KEY

TO

EXERCISES IN EUCLID.

KEY

TO

EXERCISES IN EUCLID.

BY

I. TODHUNTER, M.A., F.R.S.,

LATE HONORARY FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

London:
MACMILLAN AND CO.
1885

Cambridge:
PRINTED BY C. J. CLAY, M.A. & SON,
AT THE UNIVERSITY PRESS.

The Keys already issued to some of the Author's works have been found very useful by affording assistance to private students, and by saving the labour and time of teachers; and this has led to the issue of the present volume. Care has been taken, as in the former Keys, to present the solutions in a simple natural manner, in order to meet the difficulties which are most likely to arise, and to render the work intelligible and instructive.

November, 1880.

CONTENTS.

EXERCISES IN EUCLID.

I. 1 to 15.

1. Let AB be the given straight line on which the isosceles triangle is to be constructed; let DE be the straight line to which each side is to be equal. With centre A, and radius equal to DE, describe a circle; with centre B and radius equal to DE, describe another circle; let these circles intersect at C. Join AC and BC; then ABC will be the triangle required.

2. The given point and the vertex of the constructed triangle both fall on the circumference of the small circle.

3. Let AB and CD be two straight lines which bisect each other at right angles at the point O; so that AO is equal to OB, CO is equal to OD, and the angles at O are right angles. In CD take any point E, and join EA and EB: then EA shall be equal to EB.

For AO is equal to BO by hypothesis; EO is common to the two triangles AOE and BOE; and the angle AOE is equal to the angle BOE by Axiom 11. Therefore EA is equal to EB, by I. 4.

Similarly it may be shewn that any point in AB is equally distant from C and D.

4. The angles ABC and ACB are equal by I. 5. Hence the angles DBC and DCB are equal by Axiom 7. Therefore the sides DB and DC are equal by I. 6.

5. The angle DBA is half the angle ABC, by construction. The angle BAD is equal to half the angle ABC, by hypothesis. Therefore the angle DBA is equal to the angle BAD. Therefore BD is equal to AD by I. 6.

6. It is shewn in the demonstration of I. 5, that the angle BCF is equal to the angle CBG; therefore BH is equal to CH, by I. 6. Also it is shewn that FC is equal to GB. Therefore FH is equal to GH by Axiom 3.

7. AF is equal to AG, by construction; AH is common to the two triangles FAH and GAH; and FH is equal to GH, by Exercise 6: therefore the angle FAH is equal to the angle GAH by I. 8.

8. AB is equal to AD, by hypothesis; AC is common to the two triangles BAC, DAC; and the angle BAC is equal to the angle DAC, by hypothesis: therefore the base BC is equal to the base DC, and the angle ACB is equal to the angle ACD by I. 4.

9. The angle ACB is equal to the angle BDA by I. 8; and then the two triangles ACB and BDA are equal in all respects by I. 4; so that the angle ABC is equal to the angle BAD. Therefore AO is equal to BO by I. 6.

10. Let $ABCD$ be the rhombus, so that AB, BC, CD, DA are all equal. Join BD. Then in the two triangles BAD, BCD the base BD is common; and the two sides BA, AD are equal to the two sides BC, CD, each to each; therefore the angle BAD is equal to the angle BCD, by I. 8. Similarly it may be shewn that the angle ABC is equal to the angle ADC.

11. Let $ABCD$ be the rhombus, so that AB, BC, CD, DA, are all equal. Join BD. Then in the two triangles ABD, CBD, the side AB is equal to the side CB, the side BD is common, and the base AD is equal to the base CD; therefore the angle ABD is equal to the angle CBD by I. 8. Thus the angle ABC is bisected by BD. Similarly it may be shewn that the angle ADC is bisected by BD; and also that the angles BAD and BCD are bisected by AC.

12. Let there be two isosceles triangles ACB, ADB on the same base AB, and on *opposite* sides of it. Join CD; then CD shall bisect AB at right angles.

In the two triangles ACD, BCD the side CD is common; AC is equal to BC, by hypothesis; and the base AD is equal to the base BD, by hypothesis; therefore the angle ACD is equal to the angle BCD by I. 8.

Let AB and CD intersect at E. Then in the triangles ACE, BCE the side CE is common; the side AC is equal to the side BC; and the angle ACE has been shewn equal to the angle BCE: therefore the triangles are equal in all respects by I. 4. Thus AE is equal to BE; and the angle AEC is equal to the angle BEC, so that each of them is a right angle.

Next let the two isosceles triangles ACB, ADB be on the same base AB, and on the *same* side of it. Join CD and produce it to meet AB at E. It may be shewn as before that AE is equal to BE, and that the angles at E are right angles.

13. Let AB be the given straight line, C and D the two given points. Join CD and bisect it at E. From E draw a straight line at right angles to CD, meeting AB at F. Join CF, DF. Then CF shall be equal to DF.

For in the two triangles CEF and DEF, the side EF is common, CE is equal to DE, and the right angle CEF is equal to the right angle DEF: therefore CF is equal to DF, by I. 4.

The problem is impossible when the two points C and D are situated on the same perpendicular to the given straight line AB, and at unequal distances from that straight line.

14. Let AB be the given straight line, C and D the two given points. From D draw DE perpendicular to AB, and produce DE through E to a point F such that EF is equal to ED. Join CF, and produce it to meet AB at G; join DG: then CG and DG shall be the required straight lines.

For ED is equal to EF, and EG is common to the two triangles EDG and EFG; the right angles GEF and GED are equal: therefore by I. 4 the triangles FEG and DEG are equal in all respects, so that the angle FGE is equal to the angle DGE.

The problem is impossible when the two points C and D are equally distant from the straight line AB, and not on the same perpendicular to AB.

If C and D are equally distant from AB, and on the same perpendicular, then *any* point in AB may be taken for the point G.

15. Let the angle BAC be bisected by AD, and the angle BAG by AE: the angle DAE shall be a right angle.

Since the angle BAD is half the angle BAC, and the angle BAE is half the angle BAG, the two angles BAD and BAE together are half the two angles BAC and BAG together. But the angles BAC and BAG together are equal to two right angles, by I. 13; therefore the angles BAD and BAE together are equal to a right angle.

16. Let the four straight lines AE, BE, CE, DE meet at the point E, and make the angle AEB equal to the angle CED, and the angle BEC equal to the angle DEA: then shall AE and EC be in one straight line, and also BE and ED in one straight line.

By I. 15, Cor. 2, the four angles AEB, BEC, CED, DEA are together equal to four right angles; but the two angles AEB and BEC are equal to the two angles CED and DEA; therefore the angles AEB and BEC are together equal to two right angles; therefore AE and EC are in one straight line by I. 14. Similarly it may be shewn that BE and ED are in one straight line.

I. 16 to 26.

17. The angle BDA is greater than the angle CAD, by I. 16; the angle CAD is equal to the angle BAD, by hypothesis: therefore the angle BDA is greater than the angle BAD. Therefore the side BA is greater than the side BD, by I. 19. Similarly it may be shewn that CA is greater than CD.

18. Take any point G in BC, and join AG. The angle AGC is greater than the angle ABC, by I. 16; and the angle AGB is greater than the angle ACB, by I. 16. Therefore the angles ABC and ACB are together less than the angles AGC and AGB together; therefore the angles ABC and ACB are together less than two right angles by I. 13.

19. Join BD. The angle ABD is greater than the angle ADB, and the angle DBC is greater than the angle BDC, by I. 18; therefore the whole angle ABC is greater than the whole angle ADC. Similarly by joining AC we can shew that the angle DCB is greater than the angle DAB.

20. Let $ABCD$ be the square; on BC take any point E; join AE and produce it to meet DC produced at F: then shall AF be greater than AC.

The angle DCA is greater than the angle CFA by I. 16. The angle ACF, which is greater than a right angle, is greater than the angle DCA, which is less than a right angle. Therefore the angle ACF is greater than the angle AFC. Therefore AF is greater than AC, by I. 19.

21. Let AB be the given straight line, O the given point without it. From O draw OC perpendicular to AB; then OC shall be shorter than any other straight line OD drawn from O to AB.

For the angle OCD is a right angle; therefore the angle ODC is less than a right angle, by I. 17: therefore OC is less than OD, by I. 19.

Next, let OE be a straight line drawn from O to AB, and more remote from OC than OD is: OD shall be less than OE.

For the angle ODC is greater than the angle OEC, by I. 16; and the obtuse angle ODE is greater than the acute angle ODC: therefore the angle ODE is greater than the angle OED. Therefore OD is less than OE, by I. 19.

Lastly, from C on the straight line AB take CF equal to CD, and on the other side of C; join OF. Then OF is equal to OD by I. 4. And no other straight line can be drawn from O to AB equal to OD, besides OF.

For if this straight line were nearer to OC than OD or OF is, it would be less than OD, and if it were more remote from OC than OD or OF is, it would be greater than OD.

22. Let ABC be a triangle, and O any point. Join OA, OB, OC. Then OA and OB are together greater than AB, OB and OC are together greater than BC, and OC and OA are together greater than CA, by I. 20. Hence twice the sum of OA, OB, and OC is greater than the sum of AB, BC, and CA; therefore the sum of OA, OB, and OC is greater than half the sum of AB, BC, and CA.

23. Let $ABCD$ be a quadrilateral figure. Draw the diagonals AC and BD. Then AB and BC are greater than AC, and AD and DC are greater than AC; therefore the four sides AB, BC, CD, DA are greater than twice AC. Similarly it may be shewn that the four sides are greater than twice BD. Hence twice the sum of the four sides is greater than twice the sum of the diagonals; therefore the sum of the four sides is greater than the sum of the diagonals.

24. Let ABC be a triangle, and D the middle point of the base BC; join AD: then AB and AC together shall be greater than twice AD.

Produce AD to a point E so that DE may be equal to AD; join BE. In the two triangles ADC and EDB the two sides AD, DC are equal to the two sides ED, DB each to each; and the angle ADC is equal to the angle EDB, by I. 15: therefore AC is equal to BE, by I. 4. The two sides AB, BE are greater than AE, by I. 20; therefore the two sides AB, AC are greater than AE, that is greater than twice AD.

25. Let ABC be a triangle in which the angle C is equal to the sum of the angles A and B. At the point C in the straight line AC make the angle ACG equal to the angle A, and let CG meet AB at D. Then the triangle ACD is isosceles, by I. 6. Also as the angle ACB is equal to the sum of the angles A and B, and the angle ACG is equal to the angle A, the angle BCG is equal to the angle B: therefore the triangle BCD is isosceles, by I. 6.

26. It is shewn in the preceding Exercise that ACD is an isosceles triangle, having AD equal to CD; also that BCD is an isosceles triangle having BD equal to CD. Hence, as AD and BD are each equal to CD, the point D is the middle point of AB, and AB is equal to twice CD.

27. Let AB be the given base; at the point A make the angle DAB equal to the given angle. From AD cut off AE equal to the given sum of the sides; join EB. At the point B make the angle EBF equal to the angle AEB, and on the same side of EB; let BF meet AE at C: then ACB will be the triangle required.

For, since the angle EBC is equal to the angle BEC, the sides EC and BC are equal, by I. 6. Therefore the sum of AC and CB is equal to the sum of AC and CE, that is to the given sum of the sides. Also the base AB and the angle BAC have the required values.

28. From any point D in the straight line bisecting the angle A of a triangle draw DE perpendicular to the side AB, and DF perpendicular to the side AC. Then in the triangles DAE and DAF, the side DA is common; the angle DAE is equal to the angle DAF, by hypothesis; and the right angles DEA and DFA are equal: therefore DE is equal to DF by I. 26.

29. Let AB be the given straight line in which the point is to be found. Let CD and EF be the other two straight lines, and let them meet, produced if necessary, at O. Through O draw a straight line bisecting the angle between CD and EF; and let AB, produced if necessary, meet this straight line at K; then K will be such a point as is required: for the perpendiculars from K on the straight lines CD and EF may be shewn to be equal in the manner of the preceding Exercise.

Two straight lines can be drawn through O bisecting angles formed by the given straight lines, so that in general two solutions of the problem can be obtained, but there will be only one solution if AB is parallel to either of the bisecting straight lines.

If CD and EF are parallel the construction fails. We must then draw a straight line KL parallel to CD and EF, and midway between them: the intersection of this straight line with AB, produced if necessary, will be the required point. But if KL is parallel to AB there will be no solution.

30. Suppose A the given point through which the straight line is to be drawn, and B and C the other given points from which perpendiculars are to be drawn. Join BC, and bisect it at D; join AD: this shall be the required straight line.

For draw BE and CF perpendicular to AD, produced if necessary. Then in the triangles BDE, CDF the sides BD and CD are equal; the angles BDE and CDF are equal, by I. 15; and the right angles BED and CFD are equal: therefore BE is equal to CF, by I. 26.

31. In the triangles ADB, ADE the side AD is common; the angles BAD and EAD are equal by hypothesis; and the right angles ADB and ADE are equal: therefore BD is equal to ED, by I. 26.

32. Bisect the angle BAC by the straight line AD; from P draw PG perpendicular to AD, and produce PG both ways to meet AB at E, and AC at F: then AE will be equal to AF.

For in the two triangles AGE, AGF the side AG is common; the angles EAG and FAG are equal, by construction; and the right angles EGA and FGA are equal: therefore AE is equal to AF, by I. 26.

33. Let ABC be a triangle having the angle B a right angle, and let DEF be a triangle having the angle E a right angle; also let AC be equal to DF, and AB equal to DE: then shall the triangles ABC and DEF be equal in all respects.

Produce CB to G, so that BG may be equal to EF; and join AG. Then the angle ABG is a right angle, by I. 13, and is therefore equal to the angle DEF; also the sides AB, BG are equal to the sides DE, EF each to each: therefore the triangles ABG, DEF are equal in all respects, so that AG is equal to DF. But AC is equal to DF by hypothesis; therefore AC is equal to AG, and the angle ACG is equal to the angle AGC. Therefore the two triangles ABC, ABG are equal in all respects, by I. 26. But the triangles ABG, DEF were shewn to be equal in all respects; therefore the triangles ABC, DEF are equal in all respects.

I. 27 to 31.

34. Let ABC be a triangle having the sides AB and AC equal. Draw any straight line parallel to BC, meeting AB at D, and AC at E; then the angle ADE shall be equal to the angle AED.

For the angle ADE is equal to the angle ABC, and the angle AED is equal to the angle ACB, by I. 29. But the angle ABC is equal to the angle ACB, by I. 5. Therefore the angle ADE is equal to the angle AED.

35. Let the straight lines A and B meet at K, let the straight lines C and D meet at L, and let the straight lines A and D meet at M. The acute angle at K is equal to the acute angle at M, by I. 29; and the acute angle at M is equal to the acute angle at L also, by I. 29. Therefore the acute angle at K is equal to the acute angle at L.

36. Let the straight line AB be terminated by two parallel straight lines. Let C be the middle point of AB; through C draw any straight line DCE, terminated by the same parallel straight lines as AB, so that AD is parallel to EB. Then will ED be bisected at C.

For in the two triangles ACD, BCE the two sides AC, BC are equal by hypothesis; the angles ACD, BCE are equal by I. 15; and the angles CAD, CBE are equal by I. 29: therefore the triangles are equal in all respects by I. 26, so that CD is equal to CE.

37. Let O be a point equidistant from two parallel straight lines; through O draw one straight line AOB terminated by the parallels, and also another straight line COD terminated by the parallels, so that A and C are on one of the parallels, and B and D on the other: then will AC be equal to BD.

Since the given straight lines are parallel, a straight line can be drawn through O to meet the parallels at right angles, and this straight line will be bisected at O because O is equidistant from the parallels, by hypothesis. Therefore by Exercise 36 the straight lines AB and DC are bisected at O. Thus in the two triangles AOC, BOD the two sides AO, OC are equal to the two sides BO, OD each to each; and the angle AOC is equal to the angle BOD, by I. 15: therefore AC is equal to BD, by I. 4.

38. Let ABC be a triangle: produce BA to D; suppose that AE bisects the angle DAC, and that it is parallel to BC: then will ABC be an isosceles triangle.

For since AE is parallel to BC the angle DAE is equal to the angle ABC, by I. 29, and also the angle CAE is equal to the angle ACB, by I. 29. But the angle DAE is equal to the angle EAC, by hypothesis; therefore the angle ABC is equal to the angle ACB: therefore the side AB is equal to the side AC, by I. 6.

39. Take any point E in DC, and at the point E make the angle CEF equal to the given angle. Through A draw a straight line parallel to FE, and meeting CD at B: then B is the required point.

For the angle ABC is equal to the angle FEC by I. 29; and therefore the angle ABC is equal to the given angle.

40. Let ABC be a triangle; let a straight line be drawn bisecting the angle A, and meeting BC at D. From D draw a straight line parallel to AB, meeting AC at F, and also a straight line parallel to AC, meeting AB at E: then DE shall be equal to DF.

For in the triangles AED, AFD the side AD is common; the angle EAD is equal to the angle FAD, by construction; the angle EDA is equal to the angle DAF, and the angle FDA to the angle DAE, by I. 29; so that the angle EDA is equal to the angle FDA; hence the two triangles are equal in all respects by I. 26. Thus DE is equal to DF.

41. The angle FEC is equal to the angle ECB, by I. 29; the angle ECB is equal to the angle ECF, by hypothesis; therefore the angle FEC is equal to the angle ECF. Therefore EF is equal to FC, by I. 6. Again, the angle FGC is equal to the angle GCD, by I. 29; the angle GCD is equal to the angle FCG, by hypothesis; therefore the angle FCG is equal to the angle FGC. Therefore FG is equal to FC, by I. 6. And it has been shewn that FE is equal to FC; therefore EF is equal to FG.

42. Bisect the angle ABC by a straight line meeting AC at E; through E draw a straight line parallel to CB meeting AB at D: then D shall be the point required.

For the angle DEB is equal to the angle EBC, by I. 29; the angle DBE is equal to the angle EBC, by construction; therefore the angle DEB is equal to the angle DBE; therefore DB is equal to DE, by I. 6. And the angle DEA is equal to the angle BCA by I. 29, and is therefore a right angle; so that DE is perpendicular to AC.

43. Bisect the angle ABC by a straight line, meeting AC at E; through E draw a straight line parallel to BC, meeting AB at D. Then will BD, DE, EC be all equal.

For the angle DEB is equal to the angle EBC, by I. 29; the angle EBC is equal to the angle DBE by construction; therefore the angle DEB is equal to the angle DBE. Therefore DB is equal to DE, by I. 6. Again, the angle ADE is equal to the angle ABC, and the angle AED is equal to the angle ACB, by I. 29; also the angle ABC is equal to the angle ACB: therefore the angle ADE is equal to the angle AED. Therefore AD is equal to AE, by I. 6. But the whole AB is equal to the whole AC; therefore DB is equal to EC. Thus BD, DE, EC are all equal.

44. From A draw a straight line bisecting the angle BAC, and meeting BC at F. Then the triangles BAF and CAF are equal in all respects, by I. 4; so that the angles AFB, AFC are equal, and therefore each of them is a right angle. Therefore AF is parallel to ED, by I. 28. The angle AED is equal to the angle CAF, and the angle EDA is equal to the angle BAF, by I. 29. But the angle BAF is equal to the angle CAF, by construction; therefore the angle AED is equal to the angle ADE: therefore AE is equal to AD, by I. 6.

I. 32.

45. Let ABC be a triangle having the sides AB and AC equal. From B draw a perpendicular to AC meeting AC at D, and from C draw a perpendicular to AB meeting it at E. Then each of the angles CBD, BCE will be equal to half the angle A.

Bisect the angle A by a straight line meeting the base BC at F. Then the triangles BAF, CAF are equal in all respects, by I. 4; and therefore the angle AFB is a right angle. Then in the triangles BAF, BCE the angle ABC is common; the right angle AFB is equal to the right angle CEB: therefore the third angle BAF is equal to the third angle BCE, by I. 32. Similarly the angle CAF is equal to the angle CBD.

46. AC is equal to CE, and BC is equal to CD, by construction. The angle ACE is equal to the angle BCD, each being one third of two right angles, by I. 32; to each of these add the angle ACB: therefore the whole angle ACD is equal to the whole angle BCE. Thus in the two triangles ACD, ECB the two sides AC, CD are equal to the two sides EC, CB each to each; and the included angles are equal: therefore AD is equal to BE. Similarly AD is equal to FC.

47. The figure has eight equal sides, and eight equal angles: the interior angles of the figure together with four right angles are equal to sixteen right angles: therefore all the interior angles of the figure are equal to twelve right angles, by I. 32, Corollary 1. Hence each angle is twelve eighths of a right angle, that is, a right angle and a half.

48. Let A and B be the two given points, CD the given straight line. At the point C make the angle ECD equal to the angle of an equilateral triangle; and at the point D make the angle FDC also equal to the angle of an equilateral triangle, and on the same side of CD as the angle ECD. Through A draw a straight line parallel to EC, meeting CD, produced if necessary, at G: through B draw a straight line parallel to FD meeting CD, produced if necessary, at H. Produce GA and HB to meet at K: then GHK is the equilateral triangle required.

For since the angle at G is a third of two right angles, and so also is the angle at H, the angle at K is also a third of two right angles, by I. 32. Hence the triangle GHK is equiangular, and therefore also equilateral by I. 6.

49. Let ABC be a triangle, having AB equal to AC. Bisect the angles B and C by straight lines meeting at D. Produce CB to any point E. The angle DBE is equal to the two angles BDC, DCB by I. 32; but DCB is half the angle ACB, and is therefore equal to half the angle ABC, and is therefore equal to the angle ABD. Therefore the angle DBE is equal to the two angles ABD and BDC. Take away the common angle ABD; therefore the angle ABE is equal to the angle BDC.

50. The angle ACB is equal to the angle ABC, and the angle ACD is equal to the angle ADC, by I. 5. Therefore the angles ACB and ACD together are equal to the angles ABC and ADC together. But the angles ACB, ACD, ABC, ADC are together equal to two right angles, by I. 32. Therefore the angles ACB and ACD are together equal to one right angle.

51. Produce AB to any point H, and AC to any point K; bisect the angle CBH by the straight line BD, and bisect the angle BCK by the straight line CD: then the angle BDC, together with half the angle BAC, will make up a right angle.

Bisect the angle ABC by the straight line BE, and bisect the angle ACB by the straight line CE; then the angles EBD and ECD will be right angles. For the angle EBC is half the angle ABC, and the angle CBD is half the angle CBH; therefore the angles EBC and CBD are together half the angles ABC and CBH, that is equal to a right angle, by I. 13: thus EBD is a right angle. Similarly ECD is a right angle. The angles BEC, EBC, ECB are together equal to two right angles, by I. 32; that is BEC together with half ABC and half ACB are equal to two right angles. Join ED. Then BED and EDB are together equal to a right angle, by I. 32; CED and EDC are together equal to a right angle, by I. 32: therefore BEC and BDC are together equal to two right angles. Thus BEC and BDC are equal to BEC together with half ABC and half ACB. Therefore BDC is equal to half ABC and half ACB. Therefore BDC together with half BAC is equal to half BAC, half ABC, and half ACB, that is equal to half two right angles, that is equal to a right angle.

52. Let ABC be a triangle. Suppose the angle ABC greater than the sum of the other two angles; then twice the angle ABC is greater than the sum of the angles ABC, BCA, CAB, that is greater than two right angles; therefore the angle ABC is greater than a right angle. Again, suppose the angle ABC equal to the sum of the other two angles; then twice the angle ABC is equal to the sum of the angles ABC, BCA, CAB, that is equal to two right angles: therefore the angle ABC is a right angle. Lastly, suppose the angle ABC less than the sum of the other two angles; then twice the angle ABC is less than the sum of the angles ABC, BCA, CAB, that is less than two right angles: therefore the angle ABC is less than a right angle.

53. Construct an equilateral triangle ABC. Bisect the angle A by a straight line AD, and bisect the angle C by a straight line CD. Then ADC will be such a triangle as is required.

For the angle DAC, being half the angle of an equilateral triangle, is one sixth of two right angles; so also is the angle DCA; therefore the angle ADC is four sixths of two right angles, by I. 32. Thus the angle ADC is four times each of the angles DAC, DCA.

54. Since BC is bisected at E the two sides AE, EC are equal to the two sides FE, EB each to each; the angle AEC is equal to the angle FEB, by I. 15; therefore the triangles AEC and FEB are equal in all respects, so that the angle ACE is equal to the angle FBE.

In a similar way by comparing the triangles CGA and HGB, we see that the angle CAG is equal to the angle HBG.

Therefore the angles FBE, EBG, GBH are together equal to the angles of the triangle ABC, that is to two right angles, by I. 32. Therefore the angles ABF and ABH are together equal to two right angles: therefore HB and BF are in the same straight line, by I. 14.

55. Take any straight line AB. At the point A make the angle BAD equal to a right angle. Bisect the angle BAD by the straight line AE; and

bisect the angle DAE by the straight line AF; I. 9. Then the angle BAF is three fourths of a right angle. At the point B make the angle ABG equal to the angle BAF, and on the same side of AB; let AF and BG meet at C: then ABC will be the triangle required.

For the two angles ABC and BAC are by construction together equal to three halves of a right angle, therefore the angle ACB is half a right angle by I. 32. Thus the half of the angle ACB is a fourth of a right angle, and is therefore equal to one third of each of the angles BAC and ABC.

56. On AB measure off AD equal to the given straight line. At the point D draw DQ making with AD the angle ADQ equal to half the given angle, and meeting AC at Q. At Q draw QP, on the same side of QD as DA is, making the angle DQP equal to the angle QDP, and meeting AB at P. Then PQ is equal to PD, by I. 6; so that AP and PQ together are equal to AD, that is to the given straight line. And the angle APQ is equal to the sum of the angles PDQ and PQD, and is therefore equal to the given angle.

57. Let ABC be a triangle having the sides AB and AC equal. From B draw a straight line making the angle DBC equal to one third of the angle ABC, on the other side of BC, and meeting AC produced at D. From C draw a straight line making the angle ECB equal to one third of the angle ACB, on the other side of CB, and meeting AB produced at E. Let BD and CE intersect at F.

The triangle BFC has obviously the angles BCF and CBF equal, and is therefore isosceles by I. 6.

The angle BFE is equal to the sum of the angles BCF and CBF, by I. 32, and is therefore equal to two thirds of the angle ABC. The angle BEC is equal to the difference of the angles ABC and BCE, by I. 32, and is therefore equal to two thirds of the angle ABC. Therefore the angles BFE and BEF are equal, and the triangle BFE is isosceles, by I. 6. Similarly the triangle CFD is isosceles.

58. The angle AEC is equal to the sum of the angles ECB and EBC by I. 32, and so also is the angle DEB: therefore the angles ECB and EBC are together half the angles AEC and DEB. The angles ECF and EBF are together half the angles ECA and EBD, by construction. Hence the angles ECB, EBC, ECF, EBF are together half the angles AEC, DEB, ECA, EBD. Take the former sum from two right angles, and the remainder is the angle BFC; take the latter sum from two right angles, and the remainder is half EAC and EDB; therefore the angle BFC is half the sum of the angles EAC, EDB.

59. Let ABC be a triangle, having the angle ACB a right angle. At the point C draw a straight line CD making the angle ACD equal to the angle CAB, and meeting AB at D. Since the angle ACB is a right angle it is equal to the sum of the two angles CAB and CBA, by I. 32; the angle ACD is equal to the angle CAB, by construction: therefore the remaining angle BCD is equal to the angle CBA. Because the angles CAD and ACD are equal, the sides AD and CD are equal, by I. 6; and because the angles CBD and BCD are equal, the sides BD and CD are equal, by I. 6. Hence AD, BD, and CD are all equal; so that D is the middle point of AB, and CD is equal to half of AB.

60. Let F be the middle point of AB: then will EF be equal to DF.

For EF and DF are each equal to half of AB, by Exercise 59; therefore EF is equal to DF.

61. Use the diagram drawn for Exercise 60; from F draw FG perpendicular to ED; then will EG be equal to DG. For FE is equal to FD, by Exercise 60; therefore the angle FEG is equal to the angle FDG, by I. 5; and the right angle FGE is equal to the right angle FGD: therefore the triangle FEG is equal to the triangle FDG in all respects, by I. 26. Thus EG is equal to DG.

62. In the diagram of I. 1, let BA be produced through A to meet the circle at K. Join CK, HK, AH, CH.

The angle CAB is one third of two right angles; therefore the angle CAK is two thirds of two right angles. Also, each of the angles CAB, HAB being one third of two right angles, the whole angle CAH is two thirds of two right angles. Therefore the angle CAK is equal to the angle CAH.

In the two triangles CAK, CAH the two sides CA, AK are equal to the two sides CA, AH each to each; and the angle CAK is equal to the angle CAH: therefore CK is equal to CH, by I. 4.

Similarly it may be shewn that HK is equal to HC. Thus KC, CH, and HK are all equal.

63. Let AB and AC be the equal sides; let BD bisect the angle ABC; and let CE bisect the angle ACB. Join DE.

In the triangles BCD, CBE the angle BCD is equal to the angle CBE, and the angle DBC is equal to the angle ECB; therefore these triangles are equal in all respects, by I. 26, so that CD is equal to BE. But AC is equal to AB; therefore AD is equal to AE; therefore the angle AED is equal to the angle ADE. Also the angle ABC is equal to the angle ACB. Therefore the angle AED is equal to the angle ABC, by I. 32. Therefore ED is parallel to BC, by I. 28.

64. In AC take a point D such that AD is equal to AP; join DP; in AD produced, take a point Q such that DQ is equal to DP; join PQ: then will the angle APQ be equal to three times the angle AQP.

Since DP is equal to DQ the angle DPQ is equal to the angle DQP, by I. 5. The angle ADP is equal to the sum of the angles DPQ and DQP, by I. 32: therefore the angle ADP is twice the angle DQP. Therefore the angle APD is twice the angle DQP, by I. 5. To the former add the angle DPQ, which is equal to the angle DQP: therefore the angle APQ is three times the angle AQP.

65. Take a straight line AD equal to the given sum of the sides. At the point D draw a straight line DE making the angle ADE equal to half a right angle. With centre A, and radius equal to the given hypotenuse, describe a circle cutting DE at a point B. From B draw BC perpendicular to AD. Then ACB will be such a triangle as is required.

For ACB is a right angle by construction; and the hypotenuse AB is of the required length. Also since the angle BCD is a right angle, and the angle BDC is half a right angle, the angle CBD is half a right angle, and is therefore equal to the angle BDC: therefore BC is equal to CD, by I. 6. Thus the sum of the sides AC and CB is equal to AD, that is to the given sum.

In order that the construction may be possible the given hypotenuse must not be less than the perpendicular from A on DE; and if this condition is satisfied there will be two intersections of the circle with DE, and thus *two solutions apparently:* but it will be found on examination that there is only one distinct solution.

66. Take a straight line AD equal to the given difference of the sides. At the point D draw a straight line DE making with AD, produced through D, an angle equal to half a right angle. With centre A, and radius equal to the given hypotenuse, describe a circle cutting DE at B. From B draw BC perpendicular to AD produced. Then ACB will be such a triangle as is required.

For ACB is a right angle by construction; and the hypotenuse AB is of the required length. Also since the angle BCD is a right angle, and the angle BDC is half a right angle, the angle DBC is half a right angle, and is therefore equal to the angle BDC: therefore BC is equal to CD. Thus the difference of the sides is equal to AD, that is to the given difference.

In order that the construction may be possible the given hypotenuse must be greater than the given difference of the sides.

67. Let AB be the given hypotenuse. Bisect AB at D; from D draw a straight line at right angles to AB, and on it take DE equal to the given perpendicular. Through E draw FEG parallel to AB. From centre D, with radius equal to AD, describe a circle cutting FG at C. Join CA and CB: then ACB shall be the triangle required.

For the angle ACD is equal to the angle CAD, and the angle BCD is equal to the angle CBD, by I. 5; therefore the whole angle ACB is equal to the sum of the angles CAB and CBA: therefore the angle ACB is a right angle, by I. 32. Thus ACB is a right-angled triangle having the given hypotenuse.

From C draw CH perpendicular to AB; then will CH be equal to ED. The angles EDH and CHD are together equal to two right angles; therefore CH is parallel to ED, by I. 28; therefore the angle EDC is equal to the angle HCD, by I. 29. Therefore the two triangles EDC and HCD are equal in all respects, by I. 26; so that CH is equal to ED. Thus the perpendicular from the right angle on the hypotenuse has the given length.

68. Let DE be the given perimeter, and FGH the given angle. Draw GK at right angles to GH, and on the same side of it as GF. At the point D draw DL, making the angle LDE equal to half the angle FGH; at the point E draw EM on the same side of DE as DL, making the angle MED equal to half the angle FGK. Let DL and ME meet at C. Draw CA, meeting DE at A, and making the angle DCA equal to the angle CDA; draw CB, meeting ED at B, and making the angle ECB equal to the angle CEB: then ABC will be the required triangle.

For the angle CAB is equal to the sum of the angles ACD, ADC; that is to twice the angle ADC; that is to the angle FGH. Similarly the angle CBA is equal to the angle FGK. Thus the two angles CAB and CBA are together equal to a right angle; and therefore the angle ACB is a right angle by I. 32.

The side AC is equal to AD, and the side BC to BE, by I. 6. Therefore the sum of the sides AC, BC, and AB is equal to DE, that is to the given perimeter.

69. Let BAC be a right angle. On AB describe an equilateral triangle ADB; bisect the angle BAD by the straight line AE: then will the angles BAE, EAD, DAC be all equal.

For the angle BAC is a right angle; the angle BAD is one third of two right angles, that is two thirds of one right angle, by I. 32; therefore the angle CAD is one third of a right angle. And as the angle BAD is bisected by AE the angle BAE is equal to the angle EAD, each being one third of a right angle. Hence the angles BAE, EAD, DAC are all equal, each being one third of a right angle.

70. Let AB be the given straight line. On AB describe an equilateral triangle ABC. Bisect the angle CAB by the straight line AD, and bisect the angle CBA by the straight line BD. Through D draw a straight line parallel to CA, meeting AB at E; and through D draw a straight line DF parallel to CB, meeting AB at F. Then will AE, EF, FB all be equal.

Since DE is parallel to CA the angle EDA is equal to the angle DAC, by I. 29; but the angle DAE is equal to the angle DAC, by construction; therefore the angles EDA and DAE are equal: therefore AE is equal to ED, by I. 6. Similarly it may be shewn that BF is equal to FD.

Because DE is parallel to CA the angle DEF is equal to the angle CAB, by I. 29. Similarly the angle DFE is equal to the angle CBA. Therefore the angle EDF is equal to the angle ACB, by I. 32. Thus the triangle EDF is equiangular; and therefore it is equilateral by I. 6.

Now AE was shewn to be equal to ED; therefore AE is equal to EF. Similarly BF is equal to FE. Thus AE, EF, FB are all equal.

71. Let LM and PQ be the parallel straight lines; and A the given point. Suppose A to be *between* the parallel straight lines. Through A draw a straight line perpendicular to one of the parallel straight lines, and therefore also perpendicular to the other by I. 29. Let this straight line meet LM at B, and PQ at C. From B on LM take BD equal to AC; and from C on PQ take CE equal to AB, and on the *same* side of BC as BD is. Join AD and AE: these will be the required straight lines.

For the triangles BAD, CEA are equal in all respects by I. 4; so that AD is equal to AE, and the angle CAE is equal to the angle BDA. But the angles BAD and BDA are together equal to a right angle, by I. 32: therefore the angles BAD and CAE are together equal to a right angle. Therefore the angle EAD is a right angle, by I. 13.

If A is *not* between the parallel straight lines, CE and BD must be taken on *opposite* sides of BC.

72. Let ABC be the given triangle, and DE the given perimeter. At the point D make the angle LDE equal to half the angle ABC; and at the point E, on the same side of ED, make the angle MED equal to half the angle ACB. Let DL and EM meet at F. From F draw FG, meeting DE at G, making the angle DFG equal to the angle FDG; and from F draw FH, meeting ED at H, making the angle EFH equal to the angle FEH. Then FGH will be the triangle required.

For FG is equal to DG, and FH is equal to HE, by I. 6. Therefore the sum of the sides FG, GH, HF is equal to DE, the given perimeter.

Also the angle FGH is equal to the sum of the angles FDG, DFG, by I. 32; that is to twice the angle FDG, that is to the angle ABC. Similarly the angle FHG is equal to the angle ACB. Therefore the angle GFH is equal to the angle BAC, by I. 32.

I. 33, 34.

73. Let $ABCD$ be a quadrilateral having AB parallel to DC, and AD equal to BC. Suppose AB less than DC; from A draw a straight line parallel to BC, meeting DC at E. Then $ABCE$ is a parallelogram. Therefore the angle ABC is equal to the angle AEC, and the side AE is equal to the side BC, I. 34. Therefore AD is equal to AE, and the angle ADE is equal to the angle AED. Therefore the angles ABC and ADE are equal to the angles AEC and AED; that is they are together equal to two right angles, by I. 13.

In the same manner it may be shewn that the angles BAD and BCD are together equal to two right angles.

74. Suppose that AB and CD are equal but not parallel, and that the angle ABD is equal to the angle CDB. Produce AB and CD to meet at E. Since the angles ABD and CDB are equal EB is equal to ED. Therefore also EA is equal to EC; and the angle ECA is equal to the angle EAC. The two angles EBD and EDB are together equal to the two angles EAC and ECA, by I. 32. Therefore the angle EBD is equal to the angle EAC. Therefore BD is parallel to AC, by I. 28.

75. Let ABC be a triangle; let E be any point in AC, and D any point in BC: then AD and BE will not bisect each other.

Let AD and BE intersect at F.

If possible suppose that AF is equal to FD, and BF equal to FE. Then the triangles AFE and DFB are equal in all respects, by I. 4; so that the angle EAF is equal to the angle BDF. Therefore AE is parallel to BD, by I. 27. But this is impossible, since BD and AE, when produced, meet at C.

76. Let $ABCD$ be a quadrilateral, having AB equal to DC, and AD equal to BC: the figure shall be a parallelogram.

Join AC; then in the two triangles ABC and CDA, the sides BA, AC are equal to the sides DC, CA each to each; and the base BC is equal to the base DA: therefore the angle BAC is equal to the angle DCA. Therefore AB is parallel to DC, by I. 27.

Similarly it may be shewn that AD is parallel to BC.

77. Let $ABCD$ be a quadrilateral such that the angle A is equal to the angle C, and the angle B equal to the angle D: then the figure shall be a parallelogram.

The angle A is equal to the angle C, and the angle B is equal to the angle D; therefore the two angles A and B together are equal to the two angles C and D together. But the four angles A, B, C, D together are equal to four right angles, by I. 32; since the quadrilateral may be divided into two triangles by drawing AC or BD. Thus the angles A and B are together equal to two right angles. Therefore AD is parallel to BC, by I. 28.

Similarly AB is parallel to DC.

78. Let $ABCD$ be a parallelogram. Draw the diagonals AC and BD intersecting at E: then AC and BD shall be bisected at E.

In the triangles AED and CEB the sides AD and BC are equal, by I. 34; the angles ADE and CBE are equal, by I. 29; and the angles DAE and BCE are equal, by I. 29. Therefore the triangles are equal in all respects by I. 26. Thus AE is equal to CE, and DE is equal to BE.

79. Let $ABCD$ be a quadrilateral figure, and let AC and BD intersect at E; suppose that AC and BD are bisected at E: then $ABCD$ will be a parallelogram.

In the triangles AED and CEB the sides AE, ED are equal to the sides CE, EB each to each, by supposition; the angle AED is equal to the angle CEB, by I. 15; therefore the triangles are equal in all respects, by I. 4. Thus the angle ADE is equal to the angle CBE; therefore AD is parallel to CB, by I. 27.

Similarly AB is parallel to DC.

80. Let $ABCD$ be a parallelogram, and suppose that the straight line AC bisects the angles at A and C: then the four sides of the parallelogram will be equal.

For in the triangles ACB and ACD the side AC is common, the angle ACB is equal to the angle ACD, and the angle BAC is equal to the angle DAC; therefore the side BC is equal to the side DC, and the side BA to the side DA, by I. 26. But AB is equal to DC, and AD is equal to BC, by I. 34. Therefore the four sides AB, BC, CD, DA are all equal.

