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VOL. V. JULY, 1896. NO. 1.

Projective Groups of Perspective Collineations in the Plane, Treated Synthetically.

[A Dissertation presented to the Faculty of the University of Kansas to attain the degree of Doctor of Philosophy.]

BY ARNOLD EMCH.

Preliminary.

In presenting this paper the author has attempted to consider the subject of Perspective Collineations from the modern point of view of the theory of groups. Hitherto collineation has always been treated either by means of descriptive geometry, or analytic geometry, and without reference to group properties.

Sophus Lie's "Vorlesungen über Continuierliche Gruppen," which will be chiefly referred to, treats of the projective transformations of the plane of which collineation is only a special case. There are five types of projective transformations in the plane under which all the groups can be subordinated, and it has been the aim of Professor Newson to enumerate the projective groups of the plane by means of the synthetic method, and to classify them according to those five types.

This paper treats in the same manner the two types of transformations that are known as perspective collineations.

For suggestions in the treatment of the subject the author is thankfully indebted to his collaborator, Prof. H. B. Newson.

§1. Representation of Perspective Collineation.*

If the corresponding lines α, α^1: β, β^1: γ, γ^1; δ, δ^1; intersect each other on the line of intersection of the two planes π and π^1, and fulfill the further condition that all the connection-lines of

*Perspective Collineation has the same meaning as the German *Centrische Collineation* (Fiedler), or the French *homologie*.

corresponding points pass through one and the same point, the projectivity, thus produced, is perspective collineation.

In the general case to the line l, the line of intersection of the planes π and π^1, belonging to the plane π, corresponds a line p^1 in the plane π^1, and to the line l, belonging to the plane π^1, corresponds a line p in the plane π. Connecting the corresponding points of l and p^1, and of l and p two conics K^1 and K, in π^1 and π respectively, are produced which determine the projective transformation. In the case of a perspective collineation, however, these two conics are indeterminate; since the lines p and p^1 coincide with l. We can, therefore, choose any four points A, B, C, D, on l, in π, and connect them with their corresponding points A^1, B^1, C^1, D^1, in π^1, which coincide with the former, i. e., we can draw any four lines a^1, b^1, c^1, d^1, in π, through A, B, C, D, respectively, which with l determine the conic K^1. The conic K is determined by those lines a, b, c, d, in π, which correspond to the lines a^1, b^1, c^1, d^1, according to the original conditions.

Thus, the two conics K and K^1 are collinear and fully determine the collineation. Since K^1 touches the line l, K touches l at the same point. As it will be seen from this, the two conics K and K^1 characterizing the general projective transformation exist also in perspective collineation; but there is a multiplicity of two conics tangent to each other and tangent to the line l at the same point. As there are ∞^4 conics tangent to l one and the same perspective collineation can always be represented by ∞^4 combinations of such two conics. The line l is the axis of collineation, and the centre C of collineation is obtained by the intersection-point of the two other common tangents of the conics K and K^1.

We are now ready to make the following statement:

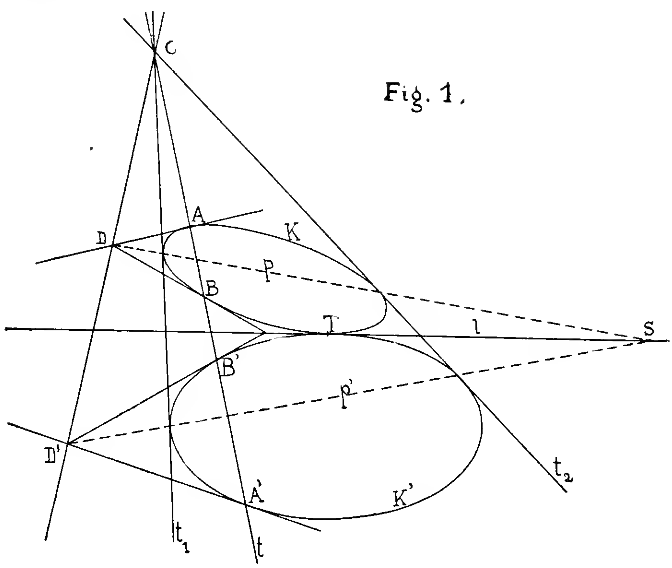
Theorem 1. Each two conics tangent to each other determine a perspective collineation with the common tangent at their point of tangency as the axis and the intersection-point of their two other common tangents as the centre of collineation.

The general theorem concerning the construction of collineation by means of two conics tangent to the same line, as well as this special theorem, are obtained in a natural way by studying the congruence of right lines (3. 1) formed by all the right lines connecting corresponding points of two collinear planes in space. The focal surface of the congruence is a developable surface of the 3. class, and its edge of regression a curve in space of the 3. order. Any two osculating planes of this curve intersect the surface in two conics which are tangent to their line of intersection. Designating the points of intersection of any ray of the congruence with the two osculating planes as corresponding points, the two planes are collinear and we have immediately our general theorem if we revolve one of the osculating planes into the other about their line of intersection. In the case of a perspective collineation the congruence is of the order 1 and the class 0. The developable surface is not determinate, so that we may choose a cone of the 2. class which with any two planes determines a perspective collineation.

As in the general case of projectivity, to each point P the corresponding P^1 is obtained by drawing two tangents from P to the conic K which will intersect the line l in two points: from these two points draw the tangents to the conic K^1. Their point of intersection gives the required point P^1.

That this construction gives perspective collineation we can also prove without referring to the general case of projectivity.

Assume the two conics K and K^1 in the required position (Fig. 1) and draw the common tangents t_1 and t_2 which intersect each other in C.



K and K^1 now belong to a system of conics tangent to t_1 and t_2 and to l at a fixed point T. The polars p, p^1 . . . . of C in regard to the conics K, K^1, ... of the system, therefore, intersect each other in one and the same point S on l, and if we draw any other line t through C which intersects K in A and B, and K^1 in A^1 and

B^1, the tangents in A and B to the conic K intersect in a point D of the polar p and those in A^1 and B^1 to the conic K^1 in a point D^1 of the polar p^1. As is well known from synthetic geometry the points D and D^1 lie on a ray through C. Moreover, the tangents in A and A^1, and in B and B^1 meet in the line l, hence, the points D and D^1 are obtained by our construction of the collineation. To every point corresponds one and only one point and both lie in a ray passing through the centre C. Two corresponding straight lines always meet in a point of the line l. These two conditions, however, constitute perspective collineation and hold for any point, or line of the plane and their corresponding elements.

In the next chapter we shall make those constructions which will be necessary in the study of group-properties of perspective collineations.

§2. Classification of Perspective Collineations.

The two conics, K and K^1, determining the perspective collineation, being given we can ask for the line q^1 which corresponds to the infinitely distant straight line q. Drawing the two parallel tangents to the conic K from each point at infinity, and from their intersection-points with the axis l of perspective collineation tangents to the conic K^1, the points of q^1 are obtained by the intersection of each such pair of tangents to the conic K^1. Conversely, there exists a straight line r whose corresponding line r^1, or what is the same, q^1, is at infinity. The lines q^1 and r may be called counter-axes (German "Gegenaxen") of perspective collineation, and are parallel to the axis of collineation.

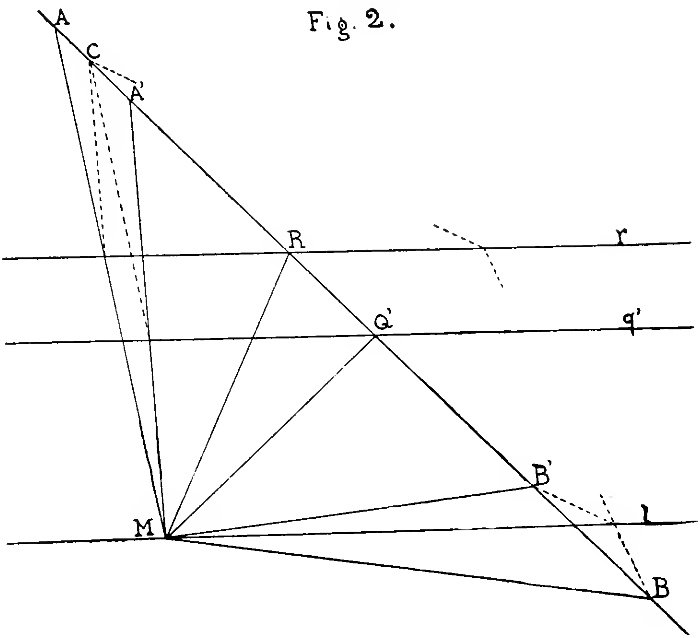
In central projection and perspective the constructions are usually made by aid of the centre and axis and the counter-axes of perspective collineation. A perspective collineation is determined by centre and axis and any one of the counter-axes, and, as immediately follows by construction, also by centre and axis and two corresponding points of the collineation.

It is now of great importance to state the connection between these two determinations.

Each perspective collineation transforms a ray through the centre into itself and also each point of the axis into itself. In other words, it leaves the points of the axis and the rays through the centre invariant. Each ray through the centre represents two coincident projective point-ranges, and each pencil of rays through a point of the axis two coincident projective pencils of rays.

Taking a ray s through the centre C, its corresponding ray s^1 is coincident with s and intersects the counter-axes q^1 and r in the two counter-points Q^1 and R of the ray (German "Gegenpunkte").* Since C and L, L being the intersection-point of s with l, correspond to themselves the following relation between these points and two pairs of corresponding points A, A^1, and B, B^1, on the ray s exists: (See Fig. 2.)

Fig. 2.



$$(CLAB)=(CLA^1B^1), \text{ or}$$

$$\frac{CA}{LA}:\frac{CB}{LB}=\frac{CA^1}{LA^1}:\frac{CB^1}{LB^1}, \text{ or}$$

$$\frac{CA}{LA}:\frac{CA^1}{LA^1}=\frac{CB}{LB}:\frac{CB^1}{LB^1}, \text{ i. e.,}$$

$$(CLAA^1)=(CLBB^1)$$

Substituting for the pair B, B^1 the pair Q, Q^1, or R, R^1, this last projectivity becomes:

$$(CLAA^1)=(CLQQ^1)=(CLRR^1), \text{ or}$$

$$(CLAA^1)=(CL\,\infty\,Q^1)=(CLR\,\infty), \text{ or}$$

$$\frac{CA}{LA}:\frac{CA^1}{LA^1}=\frac{CQ^1}{LQ^1}:\frac{CR}{LR}=\text{const.}$$

*We avail ourselves of the designation of Fiedler in his "Darstellende Geometrie," I. Band, and for the following classification especially refer to §22, page 95, of this book.

Thus, any pair of corresponding points in the perspective collineation has a constant relation to the counter-points, and we have the well known

Theorem 2. Each pair of corresponding points on a ray through the centre forms a constant anharmonic ratio with the intersection-point of the ray with the axis of perspective collineation.

Any point M on the axis l may be connected with the points C, L, Q, R, ∞, A, A^1, B, B^1, and designating these rays by small letters, there is obviously

$$(claa^1)=(clbb^1)=\text{const.}$$

This fact can be stated as the dualistic of the above theorem, viz:

Theorem 3. Each pair of corresponding rays through a point on the axis forms a constant anharmonic ratio with the ray through the centre and the axis of perspective collineation.

We call this constant the characteristic anharmonic ratio of the perspective collineation and designate it by k.* By aid of it a classification of the perspective collineation can easily be made, and so far as it will be of avail for our further consideration we will discuss the different cases of perspective collineation from this point of view. Among all the ∞^1 values of k the special case $k=-1$ deserves the greatest attention, and it shall be considered first, because it enables us at once to draw important conclusions from its combination with particular positions of the center, the axis, and the counter-axes of the perspective collineation.

From the assumption $k=-1$ follows:

$$\frac{CQ^1}{LQ^1}=\frac{CR}{LR}=-1, \text{ or}$$

$$CQ^1=-LQ^1, \text{ and } CR=-LR; \text{ i. e.,}$$

the counter-points and therefore also the counter-axes are midway between C and L, and, therefore, coincide. For every pair of corresponding points the relation exists:

$$(CLAA^1)=-1=\frac{CA}{LA}:\frac{CA^1}{LA^1}=\frac{CA^1}{LA^1}=\frac{CA}{LA}$$

From this follows

$$(CLAA^1)=-1=(CLA^1A) \text{ and}$$

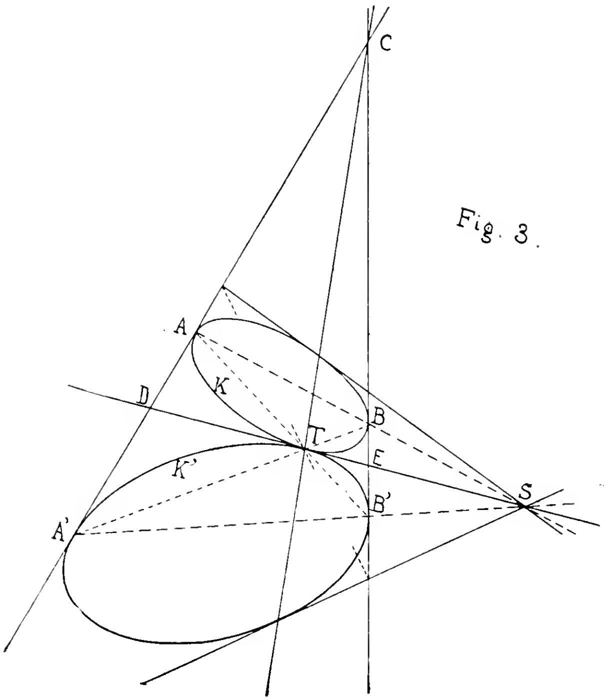
in a similar way:

$$(claa^1)=-1=(cla^1a), \text{ i. e.,}$$

in this collineation the points and rays of each pair are interchangeable. The collineation whose characteristic anharmonic ratio is $k=-1$ is therefore involutoric.

*This constant was first introduced into Geometry by Fiedler.

The same result is also obtained by starting from the two conics K and K^1 which determine the collineation. Fig. 3 represents the involutoric position of the conics $(CADA^1) = (CBEB^1) = -1$.



The polars of C in regard to the conics K and K^1 intersect each other in S on l. The polars of S in regard to K and K^1 pass therefore through T, the common point of tangency of the conics with l, and through C, i. e., they are identical. Hence the point T is the intersection-point of $\overline{AB}^1$ and $A^1\overline{B}$; $(SETD) = -1$. Thus, designating the points of tangency of the common tangents to the conics by A, A^1 and B, B^1, the conics are in involutoric position, if their common point of tangency with l, T, coincides with the intersection-point of $\overline{AB}^1$ and $A^1\overline{B}$.

Besides involution we have to consider those collineations which result from special positions of the centre and axis of perspective collineation, or what is the same, from special positions of the conics K and K^1.

In the following development it would not be necessary to take conics K and K^1 into consideration. We shall do it here in order to show how the representation of special collineations is made by means of those conics. For the arrangement we refer to the book of Fiedler already mentioned.

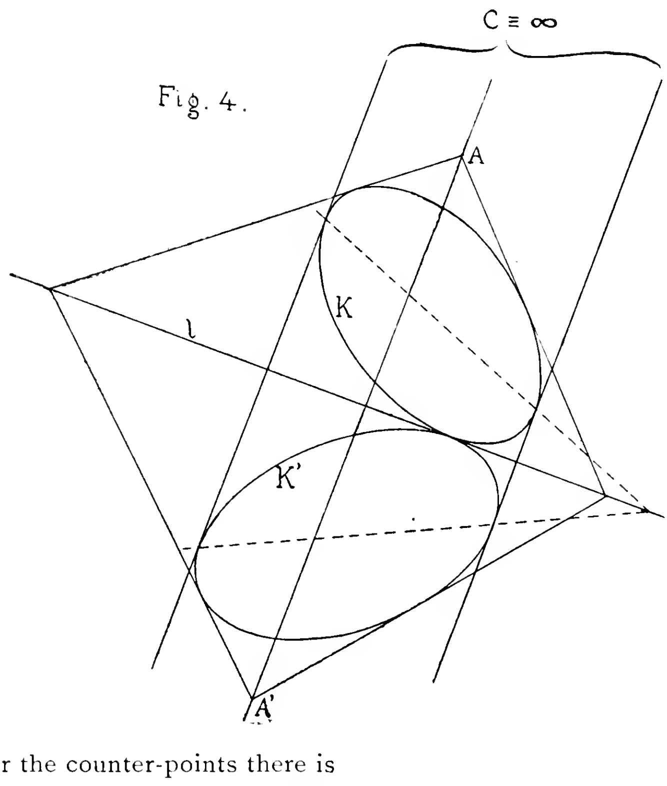
(*a*). The conics K and K^1 have two parallel common tangents, such that the centre C is at infinity (Fig. 4). There is

$$k=(\infty LAA^1)=(claa^1), \text{ or}$$

$$k=\frac{LA^1}{LA};$$

the corresponding point ranges are similar with the point of similitude in the axis.

Fig. 4.



For the counter-points there is

$$k=(\infty L \infty Q^1)=(\infty LR \infty),$$

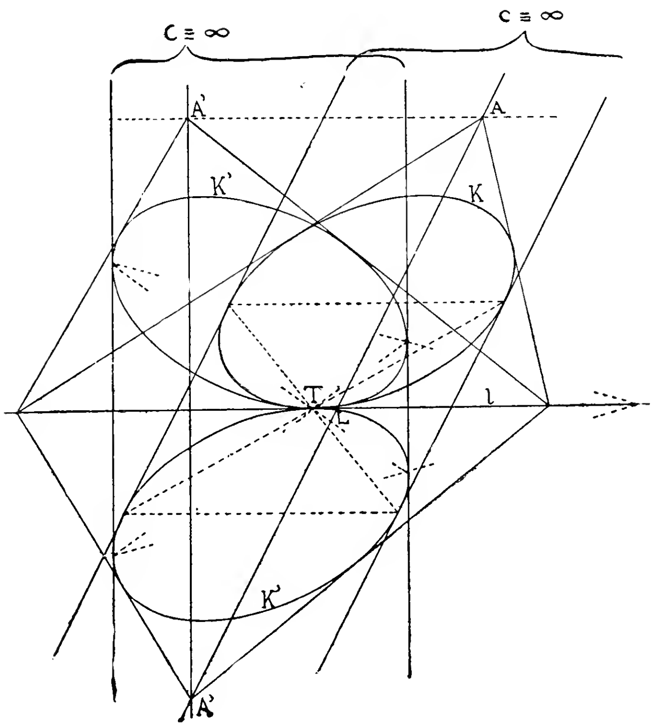
i. e., Q^1 and R are at infinity. The counter-axes q^1 and r form two coincident point-ranges of which the centre and the point at infinity of the axis are the double-points. Parallel straight lines have parallel corresponding lines. Such a collineation is called affinity, or dilation.*

*The word affinity is used in Mobius' "Barycentrischer Calcul."

(b). If K and K^1 have two parallel common tangents, and $k = -1$, i. e., if the collineation is dilation and involution (Fig. 5), there is:

$$(\infty LAA^1) = -1, \text{ or}$$
$$LA^1 = -LA; \ (claa^1) = -1.$$

Fig. 5.



Each pair of corresponding points lies in a fixed direction and is equidistant from the axis. The ranges in corresponding straight lines are symmetric with the centre of symmetry in the axis. Corresponding triangles have the same area. This collineation is called oblique, or orthogonal symmetry in regard to the axis, according as the direction of the centre is oblique, or orthogonal to the axis.

(c). K and K^1 have two parallel common tangents and $k = +1$. In this case one of the conics of involution is revolved about the axis through $180°$, (Fig. 5). The centre is at infinity and its direction is parallel to the axis. In such a collineation corresponding figures have equal areas and it is therefore called the affinity of figures of equal areas.

It is also obtained by revolving one half-plane of oblique symmetry into the other. Corresponding points lie always on the same side of the axis and are, as in the previous case, equidistant from the axis.

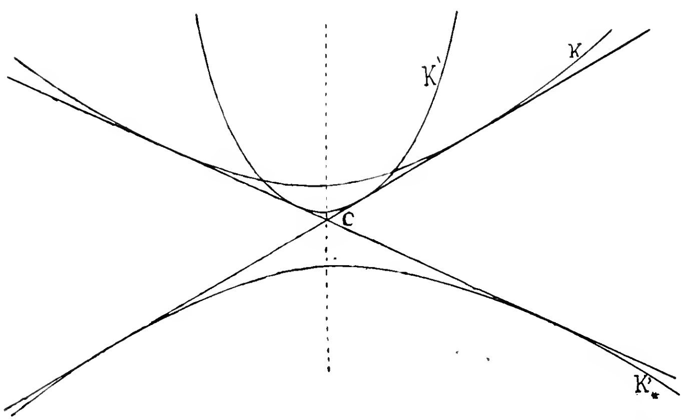
(*d*). As the previous cases, (*a*), (*b*), (*c*), were characterized by the assumption of the centre C being at affinity, there remains to consider the collineation with an infinitely distant axis. Obviously the conics K and K^1 become coaxial parabolas which intersect each other either in two finite real, or two imaginary points (Fig. 6). There is

$$k=(C \infty AA^1)=CA : CA^1=(c \infty aa^1);$$

the distances of corresponding points from the centre have a constant ratio and form similar ranges.

Corresponding straight lines are parallel, and corresponding ranges similar, so that their constant ratio is k.

Fig. 6



The collineation thus characterized is termed similarity of systems in similar position. According as k is positive or negative, real or imaginary intersection-point of the parobola, the similarity is said to be direct or inverse similarity.

(*e*). If to the former case the further condition k=—1 is added, the relation becomes

$$k=(C \infty AA^1)=(c \infty aa^1)=-1, \text{ or}$$

$$CA=CA^1.$$

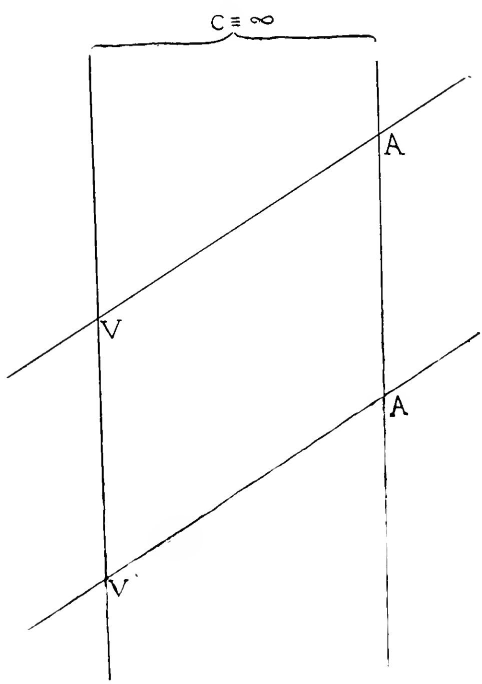
The conics K and K^1 have two imaginary intersection-points and are equal (K and K^1 in Fig. 6.). Corresponding points are in opposite directions and at equal distances from the centre.

Systems related in such a manner are said to be in central symmetry.

The value k=+1 together with an infinitely distant axis gives no collineation in the proper sense of the word. In this case the two conics K and K¹ coincide and determine what is called an identical collinear transformation. We shall meet this conception in one yet of the following chapters.

Assuming the conics K and K¹ as coaxial parabolas, and tangent at their vertices, two collineations arise according as the line at infinity or the finite common tangent is taken as the axis of collineation. In the first case the finite point of tangency of the parabolas is the centre of collineation. As is easily seen the relation of corresponding points becomes that of similar systems in similar position. In the second case the centre of collineation is the infinitely distant point of the common axis of the parabolas,

Fig. 7

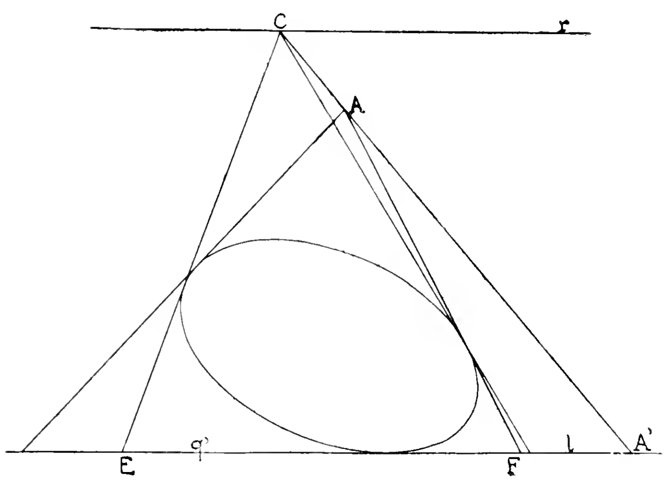


and its direction is orthogonal to the axis of collineation (the common finite tangent of the parabola). This, however, is orthogonal affinity. Adding the condition k=—1 the two cases represent central and orthogonal symmetry.

(*f*). The two conics K and K^1 may be represented by degenerate parabolas, i. e., by straight lines which coincide. Centre and axis of collineation are at infinity, and as they are coincident it follows from $k=(\infty\ \infty\ AA^1)\ k=+1$. The two systems are therefore similar and in similar position; in the relation of dilation and of corresponding equal areas. Hence they are congruent. How the construction of corresponding points is made is seen from Fig. 7. Let $V\infty$ and $V^1\infty$ represent the two coaxial parabolas. To a point A the corresponding A^1 is found by drawing AV and $A\overset{\infty}{C}$ parallel to VV^1 (the two tangents to the conic VV^1) and intersecting AC by the parallel to AV through V^1. (AV and AC intersect the axis of collineation, or the line at infinity in two points. The tangents from this point to the conic $V^1\infty$ are A^1V^1 and A^1C). We add this construction here in order to show that also in the case of singularities the construction is applicable.

We have now seen that all the common cases of perspective collineation are expressed by the characteristic constant k together with the positions of the centre and axis of collineation, or result from certain positions and relations of the two conics. What remains yet to consider are the so-called *pseudo perspective collineations.**

Fig. 8.



Here to one point may correspond a whole system of points and vice-versa. These singularities can be classified according to

*Prof. Newson introduced the term *pseudo transformation* into geometry. Singular perspective collineations can therefore also be called *pseudo perspective collineations.*

$$k=0,\ k=\infty,\ k=\frac{0}{0}$$

(*g*). For $k=0$ take for K any conic tangent to l, and for K^1 a degenerated ellipse EF in l and tangent to K^1. Obviously q^1 lies in l and C in V (Fig. 8), hence $k=0$. As the construction shows, to each point A of the plane corresponds a point A^1 of the axis.

Conversely, to each point A^1, except the points of l, corresponds the centre C, i. e., the whole plane of the other system corresponds to the centre C. To each ray g through C corresponds its point of intersection with l.

(*h*). For $k=\infty$ results an analogous case in which r coincides with l and C with q^1.

(*i*). For $k=\frac{0}{0}$, centre, axis, and counter-axis of collineation are coincident and the conics K and K^1 indeterminate. The points of each system correspond to the centre C. To each straight line in either system corresponds the axis l and to each ray through the centre in one system all the rays through the centre in the other system.

(*k*) *The last important case which is to be considered here is characterized by $k=-1$ and the centre C in the finite part of the axis l. The counter-axes q^1 and r are opposite and equidistant from l. These conditions are realized by two conics K and K^1 related by involution, but of which one conic is revolved about the axis of collineation into the half plane of the other. After having revolved one half-plane into the other, one or two common tangents of K and K^1 coincide with l. The center C becomes accordingly the intersection-point of the other common tangent with l, or the common point of tangency of K and K^1. In the first case the two conics have a contact of the second, in the other a contact of the third order. If no special assumption about the position of the centre and the conics is made the two conics will be in double contact, if, however, the connection line of the centre and the point of tangency of the conics in involution is perpendicular to the axis l the case of a triple contact arises. Figs. 9 and 10 illustrate these two cases.

What has been found here by logical deduction from the laws of collineation can also be proved by assuming one conic tangent to l, the center on l, the counter-axes q^1 and r opposite and equidistant from l and by constructing the corresponding conic. As the

*This is what Sophus Lie in his "Vorlesungen ueber continuierliche Gruppen" calls *elation*. In accordance with Lie we put this case at the end of the classification.

above cases are of great importance in the theory of groups, we state them again in a theorem:

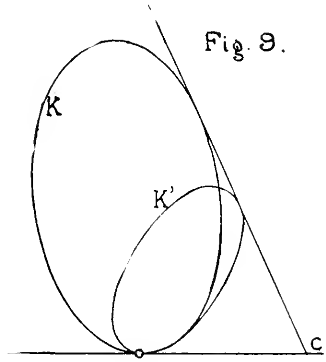


Fig. 9.

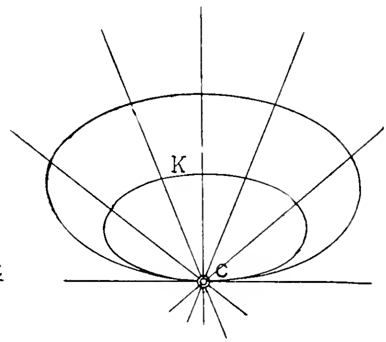


Fig. 10

*Theorem 4. Two conics with a contact of the second or third order determine a perspective collineation with the common tangent at the point of contact as its axis and a finite point in the axis as its centre.**

The centre is, of course, in both cases determined by the two conics and is found as indicated above. In fact the two cases are alike and thus it was not necessary to distinguish them in the above theorem.

§3. Number and Invariant Properties of Perspective Collineation.

From the preceding development it is known that each two conics tangent to each other determine a perspective collineation with the common tangent at their point of tangency as the axis and the intersection-point of their two other common tangents as the centre of collineation. As there are ∞^3 conics tangent to a straight line at a certain point, $\frac{1}{2}\infty^3(\infty^3-1)=\infty^6$ pairs of conics tangent to each other and to the line at that point can be formed, hence, just as many collineations. But there are discrete groups among these pairs which represent the same collineation. For, as a perspective collineation is also determined by the centre, the axis, and two corresponding points, it may be represented ∞^3 times by pairs out of those ∞^6. Hence, there are $\infty^6 : \infty^3 = \infty^3$ pairs representing different perspective collineations. Taking all the points of the axis as points of tangency for pairs of conics determining a per-

*Our theorem involves the fifth type of Lie's table, page 66, "Continuierliche Gruppen."

spective collineation, we obtain $\infty^6 \cdot \infty^1 = \infty^7$ such pairs, or perspective collineations; but we have seen above, that among these are ∞^4 that represent the same perspective collineation. Hence, there are only $\infty^7 : \infty^4 = \infty^3$ different collineations left having a certain straight line of the plane for their axis. From this follows that to each pair of tangent-conics of a system confined to a certain point of the axis corresponds a pair of tangent-conics of a system confined to any other point of the axis. As to the other perspective collineation of the plane it is sufficient to say that there are $\infty^3 \times \infty^2 = \infty^5$ different perspective collineations in a plane: each straight line of the plane giving ∞^3 such collineations. But it is obvious that each of those ∞^2 systems has the same properties as any of the rest; each and any configuration of the one system can be made coincident with a certain configuration of any of the other systems. It is of no importance for our purpose to study up relations between two different systems in a general position. However, we shall have to consider two different systems with a common centre.

This relation occurs in the study concerning the invariant properties of perspective collineations. Here we have to consider the collineations in regard to those elements which by all collineations do not change their position, or even remain invariable in their intermediate parts and as a whole. Such elements are said to be invariant in the collineation. We found that there are ∞^3 perspective collineations belonging to a straight line as their axis. Hence the

Theorem 5. There are ∞^3 perspective collineations leaving the points of a straight line invariant.

Dualistically a point and the invariant rays through it, or the centre of collineation, can be taken as the invariant element. The ∞^2 straight lines of the plane and the characteristic anharmonic ratio combined with the centre give also ∞^3 different perspective collineations with the same centre, and we have therefore.

Theorem 6. There are ∞^3 perspective collineations leaving the rays through a point invariant.

This is the relation between two collineations with a common centre to which we drew attention while considering all the perspective collineations of the plane.

The next invariant element to be considered is the line-element,* or a straight line and a point on it.

*See Lie's definition of it in his "Vorlesungen," page 202.

To this line-element all the perspective collineations can be constructed which contain it as an invariant element. This can be done in two ways, first by taking the point as the centre and the line through it as an invariant ray of the perspective collineation, second by assuming the point as a point of the axis and the ray through it as a ray through the indeterminate centre of the collineation. By the same reasoning as before we find that in each case ∞^3 perspective collineations have a line-element in common. The two cases are in a dualistic relation and may be expressed in the

Theorem 7. There are ∞^3 *perspective collineations leaving a line-element invariant.*

Combining a line-element with either the points of a straight line or the rays through a point as the invariant figure, two other cases are obtained. If the points of a straight line and another straight line through one of these points are invariant, the centre of collineation may be any point on that other invariant line. This gives ∞^1 centres, and one on the same line as the axis of collineation. Adding to each centre two corresponding points, or what is the same, a characteristic anharmonic ratio, the number of perspective collineations is multiplied by ∞^1, so that ∞^2 perspective collineations satisfy the given conditions. Hence the

Theorem 8. There are ∞^2 *perspective collineations leaving the points of a straight line and another straight line invariant.*

In the dualistic case, where the rays through a point and another point on a ray through the first point are invariant, the axis of collineation may be any straight line through that other invariant point. This gives ∞^1 axes and one and the same point as the centre of collineation.

Adding to each axis two corresponding lines, or, what is the same, a characteristic anharmonic ratio, the number of collineations is multiplied by ∞^1, so that ∞^2 collineations satisfy the given conditions. Hence the

Theorem 9. There are ∞^2 *perspective collineations leaving the rays through a point and another point on one of the rays through the first point invariant.*

Finally there exists a system of collineations in which all points of a straight line and all rays through a point are invariant. Here, a special collineation is simply characterized by two corresponding points, or the characteristic anharmonic ratio. There are just ∞^1 such collineations. This result may be stated in the

Theorem 10. There are ∞^1 *perspective collineations leaving the points of a straight line and the rays through a point invariant.*

If the straight line is a priori supposed to be the axis, the statement: there are perspective collineations leaving the points of a straight line invariant is not entirely logical. Making this assumption it is self-evident that the points of the axis are invariant. The same may be said in regard to the centre of perspective collineations. The reason for putting the above theorem into this form is to have conformity with the statements in the cases of general collineation. See for this remark Lie's "Vorlesungen," table of groups, pages 288-291.

After having found the number and invariant properties of the general perspective collineation the special perspective collineations as classified in the previous chapter can be subjected to the same investigation. As before, only one system out of the ∞^2 of the plane shall be taken into consideration. Following the division given in the classification, §2, we have first the involution. It is determined by the centre and axis, since $k=-1$ and, hence, there are just as many involutions belonging to a straight line as an axis, as there are points (centres) in the plane, i. e., ∞^2. Hence there are ∞^2 involutions leaving the points of a straight line invariant and dualistically ∞^2 involutions leaving the rays through a point invariant. The same numerical result is found in regard to a line-element. But there is only one involution leaving the points of a straight line and the rays through a point invariant. What has been said about the involution holds in general for a whole system of collineations with a constant characteristic anharmonic ratio.

Dilation is characterized by an infinitely distant centre and a characteristic k varying from $-\infty$ to $+\infty$, and includes oblique and orthogonal symmetry ($k=-1$). The centre may be one of the ∞^1 points of the line at infinity, and since k can assume ∞^1 values, there are ∞^2 dilations which leave the points of a straight line invariant. The dualistic interpretation of dilations leaving a point at infinity invariant, respectively a pencil of parallel rays, does not lead to the same numerical result.

A certain direction, i. e., the centre at infinity, being given, the ∞^2 straight lines and ∞^1 values of k may be combined to form dilations. Hence, there are ∞^3 dilations leaving a point at infinity invariant. Taking a point of the axis and a ray through it as a line-element, it follows that there are ∞^2 dilations leaving a line-element invariant.

By the same reasoning as in the general case we find that there are ∞^1 dilations leaving the points of a straight line and another straight line invariant. There are, however, ∞^2 dilations in which the rays through the centre and a finite point in one of these rays are invariant.

If $k=-1$ dilation becomes oblique and orthogonal symmetry. Obviously there are ∞^1 such symmetries leaving the points of a straight line invariant. On the other hand there are ∞^2 (straight lines of the plane) symmetries leaving one and the same point at infinity invariant. There is only one symmetry belonging to a line-element.

Revolving one system of axial symmetry through 180°, k becomes $+1$, and the centres move to infinity in the direction of the axis. The relation is that of figures of equal areas, and the number of such collineations is ∞^1. For, taking a line parallel to the axis, or, connecting two corresponding points A and A^1, one and the same collineation is determined by any two points A and A^1 which include the same length $\overline{AA^1}$. This gives ∞^1 different collineations leaving the points of a straight line invariant.

In the case of similarity the axis is at infinity, the centre finite, and k ranges from $-\infty$ to $+\infty$, thus including central symmetry. The ∞^2 positions of the centre together with ∞^1 values of k give ∞^3 similarities leaving the points of the line at infinity invariant.

The centre of similarity may lie on a finite straight line and since each finite point of the plane represents ∞^1 similarities, there are ∞^2 similarities leaving the points of the line at infinity and another straight line invariant (line-element with its point at infinity). Taking the point of the line-element in the finite part of the plane, there are just ∞^1 similarities leaving this line-element, or also a point invariant.

All similarities with $k=-1$ are, as it is known, central symmetries. As $k=-1$ their number is ∞^2.

Assuming the centre of similarity at infinity, k becomes $+1$ and the collineations are congruences. A congruence is determined by the direction of its centre and two corresponding points which subtend the same length. As there are ∞^1 directions and ∞^1 sects in the plane, the number of congruence in the plane is ∞^2. All of them leave the line at infinity invariant. The number of perspective collineations belonging to a certain direction, or leaving the centre at affinity invariant, is ∞^1.

The pseudo-collineations which are characterized by $k=0, \infty, \frac{0}{0}$, have as their numbers ∞^2, ∞^2, ∞^1, respectively. In all three cases the axis is the invariant element.

In the case of the relation of K and K^1 being in a contact of the second or third order, each point of the axis as a centre gives ∞^1 perspective collineations (determined by the centre and the ∞^1 corresponding points subtending different sects). Hence, the points of a straight line are invariant for ∞^2 such collineations.

Dualistically, the rays through a point are invariant for ∞^2 such collineations. In other words there are ∞^2 of those collineations leaving a line-element invariant and again, there are ∞^1 such collineations leaving the points of a straight and the rays through a point on this line invariant.

We have now investigated all the properties relating to number and invariants of the general and special perspective collineations which are essential from the standpoint of the theory of groups.

In the next chapter we will consider some of the infinitesimal properties of perspective collineations.

§4. Identical and Infinitesimal Transformation and W-Curves of Perspective Collineation.

