} = \tfrac{1}{4}\sec v + \frac{q\cos v}{1-q} - \frac{q^3\cos 3v}{1-q^3} + \frac{q^5\cos 5v}{1-q^5} - \dots,$$

·73₆ $$\frac{k'K}{2\pi}\operatorname{nc}\frac{2Kv}{\pi} = \tfrac{1}{4}\sec v - \frac{q\cos v}{1+q} + \frac{q^3\cos 3v}{1+q^3} - \frac{q^5\cos 5v}{1+q^5} + \dots.$$

16·74. *Within the strip* $-\frac{1}{2}\pi\,\mathrm{Rl}(K'/K) < \mathrm{Im}\,v < \frac{1}{2}\pi\,\mathrm{Rl}(K'/K)$,

·74₁ $$\frac{K}{2\pi}\mathrm{dn}\frac{2Kv}{\pi} = \frac{1}{4}+\frac{q\cos 2v}{1+q^2}+\frac{q^2\cos 4v}{1+q^4}+\frac{q^3\cos 6v}{1+q^6}+\ldots,$$

·74₂ $$\frac{kK}{2\pi}\mathrm{sn}\frac{2Kv}{\pi} = \frac{r\sin v}{1-q}+\frac{r^3\sin 3v}{1-q^3}+\frac{r^5\sin 5v}{1-q^5}+\ldots,$$

·74₃ $$\frac{kK}{2\pi}\mathrm{cn}\frac{2Kv}{\pi} = \frac{r\cos v}{1+q}+\frac{r^3\cos 3v}{1+q^3}+\frac{r^5\cos 5v}{1+q^5}+\ldots,$$

·74₄ $$\frac{k'K}{2\pi}\mathrm{nd}\frac{2Kv}{\pi} = \frac{1}{4}-\frac{q\cos 2v}{1+q^2}+\frac{q^2\cos 4v}{1+q^4}-\frac{q^3\cos 6v}{1+q^6}+\ldots,$$

·74₅ $$\frac{kK}{2\pi}\mathrm{cd}\frac{2Kv}{\pi} = \frac{r\cos v}{1-q}-\frac{r^3\cos 3v}{1-q^3}+\frac{r^5\cos 5v}{1-q^5}-\ldots,$$

·74₆ $$\frac{kk'K}{2\pi}\mathrm{sd}\frac{2Kv}{\pi} = \frac{r\sin v}{1+q}-\frac{r^3\sin 3v}{1+q^3}+\frac{r^5\sin 5v}{1+q^5}-\ldots.$$

In these formulae K, K' denote K_c, K_n/v, and therefore by the definition of v, $\mathrm{Rl}(K'/K)$ is essentially positive. In ·74, as elsewhere, r is a definite value of $q^{1/2}$.

16·8. In the formal sense, theta functions and q-series solve superbly the problem of inverting the elliptic integrals. If we express $\mathrm{dn}\,u$ as $\vartheta_d(u)/\vartheta_n(u)$ and substitute $v = \frac{1}{2}\pi$ in $\cdot 55_{3\text{-}4}$, we have

16·81 $$k' = \left(\frac{1-2q+2q^4-2q^9+\ldots}{1+2q+2q^4+2q^9+\ldots}\right)^2,$$

whence, if $h'^2 = k'$, one value of h' is connected with q by the relation

·801 $$\frac{1-h'}{1+h'} = \frac{2q+2q^9+2q^{25}+\ldots}{1+2q^4+2q^{16}+2q^{36}+\ldots}.$$

If q is given, this relation determines h', and therefore determines k', c', and c. Conversely, if the parameters are given, and if

$$\epsilon = \tfrac{1}{2}(1-h')/(1+h'),$$

the equation

·802 $$\frac{q+q^9+q^{25}+\ldots}{1+2q^4+2q^{16}+2q^{36}+\ldots} = \epsilon$$

has only one solution which vanishes with ϵ, and this solution is developable as a power series

·803 $$q = \epsilon+a_1\epsilon^5+a_2\epsilon^9+\ldots$$

which can be shown to be convergent if $|\epsilon| < \frac{1}{2}$.

With q known, the condition $\mathrm{sn}\,K_c = 1$ gives a variety of expressions for K_c; in particular, from $\cdot 73_2$

16·82 $$\frac{2K_c}{\pi} = 1+\frac{4q}{1-q}-\frac{4q^3}{1-q^3}+\frac{4q^5}{1-q^5}-\ldots,$$

and from $\cdot 55_1$ and $\cdot 55_3$

16·83 $$\frac{2K_c}{\pi} = \frac{1-3q^2+5q^6-7q^{12}+\ldots}{1+q^2+q^6+q^{12}+\ldots}\cdot\frac{1+2q+2q^4+2q^9+\ldots}{1-2q+2q^4-2q^9+\ldots}.$$

And from the definition of q, the value of K_n follows immediately from the values of q and K_c.

These developments do not touch the theoretical inversion problem, the problem of ubiquity, except by providing powerful means of attack. The problem in this form is to prove that the equation ·81, as an equation in q, possesses solutions for every finite value of k' except $k' = 0$, and that the aggregate of solutions for a given value of k' other than $k' = 1$ is an automorphic aggregate corresponding to the aggregate of values of vK_n/K_c belonging to one and the same Jacobian system.

16·9. The effects of halfperiod and quarterperiod additions on theta functions are summed up in (i) the pair of formulae

$16\cdot 91_{1-2}$ $$\mathrm{H}(u+2K_c) = -\mathrm{H}(u), \qquad \Theta(u+2K_n) = \Theta(u),$$

given above as $\cdot 51_{1-2}$, (ii) a comprehensive induction from ·42 and ·403, namely,

16·92. *If* $\Phi(u)$ *is either of the two functions* $\mathrm{H}(u)$, $\Theta(u)$, *then*

$$\Phi(u+mK_n) = v^m q^{-m^2/4} e^{-mvv^1}\Psi(u),$$

where $\Psi(u)$ *is the same function as* $\Phi(u)$ *if* m *is even and is the other of the two functions if* m *is odd,*

and (iii) the following permutations:

$16\cdot 93_1$ $\vartheta_s(u+K_c) = \{\mathrm{H}(K_c)/\mathrm{H}'(0)\}\vartheta_c(u),$

$16\cdot 93_2$ $\vartheta_s(u+K_n) = v\{q^{-1/4}\Theta(0)/\mathrm{H}'(0)\}e^{-vv}\vartheta_n(u),$

$16\cdot 93_3$ $\vartheta_c(u+K_c) = -\{\mathrm{H}'(0)/\mathrm{H}(K_c)\}\vartheta_s(u),$

$16\cdot 93_4$ $\vartheta_c(u+K_n) = \{q^{-1/4}\Theta(K_c)/\mathrm{H}(K_c)\}e^{-vv}\vartheta_d(u).$

$16\cdot 93_5$ $\vartheta_n(u+K_c) = \{\Theta(K_c)/\Theta(0)\}\vartheta_d(u),$

$16\cdot 93_6$ $\vartheta_n(u+K_n) = v\{q^{-1/4}\mathrm{H}'(0)/\Theta(0)\}e^{-vv}\vartheta_s(u),$

$16\cdot 93_7$ $\vartheta_d(u+K_c) = \{\Theta(0)/\Theta(K_c)\}\vartheta_n(u),$

$16\cdot 93_8$ $\vartheta_d(u+K_n) = \{q^{-1/4}\mathrm{H}(K_c)/\Theta(K_c)\}e^{-vv}\vartheta_c(u).$

These results are all derived from the definitions of $\mathrm{H}(u)$ and $\Theta(u)$ as series, and if doubly periodic functions with simple poles have not been constructed otherwise, theta functions provide an austere method of introduction. The variable is at first v and one quarterperiod is $\frac{1}{2}\pi$; the transition to u is again made with a view to the function $\operatorname{sn} u$, but the general lattice does not come into the picture, and the Jacobian lattice is seen rather as an alternative to one other lattice than as the canonical representative of a class.

XVII

REAL FUNCTIONS AND REAL INTEGRALS

17·1. If the parameter c is real, the six members of the anharmonic group of numbers to which c belongs are all real. Of the six numbers, two are negative, two are positive and greater than 1, and two are positive and less than 1. In dealing with Jacobian systems with a real modulus, we lose nothing by adopting as a canonical system one whose parameter and modulus satisfy the conditions

$$0 < c < 1, \qquad 0 < k < 1.$$

A system whose parameter does not satisfy the first of these conditions can be derived from a canonical system by one of the transformations of the anharmonic group, that is, by a combination of Jacobi's two transformations.

With $0 < k < 1$, the integral relation

·101 $$u = \int_x^\infty \frac{dx}{\sqrt{\{(x^2-1)(x^2-k^2)\}}}$$

is a case of the relation studied in Chapter IX. There is a basis composed of a real quarterperiod K and an imaginary quarterperiod iK', where K, K' are given by

17·11$_{1-2}$ $$K = \int_1^\infty \frac{dt}{\sqrt{\{(t^2-1)(t^2-k^2)\}}}, \qquad K' = \int_0^\infty \frac{dt}{\sqrt{\{(t^2+1)(t^2+k^2)\}}},$$

the integrations with respect to t being along the positive half of the real axis and the radicals being positive. The integral

$$\int_k^\infty \frac{dt}{\sqrt{\{(t^2-1)(t^2-k^2)\}}}$$

is mixed, whatever the path of integration.

Since K, iK' are values of u in ·101 corresponding to the values 1, 0 of x, it follows that if x is regarded as a function of u, then iK' is a zero of this function and the value of the function when $u = K$ is unity. That is to say, K, iK' is a Jacobian basis, and the integral relation ·101 is equivalent to the functional relation $x = \operatorname{ns} u$ on this basis.

In terms of the Weierstrassian function $\wp(u; K_c, K_n, K_d)$, where

$$K_c = K, \qquad K_n = iK', \qquad K_d = -K-iK',$$

we have

·102–·104 $\mathrm{cs}^2u = \wp u - e_c, \qquad \mathrm{ns}^2u = \wp u - e_n, \qquad \mathrm{ds}^2u = \wp u - e_d,$

where e_c, e_n, e_d denote $\wp K_c$, $\wp K_n$, $\wp K_d$. From the double series defining $\wp u$ in terms of K and iK', the function is real if u is either real or imaginary; in particular e_c and e_n are real, and therefore, since $e_c+e_n+e_d = 0$, e_d also is real. It follows that the two products

$$\{\wp u - e_c\}\{\wp(u+K)-e_c\}, \qquad \{\wp u - e_n\}\{\wp(u+iK')-e_n\},$$

which have constant values, are real, and therefore that $\wp(u+K)$ and $\wp(u+iK')$ are real whenever $\wp u$ is real. Hence if S, C, D, N denote the points 0, K, $K+iK'$, iK', the function $\wp u$ is real on the perimeter of the rectangle $SCDN$, and from ·102–·104 the same is true of the three functions cs^2u, ns^2u, ds^2u. The property extends algebraically to reciprocals and quotients:

17·12. *If* $0 < c < 1$, *the squares of the twelve Jacobian functions are all real on the perimeter of the fundamental rectangle* $SCDN$.

The function $\mathrm{pq}\,u$ has one of the four points S, C, D, N for a zero and one for a pole. These points, which may be denoted by P, Q, divide the perimeter $SCDNS$ into two stretches. If u describes the perimeter, it is only as u passes through P or Q that $\mathrm{pq}\,u$ can change character from real to imaginary, or can change sign from positively real or imaginary to negatively real or imaginary. At the origin, or for small positive real values of u, each of the Jacobian functions is real and positive. Hence $\mathrm{pq}\,u$ is real and positive throughout that stretch from P to Q which includes the side SC. The two sides of the rectangle which meet at P meet at right angles, and because the zero of $\mathrm{pq}\,u$ at that point is simple, the function, which is real in one direction from P, is imaginary in the perpendicular direction; the real values being positive, the imaginary values are positive or negative according as rotation from the real side to the imaginary side through one right angle is in the positive or in the negative direction. We can therefore recognize geometrically both the character and the sign of $\mathrm{pq}\,u$ on each stretch of the perimeter, and since we can write down from the classical formulae

$$\mathrm{cn}^2u = 1-\mathrm{sn}^2u, \qquad \mathrm{dn}^2u = 1-k^2\,\mathrm{sn}^2u$$

the values of pq^2u at the corners where it is not zero or infinite, we can complete without difficulty the set of diagrams on p. 290.

We can insert leading coefficients at the poles and zeros in these diagrams: at the origin in $\mathrm{cs}\,u$, $\mathrm{ns}\,u$, $\mathrm{ds}\,u$ the leading coefficient is 1, and since the derivative of each of these functions is the negative of the product of the other two, the leading coefficients at their zeros are $-k'$, k, ikk'; any other function is derivable as a reciprocal or a quotient from these three.

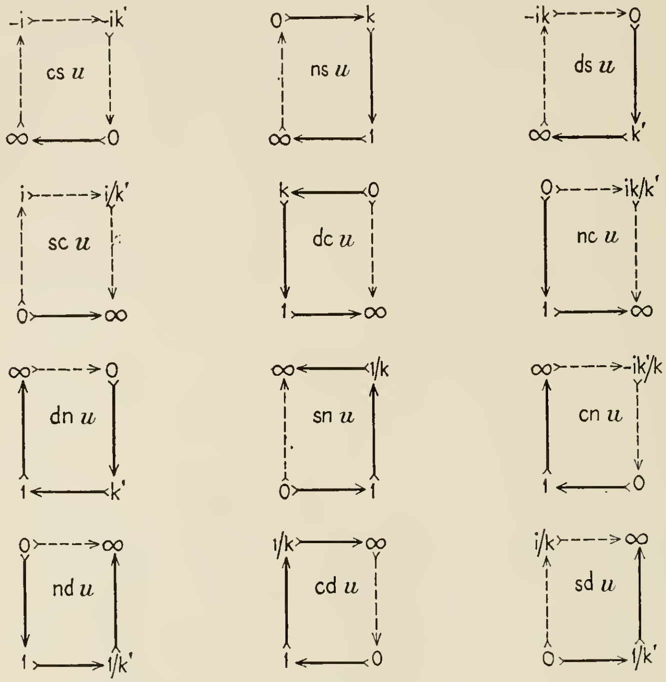


FIG. 33.

The continuous lines show the sides along which the functions are real, the dotted lines the sides along which the functions are imaginary. Arrowheads point towards zero from negative real or negatively imaginary values, away from zero towards positive real or positively imaginary values.

A function of u can not change from real to imaginary as u describes a path without a sudden change of direction, if the only singularities of the function on the path are poles, whether or not the path passes through any zeros. Hence the character of the function pq u along a side of the fundamental rectangle is maintained along the whole infinite line of which that side forms part. In particular,

17·13. *With a real parameter between* 0 *and* 1, *all the Jacobian functions are real for all real values of u; for imaginary values of u, the six even functions are real and the six odd functions are imaginary.*

The variation of a function along a produced side of the fundamental rectangle is seen most readily in terms of zeros and poles.

Suppose first that P, Q are opposite corners of the rectangle, and denote the rectangle by $PRQT$. On the lines PR and PT there are zeros and no poles; on the one line, $\text{pq}\,u$ oscillates between $-\text{pq}\,K_r$ and $+\text{pq}\,K_r$, on the other line, between $-\text{pq}\,K_t$ and $+\text{pq}\,K_t$; one set of values is real, the other imaginary. For example, along the real axis $\text{cn}\,u$ oscillates between -1 and $+1$, and along CD this function oscillates between $-ik'/k$ and $+ik'/k$. On the lines QR and QT there are poles and no zeros; $\text{pq}\,u$ remains outside the range $(-\text{pq}\,K_r, +\text{pq}\,K_r)$ on QR, outside the range $(-\text{pq}\,K_t, +\text{pq}\,K_t)$ on QT, and is real on one of these lines, imaginary on the other. Along the imaginary axis, $\text{cn}\,u$ falls from $+\infty$ to $+1$ and rises again to $+\infty$ as u increases from $-iK'$ to $+iK'$, rises from $-\infty$ to -1 and falls again to $-\infty$ as u increases from iK' to $3iK'$; along the line ND, $\text{cn}\,u$ rises from $-i\infty$ to $-ik'/k$ and falls again to $-i\infty$ as $u-iK'$ increases from 0 to $2K$, falls from $+i\infty$ to $+ik'/k$ and rises again to $+i\infty$ as $u-iK'$ increases from $2K$ to $4K$, and so on.

Secondly, let P, Q be adjacent corners of the rectangle, which can now be denoted by $PQRT$. The variations along the lines PT and QR are of the two kinds already described. For example, along the real axis $\text{sn}\,u$ oscillates between -1 and $+1$, along the imaginary axis $\text{sc}\,u$ oscillates between $-i$ and $+i$. Along ND, $\text{sn}\,u$ is real and either greater than $+1/k$ or less than $-1/k$, and along CN, $\text{sc}\,u$ is imaginary and takes no values between $-i/k'$ and $+i/k'$. But now there are two other types of variation. On the line RT, $\text{pq}\,u$ has no poles or zeros, and oscillates between $\text{pq}\,K_r$ and $\text{pq}\,K_t$, which in this case necessarily have the same sign. For example, along the real axis $\text{dn}\,u$ oscillates between $+k'$ and $+1$, along CD, $\text{sn}\,u$ oscillates between $+1$ and $+1/k$, and along ND, $\text{cs}\,u$ oscillates between $-i$ and $-ik'$. Lastly, on the line PQ the function $\text{pq}\,u$ has zeros and poles alternating, and takes all values of the right kind, that is, all real values or all imaginary values, changing sign at each pole as well as at each zero, and therefore showing the same direction of increase everywhere along the line. Thus the values of $\text{sc}\,u$ increase steadily from $-\infty$ to $+\infty$ as u increases through real values from $-K$ to K, and repeat this increase as u increases from K to $3K$, from $3K$ to $5K$, and so on; similarly $\text{sn}\,u$ increases steadily from $-i\infty$ to $+i\infty$ as u increases through imaginary values from $-3iK'$ to $-iK'$, from $-iK'$ to iK', from iK' to $3iK'$, and so on.

17·2. We made occasional use in Chapters VII and VIII of the generation of an elliptic function as a particular integral of a differential equation of the first order which when made rational in the dependent variable is of the fourth degree in the variable and of the second degree in the derivative. If we generate a copolar triad of Jacobian functions by means of simultaneous equations, the individual equations are much simpler. Writing

·201–·203 $$\operatorname{sn} u = x, \qquad \operatorname{cn} u = y, \qquad \operatorname{dn} u = z,$$

we have

·204–·206 $$dx/du = yz, \qquad dy/du = -xz, \qquad dz/du = -cxy,$$

with the initial conditions

·207–·209 $$x(0) = 0, \qquad y(0) = 1, \qquad z(0) = 1,$$

and in this definition of the functions there are no ambiguities to be resolved. The initial values of x, y, z, substituted in ·204–·206, give the initial values of the first derivatives, and by successive differentiation of ·204–·206 we obtain initial derivatives of as high an order as we wish, and so, theoretically, Taylor expansions for the three functions near the origin. From these expansions the functions can be continued analytically.