81. Let AB and CD be the parallel straight lines, and O the given point. In AB take any point E, and from E as centre, with radius equal to the given length, describe a circle meeting CD at F; join EF. Through O draw a straight line parallel to EF, meeting AB at G, and CD at H. Then $EGHF$ is a parallelogram, by construction; therefore GH is equal to EF, by I. 34: thus GH is equal to the given length.

82. Let $ABCD$ be a parallelogram; let straight lines bisecting the angles A and B meet at E: then AEB will be a right angle.

The angles EAB and EBA are together half of the angles DAB and ABC together. But the angles DAB and ABC are together equal to two right angles, by I. 29. Therefore the angles EAB and EBA are together equal to a right angle. Therefore the angle AEB is a right angle, by I. 32.

83. Let $ABCD$ be a parallelogram. Suppose that the straight lines which bisect the angles A and C are not coincident: then they shall be parallel.

Let the straight line which bisects the angle A meet BC at E; and let the straight line which bisects the angle C meet DA at F. Then, by I. 29, the angle BEA is equal to the angle DAE, that is to the half of the angle DAB, that is to the half of the angle DCB, by I. 34. Thus the angle BEA is equal to the angle BCF. Therefore EA is parallel to CF, by I. 28.

84. Let $ABCD$ be a parallelogram, and suppose that the diagonals AC and BD are equal: then all the angles of the parallelogram will be equal.

In the two triangles ABC and BAD the side AB is common; AD is equal to BC, by I. 34; and AC is equal to BD by supposition: therefore the angle ABC is equal to the angle BAD. Similarly it may be shewn that any other two adjacent angles are equal; so that all the four angles are equal.

85. Let AB and CD be the given straight lines; suppose that the required point is to be at a distance equal to E from AB, and at a distance equal to F from CD.

Draw a straight line parallel to AB, and at a distance E from it; also draw a straight line parallel to CD, and at a distance F from it; let the two straight lines thus drawn meet at O: then O will be the required point.

For the distance of O from AB will be equal to E, and the distance of O from CD will be equal to F, by I. 34.

Two straight lines can be drawn parallel to AB, and at the required distance from it, namely, one on each side of it; and in like manner two straight lines can be drawn parallel to CD, and at the required distance from it: hence *four* points can be found which will satisfy the conditions of the problem, assuming that AB and CD are not parallel.

86. Let AB and CD be the two given straight lines in which the required straight line is to be terminated. Let E be a straight line to which the required straight line is to be equal, and F that to which the required straight line is to be parallel.

From A draw a straight line parallel to F, by I. 31; and cut off from it AG equal to E, by I. 3. Through G draw a straight line parallel to AB, and let it meet CD at H. Through H draw a straight line parallel to AG, and let it meet AB at K. Then HK is the required straight line.

For HK is equal to AG, by I. 34; so that HK is of the required length: and it is parallel to AG, and therefore to F, by I. 30.

87. Let AEB, BFC, CGD be the three equilateral triangles: then will EF be equal to AC, and GF be equal to BD.

In the two triangles ABC and EBF the two sides AB, BC are equal to the two sides EB, BF each to each. The angle FBC is equal to the angle ABE, each being the angle of an equilateral triangle; to each of them add the angle ABF; therefore the angle ABC is equal to the angle EBF. Hence the triangles ABC and EBF are equal in all respects, by I. 4; so that AC is equal to EF.

Similarly it may be shewn that BD is equal to GF.

88. Let $ABCD$ be a parallelogram, and let $ABEF$ be another parallelogram having BE equal to BC, but the angle ABE greater than the angle ABC: then will the diagonal BF be less than the diagonal BD.

The two angles ABC, BCD are together equal to two right angles, and so also are the two angles ABE, BEF, by I. 29; therefore the two angles ABC, BCD are together equal to the two ABE, BEF: but the angle ABE is greater than the angle ABC, by hypothesis; therefore the angle BEF is less than the angle BCD.

In the two triangles BCD, BEF the two sides BC, CD are equal to the two sides BE, EF each to each; but the angle BCD is greater than the angle BEF: therefore the base BD is greater than the base BF, by I. 24.

89. Let AD, BE, CF be the perpendiculars from A, B, C respectively on the straight line: the sum of AD and CF shall be equal to twice BE.

The straight line DF produced does not pass between A and C; suppose that it cuts AC produced through C. Through E draw a straight line parallel to AC, and let it meet AD at G, and CF produced, through F, at H. Thus GE is equal to AB, and EH is equal to BC, by I. 34; therefore GE is equal to EH.

In the two triangles GED, HEF the angle GED is equal to the angle HEF, by I. 15; the angle DGE is equal to the angle FHE, by I. 29; and the side GE was shewn equal to the side HE: hence these triangles are equal in all respects, by I. 26; so that GD is equal to HF. Therefore AD and CF together are equal to AG and CH together, that is equal to twice BE, by I. 34.

90. Let $ABCD$ be the parallelogram; let O be the point of intersection of the diagonals AC and BD. Then by Exercise 78 the diagonals AC and BD are bisected at O. By Exercise 89 the sum of the perpendiculars from A and C on any straight line outside the parallelogram is twice the perpendicular from O; and also the sum of the perpendiculars from B and D is twice the perpendicular from O. Hence the sum of the perpendiculars from A and C is equal to the sum of the perpendiculars from B and D.

91. Let $ABCDEF$ be the six-sided figure. Then AB is by supposition equal and parallel to ED; therefore $ABDE$ is a parallelogram, by I. 33. Therefore AD passes through the middle point of BE, by Exercise 78. Similarly it can be shewn that CF passes through the middle point of BE. Thus AD, BE, and CF meet at a point.

92. Through E draw a straight line parallel to AB, and let it meet AC at F. On FC take FG equal to AF. Join GE and produce it to meet AB at H: then GEH shall be the straight line required.

Through F draw a straight line parallel to GH, and let it meet AB at K. In the triangles AFK and FGE the side AF is equal to the side FG by construction; the angle AFK is equal to the angle FGE, and the angle FAK is equal to the angle GFE, by I. 29: therefore FK is equal to GE, by I. 26. But FK is equal to EH, by I. 34: therefore GE is equal to EH, so that GH is bisected at E.

93. Let $ABCD$ be the given parallelogram, and P the given point on the side AB. On CD take CQ equal to AP; join AC and PQ intersecting at R. Through R draw a straight line at right angles to PQ, meeting AD at S, and CB at T. Then $PSQT$ will be the required rhombus.

In the triangles APR and CQR the sides AP and CQ are equal, by construction; the angle ARP is equal to the angle CRQ, by I. 15; and the angle RAP is equal to the angle RCQ, by I. 29: therefore PR is equal to QR, and AR is equal to CR, by I. 26.

In the triangles PRS and QRS the sides PR and QR are equal; RS is common; and the angles PRS and QRS are equal being right angles: therefore PS is equal to QS, by I. 4.

In the triangles CRT and ARS the sides AR and CR are equal; the angle ARS is equal to the angle CRT, by I. 15; and the angle ASR is equal to the angle CTR, by I. 29: therefore RS is equal to RT, by I. 26.

In the triangles SRP and TRP the sides SR and TR are equal; the side RP is common; and the angles SRP and TRP are equal being right angles: therefore SP is equal to TP, by I. 4.

In the same manner it may be shewn that TQ is equal to SQ, and also equal to TP. Hence $PSQT$ is a rhombus.

The construction fails if the straight line through R at right angles to PQ, instead of meeting AD and CB, meets DC and BA.

94. Let BE intersect AC at G, and DF intersect AC at H. Through G draw a straight line parallel to AD, meeting DF at K.

Then ED is equal and parallel to BF; therefore EB is equal and parallel to DF, so that $EGKD$ is a parallelogram, and GK is equal to ED, and therefore equal to AE. In the triangles AEG and GKH the sides AE and GK are equal; the angles EAG and KGH are equal, and the angles EGA and KHG are equal, by I. 29: therefore AG is equal to GH, by I. 26.

Similarly it may be shewn that CH is equal to HG: hence the three straight lines AG, GH, HC are all equal; so that AC is trisected.

I. 35 to 45.

95. Let O be the middle point of DC. Of the two straight lines AD and BC, suppose AD the less. Through O draw a straight line parallel to AB, meeting AD produced at E, and meeting BC at F.

Then in the two triangles EOD and FOC the sides DO and CO are equal; the angle EOD is equal to the angle FOC, by I. 15; and the angle OED is equal to the angle OFC, by I. 29: therefore the triangles are equal by I. 26. To each triangle add the figure $ADOFB$: thus the figure $ABFE$ is equal to the figure $ABCD$.

96. Construct a parallelogram by drawing through E a straight line parallel to AB; this parallelogram is equal to $ABCD$ by Exercise 95. The triangle AEB is half this parallelogram, by I. 41: therefore the triangle AEB is half the quadrilateral $ABCD$.

97. Let $ABCD$ be a parallelogram; let O be the middle point of the diagonal AC; through O draw any straight line meeting AB at E and CD at F: then the straight line EOF shall bisect the parallelogram.

In the two triangles AOE and COF the sides AO and CO are equal; the angle AOE is equal to the angle COF, by I. 15; and the angle OAE is equal to the angle OCF, by I. 29: therefore the triangles are equal, by I. 26. To each triangle add the figure $AOFD$; thus the figure $AEFD$ is equal to the triangle ACD. But the triangle ACD is half the parallelogram $ABCD$: therefore the figure $AEFD$ is half the parallelogram $ABCD$.

98. Let $ABCD$ be the parallelogram, and P the given point within it. Bisect AC at O; join PO and produce it to meet opposite sides of the parallelogram at E and F respectively. Then by Exercise 97, the straight line EPF bisects the parallelogram.

99. Let $ABCD$ be the parallelogram. Bisect AC at O; through O draw a straight line at right angles to AC, and let it intersect at E the straight line drawn through D parallel to AC. Produce EO through O to F, making OF equal to OE. Then $AFCE$ will be such a rhombus as is required.

For in the two triangles AOE and COE the two sides AO, OE are equal to the two sides CO, OE each to each; and the right angles AOE, COE are equal: therefore AE is equal to CE. Similarly we can shew that AF is equal to AE; and also that CF is equal to CE, and to AF. Therefore $AFCE$ is a rhombus. Also the triangle AEC is equal to the triangle ADC, by I. 37: therefore the rhombus $AFCE$ is equal to the parallelogram $ABCD$.

100. Let ABC, DEF be two triangles having the sides AB, BC equal to the sides DE, EF each to each; also the angles ABC and DEF together equal to two right angles: then the triangles shall be equal in area.

Produce CB to G, making BG equal to BC or EF; and join AG. Then in the two triangles ABG, DEF the sides AB, BG are equal to the sides DE, EF each to each; also the angles ABC and DEF are equal to two right angles by hypothesis, and the angles ABC and ABG are equal to two right angles by I. 13; therefore the angle ABG is equal to the angle DEF. Hence the two triangles ABG, DEF are equal in all respects, by I. 4.

Now the triangles ABC and ABG are equal in area by I. 38: therefore the triangles ABC and DEF are equal in area.

101. The triangle BEC is half the parallelogram $ABCD$ by I. 41: therefore the triangle BEC is equal to the figure $FDEC$. Take away the triangle FEC from both; then the remainders are equal; that is the triangle EBF is equal in area to the triangle CED.

102. Let $ABCD$ be a parallelogram; and let AC and BD intersect at O. Then, by Exercise 78, the straight lines AC and BD are bisected at O. Because AO is equal to OC the triangles AOB and COB are equal, by I. 38. Similarly BOC and DOC are equal; also COD and AOD are equal. Thus the four triangles are all equal.

103. The triangle AEC is equal to the triangle BED; to each of these add the triangle BEC; therefore the triangle ABC is equal to the triangle DBC: therefore DA is parallel to BC, by I. 39.

104. Let the diagonals intersect at O; then AO is equal to CO by Exercise 78. The triangle AOB is equal to the triangle COB, and the triangle AOP is equal to the triangle COP, by I. 38. Therefore the triangle PAB is equal to the triangle PCB.

105. Let $ABCD$ be any quadrilateral figure. Through A and C draw straight lines parallel to BD, and through B and D draw straight lines parallel to AC: thus a parallelogram is formed having two opposite sides equal to BD, and two opposite sides equal to AC, and its angles equal to those at O the intersection of AC and BD. Also this parallelogram is double the figure $ABCD$; for it consists of four parallelograms which are double of AOB, BOC, COD, DOA respectively. By drawing a diagonal of this parallelogram we obtain two triangles having two sides equal to the diagonals of the given quadrilateral, and an angle equal to one of those at O; and this triangle will be equal in area to the given quadrilateral by I. 41.

106. Let ABC be any triangle; let D be the middle point of BC, and E the middle point of AC: then ED shall be parallel to AB.

Join AD and BE. The triangle AED is equal to the triangle CED, by I. 38. The triangle BED is equal to the triangle CED, by I. 38. Therefore the triangle AED is equal to the triangle BED. Therefore AB is parallel to ED, by I. 39.

107. Let $ABCD$ be a quadrilateral; let E, F, G, H be the middle points of AB, BC, CD, DA respectively: then shall $EFGH$ be a parallelogram.

EF and GH are both parallel to AC, by Exercise 106; therefore they are parallel to each other, by I. 30. Similarly FG and HE are parallel: therefore $EFGH$ is a parallelogram.

108. The triangle BDC is equal to the triangle ABE, for each of them is half the triangle ABC, by I. 38. Take away the triangle BDF, and the

remainders are equal; that is the triangle BFC is equal to the quadrilateral $ADFE$.

109. Let ABC be a triangle; let D be the middle point of CB, and E the middle point of CA: ED shall be equal to half of AB.

Through D draw a straight line parallel to CA, and let it meet AB at F; then ED is parallel to AF, by Exercise 106: therefore AF is equal to ED, by I. 34.

In the two triangles CDE, DBF the sides CD, DB are equal by hypothesis; the angle CDE is equal to the angle DBF, and the angle ECD is equal to the angle FDB, by I. 29: therefore ED is equal to FB, by I. 26. Therefore AF is equal to FB; therefore AF is half of AB; therefore ED is half of AB.

110. By Exercise 106 the straight lines EG and FH are both parallel to BD, therefore they are parallel to each other, by I. 30; and by Exercise 109 they are each equal to half of BD, and therefore they are equal to each other.

111. Let D, E, F be the three given points; through D draw a straight line parallel to EF, through E draw a straight line parallel to FD, and through F draw a straight line parallel to DE. Let the first and second straight line meet at C, the second and third at A, and the third and first at B. Then ABC shall be the triangle required.

For by construction $BDEF$ and $DCEF$ are parallelograms, so that BD and DC are each equal to FE, by I. 34: therefore BD is equal to DC, and D is the middle point of BC. Similarly E is the middle point of CA, and F is the middle point of AB.

112. Let ABC be any triangle; let D be the middle point of BC, and E the middle point of CA: then the triangle EDC shall be one-fourth of the triangle ABC.

Join EB. The triangle EBA is equal to the triangle EBC by I. 38; therefore the triangle EBC is half the triangle ABC. Again, the triangle DEC is equal to the triangle BED by I. 38; therefore the triangle DEC is half the triangle BEC; therefore the triangle DEC is one-fourth the triangle ABC.

113. EA is equal to ED, by Exercise 59; therefore the angle EAD is equal to the angle EDA. Similarly the angle FAD is equal to the angle FDA. Therefore the whole angle EAF is equal to the whole angle EDF.

Again, the triangle EAD is equal to the triangle EBD, and the triangle FAD is equal to the triangle FCD, by I. 38. Therefore $AFDE$ is equal to EBD and FCD together. Therefore $AFDE$ is half the triangle ABC.

114. Let ABC, DBC be triangles of equal area on opposite sides of the same base BC; join AD cutting BC, or BC produced, at F: then shall AF be equal to FD.

Make the triangle BEC on the same side of BC as BDC, so that BE may be equal to BA and CE to CA; then the triangle BCE is equal to the triangle BCA in all respects, by I. 8. Therefore the triangles BCE and BCD are equal in area. Therefore ED is parallel to BC, by I. 39. Therefore the triangle FCE is equal to the triangle FCD, by I. 37.

In the two triangles CAF and CEF the side CF is common; the sides CA, CE are equal by construction; and the angles ACF and ECF are equal, since the angles ACB and ECB are equal: hence these triangles are equal in all respects, by I. 4.

Thus the area of the triangle ACF is equal to the area of the triangle ECF; and therefore the triangles ACF and DCF are equal in area. Thus AF must be equal to DF; for if these are not equal it can be shewn by the aid of I. 38, that the areas are not equal.

115. Let $ABCD$, $BEFC$, $EGHF$ be the three parallelograms. Join AF cutting BC at K; join BH cutting EF at L: then $BLFK$ shall be half of $BEFC$.

AF is parallel to BH, by I. 33. In the triangles ABK, BEL the sides AB and BE are equal, by hypothesis; the angles ABK and BEL, and the angles BAK and EBL are equal, by I. 29: therefore the triangles ABK and BEL are equal, and BK is equal to EL, by I. 26.

Thus EL is equal to LF, and the triangle ELB is half the parallelogram $LFBK$, by I. 38 and I. 41.

In the same way it may be shewn that the triangles ABK and FCK are equal, so that BK is equal to CK, and the triangle CKF is half the parallelogram $KBLF$.

Hence the triangles BEL and FCK are together equal to the parallelogram $KBLF$; so that $KBLF$ is half $BEFC$.

116. The triangle BCG is equal to the triangle BDG, by I. 37. Also the triangle BDG is equal to the sum of BFG and BFD, that is to the sum of BFG and BFA, by I. 37. And the triangle BCG is the sum of BFG and CFG. Hence the sum of BFG and BFA is equal to the sum of BFG and CFG. Therefore the triangle BFA is equal to the triangle CFG.

117. Suppose AD greater than AB. Join CD; through B draw a straight line parallel to DC, meeting AC at E; join ED: then AED will be the triangle required.

For the triangle EBD is equal to the triangle EBC, by I. 37. To each of these add the triangle ABE; then the triangle ADE is equal to the triangle ABC.

If AD is less than AB the straight line drawn through B parallel to DC will meet AC *produced;* but the demonstration will not be essentially changed.

118. Let D be the given point in BC. Join DA; through C draw a straight line parallel to DA, meeting BA produced at E; join DE: then DEB will be the triangle required.

For the triangle DEA is equal to the triangle DCA, by I. 37. To each of these add the triangle ADB: then the triangle EDB is equal to the triangle ABC.

If the given point D is in BC *produced* the process will not be essentially changed.

119. Let P be the given point in CD; join PA and PB. Through C draw a straight line parallel to PB, and through D draw a straight line parallel to PA. Through P draw a straight line parallel to AB, let it meet the straight line drawn through C at E, and the straight line drawn through D at F: then $ABEF$ is the quadrilateral required.

For the triangle PEB is equal to the triangle PCB, and the triangle PFA is equal to the triangle PDA, by I. 37. Therefore the figure $ABEF$ is equal to $ABCD$.

120. Join PA and PB. Through C draw a straight line parallel to PB, and let it meet AB produced at M; through D draw a straight line parallel to PA, and let it meet BA produced at N: then PMN is the triangle required.

For the triangle PBC is equal to the triangle PBM, and the triangle PAD is equal to the triangle PAN, by I. 37. Therefore the triangle PMN is equal to $ABCD$.

121. Let AC, produced if necessary, meet the given straight line at D; join DB; through C draw a straight line parallel to DB, meeting AB, produced if necessary, at E: then AED is such a triangle as is required.

For the triangle CED is equal to the triangle CEB, by I. 37: therefore the triangle AED is equal to the triangle ABC.

122. Let ABC be the given triangle, P the given point in the side AC. Suppose P to be nearer to A than to C. Bisect BC at D; join AD and PD. Through A draw a straight line parallel to PD, meeting BC at E. Join EP: then EP will bisect the triangle ABC.

For the triangle PED is equal to the triangle PAD, by I. 37; to each of these add the triangle PDC: therefore the triangle PCE is equal to the triangle ACD. But the triangle ACD is half the triangle ABC by I. 38 therefore the triangle PEC is half the triangle ABC.

If P be nearer to C than to A the side AB must be bisected.

123. Let $ABCD$ be the given quadrilateral, A the given angular point. Draw the diagonals AC and BD; bisect BD at E; join AE, CE. Through E draw a straight line parallel to AC; suppose this straight line to be further from B than AC is, and let it meet DC at G. Join AG: then AG will bisect the quadrilateral.

The triangle AEC is equal to the triangle AGC, by I. 37. To each add the triangle ABC; therefore the figure $ABCE$ is equal to the figure $ABCG$.

But the triangle ABE is half the triangle ABD, and the triangle CBE is half the triangle CBD, by I. 38: therefore the figure $ABCE$ is half the figure $ABCD$. Therefore the figure $ABCG$ is half the figure $ABCD$.

If the straight line drawn through E parallel to AC is *nearer* to B than AC is, it will meet BC instead of DC; but the demonstration will not be essentially changed.

124. If possible suppose that O is *not* in the diagonal AC. Let the straight line through O parallel to BC meet AB at E, and DC at F; and suppose that AC intersects EF at a point G between O and F. Through G draw a straight line parallel to AB.

Then the parallelogram GD is equal to the parallelogram GB, by I. 43. Therefore the parallelogram OD is greater than the parallelogram OB. But this is impossible, for they are equal by hypothesis. Therefore the point O cannot fall otherwise than on AC.

I. 46 to 48.

125. The angles ACD and BCF are equal, being right angles; to each add the angle ACB; thus the whole angle BCD is equal to the whole angle ACF.

In the two triangles BCD and ACF the two sides BC, CD are equal to the two sides FC, CA each to each; and the angle BCD is equal to the angle FCA: therefore FA is equal to BD, by I. 4.

126. Let BAC be a triangle, having the angle BAC *acute;* then will the square on BC be *less* than the sum of the squares on BA and AC.

From A draw a straight line at right angles to BA, and cut off AD equal to AC; join BD.

Then BD is greater than BC by I. 24. Now the square on BD is equal to the squares on BA and AD by I. 47. Hence the square on BC is less than the squares on BA and AD, that is less than the squares on BA, and AC.

127. Let BAC be a triangle, having the angle BAC *obtuse:* then will the square on BC be greater than the sum of the squares on BA and AC.

From A draw a straight line at right angles to BA, and cut off AD equal to AC; join BD.

Then BD is less than BC, by I. 24. Now the square on BD is equal to the squares on BA and AD, by I. 47. Hence the square on BC is greater than the squares on BA and AD, that is greater than the squares on BA and AC.

128. Let ABC be a triangle; and suppose the square on BC *less* than the squares on BA and AC: then the angle BAC will be an *acute* angle.

The angle BAC cannot be a right angle, for then the square on BC would be equal to the squares on BA and AC, by I. 47. The angle BAC cannot be obtuse, for then the square on BC would be greater than the squares on BA and AC, by Exercise 127. Therefore the angle BAC must be an acute angle.

Again, let ABC be a triangle; and suppose the square on BC *greater* than the squares on BA and AC: then the angle BAC will be an *obtuse* angle.

The angle BAC cannot be a right angle, for then the square on BC would be equal to the squares on BA and AC, by I. 47. The angle BAC cannot be acute, for then the square on BC would be less than the squares on BA and AC, by Exercise 126. Therefore the angle BAC must be an obtuse angle.

129. Let BAC be a triangle, having the angle A a right angle. Let a straight line meet AB at D, and AC at E: then shall the squares on BE and CD be equal to the sum of the squares on BC and DE.

The square on BE is equal to the squares on BA and AE, and the square on CD is equal to the squares on CA and AD, by I. 47. Therefore the squares on BE and CD are equal to the squares on BA, CA, AE, AD; that is to the squares on BC, AE, AD, that is to the squares on BC and DE.

130. Draw through P a straight line parallel to AD, meeting AB at K and CD at L. Draw through P a straight line parallel to AB, meeting BC at M and AD at N. Then AK is equal to DL, and KB is equal to LC, by I. 34.

The squares on PA and PC are together equal to the squares on AK, PK, CL, LP, by I. 47; that is to the squares on DL, PK, KB, LP; that is to the squares on DL, LP, PK, KB; that is to the squares on PD and PB, by I. 47.

131. Let ABC be a triangle having a right angle at C; and let the square on AC be three times the square on BC. From C draw CD to bisect AB, and CE perpendicular to AB. Then will the angles ACD, DCE, ECB be all equal.

The square on AB is equal to the squares on AC and BC, that is to three times the square on BC and the square on BC, that is to four times the square on BC. Hence it may be shewn that AB is twice BC. But AB is twice DC, by Exercise 59. Thus BC, CD, DB are all equal, so that BCD is an equilateral triangle. Hence the angle BCD is two-thirds of a right angle, and ACD is one-third of a right angle, by I. 32.

Again, in the two triangles BEC and DEC, the sides BC and DC are equal; therefore the angle CDB is equal to the angle CBD; and the angles BEC and DEC are equal being right angles: therefore the angles BCE and DCE are equal, by I. 32. Hence BCE and DCE are each one-third of a right angle; so that the three angles ACD, DCE, BCE are all equal.

132. Let ABC be a triangle, having a right angle at A; let E be the middle point of AC, and F the middle point of AB: then four times the squares on BE and CF will be equal to five times the square on BC.

For four times the square on BE is equal to four times the square on AB and four times the square on AE, by I. 47. And four times the square on CF is equal to four times the square on AC and four times the square on AF. Therefore four times the squares on BE and CF are equal to four times the square on AE, four times the square on AF, and four times the squares on AB and AC; that is to four times the square on AE, four times the square on AF, and four times the square on BC; that is to the squares on AC and AB, and four times the square on BC; that is to five times the square on BC.

133. From D draw a perpendicular DM on GB produced. The angles DBM and MBC are together equal to a right angle; and so are the angles CBA and MBC: hence the angles DBM and CBA are equal.

In the two triangles DBM, CBA the sides DB and CB are equal; the angles DBM and CBA are equal; and the right angles DMB and CAB are equal: hence BM is equal to BA, and DM is equal to CA, by I. 26. Therefore GM is equal to twice BA, and the square on GM is equal to four times the square on BA. The square on DG is equal to the squares on GM and DM, by I. 47; that is to four times the square on BA and the square on CA. Similarly it may be shewn that the square on EF is equal to four times the square on CA and the square on BA. Hence the squares on DG and EF are equal to five times the square on BA and five times the square on CA, that is to five times the square on BC, by I. 47.

II. 1 to 11.

134. Let a straight line AB be divided into two parts at C, and suppose that the squares on AC and CB are equal to twice the rectangle AC, CB: then shall AC be equal to CB.

For if AC be not equal to CB suppose AC the greater, and construct the diagram of II. 4; on CA take CX equal to CB, draw through X a straight line parallel to AD, meeting HG at Y and DF at Z. Then we have given that HF and CK are together equal to AG and GE. By our construction we make XG equal to CK, and YF equal to GE. Thus HZ and CK are together equal to AG; therefore HZ is equal to AY. But this is impossible for HD is greater than AH. Hence AC and CB cannot be unequal; that is they are equal.

135. By II. 5 the rectangle contained by the parts is always *less* than the square on half the line, except when the straight line is bisected; so that the rectangle contained by the parts is *greatest* when the straight line is bisected.

136. Take AC equal to a side of the smaller square; produce AC to D so that CD is equal to a side of the larger square: and from CD cut off CB equal to CA. Then by II. 6 the rectangle AD, DB is equal to the difference of the squares on CD and CB. Thus the required rectangle is found.

137. By II. 9 the sum of the squares on the two parts is always *greater* than double the square on half the straight line, except when the straight line is bisected; so that the sum of the squares on the parts is *least* when the straight line is bisected.

138. Take AC equal to the greater of the two straight lines; on AC produced take CD equal to the less of the two straight lines, and also take CB equal to AC. Then AD is equal to the sum of the two straight lines, and DB is equal to their difference. And it is shewn in II. 9 that the squares on AD and DB are together double of the squares on AC and CD.

139. Let AB be the given straight line to be divided; KL a side of the given square.

Make the angle ABE equal to half a right angle. With centre A and radius equal to KL describe a circle cutting BE at F. From F draw FD perpendicular to AB: then AB shall be divided at D in the manner required.

For it may be shewn as in II. 9 that FD is equal to DB. Also the squares on AD and DF are equal to the square on AF, so that the squares on AD and DB are equal to the square on KL.

A remark may be made like the last sentence of the solution of Exercise 65.

140. Let AB be the given straight line. Produce AB to C so that AC may be equal to the diagonal of the square described on AB; and from BA cut off BD equal to BC: then will the square on DA be double the square on DB.

Since the straight line CD is bisected at B and produced to A, we have the squares on CA and DA together equal to double the squares on BA and BD, by II. 10. But the square on CA is double the square on BA, by I. 47: therefore the square on DA is double the square on DB.

141. In the triangles HAC, FAB the two sides HA, AC are equal to the two sides FA, AB each to each; and the right angles HAC, FAB are equal: therefore the angle HCA is equal to the angle FBA, that is to the angle HBL. The angle LHB is equal to the angle AHC, by I. 15. Therefore the angle HLB is equal to the angle HAC, by I. 32. Thus the angle HLB is a right angle.

142. Since EB is equal to EF, the angle EBF is equal to the angle EFB; that is the angle OBL is equal to the angle CFL. Therefore the angle FCL is equal to the angle LOB, by I. 32 and Exercise 141. Thus the angle ECO is equal to the angle BOL, and therefore to the angle EOC, by I. 15. Therefore EO is equal to EC by I. 6; and therefore also equal to EA.

Thus the angles EOC and EOA are together equal to the two angles ECO and EAO; that is the angle AOC is equal to the two angles ACO and CAO: therefore the angle AOC is a right angle by I. 32.

143. In II. 11 it is shewn that the rectangle AB, BH is equal to the square on AH. Therefore the rectangle AH, HB together with the square on BH is equal to the square on AH. Thus the rectangle AH, HB is equal to the difference of the squares on AH and BH, that is to the rectangle contained by the sum and the difference of AH and BH. See the note on page 269 of the *Euclid* respecting II. 5 and II. 6.

II. 12 to 14.

144. Let ABC be a triangle having the sides AB, AC equal. From B draw BD perpendicular to CA. By II. 13 we know that the square on AC is less than the squares on AB and BC by twice the rectangle AC, CD. But the squares on AC and AB are equal, by hypothesis; therefore the square on BC is equal to twice the rectangle AC, CD.

145. See page 293 of the *Euclid*.

146. The squares on CD and AC are equal to twice the squares on AB and CB, by Exercise 145. And AC is equal to AB. Therefore the square on CD is equal to the square on AB together with twice the square on CB.

147. Let $ABCD$ be the parallelogram; join AC and BD intersecting at O; then AC and BD are bisected at O, by Exercise 78.

The squares on AB and BC are equal to twice the squares on AO and OB, by Exercise 145; and similarly the squares on AD and DC are equal to twice the squares on AO and OD. Therefore the squares on the sides of the parallelogram are equal to four times the square on AO and four times the square on BO, that is to the square on AC and the square on BD.

148. Let ABC be the triangle, and O the middle point of AB. By Exercise 145 the squares on AC and BC are equal to twice the squares on AO and OC. Now AO is given, and OC is of constant length: therefore the sum of the squares on AC and BC is invariable.

149. Let $ABCD$ be the quadrilateral; let E, F, G, H be the middle points of AB, BC, CD, DA respectively: the squares on AC and BD will be equal to twice the sum of the squares on EG and FH.

We know that $EFGH$ is a parallelogram, that EH is half BD, and HG is half AC: see Exercises 107 and 109. Therefore the squares on AC and BD are equal to four times the squares on EH and HG, that is to twice the sum of the squares on the sides of the parallelogram $EFGH$, that is twice the sum of the squares on the diagonals EG and FH, by Exercise 147.

150. Let $ABCD$ be the parallelogram; let the diagonals AC and BD intersect at O. Let a circle be described from O as centre, and let P be any point on the circumference of this circle. Then, by Exercise 145, the sum of the squares on PA and PC is equal to twice the squares on PO and AO; and this remains constant at whatever point of the circumference P may be. Similarly the sum of the squares on PB and PD remains constant. Therefore the sum of the squares on PA, PB, PC, PD remains constant.

151. Let $ABCD$ be the quadrilateral; let E be the middle point of AC, and F the middle point of BD.

The squares on AB and BC are equal to twice the squares on AE and BE by Exercise 145; and similarly the squares on AD and DC are equal to twice the squares on AE and DE: therefore the squares on the sides of the quadrilateral are equal to four times the square on AE together with twice the squares on BE and DE, that is to the square on AC together with twice the squares on BE and DE.

But the squares on BE and DE are equal to twice the squares on BF and EF by Exercise 145; and therefore twice the squares on BE and DE are equal to four times the square on BF together with four times the square on EF, that is to the square on BD together with four times the square on EF.

Therefore the squares on the sides of the quadrilateral are equal to the squares on the diagonals together with four times the square on EF.

152. Let O be the centre of the circle. By Exercise 145 the squares on EC and ED are equal to twice the squares on OE and OC, that is to twice the squares on AO and OC. Now one of the two straight lines AC and AD is the sum of AO and OC, and the other is the difference; hence the squares on AC and AD are equal to twice the squares on AO and OC: see the note on page 269 of the *Euclid* respecting II. 7; or Exercise 138.

Therefore the squares on EC and ED are together equal to the squares on AC and AD.

153. Let O be the middle point of AD. By Exercise 145 the squares on AB and BD are equal to twice the squares on BO and OD; and the squares on AC and CD are equal to twice the squares on CO and OD. But the squares on AB and BD are equal to the squares on AC and CD by supposition; hence twice the squares on BO and OD are equal to twice the squares on CO and OD. Therefore the square on BO is equal to the square on CO, so that BO is equal to CO.

154. Let AB be the base of an isosceles triangle ABC; from C draw CO perpendicular to AB: take any point D in the base, then will the square on AC exceed the square on DC by the rectangle AD, DB.

Suppose D to fall between A and O. The square on AC is equal to the squares on AO, OC by I. 47; and similarly the square on DC is equal to the squares on DO, OC. Hence the square on AC exceeds the square on DC by the difference of the squares on AO and DO, that is by the rectangle contained by the sum and the difference of AO and DO: see page 269 of the *Euclid*. Now BD is equal to the sum of AO and DO; and AD is the difference of AO and DO. Therefore the square on AC exceeds the square on DC by the rectangle AD, DB. The process is similar if D falls between B and O.

155. From D draw DF perpendicular to AB produced, and from E draw EG perpendicular to AC produced. The angles DBF and CBA are together equal to a right angle, by I. 13. The angles ACB and CBA are together equal to a right angle, by I. 32. Therefore the angle DBF is equal to the angle ACB.

In the two triangles DBF and BCA the sides DB and BC are equal; the angle DBF is equal to the angle BCA, and the right angles DFB and BAC are equal: therefore the triangles are equal in all respects. Similarly it may be shewn that the triangles CEG and BCA are equal in all respects.

The square on DA is equal to the squares on DB, BA, and twice the rectangle BA, BF, by II. 12; that is the square on DA is equal to the squares on BC, BA and twice the rectangle BA, AC: therefore the squares on DA and AC are equal to the squares on BC, BA, AC and twice the rectangle BA, AC. Similarly it may be shewn that the squares on EA and AB are equal to the squares on BC, BA, AC and twice the rectangle BA, AC. Therefore the squares on DA and AC are equal to the squares on EA and AB.

156. By II. 13 the square on BD is less than the squares on BA and AD by twice the rectangle AB, AE; and by the same Proposition the square on BD is less than the squares on BA and AD by twice the rectangle AC, AD; therefore the rectangle AB, AE is equal to the rectangle AC, AD.

157. Let ABC be an equilateral triangle; suppose AB produced to D so that the rectangle AD, DB is equal to the square on CB: then will the square on DC be equal to twice the square on CB.

From C draw CE perpendicular to AB; then the triangles CEA and CEB will be equal in all respects by I. 26. The square on DC is equal to the squares on CB, BD and twice the rectangle DB, BE, by II. 12; that is to the squares on CB, BD and the rectangle DB, BA; that is to the square on CB and the rectangle AD, DB, by II. 3; that is to twice the square on CB, by supposition.

158. Let ABC be a triangle, having the angle C a right angle; from C draw CD perpendicular to AB: then will the square on CD be equal to the rectangle AD, DB.

Bisect AB at E. Then one of the two straight lines AD and DB is equal to the sum of AE and ED, and the other is equal to their difference. Hence the rectangle AD, DB is equal to the difference of the squares on AE and ED: see the *Euclid*, page 269. But EC is equal to AE, by Exercise 59. Therefore the rectangle AD, DB is equal to the difference of the squares on EC and ED; that is to the square on CD, by I. 47.

159. Use the same diagram as in Exercise 158. Then the square on BC is equal to the squares on CD and DB, by I. 47; that is to the rectangle AD, DB and the square on DB, by Exercise 158; that is to the rectangle AB, DB, by II. 3.

Similarly the square on AC is equal to the rectangle BA, DA.

160. By II. 13 the square on AC together with twice the rectangle AB, BF is equal to the squares on AB, BC; and the square on AB together with twice the rectangle AC, CE is equal to the squares on AC, BC. Hence the squares on AC, AB, together with twice the rectangle AB, BF and twice

the rectangle AC, CE are equal to the square on AB, the square on AC, and twice the square on BC. Therefore twice the rectangles AB, BF and AC, CE are equal to twice the square on BC; and therefore the rectangles AB, BF and AC, CE are equal to the square on BC.

161. Let AB be the straight line which is to be divided. Let LM be a side of the given square. Bisect AB at C. From L draw a straight line LO at right angles to LM; with centre M and radius equal to AC describe a circle cutting LO at N. From CB cut off CD equal to LN. Then will the rectangle AD, DB be equal to the square on LM.

The rectangle AD, DB is equal to the difference of the squares on AC and CD, by II. 5; that is to the difference of the squares on MN and LN, by construction; that is to the square on LM, by I. 47.

III. 1 to 15.

162. Let A be the centre of the circle which is to be described, B the centre of the given circle. Join AB; through B draw a straight line at right angles to AB, cutting the given circle at L and M: then will AL be the radius of the circle required.

In the two triangles ABL and ABM, the side AB is common; BL is equal to BM; and the right angles ABL and ABM are equal: therefore AL is equal to AM, by I. 4. Therefore a circle described from the centre A with the radius AL will pass through M, and so will cut the given circle at the extremities of a diameter.

163. Each of the straight lines passes through the centre of the circle, as may be shewn in the manner of III. 1: thus the straight lines intersect at a fixed point.

164. Let the circles cut each other at B and E. Through B draw any straight line meeting one circle again at A and the other again at C. Through E draw a straight line parallel to ABC, meeting the first circle again at D and the second circle at F. Then shall AC and DF be equal.

Find P the centre of the circle $ABED$, and Q the centre of the circle $BCFE$. From P draw PK perpendicular to AB; and produce KP to meet DE at M: then since DE is parallel to AB the angles at M are right angles. Again, from Q draw QL perpendicular to BC, and produce LQ to meet EF at N: then the angles at N are right angles. Thus $KLNM$ is a right-angled parallelogram, and therefore KL is equal to MN.

KB is half of AB by III. 3, and BL is half of BC: therefore KL is half of AC. Similarly MN is half of DF. But KL is equal to MN: therefore AC is equal to DF.

165. Suppose that D and F are on the circumference of the circle with centre A; and that E and G are on the circumference of the circle with centre B. From A draw AL perpendicular to FC, and AP perpendicular to DC; from B draw BM perpendicular to CG and BQ perpendicular to CE. From B draw BX perpendicular to AL, so that BX is parallel to FG. Again from B draw a perpendicular BY on PA produced, so that BY is

parallel to DE. Then the angle BAY will be equal to the angle BAX, since, by supposition, FG and DE are equally inclined to AB.

Now by I. 26 it will follow that BX is equal to BY; also, as in Exercise 164, it may be shewn that BX is equal to half of FG, and BY is equal to half of DE. Therefore FG is equal to DE.

If the perpendiculars from B instead of meeting AL and PA produced, meet LA produced and AP the process is substantially unchanged.

166. Let A and B denote the centres of the two circles. From the process given in the solution of Exercise 165 it follows that the straight line drawn through the point of intersection of the two circles is always *less* than twice AB except when it is parallel to AB, and then it is equal to twice AB. Therefore the greatest possible straight line is obtained by drawing a straight line through one of the points of intersection of the circles parallel to the straight line joining the centres.

