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ABSTRACT

In this paper we argue that in the regression model to test for the possible nonlinearity of the error process, presence of ARCH and bilinearity should be tested simultaneously, and we suggest a joint test statistic. One example shows that the individual tests for ARCH and bilinearity may not be significant while a joint test rejects the linearity hypothesis. Obviously, our results are also applicable to pure time series models.

1. Introduction

The Autoregressive Conditional Heteroskedasticity (ARCH) model, introduced by Engle (1982), has become popular in econometric modeling [see Engle and Bollerslev (1986) for a survey of ARCH models and their applications]. Although the errors of the ARCH regression model are second-order white noise, the squares of the errors are serially correlated. This higher-order dependence causes large deviations in the error terms to be clustered. The errors of the ARCH model are also leptokurtic. The ARCH model is appealing for empirical work because non-normality and the clustering of large residuals is frequently observed in models of inflation, exchange rates, asset prices and other economic series which display time-varying volatility.

Granger and Andersen (1978) proposed a class of bilinear time series models with unconditional moment structure similar to ARCH. Certain bilinear models, like ARCH, are second-order white noise, but correlated in the squares. These processes are also leptokurtic. Bilinear models, however, have not been widely applied in economics. Although ARCH and bilinear processes have similar unconditional moments, the conditional moment structures are distinctly different. In the ARCH model, the conditional variance is a nonlinear function of past errors, while the conditional mean is constant. Whereas in the bilinear model, the conditional mean is a nonlinear function of past errors and the conditional variance is constant. Specifying an ARCH process for the errors when in fact the errors are generated by a bilinear process may have a significant effect on the efficiency of the estimators of the regression parameters and may lead invalid inference. Hence, when the pattern of the OLS residuals indicate that ARCH may be an appropriate specification for the error process, the alternative of a bilinear specification should also be considered and tested.

A difficulty in considering both ARCH and bilinear alternatives is that the standard Lagrange multiplier (LM) tests for the individual alternatives cannot distinguish between the two hypotheses. Luukkonen et al. (1987) study the power of the LM test for ARCH when the data are generated by a bilinear process. Although they find that the asymptotic power of the ARCH test against local bilinear alternatives does not exceed the size of the test, however, Luukkonen et al. (1987, p. 17) mention that the ARCH test will be consistent against nonlocal bilinear models for which the squares of the residuals are correlated. Hence, a ARCH test statistic may in fact be significant due to the presence of bilinearity rather than ARCH, and an insignificant ARCH statistic does not preclude the possible presence of bilinearity.

Weiss (1986) reaches a similar conclusion and asserts that ARCH and bilinearity should not be considered separately. He recommends that first bilinearity be tested allowing for the presence of ARCH. If found significant, the bilinear model is estimated and the residuals are then tested for ARCH. The test for bilinearity in the presence of ARCH, however, does not have the usual $T \cdot R^2$ form from an auxiliary regression. And because the ARCH test requires first estimating a bilinear model, this poses a computational burden for someone wanting a simple test for nonlinearity which is sensitive to both ARCH and bilinear alternatives.

In this paper we propose an easily computed simultaneous test for a ARCH and bilinearity. We expect this test to have good power properties for other alternatives as well, whenever there is time variation of the conditional mean and/or the conditional variance. We make use of recent results of Bera and McKenzie (1987) to show that a joint LM test can be constructed as the sum of the individual LM tests for ARCH and bilinearity, that is, the tests are additive. Section 2 of the paper reviews the additivity properties of the LM statistic. In Section 3 we present the joint test statistic, and Section 4 is a short conclusion. A derivation is given in an appendix.

2. Additivity of LM Statistics

The LM statistic for testing two hypotheses jointly can frequently be decomposed into the sum of the LM statistics for testing each hypothesis individually. This property was first noted by Pesaran (1979). He found that the LM test of the deterministic and stochastic parts of a dynamic linear regression model can be written as the sum of two independent parts. In a more complex situation, Bera and Jarque (1982) showed that the joint LM test for normality, homoskedasticity, serial independence, and functional form is the sum of standard LM test of each component of the joint hypothesis.

Bera and McKenzie (1987) consider this additivity property in general and find a necessary and sufficient condition for LM statistics to be additive. Let θ be a vector of parameters and assume that the null hypothesis imposes restrictions which can be written as a vector valued function of θ, H_0: $h(\theta) = 0$, where $H = \partial h(\theta)/\partial\theta$ has full column rank. Furthermore, assume H_0 naturally partitions into two separate sets of restrictions H_A: $h_1(\theta) = 0$ and H_B: $h_2(\theta) = 0$, with a corresponding partition for $H = [H_1 : H_2]$. For a test principle T, denote by T_{AB} the statistic for simultaneously testing both sets of restrcitions H_A and H_B. Let T_A denote the statistic for testing the H_A restrictions with the H_B restrictions imposed and T_B the statistic for testing the H_B restrictions with the H_A restrictions imposed. We formally define additivity with respect to the test principle T as

> Definition: The test for hypotheses H_A and H_B are additive if $T_{AB} = T_A + T_B$.