It is no part of our design to develop the subject logically and thoroughly on a fresh foundation, nor does this method offer any of the advantages of a method in which the double periodicity is known in advance, but assessed as an illustration of the manipulation of a set of equations, the examination of real and imaginary values of the functions x, y, z defined by ·204–·206, in the case in which c is real, is instructive.

For definiteness, we suppose from the first that $0 < c < 1$. Because c is real, all the derivatives, of whatever order, are real for real values of x, y, z, and the functions are real for sufficiently small real values of u. By integration,

·210–·211 $$x^2+y^2 = 1, \qquad cx^2+z^2 = 1,$$

and therefore

·212 $$u = \int_0^x \frac{dt}{\sqrt{\{(1-t^2)(1-ct^2)\}}},$$

with the positive value of the square root near $t = 0$. The formula persists as far as $x = 1$, since by hypothesis $c < 1$.

With the square root positive, the formula

·213 $$u = \int_0^\xi \frac{dt}{\sqrt{\{(1-t^2)(1-ct^2)\}}}$$

defines u as a real monotonic function of ξ from $\xi = -1$ to $\xi = 1$, and therefore defines ξ as a real monotonic function $\xi(u)$ of u from $u = -K$ to $u = K$, where

·214 $$K = \int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(1-ct^2)\}}},$$

a value easily identified, if $c = k^2$, with K as defined in $\cdot 11_1$. With $\xi(u)$ so defined, the set of equations ·204–·206 with the initial conditions ·207–·209 has for its unique solution over the range $-K \leqslant u \leqslant K$

·215 $$x(u) = \xi(u), \qquad y(u) = \sqrt{\{1-\xi^2(u)\}}, \qquad z(u) = \sqrt{\{1-c\xi^2(u)\}},$$

where the square roots are defined to be positive. Within the range,

·216 $$x(-u) = -x(u), \qquad y(-u) = y(u), \qquad z(-u) = z(u),$$

and the extreme values are

·217 $$x(-K) = -1, \qquad y(-K) = 0, \qquad z(-K) = k',$$

·218 $$x(K) = 1, \qquad y(K) = 0, \qquad z(K) = k',$$

where k' is the positive square root of $1-c$.

To extend the range, we consider the set of values at K as a set of initial values operating to maintain the identity of our set of solutions of the set of differential equations. At K, since xz is positive, dy/du is negative, and therefore for sufficiently small positive values of $u-K$, y is the negative square root of $1-x^2$, and if the radical is read as positive,

$$\frac{du}{dx} = -\frac{1}{\sqrt{\{(1-x^2)(1-cx^2)\}}}.$$

Beyond K, x decreases from 1, and we have

$$u-K = \int_x^1 \frac{dt}{\sqrt{\{(1-t^2)(1-ct^2)\}}},$$

a relation that persists until the radical becomes zero, that is, while x decreases from 1 to -1 and $u-K$ increases from 0 to $2K$. Thus for $K \leqslant u \leqslant 3K$,

$$u-K = \left(\int_0^1 - \int_0^x\right) \frac{dt}{\sqrt{\{(1-t^2)(1-ct^2)\}}} = K - \int_0^x \frac{dt}{\sqrt{\{(1-t^2)(1-ct^2)\}}},$$

that is,

·219 $$x(u) = \xi(2K-u) = -\xi(u-2K) = -x(u-2K).$$

It follows that $y(u)$, $z(u)$ are numerically equal to $y(u-2K)$, $z(u-2K)$, and since $y(u)$ is negative and $z(u)$ positive,

·220 $$y(u) = -y(u-2K), \qquad z(u) = z(u-2K).$$

At the other extreme of this second range,

·221 $$x(3K) = -1, \qquad y(3K) = 0, \qquad z(3K) = k',$$

and this is the same set of values as the set ·217 for $-K$; hence $4K$ is a period of the three functions, as functions of a real variable, and ·219–·220, which may be written

·222 $$x(u+2K) = -x(u), \qquad y(u+2K) = -y(u), \qquad z(u+2K) = z(u),$$

hold for all real values of u. The variation of the three functions along the whole of the real axis can now be described.

Next we consider the functions as functions of the real variable v, where $u = iv$. The differential equations become

·223 $$dx/dv = iyz, \qquad dy/dv = -ixz, \qquad dz/dv = -icxy,$$

and to remove i from this set of equations we have only to introduce i as a factor into one of the dependent variables. If we are not thereby to introduce i into one of the initial values ·207–·209, it is x that we must modify, and we write

·224 $$u = iv, \qquad x(u) = i\bar{x}(v), \qquad y(u) = \bar{y}(v), \qquad z(u) = \bar{z}(v).$$

We have then the set of equations

·225 $$d\bar{x}/dv = \bar{y}\bar{z}, \qquad d\bar{y}/dv = \bar{x}\bar{z}, \qquad d\bar{z}/dv = c\bar{x}\bar{y},$$

with the initial conditions

·226 $$\bar{x}(0) = 0, \qquad \bar{y}(0) = 1, \qquad \bar{z}(0) = 1.$$

On account of the changes of sign in the differential equations, ·210–·211 are replaced by

·227–·228 $$\bar{y}^2-\bar{x}^2 = 1, \qquad \bar{z}^2-c\bar{x}^2 = 1,$$

and the relation between $\bar{x}$ and v is given, for some finite range of values of v, by

·229 $$v = \int_0^{\bar{x}} \frac{dt}{\sqrt{\{(1+t^2)(1+ct^2)\}}},$$

with a positive radical. The relation

·230 $$v = \int_0^{\eta} \frac{dt}{\sqrt{\{(1+t^2)(1+ct^2)\}}}$$

defines v as a real monotonic function of η, and η inversely as a real monotonic function $\eta(v)$ of v; the range of η is unlimited, but the integral

·231 $$\int\limits_0^\infty \frac{dt}{\sqrt{\{(1+t^2)(1+ct^2)\}}}$$

has a finite value K', and it is only for the range $-K' \leqslant v \leqslant K'$ that the function $\eta(v)$ is available. The solution of ·225, ·226 for this range is

·232 $$\bar{x}(v) = \eta(v), \qquad \bar{y}(v) = \sqrt{\{1+\eta^2(v)\}}, \qquad \bar{z}(v) = \sqrt{\{1+c\eta^2(v)\}},$$

and for the same range

·233 $$x(iv) = i\eta(v), \qquad y(iv) = \sqrt{\{(1+\eta^2(v)\}}, \qquad z(iv) = \sqrt{\{1+c\eta^2(v)\}},$$

all square roots having their positive values.

At $v = K'$ the functions $\bar{x}(v)$, $\bar{y}(v)$, $\bar{z}(v)$ become infinite, and extension of the range presents a fresh problem; two solutions may be indicated. We can provide finite values by changing the functions, writing

$$\bar{x}(v) = 1/X(v), \qquad \bar{y}(v) = Y(v)/X(v), \qquad \bar{z}(v) = Z(v)/X(v)$$

and crossing over by means of the values

$$X(K') = 0, \qquad Y(K') = 1, \qquad Z(K') = k.$$

Alternatively, we have from ·230, ·231, for small values of $K'-v$,

$$\eta(v) \sim \frac{1/k}{K'-v},$$

implying

·234 $$\bar{x}(v) \sim -\frac{1/k}{v-K'}, \qquad \bar{y}(v) \sim -\frac{1/k}{v-K'}, \qquad \bar{z}(v) \sim -\frac{1}{v-K'},$$

and these asymptotic formulae must persist through the pole from negative to positive values of $v-K'$. It will be found that for the succeeding range $K' \leqslant v \leqslant 3K'$,

·235 $$\bar{x}(v) = \bar{x}(v-2K'), \qquad \bar{y}(v) = -\bar{y}(v-2K'), \qquad \bar{z}(v) = -\bar{z}(v-2K').$$

From the point of view of the complex variable, the asymptotic formulae ·234, in the form

·236 $$x(u) \sim \frac{1/k}{u-iK'}, \qquad y(u) \sim -\frac{i/k}{u-iK'}, \qquad z(u) \sim -\frac{i}{u-iK'},$$

are effective not merely along the imaginary axis but throughout the neighbourhood of iK', and they serve to identify the solutions, from whatever direction the point iK' is approached.

Just as the zero of $\operatorname{sn} u$ at the origin enables us to reduce the discussion of the functions $\operatorname{sn} iv$, $\operatorname{cn} iv$, $\operatorname{dn} iv$ for real values of v to the solution of a set of differential equations satisfied by real functions of v, so the zero of $\operatorname{cn} u$ at $u = K$ enables us to deal with the functions $\operatorname{sn}(K+iv)$, $\operatorname{cn}(K+iv)$, $\operatorname{dn}(K+iv)$. The set of equations satisfied by this set of functions is again ·223, but we must now introduce i as a factor into y, since the initial values of x and z are finite and different from zero. Writing

·237 $$x(K+iv) = \bar{x}(v), \qquad y(K+iv) = i\bar{y}(v), \qquad z(K+iv) = \bar{z}(v),$$

we have

·238 $$d\bar{x}/dv = -\bar{y}\bar{z}, \qquad d\bar{y}/dv = -\bar{x}\bar{z}, \qquad d\bar{z}/dv = c\bar{x}\bar{y},$$

with

·239 $$\bar{x}(0) = 1, \qquad \bar{y}(0) = 0, \qquad \bar{z}(0) = k',$$

and now

·240–·241 $$\bar{x}^2-\bar{y}^2 = 1, \qquad c\bar{y}^2+\bar{z}^2 = c',$$

where $c' = 1-c$. We require a monotonic function $\zeta(v)$ defined by

·242 $$v = \int\limits_0^{\zeta} \frac{dt}{\sqrt{\{(t^2+1)(c'-ct^2)\}}}$$

for $-k'/k \leqslant \zeta \leqslant k'/k$. The substitution $t^2/(c'-ct^2) = t'^2$ identifies

·243 $$\int\limits_0^{k'/k} \frac{dt}{\sqrt{\{(t^2+1)(c'-ct^2)\}}}$$

with K' as defined by ·231, and for the range $-K' \leqslant v \leqslant K'$,

·244 $$x(K+iv) = \sqrt{\{1+\zeta^2(v)\}},$$
$$y(K+iv) = -i\zeta(v),$$
$$z(K+iv) = \sqrt{\{c'-c\zeta^2(v)\}}.$$

The extreme values are finite, namely,

·245 $$x(K+iK') = 1/k, \qquad y(K+iK') = -ik'/k, \qquad z(K+iK') = 0,$$

and the extension of the range presents no difficulty. It may be observed that if we choose an auxiliary function to suit the expressions of $\bar{y}^2$ and $\bar{z}^2$ in terms of $\bar{x}^2$, the range of v is more restricted and the construction of the solution proceeds by shorter steps.

Lastly we consider the line through iK' parallel to the real axis. The differential equations ·204–·206 are unaltered, but to adapt either the asymptotic formulae ·236 at iK' or the values ·245 at $K+iK'$ to real functions we must introduce i into $y(u)$ and $z(u)$, changing two of the three functions since now it is only through the functions that i can enter. If we propose to use the values ·245, we write

·246 $$x(K+iK'+w) = \bar{x}(w),\quad y(K+iK'+w) = i\bar{y}(w),\quad z(K+iK'+w) = i\bar{z}(w).$$

The differential equations are

·247 $$d\bar{x}/dw = -\bar{y}\bar{z},\qquad d\bar{y}/dw = -\bar{x}\bar{z},\qquad d\bar{z}/dw = -c\bar{x}\bar{y},$$

with the conditions

·248 $$\bar{x}(0) = 1/k,\qquad \bar{y}(0) = -k'/k,\qquad \bar{z}(0) = 0,$$

and the quadratic relations are

·249–·250 $$c\bar{x}^2-\bar{z}^2 = 1,\qquad c\bar{y}^2-\bar{z}^2 = c'.$$

The formula

·251 $$w = \int_0^{\varpi} \frac{dt}{\sqrt{\{(t^2+1)(t^2+c')\}}}$$

defines $\varpi(w)$ over a range easily identified as $-K \leqslant w \leqslant K$, and over this range,

·252 $$\bar{x}(w) = \sqrt{\{1+\varpi^2(w)\}}/k,\quad \bar{y}(w) = -\sqrt{\{c'+\varpi^2(w)\}}/k,\quad \bar{z}(w) = \varpi(w).$$

Hence, for $-K \leqslant w \leqslant K$,

·253 $$x(K+iK'+w) = \sqrt{\{1+\varpi^2(w)\}}/k,\quad y(K+iK'+w) = -i\sqrt{\{c'+\varpi^2(w)\}}/k,\quad z(K+iK'+w) = i\varpi(w).$$

For large negative values of ϖ, $w+K$ is small and positive, and

$$\varpi(w) \sim -\frac{1}{w+K};$$

writing $K+iK'+w = u$, we have

$$\varpi(w) \sim -\frac{1}{u-iK'},$$

and since the square roots in the formulae for $x(K+iK'+w)$ and $y(K+iK'+w)$ are essentially positive, we recover ·236, for small positive real values of $u-iK'$; to reach small negative real values of $u-iK'$ we have to extend the range of w, and ·253 is no longer applicable.

It need hardly be said that the results proved in this section are established by the arguments used here only for the lines along which the functions have been studied. We have not, for example, proved that $4K$ is a period of the Jacobian functions of a complex variable.

17·3. Like 9·2, sections ·1, ·2 deal with the perimeter of the fundamental rectangle. We can appropriate the result of 9·46, since the factors which convert the functions of the earlier chapters into Jacobian functions are purely real or purely imaginary.

17·31. *If x and u are complex variables, and if the parameter c of the Jacobian function* $\mathrm{pq}\,u$ *is a real number between* 0 *and* 1, *the transformation* $x = \mathrm{pq}\,u$ *maps the fundamental rectangle and its boundary in the u plane on a quadrant and its boundary in the x plane.*

Since the real values of $\mathrm{pq}\,u$ on the boundary of the fundamental rectangle are positive, the quadrant of the x plane that is mapped is either the first or the fourth; if k as well as c is positive, the quadrant is the first for the three functions with the origin for a zero, and for the three functions $\mathrm{nc}\,u$, $\mathrm{nd}\,u$, $\mathrm{dc}\,u$.

The mapping is conformal except at the two points x_r, x_t on the boundary of the quadrant which correspond to the two corners K_r, K_t of the rectangle; the values of x_r, x_t, namely $\mathrm{pq}\,K_r$, $\mathrm{pq}\,K_t$, are the values shown explicitly in Figure 33. As in 9·4, we can distinguish three cases: x_r, x_t may be both on the real radius of the quadrant, one on the real radius and one on the imaginary radius, or both on the imaginary radius. There is, however, as the figure shows, no equality now between the third case and the first; the two exceptional points are on the real radius in six cases, on the imaginary radius in only two cases.

If x_r, x_t are given, that is, if a quadrant with a given pair of exceptional points is to be mapped, a suitable value of the parameter is seen at once from the set of diagrams: if x_r and x_t are on the same radius, the smaller of the ratios x_r/x_t, x_t/x_r can be taken either for k or for k'; if x_r and x_t are on different radii, the numerical ratio $|x_r/x_t|$ can be used either for k/k' or for k'/k.

If it is the rectangle that is given, we have the ratio of K to K'. We can construct a function $\mathrm{gj}(z; \omega_f, \omega_g, \omega_h)$ with $\omega_f : \omega_g = K : iK'$ and find the normalizing factor from this function. We have now very little choice in the numerical values of x_r and x_t, but since our choice among the twelve functions on the Jacobian basis K, iK' is still free, we can choose the radii on which the points x_r, x_t are to be found.

17·4. For real integration, the diagrams composing Figure 33 render vivid the formulae of Table XI II. The results are similar to those in

9·5, but it is the essence of the Jacobian theory that the functions are real and positive for the definite range $0 \leqslant u \leqslant K$, and differences in detail make explicit enunciations necessary.

17·41$_{1-2}$. *If* $0 \leqslant x_1$, $0 \leqslant x_4$, *the values* u_1, u_4 *of the integrals*

$$\int_{x_1}^{\infty} \frac{dt}{\sqrt{\{(t^2+1)(t^2+k'^2)\}}}, \qquad \int_{0}^{x_4} \frac{dt}{\sqrt{\{(1+t^2)(1+k'^2t^2)\}}}$$

are determined by $\quad x_1 = \operatorname{cs} u_1, \quad x_4 = \operatorname{sc} u_4$

with the conditions $0 \leqslant u_1 \leqslant K$, $0 \leqslant u_4 \leqslant K$.

17·42$_{1-2}$. *If* $1 \leqslant x_2$, $1 \leqslant x_5$, *the values* u_2, u_5 *of the integrals*

$$\int_{x_2}^{\infty} \frac{dt}{\sqrt{\{(t^2-1)(t^2-k^2)\}}}, \qquad \int_{1}^{x_5} \frac{dt}{\sqrt{\{(t^2-1)(t^2-k^2)\}}}$$

are determined by $\quad x_2 = \operatorname{ns} u_2, \quad x_5 = \operatorname{dc} u_5$

with the conditions $0 \leqslant u_2 \leqslant K$, $0 \leqslant u_5 \leqslant K$.

17·43$_{1-2}$. *If* $k' \leqslant x_3$, $1 \leqslant x_6$, *the values* u_3, u_6 *of the integrals*

$$\int_{x_3}^{\infty} \frac{dt}{\sqrt{\{(t^2+k^2)(t^2-k'^2)\}}}, \qquad \int_{1}^{x_6} \frac{dt}{\sqrt{\{(k'^2t^2+k^2)(t^2-1)\}}}$$

are determined by $\quad x_3 = \operatorname{ds} u_3, \quad x_6 = \operatorname{nc} u_6$

with the conditions $0 \leqslant u_3 \leqslant K$, $0 \leqslant u_6 \leqslant K$.

17·44$_{1-2}$. *If* $k' \leqslant x_7 \leqslant 1$, $1 \leqslant x_{10} \leqslant 1/k'$, *the values* u_7, u_{10} *of the integrals*

$$\int_{x_7}^{1} \frac{dt}{\sqrt{\{(1-t^2)(t^2-k'^2)\}}}, \qquad \int_{1}^{x_{10}} \frac{dt}{\sqrt{\{(1-k'^2t^2)(t^2-1)\}}}$$

are determined by $\quad x_7 = \operatorname{dn} u_7, \quad x_{10} = \operatorname{nd} u_{10}$

with the conditions $0 \leqslant u_7 \leqslant K$, $0 \leqslant u_{10} \leqslant K$.

17·45$_{1-2}$. *If* $0 \leqslant x_{11} \leqslant 1$, $0 \leqslant x_8 \leqslant 1$, *the values* u_{11}, u_8 *of the integrals*

$$\int_{x_{11}}^{1} \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}, \qquad \int_{0}^{x_8} \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}$$

are determined by $\quad x_{11} = \operatorname{cd} u_{11}, \quad x_8 = \operatorname{sn} u_8$

with the conditions $0 \leqslant u_{11} \leqslant K$, $0 \leqslant u_8 \leqslant K$.

