16310

KEY

TO

MR J. B. LOCK'S

ELEMENTARY TRIGONOMETRY

MACMILLAN AND CO., Limited
LONDON · BOMBAY · CALCUTTA
MELBOURNE

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO
ATLANTA · SAN FRANCISCO

THE MACMILLAN CO. OF CANADA, Ltd.
TORONTO

KEY

TO

MR J. B. LOCK'S

ELEMENTARY TRIGONOMETRY

BY

HENRY CARR, B.A.

OF THE GRAMMAR SCHOOL, LAGOS, WEST AFRICA.

MACMILLAN AND CO., LIMITED
ST MARTIN'S STREET, LONDON
1910

First Edition 1889.
Reprinted 1893, 1896, 1900, 1904, 1910.

PREFACE.

THE solutions of the elementary questions are given by Mr Carr with considerable detail in the hope that the book may be useful to students who are studying the subject without the help of a Teacher.

I have myself read the proofs and have given solutions of many of the questions, particularly of those towards the end of the book.

Any notice of misprints or inaccuracies of any kind will be gratefully acknowledged either by the Publishers or myself.

J. B. LOCK.

KEY

TO

ELEMENTARY TRIGONOMETRY.

EXAMPLES. I. PAGE 2.

1. 1 mile = 5280 feet

$$=\frac{5280}{66}\times 66 \text{ feet};$$

$$\therefore \text{ the measure required is } \frac{5280}{66}=80.$$

2. The area of the square which is the unit is 22×22 square yards, i.e. 484 square yards.

Now 1 acre = 4840 square yards

$$=\frac{4840}{484}\times 484 \text{ sq. yds.}; \quad \therefore \text{ the measure required is } \frac{4840}{484}=10.$$

3. 1 ton = 2240 lbs.

$$=\frac{2240}{14}\times 14 \text{ lbs.}=\frac{2240}{14}\times 1 \text{ stone}=\frac{2240}{14\times 10}\times 10 \text{ stones};$$

$$\therefore \text{ the required measure is } \frac{2240}{14\times 10}=16.$$

4. Length of Atlantic cable is 2300 miles $=\frac{2300}{21}\times 21$ miles,

i.e. $\frac{2300}{21}\times$ length of cable from England to France;

$$\therefore \text{ the required measure is } \frac{2300}{21}=109\tfrac{11}{21}.$$

5. Area of the field is 22×1100 square yards,

$$\text{i.e. } \frac{22\times 1100}{4840}\times 4840 \text{ square yards}$$

$$=\frac{22\times 1100}{4840}\times 1 \text{ acre}=5\times 1 \text{ acre}=5 \text{ acres}.$$

6. a miles $= a \times 1760$ yards

$$= \frac{a \times 1760}{b} \times b \text{ yards; } \therefore \text{ the measure required is } \frac{1760a}{b}.$$

7. The given distance is ac feet $= \frac{ac}{3} \times 3$ feet

$$= \frac{ac}{3} \times 1 \text{ yard} = \frac{ac}{3} \text{ yards.}$$

8. The first *measure* is 24 and the first *unit* is half a sovereign;

$\therefore$ the amount is 24×10 shillings $= 240$ shillings.

To find the second unit proceed thus:—

the given amount is 240 shillings $= 240 \times 1$ shilling;

$\therefore$ the second unit is one shilling.

To find the third unit proceed thus:—

the given amount is 240 shillings $= 960 \times \frac{240}{960}$ shillings;

$\therefore$ the unit is $\frac{240}{960}$ shillings $= \frac{240 \times 12}{960}$ *d.*, i. e. three pence.

Or we may adopt the following solution:—

The student will observe that for a given quantity if the *unit* be increased or decreased the *measure* is decreased or increased proportionally; i. e. for a given quantity the measure and unit vary inversely.

Therefore the second unit $= \frac{24}{240}$ of 10 shillings $= 1s.$,

the third unit $= \frac{24}{960}$ of 10 shillings $= \frac{1}{4}s.$

EXAMPLES. II. Pages 4, 5.

1. $\frac{\text{Area of second field}}{\text{Area of first field}} = \frac{1}{20}.$

Area of second field $= \frac{1}{20} \times$ area of first field

$$= \frac{1}{20} \times \frac{1}{2} \text{ square mile} = \frac{1}{40} \text{ sq. ml.} = \frac{3{,}097{,}600}{40} \text{ sq. yds.}$$

$= 77440$ square yards.

2. $\frac{\text{Height of first person}}{\text{Height of second person}} = \frac{9}{8}.$

Height of first person $= \frac{9}{8} \times$ height of second person

$$= \frac{9}{8} \times 5 \text{ ft. } 4 \text{ in.} = \frac{9}{8} \times 64 \text{ in.} = 72 \text{ in.} = 6 \text{ ft.}$$

3. Area of the field $=65\times3$ acres $=195$ acres $=195\times1$ acre;

$$\therefore \text{ ratio to an acre} = 195.$$

4. The first field $=6\frac{3}{4}\times2\frac{1}{2}$ acres;

$\therefore$ the measure of the first field in terms of second is $6\frac{3}{4}$;

$$\text{the ratio is } \frac{6\frac{3}{4}\times2\frac{1}{2}}{2\frac{1}{2}}=\frac{27}{4};$$

$$\text{the required fraction is } \frac{6\frac{3}{4}\times2\frac{1}{2}}{2\frac{1}{2}}=\frac{27}{4}.$$

5. $$3{\cdot}125 \text{ tons} = 3{\cdot}125\times1 \text{ ton} = 3{\cdot}125\times20 \text{ cwts.}$$

$$=62{\cdot}5 \text{ cwts.} = \frac{62{\cdot}5}{4}\times4 \text{ cwts.};$$

$$\therefore \text{ the required measure is } \frac{62{\cdot}5}{4}=15{\cdot}625.$$

Since $$3{\cdot}125 \text{ tons} = \frac{62{\cdot}5}{4}\times4 \text{ cwts.},$$

$$\therefore\ 3{\cdot}125 \text{ tons contain 4 cwts. } \frac{62{\cdot}5}{4} \text{ times} = \frac{625}{40}=15\tfrac{5}{8};$$

the ratio is $$\frac{3{\cdot}125 \text{ tons}}{4 \text{ cwts.}} = \frac{\frac{62{\cdot}5}{4}\times 4 \text{ cwts.}}{4 \text{ cwts.}} = \frac{125}{8};$$

the required fraction is $$\frac{3{\cdot}125 \text{ tons}}{4 \text{ cwts.}} = \frac{\frac{62{\cdot}5}{4}\times 4}{4} = \frac{125}{8}.$$

6. $$\frac{\text{The sum of money}}{3 \text{ guineas}} = \frac{22}{7}.$$

$$\text{The sum of money} = \frac{22}{7}\times3 \text{ gs.} = \frac{66}{7}\times1 \text{ guinea}$$

$$=\frac{66}{7}\times21 \text{ shillings} = \frac{66\times21}{7}\times1 \text{ shilling} = \frac{66\times21}{7\times20}\times20s.$$

$$=\frac{66\times21}{7\times20}\times1 \text{ pound};$$

$$\therefore \text{ the measure required is } \frac{66\times21}{7\times20}=\frac{198}{20}=9\tfrac{9}{10}.$$

$$\text{The sum of money} = \frac{22}{7}\times3 \text{ guineas} = 22\times\frac{3}{7} \text{ guinea};$$

$$\therefore \text{ when the measure is 22, the unit is } \frac{3}{7} \text{ guinea} = \frac{3\times21}{7}s. = 9s.$$

7. a miles $=1760a$ yards $=\frac{1760a}{22}\times 22$ yards

$$=\frac{1760a}{22}\times 1 \text{ chain}=\frac{1760a}{22b}\times b \text{ chains};$$

$\therefore$ the required measure is $\frac{1760a}{22b}=\frac{80a}{b}$.

a miles $=1760a$ yards $=\frac{1760a}{22}\times 22$ yards

$$=\frac{1760a}{22}\times 1 \text{ chain}=\frac{1760a}{22c}\times c \text{ chains};$$

$\therefore$ the required number of times is $\frac{1760a}{22c}=\frac{80a}{c}$.

Similarly it may be shewn that a miles $=\frac{80a}{d}\times d$ chains;

$\therefore$ the required ratio $\frac{a \text{ miles}}{d \text{ chains}}=\frac{80a}{d}$.

It may be also shewn that a miles $=\frac{80a}{k}\times k$ chains;

$\therefore$ the required fraction is $\frac{a \text{ miles}}{k \text{ chains}}=\frac{80a}{k}$.

8. £20 $=400s.\times 1$ shilling $=\frac{400}{21}\times 21$ shillings;

$\therefore$ the required unit is 21 shillings.

£25 $=500s.=500\times 1$ shilling $=\frac{500}{21}\times 21$ shillings;

$\therefore$ the required sum is 21 shillings.

$$\frac{\text{£30}}{\text{a certain sum in pounds}}=\frac{600}{21},$$

$$\therefore \frac{\text{600 shillings}}{\text{a certain sum in shillings}}=\frac{600}{21};$$

$\therefore$ the required sum is 21 shillings.

Again, £10 or 200 shillings $=\frac{200}{21}\times$ a certain sum in shillings;

$\therefore$ the required sum is 21 shillings.

EXAMPLES. III. Pages 7, 8.

1. Let the hypotenuse be x feet,

then x^2 square feet $=36$ square feet $+\,64$ square feet (Eucl. I. 47),

$\therefore x^2=36+64,$

$\therefore x^2=100,$

$\therefore x=10;$

$\therefore$ the length of the hypotenuse is 10 feet.

2. Let x yds. be the required length.

$$100^2 \text{ sq. yds.} = 60^2 \text{ sq. yds.} + x^2 \text{ sq. yds.},$$

$$\therefore\ x^2 = 100^2 - 60^2 = (100+60)(100-60) = 160 \times 40 = 6400,$$

$$\therefore\ x = 80;\ \therefore \text{ the required length is 80 yards.}$$

3. In the figure of E. T. p. 6, let AD represent the pole, AC the rope, then the required length will be represented by DC.

$$AD = 48 \text{ feet};\ AC = 52 \text{ feet},\ CD = x \text{ feet}.$$

Now (Eucl. I. 47) $AC^2 = AD^2 + CD^2$,

$$\therefore\ 52^2 = 48^2 + x^2,\ \therefore\ x^2 = 52^2 - 48^2 = (52+48)(52-48) = 100 \times 4 = 400,$$

$$\therefore\ x = 20;\ \therefore \text{ required distance is 20 feet.}$$

4. Let AD represent the height of the houses, DC the width of the street, AC will represent the length of the ladder,

$$AD = 40 \text{ ft.},\ CD = 30 \text{ ft.},\ AC = x \text{ ft.}$$

Now by (Eucl. I. 47) $AC^2 = AD^2 + CD^2$,

$$\therefore\ x^2 = 40^2 + 30^2 = 2500,\ \therefore\ x = 50;$$

$$\therefore \text{ the required length of ladder is 50 ft.}$$

5. Let AD represent height of wall, CD width of the moat, and AC length of scaling ladder,

$$AD = 72 \text{ feet},\ CD = 54 \text{ feet},\ AC = x \text{ feet}.$$

Since $AC^2 = AD^2 + CD^2$, $x^2 = 72^2 + 54^2 = 8100$,

$$\therefore\ x = 90;\ \therefore \text{ required length is 90 feet.}$$

6. Let AD represent the length of the field, and DC the width, AC will represent the length of the path.

$$AD = \tfrac{1}{4} \text{ of a mile} = \tfrac{1}{4} \times 1760 \text{ yards} = 440 \text{ yards}.$$

$$DC = \tfrac{1}{16} \text{ of a mile} = \tfrac{3}{16} \times 1760 \text{ yards} = 330 \text{ yards}.$$

$$AC = x \text{ feet}.$$

Since $AC^2 = AD^2 + DC^2$, $\therefore\ x^2 = 440^2 + 330^2 = 302500$, $\therefore\ x = 550$;

$\therefore$ length of gravel path $= 550$ yards (nearly).

Width of gravel path $= 2$ ft. $= \frac{2}{3}$ yd.

Depth of gravel path $= 2$ inches $= \frac{1}{6}$ ft. $= \frac{1}{18}$ yd.

$\therefore$ volume of gravel path $= (550 \times \frac{2}{3} \times \frac{1}{18})$ cubic yards

$$= \tfrac{550}{27} \text{ cubic yards} = 20\tfrac{10}{27} \text{ cubic yards (nearly)}.$$

7. Let AD represent the length of the field $= 4a$ ft.,

DC represent the width of the field $= 3a$ ft.,

AC represent the diameter of the field $= x$ ft.

Since $AC^2 = AD^2 + DC^2$, $\therefore\ x^2 = (4a)^2 + (3a)^2 = 16a^2 + 9a^2 = 25a^2$,

$$\therefore\ x = 5a;\ \therefore \text{ required diameter is } 5a \text{ feet.}$$

8. In the figure ABC (E. T. p. 6). Let AB, AC be the equal sides, BC the base, AD the perpendicular from A on BC.

$$AD \text{ bisects } BC, \therefore BD = \tfrac{1}{2} BC,$$

$$AB = AC = 13a \text{ yards}, \; BD = \tfrac{1}{2}BC = 5a \text{ yards}, \; AD = x \text{ yards}.$$

Since $AB^2 = AD^2 + BD^2$, $\therefore (13a)^2 = x^2 + (5a)^2$,

$x^2 = 169a^2 - 25a^2 = 144a^2$, $\therefore x = 12a$; $\therefore$ the required length is $12a$ yards.

9. In the figure E. T. p. 7, let AD, AB represent the equal sides of the isosceles right-angled triangle, DB be the base; from the vertex A to base DB draw perpendicular AK, then $DK = KB = \frac{1}{2}DB$.

Let DB be x feet.

Then the square on DB = sum of squares on DA and AB.

$$x^2 \text{ sq. ft.} = a^2 \text{ sq. ft.} + a^2 \text{ sq. ft.}, \; \therefore x^2 = 2a^2, \; \therefore x = \sqrt{2} \cdot a = DB;$$

$$\therefore DK = \tfrac{1}{2}DB = \tfrac{1}{2}\sqrt{2} \cdot a \text{ ft.} = KB.$$

From Eucl. III. 25 it may be shewn that K is the centre of the circle circumscribing ADB; $\therefore KA = KD = KB = \frac{1}{2}\sqrt{2} \cdot a$ feet;

$\therefore$ the length of required perpendicular is $\frac{1}{2}\sqrt{2} \cdot a$ feet.

Or proceed thus:—Let $AK = x$ feet. Since AK is perpendicular to DB,

$$\therefore AD^2 = AK^2 + KD^2, \; a^2 = x^2 + (\tfrac{1}{2}\sqrt{2} \cdot a)^2 = x^2 + \tfrac{1}{2}a^2;$$

$$\therefore x^2 = \frac{a^2}{2} = \frac{2a^2}{4}; \quad \therefore x = \frac{\sqrt{2}a}{2}.$$

10. In the figure E. T. p. 7, let $DB = a$ feet, $AD = DB = x$ feet.

Since $DB^2 = AD^2 + AB^2$, $\therefore a^2$ sq. ft. $= x^2$ sq. ft. $+ x^2$ sq. ft.,

$$a^2 = 2x^2, \; \therefore 4x^2 = 2a^2, \; \therefore 2x = a\sqrt{2}; \; \therefore x = \tfrac{1}{2}\sqrt{2}a.$$

11. In the equilateral triangle ABC, let AD be drawn perpendicular on BC, $\therefore BD = DC = \frac{1}{2}BC$; $\therefore AB = a$ ft., $BD = \frac{1}{2}BC = \frac{1}{2}a$ ft., $AD = x$ ft.

Now $AB^2 = AD^2 + BD^2$,

$$\therefore a^2 \text{ sq. ft.} = x^2 \text{ sq. ft.} + \left(\frac{a}{2}\right)^2 \text{ sq. ft.},$$

$$a^2 = x^2 + \frac{a^2}{4}; \; \therefore x^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}; \; \therefore x = \frac{\sqrt{3}}{2} \cdot a;$$

$\therefore$ length of perpendicular is $\frac{1}{2}\sqrt{3} \cdot a$ ft.

12. In the equilateral triangle of (11) let $AD = a$, $AB = x$ feet, and $BD = \frac{1}{2}BC = \frac{1}{2}x$ feet. Now $AB^2 = AD^2 + BD^2$,

$$\therefore x^2 \text{ sq. ft.} = a^2 \text{ sq. ft.} + \left(\frac{x}{2}\right)^2 \text{ sq. ft.}; \; \therefore x^2 = a^2 + \frac{x^2}{4};$$

$$\therefore x^2 - \frac{x^2}{4} = a^2; \; \therefore \frac{3x^2}{4} = a^2; \; \therefore x^2 = \frac{4a^2}{3} = \frac{4 \cdot 3 \cdot a^2}{9}; \; \therefore x = \frac{2\sqrt{3}}{3} \cdot a;$$

$\therefore$ required length of side is $\frac{2}{3}\sqrt{3} \cdot a$.

13. It is obvious that the diameter of the circle coincides with the diameter of the inscribed square.

Let x = diameter of circle = diameter of inscribed square,

y = side of inscribed square; $\therefore\ x^2 = y^2 + y^2 = 2y^2$; $\therefore\ x = y\sqrt{2}$;

$$\therefore\ y = \frac{x}{\sqrt{2}} = \frac{\sqrt{2}}{2}\,.\,x.$$

$$\frac{\text{The side of inscribed square}}{\text{The diameter of the circle}} = \frac{y}{x} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}};\ \therefore \text{ the required ratio is } 1 : \sqrt{2}.$$

14. Let AB be the given chord, O the centre of the circle, let the perpendicular from O on AB bisect AB in C, then

$OC = x$ feet, $OA = OB = 10$ feet, $AC = CB = \frac{1}{2}AB = 4$ feet.

Now $OA^2 = AC^2 + CO^2$,

10^2 sq. ft. $= 4^2$ sq. ft. $+ x^2$ sq. ft.;

$\therefore\ x^2 = 10^2 - 4^2 = 100 - 16 = 84$;

$\therefore\ x = \sqrt{84}$; $\therefore$ the required distance is $\sqrt{84}$ ft.

15. With the notation of (14) $OA = a$ yds. $= 3a$ feet, $OC = b$ feet, $AB = x$ feet and $AC = \frac{1}{2}x$ feet. Now $OA^2 = AC^2 + CO^2$,

$\therefore\ (3a)^2$ sq. ft. $= \left(\frac{x}{2}\right)^2$ sq. ft. $+ b^2$ sq. ft., or $9a^2 = \frac{x^2}{4} + b^2$;

$\therefore\ x^2 = 36a^2 - 4b^2 = 4(9a^2 - b^2)$; $\therefore\ x = 2\sqrt{9a^2 - b^2}$;

$\therefore$ the required length is $2\sqrt{9a^2 - b^2}$ feet.

16. Let x be the common difference of the A. P.; then since the hypotenuse is the greatest side, the other two sides are $5a - x$ and $5a - 2x$.

Eucl. I. 47, $(5a - x)^2 + (5a - 2x)^2 = (5a)^2$;

$\therefore\ 50a^2 - 30ax + 5x^2 = 25a^2$; $\therefore\ x^2 - 6ax = -5a^2$.

From this quadratic $x = a$ or $5a$,

(1) when $x = a$, $5a - x = 5a - a = 4a$,

and $5a - 2x = 5a - 2a = 3a$;

(2) when $x = 5a$, $5a - x = 5a - 5a = 0$,

and $5a - 2x = 5a - 10a = -5a$.

In the first case the other two sides are $4a$ and $3a$. In the second case the other two sides are 0 and $-5a$, i.e. the triangle becomes a straight line; $\therefore$ only the first case is admissible.

EXAMPLES. IV. Page 13.

1. Let x be the number of feet in the diameter of the square.

Then x^2 sq. ft. $= 7^2$ sq. ft. $+ 7^2$ sq. ft., or $x^2 = 7^2 + 7^2$;

$\therefore\ x^2 = 2 \times 7^2$; $\therefore\ x = 7\sqrt{2} = 7 \times 1{\cdot}4142... = 9{\cdot}899...$ ft.;

$\therefore$ the length of the diameter is $9{\cdot}899...$ ft.

2. N.B. A yard is less than the thousandth part of a mile.

Let x be the number of miles in the diameter of the square.

Then x^2 sq. mi. $= 1^2$ sq. mi. $+ 1^2$ sq. mi., $\therefore$ x^2 sq. mi. $= 2$ sq. mi.,

$\therefore$ $x = \sqrt{2} = 1{\cdot}4142...$; $\therefore$ the length of the diameter is $1{\cdot}4142...$ miles

$= 1{\cdot}4142... \times 1760$ yards $= 2489$ yards.

3. Let x be the number of inches in the hypotenuse.

Then x^2 sq. inches $= (42\frac{1}{2})^2$ sq. in. $+ (40)^2$ sq. in.;

$$\therefore\ x^2 = \left(\frac{85}{2}\right)^2 + (40)^2 = \frac{7225}{4} + \frac{6400}{4} = \frac{13625}{4};$$

$$\therefore\ x = \tfrac{1}{2}\sqrt{13625} = \tfrac{1}{2} \times 116{\cdot}726 = 58{\cdot}36;$$

$\therefore$ the length of the hypotenuse is $58{\cdot}36$ inches.

4. Let x^2 be the number of square inches in the square,

x ,, ,, of inches in each of the sides;

$$\therefore\ x^2 = 1000 \times 9 \times 144 = 1296000; \quad \therefore\ x = 1138.$$

5. Let x be the number of feet in each of the sides,

$\therefore$ x^2 sq. ft. $=$ the area of the square and the length of the diameter is $x\sqrt{2}$,

$$x^2 = (4840 \times 9 \times 10) \text{ sq. ft.} = 435600 \text{ sq. ft.};$$

$$\therefore\ x = \sqrt{435600} = 660,$$

length of diameter $= x\sqrt{2}$ ft. $= 660 \times 1{\cdot}4142$ ft. $= 933{\cdot}372$ feet;

6. Let x^2 be the number of square inches in the square,

x ,, ,, inches in each of the sides,

x^2 sq. inches $= 1$ acre $= (1 \times 43560 \times 144)$ sq. in., or $x^2 = 43560 \times 144$,

$$\therefore\ x = \sqrt{43560 \times 144} = 12\sqrt{43560} = 12 \times 208{\cdot}7 = 2504;$$

$\therefore$ the length of side is 2504 inches.

7. Let ABC be the equilateral triangle, AD perpendicular on BC, therefore $BD = DC = \frac{1}{2}BC$.

$AD = x$ feet, $AB = 10$ ft., $BD = 5$ ft.,

10^2 sq. ft. $= x^2$ sq. ft. $+ 5^2$ sq. ft.; or $x^2 = 10^2 - 5^2 = 75$;

$\therefore$ $x = \sqrt{75} = 8{\cdot}66$ ft.; $\therefore$ height of the equilateral triangle is $8{\cdot}66$ ft.

8. x^2 sq. ft. $= (5{\cdot}32)^2$ sq. ft. $- (2{\cdot}66)^2$ sq. ft.;

$$\therefore\ x^2 = (5{\cdot}32)^2 - (2{\cdot}66)^2 = 7{\cdot}98 \times 2{\cdot}66 = 21{\cdot}2268;$$

$$\therefore\ x = \sqrt{21{\cdot}2268} = 4{\cdot}607;$$

$\therefore$ height of the equilateral triangle is $4{\cdot}607$ ft.

9. x^2 sq. in. $= (24{\cdot}6)^2$ sq. in. $+ (41{\cdot}3)^2$ sq. in.,

$$x^2 = (24{\cdot}6)^2 + (41{\cdot}3)^2 = 605{\cdot}16 + 1705{\cdot}69 = 2310{\cdot}85;$$

$\therefore$ $x = \sqrt{2310{\cdot}85} = 48{\cdot}07...$; $\therefore$ the length of diameter is 48 inches.

10. x^2 sq. ft. $= 78^2$ sq. ft. $+ 36^2$ sq. ft.,

$$x^2 = 78^2 + 36^2 = 6084 + 1296 = 7380;$$

$\therefore\ x = \sqrt{7380} = 85{\cdot}9$; $\therefore$ the length of diameter is $85{\cdot}9$ ft. $= 85$ ft. 11 in.

EXAMPLES. V. Pages 21, 22.

1. The circumference $= \pi \times$ diameter

$$= \frac{22}{7} \times 1 \text{ yard} = \frac{22}{7} \text{ yds.} = 3\tfrac{1}{7} \text{ yds.}$$

2. The circumference $= \pi \times$ diameter $= \pi \times 2 \times$ radius

$$= \pi \times 8 \text{ ft.} = \frac{22}{7} \times 8 \text{ ft.} = \frac{176}{7} \text{ ft.} = 25\tfrac{1}{7} \text{ ft.}$$

3. The circumference $= \pi \times$ diameter

$$= \pi \times 48 \text{ in.} = \frac{22}{7} \times 48 \text{ in.} = \frac{1056}{7} \text{ in.} = 150\tfrac{6}{7} \text{ in.}$$

4. $\pi \times$ diameter $=$ circumference $= 10$ ft.;

$$\therefore \text{ diameter} = \frac{10 \text{ ft.}}{\pi} = \frac{7}{22} \times 10 \text{ ft.} = \frac{70}{22} \text{ ft.} = 3\tfrac{2}{11} \text{ ft.}$$

5. The circumference $= 1$ revolution $= \dfrac{1 \text{ ml.}}{220} = \dfrac{1760 \text{ yds.}}{220} = 8$ yds.;

$\therefore\ \pi \times$ diameter $= 8$ yds.,

$$\text{diameter} = \frac{8 \text{ yds.}}{\pi} = \frac{7}{22} \times 8 \text{ yds.} = \frac{56}{22} \text{ yds.} = \frac{168}{22} \text{ ft.} = 7\tfrac{7}{11} \text{ ft.}$$

6. Here the diameter is 36 inches.

Circumference $= \pi \times$ diameter $= \dfrac{22}{7} \times 36$ in. $= \dfrac{792}{7}$ in. $= \dfrac{792}{7 \times 12}$ ft. $= \dfrac{66}{7}$ ft.

The no. of revolutions in a mile $= \dfrac{1 \text{ ml.}}{\text{circumference}} = \dfrac{5280 \times 7}{66} = 80 \times 7 = 560.$

7. The diameter of first wheel is 50 inches.

Now circumference $= \pi \times$ diameter;

$$\therefore \text{ circumference} = \frac{22}{7} \times 50 \text{ inches} = \frac{1100}{7} \text{ inches.}$$

The diameter of second wheel is 52 inches.

Circumference $= \pi \times$ diameter $= \dfrac{22}{7} \times 52$ in. $= \dfrac{1144}{7}$ inches.

The no. of revolutions in a mile made by first wheel

$$= \frac{1 \text{ ml.}}{\text{circumference}} = \frac{5280 \times 7}{1100},$$

similarly no. of revolutions by second wheel $= \dfrac{5280 \times 7 \times 12}{1144}$.

Difference of no. of times $= \dfrac{5280 \times 7 \times 12}{1100} - \dfrac{5280 \times 7 \times 12}{1144}$

$$= 5280 \times 7 \times 12 \left(\frac{1144 - 1100}{1144 \times 1100}\right) = \frac{5280 \times 7 \times 12 \times 44}{1144 \times 1100} = \frac{1008}{65} = 15\tfrac{33}{65} = 15\tfrac{1}{2} \text{ nearly.}$$

8. Here diameter is 5 feet. Circumf. $= \pi \times$ dia. $= \frac{22}{7} \times 5$ ft. $= \frac{110}{7}$ ft.

No. of revolutions in 100 miles $= \frac{100 \text{ mls.}}{\text{circumference}} = \frac{100 \text{ mls.} \times 7}{110 \text{ ft.}}$

$$= \frac{52800 \times 7}{110} = 33600.$$

9. In 1 sec. the engine makes 3 revolutions,

∴ in 60×60 secs. " " $3 \times 60 \times 60$ revolutions;

i. e. in 1 hour the engine makes 10800 revolutions.

From (8) engine makes 33600 revolutions in 100 miles,

∴ engine makes 336 " " 1 mile.

∴ engine makes 1 revolution in $\frac{1}{336}$ of a mile;

∴ engine makes 10800 revolutions in $\frac{10800}{336}$ miles;

i.e. engine " 10800 " in $32\frac{1}{7}$ miles;

but engine makes 10800 revolutions in an hour;

∴ the engine runs $32\frac{1}{7}$ miles per hour.

10. In 1 second the engine makes 4 revolutions,

in 60×60 seconds " " " $4 \times 60 \times 60$ revolutions;

i.e. in 1 hour " " " 14400 revolutions.

But in 1 hour the engine goes 60 miles;

∴ the engine makes 14400 revolutions in 60 miles,

the engine makes 1 revolution in $\frac{60}{14400}$ miles, i.e. in $\frac{60 \times 5280}{14400}$ feet $=22$ feet;

therefore the circumference of the engine is 22 ft.;

$\pi \times$ diameter $=$ circumference $= 22$ ft.;

$$\therefore \text{ the diameter} = \frac{22 \text{ ft.}}{\pi} = \frac{7}{22} \times 22 \text{ ft.} = 7 \text{ ft.}$$

11. The extremity of the hand of the clock travels round the face of the clock once in an hour, and therefore it makes 24 revolutions in a day.

The diameter here is 2×11 ft. $= 22$ ft.

Circumference $= \pi \times$ diameter $= \frac{22}{7} \times 22$ ft. $= \frac{484}{7}$ ft. $=$ length of one revolution;

$$24 \text{ revolutions} = \frac{484}{7} \times 24 \text{ ft.} = \frac{484 \times 24}{7 \times 3} \text{ yds.} = 553\tfrac{1}{7} \text{ yds.};$$

∴ the extremity of the large hand travels $553\frac{1}{7}$ yds. per day.

In 1 minute the extremity of the large hand makes $\frac{1}{60}$ of a revolution,

$$\text{i.e. } \frac{1}{60} \text{ of } \frac{484}{7} \text{ ft.} = \frac{484 \times 12}{7 \times 60} \text{ in.} = \frac{484}{35} \text{ in.} = 13{\cdot}8\ldots \text{ in.}$$

12. Circumference $=\pi\times$ diameter $=\frac{22}{7}\times 108$ ft. $=\frac{2376}{7}=339\frac{3}{7}$ ft.

13. It is shewn in Euc. IV. 17 that radius of a circle $=$ side of the inscribed regular hexagon;

$\therefore$ diameter of the circle $=2\times 3$ ft. $=6$ ft.

Circumference $=\pi\times$ diameter $=\frac{22}{7}\times 6$ ft. $=\frac{132}{7}$ ft. $=\frac{132\times 12}{7}$ in. $=226\text{·}3$ in.

Perimeter of hexagon $=6\times 3$ ft. $=18$ ft. $=18\times 12$ in. $=216$ in.;

$\therefore$ total length of wire $=(226\text{·}3+216)$ in. $=442\text{·}3$ in.,

hence 443 in. are required.

14. The radius of the circle $=$ side of inscribed hexagon;

$\pi\times$ diameter $=$ circumference,

diameter $=\frac{\text{circumference}}{\pi}=\frac{113\times 10}{355}$ ft. $=\frac{226}{71}$ ft.;

$\therefore$ radius $=\frac{113\times 5}{355}$ ft. $=\frac{113}{71}$ ft. Perimeter of hexagon $=\frac{113}{71}$ ft. $\times 6=\frac{678}{71}$ ft.;

total length of wire required $=10$ ft. $+\frac{678}{71}$ ft.

$\therefore$ no. of inches necessary $=120$ in. $+115$ in. $=235$ in.

15. The diameter of the circle coincides with the diameter of the inscribed square. Let $x=$ diameter of square $=$ diameter of circle;

then, x^2 sq. in. $=24^2$ sq. in. $+24^2$ sq. in. or $x^2=24^2+24^2=2\times 24^2$;

$\therefore x=24\sqrt{2}=24\times 1\text{·}414=33\text{·}9$ in.

Circumference $=\pi\times$ diameter $=\frac{355}{113}\times 33\text{·}9$ in. $=107$ in.

Perimeter of square $=4\times 24$ in. $=96$ in.;

$\therefore$ total length of wire required $=96$ in. $+107$ in. $=203$ in.

16. The diameter of the circle coincides with diameter of the inscribed square. Also $\pi\times$ diameter $=$ circumference;

diameter $=\frac{\text{circumference}}{\omega}=\frac{113}{355}\times 12$ ft. $=\frac{113\times 144}{355}$ in. $=\frac{16272}{355}$ in. $=45\text{·}8\ldots$ in.

Let $x=$ side of inscribed square,

then $x\sqrt{2}=$ diameter of inscribed square $=$ diameter of circle $=46$ in.;

$\therefore\ x=\frac{45\text{·}8\ldots}{\sqrt{2}}$ in. $=\frac{45\text{·}8\ldots\sqrt{2}}{2}$ in. $=23\text{·}9\ldots$ in. $\times 1\text{·}4142\ldots=32\text{·}4\ldots$ in.;

$\therefore$ perimeter of square $=4\times 32\text{·}4\ldots$ in. $=130$ in. nearly;

$\therefore$ total length of wire $=144$ in. $+130$ in. $=274$ in. nearly.

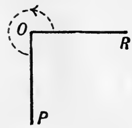
17. Circumference $=\pi\times$ diameter $=\frac{22}{7}\times\frac{5}{4}$ in. $=\frac{55}{14}$ in.

Area of surface of handle of bat $=\left(\frac{55}{14}\times 12\right)$ sq. in. $=\frac{330}{7}$ sq. in.

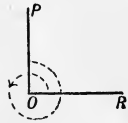
Length of string $=\frac{\text{area of surface of handle}}{\text{diameter of string}}=\frac{330}{7}\div\frac{1}{40}=\frac{330\times 40}{7}=\frac{13200}{7}$

$=1885\text{·}7\ldots$ in., $\therefore$ 1886 in. are required.

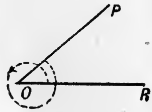
EXAMPLES. VI. Page 25.



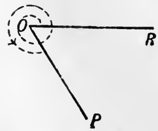
1. If OP starts from the position OR and turns about O in the direction contrary to that of the hands of the clock to the position OP (so that the angle ROP is a right angle) it will describe the angle equal to $+3$ right angles.



2. If OP turning in the direction contrary to that of the hands of the clock starts from OR, makes about the point O *one* complete revolution (i.e. four right angles) and then turns in the same direction to the position OP, so that the angle ROP is equal to a right angle, it will describe $+5$ right angles.

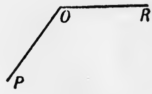


3. If OP turning in the direction contrary to that of the hands of the clock starts from OR, makes about the point O *one* complete revolution (i.e. four right angles) and then turns in the same direction to the position OP so that the angle ROP is equal to half a right angle (i.e. 45^0), it will describe $+4\frac{1}{2}$ right angles.

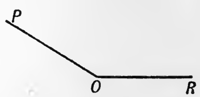


4. If OP turning in the direction contrary to that of the hands of the clock starts from OR, makes about the point O *one* complete revolution, and then turns in the same direction to the position OP so as to make the angle ROP equal to four right angles *less* $\frac{3}{4}$ of a right angle, OP will describe $+7\frac{1}{4}$ right angles.

5. In the Figure of **1**, if OP turning in the same direction as that of the hands of the clock starts from OR and turns about the point O to the position OP so that the angle ROP is equal to a right angle, OP will describe -1 right angle.



6. If OP turning in the direction contrary to that of the hands of the clock starts from OR, makes *two* complete revolutions (i.e. eight right angles) about the point O, and then turns in the same direction to the position OP so that the angle ROP is equal to $+2$ right angles and 60 degrees, OP will describe $+10\frac{2}{3}$ right angles.



7. If OP turning in the same direction as that of the hands of the clock starts from OR, makes about the point O two complete revolutions and then turns in the negative direction to the position OP so that the angle ROP is equal to -210^0, OP will describe $-10\frac{1}{3}$ right angles.

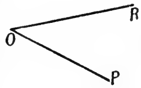
8. If OP turning in the direction contrary to that of the hands of the clock starts from OR makes *one* complete revolution about the point O, so as to be again in the position OR from which it started, it will describe $+4$ right angles.

9. If OP turning in the same direction as that of the hands of the clock, starts from OR and makes about the point O *one* complete revolution, so as to be again in the position OR from which it started, it will describe -4 right angles.

10. n represents any whole number positive or negative. $4n$ will always be divisible by 4, and $4n$ right angles will represent n complete revolutions. If therefore OP turning in the positive or negative direction, starts from the position OR and makes about the point O, n complete revolutions, so as to be again in the position OR from which it started, it will describe $4n$ right angles.

11. From **10**, when OP has described $4n$ right angles it will be in the position OR from which it started. From this position let OP turn about the point O in the positive direction to the position OP so that the angle ROP is equal to $+2$ right angles; OP will thus have described $(4n+2)$ right angles.

12. $-(4n+\frac{1}{2})$ right angles $= -4n$ right angles $+(-\frac{1}{2})$ of a right angle. When OP has described $-4n$ right angles it will again be in the position OR from which it started (**10**). From this position let OP turn about the point O in the negative direction to the position OP, so that the angle ROP is equal to half a right angle (45^0); OP will thus describe $-(4n+\frac{1}{2})$ right angles.



EXAMPLES. VII. Page 28.

1. $63^g\,21'\,18'' = \frac{63}{100} + \frac{21}{10000} + \frac{18}{1000000}$ of a right angle
$= \cdot632118$ of a right angle.

2. $104^g\,26'\,99\cdot1'' = \frac{104}{100} + \frac{26}{10000} + \frac{99\cdot1}{1000000}$ of a right angle
$= 1\cdot0426991$ of a right angle.

3. $2^g\,18'\,27'' = \frac{2}{100} + \frac{18}{10000} + \frac{27}{1000000}$ of a right angle
$= \cdot021827$ of a right angle.

4. $3^g\,29'\,48\cdot94'' = \frac{3}{100} + \frac{29}{10000} + \frac{4894}{100000000}$ of a right angle
$= \cdot03294894$ of a right angle.

5. $62'\,41'' = \frac{62}{10000} + \frac{41}{1000000}$ of a right angle
$= \cdot006241$ of a right angle.

6. $1000^g\,8'\,12'' = \frac{1000}{100} + \frac{8}{10000} + \frac{12}{100000}$ of a right angle
$= 10\cdot000812$ of a right angle.

7. $32^g\,4'\,5\cdot2'' = \frac{32}{100} + \frac{4}{10000} + \frac{52}{10000000}$ of a right angle
$= \cdot3204052$ of a right angle.

8. $1^g\ 2'\ 3{\cdot}4'' = \frac{1}{100} + \frac{2}{10000} + \frac{34}{10000000}$ of a right angle
$= {\cdot}0102034$ of a right angle.

9. $69^g\ 0'\ 7{\cdot}1'' = \frac{69}{100} + \frac{71}{10000000}$ of a right angle
$= {\cdot}6900071$ of a right angle.

10. $119^g\ 3'\ 0{\cdot}45'' = \frac{119}{100} + \frac{3}{10000} + \frac{45}{100000000}$ of a right angle
$= 1{\cdot}19030045$ of a right angle.

11. $1006^g\ 18'\ 1'' = \frac{1006}{100} + \frac{18}{10000} + \frac{1}{1000000}$ of a right angle
$= 10{\cdot}061801$ of a right angle.

12. $2^g\ 26'\ 4{\cdot}8'' = \frac{2}{100} + \frac{26}{10000} + \frac{48}{10000000}$ of a right angle
$= {\cdot}0226048$ of a right angle.

13. ·36 of a right angle = 36 grades,
·00,78 ,, ,, ,, = 78 minutes,
·00,00,91 ,, ,, ,, = 91 seconds;
∴ the angle is $36^g\ 78'\ 91''$.

14. 1 right angle = 100 grades,
·04 of a right angle = 4 grades,
·00,30 ,, ,, ,, = 30 minutes,
·00,00,21 ,, ,, ,, = 21 seconds;
∴ the angle is $104^g\ 30'\ 21''$.

15. ·01 of a right angle = 1 grade,
·00,20 ,, ,, ,, = 20 minutes,
·00,00,03 ,, ,, ,, = 3 seconds;
∴ the angle is $1^g\ 20'\ 3''$.

16. ·00,10 of a right angle = 10 minutes,
·00,00,2 ,, ,, ,, = 2 seconds;
∴ the angle is $10'\ 2''$.

17. ·06 of a right angle = 6 grades,
·00,25 ,, ,, ,, = 25 minutes;
∴ the angle is $6^g\ 25'$.

18. 3 right angles = 300 grades,
·02 of a right angle = 2 grades,
·00,12 ,, ,, ,, = 12 minutes;
00,00,50 ,, ,, ,, = 50 seconds;
∴ the angle is $302^g\ 12'\ 50''$.

19. 1 right angle = 100 grades,
·00,10 of a right angle = 10 minutes;
∴ the angle is $100^g\ 10'$.

20. ·01 of a right angle = 1 grade,
·00,01 ,, ,, ,, = 1 minute,
·00,00,001 ,, ,, ,, = ·1 second;
∴ the angle is $1^g\ 1'\ 0{\cdot}1''$.

21. 6 right angles = 600 grades,
·45 of a right angle = 45 grades,
·00,10 ,, ,, ,, = 10 minutes;
∴ the angle is $645^g\ 10'$.

22. ·02 of a right angle = 2 grades,
·00,30 ,, ,, ,, = 30 minutes;
∴ the angle is $2^g\ 30'$.

23. ·00,01 of a right angle = 1 minute,
·00,00,10 ,, ,, ,, = 10 seconds;
∴ the angle is $1'\ 10''$.

24. ·00,00,10 of a right angle = 10 seconds;
∴ the angle is $10''$.

EXAMPLES. VIII. PAGE 30.

1. 60) 27 seconds
60) 15·45 minutes
90) 8·2575 degrees
·09,17,5 of a right angle
$= 9^g\ 17'\ 50''$.

2. 60) 30 seconds
60) 4·5 minutes
90) 6·075 degrees
·06,75 of a right angle
$= 6^g\ 75''$.

3. 60) 15 seconds
60) 5·25 minutes
90) 97·0875 degrees
1·07,87,5 right angle
$= 107^g\ 87'\ 50''$.

4. 60) 19 seconds
60) 14·31666666... minutes
90) 16·23861111... degrees
$\cdot 18,04,29\dot{0}1234567\dot{9}$ of a right angle
$= 18^g\ 4'\ 29''$...

5. 60) 6 minutes
90) 132·1 degrees
$1{\cdot}46\dot{7}$ right angle
$= 146^g\ 77'\ 77{\cdot}7''$...,

6. 90) 49 degrees
$\cdot 5\dot{4}$ of a right angle
$= 54^g\ 44'\ 44{\cdot}4''$,...

7. $1^g\,37'\,50'' = \cdot01375$ of a right angle

$$\begin{array}{rl} & 90 \\ \hline 1\cdot23750 & \text{degrees} \\ 60 & \\ \hline 14\cdot2500 & \text{minutes} \\ 60 & \\ \hline 15\cdot000 & \text{seconds} \end{array}$$

∴ the result is $1^\circ\,14'\,15''$.

8. $8^g\,75' = \cdot0875$ of a right angle

$$\begin{array}{rl} & 90 \\ \hline 7\cdot8750 & \text{degrees} \\ 60 & \\ \hline 52\cdot5000 & \text{minutes} \\ 60 & \\ \hline 30\cdot0000 & \text{seconds} \end{array}$$

∴ the result is $7^\circ\,52'\,30''$.

9. $170^g\,45'\,35'' = 1\cdot704535$ of a right angle

$$\begin{array}{rl} & 90 \\ \hline 153\cdot408150 & \text{degrees} \\ 60 & \\ \hline 24\cdot489000 & \text{minutes} \\ 60 & \\ \hline 29\cdot340000 & \text{seconds} \end{array}$$

∴ the result is $153^\circ\,24'\,29\cdot34''$.

10. $24^g\,0'\,25'' = \cdot240025$ of a right angle

$$\begin{array}{rl} & 90 \\ \hline 21\cdot602250 & \text{degrees} \\ 60 & \\ \hline 36\cdot135000 & \text{minutes} \\ 60 & \\ \hline 8\cdot10000 & \text{seconds} \end{array}$$

∴ the result is $21^\circ\,36'\,8\cdot1''$.

11. $18^g\,1'\,15'' = \cdot180115$ of a right angle

$$\begin{array}{rl} & 90 \\ \hline 16\cdot210350 & \text{degrees} \\ 60 & \\ \hline 12\cdot621000 & \text{minutes} \\ 60 & \\ \hline 37\cdot260000 & \text{seconds} \end{array}$$

∴ the result is $16^\circ\,12'\,37\cdot26''$.

12. $35^g = \cdot35$ of a right angle

$$\begin{array}{rl} & 90 \\ \hline 31\cdot5 & \text{degrees} \\ 60 & \\ \hline 30\cdot0 & \text{minutes} \end{array}$$

∴ the result is $31^\circ\,30'$.

EXAMPLES. IX. PAGE 35.

1. (1) $\pi \text{ radians} = \pi \times \dfrac{2 \text{ right angles}}{\pi} = 2 \text{ right angles} = 2 \times 90^\circ = 180^\circ;$

∴ the angle contains 180°.

(2) $\dfrac{3\pi}{4} \text{ radians} = \dfrac{3\pi}{4} \times \dfrac{2 \text{ right angles}}{\pi} = \dfrac{6}{4}$ of a right angle

$= \dfrac{3}{2}$ of a right angle $= \dfrac{3}{2} \times 90^\circ = 135^\circ$; ∴ the angle contains 135°.

(3) $1 \text{ radian} = \dfrac{2 \text{ right angles}}{\pi} = \dfrac{180^\circ}{\pi} = 57^\circ\cdot2957\ldots;$

∴ the angle contains $57^\circ\cdot2957\ldots$

(4) $3 \text{ radians} = 3 \times \dfrac{2 \text{ right angles}}{\pi} = \dfrac{6}{\pi}$ right angles;

∴ the angle contains $\dfrac{6}{\pi}$ right angles.

(5) 3·14159265 etc. radians

$$=\pi \text{ radians}=\pi\times\frac{2 \text{ right angles}}{\pi}=2 \text{ right angles}=180°;$$

$\therefore$ the angle contains 180°.

(6) $\frac{2}{\pi}$ radians $=\frac{2}{\pi}\times\frac{2 \text{ right angles}}{\pi}=\frac{4}{\pi^2}$ right angles;

$\therefore$ the angle contains $\frac{4}{\pi^2}$ right angles.

(7) θ radians $=\theta\times\frac{2 \text{ right angles}}{\pi}=\frac{2\theta}{\pi}$ right angles;

$\therefore$ the angle contains $\frac{2\theta}{\pi}$ right angles.

(8) ·00314159 etc. radians

$$=\frac{\pi}{1000} \text{ radians}=\frac{\pi}{1000}\times\frac{2 \text{ right angles}}{\pi}=\frac{2}{1000} \text{ right angles}=\cdot 002 \text{ right angles.}$$

(9) 10π radians $=10\pi\times\frac{2 \text{ right angles}}{\pi}=20$ right angles;

$\therefore$ the angle contains 20 right angles.

2. Let a be the circular measure of the given angle: then

(1) $\frac{a^c}{\pi^c}=\frac{180°}{180°}=1$; $\therefore a^c=\pi^c$; $\therefore \pi$ is the required circular measure.

(2) $\frac{a^c}{\pi^c}=\frac{360°}{180°}=2$; $\therefore a^c=2\pi^c$; $\therefore 2\pi$ is the required circular measure.

(3) $\frac{a^c}{\pi^c}=\frac{60°}{180°}=\frac{1}{3}$; $\therefore a^c=\frac{\pi^c}{3}$; $\therefore \frac{\pi}{3}$ is the required circular measure.

(4) $\frac{a^c}{\pi^c}=\frac{22\frac{1}{2}°}{180°}=\frac{45°}{360°}=\frac{1}{8}$; $\therefore a^c=\frac{\pi^c}{8}$; $\therefore \frac{\pi}{8}$ is the required circular measure.

(5) $\frac{a^c}{\pi^c}=\frac{1°}{180°}=\frac{1}{180}$; $\therefore a^c=\frac{\pi^c}{180}$;

$\therefore \frac{\pi}{180}$ is the required circular measure.

(6) $\frac{a^c}{\pi^c}=\frac{57\cdot295° \text{ etc.}}{180°}=\frac{2 \text{ right angles}}{\pi}\times\frac{1}{180°}=\frac{180°}{\pi}\times\frac{1}{180°}$

$=\frac{1}{\pi}\times\frac{180°}{180°}=\frac{1}{\pi}$; $\therefore a^c=\frac{\pi^c}{\pi}=1^c$; $\therefore$ 1 is the circular measure required.

(7) $\frac{a^c}{\pi^c}=\frac{n°}{180°}=\frac{n}{180}$; $\therefore a^c=\frac{n\pi^c}{180}$;

$\therefore \frac{n}{180}\pi^c$ is the circular measure required.

(8) $\frac{\alpha^c}{\pi^c}=\frac{90^\circ}{\pi}\times\frac{1}{180^\circ}=\frac{1}{2\pi}$; $\therefore\ \alpha^c=\frac{\pi^c}{2\pi}=\frac{1^c}{2}=\frac{1}{2}^c$;

$\therefore$ $\frac{1}{2}$ is the circular measure required.

(9) $\frac{\alpha^c}{\pi^c}=\frac{A^\circ}{180^\circ}=\frac{A}{180}$; $\therefore\ \alpha^c=\frac{A\pi^c}{180}$;

$\therefore\ \frac{A\pi}{180}$ is the circular measure required.

3. (1) $33^g\ 33'\ 33{\cdot}\dot{3}''=33{\cdot}333333\ldots$ grades $=33{\cdot}\dot{3}$ grades.

Let α be the required circular measure,

$$\frac{\alpha^c}{\pi^c}=\frac{33{\cdot}\dot{3}^g}{200^g}=\frac{33{\cdot}\dot{3}}{200}=\frac{100}{3}\times\frac{1}{200}=\frac{1}{6};\quad\therefore\ \alpha^c=\frac{\pi^c}{6};$$

$\therefore\ \frac{\pi}{6}$ is the circular measure required.

(2) $\frac{\alpha^c}{\pi^c}=\frac{50^g}{200^g}=\frac{1}{4}$; $\therefore\ \alpha^c=\frac{\pi^c}{4}$; $\therefore\ \frac{\pi}{4}$ is the circular measure required.

(3) $\frac{\alpha^c}{\pi^c}=\frac{16{\cdot}\dot{6}^g}{200^g}=\frac{16{\cdot}\dot{6}}{200}=\frac{50}{3}\times\frac{1}{200}=\frac{1}{12}$; $\therefore\ \alpha^c=\frac{\pi^c}{12}$;

$\therefore\ \frac{\pi}{12}$ is the circular measure required.

(4) $\frac{\alpha^c}{\pi^c}=\frac{1^g}{200^g}=\frac{1}{200}$; $\therefore\ \alpha^c=\frac{\pi^c}{200}$; $\therefore\ \frac{\pi}{200}$ is the circular measure required.

(5) $\frac{\alpha^c}{\pi^c}=\frac{1'}{20000'}=\frac{1}{20000}$; $\therefore\ \alpha^c=\frac{\pi^c}{20000}$;

$\therefore\ \frac{\pi}{20000}$ is the circular measure required.

(6) $\frac{\alpha^c}{\pi^c}=\frac{10''}{2000000''}=\frac{1}{200000}$; $\therefore\ \alpha^c=\frac{\pi^c}{200000}$;

$\therefore\ \frac{\pi}{200000}$ is the circular measure required.

(7) $\frac{\alpha^c}{\pi^c}=\frac{n^g}{200^g}=\frac{n}{200}$; $\therefore\ \alpha^c=\frac{n\pi^c}{200}$;

$\therefore\ \frac{n\pi}{200}$ is the circular measure required.

(8) $\frac{\alpha^c}{\pi^c}=\frac{200^g}{\pi}\times\frac{1}{200^g}=\frac{1}{\pi}$; $\therefore\ \alpha^c=\frac{\pi^c}{\pi}=1^c$;

$\therefore$ 1 is the circular measure required.

(9) $\frac{\alpha^c}{\pi^c}=\frac{1000^g}{200^g}=5$; $\therefore\ \alpha^c=5\pi^c$; $\therefore\ 5\pi$ is the circular measure required.

4. (1) $180° = \pi^c$; $\therefore\ 1° = \frac{\pi^c}{180}$; $\therefore\ 45° = \frac{\pi^c}{180} \times 45 = \frac{\pi}{4}$;

$$\therefore\ 45° \div \frac{3\pi}{4} = \frac{\pi}{4} \div \frac{3\pi}{4} = \frac{\pi}{4} \times \frac{4}{3\pi} = \frac{1}{3};\ \therefore \text{ the required ratio is } \frac{1}{3}.$$

(2) If D and G be the number of degrees and grades respectively in any angle,

$$\frac{D}{180} = \frac{G}{200};\ \therefore\ \frac{60°}{180°} = \frac{G}{200} = \frac{1}{3};\ \therefore\ G = \frac{200}{3} \text{ the number of grades in } 60°;$$

$$60° \div 60^g = \frac{200^g}{3} \div 60^g = \frac{200}{60 \times 3} = \frac{10}{9};\ \therefore \text{ the required ratio is } \frac{10}{9}.$$

(3) $\frac{D}{180} = \frac{25^g}{200^g} = \frac{1}{8}$; $\therefore\ D = \frac{180°}{8} = \frac{45}{2}$ (i.e. the no. of degrees in 25^g);

$$25^g \div 22°\ 30' = 25^g \div 22\tfrac{1}{2}° = \frac{45°}{2} \div \frac{45°}{2} = 1;\ \therefore \text{ the required ratio is } 1.$$

(4) $\frac{G}{200} = \frac{\alpha}{\pi}$; $\therefore\ \frac{24}{200} = \frac{\alpha}{\pi}$; $\therefore\ \alpha = \frac{24\pi}{200}$ (i.e. the number of radians in 24^g);

$$24^g \div 2^c = \frac{24\pi}{200} \div 2 = \frac{24\pi}{200} \times \frac{1}{2} = \frac{3\pi}{50};\ \therefore \text{ the required ratio is } \frac{3\pi}{50}.$$

(5) $\frac{D}{180} = \frac{1{\cdot}75^c}{\pi^c} = \frac{175}{100\pi}$;

$$\therefore\ D = \frac{175 \times 180}{100\pi} \text{ (i.e. the number of degrees in } 1{\cdot}75^c);$$

$$\therefore\ 1{\cdot}75^c \div \frac{100°}{\pi} = \frac{175 \times 180°}{100\pi} \div \frac{100°}{\pi} = \frac{175 \times 180 \times \pi}{100\pi \times 100} = \frac{63}{20};$$

$$\therefore \text{ the required ratio is } \frac{63}{20}.$$

(6) $\frac{\alpha}{\pi} = \frac{1}{180}$; $\therefore\ \alpha = \frac{\pi}{180}$ (i.e. number of radians in $1°$);

$$1° \div 1^c = \frac{\pi}{180} \div 1 = \frac{\pi}{180};\ \therefore \text{ the required ratio is } \frac{\pi}{180}.$$

EXAMPLES. X. PAGES 37—39.

1. The angle $= \frac{\text{arc}}{\text{radius}} \times (\text{a radian}) = \frac{37\frac{1}{2}}{25}$ radians $= \frac{75}{2} \times \frac{1}{25}$ radians $= 1\tfrac{1}{2}^c$.

2. The angle $= \frac{\text{arc}}{\text{radius}} \times (\text{a radian}) = \frac{5\pi}{10} \times \frac{2 \text{ right angles}}{\pi} = \frac{5\pi}{10} \times \frac{180°}{\pi} = 90°$.

3. The angle $= \frac{\text{arc}}{\text{radius}} \times (\text{a radian}) = \frac{2 \text{ ft.}}{3\frac{2}{11} \text{ inches}} \times (\text{a radian})$

$$= \frac{24}{3\frac{2}{11}} \times \frac{2 \text{ right angles}}{\pi} = \frac{24}{1} \times \frac{11}{35} \times \frac{7}{22} \times 2 \text{ right angles} = \frac{168}{35} \text{ right angles}$$

$$= 4\tfrac{4}{5} \text{ right angles}.$$

4. The angle $=\frac{1 \text{ in.}}{8 \text{ ft. } 4 \text{ in.}} \times (\text{a radian}) = \frac{1}{100} \times \frac{20000 \text{ French minutes}}{\pi}$

$$=\frac{1}{100} \times \frac{7}{22} \times 20000 \text{ French minutes} = \frac{700}{11} \text{ French minutes.}$$

5. The angle $=\frac{\text{arc}}{\text{radius}} \times (\text{a radian})$. Let x = length of arc;

$$\therefore 4\tfrac{1}{2} \text{ radians} = \frac{x}{25}; \quad \therefore x = \frac{9}{2} \times 25 = \frac{225}{2} = 112\tfrac{1}{2} \text{ ft.}$$

6. The angle $=\frac{\text{arc}}{\text{radius}} \times (\text{a radian})$,

$$\text{i.e. } 80° = \frac{x}{4} \times \frac{180°}{\pi} = \frac{x}{4} \times \frac{7}{22} \times 180° = \frac{315x}{22}; \quad \therefore x = \frac{80° \times 22}{315} = 5\tfrac{37}{63};$$

$\therefore$ length of arc required is $5\frac{37}{63}$ ft.

7. Let x be the length of the arc.

The angle $=\frac{\text{arc}}{\text{radius}} \times (\text{a radian})$, i.e. $60^g = \frac{x}{10} \times \frac{200^g}{\pi} = \frac{x}{10} \times \frac{7}{22} \times 200^g = \frac{70x}{11}$;

$$\therefore x = \frac{66}{7} = 9\tfrac{3}{7}; \quad \therefore \text{ length of arc required is } 9\tfrac{3}{7} \text{ ft.}$$

8. The diameter of the sun can be here represented as approaching the magnitude of the arc subtended by an angle of 32′ in the centre of the circle of radius 90,000,000 miles.

Let x represent length of arc.

Then $\frac{x}{90,000,000} \times \frac{2 \text{ right angles}}{\pi} = 32'$; $\frac{7x}{90,000,000} \times \frac{180 \times 60}{22} = 32'$;

$21x = 17,600,000$; $\therefore x = 831,095$ miles; $\therefore$ length of arc is 838,095 miles.

But the diameter of the sun coincides with this arc to three significant figures; hence the diameter of the sun is equal to about 838,000 miles.

9. The train in 1 hour travels 20 miles,

" " " 1 second " $\frac{20}{60 \times 60}$ miles,

" " " 10 seconds " $\frac{20 \times 10}{60 \times 60}$ miles, i.e. $\frac{1}{18}$ of a mile.

Now number of radians $=\frac{\text{arc}}{\text{radius}} = \frac{1}{18} \times 2 = \frac{1}{9}$.

$\therefore$ the angle $=\frac{1}{9}$ of a radian;

$$\therefore \text{ the angle} = \frac{1}{9} \times \frac{2 \text{ right angles}}{\pi} = \frac{7 \times 180°}{9 \times 22} = \frac{70°}{11} = 6\tfrac{4}{11}°.$$

10. The train in 1 hour travels 60 miles,

,, ,, ,, 1 minute ,, 1 mile,

,, ,, ,, $\frac{1}{4}$ minute ,, $\frac{1}{4}$ of a mile.

Now $\frac{\text{arc}}{\text{radius}}$ = number of radians, $\frac{1}{4} \times \frac{3}{2} = \frac{3}{8}$;

$\therefore$ the angle is $\frac{3}{8} \times \frac{180°}{\pi} = \frac{3 \times 180° \times 7}{8 \times 22} = \frac{945°}{44} = 21\frac{21}{44}°$.

11. The angle $= \frac{\text{arc}}{\text{radius}} \times$ (a radian)

$$= \frac{1}{4000} \times \frac{180°}{\pi} = \frac{1}{4000} \times \frac{180 \times 60 \times 60}{\pi} \text{ seconds}$$

$$= \frac{7 \times 180 \times 60 \times 60}{4000 \times 22} \text{ secs.} = \frac{1134}{22} \text{ secs.} = 51\tfrac{6}{11}''.$$

12. The angle $= \frac{\text{arc}}{\text{radius}} \times$ (a radian). Let x represent length of arc,

$$\text{i.e. } 1'' = \frac{x}{4000} \times \frac{180°}{\pi} \times 60 \times 60;$$

$$\therefore x = \frac{4000 \times 22}{180 \times 60 \times 60} \text{ miles} = \frac{11}{567} \text{ miles} = \frac{11 \times 1760}{567} \text{ yds.} = \frac{19360}{567} \text{ yds.}$$

$= 33\cdot95$ yds., i.e. 34 yds. nearly.

13. Let π be the required ratio.

The angle $= \frac{\text{arc}}{\text{radius}} \times \frac{180°}{\pi}$, i.e. $3° = \frac{\cdot6545}{12\cdot5} \times \frac{180°}{\pi}$;

$\therefore \pi = \frac{180 \times \cdot6545}{3 \times 12\cdot5} = 3\cdot1416$; $\therefore$ the required ratio is $3\cdot1416$.

14. The angle $= \frac{\text{arc}}{\text{radius}} \times \frac{180°}{\pi}$, i.e. $7\frac{1}{2}° = \frac{1\cdot309}{10} \times \frac{180°}{\pi}$;

$$\therefore \pi = \frac{1\cdot309 \times 12}{5} = \frac{15\cdot708}{5} = 3\cdot1416.$$

15. Let π be the required ratio.

Then the angle $= \frac{\text{arc}}{\text{radius}} \times$ (a radian), i.e. $22\frac{1}{2}° = \frac{31\frac{5}{12}}{80} \times \frac{180°}{\pi}$;

$$\therefore \pi = \frac{377 \times 180 \times 2}{12 \times 80 \times 45} = \frac{37\cdot7}{12} = 3\cdot141\dot{6}.$$

16. The diameter of the moon approximates in magnitude to the arc subtended by an angle of 30′ at the centre of a circle of which the radius is equal to the distance of the eye of the observer from the centre of the moon; similarly the diameter of the sun approximates in magnitude to the arc subtended by an angle of 32′ at the centre of a circle of which the radius is the distance of the eye of the observer from the centre of the sun.