167. Let C denote the centre of the circle, A the point in the diameter, P and Q the extremities of the chord parallel to this diameter. From C draw CD perpendicular to PQ; then PQ is bisected at D, by III. 3.

By Exercise 145 the squares on AP and AQ are equal to twice the squares on AD and PD; that is to twice the squares on AC, CD, and PD, by I. 47; that is to twice the squares on AC and CP.

Now one of the segments of the diameter is the sum of AC and the radius, and the other is the difference of AC and the radius. Therefore the sum of the squares on the segments is equal to twice the square on AC and twice the square on the radius; that is to twice the squares on AC and CP; that is to the squares on AP and AQ. See Exercise 138.

168. Let O be the middle point of AB; then the sum of the squares on AP and BP is equal to twice the square on OP together with twice the square on AO. Hence we require OP to be the least possible. Join O with the centre of the circle, and let the joining line cut the circumference at Q; then Q is the required point: for OQ is less than any other straight line drawn from O to the circumference, by III. 8.

169. Let ABC be a circle, and let F be its centre; let DBE be another circle, and let G be its centre; let the second circle fall within the first, and let them touch at B. Let AC and DE be parallel diameters, A and D being towards the same parts. Then B, D, and A will be in a straight line.

Join BD and AD. If they are not in a straight line let BD produced meet AC at H. Join FG; then FG produced will pass through B, by III. 11. The angle GDB is equal to the angle FHB, by I. 29. Therefore the angle FHB is equal to the angle FBH. Therefore FH is equal to FB. Therefore FH is equal to FA which is absurd. Therefore B, D, and A cannot lie otherwise than in a straight line.

If the circles touch *externally* the process will be similar; but then the extremities of the diameters must be taken towards opposite parts.

170. Let C be the centre of the larger circle, A the centre of the other. Let FAE be a chord of the larger circle at right angles to AC; and let H be one of the points where it cuts the smaller circle. Let BHD be another chord of the larger circle, and let it be at right angles to FAE. Of the two EH and FH let FH be the less; and of the two BH and DH let DH be the less. Then will EH be equal to BH, and FH equal to DH.

From C draw CK perpendicular to BD; then $ACKH$ will be a square. The chords FE and BD are equally distant from C, and are therefore equal, by III. 14. Therefore AE and BK, the halves of these chords, are equal. But AH is equal to HK; therefore EH is equal to BH; and therefore FH is equal to DH.

171. Let A be the given point, C the centre of the circle. Join CA and through A draw a chord BAD at right angles to AC: this shall be the shortest chord through A.

For draw any other chord EAF through A; and from C draw the perpendicular CG on EAF. Then since CGA is a right angle CAG is less than a right angle, by I. 32. Therefore CG is less than CA, by I. 19. Therefore EAF is nearer to the centre than BAD; and therefore EAF is greater than BAD, by III. 15.

172. From O draw the radius OB parallel to PN, so that OB and PN are on *opposite* sides of the diameter on which ON lies. Join PB.

The angle NPB is equal to the angle OBP, by I. 29; the angle OBP is equal to the angle OPB by I. 5: therefore the angle NPB is equal to the angle OPB. Thus the angle OPN is bisected by the straight line PB; so that so long as P is on the same side of the fixed diameter the straight line bisecting the angle always passes through the fixed point B.

Produce BO to meet the circumference again at B'. Then if P is on the other side of the fixed diameter, the straight line bisecting the angle will pass through B'.

173. Take O the centre of the circle $DBCE$; join OD and OE. Take P the centre of the circle on which A and B lie, and Q the centre of the circle on which A and C lie.

The angle PAB is equal to the angle PBA, and therefore equal to the angle OBD, and therefore equal to the angle ODB, by I. 5 and I. 15. Therefore PA is parallel to DO by I. 27. Similarly QA is parallel to OE. But PA and QA form one straight line by III. 12. Therefore OD and OE form one straight line, which is a diameter of the circle $DBCE$, and is parallel to PQ.

174. Let $ABCD$ be the quadrilateral figure. Let the circles described on AB and BC as diameters intersect at E. Then the straight line which bisects EB at right angles will pass through the centres of the circles, by III. 1, so that it will bisect AB and BC. Therefore this straight line will be parallel to AE and to EC, by Exercise 106. Hence AE and EC are in one straight line; and the common chord EB is at right angles to AC. Similarly the common chord of the circles described on CD and DA as diameters is also at right angles to AC; and therefore the two common chords are parallel.

In like manner the common chord of the circles described on AD and AB as diameters is parallel to the common chord of the circles described on CB and CD as diameters.

175. Let C be the centre of the given circle, A the given point in the given straight line. On this straight line take AB equal to the radius of the given circle; join BC; at C draw the radius CD, making the angle BCD equal to the angle CBA, and on the same side of CB. Produce DC to meet the given straight line at O. Then O will be the required centre.

For the angle OCB is equal to the angle OBC, by construction; therefore OC is equal to OB, by I. 6. And BA is equal to CD, by construction; therefore OD is equal to OA; and the circle described from the centre O with the radius OA will pass through D, and will touch the given circle at D.

III. 16 to 19.

176. In the diagram of III. 17 let FD be produced to meet the outer circle at H; join EH cutting the inner circle at K; join AK: then AK will touch the circle BCD.

This may be shewn precisely as in III. 17. Also DH is equal to DF by III. 3; therefore AK will be equal to AB.

177. Let C be the centre of the given circle. From C draw a straight line perpendicular to the given straight line meeting it at A, and intersecting the circumference at B. Through B draw a straight line at right angles to CB; this touches the given circle by III. 16 Corollary; and it is parallel to the given straight line by I. 28.

178. Let C be the centre of the given circle. Draw the radius CA parallel to the given straight line. Through A draw a straight line at right angles to CA; this touches the given circle by III. 16 Corollary; and it is perpendicular to the given straight line by I. 29.

179. Let C be the centre of the circle, A the end of the diameter. Draw AD at right angles to CA, and make it of the given length. Join CD cutting the circumference at B; from B draw a straight line at right angles to CB and let it meet CA produced at E. Then EB will be the required tangent.

For in the two triangles CAD, CBE the angle C is common, the right angles CAD and CBE are equal, and the sides CA and CB are equal; hence AD is equal to BE, by I. 26. And BE touches the circle at B, by III. 16 Corollary.

180. Let O be the centre of the two circles. Let ABC be a chord of the outer circle, touching the inner circle at B; let DEF be another chord of the outer circle, touching the inner circle at E: then will AC be equal to DF.

Join OB and OE. The angles at B and E are right angles, by III. 18. Thus AC and DF are equally distant from O: therefore AC is equal to DF, by III. 14.

181. Let O be the centre of the given circle, P the given point. In the circle place a chord AB equal to the given straight line. From O draw a perpendicular OC on AB. With centre O and radius OC describe a circle, and from P draw a straight line touching this circle at E and meeting the given circle at the points D and F. Then $PDEF$ shall be the straight line required.

For by Exercise 180 the straight lines AB and DF are equal.

182. Let MCN be the diameter; let tangents be drawn at M and N; let AB be the portion of another tangent which is intercepted between the tangents at M and N: then ACB shall be a right angle.

Let AB touch the circle at E. Join AC and BC.

In the triangles CAE and CAM the sides AE and AM are equal, by Exercise 176; the sides EC and MC are equal; and the right angles AEC and AMC are equal: therefore the angles ACE and ACM are equal, by I. 4. Similarly the angles BCE and BCN are equal. Therefore the angle ACB is equal to the two angles ACM and BCN. Therefore the angle ACB is a right angle, by I. 13.

183. Let AB be the given straight line, C the centre of the given circle. Draw a straight line DE parallel to AB, and at a distance from it equal to the radius of the required circle which is given. With centre C describe a circle having its radius equal to the sum of the radii of the given circle and the required circle; let this circle cut DE at the points F and G: then either F or G may be taken as the centre of the required circle.

For the distance of F from AB is equal to the radius of the required circle, and therefore the circle described from F as centre with the given radius will touch AB. And the distance of F from C is equal to the sum of the radii of the given circle and the required circle; and therefore the circle described from F as centre will touch the given circle. Similarly G may be taken as the centre of the required circle.

If the line AB lies without the given circle, DE must be drawn on the same side of AB as C is.

184. Let O be the centre of the given circle. Let a circle described from C as centre touch the given straight line at A, and touch externally the given circle at B. From O draw a straight line parallel to CA to meet the circumference of the given circle at D, so that OD and CA may be on opposite sides of OC. Then D will be a fixed point; and D, B, A will lie in one straight line.

For OD and CA are parallel, by construction; and the angles at A are right angles, by III. 18: therefore if DO be produced to meet the given straight line the angles at the point of intersection will be right angles. Thus OD is a fixed straight line, and D is a fixed point.

Join AB, BD. The angle BOD is equal to the angle BCA by I. 29. Therefore the angles ODB and OBD together are equal to the angles CAB and CBA by I. 32. But the angle OBD is equal to the angle ODB, and the angle CBA is equal to the angle CAB, by I. 5. Therefore the angle OBD is equal to the angle CBA. Therefore DB and BA are in the same straight line.

185. See the *Euclid*, page 295.

186. Let ABC, DEF be the circles. It is required to draw a straight line touching the circle ABC, and so that the part of it intercepted by DEF may be equal to a given straight line.

In the circle DEF place the straight line DE equal to the given straight line. Find G the centre of this circle, and with G as a centre describe a circle touching DE. Draw a straight line AFH to touch the latter circle and to touch ABC, cutting the circle DEF at F and H: this shall be the straight line required.

For since DE and FH are equally distant from the centre of the circle DEF, they are equal: therefore FH is of the required length.

187. Let A and B be the centres of the two circles. In the circle having its centre at A place a chord PQ of the length which is to be

intercepted by this circle from the required straight line. In the circle having its centre at B place a chord RS of the length which is to be intercepted by this circle from the required straight line. With A and B as centres describe circles touching these chords respectively. Draw a straight line to touch the two circles thus described: this will be the straight line required.

For let this straight line meet the given circle which has A for centre at the points K and L, and the given circle which has B for centre at the points M and N. Then KL and PQ are equally distant from A, and therefore they are equal by III. 14. Similarly MN and RS are equal.

188. Let $ABCD$ be a quadrilateral, the sides of which touch a circle: then will AB and CD together be equal to BC and DA together.

Let AB touch the circle at P, let BC touch it at Q, let CD touch it at R, and let DA touch it at S. Then AP is equal to AS, BP to BQ, CQ to CR, and DR to DS, by Exercise 176. Therefore the sum of AP, PB, CR, RD is equal to the sum of AS, BQ, CQ, DS; that is AB and CD together are equal to BC and DA together.

189. Let $ABCD$ be a parallelogram which is described about a circle. Then AB and CD together are equal to BC and DA together, by the preceding Exercise. But AB is equal to CD, and BC is equal to DA, by I. 34. Hence AB, BC, CD, DA are all equal, so that the parallelogram is a rhombus.

190. If possible suppose that DE does not touch the circle but cuts it. Draw DF to touch the circle at F and produce DF to meet AE produced at G.

Then DF is equal to DB, and GF is equal to GC, by Exercise 176. Therefore DG is equal to the sum of DB and GC. But, by hypothesis, DE is equal to the sum of DB and EC; add EG to both; thus DE and EG are equal to the sum of DB, EC, and EG; that is to the sum of DB and GC. Therefore DE and EG are equal to DG; that is two sides of a triangle are equal to the third, which is impossible by I. 20. Therefore DE does not cut the circle. In the same way it may be shewn that DE does not fall without the circle. Therefore DE touches the circle.

191. Take the diagram of Exercise 188. Let O be the centre of the circle. Draw OA, OB, OC, OD, OP, OQ, OR, OS.

In the triangles OAS, OAP the side OA is common; the sides OS and OP are equal; the sides AS and AP are equal by Exercise 176. Therefore the angles AOS and AOP are equal, by I. 8. Similarly the angles BOP and BOQ are equal; also the angles COQ and COR are equal; and the angles DOR and DOS are equal. Therefore the angles AOP, BOP, COR, DOR are together equal to the angles AOS, BOQ, COQ, DOS; therefore each set of four angles are together equal to two right angles, by I. 15. Corollary 2.

192. Let CA and CB be the two radii which are at right angles. Let a straight line touch the circle, and meet CA produced at P, and CB produced at Q. From P draw PM to touch the circle, and from Q draw QN to touch the circle; then PM will be parallel to QN.

It may be shewn as in Exercise 191 that the angle MPQ is double the angle CPQ, and that the angle NQP is double the angle CQP. Therefore the two angles MPQ and NQP are double the angles CPQ and CQP. But the angles CPQ and CQP are together equal to a right angle, by I. 32:

therefore the angles MPQ and NQP are together equal to two right angles. Therefore PM and QN are parallel, by I. 28.

193. Let a straight line touch at C a circle which has its centre at A, and touch at D a circle which has its centre at B. Let DC and BA produced meet at O. Let the straight line OAB meet the first circle at E and the second circle at F, where E and F are taken on the same sides of A and B respectively. Then CE will be parallel to DF.

The angles OCA and ODB are right angles, by III. 18; therefore the angles CAO and DBO are equal, by I. 32. Therefore the two angles ACE and AEC are together equal to the two BDF and DFB, by I. 32. But the angles ACE and AEC are equal, and the angles BFD and BDF are equal, by I. 5. Therefore the angle AEC is equal to the angle BFD. Therefore CE is parallel to DF, by I. 28.

194. Let $ABCD$ be a quadrilateral. Let a circle touch AB at E, touch BC at F, and touch DA at L; let another circle touch BC at G, touch CD at H, and touch DA at K. Also let the two circles touch each other at O. Let the straight line MON be drawn touching both circles at O, and meeting AD at M, and BC at N. Then AD and BC together will exceed AB and DC together, by twice MN.

By Exercise 176 we see that AB and CD together are equal to AL, DK, CG, and FB: thus the excess of AD and BC together above AB and DC together is equal to LK and FG together. But LM and MK are each equal to MO, and FN and NG are each equal to NO. Therefore LK and FG together are equal to twice MN.

195. Let D be the point where the circle and the semicircle touch. Then D, O, C are in one straight line, III. 11. Let F be the point at which the circle touches AB, and produce FO to meet at E the tangent to the semicircle parallel to AB. Then will CO be equal to OE.

For FE is equal to the radius of the semicircle, because E is in the tangent to the semicircle which is parallel to AB. Thus FE is equal to CD. Also FO is equal to OD. Therefore OC is equal to OE.

196. From any point A without a circle having its centre at O let two straight lines AB, AC be drawn touching the circle. The angle BAC will be double the angle OBC.

The angles ABC and OBC together make up a right angle; so also do the angles ABC and BAO; therefore the angle OBC is equal to BAO. But the angle BAC is double the angle BAO; therefore the angle BAC is double the angle OBC.

197. Let AB be the diameter of a circle, C the centre. At any point D on the circumference let a tangent be drawn, meeting at E the tangent at A, and meeting at F the tangent at B. The area of the figure $ABFE$ will be half that of a rectangle contained by AB and EF.

The triangle EAC is equal to the triangle EDC, and the triangle FDC is equal to the triangle FBC, by I. 8 and Exercise 176. Hence the figure $ABFE$ is equal to twice the triangle ECF; and is therefore equal to a rectangle contained by EF and DC, and is therefore equal to half a rectangle contained by EF and AB.

198. Let $ABCD$ be the quadrilateral, having BC parallel to AD. Let K be the centre of a circle which touches AB at E, BC at F, CD at G, and DA at H. Through K draw a straight line LKM parallel to AD, and terminated by AB and DC. Then LM will be equal to one-fourth of the perimeter of $ABCD$.

It may be shewn that AB is bisected at L, and DC bisected at M: see Exercise 36.

The angle EBK is equal to the angle FBK, by Exercise 176. The angle FBK is equal to the angle LKB, by I. 29. Therefore the angle LBK is equal to the angle LKB. Therefore LK is equal to LB, by I. 6. Thus LK is half AB. Similarly MK is half DC. Thus LM is half the sum of AB and DC; that is LM is one-fourth of the perimeter of $ABCD$, by Exercise 188.

199. Let A be the fixed point in the fixed straight line. Let a circle which touches the fixed straight line at A cut the fixed parallel straight line at B and C. Draw a tangent to this circle at B, and let it meet the fixed straight line through A at F; and draw from A the perpendicular AE on this tangent: also draw from A the perpendicular AD on BC.

The angle FBA is equal to the angle FAB, by Exercise 176; and is therefore equal to the angle ABD, by I. 29. Hence in the two triangles ABE and ABD the side AB is common; the angle ABE is equal to the angle ABD; and the right angles AEB and ADB are equal: therefore AE is equal to AD, by I. 26. Hence the straight line FB will touch at E the circle described from A as a centre with a radius equal to the fixed length AD.

Similarly the tangent at C to the circle which passes through A, B, C will touch the same fixed circle.

200. Let A and B be the two given points. Let C be a point on the convex circumference of the given circle such that the tangent to the circle at C is equally inclined to AC and CB. The sum of AC and BC shall be less than the sum of the straight lines drawn from A and B to any other point on the circumference.

For take any other point on the circumference, as P; join PA and PB. Let PA cut at E the tangent at C, and join EB. Then the sum of PE and PB is greater than EB, by I. 20. Therefore the sum of PA and PB is greater than the sum of EA and EB. But the sum of EA and EB is greater than the sum of CA and CB, by page 306 of the *Euclid*. Therefore the sum of PA and PB is greater than the sum of CA and CB.

201. Join CD. The angle CDA is equal to the angle CAD, by I. 5. The angles CDB and BDE are together equal to a right angle, by III. 18. The angles CBA and CAB are together equal to a right angle, by I. 32. Therefore the angle BDE is equal to the angle CBA. Also the angle DBE is equal to the angle CBA, by I. 15. Therefore the angles BDE and DBE are equal; and the triangle DBE is isosceles, by I. 6.

202. Let O be the centre of the circle. Produce OC to a point F such that CF is equal to CO: then by comparing the triangles PCO and PCF it may be shewn that PF is equal to PO, and also OF is equal to OP, so that POF is an equilateral triangle. Therefore the angle POC is two-thirds of a right angle. Therefore the angle EPA is one-third of a right angle, by

I. 32. Therefore the angle PEA is two-thirds of a right angle, by I. 32. Therefore the angle CED is two-thirds of a right angle, by I. 15.

The angle COA is equal to the angles OBC and OCB, by I. 32, that is to twice the angle OBC, by I. 5: therefore the angle OBC is one-third of a right angle. The angle BAD is a right angle. Therefore the angle ADB is two-thirds of a right angle. Thus each of the angles CDE and CED is equal to two-thirds of a right angle; and therefore the angle ECD is also two-thirds of a right angle, by I. 32. Thus the triangle DEC is equiangular, and therefore equilateral.

III. 20 to 22.

203. The four angles ABD, BDC, DCA, CAB are equal to four right angles. The angle BAC is fixed, and the angle BDC is constant, by III. 21: therefore the sum of the angles ABD and ACD is constant.

204. The angle RAB is half the sum of the angles PAB and QAB; and the angle RBA is half the sum of the angles PBA and QBA. Thus the angle R together with half the sum of the angles PAB, QAB, PBA, QBA is equal to two right angles. Again, half the sum of the angles Q, QAB, QBA is equal to a right angle; and half the sum of the angles P, PAB, PBA is equal to a right angle. Thus half the sum of the angles P and Q together with half the sum of the angles PAB, QAB, PBA, QBA is equal to the angle R together with half the sum of the angles PAB, QAB, PBA, QBA. Therefore the angle R is equal to half the sum of the angles P and Q, and is therefore constant by III. 21.

205. In the quadrilateral $QCPD$ the angles QCP, CPD, PDQ are all constant by III. 21. The four angles of a quadrilateral are together equal to four right angles. Therefore the angle CQD is constant.

206. The angles at Q and R are constant by III. 21; therefore the third angle of the triangle formed on QR as a base by QA and RB produced is constant, by I. 32.

207. The angles ODC and ADC are together equal to two right angles by I. 13; but the angle ODC is equal to the angle ABC; therefore the angles ABC and ADC are together equal to two right angles. Therefore a circle can be described round $ABCD$: see the *Euclid*, page 276.

208. Let $ABCD$ be the quadrilateral inscribed in a circle. Let AB produced through B, and DC produced through C, meet at O.

The angles BAD and BCD are together equal to two right angles, by III. 22. The angles BCD and BCO are together equal to two right angles by I. 13. Therefore the angles BAD and BCD are together equal to the two angles BCD and BCO. Therefore the angle BCO is equal to the angle DAO. Similarly the angle CBO is equal to the angle ADO.

209. Let $ABCD$ be a parallelogram inscribed in a circle. By III. 22 the angles A and C are together equal to two right angles; by I. 34 the angles A and C are equal: therefore each of the angles A and C is a right angle.

Similarly each of the angles B and D is a right angle.

210. Let ABC be a triangle inscribed in a circle. Let D, E, F be points in the segments exterior to the triangle cut off by BC, CA, AB respectively. The angles CAB and CDB are together equal to two right angles, by III. 22; so also are the angles CBA and CEA, and the angles ACB and AFB. Therefore the angles in the exterior segments together with the angles of the triangle are together equal to six right angles; therefore the angles in the exterior segments are together equal to four right angles, by I. 32.

211. Take the diagram of III. 22. Let $ABCD$ be a quadrilateral inscribed in a circle. The angle in the segment exterior to AB and the angle ACB are together equal to two right angles, by III. 22. So also are the angle in the segment exterior to BC and the angle CAB. Also the angle in the segment exterior to CD and the angle CBD. Also the angle in the segment exterior to DA and the angle DBA. Therefore the angles in the four exterior segments together with the angles ACB, CAB, CBD, DBA are equal to eight right angles. But the angles ACB, CAB, CBD, DBA together make up the angles of the triangle ABC, and are therefore equal to two right angles. Hence the angles in the four exterior segments are equal to six right angles.

212. Let ACB be a triangle inscribed in a circle such that the angle ACB is one-third of two right angles. In the segment cut off by AB on the opposite side of C take any point D on the circumference. Then the angles ACB and ADB are together equal to two right angles, by III. 22. But the angle ACB is one-third of two right angles, by construction. Therefore the angle ADB is two-thirds of two right angles. Thus the angle in the segment ADB is double the angle in the segment ACB.

213. Let ACB be a triangle inscribed in a circle such that the angle ACB is one-sixth of two right angles. In the segment cut off by AB on the opposite side of C take any point D on the circumference. Then the angles ACB and ADB are together equal to two right angles, by III. 22. But the angle ACB is one-sixth of two right angles by construction. Therefore the angle ADB is five-sixths of two right angles. Thus the angle in the segment ADB is five times the angle in the segment ACB.

214. Let $ABCD$ be a quadrilateral. Let a side AB be produced to E, and suppose that the angle EBC is equal to the angle ADC.

The angles ABC and EBC are equal to two right angles by I. 13. But the angle EBC is equal to the angle ADC, by supposition. Therefore the angles ABC and ADC are together equal to two right angles. Therefore a circle can be circumscribed round $ABCD$: see the *Euclid*, page 276. Then the angles ADB and ACB will be equal by III. 21. In like manner any other side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

215. Let $ABCDEF$ be a hexagon inscribed in a circle; suppose that AB is parallel to DE, and BC parallel to EF: then will CD be parallel to FA.

Join AD. The angle ABC is equal to the angle FED by Exercise 35. The angles FED and FAD are together equal to two right angles by III. 22; so also are the angles ABC and ADC. Therefore the angles FED and

FAD are together equal to the angles *ABC* and *ADC*. Therefore the angle *FAD* is equal to the angle *ADC*. Therefore *FA* is parallel to *DC*, by I. 27.

216. Let the straight lines which respectively bisect the angles *APC* and *AQC* meet at *H*: the angle *PHQ* will be a right angle.

The angle *PCQ* and the angle *BAD* are together equal to two right angles, by I. 15 and III. 22. Therefore the angles *PQC* and *QPC* are together equal to the angle at *A*. Now the angle *CQH* and half the angles at *A* and *B* are equal to a right angle by I. 32; so also the angle *CPH* together with half the angles at *A* and *D* are equal to a right angle. Thus the angles *CQH*, *CPH*, together with the angle at *A*, and half the angles at *B* and *D* are equal to two right angles; therefore the angles *CQH*, *CPH*, and *A* are together equal to a right angle, by III. 22. Therefore, by what is shewn above, the angles *HQP* and *HPQ* are together equal to a right angle. Therefore the angle *PHQ* is a right angle, by I. 32.

217. Let *ABCD* be a quadrilateral inscribed in a circle. Let a straight line *EFGH* be drawn meeting the circumference at *E* and *H*, the side *AD* at *F*, and the side *BC* at *G*; and suppose the angle *AFG* to be equal to the angle *BGF*: then will *EH* also make equal angles with *AB* and *DC*.

Since the angles *AFG* and *BGF* are equal the arcs *ED* and *ABH* are together equal to the arcs *HC* and *EAB*: see the *Euclid*, page 294. Therefore the difference between the arcs *HC* and *ED* is equal to the difference between the arcs *BH* and *EA*. Therefore *HE* makes equal angles with *AB* and *DC*: see the *Euclid*, page 294.

218. Let *EFGH* be a quadrilateral which can be inscribed in a circle, so that the angles *E* and *G* are together equal to two right angles. Suppose also that a circle can be inscribed in this quadrilateral touching the sides *EF*, *FG*, *GH*, *HE* at *A*, *B*, *C*, *D* respectively. Then will the straight lines *BD* and *AC* be at right angles to each other.

Let *O* be the centre of the circle *ABCD*. The angle between *AC* and *BD* is measured by half the sum of the arcs *DA* and *BC*; see the *Euclid*, page 294, and is therefore equal to half the sum of the angles *DOA* and *BOC*. Now the angles *ODE* and *OAE* are right angles; therefore *DOA* and *DEA* are together equal to two right angles, by I. 32. But the angles *DEA* and *BGC* are together equal to two right angles, by supposition. Therefore the angle *DOA* is equal to the angle *BGC*. Similarly the angle *BOC* is equal to the angle *AED*. Therefore the angle between *AC* and *BD* is equal to half the sum of the angles *AED* and *BGC*, that is to a right angle.

III. 23 to 30.

219. Let *AC* and *BG* be chords of two equal arcs in a circle; let *AB* and *CG* join the extremities of these chords towards the same parts: then *AB* and *CG* shall be parallel.

The angle *ABC* is equal to the angle *BCG* by III. 27: therefore *CG* is parallel to *AB* by I. 27.

220. Let *AB* and *CG* be parallel chords in a circle; let them be joined towards the same parts by *AC* and *BG*, and towards opposite parts by *AG* and *BC*: then *AC* will be equal to *BG*, and *AG* to *BC*.

Because AB is parallel to CG the angle ABC is equal to the angle BCG, by I. 29: therefore the arc AC is equal to the arc BG, by III. 26: therefore the straight line AC is equal to the straight line BG, by III. 29.

Again, it has been shewn that the arc AC is equal to the arc BG; to each add the arc CG: therefore the arc ACG is equal to the arc BGC: therefore the straight line AG is equal to the straight line BC, by III. 29.

221. Suppose C and D on opposite sides of A, and C and E on opposite sides of B. Then the angle DAE is the sum of the angles ACB and AEB, which are invariable by III. 21, and is therefore invariable. Therefore the arc DE is invariable, by III. 26.

But C and D may be on the same side of A, and C and E on the same side of B. Thus the angle ECD is equal to the angle ACB, and is therefore invariable, by III. 21. Also the angle AEB is invariable, by III. 21. Therefore the angle EAD, which is the difference of ECD and AEB, by I. 32, is also invariable. Therefore the arc DE is invariable, by III. 26.

222. Let the straight line which bisects the angles meet the arc CB at a point P: then will EP be equal to BP.

The angles ECP and EBP are together equal to two right angles, by III. 22; the angles ECP and DCP are equal to two right angles, by I. 13: therefore the angle EBP is equal to the angle DCP, that is to half the angle DCB. Again, the angle BEP is equal to the angle BCP, by III. 21, that is to half the angle DCB. Thus the angles EBP and BEP are equal; therefore EP and BP are equal, by I. 6.

223. Let $ABCD$ be a quadrilateral figure inscribed in a circle; produce CD to any point E; let the straight lines which bisect the angles ABC and ADE meet at F: then F shall be on the circumference of the circle.

Let AD and BF intersect at H. The angle ADE is equal to the angle ABC, by III. 22: therefore the angle HDF is equal to the angle HBA. The angle DHF is equal to the angle BHA, by I. 15. Therefore the angle DFH is equal to the angle BAH, by I. 32. Therefore F is on the circumference of the circle. See the *Euclid*, page 276.

If BF meet AD produced at H, the proof will be similar, using the converse of III. 22 instead of that of III. 21.

224. Let C be the centre of the given circle. From D as centre with radius equal to CD describe a circle cutting AB produced at F. Join FD and produce it to meet the given circle again at E. Then will the arc AE be three times the arc BD.

For the angle ACE is equal to the two angles CEF and CFE, by I. 32; therefore the angle ACE is equal to the two angles CDE and CFD, by I. 5; therefore the angle ACE is equal to the angle DCF and twice the angle DFC, by I. 32; therefore the angle ACE is equal to three times the angle DCF, by I. 5. Hence the arc AE is three times the arc BD, by III. 26.

225. The points P and Q will always lie on a fixed segment of a circle described on AB: see the *Euclid*, page 276. Also as the angle AQB is given, and the angle C is given, the angle QBP is invariable. Hence the straight line PQ is always of the same length.

226. Let AB be the common chord of the two equal circles. Through A draw a straight line meeting one circle at C, and the other at D; draw BC and BD: these straight lines shall be equal.

For the angle ACB is equal to the angle ADB, by III. 27 and III. 28: therefore BC is equal to BD by I. 6.

227. In the two triangles BOP, BOQ the side OB is common; the angle BOP is equal to the angle BOQ, by supposition; and the right angles BPO and BQO are equal: therefore BP is equal to BQ, by I. 26.

Now AB is equal to BC, by III. 26 and III. 29. Therefore the square on AB is equal to the square on BC. Therefore the squares on AP and PB are equal to the squares on CQ and QB. But PB has been shewn equal to QB. Therefore the square on AP is equal to the square on CQ. Therefore AP is equal to CQ.

228. Suppose AB to lie between AL and AM, and AM nearer to the centre than AL. From B draw BP perpendicular to AM and BQ perpendicular to AL produced. Then, as in Exercise 227, we have PM equal to QL. Therefore the sum of AL and AM is equal to twice AP; and this is a fixed quantity, because the side AB is fixed, and so are the two angles PAB and APB.

Next let AB not lie between AL and AM, and suppose AM nearer to the centre than AL. From B draw BP perpendicular to AM and BQ perpendicular to LA produced. Now LB and MB are equal by Exercise 222; and thus we find as in Exercise 227 that LQ and PM are equal. Hence the difference of AM and AL will be found to be equal to twice AP, and this as before is a fixed quantity.

229. Let FE and OA be produced to meet at H. Bisect OF at K, and join EK.

The triangles EFK and EOK are equal in all respects by I. 8; so that EKO is a right angle: therefore KE and OH are parallel by I. 28. Therefore the angle FEK is equal to the angle EHO, and the angle KEO is equal to the angle EOH, by I. 29. Therefore the angle EHO is equal to the angle EOH.

The angle GOB is equal to the two angles OGH and OHG, by I. 32; therefore the angle GOB is equal to the two angles OEG and OHG, by I. 5; therefore the angle GOB is equal to the angle EOH and twice the angle OHG; therefore the angle GOB is equal to three times the angle EOH. Hence the arc BG is three times the arc AE, by III. 26.

230. Let AB be the given base, and ABC one of the triangles. The point C will be situated on the arc of a certain fixed segment of a circle described on AB as a base: see the *Euclid*, page 276. Let the angle at C be bisected by a straight line, and let this straight line be produced to cut at D the circumference of the circle of which the segment is part. Then since the angles ACD and BCD are equal the arcs AD and BD are equal; so that D is the middle point of the arc ADB. Hence D is a fixed point, and the straight lines which bisect the vertical angles all pass through this fixed point.

231. Let the two circles touch at A. Let K be the centre of the smaller circle, and L the centre of the larger circle. Let BDC be a chord of the larger circle, touching the smaller circle at D. Then BD and DC will subtend equal angles at A.

Produce AD to cut the larger circle at F. Join KD and LF. Then A, K, L are in one straight line, by III. 11. The angle KAD is equal to the angle KDA, and the angle LAD is equal to the angle LFA, by I. 5: therefore the angle KDA is equal to the angle LFA: therefore LF is parallel to KD, by I. 28. But KD is at right angles to BC, by III. 18; therefore LF is at right angles to BC, by I. 29. Therefore LF bisects BC, by III. 3. Therefore the arc BF is equal to the arc CF. Therefore the angle BAF is equal to the angle CAF, by III. 27.

III. 31.

232. Let AB be the given hypotenuse, and ABC one of the triangles having the angle at C a right angle. Let D be the middle point of AB. Then DC is equal to DA by Exercise 59, so that C is on the circumference of a circle having D as centre, and DA as radius, and therefore having AB as diameter.

233. Let ABC be a triangle having the sides AC and BC equal. Let D be the middle point of AB. Then the triangles ADC and BDC are equal in all respects, by I. 8; so that the angles ADC and BDC are right angles. Hence the circle described on AC as diameter will pass through D, by Exercise 232; and also the circle described on BC as diameter will pass through D. Thus the circles intersect at D.

234. Let $ABCD$ be a rectangle inscribed in a circle. Since the angle ADC is a right angle AC is a diameter of the circle. Let O be the centre of the circle. The rectangle is double of the triangle ADC by I. 34. The triangle DAC has a fixed base, namely the diameter of the circle; but its height is always less than OD except when DOA is a right angle. Therefore the greatest rectangle that can be inscribed in a circle is one which has its diameters at right angles to each other; this by I. 4 has its sides all equal, and is therefore a square.

235. Suppose C and E on opposite sides of AB, and C and A on opposite sides of FE. CD is equal to half AB, by Exercise 59. Therefore a circle described from the centre D with radius CD will pass through B, E, A, F. The angle ECB is half the angle EDB, by III. 20; that is half a right angle, and is therefore half the angle ACB, by III. 31. Thus EC bisects the angle ACB.

Similarly the angle FCA is half the angle FDA, that is half a right angle. Thus FC bisects the angle between CA and BC produced.

236. Let O be the centre of the circle. Let E and C be on the same side of AB. EF is parallel to BC; therefore the angle OEB is equal to the angle EBC, by I. 29. But the angle OEB is equal to the angle OBE, by I. 5. Therefore the angle OBE is equal to the angle EBC. Thus EB bisects the angle OBC.

Similarly if CB be produced to D it may be shewn that the angle OBD is bisected by BF.

237. A circle described on AC as diameter will pass through D and E, by Exercise 232. Then ACE and ADE will be angles in the same segment, and therefore equal by III. 21.

238. The angles ABC and ABD are both right angles, by III. 31. Therefore BC and BD are in one straight line by I. 14. That is the straight line CD passes through B.

239. Draw through A any chord AC of the larger circle, and let it intersect the smaller circle at B. Then OBA is a right angle by III. 31. Therefore OB bisects AC, by III. 3.

240. Let A, B, C be three given points in a given straight line; and let it be required to describe a circle which shall touch the straight line at B, and have the tangents drawn to the circle from A and C parallel.

On AC as diameter describe a semicircle. From B draw a straight line at right angles to the given straight line, and let it meet the semicircle at O. From O as centre with radius equal to OB describe a circle: this will be the circle required.

For this circle touches ABC at B, by III. 16 Cor. Draw AD and CE to touch this circle. Thus AD is equal to AB, by Exercise 176: therefore the triangles OAB and OAD are equal in all respects, so that the angle OAD it equal to the angle OAB, and therefore the angle DAB is double the angle OAB. Similarly the angle ECB is double the angle OCB. Therefore the angles DAB and ECB are together double the angles OAB and OCB. But the angle AOC is a right angle, by III. 31; therefore the angles OAC and OCA are together equal to a right angle, by I. 32. Therefore the angles DAB and ECB are together equal to two right angles. Therefore AD and CE are parallel, by I. 28.

241. Let A and B be the given points in the given straight line; so that the tangents from A and B to the required circle are to be parallel; let C denote the centre of the required circle. Then as in Exercise 182, or as in Exercise 240, we see that the angle ACB must be a right angle. Therefore C must lie on the circumference of a circle described on AB as a diameter, see the *Euclid*, page 276.

Again, since the circle is to have a given radius and to touch AB the centre C must lie on a straight line which is parallel to AB, and at a distance from AB equal to the given radius.

Therefore describe a circle on AB as a diameter, and draw a straight line parallel to AB and at a distance from AB equal to the given radius: then the intersection of the straight line and the circle will determine the centre of the required circle which can therefore be drawn.

242. Let ABC be a triangle; from B draw BE perpendicular to AC, produced if necessary; from C draw CF perpendicular to AB, produced if necessary. Let D be the middle point of BC.

Then DE and DF are each equal to DC, by Exercise 59. Therefore a circle described from D as centre, with radius equal to DC, will pass through B, F, and E. Therefore the perpendicular from D on EF will bisect EF, by III. 3.

243. The angle ABD is a right angle, and the angle BCD is obtuse by III. 31. The square on AD is equal to the sum of the squares on AB and BD, by I. 47. Therefore the square on AD is equal to the squares on AB, BC, CD and twice the rectangle BC, CE, by II. 12.

244. Let C be the middle point of AM. Let PM and QR intersect at D. Produce DQ to any point E.

Then $PQMR$ is a rectangle, by III. 31; therefore DQ equals DP. The angle EQC is equal to the angles EQA and AQC; and therefore equal to the angles PQD and AQC; and therefore equal to the angles QPD and AQC;

and therefore equal to the angles APM and PAM; and therefore equal to a right angle, by I. 32. Therefore QR touches the circle AQM at Q.

Similarly it may be shewn that QR touches the circle BRM at R.

245. A circle described on BC as diameter will pass through F and D; also a circle described on FD as diameter will pass through E and G; see the *Euclid*, page 276. By the aid of the latter circle we have the angle FEG equal to the angle FDG: by the aid of the former circle we have the angle FBC equal to the angle FDG. Therefore the angle FEG is equal to the angle FBC. Therefore EG is parallel to BC, by I. 28.

246. Take the case in which the chords BC and AC are *produced through* C to meet the other circumference. Let HC be a diameter of the circle ABC, and let it be produced to cut ED at K.

The angle CED is equal to the angle CAB, by III. 21; and therefore equal to the angle CHB, by III. 21. The angle ECK is equal to the angle HCB, by I. 15. Therefore the angle EKC is equal to the angle HBC, by I. 32. Therefore the angle EKC is a right angle, by III. 31.

The figure becomes modified in other cases, but the demonstration remains substantially unchanged.

247. Let ABC be a triangle having a right angle at C. Let squares be described on BC, CA, AB outside the triangle; and let D, E, F respectively be the intersections of the diagonals in these squares.

Then the angle AFB is a right angle, and therefore a circle would go round $ACBF$: see in the *Euclid* the note on III. 22. The angle FCA will be equal to the angle FBA, by III. 21. Therefore the angle FCA is half a right angle. Also the angle ACE is half a right angle. Therefore the angle FCE is a right angle. So also is FCD. Then use I. 14.

248. Draw a straight line at right angles to CA at the point A; let this meet the circumference of the given circle at B: then B will be the point required.

For a circle described on CB as diameter will go through A, and will touch the given circle at B. Let P be any other point on the circumference of the given circle. Join CP cutting at Q the circle just described; and join AQ.

Then the angle CBA is equal to the angle CQA, by III. 21. The angle CQA is greater than the angle CPA by I. 16. Therefore the angle CBA is greater than the angle CPA.

249. Let FG produced meet AB at H. The angles FDG and FEG are right angles, by III. 31: therefore a circle would go round $FDGE$, by page 276 of the *Euclid*. Therefore the angle DFG is equal to the angle DEG; and therefore equal to the angle DBA, by III. 21. Hence in the triangles AFH and ABD the angle at A is common; and the angle AFH is equal to the angle ABD; therefore the angle AHF is equal to the angle ADB, and is therefore a right angle, by III. 31.

250. Let two equal circles touch externally at B. Let AB be a diameter of one circle, and BC be a diameter of the other. Draw BD, a chord of the first circle; and BE at right angles to BD a chord of the second circle. The straight line DE shall be parallel to AC and equal to the distance between the centres of the given circles.