17·46$_{1\text{–}2}$. *If* $0 \leqslant x_9 \leqslant 1$, $0 \leqslant x_{12} \leqslant 1/k'$, *the values* u_9, u_{12} *of the integrals*

$$\int_{x_9}^{1} \frac{dt}{\sqrt{\{(k'^2+k^2t^2)(1-t^2)\}}}, \qquad \int_{0}^{x_{12}} \frac{dt}{\sqrt{\{(1+k^2t^2)(1-k'^2t^2)\}}}$$

are determined by $\quad x_9 = \operatorname{cn} u_9, \qquad x_{12} = \operatorname{sd} u_{12}$

with the conditions $0 \leqslant u_9 \leqslant K$, $0 \leqslant u_{12} \leqslant K$.

As in 9·5, the six possible forms of the radical, each associated with two natural values for the fixed limit of integration, provide twelve types of real integral, and by means of the twelve functions a standard integral of each type is given. Thus the evaluation of either of the integrals

$$\int_{x} \frac{dt}{\sqrt{\{(\kappa t^2+\lambda)(\mu t^2+\nu)\}}}, \qquad \int^{x} \frac{dt}{\sqrt{\{(\kappa t^2+\lambda)(\mu t^2+\nu)\}}},$$

for any combination of signs for which the radical can be real, requires only a substitution $t = \gamma w$ with a positive real value of γ; the necessary value of γ is obvious, and the result, in the form $x = \gamma \operatorname{pq} u$, unambiguous. A definite integral can be evaluated from either end of its range, that is, by either of two complementary functions.

The substitution $t^2 = w$ replaces the integrals in ·41–·46 by standard integrals such as

·401–·402 $$\int_{y_1}^{\infty} \frac{dw}{\sqrt{\{w(w+1)(w+c')\}}} = v_1, \qquad \int_{y_2}^{\infty} \frac{dw}{\sqrt{\{w(w-1)(w-c)\}}} = v_2,$$

with evaluations

·403–·404 $$y_1 = \operatorname{cs}^2 \tfrac{1}{2}v_1, \qquad y_2 = \operatorname{ns}^2 \tfrac{1}{2}v_2,$$

and so on. Conversely, the six integrals of each of the forms

$$\int_{y} \frac{dw}{\sqrt{\{w(\kappa w+\lambda)(\mu w+\nu)\}}}, \qquad \int^{y} \frac{dw}{\sqrt{\{w(\kappa w+\lambda)(\mu w+\nu)\}}}$$

for which the integrand is real for positive values of w are covered by a preliminary substitution $w = \delta t^2$, and the six of each form for which the integrand is real for negative values of w by a preliminary substitution $w = -\delta t^2$, where δ is positive in each case.

Although we use twelve functions in order to express each integral by means of a function appropriate to its sign-combination and to the range of integration, this does not mean that for practical applications we have to tabulate the twelve functions. If the value of one function is known, the value of any other can be inferred from the algebraic relation between the squares of two functions. For example, if the integral is of the type which provides the value x_{11} of $\operatorname{cd} u_{11}$, we have

$$\operatorname{sn}^2 u_{11} = \frac{\operatorname{sd}^2 u_{11}}{\operatorname{nd}^2 u_{11}} = \frac{1-x_{11}^2}{1-k^2 x_{11}^2},$$

and u_{11} can be identified from a table of the function $\operatorname{sn} u$. To put the same conclusion differently, the substitution

$$\frac{1-x_{11}^2}{1-k^2 x_{11}^2} = x_8^2$$

transforms the integral

$$\int_{x_{11}}^{1} \frac{dx_{11}}{\sqrt{\{(1-x_{11}^2)(1-k^2 x_{11}^2)\}}}$$

into

$$\int_{0}^{x_8} \frac{dx_8}{\sqrt{\{(1-x_8^2)(1-k^2 x_8^2)\}}}$$

and could be applied first if Legendre's integral was the only one to be recognized. But it is to be noticed that the substitution involves the modulus k and can be applied more readily to the function when the modulus is known than to the integral when the modulus has still to be found.

Fundamentally the distinction between the set of theorems 11·81–11·83 and the set ·41–·46 is that in the earlier set it is the identity of one manyvalued function with another that is affirmed—each value which occurs on one side occurs somewhere on the other side also—while in the latter set a particular value of the one function is identified with a particular value of the other. In the same way, the integrals derivable for K, namely

$$\int_0^{\infty} \frac{dt}{\sqrt{\{(t^2+1)(t^2+k'^2)\}}}, \qquad \int_0^{1} \frac{dt}{\sqrt{\{(1-t^2)(1-k^2 t^2)\}}},$$

$$\int_0^{1/k'} \frac{dt}{\sqrt{\{(1+k^2 t^2)(1-k'^2 t^2)\}}}, \qquad \int_{k'}^{1} \frac{dt}{\sqrt{\{(1-t^2)(t^2-k'^2)\}}},$$

though formally identical with those in 11·84 are now integrals from which K is determinable, since the paths of integration are assigned.

17·5. In this section we consider briefly the reduction to a standard form of the integral $\int dx/\sqrt{\phi(x)}$, where $\phi(x)$ is a general polynomial of the fourth or third degree with real coefficients. It is sufficient if we bring the polynomial in the radical to one of the forms

$$(\kappa t^2+\lambda)(\mu t^2+\nu), \qquad w(\kappa w+\lambda)(\mu w+\nu),$$

without regard to the combination of signs; the processes of the last section are then applicable.

If $\phi(x)$ is of the fourth degree, there is no loss of generality in supposing $\phi(x)$ expressed as the product of two quadratic factors $\theta(x)$, $\psi(x)$ with real coefficients. If there are real constants such that

·501–·502 $\theta(x) = \kappa(x-\alpha)^2+\lambda(x-\beta)^2, \qquad \psi(x) = \mu(x-\alpha)^2+\nu(x-\beta)^2,$

the substitution

17·51 $$\frac{x-\alpha}{x-\beta} = t$$

reduces the integral $\int dx/\sqrt{\phi(x)}$ to a multiple of $\int dt/\sqrt{\{(\kappa t^2+\lambda)(\mu t^2+\nu)\}}$.

Alternatively, the substitution

17·52 $$\left(\frac{x-\alpha}{x-\beta}\right)^2 = w$$

reduces the integral to a multiple of $\int dw/\sqrt{\{w(\kappa w+\lambda)(\mu w+\nu)\}}$, and therefore the substitution

17·53 $$\frac{\theta(x)}{\psi(x)} = \gamma y$$

is equivalent to a bilinear transformation between w and y in which the factors $\kappa w+\lambda$, $\mu w+\nu$ correspond to y, $1/y$, and reduces the integral to an integral $\int dy/\sqrt{\{y(\varpi y+\rho)(\sigma y+\tau)\}}$, where the factors $\varpi y+\rho$, $\sigma y+\tau$ correspond to $x-\alpha$, $x-\beta$, and are therefore multiples of $(x-\alpha)^2/\psi(x)$, $(x-\beta)^2/\psi(x)$. The constant γ is available for bringing the factors precisely to the standard forms, but in a numerical problem it may be best to give γ a definite value in the first place, at the cost of a second substitution when the form $\int dy/\sqrt{\{y(\varpi y+\rho)(\sigma y+\tau)\}}$ is reached.

The simultaneous expression of $\theta(x)$, $\psi(x)$ by means of real squares is possible unless† these quadratic functions both have real roots and the pairs of roots are interlaced. If both functions have real roots, then whether or not the two pairs of roots are interlaced, the transformation 13·704 is available as a real transformation. The anharmonic group of

† Geometrically, if $\theta(x, y)$, $\psi(x, y)$ denote the homogeneous functions $y^2\theta(x/y)$, $y^2\psi(x/y)$, the simultaneous reduction is possible if there are conics $\theta(x, y) = c$, $\psi(x, y) = d$ which touch each other, and the only case in which this does not occur is that in which the two families of conics $\theta(x, y) = \lambda$, $\psi(x, y) = \mu$ are both composed of hyperbolas and the two pairs of asymptotes are interlaced.

the four roots is real, and a substitution can be chosen for which the ratio is between 0 and 1. There may be a negative constant factor in the radical, but this signifies only that one of the factors must be reversed before the appropriate elliptic function can be detected.

The reduction of $\int dx/\sqrt{\phi(x)}$ when $\phi(x)$ is cubic connects the Jacobian and Weierstrassian functions, but we are not here concerned with the Weierstrassian side of the problem. We suppose $\phi(x)$ to be given in the form $(x-\alpha)\theta(x)$, where α is real. If $\theta(x)$ has real roots β, γ, a reduction to the form $\int dy/\sqrt{\{y(\varpi y+\rho)(\sigma y+\tau)\}}$ is immediate. If the roots of $\theta(x)$ are complex, the substitution

17·54 $$x-\alpha = z^2$$

converts the integral into the form $\int dz/\sqrt{\{az^4+2hz^2+b\}}$, in which real quadratic factors of the form $az^2+\sqrt{(ab)}\pm cz$ are obvious. Alternatively, regarding $x-\alpha$ as a degenerate form of a quadratic factor $\psi(x)$, we make the substitution

17·55 $$\frac{\theta(x)}{x-\alpha} = y,$$

suggested by ·53; if $\theta(x) = a(x-\alpha)^2+2h(x-\alpha)+b$, we have

$$\left\{a-\frac{b}{(x-\alpha)^2}\right\} dx = dy,$$

$$\{a(x-\alpha)^2-b\}^2 = (x-\alpha)^2\{(y-2h)^2-4ab\},$$

and the integral is a multiple of $\int dy/\sqrt{\chi(y)}$, where

$$\chi(y) = \pm y\{(y-2h)^2-4ab\}$$

and the quadratic factor has real roots because the roots of $\theta(x)$ are not real.

17·6. The numerical evaluation of the Jacobian functions is reducible for sufficiently small values of c to the evaluation of circular functions, for sufficiently small values of c' to the evaluation of hyperbolic functions; the formulae required are expansions in ascending powers of c or c', and we have seen in 15·2 how the expansions can be found. But in these expansions the functions of u which multiply the successive powers of the parameter become cumbersome very rapidly, and the series are of little practical use beyond their first or second terms. In choosing a system for use in an actual problem the immediate choice of the parameter is choice within a real anharmonic group, and we can always suppose the parameter to be positive and not greater than $\frac{1}{2}$

and the modulus to be not greater than $\frac{1}{2}\sqrt{2}$, but this restriction is altogether inadequate to the purpose of using power series.

It is the Landen transformations which enable us to diminish the value of c or the value of c' to any desired extent. In the real field, K decreases steadily to $\frac{1}{2}\pi$ and K' increases steadily without limit as $c \to 0$. We know the order of increase of K': from 15·428′,

$$K' = A \log(16/c)$$

where $A \to \frac{1}{2}$. Hence $c = 16e^{-K'/A}$, and for large values of K'

17·61₁ $$c \doteqdot 16e^{-\sigma},$$

where the symbol denotes practical indistinguishability, and where

·601 $$\sigma = \pi K'/K,$$

as in Chapter XVI.

The effect of the second Landen transformation is to double the value of σ, and therefore ultimately to replace c by approximately $c^2/16$: if $c_1 = 1/4$, then $c_2 \doteqdot 1/4^4$ and $c_3 \doteqdot 1/10^6$. With such a rate of decrease as this, it is better to repeat the transformation until c is negligible than to use series in which c and c^2 are multiplied by functions laborious to evaluate. For small values of σ,

17·61₂ $$c' \doteqdot 16e^{-1/\sigma},$$

and the effect of the first Landen transformation is to halve σ and to diminish c' accordingly.

While the asymptotic relations ·61₁₋₂ show clearly why the operation of the Landen transformations is effective, we must use an exact relation between consecutive values of the parameter until we find that this has merged in one direction or the other into the asymptotic relation. If a Jacobian system $\mathbf{U}$ has moduli k, k', and if the system $\mathscr{L}\mathbf{U}$ derived from $\mathbf{U}$ by the first Landen transformation has moduli h, h', then from 13·512, 13·515

·602–·603 $$h^2 = \frac{4k}{(1+k)^2}, \qquad k'^2 = \frac{4h'}{(1+h')^2}.$$

If we have a Landen chain

$$\ldots, \mathbf{U}_{-3}, \mathbf{U}_{-2}, \mathbf{U}_{-1}, \mathbf{U}_0, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \ldots,$$

where $\mathbf{U}_n = \mathscr{L}\mathbf{U}_{n-1}$ for all values of n, then

·604–·605 $$k_n^2 = \frac{4k_{n-1}}{(1+k_{n-1})^2}, \qquad k'^2_{n-1} = \frac{4k'_n}{(1+k'_n)^2}.$$

Since $\frac{1}{4}(1+k)^2$ decreases from 1 to $\frac{1}{4}$ as k decreases from 1 to 0, k_{n-1} is always between k_n^2 and $\frac{1}{4}k_n^2$, and k'_n is always between k'^2_{n-1} and $\frac{1}{4}k'^2_{n-1}$; in accordance with the asymptotic formulae, $k_{n-1}/k_n^2 \to \frac{1}{4}$ as $n \to -\infty$ and $k'_n/k'^2_{n-1} \to \frac{1}{4}$ as $n \to +\infty$.

Let us write

·606–·607 $$k_n = b_n/a_n, \qquad k'_n = b'_n/a'_n,$$

and consider the relations ·604–·605 in the forms

·608–·609 $$\frac{b_n}{a_n} = \frac{\sqrt{(a_{n-1}b_{n-1})}}{\frac{1}{2}(a_{n-1}+b_{n-1})}, \qquad \frac{b'_{n-1}}{a'_{n-1}} = \frac{\sqrt{(a'_n b'_n)}}{\frac{1}{2}(a'_n+b'_n)}.$$

From any pair of unequal positive numbers a_0, b_0, of which we suppose a_0 to be the larger, we can form a sequence of pairs of arithmetic and geometric means by the recurrence formulae

17·62$_{1-2}$ $$a_n = \tfrac{1}{2}(a_{n-1}+b_{n-1}), \qquad b_n = \sqrt{(a_{n-1}b_{n-1})}.$$

This pair of formulae can be reversed: since $a_n > b_n > 0$, the roots of the equation

$$x^2-2a_n x+b_n^2 = 0$$

are real, positive, and unequal; a_{n-1} is the larger and b_{n-1} the smaller of these roots. The pair of formulae ·62 therefore generates a chain of pairs of numbers which can be extended indefinitely in both directions; this chain is called an arithmetico-geometric chain. A given pair of unequal positive numbers belongs to one and only one arithmetico-geometric chain, and the chain can be developed from any one of its members.

Since $b_{n-1} < a_{n-1}$, we have

·610–·611 $$a_n < a_{n-1}, \qquad b_n > b_{n-1};$$

also $$(a_n-b_n) = \tfrac{1}{2}(\sqrt{a_{n-1}}-\sqrt{b_{n-1}})^2$$
$$= \frac{\sqrt{a_{n-1}}-\sqrt{b_{n-1}}}{\sqrt{a_{n-1}}+\sqrt{b_{n-1}}}\cdot\tfrac{1}{2}(a_{n-1}-b_{n-1}),$$

whence

·612 $$(a_n-b_n) < \tfrac{1}{2}(a_{n-1}-b_{n-1}).$$

Hence as $n \to +\infty$ the decreasing sequence $\{a_n\}$ and the increasing sequence $\{b_n\}$ have a common limit. This limit is a definite function of the initial pair of numbers a_0, b_0, called by Gauss their arithmetico-geometric mean and denoted by $M(a_0, b_0)$. Since the sequence of pairs of numbers may be developed from any of its members, $M(a_r, b_r)$ has the same value as $M(a_0, b_0)$, whether r is positive or negative; the arithmetico-geometric mean belongs in fact to the chain rather than to any one pair of numbers in the chain.

In the opposite direction, the inequality ·612 becomes

$$(a_{n-1}-b_{n-1}) > 2(a_n-b_n),$$

implying that as $n \to -\infty$, $a_n \to \infty$. Since $b_{n+1} < M(a_0, b_0)$,

$$a_n b_n < \{M(a_0, b_0)\}^2,$$

for all values of n, positive and negative, and therefore if $a_n \to \infty$, $b_n \to 0$.

17·63. *In the arithmetico-geometric chain determined by a pair of unequal positive numbers* (a_0, b_0), a_n *tends downwards to* $M(a_0, b_0)$ *and* b_n *tends upwards to* $M(a_0, b_0)$ *as* $n \to +\infty$, *while* a_n *tends upwards to* ∞ *and* b_n *tends downwards to* 0 *as* $n \to -\infty$; *the ratio* b_n/a_n *tends upwards to* 1 *as* $n \to +\infty$ *and tends downwards to* 0 *as* $n \to -\infty$.

It is to be observed that an arithmetico-geometric chain has a definite direction; we can assign the suffix 0 to any member of the chain we please, and the allocation of all other suffixes is then determined unambiguously.

We can now express the relations ·604–·605 as follows:

17·64. *In a Jacobian system* **U** *in which* k *and* k' *have real positive values, let* a_0, b_0 *and* a'_0, b'_0 *be any two pairs of positive numbers such that*

$$a_0 : b_0 = 1 : k, \qquad a'_0 : b'_0 = 1 : k',$$

and for both positive and negative values of m, *let* (a_m, b_m) *and* (a'_m, b'_m) *be the* m*th members of the arithmetico-geometric chains evolved from* (a_0, b_0) *and* (a'_0, b'_0); *then if* $\mathscr{L}$ *is the Landen transformation which doubles the ratio of* K *to* K', *the moduli* k_n, k'_n *of the system* $\mathscr{L}^n$**U** *are given by*

$$k_n = b_n/a_n, \qquad k'_n = b'_{-n}/a'_{-n},$$

whether n *is positive or negative.*

To take a_0 and a'_0 as unity would obscure slightly the completeness of the relation between the Landen chain of Jacobian systems and the two arithmetico-geometric chains; with $\mathbf{U}_0$, k_0, k'_0 written for **U**, k, k', this relation persists throughout the length of the chains, but no two values of a_m or of a'_m are equal, and to assign unit values at the particular system $\mathbf{U}_0$ is arbitrary.

The arithmetico-geometric chains giving k_n and k_n' have opposite directions; symbolically the correspondence is

$$\begin{array}{ccccccc} \dots & \mathscr{L}^{-2}\mathrm{U}_0 & \mathscr{L}^{-1}\mathrm{U}_0 & \mathrm{U}_0 & \mathscr{L}\mathrm{U}_0 & \mathscr{L}^2\mathrm{U}_0 & \dots \\ \dots & (a_{-2}, b_{-2}) & (a_{-1}, b_{-1}) & (a_0, b_0) & (a_1, b_1) & (a_2, b_2) & \dots \\ \dots & (a_2', b_2') & (a_1', b_1') & (a_0', b_0') & (a'_{-1}, b'_{-1}) & (a'_{-2}, b'_{-2}) & \dots. \end{array}$$

As $n \to +\infty$, $k_n \to 1$, $k_n' \to 0$; as $n \to -\infty$, $k_n \to 0$, $k_n' \to 1$. In other words, the Landen chain hangs between the two extremes of a system in which the functions are circular and a system in which the functions are hyperbolic, and in view of the rapidity of the convergence in each direction, in practice all but a few of the sets composing the chain are sensibly indistinguishable from one or other of the limiting forms; it is the few which are distinguishable that interest us.