The axis and centre of perspective collineation being given, ∞^1 perspective collineations can be determined which leave these elements invariant. Each of these collineations is determined by the characteristic anharmonic ratio, or a pair of conics K and K^1 touching the axis at the same point and having for the other two common tangents two rays through the centre. As is known from §3 there are ∞^1 such pairs or different collineations. If especially the two conics K and K^1 coincide, the collineation is an identical one. All the conics tangent to the axis at a certain point and tangent to two fixed rays through the centre may be taken for the representatives of the identical collineation. An infinitesimal collineation differs from an identical collineation by an infinitesimal amount, i. e., a point and its corresponding one have an infinitesimal distance and two corresponding lines include an infinitesimal angle. Thus, in order to obtain the infinitesimal from the identical collineation we have to choose for K^1 the conic infinitely close to K in the same system. This conic shall be designated by δK. According to this proposition a point A is transformed into $A+\delta A$, and a line a into $a+\delta a$.

A and $A+\delta A$ lie on a ray through the centre, and $a+\delta a$ intersect each other in a point of the axis. Applying a whole system of infinitesimal perspective collineations to each other and in a certain succession, an integral, or finite, perspective collineation is obtained. In this operation the corresponding point A^1 to A starts from A having the direction to the centres and describes a certain curve which passes through the centre. Any point A^1, or $\int A + \delta A$, as corresponding to A in regard to the conics K and $\int \delta K$, lies with A on a ray through the centre. The curve which A^1 describes is therefore a straight line through the centre and is invariant for all perspective collineations of the system.

Since any conic tangent to the axis may serve to determine a perspective collineation with a given centre, axis, and characteristic k, it is obvious that there is only one infinitesimal perspective collineation belonging to a centre and an axis, or leaving these two elements invariant. By integration all the ∞^1 finite perspective collineations may be obtained which leave the points of a straight line and the rays through a point invariant.

From the fact that the centre and axis determine an infinitesimal perspective collineation, it follows that there are ∞^2 infinitesimal perspective collineations either leaving the points of a straight line, or the rays through a point invariant. The integration gives in both cases the ∞^3 finite perspective collineations leaving the same elements invariant. Moreover, it follows that the plane has ∞^4 infinitesimal perspective collineations which by integration give the ∞^5 finite perspective collineations of the plane.

It is not necessary to enter into a study of the infinitesimal perspective collineation of the special cases, because the result is essentially the same. It is sufficient to mention that the collineation having the centre in its axis and the characteristic $k = +1$ has simply ∞^1 infinitesimal collineations leaving the points of the axis invariant. By integration the ∞^2 finite collineations of this kind are obtained.

In this last case as well as in the general case the whole of W-curves* consists of the pencil of rays through the centre. This pencil becomes a pencil of parallel rays if the centre is at infinity, as it occurs in some of the special cases of perspective collineation. We have here found the same result as Lie in his "Vorlesungen" on pages 69 and 70.

§5. Groups of Perspective Collineations.

Suppose a system of perspective collineations which is restricted by certain conditions, for instance, such as to leave given elements or combinations of elements invariant, or to be characterized by a

*"W-Curven," or "selbstprojective Curven" in Lie's Vorlesungen, page 68.

special value of the characteristic anharmonic ratio. By any of the perspective collineations belonging to such a system a point A is transformed into A^1. Taking another collineation of the same system and A^1 as an original point in it, the corresponding point will be A''. Whenever now A and A'' are related in such a manner as to be a collineation of the given system, i. e., subject to the same conditions, and, inversely, if each point A^1 can be transformed back into its corresponding A by a collineation of the system, such a system of collineations is said to be a continuous group of collineations, or simply a group of collineations. By this statement it is easy to enumerate the groups which may occur in the general and special cases of perspective collineations. We may, however, occasionally avail ourselves of a theorem of Lie concerning a criterion of groups by means of invariant properties of transformations. The theorem is:

"The system of all projective transformations of the plane leaving a certain figure invariant has the property of a group. The transformations of the system are inverse by pairs."*

In enumerating the groups we follow the order of chapters 2 and 3. Thus, we have first to consider the general perspective collineations. In Fig. 11 we assume l as the axis common to ∞^3 perspective collineations of the plane and two collineations of this system determined by ($CLAA^1$) and ($C_1L_1A'A''$), or ($claa^1$) and ($c_1la'a''$) respectively, where C and C_1 are the centres of the two perspective collineations. The transformed point to A in the first collineation is A' and the transformed point to A' in the second collineation A''. Taking an other point B on a, the points B' and B'' can be constructed, or in general, to each point on a there are coresponding ones on a' and a''.

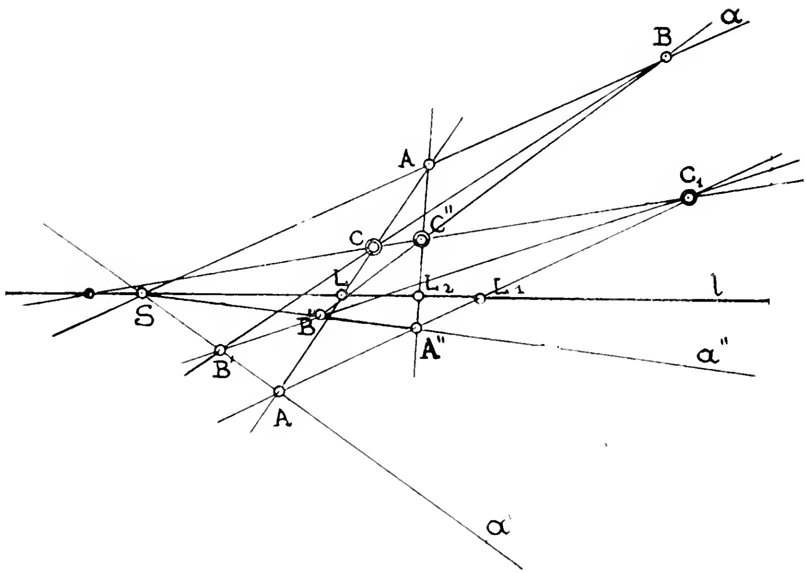
Now the point-range AB....is perspective to the point-range A'B'....with C as the centre and, again, A'B'....perspective to the point-range A''B''....with C_1 as centre. The three ranges have a self-corresponding point, S, on l. Hence, the point-ranges AB....and A''B''....are also perspective, i. e., AA'', BB''.... pass through one and the same point C'' which obviously may be taken as a new centre of collineation with l as an axis, and A, A''; B, B'';....as coresponding points. As is immediately seen, to each transformation in a perspective collineation may be found its inverse belonging to the same system. We have therefore the

Theorem 11. The system of all perspective collineations leaving the points of a straight line invariant forms a three-termed group.

*Lie's "Vorlesungen," page 113

A collineation resulting from two other collineations is related to these in such a manner that the three centres lie in a straight line. For, considering the triangles ABC and $A''B''C_1$, it is seen that

Fig. 11.



their corresponding sides intersect each other in three points S, B', A', of a', or that the triangles are homologous. Hence $\overline{AA''}$, $\overline{BB''}$, $\overline{CC_1}$, intersect each other in a point, in the centre C'' of the third collineation. Hence C, C_1, C'', lie in a straight line.

This fact leads us at once to a sub-group of perspective collineation. By choosing the centres of perspective collineation constantly on a straight line the centre of a collineation resulting from two of these collineations is a point of the same line. To each perspective collineation may also be found its inverse belonging to the same system. From §3 is known that there are ∞^2 such collineations and we have therefore:

Theorem 12. The system of perspective collineations leaving the points of a straight line invariant forms a two-termed group.

Without needing a direct proof the following dualistic statements of the two preceding theorems can be made:

Theorem 13. The system of all perspective collineations having the rays through a point invariant forms a three termed group.

And

Theorem 14. The system of all perspective collineations leaving the rays through a point and another point in one of these rays invariant forms a two-termed group.

For a proof of these theorems we refer either to Lie's theorem at the beginning of this chapter, or to the first proof of theorem 11. In the dualistic case the reasoning is exactly the same.

There is another three-termed group of perspective collineations, which is obtained by a special interpretation of the groups which leave the points of a straight line or the rays through a point invariant. The line-element, the point in which is taken as the centre, as an invariant figure, is equivalent with the rays through the centre. The system of collineations is therefore in both cases the same. On the other hand the point of the line-element may be a point of the axis. The invariant configuration is therefore that of the points of a straight line and another straight line. But the axis may be any of the rays passing through the point of the line-element, so that the number of perspective collineations is ∞^3 as before. Thus, the

Theorem 15. All the perspective collineations leaving a line-element invariant form a three-termed group.

The next and last sub-group of perspective collineation concerns the points of a straight line and the rays through a point as the invariant configuration. As usual let the point (centre) be C and the line (axis) l and two collineations represented by (CLAA') and $(CL_1A_1A_1')$. Applying the second collineation to A', the corresponding point A'' of A' is obtained which lies upon the same ray through C, as A and A'. The new collineation is therefore represented by (CLAA''), i. e., by a collineation of the same system. Since each of these collineations and its inverse belong to the system the following statement may be made:

Theorem 16. All the perspective collineations leaving the points of a straight line and the rays through a point invariant form a one-termed group.

The groups of perspective collineation are also easily obtained by the configurations in space.

Assume the two planes π and π' intersecting each other in l, and a centre (C) without these planes as a perspective collineation in space. Drawing the bisecting plane π_1 of π and π' and a perpendicular from (C) to the bisecting plane, intersecting π and π' in C and C', respectively, these two points will coincide when one of the planes π, or π' is revolved into the other about l as an axis.

From this we see that the line l is invariant for every perspective collineation having any point of space as its centre. The number of such collineations is ∞^3. A point C and a line l are invariant for ∞^1 such collineations, for there are ∞^1 centres, (C), on the perpendicular to the plane π_1 in C, and ∞^1 planes through the perpendicular. A plane through such a perpendicular intersects π and π' in two lines, which after revolving one plane into the other become an invariant line. As there are ∞^2 points (centres) in the plane perpendicular to π_1, the points of l and another straight line will be left invariant by ∞^2 perspective collineations.

In this manner we can successively deduce all the groups from intuition in space. We shall not carry on the enumeration of the other groups by this method; it is sufficient to have shown the possibility of this method, which in fact is identical with the other.

In the general case we had considered the ∞^3 perspective collineations leaving the points of a straight line, and, dualistically, the rays through a point invariant. Each collineation of the system is characterized by a certain anharmonic ratio. Two perspective collineations with the characteristics k and k_1 applied one to the other determine a new perspective collineation of the same group, whose characteristic may be designated by k. Now it is known that k'' is an algebraic relation between k and k_1, say:

$$k'' = f(k, k_1).$$

Suppose now that two perspective collineations of the same characteristic applied one to the other produce a perspective collineation with the same characteristic, such that

$$k''=f(k'', k'').$$

If this relation shall hold for all values of k'', it must be an identical one: i. e., k''=k'', as it occurs in the identical perspective collineation. From this we conclude that in general a system of perspective collineations with a constant characteristic does not form a group. From the above relation we can find the special values for which

$$k''=f(k'', k'')$$

by resolving the equation

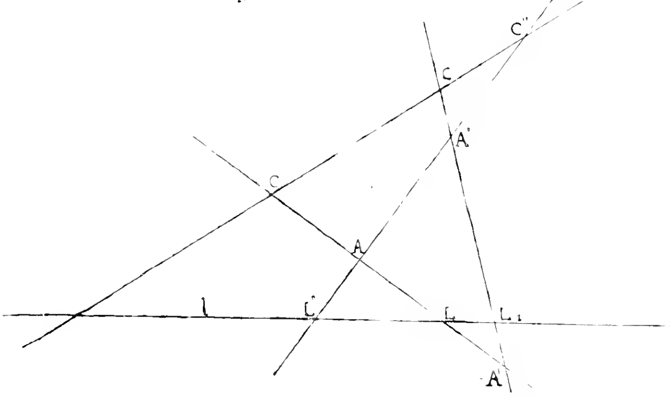
$$k'' - f(k'', k'')=0.$$

It is therefore necessary to know the form of $f(k, k_1)$.

For this purpose we consider the three perspective collineations (CLAA'), $(C_1L_1A'A'')$, (C''L''AA''), of which the last results from the two others as described before.

The sides of the triangle AA'A'' are intersected by two transversals l and the line joining the centres C, C_1, C''. Thus, according to the theorem of Menelaos we have the relations:

Fig 12



(1). $LA \cdot L_1A' \cdot L''A'' = LA' \cdot L_1A'' \cdot L''A$, or

(2). $\frac{LA}{LA'} \cdot \frac{L_1A'}{L_1A''} = \frac{L''A}{L''A''}$, and in the same way:

(3). $CA \cdot C_1A' \cdot C''A'' = CA' \cdot C_1A'' \cdot C''A$, or

(4). $\frac{CA}{CA'} \cdot \frac{C_1A'}{C_1A''} = \frac{C''A}{C''A''}$

Dividing (4) by (2) we have

$$\frac{\frac{CA}{CA'} \cdot \frac{C_1A'}{C_1A''}}{\frac{LA}{LA'} \cdot \frac{L_1A'}{L_1A''}} = \frac{\frac{C''A}{C''A''}}{\frac{L''A}{L''A''}}, \text{ or}$$

$$(CLAA')\,(C_1L_1A'A'') = (C''L''AA'')$$

Designating the characteristics respectively by k, k_1, k'', we find the required fundamental relation in the form:

$$k'' = k.\ k_1$$

If these three characteristics are equal, say equal to k, the relation becomes

$$k = k^2, \text{ or}$$
$$k^2 - k = 0\ ;$$

Whence either $k=0$, or $k=+1$. Excluding the singular case $k=0$ we can therefore say, that only those perspective collineations with a constant characteristic are liable to form a group, for which $k=+1$. If a perspective collineation $(CLAA')=k$ is given we find its inverse $(CLA'A) = \frac{1}{k}$, which is a number of the same kind. Thus, to each perspective collineation we can find its inverse. In the construction this fact is self-evident. If the characteristics of two perspective collineations are of the same sign the sign of the resulting third perspective collineation is always positive; if they are of a different sign, the sign of the third is always negative. From this we conclude, as we know already, that the system of involutions does not form a group:

$$-1 \times -1 = +1.$$

Among the general cases of perspective collineations we have to study the system of dilations in the first place. The centres are all at infinity. Hence we have only the relation

$$\frac{LA}{LA'} \cdot \frac{L_1A'}{L_1A''} = \frac{L''A}{L''A''}, \text{ while}$$

$$\frac{CA}{CA'} \cdot \frac{C_1A'}{C_1A''} = \frac{C''A}{C''A''} = 1.$$

The characteristic k'' of a dilation resulting from two other dilations of the same system is therefore expressed as before

$$k'' = k.\ k_1.$$

To each dilation can also be found its inverse, so that the respective characteristics are k and $\frac{1}{k}$. To sum up we can say:

Theorem 17. The system of dilations leaving the points of a straight line invariant forms a two-termed group.

If the centre at infinity and the axis are kept fixed the dilations differ according to their characteristics. A, A'; A', A'' being two pairs of corresponding points on a ray through the centre at infinity, A, A'', will be the corresponding pair of the resulting dilation, and there is evidently

$$\frac{LA}{LA'} \cdot \frac{LA'}{LA''} = \frac{LA}{LA''}, \text{ or}$$

$$k \cdot k_1 = k''; \text{ i. e.,}$$

the third dilation belongs to the same system. As there are ∞^1 such dilations we have

Theorem 18. The system of dilations leaving the points of a straight line and a point at infinity (centre of dilation) invariant forms a one-termed group.

The same is true for the system of dilations leaving the points of a straight line and another straight line invariant.

The dualistic interpretation, however, does not lead to the same result. From the general case it is known that all the collineations leaving the rays through a point invariant form a three-termed group. If this point is at infinity the collineations are dilations. Hence the

Theorem 19. The system of dilations leaving the rays of a parallel pencil of rays invariant forms a three-termed group.

Theorem 20. The system of dilations leaving the centre of dilation and another finite point invariant forms a two-termed group.

The one-termed dualistic group of dilation is the same as the original one: this group is self-dualistic.

In the oblique and orthogonal symmetry, which is dilation with $k=-1$, there is no group, for two symmetries applied one to the other give a collineation with the characteristic $k=+1$, $(-1 \cdot -1=+1)$.

In the case of dilation with $k=+1$ the centre is at infinity in the direction of the axis. As is known from §3, this is the relation of corresponding equal areas. Two points, A, A', on a ray parallel to the axis determine the relation

$$k=\frac{\overline{\infty A}}{\overline{\infty A'}}=+1.$$

Taking A', A'' as a corresponding pair in another collineation of this kind,

$$k_1=\frac{\overline{\infty A'}}{\overline{\infty A''}}=+1, \text{ we obtain}$$

$$\frac{\overline{\infty A}}{\overline{\infty A'}}\cdot\frac{\overline{\infty A'}}{\overline{\infty A''}}=\frac{\overline{\infty A}}{\overline{\infty A''}}=+1, \text{ or}$$

$$(+1)\times(+1)=+1.$$

Thus the

Theorem 21. The system of perspective collineations characterized by corresponding equal areas and leaving the points of a straight line invariant forms a one-termed group.

In discussing the relation $k''=k.\ k$, it was pointed out that there are groups with the characteristic $k=+1$. This assertion is now proved: but we yet shall find other groups with the characteric $k=+1$.

Instead of the line joining the centres we can suppose the line l of a system of perspective collineations to be at infinity. In this case we have similarity which is expressed by

$$\frac{CA}{CA'}\cdot\frac{C_1A'}{C_1A''}=\frac{CA}{CA''}, \text{ or again}$$

$$k''=k.\ k_1.$$

Just as in the case of dilations, to each similarity can be found its inverse belonging to the same system. There are ∞^3 such similarities, therefore

Theorem 22. The system of similarities leaving the points of the line at infinity invariant forms a three-termed group.

If the centres C, C_1, C'', are confined to a straight line the theorem follows:

Theorem 23. The system of similarities leaving the points of the line at infinity and on another straight line invariant forms a two-termed group.

If the axis at infinity and the centre are kept fixed, the similarities differ according to their characteristics. A, A': A', A'' being two pairs of corresponding points on a ray through the centre, A, A'' will be the corresponding pair of the resulting similarity, and there is evidently:

$$\frac{CA}{CA'} \cdot \frac{CA'}{CA''} = \frac{CA}{CA''}, \text{ or}$$

$$k \cdot k_1 = k'', \text{ i. e.,}$$

the third similarity belongs to the same system. As there are ∞^1 such similarities we have

Theorem 24. The system of similarities leaving the line at infinity (axis of similarity) and the rays through a point invariant forms a one-termed group.

The same is true for the system of similarities leaving the rays through a point and another point (on the line at infinity) invariant.

In central symmetry to the similarity is added the condition $k = -1$. Central symmetry as well as oblique and orthogonal symmetry does not form a group.

If the centre of similarity is at infinity we have congruence in which $k = +1$. Taking A, A' and A', A'' as corresponding pairs in two congruences, A, A'' will be a corresponding pair in a third perspective collineation, resulting from the first two. There is

$$\frac{\overline{\infty A}}{\overline{\infty A'}} \cdot \frac{\overline{\infty A}}{\overline{\infty A''}} = \frac{\overline{\infty A}}{\overline{\infty A''}} = +1$$

As there are ∞^2 such collineations we have the

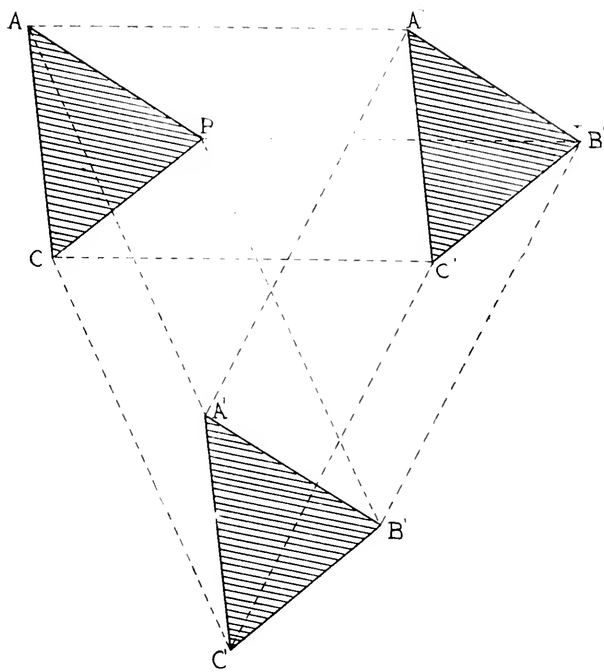
Theorem 25. The system of congruences leaving the points of the line at infinity invariant forms a two-termed group.

Supposing A, A', A'' in a straight line, the above relation still holds. The system of congruences has in this case a fixed centre at infinity and consists of ∞^1 congruences. Hence

Theorem 26. The system of congruences leaving the points of the line at infinity and the rays of a pencil of parallel rays invariant forms a one-termed group.

On account of the conspicuity in which the character of a group appears in the groups of congruences, we give an illustration in Fig. 13.

Fig 13.



The pseudo-perspective collineations which have the characteristic $k=0, =\infty, =\frac{0}{0}$ may conveniently be represented in space. The cases arise when the centre of collineation is in either one of the planes π and π'. If the centre (C) is in π then $(CLAA')=(CLR\infty)=(CLC\infty)=0$. Every point of π' is projected into C. Taking C as a point of π' (by revolving the plane π about the axis l into π') and performing the projection for another centre, say C_1 in π, the point C is projected into C_1. But the points in π' and C may be taken as a new pseudo-collineation of the system. The same can be said if the centre (C') is in π', or if $(C'LAA')=(C'L\infty C')=\pm\infty$. The inverse of a collineation for both cases $k=0$ and $k=\infty$ is indefinite, but as the other conditions for a group are satisfied, we may say that the pseudo-perspective collineations with either the characteristic 0 or ∞ form a two-termed group.

In the pseudo-perspective collineation with the centre in the axis and k indefinite there is no group at all.

With this we close the study of groups of perspective collineations, perspective collineations taken in its common meaning, and proceed to that particular kind of perspective collineations in which centre and axis coincide and k=+1. This kind of collineations might be considered as the supplement of involution. For, the involution being obtained by revolving one plane, say π, into the other π' through an angle of α^0, the collineation in question is obtained by revolving π into π' through $180-\alpha$. According to Lie (page 202 of his "Vorlesungen") we may call this collineation an elation.

But here the word does not mean entirely the same as in Lie's definition. We use it here only as a convenient word to express those collineations.

In Fig. 12 C and L, C_1 and C_2; C'' and L'' coincide, so that the characteristic of an elation is +1.

The fundamental relation is

$$(+1)\times(+1)=+1$$

and since there are ∞^2 elations restricted to a line we have

Theorem 27. The system of elations leaving the points of a straight line invariant forms a two-termed group.

and dualistically:

Theorem 28. The system of elations leaving the rays through a point invariant forms a two-termed group.

Taking A, A', A'' on the same ray through the centre, there is

$$\frac{CA}{CA'}\cdot\frac{CA'}{CA''}=\frac{CA}{CA''},$$

and the same is true for LA, LA', and so on, since L coincides with C.

Therefore the relation as above:

$$(+1)\times(+1)=+1.$$

Theorem 29. The system of elations leaving the points of a straight line and the rays through a point invariant forms a one-termed group.

The centre of elation can also be considered as the point of a line-element. From this point of view we have another theorem.

Theorem 30. The system of elations leaving a line-element invariant forms a two-termed group.

Summing up, the following groups of perspective collineations are possible. (The Roman numerals denote the types of transformations as given in Lie's "Vorlesungen über Continuierliche

Gruppen," page 510 to 512, to which the groups belong, and the Arabic numerals in brackets denote the numerals of the groups in Lie's table of projective groups of the plane (pages 288 to 291).

A. THREE-TERMED.

(1). Invariant line-element, III, (14).
(2). Invariant points of a straight line, III, (21).
(3). Invariant rays of a pencil, III, (22).

B. TWO-TERMED.

(4). Invariant line-element, V, (24).
(5). Invariant points of a straight line and another invariant straight line, III, (32).
(6). Invariant rays of a pencil and another invariant point, III, (33).
(7 and 8). See two-termed groups of type V.

C. ONE-TERMED.

(9). Invariant points of a straight line and invariant rays of a pencil being not on the straight line, III, (38).
(10). Invariant points of a straight line and invariant rays of a pencil on the straight line, V, (39).

As groups of the special cases of collineations we have within the type III, the following:

A. DILATION.

(*a*). Three-termed.—Invariant rays of a pencil of parallel rays.
(*b*). Two-termed.—(1). Invariant rays of a pencil of parallel rays and another invariant point.
(2). Invariant point of a straight line.
(*c*). One-termed.—Invariant points of a straight line and invariant rays of a pencil of parallel rays.

B. CORRESPONDING EQUAL AREAS.

(*a*). One-termed.—Invariant points of a straight line.

C. SIMILARITY.

(*a*). Three-termed.—Invariant points of the line at infinity.
(*b*). Two-termed.—Invariant points of the line at infinity and another invariant straight line.
(*c*). One-termed.—Invariant points of the line at infinity and invariant rays of a pencil.

D. CONCRUENCE.

(*a*). TWO-TERMED.—Invariant points of the line at infinity.

(*b*). ONE-TERMED.—Invariant points of the line at infinity and invariant rays of a pencil of parallel rays.

E. PSEUDO-PERSPECTIVE COLLINEATIONS.

(1). Two-termed pseudo-group.

(2). One-termed pseudo-group.

In type V we have elations and under these the following groups:

A. TWO-TERMED.

(1). Invariant line-element, (24).

(2). Invariant points of a straight line, (29).

(3). Invariant points of a pencil, (30).

B. ONE-TERMED.

Invariant points of a straight line and invariant rays of a pencil on the straight line.

Finally we will add a formal representation of groups of perspective collineations in the plane.

The whole plane contains ∞^5 perspective collineations (combinations of the ∞^2 straight line and the ∞^2 points of the plane $= \infty^4$, and ∞^1 values of the characteristic anharmonic ratio. Designating a general perspective collineation by the index c, an n-termed group by G_n and its dualistic by Γ_n, a self-dualistic group by H_n, dilation by the upper index d, corresponding equal areas by a, similarity by s, congruence or translation by t, elation by e, and the dualistic interpretation of the group $G_2{}^d$ by Γ_2^δ, the following symbolic equations between the different groups and sub-groups of collineations exist:

$$\infty^5 C = \infty^2 G_3^c + \infty^2 \Gamma_3^c + \infty^3 H_3^c{}^* + 1 G_3^s + \infty^1 \Gamma_3^d + 1 H_2^t$$

$$G_3^c = \infty^1 G_2^c + 1 G_2^d + 1 G_2^e + 1 G_1^a$$

$$\Gamma_3^c = \infty^1 \Gamma_2^c + \infty^1 \Gamma_2^\delta + 1 \Gamma_2^e$$

$$H_3^c = \infty^1 G_2^c + \infty^1 \Gamma_2^c + 1 H_2^e$$

$$G_2^c = \infty^1 H_1^c + 1 G_1^d + 1 H_1^e$$

$$\Gamma_2^c = \infty^1 H_1^c + 1 H_1^e$$

$$G_3^s = \infty^1 G_2^s + 1 H_2^t$$

$$G_2^s = \infty^1 G_1^s + 1 H_1^t$$

* These ∞^3 groups are all equivalent, especially ∞^1 groups can always be made identical by a simple translation.

$$\Gamma^d_3 = \infty^1 \Gamma^d_2 + 1 H^t_1 + 1 \Gamma^\delta_2$$
$$\Gamma^d_2 = \infty^1 \Gamma^d_1 + 1 H^t_1$$
$$\Gamma^\delta_2 = \infty^1 H^c_1 + 1 \Gamma^s_1 + 1 H^e_1$$

$$H^t_2 = \infty^1 H^t_1$$
$$H^e_2 = \infty^1 H^e_1$$

§6. Historical Sketch.

To show what position the subject of this dissertation occupies in geometry it will be necessary to give first a brief account of the development of geometrical methods which gradually lead to the modern standpoint. Projective or synthetic geometry is essentially a product of the 19th century, though it is well known that Pappus and Menelaus found some very important theorems concerning projective properties many hundred years ago and that Desargues discovered the fundamental theorems of perspective and involution in the 18th century.

The origin of projective geometry must be sought in the methods of descriptive geometry, which, by the achievements of Lambert and Monge, became at once very valuable in geometrical investigation. The first classic work on projective geometry was Poncelet's "Traité des propriétés projectives des figures," which appeared in 1822. In this great treatise the properties of figures are investigated which are unaltered by projection, or which are invariant. Poncelet introduced the so-called central-projection with a perspective-centre and a perspective-axis into the consideration of plane figures. While in France the "new geometry" was chiefly promoted by Gergonne and Chasles, in Germany its fruitfulness was shown to the scientific world by the three great investigators, Möbius, Plücker, and Steiner. The classical works of this period are:

Möbius, Barycentrischer Calcul, 1827.

Plücker, Analytisch-geometrische Untersuchungen, 1828.

Steiner, Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander, 1832.

Chasles, Aperçu historique sur l' origine et le développement des méthodes en géométrie, 1838.

From these times also dates the separation of the mathematicians into two schools. One of them, the synthetical school, was represented by Steiner, Möbius, v. Staudt, Schröter, and has as its present principal leaders: Durège, Reye, Sturm, and Fiedler.

The other, the analytical school, has as its representatives: Plücker, Hesse, Aronhold, Gordan, Cayley, and many others.

Meanwhile the brilliant results of modern synthetic and analytic geometry have had a great influence upon pure analysis. The modern theory of functions was created, and by the investigations of Jacobi, Abel, Cauchy, Riemann, Hermite, and Weierstrass, it has reached a dominant position in almost all branches of mathematics. It became more and more a prevailing opinion that in fact the synthetic and analytic methods in geometry are identical and it is now generally acknowledged that the two methods differ only in their formal representation. Fiedler in 1874 defined the homogeneous co-ordinates as anharmonic ratios which lead at once from synthetic to analytic, and from analytic to synthetic geometry. The greatest step in overcoming the difficulties between synthetic and analytic methods was however taken by Klein and Lie about 1871. Klein in his "Erlanger Programm" clearly outlined the standpoint from which the problems of modern mathematics have to be considered. The fundamental idea of higher geometry is to find all the "groups" and to investigate their properties, i. e., to find geometrical truth. As to Lie, it is well known that the achievements of this great mathematician concerning the theory of groups, since about 1874, influenced and still influence many of the most important fields of mathematics.

The old conception of invariants is abandoned and its place has been taken by the conception of groups.

In this paper it has been attempted to make a little contribution to geometry by applying the theory of groups to the well-known subject of perspective. In works on groups, which hitherto has been published, the treatment is almost exclusively analytical and it may be pointed out that our paper is the first bearing on groups in which the synthetical method is used.

Lie divided all projective transformations into five types. Before the investigations of Lie were known, only two of these types have usually been treated, the general projective transformation (collineation), and the perspective collineation. Our perspective collineations make up two of those five types: perspective, and elation. As has been said already in the preface, the other three types are being investigated in the same way by Prof. Newson.

For references we give the following list of books:

Sophus Lie, Vorlesungen über continuierliche Gruppen, Leipzig, Teubner, 1894.

Sophus Lie, Theorie der Transformationsgruppen, 3 Vols., Leipzig, Teubner.

Sophus Lie, Vorlesungen über Differential Gleichungen, Leipzig, Teubner, 1891.

Lindemann-Clebsch, Vorlesungen über Geometrie, Vols. 1 and 2, Leipzig, Teubner, 1888-93.

Klein, Vorlesungen über höhere Geometrie, 2 Vols., MS. Notes, Göttingen, 1893.

V. Staudt, Geometrie der Lage, Nürnberg, 1846.

Fiedler, Darstellende Geometrie, 3 Vols., Leipzig, Teubner, 1883-85.

Reye, Geometrie der Lage, Baumgärtner, Leipzig, 1886.

Cremona, Elements of projective geometry, Oxford, Clarendon Press, 1885.

Klein, Vorlesungen über das Ikosaeder, Leipzig, Teubner, 1885.

Mathematische Annalen, vols. 28 and 29.

Hoplophoneus occidentalis.

BY E. S. RIGGS.

(With Plate I.)

[Submitted to the Faculty of Kansas University as a thesis for the degree of A. M.]

The species *Hoplophoneus occidentalis* Leidy is based upon a fragment of a mandible from the White River Beds, described in Leidy's Extinct Mammalian Fauna of Dakota and Nebraska. It shows the lower molar dentition to be P. M. $_{\overline{2}}$, M. $_{\overline{1}}$; but of these the crowns of the third premolar and the sectorial are lost, as are both extremities of the mandible, so that even the size of the animal could only be approximated. Nothing more was ascribed to this species until 1894, when Osborn and Wortman referred two specimens to it*. In 1895 Dr. Williston published in this QUARTERLY† a preliminary description of two specimens of a large sabre-toothed cat obtained during the previous summer in the White River Beds of South Dakota by the University Geological Expedition, to which he gave the name of *Dinotomius atrox*. In January, 1896,‡ Mr. Geo. I. Adams determined a mandible (No. 1407 Amer. Mus.) as *Hoplophoneus occidentalis* and described the smaller form of Osborn and Wortman as a distinct species, suggesting also that *D. atrox* was a synonym of *H. occidentalis*, since it agreed with the American Museum specimen.

In his preliminary description of *D. atrox*, Dr. Williston, describing a complete tooth, characterized the inferior sectorial as follows: "Molar much as in the cat, save there is a well-developed internal posterior tubercle and the heel is rudimentary." An accompanying plate, although concealing the anterior portion of this tooth, represented it as described. Nevertheless, Mr. Adams in reproducing Dr. Williston's drawing for comparison with the American Museum specimen, has taken the liberty to reconstruct this tooth so as to show a prominent heel and no postero-internal cusp, and then states that the two specimens agree. Again, in the *American Journal of Science* for June, 1896, Mr. Adams reproduces

*Bulletin of the American Museum of Natural History.

†Kansas University Quarterly, Vol. III, No. 3.

‡The American Naturalist, January, 1896.

the same drawing, and, after stating that the species is best known from the Kansas University specimen, in the face of Dr. Williston's description and figure, states as a characteristic of the species, that "the postero-internal cusp is wanting." Now since this cusp is as strongly developed in the Kansas University specimen as in *H. primævus* or *H. robustus*, as specimens before me show, if Mr. Adams is correct in his statement that *H. occidentalis* does not have the cusp, then we have to do with two distinct species.

In Leidy's type there is little more than the number and size of the teeth in the lower molar series, the form of the fourth premolar, and the general size and shape of the body of the ramus, upon which comparison can be based. Below is given a series of measurements in which the Kansas University specimens are compared with Leidy's figure and with the above-mentioned American Museum specimen by means of data kindly furnished me by Dr. Osborn. All measurements are given in millimeters.

	Leidy's Type.	Am. Mus.	Kans. Univ. Smaller.	Kans. Univ. Larger.
Length from condyle to alveolar border...		170	168	
Molar series, length....................	47	75	46	
Breadth of base of third premolar........	09		09	
Breadth of base of fourth premolar........	15		16	
Breadth of base of sectorial.............	21		20	
Depth of symphysis....................		65(?)	63	72
Depth of flange from base of canine.......		63.5	67	78
Depth of mandible at base of fourth premolar		30	30	
Condyle to angle......		25	27	
Length of diastema.....................		37	41	48

It will be observed that, so far as can be determined by measurements, the Kansas University specimen agrees very closely with Leidy's type. The fourth premolar is similar in size and shape to the one remaining tooth, and, like it, is directed somewhat backward. On the other hand, the mandible of the American Museum specimen is slightly larger, has a more retreating coronoid process, a shallower flange, a shorter diastema, and is longer from the sectorial to the condyle. However, these differences are no greater than those between the two Kansas specimens, and unless the absence of the postero-internal cusp be found constant, the differences would not seem to be specific. Then, so far as can be determined by comparison, *D. atrox* and *H. occidentalis* agree as closely as individuals of the same species may be expected to agree, and, as Mr. Adams suggests, may be regarded as identical. Moreover, the dental, cranial, and skeletal characters will later be shown to be consistent with those already described in *Hoplophoneus*.

The distinctive characters of *H. occidentalis* as shown by the Kansas University specimens are: Its size, which exceeds that of any other member of the genus by one-fourth; its markedly concave sagittal crest; its strongly recurved canine, trenchant and denticulate at the point, but rounded at the margins throughout three-fourths of its length; the third lower premolar much reduced, and lower sectorial with postero-internal cusp, but heel rudimentary.

The material upon which this restoration and description is based is composed of parts of two skeletons found almost together and in exactly the same horizon, just below the *bullatus* layer of the Oreodon beds. They differ somewhat in size, but no more than might be due to age or sex. The smaller of the two shows by the less completely ossified epiphyses and the slightly worn teeth, that it was a younger animal; while the firmly ossified sacrum and the well worn teeth of the larger one indicate an older animal. Between the two specimens the skeleton is anatomically complete, save the lower incisors and canines, some whole vertebræ and many of the pophyses, the sternum and most of the ribs, half of the scapula and radius, the shaft of the fibula, half the bones of the hind feet, and nearly all those of the front feet. In the restoration I have had for comparison the skulls (more or less complete) of *Dinictis felina*, *D. paucidens*, *Hoplophoneus robustus*,* and *H. primævus*, a mandible of *Pogonodon sp.*, together with various bones of the skeleton of *D. paucidens*, and *H. robustus*. Also a skeleton of *Felis leo*, *Felis concolor*, *Felis domesticus*, and *Lynx rufus*. Where parts were wanting, they have been supplied by comparison with other members of the genus so far as the parts were present, but frequently I have had to rely upon the African Lion. In the description I have used *H. robustus* along with the lion for comparison.

The material above mentioned forms part of the paleontological collection of the University of Kansas. For the privilege of its use and for his careful direction and criticism in the preparation of this description and restoration I am indebted to Dr. Williston.

DESCRIPTION OF THE SKELETON.

The skull is complete in the smaller specimen, save the upper canines and the crowns of the lower canines and incisors. An almost complete canine from the larger one, however, shows the

*Adams. *American Naturalist*, January, 1896.

characters of this tooth. The skull is deep but narrow, the zygomatic arches, though somewhat crushed, evidently did not stand out so prominently as in *H. robustus.* The sagittal crest is concave, rising into a prominence at the occiput, which is strongly overhanging. The post-orbital processes are projecting and curve slightly forward; the supraorbital margin is less prominent than in *H. robustus.* The zygomatic processes project well below the basicranial axis as is common in species of this genus. The mastoid process is strong and much roughened for muscular attachments. The posterior nares open on a line with the posterior border of the sectorial. A median ridge extends the entire length of the bony palate. A groove leads backward from the posterior nares as far as the anterior portion of the basi-occipital, where it divides and the branches lead respectively to the precondylar foramina.

The mandible is marked by a deep descending flange, second only to that of *Eusmilus* in prominence, its long diastema, and its deep masseteric fossa sloping away on its superior border to the short, stout coronoid process. The condyle is proportionally longer and more slender than in *H. robustus*, and the angle is more projecting. The two specimens differ quite markedly in the anterior portion of the mandible. In the larger one, in which only that portion in front of the third premolar remains, the flange is eleven millimeters deeper, the chin seven millimeters broader, and does not show the constriction below the base of the lower canine which is present in the smaller specimen. The diastema is seven millimeters longer, and the rami are much thicker and stouter at the superior border. There are three mental foramina in the smaller specimen and two in the larger.