Join AD and CE. The angle ADB is a right angle, by III. 31; the angle DBE is a right angle by supposition; therefore the angles ADB and DBE are equal: therefore AD is parallel to BE, by I. 27. Therefore the angle BAD is equal to the angle CBE, by I. 29. The angle ADB is equal to the angle BEC, by III. 31. Also AB is equal to BC. Therefore BE is equal to AD, by I. 26. And it was shewn that BE is parallel to AD. Therefore DE is equal and parallel to AB, by I. 33.

251. Let $ACBD$ be the rhombus, and AB the shorter diagonal. On AB as diameter describe a circle cutting AC at H, BC at K, AD at L, BD at M. Join AK and BH, intersecting at E. Join AM and BL intersecting at F. Then $AEBF$ will be a rhombus, having its angles equal to those of $DACB$.

Since AC is equal to BC the angle CAB is equal to the angle CBA. In the triangles BAH and ABK the angles HAB and KBA are equal, as we have just shewn; the angles AHB and AKB are equal, being right angles by III. 31: therefore the angles ABH and BAK are equal, by I. 32. Therefore EA is equal to EB, by I. 5.

Also AC is parallel to BD, by Exercise 76; therefore the angle HAB is equal to the angle MBA; and the right angle AHB is equal to the right angle BMA; therefore the angle ABH is equal to the angle BAM. Therefore BE is parallel to AF. Similarly AE is parallel to BF. Thus the figure $AEBF$ is a parallelogram. And since EA has been shewn equal to EB the parallelogram is a rhombus.

Again, the angle AEB is equal to the angle HEK, by I. 15. The angles HEK and HCK are together equal to two right angles, because the angles EHC and EKC are right angles. The angles HCK and CAD are equal to two right angles, by I. 29. Therefore the angles HCK and CAD are together equal to the angles HCK and HEK. Therefore the angle CAD is equal to the angle HEK, that is to the angle AEB. This shews that the angles of one rhombus are equal to the angles of the other.

252. Let AB, CD be two chords of a circle; let them meet, produced if necessary, at E at right angles. The squares on AE, EB, CE, ED will be together equal to the square on the diameter.

Let O be the centre of the circle. Draw OG perpendicular to AB, and OH perpendicular to CD. Then AB is bisected at G, and CD is bisected at H, by III. 3. The squares on AE and EB are together double of the squares on AG and EG, and the squares on CE and ED are together double of the squares on CH and HE: see the *Euclid*, page 269. Also EG is equal to HO, and EH is equal to GO, by I. 34. Therefore the squares on AE, EB, CE, ED are equal to double the squares on AG, OG, CH, HO; that is to double the squares on OA and OC, by I. 47; that is to four times the square on OA, that is to the square on the diameter of the circle.

III. 32 to 34.

253. Produce BC to meet the circumference again at E; and produce PD to meet the circumference again at F.

Then PD is equal to FD by III. 3. The triangles BPD and BFD are equal in all respects, by I. 4; so that the angle BPD is equal to the angle BFD; and therefore equal to the angle BEP, by III. 21. The angle BPA

is equal to the angle *BEP*, by III. 32. Therefore the angle *BPA* is equal to the angle *BPD*.

254. Let two circles touch each other at the point *B*. Let a straight line *ABC* meet one circle at the points *A* and *B*, and the other circle at the points *B* and *C*. The segments cut off by *AB* will be respectively similar to those cut off by *BC*.

Draw a straight line *EBF* through *B*, at right angles to the straight line which joins the centres of the circles; then this straight line touches both circles by III. 16 Cor. The angles in the segments cut off by *AB* are equal to the angles *ABE* and *ABF* respectively, by III. 32; so also are the angles in the segments cut off by *BC*. Hence the segments cut off by *AB* are respectively similar to those cut off by *BC*, by the definition of similar segments.

255. The angle *DAP* is equal to the angle *PQA*, by III. 32; and therefore equal to the angle *QAB*, by I. 29. The angle *DPA* is equal to the angle *QBA*, by I. 13 and III. 22. Therefore the angle *PDA* is equal to the angle *BQA*, by I. 32.

256. Suppose *G* to fall between *A* and *H*. The angle *GBD* is equal to the angle *GAB*, by III. 32; the angle *GAB* is equal to the angle *HDB*, by III. 21: therefore the angle *GBD* is equal to the angle *HDB*. Therefore *HD* is parallel to *BG*, by I. 27.

The figure becomes modified in other cases but the demonstration remains substantially unchanged.

257. Produce *CA* to any point *E*, and *DA* to any point *F*. The angle *CAF* is equal to the angle *ABC*, and the angle *DAE* is equal to the angle *ABD*, by III. 32. But the angle *CAF* is equal to the angle *DAE*, by I. 15: therefore the angle *ABC* is equal to the angle *ABD*. Therefore *AB*, produced if necessary, bisects the angle *CBD*.

258. Draw *PE* the tangent at *P*, so that *PE* and *PA* are on the same side of *PB*. The angle *EPA* is equal to the angle *PBA*, by III. 32. The angle *PBA* is equal to the angle *PCD*, by I. 13 and III. 22. Therefore the angle *EPA* is equal to the angle *PCD*. Therefore *EP* is parallel to *CD*, by I. 27.

259. Let *A* be any point in the circumference of a circle. From *A* draw the chord *AB*, and the tangent *AC*. From *D*, the middle point of the arc *AB*, draw *DE* perpendicular to the chord *AB*, and *DF* perpendicular to the tangent *AC*. Then *DE* will be equal to *DF*.

The angle *DAF* is equal to the angle *DBA*, by III. 32. The angle *DBA* is equal to the angle *DAB*, by III. 27. Therefore the angle *DAF* is equal to the angle *DAE*. Then in the two triangles *DAF* and *DAE* the side *AD* is common; the angle *DAF* is equal to the angle *DAE*; and the right angle *DFA* is equal to the right angle *DEA*: therefore *DF* is equal to *DE*, by I. 26.

260. The angle *APN* is equal to the angle *AQM*, by III. 32. The right angle *ANP* is equal to the right angle *AMQ*. Therefore the angle *PAN* is equal to the angle *QAM*, by I. 32. Therefore the angle *NAM* is equal to the angle *PAQ*.

Again, since *ANP* and *AMP* are right angles a circle would go round *ANPM*; see the *Euclid*, page 276. Therefore the angle *ANM* is equal to the angle APQ, by III. 21. Therefore the angle AMN is equal to the angle AQP, by I. 32.

261. The angle RPS is a right angle, by III. 31. The angle QPS is equal to the angle PAB, by III. 32 and I. 15. But the angle PAB is equal to the angle OSB, by I. 32; that is equal to the angle PSQ. Therefore the angle QPS is equal to the angle QSP.

Again, since RPS is a right angle the angles PSR and PRS are together equal to the angle RPS, by I. 32. And it has been shewn that the angle QPS is equal to the angle QSP; therefore the angle QPR is equal to the angle QRP. Therefore by I. 6 we have QS and QR each equal to QP; so that QS is equal to QR.

262. Let AB be the given base, and D the point in the base on which the perpendicular falls. On AB describe a segment of a circle containing an angle equal to the given vertical angle. From D draw a straight line at right angles to AB, and let it cut the arc of the segment at C. Then ABC is the triangle required. For this triangle has the given base, and the given vertical angle; and the perpendicular from the vertex on the given base meets the base at the assigned point.

263. Let AB be the given base. On AB describe a segment of a circle containing an angle equal to the given vertical angle. From A draw a straight line AD at right angles to AB, and equal in length to the given altitude. Through D draw a straight line parallel to AB, and let it cut the arc of the segment at C and E. Then the triangle ACB will satisfy all the prescribed conditions; and so also will the triangle AEB.

264. Let AB be the given base. On AB describe a segment of a circle containing an angle equal to the given vertical angle. From the middle point of AB as centre with a radius equal to the given length describe a circle, and let it cut the arc of the segment at C and E. Then the triangle ACB will satisfy all the prescribed conditions, and so also will the triangle AEB.

265. Let AB be the given base. On AB describe a segment of a circle containing an angle equal to the given vertical angle. From the middle point of AB draw a straight line at right angles to AB, and let it meet the arc of the segment at C. Then the triangle ABC is the greatest triangle that can be constructed under the prescribed conditions.

For the straight line DC passes through the centre of the circle of which the segment forms part, by III. 1. Hence a straight line through C parallel to AB will be a tangent to the circle, by III. 16 Cor. If any other point except C on the arc of the segment be taken as the vertex of the triangle the vertex will be nearer to AB than C; so that the triangle will have a *less height* than ACB, and therefore will be less than ACB.

266. We suppose that B is nearer to A than C is. We may regard OB as the base of the triangle; then the height of the triangle will always be less than OC except when the angle COB is a right angle, and then the height of the triangle is OC. Hence the triangle OBC is greatest when the angle COB is a right angle. Then as the angles OBC and OCB are equal each of them will be half a right angle.

Therefore on OA describe a segment of a circle containing an angle equal to half a right angle, and take the point at which the arc of this segment cuts the given circle for the point C.

267. Suppose A and B the fixed points. Let C be a point in which the two straight lines meet. Then all the points of intersection which are on the *same* side of AB as C is will lie on the arc of a segment of a circle described on AB with an angle equal to ACB: see the *Euclid*, page 276. All the straight lines which bisect these angles pass through a fixed point, by Exercise 230.

Similarly a second fixed point is obtained by considering the straight lines bisecting the angles formed at points which lie on the side of AB *opposite* to that we have hitherto considered.

268. Let AB be the side which is opposite to the given angle. On AB describe a segment containing an angle equal to the given angle, and also a segment containing an angle equal to half the given angle. From A as centre with radius equal to the sum of the other two sides describe a circle, and let D be one of the points where it intersects the second described segment. Join AD and let it cut the first described segment at C. Then the triangle ACB will satisfy the prescribed conditions.

The angle ACB is equal to the sum of the angles CDB and CBD, by I. 32; also the angle ACB is double the angle CDB by construction: therefore the angle CDB is equal to the angle CBD: therefore CD is equal to CB, by I. 6.

Hence CA and CB together are equal to CA and CD together; that is to AD; that is to the given quantity. And AB and the angle ACB also have the prescribed values.

III. 35 to 37.

269. Let two circles cut one another at the points A and B, From any point T in AB produced suppose a tangent TP drawn to one circle, and a tangent TQ to the other: then will TP be equal to TQ.

For by III. 36 the rectangle TA, TB is equal to the square on TP, and also to the square on TQ; therefore the square on TP is equal to the square on TQ; therefore TP is equal to TQ.

270. Let PQ denote the common tangent; and let AB produced meet PQ at T. Then TP is equal to TQ, by Exercise 269: therefore PQ is bisected at T.

271. Since the angles ADC and AEC are equal a circle will go round $AEDC$. Therefore the rectangle BC, BD is equal to the rectangle BA, BE, by III. 36, Corollary.

272. Let AB be the common chord of the two circles. Through any point C in AB draw DCE a chord of one circle, and FCG a chord of the other. The rectangle AC, CB is equal to the rectangle DC, CE, and also equal to the rectangle FC, CG, by III. 35; therefore the rectangle DC, CE is equal to the rectangle FC, CG. Therefore a circle will go through D, F, E, G: see the *Euclid*, page 277.

273. Let BC be the given straight line, and C the fixed point in it. Let A be the given centre. Join AC; and describe a circle on AC as diameter. In it place a straight line CD equal to a side of the given square. With centre A and radius equal to AD describe a circle: this will be the circle required.

For let this circle cut BC at E and F. The angle ADC is a right angle, by III. 31; therefore CD touches the circle DEF, by III. 16 Cor.; therefore the rectangle CE, CF is equal to the square on CD by III. 36.

274. Let K be the middle point of BC. The square on GE is equal to the rectangle GB, GC, by III. 36. Thus the square on GE is equal to the difference of the squares on GK and BK; see the *Euclid*, page 269. Therefore the square on GK exceeds the square on GE by the square on BK. Therefore four times the square on GK exceeds four times the square on GE by four times the square on BK. Therefore four times the square on GK exceeds the square on AE by the square on BC: see Exercise 270. And it may be shewn that K is the middle point of GH; so that four times the square on GK is the square on GH. Thus the square on GH exceeds the square on AE by the square on BC.

To shew that K is the middle point of GH we may proceed thus. First AE and DF will be equal; for each of them will be a side of a right-angled triangle in which the hypotenuse is the distance of the centres of the circles, and a side is equal to the difference of the radii of the circles: see the *Euclid*, page 295. Next the square on GK is equal to the squares on GE and BK; and the square on HK is equal to the squares on HF and CK: therefore the square on GK is equal to the square on HK: therefore GK is equal to HK.

275. Let two of the circles intersect at A and B. Let T be the fixed point such that the tangent TP to one circle is equal to the tangent TQ to the other. Then A, B, and T shall be on a straight line.

For if not, join TA and let TA, produced if necessary, cut one circle at L and the other at M. Then by III. 36 the square on TP is equal to the rectangle TA, TL; and the square on TQ is equal to the rectangle TA, TM. Therefore the rectangle TA, TL is equal to the rectangle TA, TM; which is impossible. Therefore TA cannot cut the circles at any other point than B.

276. Describe a circle on BC as diameter; and let the straight line DE when produced cut this circle at N and O.

The rectangle NE, EO is equal to the difference of the squares on DN and DE; see the *Euclid*, page 269. But by III. 35 the square on DN is equal to the rectangle BD, DC; and the rectangle NE, EO is equal to the rectangle AE, EC. Therefore the rectangle AE, EC is equal to the difference of the rectangle BD, DC and the square on DE. Therefore the square on DE is equal to the difference of the rectangles BD, DC and AE, EC.

Again, the rectangle FN, FO is equal to the difference of the squares on DF and DN; see the *Euclid*, page 269. So that the square on DF is equal to the square on DN together with the rectangle FN, FO. But the square on DN is equal to the rectangle DC, DB, by III. 35; and the rectangle FN, FO is equal to the rectangle FA, FB, by III. 36, Corollary. Therefore the square on DF is equal to the sum of the rectangles BD, DC and AF, FB.

277. Let AB be the given diameter. In the circle place a straight line BP equal to a side of the given square. Join AP and produce it to meet at T the tangent to the circle at B. Then T is the required point.

For the rectangle TP, TA is equal to the square on TB, by III. 36; therefore the rectangle TP, PA together with the square on TP is equal to the square on TB, by II. 3. Therefore the rectangle TP, PA is equal to the difference of the squares on TB and TP; that is to the square on BP, by III. 31 and I. 47.

IV. 1 to 4.

278. Join AB. The straight lines through A and B to touch the circle make with AB angles which are together less than two right angles, for each of them is less than one right angle. Therefore, by Axiom 12, these straight lines will meet.

279. The angles ABC and ACB are together less than two right angles; much more therefore the angles made with BC by the straight lines which bisect ABC and ACB are together less than two right angles. Therefore, by Axiom 12, these straight lines will meet.

280. By I. 47 the square on DA is equal to the squares on DG and GA, and also equal to the squares on DE and EA; therefore the squares on DG and GA are equal to the squares on DE and EA. But the square on DG is equal to the square on DE; therefore the square on GA is equal to the square on EA; therefore GA is equal to EA.

Therefore the angle DAG is equal to the angle DAE, by I. 8, so that the angle EAG is bisected by AD.

281. Let O be the centre of the circle inscribed in the triangle ABC.

The straight line OGA bisects the angle DAE, by Exercise 280. Hence the angles AOD and AOE are equal, by I. 32. The angle ADG is equal to the angle GED, by III. 32; and is therefore half the angle DOG, by III. 20; that is half the angle GOE. Also the angle GDE is half the angle GOE, by III. 20. Therefore DG bisects the angle ADE. Similarly EG bisects the angle AED. Therefore G is the centre of the circle inscribed in the triangle ADE, by IV. 4.

282. Let it be required to describe a circle which touches the side BC of a triangle, and also touches the sides AB and AC produced.

Produce AB to any point K, and AC to any point L. Bisect the angles KBC, LCB by the straight lines DB, DC meeting at the point D. Then D will be the centre of the required circle. The demonstration will be like that of IV. 4.

In like manner suppose it required to describe a circle which touches the side CA, and also touches the sides BC and BA produced. Produce BC to any point M, and BA to any point N. Bisect the angles ACM, CAN by the straight lines CE, AE meeting at E. Then E will be the centre of the required circle.

We have now to shew that D, C, E lie on one straight line.

The angles ACB, BCL, LCM, MCA are together equal to four right angles, by I. 15, Corollary 1. The angle BCD is half the angle BCL, and the angle ECA is half the angle MCA, by construction; also the angle ACB is half the sum of the two angles ACB, LCM, by I. 15. Hence the angles BCD, ACB, ECA are together equal to two right angles. Therefore CE and CD are in one straight line, by I. 14.

283. Let the circle which is inscribed in the triangle ABC touch BC at K; let a circle touch BC at F, touch AB produced at D, and touch AC produced at E. Then FK will be equal to the difference between AB and AC. Suppose F to fall between B and K.

Let the circle inscribed in the triangle ABC touch AB at G, and AC at H. By Exercise 176 we have AE equal to AD, and AH equal to AG, so that HE is equal to GD. Now HE is equal to the sum of HC and CE, that is to the sum of CK and CF; and GD is equal to the sum of DB and BG, that is to the sum of BF and BK. Hence CK and CF are together equal to BF and BK; that is KF and twice CK are together equal to FK and twice BF: therefore BF is equal to CK.

Thus FK, which is the difference of BK and BF, is equal to the difference of BK and CK; that is to the difference of BG and CH; that is to the difference of BA and CA. Similarly the result holds if F falls between C and K.

284. Let ABC be a triangle; let a circle be inscribed in this triangle, touching BC at D, CA at E, and AB at F. Let a straight line GHK cut AB at G, and AC at K and touch the circle at H; let a straight line LMN cut BC at L, and BA at N, and touch the circle at M; let a straight line PQR cut CA at P, and CB at R, and touch the circle at Q. The sides of the three triangles so cut off are together equal to the sides of ABC.

For, by Exercise 176 we have GH equal to GF, so that AG and GH together are equal to AG and GF together, that is to AF. Similarly BN and NM together are equal to BF; BL and LM together are equal to BD; and so on. Thus the required result will be obtained.

285. The angle OBD is a right angle; the angle DBC is half the angle ABC; therefore the angle OBC is the excess of a right angle over half the angle ABC; that is half the excess of two right angles over the angle ABC: therefore BO bisects the angle between BC and AB produced.

Again the angle BDO is equal to the angles DBA and DAB, that is to half the sum of the angles ABC and BAC, by IV. 4. The angle OBD is a right angle, by construction. Therefore the angle BOD is equal to half the angle ACB, by I. 32; and is therefore equal to the angle BCD. Hence a circle will go round $BDCO$, by page 276 of the *Euclid*. Therefore the angle DCO is a right angle by III. 22. Therefore CO will bisect the angle between BC and AC produced.

Hence by the process in the solution of Exercise 282 we see that O is the centre of the circle which touches the side BC, and the sides AB, AC produced.

286. Let AB and AC produced touch the circle GDH at G and H. Then by Exercise 176 BG is equal to BD, and CH is equal to CD; also AG is equal to AH. Therefore AB and BD together are equal to AC and CD together; so that AB and BD together are equal to half the perimeter of the triangle. Similarly BA and AE together are equal to half the perimeter of the triangle. Thus AB and BD together are equal to BA and AE together. Therefore AE is equal to BD. Similarly BF is equal to CE, and CD is equal to AF.

287. Let AB and AC be two straight lines which touch a given circle at

B and C respectively: it is required to describe a circle which shall touch the two straight lines and the given circle.

Take O the centre of the given circle. Join OA cutting the circumference at D. Draw a straight line through D at right angles to OD, and let it meet AB at E, and AC at F. Bisect the angle AED by a straight line EG meeting AD at G. Then a circle described from the centre G with the radius GD will satisfy the prescribed conditions.

For from the triangles AOB and AOC we can shew that AG bisects the angle BAC. Hence the circle described from G as centre with radius GD will touch the sides of the triangle AEF, by IV. 4; and it will also touch the circle BDC.

If AO produced cut the given circle at K then another solution can be obtained by using K instead of D.

288. Take the diagram of IV. 3. Then LA is equal to LC, by Exercise 176; therefore the angle LAC is equal to the angle LCA; therefore each of them is less than a right angle, by I. 17. The angle LAC is equal to the angle ABC, by III. 32: therefore the angle ABC is less than a right angle.

Similarly BCA and CAB are each less than a right angle.

289. Let $ABCD$ be a quadrilateral, in which the sum of the sides AB and CD is equal to the sum of the sides BC and AD; and each angle is less than two right angles; then a circle can be inscribed in the quadrilateral.

Bisect the angles ABC and BCD by the straight lines BO and CO meeting at O. From O draw OE perpendicular to BC. Then a circle described with O as centre, and OE for radius will touch AB, BC, and CD. This may be shewn by the process of IV. 4. It then remains to shew that it will also touch AD.

Let BA and CD be produced to meet at S. Suppose if possible that the circle described in the manner prescribed does not *touch* AD but *cuts* it. Suppose a straight line LM drawn parallel to AD, cutting SA at L and SD at M and *touching* the circle. Then by Exercise 188 the sum of LM and BC is equal to the sum of LB and MC. But, the sum of AD and BC is equal to the sum of AB and DC, by supposition. Therefore LM and BC are equal to the sum of LA, AD, MD and BC; that is LM is equal to the three sides AL, AD, MD of the quadrilateral $ALMD$. But this is impossible: see I. 20.

Similarly we may shew that the circle described in the manner prescribed cannot fall completely within the quadrilateral $ABCD$.

And in like manner we may treat the case in which $ABCD$ is a parallelogram, so that neither pair of opposite sides will meet when produced.

290. Draw SPT touching both circles at P, cutting HK at S and LM at T. Let HL cut the straight line which joins the centres of the circles at B; and let KM cut this straight line at C. They will cut it at right angles.

LT and MT are each equal to PT, so that T is the middle point of LM. Hence by Exercise 89 BL and CM are together equal to twice PT, that is to LM. In this way we find that HL and KM are together equal to HK and LM; and then, by Exercise 289 a circle can be inscribed in $HKML$.

To shew that HL will cut BC at right angles, see the construction and figure on p. 295 of the *Euclid*. ACB is a right-angled triangle in which AC is the difference of the radii of the two circles. When the common tangent is drawn below AB similarly situated to DE, the corresponding right-angled triangle will have its sides equal to those of ACB by I. 47. Hence the radii from A to the two points of contact on the circle of which A is the centre will make equal angles with the line AB. Then by I. 26 we have the required result.

291. Let ABC be a triangle. Let D be the centre of the circle which touches BC, and also AB and AC produced. Let E be the centre of the circle which touches CA, and also BC and BA produced. Let F be the centre of the circle which touches AB, and also CA and CB produced. Then AD, BE, CF will intersect at the centre of the circle inscribed in the triangle ABC.

Let O denote this centre. Then, as in Exercise 280, we can shew that OA bisects the angle BAC. In like manner we can shew that DA bisects the angle BAC. Thus the straight line OA coincides in direction with DA; so that O falls on DA. Similarly O also falls on EB, and on FC.

292. Let ABC represent the triangle; AB and AC being the sides given in position. Describe a circle which touches AB produced at D, and AC produced at F, and touches BC at E. Then, as in Exercise 286, we can shew that AD and AF are each equal to half the perimeter of the triangle ABC; and so D and F are fixed points. Thus the circle touches the fixed straight lines at fixed points, and so it is a fixed circle.

293. Take the diagram of IV. 3. Join MK and KN. We shall shew that the angle MKN is equal to the sum of a right angle together with half the angle MLN. The angles KMA and KMB are equal, so that each of them is half the angle LMN. Therefore the angle MKB, half the angle LMN, and a right angle, are equal to two right angles by I. 32. Similarly the angle NKB, half the angle LNM, and a right angle are equal to two right angles. Therefore by addition, the angle MKN, half the angle LMN, and half the angle LNM are equal to two right angles; that is are equal to a right angle together with half the angles at L, M, N by I. 32. Therefore the angle MKN is equal to a right angle together with half the angle at L. We shall now apply this to the present Exercise. Let PQ be the given base. Draw a straight line parallel to PQ and at a distance from it equal to the given radius: then the centre of the inscribed circle will be on this straight line. Again describe on PQ a segment of a circle containing an angle equal to the sum of a right angle and half the given vertical angle: then from what is shewn above the centre of the inscribed circle will be on this segment. Hence either of the points of intersection of this segment and the straight line drawn parallel to PQ may be taken as the centre of the inscribed circle: the circle may be drawn, and then tangents to the circle from P and Q, produced to meet, will form the other two sides of the required triangle.

IV. 5 to 9.

294. From F draw FG perpendicular to BC. Since FB is equal to FC the angle FBC is equal to the angle FCB; also the right angle FGB is equal to the right angle FGC; therefore BG is equal to CG, by I. 26.

295. The tangent at A to the circle described round ADE makes with AD an angle equal to the angle AED, by III. 32. The tangent at A to the

circle described round ABC makes with ADB an angle equal to the angle ACB, by III. 32. But the angle AED is equal to the angle ACB, by I. 29. Therefore the tangent at A to the circle described round ABC coincides with the tangent at A to the circle described round ADE.

296. Let ABC be a triangle. Let O be the centre of the circle described about the triangle, and also the centre of the circle inscribed in the triangle: the triangle will be equilateral.

Because O is the centre of the circle described about the triangle OA, OB, OC are all equal. Because O is the centre of the circle inscribed in the triangle OA, OB, and OC bisect the angles at A, B, and C respectively, by IV. 4. Since OA is equal to OB the angle OAB is equal to the angle OBA; that is half the angle BAC is equal to half the angle ABC: therefore the angle BAC is equal to the angle ABC; therefore AC is equal to BC, by I. 6. Similarly BA is equal to CA. Thus the triangle ABC is equilateral.

297. Let ABC be a triangle; let P be the centre of the inscribed circle, and Q the centre of the circumscribed triangle: and suppose that P, Q, and A are on one straight line: then AB will be equal to AC.

From Q draw QM perpendicular to AB, and QN perpendicular to AC. In the two triangles AQM and AQN the angle QAM is equal to the angle QAN; the right angles AMQ and ANQ are equal; and the side AQ is common: therefore AM is equal to AN by I. 26. But AM is half AB, and AN is half AC, by III. 3: therefore AB is equal to AC.

298. Let LM denote the common chord. The rectangle PL, PM is equal to the square on PA, by III. 36; the rectangle PL, PM is equal to the rectangle PB, PC, by III. 36 Corollary: therefore the rectangle PB, PC is equal to the square on PA. Therefore PA touches the circle which passes through A, B, and C. Therefore as this circle and the circle AML have the common tangent PA they touch each other.

299. Let EP be the tangent at E to the circle described round the triangle ECD, and suppose P and C to be on the same side of ED: then EP will be parallel to AB.

The angle PEC is equal to the angle EDC, by III. 32. The angle EDC is equal to the angle ABC, by III. 22 and I. 13. Therefore the angle PEC is equal to the angle ABC. Therefore EP is parallel to AB, by I. 27.

300. Let A and B be the two given points; join AB, and produce it to meet the given straight line at C. On the given straight line take D such that the square on CD is equal to the rectangle CA, CB, II. 14. Then the circle which passes through A, B, and D is the circle required.

For since the rectangle CA, CB is equal to the square on CD the straight line CD touches the circle which passes through A, B, and D, by III. 37.

See the *Euclid*, p. 296.

301. Let A and B be the given points; join AB, and produce it to meet the given straight line at C. Bisect AB at E; from E draw a straight line at right angles to AB; and in this straight line take any point F such that AF is greater than half the given chord. From F as centre, with radius equal to FA, describe a circle; this will also pass through B. Through C draw a straight line CKL by Exercise 181 such that the length KL intercepted by this circle shall be equal to the given chord. On the given straight

line take CM equal to CK. Then the circle described round A, B, and M will satisfy the prescribed conditions.

For let this circle cut the given straight line at N. Then the rectangle CM, CN is equal to the rectangle CA, CB, by III. 36 Corollary; that is to the rectangle CK, CL. But CM is equal to CK; therefore CN is equal to CL. Therefore MN is equal to KL; that is to the given chord.

302. Let AB be the given straight line in which the centre is to lie; let AC and BC be the other two given straight lines. Bisect the angle ACB by a straight line meeting AB at D. From D draw a straight line perpendicular to AC and meeting it at E; from E on the straight line EC cut off EM equal to half the given chord. Then the circle described from the centre D, with the radius DM, is the required circle.

For let this circle cut CA again at N; then MN is bisected at E, by III. 3; therefore MN is equal to the given chord.

Again, from D draw a straight line perpendicular to BC meeting it at F. Then from the triangles CDE and CDF we see that DF is equal to DE. Therefore the chord of the circle cut from CB will be equal to that cut from CA, by III. 14.

303. Let ABC and KLM be two triangles, having the bases BC and LM equal, and the angles BAC and LKM equal. Let F be the centre of the circle circumscribing ABC, and G the centre of the circle circumscribing KLM; then FB will be equal to GL.

For, since the angle at A is equal to the angle at K, the angle BFC is equal to the angle LGM, by III. 20; therefore the angles FBC and FCB together are equal to the angles GLM and GML together; but the angles FBC and FCB are equal, and so also are the angles GLM and GML, by I. 5. Therefore the angle FBC is equal to the angle GLM, and the angle FCB is equal to the angle GML. But BC is equal to LM, by supposition; therefore FB is equal to GL, by I. 26.

304. Let A be the point from which the tangent is to be drawn; let B and C be the two given points through which the circle is to pass. Join AB; and in this straight line, produced if necessary, take the point D such that the rectangle AB, AD is equal to the square on the given tangent, by I. 45 Corollary. Then D is also a point through which the circle must pass, by III. 36. Thus we have only to describe a circle passing through the three points B, C, and D; and this can be done by IV. 5.

305. The angle APB is half the angle ACB, by III. 20; that is half a right angle; that is equal to BAC. Therefore AB touches the circle described round ANP: see III. 32.

306. Suppose that AB and CD are joined towards *opposite* parts by AD and BC. Through E draw a straight line GEH touching the circle AEB. Then the angle AEG is equal to the angle ABE, by III. 32. But the angle AEG is equal to the angle DEH, by I. 15; and the angle ABE is equal to the angle ECD, by I. 29; therefore the angle DEH is equal to the angle ECD. Therefore GEH touches the circle ECD: see III. 32. Hence the two circles have a common tangent at E, and therefore touch each other.

So also the proposition may be demonstrated when AB and CD are joined towards the *same* parts by AC and BD, and AC and BD are produced to meet. See Exercise 295.

307. By III. 14 the centre of the required circle must be equally distant from the three sides of the triangle; and hence it easily follows that it must coincide with the centre of the circle inscribed in the triangle.

308. Join BO. The angle BAF is equal to the angle CAF, by IV. 4. Therefore the arc BF is equal to the arc CF, and the straight line BF to the straight line CF, by III. 26 and III. 29. Again, the angle OBF is equal to the angles OBC and FBC; that is to the angles OBA and FAC; that is to the angles OBA and OAB; that is to the angle BOF, by I. 32. Therefore BF is equal to OF, by I. 6.

309. Let $ABCD$ be the quadrilateral figure; let BC and AD be produced to meet at P, and let BA and CD be produced to meet at Q.

The angles DRP and DCP are together equal to two right angles, by III. 22; and so also are the angles DRQ and DAQ. Therefore the angles DRP, DRQ, DCP, and DAQ are together equal to four right angles. But the angles DCP and DAQ are together equal to two right angles, by I. 13 and III. 22. Therefore the angles DRP and DRQ are together equal to two right angles. Therefore RP and RQ are in one straight line, by I. 14.

310. Describe a circle round ACB. Then the straight line which bisects the angle ACB will bisect the arc cut off by AB, by III. 26. Again the straight line drawn from the middle point of AB at right angles to AB also bisects the arc cut off by AB: see III. 30. Hence the point D is on the arc cut off by AB. Therefore the angles ACB and ADB are together equal to two right angles, by III. 22.

311. Draw the tangent to the circle at C; and let it meet AB at O. Then the angle OCE is equal to the angle CDE, by III. 32; that is equal to the angle CBA, by III. 21. Therefore OC is equal to OB, by I. 6; and O is the centre of the semicircle. Thus the tangent at C to the circle is at right angles to the tangent at C to the semicircle.

Similarly the tangent to the circle at D passes through O, and the circle cuts the semicircle at right angles at D.

312. Let P, Q, R, S be the centres of the circles described round the triangles AOB, BOC, COD, DOA respectively. Now P and S are both on the straight line which passes through the middle point of OA and is at right angles to OA, by IV. 5. Thus PS is at right angles to AC. Similarly QR is at right angles to AC. Therefore PS is parallel to QR, by I. 28.

Similarly PQ is parallel to SR.

313. The angle CED is equal to the angle ECD, by I. 5. But the angle ECD is equal to the angles ECB and BCD; that is to the angles ECB and BAC, by III. 32. And the angle CED is equal to the angles ECA and BAC, by I. 32. Therefore the angles ECB and BAC are equal to the angles ECA and BAC; therefore the angle ECB is equal to the angle ECA.

314. By the process given we see that F is the centre of the circle described round the triangle ABC, so that AF is equal to BF and equal to CF. Now in the triangle BFC the side BC is of given length; the angle BFC is twice the angle BAC, by III. 20, and is therefore a given angle; the angles FBC and FCB are equal, and therefore each of them is given, by I. 32. Hence the triangle FBC is completely known, and so FB is of constant length.

315. The angle ABD is equal to the angle ACB, by I. 5; and the angle ACB is equal to the angle AEB, by III. 21: therefore the angle ABD is equal to the angle BED. Therefore AB touches the circle described round BED: see III. 32.

316. Let E be the point of contact of the circles, TE the common tangent.

Then the angle TED is equal to the angle DBE, by III. 32, and the angle TEC is equal to the angle DAE; therefore the angle DEC is equal to the angle AEB, by I. 32.

317. Let C denote the centre of the circle. Then HK must be as near to C as possible, by III. 15; so that the angle HCK must be as great as possible: therefore the angle HPK must be as great as possible, by III. 20. Thus the problem is reduced to that which is given on page 308 of the *Euclid*.

318. The centres of both circles will lie on the straight line which bisects the angle formed by the two given straight lines. In the given circle place a chord at right angles to this straight line containing the centres, and cutting off a segment containing an angle equal to the given angle. Let PQ denote this chord. We have finally to describe a circle passing through P and Q and touching a given straight line; for this see page 296 of the *Euclid*.

319. Suppose the angles ACB and ABC to be acute. Let OA intersect EF at G. The angle AOB is equal to twice the angle ACB, by III. 20: therefore the angles BAO and ABO are together equal to the excess of two right angles over twice the angle ACB; but the angles BAO and ABO are equal, by I. 5: therefore the angle BAO is equal to the excess of a right angle over the angle ACB.

Again since BFC and BEC are right angles a circle will go round $BFEC$: see page 276 of the *Euclid*. Therefore the angles AFE and ACB are equal, by III. 22 and I. 13.

Thus the angles AFG and FAG are together equal to a right angle; and therefore the angle AGF is a right angle.

The process requires but a slight modification if either of the angles ACB and ABC is obtuse.

In like manner it can be shewn that OB is perpendicular to FD and OC to DE.

320. Let $ABCD$ be the square, P any point on the circumference. Since ABC is a right angle AC is a diameter of the circle, and APC is a right angle: see III. 31. Therefore the squares on AP and CP are together equal to the square on AC. Similarly the squares on BP and DP are together equal to the square on BD. Therefore the squares on AP, BP, CP, DP are together equal to the squares on AC and BD, that is to twice the square on AC.

321. Suppose $ABCD$ to be a rectangle described about a circle. By Exercise 188 the sum of AB and CD is equal to the sum of BC and DA; that is twice AB is equal to twice BC: therefore AB is equal to BC, and the rectangle is a square.

322. Let $ABCD$ be a rectangle. Join AC and BD, intersecting at O. By Exercise 78 the diagonals AC and BD are bisected at O; also OB is equal

to OC, by Exercise 59. Thus OA, OB, OC, OD are all equal; and the circle described from the centre O with the radius OA will pass through B, C, and D, and thus will be described about the rectangle.

323. Let O be the centre of the circle; let AOB and COD be two diameters. Draw tangents at A, B, C, D thus forming a quadrilateral figure having for sides NAK, KDL, LBM, MCN: this quadrilateral figure will be a rhombus.

The straight lines MB and MC are equal, by Exercise 176; therefore the angle MOB is equal to the angle MOC, by I. 8, so that MO bisects the angle BOC; similarly KO bisects the angle AOD: therefore the angle MOB is equal to the angle KOD.

In the two triangles MOB and KOD the angle MOB is equal to the angle KOD; the angles MBO and KDO are equal being right angles; and OB is equal to OD; therefore BM is equal to DK. Also LB is equal to LD, by Exercise 176. Therefore LM is equal to LK.

Similarly it may be shewn that LM is equal to MN, and MN to NK; so that the figure $NKLM$ is a rhombus.

IV. 10.

324. The angle ACD is equal to the two angles BDC and CBD, by I. 32. But it is shewn in IV. 10 that the angle BDC is equal to the angle BAD, and that the angle CBD is twice the angle BAD: therefore the angle ACD is equal to three times the angle BAD.

325. The triangle BCD is shewn in the course of IV. 10 to have each of the angles BCD and CBD double of the angle CDB. Also the triangle ACD has each of the angles CAD and CDA one-third of the angle ACD; see Exercise 324.

326. Suppose F the point at which the circles intersect again. Then AF is equal to AD. Also the angle AFD is equal to the angle ADB, by III. 32; the angle ADF is equal to the angle AFD, by I. 5: therefore the angle FAD is equal to the angle BAD, by I. 32. Thus the angle ADF is twice the angle DAF. Bisect the angle ADF by the straight line DG meeting the circumference of the small circle at G. Then the five angles ADG, GDF, FAD, DAC, ADC are all equal; and therefore CD is the side of a regular pentagon inscribed in the small circle.

327. Let KL be the given base. Make the angles MKL and MLK each equal to the angle CAD of IV. 10. Then the angles MKL and MLK are equal to the angles CAD and ADC; therefore the angle KML is equal to the angle ACD: therefore the angle KML is three times the angle MKL or MLK. See Exercise 324.

328. This is shewn in the course of the solution of Exercise 326.

329. The angle BAG is twice the angle BAD, by Exercise 328; but the angle ABD is twice the angle BAD, by IV. 10: therefore the angle BAG is equal to the angle ABG, and each of them is twice the angle AGB.

330. Let CA produced meet the larger circle at G, and let DC produced meet the larger circle at H: then the triangle GCH will be of the same kind as the triangle ABD.

For the angle GCH is equal to the angle BCD, by I. 15; and therefore equal to the angle ADB, by IV. 10. And the angle GHD is equal to the angle GBD, by III. 21. Therefore the angle HGC is equal to the angle BAD, by I. 32. Therefore each of the angles GHC and HCG is double the angle HGC.

331. The angle DAE is equal to the angle DAC; see Exercise 328: therefore the angle DAE is equal to the angle ADC, and therefore AE is parallel to DC, by I. 27. The angle CDB is equal to the angle CAD by IV. 10; and is therefore equal to the angle ECD: see Exercise 328: therefore BD is parallel to EC, by I. 27. Thus $CDGE$ is a parallelogram.

332. Let E denote the point at which the circles cut again. The triangles BAD and EAD are equal in all respects by Exercise 328; and the smaller circle is described round the triangle EAD: therefore an equal circle could be described round the triangle BAD.

333. Let E denote the point at which the circles cut again. The angles DFE and DAE are together equal to two right angles, by III. 22. The angle BAD is equal to the angle DAE by Exercise 328. Therefore the angle DFE is equal to the excess of two right angles over DAB; that is the angle DFE is equal to the sum of the angles CBD and BDA, that is to twice the angle CBD.

Suppose O the centre of the circle described round the triangle BCD. Then the triangles OCD and FDE are both isosceles; CD is equal to DE; and the angle COD by what has been shewn is equal to the angle DFE. Therefore DF is equal to CO, by I. 26.