The relation between the variables u, v in the systems U, $\mathscr{L}$U is $v = \mu u$, where

$$\mu = \tfrac{1}{2}(1+k) = 1/(1+k');$$

also $v = \frac{1}{2}H_c$ corresponds to $u = K_c$. If then u_n is the variable and $K_{(n)}$ is the quarterperiod K_c in the system $\mathscr{L}^n$U, we have

·613–·614 $$u_{n-1} = (1+k_n')u_n, \qquad K_{(n-1)} = \tfrac{1}{2}(1+k_n')K_{(n)}.$$

Since $$1+k_n' = \frac{a'_{-n}+b'_{-n}}{a'_{-n}} = \frac{2a'_{-(n-1)}}{a'_{-n}},$$

these relations can be written

·615–·616 $$\frac{u_{n-1}}{a'_{-(n-1)}} = \frac{2u_n}{a'_{-n}}, \qquad \frac{K_{(n-1)}}{a'_{-(n-1)}} = \frac{K_n}{a'_{-n}},$$

implying that the ratios

$$2^n u_n : K_{(n)} : a'_{-n}$$

are constant along the chain. As $n \to -\infty$,

$$K_{(n)} \to \tfrac{1}{2}\pi, \qquad a'_{-n} \to M(a_0', b_0') = a_0' M(1, k');$$

hence

17·65$_1$ $$K = \tfrac{1}{2}\pi/M(1, k'),$$

and since the relation of K' to k is the same as that of K to k',

17·65$_2$ $$K' = \tfrac{1}{2}\pi/M(1, k).$$

Applying ·65$_1$, ·65$_2$ at an arbitrary point of the chain, we have

·617–·618 $$K_{(n)} = \tfrac{1}{2}\pi/M(1, k_n'), \qquad K'_{(n)} = \tfrac{1}{2}\pi/M(1, k_n),$$

and combined with the equality of ratios

$$2^n u_n/K_{(n)} = u/K,$$

·617 and ·65$_1$ give the relation between the variable u_n and the central variable u in the form

·619 $$2^n u_n M(1, k_n') = uM(1, k'),$$

whether n is positive or negative.

Turning now to the transfer of functional values along the chain, we have to extract from the results in 13·5 formulae adapted to iteration. If we eliminate $\operatorname{ds} u$ from 13·51$_{1-2}$ and $\operatorname{cn} u$ from 13·51$_{3-4}$ we have

·620 $$\operatorname{cs} u = \tfrac{1}{2}(1+k)\operatorname{cs} v - \tfrac{1}{2}(1-k)\operatorname{sc} v,$$

·621 $$\operatorname{dn} u = \tfrac{1}{2}(1+k)\operatorname{dn} v + \tfrac{1}{2}(1-k)\operatorname{nd} v.$$

Analytically the difference between these two formulae is trivial, for $\operatorname{cs}(u+K_n) = -v \operatorname{dn} u$ and K_n in the one system corresponds to H_n in the other. Writing u_r, u_{r+1} for u, v we have

17·66$_1$ $$\operatorname{cs} u_r = \tfrac{1}{2}(1+k_r)\operatorname{cs} u_{r+1} - \frac{\tfrac{1}{2}(1-k_r)}{\operatorname{cs} u_{r+1}},$$

17·66$_2$ $$\operatorname{dn} u_r = \tfrac{1}{2}(1+k_r)\operatorname{dn} u_{r+1} + \frac{\tfrac{1}{2}(1-k_r)}{\operatorname{dn} u_{r+1}}.$$

Even these simple recurrences can be for some purposes improved, for

$$\operatorname{cs}(K_c - u) = k' \operatorname{sc} u, \qquad \operatorname{dn}(K_c - u) = k' \operatorname{nd} u,$$

and therefore, since $h' = (1-k)/(1+k)$,

·622 $$(1-k)\operatorname{sc} v = (1+k)\operatorname{cs}(H_c - v),$$

·623 $$(1-k)\operatorname{nd} v = (1+k)\operatorname{dn}(H_c - v).$$

Hence ·66$_{1-2}$ are equivalent to

17·66$_3$ $$\operatorname{cs} u_r = \tfrac{1}{2}(1+k_r)\{\operatorname{cs} u_{r+1} - \operatorname{cs}(K_{(r+1)} - u_{r+1})\},$$

17·66$_4$ $$\operatorname{dn} u_r = \tfrac{1}{2}(1+k_r)\{\operatorname{dn} u_{r+1} + \operatorname{dn}(K_{(r+1)} - u_{r+1})\}.$$

We can not reverse ·620 and ·621 rationally, to express $\operatorname{cs} v$ and $\operatorname{dn} v$ in terms of $\operatorname{cs} u$ and $\operatorname{dn} u$. That is to say, we can track the functions $\operatorname{cs} u$ and $\operatorname{dn} u$ in only one direction along the Landen chain, the direction of diminishing index, or briefly the negative direction. But from 13·51$_{5-6}$ and 13·51$_{7-8}$ we have

·624 $$\operatorname{ns} 2v = \tfrac{1}{2}(1+h')\operatorname{ns} u + \tfrac{1}{2}(1-h')\operatorname{sn} u,$$

·625 $$\operatorname{dc} 2v = \tfrac{1}{2}(1+h')\operatorname{dc} u + \tfrac{1}{2}(1-h')\operatorname{cd} u,$$

and therefore, writing v as $\frac{1}{2}u_r$ and u as $\frac{1}{2}u_{r-1}$,

17·66$_5$ $$\operatorname{ns} u_r = \tfrac{1}{2}(1+k'_r)\operatorname{ns}\tfrac{1}{2}u_{r-1}+\frac{\frac{1}{2}(1-k'_r)}{\operatorname{ns}\frac{1}{2}u_{r-1}},$$

17·66$_6$ $$\operatorname{dc} u_r = \tfrac{1}{2}(1+k'_r)\operatorname{dc}\tfrac{1}{2}u_{r-1}+\frac{\frac{1}{2}(1-k'_r)}{\operatorname{dc}\frac{1}{2}u_{r-1}}.$$

Thus in the positive direction we can track the functions $\operatorname{ns} u$ and $\operatorname{dc} u$. Modifications of ·624 and ·625 by formulae corresponding to ·622 and ·623 are of no practical value in the present connexion, for the argument introduced is K_n-u and K_n is imaginary. Formally, ·624 and ·625 are equivalent, for $\operatorname{ns}(u+K_c) = \operatorname{dc} u$ and $u = K_c$ corresponds to $2v = H_c$.

That we do not track the same function in both directions is of no consequence. In any case a function $\operatorname{pq} u$ that we require may not be one of the functions we can track, and it does not matter if we have to connect $\operatorname{pq} u$ with $\operatorname{cs} u$ for one purpose and with $\operatorname{ns} u$ for another purpose. If we are thinking of the tracking of particular functions as auxiliary to the determination of the whole system of Jacobian functions at one end or the other of a series of transformations, it is the squares of the functions with which we are concerned, and we may prefer to track the squares:

17·67$_1$ $$\operatorname{cs}^2 u_r = \tfrac{1}{4}(1+k_r)^2\operatorname{cs}^2 u_{r+1}-\tfrac{1}{2}k_r'^2+\frac{\frac{1}{4}(1-k_r)^2}{\operatorname{cs}^2 u_{r+1}},$$

17·67$_2$ $$\operatorname{cs}^2 u_r = \tfrac{1}{4}(1+k_r)^2\{\operatorname{cs}^2 u_{r+1}+\operatorname{cs}^2(K_{(r+1)}-u_{r+1})\}-\tfrac{1}{2}k_r'^2,$$

17·67$_3$ $$\operatorname{ns}^2 u_r = \tfrac{1}{4}(1+k'_r)^2\operatorname{ns}^2\tfrac{1}{2}u_{r-1}+\tfrac{1}{2}k_r^2+\frac{\frac{1}{4}(1-k'_r)^2}{\operatorname{ns}^2\frac{1}{2}u_{r-1}}.$$

From a recurrence for the square of any one Jacobian function we derive also a recurrence for an integrating function. For example, ·66$_4$ gives

17·68$_1$ $$\operatorname{Dn} u_r = \tfrac{1}{2}(1+k_r)\{\operatorname{Dn} u_{r+1}+\operatorname{Dn}(u_{r+1}-K_{(r+1)})+2k'_{r+1}u_{r+1}\};$$

by 14·74, $$E(u-K_c) = E(u)-E_c+c\operatorname{sn} u\operatorname{sn}(u-K_c),$$

and therefore

17·68$_2$ $$E(u_r) = (1+k_r)\{E(u_{r+1})-\tfrac{1}{2}E_{(r+1)}-\tfrac{1}{2}k_{r+1}^2\operatorname{sn} u_{r+1}\operatorname{cd} u_{r+1}+k'_{r+1}u_{r+1}\}.$$

When $u_{r+1} = K_{(r+1)}$, $E(u_r) = 2E(K_{(r)}) = 2E_{(r)}$; hence

·626 $$E_{(r)} = \tfrac{1}{4}(1+k_r)E_{(r+1)}+\tfrac{1}{2}(1-k_r)K_{(r+1)}.$$

Expressed for iteration along a chain, the relations 13·520, 13·521 between consecutive amplitudes take the form

·627 $$\tan(\phi_{r-1}-\phi_r) = k_r' \tan\phi_r,$$

·628 $$\sin(2\phi_{r+1}-\phi_r) = k_r \sin\phi_r.$$

The appearance of tracking the same function $\operatorname{am} u$ in both directions is deceptive, for the actual relations are between circular functions of the amplitude, and at each stage we have the problem of identifying the argument, $\phi_{r-1}-\phi_r$ or $2\phi_{r+1}-\phi_r$, from the tangent or sine. It is only for real values of the amplitude that this treatment is practicable, whereas the recurrences of ·66 and ·67 can be used if the values of the functions are complex. But unless the theory of the arithmetico-geometric mean is extended to complex pairs of numbers, by the resolution of an ambiguity at every stage, a real value of the modulus between 0 and 1 is essential to the application of this theory to the Landen chain.

The practical use of the Landen chain is to connect a system **U** with a system in which the numerical relations between the functions and the argument are known, to whatever order of accuracy may have been prescribed. Suitable systems are to be found in both directions along the chain: whatever standard of tolerance is laid down, for sufficiently large positive values of m, $\mathscr{L}^{-m}\mathbf{U}$ is a system **V** in which the modulus is negligible, the amplitude of v is indistinguishable from v, and the elliptic functions degenerate to circular functions; for sufficiently large positive values of n, $\mathscr{L}^{n}\mathbf{U}$ is a system **W** in which the complementary modulus is negligible, w is effectively the hyperbolic amplitude, and the elliptic functions degenerate to hyperbolic functions. Moreover,—and this is of course of prime importance—convergence along the chain is so rapid that the loss of accuracy in relating **U** to the nearer of the two systems **V**, **W** is negligible. The nearer system is **V** or **W** according as $c < \frac{1}{2}$ or $c > \frac{1}{2}$; with $c < \frac{1}{2}$, $c_{-2} < \frac{1}{2}.10^{-4}$, and with $c > \frac{1}{2}$, $c_2' < \frac{1}{2}.10^{-4}$.

There are two problems of evaluation: we may require the values of functions of a given argument, or conversely, as in the evaluation of elliptic integrals, it may be the value of a function that is given and the value of the argument that is to be inferred. The first step is to determine the value of m or n for which c_{-m} or c_n' is negligible; **V** or **W** is then a known system. If u is given, v or w as the case may be follows from ·619; the passage from **V** to **U** is in the positive direction along the chain, $\operatorname{ns} v$ is effectively $\csc v$, and $\operatorname{ns} u$ is found from $\operatorname{ns} v$ by repeated use of $·66_5$, or $\operatorname{ns}^2 u$ from $\operatorname{ns}^2 v$ by repeated use of $·67_3$; the passage from W to V is in the negative direction, $\operatorname{cs} w$ is identified with $\operatorname{csch} w$, and $\operatorname{cs} u$ is found from $\operatorname{cs} w$ by means of $\cdot 66_1$ or $\cdot 66_3$ or $\operatorname{cs}^2 u$ from $\operatorname{cs}^2 w$ by means of $\cdot 67_1$ or $\cdot 67_2$. Any other of the twelve functions of u is then found algebraically from $\operatorname{ns}^2 u$ or $\operatorname{cs}^2 u$. If it is the inverse calculation that is to be performed, the value of a function of u being given, we have first to calculate $\operatorname{cs} u$ if V is the intermediary system, $\operatorname{ns} u$ if W is the intermediary system; then $\operatorname{cs} v$, that is, $\cot v$, can be found from $\operatorname{cs} u$, or $\operatorname{ns} w$, that is, $\coth w$, from $\operatorname{ns} u$; it is assumed that v can be deduced from $\cot v$ or w from $\coth w$, and finally the required value of u is given by ·619.

The function $\operatorname{dn} u$, being nowhere zero or infinite for real values of u, might seem to be a 'safer' and less troublesome function to carry through a chain of operations than $\operatorname{cs} u$, but it is for that very reason a less sensitive function; if it is $\operatorname{dn} u$ that is actually wanted or given, naturally this function is used, but it is less fitted than $\operatorname{cs} u$ for the reconstruction of the whole system or for the determination of u.

We have expressed the evaluations as operations in finite terms, the standard of accuracy being premised. They may also be expressed as operations determining a convergent sequence whose limit is the required value. To illustrate this form of expression, let us enunciate two theorems in which the amplitude is introduced.

If k_{-m} is negligible, $M(1, k'_{-m})$ is indistinguishable from unity, and ·619 takes the form

·629 $$2^{-m}u_{-m} \simeq uM(1, k').$$

A trivial change of notation puts the recurrence ·627 into a clearer form for negative values of r, and we have

17·69$_1$. *If $\bar{\phi}_m$ is determined, for positive values of m, by the recurrence*

$$\tan(\bar{\phi}_{m+1} - \bar{\phi}_m) = k'_{-m} \tan \bar{\phi}_m$$

with the initial value $\bar{\phi}_0 = \phi$, then as $m \to \infty$,

$$2^{-m}\bar{\phi}_m \to M(1, k')F(\phi; k).$$

This form of the theorem reveals plainly that when k'_{-m} has become indistinguishable from unity, no further change in $2^{-m}\bar{\phi}_m$ can be effected.

From ·613, for positive values of n,

·630 $$u/u_n = (1+k'_1)(1+k'_2)\ldots(1+k'_n).$$

As $n \to \infty$, the product on the right converges to a limit Λ' which is a definite function of k', and we have for sufficiently large values of n

·631 $$u_n \simeq u/\Lambda',$$

while from 15·24,

·632 $$\phi_n \simeq \operatorname{gd} u_n.$$

Hence

17·69$_2$. *If ϕ_n is determined, for positive values of n, by the recurrence*

$$\sin(2\phi_{n+1}-\phi_n) = k_n \sin\phi_n$$

with the initial value $\phi_0 = \phi$, then as $n \to \infty$,

$$\operatorname{gd}^{-1}\phi_n \to F(\phi;k) \Big/ \prod_1^\infty (1+k_n'),$$

where the function on the left is the inverse gudermannian.

17·7. The Landen transformations are not restricted theoretically to real values of variables and parameters, but for practical purposes the simplicity of many of the formulae is deceptive in the complex field: to calculate ϕ numerically from h' and χ by means of the relation $\tan(\phi-\chi) = h'\tan\chi$ when the numbers are all complex is a formidable undertaking.

An alternative process of computation is provided by the q-series of Chapter XVI. If the value of q is known, K is given, as we have noticed in 16·8, by the substitution $v = \frac{1}{2}\pi$ in the condition $\vartheta_s(u) = \vartheta_n(u)$, and the four functions $\vartheta_s(u)$, $\vartheta_c(u)$, $\vartheta_n(u)$, $\vartheta_d(u)$ can then be computed for any value of u. The q-series in 16·55 converge very rapidly, for although they are power series, they are power series with lengthening gaps: the index of the typical effective term is either n^2 or $n(n+1)$. From the four cardinal theta functions, the twelve elliptic functions come immediately.

If it is k that is given, q is to be found as in 16·8 from the equation

·701 $$\frac{q+q^9+q^{25}+\ldots}{1+2q^4+2q^{16}+2q^{36}+\ldots} = \epsilon,$$

where, if $h'^2 = k'$,

·702 $$\epsilon = \tfrac{1}{2}(1-h')/(1+h') = \tfrac{1}{2}(1-k')/(1+h')^2 = \tfrac{1}{2}k^2/(1+k')(1+h')^2.$$

As we said previously, the solution of ·701 takes the form

$$q = \epsilon+a_1\epsilon^5+a_2\epsilon^9+\ldots.$$

No formula is known for the coefficient a_n, but the early coefficients can be found by the crudest methods:

17·71 $q = \epsilon+2\epsilon^5+15\epsilon^9+150\epsilon^{13}+1707\epsilon^{17}+20910\epsilon^{21}+O(\epsilon^{25})$.

If the parameters are complex, the q-series give the only method of computation that can be called practicable. We can not say this if the parameters are real, for Legendre's tables were in fact compiled by means of the Landen transformation; it is true that these are tables of elliptic integrals, not of elliptic functions as we now use the name, but numerical inversion is a simple operation and it is certainly possible to compute an amplitude either by inverse interpolation in Legendre's table or by direct use of the process described in ·69. For an isolated determination this method is still to be recommended, but for systematic tabulation to a moderate degree of accuracy the advantage is perhaps with the q-series. The four cardinal theta functions once recorded, the user finds by one simple division the value of any one of the twelve elliptic functions which he needs.

The problem of avoiding division by small values of $\vartheta_s(u)$ or $\vartheta_c(u)$ is solved by the use of 16·73. If the functions

$$(\pi/2K)\cot v - \mathrm{cs}\, u, \quad \mathrm{ns}\, u - (\pi/2K)\csc v, \quad (\pi/2K)\csc v - \mathrm{ds}\, u$$

are tabulated for small values of v, and the functions

$$(\pi/2k'K)\tan v - \mathrm{sc}\, u, \quad \mathrm{dc}\, u - (\pi/2K)\sec v, \quad (\pi/2k'K)\sec v - \mathrm{nc}\, u$$

for values of v near $\frac{1}{2}\pi$, interpolation in these neighbourhoods takes the familiar form of interpolation for the circular functions, the subsidiary functions tabulated being regular and tending to zero. But the series in 16·73 and 16·74 converge much less rapidly than the series in 16·55, and it is only for a special purpose that they are to be preferred in numerical work.