The ratio of these arcs will be then equal nearly to the ratio of the diameters.

Let x_1 be the length of arc subtended by angle of 30′,

x_2 " " " " " " 32′,

r_1 be the distance of the observer's eye and the centre of the moon,

$375r_1$ " " " " " " " " sun.

Then, since $\text{arc}=\dfrac{\text{angle}}{\text{2 right angles}}\times\pi\times\text{radius}$,

$$\therefore \frac{x_1}{x_2}=\frac{30'}{32}\times\frac{r_1}{375r_1}=\frac{1}{400};$$

$\therefore$ the ratio of the arcs is $\dfrac{1}{400}$ and so is the ratio of the diameters.

17. $180^\circ=\pi$ radians, $180\times60\times60$ seconds $=\pi$ radians;

$$\therefore 1 \text{ second}=\frac{\pi}{180\times60\times60} \text{ of a radian};$$

$$\therefore 10 \text{ seconds}=\frac{10\pi}{180\times60\times60} \text{ of a radian}=\frac{355}{113\times180\times360} \text{ of a radian}$$

$$=\frac{355}{7322400}=\cdot0000484\ldots \text{ radian}.$$

18. The two places evidently lie on a great circle of the globe.

The angle contained by the difference of their latitudes

$$=\frac{\text{arc of great circle}}{\text{radius}}\times(\text{a radian});$$

$$1\tfrac{1}{6}^\circ=\frac{1 \text{ inch}}{r}\times\frac{180^\circ}{\pi};\quad \therefore\ r=\frac{1080}{22}=49\tfrac{1}{11} \text{ inches}.$$

19. It is evident that the centre of the smaller circle is in the middle point of the perpendicular of the isosceles triangle; and its radius is half of the perpendicular. The radius of the greater circle is the perpendicular.

Now let θ be the number of radians in the vertical angle; then

arc of greater circle within the triangle $=2h\theta$, when $2h$ is the height of the triangle;

arc of smaller circle without the triangle $=h(2\pi-2\theta)$.

But the two arcs are equal;

$$\therefore 2h\theta=h(2\pi-2\theta);\quad \therefore 2\theta=\pi;\quad \therefore \text{the vertical angle is } \frac{\pi}{2}.$$

20. Each of the angles in the triangle ABC (fig. Eucl. I. 1) is equal to 60°.

$$\text{The angle } BAC = \frac{\text{arc } BC}{\text{radius } AB} \times \text{a radian}; \quad \therefore \text{ radian} = \frac{\text{radius } AB}{\text{arc } BC} \times 60°.$$

Let r represent AB; $BC = AB = r$. Let d represent the difference between arc BC and BC, so that arc $BC = r + d$; for the shortest distance between two points is the straight line joining them; hence the arc BC is greater than the straight line BC.

$$\therefore \text{ radian} = \frac{r}{r+d} \times 60°,$$

i.e. the radian (unit of circular measure) is less than 60°.

21. The magnitude of the diameter of the sun is constant; the change is therefore due to the approach of the sun to the observer; i.e. to the variation of the radius of the circle of which the centre is the eye of the observer, and on the circumference of which is the diameter of the sun; this diameter may be regarded as being nearly equal in magnitude to the arc it cuts off.

Let the two radii be r_1, r_2, and the arc or diameter of the sun be d;

$$\frac{\text{arc}}{\text{radius}} \times (\text{a radian}) = 32'\,36'', \quad \therefore \frac{d}{r_1} \times (\text{a radian}) = 32'\,36'',$$

$$\therefore r_1 = \frac{d \times (\text{a radian})}{32'\,36''}. \quad \text{Similarly } r_2 = \frac{d \times (\text{a radian})}{31'\,32''};$$

$$\therefore r_1 : r_2 = \frac{1}{32'36''} : \frac{1}{31'32''} = \frac{1}{1956} : \frac{1}{1892} = 1892 : 1956 = 473 : 489.$$

22. From Art. 71 it is seen that $\frac{a}{r}$ is the measure of the angle where the unit is a radian; for this angle to be represented by $\frac{\kappa \cdot a}{r}$, κ must depend solely on the unit.

(i) In order that $\frac{\kappa \cdot a}{r}$ may be the measure when the unit is a radian, since the angle $\frac{a}{r} \times (\text{a radian})$, $\therefore \kappa = 1$.

$$\text{(ii)} \quad \frac{a}{r} \times (\text{a radian}) = \frac{a}{r} \times \frac{180°}{\pi} = \frac{a}{r} \times \frac{180}{\pi} \times (\text{a degree}).$$

In order that $\frac{\kappa \cdot a}{r}$ may be the measure of this angle when a degree is unit,

$$\frac{\kappa \cdot a}{r} = \frac{a}{r} \times \frac{180}{\pi}, \text{ i.e. } \kappa = \frac{180}{\pi}.$$

EXAMPLES. XI. Pages 40—43.

1. Let the angle contain x right angles.

In x right angles there are $90x$ degrees,

" " " " " $100x$ grades;

$\therefore\ 90x+100x=38,\ \therefore\ 190x=38;$

$\therefore\ x=\frac{1}{5}$. The angle is $\frac{1}{5}$ of a right angle, i.e. $\frac{1}{5}\times\frac{\pi}{2}=\frac{\pi}{10}$.

2. Let the angles contain x and y right angles respectively, then their difference is $100(x-y)$ grades; their sum is $90(x+y)$ degrees;

$$100(x-y)=20,\quad 90(x+y)=48;$$

i.e. $x-y=\frac{1}{5}$; $x+y=\frac{8}{15}$. By addition $2x=\frac{11}{15}$;

$\therefore\ x=\frac{11}{30}$ of a right angle $=\frac{11}{30}\times 90^\circ=33^\circ$.

Now $x+y=\frac{8}{15}$;

$\therefore\ y=\frac{8}{15}-x=\frac{8}{15}-\frac{11}{30}=\frac{1}{6}$ of a right angle; $\therefore\ y=\frac{1}{6}\times 90^\circ=15^\circ$;

$\therefore$ the angles are 33° and 15°.

3. Let the 1st angle contain $2x$ right angles, i.e. $180x$ degrees,

" " 2nd " " x " " i.e. $100x$ grades;

$\therefore\ 180x+100x=140;$

$\therefore\ x=\frac{1}{2}$ of a right angle $=45^\circ$, $\therefore\ 2x=2\times 45^\circ=90^\circ$;

$\therefore$ the angles are 90° and 45°.

4. Let the angles contain $4x$ and $5x$ right angles respectively,

i.e. $400x$ grades and $450x$ degrees respectively;

$\therefore\ 450x-400x=2\frac{1}{2};\quad \therefore\ 50x=2\frac{1}{2};$

$\therefore\ x=\frac{1}{20}$ of a right angle,

$4x=\frac{1}{5}$ of a right angle $=18^\circ$, $5x=\frac{1}{4}$ of a right angle $=22\frac{1}{2}^\circ$;

$\therefore$ the angles are 18° and $22\frac{1}{2}^\circ$.

5. Let the angles contain x and y right angles respectively,

i.e. $\frac{\pi x}{2}$ and $\frac{\pi y}{2}$ radians respectively, and $90x$ and $90y$ degrees respectively;

$$\frac{\pi x}{2}-\frac{\pi y}{2}=\frac{\pi}{9};$$

$\therefore\ 9(x-y)=2,\ \therefore\ 90(x-y)=20.$ But $90(x+y)=56$;

$\therefore$ by addition $180x=76$, $x=\frac{76}{180}$ of a right angle $=\frac{76}{180}\times 90^\circ=38^\circ$;

$90x+90y=56$, $90y=56-38=18$, $y=\frac{18}{90}$ of a right angle $=18^\circ$;

$\therefore$ the angles are 38° and 18°.

6. Let the angles contain $x-y$, x, $x+y$ right angles respectively; they are in A. P.

Their sum is $3x$ right angles; but since they are the angles of a triangle their sum is 2 right angles;

$$\therefore\ 3x=2,\quad x=\frac{2}{3}\text{ of a right angle}=60^\circ.$$

7. Let the angles contain $x-y$, x, $x+y$ right angles respectively; they are in A. P.

From 6 it is seen that $x=60^\circ$.

The number of grades in $(x-y)$ right angles is $100(x-y)$;

$$\therefore\ 100(x-y) : 60 :: 5 : 6;$$

$$600(x-y)=300,\quad x-y=\tfrac{1}{2}\text{ of a right angle};$$

$$\therefore\ x-y=45^\circ,\quad 60^\circ-y=45^\circ;\quad \therefore\ y=15^\circ;$$

$$\therefore\ x+y=60^\circ+15^\circ=75^\circ;$$

$\therefore$ the angles are 75°, 60°, 45° respectively.

8. Let the angles contain $x-y$, x, $x+y$ right angles respectively,

i.e. $90(x-y)$ degrees, $90x$ degrees, $100(x+y)$ grades, respectively;

$$100(x+y) : 90x+90(x-y) :: 10 : 11,$$

$$\text{i.e. } 1100(x+y)=1800x-900y;$$

$$\therefore\ 7x=20y,\quad \therefore\ y=\frac{7}{20}x.$$

But from Example 7, $x=60^\circ$;

$$\therefore\ y=\frac{7}{20}\times 60^\circ=21^\circ,\quad x-y=60^\circ-21^\circ=39^\circ,\quad x+y=60^\circ+21^\circ=81^\circ;$$

$\therefore$ the three angles are 39°, 60°, 81° respectively.

9. Let the angles contain $x-y$, x, $x+y$ right angles respectively,

i.e. $100(x-y)$ grades, $90x$ degrees, and $(x+y)\frac{\pi}{2}$ radians, respectively;

$$100(x-y) : (x+y)\frac{\pi}{2} :: 200 : 3\pi;$$

$$\therefore\ 3(x-y)=x+y,\quad x=2y,\quad y=\frac{x}{2}.$$

But from Example 7, $x=\frac{2}{3}$ of a right angle;

$$\therefore\ y=\left(\frac{2}{3}\times\frac{1}{2}=\frac{1}{3}\right)\text{ of a right angle.}$$

$$x-y=\left(\frac{2}{3}-\frac{1}{3}=\frac{1}{3}\right)\text{ of a right angle}=\frac{1}{3}\times 100^g=33\cdot\dot{3}^g,$$

$$x=\frac{2}{3}\text{ of a right angle}=\frac{2}{3}\times 100^g=66\cdot\dot{6}^g,$$

$$x+y=\left(\frac{2}{3}+\frac{1}{3}\right)\text{ of a right angle}=100^g;$$

$\therefore$ the angles are $33\cdot\dot{3}^g$, $66\cdot\dot{6}^g$, 100^g, respectively.

10. $\frac{D}{90}=\frac{G}{100}$; $\therefore\ G=\frac{100}{90}D=\left(1+\frac{1}{9}\right)D=D+\frac{1}{9}D$; $\therefore\ G-D=\frac{1}{9}D.$

11. $\frac{M}{90\times 60}=\frac{m}{100\times 100}$; $\therefore\ M=\frac{9\times 6}{10\times 10}m=\frac{27}{50}m$; $\therefore\ \frac{50}{27}M=m$;

$$\therefore\ \frac{54}{27}M-\frac{4}{27}M=m,\quad 2M-\frac{4}{27}M=m,\quad 2M-m=\frac{4}{27}M.$$

12. From Art. 70 (El. Trig.), $\frac{G}{200}=\frac{D}{180}=\frac{C}{\pi}$.

By proportion if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$; $\therefore\ \frac{a-c}{b-d}=\frac{e}{f}$;

$$\therefore\ \frac{G-D}{200-180}=\frac{C}{\pi};\quad \therefore\ G-D=\frac{20C}{\pi}.$$

13. Let a given angle contain M English minutes, and m French minutes;

$$\therefore\ \frac{M}{90\times 60}=\frac{m}{100\times 100},\quad M=\frac{9\times 6}{10\times 10}m=\frac{54}{100}m,\quad M=\cdot 54m;$$

that is, by multiplying French minutes by ·54 we have English minutes.

14. Let one part contain x right angles and the other part y right angles;

$\therefore$ both parts contain $90\,(x+y)$ degrees;

$$90\,(x+y)=33\tfrac{1}{10}°,\quad \therefore\ 900\,(x+y)=331.$$

In x right angles there are $(x\times 90\times 60\times 60)$ English seconds,

In y " " " $(y\times 100\times 100\times 100)$ French seconds;

$$x\times 90\times 60\times 60=y\times 100\times 100\times 100,$$

$$\text{that is, } 81x=250y;\quad \therefore\ y=\frac{81}{250}x.$$

Substitute this value of y in the equation $900\,(x+y)=331$;

$$\therefore\ 900\left(x+\frac{81x}{250}\right)=331;$$

$$\therefore\ x=\frac{5}{18}\text{ of a right angle}=\frac{5}{18}\times 90°=25°,$$

$$y=\frac{81}{250}x=\frac{81}{250}\times 25°=\frac{81°}{10}=8°\ 6';$$

$\therefore$ the two parts are 25° and 8° 6′.

15. $12^g\ 50' = \cdot 125$ of a right angle.

60) 27 minutes
90) 9·45 degrees
·105 of a right angle

$$\text{Ratio}=\frac{\cdot 105}{\cdot 125}=\frac{105}{125}=\frac{21}{25}.$$

16. Let a be the number of radians;

$$\therefore\ \frac{a}{\pi}=\frac{n}{180\times 60},\quad a=\frac{n\pi}{180\times 60}=\frac{n\pi}{10800}.$$

17. It appears from Euclid I. 32, that all the interior angles of any rectilineal figure, together with four right angles are equal to twice as many right angles as the figure has sides.

Hence if n be the number of sides of any rectilineal figure, we have the sum of its n angles $+ 90° \times 4 = 90° \times 2n$; or the sum of its n angles

$$= (2n-4) \times 90° = (n-2) \times 180°.$$

If the figure be a regular polygon, its n angles are all equal and consequently each of them is $\frac{n-2}{n} \times 180°$.

(i) In the regular hexagon $n=6$;

$$\therefore \text{ each angle} = \frac{6-2}{6} \times 180° = 120° = \frac{6-2}{6} \times \pi = \frac{2\pi}{3}$$

$$= \frac{6-2}{6} \times 200^g = \frac{400^g}{3} = 133^g\ 33^`\ 33 \cdot \dot{3}''.$$

(ii) In the regular octagon $n=8$;

$$\therefore \text{ each angle} = \frac{8-2}{8} \times 180° = 135° = \frac{8-2}{8} \times \pi = \frac{3\pi}{4} = \frac{8-2}{8} \times 200^g = 150^g.$$

(iii) In the regular quindecagon $n=15$;

$$\therefore \text{ each angle} = \frac{15-2}{15} \times 180° = 156° = \frac{15-2}{15} \times \pi = \frac{13\pi}{15} = \frac{15-2}{15} \times 200^g = \frac{520^g}{3}$$

$$= 173 \cdot \dot{3}^g.$$

18. In the regular decagon $n=10$;

$$\therefore \text{ each angle} = \frac{10-2}{10} \times 180^g = 144°.$$

In the regular pentagon $n=5$;

$$\therefore \text{ each angle} = \frac{5-2}{5} \times 200° = 120^g.$$

$$\text{Ratio} = \frac{144}{120} = \frac{6}{5}.$$

19. From (**18**) the number of grades in an angle of the regular pentagon is 120; from (**17**, i) the number of degrees in an angle of the regular hexagon is 120; $\therefore$ they are equal.

20. Each angle of a regular polygon of 48 sides

$$= \left(\frac{48-2}{48} \times 180 \times 60\right) \text{ minutes (English)} = 10350 \text{ English minutes};$$

2 right angles $= 108 \times 60$ English minutes $= 10800$.

Difference $= 10800 - 10350 = 450$ English minutes.

21. From (**17**, iii) the angle of a regular quindecagon is 156°. But the exterior angle of the quindecagon together with the adjacent angle are together equal to 180°; $\therefore$ the exterior angle $= 180° - 156° = 24°$.

Let x be the required measure;

$$\therefore \text{ (i) } 90° = x \times 24°, \quad \therefore x = \frac{90}{24} = 3\tfrac{3}{4};$$

$$\text{(ii) } \frac{180°}{\pi} = x \times 24°, \quad \therefore x = \frac{180}{24\pi} = \frac{15}{2\pi}.$$

22. Let x be the measure of 1 degree, y the unit in degrees;

$$\therefore\ 1^\circ = x\,.\,y,\ \therefore\ 1^g = \frac{9x}{10}\,.\,y;\quad \therefore\ x + \frac{9x}{10} = 1,\ \therefore\ x = \frac{10}{19}.$$

$$\text{But } 1^\circ = x\,.\,y,\ \therefore\ 1^\circ = \frac{10}{19}\,.\,y,\ \therefore\ y = \frac{19^\circ}{10};$$

$$\therefore\ \text{the required unit is } \frac{19^\circ}{10}.$$

23. Let x be the measure of 1 degree, y the unit in degrees;

$$\therefore\ 1^\circ = xy^\circ;\quad 9^\circ = 9x\,.\,y^\circ;\quad 5^g = \frac{9x}{2}\,.\,y^\circ;\quad 9x + \frac{9x}{2} = \frac{3}{20};\quad \therefore x = \frac{1}{90}.$$

$$\text{But } 1^\circ = x\,.\,y^\circ,\ \therefore\ 1^\circ = \frac{1}{90}\,.\,y^\circ,\ \therefore\ y^\circ = 90^\circ,$$

$$\therefore\ \text{the required unit is } 90^\circ.$$

24. Let $1^\circ = x\,.\,y^\circ$, $\therefore\ c^\circ = cx\,.\,y^\circ$ and $b^g = \dfrac{9bx}{10}\,.\,y^\circ$,

$$\text{but } \frac{9bx}{10} = a,\quad \therefore\ x = \frac{10a}{9b},\quad \therefore\ cx = \frac{10ac}{9b} = \text{measure of } c^\circ.$$

25. Let $1^\circ = x\,.\,y^\circ$, $\therefore\ b^\circ = bx\,.\,y^\circ$, $a^g = \dfrac{9ax}{10}\,.\,y^\circ$,

$$\frac{9ax}{10} + bx = c,\quad \therefore\ x = \frac{10c}{9a+10b}.$$

$$\text{But } 1^\circ = x\,.\,y^\circ,\ \therefore\ y^\circ = \frac{1^\circ}{x} = \frac{9a+10b}{10c}\text{ degrees},$$

$$\therefore\ \text{the required unit is } \frac{9a+10b}{10c}\text{ degrees.}$$

26. Let the angle contain x right angles, i.e. $100\,x$ grades or $90x$ degrees,

$$100x - 90x = \frac{18}{\pi},\ \therefore\ x = \frac{9}{5\pi}\text{ of a right angle.}$$

Let y be the measure of a right angle when $\dfrac{9}{5\pi}$ of a right angle is the unit.

$$1\text{ right angle} = y \times \frac{9}{5\pi}\text{ of a right angle},\quad \therefore\ y = \frac{5\pi}{9}.$$

$$\therefore\ \text{the required measure is } \frac{5\pi}{9}.$$

27. Let x be the number. The three angles are x°, x^g, $x^\circ + x^g$ respectively, i.e. x°, $\dfrac{9x^\circ}{10}$, $\dfrac{19x^\circ}{10}$ respectively.

Since they are the angles of a triangle, $x^\circ + \dfrac{9x^\circ}{10} + \dfrac{19x^\circ}{10} = 180$,

$$\therefore\ x = \frac{1800^\circ}{38} = \frac{1800}{38} \times \frac{\pi}{180}\text{ radians} = \frac{5\pi}{19};$$

$$\frac{9x^\circ}{10} = \frac{5\pi}{19} \times \frac{9}{10} = \frac{9\pi}{38}.\qquad \frac{19x^\circ}{10} = \frac{5\pi}{19} \times \frac{19}{10} = \frac{\pi}{2}.$$

$$\therefore\ \text{the angles are } \frac{5\pi}{19},\ \frac{9\pi}{38},\ \frac{\pi}{2}.$$

28. Let A, B, C be the angles of the triangle, and u the measure; so that

$$A = u \text{ degrees} = u \times 1 \text{ degree}, \quad B = u \text{ grades} = \frac{9u}{10} \text{ degrees} = \frac{9u}{10} \times 1 \text{ degree},$$

$$C = u \text{ radians} = \frac{180u}{\pi} \text{ degrees} = \frac{180u}{\pi} \times 1 \text{ degree}.$$

Since they are the angles of a triangle they are equal to two right angles

$$180° = 180 \times 1 \text{ degree}; \quad \therefore \ u + \frac{9u}{10} + \frac{180u}{\pi} = 180, \quad \therefore \ u = \frac{1800\pi}{19\pi + 1800}.$$

29. Let the angles of the triangle be x degrees, $10x$ grades and $100x$ radians respectively; i.e. x degrees,

$$\frac{10x \times 9}{10} \text{ or } 9x \text{ degrees, and } \frac{100x \times 180}{\pi} \text{ degrees respectively.}$$

Of these x degrees is the smallest; and the sum of the three is equal to two right angles;

$$\therefore \ x + 9x + \frac{18000x}{\pi} = 180; \quad \therefore \ x° = \frac{180\pi}{10\pi + 18000} \text{ degrees}$$

$$= \left(\frac{180\pi}{10\pi + 18000} \times \frac{\pi}{180}\right) \text{ radians} = \frac{\pi^2}{10\pi + 18000} \text{ radians.}$$

30. Let n be the number of sides of the polygon, and $\therefore$ the number of terms of the A. P. is n;

$$\text{the sum of the series is } \frac{n}{2}\{2 \times 120° + (n-1)5\}.$$

But from (17) the sum of the angles of the polygon

$$= (n-2) \times 180°, \quad \therefore \ \frac{n}{2}\{2 \times 120 + (n-1)5\} = (n-2) \times 180°.$$

From this quadratic $n = 16$ or 9.

Both these values are admissible upon trial.

EXAMPLES XII. Pages 50—52.

1. (i) Angle of reference being ABD:

DA is the *perpendicular*, for it is opposite to ABD and is perpendicular to BD.

BD is the *base*, for it is adjacent to the angle ABD and to the right angle.

(ii) Angle of reference being BAD:

DB is the *perpendicular*, for it is opposite to BAD and is perpendicular to AD.

AD is the *base*, for it is adjacent to the angle BAD and to the right angle.

(iii) Angle of reference being ACD:

DA is the *perpendicular*, for it is opposite to ACD and is perpendicular to CD.

CD is the *base*, for it is adjacent to the angle ACD and to the right angle.

(iv) Angle of reference being DAC:

DC is the *perpendicular*, for it is opposite to DAC and is perpendicular to AD.

AD is the *base*, for it is adjacent to the angle DAC and to the right angle.

2. (i) $\sin BAD = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{DB}{AB}$, (ii) $\cos ACD = \frac{\text{base}}{\text{hypotenuse}} = \frac{CD}{CA}$,

(iii) $\tan DAC = \frac{\text{perpendicular}}{\text{base}} = \frac{DC}{DA}$, (iv) $\sin ABD = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{DA}{BA}$,

(v) $\tan BAD = \frac{\text{perpendicular}}{\text{base}} = \frac{DB}{DA}$, (vi) $\sin DAC = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{DC}{AC}$,

(vii) $\cos DCA = \frac{\text{base}}{\text{hypotenuse}} = \frac{DC}{CA}$, (viii) $\tan DCA = \frac{\text{perpendicular}}{\text{base}} = \frac{DC}{DA}$,

(ix) $\cos ABD = \frac{\text{base}}{\text{hypotenuse}} = \frac{BD}{BA}$, (x) $\sin ACD = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{DA}{CA}$.

3. It is seen from Euc. VI. 8 that the triangles ABC, BDC and BDA are similar to one another.

(i) In the right-angled triangle ABC, $\sin ACB = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{BA}{CA}$.

In the right-angled triangle BDC, $\sin ACB = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{DB}{CB}$;

$\therefore$ two values are $\frac{BA}{CA}$ and $\frac{DB}{CB}$.

(ii) In the right-angled triangle ABC, $\cos ACB = \frac{\text{base}}{\text{hypotenuse}} = \frac{CB}{CA}$.

In the right-angled triangle BDC, $\cos ABC = \frac{\text{base}}{\text{hypotenuse}} = \frac{CD}{CB}$;

$\therefore$ two values are $\frac{CB}{CA}$ and $\frac{CD}{CB}$.

(iii) In the right-angled triangle ABC, $\tan ACB = \frac{\text{perpendicular}}{\text{base}} = \frac{BA}{CB}$.

In the right-angled triangle BDC, $\tan ACB = \frac{\text{perpendicular}}{\text{base}} = \frac{DB}{CD}$;

$\therefore$ two values are $\frac{BA}{CB}$ and $\frac{DB}{CD}$.

(iv) In the right-angled triangle ABC, $\sin BAC = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{BC}{AC}$.

In the right-angled triangle BDA, $\sin BAC = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{DB}{AB}$;

$\therefore$ two values are $\frac{DB}{AB}$ and $\frac{BC}{AC}$.

(v) In the right-angled triangle ABC, $\cos BAC = \frac{\text{base}}{\text{hypotenuse}} = \frac{AB}{AC}$.

In the right-angled triangle BDA, $\cos BAC = \frac{\text{base}}{\text{hypotenuse}} = \frac{AD}{AB}$;

$\therefore$ two values are $\frac{AD}{AB}$ and $\frac{AB}{AC}$.

(vi) In the right-angled triangle ABC, $\tan BAC=\frac{BC}{AB}$.

In the right-angled triangle BDA, $\tan BAC=\frac{DB}{AD}$;

$$\therefore \text{ two values are } \frac{DB}{AD} \text{ and } \frac{BC}{AB}.$$

4. (i) $\sin BDA=\frac{DA}{BA}$ for DBA is a right angle.

(ii) $\sin BEA=\frac{BA}{EA}$ for ABE is a right angle

$=\frac{AC}{EC}$ for CAE is a right angle.

(iii) $\sin CBD=\frac{DC}{BC}$.

(iv) $\cos BAE=\frac{AB}{AE}$.

(v) $\cos BAD=\frac{AD}{AB}=\frac{AB}{AC}$.

(vi) $\cos CBD=\frac{BD}{BC}$.

(vii) $\tan BCD=\frac{DB}{CD}=\frac{BA}{CB}=\frac{AE}{CA}$.

(viii) $\tan DBA=\frac{DA}{DB}$.

(ix) $\tan BEA=\frac{BA}{EB}=\frac{AC}{EA}$.

(x) $\tan CBD=\frac{DC}{BD}$.

(xi) $\sin DAB=\frac{DB}{AB}=\frac{BC}{AC}$

(xii) $\sin BAE=\frac{BE}{AE}$.

5. In the triangle ABC CB is *perpendicular*, AC is *base* and AB *hypotenuse* when A is the angle of reference;

$$\therefore \sin A=\frac{\text{perpendicular}}{\text{hypotenuse}}=\frac{CB}{AB}=\frac{3\text{ ft.}}{5\text{ ft.}}=\frac{3}{5}.$$

$$\therefore \cos A=\frac{\text{base}}{\text{hypotenuse}}=\frac{AC}{AB}=\frac{4\text{ ft.}}{5\text{ ft.}}=\frac{4}{5}.$$

$$\therefore \tan A=\frac{\text{perpendicular}}{\text{base}}=\frac{CB}{AC}=\frac{3\text{ ft.}}{4\text{ ft.}}=\frac{3}{4}.$$

When B is the angle of reference, CA is *perpendicular*, BC is *base* and BA is *hypotenuse;*

$$\therefore \sin B=\frac{\text{perpendicular}}{\text{hypotenuse}}=\frac{CA}{BA}=\frac{4\text{ ft.}}{5\text{ ft.}}=\frac{4}{5}.$$

$$\therefore \cos B=\frac{\text{base}}{\text{hypotenuse}}=\frac{BC}{BA}=\frac{3\text{ ft.}}{5\text{ ft.}}=\frac{3}{5}.$$

$$\therefore \tan B=\frac{\text{perpendicular}}{\text{base}}=\frac{CA}{BC}=\frac{4\text{ ft.}}{3\text{ ft.}}=\frac{4}{3}.$$

$$\sin^2 A + \cos^2 A = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = 1,$$

$$\sin^2 B + \cos^2 B = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = 1.$$

6. From (5) $\sin A = \dfrac{CB}{AB} = \dfrac{a}{c}$, $\cos A = \dfrac{AC}{AB} = \dfrac{b}{c}$, $\tan A = \dfrac{CB}{AC} = \dfrac{a}{b}$.

From Euc. I. 47 $\qquad AC^2 + CB^2 = AB^2$,

$$\therefore \frac{AC^2}{AB^2} + \frac{CB^2}{AB^2} = 1, \qquad \therefore \left(\frac{AC}{AB}\right)^2 + \left(\frac{CB}{AB}\right)^2 = 1,$$

$$\therefore \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1, \qquad \therefore \sin^2 A + \cos^2 A = 1.$$

(i) Since $\dfrac{a}{c} = \sin A$, $\therefore a = c \sin A$.

(ii) ,, $\dfrac{CA}{BA} = \dfrac{b}{c} = \sin B$, $\therefore b = c \sin B$.

(iii) ,, $\dfrac{BC}{BA} = \dfrac{a}{c} = \cos B$, $\therefore a = c \cos B$.

(iv) ,, $\dfrac{AC}{AB} = \dfrac{b}{c} = \cos A$, $\therefore b = c \cos A$.

(v) $\sin A = \dfrac{CB}{AB} = \cos B$. $\qquad$ (vi) $\cos A = \dfrac{AC}{AB} = \sin B$.

(vii) $\tan A = \dfrac{CB}{AC} = \cot B$.

7. Let ABC be the right-angled triangle; C the right angle.

$$\sin A = \frac{BC}{AB} = \frac{5}{13}. \quad \cos A = \frac{AC}{AB} = \frac{12}{13}. \quad \tan A = \frac{BC}{AC} = \frac{5}{12}.$$

$$\sin B = \frac{AC}{AB} = \frac{12}{13}. \quad \cos B = \frac{BC}{AB} = \frac{5}{13}. \quad \tan B = \frac{AC}{BC} = \frac{12}{5}.$$

8. Let ABC be the right-angled triangle; C the right angle.

$$\sin A = \frac{BC}{AB} = \frac{1}{2}. \quad \cos A = \frac{AC}{AB} = \frac{\sqrt{3}}{2}. \quad \tan A = \frac{BC}{AC} = \frac{1}{\sqrt{3}}.$$

$$\sin B = \frac{AC}{AB} = \frac{\sqrt{3}}{2}. \quad \cos B = \frac{BC}{AB} = \frac{1}{2}. \quad \tan B = \frac{AC}{BC} = \sqrt{3}.$$

9. In (7) $\sin^2 A + \cos^2 A = \left(\dfrac{5}{13}\right)^2 + \left(\dfrac{12}{13}\right)^2 = \dfrac{25}{169} + \dfrac{144}{169} = 1.$

$$\sin^2 B + \cos^2 B = \left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2 = \frac{144}{169} + \frac{25}{169} = 1.$$

In (8) $\sin^2 A + \cos^2 A = \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{1}{4} + \dfrac{3}{4} = 1.$

$$\sin^2 B + \cos^2 B = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1.$$

EXAMPLES. XIII. Page 60.

1. $2 \cdot \sin D \cdot \cos D = 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1 = \sin 90^\circ = \sin A.$

2. $2 \cdot \sin C \cdot \cos C = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} = \sin 60^\circ = \sin B.$

3. $\cos^2 B - \sin^2 B = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}, \quad 1 - 2\sin^2 B = 1 - \frac{3}{2} = -\frac{1}{2};$

$$\therefore \cos^2 B - \sin^2 B = 1 - 2\sin^2 B.$$

4. $\sin B \cos C + \sin C \cos B = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = 1 = \sin 90^\circ = \sin A.$

5. $\cos^2 D - \sin^2 D = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = 0 = \cos 90^\circ = \cos A.$

6. $4 \cdot \sin^2 E + 2 \cdot \sin E = 4\left(\frac{\sqrt{5}-1}{4}\right)^2 + 2\left(\frac{\sqrt{5}-1}{4}\right) = \frac{3-\sqrt{5}}{2} + \frac{\sqrt{5}-1}{2} = 1.$

7. $\sin^2 B + \cos^2 B = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1.$

8. $\cos^2 C + \sin^2 C = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1.$

9. $\cos^2 D + \sin^2 D = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1.$

10. $\sin B \cdot \cos C - \sin C \cos B = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} = \sin 30 = \sin C.$

11. $2(\cos B \cdot \cos D + \sin B \cdot \sin D)^2 = 2 \cdot \left(\frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}\right)^2$

$$= 2 \cdot \left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)^2 = \frac{2+\sqrt{3}}{2} = 1 + \frac{\sqrt{3}}{2} = 1 + \cos 30^\circ = 1 + \cos C.$$

12. $2(\sin D \cdot \cos C - \sin C \cdot \cos D)^2 = 2\left(\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\right)^2$

$$= 2 \cdot \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2 = \frac{2-\sqrt{3}}{2} = 1 - \frac{\sqrt{3}}{2} = 1 - \cos 30^\circ = 1 - \cos C.$$

13. $\sin 30^\circ = \frac{1}{2} = \cdot 5.$

14. $\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1 \cdot 4142136}{2} = \cdot 7071068.$

15. $\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{1 \cdot 7320508\ldots}{2} = \cdot 866025\ldots$

16. $\tan 60° = \sqrt{3} = 1{\cdot}7320508\ldots$

17. $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3} = \dfrac{1{\cdot}7320508\ldots}{3} = {\cdot}57735\ldots$

18. $\sin 18° = \dfrac{\sqrt{5}-1}{4} = \dfrac{2{\cdot}2360680\ldots - 1}{4} = \dfrac{1{\cdot}2360680\ldots}{4} = {\cdot}3090170\ldots$

EXAMPLES. XIV. Pages 63—65.

1. Take fig. E.T. p. 54. Then, $OM = 179$ feet, $POM = 45°$, $PM = x$; then

$$\frac{PM}{OM} = \frac{x}{179} = \tan 45°;\ \therefore\ \frac{x}{179} = 1;\ \therefore\ x = 179 \text{ feet.}$$

2.
$$\frac{PM}{OM} = \tan 60°;\ \therefore\ \frac{x}{200} = \sqrt{3};$$

$$\therefore\ x = 200\sqrt{3} = 200 \times 1{\cdot}732 = 346 \text{ ft.}$$

3. Take fig. E. T. p. 55.

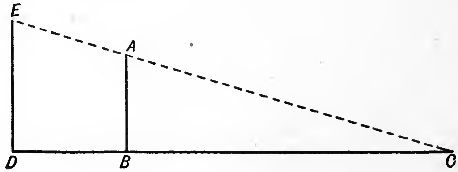
Let O be top vertical cliff; OPM angle of depression of point P, $PM = 150$ feet; $OM = x$ feet; then

$$\frac{OM}{PM} = \tan 30° = \frac{1}{\sqrt{3}};\ \therefore\ \frac{x}{150} = \frac{1}{\sqrt{3}};\ \therefore\ x = \frac{150}{\sqrt{3}} = 50\sqrt{3} = 50 \times 1{\cdot}732 = 86{\cdot}6 \text{ ft.}$$

4. Take fig. E. T. p. 55. Let O be the top of tower, OPM = angle of depression, OM height of tower above top of house $= (117 - 37)$ ft. $= 80$ ft., $PM = x$ ft.

$$\frac{PM}{OM} = \cot 30° = \sqrt{3};\ \therefore\ \frac{x}{80} = \sqrt{3},\ \therefore\ x = 80\sqrt{3} = 80 \times 1{\cdot}732 = 138{\cdot}5 \text{ ft.}$$

5. Let ED be the lamp post so that DBC is horizontal and EAC is a straight line, then $\dfrac{DE}{DC} = \tan DCE = \dfrac{BA}{BC}$.



$$\therefore\ \frac{DE}{DC} = \frac{BA}{BC};\ \therefore\ \frac{ED}{4\frac{3}{4} + 19} = \frac{6}{19};$$

$$\therefore\ ED = \tfrac{6}{19}(4\tfrac{3}{4} + 19) = \tfrac{6}{19} \times \tfrac{95}{4} = \tfrac{15}{2} = 7\tfrac{1}{2} \text{ ft.}$$

6. With the diagram of (5) if AB represent height of lamp post and BC length of its shadow, C angle of elevation of the sun,

$$\tan C = \frac{AB}{BC} = \frac{9}{3\sqrt{3}} = \sqrt{3} = \tan 60°; \ \therefore \ C = 60°.$$

Let h ft. represent height of tower;

$$\therefore \ \frac{h}{100} = \tan 60° = \sqrt{3}; \ \therefore \ h = 100 \times \sqrt{3} = 100 \times 1{\cdot}732 = 173{\cdot}2 \text{ feet.}$$

7. The breadth of the river is represented by $PQ = x$ yds.;

$$\therefore \ \frac{PQ}{QR} = \tan PRQ = \tan 32° \, 17', \ \frac{x}{100} = {\cdot}6317667, \ x = 63{\cdot}17 \text{ yds.}$$

8. Take fig. E. T. p. 62, Ex. 2.

Let PQ be height of flagstaff, i.e. 25 ft.,
MQ .... house, i.e. x ft.
Angle POM angle of elevation of the top of flagstaff, i.e. 60°,
QOM .... bottom .... i.e. 45°.

Let $OM = y$ feet.

Then $\frac{MP}{OM} = \tan 60°; \ \therefore \ \frac{x+25}{y} = \tan 60°, \ \frac{MP}{OM} = \tan 45°; \ \therefore \ \frac{x}{y} = \tan 45°.$

By division $\frac{x+25}{x} = \sqrt{3}; \ \therefore x = \frac{25}{{\cdot}732} = 34{\cdot}15;$

$\therefore$ height of house is 34·15 feet.

9. In the diagram referred to in (8) let O be the point of the cliff; P and Q the two ships. $OQM = 45°$, $OPM = 30°$, $OM = 100$ ft., $MQ = x$ feet, $PQ = y$ feet;

$$\therefore \ \frac{OM}{MQ} = \tan OQM; \ \therefore \ \frac{100}{x} = \tan 45° = 1; \ \therefore \ x = 100;$$

$$\therefore \ \frac{OM}{MP} = \tan OPM; \ \therefore \ \frac{100}{x+y} = \tan 30° = \frac{1}{\sqrt{3}}; \ \therefore \ x + y = 100\sqrt{3}.$$

Since $x + y = 100\sqrt{3}$;

$$\therefore \ y = 100\sqrt{3} - x = 100\sqrt{3} - 100 = 100(\sqrt{3} - 1) = 100 \times {\cdot}732 = 73{\cdot}2;$$

$\therefore$ the required distance is 73·2 feet.

10. In the diagram referred to in (8) let PQ be the tower 100 ft. high,

$$POM = 75°, \ QOM = 60°, \ OM = y \text{ ft.}, \ QM = x \text{ ft.};$$

$$\therefore \ \frac{MQ}{MO} = \tan 60°; \ \therefore \ \frac{x}{y} = \sqrt{3}, \ \frac{MP}{OM} = \tan 75°; \ \therefore \ \frac{100+x}{y} = 2 + \sqrt{3}.$$

By division $\frac{x+100}{x} = \frac{2+\sqrt{3}}{\sqrt{3}}; \ \therefore \ 1 + \frac{100}{x} = 1 + \frac{2}{\sqrt{3}}; \ \therefore \ \frac{100}{x} = \frac{2}{\sqrt{3}};$

$$\therefore \ x = 50\sqrt{3} = 50 \times 1{\cdot}732 = 86{\cdot}6;$$

$\therefore$ the height of the cliff is 86·6 feet.

11. In the diagram referred to in (8) let O be the position of the house, PM the direction of the road, P and Q the two consecutive milestones; $OPM=30°$, angle first observed; $OQM=60°$ angle next observed; OM ($=y$ miles) the required distance of house from the road. $MQ=x$ miles, $MP=x+1$ miles.

$$\frac{OM}{MP}=\tan OPM; \;\therefore\; \frac{y}{x+1}=\tan 30°=\frac{1}{\sqrt{3}},$$

$$\frac{OM}{MQ}=\tan OQM; \;\therefore\; \frac{y}{x}=\tan 60°=\sqrt{3}.$$

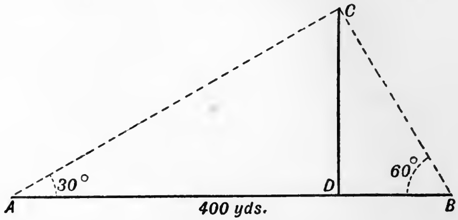
By division $\dfrac{x+1}{x}=3; \;\therefore\; 1+\dfrac{1}{x}=3, \;\therefore\; \dfrac{1}{x}=2, \;\therefore\; x=\dfrac{1}{2}.$

Since $\dfrac{y}{x}=\sqrt{3}; \;\therefore\; y=x\sqrt{3}=\dfrac{\sqrt{3}}{2}=\dfrac{1{\cdot}732}{2}=\cdot866$ of a mile

$=\cdot866\times 1760=1524$ yards;

$\therefore$ the required distance is 1524 yards.

12. The angle AC makes with AB is CAB; and BC makes with BA the angle CBA. (The student should take notice of the *directions* AB and BA.) Draw CD perpendicular to AB. $AB=400$ yards, $BD=x$ yards; $AD=(400-x)$ yards, $CD=y$ yards (the breadth of the river),



$$\frac{CD}{BD}=\tan CBD; \;\therefore\; \frac{y}{x}=\tan 60°=\sqrt{3} \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

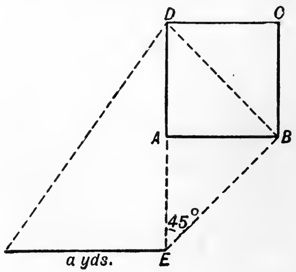
$$\frac{CD}{AD}=\tan CAD; \;\therefore\; \frac{y}{400-x}=\tan 30°=\frac{1}{\sqrt{3}} \;\ldots\ldots\ldots\text{(ii)}.$$

Divide (i) by (ii), $\therefore\; \dfrac{400-x}{x}=3, \;\dfrac{400}{x}=4, \;\therefore\; x=100.$

From (i) $y=x\sqrt{3}=100\times 1{\cdot}732=173{\cdot}2;$

$\therefore$ the required breadth is 173·2 yards.

13. Draw the figure as indicated in the problem; then, since angle $BEA=45°$ and angle $BAE=90°$; $\therefore$ angle $ABE=45°$ and $\therefore$ $AE=AB$.



Because $AB=AD$ (sides of the square), $\therefore$ angle ABD=angle $ADB=45°$. But angle $BEA=45°$; $\therefore EA=AD$. Let x=the side of the square, then $ED=2x$ and $BD=x\sqrt{2}$,

$$\frac{ED}{a}=\frac{2x}{a}=\tan^{-1}\sqrt{2}=\sqrt{2};\ \therefore\ 2x=a\sqrt{2};\ \therefore\ x\sqrt{2}=a=BD.$$

14. In fig. E. T. p. 62, Ex. 2, let O be the top of hill, MOQ and MOP angles of depression of Q and P respectively the top and bottom of flagstaff, $PQ=25$ feet=height of flagstaff, $QM=x$ ft.=height of hill above flagstaff, height of hill$=(25+x)$ ft., $OM=y$=distance of foot of hill from flagstaff.

$$\frac{QM}{OM}=\tan MOQ;\ \therefore\ \frac{x}{y}=\tan MOQ=\tan 45°\,13',$$

$$\frac{PM}{OM}=\tan MOP;\ \therefore\ \frac{x+25}{y}=\tan MOP=\tan 47°\,12'.$$

By division $\dfrac{x+25}{x}=\dfrac{\tan 47°\,12'}{\tan 45°\,13'}$, i.e. $1+\dfrac{25}{x}=\dfrac{1\cdot0799018}{1\cdot0075918}=1+\dfrac{\cdot07231}{1\cdot0075918}$;

$$\therefore\ \frac{x}{25}=\frac{1\cdot0075918}{\cdot07231}=\frac{100759}{7231};\ \therefore\ x=\frac{2518975}{7231}=348 \text{ nearly};$$

$$\therefore\ x+25=348+25=373;$$

$\therefore$ the required height of hill is 373 feet.

15. Let A be the first station and B the second so that $AB=1$ mile. Let C be the place vertically under the balloon K; then since CA has a N.W. bearing, i.e. CA is inclined westward 45° to the line AB, $\therefore$ angle $CAB=45°$; similarly $CBA=45°$, $\therefore CA=CB$, and angle $BCA=90°$,

$$\frac{AC}{AB}=\sin 45°=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}=\cdot7071;\ \therefore\ \frac{AC}{1}=\cdot7071,\ \therefore\ AC=BC=\cdot7071 \text{ miles};$$

also, since the altitude of the balloon K at A is 45°, $KC=AC$;

$\therefore$ the height of the balloon is $\cdot7071$ miles$=\cdot7071\times5280=3733$ ft.

16. Let AB be the height of the balloon, C the station due south; then angle $ACB=60°$; draw CD perpendicular to BC and west of it, make CD represent a mile, join BD and AD, then angle $ADB=45°$.

Let AB the height of balloon$=x$ miles,

$$\therefore\ BC=AB\cot ACB=x\cot 60°=\frac{x}{\sqrt{3}}.$$

Because angle $ADB=45°$ and angle $ABD=90°$;

$\therefore$ angle $BAD=45°$, and $BD=AB=x$ miles.

From Eucl. I. (47) $BD^2=BC^2+CD^2$; $\therefore\ x^2=\left(\dfrac{x}{\sqrt{3}}\right)^2+1$;

$$\therefore\ 2x^2=3;\ \therefore\ x=\tfrac{1}{2}\sqrt{6} \text{ miles}=\tfrac{1}{2}\times2\cdot4495\times5280 \text{ ft.}=6468 \text{ ft.};$$

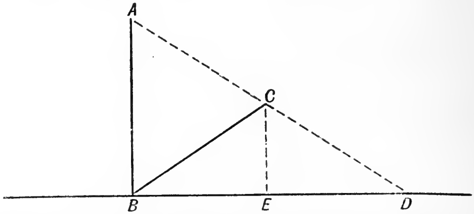
$\therefore$ the required height is 6468 feet.

17. Since the altitude of the sun is 30°, the length of the shadow of the height of the triangle will be to the height of the triangle as $\cot 30° : 1$: $\therefore$ length of shadow is $b\sqrt{3}$. The base of the triangular shadow coincides with the base of the triangle ($2a$); $\therefore$ the shadow is an isosceles triangle with base $2a$ and height $b\sqrt{3}$. The height bisects the base and the vertical angle;

$$\therefore \text{ tangent of half the vertical angle} = \frac{a}{b\sqrt{3}} = \frac{\sqrt{3}\,.\,a}{3\,.\,b}.$$

18. Let AB represent the stick and BD the length of its shadow;

$$\therefore \frac{AB}{BD} = \frac{1}{\sqrt{3}}; \quad \therefore \tan ADB = \frac{1}{\sqrt{3}}; \quad \therefore ADB = 30^\circ.$$



Let CB be the position of AB when the length of its shadow is again equal to BD. The shadow of CE, perpendicular on BD, will in that position then be equal to ED. And the angle CDE will be equal to the angle ADE. The two triangles will therefore be similar (Eucl. VI. 2).

$$\therefore \frac{CE}{ED} = \frac{AB}{BD}; \quad \therefore AC \text{ and } CD \text{ are in one and the same straight line.}$$

Because angle $ADB = 30^\circ$; $\therefore$ angle $BAD = 60^\circ$, and since $AB = BC$, $\therefore$ angle $ACB = 60^\circ$, the remaining angle ABC is also $= 60^\circ$.

Now $CBE = 90^\circ - ABC = 90^\circ - 60^\circ = 30^\circ$.

$\therefore$ the angle of inclination of stick to horizon is 30°.

19. In fig. E. T. p. 79, let P be the position of the person, then NP is radius of circle described by the person in consequence of the earth's rotation. Let $POR = 60^\circ$. Then $NP = OM = OP \cos 60^\circ = OR \cos 60^\circ = \frac{4000}{2}$ miles;

$\therefore$ the diameter of the circle described by the person is 4000 miles.

Circumference $= 4000 \times 3{\cdot}1416 = 12566{\cdot}4$ mls.

$\therefore$ the distance passed in 1 hr. $= \frac{1}{24}$ of $12566{\cdot}4$ mls. $= 523{\cdot}6$ miles.

EXAMPLES. XV. PAGE 71.

1. $\cos A \,.\, \tan A \,. = \cos A \,.\, \dfrac{\sin A}{\cos A} = \sin A.$

2. $\cot A \,.\, \tan A = \dfrac{\cos A}{\sin A} \,.\, \dfrac{\sin A}{\cos A} = 1.$

3. $\cos A = \sin A \,.\, \dfrac{\cos A}{\sin A} = \sin A \,.\, \cot A.$

4. $\sec A \,.\, \cot A = \dfrac{1}{\cos A} \,.\, \dfrac{\cos A}{\sin A} = \dfrac{1}{\sin A} = \text{cosec}\, A.$

5. $\operatorname{cosec} A \,.\, \tan A = \frac{1}{\sin A} \cdot \frac{\sin A}{\cos A} = \frac{1}{\cos A} = \sec A.$

6. $(\tan A + \cot A) \sin A \,.\, \cos A = \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right) \sin A \,.\, \cos A$

$$= \frac{\sin^2 A + \cos^2 A}{\cos A \,.\, \sin A} \,.\, \sin A \,.\, \cos A = \sin^2 A + \cos^2 A = 1.$$

7. $(\tan A - \cot A) \sin A \,.\, \cos A = \left(\frac{\sin A}{\cos A} - \frac{\cos A}{\sin A}\right) \sin A \,.\, \cos A$

$$= \frac{(\sin^2 A - \cos^2 A) \sin A \,.\, \cos A}{\sin A \,.\, \cos A} = \sin^2 A - \cos^2 A.$$

8. $\cos^2 A - \sin^2 A = 1 - \sin^2 A - \sin^2 A = 1 - 2 \sin^2 A.$

9. $(\sin A + \cos A)^2 = \sin^2 A + \cos^2 A + 2 \sin A \,.\, \cos A = 1 + 2 \sin A \cos A.$

10. $(\sin A - \cos A)^2 = \sin^2 A + \cos^2 A - 2 \sin A \,.\, \cos A = 1 - 2 \sin A \,.\, \cos A.$

11. $\cos^4 B - \sin^4 B = (\cos^2 B + \sin^2 B)(\cos^2 B - \sin^2 B) = \cos^2 B - \sin^2 B$
$= \cos^2 B - (1 - \cos^2 B) = 2 \cos^2 B - 1.$

12. $(\sin^2 B + \cos^2 B)^2 = 1^2 = 1.$

13. $(\sin^2 B - \cos^2 B)^2 = (1 - \cos^2 B - \cos^2 B)^2 = (1 - 2\cos^2 B)^2$
$= 1 - 4 \cos^2 B + 4 \cos^4 B.$

14. $1 - \tan^4 B = (1 + \tan^2 B)(1 - \tan^2 B) = \sec^2 B \{1 - (\sec^2 B - 1)\}$
$= \sec^2 B (2 - \sec^2 B) = 2 \sec^2 B - \sec^4 B.$

15. $(\sec B - \tan B)(\sec B + \tan B) = \sec^2 B - \tan^2 B$
$= 1 + \tan^2 B - \tan^2 B = 1.$

16. $(\operatorname{cosec} \theta - \cot \theta)(\operatorname{cosec} \theta + \cot \theta) = \operatorname{cosec}^2 \theta - \cot^2 \theta$
$= 1 + \cot^2 \theta - \cot^2 \theta = 1.$

17. $\sin^3 \theta + \cos^3 \theta = (\sin \theta + \cos \theta)(\sin^2 \theta + \cos^2 \theta - \sin \theta \cos \theta)$
$= (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta).$

18. $\cos^3 \theta - \sin^3 \theta = (\cos \theta - \sin \theta)(\cos^2 \theta + \sin^2 \theta + \cos \theta \sin \theta)$
$= (\cos \theta - \sin \theta)(1 + \sin \theta \cos \theta).$

19. $\sin^6 \theta + \cos^6 \theta = (\sin^2 \theta + \cos^2 \theta)(\sin^4 \theta + \cos^4 \theta - \sin^2 \theta \cos^2 \theta)$
$= \sin^4 \theta + \cos^4 \theta - \sin^2 \theta \cos^2 \theta$
$= \sin^4 \theta + \cos^4 \theta + 2 \sin^2 \theta \cos^2 \theta - 3 \sin^2 \theta \cos^2 \theta$
$= (\sin^2 \theta + \cos^2 \theta)^2 - 3 \sin^2 \theta \cos^2 \theta = 1 - 3 \sin^2 \theta \cos^2 \theta.$

20. $\sin^6 \theta - \cos^6 \theta = (\sin^2 \theta - \cos^2 \theta)(\sin^4 \theta + \sin^2 \theta \cos^2 \theta + \cos^4 \theta)$
$= (2 \sin^2 \theta - 1)\{\sin^4 \theta + \sin^2 \theta (1 - \sin^2 \theta) + (1 - \sin^2 \theta)^2\}$
$= (2 \sin^2 \theta - 1)(1 - \sin^2 \theta + \sin^4 \theta).$

21. $$\frac{\tan A+\tan B}{\cot A+\cot B}=\frac{\tan A+\tan B}{\frac{1}{\tan A}+\frac{1}{\tan B}}$$ [Art. 102.]

$$=\frac{\tan A+\tan B}{\frac{\tan A+\tan B}{\tan A\,.\,\tan B}}=\tan A\,.\,\tan B.$$

22. $$\frac{\cot\alpha+\tan\beta}{\tan\alpha+\cot\beta}=\frac{\frac{1}{\tan\alpha}+\tan\beta}{\tan\alpha+\frac{1}{\tan\beta}}=\frac{\frac{1+\tan\alpha\,.\,\tan\beta}{\tan\alpha}}{\frac{1+\tan\alpha\,.\,\tan\beta}{\tan\beta}}=\frac{\tan\beta}{\tan\alpha}=\cot\alpha\tan\beta.$$

23. $$\frac{1-\sin A}{1+\sin A}=\frac{(1-\sin A)(1-\sin A)}{(1+\sin A)(1-\sin A)}=\frac{(1-\sin A)^2}{1-\sin^2 A}=\frac{(1-\sin A)^2}{\cos^2 A}$$

$$=\left(\frac{1-\sin A}{\cos A}\right)^2=\left(\frac{1}{\cos A}-\frac{\sin A}{\cos A}\right)^2=(\sec A-\tan A)^2.$$

24. $$\frac{1+\cos A}{1-\cos A}=\frac{(1+\cos A)(1+\cos A)}{(1-\cos A)(1+\cos A)}=\frac{(1+\cos A)^2}{1-\cos^2 A}=\frac{(1+\cos A)^2}{\sin^2 A}$$

$$=\left(\frac{1+\cos A}{\sin A}\right)^2=\left(\frac{1}{\sin A}+\frac{\cos A}{\sin A}\right)^2=(\operatorname{cosec} A+\cot A)^2.$$

25. $$2\operatorname{versin}\theta-\operatorname{versin}^2\theta=2(1-\cos\theta)-(1-\cos\theta)^2$$

$$=2(1-\cos\theta)-(1-2\cos\theta+\cos^2\theta)$$

$$=1-\cos^2\theta=\sin^2\theta.$$

26. $$\operatorname{versin}\theta\,(1+\cos\theta)=(1-\cos\theta)(1+\cos\theta)=1-\cos^2\theta=\sin^2\theta.$$

EXAMPLES. XVI. Page 74.

1. In fig. E. T. p. 72 let the measure of OP be 1, and let c be the measure of OM, $\therefore\ c=\cos A$. Let x be the measure of MP;

$$\therefore\ x^2=1-c^2;\quad \therefore\ x=\pm\sqrt{1-c^2},$$

$$\sin A=\frac{MP}{OP}=\frac{\sqrt{1-c^2}}{1}=\sqrt{1-\cos^2 A},\quad \tan A=\frac{MP}{OM}=\frac{\sqrt{1-c^2}}{c}=\frac{\sqrt{1-\cos^2 A}}{\cos A},$$

$$\cot A=\frac{OM}{MP}=\frac{c}{\sqrt{1-c^2}}=\frac{\cos A}{\sqrt{1-\cos^2 A}},\quad \sec A=\frac{OP}{OM}=\frac{1}{c}=\frac{1}{\cos A},$$

$$\operatorname{cosec} A=\frac{OP}{MP}=\frac{1}{\sqrt{1-c^2}}=\frac{1}{\sqrt{1-\cos^2 A}}.$$

2. In fig. E. T. p. 73 let POM be the given angle, t the measure of OM and 1 the measure of PM;

$$\therefore\ \cot A=\frac{OM}{MP}=\frac{t}{1}\ \text{or}\ t=\cot A.$$

Let x be the measure of OP,

$$\therefore\ x^2=1+t^2,\ \therefore\ x=\sqrt{1+t^2},$$

$$\sin A=\frac{MP}{OP}=\frac{1}{\sqrt{1+t^2}}=\frac{1}{\sqrt{1+\cot^2 A}},\ \cos A=\frac{OM}{OP}=\frac{t}{\sqrt{1+t^2}}=\frac{\cot A}{\sqrt{1+\cot^2 A}},$$

$$\tan A=\frac{MP}{OM}=\frac{1}{t}=\frac{1}{\cot A},\ \sec A=\frac{OP}{OM}=\frac{\sqrt{1+t^2}}{t}=\frac{\sqrt{1+\cot^2 A}}{\cot A},$$

$$\operatorname{cosec} A=\frac{OP}{MP}=\frac{\sqrt{1+t^2}}{1}=\sqrt{1+\cot^2 A}.$$

3. In the figure E. T. p. 73 let c be the measure of OP and let 1 be the measure of OM, $\therefore\ \sec A=\dfrac{OP}{OM}=c.$

It can be shewn that the measure of MP is $\sqrt{c^2-1}$,

$$\sin A=\frac{MP}{OP}=\frac{\sqrt{c^2-1}}{c}=\frac{\sqrt{\sec^2 A-1}}{\sec A},\ \cos A=\frac{OM}{OP}=\frac{1}{c}=\frac{1}{\sec A},$$

$$\tan A=\frac{MP}{OM}=\frac{\sqrt{c^2-1}}{1}=\sqrt{\sec^2 A-1},\ \cot A=\frac{OM}{MP}=\frac{1}{\sqrt{c^2-1}}=\frac{1}{\sqrt{\sec^2 A-1}},$$

$$\operatorname{cosec} A=\frac{OP}{OM}=\frac{OP}{MP}=\frac{c}{\sqrt{c^2-1}}=\frac{\sec A}{\sqrt{\sec^2 A-1}}.$$

4. In the figure E. T. p. 73 let OP have c for measure, let 1 be the measure of MP, $\therefore\ c=\operatorname{cosec} A$ and $\sqrt{c^2-1}$ is the measure of OM,

$$\sin A=\frac{MP}{OP}=\frac{1}{c}=\frac{1}{\operatorname{cosec} A},\ \cos A=\frac{OM}{OP}=\frac{\sqrt{c^2-1}}{c}=\frac{\sqrt{\operatorname{cosec}^2 A-1}}{\operatorname{cosec} A},$$

$$\tan A=\frac{MP}{OM}=\frac{1}{\sqrt{c^2-1}}=\frac{1}{\sqrt{\operatorname{cosec}^2 A-1}},$$

$$\cot A=\frac{OM}{MP}=\frac{\sqrt{c^2-1}}{1}=\sqrt{\operatorname{cosec}^2 A-1},$$

$$\sec A=\frac{OP}{OM}=\frac{c}{\sqrt{c^2-1}}=\frac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A-1}}.$$

5. $\cos^2 A+\sin^2 A=1$ (Art. 107);

$$\therefore\ \cos^2 A=1-\sin^2 A;\ \therefore\ \cos A=\sqrt{1-\sin^2 A}.$$

$$\tan A=\frac{\sin A}{\cos A}=\frac{\sin A}{\sqrt{1-\sin^2 A}}\qquad\text{(Art. 102)},$$

$$\cot A=\frac{\cos A}{\sin A}=\frac{\sqrt{1-\sin^2 A}}{\sin A}\qquad\text{(Art. 102)},$$

$$\sec A=\frac{1}{\cos A}=\frac{1}{\sqrt{1-\sin^2 A}};\ \operatorname{cosec} A=\frac{1}{\sin A}\qquad\text{(Art. 102)}.$$

6. $\sin A = \dfrac{1}{\operatorname{cosec} A} = \dfrac{1}{\sqrt{1+\cot^2 A}}$

$$= \frac{1}{\sqrt{\left(1+\dfrac{1}{\tan^2 A}\right)}} = \frac{\tan A}{\sqrt{1+\tan^2 A}} \text{ (Arts. 105, iii.),}$$

$\cos A = \dfrac{1}{\sec A} = \dfrac{1}{\sqrt{1+\tan^2 A}}$ (Art. 105, ii.), $\cot A = \dfrac{1}{\tan A}$ (Art. 102),

$\sec A = \sqrt{1+\tan^2 A}$ (Art. 105, ii.), $\operatorname{cosec} A = \dfrac{1}{\sin A} = \dfrac{\sqrt{1+\tan^2 A}}{\tan A}$.

EXAMPLES. XVII. PAGE 75.