IV. 11 to 16.

334. Let $ABCDE$ be the regular pentagon; draw AC, BD, CE, DA, EB. Let AC and BD intersect at K; let BD and CE intersect at L; and so on; thus forming the pentagon $KLMNO$: this shall be a regular pentagon.

Since CB is equal to CD the angle CBD is equal to the angle CDB; so also the angle CAB is equal to the angle BCA. But the angle ABC is equal to the angle BCD. Therefore by I. 32 the following angles are all equal: CBD, CDB, BAC, BCA.

Then we can shew that BK is equal to CK, that CL is equal to DL, and so on. Next by comparing the triangles BKC and CLD we can shew that CK and CL are equal, and the angle BKC equal to the angle CLD; and so on.

Next by comparing the triangles OBK and KCL we can shew that OK is equal to KL; and proceeding thus we find that the pentagon is equilateral. It is also equiangular, since we have the angle BKC equal to the angle CLD, and so on; that is the angle OKL is equal to the angle KLM, and so on.

335. Suppose a circle described round the pentagon by IV. 14. The angle BFC is measured by half the sum of the arcs AE and BC; the angle FBC is measured by half the sum of the arcs CD and DE: see page 294 of the *Euclid*. Therefore the angle FBC is equal to the angle BFC; therefore BC is equal to FC. And AF is equal to BF, since the angle ABF is equal to the angle BAF. Thus AC which is equal to the sum

of AF and FC is equal to the sum of BF and CF, that is to the sum of BF and AB.

336. Let $ABCDE$ be the regular pentagon. Join AC, AD; through C draw a straight line CF parallel to AB, meeting AD at F.

A circle may be described round the pentagon by IV. 14. Then AD is parallel to BC, by Exercise 219; thus $ABCF$ is a parallelogram; and the triangle ABC is equal to the triangle AFC, by I. 34. The triangle AED is also equal to the triangle ABC. Thus the regular pentagon exceeds three times the triangle ABC, namely by the triangle FCD. Therefore the triangle ABC is less than a third of the pentagon.

Again AC is equal to AD; AB and BC are together greater than AC, by I. 20; therefore AB and BC are together greater than AD. But BC is equal to AF, by I. 34; therefore AB is greater than FD; therefore AF is greater than FD. Therefore the triangle CFD is less than the triangle CAF. Thus the regular pentagon falls short of four times the triangle ABC, namely by the excess of the triangle ACF over the triangle CFD. Therefore the triangle ABC is greater than a fourth of the pentagon.

337. Let ABC be the equilateral triangle, O the centre of the circle described round it. From O draw OD a perpendicular to BC and produce it to meet the circumference at E. Then BE and EC are sides of the hexagon.

For from the triangles BOD and COD we can shew that OD bisects the angle BOC. Therefore the angle BOD is equal to the angle BAC, by III. 20. Thus the angle BOE is equal to the angle of an equilateral triangle. But the angles OBE and OEB are equal; therefore each of these is the angle of an equilateral triangle. Therefore BE is equal to BO.

In this way we can shew that each side of the hexagon is equal to the radius of the given circle; and each angle of the hexagon is equal to twice the angle of an equilateral triangle: thus the hexagon is a *regular* hexagon.

Also the triangles OBC and EBC are equal; and in this way it follows that the regular hexagon is double the given equilateral triangle.

338. Let A_1, A_2, A_3,...A_{15} be a regular quindecagon inscribed in the given circle. Draw the straight lines A_1A_3, A_3A_8, A_8A_1, cutting off arcs which are to one another in the proportion of 2, 5, 8. The angles which stand on these arcs will also be in this proportion; and therefore $A_1A_3A_8$ will be the triangle required.

339. We can shew by the method used for Exercise 334 that a regular hexagon is obtained; we proceed to shew that the area is one-third of the area of the original hexagon. Let $ABCDEF$ be the given regular hexagon; let O be the centre of the circle described about the hexagon. O will also be the centre of the circle described about the derived hexagon. Let FB and AC intersect at G; let AC and BD intersect at H.

The angles ABG and BAG are equal; so are the angles BCH and HBC. Then it may be shewn that all the angles of the triangle GBH are equal. Thus AG, GH, HC are all equal. Also FC is parallel to AB by Exercise 219; therefore the triangle ABC is equal to the triangle ABO.

Again, the angle GOH will be one-sixth of four right angles, so that GOH will be an equilateral triangle; therefore it will be equal to the triangle GBH, that is to one-third of the triangle ABC, that is to one-third of the triangle OAB. In this way we can shew that the area of the derived hexagon is one-third of the area of the original hexagon.

340. Let A, B, C, D, E... be consecutive angular points of the equilateral figure inscribed in a circle: this figure will be equiangular. We will shew that the angles ABC and BCD are equal.

For, the angle ABC stands on an arc which consists of the whole circumference except the two smaller arcs cut off by AB and BC respectively; and the angle BCD stands on an arc which consists of the whole circumference except the two smaller arcs cut off by BC and CD respectively. Now the smaller arc cut off by AB is equal to the smaller arc cut off by CD, by III. 28. Therefore the arc ABC is equal to the arc BCD. Hence the arc on which the angle ABC stands is equal to the arc on which the angle BCD stands. Therefore the angle ABC is equal to the angle BCD, by III. 27.

VI. 1, 2.

341. In the diagram of IV. 10 take E on AD such that AE may be equal to AC: then it may be shewn that the triangle CDB is equal to the triangle CAE. Therefore the triangle CBD is to the triangle ACD as the triangle ACE is to the triangle ACD; that is as AE is to AD, by VI. 1; that is as AC is to AB. Again, the triangle ACD is to the triangle ABD as AC is to AB, by VI. 1. Therefore the triangle CBD is to the triangle ACD as the triangle ACD is to the triangle ABD. Therefore the triangle ACD is a mean proportional between the triangles CBD and ABD.

342. The triangle AFE is equal to the triangle FDE, by I. 34; the triangle FDE is equal to the triangle FDC, by I. 37: therefore the triangle AFE is equal to the triangle FDC. Hence the triangle BFD is to the triangle AFE as BD is to DC, by VI. 1. Again, the triangle AFE is to the triangle EDC as AE is to EC, by VI. 1; that is, as BD is to DC, by VI. 2. Therefore the triangle BFD is to the triangle AFE as the triangle AFE is to the triangle EDC; so that the triangle AFE is a mean proportional between the triangles BFD and EDC.

343. From a point O, within an equilateral triangle ABC, let perpendiculars OP, OQ, OR be drawn on the sides BC, CA, AB respectively. Also draw CD from C perpendicular to AB. Then in the same manner as in VI. 1. Cor. it may be shewn that triangles on equal bases are to one another as their altitudes; thus the triangle OBC is to the triangle ABC as OP is to CD; also the triangle OCA is to the triangle ABC as OQ is to CD; and the triangle OAB is to the triangle ABC as OR is to CD. Hence the sum of the triangles OBC, OCA, OAB is to the triangle ABC as the sum of OP, OQ, OR is to CD. But the sum of the triangles OBC, OCA, OAB is equal to the triangle ABC. Therefore the sum of OP, OQ, OR is equal to CD.

344. Let ABC be a triangle. From A draw AD perpendicular to BC, and from D on DA take DK equal to one-third of DA. From B draw BE perpendicular to AC, and from E on EB take EL equal to one-third of EB.

Through K draw a straight line parallel to BC; and through L draw a straight line parallel to AC: let these straight lines meet at O: this shall be the point required.

For the triangles ABC and OBC have the same base, but the height of OBC is one-third of the height of ABC; therefore the triangle OBC is one-third of the triangle ABC, as in VI. 1. In like manner the triangle OCA is one-third of the triangle ABC. Therefore the triangle OAB is also one-third of the triangle ABC.

345. CF is to FB as AE is to EB, and also DG is to GB as AE is to EB, by VI. 2. Therefore CF is to FB as DG is to GB. Therefore FG is parallel to CD, by VI. 2.

346. Let ABC be a triangle. From any point K in the base AB draw KL parallel to AC meeting CB at L, and KM parallel to BC meeting CA at M.

The diagonals CK and LM intersect at O the middle point of CK, by Exercise 78. Through O draw a straight line POQ parallel to AB, meeting AC at P, and BC at Q. Then CP is to PA as CO is to OK; but CO is equal to OK; therefore CP is equal to PA. Thus P is the middle point of CA.

Similarly Q is the middle point of CB. Thus O is on the fixed straight line which joins the middle points of AC and BC.

347. Let the straight line from D parallel to BC meet AC at E. Produce AD to meet BC at F. Then AD is equal to DF, by I. 26. And AD is to DF as AE is to EC, by VI. 2. Therefore AE is equal to EC.

348. The triangle BED is equal to the triangle CED, by I. 37; therefore the triangle DFB is equal to the triangle EFC.

The triangle ADF is to the triangle BDF as AD is to DB, by VI. 1; that is as AE is to EC, by VI. 2; that is as the triangle AEF is to the triangle EFC, by VI. 1; that is as the triangle AEF is to the triangle BDF. Thus the triangle ADF is to the triangle BDF as the triangle AEF is to the triangle BDF. Therefore the triangle ADF is equal to the triangle AEF.

349. Let AF produced meet BC at H. The triangle BFH is to the triangle BFA as FH is to FA, by VI. 1; that is as the triangle CHF is to the triangle CFA. But from what is shewn in the last Exercise we find that the triangle AFB is equal to the triangle AFC. Therefore the triangle BFH is to the triangle BFA as the triangle CFH is to the triangle BFA. Therefore the triangle BFH is equal to the triangle CFH. Therefore BH is equal to CH.

350. Let $ABCD$ be a quadrilateral figure, having the sides AB and DC parallel. Let a straight line parallel to these sides meet AD at E, and BC at F. Then DE will be to EA as CF is to FB.

Of the two sides AB and DC suppose AB the greater. From C draw a straight line CGH parallel to DA meeting EF at G, and AB at H. Then CE and GA are parallelograms; and therefore CG is equal to DE, and GH is equal to EA.

Now CG is to GH as CF is to FB, by VI. 2; therefore DE is to EA as CF is to FB.

351. Let P be the given point. Bisect PA at Q. From Q draw a straight line parallel to AC, meeting BC at R. Join PR and produce it to meet AC, produced if necessary, at S.

Then PR is to RS as PQ is to QA, by VI. 2. But PQ is equal to QA; therefore PR is equal to RS.

VI. 3, A.

352. AF is to FC as AD is to DC, by VI. 3; and so also AE is to EB as AD is to DB. But DC is equal to DB. Therefore AF is to FC as AE is to EB. Therefore EF is parallel to BC, by VI. 2.

353. The arc DBC is bisected at B: see III. 30. Therefore the angle DGC is bisected by GE, by III. 27. Therefore DG is to GC as DE is to EC. Similarly DF is to FC as DE is to EC. Therefore DG is to GC as DF is to FC. Therefore also DG is to DF as GC is to FC, by V. 16.

354. Let AB be the given straight line. From A as centre with any radius describe a circle. From B as centre with a radius equal to double the former describe another circle, cutting the former at C. Join AC and BC, and bisect the angle ACB by a straight line meeting AB at D.

Then AC is to CB as AD is to DB, by VI. 3. But BC is twice AC; therefore DB is twice AD. Bisect DB at E. Then AD, DE, EB are all equal; so that AB is trisected.

355. The angle CPD is bisected by PA; therefore CA is to AD as CP is to DP, by VI. 3. The angle APB is a right angle, by III. 31; therefore PB bisects the angle between CP produced and DP. Therefore CB is to DB as CP is to PD, by VI. A. Therefore CA is to AD as CB is to DB; therefore CA is to CB as AD is to DB, by V. 16.

356. On AB as diameter describe a circle. Bisect the arc AB at C. Join CD and produce it to meet the circumference again at E. From E draw a straight line at right angles to DE, and let it meet AB produced at P. Then P will be the required point.

For, since the arc AC is equal to the arc BC, the angle AEB is bisected by ED, by III. 27. Therefore AE is to EB as AD is to DB, by VI. 3. And since the angle DEP is a right angle EP bisects the angle between AE produced and BE. Therefore AE is to EB as AP is to BP, by VI. A. Therefore AD is to DB as AP is to BP.

357. AB is equal to AE, by supposition; therefore the angle AED is equal to the angle ABC; also the angle EAD is equal to the angle BAC; therefore ED is equal to BC, by I. 26. Now AC bisects the angle BAD, and AE is at right angles to AC; therefore BC is to CD as BA is to AD, by VI. 3; and BE is to ED as BA is to AD, by VI. A. Therefore BE is to ED as BC is to CD. But ED is equal to BC. Therefore BE is to BC as BC is to CD; so that BC is a mean proportional between BE and CD.

358. BD is to DC as BA is to AC; therefore the difference of BD and DC is to their sum as the difference of BA and AC is to their sum: see the *Euclid*, p. 310. Now the difference of BD and DC is twice DO, and their sum is twice OB. Also twice DO is to twice BO as DO is to BO, by V. 15. Therefore DO is to BO as the difference of BA and AC is to their sum.

359. Suppose E is on BC produced through C. By VI. 3 and VI. A we have BD to DC as BE is to EC. Therefore the difference of BD and DC is to DC as the difference of BE and EC is to EC. That is twice OD is to DC as twice OC is to CE. Therefore OD is to DC as OC is to CE. Therefore OD is to the sum of OD and DC as OC is to the sum of OC and CE; that is OD is to OC as OC is to OE. But OC is equal to OB; therefore OD is

to OB as OB is to OE; so that OB is a mean proportional between OD and OE.

360. Let ABC be a triangle, let D, E, F be points in BC, CA, AB, respectively such that DF and DE make equal angles with BC, ED and EF make equal angles with CA, FE and FD make equal angles with AB. Then AD, BE, CF shall be at right angles respectively to BC, CA, AB.

Let AD and FE meet at G. Since the angles GEA and DEC are equal AE bisects the angle between DE produced and GE; therefore DE is to EG as DA is to GA. Similarly DF is to FG as DA is to GA. Therefore DE is to EG as DF is to FG; therefore DE is to DF as EG is to FG. Therefore DG bisects the angle FDE, VI. 3. Thus the angle GDF is equal to the angle GDE, and the angle FDB is equal to the angle EDC; therefore the angle GDB is equal to the angle GDC, so that each of them is a right angle.

Similarly it may be shewn that BE is at right angles to AC, and CF at right angles to AB.

VI. 4 to 6.

361. Let ABC and DEF be triangles on equal bases AB and DE, and between the same parallels $ABDE$ and CF. Let a straight line be drawn parallel to $ABDE$, meeting AC at K, BC at L, DF at M, EF at N. Then the triangle CKL shall be equal to the triangle FMN.

KL is to AB as CL is to CB, and MN is to DE as FM is to FD, by VI. 4. But CL is to CB as FM is to FD, by Exercise 350. Therefore KL is to AB as MN is to DE. Therefore KL is to MN as AB is to DE. But AB is equal to DE, by supposition. Therefore KL is equal to MN. Therefore the triangle CKL is equal to the triangle FMN, by I. 38.

362. CE is to AB as FE is to FB, and ED is to AB as GD is to GB, by VI. 4. But CE is equal to ED, by supposition. Therefore FE is to FB as GD is to GB. Therefore FG is parallel to ED by VI. 2.

363. Draw any straight line through C. Draw BN and AM perpendiculars on this straight line. Then BN is to AM as CB is to CA, by VI. 4; that is BN has to AM a constant ratio.

364. Let A and B be two fixed points, and suppose a straight line to pass between them and to cut AB at C. Draw AM and BN perpendiculars on MCN. Then AM is to AC as BN is to BC, by VI. 4. Therefore AM is to BN as AC is to BC. Thus if the ratio of AM to BN is given the ratio of AC to BC is also given; and therefore C is a fixed point.

365. Let A, B, C be three given points. Suppose a straight line to pass through a point D between A and C and also through a point E between B and C. Draw AF, CG, BH perpendiculars on this straight line. Then AF is to CG as AD is to DC, by VI. 4. Therefore since the ratio of AF to CG is known the point D is known. Similarly the point E is known. Thus the required straight line is obtained by joining DE.

366. Let A and B be the points from which the perpendiculars are to be drawn, C the point through which the straight line is to be drawn.

Join AC and produce it to D, making AC to CD in the given ratio. Join BD, and through C draw a straight line ECF perpendicular to BD: then ECF will be the required straight line.

For draw AM perpendicular to ECF, and let EF and BD intersect at N. Then CM is to CN as AC is to DC, by VI. 4: thus CM has to CN the assigned ratio.

367. The angle BFD is equal to the angle EFC, by I. 15; the angle DBF is equal to the angle FEC, by I. 29. Therefore the triangle BFD is equiangular to the triangle EFC. Therefore BD is to CE as BF is to FE. But BD is equal to BA, and CA is equal to CE; therefore BA is to AC as BF is to FE. Therefore AF is parallel to CE, by VI. 2.

368. From P draw PY perpendicular to AB, and from Q draw QZ perpendicular to CD. Then the triangle PMY is equiangular to the triangle QNZ; therefore PM is to QN as PY is to QZ. Thus the ratio of PM to QN is constant.

Again, let NM and QP produced if necessary meet at R. Then the triangle PMR is equiangular to the triangle QNR; therefore RP is to RQ as PM is to QN; so that the ratio of RP to RQ is constant: therefore R is a fixed point.

369. Let $ABCD$ be a quadrilateral figure, in which AB is parallel to CD and equal to twice CD. Join AC and BD intersecting at O. Then OC will be one-third of AC.

The angle DOC is equal to the angle BOA, by I. 15; the angle OCD is equal to the angle OAB, by I. 29; therefore the triangle DOC is equiangular to the triangle BOA. Therefore AO is to CO as AB is to DC, by VI. 4. But AB is twice DC; therefore AO is twice CO. Bisect AO at P; then AP, PO, OC are all equal and OC is one-third of AC.

370. The angles CAT and CBT are right angles; therefore a circle will go round $CATB$: see page 276 of the *Euclid*. Therefore the angle CAB is equal to the angle CTB, by III. 21; therefore the angle ABN is equal to the angle CTB. Also the right angles ANB and CBT are equal. Therefore the angle NAB is equal to the angle BCT, by I. 32. Thus the triangle BAN is equiangular to the triangle BCT; and therefore BT is to BC as BN is to NA, by VI. 4.

371. Through E draw OE parallel to AB, meeting BC at O. Then AB is to AC as OE is to EC, by VI. 4; that is as OE is to BD, by supposition; that is as EF is to DF, by VI. 4.

372. Let P be the centre of the circle which passes through A, C, and any point D in BC; let Q be the centre of the circle which passes through A, B, and D.

The angle APC is equal to twice the excess of two right angles over ADC, by III. 22, 20, that is the angle APC is twice ADB; also the angle AQB is twice ADB, by III. 20; therefore the angle AQB is equal to the angle APC. Thus the isosceles triangle AQB is equiangular to the isosceles triangle APC; and therefore PA is to QA as AC is to AB.

373. Let ABC be a triangle. Suppose that the perpendicular from the required point on BC is to be to the perpendicular on CA in the ratio of X to Y.

Draw a straight line parallel to BC at the distance X from it; and draw a straight line parallel to CA at a distance Y from it; let these straight lines meet at D. Then the perpendicular from D on BC is equal to X, and the perpendicular from D on CA is equal to Y.

Join CD; take any point P on CD, and draw PM perpendicular to BC and PN perpendicular to CA. Then PM is to X as CP is to CD, by VI. 4; and also PN is to Y as CP is to CD: therefore PM is to PN as X is to Y.

Again, suppose that the perpendicular from the required point on CA is to be to the perpendicular on AB as Y is to Z. Then as before we find a point E such that the perpendicular from it on AC is Y, and the perpendicular from it on AB is Z. Join AE; then, as before, we can shew that the perpendiculars from any point in AE on CA and AB are in the ratio of Y to Z.

Let CD and AE be produced to meet at O. Then from what has been shewn it follows that the perpendiculars from O on BC, CA, AB are proportional to X, Y, Z respectively.

374. Let AEB be one triangle, and AFC the other. Suppose that AE and AF are homologous. Draw EM perpendicular to AB, and FN perpendicular to AC; and produce EM and FN to meet at P. Let D be the angle of the rectangle opposite to A.

Then the triangle AEM is equiangular to the triangle AFN; and therefore AM is to AN as AE is to AF, that is as AB is to AC. Therefore AM is to MP as AB is to BD; therefore the triangle AMP is equiangular to the triangle ABD, by VI. 6; therefore P is on the straight line AD.

If EB and AF are homologous it will be found that the perpendiculars meet on BC.

375. Let GE produced through E, and CA produced through A, meet at the point P. Then CP is to KP as CG is to KE, by VI. 4. Let FH produced through H, and CA produced through A, meet at the point Q. Then CQ is to KQ as CF is to KH, by VI. 4. But the triangle CGK is equiangular to the triangle KEA; therefore CG is to GK as KE is to EA, that is CG is to CF as KE is to KH; therefore CG is to KE as CF is to KH.

Therefore CP is to KP as CQ is to KQ: therefore the points P and Q coincide; so that GE, FH, and CA, produced, meet at a point.

376. Let PQ and AC, produced if necessary, meet at L. Then LP is to LQ as AP is to CQ, by VI. 4. Let PQ and BD, produced if necessary, meet at M. Then MP is to MQ as PB is to QD. Now AP is to PB as CQ is to QD, by supposition. Therefore LP is to LQ as MP is to MQ. Therefore L coincides with M.

If instead of having given that AP is to PB as CQ is to DQ we have AP is to PB as DQ is to QC, we can shew that PQ, AD, BC meet at a point.

377. Let the straight line parallel to AB cut AC at M, and BC at N. From M and N draw straight lines parallel to BD meeting AB at P and Q respectively. Then PA will be equal to QB.

From C draw a straight line parallel to AB meeting BD at E. Then since the straight line AD is bisected at C the straight line BD is bisected at E, by VI. 2.

The triangle MAP is equiangular to the triangle DCE; so that MP is to PA as DE is to EC, by VI. 4. Again, the triangle NBQ is equiangular to the triangle BCE, so that NQ is to QB as BE is to EC, by VI. 4; that is as DE is to EC.

Therefore MP is to PA as NQ is to QB; therefore MP is to NQ as PA is to QB. But MP is equal to NQ, by I. 34: therefore PA is equal to QB.

378. Let B and C be the centres of the two given circles. Let a circle having its centre at A touch externally the former circle at R, and the latter at S. The straight lines RS and BC when produced will meet at a fixed point.

Let the straight line RS intersect the circle with centre C again at P, and let it meet the line BC at T. The angle CPS is equal to the angle CSP by I. 5; and is therefore equal to the angle ASR, by I. 15; and is therefore equal to the angle ARS, by I. 5.

Therefore AB is parallel to CP, by I. 27. Therefore TC is to TB as CP is to BR, by VI. 4. Thus the ratio of TC to TB is a fixed quantity; and therefore T is a fixed point.

379. From E and F draw straight lines parallel to AD meeting BC at M and N respectively.

Then EM is to MC as AD is to DC, by VI. 4; that is as AD is to DB by supposition; that is as FN is to NB, by VI. 4. Therefore EM is to FN as MC is to NB.

Again, EM is to MB as PD is to DB, by VI. 4; that is as PD is to DC, by supposition; that is as FN is to NC, by VI. 4.

Therefore EM is to FN as MB is to NC.

Therefore MC is to NB as MB is to NC; therefore MC is to MB as NB is to NC. Therefore MC is to the sum of MC and MB as NB is to the sum of NB and NC. That is MC is to CB as NB is to CB; therefore MC is equal to NB.

But EM is to FN as MC is to NB. Therefore EM is equal to FN. Therefore FE is parallel to NM, by I. 33.

380. Let S be the centre of the circle of which AE is a diameter, and T the centre of the circle of which EB is a diameter. Then SL is to SP as TL is to TQ, by VI. 4; therefore the difference of SL and SP is to SP as the difference of TL and TQ is to TQ, by V. 17; that is EL is to SP as BL is to TQ. Now SP, being equal to SE, is three times ET, that is three times TQ; therefore EL is three times BL; therefore EB is twice BL; therefore BL is equal to BT.

381. The triangle AOE is equiangular to the triangle AED; for the angle EAO is common to the two triangles, and the angle AEO is equal to the angle ADE by IV. 14 and III. 27, and therefore the angle AOE is equal to the angle AED, by I. 32. Therefore AO is to AE as AE is to AD; so that AE is a mean proportional between AO and AD.

382. QR is to BR as PQ is to AB, by VI. 4; that is as PQ is to DC, by I. 34; that is as SQ is to CS, by VI. 4. Thus QR is to BR as QS is to CS; therefore RS is parallel to BC, by VI. 2.

383. BD is to EF as CD is to CF, by VI. 4; and also EF is to AC as FD is to CD: therefore BD is to AC as FD is to CF, by V. 23. Therefore BD is to FD as AC is to CF. Therefore the triangle BDF is equiangular to the triangle ACF, by VI. 6: therefore the angle BFD is equal to the angle AFC.

384. From A draw AE perpendicular to BD, from B draw BF perpendicular to AC, from C draw CG perpendicular to BD, and from D draw DH perpendicular to AC. Let AC and BD intersect at O.

The triangle OAE is equiangular to the triangle OBF, and therefore OA is to OE as OB is to OF; therefore the triangle OFE is equiangular to the triangle OAB, by VI. 6; therefore the angle OEF is equal to the angle OAB and the angle OFE to the angle OBA. Similarly the angle OEH is equal to the angle OCB, that is to the angle OAD. Therefore the angle FEH is equal to the angle BAD. And the angle EFH has been shewn equal to the angle ABD. Therefore the angle FHE is equal to the angle BDA. Therefore the triangle FEH is equiangular to the triangle BAD; and therefore similar to it, by VI. 4.

Similarly the triangle FGH is similar to the triangle BCD. Hence it will follow that $EFGH$ is similar to $ABCD$.

385. Let A be the centre of one circle, and C that of the other. Let them intersect at the given point B. Then, by supposition, BA and BC are fixed directions. Let a straight line touch the former circle at E, and the latter at F; and let EF produced meet AC produced at D. Then D is the point at which two tangents to both circles will intersect.

Then DA is to DC as AE is to CF, by VI. 4; that is as AB is to BC. Therefore DB bisects the angle between AB produced and BC, by VI. A: thus D is on a fixed straight line.

Now as BA and BC are fixed *directions* other cases may occur in which instead of A we have some point on AB *produced* through B, or in which we have instead of C some point on CB *produced* through B. Thus we obtain a second fixed straight line, namely that which bisects the angle ABC.

VI. 7 to 18.

386. Let the circles touch each other at the point B; let one circle touch the straight line at C, and let the other circle touch the straight line at D. Draw CA a diameter of the former circle, and DE a diameter of the latter. Draw the straight line BF touching the circles at B, and meeting CD at F.

Thus FB, FC, FD are all equal, by Exercise 176. Therefore a circle described with F as centre will go through B, C, and D; therefore CBD will be a right angle, by III. 31. Also EBD is a right angle, by III. 31. Thus CBE is a straight line, by I. 14. Similarly DBA is a straight line.

Now the angle BCD is equal to the angle CAB, and the angle BDC is equal to the angle DEB, by III. 32. Therefore the triangle ACD is equiangular to the triangle CDE; therefore ED is to DC as DC is to CA.

387. Let EDK be the given arc, and EGK the remaining part of the circumference. Bisect the arc EGK at F. Divide EK at H so that EH may be to HK in the given ratio. Join FH and produce it to meet EDK at L: then L will be the point required.

For since the arc EF is equal to the arc FK the angle ELF is equal to the angle FLK, by III. 27. Therefore EL is to LK as EH is to HK, by VI. 3: so that EL is to LK in the given ratio.

388. Let ABC be the triangle; draw CE parallel to AB, and make CE to CB as CB is to BA. Join BE cutting AC at D. Draw DF parallel to AB, meeting BC at F.

Then DF is to FB as CE is to CB, by VI. 4; that is as CB is to BA, by construction; that is as CF is to FD, by VI. 4. Thus DF is to FB as CF is to FD; therefore FB is to FD as FD is to CF, by V. B. Thus FD is a mean proportional between FB and FC.

389. The angles BDA and ADC are equal being right angles; also BD is to DA as DA is to DC, by supposition. Therefore the triangle BDA is equiangular to the triangle ADC, by VI. 6. Thus the angle BAD is equal to the angle ACD, and the angle ABD is equal to the angle CAD. Therefore the angle BAC is equal to the two angles ABC and ACB; therefore the angle BAC is a right angle.

390. Since BD is to BA as BA is to BC the triangle BDA is equiangular to the triangle BAC, by VI. 6; therefore the angle BAC is equal to the angle BDA; that is the angle BAC is a right angle.

391. CA is to CP as CP is to CB, by supposition; therefore the triangle ACP is equiangular to the triangle PCB, by VI. 6; therefore the angle CPA is equal to the angle CBP.

392. Let the centre of the circle in any position be at a point C, and let the circle touch the straight line OA at the point B.

PQ is a third proportional to OP and PC; so that OP is to PC as PC is to PQ; therefore the angle OCQ is a right angle, by Exercise 389. Produce QC to meet OA at N. Then in the triangles OCQ, OCN the angle COQ is equal to the angle CON because the triangles COP, COB are equal, by Exercise 176; the right angle OCQ is equal to the right angle OCN; and OC is common: therefore QC is equal to CN, by I. 26.

From Q draw QM perpendicular to OA. Then the triangle QNM is equiangular to the triangle CNB; therefore QM is to CB as QN is to CN, by VI. 4. But QN has been shewn to be twice CN; therefore QM is twice CB. Thus Q is always on a straight line which is parallel to OA and at a distance from it equal to twice the given radius.

393. Let AS and BT be the parallel straight lines, AB being a diameter of the circle; let C be the centre of the circle. Then SCT is a right angle, by Exercise 182. Therefore CP is a mean proportional between SP and PT, by VI. 8, Corollary. Therefore the rectangle SP, PT is equal to the square on CP, by VI. 17; thus the rectangle SP, PT is constant.

394. Suppose D the point in the side AB of the triangle ABC, and let DE be parallel to BC. Then the triangle ADE must be equal to the triangle DBC; therefore AD is to DB as BC is to DE, by VI. 15. But BC is to DE as AB is to AD, by VI. 4. Therefore AD is to DB as AB is to AD; therefore the rectangle AB, DB must be equal to the square on AD. Thus AB must be divided at D in the manner of II. 11.

395. The triangles ECA and BCD are equiangular; therefore EC is to CA as CB is to CD, by VI. 4; therefore the triangle ECD is equal to the triangle ACB, by VI. 15.

396. The triangle ABE is equiangular to the triangle CBF; therefore AB is to BE as CB is to BF, by VI. 4; therefore AB is to CB as BE is to BF; therefore the triangle ABF is equal to the triangle CBE, by VI. 15.

397. Let $ABCD$ be a quadrilateral figure inscribed in a circle; let AC and BD intersect at O.

The angle AOD is equal to the angle BOC, by I. 15; the angle DAO is equal to the angle CBO, by III. 21; and the angle ADO is equal to the angle BCO, by III. 21: therefore the triangle AOD is equiangular to the triangle BOC. Therefore DO is to AO as CO is to BO, by VI. 4. Therefore the rectangle DO, OB is equal to the rectangle CO, OA, by VI. 16.

Similarly it may be shewn that the triangle COD is equiangular to the triangle BOA.

398. Let EF and CD meet at M. Then GO is to EM as CO is to CM, and LO is to FM as CO is to CM, by VI. 4. Therefore GO is to EM as LO is to FM. Therefore GO is to LO as EM is to FM. Similarly KO is to HO as EM is to FM. Therefore GO is to LO as KO is to HO. Therefore the rectangle GO, HO is equal to the rectangle LO, KO, by VI. 16.

399. The angles DFC, ACD, BDC and half of ACB make up two right angles, by I. 32. The angles DGC, ACD, BDC and half of ADB also make up two right angles. Therefore the angles DFC and half of ACB are together equal to the angles DGC and half of ADB. Therefore the angle DFC is equal to the angle DGC, by III. 21. Therefore a circle will go round $DGFC$, by page 276 of the *Euclid*. Therefore the rectangle EG, ED is equal to the rectangle EF, EC, by III. 36, Corollary. Therefore EF is to EG as ED is to EC, by VI. 16.

400. Let ABC be the triangle. From A draw a straight line AD, meeting BC at D; and also draw from A a straight line AE, meeting the circumference of the circumscribing circle at E, such that the angle ACE is equal to the angle ADB.

Then the angle ABD is equal to the angle AEC, by III. 21; the angle ADB is equal to the angle ACE, by construction; therefore the angle BAD is equal to the angle EAC, by I. 32. Thus the triangle BAD is equiangular to the triangle EAC; and therefore AB is to AD as AE is to AC, by VI. 4; and therefore the rectangle AB, AC is equal to the rectangle AD, AE.

401. AC is to CE as CD is to CB; and the angle ACE is equal to the angle DCB: therefore the triangle ACE is equiangular to the triangle DCB, by VI. 6, so that the angle CEA is equal to the angle CBD. Therefore a circle will go round $CBEA$; and the point E will bisect the arc AEB, because the angle ACE is equal to the angle BCE. If AB and the angle ACB are given, this circle will be a fixed circle, and E will be a fixed point on the circumference. See p. 276 of the *Euclid*.

402. Let $DFGE$ be the square; F being on the side AC, and G on the side BC. Then the triangle ADF is equiangular to the triangle GEB; therefore EB is to EG as DF is to DA. Therefore the rectangle AD, BE is equal to the rectangle EG, DF, by VI. 16; that is to the rectangle EG, ED; that is to the square $DFGE$.

403. The triangle AFE is equiangular to the triangle CFB; therefore EF is to FB as FA is to FC, by VI. 4. In like manner from the triangles GFC and BFA we have FB to FG as FA to FC. Therefore EF is to FB as FB is to FG; therefore the rectangle EF, FG is equal to the square on FB, by VI. 17.

404. In the triangle ABC suppose that AB is equal to AC. From A draw a straight line meeting BC at D, and produce AD to meet at E the circumference of the circle described round ABC.

The angle AEB is equal to the angle ACB, by III. 21; the angle ACB is equal to the angle ABC, by I. 5: therefore the angle AEB is equal to the angle ABD. Thus the triangle AEB is equiangular to the triangle ABD, by I. 32. Therefore DA is to AB as AB is to AE, by VI. 4; therefore the rectangle DA, AE is equal to the square on AB, by VI. 17.

405. Let T be one of the points of contact of the given tangents, and P one of the points of intersection of the two circles. Then the square on ET is equal to the rectangle EA, EH, by VI. 8, Corollary; and ET is equal to EP; therefore the square on EP is equal to the rectangle EA, EH; therefore EP touches the circle HPA, by III. 37.

VI. 19 to D.

406. In the diagram of IV. 10 suppose a straight line drawn from C parallel to BD, meeting AD at F. Then FB bisects the angle ABD: see Exercise 63. The triangle ACF will be to the figure $BCFD$ as BD is to BA.

The triangle ACF is equiangular to the triangle ABD; and therefore the triangle ACF is to the triangle ABD as the square on AC is to the square on AB, by VI. 19; that is as the rectangle AB, BC is to the square on AB, by IV. 10; that is as BC is to AB. Therefore the triangle ACF is to the figure $BCFD$ as BC is to AC, see V. E. But AC is equal to BD; therefore the triangle ACF is to the figure $BCFD$ as BC is to BD; that is as BD is to BA, for the triangle BCD is equiangular to the triangle BAD.

407. Let EB be a side of the regular polygon, K the centre of the circle. Let CB be half the side of the circumscribed figure of half the number of sides; C, E, K being in one straight line. From E draw EL perpendicular to BK; then EL is half the side of the inscribed figure of half the number of sides.

Let X, Y, Z denote the areas of the three figures respectively in descending order of magnitude. Then X is to Z as the triangle CBK is to the triangle ELK; that is in the duplicate ratio of CB to EL, by VI. 19. Also X is to Y as the triangle CBK is to the triangle EBK; that is as CK is to EK, by VI. 1; that is as CB is to EL, by VI. 4. Thus X is to Z in the duplicate ratio of X to Y. Therefore Y is a mean proportional between X and Z.

408. Join EG cutting AF at P, and HK cutting FC at Q. The triangles AEF and FHC are equiangular; therefore AE is to AF as FH is to FC. But AP is half of AF, and FQ is half of FC; therefore AE is to AP as FH is to FQ. Therefore the triangle AEP is equiangular to the triangle FHQ, by VI. 6, so that the angle APE is equal to the angle FQH. Therefore EP is parallel to HQ, by I. 28.

409. Let ABC be the triangle. From C draw CH perpendicular to AB, produced if necessary; and complete the rectangle $AFCH$. Describe a rectangle $AEDG$ similar to $AHCF$, and equal to the triangle ABC, so that E may be on AH and G on AF, by VI. 25. Then D will fall on AC, by VI. 26.

Then the triangle AED is half the rectangle $AEDG$, and is therefore equal to half the triangle ABC. Thus the straight line ED satisfies the assigned conditions.

410. Let ABC, DEF be the two isosceles triangles which are to one another in the duplicate ratio of their bases BC, EF. Then ABC and DEF shall be similar triangles.

For if the angle ABC be not equal to the angle DEF, one of them must be the greater. Let ABC be the greater, and make the angle CBG equal to the angle FED. Similarly make the angle BCG equal to the angle EFD. Then GBC and ABC being isosceles triangles the point G will fall within the angle BAC, so that the triangle ABC is greater than the triangle GBC. Also DEF and GBC are similar triangles.

Now, by supposition, the triangle DEF is to the triangle ABC in the duplicate ratio of EF to BC; and the triangle DEF is, by VI. 19, to the triangle GBC in the same ratio; therefore the triangle DEF is to the triangle ABC as the triangle DEF is to the triangle GBC. Therefore the triangle ABC is equal to the triangle GBC, which is absurd. Therefore the triangle ABC is similar to the triangle DEF.

411. The rectangle contained by the two segments is known, for it is equal to that of the segments of any chord of the circle through the point. Also the ratio of the sides of the rectangle is known. Hence the rectangle can be constructed by VI. 25.

412. The rectangle contained by the line and one segment is known, for it is equal to the square on the tangent. Also one side of the rectangle is to be double the other. Hence the rectangle can be constructed by VI. 25.

413. The straight line CD is divided similarly to AB. The straight line EB is divided similarly to AB: see VI. 2. Also it is shewn in II. 11 that the rectangle CF, FA is equal to the square on AC; therefore CF is divided at A in the required manner. And KG is divided at H similarly to the way in which CF is divided at A.

414. Let BC denote the given base; on BC describe a segment of a circle containing an angle equal to the given vertical angle. Then the diameter of this circle is known.

By VI. C the perpendicular from the vertex on the base of the triangle is known. Hence we must draw a straight line parallel to BC at a distance from it equal to this known perpendicular; either of the intersections of this straight line with the arc of the described segment may be taken for the required vertex of the triangle.

415. Let ABC be an equilateral triangle; suppose a circle to be described round the triangle, and let P be any point on the circumference of this circle. Draw PA, PB, PC.

Suppose P to be between A and C. Then $APCB$ is a quadrilateral inscribed in a circle; and therefore the rectangle PB, AC is equal to the sum of the rectangles PA, BC and PC, AB, by VI. D. But AC, BC, and AB are all equal, by supposition. Therefore PB is equal to the sum of PA and PC.

416. Since the angles ABD and ACD are right angles a circle would go round $ABDC$, by page 276 of the *Euclid;* therefore the rectangle AD, BC is

equal to the sum of the rectangles AB, CD and AC, BD, by VI. D. But AB is equal to AC, therefore the angle ABC is equal to the angle ACB, by I. 5; therefore the angle DBC is equal to the angle DCB; therefore DB is equal to DC, by I. 6. Thus each of the rectangles AB, CD and AC, BD is equal to the rectangle AB, DB. Therefore the rectangle AD, BC is equal to twice the rectangle AB, DB.

417. Suppose that the straight line through A falls without the triangle ABC, and that the perpendicular CE is less than FG.

The angle AFC is a right angle by I. 8; therefore a circle would go round $AECF$, by page 276 of the *Euclid*. Therefore the rectangle AC, EF is equal to the sum of the rectangles AE, FC and AF, EC. From F draw FH perpendicular to EC produced; then EC is the excess of EH over HC, that is equal to the excess of FG over HC. Therefore the rectangle AC, EF is equal to the excess of the rectangles AE, FC and AF, FG over the rectangle AF, HC. Now the triangle AFG is equiangular to the triangle HFC, for AF and FH are respectively perpendicular to FC and FG; so that FA is to AG as FC is to HC, by VI. 4; and therefore the rectangle FA, HC is equal to the rectangle FC, AG.