The infraorbital foramina, as in *H. robustus*, are proportionally smaller and more triangular than in the lion. The post-glenoid foramina are present; but small and directed far inward. The lachrymal foramen lies well within the orbit, is small and nearer the infraorbital foramen than in *H. robustus*, and is directed more downward. The spheno-palatine foramen lies just on the median side of, and near the posterior palatine foramen, as in the last-named species, and is only a trifle larger. The optic foramen is small, laterally compressed, and is situated directly above the sphenoidal fissure. Above and in front of the optic foramen, situated about midway of the spheno-frontal suture, is the well-developed foramen spinosum. The spheno-orbital foramen, the rotund foramen and the anterior opening of the ali-sphenoid canal appear at the surface as a common, large, anteriorly directed opening. Just within their opening, however, the three diverge, forming distinct canals. The posterior opening of the ali-sphenoid canal is in front of, and near the oval foramen, but not included in a common fissure with it as in *H. robustus*. The carotid canal is represented by a groove alongside of the basi-occipital, within the optic bulla. It terminates just without the posterior end of a ridge bounding the median groove of the basi-occipital. Midway between this and the anterior border of the occipital condyle is the expanded opening of the re-condylar foramen. The dental foramen opens just back of the anterior margin of the coronoid process, and forms a groove to the base of the condyle.

Dentition I. $\frac{3}{3}$ C. $\frac{1}{1}$ P. $\frac{3}{2}$ M. $\frac{1}{1}$. **The dentition** is complete except the crown of the lower incisors and canines. The upper incisors are proportionally shorter and stouter than those of *H. robustus*. The first and second are similar in size and shape; the third is only a trifle longer, but much stouter. The canine is strong and decidedly recurved: its margins are well rounded throughout the greater part of its length, but near the point they become thinner and trenchant. The posterior edge is minutely denticulate, but the condition of the specimen does not show whether or not this was true of the anterior edge. The third premolar is removed from the canine by a wide distance. It agrees very closely, both in size and shape with the corresponding tooth of *H. robustus*. The superior sectorial has a shorter and blunter median lobe, a lower heel, and a more prominent anterior secondary lobe. The tubercular molar is two-rooted and very similar to that of *H. robustus*. The lower incisors show quite a variation in size. The first is small and compressed; the second considerably larger; the third is almost as stout as the lower canine. The third lower premolar is no larger than that of *H. robustus*. Its posterior lobe is less prominent, the anterior one has disappeared entirely. The fourth premolar is similar in every respect to Leidy's figure of the type. The median lobe is shorter, and the secondary lobe less prominent than in *H. robustus*. The lower sectorial has only a very slight heel, but the postero-internal cusp is distinctly present.

MEASUREMENTS OF SKULL.

	Smaller Specimen. mm.	Larger Specimen. mm.
Condyles to premaxillary border	240	
Occiput to premaxillary border	265	
Breadth across post-orbital processes	86	

	Smaller Specimen. mm.	Larger Specimen. mm.
Breadth across post-orbital constriction...........	38	
Premaxillary border to line of superior canines....	23	
Premaxillary border to line of front of orbits......	101	
Height of occiput above base of condyles.........	85	80
Occiput to line of post-orbital process...........	127	
Breadth of zygomata..........................	150*	
Breadth of occiput at constriction................	45	60
Breadth ocross occipital condyles................	52	
Breadth of foramen magnum....................	23	
Height of foramen magnum....................	17	
Length from condyles to anterior border of posterior nares................................	133	
Breadth across posterior margin of upper sectorials	92	
Posterior maxillary border to anterior margin of glenoid cavity..............................	45	
Greatest diameter of orbit......................	45*	
Transverse diameter of nares....................	26	
Height of nares.............................	35	
Greatest diameter of infraorbital foramen.........	15	
Breadth of external auditory meatus.............	4	5
Length of superior dental series, including canines	92	
Length of superior canine.....................		
Longitudinal diameter of base of canine..........		30
Transverse diameter of canine...................	14	
Breadth of incisor series........................	39	
Breadth of third premolar......................	14	
Breadth of sectorial premolar..................	25	
Breadth of tubercular molar....................	13	
Length of crown of upper first incisor............	11	
Transverse diameter at base..................	6	
Length of crown of third incisor................	14	
Longitudinal diameter of base..................	10	
Mandible, length from condyle to incisor.........	167	
" depth of flange from base of canine.....	67	78
" depth of symphysis..................	63	72
" greatest breadth of chin.............	42	45
" depth of ramus at base of third premolar	32	
" depth of ramus at coronoid process....	50	
Lower canine, longitudinal diameter..............	10	
Diastema, length of..........................	41	48

*Approximated.

	Smaller Specimen. mm.	Larger Specimen. mm.
Third premolar, breadth of crown	9	
Fourth premolar, breadth of crown	16	
Sectorial molar, breadth of crown	21	
Height of condyle above angle	27	
Length of condyle	39	

VERTEBRAL COLUMN.

The cervical vertebræ are represented by the atlas, axis, and the third, in the larger animal, and the seventh in the smaller one. The atlas has a strong, rounded neural arch, but the ventral arch is comparatively narrow and light. The rudimentary spine is bifurcate. The transverse processes are too badly broken to be determined. Their base is perforated posteriorly by the vertebrarterial canal, which opens on the inferior surface further back than in the lion. Here the vertebral artery ran for a short distance in a deep grove and again passed under an osseous bridge forming an atlantar foramen, as in *Dinictis* and the *Viverridæ*. On the upper surface it is again open for a short distance before passing under the anterior root of the neural arch. The internal openings are much further back from the anterior margin than in the lion. The centrum of the axis is much compressed vertically. The inferior surface is divided by a sharp median ridge, flanked by concavities. The neural arch and spine are lost. The third cervical has no spine, but a neural prominence, which is bifurcate posteriorly. The vertebrarterial foramen is very small, the anterior zygapophyses depressed, the parapophyses directed more backward than in the lion. The seventh has well marked rib-facets, and a slender spine. The transverse processes are broken.

Of the **dorsal vertebræ**, nine are preserved in the smaller animal, many of which have lost their spines and transverse processes. The centra are proportionally broader than in the lion, and are produced into rounded lateral ridges which extend backward from the base of the transverse processes, and end in the capitular facets. These facets are plainly marked, and in most instances distinct from the intervertebral surface. In the first the transverse process is proportionally longer than in the lion, the tubercular facet is concave and looks downward; in the seventh the facet is also concave, but directed more outward; in the eleventh it is concave, elongate and directed forward as well as outward, and there is a deep fossa just back of it. The spinous processes are long and slender, and instead of the sharp anterior borders found in the lion, they have a median grove at the base and are rounded near the extremity.

Seven fairly complete **lumbar vertebræ** are preserved in the two specimens, two of which are duplicates. The processes diverge less from the median line than in the lion, the postzygapophyses are short and stout, and lie close together with only a narrow notch between them. Their articular facets are directed more outward than downward. The anterior zygapophyses become somewhat longer toward the caudal end of the series. The articulating facets are nearly opposed to each other and are deeply concave vertically. The anterior margins of the lamina are less deeply concave than in the lion. The neural spines are broad, rising far back between the posterior zygapophyses and extending forward to the anterior border of the arch. The metapophyses are fairly well developed, and the anapophyses are as prominent as in the lion.

The sacrum is composed of three vertebræ. It is fourteen millimeters longer than that of the lion, but narrower at the anterior end, and the transverse processes are shorter and stouter. The centra are so completely ossified that all traces of their union have disappeared. The anterior zygapophyses are long and stout with their opposed faces concave. Those of the second and third vertebræ are also prominent. The first spine is twice as strong as that in the lion, and is directed backward.

The caudals are not only much longer and stronger than those of the lion, but the processes are better developed and a larger number have a complete neural arch. The anterior eleven are preserved in the small specimen. The first caudal has a neural spine as strong as the first sacral in the lion, and the third has a distinct rudiment. The zygapophyses are articulated as far back as between the eighth and ninth. The neural canal is present in the tenth. The transverse processes are strong in the first and second, become changed into a broad flat expansion in the sixth, which in turn gives place to an anterior and a posterior lateral expansion in the tenth. In the fifth and following vertebræ the posterior intervertebral notch is less deep, and, a short distance in front of the margin, there appears on each side a small foramen perforating the pedicle. Doubtless this foramen was for the passage of the nerve and vessels which, from the flexibility of the tail might otherwise have been subject to compression.

In the restoration of the remainder of the tail, I have figured the same number of vertebræ as in the lion, giving them as nearly as could be determined, proportions corresponding to the anterior ones. From the fact that the arch extends further back in this animal than in the lion, it would seem certain that there could not be a less number of vertebræ, and it is very probable that there were more.

MEASUREMENTS OF VERTEBRÆ.

Atlas, breadth across the anterior articulating surfaces	55
" antero-posterior breadth of arch	26
" height of neural opening	31
Axis, greatest length	64
" breadth across anterior articulating surface	49
Seventh cervical, width of posterior and of centrum	38
" " length of centrum	30
" " height of neural opening	12
" " width of neural opening	24
Seventh dorsal, expansion of transverse processes	62
" " width of neural opening	18
" " height of neural opening	9
" " length of centrum	28
Second lumbar, length of centrum	41
" " width of centrum	29
" " width of neural opening	19
" " height of neural opening	9
Sacrum, length	104
" width of anterior end	63
" least diameter of first transverse process	35
First caudal, length of centrum	32
" " width	28
Entire length of first eleven caudal vertebræ	440
Width of eleventh caudal	28
Length of same	43

Pectoral girdle. Only the lower half of the scapula is present, and one of the sternal bones. The glenoid surface of the scapula is rounder in outline than in *H. robustus* and much more so than in *Felis concolor*. The anterior part of the ventral surface is more deeply concave, and the coracoid process, like that of *H. robustus*, curves inwardly, more strongly than in the lion. The neck is less constricted and the spine is much nearer the axillary border. At the origin of the teres minor muscle the border is thickened and massive, presenting a posterior face which stands at right angles to the subscapular surface. The same is true in *H. robustus*, but in a less marked degree.

Fore Leg. The **humerus** is described from two bones, one of which lacks the head, the other the distal third. The length is determined by comparison. The great tuberosity is partly broken away, but evidently projected somewhat beyond the head. The bicipital groove is deep and narrow; the inner surface of the shaft is concave as far as to the lower extremity of the deltoid ridge. The lesser tuberosity is separated from the articulating surface by a deep groove, continuous with the concavity on the posterior surface of the shaft reaching almost to its middle. The anterior surface of the shaft is laterally compressed with the roughened, deltoid ridge unusually prominent. The supinator ridge rises above the middle of the shaft, curves outward and forward, forming a marked concavity on the antero-external surface of the shaft, and giving an unusual breadth to the distal end of the bone. The supracondyloid foramen is well rounded. The inner condyle is prominent and roughened. Back of the inner condyle and near the trochlear surface is a broad groove, a character which seems to be common in the species of *Hoplophoneus*, but which is lacking in *Dinictis*. There is only a trace of it in the species of *Felis* examined and in *Machærodus crassidens* Cragin (Williston).* The olecranal fossa is broad transversely, but shallow. The coronoid fossa extends as far outward as the exterior border of the capitellum, much as in *M. crassidens*. It is quite as deep as the olecranal fossa.

The ulna is a strong bone, rounded and convex on the posterior surface, slightly concave anteriorly and grooved on the external surface throughout the greater part of its length. The olecranon is stout, bent inward, and expanded at its roughened extremity. The great sigmoid cavity is narrowed antero-posteriorly, but broad from side to side; the beak is thin, but prominent on the outer border. The exterior border of the lesser sigmoid cavity, is not so prominent as in recent forms. On the interior border of the anterior surface, just in front of the great sigmoid cavity, is a roughened surface for muscular attachment, common to *Dinictis* but not found in the recent cats. The styloid process is short and stout, and separated from the round articulating facet for the radius by a shallow notch.

The radius. represented by the proximal half of one bone. The head is quite concave, as in *H. robustus*, and bears on its anterior margin a prominent protuberance exterior to which there is a notch separating it from the anterior prominence of the articulating surface for the ulna.

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*Kansas University Quarterly. Vol. III. No. 3.

Of the **front foot**, only the unciform and the proximal half of the fifth metacarpal, and the distal end of the second are present. The unciform, as seen from the dorsal side, is roughly a triangle in which the anterior border is slightly concave and the posterior angle rounded. The surface for the cuneiform is extended downward posteriorly, and is not bounded below by a continuous groove as in the lion. The surface for the scapho-lunar is strongly convex throughout, curving around the posterior end. The surface for articulation with the os magnum extends downward to the lower border of the anterior face, and is continuous with the scapho-lunar surface posteriorly. The proximal end of the fifth metacarpal is less expanded than in the lion, its external tuberosity is less prominent, and is not separated from the articulating surface by a groove. Its posterior end is rounded, and articulates with about half of the anterior surface of the unciform. The distal end of the second metacarpal is proportionally broad and strong. A first phalanx is short, stout, and strongly curved; the protuberances of the proximal end are shorter and the inferior surface is less deeply notched than in the lion.

MEASUREMENTS OF THE FORE LEG.

	mm.
Scapula, length of glenoid cavity	39
" from base of spine to posterior border	13
" from base of spine to anterior border	29
" thickness of posterior border	14
Humerus, length	236*
" diameter of head and great trochanter	65*
" greatest diameter of distal end	74
Ulna, length	242
" end of olecranon to beak	53
" olecranon to coronoid process	79
Radius, approximate length	177
" diameter of head	32
Unciform, length	24
" breadth	22
Second metacarpal, breadth of distal end	19
Phalanx, length	35
" breadth of proximal end	20

*Approximate.

Pelvic girdle. Between the two specimens the pelvis is almost complete. Compared with that of the lion it is less constricted at the acetabula. The iliac rami of the ischia are straighter, and less divergent posteriorly. The ischiatic rami of the pubis are stronger both relatively and actually. The anterior end of the ilium stands nearly vertical and the crest is curved strongly outward. The gluteal surface is divided longitudinally by a strong ridge, extending from the acetabulum to the crest. This character is even more prominent in *H. robustus* and is found also in *Dinictis*. Above it there is an elongated fossa, below a slightly concave surface. The muscular roughening for the *rectus femoris* is more prominent and extends further forward than in the lion. A sharp line separates the gluteal surfaces. The border below the acetabulum is thin and sharp. The ilec-pectineal eminence is scarcely noticeable. The iliac surface of the ilium is convex, the articulation with the sacrum close and admitting of little motion. The iliac ramus of the ischium is thicker, narrower and more rounded on the superior border than in the lion. The pubic ramus is thin and flat. The spine of the ischium is situated near the middle of the iliac ramus. The pubic symphysis is firmly co-ossified. The ischiatic ramus of the pubis is concave above, and is nearly as strong at posterior, as at the anterior end.

Hind Leg. The femur has more of the characters of the recent cats than has that of *H. robustus*. The shaft is straighter, the third trochanter less prominent, and not connected with the great trochanter by a sharp ridge. The head is directed less forward, and the patellar surface forms an anterior prominence. The great trochanter projects only slightly beyond the head. A marked groove extends downward from the pit for the ligamentum teres. On the outside of the shaft there is a prominent, roughened protuberance extending thirty-five millimeters above the condyle, which is not present in *H. robustus* or in the recent cats. The **patella** is rather long and narrow and is irregularly rounded on the anterior surface. The articulating face covers two-thirds of the posterior surface. It is concave vertically and convex from side to side.

The **tibia** is a strong bone, slightly curved forward, and laterally compressed. The anterior border is sharp and has a marked protuberance about midway of its length, where it makes a sharp curve inward. As seen from behind both borders are concave, that of the inner being slightly more marked. There is a distinct articulating facet for the proximal end of the fibula. The internal malleolus is strong and projects somewhat inward, ending in an angle toward the posterior border. The groove for the tendons of the *tibialis anticus* and the *flexor longus digitorum* is broad and shallow and is directed more obliquely forward than in the lion.

The astragular surface is, as in *H. robustus*, less deeply grooved and is placed more obliquely to the shaft than in recent cats. The antero-external border extends but little below the articular surface.

Of the **fibula** only the extremities remain. These indicate a stronger bone than that of *H. robustus*. The proximal end articulates with the tibia by a well-marked oval facet. The outer surface is roughened for ligamentary or muscular attachments. There is only a trace of a groove on the outer surface, instead of the deep concavity found in *H. robustus*. The distal end, as in the last-named species, is very unlike that of recent cats. It is narrower but thicker than the head and is roughly triangular in section. The internal surface bears at its lower anterior border a convex articulating facet for the astragalus which curves half way around the lower end. The posterior surface stands at a right angle with the last, and is almost as broad. There is a broad, shallow, peroneal groove at the inner side of the posterior tuberosity of the malleolus. The tendinous depression on the external surface is less marked than in the recent cats.

The foot is short and weak in the metatarsal region, as is true of all the early cats. The calcaneum is not more than two-thirds as large proportionally as that of the lion, and does not extend distally as far as the astragalus. The external process extends backward beyond the anterior margin of the superior articulating surface. Back of this and near the upper surface is a deep fossa. The sustentaculum is situated near the anterior border, opposite the external process and has a broad, shallow groove at its base. The anterior surface is quite concave; the articulating surface for the astragalus does not turn inward posteriorly, as in the recent cats.

The astragalus has a short, constricted neck, and a well-rounded head, but is markedly compressed vertically. The superior surface is but slightly concave laterally, corresponding to the slight convexity of the tibial surface, and does not extend to the posterior border as in recent cats. This does not permit of as great an angle between the foot and the tibia, and bears evidence of more plantigrade affinities. The head is less deflected from the antero-posterior axis than in the lion. There is no articulation with the cuboid, as is the case in *Dinictis*. The facet for articulation with the sustentaculum is long and deeply notched posteriorly. The posterior end is grooved for the tendon of the *flexor longus hallucis*. The groove between the inferior articulating surfaces is straight, and ends abruptly in a deep fossa.

The cuboid is narrower in proportion to its length than that of the lion. The posterior and anterior surfaces converge outwardly, making the external surface shorter than the internal. The posterior surface is convex; the groove for the tendon of the *peronæus longus* and the ligamentary prominence posterior to it extend across the entire inferior surface. The facet for the navicular is long and curved; that for the ecto-cuneiform is semilunar in outline. The anterior surface is concave to receive the fourth and fifth metatarsals.

The metatarsals are about one-half the length of those of the lion. Only the fourth, fifth, and half of the second are preserved. The fifth articulates with the cuboid by about half of its posterior end which is sloping and extends but little back of the facet. The tuberosity projects prominently outward and backward. The fourth overlaps the fifth and in turn is overlapped by the third much as in the recent cats. The shafts are sub-triangular in section and are strongly curved near the distal end. The second is about as strong as the fourth; its proximal end is laterally compressed. The superior surface is symmetrically rounded, instead of sloping outward as in recent forms and its inferior process does not project under the third. The exterior articulating surface and the ligamentary attachments indicate a fairly well-developed first toe. The proximal series of phlanges are short and stout. The second series are markedly concave above, indicating a perfectly retractile claw.

MEASUREMENTS OF PELVIS AND HIND LEG.

	Smaller. mm.	Larger. mm.
Pelvis, length	250	
" breadth between actetabula	66	
Ilium, breadth in front of actetabulum	37	36
" thickness above actetabulum	24	29
" greatest breadth	55	
" greatest diamemeter of actetabulum		42
Ischium, diameter back of actetabulum		27
Femur, length		285
" breadth of head and great trochanter	70	76
" diameter of head	34	35
" distance from head to lower margin of lesser trochanter	67	76
" breadth of condyles	59	61
Patella, length	47	
" width	31	

	Smaller. mm.	Larger. mm.
Tibia, length		234
" transverse diameter of proximal end	60	62
" transverse diameter of distal end	37	40
Fibula, approximate length	210	
" width of proximal end	28	
" width of distal end	27	
" thickness of distal end	16	
Calcaneum, length	68	
" width across processes	38	
Astragalus, length	46	48
" greatest width	33	37
Second metatarsal, length	60*	
" " width proximal end	10	
" " vertical diameter proximal end	19	
" " width of distal end	16	
Fourth metatarsal, length	61	
" " width of proximal end	17	
" " width of distal end	17	
Fifth metatarsal, length	55	
" " width of proximal end	18	
" " width of distal end	14	

SUMMARY.

In short, the characters of *Hoplophoneus occidentalis*, as shown from these specimens, are: Size similar to that of the *Felis onca* but stouter bodied and limbs shorter in proportion; skull large in proportion to body, deep but narrow, brain case small, sagittal crest concave, occiput strong and overhanging, zygomatic processes drooping, superior canine trenchant only at point, and inferior sectorial with a rudimentary heel; atlas with an atlantar foramen; zygopophysis firmly interlocked and but little diverging from the median line; sacrum long but narrow at the anterior end; caudal vertebræ stronger with processes better developed and neural canal extending to the eleventh; scapula with neck little constricted, glenoid cavity deep and rounder than in recent cats, the posterior border at the origin of the *teres minor* thickened and massive; humerus with deltoid and supinator ridges unusually developed, lesser tuberosity separated from the head by deep groove, the internal epicondyle unusually prominent and separated from the trochlea by a broad notch; ulna with the oberanon two-ninths the length of

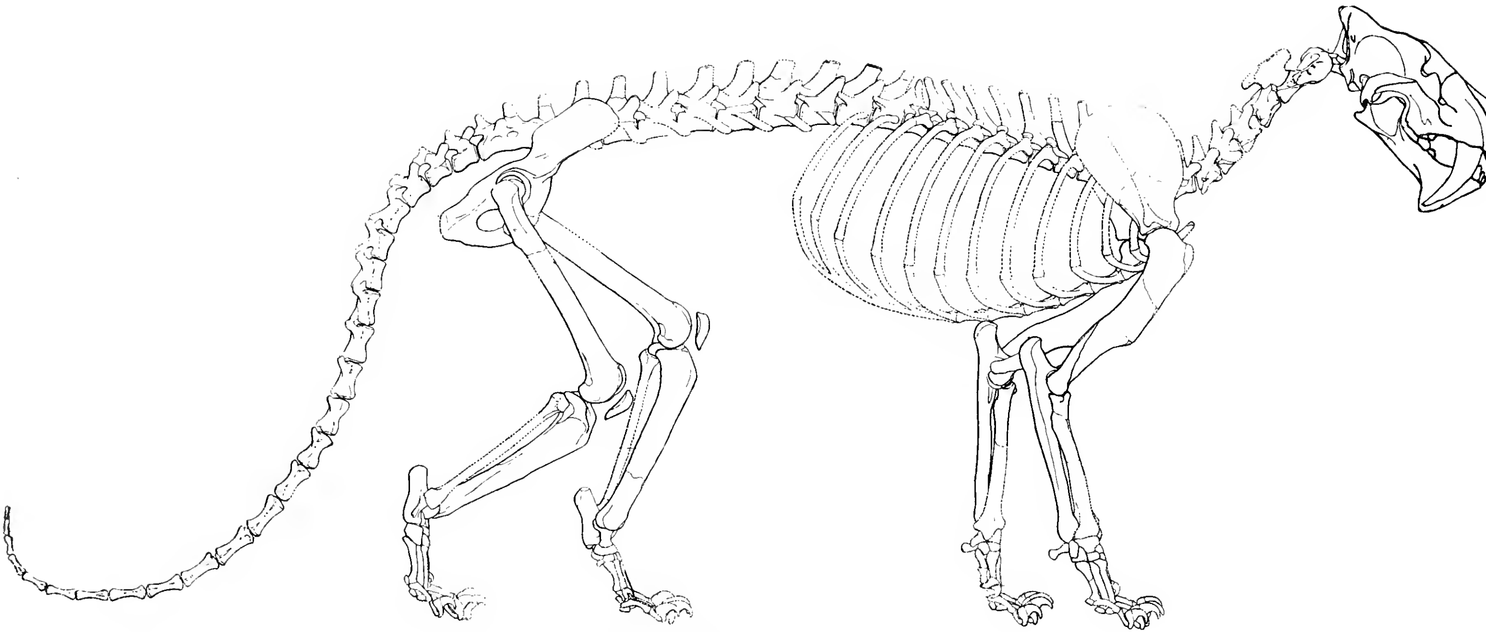
*Estimated.

the entire bone and the great-sigmoid notch narrow antero-posteriorly; pelvis articulating closely with the sacrum, ilium with a strong median dividing the gluteal surface into a superior and an inferior concavity, and the pubic ramus of the ischium unusually strong; femur with shaft nearly straight, patellar surface forming an anterior prominence, and a well-marked tubercle above the external condyle as in Felidæ; fibula not grooved on the external surface of the head, thick and articulating loosely at the distal end and having a strong posterior tubercle; astragalus only slightly grooved for the tibial articulation and the tibial surface does not extend to the posterior border, an evidence of planitigrade affinities; calcaneum short and having the sustentaculum near the anterior end; metatarsals short and curved; claws distinctly retractile.

DINICTIS PAUCIDENS.

In a recent paper on the extinct Felidæ of North America* Dr. Adams states summarily in a note that *D. paucidens*† is probably a synonym of *D. fortis*.‡ Such a statement would indicate either that Dr. Adams has not gone far enough into the description of this form to recognize the characters upon which it is based, or that *D. fortis* is a sufficiently generalized type to include whatever it may be found convenient to place under it. In the latter case, *D. felina*, the type of the genus, would fall a much easier victim, since *D. fortis* in becoming synonymous with *D. bombifrons*‡, has so far lost its distinctive characters that its dentition is essentially the same as that of the generic type, leaving as the only specific character a difference in size. However, trusting that this is due merely to oversight, I repeat here that the distinctive characters of *D. paucidens* are: "The absence of a second lower molar, the slenderness of the base, and the concave outer border of the upper sectorial as seen from above, and the presence of but two incisors in the mandible." These, together with the very "slight development of the postero-internal cusp of the lower carnassial," described as well developed in *D. bombifrons* (syn. *D. fortis*) and the "proportionate length and slenderness of the fore-arm," are differences sufficient to satisfy the most exacting.

*American Journal of Science, June, 1896.
†Riggs, Kansas University Quarterly, April, 1896
‡Adams, American Naturalist, June, 1895.



Hoplophoneus occidentalis Leidy.

From the White River Miocene, South Dakota Restored by E S. Riggs (Reduced to one-seventh)

On the Dermal Covering of Hesperornis.

BY S. W. WILLISTON.

(With Plate II.)

A specimen of *Hesperornis*, collected in western Kansas the past year by Mr. H. T. Martin and now in the University Museum, is of especial interest from the information it affords of the dermal covering of this Cretaceous toothed bird.

The specimen, which is in excellent preservation, lies upon a chalk slab, with the head doubled partly under the pelvis. Some six or eight vertebræ, together with the humeri and coracoids and many of the ribs are wanting; otherwise the specimen seems perfect. The size is distinctively less than that of *H. regalis*, and it does not seem to be due to immaturity. Possibly the species is identical with *H. gracilis*, which has been only imperfectly described.

The photographic illustration given in Plate II was taken from the fragment removed from the slab over the right tarso-metatarsal, the surface of the slab itself being less clearly, though more fully marked. I have sketched in the bone to show the relative size and position.

The podotheca is seen to be scutellate in front. The structure is shown so clearly in the photograph that I need not enter into a fuller description. The scutes are all smooth, not imbricated, and distinctly separated from each other. They are a little longer from side to side below, though not much. I count twenty-six on the slab, and to the back part of the bone, while impressions of the feathers will be seen on the opposite side.

These feathers were evidently long, reaching nearly to the phalangeal articulation, and are clearly semiplumulaceous in character, the pennaceous shaft of considerable size, the vanes long and wavy. The shaft of one feather is seen in the illustration lying close to the outline of the bone, and is of considerable size. I doubt not that the feathers throughout were of this character, or wholly plumulaceous. I find distinct impressions of the wavy vanes at the back of the head and elsewhere, but in no case is there the impression of a true feather, as I think would surely be the case had the bird possessed them.

This plumulaceous character of the plumage is not unexpected. Although Marsh nowhere mentions the plumage in his work, I know that he personally had the opinion that it was of a downy character. That the feathers of the tarsus should extend to the feet in a wading bird seems surprising, but there can be no other interpretation of the specimen.

DERMAL COVERING OF HESPERORNIS.
Enlarged about one-fourth.

The Duty of the Scholar in Politics.

BY FRANK HEYWOOD HODDER.

[Phi Beta Kappa Address, delivered at the University of Kansas, June 8, 1896.]

The duty of the scholar in politics has been the subject of so many addresses upon occasions of this character that it is difficult to say anything new respecting it. It is, however, suggested both by the occasion and by the direction of my own studies. Mr. Disraeli is reported to have once replied to an opponent in Parliament: "The honorable gentleman has said things both true and new but the things true are not new and the things new are not true." It is, after all, the things true which are not new that are important. Especially is this the case with respect to duty, whatever its direction. It rarely happens that we do not know our duty but often that, knowing it, we fail in the doing.

By the scholar, in this connection, I do not mean the specialist but rather the man of education and independence, the man who is well informed upon all important topics of current interest and who does his own thinking respecting them. This definition does not include all graduates of colleges and universities and it does include many who never had the advantage of college training. The duty in politics of the man of education and independence is then the subject. The greater the education, the greater the influence he may exert and the greater the obligation to exert it. Especially great is the obligation in the case of the young men and young women educated at the expense of the state. Upon them rests the duty of using their influence for its welfare.

But I do not intend to range at large over the whole subject. I propose instead to emphasize one particular duty—namely the duty of the scholar to use his influence for the maintenance of international peace. The discussion of this particular duty is especially appropriate to the occasion by reason of the fact that it is totally disconnected from all questions of party politics. It is a duty pre-eminently of the scholar as a man governed by reason, rather than by passion and prejudice. Recent events seem to present certain dangers to our national peace, which I shall consider in order. They are:

1st, misconstruction of the Monroe doctrine;

2d, a rising war spirit among the people; and

3d, enormous expenditures for war purposes.

First, the Monroe doctrine. I venture the assertion that the recent unwarranted construction of that doctrine is contrary to the teaching of the founders of the republic, a perversion of the true meaning of the original declaration, an encroachment upon the rights of foreign states and a menace to our peace and safety.

It is contrary to the teaching of the founders which was non-interference with the affairs of foreign nations and peace and friendship with all mankind. Three men may be called pre-eminently the founders of the republic. They were Washington, Madison and Hamilton, to whom more than to any others were due respectively the success of the revolution, the framing of the constitution and the establishment of government. The combined wisdom of these men was embodied in the farewell address issued by Washington upon his retirement from the presidency, a worthy guide to the American people for all time. In that address we find this advice:*

"Observe good faith and justice toward all nations. Cultivate peace and harmony with all..... It will be worthy of a free, enlightened and, at no distant period, great nation, to give to mankind the magnanimous and too novel example of a people guided by an exalted justice and benevolence..... The experiment, at least, is recommended by every sentiment that ennobles human nature."

"The great rule of conduct for us in regard to foreign nations is to have with them as little political connection as possible..... Europe has a set of primary interests which to us have none or a very remote relation. Hence she must be engaged in frequent controversies, the causes of which are essentially foreign to our concerns..... Our detached and distant situation invites and enables us to pursue a different course..... Why forego the advantages of so peculiar a situation? Why quit one's own to stand on foreign ground? Why entangle our peace and prosperity in the toils of European ambition, rivalry, interest, humor, or caprice?"

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*See "Statesman's Manual" for quotations from Presidential messages and addresses. Richardson's "Messages and Papers of the Presidents," now publishing by the Government, will supersede the earlier collection.

†See Wharton's "Digest of International Law," Vol. I, sects. 45 and 57, for opinions cited above.

All parties at that time agreed in counseling peace.† Jefferson, the father of democracy, expressed the same sentiment. In an official letter in 1793, while Secretary of State, he said:

"We love and value peace; we know its blessings from experience. We abhor the follies of war and are not untried in its distresses and calamities. Not meddling with the affairs of other nations, we hope that our distance will leave us free in the example and indulgence of peace with the world."

Again in writing Monroe in 1823, advising the issue of this very declaration, he said:

"I have ever deemed it fundamental for the United States never to take an active part in the quarrels of Europe. Their political interests are entirely distinct from ours. Their mutual jealousies, their balance of power, their complicated alliances, their forces and principles of government are all foreign to us. They are nations of eternal war. All their energies are expended in the destruction of the labor, property and lives of their people. On our part never had a people so favorable a chance of trying the opposite system, of peace and fraternity with all mankind and a direction of all our means and faculties to the purposes of improvement instead of destruction."

And Monroe in the very message, now made the excuse for so much warlike demonstration, took pains to repeat this doctrine of non-intervention:

"In the wars of European powers, in matters relating to themselves we have never taken part nor does it comport with our policy to do so.... With the existing colonies or dependencies of any European power we have not interfered and shall not interfere..... Our policy with regard to Europe is not to interfere with the internal concerns of any of its powers."

Statements of this character were frequently repeated by later statesmen. Van Buren in official letters, while Secretary of State, within five years of the issue of the Monroe declaration, said:

"It is the ancient and well settled policy of this government not to interfere with the internal concerns of any foreign country."

"An invariable and strict neutrality and an entire abstinence from all interference with the concerns of other nations are cardinal traits of the foreign policy of this government. The obligatory character of this policy is regarded with a degree of reverence and submission but little if anything short of that which is entertained for the Constitution itself."

Mr. Seward in 1863, at the very time he was protesting against the French occupation of Mexico, the only violation of the true Monroe doctrine ever attempted, wrote Mr. Adams:

"In regard to our foreign relations, the conviction has universally obtained that our true national policy is one of self reliance and self conduct in our domestic affairs, with *absolute non-interference* with those of other countries."

Again in 1866 Mr. Seward* in advising against interference in behalf of Chili said:

"If there is any one characteristic of the United States which is more marked than any other, it is that they have from the time of Washington adhered to the principle of non-intervention and have perseveringly declined to seek or contract entangling alliances, *even with the most friendly states.*"

Quotations of this character might be multiplied indefinitely but enough have been given to prove that the teaching of the founders from Washington to Monroe and John Quincy Adams was non-intervention and peace. Their authority cannot rightfully be invoked in support of any other policy.

Recent construction of the Monroe doctrine is a perversion of the true meaning of the original declaration. I venture this assertion without fear of contradiction by any special student of international law or of our political history. The Monroe doctrine consists of two parts corresponding to the two causes which occasioned its issue. John Quincy Adams wrote the first part, Jefferson the second, and Monroe embodied both in his annual messages for 1823 and 24. Adams, Jefferson and Monroe may therefore properly be considered its joint authors.†

The first part respects colonization. America is not subject to future European colonization. In 1821 the Czar Alexander of Russia issued a proclamation claiming the western coast of North America as far south as the 51st parallel. That territory was then claimed both by Great Britain and the United States. The proclamation of the Czar was accepted by both as evidence of an intention to establish a Russian colony in America. It is difficult for us to-day to reproduce in imagination the situation of the United States at that time. Our territory then as now extended from the Atlantic to the Pacific but that portion between the Alleghanies and the Mississippi was still sparsely settled and the vast expanse between the Mississippi and the Pacific, with the exception of

*"Works," Vol. 5, pp. 441-5.

†It is well known that Madison was consulted and advised the issue of the declaration. He, however, merely seconded Jefferson's suggestions.

Louisiana, Arkansas and Missouri, was absolutely unoccupied and almost unexplored. The territory of Mexico subsequently acquired by us was in the same condition. It would not then have been difficult for Russia to have planted a colony either in or near this territory, upon the plea that it was unoccupied. To guard against this danger President Monroe, acting upon the advice of Adams, issued this declaration:

"The American continents, by the free and independent condition which they have assumed and maintain, are henceforth not to be considered as subjects for future colonization by any European powers. . . . With their existing colonies or dependencies we have not interfered and shall not interfere."

There was not the slightest intention of assuming a protectorate over other American states for the purpose of guarding their territory from European colonization. That such was the case is absolutely proved by the language used by Mr. Adams two years later in a special message to the Senate on the subject of a Congress of American states.

"An agreement," he said, "between the parties represented at the meeting that each will guard, by its own means, against the establishment of any future European colony within its borders, may be found advisable. This was announced to the world, more than two years ago, by my predecessor, as a principle resulting from the emancipation of both the American continents."

This statement Mr. Schouler* observes is remarkable as an exposition of the Monroe doctrine from the pen of the one most competent to make it, that is from the pen of the one who originally wrote it—in effect that European exclusion from this hemisphere was to be the work not of the United States, acting as the champion of the two Americas, but of each American republic as the protector of its own rights. Mr. Webster speaking at the same time expressed the same opinion.†

"It was highly desirable to us," he said, "that new states should settle it as a part of their policy not to allow colonization within their respective territories. We did not need their aid to assist us in maintaining such a course for ourselves, but we had an interest in their assertion and support of the principle as applied to their own territories."

The Russian claim was immediately abandoned in treaties with both Great Britain and the United States. Since that time there

*"History of the United States," Vol. 3, p. 362.

†"Works," Vol. 3, pp. 200-207.

has not been the faintest suggestion of an intention on the part of any European power to establish any new colony upon either of the American continents. The rapid growth of American populations has practically resulted in the actual occupation of every part of both continents. An occasion then for an application of this part of the Monroe doctrine has not presented itself and cannot present itself.

The second part of Monroe's declaration respects intervention. It consists of two distinct propositions. European interference with American states for the purpose of subverting their governments cannot be permitted and the extension to America of the European political system cannot be permitted. At the close of the Napoleonic wars in 1815 Russia, Austria and Prussia united in the so-called Holy Alliance. Their avowed object was the maintenance of the Christian religion. Their real purpose was the preservation of their political system of absolute monarchy, based upon the divine right of kings, by a pledge of mutual assistance in case of popular insurrection. The treaty between them was offered for signature to every power in Europe except the Sultan and the Pope. All acceded to it except Great Britain whose foreign minister replied that the principles of the Alliance were inconsistent with those of the British constitution. In 1821 the allies sent an Austrian army into Italy in order to prevent the adoption of a free constitution in Naples. And in 1823 they sent a French army into Spain to suppress popular insurrection there, and re-establish the despotism of Ferdinand VII. It was then proposed that the allies call a congress to arrange for the subjugation of Spain's revolted colonies in America and the re-establishment of Spanish authority over them. Information of this design reached the United States through Great Britain. In opposition to it Monroe, acting on the advice of Jefferson, issued the second part of his famous declaration:

"With the governments who have declared their independence we could not view any interposition by any European power in any other light than as the manifestation of an unfriendly disposition toward the United States.....The political system of the allied powers is essentially different from that of America.....We should consider any attempt on their part to extend their system to any portion of this hemisphere as dangerous to our peace and safety. ....It is impossible that the allies should extend their political system to any portion of either continent without endangering our peace and happiness ...It is equally impossible, therefore, that we should behold such interposition in any form with indifference." In other words, European states could not be permitted to overthrow any American government for the purpose of establishing upon its ruins an absolute monarchy based upon the divine right of kings. There was not a word respecting intervention for any other purpose.