1. Draw a figure similar to that of E. T. p. 74. Let POM be the give angle A, take P so that the measure of OP is 5, then since

$$\sin A = \tfrac{3}{5}, \;\therefore\; MP = 3.$$

Let x be the measure of OM, $\therefore\; x^2 = 5^2 - 3^2 = 16$, $\therefore\; x = 4$,

$$\tan A = \frac{MP}{OM} = \frac{3}{4}. \quad \operatorname{cosec} A = \frac{OP}{MP} = \frac{5}{3}.$$

2. Fig. as in (1). Let $OP = 3$ and $OM = 1$ for $\cos B \left[\text{i. e. } \dfrac{OM}{OP}\right] = \dfrac{1}{3}$;

$$\therefore\; MP^2 = 3^2 - 1 = 8; \;\therefore\; MP = \sqrt{8} = 2\sqrt{2},$$

$$\sin B = \frac{MP}{OP} = \frac{2\sqrt{2}}{3}; \; \cot B = \frac{OM}{MP} = \frac{1}{2\sqrt{2}}.$$

3. Fig. as in (1). Let $MP = 4$; $\therefore\; OM = 3$ for $\tan A \left[\text{i. e. } \dfrac{MP}{OM}\right] = \dfrac{4}{3}$;

$$\therefore\; OP^2 = 4^2 + 3^2 = 25; \;\therefore\; OP = 5,$$

$$\sin A = \frac{MP}{OP} = \frac{4}{5}. \quad \sec A = \frac{OP}{OM} = \frac{5}{3}.$$

4. Fig. as in (1). Let $OP = 4$ and $OM = 1$; for $\sec\theta \left[= \dfrac{OP}{OM}\right] = 4$;

$$\therefore\; MP^2 = OP^2 - OM^2 = 4^2 - 1 = 15; \;\therefore\; MP = \sqrt{15},$$

$$\cot\theta = \frac{OM}{MP} = \frac{1}{\sqrt{15}}, \quad \sin\theta = \frac{MP}{OP} = \frac{\sqrt{15}}{4}.$$

5. Fig. as in (1). Let $MP = \sqrt{3}$ and $OM = 1$; for $\tan\theta \left[= \dfrac{MP}{OM}\right] = \sqrt{3}$

$$\therefore\; OP^2 = MP^2 + OM^2 = 3 + 1 = 4; \;\therefore\; OP = 2;$$

$$\sin\theta = \frac{MP}{OP} = \frac{\sqrt{3}}{2}, \quad \cos\theta = \frac{OM}{OP} = \frac{1}{2}.$$

6. Fig. as in (1). Let $OM=2$ and $MP=\sqrt{5}$; for $\cot\theta\left[=\frac{OM}{MP}\right]=\frac{2}{\sqrt{5}}$;

$$\therefore\ OP^2=MP^2+OM^2=5+2^2=9;\ \therefore\ OP=3,$$

$$\sin\theta=\frac{MP}{OP}=\frac{\sqrt{5}}{3},\quad \sec\theta=\frac{OP}{OM}=\frac{3}{2}.$$

7. Fig. as in (1). Let $MP=b$ and $OP=c$; for $\sin\theta\left[=\frac{MP}{OP}\right]=\frac{b}{c}$;

$$\therefore\ OM^2=OP^2-MP^2=c^2-b^2;\ \therefore\ OM=\sqrt{c^2-b^2},$$

$$\tan\theta=\frac{MP}{OM}=\frac{b}{\sqrt{c^2-b^2}}.$$

8. Fig. as in (1). Let $MP=a$, $OM=b$; for $\tan\theta\left[=\frac{MP}{OM}\right]=\frac{a}{b}$;

$$\therefore\ OP^2=OM^2+MP^2=a^2+b^2;\ \therefore\ OP=\sqrt{a^2+b^2},$$

$$\sin\theta=\frac{MP}{OP}=\frac{a}{\sqrt{a^2+b^2}};\ \cos\theta=\frac{OM}{OP}=\frac{b}{\sqrt{a^2+b^2}}.$$

9. Fig. as in (1). Let $OP=a$ and $OM=1$; for $\cos A\left[=\frac{OM}{OP}\right]=\frac{1}{a}$;

$$\therefore\ MP^2=OP^2-OM^2=a^2-1;\ \therefore\ MP=\sqrt{a^2-1},$$

$$\sin\theta=\frac{MP}{OP}=\frac{\sqrt{a^2-1}}{a},\quad \cot\theta=\frac{OM}{MP}=\frac{1}{\sqrt{a^2-1}}.$$

N.B. The required ratios in each of the above examples may be found without referring to the figure. If we combine the formulae of Art. 106, we may shew that

$$\sin\theta=\sqrt{1-\cos^2\theta},\ \text{and}\ \cot\theta=\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\qquad\text{(see Ex. XVI. 1).}$$

If $\cos\theta=\frac{1}{a}$ as in (9), we have

$$\sin\theta=\sqrt{\left(1-\frac{1}{a^2}\right)}=\frac{\sqrt{a^2-1}}{a},\quad \cot\theta=\frac{\frac{1}{a}}{\sqrt{\left(1-\frac{1}{a^2}\right)}}=\frac{1}{\sqrt{a^2-1}}.$$

10. Since $\sin\theta=a$, $\therefore\ 1-a^2=1-\sin^2\theta=\cos^2\theta$, (Art. 105, i.)

,, $\tan\theta=b$, $\therefore\ 1+b^2=1+\tan^2\theta=\sec^2\theta$; (Art. 105, ii.)

$\therefore\ (1-a^2)(1+b^2)=\cos^2\theta\,.\,\sec^2\theta=\cos^2\theta\,.\,\frac{1}{\cos^2\theta}=1$ (Art. 102).

11. $\cos\theta=h$. $\tan\theta=\frac{\sin\theta}{\cos\theta}=k$; $\therefore\ \sin\theta=k\cos\theta=hk$.

From Art. 105, i., $\sin^2\theta+\cos^2\theta=1$; $\therefore\ h^2+h^2k^2=1$; $\therefore\ h^2(1+k^2)=1$.

EXAMPLES. XVIII. Page 77.

1. With the same construction and figure as in E. T. p. 76,

$$\cos A = \frac{OM}{OP}.$$

When the angle A is 0°, OP coincides with OR and then OM is equal to OP; when A is equal to 90°, OP coincides with OM and then OM vanishes; and as A continuously increases from 0° to 90°, OM continuously decreases from OP to 0; and OP is always equal to OR.

Therefore as A approaches 0°, the fraction $\frac{OM}{OP}$ approaches $\frac{OP}{OP}$, that is 1; when $A = 90^\circ$ the fraction $\frac{OM}{OP}$ is equal to $\frac{O}{OP}$, that is 0; and as A continuously increases from 0° to 90°, the numerator of the fraction $\frac{OM}{OP}$ continuously decreases from OP to O while the denominator is constant, and therefore the fraction $\frac{OM}{OP}$, which is $\cos A$, decreases continuously from 1 to 0.

2. With the same construction and figure as in E. T. p. 76,

$$\sec\theta = \frac{OP}{OM}.$$

The changes in the secant may be traced by means of the figure in the same way as those of the cosine (1); or we may use the formula $\sec\theta = \frac{1}{\cos\theta}$; as the angle *continuously increases* from 0 to $\frac{1}{2}\pi$ the cosine continuously decreases from 1 to 0, and therefore the secant continuously increases from 1 to 'infinity.'

3. With the same construction and figure as in E. T. p. 76,

$$\sin A = \frac{MP}{OP}.$$

When the angle A is 90°, MP is equal to OP; and when A is 0°, MP is zero; as A *continuously decreases* from 90° to 0°, MP *continuously decreases* from OP to zero; and OP is always equal to OR.

Therefore when $A = 90^\circ$, the fraction $\frac{MP}{OP}$ is equal to $\frac{OP}{OP}$, that is 1; when $A = 0^\circ$ the fraction $\frac{MP}{OP}$ is equal to $\frac{O}{OP}$, that is 0; and as A *continuously decreases* from 90° to 0° the *numerator* of the fraction $\frac{MP}{OP}$ *continuously decreases* from OP to zero, while the denominator is unchanged and therefore the *fraction* $\frac{MP}{OP}$, which is $\sin A$, *decreases continuously* from 1 to 0.

4. With the construction and figure of E. T. p. 76, $\cot\theta=\frac{OM}{MP}$.

When the angle θ is 0, OM is equal to OP, and when the angle θ is $\frac{1}{2}\pi$, OM vanishes; and as the angle *continuously increases* from 0 to $\frac{1}{2}\pi$, OM continuously diminishes from OP to zero.

When the angle θ is 0, MP is equal to zero; and when θ is $\frac{1}{2}\pi$, MP is equal to OP; $\therefore$ as θ *continuously increases* from 0 to $\frac{1}{2}\pi$, MP continuously increases from zero to OP.

Therefore when θ is 0, the fraction $\frac{OM}{MP}$ is equal to $\frac{OP}{0}$, that is 'infinity'; when θ is $\frac{1}{2}\pi$ the fraction $\frac{OM}{MP}$ is equal to $\frac{0}{OP}$, that is zero; and as θ continuously increases from 0 to $\frac{1}{2}\pi$, the numerator continuously diminishes from OP to zero, while the denominator continuously increases from zero to OP; so that the fraction $\frac{OM}{MP}$, that is $\cot\theta$, *continuously* decreases from a number *greater than any assignable numerical quantity* until it is zero.

EXAMPLES. XIX. Page 82.

1. With a fig. similar to E. T. p. 79.

In OU, take ON so that the measure of ON is $\frac{1}{2}$. Draw NP parallel to OR cutting the quadrant in P. Join OP, and draw PM perpendicular to OR. Then ROP is the angle required.

For $$\sin ROP=\frac{MP}{OP}=\frac{ON}{OP}=\frac{1}{2}\div 1=\frac{1}{2}.$$

Therefore an angle POR has been drawn whose sine is $\frac{1}{2}$.

2. Since the sine of an angle is the reciprocal of its cosecant, therefore the sine is $\frac{1}{2}$ of the angle of which the cosecant is 2. The question may therefore be put into the *form*, 'Draw an angle whose sine is $\frac{1}{2}$;' which is (1).

3. With the construction of *Example* 3, p. 80.

Let $OM=1$ and $MP=2$. Then POM is the angle required.

For $$\tan POM=\frac{MP}{OM}=\frac{2}{1}=2.$$

4. Yes. Since the tangent of an angle *continuously* increases from 0 to 'infinity' as the angle continuously increases from 0° to 90°, there is therefore a value between 0° and 90° when the tangent of the angle is 431.

5. No. The cosine of an angle is never numerically greater than unity; therefore no angle can be drawn whose cosine is $\frac{4}{3}$.

6. Yes. The secant of an angle is *numerically* between 1 and infinity.

7. (i) The complement of $30° = 90° - 30° = 60°$.

(ii) ,, ,, $190° = 90° - 190° = -100°$.

(iii) ,, ,, $90° = 90° - 90° = 0°$.

(iv) ,, ,, $350° = 90° - 350° = -260°$.

(v) ,, ,, $-25° = 90° - (-25°) = 90° + 25° = 115°$.

(vi) ,, ,, $-320° = 90° - (-320°) = 90° + 320° = 410°$.

(vii) ,, ,, $\frac{3\pi}{4} = \frac{\pi}{2} - \frac{3\pi}{4} = -\frac{\pi}{4}$.

(viii) ,, ,, $-\frac{\pi}{6} = \frac{\pi}{2} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$.

8. $\sin A = \cos(90° - A)$, Art. 118; $\therefore \sin 70° = \cos(90° - 70°) = \cos 20°$.

9. $\cos(90° - 42° 44') = \sin 42° 44'$; $\therefore \cos 47° 16' = \sin 42° 44'$.

10. $\sin 79° = \cos(90° - 79°) = \cos 11°$, $\sin 11° = \cos(90° - 11°) = \cos 79°$;

$$\therefore \frac{\sin 79°}{\cos 79°} = \frac{\cos 11°}{\sin 11°}; \quad \therefore \tan 79° = \cot 11°.$$

11. $\sin 54° = \cos(90° - 54°) = \cos 36°$;

$$\therefore \frac{1}{\sin 54°} = \frac{1}{\cos 36°}, \text{ i. e. } \operatorname{cosec} 54° = \sec 36°.$$

12. In the triangle POM (E. T. p. 46), let angle $POM = A$ then angle $OPM = 90° - A$.

(i) $\sin OPM = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{OM}{OP} = \text{cosine } A$; (Art. 75, ii.)

$$\therefore \sin(90° - A) = \cos A.$$

(ii) $\cot OPM = \frac{\text{base}}{\text{perpendicular}} = \frac{MP}{OM} = \tan A$; (Art. 75, iii.)

$$\therefore \cot(90° - A) = \tan A.$$

(iii) $\operatorname{cosec} OPM = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{OP}{OM} = \sec A$; (Art. 78, v.)

$$\therefore \operatorname{cosec}(90° - A) = \sec A.$$

(iv) $\tan OPM = \frac{\text{perpendicular}}{\text{base}} = \frac{OM}{MP} = \cot A$; (Art. 78, vi.)

$$\therefore \tan(90° - A) = \cot A.$$

EXAMPLES. XX. Page 84.

1. $\sin\theta = \frac{1}{\sqrt{2}}$, $\sin 45° = \frac{1}{\sqrt{2}}$; $\therefore \theta = 45°$ is *a* solution of the equation.

2. $4\sin\theta = \operatorname{cosec}\theta$; $\therefore 4\sin\theta = \frac{1}{\sin\theta}$;

$\therefore 4\sin^2\theta = 1$; $\therefore 2\sin\theta = 1$; $\therefore \sin\theta = \frac{1}{2}$; $\therefore \theta = 30°$ is a solution.

3. $\therefore\ 2\cos\theta=\sec\theta$; $\therefore\ 2\cos\theta=\frac{1}{\cos\theta}$; $\therefore\ 2\cos^2\theta=1$; $\therefore\ \cos\theta=\frac{1}{\sqrt{2}}$.

$\therefore$ 45° is a solution.

4. $\therefore\ 4\sin\theta-3\operatorname{cosec}\theta=0$; $\therefore\ 4\sin\theta=\frac{3}{\sin\theta}$; $\therefore\ 4\sin^2\theta=3$;

$\therefore\ \sin\theta=\frac{1}{2}\sqrt{3}$; $\therefore\ \theta=60°$ is a solution.

5. $4\cos\theta-3\sec\theta=0$; $\therefore\ 4\cos\theta=\frac{3}{\cos\theta}$; $\therefore\ 4\cos^2\theta=3$;

$\therefore\ \cos\theta=\frac{1}{2}\sqrt{3}$; $\therefore\ \theta=30°$ is a solution.

6. $3\tan\theta=\cot\theta$; $\therefore\ 3\tan\theta=\frac{1}{\tan\theta}$; $\therefore 3\tan^2\theta=1$; $\therefore\ \tan\theta=\frac{1}{\sqrt{3}}$;

$\therefore\ \theta=30°$ is a solution.

7. $3\sin\theta-2\cos^2\theta=0$; $\therefore\ 3\sin\theta-2(1-\sin^2\theta)=0$;

$\therefore\ 2\sin^2\theta+3\sin\theta=2$.

From this quadratic $\sin\theta=-2$ or $\frac{1}{2}$. The value -2 is inadmissible; for there is no angle whose sine is numerically greater than 1 (Art. 115);

$\therefore\ \sin\theta=\frac{1}{2}$.

But $\sin 30°=\frac{1}{2}$; $\therefore\ \theta=30°$ is a solution.

8. $\sqrt{2}\sin\theta=\tan\theta$; $\therefore\ \sqrt{2}\sin\theta=\frac{\sin\theta}{\cos\theta}$; $\therefore$ either $\sin\theta=0$ or $\sqrt{2}=\frac{1}{\cos\theta}$.

If $\sin\theta=0$, $\theta=0$ is a solution of the equation.

If $\sqrt{2}=\frac{1}{\cos\theta}$; $\therefore\ \cos\theta=\frac{1}{\sqrt{2}}$;

$\therefore\ \theta=45°$ is a solution of the equation; $\therefore \theta=0°$ or 45°.

9. $2\cos\theta=\sqrt{3}\cot\theta$; $\therefore\ 2\cos\theta=\sqrt{3}\cdot\frac{\cos\theta}{\sin\theta}$;

$\therefore$ either $\cos\theta=0$, or $2\sin\theta=\sqrt{3}$.

If $\cos\theta=0$; $\therefore\ \theta=90°$ is a solution of the equation.

If $2\sin\theta=\sqrt{3}$; $\therefore\ \sin\theta=\frac{1}{2}\sqrt{3}$.

But $\sin 60°=\frac{1}{2}\sqrt{3}$; $\therefore$ 60° is a solution of the equation; $\therefore\ \theta=90°$ or 60°.

10. $\tan\theta=3\cot\theta$; $\therefore\ \tan\theta=\frac{3}{\tan\theta}$; $\therefore\ \tan^2\theta=3$; $\therefore\ \tan\theta=\sqrt{3}$.

But $\tan 60°=\sqrt{3}$; $\therefore\ \theta=60°$ is a solution of the equation.

11. $\tan\theta+3\cot\theta=4$; $\therefore\ \tan\theta+\frac{3}{\tan\theta}=4$; $\therefore\ \tan^2\theta-4\tan\theta+3=0$

From this quadratic $\tan\theta=3$ or 1.

Both values are admissible.

If $\tan\theta=1$; $\therefore\ \theta=45°$ is a solution of the equation.

12. $\tan\theta+\cot\theta=2$; $\therefore$ $\tan\theta+\frac{1}{\tan\theta}=2$; $\therefore$ $\tan^2\theta-2\tan\theta+1=0$.
From this quadratic $\tan\theta=1$; $\therefore$ $\theta=45^\circ$ is a solution of the equation.

13. $2\sin^2\theta+\sqrt{2}\cos\theta=2$, $2(1-\cos^2\theta)+\sqrt{2}\cos\theta=2$;

$\therefore$ $\sqrt{2}\cos\theta-2\cos^2\theta=0$; $\therefore$ either $\cos\theta=0$ or $\sqrt{2}-2\cos\theta=0$.

If $\cos\theta=0$, $\theta=90^\circ$ is a solution of the equation.

If $$\sqrt{2}-2\cos\theta=0;\ \therefore\ \cos\theta=\frac{1}{\sqrt{2}}.$$

But $$\cos 45^\circ=\frac{1}{\sqrt{2}};$$

$\therefore$ $\theta=45^\circ$ is a solution of the equation; $\therefore$ $\theta=90^\circ$ or 45°.

14. $4\sin^2\theta+2\sin\theta=1$, $\sin^2\theta+\frac{1}{2}\cdot\sin\theta=\frac{1}{4}$, $\sin^2\theta+\frac{1}{2}\cdot\sin\theta+\frac{1}{16}=\frac{5}{16}$;

$$\therefore\ \sin\theta=\tfrac{1}{4}(\pm\sqrt{5}-1).$$

But $\sin 18^\circ=\frac{1}{4}(\sqrt{5}-1)$; $\therefore$ $\theta=18^\circ$ is a solution of the equation.

15. $3\tan^2\theta-4\sin^2\theta=1$; $\therefore$ $3\frac{\sin^2\theta}{\cos^2\theta}-4\sin^2\theta=1$;

$\therefore$ $3\sin^2\theta-4\sin^2\theta(1-\sin^2\theta)=1-\sin^2\theta$; $\therefore$ $4\sin^4\theta=1$; $\therefore$ $\sin\theta=\pm\frac{1}{\sqrt{2}}$.

But $\sin 45^\circ=\frac{1}{\sqrt{2}}$; $\therefore$ $\theta=45^\circ$ is a solution of the equation.

16. $2\sin^2\theta+\sqrt{2}\sin\theta=2$; $\therefore$ $\sin^2\theta+\frac{1}{\sqrt{2}}\sin\theta=1$;

$\therefore$ $\sin^2\theta+\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{8}=\frac{9}{8}$; $\therefore$ $\sin\theta+\frac{1}{2\sqrt{2}}=\pm\frac{3}{2\sqrt{2}}$; $\therefore$ $\sin\theta=-\sqrt{2}$ or $\frac{1}{\sqrt{2}}$.

The value $-\sqrt{2}$ is inadmissible, for there is no angle of which the sine is numerically greater than 1; $\therefore$ $\sin\theta=\frac{1}{\sqrt{2}}$.

But $\sin 45^\circ=\frac{1}{\sqrt{2}}$; $\therefore$ $\theta=45^\circ$ is a solution of the equation.

17. $\cos^2\theta-\sqrt{3}\cos\theta+\frac{3}{4}=0$; $\therefore$ $\cos\theta=\frac{1}{2}\sqrt{3}$.

But $\cos 30^\circ=\frac{1}{2}\sqrt{3}$; $\therefore$ $\theta=30^\circ$ is a solution of the equation.

18. $\cos^2\theta+2\sin^2\theta-\frac{5}{2}\sin\theta=0$;

$$\therefore\ 1+\sin^2\theta-\tfrac{5}{2}\sin\theta=0;\ \therefore\ \sin\theta=2\text{ or }\tfrac{1}{2}.$$

The value 2 is inadmissible for there is no angle of which the sine is greater than 1; $\therefore$ $\sin\theta=\frac{1}{2}$.

But $\sin 30^\circ=\frac{1}{2}$; $\therefore$ $\theta=30^\circ$ is a solution of the equation.

EXAMPLES. XXI. PAGE 85.

1. $3\sin 60° - 4\sin^3 60° = 3 \cdot \frac{\sqrt{3}}{2} - 4 \cdot \left(\frac{\sqrt{3}}{2}\right)^3 = 3 \cdot \frac{\sqrt{3}}{2} - 3 \cdot \frac{\sqrt{3}}{2} = 0,$

$$4\cos^3 30° - 3\cos 30° = 4 \cdot \left(\frac{\sqrt{3}}{2}\right)^3 - 3 \cdot \frac{\sqrt{3}}{2} = \frac{3}{2} \cdot \sqrt{3} - \frac{3}{2} \cdot \sqrt{3} = 0;$$

$$\therefore\ 3\sin 60° - 4\sin^3 60° = 4\cos^3 60° - 3\cos 60°.$$

2. $\tan 30°(1 + \cos 30° + \cos 60°) = \frac{1}{\sqrt{3}}\left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{3+\sqrt{3}}{2\sqrt{3}} = \frac{1+\sqrt{3}}{2},$

$$\sin 30° + \sin 60° = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3} = \tfrac{1}{2}(1+\sqrt{3});$$

$$\therefore\ \tan 30°(1 + \cos 30° + \cos 60°) = \sin 30° + \sin 60°.$$

3. $2\cos^2\theta - 7\cos\theta + 3 = 0$, putting x for $\cos\theta$ we have

$$2x^2 - 7x + 3 = 0,\ x^2 - \tfrac{7}{2}x + \tfrac{49}{16} = \tfrac{49}{16} - \tfrac{3}{2} = \tfrac{25}{16},\ x - \tfrac{7}{4} = \pm\tfrac{5}{4};$$

$$\therefore\ x = \tfrac{7}{4} \pm \tfrac{5}{4} = 3 \text{ or } \tfrac{1}{2};\ \therefore\ \cos\theta = 3 \text{ or } \tfrac{1}{2}.$$

The value 3 is inadmissible, for there is no angle of which the cosine is greater than 1.

4. $8\cos^2\theta - 8\cos\theta + 1 = 0$ putting x for $\cos\theta$;

$$\therefore\ x^2 - x = -\tfrac{1}{8},\ x^2 - x + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{1}{8} = \tfrac{1}{8} = \tfrac{2}{16},\ \therefore\ x - \tfrac{1}{2} = \pm\tfrac{1}{4}\sqrt{2};$$

$$\therefore\ x = \tfrac{1}{2} \pm \tfrac{1}{4}\sqrt{2} = \tfrac{1}{4}(2 \pm \sqrt{2});\ \therefore\ \cos\theta = \tfrac{1}{4}(2 \pm \sqrt{2}).$$

Both values are admissible as they are neither greater than 1.

5. $8\sin^2\theta - 10\sin\theta + 3 = 0$, put x for $\sin\theta$;

$$\therefore\ 8x^2 - 10x + 3 = 0;\ \therefore\ x^2 - \tfrac{5}{4} \cdot x + (\tfrac{5}{8})^2 = \tfrac{25}{64} - \tfrac{3}{8} = \tfrac{1}{64};\ \therefore\ x - \tfrac{5}{8} = \pm\tfrac{1}{8};$$

$$\therefore\ x = \tfrac{5}{8} \pm \tfrac{1}{8} = \tfrac{3}{4} \text{ or } \tfrac{1}{2};\ \therefore\ \sin\theta = \tfrac{3}{4} \text{ or } \tfrac{1}{2};$$

$$\therefore \text{ one value of } \sin\theta \text{ is } \frac{\pi}{6}.$$

6. $12\tan^2\theta - 13\tan\theta + 3 = 0$ put x for $\tan\theta$;

$$\therefore\ 12x^2 - 13x + 3 = 0;\ \therefore\ x^2 - \tfrac{13}{12}x + (\tfrac{13}{24})^2 = (\tfrac{13}{24})^2 - 3 = \tfrac{25}{576};\ \therefore\ x - \tfrac{13}{24} = \pm\tfrac{5}{24};$$

$$\therefore\ x = \tfrac{13}{24} \pm \tfrac{5}{24} = \tfrac{3}{4} \text{ or } \tfrac{1}{3};\ \therefore\ \tan\theta = \tfrac{3}{4} \text{ or } \tfrac{1}{3}.$$

7. $3\cos^2\theta + 2 \cdot \sqrt{3} \cdot \cos\theta = 5\tfrac{1}{4}$ put $\cos\theta = x$; $\therefore\ 3x^2 + 2 \cdot \sqrt{3} \cdot x = \tfrac{21}{4}$;

$$\therefore\ x^2 + \frac{2}{\sqrt{3}} \cdot x + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{21}{12} + \frac{1}{3} = \frac{25}{12};\ \therefore\ x + \frac{1}{\sqrt{3}} = \pm\frac{5}{2\sqrt{3}};$$

$$\therefore\ x = \pm\frac{5}{2\sqrt{3}} - \frac{2}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} \text{ or } -\frac{7}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \text{ or } -\frac{7\sqrt{3}}{6}.$$

The value $-\frac{7\sqrt{3}}{6}$ is inadmissible for there can be no angle of which the cosine is numerically greater than 1; $\therefore\ \cos\theta = \tfrac{1}{2}\sqrt{3}$.

But $\cos 30° = \tfrac{1}{2}\sqrt{3}$; $\therefore\ \theta = 30°$ is a solution of the equation

i.e. $\theta = \tfrac{1}{6}\pi$,, ,, ,, ,,

8. $\sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta=(\sin^2\theta+\cos^2\theta)^2=1.$

9. $\cos^4 A+2\sin^2 A\,.\cos^2 A=\cos^2 A\,(\cos^2 A+2\sin^2 A)$

$$=\cos^2 A\,(\cos^2 A+\sin^2 A+\sin^2 A)=\cos^2 A\,(1+\sin^2 A)$$
$$=(1-\sin^2 A)\,(1+\sin^2 A)=1-\sin^4 A.$$

10. $\sin^6 A+\cos^6 A=(\sin^2 A+\cos^2 A)\,(\sin^4 A-\sin^2 A\cos^2 A+\cos^4 A)$

$$=\sin^4 A-\sin^2 A\cos^2 A+\cos^4 A=\sin^4 A-\sin^2 A\,(1-\sin^2 A)+(1-\sin^2 A)^2$$
$$=\sin^4 A-\sin^2 A+\sin^4 A+1-2\sin^2 A+\sin^4 A=1-3\sin^2 A+3\sin^4 A.$$

11. $1+\tan^4\theta=1+\dfrac{\sin^4\theta}{\cos^4\theta}=1+\dfrac{(1-\cos^2\theta)^2}{\cos^4\theta}=1+\dfrac{1-2\cos^2\theta+\cos^4\theta}{\cos^4\theta}$

$$=\frac{1-2\cos^2\theta+2\cos^4\theta}{\cos^4\theta}.$$

12. $\dfrac{\cos A+\cos B}{\sin A-\sin B}+\dfrac{\sin A+\sin B}{\cos A-\cos B}$

$$=\frac{(\cos A+\cos B)\,(\cos A-\cos B)+(\sin A+\sin B)\,(\sin A-\sin B)}{(\sin A-\sin B)\,(\cos A-\cos B)}$$
$$=\frac{\cos^2 A-\cos^2 B+\sin^2 A-\sin^2 B}{(\sin A-\sin B)\,(\cos A-\cos B)}=\frac{(\cos^2 A+\sin^2 A)-(\cos^2 B+\sin^2 B)}{(\sin A-\sin B)\,(\cos A-\cos B)}$$
$$=\frac{1-1}{(\sin A-\sin B)\,(\cos A-\cos B)}=0.$$

13. $(\sec A-\tan A)^2=\left(\dfrac{1}{\cos A}-\dfrac{\sin A}{\cos A}\right)^2=\left(\dfrac{1-\sin A}{\cos A}\right)^2$

$$=\frac{(1-\sin A)^2}{\cos^2 A}=\frac{(1-\sin A)^2}{1-\sin^2 A}=\frac{1-\sin A}{1+\sin A}.$$

14. We may proceed by figure after the manner of Examples XVIII., or since $\operatorname{cosec}\theta=\dfrac{1}{\sin\theta}$, and as θ *continuously increases* from 0 to $\frac{1}{2}\pi$, $\sin\theta$ *continuously increases* from zero to 1; $\therefore$ $\operatorname{cosec}\theta$ *continuously diminishes* from $\dfrac{1}{\text{zero}}$ to 1, that is from an infinitely large quantity to 1.

15. Since $\cot\theta=\dfrac{1}{\tan\theta}$ and as θ continuously decreases from $\frac{1}{2}\pi$ to 0, $\tan\theta$ continuously decreases from infinity to zero; $\therefore$ $\cot\theta$ continuously increases from $\dfrac{1}{\text{infinity}}$ to $\dfrac{1}{\text{zero}}$, that is from zero to infinity.

16. $\sin(\theta+\phi)=\frac{1}{2}\sqrt{3}.$

But $\quad\sin 60^\circ=\frac{1}{2}\sqrt{3}$; $\therefore$ $\theta+\phi=60^\circ$ is one solution,

$$\cos(\theta-\phi)=\tfrac{1}{2}\sqrt{3}.$$

But $\quad\cos 30^\circ=\frac{1}{2}\sqrt{3}$; $\therefore$ $\theta-\phi=30^\circ$ is one solution.

$$\theta+\phi=60^\circ \text{ and } \theta-\phi=30^\circ;\ \therefore\ 2\theta=90^\circ \text{ and } \theta=45^\circ=\tfrac{1}{4}\pi,$$
$$2\phi=30^\circ;\ \therefore\ \phi=15^\circ=\tfrac{1}{12}\pi.$$

Or, since $\sin(\theta+\phi)=\frac{1}{2}\sqrt{3}$ *and* $\cos(\theta-\phi)=\frac{1}{2}\sqrt{3}$;

$$\therefore \sin(\theta+\phi)=\cos(\theta-\phi).$$

But $\sin(\theta+\phi)=\cos\{90°-(\theta+\phi)\}$, (Art. 118); $\therefore \theta-\phi=90°-\theta-\phi$;
$\therefore 2\theta=90°$; $\therefore \theta=45°=\frac{1}{4}\pi$, and since $\theta+\phi=60°$; $\therefore \phi=15°=\frac{1}{12}\pi$.

EXAMPLES. XXII. Page 89.

Let A, B, C, D, E be points in LR, such that the measures of AB, BC, CD, DE are 1, 2, 3, 4 respectively.

1. $AB+BC+CD=1+2+3=+6$.

2. $AB+BC+CA=1+2-(2+1)=0$.

3. $BC+CD+DE+EC=+2+3+4-(4+3)=+2$.

4. $AD-CD=(1+2+3)-3=+3$.

5. $AD+DB+BE=(1+2+3)-(3+2)+2+3+4=+10$.

6. $BC-AC+AD-BD=2-(1+2)+(1+2+3)-(2+3)=0$.

7. $CD+DB+BE=3-(3+2)+(2+3+4)=+7$.

8. $CD-BD+BA+AC+CE=3-(2+3)-1+(1+2)+(3+4)=+7$.

EXAMPLES. XXIII. Page 91.

1. The angle of 270° is the angle described by OP turning about O from the position OR, in the positive direction, to the position OD thus describing 3 right angles. [The angle ROD.]

2. 370° = 360° + 10°, i. e. the angle described by OP turning about O from the position OR, in the positive direction, making *one complete* revolution *and* then turning in the positive direction through the angle of 10°.

3. 425° = 360° + 65°, i. e. the angle described by OP turning about O from the position OR, in the *positive* direction, making *one complete* revolution *and* then turning in the positive direction through the angle of 65°.

4. 590° = 360° + 180° + 50°, i. e. the angle described by OP turning about O from the position OR, in the *positive* direction, making one complete revolution, then turning in the positive direction until it has described an angle of 180° (i. e. being in the same straight line with OR), and from that position turning round about O in the positive direction, through the angle of 50°. [Fig. III. E. T. p. 96.]

5. The angle described by OP turning about O, from the position OR in the *negative* direction through the angle of 30°. [ROP, in fig. II. E. T. p. 110.]

6. −330° = −360° + 30°, i. e. the angle described by OP turning about O from the position OR, in the *negative* direction, making one complete revolution (when it has turned again to the position OR) and then turning back in the *positive* direction through an angle of 30°. [ROP, fig. I. E. T. p. 110.]

7. $-480° = -360° - 180° + 60°$, i.e. the angle described by OP turning about O from the position OR, in the negative direction, making *one* complete revolution, then turning again in the negative direction until it has described an angle of 180° (thus being in the same straight line with OR), and from that position turning back in the *positive* direction through an angle of 60°. [Fig. II. E. T. p. 96.]

8. $-750° = -720° - 30°$, i.e. the angle described by OP turning about O from the position OR in the *negative* direction, making *two* complete revolutions (when it has turned *twice* again to the position OR) *and* then turning in the *negative* direction through an angle of $-30°$. [ROP_1, fig. II. E. T. p. 110.]

9. $\frac{27}{4}\pi = 6\pi + \frac{3}{4}\pi = 6 \times 180° + 135° = 3 \times 360° + 180° - 45°$, i.e. the angle described by OP turning about O from the position OR in the *positive* direction, making *three* complete revolutions, (when it has turned *thrice* again to the position OR), then turning again in the positive direction through an angle of 180° (being thus in the same straight line with OR) and from that position turning back in the *negative* direction through an angle of $-45°$. [Fig. E. T. p. 94.]

10. $2n\pi + \frac{1}{6}\pi = n \times 360° + 30°$, i.e. the angle described by OP turning about O from the position OR in the *positive* direction, making n complete revolutions (when it has turned n times again to the position OR) *and* then turning again in the *positive* direction through the angle of $+30°$. [ROP. fig. E. T. p. 104.]

NOTE. The case considered above is when n is positive; if n is negative we may retain the same figure and in the explanation write '*negative*' for '*positive*', vid. Examples VI. (10).

11. $(2n+1)\pi + \frac{1}{3}\pi = 2n\pi + \pi + \frac{1}{3}\pi = n \times 360° + 180° + 60°$. When OP has described the angle $n \times 360°$ it is n times again in the position OR; from this position let OP turning about O, in the positive direction, describe the angle 180° (i.e. is in the same straight line with OR), for this position turning again about O in the positive direction let OP describe $+60°$, vid. Note to (10). [Fig. III. E. T. p. 96.]

12. Fig. E. T. p. 94.

$$(2n+1)\pi - \tfrac{1}{4}\pi = 2n\pi + \tfrac{3}{4}\pi = n \times 360° + 135° = n \times 360° + 180° - 45°.$$

When OP has described the angle $n \times 360°$ it is n times again in the position OR. We may proceed for the rest as in (9).

13. $2n\pi - \frac{1}{2}\pi = n \times 360° - 90°$, i.e. the angle described by OP turning about O from the position OR, making n complete revolutions (thus being n times again in the position OR) and then turning back in the *negative* direction through an angle of $-90°$. [The angle ROD.]

14. $(2n+1)\pi - \frac{1}{2}\pi = 2n\pi + \frac{1}{2}\pi = n \times 360° + 90°$. The angle described by OP turning about O from the position OR, making n complete revolutions (thus being n times again in the position OR) and then turning in the *positive* direction through an angle of $+90°$, vid. Note to (10). [The angle ROU.]

EXAMPLES. XXIV. Page 94.

1. $120° = 180° - 60°$ represents an angle in the second quadrant.

2. $340° = 360° - 20° = 2 \times 180° - 20°$ is an angle in the *fourth* quadrant. (Art. 130, ii.)

3. $490° = 540° - 50° = 3 \times 180° - 50°$ is an angle in the *second* quadrant. (Art. 130, iii.)

4. $-100° = -180° + 80°$ is an angle in the *third* quadrant. (Art. 130, iv.)

5. $-380° = -360° - 20° = -2 \times 180° - 20°$ is an angle in the *fourth* quadrant. (Art. 30, ii.)

6. $-1000° = -6 \times 180° + 80°$, i.e. an angle in the *first* quadrant. (Art. 130, i.)

7. $\frac{2}{3}\pi = \pi - \frac{1}{3}\pi$, i.e. an angle in the *second* quadrant. (Art. 130, iii.)

8. $10\pi + \frac{1}{4}\pi =$ an angle in the *first* quadrant. (Art. 130, i.)

9. $9\pi - \frac{3}{4}\pi = 8\pi + \frac{1}{4}\pi$, i.e. an angle in *first* quadrant. (Art. 130, i.)

10. $2n\pi - \frac{1}{4}\pi =$ an angle in the *fourth* quadrant. (Art. 130, ii.)

11. $(2n+1)\pi + \frac{2}{3}\pi = (2n+1)\pi + \pi - \frac{1}{3}\pi = 2m\pi - \frac{1}{3}\pi =$ an angle in the *fourth* quadrant. (Art. 130, ii.)

12. $n\pi + \frac{1}{6}\pi$. If n be even $= 2m$, then $n\pi + \frac{1}{6}\pi = 2m\pi + \frac{1}{6}\pi$, i.e. an angle in the *first* quadrant. (Art. 130, i.)

If n be odd $= 2m+1$, then $n\pi + \frac{1}{6}\pi = (2m+1)\pi + \frac{1}{6}\pi$, i.e. an angle in the *third* quadrant. (Art. 130, iv.)

EXAMPLES. XXV. Page 98.

1. The angle of 60° is in the first quadrant;

∴ the sine is positive.
the cosine ,, ,,
the tangent ,, ,, (Art. 133, i.),

i.e. the signs of the sine, cosine and tangent are +, +, +.

2. The angle of 135° is in the second quadrant;

∴ the sine is positive,
the cosine is negative,
the tangent is negative (Art. 133, ii.),

i.e. the signs of the sine, cosine and tangent are +, −, −.

3. The angle of 265° is in the third quadrant;

∴ the sine is negative,
the cosine is negative,
the tangent is positive (Art. 133, iii.),

i.e. the signs of the sine, cosine and tangent are −, −, +.

4. The angle of 275° is in the fourth quadrant;

$\therefore$ the sine is negative,
the cosine is positive,
the tangent is negative (Art. 133, iv.),

i. e. the signs of the sine, cosine and tangent are $-$, $+$, $-$.

5. The angle of $-10°$ is in the fourth quadrant;

$\therefore$ the sign is negative,
the cosine is positive,
the tangent is negative (Art. 133, iv.),

i.e. the signs of the sine, cosine and tangent are $-$, $+$, $-$.

6. The angle of $-91°$ is in the third quadrant;

$\therefore$ the sine is negative,
the cosine is negative,
the tangent is positive (Art. 133, iii.),

i. e. the signs of the sine, cosine and tangent are $-$, $-$, $+$.

7. The angle of $-193°$ is in the second quadrant;

$\therefore$ the sine is positive,
the cosine is negative,
the tangent is negative (Art. 133, II.),

i. e. the signs of the sine, cosine, and tangent are $+$, $-$, $-$.

8. The angle of $-350°$ is in the first quadrant;

$\therefore$ the sine is positive,
the cosine is positive,
the tangent is positive (Art. 133, I.),

i. e. the signs of the sine, cosine and tangent are $+$, $+$, $+$.

9. The angle of $-1000°$ is in the first quadrant [Examples XXIV. (6)];

$\therefore$ the sine is positive,
the cosine is positive,
the tangent is positive (Art. 133, I.),

i. e. the signs of the sine, cosine, and tangent are $+$, $+$, $+$.

10. The angle of $2n\pi + \pi\frac{1}{4}$ is in the first quadrant;

$\therefore$ the sine is positive,
the cosine is positive,
the tangent is positive (Art. 133, I.),

i. e. the signs of the sine, cosine and tangent are $+$, $+$, $+$.

11. The angle of $2n\pi + \frac{3}{4}\pi$ is in the second quadrant;

$\therefore$ the sine is positive,
the cosine is negative,
the tangent is negative (Art. 133, II.),

i. e. the signs of the sine, cosine, and tangent are $+$, $-$, $-$.

12. The angle of $2n\pi - \frac{1}{6}\pi$ is in the fourth quadrant;

$\therefore$ the sine is negative,
the cosine is positive,
the tangent is negative (Art. 133, IV.),

i. e. the signs of the sine, cosine, and tangent are $-$, $+$, $-$.

EXAMPLES. XXVI. PAGE 100.

1. 150° is an angle in the second quadrant.

Let the angle ROP be 150° (fig. II. E. T. p. 96).

Then the angle $POL = 180° - 150° = 30°$.

Therefore the Trigonometrical Ratios of 150° = those of 30° *numerically;* and in the second quadrant the sine is *positive* and the cosine and tangent are each *negative;*

$$\therefore \sin 150° = \tfrac{1}{2};\ \cos 150° = -\tfrac{1}{2}\sqrt{3};\ \tan 150° = -\tfrac{1}{3}\sqrt{3}.$$

2. 135° is an angle in the second quadrant.

Let the angle ROP be 135° (fig. II. E. T. p. 96).

Then the angle $POL = 180° - 135° = 45°$.

Therefore the Trigonometrical Ratios of 135° = those of 45° *numerically;* and in the second quadrant the sine is *positive*, and the cosine and the tangent are *negative;*

$$\therefore \sin 135° = \tfrac{1}{2}\sqrt{2};\ \cos 135° = -\tfrac{1}{2}\sqrt{2} \tan 135° = -1.$$

3. $-240°$ is an angle in the second quadrant.

Let the angle ROP (fig. II. E. T. p. 96) described by OP starting from the position OR and turning about O in the *negative* direction be $-240°$.

Here angle $POL = -240° - (-180°) = -60°$;

$\therefore$ the Trigonometrical Ratios of $-240°$ = those of $-60°$ *numerically;* i. e. = those of 60° *numerically;* and in the second quadrant the sine is *positive* and the cosine and tangent are *negative;*

$$\therefore \sin -240° = +\tfrac{1}{2}\sqrt{3};\ \cos -240° = -\tfrac{1}{2},\ \tan -240° = -\sqrt{3}.$$

4. 330° is an angle in the fourth quadrant.

Let the angle $ROP = 330°$ (fig. IV. E. T. p. 96).

Then the angle $POR = 330° - 360° = -30°$, i.e. an angle in the fourth quadrant.

$\therefore$ the Trigonometrical Ratios of 330° = those of 30° *numerically.*

Therefore also the Trigonometrical Ratios of 330° = those of 30° numerically; and in the fourth quadrant the cosine is positive and the sine and tangent negative;

$$\therefore \sin 330° = -\tfrac{1}{2};\ \cos 330° = +\tfrac{1}{2}\sqrt{3};\ \tan 330° = -\tfrac{1}{3}\sqrt{3}.$$

5. $-45°$ is an angle in the fourth quadrant.

The Trigonometrical Ratios of $-45°$ = those of $45°$ numerically; and in the fourth quadrant the cosine is positive and the sine and tangent negative;

$$\therefore \sin -45° = -\tfrac{1}{2}\sqrt{2};\ \cos -45° = +\tfrac{1}{2}\sqrt{2};\ \tan 45° = -1.$$

6. $-300°$ is an angle in the first quadrant.

Let the angle ROP (fig. I. E. T. p. 96) described by OP revolving about O for the negative direction, from the position, be the angle of $-300°$;

$$\therefore POR = 360° - 300° = 60°;$$

$\therefore$ the Trigonometrical Ratios of $-300°$ = those of $60°$ *numerically;* and in the first quadrant, the sine, cosine and tangent are each positive;

$$\therefore \sin -300° = +\tfrac{1}{2}\sqrt{3},\ \cos -300° = +\tfrac{1}{2};\ \tan -300° = \sqrt{3}.$$

7. $225°$ is an angle in the third quadrant.

Let the angle ROP be $225°$ (fig. III. E. T. p. 96).

Here the angle $POL = 225° - 180° = 45°$;

$\therefore$ the Trigonometrical Ratios of $225°$ = those of $45°$ *numerically;* and in the third quadrant the sine and cosine are each negative and the tangent is positive; $\therefore \sin 225° = -\tfrac{1}{2}\sqrt{2};\ \cos 225° = -\tfrac{1}{2}\sqrt{2};\ \tan 225° = +1.$

8. $-135°$ is an angle in the third quadrant.

Let the angle ROP be $-135°$ (fig. III. E. T. p. 96).

Here the angle $POL = 180° - 135° = 45°$;

$\therefore$ the Trigonometrical Ratios of $-135°$ = those of $45°$ *numerically;* and in the *third* quadrant the sine and cosine are each *negative* and the tangent is *positive;*

$$\therefore \sin -135° = -\tfrac{1}{2}\sqrt{2};\ \cos -135° = -\tfrac{1}{2}\sqrt{2};\ \tan -135° = +1.$$

9. $390°$ is an angle in the first quadrant.

Let the angle $ROP = 390°$ (fig. I. E. T. p. 96);

$\therefore$ the angle $POR = 390° - 360° = 30°$;

$\therefore$ the Trigonometrical Ratios of $390°$ = those of $30°$ *numerically;* and in the first quadrant the sine, cosine, and tangent are each positive;

$$\therefore \sin 390° = +\tfrac{1}{2};\ \cos 390° = +\tfrac{1}{2}\sqrt{3},\ \tan 390° = +\tfrac{1}{3}\sqrt{3}.$$

10. $750°$ is an angle in the first quadrant.

Let the angle $ROP = 750°$ (fig. I. E. T. p. 96).

Here the angle $POR = 750° - 720° = 30°$;

$\therefore$ the Trigonometrical Ratios of $750°$ = those of $30°$ *numerically;* and in the first quadrant the sine, cosine, and tangent are each positive;

$$\therefore \sin 750° = +\tfrac{1}{2};\ \cos 750° = +\tfrac{1}{2}\sqrt{3};\ \tan 750° = \tfrac{1}{2}\sqrt{3}.$$

11. $-840°$ is an angle in the third quadrant.

Let the angle $ROP = -840°$ (fig. III. E. T. p. 96).

Here $POL = 900° - 840° = 60°$;

$\therefore$ the Trigonometrical Ratios of $-840°$ = those of $60°$; and in the third quadrant, the sine and cosine are negative and the tangent positive;

$$\therefore \sin -840° = -\tfrac{1}{2}\sqrt{3},\ \cos -840° = -\tfrac{1}{2},\ \tan -840° = +\sqrt{3}.$$

12. $1020°$ is an angle in the fourth quadrant.

Let the angle $ROP = 1020°$ (fig. IV. E. T. p. 96).

Here $POR = 3 \times 360° - 1020° = 60°$;

$\therefore$ the Trigonometrical Ratios of $1020°$ = those of $60°$; and in the *fourth* quadrant the sine and tangent are each *negative*, and the cosine is *positive;*

$$\therefore \sin 1020° = -\tfrac{1}{2}\sqrt{3}, \ \cos 1020° = +\tfrac{1}{2}, \ \tan 1020° = -\sqrt{3}.$$

13. $2n\pi + \frac{1}{4}\pi$ is an angle in the first quadrant.

Let the angle $ROP = 2n\pi + \frac{1}{4}\pi = n \,.\, 360° + 45°$ (fig. I. E. T. p. 96).

Then angle $POR = 45°$;

$\therefore$ the Trigonometrical Ratios of $2n\pi + \frac{1}{4}\pi$ = those of $45°$ *numerically;* and in the first quadrant the sine, cosine and tangent are each *positive;*

$$\therefore \sin(2n\pi + \tfrac{1}{4}\pi) = \tfrac{1}{2}\sqrt{2}\,; \ \cos(2n\pi + \tfrac{1}{4}\pi) = \tfrac{1}{2}\sqrt{2}\,; \ \tan(2n\pi + \tfrac{1}{4}\pi) = +1.$$

14. $(2n+1)\pi - \frac{1}{3}\pi$ is an angle in the second quadrant.

Let the angle $ROP = (2n+1)\pi - \frac{1}{3}\pi$, [i.e. $n \,.\, 360° + 120°$] (fig. II. E. T. p. 96).

Then angle $POL = 180° - 120° = 60°$;

$\therefore$ the Trigonometrical Ratios of $n \,.\, 360° + 120°$ = those of $60°$; and in the *second* quadrant the sine is *positive* and the cosine and tangent are each *negative;*

$$\therefore \sin[(2n+1)\pi - \tfrac{1}{3}\pi] = +\tfrac{1}{2}\sqrt{3}\,; \ \cos[(2n+1)\pi - \tfrac{1}{3}\pi] = -\tfrac{1}{2}\,;$$
$$\tan[(2n+)\pi - \tfrac{1}{3}\pi] = -\sqrt{3}.$$

15. $(2n-1)\pi + \frac{1}{6}\pi$ is an angle in the third quadrant.

Let the angle $ROP = (2n-1)\pi + \frac{1}{6}\pi$, i.e. $n \,.\, 360° - 150°$ (fig. III. E. T. p. 96).

Then angle $POL = 180° - 150° = 30°$;

$\therefore$ the Trigonometrical Ratios of $(2n-1)\pi + \frac{1}{6}\pi$ = those of $30°$; and in the third quadrant the sine and cosine are negative and the tangent positive;

$$\therefore \sin\{(2n-1)\pi + \tfrac{1}{6}\pi\} = -\tfrac{1}{2}\,; \ \cos\{(2n-1)\pi + \tfrac{1}{6}\pi\} = -\tfrac{1}{2}\sqrt{3}\,;$$
$$\tan\{(2n-1)\pi + \tfrac{1}{6}\pi\} = +\tfrac{1}{3}\sqrt{3}.$$

EXAMPLES. XXVII. PAGE 100.

N.B. The references here are to the figures and construction of p. 96, E. T.

1.
$$\cos A = \frac{OM}{OP}.$$

As A increases from $0°$ to $90°$, OM decreases from OP to zero and is positive; $\therefore$ $\cos A$ decreases from 1 to zero and is positive.

As A increases from $90°$ to $180°$, OM increases from zero to OP and is negative; $\therefore$ $\cos A$ increases from zero to 1 and is negative.

As A increases from 180° to 270°, OM decreases from OP to zero and is negative; $\therefore$ $\cos A$ decreases from 1 to zero and is negative.

As A increases from 270° to 360°, OM increases from zero to OP and is positive; $\therefore$ $\cos A$ increases from zero to 1 and is positive.

2. $$\tan A = \frac{MP}{OM}.$$

As A increases from 0° to 90°, MP increases from zero to OP and is positive; OM decreases from OP to zero and is positive;

$\therefore$ $\tan A$ increases from zero to 'infinity' and is positive.

As A increases from 90° to 180°, MP decreases from OP to zero and is positive, OM increases from zero to OP and is negative;

$\therefore$ $\tan A$ decreases from 'infinity' to zero and is negative.

As A increases from 180° to 270°, MP increases from zero to OP and is negative, OM decreases from OP to zero, and is negative;

$\therefore$ $\tan A$ increases from zero to infinity and is positive.

As A increases from 270° to 360°, MP decreases from OP to zero and is negative; OM increases from zero to OP and is positive;

$\therefore$ $\tan A$ decreases from infinity to zero and is negative.

3. We may proceed by the figure to trace the changes in the cotangent in the same way as we have done in (1) and (2) or by means of the formula $\cot A = \frac{1}{\tan A}$, we may infer the changes of the cotangent from the known changes of the tangent.

As A increases from 0° to 90°, $\tan A$ increases from zero to infinity and is positive; $\therefore$ $\cot A$ decreases from infinity to zero and is positive.

As A increases from 90° to 180°, $\tan A$ decreases from infinity to zero and is negative;

$\therefore$ $\cot A$ increases from zero to infinity and is negative.

As A increases from 180° to 270°, $\tan A$ increases from zero to infinity and is positive;

$\therefore$ $\cot A$ decreases from infinity to zero and is positive.

As A increases from 270° to 360°, $\tan A$ decreases from infinity to zero and is negative;

$\therefore$ $\cot A$ increases from zero to infinity and is negative.

4. We may proceed by the figure or by means of the formula $\sec A = \frac{1}{\cos A}$, we may infer the changes of $\sec A$ from the known changes of $\cos A$.

As A increases from 0° to 90°, $\cos A$ decreases from 1 to 0 and is positive;

$\therefore$ $\sec A$ increases from 1 to ∞ and is positive.

As A increases from 90° to 180°, $\cos A$ increases from 0 to 1 and is negative; $\therefore$ $\sec A$ decreases from ∞ to 1 and is negative.

As A increases from 180° to 270°, $\cos A$ decreases from 1 to 0 and is negative; $\therefore$ $\sec A$ increases from 1 to ∞ and is negative.

As A increases from 270° to 360°, $\cos A$ increases from 0 to 1 and is positive; $\therefore$ $\sec A$ decreases from ∞ to 1 and is positive.

5. $$\operatorname{cosec} A = \frac{OP}{MP}.$$

As A increases from 0° to 90°, MP increases from zero to OP and is positive; $\therefore$ $\operatorname{cosec} A$ decreases from infinity to 1 and is positive.

As A increases from 90° to 180°, MP decreases from OP to zero and is positive; $\therefore$ $\operatorname{cosec} A$ increases from 1 to infinity and is positive.

As A increases from 180° to 270°, MP increases from zero to OP and is negative; $\therefore$ $\operatorname{cosec} A$ decreases from infinity to 1 and is negative.

As A increases from 270° to 360°, MP decreases from OP to zero and is negative; $\therefore$ $\operatorname{cosec} A$ increases from 1 to infinity and is negative.

6. Since $\sin A$ is never numerically greater than 1, $1-\sin A$ is never negative. Its least value is when $\sin A=0$, i.e. when $A=0$; its greatest value is when $\sin A$ is negative and numerically greatest, i.e. when $\sin A=-1$, when $A=270°$.

As A changes from 0° to 90°, $\sin A$ changes from 0 to 1 and is positive;

$\therefore$ $1-\sin A$ changes from 1 to 0 and is positive.

As A changes from 90° to 180°, $\sin A$ changes from 1 to 0 and is positive;

$\therefore$ $1-\sin A$ changes from 0 to 1 and is positive.

As A changes from 180° to 270°, $\sin A$ changes from 0 to 1 and is negative;

$\therefore$ $1-\sin A$ changes from 1 to 2 and is positive.

As A changes from 270° to 360°, $\sin A$ changes from 1 to 0 and is negative;

$\therefore$ $1-\sin A$ changes from 2 to 1 and is positive.

7. $\sin^2 A$ is never negative.

As A changes from 0° to 90°, $\sin A$ changes from 0 to 1 and is positive;

$\therefore$ $\sin^2 A$ changes from 0 to 1 and is positive.

As A changes from 90° to 180°, $\sin A$ changes from 1 to 0 and is positive;

$\therefore$ $\sin^2 A$ changes from 1 to 0 and is positive.

As A changes from 180° to 270°, $\sin A$ changes from 0 to 1 and is negative;

$\therefore$ $\sin^2 A$ changes from 0 to 1 and is positive.

As A changes from 270° to 360°, $\sin A$ changes from 1 to 0 and is negative;

$\therefore$ $\sin^2 A$ changes from 1 to 0 and is positive.

8. As A changes from 0° to 90°, $\sin A$ changes from 0 to 1 and is positive, $\cos A$ changes from 1 to 0 and is positive;

$\therefore$ $\sin A \cos A$ changes from 0 to 0 and is positive.

$\sin A \cos A$ has its greatest numerical value when $\sin A = \cos A$ *, i.e. when $A=45°$ and $\sin A \cos A = +\frac{1}{2}$.

* For it will be seen in the next Chapter that $\sin A \cos A = \frac{1}{2}\sin 2A$, and $\therefore$ $\sin A \cos A$ has its greatest value when $2A = 90°$.

As A changes from 90° to 180°, $\sin A$ changes from 1 to 0 and is positive, $\cos A$ changes from 0 to 1 and is negative;

$$\therefore \sin A \cos A \text{ changes from 0 to 0 and is negative;}$$

its greatest numerical value is when $\sin A = \cos A$, i.e. when $A = 135°$ and $\sin A \cos A = -\frac{1}{2}$.

As A changes from 180° to 270°, $\sin A$ changes from 0 to 1 and is negative, $\cos A$ changes from 1 to 0 and is negative;

$$\therefore \sin A \cos A \text{ changes from 0 to 0 and is positive,}$$

its greatest numerical value is when $\sin A = \cos A$, i.e. when $A = 225°$ and $\sin A \cos A = +\frac{1}{2}$.

As A changes from 270° to 360°, $\sin A$ changes from 1 to 0 and is positive; $\cos A$ changes from 0 to 1 and is negative;

$$\therefore \sin A \cos A \text{ changes from 0 to 0 and is negative;}$$

its greatest numerical value is when $\sin A = \cos A$, i.e. when $A = 315°$ and $\sin A \cos A = -\frac{1}{2}$.

9. As A changes from 0° to 90°, $\sin A$ changes from 0 to 1 and is positive; $\cos A$ changes from 1 to 0 and is positive;

$$\therefore \sin A + \cos A \text{ changes from 1 to 1 and is positive;}$$

its greatest numerical value is when $A = 45°$, for

$$(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2(\sin^2 A + \cos^2 A) = 2;$$

$\therefore$ $(\sin A + \cos A)^2$ is greatest when $(\sin A - \cos A)^2 = 0$, i.e. when $\sin A = \cos A$, when $A = 45°$;

$$\therefore \sin A + \cos A \text{ has its greatest numerical value when } A = 45°,$$

i.e. when $\sin A + \cos A = +\sqrt{2}$.

As A changes from 90° to 180°, $\sin A$ changes from 1 to 0 and is positive; $\cos A$ changes from 0 to 1 and is negative;

$\therefore$ $\sin A + \cos A$ changes from 1 to -1 vanishing when $\sin A = -\cos A$, i.e. when $A = 135°$.

As A changes from 180° to 270° $\sin A$ changes from 0 to 1 and is negative, $\cos A$ changes from 1 to 0 and is negative;

$$\therefore \sin A + \cos A \text{ changes from } -1 \text{ to } -1;$$

it does not change sign; $\therefore$ its greatest *numerical* value may be found above to be $\sqrt{2}$ when $A = 225°$.

As A changes from 270° to 360° $\sin A$ changes from 1 to 0 and is negative; $\cos A$ changes from 0 to 1 and is positive;

$$\therefore \sin A + \cos A \text{ changes from } -1 \text{ to } 1.$$

Its numerically smallest value being when $\sin A + \cos A = 0$, i.e. when

$$\cos A = -\sin A, \text{ i.e. when } A = 315°.$$

10. $$\tan A + \cot A = \tan A + \frac{1}{\tan A}.$$

As A changes from 0° to 90°, $\tan A$ changes from 0 to ∞ and is positive; $\frac{1}{\tan A}$ changes from ∞ to 0 and is positive;

$$\therefore\ \tan A + \frac{1}{\tan A} \text{ changes from } \infty \text{ to } \infty \text{ and is positive;}$$

the least value of $\tan A + \dfrac{1}{\tan A}$ is when $A = 45°$; for

$$\left(\tan A + \frac{1}{\tan A}\right)^2 = \left(\tan A - \frac{1}{\tan A}\right)^2 + 4;$$

$\therefore$ the least value is when $\tan A - \dfrac{1}{\tan A} = 0$, i.e. when $\tan^2 A = 1$. Therefore $\tan A + \cot A$ changes from ∞ to 2 as A changes from 0° to 45°; and then changes from 2 to ∞ as A changes from 45° to 90°.

As A changes from 90° to 180°, $\tan A$ changes from ∞ to 0 and is negative; $\cot A$ changes from 0 to ∞ and is negative;

$$\therefore\ \tan A + \frac{1}{\tan A} \text{ changes from } \infty \text{ to } \infty \text{ and is negative;}$$

its least numerical value being -2 when A is 135°.

As A changes from 180° to 270°, $\tan A$ changes from 0 to ∞ and is positive, $\dfrac{1}{\tan A}$ changes from ∞ to 0 and is positive;

$$\therefore\ \tan A + \frac{1}{\tan A} \text{ changes from } \infty \text{ to } \infty \text{ and is positive;}$$

its least numerical value being 2 when A is 225°.

As A changes from 270° to 360°, $\tan A$ changes from ∞ to 0 and is negative; $\dfrac{1}{\tan A}$ changes from 0 to ∞ and is negative;

$$\therefore\ \tan A + \frac{1}{\tan A} \text{ [i.e. } \tan A + \cot A\text{] changes from } \infty \text{ to } \infty \text{ and is negative;}$$

its least numerical value being -2 when $A = 315°$.

11. As A changes from 0° to 90°, $\sin A$ changes from 0 to 1 and is positive; $\cos A$ changes from 1 to 0 and is positive;

$$\therefore\ \sin A - \cos A \text{ changes from } -1 \text{ to } 1 \text{ vanishing when } A = 45°.$$

As A changes from 90° to 180°, $\sin A$ changes from 1 to 0 and is positive; $\cos A$ changes from 0 to 1 and is negative;

$$\therefore\ \sin A - \cos A \text{ changes from 1 to 1, remaining positive,}$$

and having its greatest numerical value when $A = 135°$.

For $\quad (\sin A - \cos A)^2 + (\sin A + \cos A)^2 = 2(\sin^2 A + \cos^2 A) = 2;$

$$\therefore\ (\sin A - \cos A)^2 \text{ has its greatest value when } \sin A + \cos A = 0,$$

$$\text{i.e. when } A = 135° \text{ and } \sin A - \cos A = +\sqrt{2}.$$

As A changes from 180° to 270°, $\sin A$ changes from 0 to 1 and is negative, $\cos A$ changes from 1 to 0 and is negative;

$\therefore\ \sin A - \cos A$ changes from 1 to -1 vanishing when $\sin A - \cos A = 0$; i.e. when $A = 225°$.