17·8. A few words may be added on the case of a real parameter and a complex variable. We can deal with this case by means of the theta functions at the cost only of computing circular functions of a complex argument. Alternatively, addition theorems reduce $\mathrm{pq}(u+iv)$ to combinations of functions of u and functions of iv, and by Jacobi's imaginary transformation of 13·2 the functions of iv are replaced by functions of v; if k is real and between 0 and 1, the complementary modulus k' which serves as primary modulus to the functions of v is subject also to these conditions, and $\mathrm{Rl}\{\mathrm{pq}(u+iv;k)\}$ and $\mathrm{Im}\{\mathrm{pq}(u+iv;k)\}$ are both determinable as combinations of real functions of u and real functions of v. A complete table, constructed from 12·31, 12·32, and Table XII 1, follows:

TABLE XVII 1

$$\left.\begin{array}{ll} \mathrm{cs}(u+iv) & \mathrm{cs}\,u\,\mathrm{cs}\,v\,\mathrm{ds}\,v - i\,\mathrm{ns}\,u\,\mathrm{ds}\,u\,\mathrm{ns}\,v \\ \mathrm{ns}(u+iv) & \mathrm{ns}\,u\,\mathrm{ns}\,v\,\mathrm{ds}\,v - i\,\mathrm{cs}\,u\,\mathrm{ds}\,u\,\mathrm{cs}\,v \\ \mathrm{ds}(u+iv) & \mathrm{ds}\,u\,\mathrm{cs}\,v\,\mathrm{ns}\,v - i\,\mathrm{cs}\,u\,\mathrm{ns}\,u\,\mathrm{ds}\,v \end{array}\right\} \div\ \mathrm{ds}^2u+\mathrm{ds}^2v$$

$$\left.\begin{array}{ll} \mathrm{sc}(u+iv) & \mathrm{sc}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v + i\,\mathrm{nc}\,u\,\mathrm{dc}\,u\,\mathrm{sn}\,v \\ \mathrm{dc}(u+iv) & \mathrm{dc}\,u\,\mathrm{dn}\,v + ic'\,\mathrm{sc}\,u\,\mathrm{nc}\,u\,\mathrm{sn}\,v\,\mathrm{cn}\,v \\ \mathrm{nc}(u+iv) & \mathrm{nc}\,u\,\mathrm{cn}\,v + i\,\mathrm{sc}\,u\,\mathrm{dc}\,u\,\mathrm{sn}\,v\,\mathrm{dn}\,v \end{array}\right\} \div\ 1+c'\,\mathrm{sc}^2u\,\mathrm{sn}^2v$$

$$\left.\begin{array}{ll} \mathrm{dn}(u+iv) & \mathrm{dn}\,u\,\mathrm{dc}\,v - ic\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{sc}\,v\,\mathrm{nc}\,v \\ \mathrm{sn}(u+iv) & \mathrm{sn}\,u\,\mathrm{nc}\,v\,\mathrm{dc}\,v + i\,\mathrm{cn}\,u\,\mathrm{dn}\,u\,\mathrm{sc}\,v \\ \mathrm{cn}(u+iv) & \mathrm{cn}\,u\,\mathrm{nc}\,v - i\,\mathrm{sn}\,u\,\mathrm{dn}\,u\,\mathrm{sc}\,v\,\mathrm{dc}\,v \end{array}\right\} \div\ 1+c\,\mathrm{sn}^2u\,\mathrm{sc}^2v$$

$$\left.\begin{array}{ll} \mathrm{nd}(u+iv) & \mathrm{nd}\,u\,\mathrm{cd}\,v + ic\,\mathrm{sd}\,u\,\mathrm{cd}\,u\,\mathrm{sd}\,v\,\mathrm{nd}\,v \\ \mathrm{cd}(u+iv) & \mathrm{cd}\,u\,\mathrm{nd}\,v - ic'\,\mathrm{sd}\,u\,\mathrm{nd}\,u\,\mathrm{sd}\,v\,\mathrm{cd}\,v \\ \mathrm{sd}(u+iv) & \mathrm{sd}\,u\,\mathrm{cd}\,v\,\mathrm{nd}\,v + i\,\mathrm{cd}\,u\,\mathrm{nd}\,u\,\mathrm{sd}\,v \end{array}\right\} \div\ 1-cc'\,\mathrm{sd}^2u\,\mathrm{sd}^2v$$

The primary modulus of the functions of v is equal to the complementary modulus of the functions of $u+iv$ and of u

The dissection of an integrating function requires little but the application of these results to the addition theorems in 14·7.

TABLE XVII 2

$$\left.\begin{array}{ll} \mathrm{Cs}(u+iv) & \mathrm{Cs}\,u - i\,\mathrm{Ns}\,v + S \\ \mathrm{Ns}(u+iv) & \mathrm{Ns}\,u - i\,\mathrm{Cs}\,v + S \\ \mathrm{Ds}(u+iv) & \mathrm{Ds}\,u - i\,\mathrm{Ds}\,v + S \end{array}\right\} \text{ where } S = \frac{\mathrm{c_1n_1d_1s^3}u - i\,\mathrm{c_1n_1d_1s^3}v}{\mathrm{ds}^2u+\mathrm{ds}^2v}$$

$$\left.\begin{array}{ll} \mathrm{Sc}(u+iv) & \mathrm{Sc}\,u - i\,\mathrm{Sn}\,v - C \\ \mathrm{Dc}(u+iv) & \mathrm{Dc}\,u + i\,\mathrm{Dn}\,v - c'C \\ \mathrm{Nc}(u+iv) & \mathrm{Nc}\,u + i\,\mathrm{Cn}\,v - C \end{array}\right\} \text{ where } C = \frac{\mathrm{n_1d_1c^1s^1}u - i\,\mathrm{c_1d_1n^1s^1}v}{\mathrm{cs}^2u\,\mathrm{ns}^2v+c'}$$

$$\left.\begin{array}{ll} \mathrm{Dn}(u+iv) & \mathrm{Dn}\,u + i\,\mathrm{Dc}\,v + cN \\ \mathrm{Sn}(u+iv) & \mathrm{Sn}\,u - i\,\mathrm{Sc}\,v - N \\ \mathrm{Cn}(u+iv) & \mathrm{Cn}\,u + i\,\mathrm{Nc}\,v + N \end{array}\right\} \text{ where } N = \frac{\mathrm{c_1d_1n^1s^1}u - i\,\mathrm{n_1d_1c^1s^1}v}{\mathrm{ns}^2u\,\mathrm{cs}^2v+c}$$

$$\left.\begin{array}{ll} \mathrm{Nd}(u+iv) & \mathrm{Nd}\,u + i\,\mathrm{Cd}\,v - cD \\ \mathrm{Cd}(u+iv) & \mathrm{Cd}\,u + i\,\mathrm{Nd}\,v + c'D \\ \mathrm{Sd}(u+iv) & \mathrm{Sd}\,u - i\,\mathrm{Sd}\,v - D \end{array}\right\} \text{ where } D = \frac{\mathrm{c_1n_1d^1s^1}u - i\,\mathrm{c_1n_1d^1s^1}v}{\mathrm{ds}^2u\,\mathrm{ds}^2v-cc'}$$

The moduli are related as in Table XVII 1

The association of real modulus with complex argument, far from being artificial, is of the utmost practical importance, since it is inevitable if conformal transformations are to be applied in detail. The fundamental property of the simple transformation $x = \mathrm{pq}\,u$ is expressed in ·31 above; we conclude with two transformations in which elliptic functions operate through a parameter.

If $z = \mathrm{ds}^2\zeta$ and $w = \mathrm{Ds}\,\zeta$, then

·801 $$dz/dw = -2\,\mathrm{cs}\,\zeta\,\mathrm{ns}\,\zeta/\mathrm{ds}\,\zeta = -2(z-c')^{1/2}z^{-1/2}(z+c)^{1/2}.$$

It follows from the Schwarz-Christoffel theorem on polygonal contours that if c and c' are real and positive, the real axis in the z plane corresponds to a 'rectangle' with one corner at infinity and with a re-entrant angle at the point $w = \mathrm{Ds}(K_c+K_n)$:

17·81. *If the variables z, w are connected through the variable ζ by the relations $z = \mathrm{ds}^2\zeta$, $w = \mathrm{Ds}\,\zeta$, with $0 < k < 1$, the halfplane $\mathrm{Im}\, z > 0$ is represented conformally on the part of the second quadrant of the w plane which lies outside the rectangle whose corners are*

$$0, \quad -(E-c'K), \quad -(E-c'K)+i(E'-cK'), \quad i(E'-cK').$$

Lastly, writing $u-\frac{1}{2}K_n$ for u in the relation

$$\mathrm{sn}\, u\, \mathrm{sn}(u+K_n) = \mathrm{sn}\, K_c\, \mathrm{sn}(K_c+K_n),$$

we have

·802 $$\mathrm{sn}(\tfrac{1}{2}K_n+u)\mathrm{sn}(\tfrac{1}{2}K_n-u) = -1/k,$$

implying, in the classical case,

·803 $$|\mathrm{sn}(\tfrac{1}{2}iK'+t)| = 1/\sqrt{k}$$

for all real values of t. Hence, if $z = \mathrm{sn}\,\zeta$, the line in the ζ plane from $\frac{1}{2}iK'+K$ to $\frac{1}{2}iK'-K$ yields in the z plane a semicircle from $1/\sqrt{k}$ to $-1/\sqrt{k}$, and since the line from K to $K+\frac{1}{2}iK'$ yields the stretch of the real axis from 1 to $1/\sqrt{k}$, the interior of the rectangle whose corners are $\pm K \pm \frac{1}{2}iK'$ corresponds to the circular area $|z| < 1/\sqrt{k}$ with slits from $1/\sqrt{k}$ to 1 and from $-1/\sqrt{k}$ to -1. But, if $w \equiv u+iv = \sin(\zeta/\lambda)$, where λ is a real constant, the interior of the ζ rectangle whose corners are $\pm\frac{1}{2}\pi\lambda \pm ih\lambda$ corresponds, for every real value of h, to the interior of the ellipse $u^2/\cosh^2 h + v^2/\sinh^2 h = 1$, slit from the vertices $(\pm\cosh h, 0)$ to the foci $(\pm 1, 0)$. We secure coincident rectangles by taking $\lambda = 2K/\pi$, $h = \pi K'/4K$, and the circular z region then corresponds to the elliptical w region. The slits can be obliterated, for they are occasioned only by discontinuities in the variable ζ, and it is easily verified that the functional values of both z and w are continuous across them:

17·82. *By the parametric relation*

$$x+iy = \mathrm{sn}\,\zeta, \qquad u+iv = \sin(\pi\zeta/2K),$$

with $0 < k < 1$, the interior and the boundary of the ellipse

$$u^2\,\mathrm{sech}^2(\pi K'/4K) + v^2\,\mathrm{csch}^2(\pi K'/4K) = 1$$

are represented conformally on the interior and the boundary of the circle $x^2+y^2 = 1/k$.

EXERCISES

For notes on these exercises see *pp.* 323–31 *below*

1. $$\mathrm{fj}\,z+\mathrm{gj}\,z+\mathrm{hj}\,z = 2\zeta\tfrac{1}{2}z-\zeta z.$$

2. $$\wp\tfrac{1}{2}z = \wp z+\mathrm{gj}\,z\,\mathrm{hj}\,z+\mathrm{hj}\,z\,\mathrm{fj}\,z+\mathrm{fj}\,z\,\mathrm{gj}\,z.$$

3. $$\mathrm{fj}^2\tfrac{1}{2}z\,\mathrm{fj}'z = \mathrm{gj}^2\tfrac{1}{2}z\,\mathrm{gj}'z = \mathrm{hj}^2\tfrac{1}{2}z\,\mathrm{hj}'z = \mathrm{fj}'z\,\mathrm{gj}'z\,\mathrm{hj}'z.$$

4. $$\mathrm{fj}^2\tfrac{1}{2}\omega_f = g_f h_f, \qquad \mathrm{gj}^2\tfrac{1}{2}\omega_f = g_f(g_f+h_f), \qquad \mathrm{hj}^2\tfrac{1}{2}\omega_f = h_f(g_f+h_f),$$
$$\mathrm{fj}\,\tfrac{1}{2}\omega_f\,\mathrm{gj}\,\tfrac{1}{2}\omega_f\,\mathrm{hj}\,\tfrac{1}{2}\omega_f = -g_f h_f(g_f+h_f).$$

5. $$e_{gh}\,\mathrm{fj}\,x\,\mathrm{fj}\,y\,\mathrm{fj}(x-y)+e_{hf}\,\mathrm{gj}\,x\,\mathrm{gj}\,y\,\mathrm{gj}(x-y)+e_{fg}\,\mathrm{hj}\,x\,\mathrm{hj}\,y\,\mathrm{hj}(x-y) = 0.$$

6. $$e_{gh}\,\mathrm{fj}\,x\,\mathrm{fj}(z-x)\mathrm{fj}\,y\,\mathrm{fj}(z-y)+e_{hf}\,\mathrm{gj}\,x\,\mathrm{gj}(z-x)\mathrm{gj}\,y\,\mathrm{gj}(z-y)+$$
$$+e_{fg}\,\mathrm{hj}\,x\,\mathrm{hj}(z-x)\mathrm{hj}\,y\,\mathrm{hj}(z-y) = -e_{gh}\,e_{hf}\,e_{fg}.$$

7. If $z_1+z_2+z_3+z_4 \equiv 0$, then

(i) $$e_{gh}\prod_r \mathrm{fj}\,z_r+e_{hf}\prod_r \mathrm{gj}\,z_r+e_{fg}\prod_r \mathrm{hj}\,z_r = -e_{gh}\,e_{hf}\,e_{fg},$$

(ii) $$e_{gh}\,\mathrm{jf}\,z_1\,\mathrm{jf}\,z_2\,\mathrm{fj}\,z_3\,\mathrm{fj}\,z_4-e_{hf}\,\mathrm{hf}\,z_1\,\mathrm{hf}\,z_2\,\mathrm{gj}\,z_3\,\mathrm{gj}\,z_4-e_{fg}\,\mathrm{gf}\,z_1\,\mathrm{gf}\,z_2\,\mathrm{hj}\,z_3\,\mathrm{hj}\,z_4$$
$$= -e_{gh}\,e_{hf}\,e_{fg},$$

(iii) $$e_{gh}\,\mathrm{fj}\,z_1\,\mathrm{jf}\,z_2\,\mathrm{hg}\,z_3\,\mathrm{gh}\,z_4+e_{hf}\,\mathrm{gj}\,z_1\,\mathrm{hf}\,z_2\,\mathrm{jg}\,z_3\,\mathrm{fh}\,z_4+e_{fg}\,\mathrm{hj}\,z_1\,\mathrm{gf}\,z_2\,\mathrm{fg}\,z_3\,\mathrm{jh}\,z_4$$
$$= -e_{gh}\,e_{hf}\,e_{fg}.$$

8. The function $(\mathrm{gj}\,x\,\mathrm{fg}\,y+\mathrm{hj}\,x\,\mathrm{jg}\,y)/(\mathrm{fj}\,x+\mathrm{hg}\,y)$ is symmetrical in x and y.

9. For any value of the constant a, the functions $\mathrm{pr}\,z\,\mathrm{pr}(z+a)$, $\mathrm{qr}\,z\,\mathrm{qr}(z+a)$ have the same periods and the same poles.

10. Unless one of the points ω_r, ω_t is a zero of $\mathrm{pq}\,z$ and the other is not, the zeros of the function $(\mathrm{pq}\,z-\mathrm{pq}\,\omega_r)(\mathrm{pq}\,z-\mathrm{pq}\,\omega_t)$ are all double or quadruple and the poles are all double.

11. If p, q, r, t are the four cardinal symbols, the integral
$$\int \frac{\mathrm{pq}\,z\,dz}{A+B\,\mathrm{pq}^2 z}$$
is reduced to an elementary integral by the substitution $\mathrm{rt}\,z = w$.

12. If α, β, γ are the values of the integral $\int dw/\sqrt{(w^4-1)}$ to infinity, (i) from the origin along the bisector of the angle between the positive halfaxes, (ii) from $w = 1$ along the positive real halfaxis, (iii) from $w = -i$ along the negative imaginary halfaxis, the relation $\alpha+\beta+\gamma = 0$ is equivalent to the relation
$$\int_0^\infty \frac{dt}{\sqrt{(t^4+1)}} = \sqrt{2}\int_1^\infty \frac{dx}{\sqrt{(x^4-1)}}$$
between positive real integrals.

13. The matrix
$$\left\|\begin{matrix} \mathrm{sn}\,u\,\mathrm{dn}\,v & \mathrm{dn}\,u\,\mathrm{sn}\,v & \mathrm{cn}\,u & \mathrm{cn}\,v \\ \mathrm{cn}\,u & \mathrm{cn}\,v & -\mathrm{sn}\,u\,\mathrm{dn}\,v & -\mathrm{dn}\,u\,\mathrm{sn}\,v \\ \mathrm{cn}\,v & \mathrm{cn}\,u & \mathrm{dn}\,u\,\mathrm{sn}\,v & \mathrm{sn}\,u\,\mathrm{dn}\,v \end{matrix}\right\|$$
is of rank two.

14. Any triad of copolar Jacobian functions x, y, z is a fundamental set of solutions of a homogeneous linear differential equation $w''' = \theta w'+\phi w$.

15. The Wronskian of any triad of copolar Jacobian functions is a non-zero constant.

16. Regarded as functions of u, the three functions

$$\operatorname{cs} u \operatorname{cs}(u+v), \qquad \operatorname{ns} u \operatorname{ns}(u+v), \qquad \operatorname{ds} u \operatorname{ds}(u+v)$$

have the same periods and the same poles, and the two combinations

$$\operatorname{cs} v \operatorname{ns} u \operatorname{ns}(u+v) - \operatorname{ns} v \operatorname{cs} u \operatorname{cs}(u+v), \qquad \operatorname{ds} v \operatorname{ns} u \operatorname{ns}(u+v) - \operatorname{ns} v \operatorname{ds} u \operatorname{ds}(u+v)$$

are constants.

17.
$$\operatorname{sn} u \operatorname{dn} v \operatorname{ns}(u+v) + \operatorname{dn} u \operatorname{sn} v \operatorname{cs}(u+v) = \operatorname{cn} v,$$
$$\operatorname{sn} u \operatorname{cn} v \operatorname{ns}(u+v) + \operatorname{cn} u \operatorname{sn} v \operatorname{ds}(u+v) = \operatorname{dn} v.$$

18. As equations in u, the four equations $\operatorname{sn} 3u = \pm 1$, $\operatorname{sn} 3u = \pm 1/k$ have only double roots; in terms of $\operatorname{sn} u$, each equation is of the ninth degree and has one simple root and four double roots.