Monroe's warning was sufficient to induce the Holy Alliance to abandon their plan of interfering in American affairs. Since that time there has been but a single violation of this part of Monroe's declaration. During our civil war the unscrupulous government of Napoleon III invaded Mexico, overthrew her government and established in its place an Empire, sustained by French arms. Immediately upon the close of our war, Secretary Seward informed France that her troops must be withdrawn. They were withdrawn and the Empire fell. Since that time there has not been the faintest suggestion of an intention upon the part of any European power to interfere in the affairs of any American state for the purpose of overthrowing its government and establishing monarchy in its place. Constitutional government has been established in every European state except Russia and the European political system of which Monroe wrote has ceased to exist. An occasion, therefore, for a second application of this part of the Monroe doctrine has not presented itself.

Briefly stated the Monroe doctrine opposed new European colonies, subjugation of American states by European powers and the system of the Holy Alliance. New colonization has never been attempted, subjugation has been tried once and failed utterly, the system of the Holy Alliance has been dead for half a century. Any statement that goes beyond these three points is unwarranted by the original declaration. Monroe's declaration was a protest against new colonies. It is now applied to colonies that antedate our national existence. Monroe's declaration was a protest against intervention. It is now made the basis for intervention. Monroe's declaration was a protest against absolutism. It is now applied to a government which, despite monarchical forms, is more thoroughly democratic than our own. Such construction is a perversion of the true meaning of the original declaration.

Let us now inquire into the origin of this misconstruction of the Monroe doctrine. With the defeat of John Quincy Adams and the election of Andrew Jackson in 1828, the era of statesman presidents came to an end and an era of military favorites and politicians began. At the same time we abandoned the founders' policy of peace and friendship with all mankind and assumed an attitude of defiance toward foreign nations. Slavery wanted more territory for its expansion and the South needed more slaves in order to keep abreast of the rapidly growing North. Longing eyes were turned toward Texas and its acquisition became the settled policy of the slave power. Jackson first tried to buy Texas but Mexico refused to sell. "To do so," Santa Anna replied, "would be to sign the death warrant of my country, for the United States would take one province after another until none remained." Jackson then sent Houston to Texas, at that time the territory of a friendly state with which we were at peace, with the understanding that he should colonize it with American citizens, foment revolution and, when a favorable opportunity presented itself, apply for admission to the United States. This conspiracy required time for its development but was carried out according to the program. The revolution came, Texas declared her independence of Mexico and applied for annexation to the United States. A treaty for the purpose failing of ratification in the Senate, President Tyler secured the passage of a joint resolution for the admission of Texas as a State in the Union.

Such was the situation when Polk became President of the United States on the 4th of March, 1845. In his inaugural address the new President said:

"I regard the question of annexation as belonging exclusively to the United States and Texas. Foreign powers do not seem to appreciate the true character of our government. Our union is a confederation of independent states, whose policy is peace with each other and all the world. To enlarge its limits, is to extend the dominion of peace over additional territories and increasing millions."

In his first annual message to Congress, again referring to Texas, he said:

"The United States cannot in silence permit any European interference on the North American continent; and should any such interference be attempted, will be ready to resist it at any and all hazards.....The nations of America are equally sovereign and independent with those of Europe. They possess the same rights, independent of all foreign interposition, to make war, to conclude peace and to regulate their internal affairs. The people of the United States cannot, therefore, view with indifference attempts of European powers to interfere with the independent action of nations on this continent.....We must ever maintain the principle that the people of this continent alone have the right to decide their own destiny. Should any portion of them, constituting an independent state, propose to unite themselves with our confederacy, this will be a question for them and us to determine, without any foreign interposition."

This is the new version of Monroe's declaration. Monroe had protested against European interference for the purpose of destroying independent states and Polk extended the protest to any interference whatever.

Within the month the annexation of Texas was completed. But the South was not satisfied. She next coveted the rich soil of California. Again Mexico was asked to sell. Again she refused and Polk precipitated a war to compel her to do so. Mexico was prostrated and compelled to part with California for fifteen million dollars. This was Polk's way of extending the blessings of peace over additional territories and increasing millions.

Before peace with Mexico had been ratified, a peculiar situation presented itself in Yucatan. The white race in that peninsula were engaged in a protracted struggle with the Indians. As the price of assistance, they simultaneously offered the dominion and sovereignty of their country to Great Britain, Spain and the United States. In a special message, advising the occupation of Yucatan, President Polk said:

"We could not consent to a transfer of this 'dominion and sovereignty' to either Spain or Great Britain or any other European power. In the language of President Monroe.... 'the American continents, by the free and independent condition which they have assumed and maintain, are henceforth not to be considered as subjects for future colonization by any European power.' ....The present is deemed a proper occasion to reiterate and reaffirm the principle avowed by Mr. Monroe and to state my cordial concurrence in its wisdom and sound policy."

Here we have the new version of the first part of Monroe's declaration. The protest against new European colonies is construed to mean that no European power shall acquire territory upon this continent in any way whatever.

Polk's two statements were glaringly inconsistent. The first declared the right of the United States to acquire territory by the free gift of an independent state, the second denied the right of Europe to acquire territory in the same way. The first denied to Europe the right of interposition; the second asserted it for the United States. The first asserted that the nations of America were sovereign and independent and alone had the right to decide their destiny; the second limited that right to a disposition conformable to our interests—in short, they might do as they pleased as long as they pleased to do as we pleased. In what mysterious way the sovereignty of the United States was suddenly extended over the entire continent was not explained. Nevertheless Polk's statement gave the Monroe doctrine its final form: Europe shall not interfere with American states and shall not acquire territory in America in any way. The United States may interfere and may acquire territory whenever her interests demand it. This, I take it, is the form in which the Monroe doctrine rests in the minds of the American people to-day.

Polk's misconstruction of the Monroe doctrine did not pass unchallenged. Mr. Calhoun was at that time the only surviving member of Monroe's cabinet. He was, therefore, of all men living the best acquainted with the circumstances and discussions attending the issue of the declaration. His pro-slavery sympathies and his own part in the annexation of Texas might have inclined him to accept Polk's construction. Instead he declared in the Senate that the case of Yucatan did not come within the Monroe declarations; that they did not furnish the slightest support for it.* It was not the extension of the European political system to this continent, for that system had already ceased to exist. It was not an interposition of an European power to oppress an American government, because that power would come, not to oppress, but to save. Even if England should assert her sovereignty over Yucatan, it would not bring the case within the Monroe doctrine because the tender of that sovereignty had voluntarily been made. It was not colonization. That word had a specific meaning. It meant the establishment by emigrants from a parent colony of a settlement in territory either uninhabited or from which the inhabitants had been partially or wholly expelled. The occupation of Yucatan could not be construed to be colonization by any forced interpretation. Yucatan might become a province or a possession of Great Britain but not a colony. In conclusion he said:

"What the President has asserted in this case is not a principle belonging to these declarations; it is a principle which, in his misconception, he endeavors to engraft upon them but which has an entirely different meaning and tendency....It goes infinitely and dangerously beyond Mr. Monroe's declaration. It puts it in the power of other countries on this continent to make us a party

*Calhoun's "Works," Vol. 4, pp. 454-66.

to all their wars.....If this broad interpretation be given to these declarations....our peace will ever be disturbed, the gates of our Janus will ever stand open, wars will never cease."

Who, then, was the author of this so-called Monroe doctrine? It was Polk, Polk the mendacious, as v. Holst has called him, the man who provoked a war of wanton conquest and based its declaration upon a lie. It is Polk's doctrine and not Monroe's. Not daring to sign his own name, he sought to give it authority by attaching that of one of the founders of the republic. When and why was it proclaimed? It was at the very time we were engaged in the annexation of Texas and the conquest of Mexico, the two acts in our national history of which we have least reason to be proud. Then it was that Polk twisted a declaration intended for the protection of free institutions into an excuse for the extension of human slavery. Its origin and purpose condemn it.

The policy which had succeded in Texas and Mexico, Polk next applied to Cuba. He first tried to buy Cuba but Spain replied that rather than sell she would see the island sunk in the ocean. Filibustering expeditions next tried to revolutionize Cuba, as Houston had revolutionized Texas, but failed. We next threatened Spain as Slidell had threatened Mexico. In the spirit of the Polk doctrine, our ministers to Great Britain, France and Spain, in the celebrated Ostend Manifesto* declared:

"After we have offered Spain a price for Cuba far beyond its present value and this shall have been refused, it will be time to consider the question 'does Cuba, in the possession of Spain, seriously endanger our internal peace and the existence of our cherished union.' Should this question be answered in the affirmative, then, by every law, human and divine, we shall be justified in wresting it from Spain if we possess the power.....We should be recreant to our duty, be unworthy of our gallant forefathers, and commit base treason against our posterity should we permit Cuba....seriously to endanger or actually to consume the fair fabric of our Union."

But anti-slavery opinion in the North was setting strongly against the slave power in its foreign as well as its domestic policy. The first republican platform in 1856 resolved that "the highwayman's plea that might makes right, embodied in the Ostend circular, was in every respect unworthy of American diplomacy and would bring shame and dishonor upon any government or people that gave it their sanction."

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*House Ex. Docs., Vol. 10, No. 93; 2d Sess., 33 Cong., pp. 127-36.

The civil war destroyed the slave power and the desire to acquire territory for slave purposes. The doctrine devised by Polk in the interest of slavery seemed to be dead. But now after nearly half a century it is revived in the interest of foreign commerce. It suggests an old epigram:

"To kill twice dead a rattlesnake,
And off his scaly skin to take,
And through his head to drive a stake,
And every bone within him break,
And of his flesh mincemeat to make,
To burn, to sear, to boil and bake,
Then in a heap the whole to rake,
And over it the besom shake
And sink it fathoms in the lake—
Whence after all, quite wide awake,
Comes back that very same old snake."

The Polk doctrine is an encroachment upon the rights of foreign states. This fact is so clear that the wonder is that it does not appeal to every one the moment it is stated. The explanation perhaps is that frequent repetition secures its acceptance much as we incline to believe a false report that is often repeated. The first and most fundamental doctrine of international law asserts the sovereignty, independence and equality of states. They are sovereign in the regulation of their internal affairs, independent of interference in their relations with other states and equal in rights. This is precisely the doctrine stated by John Quincy Adams,* when urging the declaration in the cabinet meeting.

"Considering the South Americans as independent nations," he said, "they themselves and no other nations have the *right* to dispose of their condition. *We* have no right to dispose of them, either alone or in conjunction with other nations. Neither have any other nations the right of disposing of them without their consent."

From equality of rights results a corresponding equality of obligations. The same rights belong to all—the same duties rest upon all—the greatest as well as the smallest, the strongest as well as the weakest. Strength confers no privileges and weakness grants no exemptions. If the weak state injure the strong one, it must make reparation. It is the duty of the strong state to seek it peaceably, it is her right to secure it forcibly if necessary.

In 1854 the people of Greytown, Nicaragua, insulted the American minister and destroyed American property. The United States sent a war-ship there and, failing to secure an indemnity, bom-

*"Memoirs." Vol. 6, p. 168.

barded the town. Lord Palmerston, at that time prime minister of England, in referring to the incident in Parliament, said:

"We may think that the attack was not justified by the cause which was assigned. But we have no right to judge the motives which actuated other states in vindicating wrongs which they supposed they had sustained."*

In 1855 the United States became involved in a controversy with Paraguay, in which justice appears to have been largely upon the side of the weaker state. Reparation was demanded and refused. Thereupon President Buchanan sent a fleet of nineteen vessels, which forced an apology and the payment of an indemnity. In 1890 we threatened Venezuela with force in order to collect a private claim and in 1892 we threatened Chili with war to secure an apology for an injury. No European power interfered at any time to protect the weaker state.

In 1894 the authorities at Bluefields, Nicaragua, insulted the British consul there and a mob destroyed the consulate. Great Britain demanded an indemnity of the Nicaraguan government and proposed, in default of payment, to take possession of the port of Corinto and collect the duties there until the amount claimed was realized. Immediately the American press raised the cry of "Monroe Doctrine" and in effect denied the right of Great Britain to resort to the same measures of redress in her intercourse with independent states which we had many times employed in similar cases. We might have said as Lord Palmerston did of the Greytown bombardment that we did not think the punishment was justified by the cause assigned but we were bound to add as he did, that "we had no right to judge the motives which actuated other states in vindicating wrongs which they supposed they had sustained." To deny to foreign nations the same modes of redress that we employ ourselves is an encroachment upon their sovereignty, a violation of their independence and a denial of their equality.

In 1861 the United States was confronted with the most stupendous insurrection ever organized. The rebellion began in South Carolina in December of 1860. By the 8th of February, 1861, seven states had seceded and organized an independent government as complete in all respects as was the Union government. They were subsequently joined by four more states making eleven in all, exactly one-third of the total number at that time and including nearly a third of the area and population of the Union. For five months after the beginning of this rebellion no effort was made to

*Wharton's "Digest," Vol. 2, p. 596.

check or suppress it. It was for a time even doubtful whether such an attempt would be made at all. The first conflict of arms took place in April. The President of the United States immediately called for seventy-five thousand volunteers and declared a blockade of the seceded states. A war was immediately prepared, the most regularly equipped, the most regularly conducted and the greatest of modern times. In May and June European states issued proclamations of neutrality, recognizing the fact of war and the belligerency of the parties. We considered these proclamations an unjustifiable interference in our internal affairs and an evidence of great unfriendliness and made them for years the subject of a claim for damages against a foreign state.

In the neighboring colony of a friendly state there has raged for some time an irregular guerilla war. The government of the insurgents does not approach in completeness the government of the Confederate states. It has not a tenth part of the equipment, of the regularity, or of the prospect of success that the Confederates had. And yet it is seriously proposed that we recognize these insurgents as belligerents and advise Spain to grant them independence, on the ground that she can never conquer them. In what temper would the Union government have received such advice in 1861? Interference in the affairs of foreign states, which we resent when applied to ourselves, is an encroachment upon their sovereignty, a violation of their independence and a denial of their equality.

According to well settled rules of international law, interference in the affairs of independent states is justified in only two cases: first, when demanded by self preservation and second, when necessary to prevent the commission by a government upon its subjects of crimes repugnant to humanity. The protest of President Monroe came well within the first case. It is difficult for us now to realize the comparative weakness of the United States in 1823. We had at that time a population of less than ten million people sparsely settled over a wide area. Within ten years we had come out of a war with a single European power badly beaten and glad to make peace without mention of the causes of the contest. The establishment by powerful European states of new colonies upon our borders would have been a menace to our peace and safety. The subjugation of South American states by an European alliance acting in the interest of Spain would in principle have justified the conquest of the United States by a similar alliance acting in the interest of Great Britain. The circumstances justified the protest.

Very different are the recent cases. In no one of them is there any menace to our national existence. We have no right of interference, upon the same principle of law that an individual has no standing in a controversy in which his rights are not involved. The fact that states are located in the Western hemisphere gives us no protectorate over them. Much of Europe is actually nearer to us than many South American states and all of Europe is more easily accessible than any of them. International law knows no North, no South, no East, no West. The rights and duties of states are the same everywhere. The assertion by the President that an extension of the boundary of British Guiana is dangerous to our peace and safety is an absolute absurdity. And yet, so far as I am informed, only three newspapers in the United States had the courage to say so. The only other protest came from a few college professors, who in the popular view, by reason of the special study of particular questions, become thereby incapacitated for forming intelligent opinions respecting them. These few protests were met by crushing charges: their authors were dudes and Anglomaniacs and turned up their trousers when it rained in London. And now the government has come to the college professors because no one else can read the documents upon which rests the settlement of the questions involved. Two members of the Venezuelan commission are college presidents and former professors of history and the actual study of maps and manuscripts is being carried on by Mr. Winsor, the librarian of Harvard, Professor Burr of Cornell and Professor Jameson of Brown University. I am bound to say that the moderation of Great Britain in view of our repeated interference in her affairs is truly remarkable. I do not believe that the American people would for a moment brook a similar interference by any European state in matters that concern ourselves exclusively.

The case of Cuba affects us more nearly. We cannot but sympathize with the insurgents, struggling for liberty and independence, but we have no interest that justifies interference. The interest of Great Britain in our civil war was far greater, for the blockade closed her factories and caused widespread distress and actual starvation. It is reported that the contest in Cuba is waged with great cruelty, with the use of poisonous and explosive bullets, with summary trials and barbarous executions, storming of hospitals and massacre of non-combatants, but the evidence does not show that the cruelty is much greater on one side than on the other. "As for a state's having the vocation to go forth like Hercules," says President Woolsey,* "beating down wickedness, all over the world, it is enough to say that such a principle, if carried out, would destroy the independence of states, justify nations in taking sides in regard to all national acts and lead to universal war."

A doctrine which claims a right to interfere in controversies between other states or in their internal affairs, when our national existence is in no way imperiled or even remotely involved, is a violation of international law and an encroachment upon the rights of foreign nations.

The Polk doctrine is a menace to our peace and safety. A state that interferes in matters that do not concern her does so at her peril. Especially dangerous are alliances with states so unstable and changeable as those of Central and South America. Their internal affairs are in a state of confusion. Under the forms of republican institutions their governments are in fact a succession of military dictatorships—despotisms tempered by revolution. Within a period of forty years Mexico had nearly forty revolutions and more than seventy presidents. The history of the other states is very similar. So precarious are the lives of their statesmen that a right of asylum in foreign legations is admitted in all of them upon the ground that otherwise experienced men could not be induced to engage in affairs of government.† They are continually involved in wars with each other. Their wholesale repudiation of their debts continually embroils them with Europe. The government of to-day may be overthrown to-morrow. They ask our assistance only when involved in controversies with other states. At other times they reject our advice and repel our advances. Such protection is a thankless and fruitless task. Connection with them may at any time render us responsible for acts that we cannot control. Connection with one of them recently threatened a war in which we had no interest involved or principle at stake, a war with a state to which we are bound by ties of common blood, common language, common literature and common history, a war that would have caused incalculable loss and misery, a war that would have arrested the progress of the world for a decade and disgraced the closing years of the century. Let us take warning from experience and renounce a policy fraught with so much danger to our peace and safety.

The so-called Monroe doctrine is, therefore, contrary to the teaching of the founders of the republic, a perversion of the true

*"International Law," 6th ed., p. 19.

†Wharton's "Digest," Vol. 1, p. 693.

meaning of the original declaration, an encroachment upon the rights of foreign nations and a menace to the peace and safety of our own, and it is the duty of the scholar to impress these facts upon the people through the press, in the pulpit and on the platform.

I come now to the second danger that threatens our national peace—the existence of a rising war spirit among the people. I do not by any means believe that such a spirit has become general but it has infected considerable numbers and unless checked may at any time get the upper hand. I attribute this spirit in large part to the influence of the younger men who are rapidly gaining control of public and private affairs. The older men have retained control longer than usual by reason of the prominence and claims that service in the civil war gave them. They are now passing rapidly away and their places are being filled by the generation that has grown to manhood since the war. This change is accompanied by a rise of war spirit, much as the same spirit arose during the first half of the century at the passing of the men of revolutionary times.

One cause of this spirit is to be found in a desire to extend our territory. In Europe in recent times there has been a revival of activity in colonization, indicated by the occupation of the minor islands of the Pacific and the conquests of England and Germany, France and Italy in various parts of Africa. The principal motive of this movement has been a desire to find an outlet for surplus population without incurring the loss that emigration of that surplus to the United States involves. The American people have caught the infection without having the same reason for it. The result is a revival of the doctrine that it is the manifest destiny of the United States to acquire control of the whole continent. This doctrine is illustrated by an anecdote told of a dinner given by the Americans residing in Paris during the civil war. The first speaker proposed the toast: "The United States, bounded on the North by British America, on the South by the Gulf of Mexico, on the East by the Atlantic and on the West by the Pacific Ocean." "But," said the second speaker, "this is far too limited a view of the subject. Why not look to the great and glorious future which the manifest destiny of our race prescribes for us? Here's to the United States, bounded on the North by the North Pole and on the South by the South Pole, on the East by the rising and on the West by the setting sun." "If we are going," said the third speaker, "to leave the present and take our manifest destiny into account, why restrict ourselves within the narrow limits that have just been assigned? I give you the United States, bounded on the North by the Aurora Borealis, on the South by the precession of the equinoxes, on the East by primeval chaos and on the West by the Day of Judgment."

The revival of this spirit is indicated by the frequent recurrence of articles in the magazines advocating the annexation of Canada, by a very general desire not long since for the acquisition of the Hawaiian Islands, by a strong feeling in some quarters at the present time for the occupation of Cuba and by the demand sometimes heard that we make the Isthmus canal our southern boundary. Such exuberance and enthusiasm are natural to youth. The fact seems scarcely to be considered that nearly every one of these measures involves war. I do not mean to disparage the importance of our vast extent of territory and of our boundless resources, a just source of pride to every patriotic American. The annexation of both Texas and California has been productive of incalculable good to us and to the territory involved but that does not justify the mode and motive of their acquisition. We ought not to acquire more territory by war and conquest. We ought not to annex islands so far removed from our present boundaries that a great and expensive navy would be necessary for their defense, costing more than the value of their total product. And we ought not to acquire territory of which the population is unfit to constitute a state in the Union. Quality is more important than quantity; domestic peace more valuable than foreign commerce.

A second cause of the war spirit is to be found in the existence of deep seated prejudices against particular nations, prejudices unreasoning and unreasonable. The strongest of these prejudices is directed against England. This is in part a survival of the passions of the revolution. Aversion to England and partiality to France were potent factors in our domestic politics from the revolution to the war of 1812. So strong indeed was their influence that a foreign observer was led to remark that "he found in the United States, many French and a few English but no Americans." Rightly understood the revolution furnished little reason either for hatred of England or gratitude to France. At least after the lapse of a century and especially as we were victorious, we can afford to be magnanimous. The English do not cherish the same resentment against us. An Englishman once said to me: "We don't bear you any grudge, you know, for beating us in the revolution. We are proud of you. It is just what we would have done in your place."

And I believe that this remark is characteristic of the feeling of the English people. Prejudice against England was revived by the events of our civil war. There was in truth far greater reason for hatred of France, whose government on the one hand continually urged Great Britain to interference and to a joint recognition of Southern independence and on the other tried to turn our distracted condition to her own advantage by establishing an empire in Mexico. The existence of what is called the Irish vote tends to perpetuate this prejudice and enables politicians to make capital by trading upon the passions of the people. Here again we cannot do better than turn to the advice of Washington's farewell address:

"Nothing is more essential than that permanent, inveterate antipathies against particular nations and passionate attachments for others should be excluded and that in place of them, just and amicable feelings toward all should be cultivated.....Antipathy in one nation against another disposes each more readily to offer insult and injury, to lay hold of slight causes of umbrage and to be haughty and intractable when accidental or trifling occasions of dispute occur....Hence frequent collisions and obstinate, envenomed and bloody contests. The nation, prompted by ill-will and resentment, sometimes impels the government to war contrary to the best calculations of policy. The government sometimes participates in the national propensity, and adopts through passion what reason would reject. At other times, it makes the animosity of the people subservient to projects of hostility, instigated by pride, ambition and other sinister and pernicious motives. The peace often, sometimes even the liberty of nations, has been the victim."

A third cause of the war spirit may be found in an extreme sensitiveness and a disposition to resent anything that looks like injury before the actual facts are known. The conduct of foreign relations is undoubtedly a weak point in republican institutions. Formerly they were considered the exclusive affair of government, diplomatic correspondence was secret and time was allowed for explanation or apology before definite action was threatened or taken. Now all public questions are discussed in the forum of the people and upon the first rumor of insult or injustice there arises a demand for instant apology and a threat of war. Governments like individuals dislike the appearance of yielding to pressure and a premature resort to it diminishes the chances of accommodation. The danger is that popular excitement may precipitate an unnecessary conflict. Fortunately the government has proved more moderate than the people and the danger so far has been avoided.

Nations have the rights of individuals and the same duties rest upon them—among others the duty of moderation.

"It not infrequently happens," says General Halleck,* "that what is, at first, looked upon as an injury or an insult is found, upon more deliberate examination, to be a mistake rather than an act of malice or one designed to give offense. Moreover the injury may result from the acts of inferior persons, which may not receive the approbation of their own governments. A little moderation and delay, in such cases, may bring to the offended party a just satisfaction whereas rash and precipitate measures may often lead to the shedding of innocent blood."

I would not abate one jot or tittle of our just rights but I would counsel moderation, a postponement of judgment until all the circumstances are known, an avoidance of irritating and insulting charges, a resort to peaceful measures of redress and above all no talk of war until it shall appear that war is necessary to save national honor. "He that is slow to anger is better than the mighty and he that ruleth his spirit than he that taketh a city."

The last and most important cause of the war spirit is to be found in the fact that the new generation have never known the horrors of war and are ignorant of its true character.

"Art and literature," says a recent writer on international law,† "combine to help on the work of slaughter. Poets and painters celebrate the 'pomp and circumstance of glorious war' till people come seriously to regard it as a thing of bands and banners, of glittering uniforms and burnished steel, of deeds of heroic daring and examples of lofty self-sacrifice. They forget the stern realities of cold and hunger, wounds and death, the shattered limbs, the fever thirst, the fiendish passions of cruelty and lust. They forget the demoralization it causes among both victors and vanquished and the widespread ruin that follows in its train. In the twenty-five years betwen 1855 and 1880 over two million men died in wars between civilized powers."

In our own civil war, upon the Union side alone, out of three hundred and fifty thousand dead, only sixty-seven thousand were killed in battle. Two hundred thousand died of disease, forty-three thousand died of wounds and forty thousand from accident, murder, execution, starvation or abuse. Thirty thousand one hun-

*"International Law." 3d ed., Vol. I, p. 463.

†T. J. Lawrence, "Essays on Modern International Law." 2d ed., pp. 242-4.

dred and fifty-six Union soldiers died in Southern prisons and thirty thousand one hundred and fifty-two Confederate soldiers died in Northern prisons, within four of the same number on both sides.

"Who can calculate," says the same writer, "the awful mass of human misery that these figures represent?....Comparatively few of those that perish die upon the battle field. Thousands succumb from sheer exhaustion, having endured for weeks, perhaps months, the slow agony of failing strength, under the influence of privation and over-exertion. Thousands die of disease, many of them for want of the commonest comforts of the sick. Starvation demands one host of victims, fever another, neglected wounds a third. Vice of all kinds preys upon the soldiery and exacts its terrible toll of moral and physical ruin. Even well appointed and victorious armies melt away under the influence of sickness and fatigue unless constantly reinforced. What then must be the case with a broken or retreating army, an army separated from its supplies or cooped up in a beleaguered fortress? Let the three hundred thousand French soldiers, whose bones strewed the plains of Russia from Moscow to the Niemen provide the answer. Read in the history of a more recent period how a British army was destroyed by cold and privation in the trenches before Sebastopol, while transports rocked idly in the harbor of Balaclava, almost within sight of the starved men dying like flies for want of the comforts they contained. Consult English papers for the condition of the hospitals at Plevna, when the Russians entered the town and found the wounded with broken and unset limbs twisted out of all human recognition. In records such as these you will read the true history of war. No one acquainted with them can deny that much remains to be done to correct popular ideas and sentiments on the subject. There must be a great change in the ordinary modes of thinking and speaking of war before current opinion in regard to it conforms to the standard of Christianity."

It is not death alone that makes war terrible. Worse than dead are the wrecks of men, maimed in body and shattered in mind, who live afterward, a curse to themselves and a burden to their friends. No account has yet been taken of the suffering at home. Think of the three hundred and fifty thousand dead in our last war on the Northern side alone and then think of the thousands of mothers left childless, the thousands of wives left husbandless, the thousands of children left fatherless, the heart-burnings and heart-breakings it caused, and *then* talk lightly and wantonly of war.

"The real sorrows of war," says George Cary Eggleston,* in speaking of the South, "fall most heavily upon the women. They may not bear arms. They may not even share the triumphs which compensate their brothers for toil and suffering and danger. They must sit still and endure. The poverty which war brings to them wears no cheerful face but sits down with them to empty tables and pinches them sorely in solitude. After the victory the men who have won it throw up their hats in glad huzza, while their wives and daughters await in sorest agony of suspense the news which may bring hopeless desolation to their hearts."

I have heard men say that war would be a good thing, it would raise prices and make trade brisk. Truly when such remarks can be made, much remains to be done to correct popular ideas and sentiments upon the subject of war. The duty to do this rests upon those who know and feel the evil. It rests upon all alike, teachers in the schools and professors in the colleges, writers for the press and preachers in the churches, men of business on the street and statesmen in the halls of legislation. Lord Derby has said: "The greatest of England's interests is peace." Let us echo the sentiment: The greatest American interest is peace.

I come now to the third danger that threatens our national peace—enormous expenditure for war purposes. This expenditure, as Dunning said of the influence of the crown, "has increased, is increasing and ought to be diminished." The possession of great force is a standing temptation to use it.

It has been common for great men to give accounts of their early intellectual development and of books that have helped them. I see no reason why it may not also be permitted to small men to acknowledge their indebtedness to the influences that have moulded their opinions. In the library of the school where I received my training preparatory for college, there was a copy of the "Speeches and Addresses of Charles Sumner," which I often used to read when supposed by my instructors to be studying Latin or Algebra. The first speech in that collection made a powerful impression upon my mind. It was entitled "The True Grandeur of Nations," and defended the proposition that in our age there can be no peace that is not honorable, and no war that is not dishonorable. The oration was delivered on the fourth of July, 1845, before the city corporation of Boston. Mr. Sumner was himself a notable example of the scholar in politics—not always right, to be sure, but always honest and honorable. This speech was his first public appearance, the beginning of his public

*"A Rebel's Recollections." 3d ed., p. 58.

career. I desire to quote the passage,* which, according to the testimony of those present, made the strongest impression upon his hearers:

"Within cannon range of this city stands an institution of learning which was one of the earliest cares of our forefathers. Favored child in an age of trial and struggle—carefully nursed through a period of hardship and anxiety—sustained from its first foundation by the paternal arm of the commonwealth, by a constant succession of munificent bequests and by the prayers of good men—the University of Cambridge now invites our homage as the most ancient, most interesting and most important seat of learning in the land.....It appears from the last Report of the Treasurer, that the whole available property of the University, the various accumulations of more than two centuries of generosity, amounts to $703,000."

"Change the scene and cast your eyes upon another object. There now swings idly at her moorings in this harbor a ship of the line, the Ohio, carrying ninety guns, finished as late as 1836 at an expense of $835,000—more than $130,000 beyond all the available wealth of the richest and most ancient seat of learning in the land. Choose ye, my fellow citizens of a Christian state, between the two caskets,—that wherein is the loveliness of truth, or that which contains the carrion death."

"Pursue the comparison still further. The expenditure of the University during the last year amounted to $48,000. The cost of the Ohio for one year of service, in salaries, wages and provisions is $220,000, being $172,000 above the annual expenses of the University and more than four times as much as those expenditures. In other words, for the annual sum lavished upon a single ship of the line, four institutions like Harvard University might be supported."

A similar comparison between the cost of a modern warship and a modern University would be interesting, were the material at hand for making it. The average cost in recent years of a large man-of-war, without armament, has been over three million dollars. There have recently been added to our navy six battle ships—the Indiana, the Iowa, the Maine, the Massachusetts, the Oregon and the Texas, and two armored cruisers—the Brooklyn and the New York. Their total cost, making allowance for armament, is twenty-five million dollars. This amount exceeds by ten million dollars the total income of the four hundred and seventy-six colleges and universities in the United States to-day and at the present rate would defray the current

*Sumner's "Works," Vol. I, pp. 80-2.

expenses of the University of Kansas for a period of two hundred and fifty years. And yet this twenty-five million is but a fraction of the total expenditure for war purposes which during the last five years has amounted to four hundred and twelve millions,* an average of over eighty-two millions a year—and the present Congress has surpassed all its predecessors in extravagance and voted the largest appropriations ever made and ordered the largest number of battle ships ever provided for at a single time—and all this in a period of peace abroad and commercial depression at home, with an enormous deficit in the national treasury and with widespread distress every winter in all our large cities, that has required for its relief an organization of charities hitherto unknown. Is it not time to call a halt in this enormous waste of wealth? Is there not some missionary work for educated men and women to do here at home in the way of arousing and civilizing public opinion upon this subject? "Let us," says General Walker,† "frown indignantly upon every proposed measure, upon every representative vote, upon every word of every man, whether in public or private speech, which assumes or gives countenance to the assumption that this people are to come under the curse of the war system or which threatens our friendly relations with any power on earth. Sixty-five millions, transcending in all the elements of industrial, of financial and, if you please, of military strength, the combined resources of any two of the greatest nations of the world, who shall molest us or make us afraid, who shall be so insane as to wantonly attack the greatest power on earth? Why then should we enter upon that career of competitive armament into which mutual jealousies and mutual fears have driven the nations of Europe—a career which once entered upon, has no logical stopping place short of complete exhaustion, impoverishment and financial bankruptcy and which in its turn finds that it has earned nothing but to be the object of universal dread and universal detestation? ....Let it then be our pride as it is our privilege to remain the great unarmed nation, as little fearing harm from any as desiring to wrong any. Let us follow the paths of peaceful, happy industry, developing the resources with which nature has so bounteously endowed us, reserving our giant strength for those competitions whose results are mutual benefits, and bestowing upon schools and colleges, libraries and museums, public parks and institutions of beneficence that wealth which others waste on frontier fortresses and floating castles."

*"Statistical Abstract of the United States," No. 18, p. 22.

†"The Growth of the Nation," an Address at Brown University, June 18th, 1889, printed in the Providence "Journal."

Editorial Notes.

The University of Pennsylvanvia sends out a handsome collection of the addresses at the opening of the recently purchased Bechstein Germanic Library. With this purchase—15000 volumes and 3000 pamphlets—the University of Pennsylvania at one step takes front rank among American universities for students of Germanic languages.

Dr. Geo. I Adams, late assistant on the University Geological Survey, and a Fellow of Princeton College, has printed the substance of his Dissertation in the American Journal of Science, under the tittle "The Extinct Felidæ of North America" Some reference to the paper is made in an article in this number of the QUARTERLY.

THE UNIVERSITY GEOLOGICAL SURVEY OF KANSAS, Volume I, has been issued from the office of the state printer. The work is conducted by Prof. E. Haworth assisted by J. Bennett, G. D. Adams, M. Z. Kirk, E. B. Kneer and J. G Hall. The report consists of sections, mostly in the Eastern portion of the state, with reports on certain borings, and particular deposits, as coal and salt. A vast amount of useful information has been accumulated. It is illustrated by thirty-one plates, eleven figures, and occupies 310 pages. Volume II is in preparation.

The fifth volume of the Collections of the Kansas State Historical Society, which has just been published, contains nearly 700 pages, and is a well-printed book. It contains most of the addresses delivered before the society during the past six years, including the address of Rev Doctor Cordley, on the Convention Epoch in Kansas History; that of Col C. K. Holliday, on the Freemont Campaign of 1856; of Hon. James S Emery, on History and Historical Composition; of Dr. Peter McVicar, on School Lands on the Osage Indian Reservation; of W. H. T. Wakefield, on Squatter Courts in Kansas; Mrs. Lois H. Walker's Reminiscences of Early Kansas Times; C. H. Dickson's Reminiscences of 1855; Hon. J. R. Mead's Trails in Southern Kansas; Hon. P. G. Lowe's account of Army Service on the Plains in 1852; memorial proceedings on Col. William A. Phillips; Hon Albert R. Greene's account of the Battle of Wilson Creek; Prof. O E. Olin's Romance of Kansas History; Hon. John Speer's Incidents of Pioneer Days; Doctor Cordley's discourse on Judge S. O. Thacher; and Gov Morrill's address at the annual meeting of the society, last January, on the Trials, Privations, Hardships and Sufferings of the Early Kansas Settlers Besides, this volume contains a large fund of documentary historical materials pertaining to the troublesome times in early Kansas, including the official papers of the period of the administration of Governors Robert J. Walker, James W. Denver, and Samuel Medary, and of Acting Governors Frederick P. Stanton, Hugh S. Walsh, and Geo. M. Beebe. These papers for the most part have been lying hidden in the archives of the department of state, at Washington, during a period of over 36 years. At the personal request Hon. R. W Blue, Secretary Olney directed a search to be made, which resulted in securing copies of these records. The documents complete the publication of the entire documentary history of the period of the Kansas territorial government from 1854 to 1861, the papers of former administrations having been published in the third and fourth volumes of the Historical Society's Collections.

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Continuous Groups of Projective Transformations Treated Synthetically.

BY H. B. NEWSON.

(Part II Continued.)

§2 Groups in the Plane.

I have defined a projective transformation in a plane in the sense in which the the term will be used in this paper, and have given a simple method of constructing it. Having given four points A, B, C, D, no three of which are in the same straight line, we may choose as their corresponding points A′, B′, C′, D′; thereby a projective transformation T of the plane is completely determined such that any point P is transformed into a definite point P′. If now we choose four other points A″, B″, C″, D″, as the corresponding points to A′, B′, C′, D′, we would have obtained a projective transformation T_1 transforming P directly to P″. It is clear that two transformations T and T_1 together produce the same effect as T_2. Thus it may be shown in general that any two projective transformations of the plane are together equivalent to some third. Therefore all the projective transformations of the plane form a Continuous Group of Transformations.

The number of projective transformations in the plane is likewise determined from the same considerations. Having given four points A, B, C, D, a transformation is determined when their corresponding points are chosen; and there are as many transformations of the plane as there are sets of four points in a plane. Since the plane contains ∞^2 points, we easily see that there are ∞^8 such sets of points and hence there are ∞^8 projective transformations in the plane.

Another method of determining the number of projective transformations in the plane leads to the same result. From the method of constructing a projective transformation referred to above, we see that any two conics touching a line l determine a projective transformation of the plane. Since the number of conics touching the line is ∞^4, the number of pairs of such conics is ∞^8 and hence there are ∞^8 projective transformation of the plane obtained by taking any line l as the fixed line of the construction. The line l was taken as the line of intersection of the two planes π and π', and in developing the construction of one projection of the plane upon the other the angle between the two planes was not considered. By making the planes π and π' intersect in some other line as l, we get another system of transformations which must be identical with the first system. If the angle between the two planes in the last position is not the same as in the first position, the transformations of the two systems will not be in the same order, but no new transformation will be introduced. We therefore infer that there are only ∞^8 projective transformations in the plane.

The group of the projective transformations of the plane will be called the General Projective Group and will be designated by the symbol G_8.

Theorem 4. There are ∞^8, projective transformations of the plane; these form the General Projective Group G_8 whose fundamental property is that any two transformations of the group are together equivalent to some third transformation belonging to the same group.

(For Lie's analytical proof see "Cont. Gruppen," Kapitel 2, §1.)