As A changes from 270° to 360°, $\sin A$ changes from 1 to 0 and is negative; $\cos A$ changes from 0 to 1 and is positive;

$\therefore$ $\sin A - \cos A$ changes from -1 to -1 remaining negative and having its greatest numerical value namely $-\sqrt{2}$ when $A = 315°$.

EXAMPLES. XXVIII. Page 103.

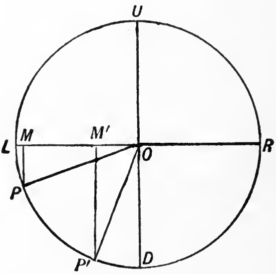
1. With the figure and construction of E. T. p. 102, let $ROP =$ the angle which OP describes from the position $OR = 30°$ and ROP' the angle described by $OP' = 90° - 30° = 60°$;

$$\therefore \sin ROP = \frac{MP}{OP} = \frac{OM'}{OP'} = \cos ROP'; \quad \therefore \sin 30° = \cos 60°.$$

2. With the figure and construction referred to above let ROP the angle which OP describes from the position $OR = 65°$, and ROP' the angle described by $OP' = 90° - 65° = 25°$;

$$\therefore \sin ROP = \frac{MP}{OP} = \frac{OM'}{OP'} = \cos ROP'; \quad \therefore \sin 65° = \cos 25°.$$

3. With the construction of p. 102 E. T., let OP and OP' be the *two revolving lines*, OP starting from the position OR and turning about O in the *positive* direction describe the angle $ROP = 195°$; OP' starting from the position OR and turning about the point O in the *negative* direction describe the angle $ROP' = -105°$.



Now angle $POL = 195° - 180° = 15°$; $\therefore$ the Trigonometrical ratios of 195° are *numerically* equal to those of 15°.

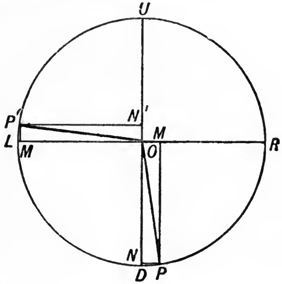
Angle $P'OD = 105° - 90° = 15°$; $\therefore$ the Trigonometrical ratios of $-105°$ = those of 15° *numerically*.

Since angle $POL =$ angle $PO'D$, $\therefore$ $MP = N'P' = OM'$;

$$\therefore \frac{MP}{OP} = \frac{N'P'}{OP'} = \frac{OM'}{OP'}; \quad \therefore \sin ROP = \cos - ROP', \text{ i.e. } \sin 195° = \cos(-105°).$$

4. Let OP, OP' be two revolving straight lines, OP starting from the position OR describe the angle $ROP = +275°$ and OP' starting from the same position describe the angle $= -185°$. Draw PM, $P'M'$ perpendiculars on OR, OL respectively, and $P'N'$ perpendicular on OU. Then angle

POD = angle $POM = (275° - 270°) = 5°$ and angle $P'OM' = (185° - 180°) = 5°$; $\therefore$ angle POM = angle $P'OM'$ and $OP = OP'$; $\therefore$ the *right-angled* triangles OPM and $OP'M'$ are equal in every respect;



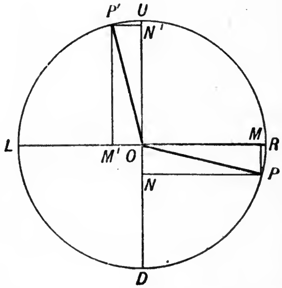
$$\therefore \frac{OM}{OP} = \frac{M'P'}{OP'} = \frac{P'N'}{OP'}; \therefore \cos 275° = \sin(-185°).$$

5. Let OP, OP' be two revolving straight lines, OP starting from the position OR describe the angle $ROP = -27°$; OP' starting from the same position describe the angle $ROP' = 117°$. Draw PM, $P'M'$ perpendiculars on OR. Angle

$OP'M'$ = angle $P'OU = (117° - 90°) = 27°$;

$\therefore$ angle $OP'M'$ = angle POM, and $OP = OP'$;
$\therefore$ *the right-angled triangles* OPM and $OP'M'$ are equal in every respect;

$$\therefore \frac{OM}{OP} = \frac{M'P'}{OP'}; \therefore \cos(-27°) = \sin 117°.$$



6. Let OP, OP' be two revolving straight lines, OP starting from the position OR describe the angle $ROP = 300°$ and OP' starting from the same position describe the angle $ROP' = -210°$. Draw PM, $P'M'$ perpendiculars on OR, OL. Angle $POM = (360° - 300°) = 60°$; $\therefore$ angle $OPM = 30°$ angle $P'OM' = (210° - 180°) = 30°$; $\therefore$ angle OPM = angle $P'OM'$ and $OP = OP'$; $\therefore$ the right-angled triangles OPM and $OP'M'$ are equal in every respect;

$$\therefore \frac{OM}{OP} = \frac{M'P'}{OP'}; \therefore \cos 300° = \sin(-210°).$$

7. $\cos \frac{1}{2}A = \sin(90° - \frac{1}{2}A)$ [Art. 140, Ex. 1.]

$= \sin\{\frac{1}{2}(A + B + C) - \frac{1}{2}A\}$ $[A + B + C = 180°]$

$= \sin \frac{1}{2}(B + C)$.

8. $\cos \frac{1}{2}B = \sin(90° - \frac{1}{2}B)$ [Art. 140, Ex. 1.]

$= \sin\{\frac{1}{2}(A + B + C) - \frac{1}{2}B\}$ $[A + B + C = 180°]$

$= \sin \frac{1}{2}(A + C)$.

9. $\sin \frac{1}{2}C = \cos(90° - \frac{1}{2}C)$ [Art. 140, Ex. 1.]
$= \cos\{\frac{1}{2}(A+B+C) - \frac{1}{2}C\}$ $[A+B+C=180°]$
$= \cos\frac{1}{2}(A+B)$.

10. $\sin \frac{1}{2}A = \cos(90° - \frac{1}{2}A)$ [Art. 140, Ex. 1.]
$= \cos\{\frac{1}{2}(A+B+C) - \frac{1}{2}A\}$ $[A+B+C=180°]$
$= \cos\frac{1}{2}(B+C)$.

EXAMPLES. XXIX. Page 105.

1. In fig. E. T. p. 104, let $ROP=60°$ and $ROP'=120°$;

$$\therefore P'OL=180°-120°=60° \text{ and } MP=M'P';$$

$$\therefore \frac{MP}{OP}=\frac{M'P'}{OP'}; \therefore \sin 60°=\sin 120°.$$

2. In fig. 11 E. T. p. 110, let OP_1 starting from the position OR describe the angle $ROP_1=+340°$ and OP_2 starting from the same position describe the angle $ROP_2=-160°$. Draw P_1M_1, P_2M_2 perpendiculars on LOR. Then angle $P_1OM_1=(360°-340°)=20°$ and angle $P_2OM_2=(180°-160°)=20°$; $\therefore$ angle P_1OM_1 = angle P_2OM_2 and $M_1P_1=M_2P_2$;

$$\therefore \frac{M_1P_1}{OP_1}=\frac{M_2P_2}{OP_2}; \therefore \sin ROP_1=\sin ROP_2; \therefore \sin 340°=\sin(-160°).$$

3. In fig. 11 E. T. p. 110, let OP_1 starting from the position OR describe the angle $ROP_1=-40°$ and OP_2 starting from the same position describe the angle $ROP_2=+220°$. Angle $P_2OM_2=(220°-180°)=40°$; $\therefore$ angle P_1OM_1 = angle P_2OM_2 and $M_1P_1=M_2P_2$;

$$\therefore \frac{M_1P_1}{OP_1}=\frac{M_2P_2}{OP_2}; \therefore \sin ROP_1=\sin ROP_2; \therefore \sin(-40°)=\sin(220°).$$

4. In fig. 11 E. T. p. 110, let OP_1 starting from the position OR describe the angle $ROP_1=320°$ and OP_2 starting from the same position describe the angle $ROP_2=-140°$. Angle $P_1OM_1=(360°-320°)=40°$; and angle $P_2OM_2=(180°-140°)=40°$; $\therefore$ angle P_1OM_1 = angle P_2OM_2; $\therefore OM_1=OM_2$ but of opposite signs;

$$\therefore \frac{OM_1}{OP_1}=-\frac{OM_2}{OP_2}; \therefore \cos ROP_1=-\cos ROP_2; \therefore \cos 320°=-\cos(-140°).$$

5. In fig. 1, p. 110 E. T., let OP_1 starting from the position OR describe the angle $ROP_1=-380°$, and OP_2 starting from the same position describe the angle $ROP_2=560°$.

Angle $P_1OM_1=(380°-360°)=20°$. Angle $ROP_2=560°=360°+200°$; $\therefore$ the *geometrical position* of angle 560° is the same as that of 200°; and angle $P_2OM_2=(200°-180°)=20°$; $\therefore$ angle $P_1OM_1=P_2OM_2$ and $OM_1=OM_2$, but of opposite signs;

$$\therefore \frac{OM_1}{OP_1}=-\frac{OM_2}{OP_2}; \therefore \cos ROP_1=-\cos ROP_2; \therefore \cos(-380°)=-\cos 560°.$$

6. In E. T. p. 110, fig. 11, let OP_2 starting from the position OR describe the angle $ROP_2 = +195°$; and OP_1 starting from the same position describe the angle $ROP_1 = -15°$. Angle $P_2OM_2 = (195° - 180°) = 15° = $ angle P_1OM_1; $\therefore OM_2 = OM_1$, but of opposite signs;

$$\therefore \frac{OM_1}{OP_1} = -\frac{OM_2}{OP_2}; \quad \therefore \cos 195° = -\cos(-15°).$$

1. Since A, B, C are the angles of a triangle therefore $A + B + C = 180°$ and $(B + C)$ is the **supplement** of A;

$\therefore \sin A = \sin(B + C)$ [Art. 141, Ex. 2.]

2. For the same reason $(A + B)$ is the **supplement** of C;

$\therefore \sin C = \sin(A + B)$ [Art. 141, Ex. 2.]

3. For the same reason $(A + C)$ is the **supplement** of B;

$$\therefore \cos B = \cos\{180° - (A + C)\} = -\cos(A + C).$$

4. $(C + B)$ is **supplement** of A;

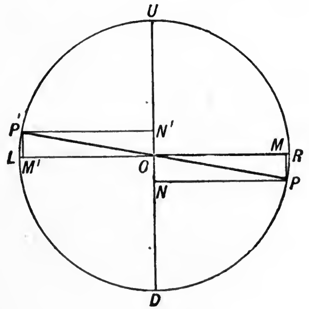
$$\therefore \cos A = \cos\{180° - (C + B)\} = -\cos(C + B).$$

MISCELLANEOUS EXAMPLES. XXX. PAGES 108—9.

1. In figure in E. T. p. 106, let the angle $ROP = +60°$; and the angle $ROP' = +240°$. Angle $P'OM' = (240° - 180°) = 60° = $ angle POM; $\therefore MP = M'P'$ but they are of opposite sign;

$$\therefore \frac{MP}{OP} = -\frac{M'P'}{OP'}; \quad \therefore \sin ROP = -\sin ROP'; \quad \therefore \sin 60° = -\sin 240°.$$

2. In the figure let the angle $ROP' = +170°$ and the angle $ROP = +350°$. Angle $P'OM' = (180° - 170°) = 10°$, angle $POM = (360° - 350°) = 10°$; $\therefore$ angle $POM = $ angle $P'OM'$ and $MP = M'P'$ but they are of opposite sign;



$$\therefore \frac{MP}{OP} = -\frac{M'P'}{OP'}; \quad \therefore \sin ROP = -\sin ROP'; \quad \therefore \sin 170° = -\sin 350°.$$

3. Draw a figure similar to that of (2). Let the angle $ROP' = -20°$ and the angle $ROP = +160°$. Then angle $POM = (180° - 160°) = 20°$; $\therefore$ angle $P'OM' =$ angle POM and $M'P' = MP$, but they are of opposite sign;

$$\therefore \frac{M'P'}{OP'} = -\frac{MP}{OP}; \quad \therefore \sin ROP' = -\sin ROP; \quad \therefore \sin(-20°) = -\sin 160°.$$

4. In fig. E. T. p. 106, let the angle $ROP = 380°$ and let the angle $ROP' = +560°$.

Now angle $POM = (380° - 360°) = 20°$ and angle

$$P'OM' = \{560° - (360° + 180°)\} = 20°;$$

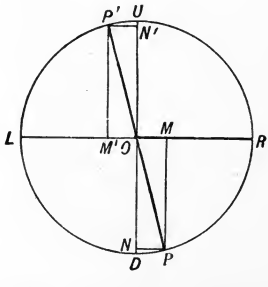
$\therefore$ angle $POM = P'OM'$ and $OM = OM'$ but they are of opposite sign;

$$\therefore \frac{OM}{OP} = -\frac{OM'}{OP'}; \quad \therefore \cos ROP = -\cos ROP'; \quad \therefore \cos 380° = -\cos 560°.$$

5. Draw a figure similar to that of (2). Let the angle $ROP = -225°$, and the angle $ROP' = -45°$. Then angle $POM = (225° - 180°) = 45° =$ angle $P'OM'$ and $OM = OM'$ but they are of opposite sign;

$$\therefore \frac{OM}{OP} = -\frac{OM'}{OP'}; \quad \therefore \cos ROP = -\cos ROP'; \quad \therefore \cos(-225°) = -\cos(-45°).$$

6. Since $1005° = 2 \times 360° + 285°$ and $1185° = 3 \times 360° + 105°$ the geometrical positions of OP and OP' are the same as for 285° and 105° respectively.



Let the angles ROP, ROP' be equal to $+285°$ and $+105°$ respectively. Draw PN, $P'N'$ perpendiculars on UOD.

Then angle $POD = (285° - 270°) = 15°$ and angle $P'OU = (105° - 90°) = 15°$; $\therefore$ angle $POD =$ angle $P'OU$ and $PN = P'N'$, but they are of opposite sign;

$$\therefore \frac{PN}{OP} = -\frac{P'N'}{OP'}; \quad \therefore \frac{OM}{OP} = -\frac{OM'}{OP};$$

$\therefore \cos ROP = -\cos ROP'$; $\therefore \cos 285° = -\cos 105°$; $\therefore \cos 1005° = -\cos 1185°$.

7. In fig. E. T. p. 107, Example 4, let the angles ROP, ROP' be equal to 60° and 150° respectively. Then angle $UOP' = 60°$ and $N'P'$, i.e. $OM' = MP$ but of opposite sign;

$$\therefore \frac{MP}{OP} = -\frac{OM'}{OP}; \quad \therefore \sin ROP = -\cos ROP'; \quad \therefore \sin 60° = -\cos 150°.$$

8. With the figure and construction of (7),

$$\frac{OM}{OP}=\frac{M'P'}{OP'};\ \therefore\ \cos ROP=\sin ROP';\ \therefore\ \cos 60^\circ=\sin 150^\circ.$$

9. Let the angles ROP, ROP' be equal to $+225^\circ$ and $+315^\circ$ respectively.
Angle $POM=(225^\circ-180^\circ)=45^\circ$ and angle $P'ON'=(315^\circ-270^\circ)=45^\circ$;
$\therefore$ angle POM = angle $P'ON'$ and $PM=P'N'$, i.e. OM' but of opposite sign;

$$\therefore\ \frac{MP}{OP}=-\frac{OM'}{OP'};\ \therefore\ \sin ROP=-\cos ROP';\ \therefore\ \sin 225^\circ=-\cos 315^\circ.$$

10. Let the angles ROP, ROP' be equal to -60° and $+30^\circ$ respectively.
Angle $POU=(90^\circ-30^\circ)=60^\circ=POM$; $\therefore$ $MP=-PN'$ and $OM=ON'$, i.e. $M'P'$;

$$\therefore\ \frac{OM}{OP}=\frac{M'P'}{OP'};\ \therefore\ \cos ROP=\sin ROP';\ \therefore\ \cos(-60^\circ)=\sin 30^\circ.$$

11. Since A, B, C, are the angles of a triangle, therefore $A+B+C=180^\circ$,

$\sin A=-\sin(180^\circ+A)$ [E. T. p. 106, Ex. 2]
$=-\sin(A+B+C+A)=-\sin(2A+B+C)$.

12. $\sin A=-\cos(90^\circ+A)$ [E. T. p. 107, Ex. 4]
$=-\cos\{\frac{1}{2}(A+B+C)+A\}=-\cos\frac{1}{2}(3A+B+C)$.

13. $\cos B=\sin(90^\circ+B)=\sin\{\frac{1}{2}(A+B+C)+B\}=\sin\frac{1}{2}(A+3B+C)$.

14. $\cos C=-\cos(180^\circ+C)=-\cos(A+B+C+C)=-\cos(A+B+2C)$.

15. $\cos\frac{1}{2}(B-C)=\sin\{90^\circ+\frac{1}{2}(B-C)\}$ [E. T. p. 107, Ex. 4]
$=\sin\{\frac{1}{2}(A+B+C)+\frac{1}{2}(B-C)\}=\sin\frac{1}{2}(A+2B)$.

16. $\sin\frac{1}{2}(C-A)=-\cos\{90^\circ+\frac{1}{2}(C-A)\}$ [E. T. p. 107, Ex. 4]
$=-\cos\{\frac{1}{2}(A+B+C)+\frac{1}{2}(C-A)\}=-\cos\frac{1}{2}(B+2C)$.

17. In fig. E. T. p. 112, let OP_1 and OP_2 be two revolving lines, and let OP starting from OR describe the angle A, and let OP_2 starting from OR describe the angle $(-A)$, then if P_1P_2 be joined P_1P_2 will always be perpendicular to OR; let P_1P_2 cut OR in M. Then by definition Arts. 81, 132

$$\sin A=\frac{M_1P_1}{OP_1};\ \sin(-A)=\frac{M_1P_2}{OP_2}.$$

Now M_1P_1 is numerically equal to M_1P_2, but of opposite sign;

$$\therefore\ \sin A=-\sin(-A).$$

18. With the figure and construction of (7),

$$\cos A=\frac{OM_1}{OP_1}=\frac{OM_1}{OP_2}=\cos(-A).$$

19. From **18** $\cos(A-90^\circ)=\cos(90^\circ-A)=\sin A$. [Ex. 1, Art. 140.]

20. From **17** $-\sin(A-90^\circ)=\sin(90^\circ-A)=\cos A$. [Ex. 1, Art. 140.]

21. $\cos(\frac{3}{2}\pi+\alpha)=\cos(\pi+\frac{1}{2}\pi+\alpha)=-\cos(\frac{1}{2}\pi+\alpha)$ [E. T. p. 106, Ex. 3]
$=\sin\alpha$ [E. T. p. 107, Ex. 4].

22. $-\sin(\frac{3}{2}\pi+\alpha)=-\sin(\pi+\frac{1}{2}\pi+\alpha)=\sin(\frac{1}{2}\pi+\alpha)$ [E. T. p. 106, Ex. 3]
$=\cos\alpha$ [E. T. p. 107, Ex. 4].

23. $-\cos(\frac{3}{2}\pi-\alpha)=-\cos(\pi+\frac{1}{2}\pi-\alpha)=\cos(\frac{1}{2}\pi-\alpha)$ [E. T. p. 106, Ex. 3]
$=\sin\alpha$ [Ex. 1, p. 102].

24. $\sin(\frac{3}{2}\pi-\alpha)=-\sin(\pi+\frac{1}{2}\pi-\alpha)=\sin(\frac{1}{2}\pi-\alpha)$ [E. T. p. 106, Ex. 3]
$=\cos\alpha$ [Ex. 1, p. 102].

25. $\sin(\frac{1}{2}\pi-\alpha)=\cos\alpha$ [Ex. 1, p. 102]
$=\sin(\frac{1}{2}\pi+\alpha)$ [E. T. p. 107, Ex. 4].

26. $\cos(\pi+\alpha)=-\cos\alpha$ [E. T. p. 106, Ex. 3]
$=\cos(\pi-\alpha)$ [E. T. p. 104, Ex. 2].

27. $\tan(90^\circ-A)=\dfrac{\sin(90^\circ-A)}{\cos(90^\circ-A)}=\dfrac{\cos A}{\sin A}$ [Ex. 1. p. 102]
$=\cot A.$

28. $\tan A=\dfrac{\sin A}{\cos A}=\dfrac{-\sin(-A)}{\cos(-A)}$ [17]
$=-\dfrac{\sin(-A)}{\cos(-A)}=-\tan(-A).$

29. $\tan(90^\circ+A)=\dfrac{\sin(90^\circ+A)}{\cos(90^\circ+A)}=\dfrac{\cos A}{-\sin A}$ [E. T. p. 107, Ex. 4]
$=-\dfrac{\cos A}{\sin A}=-\cot A.$

30. $\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}=\dfrac{\sin(\pi-\alpha)}{-\cos(\pi-\alpha)}$ [E. T. p. 104, Ex. 2]
$=-\dfrac{\sin(\pi-\alpha)}{\cos(\pi-\alpha)}=-\tan(\pi-\alpha)$ [Ex. 1, p. 102].

31. $\tan A=\dfrac{\sin A}{\cos A}=\dfrac{-\sin A}{-\cos A}=\dfrac{\sin(180^\circ+A)}{\cos(180^\circ+A)}$ [E. T. p. 106, Ex. 3]
$=\tan(180^\circ+A).$

32. $\cot(\frac{1}{2}\pi-\alpha)=\dfrac{\cos(\frac{1}{2}\pi-\alpha)}{\sin(\frac{1}{2}\pi-\alpha)}=\dfrac{\sin\alpha}{\cos\alpha}$ [Ex. 1, p. 102]
$=\tan\alpha.$

EXAMPLES. XXXI. PAGE 112.

1. (i) Since $\sin A=\frac{1}{2}$; $\therefore$ $\sin A=\sin 30^\circ$,
and $A=n\times180^\circ+(-1)^n 30^\circ$.

Put $n=0, 1, -1, -2$ successively; and we have 30°, 150°, -210°, -330°.

(ii) Since $\sin A = \frac{1}{2}\sqrt{2}$; $\therefore \sin A = \sin 45°$;

$$\therefore A = n \times 180° + (-1)^n 45°.$$

Put $n=0, 1, -1, -2$ successively; and we have $45°, 135°, -225°, -315°$.

(iii) Since $\sin A = \frac{1}{2}\sqrt{3}$; $\therefore \sin A = \sin 60°$,

$$\therefore A = n \times 180° + (-1)^n 60°.$$

Put $n=0, 1, -1, -2$ successively, and we have $60°, 120°, -240°, -300°$.

(iv) Since $\sin A = -\frac{1}{2}$; $\therefore \sin A = \sin(-30°)$,

and $$A = n \times 180° - (-1)^n 30°.$$

Put $n=0, -1, 1, 2$ successively; and we have $-30°, -150°, 210°, 330°$.

2. (i) $\sin A = \sin 20°$; $\therefore A = n \times 180° + (-1)^n 20°$.

Put $n=0, 1, 2, 3$ successively; and we have $20°, 160°, 380°, 520°$.

(ii) $\sin\theta = -\frac{1}{2}\sqrt{2}$; $\therefore \sin\theta = \sin(-\frac{1}{4}\pi)$, $\theta = n\pi - (-1)^n \frac{1}{4}\pi$.

Put $n=1, 2, 3, 4$ successively, and we have $\frac{5}{4}\pi, \frac{7}{4}\pi, \frac{13}{4}\pi, \frac{15}{4}\pi$.

(iii) $\sin\theta = -\sin\frac{1}{7}\pi = \sin(-\frac{1}{7}\pi)$; [vid. Ex. XXX. 17]

$$\therefore \theta = n\pi - (-1)^n \tfrac{1}{7}\pi.$$

Put $n=1, 2, 3, 4$ successively, and we have $\frac{8}{7}\pi, \frac{13}{7}\pi, \frac{22}{7}\pi, \frac{27}{7}\pi$.

3. (i) $\sin\theta = -\frac{1}{2}$, and $\sin(-30°) = -\frac{1}{2}$;

$$\therefore \sin\theta = \sin(-30°); \quad \therefore \theta = n\pi + (-1)^n(-\tfrac{1}{6}\pi).$$

(ii) $2\sin^2\theta + 3\sin\theta = 2$; $\therefore \sin^2\theta + \frac{3}{2}\sin\theta = 1$;

$$\therefore \sin^2\theta + \tfrac{3}{2}\sin\theta + \tfrac{9}{16} = \tfrac{25}{16}, \quad \sin\theta + \tfrac{3}{4} = \pm\tfrac{5}{4}, \quad \sin\theta = \tfrac{1}{2} \text{ or } -2.$$

The value -2 is inadmissible; for there is no angle whose sine is *numerically* greater than 1;

$$\therefore \sin\theta = \tfrac{1}{2}, \text{ but } \sin 30° = \tfrac{1}{2};$$

$$\therefore \sin\theta = \sin 30°; \quad \therefore \theta = n\pi + (-1)^n \tfrac{1}{6}\pi.$$

(iii) $\sin^2\theta = \cos^2\theta$; $\therefore 2\sin^2\theta = 1$; $\therefore \sin\theta = \pm\frac{1}{2}\sqrt{2}$,

$$\theta = n\pi + (-1)^n \tfrac{1}{4}\pi, \text{ and } n\pi + (-1)^n(-\tfrac{1}{4}\pi).$$

In the first case θ is $n\pi \pm \frac{1}{4}\pi$ as n is even or odd; in the second case θ is $n\pi \pm \frac{1}{4}\pi$ as n is odd or even; $\therefore \theta = n\pi \pm \frac{1}{4}\pi$.

4. If all the angles have the same sine as 30° they are included in the general expression $$n \times 180° + (-1)^n 30°.$$

Put $n=0, 1, -2, 2, -1$ successively, and we have the given angles. Therefore they have all the same sine.

EXAMPLES. XXXII. PAGE 115.

1. (i) $\cos\theta = \frac{1}{2}$; $\therefore \cos\theta = \cos\frac{1}{3}\pi$; $\therefore \theta = 2n\pi \pm \frac{1}{3}\pi$.

(ii) $\tan\theta = 1$; $\therefore \tan\theta = \tan\frac{1}{4}\pi$; $\therefore \theta = n\pi + \frac{1}{4}\pi$.

(iii) $\tan\theta = -1$; $\therefore \tan\theta = \tan(-\frac{1}{4}\pi)$; [See Ex. XXX. 28]

$$\therefore \theta = n\pi - \tfrac{1}{4}\pi.$$

(iv) $\tan\theta = -\sqrt{3}$; $\therefore \tan\theta = \tan(-\frac{1}{3}\pi)$; [See Ex. XXX. 28]

$$\therefore \theta = n\pi - \tfrac{1}{3}\pi.$$

(v) $\cos\theta = \cos\frac{4}{5}\pi$; $\therefore \theta = 2n\pi \pm \frac{4}{5}\pi$.

(vi) $\tan\theta = \tan\frac{3}{4}\pi$; $\therefore \theta = n\pi + \frac{3}{4}\pi$.

2. If all the angles have the same cosine as $-120°$ they are included in the general expression for the cosine of $-120°$ and therefore in the general expression for the cosine of $120°$ for $\cos(-120°) = \cos 120°$ [vid. Ex. XXX. 18].

In the general expression $2 \times n \times 180° \pm 120°$ put $n=1$, and -1; and we have the angles.

3. The angle of $60°$ is in the *first quadrant*, and the angle of $-120°$ in the *third quadrant*.

The sine and cosine, and cosecant and secant are each *positive* in the *first quadrant* and *negative* in the *third quadrant;* therefore the angles of $60°$ and $-120°$ cannot have the same sine or cosine, or the same cosecant or secant. The tangent is *positive* in the *first quadrant* and also *positive* in the *third;* therefore, if the angles of $60°$ and $-120°$ have the same tangent they are included in the general expression $n\pi + 60°$. Put $n=0$, and -1 successively and we have $60°$ and $-120°$.

4. If they have the same *sine*, they are all included in the general expression $n \times 180° + (-1)^n - 23°$, and it will be found that $+157° = 180° - 23°$ is not included, therefore it has not the same *sine* as $-23°$.

If they have the same *cosine*, they are all included in the general expression $2n \times 180° \pm 23°$, $-180° + 23°$ and $180 - 23°$ are not included in this, hence we see they have not the same *cosine*.

If they have the same *tangent* they are all included in the general expression $n \times 180° - 23°$; and $-180° + 23°$ is not included in this formula, therefore they have not the same *tangent*.

5. (i) From **1.** (i) $\theta = 2n\pi \pm \frac{1}{3}\pi$, that is $n \times 360° \pm 60°$, put $n=0$, 1, and -1 successively, and the four smallest angles are

$$60° - 60°,\ 300° - 300°.$$

(ii) From **1.** (ii) $\theta = n\pi + \frac{1}{4}\pi$, that is $n \times 180° + 45°$, put $n=0$, 1, 2, -1 successively, and we have $45°$, $225°$, $405°$, $-135°$.

(iii) From **1.** (iii) $\theta = n\pi - \frac{1}{4}\pi$, that is $n \times 180° - 45°$, put $n=0$, -1, 1, 2 successively, and we have $-45°$, $-225°$, $135°$, $315°$.

(iv) From **1.** (iv) $\theta = n\pi - \frac{1}{3}\pi$, that is $n \times 180° - 60°$, put $n=0$, -1, 1, 2 successively, and we have $-60°$, -240, $120°$, $300°$.

(v) From **1.** (v) $\theta = 2n\pi \pm \frac{4}{5}\pi$, that is $n \times 360° \pm 144°$, put $n=0$, 1, -1 successively, and we have $144°$, $-144°$, $216°$, $-216°$.

(vi) From **1.** (vi) $\theta = n\pi + \frac{3}{4}\pi$, that is $n \times 180° + 135°$, put $n=0$, -1, -2, 1 successively, and we have $135°$, $-45°$, $-225°$, $315°$.

EXAMPLES. XXXIII. PAGE 116.

1. θ is *one of the angles* represented by one or the other of the positions OP_1, OP_2 of the revolving line in fig. E. T. p. 112. We have *two different* values for the *sine* of these angles; viz.: $\frac{M_1P_1}{OP_1}$ and $\frac{M_1P_2}{OP_2}$; these two fractions are equal in magnitude and *opposite in sign.*

We also have *two different* values for the tangent of these angles: $\frac{M_1P_1}{OM_1}$ and $\frac{M_1P_2}{OM_1}$; these two fractions are equal in magnitude and *opposite in sign.*

2. Since $\tan\theta = a$, θ is *one of the angles* represented by one or the other of the positions OP_1, OP_2 of the revolving line in fig. E. T. p. 114. We have *two different* values for the sine of these angles, namely, $\frac{M_1P_1}{OP_1}$ and $\frac{M_2P_2}{OP_2}$; these two fractions are equal in magnitude and *opposite in sign.*

We also have *two different* values for the cosine of these angles; namely $\frac{OM_1}{OP_1}$ and $\frac{OM_2}{OP_2}$; these two fractions are equal in magnitude and *opposite in sign.*

3. Since a is the least angle whose sine $= a$; then in figs. i. and ii. E. T. p. 110, ROP_1 in each figure is a as a is positive or negative. In both figures $\cos A = \frac{OM_1}{OP_1}$ which is positive; and therefore in the *general* formula $\cos A = \pm\sqrt{(1-\sin^2 A)}$, only the *positive* value is admissible.

4. Since A is the least *positive* angle with the given cosine, therefore A is in the *first* Quadrant; and $\sin A$ is *positive.* Therefore only the positive value is admissible in any general formula for $\sin A$ in terms of $\cos A$;

$$\therefore \sin A = +\sqrt{(1-\cos^2 A)}.$$

EXAMPLES. XXXIV. PAGE 120.

1.
$$\cos 75° = \cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

2.
$$\sin 15° = \sin(60° - 45°) = \sin 60° \cos 45° - \cos 60° \sin 45°$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

or
$$\sin 15° = \cos(90° - 15°) = \cos 75° \qquad \text{(Art. 118)}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \text{ by (1).}$$

3. $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

4. $\tan 75^\circ = \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\cos 15^\circ}{\sin 15^\circ}$ (Art. 118),

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \div \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{4+2\sqrt{3}}{2}$$

$$= 2+\sqrt{3}.$$

5. $\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B$

$$= \sin A \,.\, \sqrt{(1-\sin^2 B)} + \sin B \,.\, \sqrt{(1-\sin^2 A)}$$

$$= \tfrac{4}{5} \cdot \tfrac{4}{5} + \tfrac{3}{5} \cdot \tfrac{3}{5} = \tfrac{9}{25} + \tfrac{16}{25} = 1.$$

$\cos(A-B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B$

$$= \tfrac{3}{5} \cdot \tfrac{4}{5} + \tfrac{4}{5} \cdot \tfrac{3}{5} = \tfrac{12}{25} + \tfrac{12}{25} = \tfrac{24}{25}.$$

6. $\sin(A-B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B$

$$= \sin A \,.\, \sqrt{(1-\sin^2 B)} - \sin B \,.\, \sqrt{(1-\sin^2 A)}$$

$$= \tfrac{3}{5} \cdot \tfrac{12}{13} - \tfrac{5}{13} \cdot \tfrac{4}{5} = \tfrac{16}{65}.$$

$\cos(A+B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B$

$$= \sqrt{(1-\sin^2 A)} \,.\, \sqrt{(1-\sin^2 B)} - \sin A \,.\, \sin B$$

$$= \tfrac{4}{5} \cdot \tfrac{12}{13} - \tfrac{3}{5} \cdot \tfrac{5}{13} = \tfrac{33}{65}.$$

7. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \sin A \sqrt{(1-\sin^2 B)} + \sin B \sqrt{(1-\sin^2 A)}$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}} \cdot \frac{2}{\sqrt{5}} = \frac{5}{\sqrt{50}} = \sqrt{\frac{25}{50}} = \frac{1}{\sqrt{2}};$$

$\therefore \sin(A+B) = \frac{1}{\sqrt{2}} = \sin 45^\circ$; $\therefore$ one value of $(A+B)$ is 45°.

8. $\sin 75^\circ = \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$ [Art. 118, and (3) supra],

$$= \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{2{\cdot}4494897 + 1{\cdot}4142136}{4} = \tfrac{1}{4}(3{\cdot}8637) = {\cdot}9659\ldots$$

9. From (2) $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$$= \frac{2{\cdot}4494897 - 1{\cdot}4142136}{4} = \frac{1{\cdot}035276}{4} = {\cdot}2588\ldots$$

10. $\tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\sqrt{3}-1}{2\sqrt{2}} \div \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$

$$= 2 - 1{\cdot}73205 = {\cdot}2679\ldots$$

EXAMPLES. XXXV. PAGE 121.

1. $$\sin(A+B)=\sin A\cos B+\cos A\sin B,$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B;$$
$$\therefore\ \sin(A+B)+\sin(A-B)=2\sin A\cos B.$$

2. $\sin(A+B)-\sin(A-B)$
$$=\sin A\cos B+\cos A\sin B-\sin A\cos B+\cos A\sin B=2\cos A\sin B.$$

3. $\cos(A+B)+\cos(A-B)$
$$=\cos A\cos B-\sin A\sin B+\cos A\cos B+\sin A\sin B=2\cos A\cos B.$$

4. $\cos(A-B)-\cos(A+B)$
$$=\cos A\cos B+\sin A\sin B-\cos A\cos B+\sin A\sin B=2\sin A\sin B.$$

5. $$\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}=\frac{2\sin A\cos B}{2\cos A\cos B}=\tan A.$$

6. $$\tan\alpha+\tan\beta=\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}.$$
$$=\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}.$$ Art. 153.

7. $$\tan\alpha-\tan\beta=\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$
$$=\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}.$$ Art. 153.

8. $$\cot\alpha+\tan\beta=\frac{\cos\alpha}{\sin\alpha}+\frac{\sin\beta}{\cos\beta}=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta}$$
$$=\frac{\cos(\alpha-\beta)}{\sin\alpha\cos\beta}.$$ Art. 153.

9. $$\cot\alpha-\tan\beta=\frac{\cos\alpha}{\sin\alpha}-\frac{\sin\beta}{\cos\beta}=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta}$$
$$=\frac{\cos(\alpha+\beta)}{\sin\alpha\cos\beta}.$$ Art. 153.

10. $$\tan\alpha+\cot\beta=\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\beta}{\sin\beta}=\frac{\sin\alpha\sin\beta+\cos\alpha\cos\beta}{\cos\alpha\sin\beta}$$
$$=\frac{\cos(\alpha-\beta)}{\cos\alpha\,.\,\sin\beta}.$$ Art. 153.

11. $$\frac{\tan\theta+\tan\phi}{\tan\theta-\tan\phi}=\left(\frac{\sin\theta}{\cos\theta}+\frac{\sin\phi}{\cos\phi}\right)\div\left(\frac{\sin\theta}{\cos\theta}-\frac{\sin\phi}{\cos\phi}\right)$$
$$=\frac{\sin(\theta+\phi)}{\cos\theta\cos\phi}\div\frac{\sin(\theta-\phi)}{\cos\theta\cos\phi}=\frac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$$

12. $$\frac{\tan\theta\,.\,\tan\phi+1}{1-\tan\theta\,.\,\tan\phi}=\left(\frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\phi}{\cos\phi}+1\right)\div\left(1-\frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\phi}{\cos\phi}\right)$$
$$=\frac{\cos(\theta-\phi)}{\cos\theta\,.\,\cos\phi}\div\frac{\cos(\theta+\phi)}{\cos\theta\,.\,\cos\phi}$$
$$=\frac{\cos(\theta-\phi)}{\cos(\theta+\phi)}=\cos(\theta-\phi)\,.\,\sec(\theta+\phi).$$

13. $$\frac{\tan\theta+\cot\phi}{\cot\phi-\tan\theta}=\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\phi}{\sin\phi}\right)\div\left(\frac{\cos\phi}{\sin\phi}-\frac{\sin\theta}{\cos\theta}\right)$$
$$=\frac{\cos(\theta-\phi)}{\cos\theta\sin\phi}\div\frac{\cos(\theta+\phi)}{\sin\phi\cos\theta}$$
$$=\frac{\cos(\theta-\phi)}{\cos(\theta+\phi)}=\cos(\theta-\phi)\,.\,\sec(\theta+\phi).$$

14. $$\frac{\cot\theta+\cot\phi}{\cot\theta-\cot\phi}=\left(\frac{\cos\theta}{\sin\theta}+\frac{\cos\phi}{\sin\phi}\right)\div\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\phi}{\sin\phi}\right)$$
$$=\frac{\sin\theta\cos\phi+\cos\theta\sin\phi}{\sin\theta\,.\,\sin\phi}\div\frac{\sin\phi\cos\theta-\cos\phi\sin\theta}{\sin\theta\sin\phi}$$
$$=\frac{\sin(\theta+\phi)}{\sin\theta\,.\,\sin\phi}\div\frac{-\sin(\theta-\phi)}{\sin\theta\sin\phi}=-\frac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$$

15. $$\frac{\tan\theta\,.\,\cot\phi+1}{\tan\theta\,.\,\cot\phi-1}=\left(\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\phi}{\sin\phi}+1\right)\div\left(\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\phi}{\sin\phi}-1\right)$$
$$=\frac{\sin(\theta+\phi)}{\cos\theta\sin\phi}\div\frac{\sin(\theta-\phi)}{\cos\theta\sin\phi}=\frac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$$

16. $$\frac{1+\cot\gamma\,.\,\tan\delta}{\cot\gamma-\tan\delta}=\left(1+\frac{\cos\gamma}{\sin\gamma}\cdot\frac{\sin\delta}{\cos\delta}\right)\div\left(\frac{\cos\gamma}{\sin\gamma}-\frac{\sin\delta}{\cos\delta}\right)$$
$$=\frac{\sin(\gamma+\delta)}{\sin\gamma\,.\,\cos\delta}\div\frac{\cos(\gamma+\delta)}{\sin\gamma\,.\,\cos\delta}=\frac{\sin(\gamma+\delta)}{\cos(\gamma+\delta)}=\tan(\gamma+\delta).$$

17. $$\frac{1-\cot\gamma\,.\,\tan\delta}{\cot\gamma+\tan\delta}=\left(1-\frac{\cos\gamma}{\sin\gamma}\cdot\frac{\sin\delta}{\cos\delta}\right)\div\left(\frac{\cos\gamma}{\sin\gamma}+\frac{\sin\delta}{\cos\delta}\right)$$
$$=\frac{\sin(\gamma-\delta)}{\sin\gamma\cos\delta}\div\frac{\cos(\gamma-\delta)}{\sin\gamma\cos\delta}=\frac{\sin(\gamma-\delta)}{\cos(\gamma-\delta)}=\tan(\gamma-\delta).$$

18. $$\frac{\tan\gamma\,.\,\cot\delta-1}{\tan\gamma+\cot\delta}=\left(\frac{\sin\gamma}{\cos\gamma}\cdot\frac{\cos\delta}{\sin\delta}-1\right)\div\left(\frac{\sin\gamma}{\cos\gamma}+\frac{\cos\delta}{\sin\delta}\right)$$
$$=\frac{\sin(\gamma-\delta)}{\cos\gamma\sin\delta}\div\frac{\cos(\gamma-\delta)}{\cos\gamma\sin\delta}=\frac{\sin(\gamma-\delta)}{\cos(\gamma-\delta)}=\tan(\gamma-\delta).$$

19. $$\frac{\tan\gamma\,.\,\cot\delta+1}{\cot\delta-\tan\gamma}=\left(\frac{\sin\gamma}{\cos\gamma}\cdot\frac{\cos\delta}{\sin\delta}+1\right)\div\left(\frac{\cos\delta}{\sin\delta}-\frac{\sin\gamma}{\cos\gamma}\right)$$
$$=\frac{\sin(\gamma+\delta)}{\cos\gamma\sin\delta}\div\frac{\cos(\gamma+\delta)}{\cos\gamma\sin\delta}=\frac{\sin(\gamma+\delta)}{\cos(\gamma+\delta)}=\tan(\gamma+\delta).$$

20. $\dfrac{\cot\delta-\cot\gamma}{\cot\gamma\,.\,\cot\delta+1}=\left(\dfrac{\cos\delta}{\sin\delta}-\dfrac{\cos\gamma}{\sin\gamma}\right)\div\left(\dfrac{\cos\gamma}{\sin\gamma}\cdot\dfrac{\cos\delta}{\sin\delta}+1\right)$

$$=\frac{\sin(\gamma-\delta)}{\sin\delta\,.\,\sin\gamma}\div\frac{\cos(\gamma-\delta)}{\sin\gamma\,.\,\sin\delta}=\frac{\sin(\gamma-\delta)}{\cos(\gamma-\delta)}=\tan(\gamma-\delta)$$

21. $\tan^2\alpha-\tan^2\beta=\left(\dfrac{\sin^2\alpha}{\cos^2\alpha}-\dfrac{\sin^2\beta}{\cos^2\beta}\right)=\dfrac{\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta}{\cos^2\alpha\cos^2\beta}$

$$=\frac{(\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{\cos^2\alpha\cos^2\beta}$$

$$=\frac{\sin(\alpha+\beta)\,.\,\sin(\alpha-\beta)}{\cos^2\alpha\cos^2\beta}.$$

22. $\cot^2\alpha-\tan^2\beta=\left(\dfrac{\cos^2\alpha}{\sin^2\alpha}-\dfrac{\sin^2\beta}{\cos^2\beta}\right)=\dfrac{\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta}{\sin^2\alpha\cos^2\beta}$

$$=\frac{(\cos\alpha\cos\beta+\sin\alpha\sin\beta)(\cos\alpha\cos\beta-\sin\alpha\sin\beta)}{\sin^2\alpha\cos^2\beta}$$

$$=\frac{\cos(\alpha-\beta)\,.\,\cos(\alpha+\beta)}{\sin^2\alpha\cos^2\beta}.$$

23. $\dfrac{\tan^2\alpha-\tan^2\beta}{1-\tan^2\alpha\,.\,\tan^2\beta}$

$$=\left(\frac{\sin^2\alpha}{\cos^2\alpha}-\frac{\sin^2\beta}{\cos^2\beta}\right)\div\left(1-\frac{\sin^2\alpha}{\cos^2\alpha}-\frac{\sin^2\beta}{\cos^2\beta}\right)$$

$$=\frac{\sin^2\alpha\,.\,\cos^2\beta-\cos^2\alpha\,.\,\sin^2\beta}{\cos^2\alpha\cos^2\beta}\div\frac{\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta}{\cos^2\alpha\cos^2\beta}$$

$$=\frac{\sin^2\alpha\,.\,\cos^2\beta-\cos^2\beta\sin^2\beta}{\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta}$$

$$=\frac{(\sin\alpha\,.\,\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\,.\,\cos\beta-\cos\alpha\sin\beta)}{(\cos\alpha\cos\beta-\sin\alpha\sin\beta)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)}$$

$$=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\cdot\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\tan(\alpha+\beta)\,.\,\tan(\alpha-\beta).$$

24. $\sin(\alpha+\beta)\,.\,\sin(\alpha-\beta)=\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta$ [Art. 153.]

$$=\sin^2\alpha-\sin^2\alpha\sin^2\beta-\sin^2\beta+\sin^2\alpha\sin^2\beta$$
$$=\sin^2\alpha-\sin^2\beta$$
$$=1-\cos^2\alpha-1+\cos^2\beta=\cos^2\beta-\cos^2\alpha.$$

25. $\cos(\alpha+\beta)\,.\,\cos(\alpha-\beta)=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta$ [Art. 153.]

$$=\cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta$$
$$=\cos^2\alpha-\sin^2\beta$$
$$=1-\sin^2\alpha-1+\cos^2\beta=\cos^2\beta-\sin^2\alpha.$$

26. $\sin(A-45^\circ)=\sin A\cos 45^\circ-\cos A\sin 45^\circ=\dfrac{1}{\sqrt{2}}(\sin A-\cos A).$

27. $\sqrt{2}\,.\,\sin(A+45°)=\sqrt{2}\,(\sin A\cos 45°+\cos A\sin 45°)$

$$=\frac{\sqrt{2}}{\sqrt{2}}(\sin A+\cos A)=\sin A+\cos A.$$

28. $\cos A-\sin A=\sqrt{2}\,.\left(\frac{\cos A}{\sqrt{2}}-\frac{\sin A}{\sqrt{2}}\right)$

$$=\sqrt{2}\,.\,(\cos A\,.\,\cos 45°-\sin A\sin 45°)$$
$$=\sqrt{2}\,.\,\cos(A+45°).$$

29. $\cos(A+45°)+\sin(A-45°)$

$$=\frac{1}{\sqrt{2}}(\cos A-\sin A)+\frac{1}{\sqrt{2}}\,.\,(\sin A-\cos A)=0.$$

30. $\cos(A-45°)=\sin(90°+A-45°)=\sin(A+45°)$ [E. T. p. 107, Ex. 4.]

or, $\cos(A-45°)=\frac{1}{\sqrt{2}}(\cos A+\sin A)=(\sin 45°\cos A+\cos 45°\sin A)$

$$=\sin(45°+A).$$

31. $\sin(\theta+\phi)\,.\,\cos\theta-\cos(\theta+\phi)\,.\,\sin\theta=\sin(\theta+\phi-\theta)$ [Art. 153.]

$$=\sin\phi.$$

32. $\sin(\theta-\phi)\,.\,\cos\phi+\cos(\theta-\phi)\sin\phi=\sin(\theta-\phi+\phi)$ [Art. 153.]

$$=\sin\theta.$$

33. $\cos(\theta+\phi)\,.\,\cos\theta+\sin(\theta+\phi)\,.\,\sin\theta=\cos(\theta+\phi-\theta)$ [Art. 153.]

$$=\cos\phi.$$

34. $\dfrac{\tan(\theta-\phi)+\tan\phi}{1-\tan(\theta-\phi)\,.\,\tan\phi}$

$$=\left(\frac{\sin(\theta-\phi)}{\cos(\theta-\phi)}+\frac{\sin\phi}{\cos\phi}\right)\div\left(1-\frac{\sin(\theta-\phi)}{\cos(\theta-\phi)}\cdot\frac{\sin\phi}{\cos\phi}\right)$$
$$=\frac{\sin(\theta-\phi)\cos\phi+\cos(\theta-\phi)\sin\phi}{\cos(\theta-\phi)\,.\,\cos\phi-\sin(\theta-\phi)\,.\,\sin\phi}$$
$$=\frac{\sin(\theta-\phi+\phi)}{\cos(\theta-\phi+\phi)}=\frac{\sin\theta}{\cos\theta}=\tan\theta.$$

35. $\dfrac{\tan(\theta+\phi)-\tan\theta}{1+\tan(\theta+\phi)\,.\,\tan\theta}$

$$=\left(\frac{\sin(\theta+\phi)}{\cos(\theta+\phi)}-\frac{\sin\theta}{\cos\theta}\right)\div\left(1+\frac{\sin(\theta+\phi)}{\cos(\theta+\phi)}\cdot\frac{\sin\theta}{\cos\theta}\right)$$
$$=\frac{\sin(\theta+\phi)\,.\,\cos\theta-\cos(\theta+\phi)\,.\,\sin\theta}{\cos(\theta+\phi)\,.\,\cos\theta+\sin(\theta+\phi)\,.\,\sin\theta}$$
$$=\frac{\sin(\theta+\phi-\theta)}{\cos(\theta+\phi-\theta)}=\frac{\sin\phi}{\cos\phi}=\tan\phi.$$

36. $2\sin(\alpha+\frac{1}{4}\pi)\,.\cos(\beta-\frac{1}{4}\pi)$

$$=2\left(\sin\alpha\,.\frac{1}{\sqrt{2}}+\cos\alpha\,.\frac{1}{\sqrt{2}}\right)\left(\cos\beta\,.\frac{1}{\sqrt{2}}+\sin\beta\,.\frac{1}{\sqrt{2}}\right)$$
$$=(\sin\alpha+\cos\alpha)(\cos\beta+\sin\beta)$$
$$=(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$$
$$=\cos(\alpha-\beta)+\sin(\alpha+\beta).$$

37. $2\sin(\frac{1}{4}\pi-\alpha)\,.\cos(\frac{1}{4}\pi+\beta)$

$$=2\left(\frac{1}{\sqrt{2}}\,.\cos\alpha-\frac{1}{\sqrt{2}}\sin\alpha\right)\left(\frac{1}{\sqrt{2}}\,.\cos\beta-\frac{1}{\sqrt{2}}\,.\sin\beta\right)$$
$$=(\cos\alpha-\sin\alpha)(\cos\beta-\sin\beta)$$
$$=(\cos\alpha\cos\beta+\sin\alpha\sin\beta)-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$$
$$=\cos(\alpha-\beta)-\sin(\alpha+\beta).$$

38. $\cos(\alpha+\beta)+\sin(\alpha-\beta)$

$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$=(\cos\alpha+\sin\alpha)(\cos\beta-\sin\beta)$$
$$=2\left(\frac{1}{\sqrt{2}}\,.\cos\alpha+\frac{1}{\sqrt{2}}\,.\sin\alpha\right)\left(\frac{1}{\sqrt{2}}\,.\cos\beta-\frac{1}{\sqrt{2}}\,.\sin\beta\right)$$
$$=2\sin(\tfrac{1}{4}\pi+\alpha)\,.\cos(\tfrac{1}{4}\pi+\beta).$$

39. $\cos(\alpha+\beta)-\sin(\alpha-\beta)$

$$=\cos\alpha\,.\cos\beta-\sin\alpha\,.\sin\beta-\sin\alpha\,.\cos\beta+\cos\alpha\,.\sin\beta$$
$$=(\cos\alpha-\sin\alpha)(\cos\beta+\sin\beta)$$
$$=2\left(\frac{1}{\sqrt{2}}\,.\cos\alpha-\frac{1}{\sqrt{2}}\,.\sin\alpha\right)\left(\frac{1}{\sqrt{2}}\,.\cos\beta+\frac{1}{\sqrt{2}}\,.\sin\beta\right)$$
$$=2\sin(\tfrac{1}{4}\pi-\alpha)\,.\cos(\tfrac{1}{4}\pi-\beta).$$

40. $\sin nA\,.\cos A+\cos nA\,.\sin A=\sin(nA+A)$ [Art. 153.]

$$=\sin(n+1)A.$$

41. $\cos(n-1)A\,.\cos A-\sin(n-1)A\,.\sin A=\cos\{(n-1)A+A\}$ [Art. 153.]

$$=\cos nA.$$

42. $\sin nA\,.\cos(n-1)A-\cos nA\,.\sin(n-1)A$ [Art. 153.]

$$=\sin\{nA-(n-1)A\}=\sin A.$$

43. $\cos(n-1)A\,.\cos(n+1)A-\sin(n-1)A\,.\sin(n+1)A$ [Art. 153.]

$$=\cos\{(n-1)A+(n+1)A\}=\cos 2nA.$$

EXAMPLES. XXXVI. Page 124.

1. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\,.\tan B}=\dfrac{\frac{1}{2}+\frac{1}{4}}{1-\frac{1}{2}\,.\frac{1}{4}}=\dfrac{\frac{3}{4}}{\frac{8}{7}}=\frac{6}{7},$

$\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\,.\tan B}=\dfrac{\frac{1}{2}-\frac{1}{4}}{1+\frac{1}{2}\,.\frac{1}{4}}=\dfrac{\frac{1}{4}}{\frac{9}{8}}=\frac{2}{9}.$

2. $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\,.\,\tan A}=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$
$$=\frac{(\sqrt{3}+1)(\sqrt{3}+1)}{3-1}=2+\sqrt{3}.$$

3. $$\tan 15^\circ=\tan(45^\circ-30^\circ)=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\,.\,\tan 30^\circ}$$
$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{3-1}=2-\sqrt{3}.$$

4. $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\,.\,\tan B}=\frac{\frac{5}{6}+\frac{1}{11}}{1-\frac{5}{6}\,.\,\frac{1}{11}}=1,$$
$\tan 45^\circ=1$; $\therefore\ \tan(A+B)=\tan 45^\circ$; $\therefore\ A+B=n\,.\,180^\circ+45^\circ.$

5. $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\,.\,\tan B}=\frac{m+\frac{1}{m}}{1-m\,.\,\frac{1}{m}}=\frac{m+\frac{1}{m}}{0}=\infty,$$
$\tan 90^\circ=\infty$; $\therefore\ \tan(A+B)=\tan 90^\circ$; $\therefore\ A+B=n\,.\,180^\circ+90^\circ.$

6. $$\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$$
$$=\frac{\frac{\cos A\cos B}{\sin A\sin B}-1}{\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}}=\frac{\cot A\,.\,\cot B-1}{\cot A+\cot B}.$$

7. $$\cot(A-B)=\frac{\cos(A-B)}{\sin(A-B)}=\frac{\cos A\cos B+\sin A\sin B}{\sin A\cos B-\cos A\sin B}$$
$$=\frac{\frac{\cos A\cos B}{\sin A\sin B}+1}{\frac{\cos B}{\sin B}-\frac{\cos A}{\sin A}}=\frac{\cot A\,.\,\cot B+1}{\cot B-\cot A}.$$

8. $$\cot\left(\theta-\frac{\pi}{4}\right)=\frac{\cos\left(\theta-\frac{\pi}{4}\right)}{\sin\left(\theta-\frac{\pi}{4}\right)}=\frac{\cos\theta\,.\,\frac{1}{\sqrt{2}}+\sin\theta\,.\,\frac{1}{\sqrt{2}}}{\sin\theta\,.\,\frac{1}{\sqrt{2}}-\cos\theta\,.\,\frac{1}{\sqrt{2}}}$$
$$=\frac{\cos\theta+\sin\theta}{\sin\theta-\cos\theta}=\frac{\frac{\cos\theta}{\sin\theta}+1}{1-\frac{\cos\theta}{\sin\theta}}=\frac{\cot\theta+1}{1-\cot\theta}.$$

9. $$\frac{\cot\theta-1}{\cot\theta+1}=\frac{\dfrac{\cos\theta}{\sin\theta}-1}{\dfrac{\cos\theta}{\sin\theta}+1}=\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\frac{\cos\theta\,.\,\dfrac{1}{\sqrt{2}}-\sin\theta\,.\,\dfrac{1}{\sqrt{2}}}{\dfrac{1}{\sqrt{2}}\,.\,\cos\theta+\dfrac{1}{\sqrt{2}}\,.\,\sin\theta}$$
$$=\frac{\cos\theta\cos\frac{1}{4}\pi-\sin\theta\sin\frac{1}{4}\pi}{\sin\frac{1}{4}\pi\cos\theta+\cos\frac{1}{4}\pi\sin\theta}=\frac{\cos(\theta+\frac{1}{4}\pi)}{\sin(\theta+\frac{1}{4}\pi)}=\cot(\theta+\tfrac{1}{4}\pi).$$

10. $\cot(\theta+\frac{1}{4}\pi)=\tan\{\frac{1}{2}\pi-(\theta+\frac{1}{4}\pi)\}=\tan(\frac{1}{4}\pi-\theta)$ [Art. 140.]
$=\tan\{-(\theta-\frac{1}{4}\pi)\}=-\tan(\theta-\frac{1}{4}\pi)$; [Ex. XXX. 28.]
$\therefore\ \tan(\theta-\frac{1}{4}\pi)+\cot(\theta+\frac{1}{4}\pi)=\tan(\theta-\frac{1}{4}\pi)-\tan(\theta-\frac{1}{4}\pi)=0.$

11. $\tan(\theta+\frac{1}{4}\pi)=\cot\{\frac{1}{2}\pi-(\theta+\frac{1}{4}\pi)\}=\cot(\frac{1}{4}\pi-\theta)$ [Art. 140.]
$=\cot\{-(\theta-\frac{1}{4}\pi)\}=-\cot(\theta-\frac{1}{4}\pi)$; [Ex. XXX. 28.]
$\therefore\ \cot(\theta-\frac{1}{4}\pi)+\tan(\theta+\frac{1}{4}\pi)=\cot(\theta-\frac{1}{4}\pi)-\cot(\theta-\frac{1}{4}\pi)=0.$

12. $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\,.\,\tan\beta}=\frac{\dfrac{m}{m+1}+\dfrac{1}{2m+1}}{1-\dfrac{m}{m+1}\cdot\dfrac{1}{2m+1}}=\frac{2m^2+2m+1}{2m^2+2m+1}=1.$$

13. $$\frac{\tan(n+1)\phi-\tan n\phi}{1+\tan(n+1)\phi\,.\,\tan n\phi}=\tan\{(n+1)\phi-n\phi\}$$ [Art. 156.]
$$=\tan\phi.$$

14. $$\frac{\tan(n+1)\phi+\tan(1-n)\phi}{1-\tan(n+1)\phi\,.\,\tan(1-n)\phi}=\tan\{(n+1)\phi+(1-n)\phi\}$$ [Art. 156.]
$$=\tan 2\phi.$$

15. $$\cos\alpha=\frac{1}{\sqrt{(1+\tan^2\alpha)}}=\frac{1}{\sqrt{1+m^2}}$$ [E. T. p. 73, Ex. 2.]
$$\sin\alpha=\frac{\tan\alpha}{\sqrt{(1+\tan^2\alpha)}}=\frac{m}{\sqrt{1+m^2}};$$ [E. T. p. 73, Ex. 2.]
similarly $$\cos\beta=\frac{1}{\sqrt{1+n^2}},\quad \sin\beta=\frac{n}{\sqrt{1+n^2}},$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\frac{1}{\sqrt{1+m^2}}\cdot\frac{1}{\sqrt{1+n^2}}-\frac{m}{\sqrt{1+m^2}}\cdot\frac{n}{\sqrt{1+n^2}}$$
$$=\frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}.$$

16. $$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{a+1-a+1}{1+(a^2-1)}=\frac{2}{a^2},$$
$$2\cot(\alpha-\beta)=\frac{2}{\tan(\alpha-\beta)}=\frac{2}{\dfrac{2}{a^2}}=a^2.$$

17. $\tan\gamma = \cot(90^0 - \gamma) = \cot(\alpha+\beta)$

$$= \frac{1}{\tan(\alpha+\beta} = \frac{1}{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\,.\,\tan\beta}} = \frac{1-\tan\alpha\,.\,\tan\beta}{\tan\alpha+\tan\beta}.$$

EXAMPLES. XXXVII. PAGE 130.

1. $\sin 60° + \sin 30° = 2\sin\frac{1}{2}(60°+30°)\cos\frac{1}{2}(60°-30°) = 2\sin 45°\cos 15°$.

2. $\sin 60° + \sin 20° = 2\sin\frac{1}{2}(60°+20°)\cos\frac{1}{2}(60°-20°) = 2\sin 40°\cos 20°$.

3. $\sin 40° - \sin 10° = 2\cos\frac{1}{2}(40°+10°)\sin\frac{1}{2}(40°-10°) = 2\cos 25°\sin 15°$.

4. $\cos\frac{1}{3}\pi + \cos\frac{1}{2}\pi = 2\cos\frac{1}{2}(\frac{1}{3}\pi+\frac{1}{2}\pi)\cos\frac{1}{2}(\frac{1}{2}\pi-\frac{1}{3}\pi) = 2\cos\frac{5}{12}\pi\cos\frac{1}{12}\pi$.

5. $\cos\frac{1}{3}\pi - \cos\frac{1}{2}\pi = 2\sin\frac{1}{2}(\frac{1}{2}\pi+\frac{1}{3}\pi)\,.\,\sin\frac{1}{2}(\frac{1}{2}\pi-\frac{1}{3}\pi)$

$= 2\sin\frac{5}{12}\pi\,.\,\sin\frac{1}{12}\pi$.

6. $\sin 3A + \sin 5A = 2\sin\frac{1}{2}(5A+3A)\cos\frac{1}{2}(5A-3A) = 2\sin 4A\,.\cos A$.

7. $\sin 7A - \sin 5A = 2\cos\frac{1}{2}(7A+5A)\sin\frac{1}{2}(7A-5A) = 2\cos 6A\,.\sin A$.

8. $\cos 5A + \cos 9A = 2\cos\frac{1}{2}(9A+5A)\cos\frac{1}{2}(9A-5A) = 2\cos 7A\cos 2A$.