We then have

> Proposition: A necessary and sufficient condition for the additivity of the LM tests of the hypotheses H_A and H_B is that $\tilde{H}_1'\tilde{\underline{I}}^{-1}\tilde{H}_2 = 0$.

where $\underline{I}$ is the information matrix and "~" denotes quantities evaluated at the maximum likelihood estimate (MLE), $\tilde{\theta}$, subject to both sets of restrictions H_0: $h(\theta) = 0$ [see Bera and McKenzie (1987)]. In the next section we make use of this result to construct a simultaneous test for ARCH and bilinearity.

3. Simultaneous Test for ARCH and Bilinearity

Consider the linear model

$$y_t = x_t'\beta + u_t \qquad t = 1, \ldots, T \tag{3.1}$$

where β is a (rxl) vector of coefficients and x_t is a vector of explanatory variables which may include lagged values of the dependent variable y_t. The error u_t is assumed to be generated by the bilinear process

$$u_t = \sum_{p=1}^{P} \sum_{q=1}^{Q} b_{pq} u_{t-p} \varepsilon_{t-q} + \varepsilon_t \tag{3.2}$$

where the innovation series ε_t is generated by the ARCH process

$$\varepsilon_t | \Phi_{t-1} \sim N(0, h_t) \tag{3.3}$$

$$h_t = \sigma^2 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_k \varepsilon_{t-k}^2 \tag{3.4}$$

where Φ_t is the information set at time t, whose elements include the current and lagged values of ε_t.

Let b denote the vector of bilinear parameters b_{pq} where p = 1, 2, ..., P and q = 1, 2, ..., Q. Similarly, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)'$, denote the vector of ARCH parameters. The joint null hypothesis is

H_A: $b = 0$

H_B: $\alpha = 0$.

We then show

Proposition: For the model (3.1)-(3.4), the hypotheses b = 0 and $\alpha = 0$ are additive with respect to the LM test.

Proof: From the joint null hypothesis

$$H = \begin{bmatrix} 0_{rxPQ} & 0_{rxk} \\ I_{PQ} & 0_{PQxk} \\ 0_{1xPQ} & 0_{1xk} \\ 0_{kxPQ} & I_k \end{bmatrix} = [H_1 : H_2]$$

where 0_{ixj} denotes a null matrix of dimension i x j and I_i denotes an identity matrix of dimension i. The log-likelihood function for a single observation is

$$L_t = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(h_t) - \frac{1}{2}\varepsilon_t^2/h_t \tag{3.5}$$

where

$$\varepsilon_t = u_t - \sum_{p=1}^{P}\sum_{q=1}^{Q} b_{pq}u_{t-p}\varepsilon_{t-q}. \tag{3.6}$$

In the appendix we show that the information matrix evaluated under the joint null hypothesis has the form

$$\tilde{I} = \begin{bmatrix} \tilde{\underline{I}}_{\beta\beta} & \tilde{\underline{I}}_{\beta b} & 0 & 0 \\ \tilde{\underline{I}}_{b\beta} & \tilde{\underline{I}}_{bb} & 0 & 0 \\ 0 & 0 & \tilde{\underline{I}}_{\sigma^2\sigma^2} & \tilde{\underline{I}}_{\sigma^2\alpha} \\ 0 & 0 & \tilde{\underline{I}}_{\alpha\sigma^2} & \tilde{\underline{I}}_{\alpha\alpha} \end{bmatrix}$$

where subscripts indicate the appropriate partition with respect to the parameters. Therefore, the sufficient condition that

$$\tilde{H}_1'\tilde{\underline{I}}^{-1}\tilde{H}_2 = [0_{PQxr}\ I_{PQ}\ 0_{PQx1}\ 0_{PQxk}]$$

$$\times \begin{bmatrix} \tilde{\underline{I}}_{\beta\beta} & \tilde{\underline{I}}_{\beta b} & 0 & 0 \\ \tilde{\underline{I}}_{b\beta} & \tilde{\underline{I}}_{bb} & 0 & 0 \\ 0 & 0 & \tilde{\underline{I}}_{\sigma^2\sigma^2} & \tilde{\underline{I}}_{\sigma^2\alpha} \\ 0 & 0 & \tilde{\underline{I}}_{\alpha\sigma^2} & \tilde{\underline{I}}_{\alpha\alpha} \end{bmatrix}^{-1} \begin{bmatrix} 0_{rxk} \\ 0_{PQxk} \\ 0_{1xk} \\ I_k \end{bmatrix}$$