Every projective transformation of the plane leaves some line or lines and some point or points of the plane unaltered in position, or as we say, invariant. There are five types of these transformations, distinguished according to the kind of plane figure which is left invariant. (See Vol. IV, page 248 K. U. Q. and "Cont. Gruppen" page 35-6). If two transformations T and T_1 both leave any plane figure invariant, e. g. a line l, the transformation T_2 which is equivalent to the combination of T and T_1 must also necessarily leave l invariant. Thus considering the totality of transformations which leave l invariant, we see that the combination of any two transformations of the system are together equivalent to a third transformation of the same system. Hence the totality of transformations leaving a line invariant have the group property and form a sub-group of the general projective group. The same reasoning applies in general to the system of transformations leaving invariant any plane figure whatever.

Theorem 5. All projective transformations of the plane leaving a plane figure invariant have the group property and form a sub-

group of the general projective group. (See "Cont. Gruppen," page 113.)

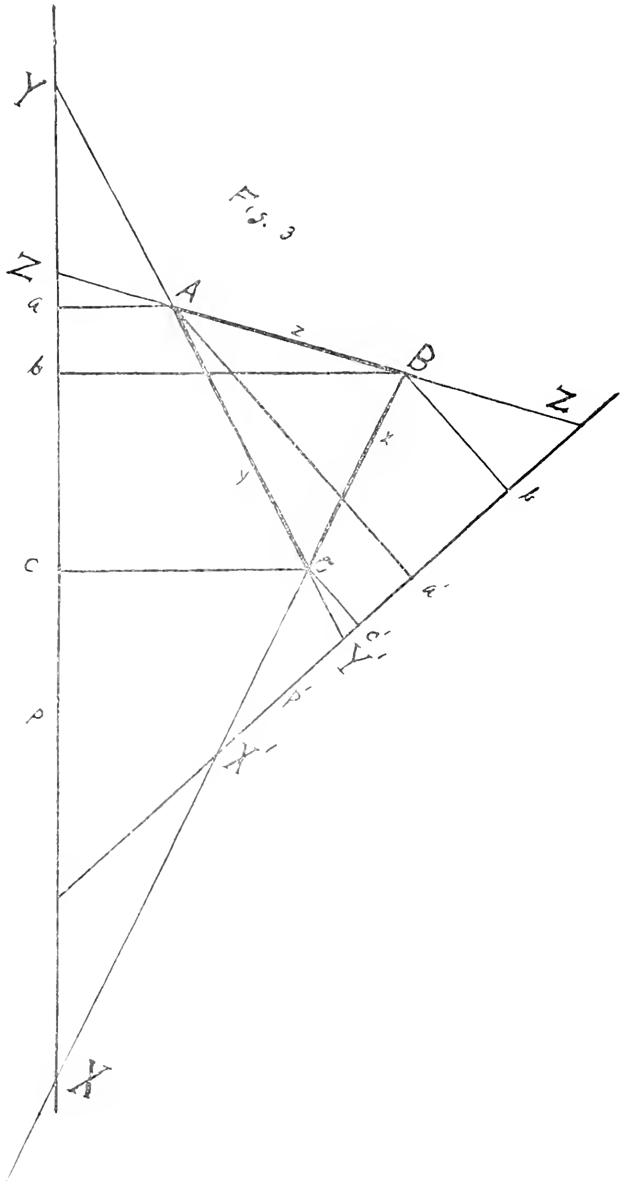
By means of this theorem many of the sub-groups of the general projective group can be readily determined.

It will be convenient to have separate symbols to designate each of the five types of transformations referred to above. We shall represent the five types of transformations whose invariant figures are

by T, T', T'', S, S', respectively.

We shall now consider more in detail these different types of transformations, beginning with the most general case (type I) whose invariant figure is a triangle. Let the vertices of the triangle be represented by A, B, C; and the opposite sides by x, y, z, respectively. By means of a transformation T the line x is transformed into itself in such a way that the points B and C on it are invariant points of the transformation. Now we know that the one-dimensional transformation of the points on a line, which leaves two points of the line invariant, is characterized by the constant anharmonic ratio of the invariant points and any pair of corresponding points. (Kansas University Quarterly, Vol. IV., page 74.) Let k_x be the characteristic anharmonic ratio of the one-dimensional projective transformation along the line x. In like manner we have projective transformations of one dimension along each of the invariant lines y and z. We shall call their characteristic anharmonic ratios k_y and k_z respectively. In reckoning these anharmonic ratios the points will be taken always in the same order around the triangle. Thus we see that every projective transformation of the kind T in the plane determines three characteristic anharmonic ratios along the three invariant lines. It is also evident that the pencil of lines through the vertex A of the invariant triangle is transformed into itself in such a way that the rays AB and AC are invariant rays of the transformation. Also the anharmonic ratio of the invariant rays and any pair of corresponding rays of the pencil is constant for all pairs of corresponding rays; this anharmonic ratio is equal to k_x, the characteristic anharmonic ratio along the opposite side x. Similar considerations apply to the pencils of rays through the invariant points B and C.

We shall now proceed to show that these three anharmonic ratios are not independent but are connected by a very simple relation. Let p and p′ (fig. 3) be a pair of corresponding lines in



the plane; and let p cut the lines x, y, z in the points X, Y, Z respectively; and let p′ cut the same lines in X′, Y′, Z′ respectively.

Since $k_z = (ABZZ')$; $k_x = (BCXX')$; and $k_y = (CAYY')$; we have

$$k_z = \frac{AZ}{BZ} : \frac{AZ'}{BZ'} ; \quad k_x = \frac{BX}{CX} : \frac{BX'}{CX'} ; \text{ and } k_y = \frac{CY}{AY} : \frac{CY'}{AY'} .$$

But by similar right triangles we have

$$\frac{AZ}{BZ} = \frac{Aa}{Bb} ; \quad \frac{AZ'}{BZ'} = \frac{Aa'}{Bb'} ; \quad \frac{BX}{CX} = \frac{Bb}{Cc} ; \quad \frac{BX'}{CX'} = \frac{Bb'}{Cc'} ; \quad \frac{CY}{AY} = \frac{Cc}{Aa} ; \quad \frac{CY'}{AY'} = \frac{Cc'}{Aa'} .$$

Multiplying together and substituting we get

$$k_x \, k_y \, k_z = \frac{AZ}{BZ} \cdot \frac{BZ'}{AZ'} \cdot \frac{BX}{CX} \cdot \frac{CX'}{BX'} \cdot \frac{CY}{AY} \cdot \frac{CY'}{AY'}$$

$$= \frac{Aa}{Bb} \cdot \frac{Bb'}{Aa'} \cdot \frac{Bb}{Cc} \cdot \frac{Cc'}{Bb'} \cdot \frac{Cc}{Aa} \cdot \frac{Aa'}{Cc'} = 1 .$$

Theorem 6. Every projective transformation of the kind T in the plane determines a characteristic anharmonic ratio along each of the invariant lines and through each of the invariant points. When these three anharmonic ratios are reckoned in the same order around the triangle their product is unity.

Thus we see that of these three anharmonic ratios only two are independent. Every transformation of T depends therefore upon 8 parameters, viz: the six co-ordinates of the three invariant points (or lines) and these two independent anharmonic ratios. Since each of these parameters may assume ∞^1 different values, we see again that there are ∞^8 transformations of the kind T in the plane.

We are also enabled to distinguish two distinct varieties of variable parameters, viz: co-ordinates of invariant points (or lines) and characteristic anharmonic ratios. This is an important distinction which will be of considerable use later on.

Theorem 7. Of the eight parameters which determine a transformation of the kind T six are coordinates of invariant points (or lines) and two are characteristic anharmonic ratios.

We proceed now to consider the system of transformations leaving a triangle invariant. In this case the six co-ordinate parameters are constant and the two anharmonic ratio parameters are variable: thus we see that there are ∞^2 transformations leaving a given triangle invariant. From another point of view we arrive at the same result. The two conics K and K^1 by means of which we can construct the transformation T touch four fixed lines l, x, y, z. K and K^1 therefore belong to a range of ∞^1 conics touching the same four lines. Any pair of conics taken from this range determines a transformation leaving the triangle (ABC) invariant. ∞^2 pairs of conics may be formed from this range, thus showing that there are

∞^2 transformations which leave the triangle invariant. By Theorem 5 these ∞^2 transformations form a two-termed group G_2 or G (ABC).

Since there are ∞^6 different triangles in the plane, it follows that there are ∞^6 such two-termed groups. Hence the general projective group G_8 is composed of ∞^6 two-termed groups G_2; thus $G_8 = \infty^6 G_2$. No two of these two-termed groups can have a transformation of the kind T in common; for if two transformations T and T_1 are identical the eight parameters of the one must be equal to the eight parameters of the other. Now the two anharmonic ratio parameters of one of the transformations may readily be equal to those of the other; but if the transformations leave different triangles invariant, all of the six co-ordinate parameters of the one can not be equal to those of the other. Hence T and T_1 can not be identical. (Later it will be shown that for particular positions of the triangles two or more groups G_2 may have common many transformations of the type S.)

Theorem 8. The general projective group G_8 is composed of ∞^6 two-termed sub-groups G_2. Each of these two-termed sub-groups has an invariant triangle. No two of these two-termed groups can have a transformation of the kind T in common.

We now proceed to show that there are other sub-groups of the general projective group G_8 that can be constructed out of these two-termed sub-groups of the type G (ABC). Suppose the vertex A of the triangle ABC to move along the side AC. It may assume ∞^1 different positions on the line and thus form ∞^1 triangles of the type $A_n BC$. To each of these triangles belongs a two-termed group of transformations. Consider any two transformations taken from different groups of this series. These two transformations both leave invariant the points B and C, and the lines x and y; and they are together equivalent to a third transformation which leaves the same figure invariant and therefore belongs to some one of these ∞^1 groups G_2. Thus we see that the ∞^3 transformations leaving the lines x and y and the points B and C invariant form a three-termed group G_3, which is made up of ∞^1 two-termed groups. Thus $G_3 = \infty^1 G_2$. It is easily seen that the general projective group G_8 contains ∞^5 such three-termed sub-groups. Two three-termed groups, whose invariant figures contain no geometric element in common, contain no transformation in common. But it is possible to chose the invariant figures so that they shall contain a common triangle; the two three-termed groups then contain a common two-termed group.

Theorem 9. The general projective group G_8 may be decomposed into ∞^5 three-termed sub-groups each of which has for invariant figure two lines, their point of intersection, and another point on one of these lines.

In a similar manner three-termed groups of the type just discussed may be put together so as to form four-termed groups; and this may be done in three different ways. First, suppose that the line y is made to revolve about the point C; it thus assumes ∞^1 different positions. Belonging to each of these positions is a three-termed group, and by the principle of Theorem 5 these form a four-termed group $G_{4.a}$, whose invariant figure is composed of two invariant points and the line joining them.

In the second place, the point B may be supposed to assume all positions on the line x; corresponding to each position of the point B is a three-termed group, and the totality of all these three-termed groups is a four-termed group $G_{4.b}$, whose invariant figure is composed of two invariant lines and their point of intersection.

Again the point C may be made to move along the line y; to each position corresponds a three-termed group, and the totality of all these three-termed groups is a four-termed group $G_{4.c}$, whose invariant figure consists of the invariant line y and the invariant point B not on the line y. These three types of four-termed groups are the only possible ones that can be compounded out of three-termed groups of the kind G_3. We shall designate these by the symbols $G_{4.a}$, $G_{4.b}$, $G_{4.c}$.

Theorem 10. There are three types of four-termed groups which may be compounded out of three-termed groups of the kind G_3 (and hence out of two-termed groups of the kind G (ABC)). Their invariant figures are respectively two points and their join; two lines and their intersection; a line and a point not on the line.

If ∞^1 four-termed groups of the kind $G_{4.a}$ be taken such that their invariant figures have common the line x and the point C on x, these form a system of ∞^5 transformations all of which leave invariant the linear element x, C. Hence these form a five-termed group. Again, if we take ∞^1 four-termed groups of the kind $G_{4.b}$, such that their invariant figures have common the line x and the point C, we have the same five-termed group as before. But if we take four-termed groups of the kind $G_{4.c}$ we can not put them together so as to form a five-termed group. This kind of a five-termed group with an invariant linear element is the only kind that can be built up out of two-termed groups G (ABC). Two such five-termed groups will generally have a two-termed group in common; for the common invariant figure is a triangle. If the two lines or the two points of the linear elements coincide, the two groups have in common a four-termed group.

Theorem 11. The ∞^5 transformations which leave a linear element invariant form a five-termed group. The general projective group contains ∞^3 such five-termed groups.

If the point C be made to move along the line x, to each position of the point belongs a five-termed group. The sum total of the transformations belonging to all these five-termed groups forms a six-termed group whose invariant figure is a straight line. It is clear that to every line in the plane belongs a six-termed group of this kind. The general projective group therefore contains ∞^2 such six-termed groups.

In like manner if the line x be made to revolve around the point C, to every position of the line x belongs a five-termed group. These ∞^1 five-termed groups make up a six-termed group whose invariant figure is a point. The general projective group contains ∞^2 of these six-termed groups.

Theorem 12. The ∞^6 transformations which leave a line or a point invariant form a six-termed group. The general projective group contains ∞^2 sub-groups of each kind.

This completes the enumeration of the sub-groups of the general projective group, which can be built up out of the two-termed subgroups of the kind G (ABC). We have a list of nine kinds of groups, as follows:

$$G_8;\ G_{6.p},\ G_{6.l};\ G_5;\ G_{4.a},\ G_{4.b},\ G_{4.c};\ G_3;\ G_2.$$

So far we have shown how to build up these groups of higher orders out of groups of lower orders. The reverse process might have been followed. We might have started with the general projective group and decomposed it into groups of lower orders. This we proceed to do briefly.

A transformation of the kind T is determined by eight parameters, six co-ordinate parameters and two anharmonic ratio parameters. When all eight of these vary they generate the general projective group. When two or more of the co-ordinate parameters are fixed quantities and the rest of them variables the various subgroups are generated. In order that a point of the plane shall remain invariant it is necessary and sufficient that two of the co-ordinate parameters shall be fixed; the variation of the other six parameters generates a six-termed sub-group $G_{6.p}$. In like manner two conditions or parameters determine a line; the variation of the other six parameters generates a six-termed group $G_{6.l}$. The general projective group G_8 contains ∞^2 sub-groups $G_{6,p}$, and also ∞^2 sub-groups $G_{6,l}$. Three conditions determine a linear element; if the co-ordinates of a linear element are fixed, the variation of the other five parameters generates a five-termed group G_5. If two points of the plane are invariant, four parameters are fixed and the variation of the remaining four produces the four-termed group $G_{4,a}$. If two lines of the plane are invariant, four parameters are fixed and the remaining four produce the four-termed group $G_{4,b}$. If a point and a line are invariant, four parameters are fixed and the remaining four produce the four-termed group $G_{4,c}$. If two points, their join and a line through one of them; or two lines, their intersection and a point on one of them are invariant, five conditions are satisfied; the variation of the remaining three parameters generates a three-termed group G_3. If three non-collinear points or three non-concurrent lines are invariant, all six co-ordinate parameters are constant and the two anharmonic ratio parameters generate a two-termed group G_2. If three collinear points are invariant, all the points of the line are invariant; but the transformations leaving all the points of a line invariant are of the kind S and S'; the same is true of three concurrent lines. Groups of this kind will be discussed later.

§3 One-Termed Groups of Transformations of the Kind T.

We shall next show that a two-termed group G (ABC) can be decomposed into one-termed sub-groups. To do this we proceed as follows:

Let T_1 be any transformation of the group G_2, and let its characteristic anharmonic ratios along the invariant lines x, y, z be $\lambda_1\ \mu_1\ \nu_1$ respectively. Let T_2 be another transformation of the group G_2, and let its characteristic anharmonic ratios be $\lambda_2\ \mu_2\ \nu_2$ respectively. These two transformations are together equivalent to another transformation of the same group G_2, whose characteristic anharmoic ratios are respectively $\lambda_3\ \mu_3\ \nu_3$. But these two-dimensional transformations each determine along the invariant lines one-dimensional transformations. The three one-dimensional transformations, one along each of the invariant lines, since they leave two points of the line invariant, belong respectively to one-termed groups of transformations of the points on a line. Hence we have by theorem 5, part I, $\lambda_1\lambda_2 = \lambda_3$; $\mu_1\mu_2 = \mu_3$; $\nu_1\nu_2 = \nu_3$. (I)
But by means of the relations $\lambda_1\mu_1\nu_1 = 1$, $\lambda_2\mu_2\nu_2 = 1$, and $\lambda_3\mu_3\nu_3 = 1$, we have $\nu_1 = (\lambda_1\mu_1)^{-1}$; $\nu_2 = (\lambda_2\mu_2)^{-1}$; $\nu_3 = (\lambda_3\mu_3)^{-1}$. Now let us put $\mu_1 = \lambda_1^{-a}$ where a is some unknown constant; let us also put $\mu_2 = \lambda_2^{-b}$, where b is another unknown constant. The three charac-

teristic anharmonic ratios of the transformation T_1 are now $\lambda_1, \lambda_1^{-a}, \lambda_1^{a-1}$: those of T_2 are $\lambda_2, \lambda_2^{-b}, \lambda_2^{b-1}$: those of T_3 can be expressed in terms of the others. The relations existing among these anharmonic ratios are given by the equations

$$\lambda_1\lambda_2 = \lambda_3; \quad \lambda_1^{-a}\lambda_2^{-b} = (\lambda_1\lambda_2)^{-a}\lambda_2^{a-b} = \lambda_3^{-a}\lambda_2^{a-b};$$

$$\lambda_1^{a-1}\lambda_2^{b-1} = (\lambda_1\lambda_2)^{a-1}\lambda_2^{b-a} = \lambda_3^{a-1}\lambda_2^{b-a}. \qquad (2)$$

If now b = a these equations reduce to

$$\lambda_1\lambda_2 = \lambda_3; \quad \lambda_1^{-a}\lambda_2^{-a} = (\lambda_1\lambda_2)^{-a} = \lambda_3^{-a}; \quad \lambda_1^{a-1}\lambda_2^{a-1} = (\lambda_1\lambda_2)^{a-1} = \lambda_3^{a-1}. \qquad (3)$$

Hence we see that if the two transformations T_1 and T_2 are so related that their characteristic anharmonic ratios along one of the invariant lines, for example along y, are each equal to the same power of the corresponding characteristic anharmonic ratios along another invariant line as x, then the resulting transformation T_3 has the same property: i. e. its corresponding anharmonic ratios have exactly the same relation. Thus we see that the two transformations T_1 and T_2 of this particular kind are together equivalent to a third T_3 of the same kind: i. e. T_3 is expressed in terms of λ_3 and a exactly as T_1 and T_2 are expressed in terms of λ_1 λ_2 and a. This is the fundamental property of a group. Hence we conclude that all the transformations of the group G_2, which have the characteristic anharmonic ratio along one of the lines as y equal to a constant power of that along another of the lines as x, form a sub-group.

This is a one-termed sub-group of G (ABC), the variable parameter of the group being the characteristic anharmonic ratio along some one of the invariant lines. This one-termed group contains ∞^1 transformations corresponding to the ∞^1 values of the variable parameter. This constant power a is the same for all transformations of the group. If we give to a different values we obtain different one-termed groups, and as many as there are values of a, viz: ∞^1. Thus we see that our two-termed group G_2 falls apart into ∞^1 one-termed groups G_1.

Theorem 13. The two-termed group of transformations G_2, which leaves a triangle invariant, consists of ∞^1 one-termed groups G_1. All the transformations belonging to one of these one-termed groups have the common property that the characteristic anharmonic ratio of each transformation along one of the invariant lines is a constant power of its characteristic anharmonic ratio along another of the invariant lines: this constant power is the same for all transformations of a one-termed group, but is different for different one-termed groups.

We shall now proceed to study in detail the properties of one of these one-termed groups. Since the variable parameter of the one-termed group in the plane is the characteristic anharmonic ratio of a one-termed group of one-dimensional transformations along one of the invariant lines, we may expect that the properties of the group of the kind G_1 on a line. (See Part I.)

The characteristic anharmonic ratios of a transformation T along the invariant lines are $\lambda, \lambda^{n-1}, \lambda^{-n}$; hereafter we shall speak of λ as the characteristic anharmonic ratio of the transformation T. We have already shown in equation (1) that λ_3, the characteristic anharmonic ratio of the resultant transformation T_3, is equal to the product of λ_1 and λ_2, the characteristic anharmonic ratios of the component transformations T_1 and T_2. By combining T_3 with T_4 we obtain T_5; so that T_5 is equivalent to the combination of T_1, T_2, T_4; thus $T_1 T_2 T_4 = T_5$. Also $\lambda_3 \lambda_4 = \lambda_5$, i. e. $\lambda_1 \lambda_2 \lambda_4 = \lambda_5$. The same reasoning may be extended to any number of transformations.

Property 1. Any two or more transformations of the group G_1 are equivalent to some single transformation of the same group; the characteristic anharmonic ratio of the resultant transformation is equal to the continued product of the characteristic anharmonic ratios of the component transformations.

If $\lambda = 1$, then $\lambda^n = 1$, and $\lambda^{n-1} = 1$; but for $\lambda = 1$ the transformation along an invariant line is an identical transformation. Hence every point on the invariant lines x, y, and z are invariant points; also all lines through A, B, and C are invariant lines; therefore every line of the plane and every point of the plane is invariant. The transformation of the group given by $\lambda = 1$ is therefore an identical transformation.

Prop. 2. The group G contains an identical transformation whose characteristic ratio is unity.

Two transformations of the group whose characteristic anharmonic ratios are reciprocals of one another are said to be inverse trans transformations. It is evident that all transformations of the group may be arranged in inverse pairs, and that the two transformations of a pair are together equivalent to the identical transformation of the group. Hence we see that if any transformation T moves P to P′, the inverse transformation T′ moves P′ back to P.

Prop. 3. The transformations of a group G may be arranged in inverse pairs; the characteristic anharmonic ratios of the transformations forming an inverse pair are the reciprocals of one another. Any transformation of the group and its inverse are together equivalent to the identical transformation of the group.

We must examine the two transformations corresponding to $\lambda = o$ and $\lambda = \infty$. We learned in one-dimensional groups to call these psuedo-transformations. In order to understand the psuedo-transformations in the plane we must consider the value of the constant a. First let a be positive between o and 1: second let a be positive between 1 and ∞; third let a be negative. Let ABC be the invariant triangle; and let the characteristic anharmonic ratio along BC be λ, along CA be λ^{-a}; and along AB be λ^{a-1} all taken in the same order around the triangle. For $\lambda = o$ and a a positive fraction these ratios are respectively o, ∞, ∞. Hence (Kansas University Quarterly Vol. IV, page 79) all points of the plane except the line AB are transformed into the point C: the line AB is indeterminately transformed. For $\lambda = \infty$ and a a positive fraction the values of these ratios are respectively ∞, o, o. Hence all points of the plane except the points on the line AC are transformed to B; the points on AC are indeterminately transformed. In the second place let a be positive between 1, and ∞: for $\lambda = o$ the three anharmonic ratios in the order mentioned above are o, ∞, o. Hence in this case all points of the plane except the line AB are transformed to the point C. For $\lambda = \infty$ and a between 1 and ∞ the values of the three ratios are respectively ∞, o, ∞. Hence all points of the plane are transformed to A except those on BC. In the third place let a be negative; for $\lambda = o$ the three values are respectively o, o, ∞. Hence all points except those on BC are transformed to A. For $\lambda = \infty$ the values are respectively ∞, ∞, o; hence all points of the plane are transformed to B except those on AC. In general it can be shown that when a is any complex quantity for $\lambda = o$ and for $\lambda = \infty$ all points of the plane are transformed into some one of the invariant points.

Prop. 4. The group G contains two pseudo-transformations whose characteristic anharmonic ratios are o and ∞ respectively. A pseudo-transformation transforms all points of the plane to one of the invariant points except the opposite side of the invariant triangle; this is indeterminately transformed.

The group G contains an identical transformation for which $\lambda = 1$. Let $\lambda = (1 + \delta)$ where δ is infinitesimally small. Since the identical transformation transforms each point of the plane into itself, it is evident that the infinitesimal transformation moves every point of the plane an infinitesimal amount. If this infinitesimal transformation be repeated n times it will give rise to a transformation of the group $\lambda = (1 + \delta)^n$. When $n = \infty$ and δ is a complex infinitesimal, λ can be made to equal any finite real or complex quantity by a proper choice of δ. Consequently any transformation of the group can be generated from the infinitesimal transformation of the group. (See Vol. IV, page 170, of this Quarterly).

Prop. 5. The group G contains one infinitesimal transformation whose characteristic anharmonic ratio is $(1+\delta)$. Any transformation of the group or the whole group itself may be generated from the infinitesimal transformation.

The foregoing properties of the most general form of a one-termed group in the plane are almost identical with the properties of the most general form of a one-termed group on a line. Both sets of properties depend upon the variation of an anharmonic ratio parameter.

We proceed now to examine certain properties of these one-termed groups of transformations and their relations to the conics K and K which are used to construct the transformation. Since four points and their four corresponding points completely determine a transformation, we should be able to construct the conics K and K' when the invariant triangle ABC and one other pair of corresponding points are given. We first show how to do this.

The conics K and K' belong to a range of ∞^1 conics touching the lines x, y, z, and l. If we take any conic K of this range S and consider the transformations formed by taking K with all conics of the range, we shall have a system of ∞^1 transformations which may be represented by T (KS). Each transformation of this system transforms any point P of the plane into points P', P'', P''', . . P^n. We wish to find the locus of these points P', P'', P''', . . P^n. The tangents from P to K intersect l in Q and R. The tangents from Q and R on the line l to the conics of the range S form two projective pencils of rays. The intersection of corresponding rays are the points P', P'', P''', . . P^n, which therefore lie on a conic through Q and R. This conic also passes through the points A, B, C; for the segments AA_1, BB_1, CC_1 are conics of the range S, and the tangents from Q and R to AA_1 intersect in A. Hence this conic which we shall call *K* passes through A and likewise through B and C.

Hence if we have given the invariant triangle and any pair of corresponding points P and P', we can construct K and K', the conics which determine the transformation, and therefore construct the whole transformation. The points A, B, C, P, and P' determine a conic *K* which cuts l in two points Q and R; connect P with Q and R; these two lines and the lines x, y, z, and l all touch a conic K of the range S. The lines joining P' to Q and R touch another conic K′ of the range S. Having found the conics K and K′ all the rest of the transformation can be constructed. If the conic *K* cuts the line l in two real points Q and R, then P and P′ are outside of K and K′ respectively; but if Q and R are a pair of conjugate imaginary points, then P and P′ are inside of K and K′ respectively; if the conic *K* should touch l, then P and P′ are on K and K′ respectively.

The conics K and K′ are each characterized by a certain numerical constant. Any tangent to the conic K cuts the four fixed tangents x, y, z, l in a constant anharmonic ratio which we shall designate by k. In like manner every tangent to K′ cuts the same four tangents in a constant anharmonic ratio which we shall designate by k′. We shall call these two anharmonic ratios the *tangential anharmonic ratios* of the conics K and K′.

Let the conics K and K′ touch l in the points L and L′; let them touch x in X and X′, y in Y and Y′, z in Z and Z′. The tangential anharmonic ratio k along the fixed tangent l is that of the four points A_1, B_1, C_1, and the point of contact L; thus $k = (A_1B_1C_1L)$; likewise $k' = (A_1B_1C_1L')$. Along the line x these same tangential anharmonic ratios are respectively $k = (XCBA_1)$ and $k' = (X'CBA_1)$. Along z they are $k = (BAZC_1)$ and $k' = (BAZ'C_1)$. Along y they are $k = (CYAB_1)$ and $k' = (CY'AB_1)$.

We shall now show that the three characteristic anharmonic ratios λ_x, λ_y, λ_z can be expressed in terms of the two tangential anharmonic ratios k and k′. X and X′ are corresponding points on the invariant line x; hence $\lambda_x = (BCXX')$. In like manner $\lambda_y = (CAYY')$ and $\lambda_z = (ABZZ')$.

Taking the ranges of points along the line z, we have

$$\lambda_z = (ABZZ') = \frac{AZ}{ZB} : \frac{AZ'}{Z'B}. \quad \text{Also } k = (BAZC_1); \text{ hence}$$

$$\frac{1}{k} = (ABZC_1) = \frac{AZ}{ZB} : \frac{AC_1}{C_1B}; \text{ and likewise we have}$$

$$k' = (BAZ'C_1) = (ABC_1Z') = \frac{AC_1}{C_1B} : \frac{AZ'}{Z'B}.$$

$$\text{Therefore } \frac{k'}{k} = \frac{AZ}{ZB} : \frac{AZ'}{Z'B} = (ABZZ') = \lambda_z. \quad \text{Thus we have}$$

expressed λ_z in terms of k and k′.

Let us next take the ranges along the invariant line y. Here we have $\lambda_y = (CAYY')$, $k = (CYAB_1)$, and $k' = (CY'AB_1)$; whence we infer that

$$1-k=(CAYB_1)=\frac{CY}{YA}:\frac{CB_1}{B_1A}$$

and $1-k'=(CAY'B_1)=\frac{CY'}{Y'A}:\frac{CB_1}{B_1A}$. Dividing the first by the second we get

$$\frac{1-k}{1-k'}=\frac{CY}{YA}:\frac{CY'}{Y'A}=(CAYY')=\lambda_y.$$

Again taking the ranges along the line x, we have $\lambda_x=(BCXX')$. $k=(XCBA_1)$, and $k'=(X'CBA_1)$: therefore

$$\frac{k}{k-1}=(BCXA_1)=\frac{BX}{XC}:\frac{BA_1}{A_1C}.$$

and $\frac{k'-1}{k'}=(BCA_1X')=\frac{BA_1}{A_1C}:\frac{BX'}{X'C}$. Multiplying together these two results we have

$$\frac{k(k'-1)}{k'(k-1)}=\frac{BX}{XC}:\frac{BX'}{X'C}=\lambda_x.$$

By multiplying together these values of λ_x, λ_y, λ_z we can verify the former theorem that the product of the three characteristic anharmonic ratios along the three invariant lines is unity: thus

$$\lambda_x\lambda_y\lambda_z=\frac{k(k'-1)}{k'(k-1)}\cdot\frac{k-1}{k'-1}\cdot\frac{k'}{k}=1.$$

Theorem 14. For any projective transformation of the kind T the three characteristic anharmonic ratios along the three invariant lines x, y, z may be expressed in terms of the two tangential anharmonic ratios of the two conics K and K' which determine the transformation: thus

$$\lambda_z=\frac{k'}{k},\ \lambda_y=\frac{k-1}{k'-1},\ \lambda_x=\frac{k(k'-1)}{k'(k-1)}.$$

If we express these in terms of λ and a, these relations are found.

$$\lambda=\frac{k'}{k}:\ \lambda^{a-1}=\frac{k-1}{k'-1}:\ \lambda^{-a}=\frac{k(k'-1)}{k'(k-1)}.$$

We wish to find out how to select the pairs of conics which produce transformations belonging to a one-termed group. We must first express k and k', the tangential anharmonic ratios of the conics K to K', in terms of λ, the characteristic anharmonic ratio of the transformation. By theorem 14 we have

$$\lambda = \frac{k'}{k} \text{ and } \mu = \lambda^{-a} = \frac{(k'-1)}{(k-1)}; \frac{k}{k'},$$

therefore $\frac{k-1}{k'-1} = \frac{k'^{a-1}}{k^{a-1}}$; by means of the relation $\lambda = \frac{k'}{k}$ we get after reduction

$$k = \frac{\lambda^{a-1} - 1}{\lambda^a - 1}, \text{ and } k' = \frac{\lambda^a - \lambda}{\lambda^a - 1}. \qquad (4)$$

When the fixed constant a is given, the conics K and K′ corresponding to a given value of λ are at once determined.

We can now determine the positions of the conics K and K′ for particular values of λ. When $\lambda = 1$, the transformation is an identical one for the whole plane; substituting this value of λ in the last equations and evaluating the indeterminate expressions we find $k = k' = \frac{a-1}{a}$. Thus in the case of the identical transformation of the group the conics K and K′ are coincident, and touch the line l at the point L, such that $(A_1 B_1 C_1 L) = \frac{a-1}{a}$. When the conics K and K′ are coincident, it is easy to see from the construction of the transformation that every point of the plane is unaltered in position: in other words the transformation in the whole plane is an identical one.

If we consider the construction of any transformation T (KK′) by means of the conics K and K′, we see that the transformation determined by the same two conics taken in the reverse order, T (K′K), is the inverse of the first: i. e. if T (KK′) transforms P to P′, then T (K′K) transforms P′ back to P: and so with every point of the plane. It is clear that every transformation of the group has an inverse belonging to the same group and that any transformation and its inverse are together equivalent to the identical transformation of the group.

In considering the positions of the conics which produce a pseudo-transformation of the group it is necessary to consider the value of the constant a. We shall consider the case where a is between 0 and 1, and a real quantity. The coincident conics producing the identical transformation of the group touch the line l between A_1 and B_1. Let λ gradually decrease in value; then the two conics separate, the point of contact of K approaching A_1 and the point of contact of K′ approaching B_1. When $\lambda = 0$, $k = -\infty$ and $k' = 0$: the conic K then becomes the degenerate conic AA_1, while K′ becomes BB_1. Thus the pseudo-transformation is produced by the two degenerate line conics AA_1 and BB_1. Let λ decrease still further and become negative; the point of contact of K approaches C_1 from one side and the point of contact of K′ approaches the same point from the other side. For some value of λ (usually an imaginary root of unity) the two conics coincide with the degenerate line conic CC_1. Let λ approach $-\infty$: the conic K then approaches its limiting form BB_1, while K′ approaches its limiting form AA_1. Thus we reach the second pseudo-transformation of the group which is produced by the same two degenerate conics BB_1 and AA_1; but now taken in the reverse order, showing that the two pseudo-transformations form an inverse pair. If a be taken not between 0 and 1, another combination of line conics will produce the pseudo-transformations.

The real group contains two real infinitesimal transformations which are inverse to one another. The conics K and K′ which determine these infinitesimal transformations differ by an infinitesimal amount from the coincident conics which produce the identical transformation of the group. (The case where the transformations of the group are not real will be discussed elsewhere.)

The analytical expressions for a one-termed group G_a can readily be written down from the properties pointed out above. Let the invariant triangle be ABC, and let the transformation T whose equations we wish to find be that one which transforms the point P (x, y, z) to the point P_1 (x_1, y_1, z_1). The anharmonic ratio of the pencil $C(ABPP_1)$ is λ; in terms of the co-ordinates of the points P and P_1 this is seen to be $\lambda = \frac{y_1}{x_1} : \frac{y}{x}$. Hence $\frac{y_1}{x_1} = \lambda \frac{y}{x}$. In like manner the ratio of the pencil $A(BCPP_1)$ is λ^{-a}; and its analytic expression in the co-ordinates of P and P_1 is $\lambda^{-a} = \frac{z_1}{y_1} : \frac{z}{y}$; hence we have $\frac{z_1}{y_1} = \lambda^{-a} \frac{z}{y}$. The anharmonic ratio of the pencil $B(CAPP_1)$ is similarly found to lead to $\frac{x_1}{z_1} = \lambda^{a-1} \frac{x}{z}$.

These three equations express the transformation T which transforms P to P_1; if we have a second transformation T_1 of the same group which transforms P_1 to P_2, its equations will be

$$\frac{y_2}{x_2} = \lambda' \frac{y_1}{x_1}; \quad \frac{z_2}{y_2} = \lambda'^{-a} \frac{z_1}{y_1}; \quad \frac{x_2}{z_2} = \lambda'^{a-1} \frac{x_1}{z_1}.$$

If we eliminate x_1, y_1, z_1 from these two sets of equations, we are able to express the co-ordinates of P_2 in terms of those of P. Setting $\lambda\lambda'=\lambda_1$, the elimination gives

$$\frac{y_2}{x_2}=\lambda_1\frac{y}{x}; \quad \frac{z_2}{y_2}=\lambda_1^{-a}\frac{z}{y} \quad : \quad \frac{x_2}{z_2}=\lambda_1^{a-1}\frac{x}{z}.$$

This shows that the two transformation T and T_1 are together equivalent to T_2: another transformation of the same group which transforms P directly to P_2.

This analytical expression for a one-termed group G_a is in fact identical with Lie's expression in homogeneous co-ordinates.

[*To be Continued.*]

Theory of Compound Curves in Railroad Engineering.

BY ARNOLD EMCH.

1. It is the purpose of this note to treat the problem of compound curves as it occurs in railroad engineering from a general geometrical stand point which enables us to discuss in an easy manner all the essential parts of the problem. It will be seen that the theory of compound curves is identical with the theory of two projective special pencils of circles. In Vol. III, No. 5, of the *American Mathematical Monthly*, the author has treated of projective pencils of circles in connection with a special complex of lines of the second degree*.

The theorem has been established:

The locus of the points of tangency of both tangent-circles of two pencils of circles is a bi-circular curve of the fourth order. The same curve is also produced by one of the pencils and the projective conjugate pencil of the other pencil.

This curve, of course, passes through the four fundamental points of the pencils of circles. Now we may take the special case where the two fundamental points of each pencil of circles coincide, or where all the circles of the pencil are tangent to a fixed line at a fixed point. This, however, represents precisely the case of compound curves in railroad engineering. Evidently the bi-circular curve of the fourth order, having also two finite double points, must degenerate into *two circles*.

2. In order to apply the previous result we will verify it directly. First we will write the equations of the two special pencils of circles in the form

$$U-2\lambda V=0, \qquad (1)$$
$$U'-2\lambda' V'=0,$$

and assume as the double points (coinciding fundamental points) of these pencils the points (0,0) and (a,b) fig. 1.

*A Special Complex of the Second Degree and its Relation with the Pencils of Circles.

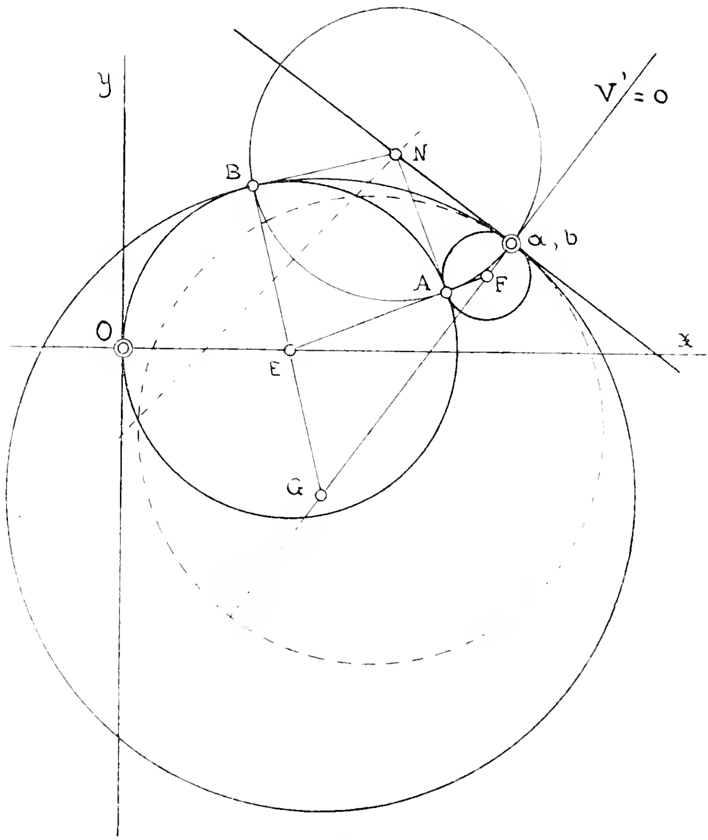


Fig. 1.