9. $\cos 5A - \cos 4A = -2\sin\frac{1}{2}(4A+5A)\,.\,\sin\frac{1}{2}(5A-4A)$ [E. T. p. 126 iii]

$= -2\sin\frac{9}{2}(A)\sin\frac{1}{2}A$.

10. $\cos A - \cos 2A = 2\sin\frac{1}{2}(2A+A)\,.\,\sin\frac{1}{2}(2A-A) = 2\sin\frac{1}{2}(3A)\,.\,\sin\frac{1}{2}A$.

11. $$\frac{\sin 2\theta+\sin\theta}{\cos\theta+\cos 2\theta} = \frac{2\sin\frac{1}{2}(2\theta+\theta)\cos\frac{1}{2}(2\theta-\theta)}{2\cos\frac{1}{2}(2\theta+\theta)\cos\frac{1}{2}(2\theta-\theta)} = \frac{\sin\frac{3}{2}\theta\cos\dfrac{\theta}{2}}{\cos\frac{3}{2}\theta\cos\dfrac{\theta}{2}}$$

$$= \frac{\sin\frac{3}{2}\theta}{\cos\frac{3}{2}\theta} = \tan\tfrac{3}{2}\theta.$$

12. $$\frac{\sin 2\theta-\sin\theta}{\cos\theta-\cos 2\theta} = \frac{2\cos\frac{1}{2}(2\theta+\theta)\sin\frac{1}{2}(2\theta-\theta)}{2\sin\frac{1}{2}(2\theta+\theta)\sin\frac{1}{2}(2\theta-\theta)} = \frac{\cos\frac{3}{2}\theta}{\sin\frac{3}{2}\theta} = \cot\tfrac{3}{2}\theta.$$

13. $$\frac{\sin 3\theta+\sin 2\theta}{\cos 2\theta-\cos 3\theta} = \frac{2\sin\frac{1}{2}(3\theta+2\theta)\,.\,\cos\frac{1}{2}(3\theta-2\theta)}{2\sin\frac{1}{2}(3\theta+2\theta)\,.\,\sin\frac{1}{2}(3\theta-2\theta)} = \frac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \cot\tfrac{1}{2}\theta.$$

14. $$\frac{\sin\theta+\sin\phi}{\cos\theta-\cos\phi} = \frac{2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)}{2\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\phi-\theta)} = \frac{\cos\frac{1}{2}(\phi-\theta)}{\sin\frac{1}{2}(\phi-\theta)} = \cot\tfrac{1}{2}(\phi-\theta),$$

$$\frac{\cos\theta+\cos\phi}{\sin\phi-\sin\theta} = \frac{2\cos\frac{1}{2}(\phi+\theta)\cos\frac{1}{2}(\phi-\theta)}{2\cos\frac{1}{2}(\phi+\theta)\sin\frac{1}{2}(\phi-\theta)} = \frac{\cos\frac{1}{2}(\phi-\theta)}{\sin\frac{1}{2}(\phi-\theta)} = \cot\tfrac{1}{2}(\phi-\theta);$$

$$\therefore \frac{\sin\theta+\sin\phi}{\cos\theta-\cos\phi} = \frac{\cos\theta+\cos\phi}{\sin\phi-\sin\theta}.$$

15. $\cos(60^\circ+A)+\cos(60^\circ-A)$
$=2\cos\frac{1}{2}\{(60^\circ+A)+(60^\circ-A)\}\cos\frac{1}{2}\{(60^\circ+A)-(60^\circ-A)\}$
$=2\cos 60^\circ\cos A=\cos A.$

16. $\cos(45^\circ+A)+\cos(45^\circ-A)$
$=2\cos\frac{1}{2}\{(45^\circ+A)+(45^\circ-A)\}\cos\frac{1}{2}\{(45^\circ+A)-(45^\circ-A)\}$
$=2\cos 45^\circ\cos A=\frac{2}{\sqrt{2}}\cos A=\sqrt{2}\,.\cos A.$

17. $\sin(45^\circ+A)-\sin(45^\circ-A)=2\cos\frac{1}{2}(90^\circ)\sin\frac{1}{2}(2A)=2\cos 45^\circ\sin A$
$=\frac{2}{\sqrt{2}}\sin A=\sqrt{2}\,.\sin A.$

18. $\cos(30^\circ-A)-\cos(30^\circ+A)=2\sin\frac{1}{2}(60^\circ)\sin\frac{1}{2}(2A)=2\sin 30^\circ\sin A$
$=\sin A.$

19. $\frac{\sin\theta-\sin\phi}{\cos\phi-\cos\theta}=\frac{2\cos\frac{1}{2}(\theta+\phi)\,.\sin\frac{1}{2}(\theta-\phi)}{2\sin\frac{1}{2}(\theta+\phi)\,.\sin\frac{1}{2}(\theta-\phi)}=\frac{\cos\frac{1}{2}(\theta+\phi)}{\sin\frac{1}{2}(\theta+\phi)}=\cot\frac{1}{2}(\theta+\phi).$

20. $\frac{\sin\theta-\sin\phi}{\sin\theta+\sin\phi}=\frac{2\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}{2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)}$
$=\frac{\cos\frac{1}{2}(\theta+\phi)}{\sin\frac{1}{2}(\theta+\phi)}\cdot\frac{\sin\frac{1}{2}(\theta-\phi)}{\cos\frac{1}{2}(\theta-\phi)}=\cot\frac{1}{2}(\theta+\phi)\,.\tan\frac{1}{2}(\theta-\phi).$

EXAMPLES. XXXVIII. PAGE 131.

1. $2\sin\theta\,.\cos\phi=\sin(\theta+\phi)+\sin(\theta-\phi).$

2. $2\cos\alpha\,.\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta).$

3. $2\sin 2\alpha\,.\cos 3\beta=\sin(2\alpha+3\beta)+\sin(2\alpha-3\beta).$

4. $2\cos(\alpha+\beta)\,.\cos(\alpha-\beta)=\cos\{(\alpha+\beta)+(\alpha-\beta)\}+\cos\{(\alpha+\beta)-(\alpha-\beta)\}$
$=\cos 2\alpha+\cos 2\beta.$

5. $2\sin 3\theta\,.\cos 5\theta=\sin(3\theta+5\theta)+\sin(3\theta-5\theta)$
$=\sin 8\theta+\sin(-2\theta)=\sin 8\theta-\sin 2\theta.$

6. $2\cos\frac{1}{2}3\theta\,.\cos\frac{1}{2}\theta=\cos(\frac{1}{2}3\theta+\frac{1}{2}\theta)+\cos(\frac{1}{2}3\theta-\frac{1}{2}\theta)=\cos 2\theta+\cos\theta.$

7. $\sin 4\theta\,.\sin\theta=\frac{1}{2}(2\sin 4\theta\,.\sin\theta)=\frac{1}{2}\{\cos(4\theta-\theta)-\cos(4\theta+\theta)\}$
$=\frac{1}{2}(\cos 3\theta-\cos 5\theta).$

8. $\cos\frac{5}{2}\theta\,.\sin\frac{3}{2}\theta=\frac{1}{2}(2\cos\frac{5}{2}\theta\,.\sin\frac{3}{2}\theta)=\frac{1}{2}\{\sin(\frac{5}{2}\theta+\frac{3}{2}\theta)-\sin(\frac{5}{2}\theta-\frac{3}{2}\theta)\}$
$=\frac{1}{2}(\sin 4\theta-\sin\theta).$

9. $2\cos 10^\circ\sin 50^\circ=\sin(50^\circ+10^\circ)+\sin(50^\circ-10^\circ)=\sin 60^\circ+\sin 40^\circ.$

10. $\cos 45^\circ \sin 15^\circ = \frac{1}{2}(2\cos 45^\circ \sin 15^\circ) = \frac{1}{2}\{\sin(45^\circ + 15^\circ) - \sin(45^\circ - 15^\circ)\}$
$= \frac{1}{2}(\sin 60^\circ - \sin 30^\circ).$

11. $2\cos 2\theta \cos\theta = \cos(2\theta + \theta) + \cos(2\theta - \theta) = \cos 3\theta + \cos\theta,$
$2\sin 4\theta \sin\theta = \cos(4\theta - \theta) - \cos(4\theta + \theta) = \cos 3\theta - \cos 5\theta;$
$\therefore\ 2\cos 2\theta\cos\theta - 2\sin 4\theta \sin\theta = \cos 3\theta + \cos\theta - \cos 3\theta + \cos 5\theta$
$= \cos\theta + \cos 5\theta = 2\cos\frac{1}{2}(5\theta + \theta)\,.\,\cos\frac{1}{2}(5\theta - \theta)$
$= 2\cos 3\theta\,.\,\cos 2\theta.$

12. $\sin\frac{5}{2}\theta\,.\,\cos\frac{1}{2}\theta = \frac{1}{2}\{\sin(\frac{5}{2}\theta + \frac{1}{2}\theta) + \sin(\frac{5}{2}\theta - \frac{1}{2}\theta)\} = \frac{1}{2}(\sin 3\theta + \sin 2\theta),$
$\sin\frac{9}{2}\theta\,.\,\cos\frac{3}{2}\theta = \frac{1}{2}\{\sin(\frac{9}{2}\theta + \frac{3}{2}\theta) + \sin(\frac{9}{2}\theta - \frac{3}{2}\theta)\} = \frac{1}{2}(\sin 6\theta + \sin 3\theta)$
$\therefore\ \sin\frac{5}{2}\theta\,.\,\cos\frac{1}{2}\theta - \sin\frac{9}{2}\theta\,.\,\cos\frac{3}{2}\theta = \frac{1}{2}(\sin 3\theta + \sin 2\theta - \sin 6\theta - \sin 3\theta)$
$= \frac{1}{2}(\sin 2\theta - \sin 6\theta) = \frac{1}{2}\{2\cos\frac{1}{2}(2\theta + 6\theta)\sin\frac{1}{2}(2\theta - 6\theta)\}$
$= \cos 4\theta \sin(-2\theta) = -\cos 4\theta\,.\,\sin 2\theta.$

13. $\sin 3\theta + \sin 2\theta = 2\sin\frac{1}{2}(3\theta + 2\theta)\cos\frac{1}{2}(3\theta - 2\theta) = 2\sin\frac{5}{2}\theta\cos\frac{1}{2}\theta;$
$\therefore\ \sin 3\theta + \sin 2\theta + 2\sin\frac{3}{2}\theta\,.\,\cos\frac{1}{2}\theta = 2\cos\frac{1}{2}\theta(\sin\frac{5}{2}\theta + \sin\frac{3}{2}\theta)$
$= 4\cos\frac{1}{2}\theta\,.\,\sin\frac{1}{2}(\frac{5}{2}\theta + \frac{3}{2}\theta)\,.\,\cos\frac{1}{2}(\frac{5}{2}\theta - \frac{3}{2}\theta)$
$= 4\cos\frac{1}{2}\theta\sin 2\theta\cos\frac{1}{2}\theta = 4\cos^2\frac{1}{2}\theta\sin 2\theta.$

14. $\sin\frac{11}{4}\theta.\sin\frac{1}{4}\theta = \frac{1}{2}\{\cos(\frac{11}{4}\theta - \frac{1}{4}\theta) - \cos(\frac{11}{4}\theta + \frac{1}{4}\theta)\} = \frac{1}{2}(\cos\frac{5}{2}\theta - \cos 3\theta)$
$\sin\frac{7}{4}\theta\,.\,\sin\frac{3}{4}\theta = \frac{1}{2}\{\cos(\frac{7}{4}\theta - \frac{3}{4}\theta) - \cos(\frac{7}{4}\theta + \frac{3}{4}\theta)\} = \frac{1}{2}(\cos\theta - \cos\frac{5}{2}\theta);$
$\therefore\ \sin\frac{11}{4}\theta\,.\,\sin\frac{1}{4}\theta + \sin\frac{7}{4}\theta\,.\,\sin\frac{3}{4}\theta = \frac{1}{2}(\cos\frac{5}{2}\theta - \cos 3\theta + \cos\theta - \cos\frac{5}{2}\theta)$
$= \frac{1}{2}(\cos\theta - \cos 3\theta) = 2\sin\frac{1}{2}(3\theta + \theta)\sin\frac{1}{2}(3\theta - \theta) = \sin 2\theta\,.\,\sin\theta.$

MISCELLANEOUS EXAMPLES. XXXIX. PAGE 132.

1. $$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,.\,\tan\beta} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\,.\,\frac{1}{3}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1.$$

2. $$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,.\,\tan\beta} = \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4}\,.\,\frac{1}{7}} = \frac{\frac{25}{28}}{\frac{25}{28}} = 1.$$

But $\tan\frac{1}{4}\pi = 1$; $\therefore\ \tan(\alpha + \beta) = \tan\frac{1}{4}\pi$; $\therefore\ \alpha + \beta = n\pi + \frac{1}{4}\pi.$
When $n = 0,\ \alpha + \beta = \frac{1}{4}\pi.$

3. $$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,.\,\tan\beta} = \frac{\dfrac{m-1}{m} + \dfrac{1}{2m-1}}{1 - \dfrac{m-1}{m}\,.\,\dfrac{1}{2m-1}}$$
$$= \frac{(m-1)(2m-1) + m}{m(2m-1) - (m-1)} = \frac{m(2m-1) - (m-1)}{m(2m-1) - (m-1)} = 1.$$

But $\tan\frac{1}{4}\pi = 1$; $\therefore\ \tan(\alpha + \beta) = \tan\frac{1}{4}\pi$;
$\therefore\ \alpha + \beta = n\pi + \frac{1}{4}\pi,$ when $n = 0\ \alpha + \beta = \frac{1}{4}\pi.$

4. $\dfrac{\cos\alpha-\cos 5\alpha}{\sin\alpha+\sin 5\alpha}=\dfrac{2\sin\frac{1}{2}(\alpha+5\alpha)\,.\,\sin\frac{1}{2}(5\alpha-\alpha)}{2\sin\frac{1}{2}(\alpha+5\alpha)\cos\frac{1}{2}(5\alpha-\alpha)}=\dfrac{\sin 2\alpha}{\cos 2\alpha}=\tan 2\alpha.$

5. $\dfrac{\sin 5x-\sin 3x}{\cos 5x+\cos 3x}=\dfrac{2\cos\frac{1}{2}(5x+3x)\,.\,\sin\frac{1}{2}(5x-3x)}{2\cos\frac{1}{2}(5x+3x)\cos\frac{1}{2}(5x-3x)}=\dfrac{\sin x}{\cos x}=\tan x.$

6. $\dfrac{\cos A+\cos 3A}{\cos 3A+\cos 5A}=\dfrac{2\cos\frac{1}{2}(A+3A)\,.\,\cos\frac{1}{2}(3A-A)}{2\cos\frac{1}{2}(3A+5A)\cos\frac{1}{2}(5A-3A)}=\dfrac{\cos 2A\cos A}{\cos 4A\cos A}$

$$=\frac{\cos 2A}{\cos 4A}.$$

7. $\dfrac{\sin 3x-\sin x}{\cos 3x+\cos x}=\dfrac{2\cos\frac{1}{2}(3x+x)\,.\,\sin\frac{1}{2}(3x-x)}{2\cos\frac{1}{2}(3x+x)\,.\,\cos\frac{1}{2}(3x-x)}=\dfrac{\sin x}{\cos x}=\tan x,$

$\dfrac{\sin 3x+\sin x}{\cos 3x-\cos x}=\dfrac{2\sin\frac{1}{2}(3x+x)\cos\frac{1}{2}(3x-x)}{-2\sin\frac{1}{2}(3x+x)\sin\frac{1}{2}(3x-x)}=\dfrac{\cos x}{-\sin x}=-\cot x;$

$\therefore\ \dfrac{\sin 3x-\sin x}{\cos 3x+\cos x}+\dfrac{\sin 3x-\sin x}{\cos 3x-\cos x}$

$$=\tan x-\cot x=\tan x-\frac{1}{\tan x}=\frac{\tan^2 x-1}{\tan x}$$

$$=-\frac{2(1-\tan^2 x)}{2\tan x}=-\frac{2}{\tan 2x}=-2\cot 2x.$$

8. $\dfrac{(\sin 4A-\sin 2A)(\cos A-\cos 3A)}{(\cos 4A+\cos 2A)(\sin A+\sin 3A)}$

$$=\frac{2\cos\frac{1}{2}(4A+2A)\,.\,\sin\frac{1}{2}(4A-2A)\,.\,2\sin\frac{1}{2}(3A+A)\,.\,\sin\frac{1}{2}(3A-A)}{2\cos\frac{1}{2}(4A+2A)\,.\,\cos\frac{1}{2}(4A-2A)\,.\,2\sin\frac{1}{2}(3A+A)\,.\,\cos\frac{1}{2}(3A-A)}$$

$$=\frac{4\cos 3A\,.\,\sin A\,.\,\sin 2A\,.\,\sin A}{4\cos 3A\,.\,\cos A\,.\,\sin 2A\,.\,\cos A}=\frac{\sin^2 A}{\cos^2 A}=\tan^2 A.$$

9. $2\sin 2\alpha\,.\,\cos\alpha=\sin(2\alpha+\alpha)+\sin(2\alpha-\alpha)=\sin 3\alpha+\sin\alpha,$

$2\cos 4\alpha\,.\,\sin\alpha=\sin(4\alpha+\alpha)-\sin(4\alpha-\alpha)=\sin 5\alpha-\sin 3\alpha;$

$\therefore\ 2\sin 2\alpha\,.\,\cos\alpha+2\cos 4\alpha\,.\,\sin\alpha=\sin 3\alpha+\sin\alpha+\sin 5\alpha-\sin 3\alpha$

$$=\sin 5\alpha+\sin\alpha.$$

10. $2\cos 2\alpha\,.\,\cos\alpha=\cos(2\alpha+\alpha)+\cos(2\alpha-\alpha)=\cos 3\alpha+\cos\alpha,$

$2\sin 4\alpha\,.\,\sin\alpha=\cos(4\alpha-\alpha)-\cos(4\alpha+\alpha)=\cos 3\alpha-\cos 5\alpha;$

$\therefore\ 2\cos 2\alpha\,.\,\cos\alpha-2\sin 4\alpha\,.\,\sin\alpha=(\cos 3\alpha+\cos\alpha)-(\cos 3\alpha-\cos 5\alpha)$

$$=\cos\alpha+\cos 5\alpha=2\cos 3\alpha\cos 2\alpha$$

$$=2\cos 3\alpha\cos 2\alpha.$$

11. $\tan 5A-\tan 3A-\tan 2A$

$$=\tan(3A+2A)-(\tan 3A+\tan 2A)$$

$$=\frac{\tan 3A+\tan 2A}{1-\tan 3A\,.\,\tan 2A}-(\tan 3A+\tan 2A)$$

$$=\frac{\tan 3A+\tan 2A-(1-\tan 3A\,.\,\tan 2A)(\tan 3A+\tan 2A)}{1-\tan 3A\,.\,\tan 2A}$$

$$=\frac{(\tan 3A+\tan 2A)\,.\,\tan 3A\,.\,\tan 2A}{1-\tan 3A\,.\,\tan 2A}$$

$$=\frac{\tan 3A+\tan 2A}{1-\tan 3A\,.\,\tan 2A}\,.\,\tan 3A\,.\,\tan 2A$$

$$=\tan 5A\,.\,\tan 3A\,.\,\tan 2A$$ [Art. 156.]

12. $4\sin(\theta+\phi)\,.\,\cos(\theta-\phi)=3,\ 4\cos(\theta+\phi)\,.\,\sin(\theta-\phi)=1.$

By addition $4\{\sin(\theta+\phi)\,.\,\cos(\theta-\phi)+\cos(\theta+\phi)\,.\,\sin(\theta-\phi)\}=4;$

$\therefore\ \sin\{(\theta+\phi)+(\theta-\phi)\}=1;\ \therefore\ \sin 2\theta=1;\ \therefore\ 2\theta=90°;\ \therefore\ \theta=45°.$

By subtraction $4\{\sin(\theta+\phi)\,.\,\cos(\theta-\phi)-\cos(\theta+\phi)\,.\,\sin(\theta-\phi)\}=2;$

$\therefore\ \sin\{(\theta+\phi)-(\theta-\phi)\}=\frac{1}{2};\ \therefore\ \sin 2\phi=\frac{1}{2};\ \therefore 2\phi=30°;\ \therefore\ \phi=15°.$

13. $$\frac{\sin A\,.\,\sin 2A+\sin 2A\,.\,\sin 5A+\sin 3A\,.\,\sin 10A}{\cos A\,.\,\sin 2A+\sin 2A\,.\,\cos 5A-\cos 3A\,.\,\sin 10A}$$

$$=\frac{\sin 2A(\sin A+\sin 5A)+\sin 3A\,.\,\sin 10A}{\sin 2A(\cos A+\cos 5A)-\cos 3A\,.\,\sin 10A}$$

$$=\frac{2\sin 3A\,.\,\cos 2A\,.\,\sin 2A+\sin 3A\,.\,\sin 10A}{2\cos 3A\,.\,\cos 2A\,.\,\sin 2A-\cos 3A\,.\,\sin 10A}$$

$$=\frac{\sin 3A(2\cos 2A\,.\,\sin 2A+\sin 10A)}{\cos 3A(2\cos 2A\,.\,\sin 2A-\sin 10A)}$$

$$=\frac{\sin 3A(\sin 4A+\sin 10A)}{\cos 3A(\sin 4A-\sin 10A)}=-\frac{\sin 3A(\sin 4A+\sin 10A)}{\cos 3A(\sin 10A-\sin 4A)}$$

$$=-\frac{\sin 3A\,.\,2\sin 7A\,.\,\cos 3A}{\cos 3A\,.\,2\sin 3A\,.\,\cos 7A}=-\frac{\sin 7A}{\cos 7A}=-\tan 7A.$$

14. $$\tan\frac{A+B}{2}-\tan\frac{A-B}{2}$$

$$=\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}-\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A-B)}$$

$$=\frac{\sin\frac{1}{2}(A+B)\,.\,\cos\frac{1}{2}(A-B)-\cos\frac{1}{2}(A+B)\,.\,\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)\,.\,\cos\frac{1}{2}(A-B)}$$

$$=\frac{\sin\{\frac{1}{2}(A+B)-\frac{1}{2}(A-B)\}}{\cos\frac{1}{2}(A+B)\,.\,\cos\frac{1}{2}(A-B)}=\frac{\sin B}{\cos\frac{1}{2}(A+B)\,.\,\cos\frac{1}{2}(A-B)}$$

$$=\frac{2\sin B}{2\cos\frac{1}{2}(A+B)\,.\,\cos\frac{1}{2}(A-B)}=\frac{2\sin B}{\cos A+\cos B}.$$ [Art. 161.]

EXAMPLES. XL. PAGE 135.

1. $$2\operatorname{cosec} 2A=\frac{2}{\sin 2A}=\frac{2}{2\sin A\,.\,\cos A}=\frac{1}{\sin A\,.\,\cos A}$$

$$=\sec A\,.\,\operatorname{cosec} A.$$ [Art. 164, (1).]

2. $\dfrac{\operatorname{cosec}^2 A}{\operatorname{cosec}^2 A - 2} = \dfrac{1}{\sin^2 A} \div \left(\dfrac{1}{\sin^2 A} - 2\right) = \dfrac{1}{1 - 2\sin^2 A}$

$= \dfrac{1}{\cos 2A}$ [Art. 164, (4).]

$= \sec 2A.$

3. $\dfrac{2 - \sec^2 A}{\sec^2 A} = \left(2 - \dfrac{1}{\cos^2 A}\right) \div \dfrac{1}{\cos^2 A} = 2\cos^2 A - 1 = \cos 2A.$ [Art. 164, (3).]

4. $\cos^2 A\,(1 - \tan^2 A) = \cos^2 A\left(1 - \dfrac{\sin^2 A}{\cos^2 A}\right) = \cos^2 A - \sin^2 A$

$= \cos 2A.$ [Art. 164, (2).]

5. $\cot 2A = \dfrac{\cos 2A}{\sin 2A} = \dfrac{\cos^2 A - \sin^2 A}{2\sin A \,.\, \cos A}$ [Art. 164, (2), (1).]

$= \dfrac{\dfrac{\cos^2 A}{\sin^2 A} - 1}{\dfrac{2\sin A\cos A}{\cos^2 A}} = \dfrac{\cot^2 A - 1}{2\cot A}.$

6. $\dfrac{2\tan B}{1 + \tan^2 B} = \dfrac{2\tan B}{\sec^2 B} = \dfrac{2\sin B}{\cos B} \,.\, \cos^2 B = 2\sin B \,.\, \cos B$

$= \sin 2B.$ [Art. 164, (1).]

7. $\tan B + \cot B = \tan B + \dfrac{1}{\tan B} = \dfrac{\tan^2 B + 1}{\tan B}$

$= \dfrac{\sec^2 B}{\dfrac{\sin B}{\cos B}} = \dfrac{\cos B}{\sin B \,.\, \cos^2 B} = \dfrac{2}{2\sin B \,.\, \cos B}$

$= \dfrac{2}{\sin 2B} = 2\operatorname{cosec} 2B.$ [Art. 164, (1).]

8. $\dfrac{1 - \tan^2 B}{1 + \tan^2 B} = \dfrac{1 - \dfrac{\sin^2 B}{\cos^2 B}}{\sec^2 B} = \dfrac{\dfrac{\cos^2 B - \sin^2 B}{\cos^2 B}}{\dfrac{1}{\cos^2 B}} = \cos^2 B - \sin^2 B$

$= \cos 2B.$ [Art. 164, (2).]

9. $\cot B - \tan B = \dfrac{\cos B}{\sin B} - \dfrac{\sin B}{\cos B} = \dfrac{\cos^2 B - \sin^2 B}{\sin B\cos B}$

$= \dfrac{\cos 2B}{\sin B \,.\, \cos B} = \dfrac{2\cos 2B}{2\sin B \,.\, \cos B} = \dfrac{2\cos 2B}{\sin 2B} = 2\cot 2B.$

10. $$\frac{\cot^2 B+1}{\cot^2 B-1}=\frac{\operatorname{cosec}^2 B}{\frac{\cos^2 B}{\sin^2 B}-1}=\frac{\frac{1}{\sin^2 B}}{\frac{\cos^2 B-\sin^2 B}{\sin^2 B}}=\frac{1}{\cos^2 B-\sin^2 B}$$
$$=\frac{1}{\cos 2B}=\sec 2B.$$ [Art. 164, (2).]

11. $$(\sin\tfrac{1}{2}\theta+\cos\tfrac{1}{2}\theta)^2=\sin^2\tfrac{1}{2}\theta+\cos^2\tfrac{1}{2}\theta+2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$
$$=1+2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta=1+\sin\theta.$$ [Art. 164, (1).]

12. $$(\sin\tfrac{1}{2}\theta-\cos\tfrac{1}{2}\theta)^2=\sin^2\tfrac{1}{2}\theta+\cos^2\tfrac{1}{2}\theta-2\sin\tfrac{1}{2}\theta\,.\cos\tfrac{1}{2}\theta$$
$$=1-2\sin\tfrac{1}{2}\theta\,.\cos\tfrac{1}{2}\theta=1-\sin\theta.$$ [Art. 164, (1).]

13. $$\cos^2\tfrac{1}{2}\theta\,(1+\tan\tfrac{1}{2}\theta)^2=\left(\cos\tfrac{1}{2}\theta+\cos\tfrac{1}{2}\theta\,.\frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\right)^2=(\cos\tfrac{1}{2}\theta+\sin\tfrac{1}{2}\theta)^2$$
$$=\cos^2\tfrac{1}{2}\theta+\sin^2\tfrac{1}{2}\theta+2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$
$$=1+2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta=1+\sin\theta.$$ [Art. 164, (1).]

14. $$\sin^2\tfrac{1}{2}\theta\,(\cot\tfrac{1}{2}\theta-1)^2=\left(\sin\tfrac{1}{2}\theta\,.\frac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}-\sin\tfrac{1}{2}\theta\right)^2=(\cos\tfrac{1}{2}\theta-\sin\tfrac{1}{2}\theta)^2$$
$$=\cos^2\tfrac{1}{2}\theta+\sin^2\tfrac{1}{2}\theta-2\sin\tfrac{1}{2}\theta\,.\cos\tfrac{1}{2}\theta$$
$$=1-2\sin\tfrac{1}{2}\theta\,.\cos\tfrac{1}{2}\theta=1-\sin\theta.$$ [Art. 164, (1).]

15. $$\left(\frac{\tan\frac{1}{2}\theta+1}{\tan\frac{1}{2}\theta-1}\right)^2=\left(\frac{\frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}+1}{\frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}-1}\right)^2=\left(\frac{\sin\frac{1}{2}\theta+\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta-\cos\frac{1}{2}\theta}\right)^2$$
$$=\frac{\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{2}\theta+2\sin\frac{1}{2}\theta\,.\cos\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{2}\theta-2\sin\frac{1}{2}\theta\,.\cos\frac{1}{2}\theta}$$
$$=\frac{1+2\sin\frac{1}{2}\theta\,.\cos\frac{1}{2}\theta}{1-2\sin\frac{1}{2}\theta\,.\cos\frac{1}{2}\theta}=\frac{1+\sin\theta}{1-\sin\theta}.$$

16. $$\frac{\sin\beta}{1+\cos\beta}=\frac{2\sin\frac{1}{2}\beta\cos\frac{1}{2}\beta}{2\cos^2\frac{1}{2}\beta}=\frac{\sin\frac{1}{2}\beta}{\cos\frac{1}{2}\beta}=\tan\tfrac{1}{2}\beta.$$ [Art. 162, (2).]

17. $$\frac{\sin\beta}{1-\cos\beta}=\frac{2\sin\frac{1}{2}\beta\,.\cos\frac{1}{2}\beta}{2\sin^2\frac{1}{2}\beta}=\frac{\cos\frac{1}{2}\beta}{\sin\frac{1}{2}\beta}=\cot\tfrac{1}{2}\beta.$$ [Art. 162, (2).]

18. $$\frac{1-\cos\beta}{1+\cos\beta}=\frac{2\sin^2\frac{1}{2}\beta}{2\cos^2\frac{1}{2}\beta}=\tan^2\tfrac{1}{2}\beta.$$ [Art. 162, (2.)]

19. $$\frac{1+\sec\beta}{\sec\beta}=\left(\frac{1}{\sec\beta}+1\right)=\cos\beta+1=2\cos^2\tfrac{1}{2}\beta.$$ [Art. 162, (2).]

20. $$\operatorname{cosec}\beta-\cot\beta=\frac{1}{\sin\beta}-\frac{\cos\beta}{\sin\beta}=\frac{1-\cos\beta}{\sin\beta}=\frac{2\sin^2\frac{1}{2}\beta}{2\sin\frac{1}{2}\beta\,.\cos\frac{1}{2}\beta}$$
$$=\frac{\sin\frac{1}{2}\beta}{\cos\frac{1}{2}\beta}=\tan\tfrac{1}{2}\beta.$$ [Art, 162, (2).]

21. $$\frac{\cos 2x}{1+\sin 2x}=\frac{\cos^2 x-\sin^2 x}{\sin^2 x+\cos^2 x+2\sin x\,.\cos x}=\frac{(\cos x+\sin x)(\cos x-\sin x)}{(\cos x+\sin x)^2}$$
$$=\frac{\cos x-\sin x}{\cos x+\sin x}=\frac{1-\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin x}{\cos x}}=\frac{1-\tan x}{1+\tan x}.$$

22. $$\frac{\cos x}{1-\sin x}=\frac{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}{\sin^2\frac{1}{2}x+\cos^2\frac{1}{2}x-2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$
$$=\frac{(\cos\frac{1}{2}x+\sin\frac{1}{2}x)(\cos\frac{1}{2}x-\sin\frac{1}{2}x)}{(\cos\frac{1}{2}x+\sin\frac{1}{2}x)^2}=\frac{\cos\frac{1}{2}x-\sin\frac{1}{2}x}{\cos\frac{1}{2}x+\sin\frac{1}{2}x}$$
$$=\frac{1-\dfrac{\sin\frac{1}{2}x}{\cos\frac{1}{2}x}}{1+\dfrac{\sin\frac{1}{2}x}{\cos\frac{1}{2}x}}=\frac{1-\tan\frac{1}{2}x}{1+\tan\frac{1}{2}x}.$$

23. $$\frac{\cos x}{1+\sin x}=\frac{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}{\sin^2\frac{1}{2}x+\cos^2\frac{1}{2}x+2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$
$$=\frac{(\cos\frac{1}{2}x+\sin\frac{1}{2}x)(\cos\frac{1}{2}x-\sin\frac{1}{2}x)}{(\cos\frac{1}{2}x+\sin\frac{1}{2}x)^2}=\frac{\cos\frac{1}{2}x-\sin\frac{1}{2}x}{\cos\frac{1}{2}x+\sin\frac{1}{2}x}$$
$$=\frac{\dfrac{\cos\frac{1}{2}x}{\sin\frac{1}{2}x}-1}{\dfrac{\cos\frac{1}{2}x}{\sin\frac{1}{2}x}+1}=\frac{\cot\frac{1}{2}x-1}{\cot\frac{1}{2}x+1}.$$

24. $$\frac{\cos x}{1-\sin x}=\frac{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}{\cos^2\frac{1}{2}x+\sin^2\frac{1}{2}x-2\sin\frac{1}{2}x\,.\cos\frac{1}{2}x}$$
$$=\frac{(\cos\frac{1}{2}x+\sin\frac{1}{2}x)(\cos\frac{1}{2}x-\sin\frac{1}{2}x)}{(\cos\frac{1}{2}x-\sin\frac{1}{2}x)^2}=\frac{\cos\frac{1}{2}x+\sin\frac{1}{2}x}{\cos\frac{1}{2}x-\sin\frac{1}{2}x}$$
$$=\frac{\dfrac{\cos\frac{1}{2}x}{\sin\frac{1}{2}x}+1}{\dfrac{\cos\frac{1}{2}x}{\sin\frac{1}{2}x}-1}=\frac{\cot\frac{1}{2}x+1}{\cot\frac{1}{2}x-1}.$$

25. $$\frac{1+\sin x+\cos x}{1+\sin x-\cos x}=\frac{(1+\cos x)+\sin x}{(1-\cos x)+\sin x}=\frac{2\cos^2\frac{1}{2}x+2\sin\frac{1}{2}x\cos\frac{1}{2}x}{2\sin^2\frac{1}{2}x+2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$
$$=\frac{\cos\frac{1}{2}x(\cos\frac{1}{2}x+\sin\frac{1}{2}x)}{\sin\frac{1}{2}x(\cos\frac{1}{2}x+\sin\frac{1}{2}x)}=\frac{\cos\frac{1}{2}x}{\sin\frac{1}{2}x}=\cot\tfrac{1}{2}x.$$

26. $$\frac{\cos^3\alpha+\sin^3\alpha}{\cos\alpha+\sin\alpha}=\frac{(\cos\alpha+\sin\alpha)(\cos^2\alpha+\sin^2\alpha-\sin\alpha\cos\alpha)}{\cos\alpha+\sin\alpha}$$
$$=\cos^2\alpha+\sin^2\alpha-\sin\alpha\cos\alpha=1-\sin\alpha\cos\alpha$$
$$=\tfrac{1}{2}(2-2\sin\alpha\cos\alpha)=\tfrac{1}{2}(2-\sin 2\alpha).$$ [Art. 164, (1.)]

27. $\frac{\cos^3\alpha-\sin^3\alpha}{\cos\alpha-\sin\alpha}=\frac{(\cos\alpha-\sin\alpha)(\cos^2\alpha+\sin^2\alpha+\sin\alpha\cos\alpha)}{\cos\alpha-\sin\alpha}$

$=\cos^2\alpha+\sin^2\alpha+\sin\alpha\cos\alpha=1+\sin\alpha\cos\alpha$

$=\frac{1}{2}(2+2\sin\alpha\cos\alpha)=\frac{1}{2}(2+\sin 2\alpha).$

28. $\cos^4\alpha-\sin^4\alpha=(\cos^2\alpha+\sin^2\alpha)(\cos^2\alpha-\sin^2\alpha)$

$=\cos^2\alpha-\sin^2\alpha=\cos 2\alpha.$ [Art. 164, (2).]

29. $\cos^6\alpha+\sin^6\alpha=(\cos^2\alpha+\sin^2\alpha)(\cos^4\alpha-\cos^2\alpha\sin^2\alpha+\sin^4\alpha)$

$=(\cos^4\alpha-\cos^2\alpha\sin^2\alpha+\sin^4\alpha)$

$=\{(\cos^2\alpha-\sin^2\alpha)^2+\sin^2\alpha\cos^2\alpha\}$

$=\cos^2 2\alpha+\sin^2\alpha\cos^2\alpha$ [Art. 162, (2).]

$=\frac{1}{4}\{4\cos^2 2\alpha+(2\sin\alpha\cos\alpha)^2\}=\frac{1}{4}(4\cos^2 2\alpha+\sin^2 2\alpha)$

$=\frac{1}{4}(4\cos^2 2\alpha+1-\cos^2 2\alpha)=\frac{1}{4}(1+3\cos^2 2\alpha).$

30. $\cos^6\alpha-\sin^6\alpha=(\cos^2\alpha-\sin^2\alpha)(\cos^4\alpha+\cos^2\alpha\sin^2\alpha+\sin^4\alpha)$

$=\cos 2\alpha\{(\cos^2\alpha+\sin^2\alpha)^2-\sin^2\alpha\cos^2\alpha\}$ [Art. 164, (2).]

$=\cos 2\alpha(1-\sin^2\alpha\cos^2\alpha)=\cos 2\alpha-\cos 2\alpha\sin^2\alpha\cos^2\alpha$

$=\frac{1}{4}\{4\cos 2\alpha-\cos 2\alpha(2\sin\alpha\,.\,\cos\alpha)^2\}$

$=\frac{1}{4}(4\cos 2\alpha-\cos 2\alpha\,.\,\sin^2 2\alpha)$

$=\frac{1}{4}\{4\cos 2\alpha-\cos 2\alpha(1-\cos^2 2\alpha)\}=\frac{1}{4}(3\cos 2\alpha+\cos^3 2\alpha)$

$=\frac{1}{4}\{(3+\cos^2 2\alpha)\cos 2\alpha\}.$

31. $\frac{\sin 3\beta}{\sin\beta}-\frac{\cos 3\beta}{\cos\beta}=\frac{\sin 3\beta\cos\beta-\cos 3\beta\sin\beta}{\sin\beta\cos\beta}=\frac{\sin 2\beta}{\sin\beta\,.\,\cos\beta}$

$=\frac{2\sin 2\beta}{2\sin\beta\,.\,\cos\beta}=\frac{2\sin 2\beta}{\sin 2\beta}=2.$

32. $\frac{\cos 3\beta}{\sin\beta}+\frac{\sin 3\beta}{\cos\beta}=\frac{\cos 3\beta\cos\beta+\sin 3\beta\sin\beta}{\sin\beta\cos\beta}=\frac{\cos 2\beta}{\sin\beta\cos\beta}$

$=\frac{2\cos 2\beta}{2\sin\beta\cos\beta}=\frac{2\cos 2\beta}{\sin 2\beta}=2\cot 2\beta.$

33. $\frac{\sin 4\beta}{\sin 2\beta}=\frac{2\sin 2\beta\cos 2\beta}{\sin 2\beta}=2\cos 2\beta.$ [Art. 164, (1).]

34. $\frac{\sin 5\beta}{\sin\beta}-\frac{\cos 5\beta}{\cos\beta}=\frac{\sin 5\beta\cos\beta-\cos 5\beta\sin\beta}{\sin\beta\,.\,\cos\beta}=\frac{\sin 4\beta}{\sin\beta\cos\beta}$

$=\frac{4\sin 2\beta\cos 2\beta}{2\sin\beta\cos\beta}=\frac{4\sin 2\beta\cos 2\beta}{\sin 2\beta}=4\cos 2\beta.$

35. $\frac{\sin\frac{5}{12}\pi}{\sin\frac{1}{12}\pi}-\frac{\cos\frac{5}{12}\pi}{\cos\frac{1}{12}\pi}=\frac{\sin\frac{5}{12}\pi\,.\,\cos\frac{1}{12}\pi-\cos\frac{5}{12}\pi\,.\,\sin\frac{1}{12}\pi}{\sin\frac{1}{12}\pi\,.\,\cos\frac{1}{12}\pi}$

$$=\frac{\sin\left(\frac{5}{12}\pi-\frac{1}{12}\pi\right)}{\sin\frac{1}{12}\pi\,.\,\cos\frac{1}{12}\pi}=\frac{2\sin\frac{1}{3}\pi}{2\sin\frac{1}{12}\pi\,.\,\cos\frac{1}{12}\pi}$$

$$=\frac{2\sin\frac{1}{3}\pi}{\sin\frac{1}{6}\pi}=\frac{4\sin\frac{1}{6}\pi\,.\,\cos\frac{1}{6}\pi}{\sin\frac{1}{6}\pi}$$

$$=4\cos\tfrac{1}{6}\pi=4\times\tfrac{1}{2}\sqrt{3}=2\sqrt{3}.$$

36. $\tan(45^\circ+A)-\tan(45^\circ-A)=\dfrac{1+\tan A}{1-\tan A}-\dfrac{1-\tan A}{1+\tan A}$ [Art. 156.]

$$=\frac{(1+\tan A)^2-(1-\tan A)^2}{1-\tan^2 A}=\frac{4\tan A}{1-\tan^2 A}$$

$$=2\,.\,\frac{2\tan A}{1-\tan^2 A}=2\tan 2A.$$ [Art. 163, (5).]

37. $\tan(45^\circ-A)+\cot(45^\circ-A)$

$$=\tan(45^\circ-A)+\frac{1}{\tan(45^\circ-A)}=\frac{1-\tan A}{1+\tan A}+\frac{1}{\dfrac{1-\tan A}{1+\tan A}}$$

$$=\frac{1-\tan A}{1+\tan A}+\frac{1+\tan A}{1-\tan A}$$

$$=\frac{(1-\tan A)^2+(1+\tan A)^2}{1-\tan^2 A}=\frac{2(1+\tan^2 A)}{1-\tan^2 A}$$

$$=\frac{2\sec^2 A}{1-\tan^2 A}=\frac{2\,.\,\dfrac{1}{\cos^2 A}}{1-\dfrac{\sin^2 A}{\cos^2 A}}=\frac{2}{\cos^2 A-\sin^2 A}$$

$$=\frac{2}{\cos 2A}=2\sec 2A.$$

38. $\dfrac{\tan^2(45^\circ+A)-1}{\tan^2(45^\circ+A)+1}$

$$=\frac{\dfrac{\sin^2(45^\circ+A)}{\cos^2(45^\circ+A)}-1}{\sec^2(45^\circ+A)}=\frac{\dfrac{\sin^2(45^\circ+A)-\cos^2(45^\circ+A)}{\cos^2(45^\circ+A)}}{\dfrac{1}{\cos^2(45^\circ+A)}}$$

$$=\sin^2(45^\circ+A)-\cos^2(45^\circ+A)$$

$$=\{\sin(45^\circ+A)+\cos(45^\circ+A)\}\{\sin(45^\circ+A)-\cos(45^\circ+A)\}$$

$$=\left(\frac{\cos A+\sin A}{\sqrt{2}}+\frac{\cos A-\sin A}{\sqrt{2}}\right)\left(\frac{\cos A+\sin A}{\sqrt{2}}-\frac{\cos A-\sin A}{\sqrt{2}}\right)$$

$$=\frac{4\sin A\,.\,\cos A}{\sqrt{2}\,.\,\sqrt{2}}=2\sin A\cos A=\sin 2A.$$

39. $$\frac{\sec A+\tan A}{\sec A-\tan A}=\frac{\dfrac{1}{\cos A}+\dfrac{\sin A}{\cos A}}{\dfrac{1}{\cos A}-\dfrac{\sin A}{\cos A}}=\frac{1+\sin A}{1-\sin A}=\frac{1+2\sin\frac{1}{2}A\,.\cos\frac{1}{2}A}{1-2\sin\frac{1}{2}A\,.\cos\frac{1}{2}A}$$

$$=\frac{\dfrac{1}{\cos^2\frac{1}{2}A}+\dfrac{2\sin\frac{1}{2}A\cos\frac{1}{2}A}{\cos^2\frac{1}{2}A}}{\dfrac{1}{\cos^2\frac{1}{2}A}-\dfrac{2\sin\frac{1}{2}A\,.\cos\frac{1}{2}A}{\cos^2\frac{1}{2}A}}=\frac{\sec^2\frac{1}{2}A+2\tan\frac{1}{2}A}{\sec^2\frac{1}{2}A-2\tan\frac{1}{2}A}$$

$$=\frac{1+\tan^2\frac{1}{2}A+2\tan\frac{1}{2}A}{1+\tan^2\frac{1}{2}A-2\tan\frac{1}{2}A}=\frac{(1+\tan\frac{1}{2}A)^2}{(1-\tan\frac{1}{2}A)^2}$$

$$=\frac{\dfrac{1+\tan\frac{1}{2}A}{1-\tan\frac{1}{2}A}}{\dfrac{1-\tan\frac{1}{2}A}{1+\tan\frac{1}{2}A}}=\frac{\tan(45^\circ+\frac{1}{2}A)}{\tan(45^\circ-\frac{1}{2}A)}$$ [Art. 156.]

$$=\tan(45^\circ+\tfrac{1}{2}A)\,.\cot(45^\circ-\tfrac{1}{2}A).$$

40. $$\frac{\cos(A+45^\circ)}{\cos(A-45^\circ)}=\frac{\dfrac{\cos A-\sin A}{\sqrt{2}}}{\dfrac{\cos A+\sin A}{\sqrt{2}}}=\frac{\cos A-\sin A}{\cos A+\sin A}$$

$$=\frac{(\cos A-\sin A)^2}{\cos^2 A-\sin^2 A}=\frac{\cos^2 A+\sin^2 A-2\sin A\,.\cos A}{\cos 2A}$$

$$=\frac{1-\sin 2A}{\cos 2A}=\frac{1}{\cos^2 A}-\frac{\sin 2A}{\cos 2A}=\sec 2A-\tan 2A.$$

41. $$\frac{\sin B+\sin 2B}{1+\cos B+\cos 2B}=\frac{\sin B+2\sin B\,.\cos B}{1+\cos B+2\cos^2 B-1}$$ [Art. 164, (1), (3).]

$$=\frac{\sin B\,(1+2\cos B)}{\cos B\,(1+2\cos B)}=\frac{\sin B}{\cos B}=\tan B.$$

42. $$\frac{\sin 2B-\sin B}{1-\cos B+\cos 2B}=\frac{2\sin B\,.\cos B-\sin B}{1-\cos B+2\cos^2 B-1}$$ [Art. 164, (1), (3).]

$$=\frac{\sin B\,(2\cos B-1)}{\cos B\,(2\cos B-1)}=\frac{\sin B}{\cos B}=\tan B.$$

EXAMPLES. XLI. PAGE 139.

1. $$\frac{\sin 3A}{\sin A}=\frac{3\sin A-4\sin^3 A}{\sin A}=3-4\sin^2 A$$

$$=3-2\,(1-\cos 2A)=2\cos 2A+1.$$ [Art. 164, (4).]

2. $\frac{\cos 3A}{\cos A} = \frac{4\cos^3 A - 3\cos A}{\cos A} = 4\cos^2 A - 3$

$= 2(1+\cos 2A) - 3 = 2\cos 2A - 1.$ [Art. 164, (4).]

3. $\frac{3\sin A - \sin 3A}{\cos 3A + 3\cos A} = \frac{3\sin A - (3\sin A - 4\sin^3 A)}{4\cos^3 A - 3\cos A + 3\cos A} = \frac{4\sin^3 A}{4\cos^3 A} = \tan^3 A.$

4. $\cot 3A = \frac{\cos 3A}{\sin 3A} = \frac{4\cos^3 A - 3\cos A}{3\sin A - 4\sin^3 A}.$

Divide both numerator and denominator by $\sin^3 A$.

$$\cot 3A = \frac{4\cot^3 A - \frac{3\cot A}{\sin^2 A}}{\frac{3}{\sin^2 A} - 4} = \frac{4\cot^3 A - 3\cot A \,.\, \text{cosec}^2 A}{3\,\text{cosec}^2 A - 4}$$

$$= \frac{4\cot^3 A - 3\cot A\,(1+\cot^2 A)}{3(1+\cot^2 A) - 4}$$ [Art. 105, iii.]

$$= \frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}.$$

5. $\frac{\sin 3A - \sin A}{\cos 3A + \cos A} = \frac{3\sin A - 4\sin^3 A - \sin A}{4\cos^3 A - 3\cos A + \cos A}$

$$= \frac{\sin A\,(2 - 4\sin^2 A)}{\cos A\,(4\cos^2 A - 2)} = \frac{\sin A\,(2 - 4\sin^2 A)}{\cos A\,(2 - 4\sin^2 A)}$$

$$= \frac{\sin A}{\cos A} = \tan A.$$

Or $\frac{\sin 3A - \sin A}{\cos 3A + \cos A} = \frac{2\cos 2A \sin A}{2\cos 2A\cos A} = \frac{\sin A}{\cos A} = \tan A.$

6. $\frac{\sin 3A - \cos 3A}{\sin A + \cos A} = \frac{3(\sin A + \cos A) - 4(\sin^3 A + \cos^3 A)}{\sin A + \cos A}$

$= 3 - 4(\sin^2 A + \cos^2 A - \sin A\cos A)$

$= 4\sin A \,.\, \cos A - 1 = 2\sin 2A - 1.$

7. $\frac{\sin 3A + \cos 3A}{\cos A - \sin A} = \frac{4(\cos^3 A - \sin^3 A) - 3(\cos A - \sin A)}{\cos A - \sin A}$

$= 4(\cos^2 A + \sin^2 A + \sin A\cos A) - 3$

$= 1 + 4\sin A \,.\, \cos A = 2\sin 2A + 1.$

8. $\frac{1}{\tan 3A - \tan A} + \frac{1}{\cot A - \cot 3A}$

$$= \frac{1}{\frac{\sin 3A}{\cos 3A} - \frac{\sin A}{\cos A}} + \frac{1}{\frac{\cos A}{\sin A} - \frac{\cos 3A}{\sin 3A}}$$

$$= \frac{\cos A\cos 3A}{\sin 2A} + \frac{\sin A \sin 3A}{\sin 2A}$$

$$= \frac{\cos A\cos 3A + \sin A\sin 3A}{\sin 2A} = \frac{\cos 2A}{\sin 2A} = \cot 2A.$$

9. $$\left(\frac{3\sin A-\sin 3A}{3\cos A+\cos 3A}\right)^2=\left\{\frac{3\sin A-(3\sin A-4\sin^3 A)}{3\cos A+4\cos^3 A-3\cos A}\right\}^2=\frac{\sin^6 A}{\cos^6 A}$$
$$=\left(\frac{\sin^2 A}{\cos^2 A}\right)^3=\left(\frac{1-\cos 2A}{1+\cos 2A}\right)^3. \quad \text{[Art. 164, (4), (3).]}$$
Divide both numerator and denominator by $\cos 2A$.
$$\therefore \left(\frac{3\sin A-\sin 3A}{3\cos A+\cos 3A}\right)^2=\left\{\frac{\dfrac{1}{\cos 2A}-1}{\dfrac{1}{\cos 2A}+1}\right\}^3=\left(\frac{\sec 2A-1}{\sec 2A+1}\right)^3.$$

10. $$\frac{1-\cos 3A}{1-\cos A}=\frac{1+3\cos A-4\cos^3 A}{1-\cos A}.$$
Divide numerator by denominator.
$$\therefore \frac{1-\cos 3A}{1-\cos A}=1+4\cos A+4\cos^2 A=(1+2\cos A)^2.$$

MISCELLANEOUS EXAMPLES. XLII. PAGE 140.

1. $$\frac{\sin A+\cos A}{\cos A-\sin A}=\frac{\sin^2 A+\cos^2 A+2\sin A.\cos A}{\cos^2 A-\sin^2 A}$$
$$=\frac{1+\sin 2A}{\cos 2A}=\frac{1}{\cos 2A}+\frac{\sin 2A}{\cos 2A}=\sec 2A+\tan 2A.$$

2. $$\frac{\tan\frac{1}{2}A+1}{1-\tan\frac{1}{2}A}=\frac{\sin\frac{1}{2}A+\cos\frac{1}{2}A}{\cos\frac{1}{2}A-\sin\frac{1}{2}A}=\frac{\sin^2\frac{1}{2}A+\cos^2\frac{1}{2}A+2\sin\frac{1}{2}A\cos\frac{1}{2}A}{\cos^2\frac{1}{2}A-\sin^2\frac{1}{2}A}$$
$$=\frac{1+\sin A}{\cos A}=\tan A+\sec A.$$

3. $$2\sin(n+1)\alpha.\cos(n-1)\alpha-2\sin 2\alpha$$
$$=\sin 2n\alpha+\sin 2\alpha-2\sin 2\alpha=\sin 2n\alpha-\sin 2\alpha$$
$$=2\cos(n+1)\alpha.\sin(n-1)\alpha=2\sin(n-1)\alpha\cos(n+1)\alpha.$$

4. $$\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\frac{2\sin\frac{1}{2}(\alpha+\beta).\cos\frac{1}{2}(\alpha-\beta)}{2\cos\frac{1}{2}(\alpha+\beta).\cos\frac{1}{2}(\alpha-\beta)}=\frac{\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha+\beta)}=\tan\tfrac{1}{2}(\alpha+\beta).$$

5. $$\frac{\cos 2\alpha+\cos 12\alpha}{\cos 6\alpha+\cos 8\alpha}=\frac{2\cos 7\alpha.\cos 5\alpha}{2\cos 7\alpha.\cos\alpha}=\frac{\cos 5\alpha}{\cos\alpha}$$
$$\frac{\cos 7\alpha-\cos 3\alpha}{\cos\alpha-\cos 3\alpha}=\frac{-2\sin 5\alpha.\sin 2\alpha}{2\sin 2\alpha.\sin\alpha}=-\frac{\sin 5\alpha}{\sin\alpha};$$
$$\therefore \frac{\cos 2\alpha+\cos 12\alpha}{\cos 6\alpha+\cos 8\alpha}+\frac{\cos 7\alpha-\cos 3\alpha}{\cos\alpha-\cos 3\alpha}=\frac{\cos 5\alpha}{\cos\alpha}-\frac{\sin 5\alpha}{\sin\alpha}$$
$$=\frac{\cos 5\alpha\sin\alpha-\sin 5\alpha\cos\alpha}{\cos\alpha\sin\alpha}=-\frac{\sin 4\alpha}{\sin\alpha\cos\alpha}=-\frac{2\sin 4\alpha}{\sin 2\alpha};$$
$$\therefore \frac{\cos 2\alpha+\cos 12\alpha}{\cos 6\alpha+\cos 8\alpha}+\frac{\cos 7\alpha-\cos 3\alpha}{\cos\alpha-\cos 3\alpha}+2\frac{\sin 4\alpha}{\sin 2\alpha}=-2\frac{\sin 4\alpha}{\sin 2\alpha}+2\frac{\sin 4\alpha}{\sin 2\alpha}=0.$$

6. Since $A=18°$ therefore $2A=36°$ and $3A=54°$.

$2A+3A=90°$ therefore $2A$ is the complement of $3A$;

and $\therefore\ \sin 2A=\cos 3A$. [Art. 118.]

Because $\sin 2A=\cos 3A$,

therefore $2\sin A\cos A=4\cos^3 A-3\cos A$;

divide both sides by $\cos A$, $\therefore\ 2\sin A=4\cos^2 A-3$;

$\therefore\ 4\sin^2 A+2\sin A-1=0$.

From this quadratic we have $\sin A=\frac{1}{4}(-1\pm\sqrt{5})$,

i. e. $\sin 18°=\frac{1}{4}(-1\pm\sqrt{5})$.

But as the angle of 18° is in the first quadrant, its sine is positive,

$$\therefore\ \sin 18°=\tfrac{1}{4}(\sqrt{5}-1).$$

7. $$\frac{\sin\alpha+\sin\beta+\sin(\alpha+\beta)}{\sin\alpha+\sin\beta-\sin(\alpha+\beta)}$$
$$=\frac{2\sin\frac{1}{2}(\alpha+\beta)\,.\,\cos\frac{1}{2}(\alpha-\beta)+2\sin\frac{1}{2}(\alpha+\beta)\,.\,\cos\frac{1}{2}(\alpha+\beta)}{2\sin\frac{1}{2}(\alpha+\beta)\,.\,\cos\frac{1}{2}(\alpha-\beta)-2\sin\frac{1}{2}(\alpha+\beta)\,.\,\cos\frac{1}{2}(\alpha+\beta)}$$
$$=\frac{\cos\frac{1}{2}(\alpha-\beta)+\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)-\cos\frac{1}{2}(\alpha+\beta)}=\frac{2\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta}{2\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta}=\cot\tfrac{1}{2}\alpha\,.\,\cot\tfrac{1}{2}\beta.$$

8. $\sin 2A\,.\,\sin 2B=\frac{1}{2}\{\cos 2(A-B)-\cos 2(A+B)\}$
$=\frac{1}{2}[1-2\sin^2(A-B)-\{1-2\sin^2(A+B)\}]$ [Art. 164, (4).]
$=\sin^2(A+B)-\sin^2(A-B)$.

9. $\cos 4A=2\cos^2 2A-1=2(2\cos^2 A-1)^2-1$ [Art. 164, (4).]
$=2(4\cos^4 A-4\cos^2 A+1)-1=8\cos^4 A-8\cos^2 A+1$.

10. $$\frac{\sin 50°}{\cos 50°}+\frac{\cos 50°}{\sin 50°}=\frac{\sin^2 50°+\cos^2 50°}{\sin 50°\cos 50°}=\frac{2}{2\sin 50°\cos 50°}=\frac{2}{\sin 100°}$$
$$=\frac{2}{\sin(90°+10°)}=\frac{2}{\cos 10°}=2\sec 10°$$
[E. T. p. 107, Ex. 4.]

11. $4\sin A\,.\,\sin(60°+A)\sin(60°-A)$
$=4\sin A(\sin^2 60°-\sin^2 A)=4\sin A(\frac{3}{4}-\sin^2 A)$ Examples XXXV. 24.
$=3\sin A-4\sin^3 A=\sin 3A$. [Art. 167.]

12. $$(\cot\tfrac{1}{2}A-\tan\tfrac{1}{2}A)^2=\left(\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}-\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}\right)^2=\left(\frac{\cos^2\frac{1}{2}A-\sin^2\frac{1}{2}A}{\sin\frac{1}{2}A\,.\,\cos\frac{1}{2}A}\right)^2$$
$$=\left(\frac{\cos A}{\sin\frac{1}{2}A\,.\,\cos\frac{1}{2}A}\right)^2=\left(\frac{2\cos A}{\sin A}\right)^2=\frac{4\cos^2 A}{\sin^2 A},$$
$$\cot A-2\cot 2A=\frac{\cos A}{\sin A}-\frac{2\cos 2A}{\sin 2A}$$
$$=\frac{2\cos^2 A}{2\sin A\cos A}-\frac{2(\cos^2 A-\sin^2 A)}{2\sin A\,.\,\cos A}$$

$$=\frac{\sin^2 A}{\sin A\cos A}=\frac{\sin A}{\cos A};$$

$$\therefore\ (\cot\tfrac{1}{2}A-\tan\tfrac{1}{2}A)^2(\cot A-2\cot A)=\frac{4\cos^2 A}{\sin^2 A}\cdot\frac{\sin A}{\cos A}=4\frac{\cos A}{\sin A}=4\cot A.$$

13. $\cos 3\alpha-\sin\beta\,.\,\sin 5\alpha-\cos 7\alpha$

$$=\cos 3\alpha-\cos 7\alpha-\sin\beta\,.\,\sin 5\alpha=2\sin 5\alpha\,.\,\sin 2\alpha-\sin\beta\sin 5\alpha$$
$$=\sin 5\alpha\,(2\sin 2\alpha-\sin\beta),$$

$\sin 3\alpha+\sin\beta\,.\,\cos 5\alpha-\sin 7\alpha$

$$=\sin 3\alpha-\sin 7\alpha+\sin\beta\,.\,\cos 5\alpha=-2\cos 5\alpha\sin 2\alpha+\sin\beta\cos 5\alpha$$
$$=-\cos 5\alpha\,(2\sin 2\alpha-\sin\beta),$$

$$\frac{\cos 3\alpha-\sin\beta\,.\,\sin 5\alpha-\cos 7\alpha}{\sin 3\alpha-\sin\beta\,.\,\cos 5\alpha-\sin 7\alpha}$$
$$=-\frac{\sin 5\alpha\,(2\sin 2\alpha-\sin\beta)}{\cos 5\alpha\,(2\sin 2\alpha-\sin\beta)}=-\tan 5\alpha \text{ a value independent of } \beta.$$

14. $(\cos x+\cos y)^2+(\sin x+\sin y)^2$

$$=(\cos^2 x+\sin^2 x)+(\cos^2 y+\sin^2 y)+2(\cos x\,.\,\cos y+\sin x\,.\,\sin y)$$
$$=2+2\cos(x-y)=2\,\{1+\cos(x-y)\}$$
$$=4\cos^2\tfrac{1}{2}(x-y).$$
[Art. 164, (3).]