$$= \begin{bmatrix} 0_{PQxr} & I_{PQ} \end{bmatrix} \begin{bmatrix} \tilde{\underline{I}}_{\beta\beta} & \tilde{\underline{I}}_{\beta b} \\ \tilde{\underline{I}}_{b\beta} & \tilde{\underline{I}}_{bb} \end{bmatrix}^{-1} \begin{bmatrix} 0_{rxk} \\ 0_{PQxk} \end{bmatrix}$$

$$+ \begin{bmatrix} 0_{PQx1} & 0_{PQxk} \end{bmatrix} \begin{bmatrix} \tilde{\underline{I}}_{\sigma^2\sigma^2} & \tilde{\underline{I}}_{\sigma^2\alpha} \\ \tilde{\underline{I}}_{\alpha\sigma^2} & \tilde{\underline{I}}_{\alpha\alpha} \end{bmatrix}^{-1} \begin{bmatrix} 0_{1xk} \\ I_k \end{bmatrix}$$

$$= 0$$

is met, and hence, the statistics are additive.

We have shown that the LM statistic for simultaneously testing ARCH and bilinearity (LM_{AB}) can be constructed as the sum of the standard LM tests for ARCH and bilinearity. The LM test for ARCH (LM_A) can be computed as T times the squared multiple correlation coefficient from regressing $\hat{\varepsilon}_t^2$ on an intercept and k lagged values of $\hat{\varepsilon}_t^2$, where $\hat{\varepsilon}_t = y_t - x_t'\hat{\beta}$ and $\hat{\beta}$ is the least squares estimate of β [see Engle (1982)]. Similarly, letting $m = (\beta_1,\ldots,\beta_r,b_{11},\ldots,b_{PQ})'$, the LM test for bilinearity (LM_B) can be constructed as $T \cdot R^2$ from the regression of ε_t on $\partial\varepsilon_t/\partial m$, where $\partial\varepsilon_t/\partial m$ is computed from (3.6) assuming $\varepsilon_t \sim$ IID $N(0,\sigma^2)$ and is evaluated at $\hat{\beta}$ and $b = 0$ [see Pagan (1978)]. Letting $\hat{z}_t = (1,\hat{\varepsilon}^2_{t-1}, \ldots, \hat{\varepsilon}^2_{t-k})'$, the simultaneous test statistic can be written as

$$LM_{AB} = LM_A + LM_B = T \cdot \frac{(\Sigma\hat{\varepsilon}^2_t\hat{z}_t)'(\Sigma\hat{z}_t\hat{z}'_t)^{-1}(\Sigma\hat{\varepsilon}^2_t\hat{z}_t)}{\Sigma\hat{\varepsilon}^4_t}$$

$$+ T \cdot \frac{[\Sigma\hat{\varepsilon}_t(\partial\hat{\varepsilon}_t/\partial m)]'[\Sigma(\partial\hat{\varepsilon}_t/\partial m)(\partial\hat{\varepsilon}_t/\partial m)']^{-1}[\Sigma\hat{\varepsilon}_t(\partial\hat{\varepsilon}/\partial m)]}{\Sigma\hat{\varepsilon}^2_t}$$

and under the null hypothesis will be distributed as a chi-squared random variable with $k + PQ$ degrees of freedom (χ^2_{k+PQ}).

The joint test should be useful for detecting nonlinearity when only a mild form of ARCH and bilinearity is present in the error process. It is easy to imagine a situation in which both of the individual tests for ARCH and bilinearity just fail to be significant, while the joint test leads to the rejection of linearity. To illustrate this possibility, we generated 100 observations from the model

$$y_t = 25 + 5x_t + u_t \qquad t = 1,\ldots,100$$

$$u_t = .09u_{t-2}\varepsilon_{t-1} + \varepsilon_t$$

$$\varepsilon_t|\Phi_{t-1} \sim N(0,h_t)$$

$$h_t = 7 + .5\varepsilon^2_{t-1}$$

where x_t was generated as N(20,9). We computed LM_A as $100 \cdot R^2$ from the regression of $\hat{\varepsilon}_t^2$ on an intercept and $\hat{\varepsilon}_{t-1}^2$. Under the null hypothesis of no ARCH or bilinearity, this statistic is distributed as χ_1^2. LM_B was computed as $100 \cdot R^2$ from the regression of $\hat{\varepsilon}_t$ on an intercept, x_t, and $\hat{\varepsilon}_{t-1} \cdot \hat{\varepsilon}_{t-2}$, and under the null hypothesis LM_B is also distributed as χ_1^2. LM_{AB} is given by the sum of LM_A and LM_B, and is distributed as χ_2^2. The computed values of the statistics and the 10 percent critical values are given in Table 1. For the particular sample that was generated, the individual tests were insignificant at the 10 percent level, while the joint test was significant.