Therefore the rectangle AC, EF is equal to the excess of the rectangles AE, FC and AF, FG over the rectangle FC, AG, that is equal to the rectangles AF, FG and FC, EG.

The demonstration will remain substantially the same for other forms of the diagram.

XI. 1 to 12.

418. Let PA, PB be two equal straight lines drawn from a point P to a plane. Let PN be perpendicular to the plane.

The angles PNA and PNB are right angles; therefore the square on PA is equal to the sum of the squares on PN and NA, and the square on PB is equal to the sum of the squares on PN and NB. But PA is equal to PB; therefore NA is equal to NB. Therefore the angle PAN is equal to the angle PBN, by I. 8.

419. Suppose AB, AC to be two straight lines in one plane equally inclined to another plane; and let the planes intersect in BC. From A draw AD perpendicular to the second plane. Then the angle ABD is equal to the angle ACD by supposition; the right angle ADB is equal to the right angle ADC; and AD is common to the two triangles ADB and ADC. Therefore AB is equal to AC, by I. 26; and therefore the angle ABC is equal to the angle ACB, by I. 5.

If the point A is in the line of intersection of the two planes this method does not apply. Then take AB equal to AC; from B and C draw BF, CG perpendicular to the line of intersection of the planes, and from F and G draw FD, GE in the second plane perpendicular to the line of intersection of the planes; and from B and C draw BD, CE perpendicular respectively to the straight lines FD and GE. Then BD, CE will be perpendicular to the second plane by the construction in XI. 11; and the angles BFD, and CGE will each measure the inclination of the two given planes, by XI. Def. 6. Join AD, AE. Then AB is equal to AC; the angle BAD is equal to the angle CAE, by supposition; and the right angle BDA is equal to the right angle CEA: therefore BD is equal to CE, by I. 26.

The angle BFD is equal to the angle CGE, for each measures the inclination of one plane to the other; therefore BF is equal to CG, by I. 26. Also AB is equal to AC; therefore AF is equal to AG: see I. 47. Therefore the angle BAF is equal to the angle CAG, by I. 8.

420. Let DCE be the straight line in the plane. Through B draw in the plane a straight line FBG at right angles to BC. Then GBF is at right angles to both BC and BA, and is therefore at right angles to the plane ABC, by XI. 4. Now DE and GF are both in one plane, and both at right angles to BC; therefore DE is parallel to GF, by I. 28. Therefore DE is at right angles to the plane ACB, by XI. 8. Therefore DE is at right angles to AC; that is AC is at right angles to DE.

421. In the plane ABC draw GCF parallel to AB; produce CD to meet AB at K; and from K draw KL parallel to DE.

Then DE is perpendicular to the plane ABC by supposition; therefore LK is perpendicular to this plane, by XI. 8; therefore BKL is a right angle. And BKC is a right angle, see the *Euclid*, p. 313. Therefore BK is perpendicular to the plane containing LK, DE, and KC. Therefore GCF is perpendicular to this plane, by XI. 8. Therefore GC is at right angles to CE.

422. Let A and B be the two given points. From A draw AC perpendicular to the plane, and produce AC to D making CD equal to CA. Join DB intersecting the given plane at P. Then AP and BP are together less than any two other straight lines which can be drawn from A and B to meet in the plane.

For take any point Q in the plane; join AQ and DQ. Then in the two triangles ACQ and DCQ the sides AC and DC are equal; the side CQ is common; and the right angle ACQ is equal to the right angle DCQ: therefore AQ is equal to DQ, by I. 4. In the same manner it can be shewn that AP is equal to DP. Therefore the sum of AQ and BQ is equal to the sum of DQ and BQ; and is therefore greater than DB, by I. 20. But DB is the sum of DP and PB, and is therefore equal to the sum of AP and BP. Thus the sum of AP and BP is less than the sum of AQ and BQ.

423. Let OA, OB, OC be the three equal straight lines, meeting at O. From O draw OP perpendicular to the plane ABC; join PA, PB, PC.

The angles OPA, OPB, OPC are right angles. Hence by I. 47 it can be shewn that PA, PB, PC are all equal; so that P is the centre of the circle described round the triangle ABC.

424. Let the three straight lines meet at O. Take on these straight lines equal lengths OA, OB, OC. From O draw OP perpendicular to the plane ABC. Then OP is the required straight line.

For PA, PB, PC are all equal, by Exercise 423; therefore the angles POA, POB, POC are all equal, by I. 8.

425. Since EC, DF are perpendicular to the same plane they are parallel, by XI. 6: therefore the points E, C, D, F are in one plane.

Let CF, produced if necessary, meet AB at G. Draw the straight line GH parallel to EC or DF. Then GH is at right angles to the plane CAB, by XI. 8; and therefore the angle AGH is a right angle. Similarly a straight line GK drawn parallel to ED will lie in the plane ECD, and will be at right angles to the plane DAB; therefore the angle AGK is a right angle. Thus the straight line AB will be at right angles to the plane in which GH and GK lie, by XI. 4; and will therefore be at right angles to the straight line CFG which lies in that plane.

426. From a point O draw a straight line OP perpendicular to a given plane, and a straight line OQ perpendicular to the straight line AB lying in that plane. Then PQ will be perpendicular to AB.

From Q draw QL parallel to OP; then QL is at right angles to the plane, by XI. 8; so that the angle LQA is a right angle. Also AQO is a right angle, by supposition. Therefore AQ is at right angles to the plane containing OQ and LQ, by XI. 4: so that the angle AQP is a right angle.

XI. 13 to 21.

427. AB is perpendicular to the plane BED, and AC is perpendicular to the plane CED; therefore the straight line ED is perpendicular to the plane $ABCE$, by XI. 18, 19. Therefore the angles DEB and DEC are right angles. Since ABE and ACE are right angles a circle would pass round $ACEB$; and therefore CAB and CEB are together equal to two right angles: see page 276 of the *Euclid*. Thus the four angles CAB, CEB, DEB, DEC are together equal to four right angles.

428. Suppose ABC to denote a triangle, and KLM another triangle lying within the former: then it may be shewn that the *perimeter* of the second triangle is less than the *perimeter* of the former. The demonstration will be a series of steps of the following kind: produce KL to cut a side of the triangle ABC, say to cut the side AC at P: then LP and PM together are greater than LM, by I. 20; therefore the perimeter of KPM is greater than the perimeter of KLM. In this way we finally obtain the required result by repeated application of I. 20.

Now in the problem at present under consideration we must use XI. 20 instead of I. 20. Thus let O denote the point not in the plane of the triangles: then the sum of the angles LOP and POM is greater than the angle LOM; therefore the sum of the angles subtended at O by the sides of KPM is greater than the sum of the angles subtended at O by the sides of KLM. In this way we finally obtain the required result by repeated application of XI. 20.

429. Draw AE parallel to ab, meeting Bb, produced if necessary, at E; draw CF parallel to cd, meeting Dd, produced if necessary, at F. Then ab and cd are parallel, by XI. 15 and XI. 16. Therefore AE and CF will be parallel, by XI. 9. Therefore the triangle AEB is equiangular to the triangle CFD; so that AB is to CD as AE is to CF. But AE is equal to ab, and CF is equal to cd, by I. 34. Therefore AB is to CD as ab is to cd.

430. Let $ABCD$ be the regular tetrahedron. From A draw AF perpendicular to the plane BCD. Then F is the centre of the circle which would go round BCD; so that FB, FC, FD are all equal: see Exercise 423. Produce BF to meet CD at G; then it may be shewn that BG is at right angles to CD. Also FG will be one third of BG: see Exercise 343.

From F draw FK perpendicular to AG, and from B draw BL perpendicular to AG. Then the triangle BGL is equiangular to the triangle FKG; and BL is three times FK, since BG is three times FG, by VI. 4. Now FK is perpendicular to the plane ACD, by XI. 11; and so also is BL. Thus the perpendicular from B on the face ACD is three times the perpendicular from F on the face. And the perpendicular from B on the face ACD is equal to the perpendicular from A on the face BCD by reason of the symmetry of the regular tetrahedron.

431. Since the angles at the vertex are right angles each of the three faces meeting at the vertex is at right angles to the other two. Let BAC be the equilateral base, and D the vertex. Then since BA is equal to BC, and BD is common to the triangles BDA and BDC, we have AD equal to CD, by I. 47. Similarly AD is equal to BD.

Now let P be any point of the base. From P draw Pm perpendicular to the plane ADC, and Pn perpendicular to the plane ADB. Then the plane mPn will by XI. 18 be perpendicular to both the planes ADC, ADB, and therefore also to AD their line of intersection. Also the plane mPn is parallel to BDC by XI. 14; let it cut AC in c, AB in b, AD in a. Then ba is parallel to BD, and ca is parallel to CD by XI. 16. Also aD is equal to the perpendicular from P on the plane BDC.

The triangle Pmc is equiangular to the triangle BDC; therefore Pm is equal to mc, by VI. 4. Thus the sum of Pm and Pn is equal to the sum of mc and ma; that is equal to ac; that is equal to aA, since ac is parallel to DC. Therefore the sum of Pm, Pn, and aD is equal to AD, and is therefore constant.

432. Let OA, OB, OC be the three straight lines which meet at O. Through O draw any straight line OQ meeting the plane ABC at a point Q within the triangle ABC.

Then by the aid of I. 20 we can shew that the sum of QA, QB, and QC is less than the sum of AB, BC, and CA, but greater than half the sum of AB, BC, and CA: see Exercises 22 and 441. Now in precisely the same manner the present Exercise may be established, using XI. 20 instead of I. 20; and instead of a straight line, as QA, the corresponding angle QOA.

433. Let one plane cut the three straight lines at A, B, C respectively; let another plane, parallel to this, cut the straight lines at K, L, M, respectively; let a third plane cut the straight lines at P, Q, R, respectively, where P, Q, R are not in a straight line. And suppose AKP, BLQ, CMR to be cut in the same ratio by the planes. Then the plane PQR will be parallel to the other two planes.

For if not draw a plane through PQ parallel to the planes ABC and KLM; and let this cut the straight line CMR at S.

Then CR is to RM as BQ is to QL, by supposition; and CS is to SM as BQ is to QL, by XI. 17: therefore CR is to RM as CS is to SM; which is impossible. The condition that P, Q, R should not be on one straight line is necessary; for otherwise an indefinite number of planes could pass through P, Q, R.

434. Let AB and CD be the two straight lines. At the point A draw AK parallel to CD; and at the point C draw CL parallel to AB. Thus the planes BAK, LCD are the planes required.

For these planes are parallel, by XI. 15; and the former plane passes through the straight line AB, and the latter through the straight line CD.

435. Let the planes $ABCD$, $KLMN$ be parallel. Let a plane cut $ABCD$ in OP, and $KLMN$ in RS. Let another plane cut $ABCD$ in OQ, and $KLMN$ in RT. Then the angle POQ will be equal to the angle SRT.

For RS is parallel to OP, and RT is parallel to OQ, by XI. 16; therefore the angle POQ is equal to the angle SRT, by XI. 10.

436. The plane ABC is at right angles to both planes, by XI. 18; and therefore to their common intersection, by XI. 19. Therefore any straight line in the plane ABC is perpendicular to their common intersection; and BC is such a straight line; therefore BC is perpendicular to their common intersection.

437. Let AB and BC be two consecutive sides of a polygon obtained by cutting a prism by a plane; let ab and bc be the corresponding sides of the polygon obtained by cutting the prism by a plane parallel to the former. Then Aa and Bb are parallel by the definition of a prism; and AB and ab are parallel, by XI. 16; therefore $ABba$ is a parallelogram: therefore AB is equal to ab, by I. 34. Similarly BC is equal to bc; and so on. Also the angle ABC is equal to the angle abc by XI. 10.

438. Let AB and BC be two consecutive sides of a polygon obtained by cutting a pyramid by a plane; let ab and bc be the corresponding sides of the polygon obtained by cutting the pyramid by a plane parallel to the former. Let O be the vertex of the pyramid.

Then AB is parallel to ab, and BC is parallel to bc, by XI. 16; therefore AB is to ab as OB is to Ob, by VI. 4. Similarly BC is to bc as OB is to Ob. Therefore AB is to ab as BC is to bc. Also the angle ABC is equal to the angle abc, by XI. 10. Since these results hold for any corresponding pair of consecutive sides the polygons are similar.

439. The angle ABC is equal to the angle abc, by XI. 10. Also AB is to ab as PB is to Pb, by VI. 4; that is as pb is to pB, by supposition; that is as bc is to BC, by VI. 4. Therefore the triangle ABC is equal to the triangle abc, by VI. 15.

440. Let OC be the line of intersection, where O is in EF, or EF produced. The plane AOE contains AB and is therefore perpendicular to the plane AOC, by XI. 18; similarly the plane AOE contains AE and is therefore perpendicular to the plane EOC. Therefore OC is perpendicular to the plane AOE, by XI. 19; therefore OC is perpendicular to the straight line EF in that plane.

I. 1 to 48.

441. The sum of BP and CP is less than the sum of BA and CA, by I. 21; similarly the sum of CP and AP is less than the sum of CB and AB; and the sum of AP and BP is less than the sum of AC and BC. Thus twice the sum of AP, BP, and CP is less than twice the sum of AC, BA, and CB; and therefore the sum of AP, BP, and CP is less than the sum of AC, BA, and CB.

442. The angle APR is equal to the angle BQS, by I. 29. The angle APR is equal to the angle ARP, and the angle BQS is equal to the angle BSQ, by I. 5; therefore the angle ARP is equal to the angle BSQ. Therefore AR is parallel to BS, by I. 28.

443. Let $ABCD$ be a parallelogram, and P any point within it. The triangles PAB and PDC will be together half the parallelogram.

Through P draw a straight line parallel to AB and DC, meeting AD at K, and BC at M. The triangle PDC is half the parallelogram $KDCM$, by I. 41; and also the triangle PAB is half the parallelogram $KABM$. Therefore the triangles PDC and PAB together are equal to half the sum of the parallelograms $KDCM$ and $KABM$; that is to half the parallelogram $ABCD$.

444. Let $ABCD$ be a quadrilateral, such that the diagonal AC bisects it. Then the triangle ADC is equal to the triangle ABC : therefore BD is bisected by AC, by Exercise 114.

445. Let $ABCD$ be the quadrilateral figure. The triangle ABC is equal to the triangle ABD; for each is half the quadrilateral by supposition. Therefore DC is parallel to AB, by I. 39.

Similarly BC is parallel to AD.

446. Produce CA to M, and BA to N, making CM equal to BN.

In the triangles MCB and NBC the side CB is common; the side MC is equal to the side NB, by construction; and the angle MCB is equal to the angle NBC, by I. 5. Therefore the side MB is equal to the side NC, and the angle BMC is equal to the angle CNB.

In the triangles BMA and CNA we have the side MB equal to the side NC, and the angle BMA equal to the angle CNA, as just shewn; also the side MA equal to the side NA; therefore the angle MAB is equal to the angle NAC, by I. 4.

447. From the given straight line cut off BC equal to the given length. Join AC; draw from A a straight line AP meeting CB at P, and making the angle CAP equal to the angle ACP. Suppose that P falls between B and C.

Then AP is equal to CP, by I. 6; and therefore the sum of AP and PB is equal to CB.

If P does not fall between B and C the problem is impossible. But then the *difference* between AP and PB may be made equal to the given length.

448. In the first case of I. 26 suppose the triangle DEF applied to the triangle ABC, so that EF may be on BC, and the triangle DEF on the same side of BC as the triangle ABC is. Then since the angle DFE is equal to the angle ACB, and the angle DEF is equal to the angle ABC, the triangle DEF will coincide with the triangle ABC, and therefore be equal to it.

In the second case of I. 26 suppose the triangle DEF applied to the triangle ABC, so that DE may be on AB, and the triangle DEF on the same side of AB as the triangle ABC. Then since the angle DEF is equal to the angle ABC the straight line EF will fall on BC. Also DF will fall on AC; for if not let it take a different position as AH. Then the angle AHB coincides with the angle DFE and is equal to it; but the angle DFE is equal to the angle ACB, by supposition; therefore the angle AHB is equal to the angle ACB. But this is impossible, by I. 16. Therefore DF cannot fall otherwise than on AC. Therefore the triangle DEF coincides with the triangle ABC, and is equal to it.

449. Let ABC be a triangle having the sides AB and AC equal. Let a straight line EDF meet AB at E, meet BC at D, and meet AC produced at

F; and let EF be bisected at D: then AE and AF together will be equal to AB and AC together.

From F draw FG parallel to AB, meeting BC produced at G. In the triangles EDB and FDG the sides ED and FD are equal, by supposition; the angles EDB and FDG are equal, by I. 15; and the angles EBD and FGD are equal, by I. 29: therefore FG is equal to EB, by I. 26.

Also the angle FCG is equal to the angle ACB, by I. 15; and is therefore equal to the angle EBD, by I. 5. Thus the angle FGC is equal to the angle FCG; and therefore FG is equal to FC, by I. 6. Thus FC is equal to EB; therefore the sum of AB and AC is equal to the sum of AE and AF.

450. Suppose the straight line DME to meet AB at D, and to meet AC produced at E; and let AD be equal to AE: then will BD be equal to CE.

Through C draw CF parallel to AB, meeting DE at F. In the triangles DMB and MFC the sides BM and CM are equal, by supposition; the angles DMB and FMC are equal, by I. 15; and the angles DBM and MCF are equal, by I. 29: therefore DB is equal to FC, by I. 26.

The angle CFE is equal to the angle ADE, by I. 29; the angle ADE is equal to the angle AED, by I. 5: therefore the angle CFE is equal to the angle CEF. Therefore CF is equal to CE, by I. 6. Thus CE is equal to BD.

451. Let AB be one of the diagonals, and the triangle ABC half of one of the parallelograms. Let O be the middle point of AB and join OC. Then OC is half of the other diagonal, by Exercise 78. Thus AB and OC are given; and it is manifest that the triangle ABC has its greatest possible value when OC is at right angles to AB; and then AC is equal to BC, by I. 4. Similarly any other two adjacent sides must be equal, so that the parallelogram will be a rhombus.

452. If AD be not equal to BD and CD it must be either greater or less than them. If possible suppose AD greater. Then the angles ABD and ACD are together greater than BAD and CAD together, by I. 18; therefore the three angles of the triangle ABC are greater than two right angles; but this is impossible, by I. 32. Therefore AD cannot be greater than BD.

Similarly it may be shewn that AD cannot be less than BD.

453. Let AB and CD be two equal straight lines which intersect at right angles. The quadrilateral $ACBD$ will be equal to half the square on AB or CD.

Through C and D draw straight lines parallel to AB, and through A and B draw straight lines parallel to CD; thus a parallelogram is formed; and it has all its sides equal, and all its angles right angles, so that it is a square. Also each triangle ABC, ABD is half the corresponding rectangle of which AB is a side. Therefore the sum of the two triangles is half the sum of the two rectangles, that is half the square on AB.

454. Let ABC be the given triangle, D the given point within it. Draw AD and produce it to meet BC at E. On DA take DF equal to DE; and through F draw a straight line parallel to BC, meeting AB at G, and AC at H. Produce GD to meet BC at K, and produce HD to meet BC at L: then will $GHKL$ be a parallelogram.

In the triangles GDF and KDE the sides DF and DE are equal, by construction; the angles GDF and KDE are equal, by I. 15; and the angles

FGD and *EKD* are equal, by I. 29. Therefore *GF* is equal to *KE*, by I. 26. In like manner by comparing the triangles *HDF* and *LDE* we find that *HF* is equal to *LE*. Thus *GH* is equal to *LK*; and therefore *GL* is equal and parallel to *HK*, by I. 33.

Instead of joining the point *A* with *D* we might join *B* or *C* with *D*. Thus three solutions occur; but we must have *AD* greater than *DE* in order that the first solution may be possible; and similar conditions hold with respect to the other two solutions.

455. Let *AB* be one of the given sides. On *AB* describe a triangle *ABD* having its area equal to the given area. Through *D* draw a straight line *DE* parallel to *AB*. With centre *A* and radius equal to the other given side describe a circle cutting *DE* at *C*. Then *ACB* will be the triangle required.

For the sides *AB* and *AC* have the prescribed lengths; and the area of *ABC* is equal to the area of *ABD*, by I. 37, and therefore has the prescribed value.

456. Let *AB* be the given base. At the point *B* make the angle *ABD* equal to half the difference of the angles at the base. With centre *A* and radius equal to the given difference of the sides describe a circle meeting *BD* at *D* and *E*, and let *E* be the nearer of the two points to *B*. Produce *AE* to any point *F*; at the point *B* make the angle *EBG* equal to the angle *FEB*; and let *BG* meet *EF* at *C*. Then *ABC* will be the triangle required.

For *EC* is equal to *BC*, by I. 5; therefore the difference of *AC* and *BC* is equal to *AE*. The angle *ABC* is the sum of the angles *EBC* and *ABE*; the angle *CAB* is the difference of the angles *CEB* and *ABE*, by I. 32. Thus the angle *ABC* exceeds the angle *CAB* by twice the angle *ABE*, that is by the prescribed excess. And *AB* is the given base.

457. On *AB* take *AF* equal to the given straight line. Bisect the angle *BAC* by the straight line *AE*. From *F* draw a straight line at right angles to *AB*, meeting *AE* at *G*. From *G* draw a straight line meeting *AB* at *P*, making the angle *AGP* equal to the angle *GAP*. From *P* draw *PQ* perpendicular to *AC*.

The angle *AGP* is equal to the angle *GAP* by construction; therefore *AP* is equal to *GP*, by I. 5. The angle *GPF* is equal to the sum of the angles *AGP* and *GAP*, by I. 32; that is to twice the angle *GAP*; that is to the angle *PAQ*. The angles *PFG* and *AQP* are equal being right angles. Therefore *PF* is equal to *AQ*, by I. 26. Therefore the sum of *AQ* and *AP*, is equal to *AF*; that is to the prescribed sum.

458. Let *BC* be the base of a triangle, *D* the middle point of the base; and let the angle *BAC* of the triangle be a right angle; then *AD* is equal to *BD*: see Exercise 59 or Exercise 452.

Next let *BEC* be the triangle, and suppose *BEC* an acute angle. From *C* draw *CA* perpendicular to *BE*; then *CA* falls within the triangle *CBE*. The angle *DAE* is greater than a right angle, and the angle *DEA* is less than a right angle; therefore *DE* is greater than *DA*, by I. 19. But *DA* is equal to *BD*, by the first case; therefore *DE* is greater than *BD*.

Finally let *BFC* be the triangle, and suppose *BFC* an obtuse angle. From *C* draw *CA* perpendicular to *BF* produced; then *CA* falls without the triangle *CBF*. The angle *AFD* is greater than the angle *FBD*, by I. 32; and therefore greater than the angle *BAD*, by I. 5; therefore *DF* is less than *DB*.

459. Let $ABCD$ be a square. Take on AB a point E, on BC the point F, on CD the point G, and on DA the point H, such that AE, BF, CG, and DH are all equal: then will $EFGH$ be a square.

For we have EB, FC, GD, and HA all equal. Thus the triangles HAE, EBF, FCG, and GDH are all equal, by I. 4: therefore the figure $EFGH$ is equilateral. It is also rectangular. For the angle HEA is equal to the angle EFB; therefore the angles HEA and FEB are together equal to the angles EFB and FEB together; that is to a right angle, by I. 32. Therefore the angle HEF is a right angle by I. 13. Similarly the other angles of the figure $EFGH$ are right angles.

460. Let AB be the given base, P the point through which a side is to pass. Join AP and produce it to E, and cut off a part AD equal to the given difference of the sides. Join BD. At the point B make the angle DBF equal to the angle BDE; let DE intersect BF at C. Then ABC is the required triangle.

For the angle BDC is equal to the angle DBC, by construction; therefore BC is equal to DC, by I. 6. But AC exceeds CD by AD; therefore AC exceeds BC by AD. Thus AC exceeds BC by the prescribed length; and AC passes through the given point P.

461. From AB cut off AE equal to AC, and join ED. Thus the triangles AED and ACD are equal in all respects by I. 4. The angle BED is greater than the angle ADE, by I. 16; therefore the angle BED is greater than the angle ADC. The angle ADC is greater than the angle ABD, by I. 16; therefore the angle BED is greater than the angle EBD; therefore BD is greater than DE, by I. 19. But DE is equal to DC; therefore BD is greater than DC.

462. Let ABC be a triangle, having the angle BAC triple the angle ABC.

Make the angle BAE equal to the angle ABC, and let AE meet BC at D. Then the angle DAC is double the angle ABC. And the angle ADC is equal to the angles ABD and BAD, by I. 32; therefore the angle ADC is equal to twice the angle ABD; therefore the angle ADC is equal to the angle DAC. Thus BAD and CAD are isosceles triangles.

463. Let ABC be a triangle, having the angle BAC equal to double the angle ABC.

From C as centre, with radius equal to BC describe a circle meeting BA produced at D. Then BCD is an isosceles triangle; and therefore the angle ADC is equal to the angle ABC. The angle BAC is equal to the two angles ADC and ACD, by I. 32; and it is also equal to twice the angle ABC, that is to twice the angle ADC. Therefore the angle ADC is equal to the angle ACD; therefore AD is equal to AC, by I. 6. Thus BCD and CAD are isosceles triangles.

464. Let ABC be the triangle having AB equal to AC. Let D be the middle point of AB. Produce AB to E so that EB is equal to BA. Then CE will be equal to twice CD.

Produce CD to F making DF equal to CD; and join BF. Then the two triangles ADC and BDF are equal in all respects by I. 4; so that FB is equal to CA, and the angle DBF equal to the angle DAC.

In the two triangles CBF and CBE the side CB is common; the side BF is equal to AC, that is to AB, that is to BE; the angle FBC is equal to the angles FBA and ABC, that is to the angles DAC and ACB, that is to the angle EBC, by I. 32; therefore FC is equal to EC, by I. 4. But FC is equal to twice DC, by construction; therefore EC is equal to twice DC.

465. Suppose D and E of the preceding Exercise to denote the fixed points; then every point C is a point on the required locus, so that the locus is a circle having its centre at A and its radius equal to AB.

466. Let H be opposite C in the parallelogram $DCFH$, and let K be opposite C in the parallelogram $GCEK$. Draw AH, HB, BK, and KA. Then HD is equal and parallel to BG, and DA is equal and parallel to GK; thus the angle HDA is equal to the angle BGK, and HA is equal and parallel to BK. Similarly HB is equal and parallel to AK. Therefore $AHBK$ is a parallelogram; and its diagonals bisect each other by Exercise 78. But C is the middle point of AB, and therefore also of HK.

467. Through F draw a straight line parallel to BC, meeting AB at G, and join EG. From BA cut off BH equal to DF, and join EH. Then the triangle BHE is half the rectangle BE, BH; and is therefore equal to half the rectangle BE, DF. We have then to shew that the triangles AEF and BEH are together equal to half the rectangle $ABCD$.

The triangle BEH is equal to the triangle EGA, by I. 38; therefore the triangles AEF and BEH together are equal to the figure $EGAF$; and therefore equal to the two triangles EGF, and GFA; that is to half the rectangle $BCFG$ together with half the rectangle $GFDA$; that is to half the rectangle $ABCD$.

468. Take the case in which D is without the triangle ABC, and BD between BA and BC; also suppose F to be on BA produced and E on CA produced.

Then ED is equal to EC, as they are radii of the same circle; and similarly FD is equal to FB. Then ED and AF together are equal to EC and AF together; that is to EA, AF, and AC together. Again FD and EA together are equal to FB and EA together; that is to EA, FA, and AB together, that is to EA, AF, and AC together. Therefore ED and AF together are equal to FD and EA together.

Similarly the other cases which arise from modifications of the diagram may be treated. For instance, if D be within the triangle ABC, the point F on AB, and the point E on AC, we shall find that EA and ED together are equal to FA and FD together.

In all cases the sides of the quadrilateral taken in order are AE, ED, DF, FA; and if one of the sides BA, CA, requires to be produced, the other also will have to be produced.

469. From AB cut off AE equal to the required length. From E draw EH perpendicular to AC. Bisect the angle HEB by the straight line EM meeting AC at M; from M draw MP at right angles to AC, meeting AB at P; then P will be the required point.

For the angle EMP is equal to the angle MEH, by I. 29; and the angle MEH is equal to the angle MEP, by construction; therefore the angle EMP is equal to the angle MEP; therefore PE is equal to PM by I. 6. Thus the excess of AP above PM is equal to the excess of AP above EP, that is equal to AE: and therefore it has the prescribed value.

470. Let $ABCDEF$ be the equiangular hexagon. Join AD.

The two angles FAD and ADE together are equal to the excess of four right angles above AFE and FED together; that is to the excess of four right angles above two of the angles of the hexagon. Similarly the two angles BAD and ADC together are equal to the excess of four right angles above two of the angles of the hexagon. Therefore the angles FAD and ADE together are equal to the angles BAD and ADC together.

Again the angles FAD and DAB together are equal to an angle of the hexagon, and are therefore equal to EDA and ADC together. Therefore the angles FAD, ADE, EDA and ADC together are equal to the angles BAD, ADC, FAD, and DAB together. Therefore the angle EDA is equal to the angle BAD. Therefore AB is parallel to ED, by I. 27.

Produce AF and DE to meet at G, and produce AB and DC to meet at H. Then $AHGD$ is a parallelogram, by what has been shewn. Therefore GD is equal to AH; that is GE and ED together are equal to AB and BH together. Now since the hexagon is equiangular each angle is equal to two-thirds of two right angles, by I. 32, Corollary 1; therefore each of the angles GFE and GEF is one-third of two right angles; therefore the angle FGE is also one-third of two right angles, by I. 32. Thus the triangle FGE is equiangular; and therefore equilateral, by I. 6. Therefore GE is equal to EF. Similarly BC is equal to BH. Therefore FE and ED together are equal to AB and BC together.

471. From D draw DK perpendicular to AB produced. Then $DKAM$ is a rectangular parallelogram, by I. 28; and therefore DK is equal to MA, by I. 34.

The angles DBK and CBA are together equal to a right angle, by I. 13. The angles CBA and BCA are together equal to a right angle, by I. 32. Therefore the angle DBK is equal to the angle BCA. Also the right angle DKB is equal to the right angle BAC; and the side DB is equal to the side BC. Therefore DK is equal to BA, by I. 26. Therefore MA is equal to BA.

Similarly NA is equal to AC.

472. Through P draw a straight line parallel to AC, meeting AB at Q. From QB cut off QM equal to AQ. Join MP and produce it to meet AC at N. Then AMN is the triangle required.

For draw any other straight line KPL through P, meeting AB at K and AC at L: then the triangle AKL will be greater than the triangle AMN. Suppose AK greater than AM. From M draw MR parallel to AC meeting KL at R. Then MP is equal to NP; see Exercise 106; the angle MPR is equal to the angle NPL, by I. 15; and the angle PMR is equal to the angle PNL, by I. 29; therefore the triangle PNL is equal to the triangle PMR, by I. 26. Hence the triangle AKL exceeds the triangle AMN by the triangle KMR.

The proof is similar, when AK is less than AM.

473. Let a straight line parallel to the given straight line meet CA at L and CB at M. At C make the angle LCK equal to MLC, and let CK cut AB at D. Through D draw a straight line parallel to LM meeting CA at E and CB at F. Then will EF be bisected at D.

For the angle DEC is equal to the angle MLC, by I. 29; and the angle DCE is equal to the angle MLC by construction; therefore the angle DCE is equal to the angle DEC. The angles DCE and DCF are together equal to a right angle; the angles DEC and DFC are together equal to a right angle, by I. 32; therefore the angle DCF is equal to the angle DFC. Therefore DE and DF are each equal to DC, by I. 6; therefore DE is equal to DF, so that FE is bisected at D.

474. Bisect BD at E. Then BE is equal to BC. The angle ABC is equal to the angles BEC and BCE, by I. 32. Therefore the angle ABC is twice the angle BEC. Therefore the angle BEC is half the angle ABC, and is therefore equal to twice the angle BAC, by supposition. And the angle EBC is equal to the angles BAC and BCA, by I. 32. Therefore the angle EBC is equal to twice the angle BAC. Therefore the angle EBC is equal to the angle BEC; therefore EC is equal to BC, by I. 6. Thus EBC is equilateral.

Again, EC is equal to ED; therefore the angle BEC is equal to twice the angle BDC. Therefore the angle BDC is equal to the angle BAC. Therefore the angle ACD is equal to the angle ABC, by I. 32. Then the triangle ABC is equiangular to the triangle ACD.

475. Take the diagram of I. 43. Let BD and HG intersect at L. Through L draw a straight line parallel to AD, meeting AB at M and DC at N.

The complements AL and LC are equal, by I. 43; to each add KN. Then the parallelograms AK and MF together are equal to the parallelogram KC. Therefore the difference of KC and AK is equal to MF.

Now the parallelogram MK is twice the triangle BLK, and the parallelogram LF is twice the triangle DLK, by I. 41: therefore the parallelogram MF is twice the triangle BKD. Therefore the difference of the parallelograms KC and AK is equal to twice the triangle BKD.

476. Take BC equal to the given side; draw CD at right angles to BC, and make it equal to the difference between the hypotenuse and the other side. At the point B in the straight line BD make the angle DBE equal to the angle BDC, and let BE meet DC produced at A: then ABC is the triangle required.

For the angle ABD is equal to the angle ADB, by construction; therefore AD is equal to AB, by I. 6; but AD exceeds AC by CD; therefore AB exceeds AC by CD. Thus the hypotenuse AB exceeds the side AC by the prescribed length. Also the side CB has the prescribed length.

The process requires the angle CBD to be less than the angle CDB; it will be found that this leads to the condition that the hypotenuse must be less than the sum of the sides, which is of course necessary by I. 20.

477. The triangle EBC is half the triangle ABC, by I. 38; the triangle BED is half the triangle EBC, and is therefore one-fourth of the triangle ABC. Bisect AG at H; then the triangle ABH is equal to the triangle GBH, and the triangle AHE is equal to the triangle GHE: therefore the

triangle BHE is half the triangle BAE, and is therefore equal to the triangle BDE. Therefore GH is equal to GD, by Exercise 114. Thus AG is double of GD.

478. Let BC meet AE at F. The angle DAC is equal to the angle DCA; see Exercise 59. Suppose F to be between D and C. The angle EFC is equal to the angles EDF and DEF, by I. 32. Also the angle EFC is equal to the angles DCA and FAC, by I. 32; that is to the angles DAC and FAC; that is to twice the angle FAC together with the angle DAF; that is to the angles BAC and DAF. Thus the sum of the angles EDF and DEF is equal to the sum of the angles BAC and DAF. But the right angles EDF and BAC are equal; therefore the angle DAF is equal to the angle DEF; therefore DA is equal to DE, by I. 6.

479. The triangle ACF is half the rhombus, and the triangle ABC is half the square; therefore these triangles are equal; therefore FEB is a straight line parallel to AC, by I. 39. Draw BH and EK perpendicular to AC. Then BH is equal to AH by I. 6; and therefore equal to half AC; and therefore equal to half AE. Therefore EK is half of AE, by I. 34. And the angle AKE is a right angle. Therefore if EK be produced to a point L such that KL is equal to KE, the triangle ALE is equilateral. Thus AEK is an angle of an equilateral triangle, and therefore it is two-thirds of a right angle.

Therefore the angle KAE is one-third of a right angle, by I. 32. Also the angle BAC is half a right angle; therefore the angle BAE is one-sixth of a right angle. The angles CAF and EAF are equal by Exercise 11; therefore each of them is one-sixth of a right angle. Thus the angles BAE, EAF, and FAC are all equal.

480. From G draw GM perpendicular to AB, and GN perpendicular to AC. Then the angle NGM is a right angle. The angle EGD is also a right angle. Therefore the angle EGN is equal to the angle DGM.

In the two triangles EGN and DGM the side EG is equal to the side DG; the angles EGN and DGM have been shewn to be equal; and the right angles ENG and DMG are equal: therefore GM is equal to GN by I. 26. Thus $AMGN$ is a square, and the diagonal AG bisects the angle BAC; so that the locus of G is the straight line which bisects the angle BAC.

481. We shall first shew that a rectangle is greater than any parallelogram on the same base with the same perimeter.

Let $ABCD$ be a parallelogram, and $ABEF$ a rectangle on the same base AB, and having the same perimeter; then AF is equal to AD and therefore the perpendicular from D on AB is less than AF. Let DC produced cut AF at G and BE at H. Then the parallelogram $ABCD$ is equal to $ABHG$, by I. 35; and is therefore less than the rectangle $ABEF$.

Next we shall shew that a square is greater than any rectangle having the same perimeter.

Let $ABCD$ be a rectangle, AB being longer than AD. Let $AEFG$ be a square having the same perimeter, E being on AB, and G on AD produced.

Since the perimeters are equal EB is equal to GD, and BC is less than GF; therefore the rectangle EC is less than the rectangle DF; therefore the rectangle AC is less than the square EG.

482. Let $ABCD$ be the square in which the square of given area is to be inscribed. Join AC; and bisect it at E. From E as centre with a radius equal to half the diagonal of the square of given area describe a circle cutting AB at F; and produce FE to meet CD at G. Through E draw a straight line at right angles to FG, meeting BC at H, and AD at K. Then $FHGK$ will be the square required.

For FE is equal to GE and HE is equal to KE, by Exercise 36. Then from the triangles FEH and GEH we shew that FH is equal to GH; and from the triangles HEG and KEG we shew that HG is equal to KG. In this way we shew that the figure $FHGK$ is equilateral.

The angle AEB is a right angle, and so also is the angle FEH; therefore the angle AEF is equal to the angle BEH. Also the angle EAF is equal to the angle EBH, each being half a right angle. And EB is equal to EA. Therefore EF is equal to EH; therefore the angle EFH is equal to the angle EHF; therefore each of them is half a right angle. In this way we shew that the figure $FHGK$ is rectangular.

483. The triangle BDC is twice the triangle ADC, and the triangle BDF is twice the triangle ADF; see I. 38. Therefore the triangle BFC is twice the triangle FAC. Similarly the triangle BFC is twice the triangle FAB. Therefore the triangle BFC is equal to the sum of the triangles FAC and FAB; so that the triangle BFC is half the triangle BAC.

Again the triangle DAC is equal to the triangle EAB, each being one-third of the triangle ABC. Take away from each the figure $AEFD$; thus the triangle FEC is equal to the triangle FDB. But the triangle FAC was shewn to be equal to the triangle FAB; therefore the triangle AEF is equal to the triangle ADF. Therefore the figure $ADEF$ is twice the triangle ADF. But the triangle BDF is twice the triangle ADF. Therefore the figure $ADEF$ is equal to the triangle BDF.

484. On AB take AF equal to AE, and on BF take BG equal to BD. Then the triangle AOF is equal to the triangle AOE, and the triangle BOG is equal to the triangle BOD, by I. 4.

We shall now shew that the triangle FOG is equal in area to the triangle DOE.

The angle EOA is equal to the angles OAB and OBA, by I. 32; that is to half the sum of the angles BAC and ABC, that is to half a right angle. Therefore the angle AOF is half a right angle. Similarly the angle BOD is half a right angle, and also the angle BOG. Hence the angle GOF is half a right angle, and the angle DOE is three halves of a right angle. On OA take OH equal to OD, and join EH. Then in the triangles EOH and FOG the side EO is equal to the side FO; the side HO is equal to the side GO; and the angle HOE is equal to the angle GOF, each being half a right angle. Therefore the triangles EOH and FOG are equal in all respects. But the triangle EOH is equal in area to the triangle EOD, by I. 38; therefore the triangle FOG is equal in area to the triangle EOD.

Thus the triangle AOF is equal to the triangle AOE, the triangle BOG is equal to the triangle BOD, and the triangle GOF is equal to the triangle DOE; therefore the triangle AOB is half the quadrilateral $ABDE$.

485. Let ABC be the scalene triangle. If possible let BD be the dividing straight line. The angle BCD is not equal to the angle BAD, since by supposition the triangle is scalene; the angle BCD cannot be

equal to the angle BDA, by I. 32; and thus the only possible case is that the angle BCD should be equal to the angle ABD, and the angle BAD equal to the angle CBD. This requires AB to be equal to BC, and is contrary to the supposition that the triangle is scalene.