15. $2\cos^2 A\cos^2 B+2\sin^2 A\,.\,\sin^2 B$

$$=\tfrac{1}{2}(2\cos^2 A\,.\,2\cos^2 B+2\sin^2 A\,.\,2\sin^2 B)$$
$$=\tfrac{1}{2}\{(1+\cos 2A)(1+\cos 2B)+(1-\cos 2A)(1-\cos 2B)\}$$
[Art. 164, (3), (4).]
$$=\tfrac{1}{2}(2+2\cos 2A\,.\,\cos 2B)=1+\cos 2A\,.\,\cos 2B.$$

16. $\cot\tfrac{1}{8}\pi-\tan\tfrac{1}{8}\pi=\dfrac{\cos\frac{1}{8}\pi}{\sin\frac{1}{8}\pi}-\dfrac{\sin\frac{1}{8}\pi}{\cos\frac{1}{8}\pi}=\dfrac{\cos^2\frac{1}{8}\pi-\sin^2\frac{1}{8}\pi}{\sin\frac{1}{8}\pi\,.\,\cos\frac{1}{8}\pi}$

$$=\frac{\cos\frac{1}{4}\pi}{\sin\frac{1}{8}\pi\,.\,\cos\frac{1}{8}\pi}=\frac{2\cos\frac{1}{4}\pi}{\sin\frac{1}{4}\pi}=2\cot\tfrac{1}{4}\pi=2.$$

17. $\tan 4\theta=\dfrac{2\tan 2\theta}{1-\tan^2 2\theta}$ [Art. 164, (5).]

$$=\frac{\dfrac{4\tan\theta}{1-\tan^2\theta}}{1-\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)^2}=\frac{\dfrac{4\tan\theta}{1-\tan^2\theta}}{\dfrac{1-6\tan^2\theta+\tan^4\theta}{(1-\tan^2\theta)^2}}=\frac{4\tan\theta\,(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}.$$

18. $\cos^2\tfrac{1}{8}\pi-\sin^2\tfrac{1}{8}\pi=\cos\tfrac{1}{4}\pi=\dfrac{1}{\sqrt{2}},\ \cos^2\tfrac{1}{8}\pi+\sin^2\tfrac{1}{8}\pi=1.$

By addition $\quad 2\cos^2\tfrac{1}{8}\pi=1+\dfrac{1}{\sqrt{2}}=\dfrac{1+\sqrt{2}}{\sqrt{2}};$

$$\therefore\ 4\cos^2\tfrac{1}{8}\pi=\frac{2(1+\sqrt{2})}{\sqrt{2}}=2+\sqrt{2};\ \therefore\ 2\cos\tfrac{1}{8}\pi=\sqrt{2+\sqrt{2}}.$$

19. $\cos^2 11^\circ 15' - \sin^2 11^\circ 15' = \cos 22^\circ 30' = \cos \frac{1}{8}\pi = \frac{1}{2}\sqrt{2+\sqrt{2}}$ (18),

$\cos^2 11^\circ 15' + \sin^2 11^\circ 15' = 1.$

By addition $2\cos^2 11^\circ 15' = 1 + \frac{1}{2}\sqrt{2+\sqrt{2}}$;

$\therefore\ 4\cos^2 11^\circ 15' = 2 + \sqrt{2+\sqrt{2}}$;

$\therefore\ 2\cos 11^\circ 15' = \sqrt{2+\sqrt{2+\sqrt{2}}}.$

20. $$\frac{\sin A \,.\, \sin 2A + \sin A \,.\, \sin 4A + \sin 2A \,.\, \sin 7A}{\sin A \,.\, \cos 2A + \sin 2A \,.\, \cos 5A + \sin A \,.\, \cos 8A}$$

$$= \frac{\sin A\,(\sin 2A + \sin 4A) + \sin 2A \,.\, \sin 7A}{\sin A\,(\cos 2A + \cos 8A) + \sin 2A \,.\, \cos 5A}$$

$$= \frac{2\sin A \,.\, \cos A \,.\, \sin 3A + \sin 2A \,.\, \sin 7A}{2\sin A \,.\, \cos 5A \,.\, \cos 3A + \sin 2A \,.\, \cos 5A}$$

$$= \frac{\sin 2A\,(\sin 3A + \sin 7A)}{2\sin A \,.\, \cos 5A\,(\cos 3A + \cos A)} = \frac{2\sin 2A \,.\, \cos 2A \,.\, \sin 5A}{2\sin A \cos A \,.\, 2\cos 2A \,.\, \cos 5A}$$

$$= \frac{\sin 4A \,.\, \sin 5A}{2\sin 2A \cos 2A \cos 5A} = \frac{\sin 4A \,.\, \sin 5A}{\sin 4A \,.\, \cos 5A} = \tan 5A.$$

21. $$\frac{\sin\theta + \sin(\theta+\phi) + \sin(\theta+2\phi)}{\cos\theta + \cos(\theta+\phi) + \cos(\theta+2\phi)}$$

$$= \frac{\sin\theta + \sin(\theta+2\phi) + \sin(\theta+\phi)}{\cos\theta + \cos(\theta+2\phi) + \cos(\theta+\phi)} = \frac{2\sin(\theta+\phi)\cos\phi + \sin(\theta+\phi)}{2\cos(\theta+\phi)\cos\phi + \cos(\theta+\phi)}$$

$$= \frac{\sin(\theta+\phi)\,\{2\cos\phi + 1\}}{\cos(\theta+\phi)\,\{2\cos\phi + 1\}} = \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} = \tan(\theta+\phi).$$

22. $2\cos^8 A - 2\sin^8 A = 2(\cos^4 A + \sin^4 A)(\cos^2 A - \sin^2 A)(\cos^2 A + \sin^2 A)$

$= 2(\cos^4 A + \sin^4 A)(\cos^2 A - \sin^2 A)$

$= 2\{(\cos^2 A + \sin^2 A)^2 - 2\sin^2 A \cos^2 A\}\cos 2A$

$= 2(1 - 2\sin^2 A \cos^2 A)\cos 2A$

$= (2 - 2\sin^2 A \,.\, 2\cos^2 A)\cos 2A$

$= \{2 - (1 - \cos 2A)(1 + \cos 2A)\}\cos 2A$ [Art. 164.]

$= (1 + \cos^2 2A)\cos 2A.$

23. $(3\sin A - 4\sin^3 A)^2 + (4\cos^3 A - 3\cos A)^2 = \sin^2 3A + \cos^2 3A = 1.$

24. $$\frac{\sin 2\alpha \,.\, \cos\alpha}{(1+\cos 2\alpha)(1+\cos\alpha)} = \frac{2\sin\alpha\cos^2\alpha}{2\cos^2\alpha\,(1+\cos\alpha)} = \frac{\sin\alpha}{1+\cos\alpha}$$ [Art. 164.]

$$= \frac{2\sin\frac{1}{2}\alpha \,.\, \cos\frac{1}{2}\alpha}{2\cos^2\frac{1}{2}\alpha} = \frac{\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha} = \tan\tfrac{1}{2}\alpha.$$

25. $$2\frac{\cot(n-2)\alpha\,.\,\cot n\alpha+1}{\cot(n-2)\alpha-\cot n\alpha}=2\frac{\dfrac{\cos(n-2)\alpha\,.\,\cos n\alpha}{\sin(n-2)\alpha\,.\,\sin n\alpha}+1}{\dfrac{\cos(n-2)\alpha}{\sin(n-2)\alpha}-\dfrac{\cos n\alpha}{\sin n\alpha}}$$

$$=2\frac{\cos(n-2)\alpha\cos n\alpha+\sin(n-2)\alpha\,.\,\sin n\alpha}{\cos(n-2)\alpha\sin n\alpha-\sin(n-2)\alpha\cos n\alpha}$$

$$=2\frac{\cos\{(n-2)\alpha-n\alpha\}}{\sin\{n\alpha-(n-2)\alpha\}}=2\frac{\cos 2\alpha}{\sin 2\alpha}$$

$$=\frac{2(\cos^2\alpha-\sin^2\alpha)}{2\sin\alpha\cos\alpha}=\frac{\cos\alpha}{\sin\alpha}-\frac{\sin\alpha}{\cos\alpha}$$

$$=\cot\alpha-\tan\alpha.$$

26. $$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{\frac{2}{7}}{1-(\frac{1}{7})^2}=\frac{\frac{2}{7}}{\frac{48}{49}}=\tfrac{7}{24},$$

$$\tan(2\alpha+\beta)=\frac{\tan 2\alpha+\tan\beta}{1-\tan 2\alpha\,.\,\tan\beta}=\frac{\frac{7}{24}+\frac{2}{11}}{1-\frac{7}{24}\cdot\frac{2}{11}}=\frac{\frac{125}{264}}{\frac{250}{264}}=\tfrac{125}{250}=\tfrac{1}{2}.$$

27. We have to shew that $(x-\tan\frac{1}{2}A)(x-\cot\frac{1}{2}A)$ is identical with

$$x^2-2x\operatorname{cosec}A+1.$$

The first expression is $x^2-(\tan\frac{1}{2}A+\cot\frac{1}{2}A)x+1$.

Hence we have to shew that $\tan\frac{1}{2}A+\cot\frac{1}{2}A=2\operatorname{cosec}A$,

$$\tan\tfrac{1}{2}A+\cot\tfrac{1}{2}A=\tan\tfrac{1}{2}A+\frac{1}{\tan\frac{1}{2}A}=\frac{\tan^2\frac{1}{2}A+1}{\tan\frac{1}{2}A}$$

$$=\frac{\sin^2\frac{1}{2}A+\cos^2\frac{1}{2}A}{\sin\frac{1}{2}A\cos\frac{1}{2}A}=\frac{2}{\sin A}=2\operatorname{cosec}A.$$ Q. E. D.

Or, proceed to solve the given equation

$$x^2-2x\,.\operatorname{cosec}A+1=0;$$

$$\therefore\ x^2-2x\,.\operatorname{cosec}A+\operatorname{cosec}^2A=\operatorname{cosec}^2A-1=\frac{1}{\sin^2A}-1=\frac{\cos^2A}{\sin^2A}.$$

$$\therefore\ x-\operatorname{cosec}A=\pm\frac{\cos A}{\sin A};$$

$$\therefore\ x=\frac{1}{\sin A}\pm\frac{\cos A}{\sin A}=\frac{1+\cos A}{\sin A}\text{ or }\frac{1-\cos A}{\sin A}$$

$$=\frac{2\cos^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}\text{ or }\frac{2\sin^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}=\cot\tfrac{1}{2}A\text{ or }\tan\tfrac{1}{2}A.$$

28. If $\tan B=\dfrac{b}{a}$, $\therefore\ b=a\tan B$;

$$\therefore\ \sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}=\sqrt{\frac{1+\tan B}{1-\tan B}}+\sqrt{\frac{1-\tan B}{1+\tan B}}$$

$$=\frac{2}{\sqrt{(1-\tan^2B)}}=\frac{2}{\sqrt{\left(1-\dfrac{\sin^2B}{\cos^2B}\right)}}=\frac{2\cos B}{\sqrt{(\cos^2B-\sin^2B)}}=\frac{2\cos B}{\sqrt{\cos 2B}}.$$

EXAMPLES. XLIII. Page 142.

1. $\cos(\alpha+\beta+\gamma) = \cos(\alpha+\beta)\cos\gamma - \sin(\alpha+\beta)\sin\gamma$
$= \cos\alpha \, . \cos\beta \, . \cos\gamma - \cos\alpha \, . \sin\beta \, . \sin\gamma - \cos\beta\sin\gamma\sin\alpha - \cos\gamma \, . \sin\alpha \, . \sin\beta.$

2. $\sin(\alpha+\beta-\gamma) = \sin(\alpha+\beta)\cos\gamma - \cos(\alpha+\beta)\sin\gamma$
$= \sin\alpha \, . \cos\beta \, . \cos\gamma + \sin\beta \, . \cos\gamma \, . \cos\alpha - \sin\gamma \, . \cos\alpha\cos\beta + \sin\alpha\sin\beta\sin\gamma.$

3. $\cos(\alpha-\beta+\gamma) = \cos(\alpha-\beta)\cos\gamma - \sin(\alpha-\beta)\sin\gamma$
$= \cos\alpha \, . \cos\beta \, . \cos\gamma + \cos\alpha \, . \sin\beta \, . \sin\gamma - \cos\beta \, . \sin\alpha \, . \sin\gamma + \cos\gamma \, . \sin\beta \, . \sin\alpha.$

4. $\sin\alpha - \sin(\alpha+\beta-\gamma) = -2\cos\tfrac{1}{2}(2\alpha+\beta-\gamma)\sin\tfrac{1}{2}(\beta-\gamma).$
And $\sin\beta - \sin\gamma = 2\cos\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\beta-\gamma);$ [Art. 158.
$\therefore \sin\alpha + \sin\beta - \sin\gamma - \sin(\alpha+\beta-\gamma)$
$= 2\cos\tfrac{1}{2}(\beta+\gamma) \, . \sin\tfrac{1}{2}(\beta-\gamma) - 2\cos\tfrac{1}{2}(2\alpha+\beta-\gamma) \, . \sin\tfrac{1}{2}(\beta-\gamma)$
$= 2\sin\tfrac{1}{2}(\beta-\gamma)\{\cos\tfrac{1}{2}(\beta+\gamma) - \cos\tfrac{1}{2}(2\alpha+\beta-\gamma)\}$
$= 2\sin\tfrac{1}{2}(\beta-\gamma) \, . \, 2\sin\tfrac{1}{2}(\alpha+\beta) \, . \sin\tfrac{1}{2}(\alpha-\gamma)$ [Art. 158.
$= 4\sin\tfrac{1}{2}(\alpha-\gamma) \, . \sin\tfrac{1}{2}(\beta-\gamma) \, . \sin\tfrac{1}{2}(\alpha+\beta).$

5. $\sin(\alpha-\beta-\gamma) - \sin\alpha = -2\cos\tfrac{1}{2}(2\alpha-\beta-\gamma)\sin\tfrac{1}{2}(\beta+\gamma),$
$\sin\beta + \sin\gamma = 2\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta-\gamma);$ [Art. 158.
$\therefore \sin(\alpha-\beta-\gamma) - \sin\alpha + \sin\beta + \sin\gamma$
$= 2\sin\tfrac{1}{2}(\beta+\gamma) \, . \cos\tfrac{1}{2}(\beta-\gamma) - 2\cos\tfrac{1}{2}(2\alpha-\beta-\gamma) \, . \sin\tfrac{1}{2}(\beta+\gamma)$
$= 2\sin\tfrac{1}{2}(\beta+\gamma)\{\cos\tfrac{1}{2}(\beta-\gamma) - \cos\tfrac{1}{2}(2\alpha-\beta-\gamma)\}$
$= 2\sin\tfrac{1}{2}(\beta+\gamma) \, . \, 2\sin\tfrac{1}{2}(\alpha-\gamma) \, . \sin\tfrac{1}{2}(\alpha-\beta)$
$= 4\sin\tfrac{1}{2}(\alpha-\beta) \, . \sin\tfrac{1}{2}(\alpha-\gamma) \, . \sin\tfrac{1}{2}(\beta+\gamma).$

6. $\sin 2\alpha - \sin 2(\alpha+\beta+\gamma) = -2\cos(2\alpha+\beta+\gamma)\sin(\beta+\gamma)$
$\sin 2\beta + \sin 2\gamma = 2\sin(\beta+\gamma) \, . \cos(\beta-\gamma);$ [Art. 158.
$\therefore \sin 2\alpha + \sin 2\beta + \sin 2\gamma - \sin 2(\alpha+\beta+\gamma)$
$= 2\sin(\beta+\gamma) \, . \cos(\beta-\gamma) - 2\cos(2\alpha+\beta+\gamma) \, . \sin(\beta+\gamma)$
$= 2\sin(\beta+\gamma)\{\cos(\beta-\gamma) - \cos(2\alpha+\beta+\gamma)\}$
$= 2\sin(\beta+\gamma) \, . \, 2\sin(\alpha+\beta) \, . \sin(\gamma+\alpha)$
$= 2\sin(\alpha+\beta) \, . \sin(\beta+\gamma) \, . \sin(\gamma+\alpha).$

7. $\sin(\beta-\gamma) + \sin(\gamma-\alpha) = -2\sin\tfrac{1}{2}(\alpha-\beta) \, . \cos\tfrac{1}{2}(\alpha+\beta-2\gamma)$
$\sin(\alpha-\beta) = 2\sin\tfrac{1}{2}(\alpha-\beta) \, . \cos\tfrac{1}{2}(\alpha-\beta);$
$\therefore \sin(\beta-\gamma) + \sin(\gamma-\alpha) + \sin(\alpha-\beta)$
$= 2\sin\tfrac{1}{2}(\alpha-\beta)\{\cos\tfrac{1}{2}(\alpha-\beta) - \cos\tfrac{1}{2}(\alpha+\beta-2\gamma)\}$
$= 2\sin\tfrac{1}{2}(\alpha-\beta) \, . \, 2\sin\tfrac{1}{2}(\alpha-\gamma) \, . \sin\tfrac{1}{2}(\beta-\gamma)$
$= 4\sin\tfrac{1}{2}(\alpha-\beta) \, . \sin\tfrac{1}{2}(\beta-\gamma) \, . \sin\tfrac{1}{2}(\alpha-\gamma)$
$= -4\sin\tfrac{1}{2}(\alpha-\beta) \, . \sin\tfrac{1}{2}(\beta-\gamma) \, . \sin\tfrac{1}{2}(\gamma-\alpha);$
$\therefore \sin(\beta-\gamma) + \sin(\gamma-\alpha) + \sin(\alpha-\beta)$
$+ 4\sin\tfrac{1}{2}(\beta-\gamma) \, . \sin\tfrac{1}{2}(\gamma-\alpha) \, . \sin\tfrac{1}{2}(\alpha-\beta) = 0.$

8. $\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)=2\sin\gamma\cos(\alpha-\beta)$

$\sin(\alpha+\beta-\gamma)-\sin(\alpha+\beta+\gamma)=-2\sin\gamma\cos(\alpha+\beta)$;

$\therefore \sin(\alpha+\beta+\gamma)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)-\sin(\alpha+\beta-\gamma)$

$=2\sin\gamma\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}$

$=2\sin\gamma\,.\,2\sin\alpha\,.\,\sin\beta=4\sin\alpha\,.\,\sin\beta\,.\,\sin\gamma.$

9. $\sin(\alpha+\beta+\gamma)+\sin(\beta+\gamma-\alpha)=2\sin(\beta+\gamma)\cos\alpha$

$\sin(\gamma+\alpha-\beta)-\sin(\alpha+\beta-\gamma)=-2\cos\alpha\sin(\beta-\gamma)$;

$\therefore \sin(\alpha+\beta+\gamma)+\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)-\sin(\alpha+\beta-\gamma)$

$=2\cos\alpha\{\sin(\beta+\gamma)-\sin(\beta-\gamma)\}$

$=2\cos\alpha\,.\,2\cos\beta\,.\,\sin\gamma=4\cos\alpha\,.\,\cos\beta\,.\,\sin\gamma.$

10. $\cos x+\cos y=2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$

$\cos z+\cos(x+y+z)=2\cos\frac{1}{2}(2z+x+y)\,.\,\cos\frac{1}{2}(x+y)$;

$\therefore \cos x+\cos y+\cos z+\cos(x+y+z)$

$=2\cos\frac{1}{2}(x+y)\{\cos\frac{1}{2}(x-y)+\cos\frac{1}{2}(2z+x+y)\}$

$=2\cos\frac{1}{2}(x+y)\,2\cos\frac{1}{2}(z+x)\cos\frac{1}{2}(y+z)$

$=4\cos\frac{1}{2}(x+y)\,.\,\cos\frac{1}{2}(y+z)\,.\,\cos\frac{1}{2}(z+x).$

11. $\cos 2x+\cos 2y=2\cos(x+y)\,.\,\cos(x-y)$

$\cos 2z+\cos 2(x+y+z)=2\cos(2z+x+y)\,.\,\cos(x+y)$;

$\therefore \cos 2x+\cos 2y+\cos 2z+\cos 2(x+y+z)$

$=2\cos(x+y)\{\cos(x-y)+\cos(2z+x+y)\}$

$=2\cos(x+y)\,.\,2\cos(x+z)\,.\,\cos(y+z)$

$=4\cos(x+y)\,.\,\cos(y+z)\,.\,\cos(z+x).$

12. $\cos(y+z-x)+\cos(z+x-y)=2\cos z\,.\,\cos(x-y)$

$\cos(x+y-z)+\cos(x+y+z)=2\cos z\,.\,\cos(x+y)$;

$\therefore \cos(y+z-x)+\cos(z+x-y)+\cos(x+y-z)+\cos(x+y+z)$

$=2\cos z\{\cos(x-y)+\cos(x+y)\}$

$=2\cos z\,.\,2\cos x\,.\,\cos y=4\cos x\,.\,\cos y\,.\,\cos z.$

13. $\cos^2 x+\cos^2 y+\cos^2 z+\cos^2(x+y+z)$

$=\frac{1}{2}\{1+\cos 2x+1+\cos 2y+1+\cos 2z+1+\cos 2(x+y+z)\}$

$=\frac{1}{2}\{4+\cos 2x+\cos 2y+\cos 2z+\cos 2(x+y+z)\}$

$=\frac{1}{2}\{4+4\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y)\}$ [Example 11.

$=2\{1+\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y)\}.$

14. $\sin^2 x+\sin^2 y+\sin^2 z+\sin^2(x+y+z)$

$=\frac{1}{2}\{1-\cos 2x+1-\cos 2y+1-\cos 2z+1-\cos 2(x+y+z)\}$

$=\frac{1}{2}[4-\{\cos 2x+\cos 2y+\cos 2z+\cos 2(x+y+z)\}]$

$=\frac{1}{2}\{4-4\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y)\}$ [Example 11.

$=2\{1-\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y)\}.$

15. $\cos^2 x + \cos^2 y + \cos^2 z + \cos^2 (x+y-z)$

$$= \tfrac{1}{2}\{1 + \cos 2x + 1 + \cos 2y + 1 + \cos 2z + 1 + \cos 2(x+y-z)\}$$
$$= \tfrac{1}{2}\{4 + \cos 2x + \cos 2y + \cos 2z + \cos 2(x+y-z)\}.$$

Now $\cos 2x + \cos 2y = 2\cos(x+y) \cdot \cos(x-y)$,

and $\cos 2z + \cos 2(x+y-z) = 2\cos(x+y) \cdot \cos(x+y-2z)$;

$\therefore \cos 2x + \cos 2y + \cos 2z + \cos 2(x+y-z)$

$$= 2\cos(x+y)\{\cos(x-y) + \cos(x+y-2z)\}$$
$$= 2\cos(x+y) \cdot 2\cos(x-z) \cdot \cos(y-z)$$
$$= 4\cos(x-z) \cdot \cos(y-z) \cdot \cos(x+y);$$

$\therefore \cos^2 x + \cos^2 y + \cos^2 z + \cos^2(x+y-z)$

$$= \tfrac{1}{2}\{4 + 4\cos(x-z) \cdot \cos(y-z) \cdot \cos(x+y)\}$$
$$= 2\{1 + \cos(x-z) \cdot \cos(y-z) \cdot \cos(x+y)\}.$$

16. $\cos\alpha \cdot \sin(\beta-\gamma) + \cos\beta \cdot \sin(\gamma-\alpha) + \cos\gamma \cdot \sin(\alpha-\beta)$

$$= \tfrac{1}{2}\{2\cos\alpha \cdot \sin(\beta-\gamma) + 2\cos\beta \cdot \sin(\gamma-\alpha) + 2\cos\gamma \cdot \sin(\alpha-\beta)\}$$
$$= \tfrac{1}{2}\{\sin(\alpha+\beta-\gamma) - \sin(\alpha-\beta+\gamma) + \sin(\beta+\gamma-\alpha) - \sin(\beta-\gamma+\alpha)$$
$$+ \sin(\gamma+\alpha-\beta) - \sin(\gamma-\alpha+\beta)\} = 0.$$

17. $\sin\alpha \cdot \sin(\beta-\gamma) + \sin\beta \cdot \sin(\gamma-\alpha) + \sin\gamma \cdot \sin(\alpha-\beta)$

$$= \tfrac{1}{2}\{2\sin\alpha \cdot \sin(\beta-\gamma) + 2\sin\beta \cdot \sin(\gamma-\alpha) + 2\sin\gamma \cdot \sin(\alpha-\beta)\}$$
$$= \tfrac{1}{2}\{\cos(\alpha-\beta+\gamma) - \cos(\alpha+\beta-\gamma) + \cos(\beta-\gamma+\alpha) - \cos(\beta+\gamma-\alpha)$$
$$+ \cos(\gamma-\alpha+\beta) - \cos(\gamma+\alpha-\beta)\} = 0.$$

18. $\cos(\alpha+\beta) \cdot \cos(\alpha-\beta) + \sin(\beta+\gamma) \cdot \sin(\beta-\gamma) - \cos(\alpha+\gamma) \cdot \cos(\alpha-\gamma)$

$= \cos^2\alpha - \sin^2\beta + \sin^2\beta - \sin^2\gamma - \cos^2\alpha + \sin^2\gamma$ [Examples XXXV. 24, 25.

$= 0.$

19. $\cos(\delta-\alpha) \cdot \sin(\beta-\gamma) + \cos(\delta-\beta) \cdot \sin(\gamma-\alpha) - \cos(\delta-\gamma) \cdot \sin(\beta-\alpha)$

$$= \tfrac{1}{2}\{2\cos(\delta-\alpha) \cdot \sin(\beta-\gamma) + 2\cos(\delta-\beta) \cdot \sin(\gamma-\alpha) - 2\cos(\delta-\gamma) \cdot \sin(\beta-\alpha)\}$$
$$= \tfrac{1}{2}\{\sin(\delta-\alpha+\beta-\gamma) - \sin(\delta-\alpha-\beta+\gamma) + \sin(\delta-\beta+\gamma-\alpha)$$
$$- \sin(\delta-\beta-\gamma+\alpha) = \sin(\delta-\gamma+\beta-\alpha) + \sin(\delta-\gamma-\beta+\alpha)\} = 0.$$

20. $8\cos\tfrac{1}{2}(\theta+\phi+\chi) \cdot \cos\tfrac{1}{2}(\phi+\chi-\theta) \cdot \cos\tfrac{1}{2}(\chi+\theta-\phi) \cdot \cos\tfrac{1}{2}(\theta+\phi-\chi)$

$$= 2\{2\cos\tfrac{1}{2}(\theta+\phi+\chi) \cdot \cos\tfrac{1}{2}(\phi+\chi-\theta)\} \times \{2\cos\tfrac{1}{2}(\chi+\theta-\phi) \cdot \cos\tfrac{1}{2}(\theta+\phi-\chi)\}.$$

Now $2\cos\tfrac{1}{2}(\theta+\phi+\chi) \cdot \cos\tfrac{1}{2}(\phi+\chi-\theta) = \cos(\phi+\chi) + \cos\theta$,

and $2\cos\tfrac{1}{2}(\chi+\theta-\phi) \cdot \cos\tfrac{1}{2}(\theta+\phi-\chi) = \cos(\phi-\chi) + \cos\theta$;

$\therefore 2\{2\cos\tfrac{1}{2}(\theta+\phi+\chi) \cdot \cos\tfrac{1}{2}(\phi+\chi-\theta)\} \times \{2\cos\tfrac{1}{2}(\chi+\theta-\phi) \cdot \cos\tfrac{1}{2}(\theta+\phi-\chi)\}$

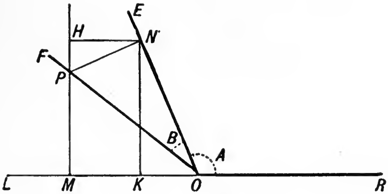
$$= 2\{\cos(\phi+\chi) + \cos\theta\} \times \{\cos(\phi-\chi) + \cos\theta\}$$
$$= 2[\cos(\phi+\chi)\cos(\phi-\chi) + \cos\theta\{\cos(\phi+\chi) + \cos(\phi-\chi)\} + \cos^2\theta]$$
$$= 2\cos(\phi+\chi)\cos(\phi-\chi) + 4\cos\theta\cos\phi\cos\chi + 2\cos^2\theta$$
$$= \cos 2\phi + \cos 2\chi + 4\cos\theta \cdot \cos\phi\cos\chi + 1 + \cos 2\theta.$$

Therefore

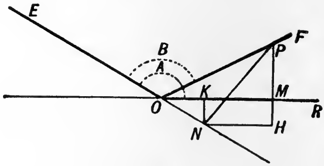
$$8\cos\tfrac{1}{2}(\theta+\phi+\chi) \cdot \cos\tfrac{1}{2}(\phi+\chi+\theta) \cdot \cos\tfrac{1}{2}(\chi+\theta-\phi) \cdot \cos\tfrac{1}{2}(\theta+\phi-\chi)$$
$$= \cos 2\theta + \cos 2\phi + \cos 2\chi + 4\cos\theta \cdot \cos\phi \cdot \cos\chi + 1.$$

EXAMPLES. XLIV. Page 145.

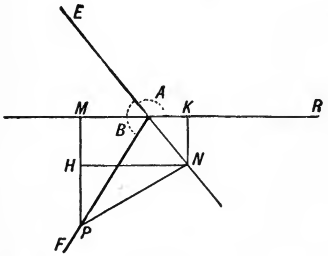
1.



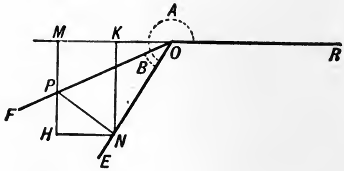
2.



3.



4.



5. $\cos(90°+A) = \cos 90° \cos A - \sin 90° \sin A$
$= 0 \times \cos A - 1 \times \sin A = -\sin A.$

6. $\sin(90°-A) = \sin 90° \cos A + \cos 90° \sin A$
$= 1 \times \cos A + 0 \times \sin A = \cos A.$

7. $\cos(90°-A) = \cos 90° \cos A + \sin 90° \sin A$
$= 0 \times \cos A + 1 \times \sin A = \sin A.$

8. $\sin(180°-A) = \sin 180° \cos A - \cos 180° \sin A$
$= 0 \times \cos A - (-1 \times \sin A) = \sin A.$

9. $\cos(180°-A) = \cos 180° \cos A + \sin 180° \sin A$
$= -1 \times \cos A + 0 \times \sin A = -\cos A.$

10. $\sin(180°+A) = \sin 180° \cos A + \cos 180° \sin A$
$= 0 \times \cos A + (-1 \times \sin A) = -\sin A.$

11. $\cos(A+B) = \sin(90°+A+B)$ [See p. 107.
$= \sin(90°+A)\cos B + \cos(90°+A)\sin B$
$= \cos A \cos B - \sin A \sin B.$

$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$
$= \sin A \cos B - \cos A \sin B.$

$\cos(A-B) = \sin(90°+A-B)$
$= \sin(90°+A)\cos(-B) + \cos(90°+A)\sin(-B)$
$= \cos A \cos B + \sin A \sin B.$

EXAMPLES. XLV. Page 147.

1. $\sin 180° = \sin(90° + 90°) = \sin 90° \cos 90° + \cos 90° \sin 90°$
$= 1 \times 0 + 1 \times 0 = 0.$
$\cos 180° = \cos(90° + 90°) = \cos 90° \cos 90° - \sin 90° \sin 90°$
$= 0 \times 0 - 1 \times 1 = -1.$

2. In fig. E. T. p. 72, let $OM = 2$, $PM = 1$; $\therefore OP = \sqrt{5}$;

$$\therefore \tan POM = \tfrac{1}{2},\ \sin POM = \frac{1}{\sqrt{5}},\ \cos POM = \frac{2}{\sqrt{5}}.$$

Since A is between 180° and 270°, $\tan A$ is positive and $\sin A$ and $\cos A$ are negative, $\therefore \sin A = -\dfrac{1}{\sqrt{5}}$, $\cos A = -\dfrac{2}{\sqrt{5}}$,

$$\sin 2A = 2 \sin A \cos A = 2 \times -\frac{1}{\sqrt{5}} \times -\frac{2}{\sqrt{5}} = \frac{4}{5},$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A = -\frac{3}{\sqrt{5}} + \frac{4}{5\sqrt{5}} = -\frac{11}{5\sqrt{5}} = -\frac{11}{5}\sqrt{5}.$$

3. Since θ is the fourth quadrant $\sin \theta$ is negative,

$$\sin \theta = -\sqrt{(1 - \cos^2 \theta)} = -\frac{\sqrt{15}}{4},$$

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \times -\frac{\sqrt{15}}{4} \times \tfrac{1}{4} = -\tfrac{1}{8}\sqrt{15},$$

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta = -\frac{3\sqrt{15}}{4} + \frac{15\sqrt{15}}{16} = \tfrac{3}{16}\sqrt{15},$$

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta = \tfrac{1}{16} - \tfrac{3}{4} = -\tfrac{11}{16}.$$

To determine in what quadrant 3θ lies, it should be noticed that $\cos 3\theta$ is negative and therefore 3θ may lie in the second or the third quadrant, but $\sin 3\theta$ is positive therefore 3θ *must* lie in the second quadrant.

4. $\cos p\theta + \cos q\theta = 0$; $\therefore 2 \cos \tfrac{1}{2}(p+q)\theta \cdot \cos \tfrac{1}{2}(p \sim q)\theta = 0$;
$\therefore \cos \tfrac{1}{2}(p+q)\theta = 0$ or $\cos \tfrac{1}{2}(p \sim q)\theta = 0$.

If $\cos \tfrac{1}{2}(p+q)\theta = 0$, then $(p+q)\theta = (2n+1)\pi$; $\therefore \theta = \dfrac{2n+1}{p+q}\pi$.

If $\cos \tfrac{1}{2}(p \sim q)\theta = 0$, then $(p \sim q)\theta = (2n+1)\pi$; $\therefore \theta = \dfrac{2n+1}{p \sim q}\pi$ by giving integral values to n in order $\dfrac{2n+1}{p+q}\pi$ and $\dfrac{2n+1}{p \sim q}\pi$ are evidently two series in A. P. with common differences $\dfrac{2\pi}{p+q}$ and $\dfrac{2\pi}{p \sim q}$.

EXAMPLES. XLVI. Page 149.

1. When A lies between $-180°$ and $180°$, $\frac{1}{2}A$ lies between $-90°$ and $90°$, i.e. $\cos\frac{1}{2}A$ is positive; $\therefore \cos\frac{1}{2}A = +\sqrt{\frac{1+\cos A}{2}}$.

2. When A lies between $180°$ and $540°$, $\frac{1}{2}A$ lies between $90°$ and $270°$, i.e. $\cos\frac{1}{2}A$ is negative; $\therefore \cos\frac{1}{2}A = -\sqrt{\frac{1+\cos A}{2}}$.

3. When A lies between $180°$ and $360°$, $\frac{1}{2}A$ lies between $90°$ and $180°$, i.e. $\sin\frac{1}{2}A$ is positive; $\therefore \sin\frac{1}{2}A = +\sqrt{\frac{1-\cos A}{2}}$.

4. When A lies between $(4n+1)\pi$ and $(4n+3)\pi$ its trigonometrical ratios have the same signs as when it lies between π and 3π; $\therefore$ the trigonometrical ratios of $\frac{1}{2}A$ have the same signs as when it is between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, i.e. $\cos\frac{1}{2}A$ is negative;

$$\therefore \cos\frac{1}{2}A = -\sqrt{\frac{1+\cos A}{2}}.$$

5. When A lies between $4n\pi$ and $(4n+2)\pi$ its trigonometrical ratios have the same signs as when it lies between 0 and 2π; therefore the trigonometrical ratios of $\frac{1}{2}A$ have the same signs as when it lies between 0 and π, i.e. $\sin\frac{1}{2}A$ is positive;

$$\therefore \sin\frac{1}{2}A = +\sqrt{\frac{1-\cos A}{2}}.$$

EXAMPLES. XLVII. Page 154.

1. (i) When $\frac{1}{2}A$ is $22°$ $\sin\frac{1}{2}A$ is positive and less than $\cos\frac{1}{2}A$, $\cos\frac{1}{2}A$ is also positive; $\therefore \sin\frac{1}{2}A + \cos\frac{1}{2}A$ is positive,

$\sin\frac{1}{2}A - \cos\frac{1}{2}A$ is negative.

(ii) $\sin 191° = \sin(180° - 191°) = -\sin 11°$; $\therefore \sin 191°$ is negative and numerically less than $\cos 191°$,

$\cos 191° = -\cos(180° - 191°) = -\cos 11°$; $\therefore \cos 191°$ is also negative;

$\sin 191° + \cos 191°$ is negative,

$\sin 191° - \cos 191°$ is positive.

(iii) $\sin 290° = \sin(360° - 70°) = -\sin 70°$; $\therefore \sin 290°$ is negative and numerically greater than $\cos 290°$,

$\cos 290° = \cos(360° - 70°) = \cos 70°$; $\therefore \cos 270°$ is positive;

$\therefore \sin 290° + \cos 290°$ is negative,

$\sin 290° - \cos 290°$ is negative.

(iv) $\sin 345° = \sin(360° - 15°) = -\sin 15°$; $\therefore \sin 345°$ is negative and numerically less than $\cos 345°$,

$\cos 345° = (360° - 15°) = \cos 15°$ is positive;

$\therefore \sin 345° + \cos 345°$ is positive,

$\sin 345° - \cos 345°$ is negative.

(v) $\sin -22° = -\sin 22°$ is negative and numerically less than $\cos -22°$; $\cos -22° = \cos 22°$ is positive;

$$\therefore \sin -22° + \cos -22° \text{ is positive,}$$
$$\sin -22° - \cos -22° \text{ is negative.}$$

(vi) $\sin -275° = \sin(360° - 275°) = \sin 85°$; $\therefore$ $\sin -275°$ is positive and numerically greater than $\cos -275°$;

$$\cos -275° = \cos(360° - 275°) = \cos 85° \text{ is positive;}$$
$$\therefore \sin -275° + \cos -275° \text{ is positive,}$$
$$\sin -275° - \cos -275° \text{ is positive.}$$

(vii) $\sin -470° = \sin(360° - 470°) = \sin -110° = -\sin(180° - 110°)$ $= -\sin 70°$; $\cos -470° = \cos(360° - 470°) = \cos -110° = -\cos(180° - 110°)$ $= -\cos 70°$; $\therefore$ $\sin -470°$ is negative and numerically greater than $\cos -470°$; $\cos -470°$ is also negative;

$$\therefore \sin -470° + \cos -470° \text{ is negative,}$$
$$\sin -470° - \cos -470° \text{ is negative.}$$

(viii) $\sin 1000° = \sin(3 \times 360° - 80°) = -\sin 80°$;

$$\cos 1000° = \cos(3 \times 360° - 80°) = \cos 80°;$$

$\therefore$ $\sin 1000°$ is negative and numerically greater than $\cos 1000°$; $\cos 1000°$ is positive, $\sin 1000° + \cos 1000°$ is negative,

$$\sin 1000° - \cos 1000° \text{ is negative.}$$

2. Consider the values of $(\sin \frac{1}{2}A + \cos \frac{1}{2}A)$ and $(\sin \frac{1}{2}A - \cos \frac{1}{2}A)$ when
(i) $A = 92°, 268°, 900°, 4n\pi + \frac{3}{4}\pi, (4n+2)\pi - \frac{3}{4}\pi$.

We see that when $\frac{1}{2}A = 46°$ $\sin \frac{1}{2}A$ is $>$ $\cos \frac{1}{2}A$ and positive,
also when $\frac{1}{2}A = 134°$, when $\frac{1}{2}A = (360° + 90°)$, when $\frac{1}{2}A = 2n\pi + \frac{3}{8}\pi$,
and when $\frac{1}{2}A = (2n+1)\pi - \frac{3}{8}\pi$ the same is true;
hence in all these cases $\sin \frac{1}{2}A + \cos \frac{1}{2}A$ is positive,
and $\sin \frac{1}{2}A - \cos \frac{1}{2}A$ is positive,
and $\therefore$ the *formulæ* for $\sin \frac{1}{2}A$ and for $\cos \frac{1}{2}A$ in terms of $\sin A$ are unaltered.

When (ii) $A = 88°, -88°, 770°, -770°$, or $4n \pm \frac{1}{8}\pi$,
it may be shewn that when $\frac{1}{2}A = 44°$, $\cos \frac{1}{2}A$ is greater than $\sin \frac{1}{2}A$ and is positive; the same statement is also true when $\frac{1}{2}A = -44°$,
when $\frac{1}{2}A = (360° + 25°)$, when $\frac{1}{2}A = -(360° + 25°)$ and when $\frac{1}{2}A = 2n\pi \pm \frac{1}{16}\pi$.

Hence in all these cases $\sin \frac{1}{2}A + \cos \frac{1}{2}A$ is positive,

$$\sin \tfrac{1}{2}A - \cos \tfrac{1}{2}A \text{ is negative,}$$

and the formulæ for $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$ in terms of $\sin A$ have the same form in each case.

3. $\sin 9°$ is positive and numerically less than $\cos 9°$; $\cos 9°$ is positive;

$$\therefore \sin 9° + \cos 9° = +\sqrt{(1 + \sin 18°)} = \sqrt{\{1 + \tfrac{1}{4}(\sqrt{5} - 1)\}} = \tfrac{1}{2}\sqrt{(3 + \sqrt{5})}$$
$$\sin 9° - \cos 9° = -\sqrt{(1 - \sin 18°)} = -\sqrt{\{1 - \tfrac{1}{4}(\sqrt{5} - 1)\}} = \tfrac{1}{2}\sqrt{(5 - \sqrt{5})};$$

$$\therefore \quad \text{(i)} \quad \sin 9° = \tfrac{1}{4}\{\sqrt{(3 + \sqrt{5})} - \sqrt{(5 - \sqrt{5})}\}.$$

$$\text{(ii)} \quad \cos 9° = \tfrac{1}{4}\{\sqrt{(3 + \sqrt{5})} + \sqrt{(5 - \sqrt{5})}\}.$$

(iii) $\sin 81° = \cos 9°$.

(iv) $\cos 189° = \cos(180° + 9°) = -\cos 9°$.

(v) $\tan 202\frac{1}{2}° = \tan(180° + 22\frac{1}{2}°) = \tan 22\frac{1}{2}°$.

$22\frac{1}{2}°$ is in the first quadrant, therefore its tangent is positive.

From E. T. Art. 181, $\tan 22\frac{1}{2} = \dfrac{-1 + \sqrt{(1 + \tan^2 45°)}}{\tan 45°} = \sqrt{2} - 1$.

(vi) $\tan 97\frac{1}{2}° = -\cot 7\frac{1}{2}° = -\dfrac{1}{\tan(7\frac{1}{2}°)}$,

$$\tan 7\frac{1}{2}° = \frac{\sin 15°}{1 + \cos 15°} = \frac{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}{1 + \dfrac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}$$

$$= \frac{(\sqrt{3}-1)(2\sqrt{2}+1-\sqrt{3})}{(2\sqrt{2}+\sqrt{3}+1)(2\sqrt{2}+1-\sqrt{3})} = \frac{2\sqrt{6}-2\sqrt{2}-4+2\sqrt{3}}{6+4\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}-2+\sqrt{3}}{3+2\sqrt{2}} = \frac{(\sqrt{6}-\sqrt{2}-2+\sqrt{3})(3-2\sqrt{2})}{(3+2\sqrt{2})(3-2\sqrt{2})}$$

$$= \sqrt{6}-\sqrt{3}+\sqrt{2}-2;$$

$$\therefore \tan 97\frac{1}{2}° = -\frac{1}{\sqrt{6}-\sqrt{3}+\sqrt{2}-2} = -\frac{1}{(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)}$$

$$= -\frac{(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)}{(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)} = -(\sqrt{3}+\sqrt{2})(\sqrt{2}+1).$$

4. (i) If $A = 200°$, $\frac{1}{2}A = 100°$. Now $\sin 100° = \sin(180° - 80°) = \sin 80°$, $\sin 80°$ is positive and numerically greater than $\cos\frac{1}{2}A$;

$$\therefore \sin\tfrac{1}{2}A + \tfrac{1}{2}\cos A = +\sqrt{(1+\sin A)},\ \sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A = +\sqrt{(1-\sin A)};$$

$$\therefore 2\sin\tfrac{1}{2}A = +\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)}.$$

(ii) The tangent of the angle 100° is negative; therefore we have to take the *negative value* in the formula $\tan\frac{1}{2}A = \dfrac{-1 \pm \sqrt{(1+\tan^2 A)}}{\tan A}$; the negative value is $\tan\frac{1}{2}A = \dfrac{-1-\sqrt{(1+\tan^2 A)}}{\tan A} = \dfrac{-\{1+\sqrt{(1+\tan^2 A)}\}}{\tan A}$. (Since $\tan A = \tan 200°$ is *positive*.)

5. (i) If A lies between 270° and 360°, $\frac{1}{2}A$ lies between 135° and 180°; $\cos\frac{1}{2}A$ is negative and numerically greater than $\sin\frac{1}{2}A$;

$$\therefore \sin\tfrac{1}{2}A + \cos\tfrac{1}{2}A = -\sqrt{(1+\sin A)},\ \sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A = +\sqrt{(1-\sin A)};$$

$$\therefore 2\sin\tfrac{1}{2}A = \sqrt{(1-\sin A)} - \sqrt{(1+\sin A)}.$$

(ii) Now $\tan A$ is negative when A is between 270° and 360°; also $\tan\frac{1}{2}A$ is negative when $\frac{1}{2}A$ is between 135° and 180°; we have therefore to take the negative value for $\tan\frac{1}{2}A$ in the formula

$$\tan\tfrac{1}{2}A=\frac{-1\pm\sqrt{(1+\tan^2 A)}}{\tan A}\text{; the negative value is}$$

$$\frac{\sqrt{(\tan^2 A+1)}-1}{\tan A}\text{ (for }\tan A\text{ is negative)}=-\frac{1}{\tan A}+\frac{\sec^2 A}{\tan A}=-\cot A+\operatorname{cosec} A.$$

6. If A lies between 450° and 630°, $\frac{1}{2}A$ lies between 225° and 315°.

When $\frac{1}{2}A$ is between 225° and 270° $\sin\frac{1}{2}A$ is greater than $\cos\frac{1}{2}A$; and is negative;

$$\therefore\ \sin\tfrac{1}{2}A+\cos\tfrac{1}{2}A\text{ is negative, }\sin\tfrac{1}{2}A-\cos\tfrac{1}{2}A\text{ is negative.}$$

When $\frac{1}{2}A$ is between 270° and 315°, $\sin\frac{1}{2}A$ is negative and greater than $\cos\frac{1}{2}A$; $\cos\frac{1}{2}A$ is positive and less than $\sin\frac{1}{2}A$;

$$\therefore\ \sin\tfrac{1}{2}A+\cos\tfrac{1}{2}A\text{ is negative, }\sin\tfrac{1}{2}A-\cos\tfrac{1}{2}A\text{ is negative;}$$

$$\therefore\ 2\sin\tfrac{1}{2}A=-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}.$$

7. $2\sin\frac{1}{2}A=\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}$.

When $\sin\frac{1}{2}A+\cos\frac{1}{2}A=+\sqrt{(1+\sin A)}$, $\sin\frac{1}{2}A-\cos\frac{1}{2}A=-\sqrt{(1-\sin A)}$.

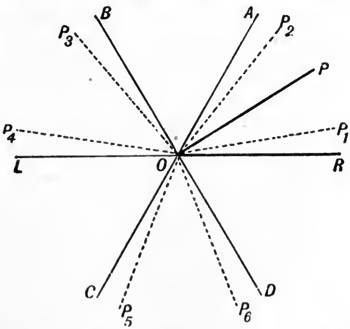
These two statements are each satisfied when $\cos\frac{1}{2}A$ is $>\sin\frac{1}{2}A$ and is positive; that is, when the revolving line OP turning in the positive direction is between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$.

8. See (6) above, by subtraction,

$$2\cos\tfrac{1}{2}A=-\sqrt{1+\sin A}-\sqrt{1-\sin A}.$$

9. When A lies between $n\times 360°-90°$ and $n\times 360°$ $\tan A$ is negative and $\tan\frac{1}{2}A$ is negative. When A lies between $n\times 360°$ and $n\times 360°+90°$ $\tan A$ is positive and so also is $\tan\frac{1}{2}A$. So that when A lies between $n\times 360°-90°$ and $n\times 360°+90°$ $\tan A$ and $\tan\frac{1}{2}A$ have the same sign, so that $\tan\frac{1}{2}A\times\tan A$ is *positive* and $\therefore\ =\sqrt{(1+\tan^2 A)}-1$. Similarly it may be shewn that when A lies between $n\times 360°+90°$ and $n\times 360°+270°$ $\tan A$ and $\tan\frac{1}{2}A$ have opposite signs, so that $\tan\frac{1}{2}A\times\tan A$ is *negative* and $\therefore=-\sqrt{(1+\tan^2 A)}-1$.

10. When the sine of an angle is given by Art. 144 if A is the least positive angle which has the given sine, then the angle may be any one of



the angles $n\times 360°+A$ or $n\times 360°+180°-A$; $\therefore\ \sin\frac{1}{3}A$ is the sine of any of the angles $n\times 120°+\frac{1}{3}A$ or $n\times 120°+60°-\frac{1}{3}A$; that is the sine of any of the angles given by the dotted lines OP_1, OP_2, OP_3, OP_4, OP_5, OP_6 in the figure.

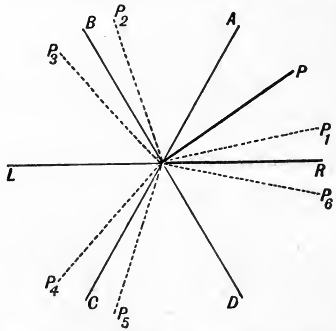
The lines OR, OA, OB, OL, OC, OD divide the four right angles at O into equal angles each of $60°$; the angle $ROP = A$, and the angles ROP_1, AOP_2, BOP_3, LOP_4, COP_5, DOP_6 are each equal to $\frac{1}{3}A$. From the symmetry of the figure it will be seen that $\sin \frac{1}{3}A$ may have any one of *three* different values and no more, for

$$\sin ROP_1 = \sin ROP_4, \ \sin ROP_2 = \sin ROP_3 \text{ and } \sin ROP_5 = \sin ROP_6;$$

but $\cos ROP_1 = -\cos ROP_4$ etc., $\therefore$ there are six different values for $\cos \frac{1}{3}A$.

11. When the cosine of an angle is given, by Art. 146 if A is the least positive angle which has the given cosine, then the angle may be any one of the angles $n \times 360° + A$ or $n \times 360° - A$.

Therefore $\frac{1}{3}A$ may be any one of the angles $n \times 120° + \frac{1}{3}A$ or $n \times 120° - \frac{1}{3}A$, that is any one of the angles given by the dotted lines in the figure

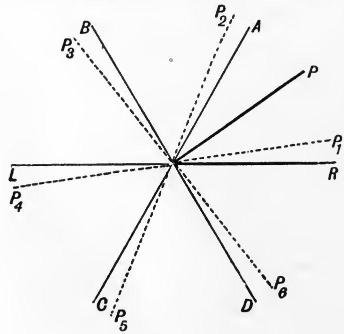


OP_1, OP_2, OP_3, OP_4, OP_5, OP_6. From the symmetry of the figure it will be seen that $\cos \frac{1}{3}A$ may have any one of *three* different values and no more, for

$$\cos ROP_1 = \cos ROP_6, \ \cos ROP_2 = \cos ROP_5, \ \cos ROP_3 = \cos ROP_4.$$

But $\sin ROP_1 = -\sin ROP_6$ and so on, $\therefore$ there are six different values for $\frac{1}{3}A$.

12. When $\tan A$ is given the angle may be (by Art. 148) any one of the angles $n \times 180° + A$, and $\therefore$ $\frac{1}{3}A$ may be any one of the angles $n \times 60° + \frac{1}{3}A$;



hence $\tan\frac{1}{3}A$ is the tangent of any one of the angles given by the dotted lines OP_1, OP_2, OP_3, OP_4, OP_5, OP_6; hence it will be seen that $\tan\frac{1}{3}A$ has three different values and no more, for

$$\tan ROP_1=\tan ROP_4,\ \tan ROP_2=\tan ROP_5,\ \tan ROP_3=\tan ROP_6.$$

13. In the figure of Art. 182 the sines of the angles ROP_3, ROP_4, ROP_5, ROP_6 (which are the possible values of $\sin\frac{1}{2}A$ when $\tan A$ is given) are all different and the result follows. Also the values of the cosines of these angles are all different.

14. In the figure of Art. 179,

$$\tan ROP_3=\tan ROP_5 \ \text{ and } \ \tan ROP_4=\tan ROP_6;$$

hence the result follows.

EXAMPLES. XLVIII. PAGE 157.

1.
$$2\sin\theta+2\cos\theta=\sqrt{2}.$$

Divide both sides by $2\sqrt{2}$, then $\dfrac{\sin\theta}{\sqrt{2}}+\dfrac{\cos\theta}{\sqrt{2}}=\frac{1}{2}$;

$$\therefore\ \sin\theta\cos\tfrac{1}{4}\pi+\cos\theta\sin\tfrac{1}{4}\pi=\tfrac{1}{2},\ \sin(\theta+\tfrac{1}{4}\pi)=\tfrac{1}{2}=\sin\tfrac{1}{6}\pi;$$
$$\therefore\ \theta+\tfrac{1}{4}\pi=n\pi+(-1)^n\tfrac{1}{6}\pi;\ \therefore\ \theta=-\tfrac{1}{4}\pi+n\pi+(-1)^n\tfrac{1}{6}\pi.$$

2.
$$\sin\theta+\sqrt{3}\,.\cos\theta=1.$$

Divide both sides by 2, then $\dfrac{\sin\theta}{2}+\dfrac{\sqrt{3}}{2}.\cos\theta=\frac{1}{2}$;

$$\therefore\ \sin\theta\cos\tfrac{1}{3}\pi+\sin\tfrac{1}{3}\pi\cos\theta=\tfrac{1}{2},\ \sin(\theta+\tfrac{1}{3}\pi)=\tfrac{1}{2}=\sin\tfrac{1}{6}\pi;$$
$$\therefore\ \theta+\tfrac{1}{3}\pi=n\pi+(-1)^n\tfrac{1}{6}\pi;\ \therefore\ \theta=-\tfrac{1}{3}\pi+n\pi+(-1)^n\tfrac{1}{6}\pi.$$

3.
$$\sqrt{2}\sin\theta+\sqrt{2}\cos\theta=\sqrt{3}.$$

Divide both sides by 2, then $\dfrac{1}{\sqrt{2}}.\sin\theta+\dfrac{1}{\sqrt{2}}.\cos\theta=\dfrac{\sqrt{3}}{2}$;

$$\therefore\ \sin(\tfrac{1}{4}\pi+\theta)=\frac{\sqrt{3}}{2}=\sin\tfrac{1}{3}\pi,\ \tfrac{1}{4}\pi+\theta=n\pi+(-1)^n\tfrac{1}{3}\pi;$$
$$\therefore\ \theta=-\tfrac{1}{4}\pi+n\pi+(-1)^n\tfrac{1}{3}\pi.$$

4.
$$\sin\theta-\cos\theta=1.$$

Divide both sides by $\sqrt{2}$, then $\dfrac{1}{\sqrt{2}}\sin\theta-\dfrac{1}{\sqrt{2}}\cos\theta=\dfrac{1}{\sqrt{2}}$;

$$\therefore\ \sin(\theta-\tfrac{1}{4}\pi)=\frac{1}{\sqrt{2}}=\sin\tfrac{1}{4}\pi;\ \therefore\ \theta-\tfrac{1}{4}\pi=n\pi+(-1)^n\tfrac{1}{4}\pi;$$
$$\therefore\ \theta=\tfrac{1}{4}\pi+n\pi+(-1)^n\tfrac{1}{4}\pi.$$

5.
$$\sin\theta+\cos\theta=1.$$

Divide both sides by $\sqrt{2}$, then $\dfrac{1}{\sqrt{2}}.\sin\theta+\dfrac{1}{\sqrt{2}}.\cos\theta=\dfrac{1}{\sqrt{2}}$,

$$\cos(\theta-\tfrac{1}{4}\pi)=\frac{1}{\sqrt{2}}=\cos\tfrac{1}{4}\pi;$$
$$\therefore\ \theta-\tfrac{1}{4}\pi=2n\pi\pm\tfrac{1}{4}\pi;\ \therefore\ \theta=\tfrac{1}{4}\pi+2n\pi\pm\tfrac{1}{4}\pi.$$

6. $$\sqrt{3}\sin\theta-\cos\theta-\sqrt{2}=0.$$

Divide both sides by 2, then $\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}.\cos\theta=\frac{1}{\sqrt{2}}$;

$$\therefore\ \sin(\theta-\tfrac{1}{6}\pi)=\frac{1}{\sqrt{2}}=\sin\tfrac{1}{4}\pi;\ \therefore\ \theta-\tfrac{1}{6}\pi=n\pi+(-1)^n\tfrac{1}{4}\pi;$$
$$\therefore\ \theta=\tfrac{1}{6}\pi+n\pi+(-1)^n\tfrac{1}{4}\pi.$$

7. $2\sin x+5\cos x=2,\ \sin x+2\cdot5\cos x=1,\ \sin x+\tan 68°\,12'\cos x=1,$
$$\sin x\cos 68°\,12'+\sin 68°\,12'\cos x=\cos 68°\,12',$$
$$\sin(x+68°\,12')=\cos 68°\,12'=\sin 21°\,48';$$
$$\therefore\ x+68°\,12'=n\times180°+(-1)^n(21°\,48');$$
$$\therefore\ x=-68°\,12'+n\times180°+(-1)^n(21°\,48').$$

8. $$3\cos x-8\sin x=3,\ \cos x-2\cdot6\sin x=1,$$
$$\cos x-\tan 69°\,26'\,30''\sin x=1,$$
$$\cos x\cos 69°\,26'\,30''-\sin 69°\,26'\,30''\sin x=\cos 69°\,26'\,30'',$$
$$\cos(x+69°\,26'\,30'')=\cos(69°\,26'\,30''),$$
$$x+69°\,26'\,30''=2n\times180°\pm69°\,26'\,30'';$$
$$\therefore\ x=-69°\,26'\,30''+2n\times180°\pm69°\,26'\,30''.$$

9. $$4\sin x-15\cos x=4,\ \sin x-3\cdot75\cos x=1,$$
$$\sin x-\tan 75°\,4'\cos x=1,$$
$$\sin x\cos 75°\,4'-\sin 75°\,4'\cos x=\cos 75°\,4',$$
$$\sin(x-75°\,4')=\cos 75°\,4'=\sin 14°\,56';$$
$$\therefore\ x-75°\,4'=n\times180°+(-1)^n(14°\,56');$$
$$\therefore\ x=75°\,4'+n\times180°+(-1)^n(14°\,56').$$

10. $$\cos(a+x)-\sin(a+x)=\sqrt{2}\cos\beta.$$

Divide both sides by $\sqrt{2}$, then $\frac{1}{\sqrt{2}}\cos(a+x)-\frac{1}{\sqrt{2}}\sin(a+x)=\cos\beta$;

$$\therefore\ \cos(a+x+\tfrac{1}{4}\pi)=\cos\beta;\ \therefore\ a+x+\tfrac{1}{4}\pi=2n\pi\pm\beta;$$
$$\therefore\ x=-a-\tfrac{1}{4}\pi+2n\pi\pm\beta.$$

EXAMPLES. XLIX. PAGE 158.

1. Let $a=\sin^{-1}\frac{3}{5}$; $\therefore\ \sin a=\frac{3}{5}$, $\cos a=\sqrt{1-(\frac{3}{5})^2}=\frac{4}{5}$, i.e. $a=\cos^{-1}\frac{4}{5}$,
$$\tan a=\frac{\sin a}{\cos a}=\tfrac{3}{5}\div\tfrac{4}{5}=\tfrac{3}{4},\ \text{i.e. } a=\tan^{-1}\tfrac{3}{4};$$
$$\therefore\ \sin^{-1}\tfrac{3}{5}=\cos^{-1}\tfrac{4}{5}=\tan^{-1}\tfrac{3}{4}.$$

2. Let $a=\sin^{-1}\frac{1}{2}$; $\therefore\ \sin a=\frac{1}{2}$ and $\cos a=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$, i.e. $a=\cos^{-1}\frac{\sqrt{3}}{2}$;
$$\cot a=\frac{\cos a}{\sin a}=\frac{\sqrt{3}}{2}\div\tfrac{1}{2}=\sqrt{3},\ \text{i.e. } a=\cot^{-1}\sqrt{3};$$
$$\therefore\ \sin^{-1}\tfrac{1}{2}=\cos^{-1}\frac{\sqrt{3}}{2}=\cot^{-1}\sqrt{3}.$$

3. Let $\sin^{-1} a=\theta$; $\therefore \sin\theta=a$ and $\cos\theta=\sqrt{(1-\sin^2\theta)}=\sqrt{(1-a^2)}$,

i.e. $\theta=\cos^{-1}\sqrt{(1-a^2)}$; $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{a}{\sqrt{(1-a^2)}}$; i.e. $\theta=\tan^{-1}\dfrac{a}{\sqrt{(1-a^2)}}$;

$$\therefore \sin^{-1} a=\cos^{-1}\sqrt{(1-a^2)}=\tan^{-1}\frac{a}{\sqrt{(1-a^2)}}.$$

4. $\alpha=\sin^{-1}\frac{3}{5}$, i.e. $\sin\alpha=\frac{3}{5}$. $\beta=\cos^{-1}\frac{3}{5}$, i.e. $\cos\beta=\frac{3}{5}$;

$\therefore \sin\alpha=\cos\beta=\sin(90°-\beta)$; $\therefore \alpha=\frac{1}{2}\pi-\beta$, i.e. $\alpha+\beta=\frac{1}{2}\pi$.

5. $A=\sin^{-1} a$, i.e. $\sin A=a$. $B=\cos^{-1} a$, i.e. $\cos B=a$;

$\therefore \sin A=\cos B=\sin(90°-B)$; $\therefore A=90°-B$, i.e. $A+B=90°$.

6. Let $\alpha=\tan^{-1}\frac{5}{7}$, i.e. $\tan\alpha=\frac{5}{7}$, $\beta=\tan^{-1}\frac{1}{6}$, i.e. $\tan\beta=\frac{1}{6}$.