The circles of the first pencil we assume tangent to the y—axis ($x=o$), and those of the second tangent to the line

$$\frac{x-a}{y-b}=k.$$

$$\text{or}\quad x-yk-a+bk=o \qquad (2)$$

at the point (a,b).

Now any two circles of a pencil of circles determine all the other circles of the pencil, as indicated in the formulæ (1). We may also choose special circles of these pencils, for instance, the tangents at their double points and the zero-circles in these points. Thus we have

$$U=x^2+y^2, \qquad V=x$$

$$U'=(x-a)^2+(y-b)^2, \quad V'=x-yk-a+bk.$$

The equations of our special pencils of circles are therefore

$$x^2 + y^2 - 2\lambda x = 0. \tag{3}$$

$$(x-a)^2 + (y-b)^2 - 2\lambda'(x + yk - a - bk) = 0. \tag{4}$$

It is required that any of the circles (3) is orthogonal to as many circles of (4) as is possible. Designating the co-ordinates of the centers of any two circles by (α, β) and (α', β') and their radii respectively by ρ and ρ', the condition for the orthogonality of the two circles is

$$(\alpha - \alpha')^2 + (\beta - \beta')^2 = \rho^2 + \rho'^2. \tag{5}$$

Associating the values α, β, ρ for a special value of λ with the corresponding circle of the pencil (3) we have

$$\alpha = \lambda, \quad \beta = 0, \quad \rho = \lambda.$$

In the same way we associate the values α', β', ρ' with the pencil (4) and have

$$\alpha' = a + \lambda', \quad \beta' = b + \lambda' k, \quad \rho' = \lambda' \sqrt{(1 + k^2)}.$$

Substituting these values in equation (5) there is

$$(\lambda - a - \lambda')^2 + (\lambda' k + b)^2 = \lambda^2 + \lambda'^2 (1 + k^2),$$

or after some reductions

$$2\lambda'(a - \lambda + bk) = 2\lambda a - a^2 - b^2,$$

whence

$$\lambda' = \frac{2\lambda a - a^2 - b^2}{2a - 2\lambda + 2bk}. \tag{6}$$

According to this condition, to each value of λ belongs one and only one value of λ', i. e., taking any circle of the pencil (3), there is one and only one circle in the pencil (4) orthogonal to that circle. If we substitute in formula (6) for λ' and λ successively the values:

$$\lambda' = \alpha' - a, \ \lambda = \alpha, \text{ and } \lambda' = \frac{\beta' - b}{k}, \ \lambda = \alpha,$$

we obtain the two expressions

$$\alpha' = a + \frac{2a\alpha - a^2 - b^2}{2a - 2\alpha + 2bk},$$

and

$$\beta' = b + \frac{k(2a\alpha - a^2 - b^2)}{2a - 2\alpha + 2bk},$$

or

$$\alpha' = \frac{a^2 - b^2 + 2abk}{2a - 2\alpha - 2bk}.$$

$$\beta' = \frac{2a(b+ak)-2ab-a^2k}{2a-2a+2bk}.$$

From this is seen that the centers of corresponding orthogonal circles in the two pencils form projective point-ranges. The two pencils are, therefore, also projective and their product is a bi-circular curve of the fourth order which degenerates into two circles. To obtain the equation of these circles we have to eliminate λ and λ' from the following equations:

$$x^2+y^2-2\lambda x=0 \qquad \text{(I)}$$

$$(x-a)^2+(y-b)^2-2\lambda'(x-yk-a+bk)=0 \qquad \text{(II)}$$

$$\lambda'=\frac{2\lambda a-a^2-b^2}{2a-2\lambda-2bk}. \qquad \text{(III)}$$

From (I) follows

$$\lambda=\frac{x^2+y^2}{2x},$$

hence

$$\lambda'=\frac{a(x^2+y^2)-x(a^2+b^2)}{2x(a-bk)-(x^2+y^2)}.$$

Substituting this value in II, there is

$$\Big[(x-a)^2+(y-b)^2\Big]\Big[2x(a-bk)-(x^2+y^2)\Big]-$$
$$-2\Big[x-yk+bk-a\Big]\Big[a(x^2+y^2)-x(a^2+b^2)\Big]=0.$$

After some transformations and reductions this equation may be written in the conspicuous form

$$\Big[x^2+y^2-x(a-bk+b\sqrt{1+k^2})-y(b+ak-a\sqrt{1+k^2})\Big]\times$$
$$\Big[x^2+y^2-x(a-bk-b\sqrt{1+k^2})-y(b+ak+a\sqrt{1+k^2})\Big]=0. \qquad (7)$$

This is the equation of the product of the two projective special pencils of circles and, evidently, represents two circles

$$x^2+y^2-x(a-bk-b\sqrt{1+k^2})-y(b+ak+a\sqrt{1+k^2})=0, \qquad (8)$$

$$x^2+y^2-x(a-bk+b\sqrt{1+k^2})-y(b+ak-a\sqrt{1+k^2})=0, \qquad (9)$$

which both pass through the origin and through the point (a,b), the two finite double points.

The co-ordinates of the center of the first circle are

$$m = \frac{a - bk - b\sqrt{1+k^2}}{2},$$

$$n = \frac{b + ak + a\sqrt{1+k^2}}{2}. \qquad (10)$$

and of the second

$$m' = \frac{a - bk + b\sqrt{1+k^2}}{2},$$

$$n' = \frac{b + ak - a\sqrt{1+k^2}}{2}. \qquad (11)$$

The radii of these circles respectively are $\sqrt{m^2+n^2}$ and $\sqrt{m'^2+n'^2}$. It is easily verified that

$$(m-m')^2+(n-n')^2 = m^2+n^2+m'^2+n'^2.$$

This, however, is the condition that two circles are normal to each other. Hence:

The two circles forming the locus intersect each other at right angles.

From this follows, that the points P, Q, O, M, T in fig. 2, all lie on the same circle with the line $\overline{PQ}$ as a diameter.

3. The normal pencil of circles of the pencil (4) is obtained by considering the normal to the straight line (2), $\frac{x-a}{y-b} = k$, which is

$$xk + y - ak - b = 0,$$

and the zero-circle at the point (a,b) as two circles of the pencil. The required normal pencil is therefore given by the equation

$$(x-a)^2+(y-b)^2-2\lambda''(xk+y-ak-b) = 0. \qquad (12)$$

For a fixed value of λ'' the co-ordinates of the center of the corresponding circle are

$$\alpha'' = a+\lambda''k,$$

$$\beta'' = b+\lambda'',$$

and

$$\rho'' = \lambda''\sqrt{1+k^2}.$$

The condition for the tangency of the circle (12) and of the original circle (3) is

$$(\alpha-\alpha'')^2+(\beta-\beta'')^2=(\rho\pm\rho'')^2.$$

Substituting the values of α, β, ρ and α'', β'', ρ'' and developing we find the expression

$$\lambda''=\frac{a^2+b^2-2a\lambda}{2\lambda(k\pm\sqrt{1+k^2})-2(ak+b)}.$$

which shows that to each value of λ belong two values of λ'', or that each circle of the pencil

$$x^2+y^2-2\lambda x=0$$

is touched by two and only two circles out of the pencil

$$(x-a)^2+(y-b)^2-2\lambda''(xk-y-ak+b)=0.$$

These results are all well known from the theory of pencils of circles and it is for the present purpose not necessary to develope further details.

We will now show that any circle C' of the pencil

$$(x-a)^2+(y-b)^2-2\lambda'(x+yk-a-bk)=0$$

which is normal to a certain circle C of the pencil

$$x^2+y^2-2\lambda x=0$$

cuts the latter circle in two points, A and B, which are precisely the points of tangency of the two possible tangent circles C_1'' and C_2'' out of the normal pencil of circles

$$(x-a)^2+(y-b)^2-2\lambda''(xk-y-ak+b)=0.$$

In fig. 2, C_1'' and C_2'' are the two circles tangent to the circle C.

Now the tangent to C or C_2'' at B, intersects the tangent V_1 in the point (α', β'), or N, such that NB = NM = NA. Hence the normal circle of C', C_1'', and C_2'' pass through A and B. q. e. d.

The locus of the points of tangency of the circles of our special pencils of circles is, therefore, the same as the product of projectivity of one of the pencils with the normal pencil of the other, i. e., consists of two circles which both pass through M and O.

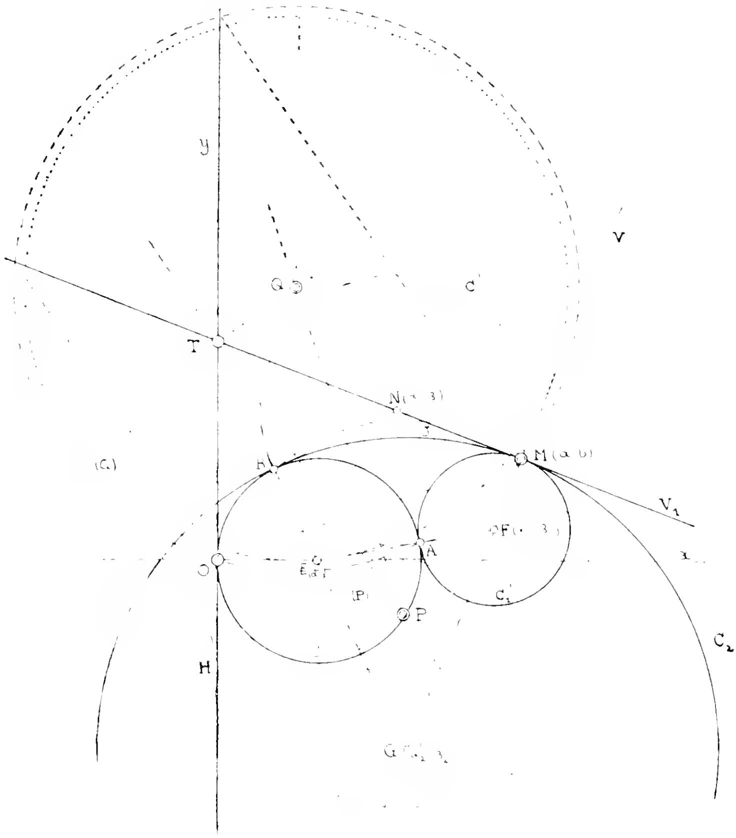


Fig. 2.

The equation of the line V' is

$$xk-y-ak+b=0,$$

or

$$\frac{xk}{\sqrt{1+k^2}}-\frac{y}{\sqrt{1+k^2}}-\frac{ak-b}{\sqrt{1+k^2}}=0,$$

and of the y-axis

$$x=0.$$

The equations of the inner and outer bi-sector are, therefore, given by the expressions

$$x(k-\sqrt{1+k^2})-y-ak-b=0, \qquad (13)$$

and

$$x(k - \sqrt{1 + k^2}) - y - ak - b = 0. \qquad (14)$$

As is easily verified, the co-ordinates (10) which represent the center of one of the circles of the locus, satisfy the first of these equations and those of (11) the second.

If we now use the technical terminology, i. e. designate the arcs MB, OB, MA, AO, etc., as arcs of compound curves and their points of tangency as points of compound curves we may state the theorem:

The locus of all points of compound curves between two tangents and points consists of two circles which pass through the two given points on two given tangents and whose centers lie on the bi-sectors of the two given tangents.

To construct these centers we may, therefore, connect A with B, erect a perpendicular to AB in the middle of AB, which will intersect the bi-sectors in the required points P and Q.

Considering any point of compound curve as B, then it lies on the same right line with the centers of the corresponding arcs of compound curves OB and MB. Since O and B lie also on the circle of the locus of points of compound curves with the center P, the perpendicular to the chord OB through E passes through P. Hence

$$PEX = PDG.$$

This means that every ray connecting the centers of two compound curves whose point of tangency, or point of compound curve, lies constantly on one of the circles of the locus, is tangent to a fixed circle which is concentric with the circle of the locus. The same can be proved for the ray EF. The two concentric circles one with P, the other with Q as a center, in fig. 2, are designated by (P) and (Q).

The circle of the locus with P as a center intersects the y-axis and the line V' in two other points J and H, such that JM = OH, and TM + TO = OH. Evidently OH is equal to the diameter of the circle (P). In a similar manner it is proved that the diameter of the circle (Q) is equal to TM − TO. To sum up we may say:

The locus of points of compound curves of all compound curves between two tangents, TM and TO, and two tangent points, M and O, consists of two circles which pass through the points M and O and whose centers lie on the bi-sectors of the tangents TM and TO.

By this condition, the centers P and Q of these circles and, therefore, the circles themselves are perfectly determined. In all compound curves the radial lines through the points of compound curves belonging to this *system are all tangent to either one or the other of two fixed circles having as their centers the points P and Q and for their radii the values*

$$\frac{TM + TO}{2} \quad \text{and} \quad \frac{TM - TO}{2}.^{*}$$

The points P, Q, O, M, T, in fig. 2, all lie on the same circle, having the line PQ as a diameter.

4. Among the great number of special cases we will consider the problem where the two tangents are parallel. The general theorem and construction still hold, so that the solution is simply a matter of reduction for special values. To find the equations of the locus we have to put $k = \infty$ in formulæ (8) and (9). Observing that for an indefinitely large value of k

$$\lim(bk - b\sqrt{1 + k^2})_{k = \infty} = 0$$

$$\lim(ak - a\sqrt{1 + k^2})_{k = \infty} = 0$$

these equations become respectively

$$bx - ay = 0 \qquad (15)$$

and

$$x^2 + y^2 - ax - by = 0, \qquad (16)$$

or

$$(x - \frac{a}{2})^2 + (y - \frac{b}{2})^2 = \frac{a^2 + b^2}{4}. \qquad (17)$$

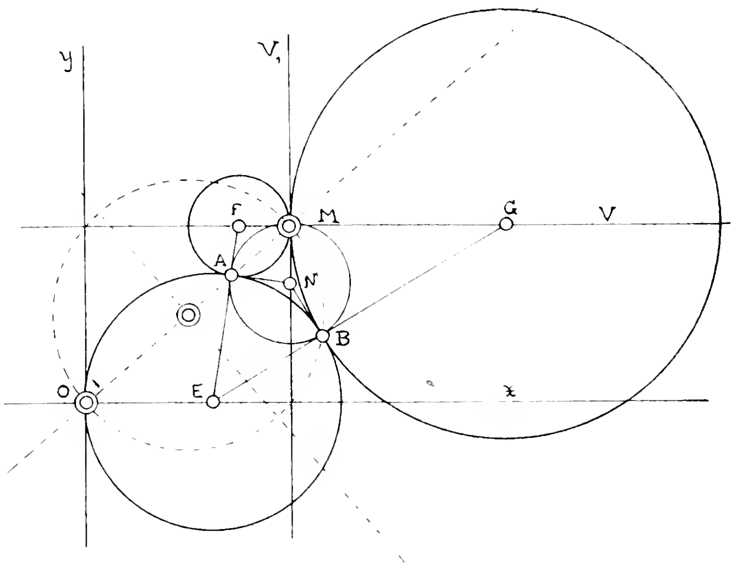
The meaning of the equations (15) and (16) is clear: The first represents a straight line through the point (a,b) and the origin; the second a circle with the point $(\frac{a}{2}, \frac{b}{2})$ as a center, and OM as a diameter, fig. 3.

*In practical treatises on this subject the conception of compound curves is not given under this general point of view. Thus in Mr. W. H. Searles's treatise on Field Engineering the following restriction is made:

"A compound curve consists of two or more consecutive circular arcs of different radii, having their centers on the same side of the curve; but any two consecutive arcs must have a common tangent at their meeting point, or their radii at this point must coincide in position."

This result is also obtained from the expressions (10) and (11). For $k=\infty$ the first indicates that the center of the circle (8) is at

Fig. 3.



an infinite distance in a direction whose trigonometric tangent is

$$\lim \left[\frac{b - ak - a\sqrt{1+k^2}}{a - bk - b\sqrt{1+k^2}}\right]_{k=\infty} = \frac{a}{b}.$$

The second expression gives for the co-ordinates of the circle (9)

$$x = \frac{a}{2},\ y = \frac{b}{2}.$$

The Visual Perception of Distance.

BY JOHN F. ROUSE.

If we omit Descartes, the scientific study of the perception of distance began with Bishop Berkeley. Assuming that a difference in the distance of a point can make no difference in the nature of the retinal image, since "distance being a line directed endwise to the eye projects only one point upon the fund of the eye—which point remains invariably the same, whether the distance be greater or smaller," he concluded that distance could not be a visual sensation, but must be an intellectual "suggestion," due to some non-visual experience, and this experience he considered *tactile*.

According to his view, visual perception of distance is the acquired interpretation of light and color differences in terms of distance already gained by skin and muscle. To say that an object is a certain distance, is to assert that so much sensation of skin and muscle must be had before the object can be touched.

But the notion that distance is not a visual, but a tactile form of consciousness, suggested by visual signs, though endorsed by many later psychologists, is by no means *generally* accepted. Some argue that the estimation of distance by the eye, is, as Berkeley said, a result of suggestion and experience, but that visual experience alone is adequate, and this Berkeley denied. It is further maintained that depth feeling is just as optical in its nature as either hight or breadth, and that in the absence of motion of the body, or any part of it, toward or away from objects observed, the movement of the objects themselves may be substituted, with similar experience resulting.

Persons blind from birth and acquiring their sight in later years, have thus *at first* experienced distance by touch, and *afterward* both by touch and sight. As these persons (about twelve cases having been reported) *generally* maintain that all objects seemed to be, when first seen, in one plane near the globe of the eye, and that optical perception of their distance was learned by "association" with the tactile sense, it seems that a strong argument for Berkeley's theory has been found. But there have been a few people of the above class to whom objects, when first seen, did not appear in the same plane, but nearer and farther, although, of course, experience enabled them to locate the objects more accurately.

The difficulty of finally settling the question of whether or not distance is a visual as well as a tactile form of consciousness, is greatly augmented by the fact that, although we find persons who at one time perceived distance tactually, and later both tactually and visually, as above, we do not have at hand to compare with them, persons who at first have no tactile sensations, but only visual, and then later both kinds of sensations.

Numerous experiments and observations have led psychologists to conclude that distance perception may be regarded as the product of three ever varying factors: *retinal, muscular*, and *intellectual*, as may be seen in the following so-called "clues"—accommodation, double and disparate images, difference in parallactic displacement of objects when the head is moved, faintness of tint, dimness of outline, and smallness of retinal images of objects named and known, together with various comparisons and allowances made, voluntarily and involuntarily. All of the above have *something to do* with our notions of "far" and "near;" but when we consider that these "aids" have a way of overcoming and overbalancing each other, especially when influenced by the presence of some other sensible quality in the object, and that definite tactile and retinal modifications do not accompany differences in distance, and further, that there are many other irregularities, it then becomes evident to us that the act of judging distance follows no simple law. But that there are certain tendencies shown in our acts of judgment, a number of psychological experiments have plainly indicated; and it was to continue the examination of various estimations of distance that the following investigation was made. The accompanying drawing is intended to represent the large room in which the tests of judging given distances were made, and to show the mechanism and arrangement of the apparatus used.

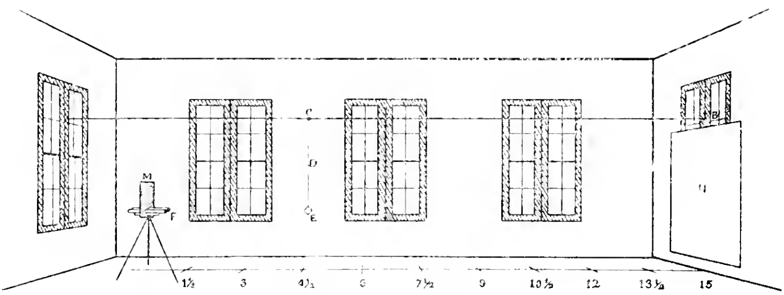


Fig. 1.

In the figure above, E represents a light-brown paste-board box, suspended with an invisible wire (D) to a larger wire (AB), by means of a smooth ring (C) capable of sliding back and forth. Four such cubical boxes were employed, their edges measuring, respectively, 2, 3, 4, and 5 centimeters. The larger wire (AB), supporting them, was tightly drawn in a horizontal position overhead and extended the entire length of the 52 foot room. Two tin tubes (F), well smoked inside, were so fixed at one end of the room that they were directly below the horizontal wire (AB), parallel with it, and in the same horizontal plane with the boxes when suspended successively. A curtain of plain dark material (H) was placed at the further end of the room, just below and at right angles to the wire drawn above.

Two persons were required (besides the subject) to perform the experiment, one to move the boxes back and forth (with a long stick or pointer) to correspond with divisions of a chalk line drawn upon the floor directly below the horizontal wire (AB), and another to give the subject views of the boxes when placed at proper positions, not permitting him to see them moved, or to know when one box was exchanged for another, and to keep account of estimations made, thus leaving the subject free to judge the distance of the objects at different positions. Views at the proper time were given by uncovering the farther ends of the tubes, a large piece of paste-board (M), perforated with holes through which the nearer ends of the tubes passed, cutting off all view in front of the subject except through the tubes.

In the experiment ten young men from the higher classes of the university were used as subjects, each sitting at the opposite end of the room from the curtain, and judging the distance of each of the four boxes, when placed in a definite series of positions. When judging distance at the first of the experiment the subject looked through the tubes and saw nothing but the suspended boxes and the screen (H), without knowing the size of the former or the distance of the latter. Then afterwards he was shown the boxes, and allowed to handle them and to learn their respective dimensions, informed of the distance of the curtain, and permitted to judge the distance of the objects, when again suspended, but without looking through the tubes. In this way the judgment was assisted in every way possible, except in seeing the objects moved, which was in no case permitted.

In the former case, when the tubes were used, the same definite series of positions was estimated in three ways: with *right* eye, with *left* eye, and with *both* eyes, one tube being closed for monocular vision. In seeing directly (without the tubes) only binocular vision was used.

In addition to the above series of tests a shorter one was given, using the tubes and binocular vision (with 2 in. box), to illustrate Wundt's Law regarding judgment of the distance of objects when moving closer and closer, or farther and farther.

Each of the ten subjects made observations requiring a sitting of an hour or more. Care was taken to have the room lighted evenly in different parts, and the same set of tests was given to each subject in as nearly the same manner as possible.

Below is shown a tabulated report of 160 average judgments, made from 1,600 tests upon ten subjects. Arabic numerals at the top of each of the four columns are used to indicate different sized boxes used, the boxes being named in the order of their sizes, beginning with the smallest box.

TABLE I.

	Using the tubes, and seeing only objects and curtain beyond, size of former and distance of latter not being known.												Without tubes, size of objects and distance of curtain known.			
Real Dist	(I.) Right Eye				(II.) Left Eye				(III.) Both Eyes.				(IV.) Both Eyes (free)			
Meters	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
$1\frac{1}{2}$	1.95	1.50	1.05	1.10	2.05	1.55	1.55	1.30	1.75	1.65	1.40	1.25	1.50	1.40	1.45	1.35
3	3.75	2.60	2.45	2.35	3.65	3.05	2.65	2.15	3.45	2.80	2.65	2.55	2.80	3.05	3.00	2.85
$4\frac{1}{2}$	5.60	3.75	3.65	3.15	5.35	4.05	3.65	3.40	4.25	3.70	3.95	3.25	4.05	4.40	4.30	4.25
6	7.55	5.20	5.05	4.55	7.00	5.25	4.70	4.65	5.45	4.65	4.90	4.45	5.55	5.70	5.80	5.50
$7\frac{1}{2}$	8.75	6.75	6.15	5.60	8.[illegible]5	6.45	6.00	5.7[illegible]	6.55	5.95	6.45	5.50	6.35	6.90	7.10	6.85
9	10.50	8.00	7.45	6.55	10.15	[illegible].90	7.20	6.60	[illegible].85	7.25	7.45	6.75	7.65	8.35	8.40	8.00
$10\frac{1}{2}$	11.55	9.40	8.5[illegible]	7.65	11.35	8.85	8.45	7.60	9.15	8.35	8.50	8.20	9.55	9.55	9.10	9.35
12	12.45	10.55	9.[illegible]0	8.[illegible]5	12.45	10.15	9.50	8.70	10.45	9.45	9.65	9.30	[illegible]0.[illegible]5	11.10	11.55	11.40
$13\frac{1}{2}$	13.25	11.30	10.80	9.55	13.15	11.00	10.65	9.50	11.55	10.75	10.75	10.50	12.80	12.90	13.15	13.00
15	14.85	13.00	12.30	10.85	13.60	12.[illegible]0	11.90	10.90	12.90	12.40	12.0[illegible]	11.20	14.80	14.80	14.90	14.80
$82\frac{1}{2}$	99.20	72.05	67.15	60.00	87.50	70.95	66.25	60.60	73.35	66.95	67.70	62.95	76.10	78.15	78.75	77.35

It will be seen that the average estimations generally vary inversely as the size of the object observed, i. e., as the box used is larger, the distance judged is shorter, and vice versa. E. g., for 3 meters in (I), the smallest box (1) was thought to be 3.75 meters distant, and the next larger ones, (2), (3) and (4), 2.60, 2.45 and 2.35 meters, respectively. This is a common illusion, and it is natural that it should be shown here. To this tendency there is but one exception in (I) and one exception in (II), while in (III) there are a half dozen exceptions, and in (IV) the illusion almost wholly disappears, showing that in unassisted binocular vision there is a *slight* tendency to overcome the mistake of judging a larger object to be nearer, and a smaller object to be farther, while in binocular vision assisted in different ways the illusion, in a great measure, disappears. A comparison of the sums of the averages for each box in different columns will show the same relation more plainly. Observing the same order of boxes, the smallest first, we find the sums as follows:

TABLE II.

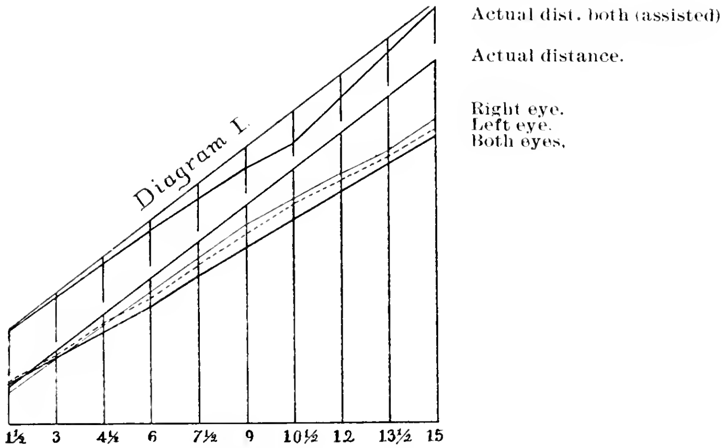
	1	2	3	4
(I)	90.20	72.05	67.15	69.00
(II)	87.70	70.45	66.25	60.60
(III)	73.35	64.95	6[illegible].70	62.45
(IV)	76.10	78.15	75.75	77.35

By averaging (1), (2), (3) and (4) of each of the four columns of table No. 1, the final average estimations (with different kinds of vision) appear as follows:

TABLE III.

Real Dist.	(I.) Right.	(II.) Left.	(III.) Both	(IV.) Both (free)
1½	1.3750	1.6125	1.5125	1.4250
3	2.7875	2.9500	2.8625	2.9250
4½	4.0375	4.1125	3.7875	4.2500
6	5.5875	5.1000	4.8625	5.6375
7½	6.8125	6.7250	6.1125	6.8000
9	8.4250	7.9625	7.3250	8.1000
10½	9.2875	9.0125	8.5500	9.3875
12	10.3625	10.1500	9.[illegible]125	11.2750
13½	11.2250	11.0750	10.8875	12.9625
15	12.7500	12.2750	12.1250	14.8250
82½	72.3500	71.3250	67.[illegible]	77.5875

Table III seem to indicate that, within a scope of 15 meters, distance is nearly always *underestimated*, and appears *less* to the *left* eye than to the *right* eye, and less to *both* eyes (unassisted) than to the *left* eye, as is more plainly shown by the following diagram. The perpendicular lines represent on a small scale the actual distances indicated at their lower extremities: and the *length* of these perpendiculars *from the horizontal base line to where they are cut* by the different curves, shows the *respective estimates* of these "actual distances."



Comparing the sum of the actual distances (82½ m.), shown in table III, with the sums of the estimates in columns (I), (II) and (III), the order of accuracy is shown as follows:

Real Distance.	Right Eye.	Left Eye.	Both, unassisted.
82½	72.35	71.325	67.7375

A previous experiment was made in nearly the same manner as this one, except that no distance greater than 10 meters was shown (instead of 15, as in this), and that a greater number of subjects were used with fewer tests each, and finally, that instead of giving the tests in the order of right eye, left eye and both eyes, it was given in this order: both eyes, right eye and left eye; so the order of giving the tests could not have influenced the results of the two experiments to be similar to each other.

The averages of the results obtained from the preceding experiment were as follows:

Real Distance.	Right Eye.	Left Eye.	Both, unassisted.
92½	87.96¼	86.03¾	79.72½

Simplifying these two sets of results by reducing the real distances (82½ and 92½) to unity, we have the following comparisons:

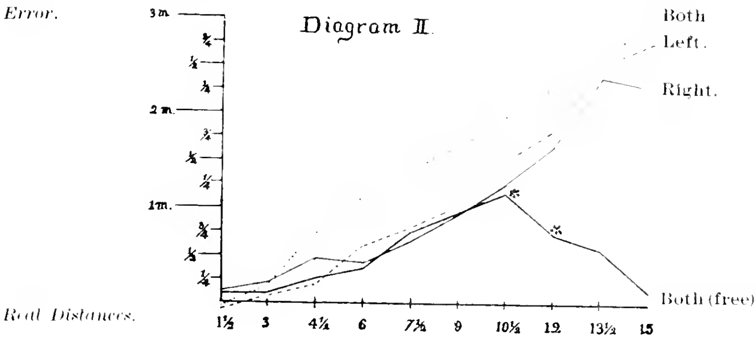
		Estimations.		
	Real Distance.	Right.	Left.	Both.
For 15 meters,	1 meter.	.87	.86	.82
For 10 meters,	1 meter.	.95	.93	.86

These figures show the accuracy of judgment to be greater within a scope of 10 meters than 15, which might have been expected, since in the curve in diagram I the oblique lines representing the relative judgments, diverge more and more from the true line as the distance increases. This is better shown below:

TABLE IV.

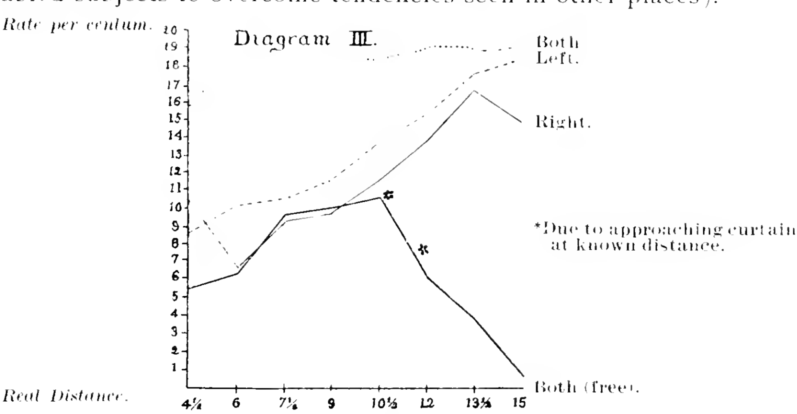
Dist.	Actual Errors.				Per centum of Errors.			
	Right.	Left.	Both.	Both(free)	Right.	Left.	Both.	Both(free)
1½	.12½	—.11¼	—.01½	.07½	8 33	—7.50	—.83	5.00
3	.21¼	.05	.13¾	07½	7.08	1.66	4.58	2.50
4½	.46¼	.38¾	.71¼	25	10.27	8.61	15.83	5.55
6	.41¼	.60	1.13¼	.36¼	6 87	10 00	18 78	6.04
7½	.68¾	.77½	1.38¾	.70	9.16	10.33	18.50	9 33
9	.87½	1.03¾	1.67½	.90	9.72	11.52	18.61	10 00
10½	1.21¼	1.43¼	1.95	1.11¼	11.54	13.69	18.57	10.59
12	1.63¾	1.85	2.28¼	.72½	13 64	15.41	19.06	6.04
13½	2.27½	2 42½	2.61¼	.53¾	16.85	17.96	19.35	3 98
15	2.25	2.72½	2.87½	11½	15 00	18.16	19.16	.76
82½	10.15	11.17½	14 96¼	4.91¼				

The curves in the following diagram show the *actual error* for distances marked below:



*Due to approaching curtain at known distance.

Below is a plot showing *per centum* of errors (omitting first two positions, where close view and some imperfections of method enabled subjects to overcome tendencies seen in other places).



The above shows that the error has a strong tendency to increase as the distance increases, with few exceptions.

The following is a report (illustrated by curve) of a series of 190 tests to explain Wundt's Law:

Diagram IV.

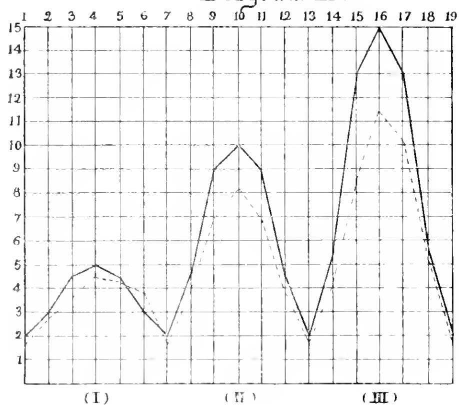


TABLE V.

Dist..	2.00	3.00	4.50	5.00	4.50	3.00	2.00	4.50	9.00	10.00	9.00	4.50	2.00	3.50	13.00	15.00	13.00	5.50	2.00
Est..	1.60	2.[illegible]5	3.65	4.45	4.40	3.75	1.65	3.95	6.90	8.15	6.90	3.90	1.55	1.15	8.60	11.35	10.10	5.30	1.55

In the table a series of positions is shown, such that the object observed is part of the time approaching the observer and part of the time receding, at equal distances each. In the plot the heavy line represents the true distances and the dotted line the judgments, the left hand side of the figures showing the forward movement, the right hand side the backward.

Remembering that the relative accuracy is shown in the above diagram by the tendency of the two lines to come together, we conclude that the accuracy in (I) and (III) is greater when the object seen approaches than when it recedes, although in (II) there seems to be no marked tendency either way. A numerical statement of all the different judgments is shown in the following table.

In the following table Roman numerals correspond to those in the preceding diagram.

TABLE VI.

Parts.	Real Distance.	Estimations. Approaching.	Estimations. Receding.
(I.)	14.50	14.25	12.45
(II.)	25.50	20.50	20.65
(III.)	35.50	28.30	25.65
Total	75.50	60.05	58.75

From the above numerical statement it would appear that, within a range of 15 meters. the accuracy of judging the distance of an approaching object is greater than in determining that of a receding one by the ratio of 63.05 to 58.75.

In making estimates it was noticed that subjects quite generally moved the head in different ways while looking at the objects, apparently to give motion to the eye. Many were unable to tell with which eye they saw the object, often mistaking monocular for binocular vision, and one kind of monocular vision for another. Some subjects made their estimates quickly, others slowly. Some quickly at one time, but slowly at another. A comparison of the results of these different subjects did not indicate that the time element entered into the problem at all. Each subject required at least an hour for the whole series of tests.

GENERAL CONCLUSIONS.

1. There is a strong tendency to *underestimate* visible distances.

2. The illusion of judging a large object to be nearer and a small one farther, is *less common* in *binocular* than in *monocular* vision, although the value of this advantage must not be considered too high, as it only shows a less variable relation between different estimates of binocular vision, all of which may be farther wrong than the aggregate of monocular estimates, though severally more variable.

3. The greatest accuracy of judgment is that attained in *binocular vision, assisted* by various clues (as size of objects, comparison of other distances).

4. When vision is in no way assisted, the order of accuracy is: *right eye, left eye*, and *both eyes*, in the ratio of .87: .86: .82 (true distance being *unity*).

5. The distance of *approaching* objects is more truly judged than that of *receding* ones, in the ratio of 63.05 to 58.75 (true distance being 75.50).

6. As the true distance increases the *error steadily increases* also.

7. Distance perception has *little dependence* upon the *time* consumed in the process.

8. *Movement of head* to give motion to eye appears to be a factor in distance perception.

The Limitations of the Composition of Verbs with Prepositions in Thucydides.

DAVID H. HOLMES.

In Greek the subject of composition in general has received but little attention. So far as I know, the particular chapter which I have chosen has not been treated at all. But no attempt has been made in this paper to discuss such problems as the change of meaning caused by composition, or the case-constructions of compounds, or the influence of the preposition on the voice of the verb. As these subjects have been uniformly passed over by grammarians, we cannot reproach ourselves for treating them with the same respect.

The object of the present investigation is rather to see from an examination of the material offered by Thucydides what the indications are from this source regarding the principles underlying the composition of verbs with prepositions, and the limitations affecting the operation of these principles.

If any justification is needed for the undertaking of such a task, it is found in the interest and instruction which attach to the answers to such questions as: the range of prepositions of the different verbs; the relative affinity of verbs for prepositions; the lines of favoritism between verbs and prepositions; causes and results of loss of color in the preposition. The same inquiries will be extended to diprothetics and triprothetics.

Such is the modest aim of this paper. But whatever the results may be, they can not of course be considered binding, except so far as the language of Thucydides is concerned, until, at least, other authors are investigated in the same manner.

With this aim in view, then, I shall present the following material: First, A consideration of the individual prepositions; Second, Statistical tables for monoprothetics, diphrothetics and triprothetics; Third, An examination of the statistics.

I. A Consideration of the Individual Prepositions.

The test of a proper preposition is its ability to combine with verbs. It is only necessary to strike ἀμφὶ out of the list of proper prepositions to get the range of combinable prepositions in Thucydides. They all occur in their simple form (ἀνὰ and ἀμφὶ twice each). The compounds of ἀμφὶ like the preposition, are mostly confined to poetry.

ἀνά. The case of ἀνὰ is different. While the simple preposition is confined mostly to phrases and poetry, it survives in composition, having a range of 77 verbs in Thucydides. Its favorite verb is χωρέω with which it occurs 144 times. It is the favorite preposition of 5 verbs, not counting its exclusives. It combines exclusively with 17 verbs, of which 9 are ἅπαξ εἰρημένα. In one of these, ἀνοίγνυμι, the place of the simple has been usurped by the compound in prose. The simple οἴγνυμι belongs to poetry. In ἀναλίσκω and ἀναλόω, we have probable usurpations of old simples which had passed out of the language in pre-historic times. Ἀνὰ does not occur as first element in diprothetics or triprothetics. The range of the simple ἀνά, like ἀμφί, is largely poetic.