486. Let AE and CD intersect at G. Then CG is equal to GD, by Exercise 78. Therefore the triangle CGF is equal to the triangle DGF, and the triangle CGE is equal to the triangle DGE, by I. 38. Therefore the triangle FCE is equal to the triangle DFE. But BC is equal to CE, by supposition; therefore the triangle BCF is equal to the triangle ECF, by I. 38. Therefore the triangle BFE is double the triangle DFE. Therefore BF is twice DF.

487. BE and AD are each equal and parallel to CZ; therefore BE and AD are equal and parallel; therefore AB and ED are equal and parallel, by I. 33, and $ADEB$ is a parallelogram.

Produce ZC to meet AB at L. Then the parallelogram DL is equal to the parallelogram AZ by I. 35, which is equal to the parallelogram FC. Similarly EL is equal to KC; thus $ADEB$ is equal to the sum of FC and CK.

488. In the quadrilateral $ABCD$ suppose that AB is parallel to DC. Let AC and BD intersect at E. Through E draw a straight line parallel to AB meeting AD at M, and BC at N. Then MN will be bisected at E. For if ME be not equal to NE one of them must be the greater; suppose ME greater than NE. Then the triangle MAE will be greater than the triangle NBE, and the triangle MDE will be greater than the triangle NCE: see I. 38. Therefore the triangle AED is greater than the triangle BEC.

Again, the triangle ABC is equal to the triangle ABD, by I. 37; therefore the triangle AED is equal to the triangle BEC. But the triangle AED was shewn to be greater than the triangle BEC. Therefore ME and NE cannot be unequal; that is, they are equal.

489. Let ABC and DEF be two triangles; let the bases AB and DF be equal, and in the same straight line, and let CE be parallel to this straight line. Let a straight line be drawn parallel to CE, meeting CA at G, CB at H, ED at K, and EF at L. Then GH will be equal to KL.

For if GH be not equal to KL one of them must be the greater; suppose GH the greater, and from it cut off GM equal to KL. Join AM, BM, CM.

Then the triangle GCM is equal to the triangle KEL, the triangle GMA is equal to the triangle KLD, and the triangle AMB is equal to the triangle DLF, by I. 38. Therefore the triangle DEF is equal to the sum of the triangles GCM, GMA, and AMB; but the triangle ACB is equal to the triangle DEF; therefore the triangle ACB is equal to the sum of the triangles GCM, GMA, and AMB; that is the whole is equal to a part, which is absurd. Therefore GH is not unequal to KL, that is GH is equal to KL.

490. Let O be the middle point of BC; then OA, OB, and OC are all equal, by Exercise 59. Now AB is equal to half AC, and is therefore less than OB, which is half BC; therefore the angle OBA is greater than the angle BOA, by I. 18. But the angle BOA is equal to the angles OCA and OAC, by I. 32; therefore the angle BOA is equal to twice the angle OCA, by I. 5. Therefore the angle OBA is greater than twice the angle OCA.

491. Let $ABCD$ be the parallelogram. Trisect DC at E and F; and trisect BC at H and G: see Exercise 70. Then the triangles ADC and ABC are equal, by I. 34. Thus each of the triangles DAE, EAF, FAC, CAG, GAH, HAB is one-sixth of the parallelogram $ABCD$. Thus the triangle DAF, the quadrilateral $FAGC$, and the triangle GAB are all equal, each being one-third of the parallelogram $ABCD$.

492. The angles DAB and ABH are together equal to two right angles, by I. 29; therefore the angles DAB and AHB are together equal to two right angles, by I. 5. But the angles AHC and AHB are together equal to two right angles, by I. 13; therefore the angle DAB is equal to the angle AHC. Similarly the angle DAB is equal to the angle AKC. Also the angle DAB is equal to the angle HCK, by I. 34. Therefore three times the angle BAD is equal to the sum of the angles AHC, AKC, HCK; that is to the sum of the angles of the triangle HKC together with the angles AHK and AKH; that is to two right angles together with two angles of an equilateral triangle; that is to ten-thirds of a right angle. Therefore the angle DAB is ten-ninths of a right angle.

493. Let ABC be the given triangle; D a given point in the side AC. Draw a straight line DE meeting AC at D and AB at E, so that the triangle DAE may be one-third of the given triangle: this can be done by aid of I. 44, DA being the given straight line, A the given angle, and DE the diagonal of the parallelogram so described, which will be two-thirds of the triangle ABC. Then bisect the quadrilateral $DEBC$ by a straight line drawn from D, by Exercise 123. Thus the triangle ABC is divided into three equal parts by straight lines drawn from D.

If the triangle DAB be less than a third of the triangle ABC, then DC and the angle C must be used in the first part of the construction instead of DA and the angle A.

494. Let AC and BF intersect at H, let BE and CD intersect at G. Let GH, produced if necessary, meet BC at K. Then will BK be equal to CK.

Through H draw a straight line parallel to BC, meeting BE at M and CD at N. The triangles ACD and EBF are equal, by I. 38. Therefore HN is equal to HM, by Exercise 489. Therefore the triangle GHN is equal to the triangle GHM, by I. 38. Therefore BK is equal to CK, by Exercise 489.

II. 1 to 14.

495. Let ABC be the given triangle, BC the side to be produced. Suppose that CA is not less than BA. From A draw AD perpendicular to BC or BC produced. Produce CD to E so that DE may be equal to DC. Then the difference of the squares on CA and BA is equal to the difference of the squares on CD and BD, by I. 47; that is to the rectangle of the sum and difference of CD and BD, by page 269 of the *Euclid*. Now if the perpendicular falls *within* the triangle ABC the sum of CD and BD is BC, and the difference is BE; and if the perpendicular falls *without* the triangle ABC the sum of CD and BD is BE, and the difference is BC. Thus in each case the difference of the squares on CD and BD is equal to the rectangle BC, BE; so that BE is the produced part required.

496. Let AB be the given straight line; let BC represent the produced part. Then we require that the squares on AB and BC shall be equal to twice the rectangle AC, BC; so that the square on BC together with twice the square on AB must be equal to the square on AB together with twice the rectangle AC, BC; that is the square on BC together with twice the square on AB must be equal to the squares on AC and BC, by II. 7; therefore the square on AC must be equal to twice the square on AB. This determines AC, and shews that AC must be equal to the diagonal of a square described on the side AB.

497. Let AB be the given straight line; let BC represent the produced part. Then we require that the squares on AB and AC shall be equal to twice the rectangle AC, BC; so that the square on AC together with twice the square on AB must be equal to the square on AB together with twice the rectangle AC, BC; that is the square on AC together with twice the square on AB must be equal to the squares on AC and BC, by II. 7; therefore the square on BC must be equal to twice the square on AB. This determines BC, and shews that BC must be equal to the diagonal of a square described on the side AB.

498. In the diagram and proof of II. 11 it is shewn that the rectangle CF, FA is equal to the square on CA. If then CA be the given straight line describe a square on CA, and proceed as in II. 11; then F will be the point required.

499. Take any straight line AD; produce AD to B so that DB may be equal to twice AD. From A draw a straight line AE at right angles to AB. From the centre D with radius equal to DB describe a circle cutting AE at C. Then DBC will be such a triangle as is required.

For the square on BC is equal to the squares on CD, DB and twice the rectangle AD, DB. But twice the rectangle AD, DB is equal to the square on DB, because DB is equal to twice AD. Also CD is equal to DB. Therefore the square on BC is equal to three times the square on DB.

500. Let ABC be a triangle, and AD a perpendicular to the base BC produced; and suppose that the square on AB exceeds the squares on BC, CA by the rectangle BC, CA.

The square on AB exceeds the squares on BC, CA by twice the rectangle BC, CD; therefore the rectangle BC, CA is equal to twice the rectangle BC, CD; therefore CA is equal to twice CD. Produce CD to E, so that DE is equal to CD; and join AE. Then AC is equal to CE. Also AE is equal to AC, by I. 4. Therefore ACE is an equilateral triangle. Therefore the angle ACE is one-third of two right angles, by I. 32. Therefore the angle ACB is two-thirds of two right angles, by I. 13.

501. Let the straight line AB be the *sum* of two adjacent sides of the rectangle; let CD represent the *difference* of the two sides. Then AB is known, and we proceed to find CD.

The difference of the squares on AB and CD is equal to the rectangle contained by the sum and the difference of AB and CD; that is to four times the rectangle which is required; that is to four times the given square; that is to a known quantity. But the square on AB is known; thus the square on CD is known; and therefore CD is known. Hence the sides of the required rectangle are found; for one side is half the sum of AB and CD, and the other is half the difference of AB and CD.

502. Use the same notation as in Exercise 501. Then the difference of the squares on AB and CD is equal to four times the given square. But CD is known; hence the square on AB is known, and therefore AB is known. Hence the sides of the required rectangle are found.

503. Let $ABCD$ be the given square; let H, E, F, G be the middle points of the successive sides, so that $HEFG$ is a square. Let $PQRS$ be any other inscribed square. In each square the diagonals intersect at the same point, say T: see Exercise 36.

Then the square on PQ is equal to the squares on TP and TQ, and the square on HE is equal to the squares on TH and TE, by I. 47. But TP is greater than TH, and TQ is greater than TE, by I. 19; therefore the square on PQ is greater than the square on HE.

504. Let AB be the given straight line; it is required to divide it at C, so that the squares on AB and BC may be equal to twice the square on AC.

We require then that three times the square on AB, together with the square on BC, may be equal to twice the square on AB together with twice the square on AC. Produce BA to D, so that AD may be equal to AC. Then DB is the sum of AB and AC, and CB is the difference; so that the square on DB together with the square on CB is equal to twice the square on AB together with twice the square on AC, by page 269 of the *Euclid*. Hence we must have the square on DB equal to three times the square on AB. Thus DB is known; then we take AC equal to AD; and the straight line AB is divided at C in the manner required.

505. Let $ABDC$ and $AGFE$ be rectangles of equal areas and perimeters. Place them so as to have a common angle at A, and let AE fall on AC, and AB on AG. Let BD and EF intersect at H.

Then since the perimeters are equal we have twice EC equal to twice HF, and therefore EC is equal to HF. Also since the areas are equal the rectangle $BGFH$ is equal to the rectangle $ECDH$. But EC is equal to HF. Therefore CD is equal to FG. Therefore by using either the condition that the areas are equal or that the perimeters are equal we find that AC is equal to AG. Thus the two rectangles are equal in all respects.

506. Let O be the centre of the rectangle. Since the sum of PA and PC is equal to the sum of PB and PD, the squares on PA and PC together with twice the rectangle PA, PC are equal to the squares on PB and PD together with twice the rectangle PB, PD, by II. 4. But the squares on PA and PC are equal to twice the square on PO and twice the square on AO; and the squares on PB and PD are equal to twice the square on PO and twice the square on BO, see the *Euclid*, p. 293: so that the squares on PA and PC are equal to the squares on PB and PD. Hence the rectangle PA, PC is equal to the rectangle PB, PD.

Thus these two rectangles have equal areas and equal perimeters; and therefore, by Exercise 505, must be equal in all respects. Hence PA must be equal either to PB or PD. If we take PA equal to PB the point P falls on the straight line which is parallel to BC and passes through the centre of the rectangle. If we take PA equal to PD the point P falls on the straight line which is parallel to AB and passes through the centre of the rectangle.

III. 1 to 37.

507. Let BC be the straight line which is to be touched at the point A. Let D be the other given point. From A draw AE at right angles to BC. At the point D make the angle ADF equal to the angle DAE, and let DF intersect AE at G.

Then GA is equal to GD by I. 6. Therefore a circle described from the centre G with radius GA will pass through D; and it will touch the straight line BC by III. 16, Corollary.

508. Let C be the centre of the given circle, A the point at which it is to be touched; let B be the other given point through which the required circle is to pass.

Join AB and bisect it at right angles by the straight line EF; the centre of the required circle must be on the straight line EF, by III. 1. Also the centre of the circle must be on the straight line CA, or CA produced, by III. 11 and III. 12. Therefore the centre of the circle must be at the intersection of CA and EF. Thus the circle is determined.

509. Let A be the given point, AB a tangent to the given circle at that point, CD the given straight line. The centre of the required circle must lie on the straight line AE drawn through A at right angles to AB. As the circle is to touch AB and CD its centre must lie on the straight line which bisects one of the two angles made by AB and CD. See the note on III. 17 in the *Euclid*, page 275. Thus by the intersection of the straight line AE with the straight lines bisecting these two angles the position of the centre is found and the circle determined. We see that there are two solutions.

510. A circle will go round $ABDE$, by page 276 of the *Euclid;* therefore the angle ABE is equal to the angle ADE, by III. 21. Now the angle FBD is the difference between a right angle and the angle BDF, by I. 32; that is the difference between a right angle and the angle EDC, by I. 15; so that the angle FBD is equal to the angle ADE, and therefore to the angle ABE.

511. A circle described on BC as diameter will go through E and F, by page 276 of the *Euclid*. Then EK and FK will each be equal to half BC, by Exercise 59; so that K is the centre of this circle.

The angle FKE is equal to twice the angle FCE by III. 20. Suppose a perpendicular drawn from K on FE; thus the triangle FKE is divided into two triangles equal in all respects; and the angle FEK is seen to be equal to the difference of a right angle and half the angle FKE; that is to the difference of a right angle and the angle FCE. But the angle FAC is also equal to the difference of a right angle and the angle FCE. Therefore the angle FAC is equal to the angle FEK; and therefore also to the angle EFK.

512. Let AC, AD be on the same side of AB, and AC the nearer to AB. The angle CDF is equal to the angle CBA, by I. 13 and III. 22. The angle CEF is equal to the angles CAB and ABE, by I. 32. Therefore the sum of the angles CDF and CEF is equal to the sum of the angles CBA, CAB and ABE. But the angles CBA and CAB are together equal to a right angle, by I. 32 and III. 31; and the angle ABE is a right angle by III. 16. Therefore the sum of the angles CDF and CEF is equal to two right angles.

Therefore a circle will go round $FDCE$, by page 276 of the *Euclid;* therefore the angle FDE is equal to the angle FCE, by III. 21.

The proof is very similar for other forms of the diagram.

513. Let $ABCD$ be a quadrilateral. Bisect the angles A, B, C, D by straight lines which meet and form a quadrilateral: then a circle can be described round this quadrilateral.

Let AP and BP be two of the straight lines, and CQ and DQ the other two straight lines. Then the angle APB, half the angle A, and half the angle B are together equal to two right angles, by I. 32. So also the angle CQD, half the angle C, and half the angle D are together equal to two right angles. Therefore the angles APB and DQC together with half the sum of the angles A, B, C, D are together equal to four right angles. But half the sum of the angles A, B, C, D is equal to two right angles. Hence the angles APB and DQC are together equal to two right angles. Hence a circle can be described round the quadrilateral of which P and Q are opposite corners, by page 276 of the *Euclid.*

514. Let A be the centre of one circle, B the centre of the other; join AB meeting the first circle at G and the second at H. Then GH is the shortest distance between the circles.

For draw the straight line CD from any point C on the first circle to any point D on the second. Join AC, AD, and DB. Then AC and CD are together greater than AD, by I. 20; therefore AC, CD, and DB are together greater than AD and DB together. But AD and DB together are greater than AB; therefore AC, CD and DB together are greater than AB. But AC is equal to AG, and BD is equal to BH; therefore CD is greater than GH.

515. Suppose the straight line drawn; let BAC denote it. Join B and C with D the other extremity of the common chord. Then the angles ABD and ACD, being the angles in known segments of circles are known; also the side BC is known. Hence the triangle BCD can be constructed. Then the angle BAD becomes known, being the angle subtended by a known chord BD in a known circle. This determines the situation of the straight line BAC.

There will be in general two solutions, in one of which the segment BAD will be greater than a semicircle, and in the other, less.

516. Suppose the polygon to be a quadrilateral. Then the sum of the alternate angles is equal to two right angles, by III. 22; and the sum of the alternate angles, together with two right angles, is equal to four right angles.

Suppose the polygon to be a hexagon, as $ABCDEF$. Draw AD. The sum of the angles ABC and ADC is two right angles, by III. 22; so also is the sum of the angles ADE and AFE. Therefore the sum of the angles ABC, CDE, EFA is equal to four right angles; and the sum of these angles, together with two right angles, is equal to six right angles.

Again, suppose the polygon to be an octagon, as $ABCDEFGH$. Draw the straight lines AD and AF. Then the angles ABC and ADC are equal to two right angles, so are the angles ADE and AFE, so are the angles AFG and AHG. Then as before we obtain the required result.

In this way the proposition may be established for a polygon of any even number of sides.

517. Let C be the centre of the given circle, A the given point on the circumference, HK the given chord. On AC as diameter describe a semicircle, and let it cut HK at B. Join AB and produce it to meet the given circle at D.

The angle ABC is a right angle, by III. 31; therefore AD is bisected at B, by III. 3.

There will be two solutions in general, as the semicircle may cut HK at two points; but no solution if the semicircle does not meet HK.

518. Let A, B, C, D, E,... be the successive angular points of the polygon; O the centre of the circle.

The angles at A, B, C, D,... are bisected respectively by the straight lines OA, OB, OC, OD,...: see the note on III. 17 on page 275 of the *Euclid*. Then in the triangles OBA, OBC the side OB is common, the angle OBA equal to the angle OBC, and the side BA equal to the side BC, by supposition: therefore the angle BAO is equal to the angle BCO by I. 4. Therefore the doubles of these are equal; so that the whole angle at A is equal to the whole angle at C.

In this way we shew that the alternate angles are equal; and so if the number of them is *odd* they are all equal. If the number of the angles is even they are not necessarily equal; for instance, a circle might be inscribed in any rhombus.

519. If AE and BD intersect *within* the circle the angle APB is measured by half the *sum* of the arc DE and a semicircumference, by page 294 of the *Euclid*, and is therefore constant. Similarly if AE and BD intersect *without* the circle the angle APB is measured by half the *difference* of the arc DE and a semicircumference and is therefore constant.

520. Let ABC be one of the triangles. The four angles of the quadrilateral AD are together equal to four right angles; hence the angle BDC is equal to the excess of two right angles above the angle BAC and is therefore a constant angle. Hence the locus of D is a segment of a circle by page 276 of the *Euclid*. The straight lines which bisect the angle BDC all meet at a point, by Exercise 230.

521. The angle OBA is equal to the angle OAB, by I. 5; that is the angle OBC is a constant angle, by III. 21. Hence the locus of B is a circle, by page 276 of the *Euclid*. This circle is made up of two segments each equal to the corresponding segment of the given circle cut off by OC: see III. 24.

522. Suppose PE drawn perpendicular to AB, and produce EP to meet DC; then these straight lines will meet at right angles by I. 29. Thus PE and PG form one straight line.

Since the angles PFB and PEB are right angles a circle will go round $PFBE$, by page 276 of the *Euclid;* therefore the angle PEF is equal to the angle PBF. Similarly the angle PGH is equal to the angle PDH. But BC is parallel to AD; and therefore the angle PBF is equal to the angle PDH, by I. 29.

Therefore the angle PEF is equal to the angle PGH. Therefore EF is parallel to GH, by I. 27.

523. Let AB be a fixed chord of a circle, C a fixed point in it. Let PCQ be any other chord of the circle. Let D be the middle point of AB and R the middle point of PQ. Let O be the centre of the circle.

The angles ODC and ORC are right angles, by III. 3; therefore a circle will go round $ORCD$, by page 276 of the *Euclid;* therefore the angle CRD is equal to the angle COD, by III. 21. Thus the angle CRD is a constant angle.

524. The angle BFA is equal to the angle BEC, because the segments are similar. For the same reason the angle ADB is equal to the angle CDB; therefore the angle BDE is equal to the angle BDF, by I. 13. Therefore the angle BCE is equal to the angle BAF, by III. 22. Thus the angle BFA is equal to the angle BEC, and the angle BAF is equal to the angle BCE; therefore the angle ABF is equal to the angle CBE, by I. 32. Thus the triangle BEC is equiangular to the triangle BFA.

Also the angle BFA is equal to the angle BDA, by III. 21; that is to the angle BCE, by I. 13 and III. 22; that is to the angle BAF, as shewn above. Thus the triangle BAF is isosceles by I. 6. Therefore also the triangle BCE is isosceles.

525. Let A and B be the centres of the circles, AC, BD perpendiculars on the common tangent. From O the middle point of AB draw OQ perpendicular to CD. Then OQ is equal to half the sum of CA and BD, by Exercise 89; that is OQ is equal to half AB, and is therefore constant. Hence CD touches at Q the circle described from O as centre with radius equal to half AB.

526. Suppose A the given point, and BC the given straight line. Suppose that P and Q are two points in BC, such that PQ is of the given length and the angle PAQ equal to the given angle. Suppose that a circle goes round PAQ, and that O is the centre of the circle. The radius of this circle is known, for it is the circle in which a chord of given length subtends a given angle at the circumference. Since the radius and the length of PQ are known the distance of O from the fixed straight line becomes known.

Draw a straight line parallel to BC, and at a distance from it equal to that just determined. With A as a centre, and radius equal to that already determined describe a circle. The intersection of the circle and this straight line will determine the position of O; and then the positions of AP and AQ become known.

527. We shall first shew that a certain straight line can be found the tangents drawn from any point of which to the two circles are equal: then the intersection of this straight line with the given straight line determines the required point.

Let A and B be the centres of the given circles. Draw any circle intersecting the two given circles; let EF be the common chord of the first circle and the third circle; let CD be the common chord of the second circle and the third circle. Let FE and DC produced meet at G; from G draw GK perpendicular to AB. Then the tangents drawn to the two given circles from any point in GK will be equal. For take any point P in that straight line. Also from G draw GS to touch the first given circle, and GT to touch the second. The rectangle GE, GF is equal to the rectangle GC, GD, by III. 36 Corollary. Therefore the square on GS is equal to the square on GT, by III. 36; that is the excess of the square on GA over the square on SA is equal to the excess of the square on GB over the square on TB, by I. 47. Therefore the squares on GK and KA diminished by the square on SA are equal to the squares on GK and KB diminished by the square on TB, by I. 47. Therefore the excess of the square on KA over the square on SA is equal to the excess of the square on KB over the square on TB. Add the square on PK to both these equals; and then we find that the excess of the square on PA over the square on SA is equal to the excess of the square on PB over the square on TB. Thus the square on the tangent from P to one circle is equal to the square on the tangent from P to the other circle.

Therefore the intersection of GK with the given straight line is the point required.

528. Let AB be the fixed chord, CD the other chord. Let AD and BC intersect at O. The angle AOB is measured by half the sum of the arcs AB and CD, by page 294 of the *Euclid*, and is therefore constant. Hence the locus of O is a certain segment of a circle, described on AB, by page 276 of the *Euclid*.

529. The points C, A, B, are in one straight line; so are A, E, D, and also B, D, F; III. 11 and III. 12. Join CE, and produce it to meet BF at H. The angle BHC is equal to the angles ADB and DEH, and also equal to the angles BFC and HCF, by I. 32; therefore the angles ADB and DEH are together equal to the angles HCF and BFC together. But the angle BFC is equal to the angle BCF, by I. 5; and the angle DEH is equal to the angle BCE, by I. 15 and I. 5. Therefore the angles ADB and BCE are together equal to the angles BCF and HCF together; that is to the angle BCE and twice the angle ECF. Take away the common angle BCE; then the angle ADB is equal to twice the angle ECF.

530. By *Euclid*, page 269, the square described on the sum of AP and CP together with the square described on the difference of AP and CP is equal to twice the sum of the squares on AP and CP, that is to twice the square on AC, by I. 47. Hence the square described on the sum of AP and CP is always *less* than twice the square on AC, except when AP and CP are equal. Therefore the greatest value of the sum of AP and CP occurs when CP is equal to AP.

531. The angle BPQ is measured by half the sum of the arcs AL and BM, by page 294 of the *Euclid;* similarly the angle BQP is measured by half the sum of the arcs LB and CM. But the arc AL is equal to the arc BL, and the arc BM is equal to the arc CM. Therefore the angle BPQ is equal to the angle BQP.

Again, let MN cut BC at R and CD at S, then in like manner CRS is an isosceles triangle. Now by I. 32 the angles of the two triangles BPQ, CRS, are together double the angles of the triangle MQR; and BPQ and BQP are together double of MQR, by I. 15; so also CRS and CSR are together double of MRQ; therefore the two angles ABC and BCD are together double of the angle LMN.

532. Let AB be the given chord of the circle. Suppose C to be the required point on the circumference; join CA and CB and from AC cut off AD equal to the given difference. Join DB. Then CDB is an isosceles triangle; and as the angle C, being that subtended by a known chord, is known the angle CDB is known; and therefore also the angle ADB is known.

Hence the point D can be found. For it is on a segment of a circle described on AB containing a known angle; and it is on a circle described from A as centre with a known radius. When D is known, by joining AD and producing it to meet the given circle C can be found.

533. Let AB denote the base of the triangle, and CD the perpendicular from the vertex on the base. Suppose DA greater than DB, and from DA cut off DE equal to DB, and join EC. Then we have given AE which is the difference of DA and DB; also the sum of CA and CE, for this is equal to the sum of CA and CB; and the angle ACE, for this is the difference of the angles CED and CAB, by I. 32, that is the difference of the angles CBD and CAB. Hence the triangle ACE can be constructed by Exercise 268; and then the triangle ACB.

534. Suppose the segment APB to fall within the segment AQB. Let AT within the segment AQB be a tangent at A to APB, and produce TA to R. Let AS be a tangent at A to AQB.

Then the angle PAT is equal to the angle PBA, and the angle BAS is equal to the angle BQA; by III. 32. The angle PAR will be equal to the sum of the angles PAB and APB, by I. 13, III. 32 and I. 32. That is the angles PAB, BAS, SAR are together equal to the angles PAB, and APB. Therefore the angles BAS and SAR together are equal to the angle APB, that is to PAQ and AQB. But the angle BAS was shewn to be equal to the angle AQB; therefore the angle SAR is equal to the angle PAQ. That is the angle PAQ is equal to the angle between the tangents.

535. Let C be the centre of the given circle. Let T be the middle point of KL, M the middle point of PQ, and N the middle point of RS. The angles CNA, CTA, CMA are all right angles by III. 3; therefore N, T, and M are on a circle described on AC as diameter, by page 276 of the *Euclid*. The angle between MN and AL is measured by half the sum of the arcs AN and MT, by page 294 of the *Euclid;* that is by half the sum of the arcs AN and NT, by III. 26; that is by half the arc AT, which is a fixed arc. Thus the angle between MN and AL is *constant*, so that MN always remains parallel to itself.

536. Let $EFGH$ be a quadrilateral, such that round it the quadrilateral $ABCD$ can be described, so that the angle BEF is equal to the angle BFE, the angle CFG to the angle CGF, the angle DGH to the angle DHG, and the angle AHE to the angle AEH. Then a circle may be described about the quadrilateral $EFGH$.

The angles HEF, AEH, BEF are together equal to two right angles, by I. 13; and so are the angles HGF, DGH, CGF. Therefore the angles HEF, AEH, BEF, HGF, DGH, CGF are together equal to four right angles. Now in the four triangles AEH, BFE, FCG, and GDH the sum of all the angles is eight right angles; also the sum of the four angles at A, B, C, D is four right angles; therefore the sum of the remaining angles is four right angles. Hence we find that the sum of the four angles AEH, BEF, DGH, CGF is two right angles. Therefore the angles HEF and HGF are together equal to two right angles; and therefore a circle would go round $HEFG$, by page 276 of the *Euclid*.

537. From A draw the straight line ABC passing through the centres of the two circles, meeting the inner circle at B, and the outer circle at C. Suppose that the straight line AED is such as is required, meeting the inner

circle at E, and the outer circle at D, and making ED of the given length. Join CD, EB, and from B draw BF perpendicular to CD.

The angles AEB and ADC are right angles, by III. 31; and the angle BFD is a right angle, by construction. Therefore $EBFD$ is a rectangle, and BF is equal to ED.

Thus to solve the problem we describe a semicircle on BC and in it place the straight line BF equal to the given straight line, and then draw AED parallel to BF.

538. CD is parallel to AB, and therefore CD produced will cut AE at right angles, by I. 29: similarly AD produced will cut CE at right angles. Hence the perpendicular from E on AC will pass through D, by page 313 of the *Euclid;* that is if ED be joined and produced it will cut AC at right angles.

539. Let ABC be a triangle; let AD, BE, CF be the perpendiculars from the angular points on the opposite sides. By the *Euclid*, page 313, these perpendiculars meet at a point; denote the point by O.

Since the angles AFC and ADC are right angles a circle will go round $AFDC$, by the *Euclid*, page 276. Then the rectangle AO, OD is equal to the rectangle CO, OF, by III. 35. Similarly the rectangle AO, OD is equal to the rectangle BO, OE.

540. Let ABC be a triangle; let BF bisect the angle ABC, and meet AC at F; let CG bisect the angle ACB, and meet AB at G. Let BF and CG intersect at O. From A draw AD perpendicular to BF, and AE perpendicular to CG. Then ED will be parallel to BC, and ED, produced if necessary, will bisect AC and AB.

The straight line AO will bisect the angle BAC, by page 312 of the *Euclid*. The angle OAF is half the angle BAC; the angle AFB is equal to the angles FCB, and FBC, by I. 32; that is to the angle FCB and half the angle ABC; therefore the angle AOF is equal to the difference of a right angle and half the angle ACB. But a circle would go round $AEOD$; therefore the angle AED is equal to the angle AOF, that is to the difference of a right angle and half the angle ACB. But the angle AEC is a right angle; therefore the angle DEC is equal to half the angle ACB; that is to the angle ECB. Therefore ED is parallel to BC, by I. 27.

Produce ED to meet AC at K. The angle KEC has been shewn equal to the angle KCE; and AEC is a right angle; therefore the angle KEA is equal to the angle KAE, by I. 32; therefore KA and KC are each equal to EK, by I. 6; therefore KA is equal to KC.

Similarly DE, produced if necessary, will bisect AB.

541. Let AB be the diameter of the given circle. On AB describe a triangle ABD equal to half the given rectilineal figure: see I. 45, Corollary. Through D draw a straight line parallel to AB, meeting the circle at C. Then the triangle ACB is equal to the triangle ADB, by I. 37; and is therefore equal to half the given rectilineal figure. Also the angle ACB is a right angle by III. 31.

From A draw the chord AE parallel to CB; and join BE. Then the angle BAE is equal to the angle ABC, by I. 29; the right angle AEB is equal to the right angle BCA; therefore the angle ABE is equal to the angle BAC. Thus the triangle ABE is equal to the triangle BAC in all respects; and therefore the figure $ACBE$ is equal to the given rectilineal figure.

Also the angle CAE is a right angle; for it is equal to the two angles CAB and BAE; that is to the two angles CAB and CBA, by I. 29; that is to a right angle. Similarly the angle CBE is a right angle. Therefore the figure $ACBE$ is a rectangle.

542. Let O be the point of intersection of AD and BE; join CO and produce it to meet AB at L. Then CL is perpendicular to AB, by page 313 of the *Euclid*. The two circles both pass through L, by III. 31; thus the rectangles HO, OK, and CO, OL, and FO, OG are all equal. Therefore a circle will go through F, G, H, K by page 277 of the *Euclid*.

543. Let AB be one diameter; and CD another diameter, at right angles to the former. Let AE, CG, BF, DH be four parallel chords. Then the arcs EAG, GDF, FBH, HCE will all be equal.

Join AC, the angle EAC is equal to the angle ACG, by I. 29; therefore the arc EC is equal to the arc AG, by III. 26. Therefore the arc EAG is equal to the arc AEC, that is to a quarter of the circumference. Similarly the arc HBF is a quarter of the circumference.

In like manner the arcs HCE and FDG are equal; and as they are together equal to half the circumference each of them is a quarter of the circumference.

544. From A draw AM perpendicular to EC, from B draw BN perpendicular to EC. Then we must shew that AM and BN are together equal to EC. For then the square described on EC will be equal to the rectangle EC, AM together with the rectangle EC, BN; that is to twice the triangle AEC together with twice the triangle EBC; that is to twice the figure $AEBC$.

Through O the centre of the circle draw the diameter POQ parallel to EC; let BN intersect PQ at R; and from O draw OS perpendicular to EC. Then OS is equal to RN, by I. 34; therefore the sum of BN and AM is twice BR; and EC is twice ES. Therefore we have to shew that ES is equal to BR.

Now the angle EDO is equal to the angle ROB, by I. 29; the right angle EOD is equal to the right angle BRO; therefore the angle DEO is equal to the angle OBR.

In the two triangles EOS and BOR the angle OES is equal to the angle OBR, as just shewn; the right angles OSE and ORB are equal; and the side OE is equal to the side OB; therefore ES is equal to BR, by I. 26.

545. Let O be the centre of the circle. Bisect BC at D and draw DE parallel to OC, meeting OB at E. Then E is the middle point of OB, and ED is half of OC: see Exercises 106 and 109. And D is the middle point of the diagonals of the parallelogram, by Exercise 78.

Thus the required locus is the circle having its centre at E and its radius equal to half OC, that is to half the radius of the given circle.

546. Describe the circle which is obtained in the solution of Exercise 545 for the required locus. Join AE and produce it to meet the circumference of this circle at H. Then AH is the required direction in order that the diagonal may have its greatest possible length; and its length is twice AH.

547. Let A and B be the centres of the circles. Suppose that AD touches the circle having its centre at B, and that BC touches the circle having its centre at A, where C and D are on opposite sides of AB. Let

AD cut at K the circle which has its centre at A; and let BC cut at H the circle which has its centre at B. Join AC, BD, HK.

Then AC is equal to BD, and CB is equal to AD, by supposition; and the angles at C and D are right angles. Therefore $ACBD$ is a rectangle. Also CB is equal to twice BH, and AD is equal to twice AK by supposition. Hence $CHKA$ is a square, and HK touches both circles, and is equal to AC.

548. Let A and B be the centres of the two circles. From A as centre, with radius equal to the diameter of either of the given circles, describe a circle cutting at P the circle which has its centre at B. From P draw two tangents PT, PS to the circle which has its centre at A. Produce TP to meet at K the circle which has its centre at B; and produce SP to meet this circle at L. Join BL, BK.

The angle TAP is equal to the angle of an equilateral triangle; this may be shewn by producing AT to R, so that RT may be equal to TA, and joining PR. Similarly the angle SAP is equal to the angle of an equilateral triangle. Thus the angles TPA and SPA are each equal to half the angle of an equilateral triangle; and therefore the angle LPK, that is the angle TPS, is equal to the angle of an equilateral triangle. Hence the angle LBK is twice the angle of an equilateral triangle.

In the triangles TAS and LBK the sides TA, SA are equal to the sides LB, KB each to each, and the angle TAS is equal to the angle LBK; therefore TS is equal to LK.

It may happen that TP or SP cuts the circle which has its centre at B, and so does not require to be produced. The demonstration is not essentially changed.

It is necessary that the distance between A and B should not be greater than three times the radius, in order that the preceding solution may hold. There are two solutions if the described circle cuts the circle which has its centre at B, one if these circles touch, and none if one circle falls without the other.

549. The angle BCG is a right angle, by construction; the angle BFG is a right angle, by III. 31: therefore a circle will go round $BFGC$. Therefore the rectangle BA, AC is equal to the rectangle FA, AG.

Again, if a circle be described on BD as diameter it will pass through C, and AD will be a tangent to it because the angle ADB is a right angle. Therefore the rectangle BA, AC will be equal to the square on AD, by III. 36.

550. Let AB be the given base. On AB describe a segment of a circle ACB containing an angle equal to the given vertical angle. Bisect AB at E; through E draw ED at right angles to AB, meeting the segment at D; and complete the circle. Produce DE to meet the circle again at F.

Suppose C the required vertex of the triangle: join FC, cutting AB at G. Then, as in Exercise 549, the rectangle FG, FC is equal to the square on FB, that is to a known quantity. And FC bisects the angle ACB, so that GC is known by supposition; then FC may be found by the aid of Exercise 502. From F as centre with the radius just determined describe a circle; then the intersection of this with the segment ADB will determine the point C.

551. The angle APB is measured by half the sum of the arcs AB and DE, and is therefore a known angle. Similarly the angle BPC is known. Join AB, and on it describe a segment of a circle containing an angle equal to

the former; join BC, and on it describe a segment containing an angle equal to the latter. Then the intersection of the two segments will determine the point P.

552. Let A be any point on the circumference of the circle. From A draw AB perpendicular to OB one of the given straight lines, and AC perpendicular to OC the other given straight line. Also through A draw a straight line meeting OB at E, and OC at D; and equally inclined to the two given straight lines.

Then the angle OED is equal to the angle ODE by construction, and therefore the angle CAD is equal to the angle CDA, by I. 29; therefore AC is equal to DC. Hence the sum of AB and AC is equal to the sum of OC and CD, that is equal to OD.

Thus if the straight line DA *cuts* the circle we can obtain a *less* value of the sum of AB and AC by taking instead of A some point on the arc between DE and O. In this way we see that when the sum of AB and AC is least the point A must be such that the *tangent* to the circle there is equally inclined to the fixed straight lines, and is *between* O and the circle.

Similarly in order that the sum of AB and AC may be greatest the tangent at A must be equally inclined to the fixed straight lines, and the circle and the point O be on the *same* side of the tangent.

553. Let the segments described on AC and AB intersect at D. Then the angles ADB and ACB are together equal to two right angles, so are the angles ADC and ABC; therefore the angles BDC and BAC are together equal to two right angles by I. 15, Corollary 2. Therefore the segment described on BC will pass through D: thus the segments all pass through one point.

The angles ADB and ACB are together equal to two right angles; therefore the angle ACB is equal to the angle contained by the remaining part of the circle of which ADB is a segment, by III. 22. Hence it follows that the circle of which ADB is a segment is equal to the circle which could be described round the triangle ABC. Similarly this holds for the circles of which ADC and BDC are segments. Hence the three circles are equal. Produce BD to meet AC at M, and produce CD to meet AB at N. Since the circles ADB and ADC are equal the angles NBD and MCD are equal, by III. 28 and III. 27; the angles NDB and MDC are equal, by I. 15; therefore the angle DNB is equal to the angle DMC, by I. 32; therefore the angle AND is equal to the angle AMD. The angles NDM and NAM are together equal to the angles BDC and NAM, that is to two right angles. Hence the angles AND and AMD are together equal to two right angles; and as they are equal each of them is a right angle. Similarly AD produced meets BC at right angles.

IV. 1 to 16.

554. The three perpendiculars meet at a point, by page 313 of the *Euclid*: denote this point by O. In the triangles AFC and OEC the angle at C is common; the right angle AFC is equal to the right angle OEC: therefore the angle FAC is equal to the angle EOC.

Since the angles BEC and ADC are right angles a circle will go round $OECD$, by the *Euclid*, page 276; therefore the angle EDC is equal to the angle EOC; therefore the angle EDC is equal to the angle BAC.

Similarly the angle FDB is equal to the angle BAC. Therefore DE and DF are equally inclined to BC, and therefore to AD.

555. Let ABC be a triangle; suppose that the inscribed circle touches BC at D, CA at E, and AB at F. Construct the triangle FDE; from D draw DP perpendicular to EF, from E draw EQ perpendicular to FD, and from F draw FR perpendicular to ED. Then will PQ, QR, RP be parallel to AB, BC, CA respectively.

AB touches the inscribed circle DEF; therefore the angle AFP is by III. 32 equal to the angle FDE. But in the solution of Exercise 554 it is shewn that the angle FPQ is equal to the angle FDE. Therefore the angle FPQ is equal to the angle AFP; therefore PQ is parallel to AB, by I. 27.

Similarly QR is parallel to BC, and PR is parallel to CA.

556. Draw the circumscribing circle; then as one angle is given the side opposite this angle is given in magnitude. For if from any point on the circumference of the circumscribing circle we draw two straight lines containing an angle equal to the given angle, the chord which they intercept will be equal in magnitude to the side. Thus the problem is reduced to that of Exercise 293.

557. Let AB be the base; then the vertices of all the triangles lie on a segment of a circle described on AB; see the *Euclid*, p. 276.

Let ACB be one of the triangles. Produce AB, AC to H and K, and bisect the angles KCB, CBH, by the straight lines CE, BE meeting at E. Then as in Exercise 282, E is the centre of the circle touching BC and AB, AC produced; and the straight line AE bisects the angle CAB as in Exercise 280.