Now $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\,.\,\tan\beta}=\frac{\frac{5}{7}+\frac{1}{6}}{1-\frac{5}{7}\,.\,\frac{1}{6}}=1=\tan\tfrac{1}{4}\pi;$$

$$\therefore \alpha+\beta=\tfrac{1}{4}\pi,\ \text{i.e. } \tan^{-1}\tfrac{5}{7}+\tan^{-1}\tfrac{1}{6}=\tfrac{1}{4}\pi.$$

7. Let $\alpha=\tan^{-1}\frac{2}{11}$, i.e. $\tan\alpha=\frac{2}{11}$, and $\beta=\tan^{-1}\frac{1}{7}$, i.e. $\tan\beta=\frac{1}{7}$;

$$\therefore 2\beta=2\tan^{-1}\tfrac{1}{7},\ \tan 2\beta=\frac{2\tan\beta}{1-\tan^2\beta}=\frac{\frac{2}{7}}{1-\frac{1}{49}}=\tfrac{7}{24},$$

$$\text{i.e. } 2\beta=\tan^{-1}\tfrac{7}{24}=2\tan^{-1}\tfrac{1}{7}.$$

Now $$\tan(\alpha+2\beta)=\frac{\tan\alpha+\tan 2\beta}{1-\tan\alpha\,.\,\tan 2\beta}=\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\,.\,\frac{7}{24}}=\tfrac{1}{2};$$

$$\therefore \alpha+2\beta=\tan^{-1}\tfrac{1}{2},\ \text{i.e. } \tan^{-1}\tfrac{2}{11}+2\tan^{-1}\tfrac{1}{7}=\tan^{-1}\tfrac{1}{2}.$$

8. Let $\alpha=\tan^{-1} m_1$, i.e. $\tan\alpha=m_1$, $\beta=\tan^{-1} m_2$, i.e. $\tan\beta=m_2$,

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\,.\,\tan\beta}=\frac{m_1+m_2}{1-m_1m_2};$$

$$\therefore \alpha+\beta=\tan^{-1}\frac{m_1+m_2}{1-m_1m_2},\ \text{i.e. } \tan^{-1} m_1+\tan^{-1} m_2=\tan^{-1}\frac{m_1+m_2}{1-m_1m_2}.$$

9. Let $x=\sin^{-1} a$, i.e. $\sin x=a$, $\cos x=\sqrt{(1-a^2)}$,

$\sin 2x=2\sin x\cos x=2a\sqrt{(1-a^2)}$, i.e. $2x=\sin^{-1} 2a\sqrt{(1-a^2)}$;

$$\therefore 2\sin^{-1} a=\sin^{-1} 2a\sqrt{(1-a^2)},$$

$$\sin(2\sin^{-1} a)=\sin(\sin^{-1} 2a\sqrt{(1-a^2)})=2a\sqrt{(1-a^2)}.$$

10. Let $\cos^{-1} a=x$, i.e. $\cos x=a$; $\cos 2x=2\cos^2 x-1=2a^2-1$;

$$\therefore 2x=\cos^{-1}(2a^2-1),\ \text{i.e. } 2\cos^{-1} a=\cos^{-1}(2a^2-1).$$

11. From (9) we obtain $2\sin^{-1} a=\sin^{-1} 2a\sqrt{(1-a^2)}$ putting $\frac{1}{2}$ for a we have $$2\sin^{-1}\tfrac{1}{2}=\sin^{-1}\frac{\sqrt{3}}{2};$$

$$\therefore \cos^{-1}\tfrac{1}{2}+2\sin^{-1}\tfrac{1}{2}=\cos^{-1}\tfrac{1}{2}+\sin^{-1}\frac{\sqrt{3}}{2}.$$

Let $\alpha=\cos^{-1}\frac{1}{2}$, i.e. $\cos\alpha=\frac{1}{2}$; $\therefore \alpha=60°$.

Let $\beta = \sin^{-1}\frac{\sqrt{3}}{2}$, i.e. $\sin\beta = \frac{\sqrt{3}}{2}$; $\therefore\ \beta = 60°$;

$$\therefore\ \cos^{-1}\tfrac{1}{2} + 2\sin^{-1}\tfrac{1}{2} = \alpha + \beta = 60° + 60° = 120°.$$

12. From (9) $2\sin^{-1}a = \sin^{-1} 2a\sqrt{(1-a^2)}$, write $\frac{4}{5}$ for a;

$$\therefore\ 2\sin^{-1}\tfrac{4}{5} = \sin^{-1}\tfrac{8}{5}\sqrt{\{1-(\tfrac{4}{5})^2\}} = \sin^{-1}\tfrac{24}{25};$$

$$\therefore\ 2\sin^{-1}\tfrac{4}{5} - \sin^{-1}\tfrac{24}{25} = \sin^{-1}\tfrac{24}{25} - \sin^{-1}\tfrac{24}{25} = 0 = \cos^{-1} 1.$$

13. Let $\tan^{-1}(\cos 2\alpha) = x$, i.e. $\tan x = \cos 2\alpha$.

Now $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x} = \dfrac{2\cos 2\alpha}{1-\cos^2 2\alpha} = \dfrac{2\cos 2\alpha}{\sin^2 2\alpha}$

$$= \frac{2\cos 2\alpha}{4\sin^2\alpha\cos^2\alpha} = \frac{(\cos^2\alpha - \sin^2\alpha)(\cos^2\alpha + \sin^2\alpha)}{2\sin^2\alpha\cos^2\alpha}$$

$$= \frac{\cos^4\alpha - \sin^4\alpha}{2\sin^2\alpha\cos^2\alpha} = \frac{1}{2}\left(\frac{\cos^2\alpha}{\sin^2\alpha} - \frac{\sin^2\alpha}{\cos^2\alpha}\right) = \frac{\cot^2\alpha - \tan^2\alpha}{2};$$

$$\therefore\ 2x = \tan^{-1}\left(\frac{\cot^2\alpha - \tan^2\alpha}{2}\right),$$

$$\text{i.e. } 2\tan^{-1}(\cos 2\alpha) = \tan^{-1}\left(\frac{\cot^2\alpha - \tan^2\alpha}{2}\right).$$

14. $$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}\frac{1-x-y-xy}{1+x+y-xy}$$

$$= \tan^{-1}\frac{\dfrac{x+y}{1-xy} + \dfrac{1-x-y-xy}{1+x+y-xy}}{1 - \dfrac{(x+y)(1-x-y-xy)}{(1-xy)(1+x+y-xy)}}$$

$$= \tan^{-1}\frac{(x+y)(1+x+y-xy) + (1-xy)(1-x-y-xy)}{(1-xy)(1+x+y-xy) - (x+y)(1-x-y-xy)}$$

$$= \tan^{-1}\frac{(x+y)\{(x+y)+(1-xy)\} + (1-xy)\{(1-xy)-(x+y)\}}{(1-xy)\{(1-xy)+(x+y)\} - (x+y)\{(1-xy)-(x+y)\}}$$

$$= \tan^{-1}\frac{(x+y)^2 + (1-xy)^2}{(x+y)^2 + (1-xy)^2} = \tan^{-1} 1 = \tfrac{1}{4}\pi.$$

15. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = 2\tan^{-1}\frac{1}{5} + 2\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$$= \tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}} + \tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}} - \tan^{-1}\tfrac{1}{239}$$

$$= \tan^{-1}\tfrac{5}{12} + \tan^{-1}\tfrac{5}{12} - \tan^{-1}\tfrac{1}{239}$$

$$= \tan^{-1}\frac{\frac{5}{6}}{1-\frac{25}{144}} - \tan^{-1}\tfrac{1}{239}$$

$$= \tan^{-1}\tfrac{120}{119} - \tan^{-1}\tfrac{1}{239} = \tan^{-1}\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\cdot\frac{1}{239}}$$

$$= \tan^{-1}\tfrac{28561}{28561} = \tan^{-1} 1 = \tfrac{1}{4}\pi.$$

16. Let $\alpha=\sin^{-1}\frac{4}{5}$, i.e. $\sin\alpha=\frac{4}{5}$; $\cos\alpha=\sqrt{(1-\sin^2\alpha)}=\sqrt{\{1-(\frac{4}{5})^2\}}=\frac{3}{5}$.
Let $\beta=\sin^{-1}\frac{8}{17}$, i.e. $\sin\beta=\frac{8}{17}$; $\cos\beta=\sqrt{(1-\sin^2\beta)}=\sqrt{\{1-(\frac{8}{17})^2\}}=\frac{15}{17}$.
Let $\gamma=\sin^{-1}\frac{13}{85}$, i.e. $\sin\gamma=\frac{13}{85}$ $\cos\gamma=\sqrt{(1-\sin^2\gamma)}=\sqrt{\{1-(\frac{13}{85})^2\}}=\frac{84}{85}$,

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=\frac{4}{5}\cdot\frac{15}{17}+\frac{3}{5}\cdot\frac{8}{17}=\frac{84}{85}=\cos\gamma=\sin(\tfrac{1}{2}\pi-\gamma),$$

i.e. $\alpha+\beta+\gamma=\frac{1}{2}\pi$ or $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{13}{85}=\frac{1}{2}\pi$.

17. Let $\tan^{-1}\sqrt{5}(2-\sqrt{3})=\alpha$, i.e. $\tan\alpha=\sqrt{5}(2-\sqrt{3})$,
and $\cot^{-1}\sqrt{5}(2+\sqrt{3})=\beta$, i.e. $\cot\beta=\sqrt{5}(2+\sqrt{3})$;

$$\therefore \tan\beta=\frac{1}{\sqrt{5}(2+\sqrt{3})}=\frac{\sqrt{5}(2-\sqrt{3})}{5}.$$

Now $$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\,.\,\tan\beta}$$

$$=\frac{\sqrt{5}(2-\sqrt{3})-\frac{\sqrt{5}(2-\sqrt{3})}{5}}{1+(2-\sqrt{3})^2}=\frac{4\sqrt{5}(2-\sqrt{3})}{20(2-\sqrt{3})}=\frac{1}{\sqrt{5}};$$

$$\therefore \cot(\alpha-\beta)=\sqrt{5},\text{ i.e. }\alpha-\beta=\cot^{-1}\sqrt{5},$$

or $$\tan^{-1}\sqrt{5}(2-\sqrt{3})-\cot^{-1}\sqrt{5}(2+\sqrt{3})=\cot^{-1}\sqrt{5}.$$

18. Let $\alpha=\sin^{-1}m$, i.e. $\sin\alpha=m$ and $\cos\alpha=\sqrt{(1-m^2)}$,
and $\beta=\sin^{-1}n$, i.e. $\sin\beta=n$ and $\cos\beta=\sqrt{(1-n^2)}$,

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=m\sqrt{(1-n^2)}+n\sqrt{(1-m^2)},$$

$$\text{i.e. }\alpha+\beta=\sin^{-1}\{m\sqrt{(1-n^2)}+n\sqrt{(1-m^2)}\},$$

or $$\sin^{-1}m+\sin^{-1}n=\sin^{-1}\{m\sqrt{(1-n^2)}+n\sqrt{(1-m^2)}\}.$$

If $$\sin^{-1}m+\sin^{-1}n=\tfrac{1}{2}\pi\text{ or }\sin^{-1}1;$$

$$\therefore \sin^{-1}\{m\sqrt{(1-n^2)}+n\sqrt{(1-m^2)}\}=\sin^{-1}1;$$

$$\therefore m\sqrt{(1-n^2)}+n\sqrt{(1-m^2)}=1.$$

MISCELLANEOUS EXAMPLES. L. PAGE 159.

1. $$\tan^{-1}\frac{1}{1+a}+\tan^{-1}\frac{1}{1-a}=\tan^{-1}\frac{\frac{1}{1+a}+\frac{1}{1-a}}{1-\frac{1}{1+a}\cdot\frac{1}{1-a}}=\tan^{-1}-\frac{2}{a^2};$$

$$\therefore \tan^{-1}\frac{1}{1+a}+\tan^{-1}\frac{1}{1-a}+\tan^{-1}\frac{2}{a^2}$$

$$=\tan^{-1}-\frac{2}{a^2}+\tan^{-1}\frac{2}{a^2}=\tan^{-1}\frac{-\frac{2}{a^2}+\frac{2}{a^2}}{1+\frac{2}{a^2}\cdot\frac{2}{a^2}}=\tan^{-1}0.$$

The least angle whose tangent is 0 is 0°, therefore (Art. 149, E. T.) all the angles whose tangent is 0 are included in the expression $n\pi$.

$$\therefore \tan^{-1}\frac{1}{1+a}+\tan^{-1}\frac{1}{1-a}+\tan^{-1}\frac{2}{a^2}=n\pi.$$

2. $$\tan^{-1}\frac{a-1}{a}+\tan^{-1}\frac{1}{2a-1}$$

$$=\tan^{-1}\frac{\frac{a-1}{a}+\frac{1}{2a-1}}{1-\frac{a-1}{a(2a-1)}}=\tan^{-1}\frac{a(2a-1)-(a-1)}{a(2a-1)-(a-1)}=\tan^{-1}1.$$

The least angle whose tangent is 1 is $\frac{1}{4}\pi$; therefore all the angles whose tangent is 1 are included in the expression $n\pi+\frac{1}{4}\pi$;

$$\therefore\ \tan^{-1}\frac{a-1}{a}+\tan^{-1}\frac{1}{2a-1}=n\pi+\tfrac{1}{4}\pi.$$

3. $\sin\alpha=x$, $\sin\beta=y$, $\therefore\ \cos\alpha=\pm\sqrt{1-x^2}$, $\cos\beta=\pm\sqrt{1-y^2}$,

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$=xy\pm\sqrt{\{(1-x^2)(1-y^2)\}}=xy\pm\sqrt{(1-x^2-y^2+x^2y^2)};$$

$$\therefore\ \alpha-\beta=\cos^{-1}\{xy\pm\sqrt{(1-x^2-y^2+x^2y^2)}\},$$

i.e. $$\sin^{-1}x-\sin^{-1}y=\cos^{-1}\{xy\pm\sqrt{(1-x^2-y^2+x^2y^2)}\}.$$

4. $$\tan^{-1}\frac{x+1}{x+2}+\tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\frac{\frac{x+1}{x+2}+\frac{x-1}{x-2}}{1-\frac{x+1}{x+2}\cdot\frac{x-1}{x-2}}=\tan^{-1}\frac{4-2x^2}{3}.$$

Now $$\tan^{-1}\frac{4-2x^2}{3}=\tfrac{1}{4}\pi=\tan^{-1}1;\ \therefore\ \frac{4-2x^2}{3}=1;\ \therefore\ x^2=\tfrac{1}{2}.$$

5. $$\tan^{-1}a+\cot^{-1}a=\tan^{-1}a+\tan^{-1}\frac{1}{a}=\tan^{-1}\frac{a+\frac{1}{a}}{1-1}=\tan^{-1}\infty.$$

Let $$x=\tan^{-1}\infty;\ \therefore\ \tan x=\infty;$$

$$\therefore\ x=n\pi+\tfrac{1}{2}\pi=(2n+1)\tfrac{1}{2}\pi.$$

6. $$\tan^{-1}\alpha+\tan^{-1}\beta=\tan^{-1}\frac{\alpha+\beta}{1-\alpha\beta};$$

$$\therefore\ \tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma=\tan^{-1}\frac{\frac{\alpha+\beta}{1-\alpha\beta}+\gamma}{1-\frac{\alpha+\beta}{1-\alpha\beta}\cdot\gamma}$$

$$=\tan\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-\alpha\beta-\alpha\gamma-\beta\gamma}=\pi=\tan^{-1}0;$$

$$\therefore\ \frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-\alpha\beta-\alpha\gamma-\beta\gamma}=0,\ \text{i.e. } \alpha+\beta+\gamma-\alpha\beta\gamma=0,\ \text{or } \alpha+\beta+\gamma=\alpha\beta\gamma.$$

7. $$\tan^{-1}\frac{1}{x-1}-\tan^{-1}\frac{1}{x+1}=\tan^{-1}\frac{\dfrac{1}{x-1}-\dfrac{1}{x+1}}{1+\dfrac{1}{x-1}\cdot\dfrac{1}{x+1}}=\tan^{-1}\frac{2}{x^2}.$$

Now $$\tan^{-1}\frac{2}{x^2}=\frac{\pi}{12};\ \therefore\ \tan\frac{\pi}{12}=\frac{2}{x^2},\ \frac{\surd 3-1}{\surd 3+1}=\frac{2}{x^2};$$

$$\therefore\ x=\pm(1+\surd 3).$$

8. $$\tan^{-1}(x+1)-\tan^{-1}(x-1)=\tan^{-1}\frac{(x+1)-(x-1)}{1+(x+1)(x-1)}=\tan^{-1}\frac{2}{x^2}.$$

Now $$\tan^{-1}\frac{2}{x^2}=\cot^{-1}(x^2-1)=\tan^{-1}\frac{1}{x^2-1};$$

$$\therefore\ \frac{2}{x^2}=\frac{1}{x^2-1};\ \therefore\ \frac{x^2}{2}=x^2-1,\ x^2=2,\ x=\pm\surd 2.$$

9. $$\sin^{-1}\frac{2x}{1+x^2}+\tan^{-1}\frac{2x}{1-x^2}=\pi.$$

Let $\tan^{-1}x=a$, i.e. $\tan a=x$; and $\sin 2a=\dfrac{2\tan a}{1+\tan^2 a}$.

i.e. $$\sin 2a=\frac{2x}{1+x^2};\ \therefore\ \sin^{-1}\frac{2x}{1+x^2}=2a=2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}.$$

Therefore the equation may be written

$$2\tan^{-1}\frac{2x}{1-x^2}=\pi;\ \therefore\ \tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{2};$$

$$\therefore\ \frac{2x}{1-x^2}=\infty;\ \therefore\ x^2-1=0,\ x=\pm 1.$$

10. $$\tan^{-1}\frac{a-\surd(a^2-4)}{2\surd(a+1)}+\tan^{-1}\frac{a+\surd(a^2-4)}{2\surd(a+1)}$$

$$=\tan^{-1}\frac{\dfrac{a-\surd(a^2-4)}{2\surd(a+1)}+\dfrac{a+\surd(a^2-4)}{2\surd(a+1)}}{1-\dfrac{a-\surd(a^2-4)}{2\surd(a+1)}\cdot\dfrac{a+\surd(a^2-4)}{2\surd(a+1)}}=\tan^{-1}\surd(a+1);$$

$$\therefore\ \tan^{-1}\frac{a-\surd(a^2-4)}{2\surd(a+1)}+\tan^{-1}\frac{1}{\surd(a+1)}+\tan^{-1}\frac{a+\surd(a^2-4)}{2\surd(a+1)}$$

$$=\tan^{-1}\surd(a+1)+\tan^{-1}\frac{1}{\surd(a+1)}=\tan^{-1}\frac{\surd(a+1)+\dfrac{1}{\surd(a+1)}}{1-\dfrac{\surd(a+1)}{\surd(a+1)}}$$

$$=\tan^{-1}\frac{\surd(a+1)+\dfrac{1}{\surd(a+1)}}{0}=\tan^{-1}\infty.$$

The least angle whose tangent is ∞ is $\frac{1}{2}\pi$; therefore all the angles whose tangent is ∞ are included in the expression $n\pi+\frac{1}{2}\pi$.

11. All the angles whose sine is a are included in the expression $n\pi+(-1)^n a$; and all the angles whose cosine is a are included in the expression $2n\pi\pm(\frac{1}{2}\pi-a)$; for $\sin a=\cos(\frac{1}{2}\pi-a)$;

$\therefore \sin^{-1}a+\cos^{-1}a=n\pi+(-1)^n a+2n\pi\pm(\frac{1}{2}\pi-a)=3n\pi+(-1)^n a\pm(\frac{1}{2}\pi-a)$.

The expression $3n\pi+(-1)^n a\pm(\frac{1}{2}\pi-a)$ is included in the expression $n\pi+(-1)^n a\pm(\frac{1}{2}\pi-a)$ whatever integer odd or even n may be, for when $3n$ is odd n is odd and when $3n$ is even n is even;

$$\therefore \sin^{-1}a+\cos^{-1}a=n\pi+(-1)^n a\pm(\tfrac{1}{2}\pi-a).$$

12. $\tan A=\dfrac{\sin A}{\cos A}$; therefore its sign depends on the signs of $\sin A$ and $\cos A$; being positive when they have like signs; and negative when they have unlike signs.

$\sin 2A=2\sin A\cos A$; therefore its sign depends on the signs of $\sin A$ and $\cos A$; being positive when they have like signs, and negative when they have unlike signs.

13. $\cos A+\cos 3A+\cos 5A=0$; $\therefore \cos 3A+(\cos 5A+\cos A)=0$;

$$\therefore \cos 3A+2\cos 3A\cos 2A=0;\ \therefore \cos 3A(1+2\cos 2A)=0;$$

$$\therefore \cos 3A=0, \text{ or } 1+2\cos 2A=0.$$

If $\cos 3A=0$, then $\cos 3A=\cos\frac{1}{2}\pi$;

$$\therefore 3A=2n\pi\pm\tfrac{1}{2}\pi, \text{ i.e. } n\pi+\tfrac{1}{2}\pi;\ \therefore A=\tfrac{1}{3}(n\pi+\tfrac{1}{2}\pi)=\tfrac{1}{6}(2n+1)\pi.$$

If $1+2\cos 2A=0$, $\therefore \cos 2A=-\frac{1}{2}=\cos\frac{2}{3}\pi$;

$$\therefore 2A=2n\pi\pm\tfrac{2}{3}\pi, \text{ i.e. } A=\tfrac{1}{3}(3n\pm1)\pi.$$

14.

$$\sin 5\theta+\sin 3\theta+\sin\theta=3-4\sin^2\theta,$$
$$\sin 3\theta+(\sin 5\theta+\sin\theta)+4\sin^2\theta-3=0,$$
$$\sin 3\theta+2\sin 3\theta\cos 2\theta+4\sin^2\theta-3=0,$$
$$\sin 3\theta(1+2\cos 2\theta)+4(1-\cos^2\theta)-3=0,$$
$$\sin 3\theta(4\cos^2\theta-1)-(4\cos^2\theta-1)=0,$$
$$(4\cos^2\theta-1)(\sin 3\theta-1)=0;$$
$$\therefore 4\cos^2\theta=1 \text{ or } \sin 3\theta=1.$$

If $4\cos^2\theta=1$; $\therefore \cos^2\theta=\frac{1}{4}=\cos^2\frac{1}{3}\pi$; $\therefore \cos\theta=\pm\cos\frac{1}{3}\pi$.

If $\cos\theta=\cos\frac{1}{3}\pi$, $\theta=2n\pi\pm\frac{1}{3}\pi$.

If $\cos\theta=-\cos\frac{1}{3}\pi$, then $\theta=\pi-\frac{1}{3}\pi$; $\therefore \theta=2n\pi\pm(\pi-\frac{1}{3}\pi)$.

The two expressions are included in the expression $n\pi\pm\frac{1}{3}\pi$, when n is any integer whatever.

If $\sin 3\theta-1=0$, $\sin 3\theta=\sin\frac{1}{2}\pi$; $\therefore 3\theta=n\pi+(-1)^n\frac{1}{2}\pi$.

15.

$$2\sin^2 3A+\sin^2 6A=2;$$

$$\therefore \sin^2 6A=2(1-\sin^2 3A),\ 4\sin^2 3A\cos^2 3A=2\cos^2 3A;$$

therefore either $\cos^2 3A=0$, or $\sin^2 3A=\frac{1}{2}$.

If $\cos^2 3A=0$, $3A=n\pi+\frac{1}{2}\pi$.

If $\sin^2 3A=\frac{1}{2}=\sin^2\frac{1}{4}\pi$, $3A=n\pi+(-1)^n\frac{1}{4}\pi$.

16. $$a(\cos 2x - 1) + 2b(\cos x + 1) = 0;$$
$$\therefore\ 2a(\cos^2 x - 1) + 2b(\cos x + 1) = 0;$$
$$\therefore\ 2(\cos x + 1)(a\cos x - a + b) = 0;$$
therefore either $\cos x + 1 = 0$, or $a\cos x - a + b = 0$.

If $\cos x + 1 = 0$, $\cos x = -1$; $\therefore\ x = 2n\pi + \pi$.

If $a\cos x - a + b = 0$, $\cos x = \dfrac{a-b}{a}$, $\therefore\ x = \cos^{-1}\dfrac{a-b}{a}$.

17. $$\sin(m+n)\theta + \sin 2m\theta + \sin(m-n)\theta = 0;$$
$$\therefore\ \{\sin(m+n)\theta + \sin(m-n)\theta\} + \sin 2m\theta = 0;$$
$$\therefore\ 2\sin m\theta\cos n\theta + 2\sin m\theta\cos m\theta = 0;$$
$$\therefore\ 2\sin m\theta(\cos n\theta + \cos m\theta) = 0;$$
therefore either $\sin m\theta = \theta$, or $\cos n\theta + \cos m\theta = 0$.

If $\sin m\theta = 0$, $m\theta = n\pi$, $\therefore\ \theta = \dfrac{n\pi}{m}$.

If $$\cos n\theta + \cos m\theta = 0,$$
then $$2\cos\tfrac{1}{2}\theta(m+n)\cos\tfrac{1}{2}\theta(m-n) = 0;$$
therefore either $\cos\frac{1}{2}\theta(m+n) = 0$, or $\cos\frac{1}{2}\theta(m-n) = 0$.

If $\cos\frac{1}{2}\theta(m+n) = 0$, $\frac{1}{2}\theta(m+n) = r\pi + \frac{1}{2}\pi$;
$$\therefore\ \theta = \frac{2r+1}{m+n}\pi.$$

If $\cos\frac{1}{2}\theta(m-n) = 0$, $\frac{1}{2}\theta(m-n) = r\pi + \frac{1}{2}\pi$;
$$\therefore\ \theta = \frac{2r+1}{m-n}\pi.$$

Both expressions are included in the expression
$$\theta = \frac{2r+1}{m \pm n}\pi.$$

18. $$\sin\{\pi x(x+y)\} + \sin\{\pi y(x+y)\} = 0 \quad \text{(i)},$$
$$\text{i.e. } 2\sin\tfrac{1}{2}\pi(x+y)^2\cos\tfrac{1}{2}\pi(x^2-y^2) = 0 \quad \text{(ii)},$$
$$\sin\pi x^2 + \sin\pi y^2 = 0 \quad \text{(iii)},$$
$$\text{i.e. } 2\sin\tfrac{1}{2}\pi(x^2+y^2)\cos\tfrac{1}{2}\pi(x^2-y^2) = 0 \quad \text{(iv)}.$$

Now (i) and (iii) are simultaneously true if (ii) and (iv) are simultaneously true; that is, if the same values of x and y satisfy the two equations,
$$\sin\tfrac{1}{2}\pi(x+y)^2\cos\tfrac{1}{2}\pi(x^2-y^2) = 0,$$
$$\sin\tfrac{1}{2}\pi(x^2+y^2)\cos\tfrac{1}{2}\pi(x^2-y^2) = 0.$$

Both equations *together* become zero if either
$\cos\frac{1}{2}\pi(x^2-y^2) = 0$ or if $\sin\frac{1}{2}\pi(x+y)^2 = 0$ *and* $\sin\frac{1}{2}\pi(x^2+y^2) = 0$.

If $\cos\frac{1}{2}\pi(x^2-y^2) = 0$, then $\frac{1}{2}\pi(x^2-y^2) = n\pi + \frac{1}{2}\pi$;
$\therefore\ x^2 - y^2 = 2n + 1$, i.e. an odd number.

If $\sin\frac{1}{2}\pi(x+y)^2 = 0$, $\frac{1}{2}\pi(x+y)^2 = n\pi$ and $(x+y)^2 = 2n$.

If $\sin\frac{1}{2}\pi(x^2+y^2) = 0$, $\frac{1}{2}\pi(x^2+y^2) = m\pi$ and $x^2 + y^2 = 2m$.

Combine these two last equations, for in order that the general equations should be true, these two are to be true *together*.

$$(x+y)^2-(x^2+y^2)=2n-2m=2xy,$$

$$(x-y)^2=x^2+y^2-2xy=2m-(2n-2m)=4m-2n.$$

For 19, 20, 21 see Ans. E. T.

EXAMPLES. LI. PAGE 162.

1. If $m=a^h$, $n=a^k$.

(i) $m^2\times n^3=(a^h)^2\times(a^k)^3=a^{2h}\times a^{3k}=a^{2h+3k}$.

(ii) $m^4\div n^5=(a^h)^4\div(a^k)^5=a^{4h}\div a^{5k}=a^{4h-5k}$.

(iii) $\sqrt[3]{(m^4\times n^5)}=\sqrt[3]{\{(a^h)^4\times(a^k)^5\}}=\sqrt[3]{(a^{4h}\times a^{5k})}=\sqrt[3]{(a^{4h+5k})}=(a^{4h+5k})^{\frac{1}{3}}=a^{\frac{4h+5k}{3}}$.

(iv) $\{\sqrt[4]{(m^5\times n^3)}\}^2=\{\sqrt[4]{[(a^h)^5\times(a^k)^3]}\}^2=\{\sqrt[4]{(a^{5h}\times a^{3k})}\}^2=\{\sqrt[4]{a^{5h+3k}}\}^2$

$$=\{(a^{5h+3k})^{\frac{1}{4}}\}^2=(a^{5h+3k})^{\frac{1}{2}}=a^{\frac{5h+3k}{2}}.$$

2. (i) $453\times650=10^{2\cdot6560982}\times10^{2\cdot8129134}=10^{2\cdot6560982+2\cdot8129134}=10^{5\cdot4690116}$.

(ii) $(453)^4=(10^{2\cdot6560982})^4=10^{2\cdot6560982\times4}=10^{10\cdot6243928}$.

(iii) $(650)^3\times(453)^2=(10^{2\cdot8129134})^3\times(10^{2\cdot6560982})^2=10^{8\cdot4387402}\times10^{5\cdot3121964}$

$$=10^{8\cdot4387402+5\cdot3121964}=10^{13\cdot7509366}.$$

(iv) $\sqrt[3]{453}=\sqrt[3]{10^{2\cdot6560982}}=(10^{2\cdot6560982})^{\frac{1}{3}}=10^{\frac{1}{3}(2\cdot6560982)}=10^{\cdot8853661}$.

(v) $\sqrt{453}\times\sqrt[6]{650}=\sqrt{10^{2\cdot6560982}}\times\sqrt[6]{10^{2\cdot8129134}}$

$$=(10^{2\cdot6560982})^{\frac{1}{2}}\times(10^{2\cdot8129134})^{\frac{1}{6}}$$

$$=10^{\frac{1}{2}(2\cdot6560982)}\times10^{\frac{1}{6}(2\cdot8129134)}=10^{1\cdot3280491}\times10^{\cdot4688189}$$

$$=10^{1\cdot3280491+\cdot4688189}=10^{1\cdot7968680}.$$

(vi) $\sqrt[5]{453}\times(1650)^3=\sqrt[5]{10^{2\cdot6560982}}\times(10^{2\cdot8129134})^3$

$$=(10^{2\cdot6560982})^{\frac{1}{5}}\times(10^{2\cdot812134})^3$$

$$=10^{\frac{1}{5}(2\cdot6560982)}\times10^{2\cdot812134\times3}=10^{\cdot53121964}\times10^{8\cdot4387402}$$

$$=10^{\cdot5312196+8\cdot4387402}=10^{8\cdot9699598}.$$

(vii) $\sqrt{(453\times650)}=\sqrt{(10^{2\cdot6560982}\times10^{2\cdot8129134})}=\sqrt{(10^{2\cdot6560982+2\cdot8129134})}$

$$=\sqrt{10^{5\cdot4690116}}=(10^{5\cdot4690116})^{\frac{1}{2}}=10^{\frac{1}{2}(5\cdot4690116)}=10^{2\cdot7345058}.$$

3. $8=2\times2\times2=2^3$, $32=8\times4=2^3\times2^2=2^{3+2}=2^5$,

$\frac{1}{2}=\frac{2}{4}=\frac{2}{2^2}=2\div2^2=2^{1-2}=2^{-1}$, $\frac{1}{16}=\frac{2}{32}=\frac{2}{2^5}=2\div2^5=2^{1-5}=2^{-4}$,

$\cdot125=\frac{125}{1000}=\frac{1}{8}=\frac{2}{16}=\frac{2}{2^4}=2\div2^4=2^{1-4}=2^{-3}$,

$128=16\times8=2^4\times2^3=2^{4+3}=2^7$.

4. $9=3\times3=3^2,\ 81=9\times9=3^2\times3^2=3^{2+2}=3^4$

$\frac{1}{3}=\frac{3}{9}=\frac{3}{3^2}=3\div3^2=3^{1-2}=3^{-1},\ \frac{1}{27}=\frac{3}{81}=\frac{3}{3^4}=3\div3^4=3^{1-4}=3^{-3},$

$\cdot\dot{1}=\frac{1}{9}=\frac{3}{27}=\frac{3}{3^3}=3\div3^3=3^{1-3}=3^{-2}$

$\frac{1}{81}=\frac{3}{81\times3}=\frac{3}{3^4\times3}=\frac{3}{3^{4+1}}=\frac{3}{3^5}=3\div3^5=3^{1-5}=3^{-4}.$

EXAMPLES. LII. Page 163.

1. $2^2=(10^{\cdot3010300})^2=10^{\cdot60206},\ 3^2=(10^{\cdot4771213})^2=10^{\cdot9542426},$

$2^3=(10^{\cdot3010300})^3=10^{\cdot90309},$

$2\times3=10^{\cdot30103}\times10^{\cdot4771213}=10^{\cdot30103+\cdot4771213}=10^{\cdot7781513},$

$2^4=(10^{\cdot3010300})^4=10^{1\cdot2041200},\ 7^2=(10^{\cdot8450980})^2=10^{1\cdot690196}.$

2. $14=7\times2=10^{\cdot8450980}\times10^{\cdot30103}=10^{\cdot8450980+\cdot3010300}=10^{1\cdot146128},$

$16=2^4=(10^{\cdot30103})^4=10^{1\cdot20412},$

$18=9\times2=3^2\times2=(10^{\cdot4771213})^2\times10^{\cdot30103}$

$=10^{\cdot9542426}\times10^{\cdot30103}=10^{\cdot9542426+\cdot30103}=10^{1\cdot2552726},$

$24=3\times8=3\times2^3=10^{\cdot4771213}\times(10^{\cdot30103})^3$

$=10^{\cdot4771213}\times10^{\cdot90309}=10^{\cdot4771213+\cdot90309}=10^{1\cdot3802113},$

$27=3^3=(10^{\cdot4771213})^3=10^{1\cdot4313639},$

$42=7\times6=7\times2\times3=10^{\cdot845098}\times10^{\cdot30103}\times10^{\cdot4771213}$

$=10^{\cdot845098+\cdot30103+\cdot4771213}=10^{1\cdot6232493}.$

3. $10=10',\ 5=10\div2=10'\div10^{\cdot30103}=10^{1-\cdot30103}=10^{\cdot69897},$

$15=10\times3\div2=10'\times10^{\cdot4771213}\div10^{\cdot30103}=10^{1+\cdot4771213-\cdot30103}=10^{1\cdot1760913},$

$25=100\div4=10^2\div2^2=10^2\div(10^{\cdot30103})^2=10^{2-\cdot60206}=10^{1\cdot39794},$

$30=10\times3=10'\times10^{\cdot4771213}=10^{1\cdot4771213},$

$35=7\times10\div2=10^{\cdot845098}\times10'\div10^{\cdot30103}=10^{\cdot845098+1-\cdot30103}=10^{1\cdot544068}.$

4. $36=9\times4=3^2\times2^2=(10^{\cdot4771213})^2\times(10^{\cdot30103})^2$

$=10^{\cdot9542426}\times10^{\cdot60206}=10^{\cdot9542426+\cdot60206}=10^{1\cdot5563026},$

$40=10\times4=10\times2^2=10^1\times(10^{\cdot30103})^2=10^{1+\cdot60206}=10^{1\cdot60206},$

$50=100\div2=10^2\div10^{\cdot30103}=10^{2-\cdot30103}=10^{1\cdot69897},$

$200=2\times100=2\times10^2=10^{\cdot30103}\times10^2=10^{2\cdot30103},\ 1000=10^3.$

5. $3^{10}\times7^{10}\div2^{20}=(10^{\cdot4771213})^{10}\times(10^{\cdot845098})^{10}\div(10^{\cdot30103})^{20}$

$=10^{4\cdot771213}\times10^{8\cdot45098}\div10^{6\cdot0206}$

$=10^{4\cdot771213+8\cdot45098-6\cdot0206}=10^{7\cdot201593}$

$2^{12}\times3^{20}\div7^{11}=(10^{\cdot30103})^{12}\times(10^{\cdot4771213})^{20}\div(10^{\cdot845098}$

$=10^{3\cdot61236}\times10^{9\cdot542426}\div10^{9\cdot296078}$

$=10^{3\cdot61236+9\cdot542426-9\cdot296078}=10^{3\cdot858708}.$

6. $\sqrt[3]{21} \times \sqrt[4]{18} = \sqrt[3]{(7 \times 3)} \times \sqrt[4]{(9 \times 2)}$

$$= \sqrt[3]{(10^{\cdot 845098} \times 10^{\cdot 4771213})} \times \sqrt[4]{\{(10^{\cdot 4771213})^2 \times 10^{\cdot 30103}\}}$$
$$= \sqrt[3]{10^{\cdot 845098 + \cdot 4771213}} \times \sqrt[4]{10^{\cdot 9542426 + \cdot 30103}}$$
$$= (10^{1 \cdot 3222193})^{\frac{1}{3}} \times (10^{1 \cdot 2552726})^{\frac{1}{4}} = 10^{\cdot 44073976 + \cdot 36381815} = 10^{\cdot 7545579},$$

$\sqrt[2]{(\cdot 49 \times 4^5)} \times \sqrt[3]{(3^4 \times 2^{10})}$

$$= \sqrt[2]{(7^2 \div 10^2 \times 2^{10})} \times \sqrt[3]{(3^4 \times 2^{10})}$$
$$= \sqrt[2]{\{(10^{\cdot 845098})^2 \div 10^2 \times (10^{\cdot 30103})^{10}\}} \times \sqrt[3]{\{(10^{\cdot 4771213})^4 \times (10^{\cdot 30103})^{10}\}}$$
$$= \sqrt[2]{10^{\cdot 845098 - 2 + 3 \cdot 0103}} \times \sqrt[3]{10^{1 \cdot 9084852 + 3 \cdot 0103}}$$
$$= (10^{2 \cdot 700496})^{\frac{1}{2}} \times (10^{4 \cdot 9187852})^{\frac{1}{3}} = 10^{1 \cdot 350248} \times 10^{1 \cdot 639595}$$
$$= 10^{1 \cdot 350248 + 1 \cdot 639595} = 10^{2 \cdot 989843}.$$

7. $\sqrt[10]{42} = \sqrt[10]{(7 \times 3 \times 2)} = \sqrt[10]{(10^{\cdot 845098} \times 10^{\cdot 4771213} \times 10^{\cdot 30103})}$

$$= \sqrt[10]{10^{\cdot 845098 + \cdot 4771213 + \cdot 30103}}$$
$$= \sqrt[10]{10^{1 \cdot 6232493}} = (10^{1 \cdot 6232493})^{\frac{1}{10}} = 10^{\cdot 1623249}.$$

But $10^{\cdot 1623249} = 1 \cdot 4532$, $\therefore \sqrt[10]{42} = 1 \cdot 4532$.

8. $\sqrt[3]{(42)^4} \times \sqrt[4]{(42)^3}$

$$= \sqrt[3]{(7 \times 3 \times 2)^4} \times \sqrt[4]{(7 \times 3 \times 2)^3}$$
$$= \sqrt[3]{(10^{\cdot 845098} \times 10^{\cdot 4771213} \times 10^{\cdot 30103})^4}$$
$$\times \sqrt[4]{(10^{\cdot 845098} \times 10^{\cdot 4771213} \times 10^{\cdot 30103})^3}$$
$$= \sqrt[3]{(10^{\cdot 845098 + \cdot 4771213 + \cdot 30103})^4} \times \sqrt[4]{(10^{\cdot 845098 + \cdot 4771213 + \cdot 30103})^3}$$
$$= \sqrt[3]{(10^{1 \cdot 623249})^4} \times \sqrt[4]{(10^{1 \cdot 623249})^3} = (10^{6 \cdot 492996})^{\frac{1}{3}} \times (10^{4 \cdot 869747})^{\frac{1}{4}}$$
$$= 10^{2 \cdot 164332} \times 10^{1 \cdot 21743675} = 10^{2 \cdot 164332 + 1 \cdot 21743675} = 10^{3 \cdot 38177}.$$

But $10^{3 \cdot 38177} = 2408 \cdot 6$, $\therefore \sqrt[3]{(42)^4} \times \sqrt[4]{(42)^3} = 2408 \cdot 6$.

9. (i) $\sqrt[3]{6} \times \sqrt[4]{7} + \sqrt[5]{9}$

$$= \sqrt[3]{(3 \times 2)} \times \sqrt[4]{7} \times \sqrt[5]{3^2}$$
$$= \sqrt[3]{(10^{\cdot 4771213} \times 10^{\cdot 30103})} \times \sqrt[4]{10^{\cdot 845098}} \times \sqrt[5]{(10^{\cdot 4771213})^2}$$
$$= \sqrt[3]{10^{\cdot 4771213 + \cdot 30103}} \times \sqrt[4]{10^{\cdot 845098}} \times \sqrt[5]{10^{\cdot 9542426}}$$
$$= (10^{\cdot 7781513})^{\frac{1}{3}} \times (10^{\cdot 845098})^{\frac{1}{4}} \times (10^{\cdot 9542426})^{\frac{1}{5}}$$
$$= 10^{\cdot 25938377} \times 10^{\cdot 2112745} \times 10^{\cdot 19084832}$$
$$= 10^{\cdot 25938377 + \cdot 2112745 + \cdot 19084832} = 10^{\cdot 6615067}.$$

But $10^{\cdot 6615067} = 4 \cdot 5868$; $\therefore \sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9} = 4 \cdot 5868$.

(ii) $\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}$

$$= (10^{\cdot 30103})^{\frac{1}{10}} \times (10^{\cdot 4771213})^{-\frac{5}{4}} \times (10^{\cdot 845098})^{\frac{7}{11}}$$
$$= 10^{\cdot 030103} \times 10^{-\cdot 5964016} \times 10^{\cdot 5377896}$$
$$= 10^{\cdot 030103 - \cdot 5964016 + \cdot 5377896} = 10^{-\cdot 028509}.$$

But $10^{-\cdot 028509} = \cdot 93646$, $\therefore \sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}} = \cdot 93646$.

10. $(67\cdot21)^{\frac{3}{5}} \times (49\cdot62)^{\frac{1}{5}} \times (3\cdot971)^{-\frac{7}{5}}$

$$=(10^{1\cdot8274339})^{\frac{3}{5}} \times (10^{1\cdot6956568})^{\frac{1}{5}} \times (10^{\cdot5988999})^{-\frac{7}{5}}$$
$$=10^{1\cdot09646034} \times 10^{\cdot33913136} \times 10^{-\cdot83845986}$$
$$=10^{1\cdot09646034+\cdot33913136-\cdot83845986}=10^{\cdot59713184}.$$

But $10^{\cdot5971310}=3\cdot9549$, $\therefore\ (67\cdot21)^{\frac{3}{5}} \times (49\cdot62)^{\frac{1}{5}} \times (3\cdot971)^{-\frac{7}{5}}=3\cdot9549$ nearly.

11. Area of field $=640\cdot12 \times 640\cdot12=(640\cdot12)^2=(10^{2\cdot8062614})^2=10^{5\cdot6125228}$.

But $10^{5\cdot6125228}=40975\cdot3$;

$\therefore$ area of field $=40975\cdot3$ square feet.

12. Edge of cube $=\sqrt[3]{42601}$ inches

$$=\sqrt[3]{10^{4\cdot6294198}}=10^{\frac{1}{3}(4\cdot6294198)}=10^{1\cdot5431399}.$$

But $10^{1\cdot5431399}=34\cdot925$;

$\therefore$ the edge of the cube $=34\cdot925$ inches.

13. The edge of the cube $=\sqrt[3]{34\cdot701}$ inches

$$=\sqrt[3]{10^{1\cdot5403420}}=10^{\frac{1}{3}(1\cdot5403420)}=10^{\cdot5134473}.$$

But $10^{\cdot5134473}=3\cdot2617$;

$\therefore$ edge of the cube $=3\cdot2617$ inches.

14. Volume of cube $=(47\cdot931)^3$ cubic yards $=(10^{1\cdot6806165})^3=10^{5\cdot0418495}$.

But $10^{5\cdot0418495}=110115$;

$\therefore$ volume of cube $=110115$ cubic yards.

EXAMPLES. LIII. Page 166.

1. The index of the power of a which $=a^3$ is 3, $\therefore\ 3 =\log_a a^3$,

„ „ „ „ $=a^{\frac{10}{3}}$ „ $\frac{10}{3}$, $\therefore\ 3\cdot\dot{3}=\log_a a^{\frac{10}{3}}$,

„ „ „ „ $=\sqrt[4]{a}$ (i.e. $a^{\frac{1}{4}}$) is $\frac{1}{4}$, $\therefore\ \frac{1}{4}=\log_a \sqrt[4]{a}$,

„ „ „ „ $=\sqrt[3]{a^2}$ (i.e. $a^{\frac{2}{3}}$) is $\frac{2}{3}$, $\therefore\ \frac{2}{3}=\log_a \sqrt[3]{a^2}$,

„ „ „ „ $=\frac{1}{a^{\frac{5}{2}}}$ (i.e. $a^{-\frac{5}{2}}$ is $-\frac{5}{2}$, $\therefore\ -\frac{5}{2}=\log_a \frac{1}{a^{\frac{5}{2}}}$.

2. $8=2^3$, $\therefore\ 3=\log_2 8$, $64=2^6$, $\therefore\ 6=\log_2 64$,

$\frac{1}{2}=2^{-1}$, $\therefore\ -1=\log_2 \frac{1}{2}$,

$$\cdot125=\frac{125}{1000}=\frac{1}{8}=\frac{1}{2^3}=2^{-3},\ \therefore\ -3=\log_2 \cdot125,$$

$$\cdot015625=\frac{1}{64}=\frac{1}{2^6}=2^{-6},\ \therefore\ -6=\log_2 \cdot015625,$$

$$\sqrt[3]{64}=(64)^{\frac{1}{3}}=(2^6)^{\frac{1}{3}}=2^2,\ \therefore\ 2=\log_2 \sqrt[3]{64}.$$

3. $9=3^2, \ \therefore\ \log_3 9=2;\ 81=3^4,\ \therefore\ \log_3 81=4,$

$$\frac{1}{3}=3^{-1},\ \therefore\ \log_3\frac{1}{3}=-1;\ \frac{1}{27}=\frac{1}{3^3}=3^{-3},\ \therefore\ \log_3\frac{1}{27}=-3,$$

$$\cdot\dot{1}=\frac{1}{9}=\frac{1}{3^2}=3^{-2},\ \therefore\ \log_3\cdot\dot{1}=-2;\ \frac{1}{81}=\frac{1}{3^4}=3^{-4},\ \therefore\ \log_3\frac{1}{81}=-4.$$

4. $8=\sqrt{64}=(64)^{\frac{1}{2}}=(4^3)^{\frac{1}{2}}=4^{\frac{3}{2}},\ \therefore\ \log_4 8=\frac{3}{2},$

$$\sqrt[3]{16}=\sqrt[3]{4^2}=(4^2)^{\frac{1}{3}}=4^{\frac{2}{3}},\ \therefore\ \log_4\sqrt[3]{16}=\frac{2}{3},$$

$$\sqrt[4]{\cdot 5}=\sqrt[4]{\frac{1}{2}}=\left(\frac{1}{2}\right)^{\frac{1}{4}}=\left(\frac{1}{\sqrt{4}}\right)^{\frac{1}{4}}=(4^{-\frac{1}{2}})^{\frac{1}{4}}=4^{-\frac{1}{8}},\ \therefore\ \log_4\sqrt[4]{\cdot 5}=-\frac{1}{8},$$

$$\sqrt[3]{\cdot 015625}=\sqrt[3]{\frac{1}{64}}=\left(\frac{1}{4^3}\right)^{\frac{1}{3}}=(4^{-3})^{\frac{1}{3}}=4^{-1},\ \therefore\ \log_4\sqrt[3]{\cdot 015625}=-1.$$

5.

$$\log_2 8=\log_2 2^3=3\log_2 2=3,$$
$$\log_2\cdot 5=\log_2\tfrac{1}{2}=\log_2 2^{-1}=-1,$$
$$\log_3 243=\log_3 3^5=5,$$
$$\log_5(\cdot 04)=\log_5\tfrac{1}{25}=\log_5 5^{-2}=-2,$$
$$\log_{10}1000=\log_{10}10^3=3,$$
$$\log_{10}(\cdot 100)=\log_{10}10^{-3}=-3.$$

6.

$$\log_a a^{\frac{4}{3}}=\tfrac{4}{3}\log_a a=\tfrac{4}{3};\ \text{for } \log_a a=1,$$
$$\log_b\sqrt[3]{b^2}=\log_b b^{\frac{2}{3}}=\tfrac{2}{3}\log_b b=\tfrac{2}{3},$$
$$\log_8 2=\log_8 8^{\frac{1}{3}}=\tfrac{1}{3}\log_8 8=\tfrac{1}{3},$$
$$\log_{27}3=\log_{27}27^{\frac{1}{3}}=\tfrac{1}{3}\log_{27}27=\tfrac{1}{3},$$
$$\log_{100}10=\log_{100}100^{\frac{1}{2}}=\tfrac{1}{2}\log_{100}100=\tfrac{1}{2}.$$

7. $\log_{10}6=\log_{10}(3\times 2)=\log_{10}2+\log_{10}3=\cdot 4771213+\cdot 30103=\cdot 7781513,$

$$\log_{10}42=\log_{10}(2\times 3\times 7)=\log_{10}2+\log_{10}3+\log_{10}7$$
$$=\cdot 30103+\cdot 4771213+\cdot 845098=1\cdot 6232493,$$

$$\log_{10}16=\log_{10}(2^4)=4\log_{10}2=4\times(\cdot 30103)=1\cdot 2041200.$$

8. $\log_{10} 49 = \log_{10} 7^2 = 2 \log_{10} 7 = 2 \times (\cdot 845098) = 1\cdot 690196,$

$\log_{10} 36 = \log_{10} (4 \times 9) = \log_{10} (2^2 \times 3^2) = \log_{10} (2^2) + \log_{10} (3^2) = 2 \log_{10} 2 + 2 \log_{10} 3$
$= \cdot 6020600 + \cdot 9542426 = 1\cdot 5563026,$

$\log_{10} 63 = \log_{10} (9 \times 7) = \log_{10} (3^2) + \log_{10} 7 = 2 \log_{10} 3 + \log_{10} 7$
$= \cdot 954246 + \cdot 8450980 = 1\cdot 7993406.$

9. $\log_{10} 200 = \log_{10} (100 \times 2) = 2 \log_{10} 10 + \log_{10} 2 = 2 + \cdot 30103 = 2\cdot 30103,$

$\log 600 = \log (100 \times 3 \times 2) = \log 100 + \log 3 + \log 2$
$= 2 + \cdot 4771213 + \cdot 3010300 = 2\cdot 7781513,$

$\log 70 = \log 10 \times 7 = \log 10 + \log 7 = 1 + \cdot 8450980 = 1\cdot 8450980.$

10. $\log 5 = \log (10 \div 2) = \log 10 - \log 2 = 1 - \cdot 30103 = \cdot 6989700,$

$\log 3\cdot\dot{3} = \log \frac{10}{3} = \log (10 \div 3) = \log 10 - \log 3$
$= 1 - \cdot 4771213 = \cdot 5228787,$

$\log 50 = \log (100 \div 2) = \log 100 - \log 2$
$= 2 - \cdot 3010300 = 1\cdot 6989700.$

11. $\log 35 = \log (70 \div 2) = \log (10 \times 7 \div 2) = \log 10 + \log 7 - \log 2$
$= 1 + \cdot 845098 - \cdot 3010300 = 1\cdot 544068,$

$\log 150 = \log (100 \times 3 \div 2) = \log 100 + \log 3 - \log 2$
$= 2 + \cdot 4771213 - \cdot 30103 = 2\cdot 1760913,$

$\log (\cdot 2) = \log (2 \div 10) = \log 2 - \log 10 = \cdot 30103 - 1 = -1 + \cdot 3010300.$

12. $\log 3\cdot 5 = \log (7 \div 2) = \log 7 - \log 2$
$= \cdot 8459800 - \cdot 3010300 = \cdot 5440680,$

$\log 7\cdot 29 = \log (729 \div 100) = \log (3^6 \div 10^2)$
$= 6 \log 3 - 2 \log 10 = 2\cdot 8627278 - 2 = \cdot 8627278,$

$\log \cdot 081 = \log (81 \div 1000) = \log (3^4 \div 10^3)$
$= 4 \log 3 - 3 \log 10 = 1\cdot 9084852 - 3 = -2 + \cdot 9084852.$

13. $\log_{10} \{\sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9}\} = \log_{10} \{(3 \times 2)^{\frac{1}{3}} \times 7^{\frac{1}{4}} \times (3^2)^{\frac{1}{5}}\}$
$= \frac{1}{3} \log_{10} (3 \times 2) + \frac{1}{4} \log_{10} 7 + \frac{2}{5} \log_{10} 3$
$= \frac{1}{3} \log_{10} 3 + \frac{1}{3} \log_{10} 2 + \frac{1}{4} \log_{10} 7 + \frac{2}{5} \log_{10} 3$
$= \frac{1}{3} \log_{10} 2 + \frac{11}{15} \log_{10} 3 + \frac{1}{4} \log_{10} 7$
$= \frac{\cdot 30103}{3} + \frac{11 \times \cdot 4771213}{15} + \frac{\cdot 845098}{4}$
$= \cdot 1003433 + \cdot 3498889 + \cdot 2112745 = \cdot 6615067.$

But $\cdot 6615067 = \log_{10} 4\cdot 5868$; $\therefore \log_{10} \{\sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9}\} = \log_{10} 4\cdot 5868$;

$\therefore \sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9} = 4\cdot 5868,$

$\log_{10} \{\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}\} = \log 2^{\frac{1}{10}} + \log 3^{-\frac{5}{4}} + \log 7^{\frac{7}{11}}$
$= \frac{1}{10} \log_{10} 2 - \frac{5}{4} \log_{10} 3 + \frac{7}{11} \log_{10} 7$
$= \frac{\cdot 30103}{10} - \frac{5 \times \cdot 4771213}{4} + \frac{7 \times \cdot 845098}{11}$
$= \cdot 030103 - \cdot 5964016 + \cdot 5377896$
$= -\cdot 028509 = \log_{10} \cdot 93646;$

$\therefore \sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}} = \cdot 93646,$

14. (i) $\log\{\sqrt[3]{2}\times\sqrt[4]{7}\div\sqrt[5]{9}\}$

$=\log\{2^{\frac{1}{3}}\times 7^{\frac{1}{4}}\div 9^{\frac{1}{5}}\}=\log\{2^{\frac{1}{3}}\times 7^{\frac{1}{4}}\div(3^2)^{\frac{1}{5}}\}$

$=\log\{2^{\frac{1}{3}}\times 7^{\frac{1}{4}}\div 3^{\frac{2}{5}}\}=\log 2^{\frac{1}{3}}+\log 7^{\frac{1}{4}}-\log 3^{\frac{2}{5}}$

$=\frac{1}{3}\log 2+\frac{1}{4}\log 7-\frac{2}{5}\log 3.$

(ii) Vid. (13).

15. (i) $\log_{10} ab=\log_{10} a+\log_{10} b$

$=2{\cdot}6560982+2{\cdot}8129134=5{\cdot}4690116.$

(ii) $\log_{10} a^4=4\log_{10} a=4\times 2{\cdot}6560982$

$=10{\cdot}6243928.$

(iii) $\log_{10} a^2b^3=2\log a+3\log b$

$=5{\cdot}3121964+8{\cdot}4387402=13{\cdot}7509366.$

(iv) $\log_{10}\sqrt[3]{a}=\frac{1}{3}\log_{10} a=\dfrac{2{\cdot}6560982}{3}$

$={\cdot}8853661.$

(v) $\log_{10}(a^3b)^{\frac{1}{6}}=\frac{1}{6}(3\log_{10} a+\log_{10} b)$

$=\dfrac{7{\cdot}9682946+2{\cdot}8129134}{6}$

$=\dfrac{10{\cdot}7812080}{6}=1{\cdot}7968680.$

(vi) $\log_{10} a^{\frac{1}{5}}b^3=\log_{10} a^{\frac{1}{5}}+\log_{10} b^3$

$=\frac{1}{5}\log_{10} a+3\log_{10} b$

$=\dfrac{2{\cdot}6560982}{5}+3\times 2{\cdot}8129134$

$=8{\cdot}9699598.$

16. (i) $\log_{10}(\sqrt[3]{21}\times\sqrt[4]{18})=\log_{10}\sqrt[3]{7\times 3}+\log_{10}\sqrt[4]{3^2\times 2}$

$=\dfrac{\log_{10} 7+\log_{10} 3}{3}+\dfrac{2\log_{10} 3+\log_{10} 2}{4}$

$=\dfrac{1{\cdot}3222193}{3}+\dfrac{1{\cdot}2552726}{4}$

$={\cdot}4407397+{\cdot}3138182={\cdot}7545579.$

(ii) $\log_{10}\sqrt[2]{({\cdot}49\times 4^5)}\times\sqrt[3]{(3^4\times 2^{10})}$

$=\frac{1}{2}(2\log_{10} 7-\log_{10} 10^2+10\log_{10} 2)+\frac{1}{3}(4\log_{10} 3+10\log_{10} 2)$

$=1{\cdot}3502480+1{\cdot}6395951=2{\cdot}989483.$

EXAMPLES. LIV. Pages 169, 170.

1. 17601 is between 10^4 and 10^5; $\therefore \log_{10} 17601=4+$ a decimal.

361·1 is between 10^2 and 10^3; $\therefore \log_{10} 361{\cdot}1=2+$ a decimal.

4·01 is between 10^0 and 10; $\therefore \log_{10} 4{\cdot}01$ is a decimal.

Integral part $=0$.

723000 is between 10^5 and 10^6; $\therefore \log_{10} 723000$ is $5+$ a decimal.

29 is between 10 and 10^2; $\therefore \log_{10} 29$ is $1+$ a decimal.

2. $\cdot 04$ is between 10^{-2} and 10^{-1}; $\therefore$ $\log_{10} \cdot 04 = -2 +$ a positive decimal;

$\therefore$ $-2 =$ integral part.

$\cdot 0000612$ is between 10^{-5} and 10^{-4};

$\therefore$ $\log_{10} \cdot 0000612 = -5 +$ a positive decimal; $\therefore$ $-5 =$ integral parts.

$\cdot 7963$ is between 10^{-1} and 10^{0} (i.e. 1);

$\therefore$ $\log_{10} \cdot 7963 = -1 +$ a positive decimal; $\therefore$ integral part $= -1$.

$\cdot 001201$ is between 10^{-3} and 10^{-2}; $\therefore$ $\log_{10} \cdot 001201 = -3 +$ a positive decimal;

$\therefore$ integral part $= -3$.

3. 7963 is between 10^{3} and 10^{4};

$\therefore$ $\log_{10} 7963 = 3 +$ a decimal; $\therefore$ integral part $= 3$.

$\cdot 1$ is between 10^{-1} and 10^{0}; $\therefore$ $\log_{10} \cdot 1 = -1 +$ a positive decimal;

$\therefore$ integral part $= -1$.

$2{\cdot}61$ is between 10^{0} and 10^{1}; $\therefore$ $\log_{10} 2{\cdot}61 = 0 +$ a decimal;

$\therefore$ integral part $= 0$.

$79{\cdot}6341$ is between 10^{1} and 10^{2};

$\therefore$ $\log_{10} 79{\cdot}6341 = 1 +$ a decimal; $\therefore$ integral part $= 1$.

$1{\cdot}0006$ is between 10^{0} and 10^{1}; $\therefore$ $\log_{10} 1{\cdot}0006 = 0 +$ a decimal;

integral part $= 0$.

$\cdot 00000079$ is between 10^{-7} and 10^{-6};

$\therefore$ $\log_{10} \cdot 00000079 = -7 +$ a positive decimal; $\therefore$ integral part $= -7$.

4. $10^{3 \cdot 461}$ is between 10^{3} and 10^{4};

$\therefore$ there are 4 digits in integral part of the number.

$10^{\cdot 30203}$ is between 10^{0} and 10^{1};

$\therefore$ there is 1 digit in integral part of the number.

$10^{5 \cdot 4712301}$ is between 10^{5} and 10^{6};

$\therefore$ there are 6 digits in integral part of the number.

$10^{2 \cdot 6710100}$ is between 10^{2} and 10^{3};

$\therefore$ there are 3 digits in integral part of the number.

5. The logarithm of the number is between -2 and -1;

$\therefore$ the number is between 10^{-2} and 10^{-1},

" " " $\cdot 01$ and $\cdot 1$;

$\therefore$ the first *significant* figure is in second decimal place.

The logarithm of the number is between -1 and 0;

$\therefore$ the number is between 10^{-1} and 10^{0},

" " " $\cdot 1$ and 1;

$\therefore$ the first significant figure is in first decimal place.

The logarithm of the number is between -6 and -5;

$\therefore$ the number is between 10^{-6} and 10^{-5},

" " " $\cdot 000001$ and $\cdot 00001$;

$\therefore$ the first significant figure is in the sixth decimal place.

6. The number is $\log_{10} 10^{4\cdot2990713}$;

$\therefore$ the number is between 10^4 and 10^5,

,, ,, ,, 10000 and 100000;

$\therefore$ the first significant figure is ten thousands.

The given number is $10^{\cdot3040595}$;

$\therefore$,, ,, ,, is between 10^0 and 10^1,

,, ,, ,, 1 and 10;

$\therefore$ first significant figure is in units place.

The given number is $10^{2\cdot5860244}$;

$\therefore$ the number is between 10^2 and 10^3,

,, ,, ,, 100 and 1000.

The given number is $10^{-3+\cdot1760913}$;

$\therefore$ the number is between 10^{-3} and 10^{-2};

,, ,, ,, ·001 and ·01.

The given number is $10^{-1+\cdot3180633}$;

$\therefore$ the number is between 10^{-1} and 10^0,

,, ,, ,, ·1 and 1.