ἀντί. The simple preposition ἀντὶ occurs 52 times in Thucydides. It is found in composition with 80 verbs, of which 48 are monoprothetic, 27 diprothetic and 5 triprothetic. No other preposition occurs more than once in triprothetics Its favorite verb is ἔχω with which it combines 41 times. Other favorites are ἵστημι and εἶπον. It combines exclusively with 10 verbs, of which 7 are ἅπαξ εἰρημένα.

ἀπό. The simple ἀπὸ occurs 634 times. It has a combinable range of 114 verbs, of which 112 are monoprothetic, and 2 diprothetic. The favorite verb is ἱκνέομαι, in composition with which it occurs 192 times. It is the favorite preposition of 22 verbs, not counting its exclusives. It is the exclusive preposition of 23 verbs, of which 15 are ἅπαξ εἰρημένα. In ἀπαντάω, we have a usurpation of the simple ἀντάω, which is limited to poetry. The compounds ἀποκτείνω, its passive ἀποθνήσκω, and ἀφικνέομαι are equivalents of their respective simples, except in the perfect and pluperfect of θνήσκω, which are rarely compounded in Attic Greek, never in Thucydides. In ἀπόλλυμι, we have a complete usurpation, the form ὄλλυμι being restricted to poetry. Homer has ἄπο....ὄλλυμι in so-called tmesis, where the prepositional element was strongly felt. To say, however, with Liddell and Scott, that ἀπόλλυμι is a stronger form of ὄλλυμι, presupposes a weaker ὄλλυμι for Attic prose, which does not exist. ἀπόλλυμι is stronger than ἀποκτείνω, just as ὄλλυμι is stronger than κτείνω. Ἀπεχθάνομαι is a usurpation of the poetic ἔχθω.

διά. In the simple form διὰ occurs 534 times. It has a range of 101 verbs, 98 of which are monoprothetic and 3 diprothetic. Φθείρω is its favorite verb with which it combines 151 times. It is the favorite of 14 verbs, though the favoritism is not so sharply defined as in the prepositions treated above. It has an exclusive range of 18 verbs, of which 10 are ἅπαξ εἰρημένα. In διαφθείρω, we have an effort to usurp φθείρω, the proportion standing 3.75:1. The place of νοέω, largely confined to poetry, is taken in prose by its compounds; διὰ being its favorite preposition by more than 4 to 1.

ἐξ. The preposition ἐξ occurs in simple form 897 times. It has a range of 89 verbs of which 85 are monoprothetic, 4 diprothetic. Ἔρχομαι (ἐλθεῖν) is its favorite verb, with which it combines 47 times. Πέμπω is also a marked favorite. Ἐξ is favorite preposition of eight verbs, not counting its exclusives. The favoritism of ἐξ for verbs or of verbs for ἐξ is not strongly marked. Its exclusive range consists of 17 verbs of which 9 are ἅπαξ εἰρημένα. The simple ἀρτύω is superseded by the compounds in κατὰ and ἐξ, ἐξ alone occurring in Thucydides.

ἐν. The preposition ἐν occurs 1794 times, in which respect it stands first in the list. This fact is rather remarkable considering that it governs but one case. It has a combinable range of 67 verbs, 55 being monoprothetic, and 12 diprothetic. Its favorite verb is δίδωμι with which it combines 38 times. It is the favorite preposition of three verbs, the preference being marked with πίμπρημι which is superseded in prose by ἐμπίμπρημι. Its exclusive range consists of 13 verbs of which 11 are ἅπαξ εἰρημένα. In ἐναντιόομαι and ἐμπίμπρημι we have usurpations, the simple of the former being restricted to Ionic Greek, of the latter to poetry.

ἐπί. The simple ἐπὶ occurs 1216 times. It has a range of 156 verbs in which respect it heads the list of prepositions. 117 are monoprothetic, 39 diprothetic. Its favorite verb is εἶμι; γίγνομαι and ἐλθεῖν are also favorites, all three having ἐπὶ for their favorite preposition. It is the favorite prepositional element of 23 verbs. Here as in all cases exclusives are not counted. It has an exclusive range of 20 verbs, of which 5 are ἅπαξ εἰρημένα. There is no case of complete usurpation with ἐπὶ in Thucydides. Though the simple of ἐπιμελέομαι or ἐπιμέλομαι does not occur, yet its meaning is sharply differentiated from that of the simple. The spheres are different.

ἐς. Ἐς occurs 1692 times in Thucydides, ranking next to ἐν, and like ἐν, governing but one case. Its range of verbs is limited to 23, all of which are monoprothetic. Its combinable range is less than that of any other preposition in proportion to the number of its occurrences as a simple preposition. Its favorite verb is βάλλω, with which it is found 65 times, and of which it is also a favorite preposition, ranking next to πρός. It is the favorite preposition of only one verb, ἀκοντίζω, and has no exclusives and no usurpations.

κατά. The preposition κατὰ occurs in simple form 861 times. It has a range of 105 verbs, 104 monoprothetic and 1 diprothetic. Ἵστημι is its favorite verb with which it occurs 260 times, and of which it is also the favorite preposition. It is the favorite preposition of 16 verbs and has an exclusive range of 25 verbs, of which 12 are ἅπαξ εἰρημένα. In κατάγνυμι we have a usurpation in the active voice. Καθέζομαι. κάθημαι and καθίζω are usurpations. The simples are poetic, ἕζομαι and ἵζω are late Greek.

μετά. Μετὰ occurs 619 times. It is restricted in the range of its verbs to 24, of which 22 are monoprothetic and 2 diprothetic. Its favorite verb is ἵστημι. Leaving out its only exclusive, μεταμέλει, it can not be said to be the favorite preposition of any verb. Μετὰ is not a general favorite in composition.

ξύν. Ξὺν occurs 35 times. It is not, strictly speaking, an Attic preposition, surviving chiefly in legal and religious phrases. It has a range of 153 verbs, of which 102 are monoprothetic, 50 diprothetic and 1 triprothetic. In respect of range of combinable verbs, it stands second in the list of prepositions, being next to ἐπί. Its favorite verb is βαίνω, with which it combines 130 times and of which it is the favorite preposition. It is the favorite preposition of 10 verbs. It has an exclusive range of 19 verbs, of which 13 are ἅπαξ εἰρημένα. There are no usurpations with ξύν.

παρά. The preposition παρὰ occurs in simple form 282 times. It combines with 54 verbs of which 48 are monoprothetic and 6 diprothetic. Εἶμι is its favorite verb, with which it occurs 173 times and of which it is the favorite preposition. It is the favorite preposition of 7 verbs and has a range of 8 exclusives, 4 being ἅπαξ εἰρημένα. Παρὰ has no usurpations. While αἰνέω is found in Attic prose only in composition (except twice in Plato), and in Thucydides only with παρὰ and ἐπί (κατά, once), yet the spheres of each are sharply defined.

περί. Περὶ occurs 478 times. It has a range of 43 verbs, all of which are monoprothetic. Its favorite verb is γίγνομαι, with which it combines 48 times. It is the favorite preposition of 3 verbs and is the exclusive of 2, both of which are ἅπαξ εἰρημένα. Περὶ has no usurpations.

πρό. Πρὸ occurs 80 times. It has a combinable range of 105 verbs, 69 being monoprothetic, 35 diprothetic and 1 triprothetic. Χωρέω is its favorite verb with which it combines 37 times. It is the favorite preposition of 7 verbs and is the exclusive preposition of 6, one of which is ἅπαξ εἰρημένον. Πρὸ has no usurpations.

πρός. The preposition πρὸς occurs in simple form 861 times. It has a combinable range of 74 verbs of which 56 are monoprothetic, 17 diprothetic and 1 triprothetic. Its favorite verb is βάλλω, with which it occurs 67 times and of which it is also the favorite preposition, being a little in advance of ἐς. It is claimed by 11 verbs as a favorite and by 2 as an exclusive. No ἅπαξ εἰρημένα and no usurpations occur with πρός.

ὑπέρ. Ὑπὲρ occurs 64 times and has a range of 11 verbs, all of which are monoprothetic. Its favorite verb is βαίνω, with which it occurs 9 times. It is not a favorite of any verb and has but 1 exclusive which is ἅπαξ εἰρημένον. No usurpations.

ὑπό. The simple ὑπὸ occurs 422 times. Its range of combinable verbs consists of 58, of which 45 are monoprothetic, 12 diprothetic and 1 triprothetic. Its favorite verb is ἄρχω, with which it combines 94 times and of which it is the favorite preposition. 3 verbs claim it as their favorite preposition and 5 as an exclusive, of which 1 is ἅπαξ εἰρημένον. In ὑποπτεύω and ὑποτοπέω we have usurpations of ὀπτεύω found only in Aristophanes, and τοπέω used once by Eustathius, the Homeric commentator.

II. Statistical Tables.

This portion of the work consist of four tables. The first shows all the simple verbs in Thucydides which combine with prepositions to form other verbs. It indicates the prepositions so used and the number of occurrences of both compounds and simples. It gives the complete statistics for monoprothetics based on simple verbs. I have taken no account of compounds whose verbal elements are not referable to simple verbs. Accordingly I have omitted verbs like ἐπικουρέω, προθυμέομαι, ἐγχειρέω referable to ἐπίκουρος, θυμὸς and χεὶρ respectively. On the other hand such verbs as ἐνδίδωμι, ξυμπροθυμέομαι, are included, being referable to the simples δίδωμι and προθυμέομαι. A compound like κατηγορέω, although the verbal element *ηγορέω does not exist, is included, since *ηγορέω is referable to ἀγορεύω. Another example is ἐκδιαιτάομαι (διαιτάω). Such verbs are starred. The second table shows the same facts for the diprothetics and triprothetics as the first table for the monoprothetics. The third table shows the different combinations of prepositions as seen in diprothetics and triprothetics. The fourth shows the relative range of the prepositions, their favorite verbs and statistics. It also combines for the sake of convenience some of the more salient points of the other tables.

It is impossible that the statistics shown by the appended tables should be absolutely without error. Infallibility belongs only to the enthusiasm of youth. But it is believed that no false impressions can be gotten from the figures indicated.

III. An Examination of the Statistics.

INTRODUCTORY.

The preposition is a local adverb.

The prevalent definition of the verb is predication.

There is no kind of predication that does not imply motion, actual or potential. At any rate in the consideration of the preposition or its relation to the verb, we are justified in making that element predominant which is necessarily the most fundamental. Motion in a verb, then, is that quality in a verb which is capable of direction.

The fundamental notion of the preposition is one of place. The deviations from this notion, the transfers from place to time, or the paling out of the original color, all have their basis in the primal notion of place.

It is unnecessary to demonstrate the interdependence and kinship of the notions of motion and place. Place involves motion just as the preposition involves the verb. It also lies implicitly in the nature of the subject that certain forms of motion will have a natural affinity for certain relations of place, while some forms of both motion and place will absolutely refuse to coalesce. This is due to the different modifications of motion assumed by the verb. By modification of motion, we mean: *the alteration of its color, the definition of its kind, or the indication of its direction.* Absolutely pure motion is free from such modification. If there were a verb which designated motion without reference to color, direction, or kind, it could be said to express pure motion. But pure motion does not exist in language. Language begins with concrete notions, however general the application which the expression of that notion may have had, after the notion had once taken form. Thus there are verbs which express motion in a more general way than others. E. g., εἶμι, however concrete the notion for which it originally stood, is used for so many different kinds of motion, that, for purposes of this paper, it can be said to express relatively pure motion.

The motion in a verb may be modified either internally or externally.

Internal Modification.

For purposes of the present paper, verbs may be divided into two classes: those expressing actual motion, and those expressing potential motion. Verbs of actual motion include those verbs which express motion with its kind, direction or color more or less distinctly marked. Verbs of potential motion include verbs of existence, speech, thought, perception.

Verbs expressing relatively pure motion are rare, but language does not require many. The verbs εἶμι, ἔρχομαι (ἐλθεῖν) and more remotely, βαίνω, furnish the best examples of relatively pure motion in the language.

That εἶμι is well selected is attested by the following considerations: I. It is used for various kinds of motion without distinction. Thus, for *walking:* Il. 7, 213: ποσσὶν ἤϊε μακρὰ βιβάς; for *hastening:* Od. 15, 213: ἀλλ' αὐτὸς καλέων δεῦρ' εἴσεται; for *flight of birds:* Il. 17, 756; for the *motion of things:* Il. 3, 611: πέλεκυς εἶσιν διὰ δουρός; &c., II. It is shown by the almost equal balance of the "whither" and "whence" relations as seen in the composition of the verb with the prepositions ἀπὸ and πρός, ἄπειμι occurring 33 times and πρόσειμι 29 times. This consideration is not set aside by the fact that ἐπὶ occurs 83 times in composition with this verb, because ἐπὶ is hostile, the sphere of ἐπιέναι in Thucydides being military—a fact constantly to be borne in mind. Hence the preponderance of ἐπὶ is of no account in this connection.

Ἔρχομαι (ἐλθεῖν) is a good example also, as shown by the following facts: I. It is frequently used with a supplementary participle showing the manner or the kind of the motion. Thus, Il. 11, 715: ἦλθε θέουσα; id. 10, 510: πεφοβημένος ἔλθῃς; Od. 6, 40: πόδεσσιν ἔρχεσθαι; Il. 5, 204: πεζὸς εἰλήλουθα; of *flying:* Od. 14, 334. In fact the use of this verb of the motion of spears, javelins, or of natural phenomena such as rivers, wind and storm, clouds and stars, time and sound, is too frequent to need confirmatory references and quite sufficient to denote the relative purity of the idea of motion contained in it. II. Another evidence is furnished by the fact that ἔρχομαι plays the part of present to both ἥκω and οἴχομαι, two verbs of motion with exactly opposite points of view. III. Here again we find that same prepositional balance as in the case of εἶμι, except that in this case the prepositions are ἀπὸ and ἐπί, ἀπέρχομαι (ἀπελθεῖν) and ἐπέρχομαι (ἐπελθεῖν) each occurring 76 times.

Next to εἶμι and ἔρχομαι (ἐλθεῖν), though by a considerable interval, ranks βαίνω. In βαίνω at least the color becomes visible. Yet no little freedom is also here manifest, as a participle often accompanies the verb to show the kind of motion. Thus, Il. 2, 167: βῆ ἀΐξασα; and id. 2, 665: βῆ φεύγων. Another evidence is that certain tenses of βαίνω are represented by εἶμι and ἔρχομαι (ἐλθεῖν).

These three verbs, εἶμι, ἔρχομαι (ἐλθεῖν) and βαίνω, sustain very much the same relation to what are ordinarily classed in the grammars as verbs of motion, as ποιέω does to what are more broadly termed verbs of action.

The moment color is given to the motion of a verb, that moment internal modification sets in and the sphere of the verb is narrowed.

The first curtailment is given to the idea of motion in the expression of its character or kind. Thus, *βάλλω, πέμπω, πίπτω, φέρω, ἵστημι, τίθημι, ἔχω*; and *πλέω, θέω, τρέφω*, &c. Still further curtailment, and more important in this connection, is seen in verbs which express with greater or less definiteness, the direction of their motion. Thus, *ἥκω, οἴχομαι, διώκω, ἀκολουθέω*, &c. Verbs in which the idea of motion is obscured or even lost in the color of the action, form another group, by far the largest, owing to the almost endless varieties of activity. As soon as a new activity is introduced into life, a new verb is created in language. Thus the history of the verb becomes the history of civilization. It is evident that verbs like *τειχίζω, βοηθέω, μάχομαι*, &c., have more color or are more picturesque than *εἶμι, πέμπω* or *ἥκω*; while verbs like *ἄρχω, κλέπτω, ὄλλυμι, καίω*, &c., possess still less motion if not indeed also still more color. Thus, the idea of motion may be almost wholly supplanted as in verbs like *εὕδω* and *θνήσκω*. Thus we see that the idea of motion in a verb is modified internally in color, kind or direction.

External Modification.

In external modification the problem is simpler. It is not germane to our subject to discuss here the external limitations of motion effected by adverbial or adnominal means. Such influences do not effect any change in the character of the motion expressed by the verb. I have already defined what I mean by the term modification. External modification is limited to direction and hence to the prepositions. We have to do here with prepositions in composition only. Our subject might be stated thus: The limits set to external modification by internal modification. It is evident that certain kinds of motion are inconsistent with certain varieties of direction. Such limitations are natural. Again certain other kinds of motion may be so characteristic of certain departments of literature as to be confined more or less strictly to these departments. On the other hand, the department may be of such a nature as to exclude certain varieties of direction or of modification. Again, the affiliation of a certain kind of motion for a certain direction may be so strong as by that very fact to refuse affiliation with other directions in no way hostile in themselves, thus bringing about *usurpation* from the point of view of the direction, and *exclusion* from the point of view of the motion. Such limitations are empirical and artificial.

Having thus seen that the principal elements at the basis of verb and preposition are motion, place, direction, let us see how these elements affect the composition of verbs with prepositions, so far as indicated by the language of Thucydides; and what light they throw on the questions of range, affinity, favoritism, loss of color, &c., announced at the beginning of our discussion.

Perhaps the most practical way of getting at a result is to collect all the verbs having the greatest combinable range of prepositions together, and place side by side with them those verbs having the next highest range, and so on to a point where a clear observation can be made of the change which takes place in the kind, direction or character of the motion expressed by them, as their prepositional ranges become narrower. See page 16 of the accompanying statistical tables for a list arranged for this purpose.

As I have already shown, relatively pure motion is best seen in εἶμι, ἔρχομαι (ἐλθεῖν) and βαίνω. This motion is stamped with a certain character in the verbs βάλλω, ἄγω, ἔχω, φέρω, &c., is given manner in πλέω, πίπτω, ἵστημι, θέω, &c., direction in ἥκω, λείπω, ἕπομαι, διώκω, &c., while in verbs like μάχομαι, ἀναγκάζω, &c., the color of the action is more prominent than the notion of motion, which continues to grow less in ἄρχω, δέω, γελάω, and is scarcely felt at all in ἀδικέω, εὕδω, θνήσκω.

The same variation in color is also seen in verbs expressing potential motion. Thus, in verbs of existence, εἰμὶ and γίγνομαι may be taken as being most nearly colorless. The metaphysical idea of motion in such verbs often becomes physical when given direction. But the idea of motion fades out as the idea of existence gives place to condition. Cf. ζάω, εὐδαιμονέω.

In like manner, in the case of verbs of speech, ἀγορεύω, εἶπον and λέγω (φημὶ not occurring in composition) may be said to be most nearly colorless. But the idea of speech assumes character in καλέω and γράφω,* still more so in βοάω, δείκνυμι, still more so in ψηθίζω ομαι, ὄμνυμι, μαρτυρέω-ομαι, and becomes faint in διδάσκω, ὁμολογέω.

Again in verbs of thought and perception. This variety of potential motion finds its purest expression in the verbs νοέω ομαι, γιγνώσκω (οἴομαι not being used in composition), becoming colored in κρίνω-ομαι on the one hand, and in εἶδον, ὁράω and ἀκούω on the other; while in μιμνήσκω, φοβέω and ἐλπίζω the mobility of the thought is replaced by color, and in αἰσθάνομαι and μανθάνω the notions of thought and perception are mixed.

It appears therefore from this general survey of the combinable verbs, with the aid of the statistical tables given below, that the

*The constructions of γράφω justify this classification.

range of prepositions is largest in the case of those verbs which express motion most nearly in its purity, actual or potential, physical or in the form of existence, speech, thought, or perception: and as those notions give place to definition of color, kind or direction, the range of prepositions grows less. That is to say:

In general, the range of combinable prepositions of a verb is in direct ratio to the nearness with which the verb expresses pure motion.

Until other authors are examined in the same way, however, we cannot safely go further than to say that the indications for Thucydides point in this direction, and even here there are a few possible objections. These are not many and not difficult to answer.

1. It may be urged that βάλλω, although not expressing pure motion as we have defined it, inasmuch as the character of the motion is designated, nevertheless has a larger range of prepositions than any other verb including any of those instanced as verbs of relatively pure motion. That is to say, βάλλω heads the list with a range of 16 prepositions, no other *single* verb in Thucydides having more than 14. Ὑπὸ is the only preposition out of the 17 proper prose prepositions with which it does not combine. But both ἀμφὶ and ὑπὸ are in its Homeric range. On the other hand, εἶμι has a range of only 12, ἔρχομαι (ἐλθεῖν) 13, and βαίνω 13. In reply, there are three considerations that must not be overlooked: (1) Not one of the verbs εἶμι, ἔρχομαι (ἐλθεῖν) and βαίνω has in its simple form a complete tense-system, and hence they supply each other's deficiencies. Take the three verbs as one, however, and the range of prepositions increases to 15. (2) The absence of ἀντὶ from the range of εἶμι, ἔρχομαι (ἐλθεῖν) and βαίνω is significant. It is due to the intense feeling of ἀντί. This consciousness of ἀντὶ shows itself in other ways to be noticed later on. Its sensitiveness is so marked as to attract a verb of more feeling or color than mere motion, and hence it is found with verbs like ἀγωνίζομαι, εἶπον, ἵστημι, τάσσω, &c. This community of feeling between verb and preposition we shall have occasion to notice again in still other manifestations. That the feeling of ἀντὶ in composition is stronger than that of any other preposition, appears in diprothetics and triprothetics. Its range of diprothetics relative to its whole combinable range is greater than that of any other preposition and it is first element in 5 out of 9 triprothetics. (3) βάλλω is a military term. Thucydides' is a military history. Every possible turn to perhaps the most comprehensive military term in the whole range of the language, would most naturally be necessary, owing to the military character of the department. This would account for the large prepositional

range of βάλλω in Thucydides. In Homer on the other hand, where the department is the same, its range is limited to 14 including the poetic ἀφμί, while βαίνω alone has a range of 15, which increases to 17, counting ἀμφί, in connection with ἔρχομαι (ἐλθεῖν) and εἶμι, ἀντὶ still being the missing link. This influence of department manifests itself again in a negative way in Demosthenes, where βάλλω stands 15 (εἰσβάλλω, the most military of all military terms, naturally being missing), while the εἶμι-ἔρχομαι- and βαίνω-combination stands 16.

II. Another objection of very much the same sort might be raised from the fact that γράφω in the verbs of speech, has a range of 9 prepositions, which is larger than that of any of the verbs cited as expressing relatively pure speech, ἀγορεύω, εἶπον or λέγω. True, ἀγορεύω has a range of but 4, εἶπον 7, and λέγω 5. But here again, as in the case of verbs of relatively pure physical motion, no one of the verbs makes a complete system of Attic tenses. Taken collectively they have a range of 10 prepositions. Γράφω had the advantage in that it started life as a verb of actual motion. Its later legal sphere was again in its favor. That γράφω should get the better of the verbs most nearly colorless in the orators, is what would be expected from the legal technique employed in that department. Accordingly, in Demosthenes, the proportion is 13 for γράφω, as against 8 for the group ἀγορεύω, εἶπον and λέγω.

III. A third objection may be found in the narrow range of prepositions of the verbs ἱκνέομαι and στέλλω, in which the notion of motion clearly predominates. Here again the community of feeling between verb and preposition comes into play, especially in the case of ἱκνέομαι. In ἱκνέομαι, "arrive", the point of view of the motion is "whence". The notion is not so much "come to", as "come from —— to". Hence ἀπὸ is the preposition for which ἱκνέομαι has the strongest affinity. But the addition of ἀπὸ did not create any modification in the idea of the verb. The notion was still "arrive", the point of view of the motion being simply reinforced. Now began a race between ἱκνέομαι and ἀφικνέομαι in which ἀφικνέομαι was the winner, debarring its rival entirely from the track of prose. The problem which the language then had to solve had changed from defining the direction of ἱκνέομαι to defining the direction of ἀφικνέομαι, that is, from defining the direction of a simple, to defining the direction of a compound. But the language does not take so kindly to diprothetic composition as to monoprothetic, and although attempt was made even here toward diprothetic composition, of which occasional evidence remains (εἰσαφικνέομαι Od. 12, 40; 8 times in Homer: προ- and προσαφικνέομαι, see Table II.), yet it preferred in this case to show the direction by prepositions in the simple form. The combination ἀφικνέομαι made the loss of color of ἀπὸ merely a matter of time. Thus in ἀφικνέομαι the compound has usurped the place of the simple, the preposition ἀπὸ having come in to the exclusion (or nearly so) of other prepositions, though a few cases exist of ἱκνέομαι in composition with διά, ἐκ (one each in Thuc.), with διά, ἐκ and κατὰ (Hom.) and ἐπί (Dem.).

The case of στέλλω is of the same kind with the additional circumstance that the official character of στέλλω gives it a much narrower range of prepositions (see Table I.). When στέλλω fails, πέμπω supplies the deficiencies.

Additional evidence for the truth of our main thesis is derived from a consideration of the diprothetics and triprothetics. Here as in monoprothetic composition, where there exists most mobility, there exists also most modification. The more nearly the idea of the simple verb approaches pure motion, the wider its range of diprothetic combinations. Pursuing the same method as in monoprothetics, we find that, with reference to range of diprothetic combinations, the verbs run as follows:

ἵστημι	13	πέμπω	5
εἶμι	11	χωρέω	5
ἔρχομαι (ἐλθεῖν)	10	λείπω	4
ἄγω	9	στέλλω	4
βαίνω	9	ἁλίσκω	3
βάλλω	8	βιβάζω	3
αἱρέω	6	γιγνώσκω	3
ἔχω	6	δίδωμι	3
ἧμαι	6	ἕζομαι	3
πλέω	6	ὄλλυμι	3
λαμβάνω	5	τίθημι	3
		φέρω	3

For further particulars see Table II.

The εἶμι—ἔρχομαι (ἐλθεῖν)—βαίνω—combination gives us here a remarkable range of 22 prepositional doublets. The prominence of ἵστημι in this connection is interesting. The large number of combinations possible with ἵστημι is due to the predominance of κατὰ and ἀνὰ as second elements in its diprothetics. The modification produced in the motion of ἵστημι by κατὰ and ἀνὰ in composition with it, is not so much a change in its direction as a reinforcement and an extension of it, from opposite points of view. "Up" and "down", like "high" and "deep", are the same idea logically, but from exactly opposite points of view. So the diprothetic compounds of ἵστημι which have κατὰ or ἀνὰ as second element, give, in feeling, practically a monoprothetic resultant. In this same way κάθημαι astonishes us with a range of 6 combinations, although both a usurpation and an exclusive in its monoprothetic form. The explanation, however, is easy, as the diprothetics of ἧμαι are practically monoprothetic in feeling, owing to loss of color in κατά, the second element in composition.

In the case of triprothetics, 7 in all, the range of verbs is too narrow for valuable results from comparison, but so far as they go, they fall into line with the views advanced in this paper. Four of them, ἄγω, εἶμι, ἔρχομαι (ἐλθεῖν), ἵστημι, all in the foreground as verbs of motion, will also be remembered as the most prominent diprothetics and among the most prominent monoprothetics. It is a curious fact that the remaining three, ἐλπίζω, σείω and εὑρίσκω, in which the idea of motion is by no means secondary, are not found among the diprothetics of Thucydides. The discussion of these verbs in their triprothetic relation belongs to the ἅπαξ εἰρημένα of Thucydides, a treatment of which is not germane to this investigation.

Suggested Corollaries.

Growing out of the above discussion are several special phenomena, from a consideration of which can be deduced corollaries to the main theorem. Within the limits of the present study we cannot hope to be exhaustive or more than suggestive, as many of the points alluded to could, of themselves, be carried out to the point of special monographs.

Favoritism of Verbs for Certain Prepositions.

One of the first things to strike the eye in an examination of the foregoing statistical tables, is the great preponderance of some prepositions over others with certain verbs. Let us see if there is any principle underlying such favoritism, and what light it throws on the general subject of the composition of verbs with prepositions. It is not our purpose to discuss each individual case, but merely to point out general tendencies. A few examples from Table I. bearing on each point will suffice.

I. Extension and Reinforcement.

Ἀλλάσσω combines with ἀπὸ 27 times to 24 times in all with 6 other prepositions. The idea of "change", "alter", naturally carries with it a very strong feeling for the relation "from", and hence the marked preference for ἀπό.

Βοηθέω, as would naturally be expected from its meaning, is found with ἐπὶ 27 times, with πρὸς 25 times, twice as frequently as with any other preposition. In like manner δέχομαι favors πρός, the ratio being πρός : 5 others : : 55 : 36; thus διώκω favors κατὰ and ἐπί; ἥκω favors πρός; θνήσκω favors ἀπό; ἵστημι favors κατά; πέμπω favors ἀπὸ and ἐκ; στέλλω favors ἀπό; ἐλαύνω favors ἐκ; ἕπομαι favors ἐπί; &c. Thus we see that the first movement between the verb and the preposition is in the line of the least resistance—*extension* and *reinforcement*. The nature of a verb can be best appreciated by a study of its favorite prepositions, the nature of a preposition, from its favorite verbs.

II. Exclusion.

This preference of a verb for a preposition may be so strong as to drive out all other prepositions, as in the case of ἀγείρω with σὺν, after Homer. In like manner, καίω with κατά; κτείνω with ἀπό; φθείρω with διά (with ἀπὸ once in Thuc.). This gives rise to what we may term *exclusion*. Verbs which combine with only one preposition may be called exclusives. Exclusives, however, are to be sharply distinguished from ἅπαξ εἰρημένα, since a single occurrence would not generate sufficient force to produce exclusion.

III. Usurpation.

Again the preference of the verb for the preposition may be so marked as to bring about *usurpation*, or a complete effacement of the simple by the compound. Such usurpations are most notable among exclusives, though cases are not infrequent where the different compounds have acted conjointly in the displacement of the simple. Thus of the first sort are ἀνοίγνυμι, ἀναλόω, ἐναντιόομαι, καθέζομαι, κάθημαι, &c. Examples of the latter are: the compounds of αἰνέω, νοέω, &c.

IV. Phraseological Expressions.

This preference for a certain preposition is often due merely to a transferred signification imported by the prepositional element which gives a phraseological resultant. Thus, ξυμβαίνω, ὑπάρχω, πάρειμι, παρέχω, &c.

V. Loss of Color of Prepositions.

Another natural concomitant of this principle of favoritism is the loss of color of the preposition. This has already been incidentally alluded to. This loss of color is most prominent in compounds which are mere reinforcements of the meanings of the simples. Where least needed, the feeling is least. We look for loss of color, therefore, first in extensions, exclusions, and usurpations. In extensions, the similarity in meaning, which was the basis of the attraction, became the cause of the fading out of the color. What became the life of the compound became the death of the preposition in the compound. In exclusions and usurpations the loss of color became easier by reason of the absence of contrast with other prepositions which would have operated to some extent in keeping up the difference in feeling. The function of the simple becomes the function of the compound, the simple often being relegated to poetry, while the compound does duty in prose. The simple often reappears in late Greek, a striking parallel to which is found in the Silver Latinity. Thus, καθέζομαι, ἕζομαι being poetic and late Greek. Cf. also ἀφικνέομαι, ἀνοίγνυμι and ἀπόλλυμι. The preposition is sometimes ignored in augment. Thus, ἠνέῳγμαι, N. T. Rev. 10, 8; Heliodor. 9, 9; ἠνεῴχθην, Dio Cass. 44, 17; ἐκαθεζόμην Xen. An. 1, 5. 9; and frequently in Attic. The emergence in late Greek of strengthened compounds often follows loss of color in the preposition. Thus, the strengthened combinations προσεπι-, ἐπιπροσ-: ἐξαπο-, ἀπεξ-; συμμετα-, μετασυν-: προσεισ-; κατaντι- and ἀντικατα-, are not uncommon in late Greek, but rare in classical Greek. Cf. Table III.

VI. Relative Consciousness of Prepositions.

The loss of color in the preposition naturally suggests the relative consciousness of the prepositions. Here again we cannot hope to be more than suggestive. Valuable service is rendered in this connection by the diprothetics. A careful examination of Tables II. and III. will show the operation of two principles in diprothetic composition. First—a desire for reinforcement—the extension side. Second—a desire for modification—the plastic side. Now reinforcement implies weakness. Language is continually building itself up where long use or abuse has broken it down. In the case of monoprothetics it is evident that most weakness is found in extensions and usurpations. A monoprothetic whose prepositional element has faded out is felt as a simple. This leads either to a discarding of the preposition altogether and a restoration of the simple, as actually occurs in late Greek, or to reinforcement. Reinforcement of such monoprothetics gives a diprothetic form but a monoprothetic feeling. The language of Thucydides presents us with a range of 387 separate monoprothetics, but only 86 diprothetics. It is fair then to conclude that the language may consent to a single union but resist a double one, and since the growth of language is along the line of the least resistance, we find that diprothetics having reinforcement as their cause, greatly preponderate over the plastic use. Now when language reinforces it brings to bear the most powerful means at its command. This is seen in the predominant prepositions in diprothetic composition. Those prepositions hold their color longest which play the most prominent role in diprothetics and triprothetics. Thus, ἀντὶ appears as first element in 27, ἐπὶ in 39, ξὺν in 50, πρὸ in 35, and πρὸς in 17 diprothetics, while of triprothetics, ἀντὶ has 5, and ξύν, πρό, πρὸς and ὑπό, have each one. The absence of ἐπὶ in triprothetics would seem to militate against this view, but coincident with this absence, it occurs as a second element in 8 out of the 9 triprothetics in Thucydides, reinforced by ἀντὶ in 5 out of the 8 cases and by πρὸς in one, thus indicating the fading out of the color of ἐπὶ in diprothetics.

This tendency to make that combination in which there will be the most strength, shows itself also in another way. In the formation of a diprothetic, when there exists a choice between monoprothetics in ἐκ or ἀπό, or between εἰς and πρός, or between κατὰ and ἀντί, the forms in ἐκ, εἰς and κατὰ are chosen. The exceptions can usually be explained. Thus, ἄγω (see Table II.) has ἐκ instead of ἀπὸ as second element in diprothetics; εἶμι, ἐκ 3 times, ἀπὸ once; ἔρχομαι, ἐκ instead of ἀπό; ἵστημι does not count, as other considerations are involved, such as loss of color of κατὰ and the military character of ἀφίστημι, accounting for the preponderance of these elements here. In this phenomenon we are limited to the class of diprothetics which represent the plastic side. Naturally enough, those simples predominate here in which the motion is least obscured. Where modification is necessary, room and mobility are needed. It follows that the second elements of diprothetics represent two opposite conditions of things: 1st, loss of color of the preposition; 2nd, vividness of preposition. In the first case, reinforcement was aimed at; in the second, modification of the idea of the verb. Hence there is greater diprothetic feeling in the latter class than in the former, and from this follows the comparative ease with which diprothetics of the former class were formed and their consequent preponderance over the latter class.

In triprothetics, the principle of reinforcement again is chiefly operative, and here naturally enough, the second element is the least conscious. It is noticeable that ἐπὶ is second element in 8 of the 9 triprothetics in Thucydides.

Summary.

In the foregoing discussion I have endeavored to prove the theorem for Thucydides:

In general the range of combinable prepositions of a verb is in direct ratio to the nearness with which the verb expresses pure motion.

From the demonstration of this theorem can be deduced the following corollaries:

1. A verb unites most readily and first with that preposition which is in a sense an extension of its own meaning.

2. The converse is also true, that that preposition has the greatest affinity for those verbs which are in line with its own direction.

3. The character of a verb is best shown by its favorite prepositions, or more narrowly, the best index of a verb is its favorite preposition.

4. The converse is also true, that the character of a preposition is best shown by its favorite verbs.

5. Favoritism is extension, extension leads to exclusion, exclusion leads to usurpation. All contribute toward the loss of color of the preposition.

6. Loss of color in the preposition is attended with a decline of the simple, a narrow range of combinable prepositions, followed, perhaps, by emergence in late Greek of the simple or of a strengthened compound.

7. Those monoprothetics which are extensions of their simples or which reinforce the point of view of the simple, enter most into diprothetic composition.

8. Those prepositions which preponderate in monoprothetics, preponderate also as second elements in diprothetics.

9. Those prepositions have lost most color which appear most as second elements in diprothetics.

10. Those prepositions are most conscious which appear most as first elements in diprothetics.

11. In general, in the formation of diprothetics from a given simple, the formation is made on the basis of the monoprothetics in ἐκ, εἰς and κατά, instead of in ἀπό, πρὸς and ἀντί, where choice is possible.

12. In triprothetics, the first element is the most conscious, the second the least, while the third is variable.

It is the operation of the above principles that defines the Limitations of the Composition of Verbs with Prepositions in Thucydides.

Tables to "Composition of Verbs in Thucydides."

STATISTICS

for the Composition of Verbs with Prepositions in Thukydides.

Prepared by D. H. HOLMES, Ph. D.

Tafel I.

Statistik für monoprothetische Verben.

Sie giebt an:

a) alle kombinierbaren Verben bei Thukydides;
b) die Reihe der Präpositionen eines jeden Verbums und umgekehrt die Verbenreihe einer jeden Präposition;
c) die erforderlichen Zahlenangaben.

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
ἀγγέλλω*)	1		20	1	1		15	8				13	10	1				9	ἀπό	43
ἀγείρω											6							1	ξύν	6
ἄγνυμι									2									1	κατά	
ἀγορεύω									2		3			5	2			4	πρό	1
ἄγχω			2															1	ἀπό	
ἄγω	36		33	7	18	9	43	13	17		14	8	1	8	45		7	14	πρός, ἐπί	110
ἀγωνίζομαι		2		3		1					4			1				5	ξύν	22
ἀδικέω											2							1	ξύν	119
ἀθρέω	1																	1	ἀνά	1
ἀθροίζω											2							1	ξύν	8
αἰνέω							14		1			40						3	παρά	
*αἱρέομαι		2					2								3			3		
αἱρέω	32		28	17	11				42		7	1	5	1			3	9	κατά	154
αἴρω		2	10				22		7		4							5	ἐπί	58
αἰσθάνομαι														10				1	πρό	120
αἰσχύνω									1									1	κατά	8
*αἰτέομαι												1						1	παρά	
αἰτέω		1	6															2	ἀπό	21
αἰτιάομαι							6		2									2	ἐπί	15
ἀκολουθέω							7				3							2	ἐπί	13
ἀκοντίζω								8	2									2	ἐς	4
ἀκούω							2	13	2								27	4	ὑπό	74
ἀλγέω			1										2					2		2
ἀλείφω					2													1	ἐκ	3
ἀληθεύω							2											1	ἐπί	
*ἁλίσκω	2																	1	ἀνά	
*ἀλλαξείω			1															1	ἀπό	
ἀλλάσσω		2	26	9	1	1			3	1	7							8	ἀπό	1
*ἁλόω	16																	1	ἀνά	
ἁμαρτάνω				3	2													2	διά	41

*) In der alphabetischen Anordnung sind dieselben Prinzipien wie in den n Indices und Wörterbüchern befolgt worden.