19. (i) $\dfrac{1-\operatorname{cn} 2u}{1+\operatorname{cn} 2u} = \left(\dfrac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u}\right)^2,$

(ii) $\dfrac{\operatorname{sc} 3u - i}{\operatorname{sc} 3u + i} = \dfrac{\operatorname{sc} u + i}{\operatorname{sc} u - i}\left(\dfrac{c'\operatorname{sc}^4 u - 2ic'\operatorname{sc}^3 u - 2i \operatorname{sc} u - 1}{c'\operatorname{sc}^4 u + 2ic'\operatorname{sc}^3 u + 2i \operatorname{sc} u - 1}\right)^2.$

20. (i) $\mathrm{s_1 d_1 c^1 n^1} u = \tan(\tfrac{1}{2}\operatorname{am} 2u)$; (ii) $\operatorname{sc}(\tfrac{1}{2}K_c + u)\operatorname{sc}(\tfrac{1}{2}K_c - u) = 1/k'$.

(iii) If $\beta = \operatorname{am} \tfrac{1}{2}K_c$, then $\operatorname{dn} \tfrac{1}{2}K_c = \cot\beta$, $k' = \cot^2\beta$, and

$$\operatorname{cd}(u+\tfrac{1}{2}K_c) = -\csc\beta(\operatorname{dn} u - \csc^2\beta \operatorname{sn} u \operatorname{cn} u)/(\operatorname{cn} u - \operatorname{sn} u \operatorname{dn} u).$$

21. Functional equivalents of

$$\text{(i)}\ v = \int_0^t \frac{dt}{\sqrt{\{(3t^2+4)(2t^2+11)\}}}, \qquad \text{(ii)}\ v = \int_t^\infty \frac{dt}{\sqrt{\{(3t^2+4)(2t^2+11)\}}}$$

are (i) $t = \sqrt{(4/3)}\operatorname{sc}(v\sqrt{33})$, (ii) $t = \sqrt{(11/2)}\operatorname{cs}(v\sqrt{33})$, with $c = 25/33$ in each case.

22. Functional equivalents of

$$\text{(i)}\ v = \int_t^{3/2} \frac{dt}{\sqrt{\{(9-4t^2)(5t^2+7)\}}}, \qquad \text{(ii)}\ v = \int_{3/2}^t \frac{dt}{\sqrt{\{(4t^2-9)(5t^2+7)\}}}$$

are (i) $t = \tfrac{3}{2}\operatorname{cn}(v\sqrt{73}, 45/73)$, (ii) $t = \tfrac{3}{2}\operatorname{nc}(v\sqrt{73}, 28/73)$.

23. If
$$I = \int_0^\infty \frac{dt}{\sqrt{\{(t+a^2)(t+a^2\sin^2\beta)(t+a^2\sin^2\gamma)\}}},$$
then
$$\operatorname{cn}(\tfrac{1}{2}aI\cos\gamma; \cos\beta\sec\gamma) = \sin\gamma.$$

24. The relations

$$\text{(i)} \quad v = \int_t^\infty \frac{dt}{\sqrt{\{(t-1)(t-4)(t-6)(t-9)\}}}, \qquad 9 \leqslant t,$$

$$\text{(ii)} \quad v = \int_5^t \frac{dt}{\sqrt{\{(t-1)(t-4)(6-t)(9-t)\}}}, \qquad 4 \leqslant t \leqslant 6,$$

$$\text{(iii)} \quad v = \int_t^1 \frac{dt}{\sqrt{\{(1-t)(4-t)(6-t)(9-t)\}}}, \qquad t \leqslant 1,$$

are equivalent to

$$\text{(i) } t = 5+4\,\mathrm{ns}\,4v, \qquad \text{(ii) } t = 5+\mathrm{sn}\,4v, \qquad \text{(iii) } t = 5-4\,\mathrm{dc}\,4v,$$

with $k = 1/4$ in each case.

25. The relations

$$\text{(i)} \quad v = \int_0 \frac{dt}{\sqrt{\{(2t+1)t(5-2t)(4-t)\}}}, \qquad 0 \leqslant t \leqslant 5/2,$$

$$\text{(ii)} \quad v = \int_4 \frac{dt}{\sqrt{\{(2t+1)t(2t-5)(t-4)\}}}, \qquad 4 \leqslant t,$$

are equivalent to

$$\text{(i)} \quad t = \frac{5-5\,\mathrm{cd}\,\alpha v}{5+\mathrm{cd}\,\alpha v}, \qquad \text{(ii)} \quad t = \frac{5\,\mathrm{dc}\,\alpha v+3}{3-\mathrm{dc}\,\alpha v},$$

with $k = 3/5$, $\alpha = 5\sqrt{3}/2$ in each case.

26. The relations

$$\text{(i)} \quad v = \int_t^1 \frac{dt}{\sqrt{(1-t^3)}}, \qquad \text{(ii)} \quad v = \int_t^\infty \frac{dt}{\sqrt{(1+t^3)}},$$

are equivalent to

$$\text{(i)} \quad 2\sqrt{3}/(\sqrt{3}+1-t) = 1+\mathrm{cn}\{v\sqrt[4]{3}, (2+\sqrt{3})/4\},$$
$$\text{(ii)} \quad 2\sqrt{3}/(t+1-\sqrt{3}) = \mathrm{nc}\{v\sqrt[4]{3}, (2+\sqrt{3})/4\}-1.$$

27. The relation
$$v = \int_0^t \frac{dt}{\sqrt{(1+t^4)}}$$

is equivalent to

(i) $(1-t^2) = (1+t^2)\mathrm{cn}(2v, \tfrac{1}{2})$, (ii) $t = \mathrm{s_1 d_1 c^1 n^1}(v, \tfrac{1}{2})$,

(iii) $t = \tfrac{1}{2}(1+i)\mathrm{sd}\{(1-i)v, \tfrac{1}{2}\}$,

(iv) $(1-t)/(1+t) = (\sqrt{2}-1)\mathrm{sc}\{\tfrac{1}{2}(2+\sqrt{2})(v_1-v), 1-(\sqrt{2}-1)^4\}$,

where v_1 is a value of the integral when the upper limit is 1.

28. The integrals

$$\text{(i)} \int^x \frac{dx}{\sqrt{(1-x^4)}}, \qquad \text{(ii)} \int_x \frac{dx}{\sqrt{(6x^4+19x^2+15)}}, \qquad \text{(iii)} \int^x \frac{dx}{\sqrt{(6x^4-19x^2+15)}},$$
$$x^2 > 5/3$$

are converted into Legendre's form by the substitutions

(i) $x^2 = y^2/(2-y^2)$, (ii) $x^2 = 5(1-y^2)/3y^2$, (iii) $x^2 = (10-9y^2)/6(1-y^2)$.

29. The substitution $(3x+2)^2/(2x-1)(x-4) = \tfrac{4}{3}y^2$ converts

$$\int^x \frac{dx}{\sqrt{\{(2x-1)(x-4)(5x^2+4)\}}}$$

into a multiple of
$$\int^y \frac{dy}{\sqrt{\{(1+y^2)(1+\tfrac{1}{6}y^2)\}}}.$$

30. The interior of an isosceles rightangled triangle is represented conformally on a halfplane by the transformation $z = \mathrm{dc}\,w\,\mathrm{dn}\,w$ with parameter 1/2.

31. The interior of an equilateral triangle is represented conformally on a halfplane by the transformation

$$z = (\mathrm{cs}\, w + \mathrm{ns}\, w)(\mathrm{cd}\, w + \mathrm{nd}\, w)$$

with parameter $(2+\surd 3)/4$.

32. The interior of a rightangled triangle which is half of an equilateral triangle is represented conformally on a halfplane by the transformation

$$(1-z)/(1+z) = (1-\surd 3\mathrm{s}_2\mathrm{d}_2\mathrm{c}^2\mathrm{n}^2 w)^3,$$

with parameter $(2+\surd 3)/4$.

33. The interior of an isosceles triangle each of whose base angles is one-third of a right angle is represented conformally on a halfplane by the transformation

$$z^{-2} = 1-(1+\surd 3\mathrm{c}_2\mathrm{n}_2\mathrm{d}^2\mathrm{s}^2 w)^{-3},$$

with parameter $(2-\surd 3)/4$.

34. If p, q, r, t are the four cardinal symbols,

$$\int \frac{\mathrm{pq}\, u\, du}{\mathrm{pq}\, u + \mathrm{pq}\, K_t} = \alpha\, \mathrm{Pt}\, u + \beta\, \mathrm{rt}\, u, \qquad \int \frac{\mathrm{rq}\, u\, du}{\mathrm{pq}\, u + \mathrm{pq}\, K_t} = \gamma\, \mathrm{pt}\, u + \delta\, \mathrm{qt}\, u,$$

where α, β, γ, δ are constants.

35.
$$k \int \frac{\mathrm{cn}\, u\, du}{1-k\, \mathrm{sn}\, u} = k\, \mathrm{sd}\, u + \mathrm{nd}\, u, \qquad k^2 \int \frac{\mathrm{dn}\, u\, du}{\mathrm{dn}\, u + k'} = \mathrm{Dc}\, u - k'\mathrm{sc}\, u,$$

$$\int \frac{du}{\mathrm{nc}\, u - 1} = \mathrm{Cs}\, u - \mathrm{ds}\, u, \qquad k'^2 \int \frac{\mathrm{dc}^2 u\, du}{1+k\, \mathrm{sn}\, u} = D(u) - k^2 u - k\, \mathrm{dc}\, u.$$

36.
$$k^2 \int_0^u \frac{du}{(1+\mathrm{cn}\, u)(\mathrm{dn}\, u + k')} = \frac{\mathrm{nc}\, u + 2}{\mathrm{cs}\, u + \mathrm{ns}\, u} - \log \frac{\mathrm{ns}\, u + 1}{\mathrm{ds}\, u + k'} - k'(\mathrm{Nc}\, u + \mathrm{Ns}\, u + \mathrm{ds}\, u).$$

37. (i) Writing $\Delta = 1 - c\, \mathrm{sn}^2 u\, \mathrm{sn}^2 v$, let

$$\partial(\Delta^{-1}\mathrm{s}_1\mathrm{c}_1\mathrm{d}_1\mathrm{n}^3 u)/\partial u = \Delta^{-2} N_u, \qquad \partial(\Delta^{-1}\mathrm{s}_1\mathrm{c}_1\mathrm{d}_1\mathrm{n}^3 v)/\partial v = \Delta^{-2} N_v;$$

if $\phi(v) = c\, \mathrm{sn}^4 v$, then $\phi N_u - N_v$, as a function of sn u, is divisible by Δ, the quotient being $\Delta^2 - 2\, \mathrm{cn}^2 u\, \mathrm{dn}^2 u$. If further $f(v) = \mathrm{ns}^2 v$, then

$$f\phi\, \partial(\Delta^{-1}\mathrm{s}_1\mathrm{c}_1\mathrm{d}_1\mathrm{n}^3 u)/\partial u - \partial(f\Delta^{-1}\mathrm{s}_1\mathrm{c}_1\mathrm{d}_1\mathrm{n}^3 v)/\partial v = f\Delta.$$

(ii) For an appropriate range of values of v,

$$\mathrm{c}_1\mathrm{d}_1\mathrm{s}^1\mathrm{n}^1 v \int_0^K \frac{du}{1 - c\, \mathrm{sn}^2 u\, \mathrm{sn}^2 v} = -K\, \mathrm{Cs}\, v - Ev.$$

38. If the functions of v have for parameter the complement of the parameter c of the functions of u, then for appropriate ranges of values of v,

(i)
$$c'\mathrm{s}_1\mathrm{c}_1\mathrm{d}^1\mathrm{n}^1 v \int_0^K \frac{du}{1-\mathrm{sn}^2 u\, \mathrm{dn}^2 v} = c'K\, \mathrm{Cd}\, v - Ev,$$

(ii)
$$\mathrm{d}_1\mathrm{n}_1\mathrm{s}^1\mathrm{c}^1 v \int_0^K \frac{du}{1+\mathrm{sn}^2 u\, \mathrm{cs}^2 v} = K\, \mathrm{Dc}\, v - Ev + \tfrac{1}{2}\pi.$$

39. If $\sin\psi = \sin\alpha \sin\theta$, then, with modulus $\sin\alpha$,

$$\int_0^{\pi/2} \frac{1-\cos^3\psi}{1-\cos^2\psi}\, d\theta = 3\int_0^{\pi/2} \cos\psi \cos^2\theta\, d\theta = (\csc^2\alpha + 1)E - K\cot^2\alpha.$$

40. If (X, iX') is the fundamental Jacobian basis, that is, the basis such that $X \to \frac{1}{2}\pi$ as $c \to 0$ and $X' \to \frac{1}{2}\pi$ as $c' \to 0$, the series

$$1+(\tfrac{1}{2})^2(cc')+\left(\frac{1.5}{2.4}\right)^2(cc')^2+\left(\frac{1.5.9}{2.4.6}\right)^2(cc')^3+\dots$$

converges to $(2/\pi)X$ inside the loop of the lemniscate $|cc'| = \frac{1}{4}$ that surrounds $c = 0$ and to $(2/\pi)X'$ inside the loop that surrounds $c' = 0$.

41. Near $c = 1$, if $0 < \alpha < 1$, then

$$\int_0^1 \frac{dt}{(1-t)^{1-\alpha}(1-ct)^\alpha} = \log\frac{1}{1-c}+A+O(1-c)^{1/2},$$

where
$$A = \int_0^1 \left\{\frac{1}{t^{1-\alpha}(1+t)^\alpha}-\frac{1}{t}\left(1-\frac{1}{(1+t)^\alpha}\right)\right\}dt;$$

integration is along the real axis of the t plane, but c may be complex.

42. With the notation of Ex. 41,

$$\int_0^1 \frac{f(t,c)\,dt}{(1-t)^{1-\alpha}(1-ct)^\alpha} \sim f(1,1)\left\{\log\frac{1}{1-c}+A\right\}-\int_0^1 \frac{f(1,1)-f(t,1)}{1-t}\,dt,$$

if the integral on the right exists.

43. In the notation of hypergeometric functions,

$$X(c) = \tfrac{1}{2}\pi F(\tfrac{1}{2},\tfrac{1}{2};1;c).$$

44. Inside the circle which has $c = 0$, $c' = 0$ for the ends of a diameter,

$$F\{\tfrac{1}{4},\tfrac{1}{4};\tfrac{1}{2};(c'-c)^2\} = (X+X')/B,$$
$$(c'-c)F\{\tfrac{3}{4},\tfrac{3}{4};\tfrac{3}{2};(c'-c)^2\} = (X'-X)/C,$$

where

$$B = 2K_{1/2} = \tfrac{1}{2}\int_0^1 \frac{dt}{t^{3/4}(1-t)^{3/4}}, \qquad C = 2E_{1/2}-K_{1/2} = \tfrac{1}{2}\int_0^1 \frac{dt}{t^{1/4}(1-t)^{1/4}}.$$

45. If $f(u)$, $g(u)$ satisfy the conditions

$$f(u+2K_c) = f(u), \qquad g(u+2K_c) = -g(u),$$

and are regular throughout the parallelogram whose vertices are $\mp K_c \mp K_n$, then

$$\int_{-K_c}^{K_c} \{f(u+K_n)+f(u-K_n)\}\operatorname{dn} u\,du = 2\pi f(0),$$
$$\int_{-K_c}^{K_c} \{g(u+K_n)-g(u-K_n)\}\operatorname{sn} u\,du = -2\pi v g(0)/k,$$
$$\int_{-K_c}^{K_c} \{g(u+K_n)+g(u-K_n)\}\operatorname{cn} u\,du = 2\pi g(0)/k,$$

where v is the signature of the basis (K_c, K_n).

46. In the notation of Ch. XVI, with a rectilinear path of integration and for integral values of n,

$$\int_0^{K_c} \operatorname{dn} u \cos 2nv\,du = \tfrac{1}{2}\pi \operatorname{sech} n\tau = \frac{\pi q^n}{1+q^{2n}},$$

$$\int_0^{K_c} \operatorname{sn} u \sin(2n+1)v\, du = \tfrac{1}{2}\pi k^{-1} \operatorname{csch}(n+\tfrac{1}{2})\tau = \frac{\pi r^{2n+1}}{k(1-q^{2n+1})},$$

$$\int_0^{K_c} \operatorname{cn} u \cos(2n+1)v\, du = \tfrac{1}{2}\pi k^{-1} \operatorname{sech}(n+\tfrac{1}{2})\tau = \frac{\pi r^{2n+1}}{k(1+q^{2n+1})}.$$

47. If the line from $-K_c$ to K_c is indented to avoid the origin, the integral of $\operatorname{cs} u$ along the path formed is $-\pi v$ or πv according as the indent does or does not pass between 0 and K_n on the line joining these points.

48. If the parallelogram whose vertices are $K_n \mp K_n \mp K_c$ has parallel indentations at 0 and $2K_n$, and if the indented contour surrounds the origin, then the integral of $(1-e^{2nvv})\operatorname{cs} u$ round the contour is expressible as

$$(1+q^{2n}) \int_{-K_c}^{K_c} (1-e^{2nvv})\operatorname{cs} u\, du + \pi v(1-q^{2n}),$$

where the path of integration may be rectilinear.