Since the angle AEB is the difference of the angles EBH and EAB by I. 32; and CBH is double of EBH, and CAB double of EAB, therefore the angle ACB is double of the angle AEB, and the point E lies on the segment of a circle described on AB and containing an angle AEB equal to half the given angle ACB. Let this segment AEB be described.

Bisect AB at G, and draw GDF at right angles to AB meeting the segment ACB at D, and the segment AEB at F, and join AD, DB, FA, FB. Then the angle ADB is double of the angle AFB by construction; also by I. 4 these angles are bisected by GF, therefore the angle ADG is double of AFD and AD is equal to DF. Similarly DB is equal to DF, and therefore D is the centre of the segment AEB, by III. 9.

558. Let AD meet BC at M. The angles ACM and CAM are together equal to a right angle; the angles ACB and CBE are together equal to a right angle; therefore the angle CAD is equal to the angle CBE. But the angle CAD is equal to the angle CBD, by III. 21. Therefore the angle MBD is equal to the angle MBL. Also the right angle DMB is equal to the right angle LMB; and BM is common to the two triangles BMD and BML: therefore MD is equal to ML, by I. 26.

559. A circle may be described round the regular pentagon; then the angle DAC standing on DC, is equal to the angle ADB standing on AB, which is equal to DC. Therefore AO is equal to OD, by I. 6.

Again, the angle COD is equal to the sum of the angles CBO and BCO, that is to twice the angle CBO; and the angle ACD is equal to the sum of the angles ACE and ECD, that is to twice the angle ACE: therefore the angle COD is equal to the angle OCD; therefore OD is equal to DC; and therefore OA is equal to CD.

Thus the triangle ACD is exactly like the triangle ABD of IV. 10; the sides AC and AD are equal; the angles ADC and ACD are each double of the angle DAC; and AO is equal to DC. Therefore as in IV. 10 we have the rectangle AB, BC equal to the square on BD, so here we have the rectangle AC, CO equal to the square on DC.

560. The angles CQR and CPR are right angles; therefore a circle described on CR as diameter will go round $CQRP$. This circle will be of constant magnitude; for PQ is of constant length, and the angle PCQ is constant. Thus the distance of R from C is constant.

Again, let PM be perpendicular to CQ, and QN perpendicular to CP. Then a circle will go round $QMNP$; and this circle will be of constant magnitude, for its diameter PQ is of constant magnitude. Also MN will be of constant magnitude, for the angle MQN is the difference between a right angle and the angle C, and is therefore constant.

Finally a circle described on CS as diameter will go round $CMSN$; and it will be of constant magnitude; for MN is of constant length, and the angle MCN is constant. Thus the distance of S from C is constant.

561. Let AB be the hypotenuse, ACB one of the right-angled triangles. It follows from the solution of Exercise 293 that the required locus is a segment of a circle ADB containing an angle equal to a right angle and a half. Complete the circle of which this segment is part; let O be the centre. Then the angle contained in the remaining part of the circle is half a right angle, by III. 22; therefore the angle AOB is a right angle, by III. 20. Therefore the arc ADB is a quarter of the circumference.

562. D is the centre of the circle described round ABC, by IV. 5; therefore the locus of D is the straight line which bisects AB at right angles: see III. 1.

563. Let AB be the given base; let ACB represent the triangle, O the centre of the inscribed circle, P the centre of the circle which touches AB, and touches CA and CB produced. Then OP is a known length. Suppose the angle CAB given.

OA bisects the angle CAB, and PA bisects the angle between AB and CA produced; therefore OAP is a right angle. Similarly OBP is a right angle. A circle may be described on the known length OP as a diameter. Then the angle AOB subtended in such a circle by a known chord AB becomes known; also the angle OAB is known, for it is half the angle CAB. Thus all the angles of the triangle OAB are known, and the triangle can be constructed. Then make the angle CAB equal to twice the angle OAB, and the angle CBA equal to twice the angle OBA: thus we obtain the required triangle ACB.

564. Let A be the given point in the given straight line AE; then the centre of the required circle must be on the straight line drawn through A at right angles to AE. Let C be the centre of the given circle, and suppose the required circle to cut it at the points L and M: then by supposition L, C, and M are in one straight line; suppose AC produced to meet the required circle again at D: then the rectangle AC, CD is equal to the rectangle LC, CM, by III. 35; that is to the square on LC. Now as LC and AC are

known this determines CD: thus D, a point through which the required circle is to pass, is known, and the required centre must be on the straight line which bisects AD at right angles. The centre of the required circle is therefore at the point of intersection of two known straight lines, and therefore its position is determined.

565. Let A be the given point, B the centre of one of the given circles, C the centre of the other given circle. Join AB and produce it to a point H such that the rectangle AB, BH is equal to the square on the radius of the circle which has its centre at B; then by the solution of Exercise 564 we see that H is a point through which the required circle must pass. Similarly join AC and produce it to a point K such that the rectangle AC, CK is equal to the square on the radius of the circle which has its centre at C; then K is a point through which the required circle must pass. Thus we have only to describe a circle round the triangle AHK, which we do by IV. 5.

566. Describe an equilateral triangle ABC in the given circle. Bisect the sides BC, CA, AB at the points D, E, F, respectively. From the points D, E, F, draw DO, EO, FO at right angles to the straight lines BC, CA, AB respectively meeting one another at the point O. Describe a circle in the triangle formed by the tangent at A and the straight lines OE, OF produced. This will be one of the three required circles. Similarly the other two required circles can be drawn.

567. This can be shewn from IV. 10. There the straight line ACB is divided at C in the manner of II. 11. Also BD is equal to AC. The angle BAD is one-fifth of two right angles, that is one-tenth of four right angles. This is the angle subtended at the centre of a circle by the side of a regular decagon inscribed in the circle: therefore BD is the side of a regular decagon inscribed in the circle.

568. Let $ABCDE$ be a regular pentagon inscribed in a circle. Let F be the centre of the circle. From any angle D draw the diameter DH cutting the straight line AB at the point I. This will be perpendicular to the opposite side AB. Join DA, DB. Then the triangle ADB is isosceles; and DH will bisect AB. Join FA, AH; then AH will be the side of a regular decagon inscribed in the circle.

Now the angle AFH is equal to the angle ADB by III. 20, therefore the isosceles triangle AFH is equiangular to the isosceles triangle ADB by I. 32, and the angle FAH is double of the angle AFH. But by Exercise 567 the side AH is equal to the greater segment of the radius HF cut as in II. 11, therefore if the radius FH be cut at G so that the rectangle FH, HG is equal to the square on FG, then AH is equal to FG.

Join AG. The square on AB is equal to four times the square on AI. Therefore the square on AB together with four times the square on IH is equal to four times the square on AH. But as in IV. 10 we have AG equal to AH, therefore GH is bisected at I, therefore the square on GH is equal to four times the square on HI: therefore the square on AB together with the square on GH is equal to four times the square on FG. But since the square on FG is equal to the rectangle FH, HG, the sum of the squares on FH, HG is equal to three times the square on FG by II. 7. Therefore the square on AB is equal to the squares on FH, FG.

569. Let ABC be the given triangle, B the vertex. Describe a circle about it. Let F be the centre of this circle. Join FB. On FB as diameter describe a circle cutting AC at D. Join BD. This shall be the required straight line. Produce BD to meet the circumference of the circumscribed circle at E. Join FD. Then FDB is a right angle by III. 31, therefore ED is equal to DB by III. 3. The rectangle AD, DC is equal to the rectangle ED, DB by III. 35, that is to the square on DB.

570. Draw a quadrilateral $DBEF$ and let BD, EF produced intersect at A, and let BE, DF produced intersect at C. Describe a circle about the triangle AEB and another about the triangle BDC. Let BO be the common chord of the circles. Join AO and DO. Then the angles AOB and AEB are equal by III. 21; and the angle AOB is equal to the sum of the angles AOD and DOB; also the angle AEB is equal to the sum of the angles BCD and DFA by I. 32 and I. 15. Therefore the angle AEB is equal to the sum of the angles DOB and DFA by III. 21.

Therefore the angle AOD is equal to the angle DFA. Therefore the circle described about the triangle AFD passes through the point O by page 276 of the *Euclid*.

Similarly it may be shewn that the circle which circumscribes EFC also passes through O.

571. Let ABC be a triangle. From A draw a perpendicular on BC. From B draw a perpendicular on AC; let these perpendiculars cut at D. From C draw CG perpendicular to AB. This will pass through D by p. 313 of the *Euclid*.

The rectangle GA, GE is equal to the rectangle GC, GD and therefore equal to the rectangle GB, GF by III. 36 Corollary.

The angle DAG is equal to the angle ECG by III. 21. The angle DAG is also equal to the angle BCG by I. 32. Therefore the angle ECG is equal to the angle BCG. Therefore the triangles ECG and BCG are similar, and CG is common, therefore EG is equal to BG by I. 26. Now the rectangle GA, GE is equal to the rectangle GB, GF, therefore GA is equal to GF, therefore AE is equal to FB.

572. Let O be the centre of the circle inscribed in the triangle ABC and let D, E, F be the centres of the escribed circles. Then AO is perpendicular to FE; see Exercise 282. The result required follows as in Exercise 553.

573. Let $ABDC$ be the quadrilateral. Let EA bisect the angle BAC and let EB bisect the angle ABD. Then E is the centre of the circle which touches internally the straight lines DB, BA, AC. Similarly let F be the centre of the circle which touches internally AC, CD, DB. In like manner let G, H be the centres of the other two circles. Then it can be easily shewn that the angles at E and F are together equal to two right angles. Therefore a circle can be described round the quadrilateral $EFGH$ by page 276 of the *Euclid*. See Exercise 513.

574. Take the point P in the arc AB. Because DP, PF are respectively perpendicular to BC, BA, therefore the angle ABC is equal to the angle DPF, and the angle DPF is equal to the angle DEF by III. 21, therefore the angle ABC is equal to the angle DEF. Similarly the angle ACB is equal to the angle DFE and the angle BAC is equal to the angle EDF.

Therefore the sides of the triangle ABC are equal to the sides of the triangle DEF respectively by III. 26 and III. 29. Again, the arc BC is equal to the arc EF; take away if necessary the common arc FC, then BE is parallel to CF by Exercise 219. Similarly it may be shewn that AD is parallel to either BE or CF.

575. We will take the case in which the point D falls within the given circle; the case in which D falls without the given circle can be treated in substantially the same manner. Let C be the centre of the described circle. Join QB, BC, CA, QC, CD. The two angles QBC, QAC are together equal to two right angles by III. 22. Now since CD is equal to CA, the angle CDA is equal to the angle CAD: therefore the two angles QDC, QAC are together equal to two right angles; therefore the angle QDC is equal to the angle QBC; and the triangle DCB is equilateral, therefore the angle QBD is equal to the angle QDB, therefore the side QD is equal to the side QB by I. 6; therefore QC bisects the angle DCB by I. 8, therefore QB subtends at the centre of the given circle an angle equal to two-thirds of a right angle by III. 20, therefore QB is equal to the radius of this circle by IV. 15 Corollary, but QD has been shewn to be equal to QB. Therefore QD is equal to the radius of the circle.

576. Let $ABCD$ be the given square, and P a point without it such that the angles APB, BPC, CPD are all equal. Then PAB will be greater than a right angle and PCB less than a right angle.

Draw BM, BN perpendiculars on PA, PC produced if necessary; then since BP bisects the angle APC, BM will be equal to BN. Hence as BA is equal to BC the triangles BAM, BCN will be equal in every respect. Thus the angles BAP, BCP are together equal to two right angles, and therefore P lies on the circle described so as to pass through the points A, B, C; that is the circle described about the square $ABCD$.

577. Let ABC be a triangle. Draw AP perpendicular to BC. Let the circle inscribed in the triangle APB touch AP, BP, AB at M, N, E respectively. Then the sum of the straight lines AP, BP is equal to the sum of the straight lines AM, MP, BN, NP, but AM is equal to AE, PM is equal to PN and BN is equal to BE by Exercise 176, therefore the sum of the straight lines AP and BP is equal to the sum of the straight lines AE, BE together with twice the straight line MP, that is equal to the sum of AB and twice the radius of the circle. Hence if X denote the diameter of the circle inscribed in ABP, the sum of X and AB is equal to the sum of AP and BP. Similarly if Y denote the diameter of the circle inscribed in ACP, the sum of Y and AC is equal to the sum of AP and CP. Therefore the sum of the straight lines X, Y, AB, AC is equal to the straight line BC together with twice the straight line AP. Two other results like this can be found; and then by addition the required result can be obtained.

578. Let O be the common centre. Take any point P on the circumference of the middle circle; join OP and produce it to Q making PQ equal to OP. With centre Q and radius equal to that of the smallest circle, describe a circle, and let one of the points at which it meets the outermost circle be S. Join QS; from O draw OR a radius of the inner circle parallel to QS. Join SP, PR.

By construction SQ is equal to RO and QP is equal to PO. Then in the two triangles SPQ, RPO the two sides SQ, QP are equal to the two sides

RO, OP each to each, and the included angle SQP is equal to the included angle ROP by I. 29, therefore the base SP is equal to the base PR and the triangles are equal in all respects by I. 4, therefore the angle SPQ is equal to the angle OPR, therefore SP, PR are in the same straight line; See **I. 15.**

VI. 1 to D.

579. Draw PM perpendicular to AB. Then CD is to PM as AC is to AM, and CE is to PM as BC is to BM by VI. 4. Therefore the rectangle CD, CE is to the square on PM as the rectangle AC, BC is to the rectangle AM, MB. But the square on PM is equal to the rectangle AM, BM by III. 35, therefore the rectangle CD, CE is equal to the rectangle AC, BC, which is equal to the square on CF by III. 35. Therefore since the rectangle CD, CE is equal to the square on CF, CD is a third proportional to CE and CF by VI. 17.

580. From the middle point P of BD draw a straight line at right angles to BD meeting AB produced through C at the point O; then OB is equal to OD by I. 4. Therefore the angle ODB is equal to the angle OBD by I. 5, which is equal to the sum of the angles OAD and ADB by I. 32. But the angle ADB is equal to the angle CDB, therefore the angle ODB is equal to the sum of the angles OAD and CDB, therefore the angle ODC is equal to the angle OAD. Also the angle AOD is common to the two triangles OCD, ODA. Hence these triangles are similar by I. 32; therefore OD is to OC as AD is to DC, therefore OD is to OC as AB is to BC by VI. 3. But OD is equal to OB, therefore OB is to OC as AB is to BC, which shews that O is a fixed point. Hence the locus of D is a circle whose centre is O and radius OB.

581. Let $ABCD$ be a square. Take BE equal to a fourth of BD and therefore equal to a third of DE. Join AE and produce it to meet BC at F. Then by the similar triangles ADE and BFE we have BF a third of AD or BC. Let the straight lines Bbc, Ccd, Dda be similarly drawn from B, C, D. Then by the symmetry of the construction it is evident that the figure $abcd$ is both equilateral and equiangular, that is it is a square. And since the straight line BC is equal to three times the straight line BF the straight line Bc is equal to three times the straight line Bb and therefore the straight line bc is equal to twice the straight line Bb. And the square on BC is equal to the sum of the squares on Bc, Cc by I. 47. But the square on Cc is equal to the square on Bb, therefore the square on BC is equal to the sum of the squares on Bc, Bb, which is equal to ten times the square on Bb. Therefore the square $abcd$ is equal to four times the square on Bb or four-tenths of the square on BC which is equal to two-fifths of $ABCD$.

582. AF bisects the angle A of the triangle by VI. 2, 3.

583. On the straight line AC describe a semicircle; from B draw BQ meeting the semicircle at Q so that the angle ABQ may be equal to half a right angle. From Q draw QP perpendicular to AC.

Since the angle PBQ is half a right angle, and the angle QPB is a right angle, therefore the remaining angle PQB is half a right angle, and is therefore equal to the angle ABQ; therefore the side PB is equal to the side PQ, by I. 6. Therefore the square on PB is equal to the square on PQ, that is to the rectangle AP, PC, by III. 35. Therefore PB is a mean proportional between PA and PC, by VI. 17.

584. The angle APB is constant and PA to PD is a known ratio, thus the angles PDA, PAD are constant by VI. 6. Then the angle PDA is equal to the angle PEB and the angle PAD is equal to the angle PBE. A circle will thus go round $ABDE$; this is a fixed circle and DE subtends a constant angle in it, so that DE is of constant magnitude, and therefore at a fixed distance from its centre, that is DE always touches a fixed circle.

585. From A draw AF perpendicular to BC. Then the angle BAF is equal to two-thirds of a right angle by I. 26 and I. 32, therefore AF is half of AB as may be shewn by producing AF to double its length, therefore AF is to AB as DF is to DB. Therefore AD bisects the angle BAF by VI. 3, therefore the angle BAD is equal to the angle DBA, and the angle ADE is therefore equal to two-thirds of a right angle. Similarly also the angle DEA.

586. Let $ABCD$ be a rectangle having the square on the side AB double of the square on the side BC; also the square on the side DC double of the square on the side AD. Let AF be drawn from the angle at A, and CE from the angle at C perpendicular to the diagonal DB.

Now BE is to BC as BC is to BD by VI. 4. Therefore the rectangle BE, BD is equal to the square on BC by VI. 17, that is to one-third of the square on BD by I. 47, therefore BE is equal to one-third of BD. Similarly DF can be shewn to be equal to one-third of BD. Therefore the remainder FE is equal to a third of BD. Thus BE, EF, FD are all equal.

587. By IV. 4 the angles CAH, CAG are equal to one another, therefore the angle GAH is equal to the angle DAC; and the isosceles triangles AGD, CAH are similar, therefore GA is to AH as DA is to AC; thus the angle ADC is equal to the angle AGH by VI. 6, and GA is to GH as DA is to DC. Similarly it may be shewn that the angles BGK, BDC are equal to one another and that BD is to DC as GB is to GK. Again since DA is equal to DB, therefore GA is to GH as DB is to DC, therefore GA is to GH as GB is to GK. But AG is equal to GB, therefore GH is equal to GK.

588. Let ABC be a right-angled triangle and AB, BC the sides containing the right angle ABC on which are described the squares DB, BE. Let CD, AE be joined cutting the sides at H and G. Join DB, BE which form a straight line by I. 14, and HG. From the similar triangles DHA, BHC, HA is to HB as AD is to BC; from the similar triangles BAG, CEG, BG is to GC as BA is to CE. By comparing these two ratios HA is to HB as BG is to GC. But by VI. 4 from the similar triangles BAG, CEG, BG is to GC as AG is to GE, therefore HA is to HB as AG is to GE, therefore HG is parallel to BE by VI. 2, therefore the angle BGH is equal to the angle GBE by I. 29, that is to half a right angle, and is therefore equal to the angle BHG, therefore BH is equal to BG by I. 6, therefore from the third mentioned proportion the rectangle AH, CG is equal to the square on BG by VI. 17, which is equal to the square on BH, therefore BG and BH are both mean proportionals between AH and CG.

589. Let OA, OB be the two given straight lines, and P the given point. The ratio of the straight lines from P is known and the angle they include.

Hence if the points where they meet the given straight lines be joined we have a triangle with known angles.

Make such a triangle pqr: on qr describe a segment of a circle having an angle equal to the angle O, and on pq a segment having an angle equal to the angle POB. Let o be the point where these segments meet. Join op. Then draw PQ making the angle OPQ equal to the angle opq and PR making the angle OPR equal to the angle opr.

590. Produce PE to meet the arc BO at G. Join AG.

Then the rectangle DE, EA is equal to the rectangle BE, EC, that is to the rectangle PE, EG by III. 35, therefore DE is to EP as GE is to EA by VI. 16; therefore the angle OAG is equal to the angle DPE by VI. 6. But the angle OAG is equal to twice the angle OPG by III. 20, therefore the angle DPE is bisected by the straight line OP, therefore the angles DPO, EPO are equal to one another.

591. Join CF and produce it to meet AB at H. Join HG.

Now HF is equal to FC; see VI. 1; thus HG is equal and parallel to EC by I. 4, 27. Again the ratio of BO to BE is half that of BG to BE, that is half that of HG to AE by VI. 4, that is half that of HB to AB by VI. 2. Draw HK parallel to DC; then DF is equal to FK by I. 26. Therefore the ratio of DF to DA is half that of DK to DA, that is half that of HB to AB by VI. 2. Therefore DF is to DA as BO is to BE.

592. Join AB, AF. First we shall shew that the square on AB is equal to the rectangle AE, AD. We have the square on AB equal to the square on AE together with the rectangle BE, EC by the note on III. 35 and III. 36 on page 277 of the *Euclid*. Also the rectangle BE, EC is equal to the rectangle AE, ED by III. 35, therefore the square on AB is equal to the square on AE together with the rectangle AE, ED, that is it is equal to the rectangle AE, AD. But AB is equal to AF: therefore the square on AF is equal to the rectangle AE, AD, therefore by the converse of VI. 8 we have the angle AEF a right angle. Similarly the angle AEG is a right angle, therefore GE, EF are in one straight line by I. 14.

593. The triangle ADE is equal to the triangle BFE by I. 38, therefore the triangle ADE together with the triangle ABE is equal to the triangle AFE; therefore by I. 38 the triangle ADE together with the triangle BCG is equal to the triangle FCG. Take away the common part, the triangle GHC; therefore the triangle ADE together with the triangle BHC is equal to the triangle FHG.

594. Let CPQ be the straight line drawn from the angle C meeting the intersection of the two straight lines at P and the side AB at Q.

The triangle PEC is equal to one-third of the triangle PAE by VI. 1; also the triangle BEC is equal to one-third of the triangle BAE; therefore the triangle BPA is equal to three times the triangle BPC, that is to twelve times the triangle BPD, therefore the straight line PA is equal to twelve times the straight line PD, therefore the triangle PDC is equal to one-twelfth of the triangle PAC by VI. 1, therefore the triangle BPD is equal to one thirty-sixth of the triangle PAC: therefore the triangle BPC is equal to one-ninth of the triangle PAC. Therefore this is also the ratio of the triangle BPQ to the triangle APQ, therefore it is the ratio of BQ to AQ.

595. The tangents so drawn are parallel to the sides of the inscribed figure, as may be shewn by drawing radii to the points of contact. Then the required result follows by Exercise 35.

596. Let ABC, CBD be the two right-angled triangles, then the angle ACD is a right angle. Make BE equal to BC. Join EC: then BEC is the required triangle, for the three triangles are as AB, BD, EB by VI. 1, and EB is a mean proportional between AB and BD by construction and VI. 8 Corollary.

597. Join DE. Then DE is parallel to AB by VI. 2. From the similar triangles DCE, CAB, CD is to DE as CA is to AB, but CD is equal to one-third of CA, therefore ED is equal to one-third of AB. From the similar triangles OED, BOA, OE is to ED as OA is to BA, but ED is equal to one-third of BA, therefore OE is equal to one-third of OA, that is to one-fourth of EA. Similarly OD is equal to one-fourth of DB.

598. Let O and P be the centres of the two circles. Join CD, PE. Then the angle CDA is a right angle by III. 31; also the angle PED is a right angle by III. 18, therefore PE is parallel to CD by I. 28, therefore AD is to DE as AC is to CP by VI. 2, therefore AD is to twice DE as AC is to twice CP by V. 4. Similarly BF is to twice FG as twice CP is to twice CO. Hence AD is to twice DE, as twice GF is to FB, by V. 11: therefore the rectangle AD, FB is equal to four times the rectangle FG, DE by VI. 16.

599. When circles cut at right angles the tangents at a point of intersection are at right angles to each other; and thus the radius of each circle is a tangent to the other circle at this point. Let D be the point where the circles which have B and C as centres meet; we have only to shew that the angle BDC is a right angle.

Since BD is equal to the tangent from B to the circle AC, the square on BD is equal to the rectangle BA, BC by III. 36. Similarly the square on CD is equal to the rectangle CA, CB. Therefore by II. 2 the square on BC is equal to the sum of the squares on BD, DC, therefore the angle BDC is a right angle by I. 48.

600. Let BF cut AD at P. Then it may be shewn that the angle APB is a right angle and that the angle PAB is two-thirds of a right angle. Hence it follows that PA is one-half of AB. Also AD is twice AB: See IV. 15 Corollary. Hence AP is one-fourth of AD and therefore AP is one-third of PD.

601. Since the angle A is equal to the angle D, and AB is equal to DF, therefore the perpendicular from B on AC is equal to the perpendicular from F on DE. Therefore by VI. 1, the triangles are as AC to DE.

602. Let E be the centre of the inscribed, D of the escribed circle, then B, E, D, are in one straight line. Draw EF, DG perpendicular to BA. Then BE is to BD as FE is to DG, that is as EM is to DP, therefore PD is parallel to EM by VI. 7 and I. 28. And ND is parallel to EM. Therefore PD and DN must lie in a straight line.

603. The triangle DMA is similar to the triangle BAC; therefore AM is to MD as CA is to AB by VI. 4. Therefore by V. 16, AM is to CA as MD is

to AB, that is as NA is to AB, therefore the triangle CAM is similar to the triangle BAN, by VI. 6, therefore the angle AMC is equal to the angle ANB, therefore the angle BMC is equal to the angle BNC by I. 13.

604. Let B be the middle point of the arc ABC. From B draw any two straight lines BF, BG, meeting the circumference at F and G, and the chord AC at D and E, respectively.

The angle BED is equal to the sum of the angles BCE, EBC, that is to the sum of the angles BFC, CFG by III. 27 and III. 21, that is to the angle BFG. Now BED and DEG are equal to two right angles; therefore BFG and DEG are equal to two right angles, therefore the points F, D, E, G are on the circumference of a circle. See page 276 of the *Euclid*.

605. Let H be the centre of the inscribed K of the escribed circle. The angle HAK is equal to the angle HBK, that is to a right angle by IV. 4; therefore a circle would go round $HAKB$: see page 276 of the *Euclid*. Produce HD to meet the circumference of this circle again at L; then the angle KLH is a right angle by III. 31; therefore $DEKL$ is a rectangle and EK is equal to DL. And the rectangle AD, DB is equal to the rectangle HD, DL, that is to HD, EK. Similarly the rectangle AE, EB is equal to the rectangle HD, EK.

606. Let ABC be any triangle. Let DE be drawn parallel to the base BC. Let F be the middle point of DE. Join AF and produce it to meet BC at G.

Then AB is to AD as CA is to AE by VI. 2; but BA is to AD as BG is to DF and CA is to AE as CG is to FE by VI. 4, therefore DF is to FE as BG is to GC, but DF is equal to FE, therefore BG is equal to GC. Hence the locus of F is the straight line drawn from A to the middle point of BC.

607. Let ABC be a triangle, AC the base and D the middle point of AC; then BD bisects every straight line parallel to AC: see Exercise 606. Let $EHLG$ be the parallelogram having the side HL in AC, and the sides EH, GL parallel to the fixed direction. Then F, the middle point of EG, is on BD. Draw FK parallel to the fixed direction; then P the middle point of FK is the intersection of the diagonals of the parallelogram $EHLG$. Now the straight line FK moves always parallel to itself and so the locus of its middle point is a fixed straight line passing through D.

608. The three bisectors meet at a point, see page 311 of the *Euclid;* the locus is a circle having its centre at the middle point of AB and its radius equal to one-sixth of AB. For let C be the right angle. Let the bisectors be BD, AE, CF and let them intersect at G. Because AF is equal to FB, the triangle AGF is equal to the triangle GFB. Again because AD is equal to DC the triangle ADB is equal to the triangle DCB and the triangle ADG is equal to the triangle DCG by I. 38, therefore the triangle AGB is equal to the triangle CGB, therefore the triangle FGB is equal to half the triangle CGB, therefore GF is equal to half GC, that is to one-third of FC. See VI. 1. But CF is equal to FA by Exercise 59, therefore the radius FG is equal to one-sixth of AB.

609. Draw a common tangent; let this meet the straight line which joins the centres at A. Then a straight line drawn through A and the

given point is the one required, for the parts intercepted by the circles cut off arcs bearing in both circles the same ratio to the whole circumference. See page 303 of the *Euclid*.

610. Let ABC be the given triangle and AC the base. With centre A and radius AC describe a circle; join B to the given point D and produce BD to meet the circumference of the circle at E. Join AE: draw DF parallel to AE meeting AB at F, and FG parallel to AC meeting BC at G. Then FD is to AE as BF is to BA, that is as FG is to AC, by VI. 4, but AE is equal to AC, therefore FD is equal to FG; and DF, FG are two adjacent sides of the required rhombus.

611. The triangle AFH is similar to the triangle EAC: for the angle FAH is equal to the angle BAC together with one right angle by IV. 9, that is to the angle EAC: and FA is to AH as EA is to AC, therefore the angle AFH is equal to the angle AEC by VI. 6. Similarly the angle BFG is equal to the angle CDB and the angle ECD is equal to the sum of the angles AEC and BDC as may be seen by drawing through C a straight line parallel to AE, therefore the angle ECD together with the angle HFG is equal to the angle AFB, that is to a right angle by IV. 9.

Miscellaneous.

612. Draw OD perpendicular to the fixed straight line. In OD take a point E such that the rectangle OE, OD may be equal to the rectangle OQ, OP. Join EQ.

Since OE is to OQ as OP is to OD and since the triangles EOQ, POD have a common angle DOP, they are similar by VI. 6; therefore the angle OQE is equal to the angle ODP, that is it is a right angle. Therefore the locus of Q is a circle having OE for its diameter. See page 276 of the *Euclid*.

613. Draw the diameter OD. In OD take a point E such that the rectangle OE, OD may be equal to the rectangle OQ, OP. Join EQ, PD. Then as in Exercise 612 the triangles OEQ, OPD are similar. But the angle OPD is a right angle by III. 31; therefore the angle OEQ is a right angle. Now E is a fixed point, therefore EQ is a fixed straight line. Hence the locus of Q is a straight line.

614. Let $ABCD$ be a quadrilateral figure inscribed in a circle. Let BA and CD produced meet at P, and AD and BC produced, meet at Q. From B draw BE meeting PQ at E making the angle PBE equal to the angle PQD. Then the angle QBE is equal to the angle QPD: because the angles ADC, ABC are together equal to two right angles by III. 22. Then the triangle PBE is similar to the triangle PQA, therefore the rectangle PA, PB is equal to the rectangle PE, PQ by VI. 4 and VI. 16. Similarly the rectangle QC, QB is equal to the rectangle QE, QP. Therefore by addition the rectangle PA, PB together with the rectangle QC, QB is equal to the rectangle PQ, PE together with the rectangle QE, PQ, that is to the square

on PQ. But the rectangle PA, PB is equal to the square on the tangent from P, and the rectangle QC, QB is equal to the square on the tangent from Q by III. 36.

615. Let H be the centre of the given circle, G the middle point of EF. Join HA, HE, HG, HF.

Then twice the square on EG together with twice the square on GH is equal to the sum of the squares on EH, FH by page 293 of the *Euclid*, that is to the rectangle EA, ED together with the rectangle FB, FA, together with twice the square on HA; that is to the square on EF, together with twice the square on HA by Exercise 614, that is to four times the square on EG together with twice the square on HA. Therefore the square on GH is equal to the sum of the squares on HA, EG, and thus the radii drawn from H and G to a point of intersection of the two circles are at right angles by I. 48.

616. Let ABC be a right-angled triangle having the angle at B a right angle. Let BD be drawn from the angle at B perpendicular to AC: from D let DE, DF be drawn perpendicular to CB, BA. Join FE. Then FED is a triangle having the perpendiculars DF, DE as two of its sides. Also the angle EDF is a right angle since $FDEB$ is a right-angled parallelogram.

Since D is on a circle of which BC is a diameter by III. 31, DE is not greater than one-half of BC. Similarly DF is not greater than one-half of AB.

617. Let PQ, RS be the two straight lines. Let them intersect at O. Let A be the middle point of RP, B of RS, C of QS, D of PQ. Then $ABCD$ shall be a parallelogram. For AB and DC are both parallel to PS by Exercise 106. Therefore they are parallel to one another. Similarly AD and BC are parallel.

Now DC is half PS by Exercise 109; and by VI. 4 it may be shewn that the perpendicular from D on PS is half the perpendicular from Q on PS, and that the perpendicular from B on PS is half the perpendicular from R on PS. The parallelogram $ABCD$ is equal to the difference of two parallelograms, each having DC for base, one having for height the perpendicular from B on PS, and the other having for height the perpendicular from D on PS. Thus the parallelogram $ABCD$ is half the difference of two parallelograms each having DC for base, one having for height the perpendicular from R on PS, and the other having for height the perpendicular from Q on PS. Thus the parallelogram $ABCD$ is half the difference of the triangles RPS and QPS, that is half the difference of the triangles ROP and QOS.

618. Let Q be the middle point of AB, and S the middle point of AC. Let O be the centre of the circle. Join AO cutting QS at P.

The difference of the squares on QO and OB is equal to the square on QB by I. 47, that is to the square on QA, that is to the squares on AP, PQ: therefore the difference of the squares on PO and OB is equal to the square on PA: therefore the difference of the squares on RO, OB is equal to the square on RA, that is the square on the tangent from R is equal to the square on RA: see III. 36. Therefore RA is equal to the tangent from R.

619. By the preceding Exercise the rectangle RQ, RP is equal to the square on RA, therefore RQ is to RA as RA is to RP by VI. 17; therefore

the triangle QRA is similar to the triangle ARP by VI. 6, therefore the angle AQR is equal to the angle RAP.

620. Let $ABCD$ be the quadrilateral. Let a, b, c, d be the middle points of AB, BC, CD, DA, respectively; let E, F be the middle points of AC, DB respectively. Let O be the intersection of the circles round the triangles aFd, aEb; we will prove that a circle will pass through $dcEO$.

The angle FOa is the difference of two right angles and the angle adF; that is the difference of two right angles and the angle ABD. The angle aOE is the difference of two right angles and the angle abE, that is the difference of two right angles and the angle BAC. We shall obtain the angle dOE by subtracting from four right angles, the excess of the angles aOE, and aOF above the angle FOd: thus the angle dOE is equal to the sum of the angles ABD, BAC, FOd; that is to the sum of the angles ABD, BAC, Fad; that is to the sum of the angles ABD, BAC, ADB; that is to the excess of two right angles and the angle BAC over the angle BAD; that is to the difference of two right angles and the angle DAC; that is to the difference of two right angles and the angle dcE. Therefore a circle can be described round $dcEO$, by page 276 of the *Euclid*.

Similarly a circle can be described round $cFOb$.

621. Let OA, OB, OC be the three straight lines which bisect the angles of an equilateral triangle. They meet at one point O, by page 312 of the *Euclid*. Let P be the given point from which PA, PB, PC are drawn perpendicular to OA, OB, OC respectively. Then the straight line PA shall be equal to the sum of the straight lines PB and PC, supposing PA to be the longest perpendicular. Draw PD parallel to OB meeting OC at D. Draw DH from D perpendicular to OA, and DF perpendicular to PA. Then DH is equal to BP because OC bisects the angle BOH.

From the triangles PCD, PFD, PC is equal to PF. Also BP has been shewn to be equal to DH, that is to AF. But AF, FP are together equal to AP, therefore BP, PC are together equal to AP.

622. Draw the straight line AF bisecting the exterior angle between AC and AD, and meeting the circle ABC at E; and draw the tangents TE, TF meeting at T. Then the angles EAC, FAD are equal; therefore FBD, EBC are equal by III. 21. Therefore the complements of these angles are equal, that is ABF is equal to ABE, and thus by III. 32 the angles TEF and TFE are equal. Therefore T is on the line AB produced; see Exercise 527.

If the interior angle between AC and AD be bisected the proof is substantially the same.

623. Let ABC be the given triangle. Through E and D the middle points of AC and AB respectively draw parallel straight lines EF, DG meeting BC at the given angle. Through A draw a straight line parallel to BC and let GD, FE produced meet this straight line at K and H respectively. Then $FGKH$ is the parallelogram required; for it is obvious that the triangle AEH is equal to the triangle FEC, and that the triangle KAD is equal to the triangle BDG.

624. Draw the diagonals AC and BD intersecting at O. Suppose P to be within the triangle DOC.

We have to shew that the triangle PAC is equal to the difference of the triangles PAB and PAD; this we will do by shewing that the sum of the triangles PAD and PAC is equal to the triangle PAB.

The sum of the triangles PAD and PAC is equal to the sum of the triangles POD, DOA, POC. The triangle PAB is equal to the sum of the triangles POB, BOA, POA. Now AC and BD bisect each other by Exercise 78, therefore by I. 38 the triangle POD is equal to the triangle POB; the triangle DOA is equal to the triangle BOA and the triangle POC is equal to the triangle POA. Thus the required result is obtained.

In a similar way any other case can be treated.

625. Let R, Q be the two points where the circles cut one another.

The rectangle AC, CD is equal to the rectangle RC, CQ, that is to the rectangle EC, CB by III. 35; therefore AC is to CE as CB is to CD by VI. 16; therefore by V. 18 AE is to CE as BD is to CD: therefore AE is to BD as CE is to CD by V. 16. Similarly AE is to BD as CA is to CB; therefore the square on AE is to the square on BD as the rectangle AC, CE is to the rectangle BC, CD, therefore the square on BD is to the square on AE as the rectangle BC, CD is to the rectangle AC, CE.

CAMBRIDGE: PRINTED BY C. J. CLAY M.A. & SON, AT THE UNIVERSITY PRESS.

WORKS

BY

I. TODHUNTER, M.A., F.R.S.

Natural Philosophy for Beginners. In two parts.
Part I. The Properties of Solid and Fluid Bodies. With numerous Examples. 18mo. 3s. 6d. Part II. Sound, Light, and Heat. 3s. 6d.

Euclid for Colleges and Schools. New Edition.
18mo. cloth. 3s. 6d.

Key to Exercises in Euclid. Crown 8vo. cloth.
6s. 6d.

Mensuration for Beginners. With numerous Examples. New Edition. 18mo. cloth. 2s. 6d.

Algebra for Beginners. With numerous Examples. New Edition. 18mo. cloth. 2s. 6d.

Key to the Algebra for Beginners. New Edition. Crown 8vo. cloth. 6s. 6d.

Trigonometry for Beginners. With numerous Examples. New Edition. 18mo. cloth. 2s. 6d.

Key to the Trigonometry for Beginners. Crown 8vo. cloth. 8s. 6d.

Mechanics for Beginners. With numerous Examples. New Edition. 18mo. cloth. 4s. 6d.

Key to the Mechanics for Beginners. Crown 8vo. cloth. 6s. 6d.

Algebra for the use of Colleges and Schools. With numerous Examples. New Edition. Crown 8vo. cloth. 7s. 6d.

Key to the Algebra for the use of Colleges and Schools. New Edition. Crown 8vo. cloth. 10s. 6d.

A Treatise on the Theory of Equations. New Edition. Crown 8vo. cloth. 7s. 6d.

Plane Trigonometry for Colleges and Schools. With numerous Examples. New Edition. Crown 8vo. cloth. 5*s.*

Key to the Plane Trigonometry for Colleges and Schools. Crown 8vo. cloth. 10*s.* 6*d.*

A Treatise on Spherical Trigonometry for the use of Colleges and Schools. With numerous Examples. New Edition. Crown 8vo. cloth. 4*s.* 6*d.*

A Treatise on Conic Sections. With numerous Examples. New Edition. Crown 8vo. cloth. 7*s.* 6*d.*

A Treatise on the Differential Calculus. With numerous Examples. New Edition. Crown 8vo. cloth. 10*s.* 6*d.*

A Treatise on the Integral Calculus. With numerous Examples. New Edition. Crown 8vo. cloth. 10*s.* 6*d.*

Examples of Analytical Geometry of Three Dimensions. New Edition. Crown 8vo. cloth. 4*s.*

An Elementary Treatise on Laplace's, Lamé's, and Bessel's Functions. Crown 8vo. 10*s.* 6*d.*

A History of the Mathematical Theory of Probability, from the time of Pascal to that of Laplace. 8vo. cloth. 18*s.*

The Conflict of Studies, and other Essays on Subjects connected with Education. 8vo. 10*s.* 6*d.*

William Whewell, D.D. Master of Trinity College, Cambridge. An account of his writings with selections from his literary and scientific correspondence. In Two Volumes. Demy 8vo. cloth. 25*s.*

MACMILLAN AND CO. LONDON.