Table 1

Individual and Joint for ARCH and
Bilinearity using Simulated Data

Test	Computed value	Critical value
LM_A	2.09	2.706
LM_B	2.64	2.706
LM_{AB}	4.73	4.605

4. Conclusion

Given the inadequacies of the standard LM tests for ARCH and bilinearity discussed above, we argue that ARCH and bilinearity should be considered jointly as an alternative hypothesis. In this paper we propose a statistic for simultaneously testing for ARCH and bilinearity which can be easily computed using standard regression programs.

It should serve as a useful pretest for the model building strategy outlined by Weiss (1986), providing a safeguard against too eager of a rejection of nonlinearity in the error process. We provide an example where both of the individual tests for ARCH and bilinearity fail to be significant, while the joint test leads to the acceptance of non-linearity. Recently, a number of tests for nonlinearity have been proposed in the time series literature [see for example Keenan (1985); Petruccelli and Davies (1986); Luukkonen et al. (1987) and O'Brien (1987)]. An extensive Monte Carlo study is underway to investigate the finite sample distribution and power of the available tests for nonlinearity under a variety of alternatives.

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Appendix

To prove the additivity result, it is sufficient to show that the information matrix is block diagonal between the conditional mean parameters $m = (\beta_1, \ldots, \beta_r, b_{11}, \ldots, b_{PQ})'$ and the conditional variance parameters $v = (\sigma^2, \alpha_1, \ldots, \alpha_k)'$ when evaluated under the null hypothesis. Since explicit expressions for the non-zero elements of the information matrix are not required, they will not be considered here.

From (3.5) and (3.6), the first partial derivative of the log-likelihood function is

$$\frac{\partial L_t}{\partial m_i} = -\frac{1}{2h_t}\frac{\partial h_t}{\partial m_i} - \frac{\varepsilon_t}{h_t}\frac{\partial \varepsilon_t}{\partial m_i} + \frac{\varepsilon_t^2}{2h_t}\frac{\partial h_t}{\partial m_i}$$

and the second partial is

$$\frac{\partial^2 L_t}{\partial v_j \partial m_i} = \frac{1}{2h_t^2}\frac{\partial h_t}{\partial v_j}\frac{\partial h_t}{\partial m_i} - \frac{1}{2h_t}\frac{\partial^2 h_t}{\partial v_j \partial m_i} + \frac{\varepsilon_t}{h_t^2}\frac{\partial h_t}{\partial v_j}\frac{\partial \varepsilon_t}{\partial m_i}$$

$$- \frac{\varepsilon_t^2}{h_t^3}\frac{\partial h_t}{\partial v_j}\frac{\partial h_t}{\partial m_i} + \frac{\varepsilon_t^2}{2h_t^2}\frac{\partial^2 h_t}{\partial v_j \partial m_i} .$$

The elements of the information matrix are given by the negative of the expectations of these second partial derivatives summed over t. The expectations are simplified by taking iterative expectations on the information set Φ_{t-1} and recalling from (3.3) and (3.4) that

$$E(\varepsilon_t|\Phi_{t-1}) = 0, \quad E(\varepsilon_t^2|\Phi_{t-1}) = h_t$$

and
$$-E\left(\frac{\partial^2 h_t}{\partial v_j \partial m_i}\right) = -E\left[E\left(\frac{\partial^2 h_t}{\partial v_j \partial m_i} \mid \Phi_{t-1}\right)\right]$$

$$= E\left(\frac{1}{h_t^2} \frac{\partial h_t}{\partial v_j} \frac{\partial h_t}{\partial m_i}\right)$$

Noting that

$$\frac{\partial h_t}{\partial m_i} = 2\alpha_1 \varepsilon_{t-1} \frac{\partial \varepsilon_{t-1}}{\partial m_i} + \ldots + 2\alpha_k \varepsilon_{t-k} \frac{\partial \varepsilon_{t-k}}{\partial m_i}$$

is linear in the α's, the above expectation is readily seen to be zero when evaluated under the null hypothesis, i.e., at $\alpha = 0$ and $b = 0$. Hence, the information matrix will be block diagonal under the null hypothesis between the conditional mean parameters and the conditional variance parameters as required.