49. $$\int_0^{K_c} \operatorname{cs} u \sin 2nv\, du = \frac{\pi}{2} \cdot \frac{1-q^{2n}}{1+q^{2n}}.$$

50. $$\frac{K^2}{\pi^2} \operatorname{dn}^2 \frac{2Kv}{\pi} = \frac{KE}{\pi} + \frac{2q\cos 2v}{1-q^2} + \frac{4q^2\cos 4v}{1-q^4} + \frac{6q^3\cos 6v}{1-q^6} + \dots.$$

51. If $\vartheta_p(u_r)$ is denoted by p_r, and if $u_1+u_2+u_3+u_4 = 0$, then

$$cc'\,\mathrm{s_1s_2s_3s_4} - c\,\mathrm{c_1c_2c_3c_4} - c'\,\mathrm{n_1n_2n_3n_4} + \mathrm{d_1d_2d_3d_4} = 0,$$
$$c\,\mathrm{c_1c_2s_3s_4} - c\,\mathrm{s_1s_2c_3c_4} + \mathrm{d_1d_2n_3n_4} - \mathrm{n_1n_2d_3d_1} = 0,$$
$$c'\,\mathrm{n_1n_2s_3s_4} - \mathrm{d_1d_2c_3c_4} - c'\,\mathrm{s_1s_2n_3n_4} + \mathrm{c_1c_2d_3d_4} = 0,$$
$$\mathrm{d_1d_2s_3s_4} - \mathrm{n_1n_2c_3c_4} + \mathrm{c_1c_2n_3n_4} - \mathrm{s_1s_2d_3d_4} = 0;$$

also $$\mathrm{s_1c_2n_3d_4} + \mathrm{c_1s_2d_3n_4} + \mathrm{n_1d_2s_3c_4} + \mathrm{d_1n_2c_3s_4} = 0.$$

52. If p_1, p_2, p denote $\vartheta_p(u)$, $\vartheta_p(v)$, $\vartheta_p(u+v)$, the sets of ratios c:n:d, s:n:d, s:c:d, s:c:n satisfy four sets of equations, as follows:

$$c\,\mathrm{c_1c_2.c} + c'\,\mathrm{n_1n_2.n} - \mathrm{d_1d_2.d} = 0, \qquad c\,\mathrm{s_1c_2.s} - \mathrm{n_1d_2.n} + \mathrm{d_1n_2.d} = 0,$$
$$c\,\mathrm{s_1s_2.c} - \mathrm{d_1d_2.n} + \mathrm{n_1n_2.d} = 0, \qquad c\,\mathrm{c_1s_2.s} - \mathrm{d_1n_2.n} + \mathrm{n_1d_2.d} = 0,$$
$$\mathrm{d_1d_2.c} + c'\,\mathrm{s_1s_2.n} - \mathrm{c_1c_2.d} = 0, \qquad \mathrm{n_1d_2.s} - \mathrm{s_1c_2.n} - \mathrm{c_1s_2.d} = 0,$$
$$\mathrm{n_1n_2.c} - \mathrm{c_1c_2.n} + \mathrm{s_1s_2.d} = 0; \qquad \mathrm{d_1n_2.s} - \mathrm{c_1s_2.n} - \mathrm{s_1c_2.d} = 0;$$

$$c'\,\mathrm{s_1n_2.s} + \mathrm{c_1d_2.c} - \mathrm{d_1c_2.d} = 0, \qquad \mathrm{s_1d_2.s} + \mathrm{c_1n_2.c} - \mathrm{n_1c_2.n} = 0,$$
$$c'\,\mathrm{n_1s_2.s} + \mathrm{d_1c_2.c} - \mathrm{c_1d_2.d} = 0, \qquad \mathrm{d_1s_2.s} + \mathrm{n_1c_2.c} - \mathrm{c_1n_2.n} = 0,$$
$$\mathrm{c_1d_2.s} - \mathrm{s_1n_2.c} - \mathrm{n_1s_2.d} = 0, \qquad \mathrm{c_1n_2.s} - \mathrm{s_1d_2.c} - \mathrm{d_1s_2.n} = 0,$$
$$\mathrm{d_1c_2.s} - \mathrm{n_1s_2.c} - \mathrm{s_1n_2.d} = 0; \qquad \mathrm{n_1c_2.s} - \mathrm{d_1s_2.c} - \mathrm{s_1d_2.n} = 0.$$

53. If $\mathfrak{L}\{f\}$ denotes the Laplace-transform of the function $f(t)$ of the positive real variable t, that is, the function of the positive real variable p defined by

$$\mathfrak{L}\{f(t)\} = \int_0^\infty e^{-pt} f(t)\, dt,$$

then

(i) $$\mathfrak{L}\{e^{-\alpha^2/4t}/\sqrt{(\pi t)}\} = e^{-\alpha\sqrt{p}}/\sqrt{p},$$

(ii) $$\mathfrak{L}\{[\sqrt{t}/2\pi]\} = \{e^{-4\pi^2 p} + e^{-16\pi^2 p} + e^{-36\pi^2 p} + \dots\}/p.$$

54. If the relation of the several functions to the lattice, and their dependence on the parameter σ (where $q = e^{-\sigma}$), are indicated by the notation

$$\mathrm{H}(u) = \mathrm{H}_s(v;\sigma), \qquad \mathrm{H}(u+K_c) = \mathrm{H}_c(v;\sigma),$$
$$\Theta(u) = \Theta_n(v;\sigma), \qquad \Theta(u+K_c) = \Theta_d(v;\sigma),$$

then for $-\frac{1}{2} \leqslant v \leqslant \frac{1}{2}$,

$$\mathfrak{L}\{\mathrm{H}_s(\pi v;\pi^2 t)\} = -(1/\sqrt{p})\sinh(2v\sqrt{p})\operatorname{sech}\sqrt{p},$$
$$\mathfrak{L}\{\Theta_n(\pi v;\pi^2 t)\} = (1/\sqrt{p})\cosh(2v\sqrt{p})\operatorname{csch}\sqrt{p},$$

and for $0 \leqslant v \leqslant 1$,

$$\mathfrak{L}\{\mathrm{H}_c(\pi v;\pi^2 t)\} = -(1/\sqrt{p})\sinh\{(2v-1)\sqrt{p}\}\operatorname{sech}\sqrt{p},$$
$$\mathfrak{L}\{\Theta_d(\pi v;\pi^2 t)\} = (1/\sqrt{p})\cosh\{(2v-1)\sqrt{p}\}\operatorname{csch}\sqrt{p}.$$

55. If a'_m, b'_m is the typical member of an arithmetico-geometric chain in which $a'_0:b'_0 = 1:k'$, and if $\tan\bar{\chi}_m = \sqrt{(a'_m/b'_m)}\tan 2\bar{\chi}_{m-1}$ with $\tan\bar{\chi}_0 = \sqrt{k'}\tan\phi$, then as $m \to \infty$,

$$2^{-m}\bar{\chi}_m \to M(1,k')F(\phi;k).$$

56. If in the transformation $x = \mathrm{pq}\,u$ the Jacobian parameter c is a real number between 0 and 1, and the points P, Q are adjacent corners of the fundamental rectangle, then the line which joins the midpoint of PQ to the midpoint of the opposite side of the rectangle is represented in the x plane by a circular quadrant of radius $|\mathrm{pq}\,\frac{1}{2}(K_q - K_p)|$ whose centre is the origin.

57. If two variables z, w are connected through an intermediary by the relations $z = \mathrm{pq}^2\zeta$, $w = \mathrm{Pq}\,\zeta$, with $0 < k < 1$, and if the pole and the zero of $\mathrm{pq}\,\zeta$ are adjacent vertices of the fundamental rectangle, the halfplane $\mathrm{Im}\,z > 0$ corresponds to a w quadrant enlarged by the addition of a rectangle in the corner of an adjacent quadrant.

NOTES ON THE EXERCISES

1. By comparison of periods and principal parts; since the functions are odd, the additive constant is zero. Cf. 14·81 on p. 244.

2. From Ex. 1 by differentiation.

3. Trivially from Ex. 2,

$$\frac{\mathrm{fj}^2\tfrac{1}{2}z}{(\mathrm{fj}\,z+\mathrm{gj}\,z)(\mathrm{fj}\,z+\mathrm{hj}\,z)}=\frac{\mathrm{gj}^2\tfrac{1}{2}z}{(\mathrm{gj}\,z+\mathrm{hj}\,z)(\mathrm{gj}\,z+\mathrm{fj}\,z)}=\frac{\mathrm{hj}^2\tfrac{1}{2}z}{(\mathrm{hj}\,z+\mathrm{fj}\,z)(\mathrm{hj}\,z+\mathrm{gj}\,z)}=1.$$

4. Ex. 2 gives $\wp'\tfrac{1}{2}\omega_f$ unambiguously.

5. Since $2\,\mathrm{fj}\,x\,\mathrm{fj}\,y\,\mathrm{fj}(x-y)=\{\wp'x(\wp y-e_f)+\wp'y(\wp x-e_f)\}/(\wp x-\wp y)$.

6. For arbitrary values of x and y, the function of z on the left can have no poles that are not simple, and the possible residues are all zero by Ex. 5. To find the constant, put $z=0$.

Alternatively, by elementary algebra in terms of the $\wp$ function.

7. (i) An alternative enunciation of Ex. 6.

(ii) Replacing z_1, z_2 by $z_1+\omega_f$, $z_2-\omega_f$.

(iii) Adding ω_f, ω_g, ω_h to z_2, z_3, z_4.

Writing $z_4=0$ we have a multitude of pairs of formulae from which addition theorems are deducible; see Ex. 51, 52.

Results equivalent to those in this exercise are given by Tannery and Molk.

8. In terms of primitive functions the relation to be verified is

$$\mathrm{gj}^2x\,\mathrm{fj}^2y-f_g^2\,\mathrm{hj}^2x=\mathrm{fj}^2x\,\mathrm{gj}^2y-f_g^2\,\mathrm{hj}^2y.$$

The function given is $\mathrm{jg}(x+y)+\mathrm{fg}(x+y)$, found from $4{\cdot}71_1$ by the substitution of $y-\omega_g$ for y.

9. Addition theorems can be deduced; see Ex. 16.

10. The only case of quadruplicity is that in which $\omega_t=\omega_r$ and $\mathrm{pq}\,\omega_r\neq 0$.

11. Since the difference between the two fractions $1/(a-b\,\mathrm{pq}\,z)$, $1/(a+b\,\mathrm{pq}\,z)$ is integrable, the integration of the fraction $1/(a-b\,\mathrm{pq}\,z)$, and therefore of any rational function of $\mathrm{pq}\,z$, can be effected in terms of the integral of the sum of the two fractions, that is, in terms of the integral of a function of the form $1/(A+B\,\mathrm{pq}^2z)$. This is Legendre's standard integral of the third kind. The integral of the first kind is the integral of pq^2z and is in effect standardized by the Weierstrassian function ζz and by the functions we have called the integrating functions; the integral of the second kind is the inverse of a Jacobian function. See further Ex. 37, 38.

12. For direct verification we have, by the substitution $t=1/y$,

$$\int_0^1\frac{dt}{\sqrt{(t^4+1)}}=\int_1^\infty\frac{dt}{\sqrt{(t^4+1)}}=\tfrac{1}{2}\int_0^\infty\frac{dt}{\sqrt{(t^4+1)}},$$

and by the substitution $t^2=(x-1)/(x+1)$,

$$\int_0^1\frac{dt}{\sqrt{(t^4+1)}}=\int_1^\infty\frac{dx}{\sqrt{\{2(x^4-1)\}}}.$$

The relation $\alpha+\beta+\gamma=0$ is that between quarterperiods; see 6·503.

13. The source of this result is given in Ex. 52, where it can be seen that addition of quarterperiods yields only three results essentially distinct from this.

14,15. Independent proofs are easy, but either result may be inferred from the other, since the determinant $|xy'z'''|$ is the derivative of the Wronskian $|xy'z''|$.

The functions x, y, z satisfy a set of simultaneous differential equations

$$x' = ayz, \qquad y' = bzx, \qquad z' = cxy,$$

where a, b, c are constants. Hence $x'' = ax(bz^2+cy^2)$, and therefore $x''' = \theta x'+\phi x$, where $\theta = bcx^2+cay^2+abz^2$, $\phi = 3abcxyz$. Alternatively,

$$x^2 = a(\psi+A), \qquad y^2 = b(\psi+B), \qquad z^2 = c(\psi+C),$$

where $\psi' = 2xyz$ and A, B, C are simultaneous values of x^2/a, y^2/b, z^2/c, whence the value of the determinant $|xy'z''|$ is $a^2b^2c^2(B-C)(C-A)(A-B)$.

16. Putting $u = K_c$ in the first difference, $u = K_d$ in the second, we have

$$\operatorname{cs} v \operatorname{ns} u \operatorname{ns}(u+v) - \operatorname{ns} v \operatorname{cs} u \operatorname{cs}(u+v) = \operatorname{ds} v,$$
$$\operatorname{ds} v \operatorname{ns} u \operatorname{ns}(u+v) - \operatorname{ns} v \operatorname{ds} u \operatorname{ds}(u+v) = c \operatorname{cs} v.$$

Combining these formulae with the two derived from them by interchange of u and v we have two pairs of simultaneous equations from which $\operatorname{cs}(u+v)$, $\operatorname{ns}(u+v)$, $\operatorname{ds}(u+v)$ can be found.

17. By adding K_n to both u and v in Ex. 16.

By adding quarterperiods to u, v independently, or by arguing on any two functions $\operatorname{pr} u \operatorname{pr}(u+v)$ and $\operatorname{qr} u \operatorname{qr}(u+v)$, we obtain a profusion of equations of which sixteen in all are essentially distinct; these sixteen, found by an alternative process which organizes them, are given in Ex. 52.

18. The argument is functional: $\operatorname{sn}' 3u$ is zero in all four cases. In algebraical verification, evaluating $\operatorname{sn} 3u$ as $\operatorname{sn}(2u+u)$ we find, if $\operatorname{sn} u = x$,

$$\frac{1-\operatorname{sn} 3u}{1+\operatorname{sn} 3u} = \frac{1+x}{1-x}\left(\frac{1-2x+2k^2x^3-k^2x^4}{1+2x-2k^2x^3-k^2x^4}\right)^2,$$

$$\frac{1-k\operatorname{sn} 3u}{1+k\operatorname{sn} 3u} = \frac{1+kx}{1-kx}\left(\frac{1-2kx+2kx^3-k^2x^4}{1+2kx-2kx^3-k^2x^4}\right)^2.$$

The roots of $\operatorname{sn} 3u = 1$ which are double in terms of $\operatorname{sn} u$ are congruent, mod $4K_c$, $2K_n$, with $K_c+\frac{1}{3}K_n \mp \frac{2}{3}K_c \mp \frac{1}{3}K_n$, the roots which are simple are congruent with $3K_c$.

20. (i) The functions are uniform, their squares are equal, by Ex. 19 (i), and they both resemble u near $u = 0$.

(ii) A version of $\operatorname{sc} u \operatorname{sc}(K_c-u) = \operatorname{sn} K_c \operatorname{sd} K_c$. If k and k' are real and positive, we have $|\operatorname{sc}(\frac{1}{2}K_c+it)| = 1/\sqrt{k'}$. See Ex. 56.

(iii) In this form of the results there are no ambiguities to be resolved. The values of $\operatorname{dn} \frac{1}{2}K_c$ and k' are given by (i) and (ii), and the formula for $\operatorname{cd}(u+\frac{1}{2}K_c)$ is a case of $12{\cdot}44_4$.

21. We can scan Table XI 11 for the sign pattern, but a moment's preliminary consideration discovers the functions wanted. The critical values are both imaginary, and therefore the points P, Q are the corners of the fundamental rectangle which are on the real axis, and the functions available are $\operatorname{sc} u$ and $\operatorname{cs} u$, of which the first, with zero at the origin, is the better suited to (i), and the second, with pole there, to (ii). We have now only to assimilate (i) to $(dx/du)^2 = (1+c'x^2)(1+x^2)$ and (ii) to $(dx/du)^2 = (x^2+1)(x^2+c')$.

Alternatives still remain, for we could take $t^2 = (11/2)x^2$ in (i) or $t^2 = (4/3)x^2$ in (ii); the results are functionally accurate, but they imply a negative value for c.

There is no real need to use different functions, for the relation (ii) is equivalent, from (i), to $t = \sqrt{(4/3)}\,\mathrm{sc}\{(v_\infty - v)\sqrt{33}\}$, but when the fixed limit has one of the four natural values there is one specially appropriate function.

22. In each case one critical value is real and is at the origin, and the other is imaginary; hence P, Q are diagonally opposite and have the origin between them, and the functions available are $\mathrm{cn}\,u$ and $\mathrm{nc}\,u$. In (i) the real values of the function are less than the value at the origin and the function is $\mathrm{cn}\,u$; in (ii) the real values increase with t and the function is $\mathrm{nc}\,u$. After making the substitution which reduces the first factor of the radical to $1-x^2$ or x^2-1, we have only to divide the second factor by the sum of its coefficients to reduce it to $c'+cx^2$ or $c'x^2+c$.

24. Since the two critical values are real, P, Q are adjacent corners of the fundamental rectangle.

(i) In general, ∞ as a limit is of no significance before the radical has been transformed, but here it is clear in advance that the substitution will be of the form $t-5 = \lambda x$, and therefore that ∞ has the same significance for the function before transformation as afterwards. Hence the function, with a pole at the origin and two real critical values, is $\mathrm{ns}\,u$. To vary the procedure, we substitute $t-5 = \lambda\,\mathrm{ns}\,\alpha v$, and compare the given relation

$$(dt/dv)^2 = \{(t-5)^2-4^2\}\{(t-5)^2-1^2\}$$

with the relation

$$(dt/dv)^2 = (\alpha/\lambda)^2\{(t-5)^2-\lambda^2\}\{(t-5)^2-c\lambda^2\}.$$

(ii) The origin becomes a zero after transformation; we substitute

$$t-5 = \lambda\,\mathrm{sn}\,\alpha v,$$

and compare the two relations

$$(dt/dv)^2 = \{4^2-(t-5)^2\}\{1^2-(t-5)^2\},$$
$$(dt/dv)^2 = (\alpha/\lambda)^2\{\lambda^2-(t-5)^2\}\{\lambda^2-c(t-5)^2\}.$$

(iii) The integral is the same as in (i), with $10-t$ for t, that is, with the sign of $t-5$ changed, but the fixed limit is now a zero under the radical; that is, the function acquires one of its critical values at the origin, and the form in which the result is given is more compact than $t = 5-4\,\mathrm{ns}\,4(\bar{v}-v)$, in which $\bar{v}$ is the complete integral for case (i) and $4\bar{v}$ is K_c.

25. The range of zeros under the radical is not symmetrical, and the homographic transformations must be constructed. In each case the function required has two real critical values of which one is at the origin, and since a critical value at the origin is necessarily 1, the association is of $x = 1$ with $t = 0$ in (i) and of $x = 1$ with $t = 4$ in (ii). Hence in (i) the values $-1/k$, -1, 1, $1/k$ of x correspond to the values 4, 5/2, 0, $-1/2$ of t, and in (ii) the values -1, $-k$, k, 1 of x correspond to the same values of t in the original order $-1/2$, 0, 5/2, 4. In each case $\{(1-k)/(1+k)\}^2$ can be equated to the cross-ratio $(-1/2, 4; 0, 5/2)$, and $k = 3/5$.

The two transformations are

$$\text{(i)}\ \frac{1}{6}\cdot\frac{5-2t}{2t} = \frac{1}{4}\cdot\frac{1+x}{1-x}, \qquad \text{(ii)}\ \frac{1}{6}\cdot\frac{2t-5}{2t} = \frac{1}{4}\cdot\frac{5x-3}{5x+3}.$$

and the two functional relations are (i) $x = \mathrm{cd}\,\alpha v$, (ii) $x = \mathrm{dc}\,\alpha v$. Substitution gives the two values of α, which are necessarily identical.

26. (i) By finding the values of λ for which $(1+t+t^2)+\lambda(1-t)$ is a perfect square in t, we construct the identities

$$4(1+t+t^2) = (2-\sqrt{3})(\sqrt{3}+1-t)^2+(2+\sqrt{3})(\sqrt{3}-1+t)^2,$$
$$4\sqrt{3}(1-t) = (\sqrt{3}+1-t)^2-(\sqrt{3}-1+t)^2.$$

The function required has one real critical value and one imaginary critical value, and the real critical value corresponds to a zero value of the integral, that is, is at the origin. Hence the function is either $\mathrm{cn}\,u$, with factors $c'+cx^2$, $1-x^2$, or $\mathrm{nc}\,u$, with factors $c'x^2+c$, x^2-1.

From Ex. 19 (i), the relation can be written explicitly in the simple form

$$t = 1-\sqrt{3}\,\mathrm{s_2 d_2 c^2 n^2}\,\tfrac{1}{2}u.$$

Remark that $t = 0$ has no significance for the function and is not a natural limit for the integral; in other words, if we invert the relation

$$v = \int\limits_0^t \frac{dt}{\sqrt{(1-t^3)}},$$

the argument of the elliptic function must take the form $\alpha(v-v_0)$ with $v_0 \neq 0$. On the other hand, ∞ is a natural limit, and if we take $1/\sqrt{(t^3-1)}$ for integrand, the integral from t to ∞ inverts economically in the real field.