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
ἀμάω				1														1	διά	
ἀμελέω												1						1	παρά	5
*ἀμύνομαι		2												2				2		
ἀμύνω							6											1	ἐπί	117
ἀμφοτερίζω							1											1		
ἀναγκάζω							1		5						8			3	πρός	67
ἀντάω			18															1	ἀπό	
ἀντιόομαι						25												1	ἐν	
ἀξιόω		1	1															2		98
ἀπατάω					4													1	ἐκ	8
*ἅπτομαι		2																1	ἀντί	
ἅπτω									2		2						1	3		19
ἀράσσω			1						1									2		
*ἀργυρίζω					1													1	ἐκ	
ἀρέσκω			1															1	ἀπό	21
ἀρκέω		1																1	ἀντί	12
ἁρμόζω											1							1	ξύν	
ἀρνέομαι			1															1	ἀπό	2
ἁρπάζω				4														1	διά	3
ἀρτάω			1		1						1							3		2
ἀρτύω					16													1	ἐκ	
ἄρχω											9						94	2	ὑπό	277
αὐλίζω						3	2											2	ἐν	17
αὐξάνω							2											1	ἐπί	10
αὐτομολέω			1															1	ἀπό	7
αὐχέω																1		1	ὑπέρ	1
βαδίζω				1														1	διά	
βαίνω	26		48	34	7	1	11	20	25		130	19		2	4	9		13	ξύν	1
βάλλω	5	1	3	22	8	12	8	65	9	10	9	9	4	6	67	6		16	πρός, ἐς	24
*βασείω											1							1	ξύν	
*βιάζομαι															1	1		2		
βιάζω									1									1	κατά	54
βιβάζω	2		3	3	5	2	2	2	2		1							9	ἐκ	
βιόω							2											1	ἐπί	
βλέπω			1															1	ἀπό	2
βοάω	1			2		2	9		3									5	ἐπί	5
βοηθέω		2			1		27				11	13			25			6	ἐπί, πρός	136
*βουλεύομαι				3														1	διά	

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
βουλεύω							34				2			2				3	ἐπὶ	109
βυρσόω									1									1	κατὰ	
γελάω									1									1	κατὰ	
γηράσκω						1												1	ἐν	
γίγνομαι			5	1	1	9	74				5	44	48	7	32			10	ἐπὶ	834
γιγνώσκω	3			10			4		8	7	3			3				7	διὰ	130
γράφω	2	1				3	3	1		2	16			1			1	9	ξὺν	19
δαμάζω									1									1	κατὰ	
δαρθάνω									1									1	κατὰ	
δατέομαι	1																	1	ἀνὰ	
δείδω									3									1	κατὰ	140
δείκνυμι			14			2	4							1			2	5	ἀπὸ	19
δέχομαι	1		9			13		2							55		11	6	πρὸς	110
δέω	9								3								1	3	ἀνὰ	10
δέω			2												11			2	πρὸς	181
δηλόω														2				1	πρὸ	59
*διαιτάομαι					1	1					1							3		6
διδάσκω				1										1				2		26
διδράσκω	3		5	1	2													4	ἀπὸ	
δίδωμι	2	5	81	2	4	38	6			3		82		31				10	παρὰ ἀπὸ	113
δικάζω									2									1	κατὰ	7
δικαιόω				1														1	διὰ	11
διώκω			2		1		11		14		2			1				6	κατὰ	24
δοιάζω						3												1	ἐν	
δοκέω											3			2				2	ξὺν	309
δουλόω									2									1	κατὰ	29
δράω											1							1	ξὺν	84
δυναστεύω												1						1	παρὰ	2
δύω			1	1		1			11									4	κατὰ	
δωσείω												1						1	παρὰ	
ἐγγυάω				1														1	διὰ	
ἕζομαι									17									1	κατὰ	
ἐθίζω											1							1	ξὺν	1
εἶδον			2			3	4		6				10	10		6		7	περὶ πρὸ	64
εἴκω												2					2	2		
εἶλλω						1												1	ἐν	
εἰμὶ			10		35	28	1			3	3	173	25		8		2	10	παρὰ	2389
εἶμι			33	6	25		83	11	7	3	11	13	1	15	29			12	ἐπὶ	116

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
εἶπον	3	15	5		1		1							20			3	7	πρὸ	144
εἴργω od. εἵργω			4	3	4				6				2					5	κατὰ	33
ἐλαύνω		2		17									1		4			4	διά	12
ἐλέγχω					1													1	ἐκ	3
ἐλευθερόω											4							1	ξὺν	31
ἕλκω	12	1	2				2		7								1	6	ἀνά	4
ἐλπίζω		1					2											2		53
ἐπείγω									1									1	κατὰ	26
ἔοικα			1															1		
ἑορτάζω				1														1	διά	1
ἐπιχειρέω														1				1	πρὸ	
ἕπομαι							11				8							2	ἐπὶ	15
ἐργάζομαι					9				8		1			3	1			5	ἐκ / κατὰ	17
ἐρεθίζω	1																	1	ἀνά	
ἔρομαι							3											1	ἐπὶ	3
ἔρχομαι (ἐλθεῖν)	1		76	20	47		76	34	12	3	18	41	3	26	18			13	ἀπὸ / ἐπὶ	172
ἐρωτάω							4											1	ἐπὶ	13
ἐτάζω					3													1	ἐκ	
εὐδαιμονέω						1												1	ἐν	1
εὕδω									1									1	κατὰ	
εὑρίσκω	1				2													2		42
ἐρῶ	1	5	3											8				4	πρὸ	5
*ἐχθάνομαι			4															1	ἀπὸ	
ἔχω	16	41	49	6			29		48	24	3	126	9	31	12	4	4	14	παρά	767
ζάω			1	1														2		20
ζεύγνυμι	1																	1	ἀνά	1
ζητέω	2																	1		7
ἡγέομαι				1	11			4	1								1	5	ἐκ	127
*ἡγορέω									10						2			2	κατὰ	
ἥκω	1			1					4		1	1		1	34			7	πρὸς	86
ἧμαι									13									1	κατὰ	
ἡσσάομαι		1																1	ἀντὶ	34
θάπτω											1							1	ξὺν	14
θαρσέω	6																	1	ἀνά	39
θαρσύνω												5						1	παρά	7
θειάζω							2											1	ἐπὶ	1
θεραπεύω							2											1	ἐπὶ	22
θέω				1					2						1			3		5

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
θνήσκω			65											1				2	ἀπὸ	32
θορυβέω				1														1	διά	18
θροέω				2														1	διά	
ἵζω									12									1	κατά	
ἵημι	21		47		2	2	15		6		1	5		11	2			10	ἀπὸ	1
ἱκνέομαι			192	1	1													3	ἀπὸ	2
ἱππεύω												2			1			2		
*ἰσόομαι		1																1	ἀντὶ	
ἰσόω					3													1	ἐκ	
ἵστημι	43	24	149	9	3	2	5		260	25	31	14	23	11			12	14	κατά	78
*ἰσχνέομαι																	26	1	ὑπὸ	
ἰσχυρίζω		1	1															2		7
ἴσχω							3		2				1	5	2			5	πρὸ	5
καίω									12									1	κατά	5
καλέω	5					4	30		1	1	12	23		22	3			9	ἐπὶ	78
κάμνω					1													1	ἐκ	5
καρπόω					1													1	ἐκ	2
καρτερέω						1												1	ἐν	4
κεῖμαι	2			5		12	30	1			14		1	4	18		1	10	ἐπὶ	38
*κελεύομαι				1								24						2	παρά	
κελεύω		2					2				1							3		148
κεράννυμι											1							1	ξὺν	2
*κηρυκεύομαι				1			17								1			3	ἐπὶ	
κηρύσσω	1																	1		9
κινδυνεύω			3	12							3	2		2				5	διά	55
κινέω				1														1	διά	
κλάω	2						3											2		
κλέπτω				1														1	διά	
κληρόω			2	1														2		1
κλῄω			11						4		6		3					4	ἀπὸ	5
κλίνω					1		1											2		
κλύζω							1		1				1					3		
*κοινόομαι											1							1	ξὺν	
κολάπτω					1													1	ἐκ	
κολπίζω									1									1	κατά	
κολυμβάω			1						1									2		
κομίζω	4		9	14	3			13	2		2	6	2		5			10	διά / ἐς	90
κομπέω							2											1	ἐπὶ	1
κόπτω				2	1				4				1	2				5	κατά	8

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
κοσμέω				5														1	διά	7
κράζω						1												1	ἐν	
κρατέω							17		1									2	ἐπί	72
κρεμάννυμι					1		4											2	ἐπί	
*κρίνομαι			36	10													1	3	ἀπό	
κρίνω	1				2									1				3		38
κρούω	3	1	5		4						2							5	ἀπό	7
κρύπτω			3				1											2	ἀπό	6
κτάομαι	1						3				2				5			4	πρός	46
κτείνω			68															1	ἀπό	20
κτίζω											1							1	ξύν	14
κύπτω						1												1	ἐν	
κυρόω							2											1	ἐπί	2
κωλύω			3	13			1											3	διά	66
λαμβάνω	27	14	27	1			5		67	5	27	36	1	2	24		11	13	κατά	207
λαμπρύνομαι						1												1	ἐν	1
λανθάνω				2														1	διά	60
*λαύω			3															1	ἀπό	
λέγω		18	3	3			2							1				5	ξύν	261
λέγω					4				3		37							3		
λείπω			23	4	27	7	5		34			4		3			23	9	κατά	21
λεύω									1									1	κατά	1
λιμνάζω													1					1	περί	
λιμπάνω									1									1	κατά	
λογίζομαι	2				4													2	ἐκ	14
λοχίζω														3				1	πρό	2
λυπέω												2						1	παρά	10
λύω			14	35					44			3					1	5	κατά / διά	46
μανθάνω							1							1				2		15
μαρτυρέω											1							1	ξύν	
μαρτύρομαι							1											1	ἐπί	2
*μαχέω							1				15							2	ξύν	
μάχομαι	1		1	5							2			1				5	διά	51
*μελέομαι							7											1	ἐπί	
μέλομαι							2											1	ἐπί	
μέλω										7								1	μετά	1
μέλλω		1		10														2	διά	181
μέμφομαι				1					3									2	κατά	13
μένω	6					17	4				4	11	15		1		29	8	ὑπό	57

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
μετρέω											2							1	ξύν	
μηχανάομαι		1																1	ἀντί	7
μίγνυμι							3				11				25		1	4	πρός	
μιμνήσκω	4		1				2										6	4	ὑπό	21
μίσγω							3				3				4			3		
μισθόω			1												2			2		4
μνημονεύω				1														1	διά	2
μονόω			1															1	ἀπό	7
μυθέομαι												5						1	παρά	
ναυμαχέω				4							1							2	διά	58
ναυπηγέω		2																1	ἀντί	5
νείφω																	2	1	ὑπό	
νέμω				2			1		1									3		39
*νεύομαι	9																	1	ἀνά	
νεύω					1										1			2		
νέω								1			1							2		2
νικάω					2									2				2		109
*νοέομαι			1	63													8	3	διά	
νοέω							15		6					7				3	ἐπί	
νοστέω			1														1	2		
ξηραίνω			1															1	ἀπό	
οἴγνυμι	15																	1	ἀνά	
οἶδα											11			8				2	ξύν	123
οἰκέω			1	1	1	8	2		8		3	3			3			9	κατά	85
*οἰκίζομαι	4					1												2	ἀνά	
οἰκίζω			1		2				21		7							4	κατά	26
οἰκοδομέω	2		2	2		6	1	1			1	4	1		2			10	ἐν	22
οἰκτίζω		1																1	ἀντί	1
οἰμώζω	1																	1	ἀνά	
ὀκέλλω							2											1	ἐπί	3
ὀκνέω			9						2									2	ἀπό	4
ὀλισθάνω			1															1	ἀπό	
ὄλλυμι			41	3														2	ἀπό	1
ὀλοφύρομαι	1		1												1			3		4
ὁμιλέω															2			1	πρός	5
ὄμνυμι			1				1				12							3	ξύν	20
ὁμολογέω											1							1	ξύν	18
ὀνομάζω		1					4											2	ἐπί	17
ὀξύνω												4						1	παρά	
ὀπτεύω																	16	1	ὑπό	

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
ὁράω						1	3		6				28	6		3	1	7	περὶ	155
ὀρθόω	1								21									2	κατὰ	10
ὀρκόω					4													1	ἐκ	2
ὁρμάω			6		3													2	ἀπὸ	28
ὁρμέω		5					22						2					3	ἐπὶ	19
ὁρμίζω							1		7	1			1	1			1	6	κατὰ	15
ὀρύσσω				1					1			1						3		1
ὀτρύνω					2		2											2		
ὀφείλω		1					1							1	2			4		
παλάσσω						1												1	ἐν	
πάσχω		2												3				2		91
παταγέω		1																1	ἀντὶ	
πατέω									2									1	κατὰ	
παύω	4				1				4									3	ἀνὰ / κατὰ	41
πείθω	15										1							2	ἀνὰ	213
πειράω	3		9															2	ἀπὸ	86
πέμπω	1	4	45	9	42		2	9		21	10	5	6	14	6		1	14	ἀπὸ / ἐκ	203
περαιόω				3														1	διὰ	27
πήγνυμι									2									1	κατὰ	4
πηλακίζω														2				1	πρὸ	
πίμπλημι	1		1	1	1	1												5		1
πίμπρημι						15							1					2	ἐν	1
πίπτω	1		1		22	11	18	16	2	1	9	1	8		53		1	13	πρὸς	9
πλέκω											1							1	ξὺν	
πλέω	1	2	59	11	37	1	42	24	29		15	61	35	1	22			14	παρὰ / ἀπὸ	203
*πλήγνυμι					1													1	ἐκ	
πληρόω	1	3					1				2				3			5		51
πλήσσω					18				13									2	ἐκ	5
πνέω					2	2												2		
ποιέω		2				4				2		1	8	1	17			7	πρὸς	433
πολεμέω		2		4	1				5		16				3			6	ξὺν	109
πολεμόω					2										1			2		4
πολιορκέω		1		1	11						3							4	ἐκ	59
πολιτεύω						2					3							2		14
πονέω					1													1	ἐκ	9
πορεύω				2		1					1							3		47
πορθέω				3	2													2		14

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
πορίζω					4						3							2		20
πράσσω	2			6	1						13							4	ξὺν	191
πρεσβεύω		1									2							2		20
προθυμέομαι											3							1	ξὺν	15
πυνθάνομαι														2				1	πρὸ	66
ῥάσσω											1							1	ξὺν	
ῥέω									1				2					2		5
ῥήγνυμι	3		2						1		1	3						5		
ῥίπτω	4																	1	ἀνὰ	3
ῥώννυμι	1						8											2	ἐπὶ	5
σαλεύω			3															1	ἀπὸ	
σείω	1								1					1				3		2
σημαίνω			1				1										2	3		17
σιμόω			1															1	ἀπὸ	
σιτίζω							3											1	ἐπὶ	
σκάπτω									4									1	κατὰ	1
σκεδάννυμι			1	2														2		7
σκέπω									3									1	κατὰ	
σκευάζω	2			1			8		10		1	164						6	παρὰ	3
σκήπτω							2		1									2		1
σκοπέω	2			5									2	5				4		37
σκοτάζω											2							1	ξὺν	
σπάω	3		2	6			5		1									5	διὰ	
σπείρω				2														1	διὰ	
σπένδω							1											1	ἐπὶ	41
σπέργω							1		1									2		
σπεύδω			1															1	ἀπὸ	10
σταυρόω	1		3	1									1	1	1			6	ἀπὸ	2
στέλλω	3		64				12				1			1				5	ἀπὸ	4
στερέω			6															1	ἀπὸ	16
στρατεύω					8		8				28							3	ξὺν	111
στρατοπεδεύομαι		2			1	1												3		27
στρέφω	5		6				3		19		9						2	6	κατὰ	
σύρω			1															1	ἀπὸ	
σφάζω			2															1	ἀπὸ	2
σῴζω				23							1							2	διὰ	54
ταλαιπωρέω							1								1			2		16
ταράσσω											1							1	ξὺν	13
τάσσω		10		6			11			1	14	19		3	22			8	πρὸς / παρὰ	65

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d. einfach. V.
ταχύνω							1											1	ἐπί	
τείνω												3				1	2	3		
τειχίζω			19		4	1	2				2		11		1		1	8	ἀπό	61
τεκμαίρομαι											1							1	ξύν	3
τελέω			3	3			10				2				1		1	6	ἐπί	4
τελευτάω						1												1	ἐν	89
τέμνω			1			1					3							3	ξύν	35
τεχνάομαι					1													1	ἐκ	3
τηρέω							3											1	ἐπί	16
τίθημι	13	3	1	3			39	1	19	2	18	1	3	19	36		4	14	ἐπί πρός	39
τίκτω						1												1	ἐν	
τιμάω							3							6				2	πρό	24
τιμωρέω		1												2				2		25
τίνω					2													1	ἐκ	
τολμάω		2	1															2		33
τοξεύω									1									1	κατά	4
*τοπεύω																	1	1	ὑπό	
τοπέω																	8	1	ὑπό	
τραυματίζω									5									1	κατά	4
τρέπω			22		1		24					1		2				5	ἐπί ἀπό	105
τρέφω				1														1	διά	11
τρέχω	1			2	1		3	2	4		1							7	κατά	
τρίβω				7					1		1							3	διά	7
τρυχόω					2													1	ἐκ	3
τυγχάνω		1				11	7				2	10	12	1				7	περί	135
ὑβρίζω					2		1											2		7
ὑστερίζω							1											1	ἐπί	1
φαίνω	1		19	4	1		1		1							2	1	8	ἀπό	101
φέρω	5		7	24	3		40	10	9	1	42	3	1	12	20	5		14	ξύν ἐπί	78
φεύγω			6	44	1				34								4	5	διά	58
φθάνω														3				1	πρό	50
φθείρω			1	151														2	διά	40
φθλέγω							1		1									2		
φοβέω					8				2	1								3	ἐκ	108
φοιτάω							2											1	ἐπί	7
φορέω	1			2				1			1							4		4
φράγνυμι			1															1	ἀπό	

Verben	ἀνά	ἀντί	ἀπό	διά	ἐκ	ἐν	ἐπί	ἐς	κατά	μετά	ξύν	παρά	περί	πρό	πρός	ὑπέρ	ὑπό	Zahl der Präpos.	Bevorzugte Pr.	Zahl des Vorkomm. d einfach. V.
φράσσω			1			2												2		4
*φρέω				1														1	διά	
φρονέω									13				1			3		3	κατά	12
φροντίζω					1													1	ἐκ	
φρουρέω						2							1					2		22
φυγγάνω				1														1	διά	
φυλάσσω				1										3				2	πρὸ	
φωράω									2									1	κατὰ	1
χειμάζω				4			1											2	διά	8
χειρίζω						2				5								2	μετὰ	
χειροτονέω		2																1	ἀντὶ	
χέω				2														1	διά	
χόω															2			1	πρὸς	4
χράω			3	3			1											3		143
χρηματίζομαι					1													1	ἐκ	
χρονίζω						1												1	ἐν	2
χωλόω			1															1	ἀπὸ	
χωρέω	144		52				1			2	25			37	45		30	8	ἀνά	103
ψηφίζομαι				1					1									2		48
ψηφίζω	1						4											2	ἐπὶ	
ψύχω			1	1														2		
ὠθέω	1		17	2	9								2					5	ἀπὸ	6
(397)																				

Tafel II.

Statistik für diprothetische Verben.

Verben	Zahl der Kombin.	Präpositionale Kombinationen	Zahl d. Vork.	Verben	Zahl der Kombin.	Präpositionale Kombinationen.	Zahl d. Vork.
ἀγορεύω	1	προσκατα	1			προανα	1
ἄγω	10	ἀνθυπο	1			προεκ	2
		ἀντανα	12	αἰνέω	2	ξυνεπι	1
		ἀντεπι	2			προεπι	1
		ἐξανα	2	αἱρέω	6	ἐπικατα	1
		ἐπανα	2			ξυνανα	1
		ἐπεκ	3			ξυναπο	1
		ἐπικατα	2			ξυγκατα	5
		ξυνεπι	4			προσανα	1

Verben	Zahl d. Kombin.	Präpositionale Kombinationen	Zahl d. Vork.	Verben	Zahl d. Kombin.	Präpositionale Kombinationen	Zahl d. Vork.
		ὑπεκ	1			προδια	2
αἴρω	1	ἐπανα	1			προκατα	1
αἰτέω	1	ἀνταπο	1	δαρθάνω	1	ἐπικατα	1
αἰτιάομαι	1	ξυνεπι	1	δίδωμι	3	ἀνταπο	7
*ἁλίσκω	3	ἀπανα	4			καταπρο	7
		προανα	2			ὑπεν	1
		ὑπανα	2	διώκω	1	ξυγκατα	1
ἀλλάσσω	1	ἐξαπο	1	δουλόω	1	ξυγκατα	2
ἁμαρτάνω	1	ξυνεκ	1	ἕζομαι	3	ἀντικατα	2
ἀμύνω	1	ξυνεπι	1			ἐγκατα	2
ἀντάω	1	προαπο	3			προσκατα	6
*ἄρχομαι	1	προκατα	1	εἰμί	2	ἐπιπαρα	1
ἄρχω	1	προϋπο	5			ξυμπαρα	1
ᾄσσω	1	προεκ	1	εἶμι	11	ἀντανα	1
βαίνω	9	ἐπανα	1			ἀντεπι	5
		ἐπεκ	2			ἀντιπρο	1
		ἐπεσ	2			ἐπεκ	20
		ἐπιδια	1			ἐπικατα	1
		ἐπικατα	5			ἐπιπαρα	5
		ξυνδια	1			ξυνεκ	1
		ξυγκατα	1			ξυνεπι	1
		προανα	1			προεκ	1
		ὑποκατα	1			προσανα	1
βάλλω	8	ἐπεσ	1			ὑπαπο	2
		ξυνδια	2	εἶπον	1	ἐπανα	1
		ξυνεσ	2	ἐργάζομαι	1	ξυγκατα	1
		προδια	1	ἔρχομαι	10	διεκ	3
		προεν	1			ἐπανα	3
		προπαρα	1			ἐπεκ	23
		προσξυν	1			ἐπεσ	2
		προσπερι	2			ξυνεκ	1
βιβάζω	3	ἀντεν	1			ξυνεσ	2
		ἐπανα	1			παρεκ	1
		μετεν	1			προαπο	2
βοηθέω	1	ἐπεκ	1			προεκ	1
βουλεύω	2	ἀντεπι	3			ὑπεκ	5
		προεπι	1	εὔχομαι	1	ξυνεπι	1
γίγνομαι	2	ξυμπαρα	2	ἔχω	6	ἀντιπαρα	1
		προξυν	2			ἐμπαρα	2
γιγνώσκω	3	ξυνδια	2			παρακατα	1

Verben	Zahl d. Kombin.	Präpositionale Kombinationen	Zahl d. Vork.	Verben	Zahl d. Kombin.	Präpositionale Kombinationen	Zahl d. Vork.
		προανα	1			ξυνεπι	5
		προκατα	1			ξυγκατα	2
		προσπαρα	1			προκατα	16
ἧμαι	6	ἀντικατα	1	λέγω	1	ἐγκατα	1
		ἐπικατα	1	λείπω	4	ἐγκατα	11
		ξυγκατα	2			ξυγκατα	1
		παρακατα	1			παρακατα	1
		προκατα	1			προσκατα	2
		προσκατα	7	λυπέω	1	ἀντιπαρα	1
θέω	2	ἐπεκ	2	μελέομαι	1	ξυνεπι	1
		προεκ	1	μένω	2	ἀντανα	1
θνήσκω	1	ἐναπο	2			ξυμπαρα	1
ἵημι	2	διεκ	1	νέμω	1	ξυγκατα	1
		ἐπανα	1	νεύω	1	ξυναπο	1
ἱκνέομαι	2	προαπο	2	νέω	1	ἐπιπαρα	1
		προσαπο	1	οἰκέω	1	προσξυν	3
ἰσόω	1	ἐπανα	1	οἰκίζω	1	ξυγκατα	4
ἵστημι	13	ἀντικατα	7	οἰκοδομέω	1	ἐγκατα	1
		ἀπανα	7	ὄλλυμι	3	ξυναπο	2
		ἐγκατα	3			προαπο	2
		ἐξανα	4			προσαπο	2
		ἐπανα	5	ὀπτεύω	1	ἀνθυπο	1
		ἐπικατα	1	ὀρθόω	1	ἐπανα	1
		μετανα	3	ὁρμίζω	1	ἐγκατα	1
		ξυναπο	8	πείθω	1	ξυνανα	1
		ξυνεπι	1	πέμπω	5	ἐπιμετα	2
		ξυγκατα	3			ξυμπρο	1
		προαπο	1			προαπο	1
		προκατα	1			προσμετα	2
		προσαπο	1			ὑπεκ	1
ἴσχω	1	παρανα	1	πήγνυμι	1	παρακατα	1
καλέω	2	ἀντιπαρα	1	πίπτω	1	ξυνεσ	2
		προσπαρα	3	πλέω	6	ἀντεκ	1
καλεύομαι	1	ἀντιπαρα	1			ἀντιπαρα	1
κεῖμαι	1	ὑπεκ	2			διεκ	2
κλάω	1	ἐναπο	1			ἐπεκ	1
κομίζω	2	ξυμπαρα	2			ἐπεσ	2
		ὑπεκ	1			ξυνεκ	1
λαμβάνω	5	ἐγκατα	8	πολεμέω	1	ξυνδια	1
		ἐπικατα	2	σκευάζομαι	1	ἀντιπαρα	3

Verben	Zahl d. Kombin.	Präpositionale Kombinationen	Zahl d. Vork.
σκευάζω	2	ξυγκατα	1
		προπαρα	3
σκήπτω	1	ἐγκατα	1
στέλλω	4	ξυναπο	1
		προαπο	3
		προσαπο	1
		προσεπι	2
στρατεύω	1	ξυνεπι	1
στρέφω	2	ἐπανα	2
		ξυγκατα	1
σῴζω	1	ξυνδια	3
τάσσω	2	ἀντεπι	1
		ἀντιπαρα	6
τειχίζω	1	ἀντεπι	1
τίθημι	3	ἀντεπι	1
		ξυνεπι	5
		ὑπεκ	1
τρίβω	1	ἐνδια	5
φαίνω	1	ἀνταπο	2
φέρω	3	ἐπεσ	1
		ἐπιδια	1

Verben	Zahl d. Kombin.	Präpositionale Kombinationen	Zahl d. Vork.
		ξυνεκ	1
φεύγω	2	πρ κατα	3
		ὑπεκ	2
φθείρω	1	προδια	2
χωρέω	5	ἐξανα	11
		ἐπανα	15
		προανα	1
		προαπο	1
		ὑπανα	1
ψεύδομαι (86)	1	ἐπικατα	1

Triprothetische Verben.

Verben	Zahl d. Kombin.	Präpositionale Kombinationen	Zahl d. Vork.
ἄγω	3	ἀντεπανα	1
		ἀντεπεκ	1
		ὑπεξανα	1
εἶμι	1	ἀντεπεκ	2
ἐλαύνω	1	ἀντεπεκ	1
ἔρχομαι	1	ἀντεπεκ	1
εὑρίσκω	1	προσεπεκ	1
ἵστημι	1	ξυνεπανα	1
σείω (7)	1	προεπανα	1

Tafel III

Präpositionale Kombinationen.

a) Diprothetische Verben.

Kombin.	Zahl d. Verben	Kombin.	Zahl d. Verben	Kombin.	Zahl d. Verben	Kombin.	Zahl d. Verben
		ἀντιπρο	1	ἐπιδια	2	ξυναπο	5
		ἀπανα	2	ἐπεκ	7	ξυνδια	5
ἀνθυπο	2	διεκ	3	ἐπες	5	ξυνεκ	5
ἀντανα	4	ἐξανα	3	ἐπικατα	9	ξυνες	3
ἀνταπο	3	ἐξαπο	1	ἐπιμετα	1	ξυνεπι	11
ἀντεπ	1	ἐναπο	2	ἐπιπαρα	3	ξυγκατα	13
ἀντεν	1	ἐνδια	1	καταπρο	1	ξυμπαρα	4
ἀντεπι	6	ἐγκατα	8	μετανα	1	ξυμπρο	1
ἀντικατα	3	ἐμπαρα	1	μετεν	1	παρανα	1
ἀντιπαρα	7	ἐπανα	12	ξυνανα	2	παρεκ	1

Kombin.	Zahl d. Verben
παρακατα	4
προανα	5
προαπο	8
προδια	3
προεκ	5
προεν	1
προεπι	2
προκατα	7
προξυν	1
προπαρα	2
προϋπο	1
προσανα	2
προσαπο	4
προσεπι	1
προσκατα	4
προσμετα	1
προσξυν	2
προσπαρα	2
προσπερι	1
ὑπανα	2
ὑπαπο	1
ὑπεκ	7
ὑπεν	1
ὑποκατα	1

(66)

b) Triprothetische Verben.

Kombin.	Zahl d. Verben
ἀντεπανα	1
ἀντεπεκ	4
ξυνεπανα	1
προεπανα	1
προσεπεκ	1
ὑπεξανα	1

(6)

Tafel IV.

Statistik für die Präpositionen.

Präpositionen	Zahl d. Vork. d. einfach. Pr.	Zahl d. Verb. in Komposit.	Zahl d. monoproth. Verb.	Zahl d. diproth. Verb.	Zahl d. triproth. Verb.	Zahl der Exklusiva	Zahl der ἅπαξ εἰρ.	Zahl d. Vork. m. bevorzugtem Verb	Zahl d. Vork. m. bevorzugt. Präposition	Stelle in der Rangordn. d. verbund. Pr.	Stelle in der Rangordn. d. unverb. Pr.	Bevorzugte Verben
ἀμφὶ	2											
ἀνά	2	77	77			17	9	144	5	9	16	χωρέω
ἀντὶ	52	80	48	27	5	10	7	41		8	14	ἔχω, ἵστημι, εἶπον
ἀπὸ	634	114	112	2		23	15	192	22	3	6	ἱκνέομαι
διά	534	101	98	3		18	10	151	14	6	8	φθείρω
ἐκ	897	89	85	4		17	9	47	8	7	4	ἔρχομαι, πέμπω
ἐν	1794	67	55	12		13	11	38	3	11	1	δίδωμι
ἐπὶ	1216	156	117	39		20	5	83	23	1	3	εἰμι, γίγνομαι, ἔρχομαι
ἐς	1692	23	23					65	1	16	2	βάλλω
κατά	861	105	104	1		25	12	260	16	5	5	ἵστημι
μετά	619	24	22	2		1		25		15	7	ἵστημι
ξὺν	35	154	103	50	1	19	13	130	10	2	15	βαίνω
παρὰ	282	54	48	6		8	4	173	7	13	11	εἰμί
περὶ	478	43	43			2	2	48	3	14	9	γίγνομαι
πρὸ	80	105	69	35	1	6	1	37	7	5	12	χωρέω
πρὸς	861	74	56	17	1	2		67	11	10	5	βάλλω
ὑπὲρ	64	11	11			1	1	9		17	13	βαίνω
ὑπὸ	422	58	45	12	1	5	1	94	3	12	10	ἄρχω

Tafel V.

16
Präpositionen
βάλλω
(1)

14
Präpositionen
ἄγω
ἔχω
ἵστημι
πέμπω
πλέω
φέρω
(6)

13
Präpositionen
βαίνω
ἔρχομαι (ἐλ-
θεῖν)
λαμβάνω
πίπτω
τίθημι
(5)

12
Präpositionen
(9) (3)
αἱρέω -ομαι
εἶμι
(2)

10
Präpositionen
γίγνομαι
δίδωμι
εἰμί
ἵημι
κεῖμαι
κομίζω
οἰκοδομέω
(7)

9
Präpositionen
ἀγγέλλω
βιβάζω
γράφω
καλέω
λείπω
οἰκέω
(6)

8
Präpositionen
μένω
τάσσω
τειχίζω
φαίνω
χωρέω
(5)

7
Präpositionen
ἀλλάσσω
γιγνώσκω
εἶδον
εἶπον
ποιέω
τρέχω
τυγχάνω
(7)

6
Präpositionen
βοηθέω
δέχομαι
διώκω
ἕλκω
ἥκω
(3) (3)
κρίνω -ομαι
(3) (3)
νοέω -ομαι
(4) (2)
οἰκίζω -ομαι
ὁράω
ὁρμίζω
πολεμέω
σκευάζω
σταυρόω
στρέφω
τελέω
(15)

5
Präpositionen
ἀγωνίζομαι
αἴρω
βοάω
δείκνυμι
εἴργω
ἐργάζομαι
ἡγέομαι
ἴσχω
(3) (2)
κελεύω -ομαι
κινδυνεύω
κόπτω
κρούω
λέγω
λύω
μάχομαι
πίμπλημι
πληρόω
ῥήγνυμι
σπάω
στέλλω
τρέπω
φεύγω
ὠθέω
(23)

4
Präpositionen
ἀγορέω
ἀκούω
(3) (1)
ἅπτω -ομαι
(3) (1)
βουλεύω -ομαι
διδράσκω
δύω
ἐλαύνω
ἐρῶ
(3) (1)
ἰσόω -ομαι
κλῄω
κτάομαι
μίγνυμι
μιμνῄσκω
ὀφείλω
πολιορκέω
πράσσω
σκοπέω
φορέω
(18)

3
Präpositionen
(1) (2)
βιάζω -ομαι
αἰνέω
(2) (1)
αἰτέω -ομαι
(1) (2)
ἀμύνω -ομαι
ἀναγκάζω
ἀρτάω
δέω
*διαιτάομαι
θέω
ἱκνέομαι
κηρυκεύομαι
κλύζω
κωλύω
λέγω
μίσγω
νέμω
ὀλοφύρομαι
ὄμνυμι
ὁρμέω
ὀρύσσω
παύω
πορεύω
σείω
σημαίνω
στρατεύω
στρατοπεδεύ-
ομαι
τείνω
τέμνω
τρίβω
φοβέω
φρονέω
χράω
(32)

2
Präpositionen
(97)
s. Tafel I.

1
Präposition
(181)
s. Tafel I.

Summa: 387, wobei zehn Media nicht gezählt sind.

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Editorial Notes.

ANNOUNCEMENT— Beginning with January 1897, the Kansas University Quarterly will be published in two series Series A. for Science and Mathematics; Series B. for Literature and History The management is assured that this arrangement will be much more satisfactory to those who consult the Quarterly and exchange with it.

Volume V will close with the present issue, having, therefore, but two numbers. A complete file of the Quarterly for the scientist will therefore be vols I V, and from Vol VI on in series A For students in history the file will run from Vol I to Vol V, and then via Series B Subscribers will for the present receive both series. Exchanges will receive the series suited to their character Libraries and institutions will receive both.

During the summer of 1896, Prof. Haworth, of the Department of Physical Geology and Mineralogy of the University of Kansas, was engaged in making investigations principally in the western part of Kansas He had a total of eleven helpers during the greater part of the summer. Early in the season Mr. W R. Crane spent about four weeks in a further study of the coal beds, and now has the field work principally done for a Report on the Coal Deposits of Kansas. Later in the season he did work in the extreme northwestern part of the state in connection with the investigations of the water supply of the state conducted by Prof. Haworth for the State Board of Irrigation Mr. W N Logan took the field in May to continue investigations begun last season on the general stratigraphy of the Benton and Niobrara After devoting about six weeks to this work he spent from three to four more weeks in the northern tier of counties investigating the underground water Dr George I Adams, who had spent the greater portion of two previous summer vacations in connection with the field work of the University Geological Survey, devoted ten weeks to field work in west central Kansas. He had with him one assistant all the time and two part of the time.

Dr G. P. Grimsley, of Washburn College, has undertaken the task of making a careful study of the Gypsum deposits of Kansas and. in connection with Prof Bailey of the Dept of Chemistry, is preparing a Report on the Gypsum of Kansas which will treat the subject from the standpoints of Geology, Chemistry and economic value.

Prof. C. S. Prosser, of Union College, Schenectady, New York, devoted eight weeks to a study of the Lower Cretaceous in Kansas, and to the "Red Beds," which are of somewhat doubtful age. It is hoped that he will be able to decide many mooted questions now connected with this interesting area.

Prof. Haworth himself took up field work in April by continuing his investigations of the lead and zinc deposits in the southern part of the state, and later made a detailed study of an area covered by four U. S. Topographic sheets lying principally south of the Arkansas river and including Dodge City and Garden City. This latter work was done under the auspices of the United States Geological Survey, the report on which will be published by that bureau.

The draughting and literary work is now being done for Vol II of the University Geological Survey of Kansas, a volume to be devoted to the general stratigraphy of the Cretaceous and the Tertiary of the state with such other matters as are closely allied. Also laboratory and literary work is being carried rapidly forward on the several chapters on coal, gypsum, etc. to constitute Vol. III of the Survey, a volume to be devoted to economic geology entirely.

It is hoped that the Legislature at its next session will make provision for the proper illustration and publication of each of these volumes.

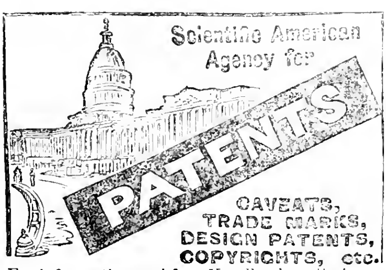
The Elements of Physics, by E. L. Nichols and W. S. Franklin, Vol. II., Electricity and Magnetism. This book is of a decidedly higher grade than the majority of text-books upon the subject, and really forms a connecting link between them and the more elaborate treatises upon special departments of physics. Starting with a chapter upon the properties and analysis of distributed scalars and vectors, the work discusses clearly and fully the topics usually treated under the subjects of electricity and magnetism.

The proofs of the various propositions are in general clear and as simple as the difficulties of the subject permit. A good knowledge of calculus is required, and the student should be especially familiar with the idea of infinitesimals.

The book is thoroughly up to date, even such a recent subject as Roentgen's discovery being treated. It may be highly recommended to those students possessing the requisite mathematical knowledge, as a thoroughly scientific and accurate presentation of the subject. In order, however, to derive the greatest benefit from its study, the student should do a large amount of outside reading describing, in detail, the phenomena treated. A. St. C. D.

With this number Mr. George Wagner is added to the Editorial Board of the QUARTERLY. He has already been helpful in rectifying the exchange list, and will in future be in charge of the circulation.

The Editor of the UNIVERSITY QUARTERLY has studied the type-writer question with some care. He has come to the conclusion that every professional scholar should [illegible] a typewriter, for the sake of accuracy, neatness and economy of time. What machine is the best for a professional stenographer he does not pretend to know, but he has concluded that the Hammond is the best for the professional man who is to operate his own machine. And this from the following grounds: Having watched the work of four or five standard machines in the hands of his colleagues he finds that the Hammond work is vastly superior in alignment and uniformity of impression; it is lighter and less bulky than other standard machines; the single keyboard with shift-keys is much sooner learned, and as rapidly worked by any but professionals; it makes less noise than some; the quick and inexpensive change of font makes it convenient for all, and especially desirable for language men. For these reasons the Editor uses and recommends the Universal Hammond.



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