27. (i) Writing $t^2 = y$ and transforming the new radical by means of the identities $4y = (1+y)^2-(1-y)^2$, $1+y^2 = \frac{1}{2}(1+y)^2+\frac{1}{2}(1-y)^2$.

(ii) Immediately from (i); see Ex. 19. While (i) is the more useful for computation, (ii) presents t as a *uniform* function of v.

(iii) Actually more obvious than (i), using a fourth root of -1; transformation to (ii) is a straightforward exercise in separation of real and imaginary parts.

(iv) Applying the standard process to the factorized quartic by means of the identities

$$2\sqrt{2}(1\pm t\sqrt{2}+t^2) = (\sqrt{2}\pm 1)(1+t)^2+(\sqrt{2}\mp 1)(1-t)^2.$$

A transformation of functions whose modulus is $1/\sqrt{2}$ to functions whose complementary modulus is $(\sqrt{2}-1)^2$ is a Landen transformation.

28. (i) The sign-pattern of $(1-x^2)(1+x^2)$ is that of $\mathrm{sd}\,u$ and $\mathrm{cn}\,u$, but with $\mathrm{cn}\,u$ it is the lower limit of the integral that is variable. To render the factors multiples of $1-c'\,\mathrm{sd}^2u$, $1+c\,\mathrm{sd}^2u$, we must take $c = c' = \frac{1}{2}$; then

$$x^2 = \tfrac{1}{2}\mathrm{sd}^2u = \tfrac{1}{2}\mathrm{sn}^2u/(1-\tfrac{1}{2}\mathrm{sn}^2u).$$

(ii) The sign-pattern and the position of the variable limit indicate $\mathrm{cs}\,u$, and the factors $2x^2+3$, $3x^2+5$ are to become multiples of cs^2u+1, cs^2u+c'. To secure $c' < 1$ we associate the factor cs^2u+1 with the larger of the two numbers 3/2, 5/3. Thus $3x^2 = 5\,\mathrm{cs}^2u = 5(1-\mathrm{sn}^2u)/\mathrm{sn}^2u$. As it happens, we have not needed to determine the parameter, but from the identity

$$3(2x^2+3) = 10\,\mathrm{cs}^2u+9 = 10(\mathrm{cs}^2u+c')$$

we have $c' = 9/10$.

(iii) The function is $\mathrm{dc}\,u$, the factors are to become multiples of dc^2u-1, dc^2u-c, and to have $c < 1$ we associate dc^2u-1 with the larger of 3/2, 5/3. Thus $3x^2 = 5\,\mathrm{dc}^2u = 5(1-c\,\mathrm{sn}^2u)/(1-\mathrm{sn}^2u)$, while $c = 9/10$.

29. From the identities

$$14(5x^2+4) = (5x-6)^2+5(3x+2)^2, \qquad 8(2x-1)(x-4) = (5x-6)^2-(3x+2)^2,$$

the substitution $\sqrt{6}(3x+2)/(5x-6) = t$ converts the integral v into the form associated with $\operatorname{sd} u$, with $c' = 1/6$; that is, $t = \operatorname{sd}(\alpha v, 5/6)$ where α is a constant. The integral in y belongs to $\operatorname{sc} u$, and is therefore the result of the transformation implied in the relation $\operatorname{sc}^2 u = \operatorname{sd}^2 u/(1-c'\operatorname{sd}^2 u)$, namely,

$$y^2 = t^2/(1-\tfrac{1}{6}t^2) = 6(3x+2)^2/\{(5x-6)^2-(3x+2)^2\}.$$

30. If the origin in the z plane is to correspond to the right angle and the points ± 1 to the base angles, the Schwarz-Christoffel form of the transformation may be taken as

$$2w = \int_1^z \frac{dz}{z^{1/2}(z^2-1)^{3/4}},$$

the factor 2 being introduced for convenience. The substitution $z^2-1 = z^2t^4$ gives

$$w = \int_0^t z\,dt = \int_0^t \frac{dt}{\sqrt{(1-t^4)}},$$

implying $t\sqrt{2} = \operatorname{sd} w$, with $c = \tfrac{1}{2}$.

Although z is not a singlevalued function of t, we have shown incidentally that, with the tacit choice of radicals that we have made, $z = dw/dt = \sqrt{2}\operatorname{dc} w \operatorname{dn} w$. It follows that z, as a function of w, has no branchpoints, and that in fact the expression of z by way of t defines two separate uniform functions of w which are equally effective for the representation.

To drop the factor $\sqrt{2}$ from z now only alters the scale in the z plane.

31. The Schwarz-Christoffel transformation

$$\tfrac{3}{2}w = \int_z^\infty \frac{dz}{z^{2/3}(z^2-1)^{2/3}}$$

is converted by the substitution $z^2-1 = z^2t^3$ into

$$w = \int_t^1 z\,dt = \int_t^1 \frac{dt}{\sqrt{(1-t^3)}}.$$

Since $1/z = dt/dw$, and scale factors are unimportant, it follows from Ex. 26 (i) that one solution is $z = (1+\operatorname{cn} w)^2/\operatorname{sn} w \operatorname{dn} w$, with the parameter given.

32. With appropriate values of the constant C and of the constant of integration, the transformation

$$dz/dw = Cz^{1/2}(1-z)^{2/3}(1+z)^{5/6}$$

is converted by the substitution $1-z = (1+z)t^3$ into

$$2w = \sqrt[4]{3} \int_t^1 \frac{dt}{\sqrt{(1-t^3)}},$$

which, by Ex. 26 (i), is equivalent to

$$t = 1-\sqrt{3}\,\frac{1-\operatorname{cn} 2w}{1+\operatorname{cn} 2w}, \qquad c = \frac{2+\sqrt{3}}{4}.$$

33. The substitution $z^{-2} = 1-t^{-3}$ converts the Schwarz-Christoffel transformation, in this case

$$dz/dw = Cz^{1/3}(z^2-1)^{5/6},$$

to the form $dt/dw = -(2/\sqrt[4]{3})\sqrt{(t^3-1)}$. But it is to be noted that z is not now a uniform function of w. This is not mere want of ingenuity. It follows from Briot and Bouquet's discussion, *J. de l'Éc. Poly.* (1856), of the differential equation afterwards shown by Schwarz and Christoffel to be at the heart of the problem, that the only triangles for which uniform conformal representations exist are the three considered in Ex. 30–32.

34. The integrands are multiples of

$$\mathrm{pq}\,u(\mathrm{pq}\,u-\mathrm{pq}\,K_t)\mathrm{qt}^2u, \qquad \mathrm{rq}\,u(\mathrm{pq}\,u-\mathrm{pq}\,K_t)\mathrm{qt}^2u.$$

36. Since $\mathrm{Ns}\,u+1/u$ tends to zero with u, so also does $\mathrm{Ns}\,u+\mathrm{ds}\,u$. In terms of $E(u)$,

$$\mathrm{Ns}\,u+\mathrm{ds}\,u = \mathrm{sn}\,u/(\mathrm{cd}\,u+\mathrm{nd}\,u)+u-E(u).$$

37. (ii) By repeated integration from (i). On each side the function vanishes when $v = K$ and is an odd function resembling K/v near $v = 0$.

Let γ denote the path of integration for u from 0 to K, let γ' denote the reflection of γ in the origin, and let Γ denote the curve obtained by translation of $\gamma'+\gamma$ repeatedly through a step $2K$ in either direction. Then there must be a path of integration for v from 0 which does not cross either of the curves obtained by the translation of Γ through $\pm iK'$. In favourable cases, and in particular if the u path is straight, the two v barriers are the edges of an infinite strip; restriction of v to a strip of this kind is apparent in the form of the result, for the function on the right is periodic in $2K$ but not in $2iK'$.

38. Identities similar to that in Ex. 37 (i) are constructed, the v numerator being $\mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,v$ in (i) and $\mathrm{cs}\,v\,\mathrm{ns}\,v\,\mathrm{ds}\,v$ in (ii). In (ii) one v barrier is obtained by rotating Γ through a right angle round the origin, and the function on the left is discontinuous at $v = 0$, tending to $+\frac{1}{2}\pi$ or to $-\frac{1}{2}\pi$ according to the relation between the directions from which u and v approach the origin.

Except in notation, the results of Ex. 37 (ii) and Ex. 38 are due to Legendre, and the method is his. The integral is the complete integral of the third kind, in the three forms possible with real values. It is only the complete integral that is reducible; the indefinite integrals of Ex. 34–36 do not involve an arbitrary constant independent of the Jacobian system, and if they are regarded as involving integrals of the third kind, these integrals are degenerate. Substitution of K' for v in 38 (ii) is equivalent to one of Legendre's proofs of his identity 14·62.

39. Substitute at once $\theta = \mathrm{am}\,u$. The product by $\frac{8}{3}R^3\sin^2\alpha$ is the volume common to two circular cylinders of radii R, $R\sin\alpha$ with perpendicular axes which intersect; the first integral uses polar elements of area, the second cartesian elements of area, in a plane perpendicular to the axis of the more slender solid.

40. The series satisfies the quarterperiod differential equation.

41. The integral is identically $\log 1/(1-c)-I_1+I_2$, where

$$I_1 = \int_0^c \left\{\frac{1}{1-t}-\frac{1}{(1-t)^{1-\alpha}(1-ct)^{\alpha}}\right\}dt, \qquad I_2 = \int_c^1 \frac{dt}{(1-t)^{1-\alpha}(1-ct)^{\alpha}}.$$

With the substitution $c't/(1-t) = u$, where $c' = 1-c$,

$$I_1 = \int_0^c \left\{1-\frac{1}{(1+u)^\alpha}\right\}\frac{du}{c'+u} = \int_0^1 \left\{1-\frac{1}{(1+u)^\alpha}\right\}\frac{du}{u} - I_3 - I_4,$$

where

$$I_3 = \int_c^1 \left\{1-\frac{1}{(1+u)^\alpha}\right\}\frac{du}{u} = O(c'),$$

$$I_4 = \int_0^c \left\{1-\frac{1}{(1+u)^\alpha}\right\}\frac{c'\,du}{u(c'+u)}.$$

Without the factor c', the integral I_4 would be divergent for $c' = 0$; that is, I_4 is not $O(c')$. With the classical restriction, $c'+u \geqslant 2c'^{1/2}u^{1/2}$, and since

$$\int_0 \left\{1-\frac{1}{(1+u)^\alpha}\right\}\frac{du}{u^{3/2}}$$

is finite, $I_4 = O(c'^{1/2})$ immediately. For a complex value of c', let the half-line from 0 through c' cut some fixed circle whose radius is independent of c and less than 1 in b, and deform the path $0c$ into $0b+bc$; the integral J along $0b$ is $O(c'^{1/2})$, by a slight modification of the argument just used, and the integral I_5 along bc is $O(c')$.

With the substitution $1-t = c'u$,

$$I_2 = \int_0^1 \frac{du}{u^{1-\alpha}(1+cu)^\alpha} = \int_0^1 \frac{du}{u^{1-\alpha}(1+u)^\alpha} + O(c'),$$

and I_3, I_5, and the unevaluated part of I_2 are all dominated by J.

44. The functions satisfy the quarterperiod equation, the first is unchanged, the second changed only in sign, if c and c' are interchanged.

The values of B, C in terms of $K_{1/2}$ and $E_{1/2}$ are obvious, since the hypergeometric functions become unity when $c = c'$. For the integral forms, we have, from Ex. 41, near $z = 1$,

$$\int_0^1 \frac{dt}{t^{3/4}(1-t)^{3/4}}F(\tfrac{1}{4},\tfrac{1}{4};\tfrac{1}{2};z) = \int_0^1 \frac{dt}{t^{3/4}(1-t)^{3/4}(1-zt)^{1/4}} = \log\frac{1}{1-z} + O(1),$$

$$\int_0^1 \frac{dt}{t^{1/4}(1-t)^{1/4}}F(\tfrac{3}{4},\tfrac{3}{4};\tfrac{3}{2};z) = \int_0^1 \frac{dt}{t^{1/4}(1-t)^{1/4}(1-zt)^{3/4}} = \log\frac{1}{1-z} + O(1),$$

giving, when $z = (1-2c)^2$, the dominant term as $\log(1/c)$ in each case; on the other hand, $X'+X = \tfrac{1}{2}\log(1/c)+O(1)$.

Appeal to symmetry and skewsymmetry avoids the evaluation of constants in the application of Ex. 42.

45. By integrating the functions $f(u)\,\mathrm{cs}\,u$, $g(u)\,\mathrm{ns}\,u$, $g(u)\,\mathrm{ds}\,u$ round the parallelogram. The difference in postulated behaviour between $f(u)$ and $g(u)$ is wanted because $\mathrm{cs}\,u$ is not negatived by the addition of $2K_c$ to u, and the form of the integrand differs in the second and third integrals because the addition of $2K_n$ to u negatives $\mathrm{ds}\,u$ but not $\mathrm{ns}\,u$.

The parallelogram $\mp K_c \mp K_d$ provides similar theorems for the functions $\mathrm{nd}\,u$, $\mathrm{cd}\,u$, $\mathrm{sd}\,u$. Parallelograms with centres at other cardinal points produce the same six theorems. A function with a pole at 0 or K_c can not occur in theorems equally general if the path of integration is to be from $-K_c$ to K_c.

46. By taking $f(u) = e^{2nvv}$, $g(u) = e^{(2n+1)vv}$ in Ex. 45. The results, with trivial additions, are equivalent to the Fourier expansions $16{\cdot}74_{1-3}$, and $16{\cdot}74\ _{4-6}$ can be found in the same way; the factor $K/2\pi$ enters because the Fourier integration is with respect to v, not to u.

47. The integral is a value of $\log(-1)$, but instead of examining the various configurations, integrate $\mathrm{cs}\, u$ round tho boundary composed of the given path, a congruent arc joining $-K_c+2K_n$ to K_c+2K_n, and the lines from $\mp K_c$ to $\mp K_c+2K_n$. The integral is doubled, and the value of the contour integral is $2\pi v$ times the residue at the included pole, which is at 0 or at $2K_n$ according to the lie of the indent.

48. Immediately from Ex. 47.

49. From Ex. 48, by changing the sign either of n or of u and combining. The formula does not give a Fourier series, since $\sum (1-q^{2n})/(1+q^{2n})$ is divergent, nor is such a series to be expected, since $\mathrm{cs}\, u$ has a pole at the origin, but $16{\cdot}73_1$ follows immediately if the fraction is written as $1-2q^{2n}/(1+q^{2n})$. There are similar proofs of $16{\cdot}73_{2-6}$.

51. The first formula is the result of Ex. 7 (i) rewritten. Then u_1, u_2 are replaced by u_1+K_c, u_2-K_c, by u_1+K_n, u_2-K_n, and by u_1+K_d, u_2-K_d, in turn. Lastly u_2, u_3, u_4 are replaced by u_2+K_c, u_3+K_n, u_4+K_d.

There are no other typical forms, but when the arguments are permuted, a total of sixteen formulae, distinct for assigned arguments, is obtained. Each formula may be divided by a product $\mathrm{p}_1\mathrm{q}_2\mathrm{r}_3\mathrm{t}_4$ to provide a relation between Jacobian functions, but if the results are presented in this form duplication is harder to avoid and the structure of the group becomes harder to appreciate.

52. From the complete set of sixteen formulae implied in Ex. 51, by writing $u_4 = 0$, $u_3 = -(u_1+u_2)$. Any two formulae in the same set can be utilized, in three distinct ways, as a pair of simultaneous equations giving addition theorems for two copolar Jacobian functions. For example, $\mathrm{sn}(u+v)$ and $\mathrm{cn}(u+v)$ can be found algebraically from

$$\mathrm{sn}\, u\, \mathrm{dn}\, v\, \mathrm{sn}(u+v)+\mathrm{cn}\, u\, \mathrm{cn}(u+v) = \mathrm{cn}\, v,$$
$$\mathrm{dn}\, u\, \mathrm{sn}\, v\, \mathrm{sn}(u+v)+\mathrm{cn}\, v\, \mathrm{cn}(u+v) = \mathrm{cn}\, u.$$

Since an addition theorem has been used to establish Ex. 5, this process is not an independent proof of addition theorems from first principles.

The Jacobian equivalents of the individual formulae in this example can all be established by an examination of poles; they provide excellent material for practice in this kind of analysis, and an attractive short cut to the addition theorems themselves. Some of the results can be anticipated in form and constructed in detail; see Ex. 16, 17.

53. In (ii), the function operated on is the greatest integer in $\sqrt{t}/2\pi$; on the right, the numerical coefficients in the indices are the squares of the even numbers, zero excluded, and the series within the brackets is $\frac{1}{2}\{\Theta(K)-1\}$ for $q = e^{-4\pi^2 p}$, that is, for $K'/K = 4\pi p$.

54. The notation is improvised. For results in this field, see Doetsch, *Theorie und Anwendung der Laplace-Transformation* (Springer, 1937).

55. The substitution $\tan\bar{\phi}_m = \lambda_m \tan\bar{\chi}_m$ converts the recurrence

$$\tan(\bar{\phi}_{m+1} - \bar{\phi}_m) = h_m \tan\bar{\phi}_m$$

into the form $\tan\bar{\chi}_{m+1} = \mu_m \tan 2\bar{\chi}_m$ if $h_m \lambda_m^2 = 1$. This formal simplification of the Landen recurrence is due to Gauss. The auxiliary variable $\bar{\chi}$ seems to have no other part to play. The recurrence for the hyperbolic amplitude θ is modified in the same way: if $\tanh\psi_n = \sqrt{(a_n b_n)}\tanh 2\psi_{n-1}$ with $\tanh\psi_0 = \sqrt{k}\tanh\theta$, then as $n \to \infty$,

$$2^{-n}\psi_n \to M(1,k)G(\theta;k'),$$

where $G(\theta;k')$ is the integral in 13·604.

A wealth of arithmetico-geometric formulae is given in L. V. King's monograph *On the Direct Numerical Calculation of Elliptic Functions and Integrals* (Cambridge, 1924). His explicit recurrences all follow the positive half of the chain based on $(1,k')$, but since he deals with functions whose modulus is k' as well as with functions whose modulus is k, he does in effect use the positive half of the $(1,k)$ chain also. His serious handicap is the restriction to the classical functions.

56. No formulae are needed: see Ex. 20 (ii) and compare the proof of 17·803 in the text.

57. Compare Th. 17·81. The accessible re-entrant angle of the infinite 'rectangle' is now on one of the axes: a trench is dug at the foot of a wall.

Unlike the transformations in Ex. 30–33, this transformation and that of Th. 17·81 have a variable element in addition to scale factors. To apply 17·81 to a rectangle of given proportions we have to determine c from the ratio of $\mathrm{Ds}\,K_c$ to $\mathrm{Ds}\,K_n$, that is, in effect, of $c'K - E$ to $cK' - E'$.

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