The given number is $10^{\cdot4980347}$;

$\therefore$ the number is between 10^0 and 10^1,

,, ,, ,, 1 and 10.

7. $\log 8^{10} = 10 \log 8 = 30 \log 2 = 9\cdot0309$; $\therefore$ $8^{10} = 10^{9\cdot0309}$;

$\therefore$ 8^{10} is between 10^9 and 10^{10}.

$\log 2^{12} = 12 \log 2 = 3\cdot61236$; $\therefore$ 2^{12} is between 10^3 and 10^4.

$\log 16^{20} = 80 \log 2 = 24\cdot0824$; $\therefore$ 16^{20} is between 10^{24} and 10^{25}.

$\log 2^{100} = 30\cdot103$; $\therefore$ 2^{100} is between 10^{30} and 10^{31}.

8. $\log 7^{10} = 8\cdot45098$; $\therefore$ 7^{10} is between 10^8 and 10^9.

$\log 49^6 = 12 \log 7 = 10\cdot141176$; $\therefore$ 49^6 is between 10^{10} and 10^{11}.

$\log 343^{\frac{100}{3}} = \log (7^3)^{\frac{100}{3}} = \log 7^{100} = 84\cdot5098$;

$\therefore$ $343^{\frac{100}{3}}$ is between 10^{84} and 10^{85}.

$\log (\frac{10}{7})^{20} = 20 \log 10 - 20 \log 7 = 3\cdot09804$; $\therefore$ $(\frac{10}{7})^{20}$ is between 10^3 and 10^4

$\log (4\cdot9)^{12} = \log (\frac{49}{10})^{12} = 24 \log 7 - 12 \log 10 = 8\cdot282352$;

$\therefore$ $(4\cdot9)^{12}$ is between 10^8 and 10^9.

$\log (3\cdot43)^{10} = \log (\frac{343}{100})^{10} = 30 \log 7 - 20 \log 10 = 5\cdot35294$;

$\therefore$ $(3\cdot43)^{10}$ is between 10^5 and 10^6.

9. $\log_{10}\sqrt[10]{2}=\frac{1}{10}\log_{10}2=\cdot030103$; $\therefore\ \sqrt[10]{2}=10^{\cdot030103}$;

$\therefore\ \sqrt[10]{2}$ is between 10^0 and 10^1;

" " 1 and 10.

$\log(\frac{1}{2})^{10}=-10\log 2=-4+\cdot9897$;

$\therefore\ (\frac{1}{2})^{10}\ (=10^{-4+\cdot9897})$ is between 10^{-4} and 10^{-3}.

$\log(\frac{10}{7})^{20}=20\log 10-20\log 7=3\cdot09804$;

$\therefore\ (\frac{10}{7})^{20}=10^{3\cdot09804}$; $\therefore\ (\frac{10}{7})^{20}$ is between 10^3 and 10^4.

$\log(\cdot02)^4=4\log\cdot2-4\log 100=-7+\cdot20412$;

$\therefore\ (\cdot02)^4=10^{-7+\cdot20412}$; $\therefore\ (\cdot02)^4$ is between 10^{-7} and 10^{-6}.

$\log(\cdot49)^6=\log(49\div100)^6=12\log 7-12\log 10=-2+\cdot141176$;

$\therefore\ (\cdot49)^6$ is between 10^{-2} and 10^{-1}.

10. $\log(20)^7=7\log 2+7\log 10=9\cdot10721$;

$\therefore\ 20^7$ is between 10^9 and 10^{10}.

$(\cdot02)^7=(2\div100)^7=(10^{\cdot30103}\div10^2)^7=(10^{-2+\cdot30103})^7=10^{-14+2\cdot10721}=10^{-12+\cdot10721}$

$\therefore\ (\cdot02)^7$ is between 10^{-12} and 10^{-11}.

$\log(\cdot007)^2=\log(7\div1000)^2\ 2\log 7-6\log 10=-5+\cdot690196$;

$\therefore\ (\cdot007)^2$ is between 10^{-5} and 10^{-4}.

$\log(3\cdot43)^{\frac{1}{10}}=\log(343\div100)^{\frac{1}{10}}=\cdot0535294$;

$\therefore\ (3\cdot43)^{\frac{1}{10}}$ is between 1 and 10.

$\log(\cdot0343)^8=\log\{7^3\div10^4\}^8=-12+\cdot282352$;

$\therefore\ (\cdot0343)^8$ is between 10^{-12} and 10^{-11}.

$\log(\cdot0343)^{\frac{1}{10}}=\log(7^3\div10^4)^{\frac{1}{10}}=-1+\cdot8535294$;

$\therefore\ (\cdot0343)^{\frac{1}{10}}$ is between $\cdot1$ and 1.

EXAMPLES. LV. Pages 172, 173.

1. $776\cdot43=\frac{77643}{100}=\log 77643-2=2\cdot8901023$;

$7\cdot7643=\frac{77643}{10000}=\log 77643-4$, $\cdot00077643=\frac{77643}{100000000}=\log 77643-8$,

$\log_{10}776430=5\cdot8901023$.

2. $\log_{10}\ 5908200\ =6\cdot7714552$, $\log_{10}5\cdot9082=\cdot7714552$,
$\log_{10}\cdot00059082=\bar{4}\cdot7714552$, $\log_{10}590\cdot82=2\cdot7714552$,
$\log_{10}\ 5908\cdot2\ =3\cdot7714552$.

3. $\log\sqrt[4]{(\cdot0059082)}=\frac{1}{4}\{\log\cdot0059082-\log 10^7\}$
$=\frac{1}{4}\{-3+\cdot7714552\}=\frac{1}{4}\{-4+1\cdot7714552\}$
$=-1+\cdot4428638=-1+\log 2\cdot7724=\log\cdot27724$.

4. $\log\{\cdot00059082\times\cdot027724\}=-4+\cdot7714552-2+\cdot4428638$
$=-5+\cdot2143190=\log\cdot00001638$.

5. $\log\sqrt[10]{(\cdot077643)}=\frac{1}{10}\{-2+\cdot8901023\}=\frac{1}{10}\{-10+8\cdot890123\}$
$=-1+\cdot88901023=-1+\log 7\cdot7448=\log\cdot77448$.

6. $\log\{(\cdot27724)^2\times\cdot077643\} = -2+\cdot8857276-2+\cdot8901023$
$= -3+\cdot7758299 = -3+\log 5\cdot9680$
$=\log\cdot005968.$

EXAMPLES. LVI. Page 175.

1. Let m be any number, and let x be its log to base 8. The log of m to base 2 is supposed to be known.

Now $m=8^x=(2^3)^x=2^{3x}$, or $3x=\log_2{}^m$;

$\therefore\ x=\frac{1}{3}$ of the log of m to base 2.

2. Let m be any number, and let x be its log to base 3. The log of m to base 9 is supposed to be known.

Now $m=3^x=(9^{\frac{1}{2}})^x=9^{\frac{x}{2}}$, or $\frac{x}{2}=\log_9{}^m$;

$\therefore\ x=2\times$ the log of m to base 9.

3. Let m be any number, and let x be its log to base 2. The log of m to base 10 is given.

Now $m=2^x=(10^{\cdot30103\cdots})^x=10^{x\times\cdot30103\cdots}$, or $x\times\log_{10}2=\log_{10}{}^m$;

$\therefore\ x=$log of m to base 10 divided by $\log_{10}{}^2$.

4. Let m be any number, and let x be its log to base 10. The log of m to base 3 is known.

Now $m=10^x=2\,(3^{\log_3 10})^x=3^{x\times\log_3 10}$, or $x\times\log_3 10=\log_3{}^m$;

$\therefore\ x=$log of m to base 3 divided by $\log_3{}^{10}$.

5. Let m be any number, and let x be $\log_3{}^m$. The log of m to the base 10 is known.

Now, $m=3^x=(10^{\log_{10}3})^x=10^{x\log_{10}3}$; $\therefore\ x\log_{10}3=\log_{10}{}^m$;

$\therefore$ the log of m to base 3 = log of m to base 10 divided by $\log_{10}{}^3$.

6. Let $x=\log_2 10$; then $2^x=10$; $\therefore\ 2=10^{\frac{1}{x}}$.

But $2=10^{\cdot3010300}$; $\therefore\ \frac{1}{x}=\cdot30103$.

7. Let $x=\log_7 10$; then $7^x=10$; $\therefore\ 7=10^{\frac{1}{x}}$.

But $7=10^{\cdot8450980}$; $\therefore\ \frac{1}{x}=\cdot8450980$.

8. Let $x=\log_8 10$; then $8^x=10$; $\therefore\ 2^{3x}=10$, or $2=10^{\frac{1}{3x}}$.

But $2=10^{\cdot30103}$; $\therefore\ \frac{1}{3x}=\cdot30103$.

Let $y=\log_{32}10$; then $32^y=10$ or $2=10^{\frac{1}{5y}}$;

$\therefore\ \frac{1}{5y}=\cdot30103.$

MISCELLANEOUS EXAMPLES. LVII. Pages 175, 176.

1. Let $\log_2 8 = x$; $\therefore$ $8 = 2^x$, but $8 = 2^3$; $\therefore$ $2^x = 2^3$;

$\therefore$ $x = 3$ and $\log_2 8 = 3$.

Let $\log_5 1 = x$; $\therefore$ $1 = 5^x$; but $1 = 5^0$; $\therefore$ $5^x = 5^0$;

$\therefore$ $x = 0$, and $\log_5 1 = 0$.

Let $x = \log_8 2$, then $8^x = 2$; but $8^{\frac{1}{3}} = 2$; $\therefore$ $x = \frac{1}{3}$.

Let $\log_7 1 = x$; $\therefore$ $1 = 7^x$; but $1 = 7^0$; $\therefore$ $7^x = 7^0$;

$\therefore$ $x = 0$ and $\log_7 1 = 0$.

Let $x = \log_{32} 128$; then $32^x = 128$; or $2^{5x} = 2^7$; $\therefore$ $x = \frac{7}{5}$.

2. In every system of logarithms the log of 1 is 0; for $a^0 = 1$,

$\log 2$, and $\log 3$ are given,

$4 = 2^2$; $\therefore$ $\log 4 = 2 \log 2$,

$5 = 10 \div 2$; $\therefore$ $\log 5 = \log 10 - \log 2$,

$6 = 3 \times 2$; $\therefore$ $\log 6 = \log 3 + \log 2$,

$\log 7$ is given,

$8 = 2^3$; $\therefore$ $\log 8 = 3 \log 2$,

$9 = 3^2$; $\therefore$ $\log 9 = 2 \log 3$,

$\log 10 = 1$, $\log 11$ cannot be found from our data,

$12 = 2^2 \times 3$; $\therefore$ $\log 12 = 2 \log 2 + 3$,

$\log 13$ cannot be found from our data,

$14 = 7 \times 2$; $\therefore$ $\log 14 = \log 7 + \log 2$,

$15 = 10 \times 3 \div 2$; $\therefore \log 15 = \log 10 + \log 3 - \log 2$,

$16 = 2^4$; $\therefore$ $\log 16 = 4 \log 2$,

$\log 17$ cannot be found from our data,

$18 = 3^2 \times 2$; $\therefore$ $\log 18 = 2 \log 3 + \log 2$,

$\log 19$ cannot be found from our data,

$20 = 10 \times 2$; $\therefore$ $\log 20 = \log 10 + \log 2$,

$21 = 7 \times 3$; $\therefore$ $\log 21 = \log 7 + \log 3$,

$\log 22$ cannot be found from our data,

$\log 23$ cannot be found from our data,

$24 = 2^3 \times 3$; $\therefore$ $\log 24 = 3 \log 2 + \log 3$,

$25 = 10^2 \div 2^2$; $\therefore$ $\log 25 = 2 \log 10 + 2 \log 2$,

$\log 26$ cannot be found from our data,

$27 = 3^3$; $\therefore$ $\log 27 = 3 \log 3$,

$\log 29$ cannot be found from our data,

$30 = 10 \times 3$; $\therefore$ $\log 30 = \log 10 + \log 3$.

The eight numbers are 11, 13, 17, 19, 22, 23, 26, 29.

3. $\log 1 = 0$,

$\log 2 = \log \sqrt[3]{8} = \frac{1}{3} \log 8$,

$$\log 3 = \log \frac{21 \times \sqrt[3]{8}}{14} = \log 21 + \tfrac{1}{3} \log 8 - \log 14,$$

$$\log 4 = 2 \log 2 = \tfrac{2}{3} \log 8,$$

$$\log 5 = \log \tfrac{10}{2} = 1 - \log 2 = 1 - \tfrac{1}{3} \log 8,$$

$$\log 6 = \log (3 \times 2) = \log 3 + \log 2$$

$$= \log 21 + \tfrac{2}{3} \log 8 - \log 14,$$

$$\log 7 = \log \tfrac{14}{2} = \log 14 - \log 2 = \log 14 - \tfrac{1}{3} \log 8,$$

$$\log 9 = 2 \log 3 = 2 (\log 21 + \tfrac{1}{3} \log 8 - \log 14).$$

4.
$$\log 85762 = 4{\cdot}9332949;$$

$$\therefore \log {\cdot}0085762 = \bar{3}{\cdot}9332949,$$

$$\log \sqrt[11]{{\cdot}0085762} = \tfrac{1}{11} \log {\cdot}0085762 = \frac{\bar{3}{\cdot}9332949}{11}$$

$$= \frac{-11 + 8 + {\cdot}9332949}{11}$$

$$= -1 + {\cdot}8121177$$

$$= \bar{1}{\cdot}8121177,$$

$$\log (85762)^{11} = 11 \log 85762 = 11 \times 4{\cdot}9332949$$

$$= 54{\cdot}2662439;$$

$$\therefore (85762)^{11} = 10^{54{\cdot}2662439};$$

$\therefore$ $(85762)^{11}$ is between 10^{54} and 10^{55};

$\therefore$ there are 55 figures in the integral part of this power.

5.
$$\log \{47{\cdot}609 \times 476{\cdot}09 \times {\cdot}47609 \times {\cdot}000047609\}$$

$$= \log \{4{\cdot}7609^4 \times 10 \times 10^2 \times 10^{-1} \times 10^{-5}\}$$

$$= 4 \log 4{\cdot}7609 + 1 + 2 - 1 - 5$$

$$= 1 + 2 - 1 - 5 + 4 \times {\cdot}6776891$$

$$= -3 + 2{\cdot}7107564$$

$$= -1 + {\cdot}7107564 = \bar{1}{\cdot}7107564 = \log {\cdot}51375;$$

$$\therefore 47{\cdot}609 \times 476{\cdot}09 \times {\cdot}47609 \times {\cdot}000047609 = {\cdot}51375.$$

6. By trial $3^7 = 3087$ and $3^8 = 9261$;

$\therefore$ 3742 is between 3^7 and 3^8;

$\therefore$ $\log_3 3742 = 7 +$ a decimal.

By trial $6^4 = 1296$ and $6^5 = 7776$;

$\therefore$ 3742 is between 6^4 and 6^5;

$\therefore$ $\log_6 3742 = 4 +$ a decimal.

By trial $10^3 = 1000$ and $10^4 = 10000$;

$\therefore$ 3742 is between 10^3 and 10^4;

$\therefore$ $\log_{10} 3742 = 3 +$ a decimal.

By trial $12^3 = 1728$ and $12^4 = 20736$;

$\therefore$ 3742 is between 12^3 and 12^4;

$\therefore$ $\log_{12} 3742 = 3 +$ a decimal.

7. (i)
$$2^x \times 3^{4x} = 7^2,$$
$$\therefore \log(2^x \times 3^{4x}) = \log 7^2,$$
$$\therefore \log 2^x + \log 3^{4x} = \log 7^2,$$
$$\therefore x \log 2 + 4x \log 3 = 2 \log 7,$$
$$\therefore x(\log 2 + 4 \log 3) = 2 \log 7,$$
$$\therefore x = \frac{2 \log 7}{\log 2 + 4 \log 3}.$$

(ii)
$$3^{2x} = 128 \times 7^{4-x},$$
$$\therefore \log 3^{2x} = \log(128 \times 7^{4-x}) = \log(2^7 \times 7^{4-x});$$
$$\therefore 2x \log 3 = 7 \log 2 + (4 - x) \log 7;$$
$$\therefore x(2 \log 3 + \log 7) = 7 \log 2 + 4 \log 7;$$
$$\therefore x = \frac{7 \log 2 + 4 \log 7}{2 \log 3 + \log 7}.$$

(iii) $12^x = 49 = 7^2$, $\quad \therefore \log 12^x = \log 7^2$, $\quad \therefore x \log(4 \times 3) = 2 \log 7$;
$$\therefore x(\log 4 + \log 3) = 2 \log 7;$$
$$\therefore x(2 \log 2 + \log 3) = 2 \log 7;$$
$$\therefore x = \frac{2 \log 7}{2 \log 2 + \log 3}.$$

(iv) $2^{8x} = 21^{4-3x}$, $\quad \therefore \log 2^{8x} = \log 21^{4-3x}$,
$$\therefore 8x \log 2 = (4 - 3x) \log 21 = (4 - 3x)(\log 3 + \log 7);$$
$$\therefore 8x \log 2 = 4 \log 3 - 3x \log 7 - 3x \log 3 + 4 \log 7;$$
$$\therefore x\{8 \log 2 + 3 \log 7 + 3 \log 3\} = 4\{\log 3 + \log 7\};$$
$$\therefore x = \frac{4(\log 3 + \log 7)}{8 \log 2 + 3 \log 7 + 3 \log 3}.$$

8. Let $x = \log_7 490$, then $7^x = 490$,
$$\therefore 7^x = 7^2 \times 10; \quad \therefore 7^{x-2} = 10.$$
Equate the logs of each; $\therefore (x - 2) \log_{10} 7 = \log_{10} 10 = 1$;
$$\therefore x = 2 + \frac{1}{\log_{10} 7}.$$

9. Let $x = \log_9 270$; then $9^x = 270$,
$$\therefore 3^{2x} = 3^3 \times 10 \text{ or } 3^{2x-3} = 10.$$
Equate the logs of each; $\therefore (2x - 3) \log_{10} 3 = \log_{10} 10 = 1$;
$$\therefore x = \frac{3}{2} + \frac{1}{2 \log_{10} 3}.$$

10. Let $x = \log_5 10$, then $5^x = 10$,
$$\therefore 5^x \times 2^x = 10 \times 2^x; \quad \therefore 10^x = 10 \times 2^x.$$
Equate the logs of each, then $x \log_{10} 10 = \log_{10} 10 + x \log_{10} 2$;
$$\therefore x = 1 + x \log_{10} 2, \quad \therefore x = \frac{1}{1 - \log_{10} 2}.$$

11. $\log_8 9 = a$, i.e. $9 = 8^a$; $\log_2 5 = b$, i.e. $5 = 2^b$.

Since $5 = 2^b$, $\therefore 10 = 2^{b+1}$, $\therefore 2 = 10^{\frac{1}{b+1}}$,

$$\therefore \log_{10} 2 = \frac{1}{b+1}.$$

$$\text{Again } 3 = \sqrt{9} = \sqrt{8^a} = 8^{\frac{a}{2}} = 2^{\frac{3a}{2}}.$$

But $2 = 10^{\frac{1}{b+1}}$, $\therefore 3 = 2^{\frac{3a}{2}} = 10^{\frac{3a}{2b+2}}$;

$$\therefore \log_{10} 3 = \frac{3a}{2b+2}.$$

Now $\log_{10} 4 = 2\log_{10} 2 = \frac{2}{b+1}$,

$$\log_{10} 5 = \log_{10} 10 - \log_{10} 2 = 1 - \frac{1}{b+1} = \frac{b}{b+1},$$

$$\log_{10} 6 = \log_{10} 3 + \log_{10} 2 = \frac{3a}{2(b+1)} + \frac{2}{2(b+1)} = \frac{3a+2}{2(b+1)},$$

$$\log_5 7 = c, \quad 7 = 5^c, \quad \therefore \log_{10} 7 = c\log_{10} 5 = \frac{cb}{b+1}.$$

12. $2^5 = 32$ and $2^6 = 64$, $\therefore$ all the integers from 32 to 63 have 5 for the characteristic of their logarithms; all these integers will be found to be $(63 - 31)$, i.e. 32.

13. All the integers having 10 for the characteristic of their logarithms begin with a^{10} and the integer next before a^{11}; all these integers will be found to be of the number $(a^{11} - a^{10})$.

14. $\log \sqrt[11]{\{(39\cdot 2)^2\}} = \frac{2}{11}\log\frac{392}{10} = \frac{2}{11}\{\log(7^2 \times 2^3) - 1\}$

$$= \tfrac{2}{11}\{2 \times \cdot 845098 + 3 \times \cdot 30103 - 1\}$$

$$= \tfrac{1}{11}(3\cdot 18572) = \cdot 289688 = \log 1\cdot 9485.$$

15. The expression

$= 7(\log 15 - \log 16) + 6(\log 8 - \log 3) + 5(\log 2 - \log 5) + \log 32 - \log 35$

$= 7\log 3 + 7\log 5 - 28\log 2 + 18\log 2 - 6\log 3$

$\qquad + 5\log 2 - 5\log 5 + 5\log 2 - 2\log 5$

$= \log 3.$

16. The expression $= 2\{\log a + \log a^2 + \log a^3 + \dots + \log a^n\}$

$= 2\{\log a + 2\log a + 3\log a \dots + n\log a\}$

$= 2\log a\{1 + 2 + 3 + \dots + n\}$

$= \frac{1}{2}\{n(n+1)\}\, 2\log a$ (by A.P.)

$= n(n+1)\log a.$

17. Let $\log_a b = x$, $\therefore\ b = a^x$; $\therefore\ b^{\frac{1}{x}} = a$, $\quad \therefore\ \frac{1}{x} = \log_b a$.

Thus
$$\log_a b \times \log_b a = x \times \frac{1}{x} = 1.$$

Let $x = \log_a b$ and $y = \log_b c$.

Then $b = a^x$, $c = b^y = a^{xy}$, $\therefore\ xy = \log_a c$.

But
$$\log_a c \times \log_c a = 1, \ \therefore\ \log_c a = \frac{1}{xy};$$

$$\therefore\ \log_a b \times \log_b c \times \log_c a = x \times y \times \frac{1}{xy} = 1.$$

18. By continuing as in **17**, it can be shewn that
$$\log_a b \,.\, \log_b c \,.\, \log_c d \ldots\ldots \log_q r \,.\, \log_r a = 1;$$

$$\therefore\ \log_a b \,.\, \log_b c \,.\, \log_c d \ldots\ldots \log_q r = \frac{1}{\log_r a} = \log_a r, \text{ since } \log_a r \times \log_r a = 1.$$

19. Let $\log 3{\cdot}456 = x$, then $3{\cdot}456 = 10x$,

$\therefore\ (3{\cdot}456)^{100000} = (10^x)^{100000} = 10^{100000 \times x}$,

but $(3{\cdot}456)^{100000}$ lies between 10^{53856} and 10^{53855},

$\therefore\ 100000 \times x = 53855 +$ a proper fraction;

$\therefore\ x = {\cdot}53855\ldots$; $\quad \therefore\ \log 345{\cdot}6 = 2{\cdot}53855$.

20. Let $\log 3{\cdot}981 = x$, then $3{\cdot}981 = 10^x$; $\therefore\ (3{\cdot}981)^{100000} = (10^x)^{100000}$,

but $(3{\cdot}981)^{100000}$ lies between 10^{60000} and 10^{59999};

$\therefore\ 100000 \times x = 59999 +$ a proper fraction;

$\therefore\ x = {\cdot}59999\ldots$; $\therefore\ \log 39810 = 4{\cdot}59999$.

21. Let P be the number of people living at the beginning of any year; then at the end of the year the number will be $P + \frac{P}{48} - \frac{P}{60}$, i.e. $\frac{241}{240}P$. Similarly at the end of two years the number is $\frac{241}{240} \times \frac{241}{240}P = (\frac{241}{240})^2 P$; and at the end of x years the number will be $(\frac{241}{240})^x P$.

Let the number be doubled at the end of x years,

$$\therefore\ (\tfrac{241}{240})^x P = 2P; \text{ or, } (\tfrac{241}{240})^x = 2;$$

$$\therefore\ x \log \tfrac{241}{240} = \log 2; \quad \therefore\ x(\log 241 - \log 240) = \log 2,$$

$$\therefore\ x = \frac{\log 2}{\log 241 - \log 240} = \frac{\log 2}{\log 241 - \log 3 - 3\log 2 - 1}$$

$$= \frac{{\cdot}30103}{{\cdot}0018057} = 166{\cdot}7;$$

$\therefore$ the population will be doubled within 167 years.

22. $\log s + \log(s-a) - \log b - \log c = \log s(s-a) - \log bc$

$$= \log \frac{s(s-a)}{bc} = 2 \log \left\{\frac{s(s-a)}{bc}\right\}^{\frac{1}{2}}.$$

23. $\log(a^2+x^2)+\log(a+x)+\log(a-x)$
$$=\log(a^2+x^2)(a+x)(a-x)=\log(a^4-x^4).$$

24. $\log\sin 4A=\log 2\sin 2A\cos 2A$
$$=\log 4\sin A\cos A\cos 2A$$
$$=\log 4+\log\sin A+\log\cos A+\log\cos 2A.$$

EXAMPLES. LVIII. Pages 181, 182.

1. $\left.\begin{array}{r}\cdot 8839112\\ \cdot 8839055\\ \hline \cdot 0000057\end{array}\right\}$ The differences in the numbers are ·0001 and ·00002, the differences in the logs ·0000057 and d;

$$\therefore d=\frac{\cdot 00002}{\cdot 0001}\text{ times }\cdot 0000057=\cdot 0000011\ldots$$

$$\therefore \log 7\cdot 6432=\cdot 8839055+\cdot 0000011=\cdot 8839066.$$

2. Here, $d=\frac{3}{10}$ of $\cdot 0000077=\cdot 0000023\ldots$;
$$\log 5\cdot 64123=\cdot 7513715+\cdot 0000023,$$
$$\therefore \log 564\cdot 123=2\cdot 7513738.$$

3. Here, $d=\frac{16}{100}$ of $\cdot 0000050=\cdot 0000008\ldots$;
$$\therefore \log 8\cdot 736416=\cdot 9413325+\cdot 0000008.$$

4. Here, $d=\frac{25}{100}$ of $\cdot 0000067=\cdot 0000017\ldots$;
$$\therefore \log 6\cdot 437125=\cdot 8086903+\cdot 0000017.$$

5. Here, $d=\frac{6}{10}$ of $\cdot 0000116=\cdot 000007\ldots$;
$$\therefore \log 37245\cdot 6=4\cdot 5710680+\cdot 000007.$$

6. $\begin{array}{r}\cdot 5686827\\ \cdot 5686710\\ \hline \cdot 0000117\end{array}$ $\begin{array}{r}\cdot 5686760\\ \cdot 5686710\\ \hline \cdot 0000050\end{array}$ Here, $d=\frac{50}{117}$ of $\cdot 0001=\cdot 00004\ldots$;
$$\therefore \cdot 5686760=\log(3\cdot 7040+\cdot 00004)$$
$$=\log 3\cdot 70404.$$

7. Here, $d=\dfrac{\cdot 0000025}{\cdot 0000095}$ of $\cdot 0001=\cdot 000026\ldots$;
$$\therefore \cdot 6602987=\log(4\cdot 5740+\cdot 000026);$$
$$\therefore 4\cdot 6602987=\log 45740\cdot 26.$$

8. Here, $d=\frac{64}{175}$ of $\cdot 0001=\cdot 000037\ldots$;
$$\therefore \cdot 3966938=\log(2\cdot 4928+\cdot 000037);$$
$$\therefore 6\cdot 3966938=\log 2492837.$$

9. Here, $d=\frac{78}{99}$ of $\cdot 0001=\cdot 00008\ldots$;
$$\therefore \cdot 6431150=\log(4\cdot 3965+\cdot 00008);$$
$$\therefore \bar{4}\cdot 6431150=\log\cdot 000439658.$$

10. $d=\frac{44}{76}$ of $\cdot 0001=\cdot 000058\ldots$;
$$\therefore \log\cdot 7550480=\log(5\cdot 6891+\cdot 000058).$$

EXAMPLES. LIX. Pages 184, 185.

1.

$$\begin{array}{r} \cdot6738727 \\ \cdot6736577 \\ \hline \cdot0002150 \end{array} \qquad d=\frac{30''}{60''} \text{ of } \cdot0002150=\cdot0001075;$$

$$\therefore \sin 42^\circ\, 21'\, 30''=\cdot6736577+\cdot0001075.$$

2. Here, $d=\frac{30''}{60''}$ of $\cdot0002150=\cdot0001075$;

$$\therefore \cos 47^\circ\, 38'\, 30''=6738727-\cdot0001075.$$

N.B. Corresponding to an *increase* in the angle there is a *diminution* in the *cosine*.

3. Here, $d=\frac{45''}{60''}$ of $\cdot0001064=\cdot0000798$;

$$\therefore \cos 21^\circ\, 27'\, 45''=\cdot9307370-\cdot0000798.$$

4.

$$\begin{array}{r} \cdot6667493 \\ \cdot6665325 \\ \hline \cdot0002168 \end{array} \quad \begin{array}{r} \cdot6666666 \\ \cdot6665325 \\ \hline \cdot0001341 \end{array} \qquad D=\tfrac{1341}{2168} \text{ of } 60''=37'';$$

$$\cdot6666666=\sin(41^\circ\, 48'+D'').$$

5. Here, $D=\frac{749}{2742}$ of $60''=16\cdot4''$;

$$\cdot3333333=\cos(70^\circ\, 32'-16\cdot4'').$$

N.B. The angle diminishes as the cosine increases.

6. Here, $D=\frac{1833}{2817}$ of $60''=39''$;

$$\therefore \cdot25=\cos(75^\circ\, 32'-39'').$$

7. Here, $d=\frac{30''}{60''}$, of $\cdot0001251=\cdot0000625$;

$$\therefore L\sin 45^\circ\, 16'\, 30''=9\cdot8514969+\cdot0000625.$$

8. Here, $d=\frac{45''}{60''}$ of $\cdot0003106=\cdot0002329$;

$$\therefore L\tan 27^\circ\, 13'\, 45''=9\cdot7112148+\cdot0002329.$$

9. Here, $d=\frac{20''}{60''}$ of $\cdot0002647=\cdot0000882$;

$$\therefore L\cot 36^\circ\, 18'\, 20''=10\cdot1339650-\cdot0000882.$$

N.B. The cotangent diminishes as the angle increases.

10.

$$\begin{array}{r} 9\cdot8465705 \\ 9\cdot8463018 \\ \hline \cdot0002687 \end{array} \quad \begin{array}{r} 9\cdot8464028 \\ 9\cdot8463018 \\ \hline \cdot0001010 \end{array} \qquad \therefore d=\tfrac{1010}{2687} \text{ of } 60''=23'';$$

$$9\cdot8464028=L\tan(35^\circ\, 4'+23'').$$

11. Here, $D=\frac{368}{679}$ of $60''=32\cdot5''$;

$$\therefore 9\cdot9448230=L(\cos 28^\circ\, 17'-32\cdot5'').$$

N.B. The cosine diminishes as the angle increases.

12. Here, $D = \frac{985}{3136}$ of $60'' = 19''$;

$$10\cdot4274623 = L \text{ cosec } (21^\circ\, 57' - 19'').$$

N.B. The cosec diminishes as the angle increases.

EXAMPLES. LX. Pages 188, 189.

1.
$$\sin A = \frac{a}{c} = \frac{1046\cdot7}{1856\cdot2}.$$

$$\log \sin A = \log \frac{a}{c} = \log a - \log c$$

$$= 3\cdot019822 - 3\cdot2686248 = \bar{1}\cdot7511974.$$

Here, $d = \frac{983}{1851}$ of $60''$; $\therefore\ d = 31\cdot8''$;

$$\therefore\ 9\cdot7511974 = L \sin (34^\circ\, 19' + 31\cdot8'');$$

$$\therefore\ A = 34^\circ\, 19'\, 31\cdot8''.$$

2.
$$\frac{c}{a} = \text{cosec } A = \text{cosec } 34^\circ\, 15';$$

$$\log c = \log a + L \text{ cosec } 34^\circ\, 15' - 10$$
$$= 2\cdot9259306 + 10\cdot2496421 - 10$$
$$= 3\cdot1755727 = \log 1498\cdot2;$$
$$\therefore\ c = 1498\cdot2.$$

3.
$$\tan A = \frac{a}{b} = \frac{4845}{4742},$$

$$L \tan A = 10 + \log a - \log b = 10\cdot0093323.$$

Here $d = \frac{2358}{2527}$ of $60'' = 56''$;

$$\therefore\ 10\cdot0093323 = L \tan (45^\circ\, 36' + 56'');$$

$$\therefore\ A = 45^\circ\, 36'\, 56''.$$

4.
$$\frac{a}{c} = \sin A;$$

$$\therefore\ \log a = \log 8762 + L \sin 37^\circ\, 10' - 10$$
$$= 3\cdot9426032 + 9\cdot7811344 - 10$$
$$= 3\cdot7237376 = \log 5293\cdot4;$$
$$\therefore\ a = 5293\cdot4.$$

$$\frac{b}{c} = \cos A;$$

$$\log b = \log c + \log \cos 37^\circ\, 10'$$
$$= 3\cdot9426032 + 9\cdot9013938 - 10$$
$$= 3\cdot843997 = \log 6982\cdot3.$$

5.
$$\frac{b}{a} = \cot A;$$
$$\therefore \log a = \log b - \log \cot A$$
$$= 3{\cdot}2289647 - 10{\cdot}4683893 + 10$$
$$= 2{\cdot}7605754 = \log 576{\cdot}2.$$

6.
$$b^2 = (c^2 - a^2) = (c+a)(c-a),$$
$$2 \log b = \log (c+a) + \log (c-a)$$
$$= 3{\cdot}7723951 + 3{\cdot}5771470 = 7{\cdot}3495421;$$
$$\therefore \log b = 3{\cdot}67477 = \log 4729;$$
$$\therefore b = 4729 \text{ chains.}$$

7.
$$a^2 = c^2 - b^2;$$
$$\therefore 2 \log a = \log (c+b) + \log (c-b)$$
$$= 3{\cdot}6630410 + 3{\cdot}4655316 = 7{\cdot}1285726;$$
$$\therefore \log a = 3{\cdot}5642863 = \log 3666{\cdot}8.$$

8.
$$\log \tan A = \log a - \log b,$$
$$L \tan A = 10 + \log 7694{\cdot}5 - \log 8471$$
$$= 10 + 3{\cdot}8861804 - 3{\cdot}9279347$$
$$= 9{\cdot}9582457 = L \tan 42^\circ 15'.$$
$$\frac{c}{a} = \operatorname{cosec} A = \operatorname{cosec} 42^\circ 15';$$
$$\therefore \log c = \log a + L \operatorname{cosec} 42^\circ 15' - 10$$
$$= \log 7694{\cdot}5 + L \operatorname{cosec} 42^\circ 15' - 10$$
$$= 3{\cdot}8861804 + 10{\cdot}1723937 - 10$$
$$= 4{\cdot}058574 = \log 11444.$$

EXAMPLES. LXI. Page 190.

1. Vide fig. E. T. p. 186. Let AB be the distance; then, since
$$c = a \operatorname{cosec} A,$$
$$\therefore \log c = \log a + \log \operatorname{cosec} A$$
$$= \log 2500 + L \operatorname{cosec} A - 10$$
$$= 3{\cdot}3979400 + 10{\cdot}1867171 - 10$$
$$= 3{\cdot}5846571 = \log 3842{\cdot}9;$$
$$\therefore c = 3842{\cdot}9.$$

2. Vide fig. E. T. p. 186. Let the height be BC; then since $a = b \tan A$;
$$\therefore \log a = \log 369{\cdot}5 + L \tan 37^\circ 19' 30'' - 10$$
$$= 2{\cdot}5676144 + 9{\cdot}8822317 - 10$$
$$= 2{\cdot}4498461 = \log 281{\cdot}74.$$

3. Figure as above. Let a be the height, then since

$$a = b \tan A = b \tan 32^\circ 12'$$

$$\therefore \log a = \log 176{\cdot}23 + L \tan 32^\circ 12' - 10$$

$$= 2{\cdot}2460798 + 9{\cdot}8158311 - 10$$

$$= 2{\cdot}061910 = \log 115{\cdot}32.$$

4. Figure as above. Let b be the required distance.

$$\text{Then } b = a \cot A = 163{\cdot}5 \times \cot 29^\circ 47' 18'';$$

$$\log b = \log 163{\cdot}5 + L \cot 29^\circ 47' 18'' - 10$$

$$= 2{\cdot}2135178 + 10{\cdot}2422738 - 10$$

$$= 2{\cdot}4557916 = \log 285{\cdot}6;$$

$$\therefore b = 285{\cdot}6.$$

$$L \cot 29^\circ 47' = 10{\cdot}2423617, \quad L \cot 29^\circ 48' = 10{\cdot}2420687.$$

Hence, $$d = \frac{18''}{60''} \text{ of } {\cdot}0002930 = {\cdot}0000879\ldots$$

$$\therefore L \cot 29^\circ 47' 18'' = 10{\cdot}2423617 - {\cdot}0000879 = 10{\cdot}2422738.$$

The logarithmic cotangent diminishes as the angle increases.

5. Figure as above. $\frac{a}{b} = \frac{673{\cdot}12}{415{\cdot}89} = \tan A;$

$$\therefore L \tan A = 10 + \log 673{\cdot}12 - \log 415{\cdot}89$$

$$= 10 + 2{\cdot}8280925 - 2{\cdot}6189785$$

$$= 10{\cdot}2091140.$$

From the tables $10{\cdot}2090013 = L \tan 58^\circ 17'$,

$10{\cdot}2092839 = L \tan 58^\circ 18'$.

$$\text{Hence, } d = \tfrac{1127}{2826} \text{ of } 60'' = 24''\ldots;$$

$$\therefore L \tan A = L \tan 58^\circ 17' 24'', \quad A = 58^\circ 17' 24''.$$

$$B = 90^\circ - A = 90^\circ - 58^\circ 17' 24'' = 31^\circ 42' 36''.$$

6. Figure as above. $\sin A = \frac{a}{c} = \frac{576{\cdot}12}{873{\cdot}14}.$

$$L \sin A = 10 + \log 576{\cdot}12 - \log 873{\cdot}14$$

$$= 10 + 2{\cdot}7605130 - 2{\cdot}9410839$$

$$= 9{\cdot}8194291.$$

From Tables, $9{\cdot}8194012 = L \sin 41^\circ 17'$, $9{\cdot}8195450 = L \sin 41^\circ 18'$,

$$d = \tfrac{279}{1438} \text{ of } 60'' = 11{\cdot}6'';$$

$$\therefore 98194291 = L \sin 41^\circ 17' 11{\cdot}6'',$$

$$b^2 = c^2 - a^2 = (c + a)(c - a);$$

$$\therefore 2 \log b = \log 1449{\cdot}26 + \log 297{\cdot}02$$

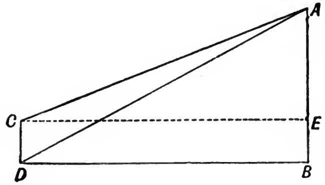
$$= 3{\cdot}1611463 + 2{\cdot}4727857 = 5{\cdot}63393;$$

$$\therefore \log b = 2{\cdot}816966 = \log 656{\cdot}1.$$

7. Vid. fig. E. T. p. 62. Let OM be the lighthouse, Q and P the two ships, $\angle OQM = 27^\circ 18'$, $\angle OPM = 20^\circ 36'$,

$$\begin{aligned} PQ = PM - QM = OM \cot OPM - OM \cot OQM \\ = 112{\cdot}5 (\cot 20^\circ 36' - \cot 27^\circ 18') \\ = 112{\cdot}5 (2{\cdot}6604569 - 1{\cdot}9374645) \\ = 112{\cdot}5 \times {\cdot}72 = 81 \text{ feet.} \end{aligned}$$

8. Let AB be the cliff, CD the lighthouse, angle $ACE = 23^\circ 17'$, $CD = 97{\cdot}25$ and $\angle ADB = 24^\circ 19'$.



$$\frac{AC}{CD} = \frac{\sin ADC}{\sin DAC},$$

$$\sin ADC = \sin (90^\circ - ADB) = \cos ADB.$$

Since $\quad ADC = 90^\circ - ADB$ and $ACD = 90^\circ + ACE$;

$$\begin{aligned} \therefore\ CAD = 180^\circ - (ACD + ADC) \\ = 180^\circ - (90^\circ + 23^\circ 7' + 90^\circ - 24^\circ 19') = 1^\circ 2'; \end{aligned}$$

$$\therefore \frac{AC}{CD} = \frac{\cos 24^\circ 19'}{\sin 1^\circ 2'},$$

$$AC = \frac{97{\cdot}25 \times \cos 24^\circ 19'}{\sin 1^\circ 2'}.$$

But AE the height of the cliff above the light-house

$$= AC \sin 23^\circ 17' = \frac{97{\cdot}25 \cos 24^\circ 19' \sin 23^\circ 17'}{\sin 1^\circ 2'};$$

$$\begin{aligned} \therefore \log AE = \log 97{\cdot}25 + L \cos 24^\circ 19' + L \sin 23^\circ 17' - L \sin 1^\circ 2' - 10 \\ = 1{\cdot}9878896 + 9{\cdot}9596535 + 9{\cdot}596903 - 8{\cdot}2560943 - 10 \\ = 3{\cdot}2883518 = \log 1942{\cdot}4; \\ \therefore\ AE = 1942 \text{ ft.} \end{aligned}$$

9. Draw a figure similar to figure on p. 79 E. T., and let $POM = 51^\circ 25'$; the diameter of the circle described by St Paul's

$$= 2NP = 2OP \cos POM = 7914 \cos 51^\circ 25';$$

the circumference of this circle is $3{\cdot}1416 \times 7914 \cos 51^\circ 25'$. Let x be the no. of miles travelled by the cathedral in an hour in consequence of the revolution of the earth; since the cathedral makes a complete revolution in a day

$$24x = 3{\cdot}1416 \times 7914 \cos 51^\circ 25';$$

$\therefore \log x = \log 3{\cdot}1416 + \log 7914 + L\cos 51°\,25' - \log 24 - 10$
$= {\cdot}4971509 + 3{\cdot}8983960 + 9{\cdot}7949425 - 1{\cdot}3802112 - 10$
$= 2{\cdot}7102782 = \log 646{\cdot}7$ nearly;
$\therefore x = 646{\cdot}7$ miles.

10. Let h = the height of the balloon, x the horizontal distance of the first station from the vertical line through the balloon, and y the horizontal distance of the second station from the vertical line through the balloon.

Then $\frac{h}{x} = \tan 47°\,18'\,30''$, $y^2 = x^2 + (671{\cdot}38)^2$, $\frac{h}{y} = \tan 41°\,14'$,

$\tan 47°\,18 = 1{\cdot}0836896$; $\tan 47°\,19' = 1{\cdot}084323$; difference for $60''$ is ${\cdot}0006327$;

$$\therefore 60'' : 30'' :: {\cdot}0006327 : {\cdot}0003164;$$

$\therefore \tan 47°\,18'\,30'' = 1{\cdot}0836896 + {\cdot}0003164 = 1{\cdot}084$, $\tan 41°\,14' = {\cdot}876462$.

Now $\frac{h}{x} = 1{\cdot}084$; $\therefore x = \frac{h}{1{\cdot}084}$; $\frac{h}{y} = {\cdot}876462$, $\therefore y = \frac{h}{{\cdot}876420}$.

Since the second station is due west of the first,

$$\therefore x^2 + (671{\cdot}38)^2 = y^2.$$

$$\therefore h^2\left\{\frac{1}{({\cdot}87642)^2} - \frac{1}{(1{\cdot}084)^2}\right\} = (671{\cdot}38)^2;$$

$$\therefore h^2\{(1{\cdot}084)^2 - ({\cdot}87642)^2\} = (671{\cdot}38)^2 \times ({\cdot}87642)^2 \times (1{\cdot}084)^2;$$

$$\therefore h^2\{(1{\cdot}084 + {\cdot}87642)(1{\cdot}084 - {\cdot}87642)\} = (671{\cdot}38)^2 \times ({\cdot}87642)^2 \times (1{\cdot}084)^2;$$

$$\therefore h^2 \times 1{\cdot}96042 \times {\cdot}20758 = (671{\cdot}38)^2 \times ({\cdot}87642)^2 \times (1{\cdot}084)^2;$$

$\therefore 2\log h = 2(\log 671{\cdot}38 + \log {\cdot}87642 + \log 1{\cdot}084) - \log 1{\cdot}96042 - \log {\cdot}20758$
$= 2(2{\cdot}8269684 + \bar{1}{\cdot}9427123 + {\cdot}0350293) - {\cdot}2923447 - \bar{1}{\cdot}3171855 = 6$;
$\therefore \log h = 3 = \log 1000$; $\therefore h = 1000$ ft.

EXAMPLES. LXI. a. PAGE 190 (VI).

1. $\log \sqrt[3]{451} = \frac{1}{3}\log 451 = \frac{1}{3}(2{\cdot}65418) = {\cdot}88472$
$= \log 7{\cdot}669$ [from the Tables];
$\therefore \sqrt[3]{451} = 7{\cdot}669$.

2. $\log \sqrt[5]{802} = \frac{1}{5}\log 802 = \frac{1}{5} \times (2{\cdot}90417) = {\cdot}58083$
$= \log 3{\cdot}809$ [from the Table].

3. The log of the expression $= \frac{4}{9} \times \log 273 + \frac{1}{4}\log 234$
$= \frac{4}{9} \times 2{\cdot}43616 + \frac{1}{4} \times 2{\cdot}36922$
$= 1{\cdot}08274 + {\cdot}59230$
$= 1{\cdot}67504 = \log 47{\cdot}32$.

4. The log of the expression $= \frac{3}{5} \times \log 451 + \frac{4}{3}\log 231$
$= \frac{3}{5} \times 2{\cdot}65418 + \frac{4}{3} \times 2{\cdot}36361$
$= 1{\cdot}59250 + 3{\cdot}15148$
$= 4{\cdot}74390 = \log 55460$.

5. The log of the expression $= 3(\log 192{\cdot}5 - \log 84)$
$= 3 \times 2{\cdot}28443 - 3 \times 1{\cdot}92428$
$= 6{\cdot}85329 - 5{\cdot}77284$
$= 1{\cdot}08945 = \log 12{\cdot}03$ [by the Table].

6. The log of the expression $= \frac{2}{3} \times \log 34{\cdot}79 - \frac{3}{2} \times \log 41{\cdot}25$
$= \frac{2}{3} \times (1{\cdot}54145) - \frac{3}{2} \times (1{\cdot}61542)$
$= 1{\cdot}02763 - 2{\cdot}42313 = \bar{2}{\cdot}6045$
$= \log {\cdot}04023.$

7. The log of the expression $= \frac{2}{7} \times \log 24{\cdot}76 - \frac{3}{2} \times \log {\cdot}0045$
$= \frac{2}{7} \times (1{\cdot}39375) - \frac{3}{2} \times (\bar{3}{\cdot}653213)$
$= {\cdot}39821 - \bar{4}{\cdot}47981 = 3{\cdot}91840$
$= \log 8287.$

8. The log of the expression $= \log 7{\cdot}89 - \log {\cdot}0345 + \frac{1}{7} \log 89130$
$= 1{\cdot}89707 - \bar{2}{\cdot}53781 + \frac{1}{7} \times 4{\cdot}95002$
$= 3{\cdot}0664 = \log 1165.$

9. The log of the expression
$= \log 3 - \log 2 + \frac{1}{2} \log 5{\cdot}2 - \log 5 - \frac{1}{2} \log 11{\cdot}31 - \frac{1}{2} \log 3 + \frac{1}{2} \log 7$
$= \frac{1}{2}({\cdot}47712) - {\cdot}30103 + \frac{1}{2} \times {\cdot}71600 - {\cdot}69897 - \frac{1}{2}(1{\cdot}05346) + \frac{1}{2}({\cdot}84509)$
$= {\cdot}23856 - {\cdot}30103 + {\cdot}35800 - {\cdot}69897 - {\cdot}52673 + {\cdot}42254$
$= \bar{1}{\cdot}49237 = \log {\cdot}3107.$

10. The log of the expression $= \frac{1}{5}\{\log 2 + \frac{1}{2} \log 34 - \log 3 - \frac{1}{2} \log 791\}$
$\frac{1}{5} \times \{{\cdot}30103 + \frac{1}{2}(1{\cdot}53147) - {\cdot}47712 - \frac{1}{2}(2{\cdot}89817)\}$
$= \frac{1}{5} \times \{{\cdot}30103 + {\cdot}76573 - {\cdot}47712 - 1{\cdot}44908\}$
$= \bar{1}{\cdot}8281 = \log {\cdot}6731.$

11. The log of the expression $= \frac{1}{4} \log 3 - \frac{1}{6} \log 3$
$= \frac{1}{12}(\log 3) = {\cdot}03976 = \log 1{\cdot}096$ nearly.

12. The log of the expression $= \frac{1}{2} \times \{3 \log 21 + 5 \log 45 - 7 \log 2 - 9 \log 3\}$
$= \frac{1}{2} \times \{3 \times 1{\cdot}32221 + 5 \times 1{\cdot}65321 - 7 \times {\cdot}30103 - 9 \times {\cdot}47712\}$
$= \frac{1}{2} \times \{3{\cdot}96663 + 8{\cdot}26605 - 2{\cdot}10721 - 4{\cdot}29408\}$
$= 2{\cdot}91569 = \log 823{\cdot}6.$

13. $10^x = 421, \; x = \log 421 = 2{\cdot}624.$

14.
$$\left(\tfrac{21}{20}\right)^x = 3,$$
$$x(\log 21 - \log 20) = \log 3, \quad x \times {\cdot}02118 = {\cdot}47712;$$
$$x = \frac{{\cdot}47712}{{\cdot}02118} = 22{\cdot}52.$$

15. $$\left(\tfrac{203}{200}\right)^{2x}=2;$$
$$\therefore\ 2x(\log 203-\log 200)=\log 2,$$
$$2x\times \cdot 00646=\cdot 30103,\quad x=\frac{\cdot 30103}{\cdot 01292}=23\cdot 29.$$

16. $$\left(\tfrac{26}{25}\right)^{x}=3,$$
$$x(\log 26-\log 25)=\log 3,\quad x\times \cdot 01703=\cdot 47712;$$
$$x=\frac{\cdot 47712}{\cdot 01703}=28\cdot 01.$$

17. $$\log 37^{x+3}=3\cdot 412,$$
$$(x+3)\log 37=3\cdot 412,\quad (x+3)\times 1\cdot 56820=3\cdot 412,$$
$$x+3=\frac{3\cdot 412}{1\cdot 5682};$$
$$x=2\cdot 1757-3=-\cdot 8243.$$

18. $$x=10\sqrt[3]{31\cdot 2},$$
$$\log 10\sqrt[3]{31\cdot 2}=\log 10+\tfrac{1}{3}\log 31\cdot 2$$
$$=1+\tfrac{1}{3}\times 1\cdot 49415$$
$$=1+\cdot 49805=1\cdot 49805$$
$$=\log 31\cdot 48.$$

EXAMPLES. LXI. b. PAGE 190 (VII).

1. The amount for 10 years $=\left(\tfrac{104}{100}\right)^{10}\times 100=\left(\tfrac{26}{25}\right)^{10}\times 100,$
$$\log\left(\tfrac{26}{25}\right)^{10}\times 100=10(\log 26-\log 25)+\log 100$$
$$=10\times(1\cdot 41497-1\cdot 39794)+\log 100$$
$$=10\times \cdot 01703+2=2\cdot 1703=\log 148;$$
$\therefore$ amount for 10 years $=$ £148,
or compound Interest $=$ £48.

2. The amount $=\left(\tfrac{21}{20}\right)^{8}\times$ £1,
$$\log\left(\tfrac{21}{20}\right)^{8}\times 1=8(\log 21-\log 20)=\cdot 16852=\log 1\cdot 474;$$
$\therefore$ compound Interest $=$ £1·474 $-$ £1 $=$ £·474 $=$ 9*s.* $5\frac{3}{4}$*d.*

3. $x=$ the no. of years;
$\left(\tfrac{103}{100}\right)^{x}$ is the amount at end of x years, $\left(\tfrac{103}{100}\right)^{x}=2;$
$$\therefore\ x(\log 103-\log 100)=\log 2,\quad x\times(2\cdot 01283-2)=\cdot 30103,$$
$$x=\frac{\cdot 30103}{\cdot 01283}=23\cdot 4 \text{ years}.$$

4. Let $x=$ the no. of years, $\left(\tfrac{104}{100}\right)^{x}$, or $\left(\tfrac{26}{25}\right)^{x}=2,$
$$x\times(\log 26-\log 25)=\log 2,\quad x\times(1\cdot 41497-1\cdot 39794)=\cdot 30103,$$
$$x=\frac{\cdot 30103}{\cdot 01703}=17\cdot 7.$$

5. The present value $=(\frac{100}{104})^8\times £100$,

$$\log(\tfrac{100}{104})^8\times 100=8(\log 25-\log 26)+\log 100$$
$$=8\times(1\cdot39794-1\cdot41497)+2$$
$$=8(\bar{1}\cdot98297)+2=\bar{1}\cdot86376+2$$
$$=1\cdot86376=\log 73\cdot07;$$

$\therefore$ the present value $=(\frac{100}{104})^8\times £100=£73\cdot07$.

6. Let x = the no. of years,

$(\frac{1005}{1000})^x$ = population at end of x years, $(\frac{1005}{1000})^x=2$,

$x(\log 1005-\log 1000)=\log 2$, $x\times\cdot00216=\cdot30103$;

$$\therefore\ x=\frac{\cdot30103}{\cdot00216}=140 \text{ years nearly.}$$

7. To find the amount for 1 half year we multiply £1000 by

$$\frac{101\frac{1}{2}}{100}\text{ or }\frac{203}{200},$$

the amount for 1 year $=(\frac{203}{200})^2\times £1000$,

.... 21 years $=(\frac{203}{200})^{42}\times £1000$;

$$\log(\tfrac{203}{200})^{42}\times 1000=42\log(203-\log 200)+\log 1000$$
$$=42\times(2\cdot30749-2\cdot30103)+3$$
$$=42\times\cdot00646+3=3\cdot27132=\log 1868;$$

$\therefore$ the amount required $=£1868$.

8. Let x = the no. of years, $(\frac{203}{200})^{2x}=3$,

$$2x\times(\cdot00646)=\log 3=\cdot47712,$$
$$x=\frac{\cdot47712}{\cdot01292}=36\cdot9.$$

9. Interest at 1*d*. for 1*s*. per month is $8\frac{1}{3}$ per cent.

To find the amount for the first month we multiply 1*s*. by $\frac{108\frac{1}{3}}{100}$ or $\frac{325}{300}$;

the amount for 2 months $=(\frac{325}{300})^2 s.$,

.... 144 months $=(\frac{325}{300})^{144} s.$;

$$\log(\tfrac{325}{300})^{144}=144\times(\log 325-\log 300)$$
$$=144\times(2\cdot51188-2\cdot47712)=5\cdot00544$$
$$=\log 101300 \text{ nearly};$$

$\therefore$ the amount $=101300s.=£5065$ nearly.

10. The man puts by $2^1d.$ at the end of the 2^{nd} week,

.... $2^2d.$ .... 4^{th} ......

.... $2^3d.$ .... 6^{th} ......

$\therefore$ .... $2^{26}d.$ ....52^{nd} ......

$$\log 2^{26}=26\times\log 2=26\times\cdot30103=7\cdot82678d.$$
$$=\log 6711000 \text{ nearly};$$

$\therefore$ $2^{26}d.=6711000d.=£27962$.

11. The velocity at the end of 1^{st} sec. $= \cdot 001$ ft. per sec.

.... 2^{nd} sec. $= \cdot 001 \times \frac{4}{3}$ ft. ....

.... 3^{rd} sec. $= \cdot 001 \times (\frac{4}{3})^2$ ft. ....

.... 25^{th} sec. $= \cdot 001 \times (\frac{4}{3})^{24}$ ft. ....

$$\log \cdot 001 \times (\tfrac{4}{3})^{24} = \log \cdot 001 + 24 (\log 4 - \log 3)$$
$$= \bar{3} + 24 \times (\cdot 60206 - \cdot 47712)$$
$$= \bar{3} + 24 \times \cdot 12494 = \bar{1}\cdot 99856 = \log \cdot 9967;$$

$\cdot 001 \times (\frac{4}{3})^{24}$ ft. per sec. $= \cdot 9967$ ft. per sec. $= \cdot 679$ miles per hr.

12. Let $2x =$ the diameter of sphere,

$$\tfrac{4}{3}\pi,\ x^3 = 1 \text{ c. yd.},\quad x^3 = 1 \times \tfrac{3}{4} \times \tfrac{7}{22},$$
$$3 \log x = \log 3 - \log 4 + \log 7 - \log 22$$
$$= \cdot 47712 - \cdot 60206 + \cdot 84509 - 1\cdot 34242 = \bar{1}\cdot 37773,$$
$$\log x = \bar{1}\cdot 79257 = \log \cdot 622,$$
$$x = \cdot 622 \text{ yds.};\ \therefore\ \text{diameter} = 2x = 1\cdot 24 \text{ yds.}$$

13. The required present value is

$$£125 \{r + r^2 + \ldots\ldots r^{12}\} \text{ where } r = \tfrac{100}{102}$$
$$= £125 \frac{1 - r^{13}}{1 - r} - £125 = £125 \{1 - (\tfrac{100}{102})^{13}\} \times 51 - £125,$$
$$\log (\tfrac{100}{102})^{13} = 13 (\log 100 - \log 102) = 13 \times (2 - 2\cdot 00860)$$
$$= \bar{1}\cdot 99140 \times 13 = \bar{1}\cdot 88820 = \log \cdot 773;$$
$$\therefore\ (\tfrac{100}{102})^{13} = \cdot 773.$$

Hence the required present value $= £125 \{1 - \cdot 773\} \times 51 - £250$

$$= £125 \{\cdot 227 \times 51 - 1\} = £125 \{10\cdot 577\} = £1322.$$

EXAMPLES. LXII. Pages 193, 194.

1. $\cos 60° = \frac{1}{2}$; $\therefore A = 60°$.

2. $\cos 120° = -\frac{1}{2}$; $\therefore A = 120°$.

3. $\sin 30° = \sin (180° - 30°) = \frac{1}{2}$; $\therefore A = 30°$, or $150°$.

4. $\tan 135° = -1$; $\therefore A = 135°$.

5. $\sin 45° = \sin (180° - 45°) = \frac{1}{\sqrt{2}}$; $\therefore A = 45°$, or $135°$.

6. $\tan 120° = -\sqrt{3}$; $\therefore A = 120°$.

7. $\sin (A + B + C) = \sin 180° = 0$.

8. $\cos (A + B + C) = \cos 180° = -1$.

9. $\sin \frac{1}{2} (A + B + C) = \sin 90° = 1$.

10. $\cos\frac{1}{2}(A+B+C)=\cos 90°=0.$

11. $\tan(A+B)=\tan(A+B+C-C)=\tan(180°-C)=-\tan C.$

12. $\cot\frac{1}{2}(B+C)=\cot\{\frac{1}{2}(A+B+C)-\frac{1}{2}A\}=\cot(90°-\frac{1}{2}A)=\tan\frac{1}{2}A.$

13. $\cos(A+B)=\cos(180°-C)=-\cos C.$

14. $\cos(A+B-C)=\cos(A+B+C-2C)=\cos(180°-2C)=-\cos 2C.$

15. $$\tan A-\cot B=\frac{\sin A\sin B-\cos A\cos B}{\cos A\sin B}$$
$$=-\frac{\cos(A+B)}{\cos A\sin B}=\frac{\cos C}{\cos A\sin B}.$$

16. $$\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)},$$
and
$$\cos\frac{1}{2}(A+B)=\cos(90°-\frac{1}{2}C)=\sin\frac{1}{2}C,$$
$$\sin\frac{1}{2}(A+B)=\sin(90°-\frac{1}{2}C)=\cos\frac{1}{2}C;$$
$$\therefore\ \frac{\sin A-\sin B}{\sin A+\sin B}=\frac{\sin\frac{1}{2}C\,.\,\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C\,.\,\cos\frac{1}{2}(A-B)}.$$

17. $$\frac{\sin 3B-\sin 3C}{\cos 3C-\cos 3B}=\frac{2\cos\frac{3}{2}(B+C)\,.\,\sin\frac{3}{2}(B-C)}{2\sin\frac{3}{2}(B+C)\,.\,\sin\frac{3}{2}(B-C)}=\frac{\cos\frac{3}{2}(B+C)}{\sin\frac{3}{2}(B+C)};$$
$$\therefore\ \tfrac{3}{2}(B+C)=\tfrac{3}{2}\{180°-A\}=270°-\tfrac{3}{2}A;$$
$$\therefore\ \cot\tfrac{3}{2}(B+C)=\cot(180°+90°-\tfrac{3}{2}A)=\tan\tfrac{3}{2}A.$$

18. $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B),$

but
$$\sin\frac{1}{2}(A+B)=\sin(90°-\frac{1}{2}C)=\cos\frac{1}{2}C,$$
and
$$\sin C=2\sin\frac{1}{2}C\cos\frac{1}{2}C=2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}C;$$
$$\therefore\ \sin A+\sin B-\sin C=2\cos\frac{1}{2}C\cos\frac{1}{2}(A-B)-2\cos\frac{1}{2}(A+B)\,.\,\cos\frac{1}{2}C$$
$$=2\cos\frac{1}{2}C\{\cos\frac{1}{2}(A-B)-\cos\frac{1}{2}(A+B)\}$$
$$=2\cos\frac{1}{2}C\,.\,2\sin\frac{1}{2}A\,.\,\sin\frac{1}{2}B.$$

19. $$\therefore\ (\sin A-\sin B)+\sin C=2\sin\frac{1}{2}C\{\sin\frac{1}{2}(A-B)+\sin\frac{1}{2}(A+B)\}$$
$$=2\sin\frac{1}{2}C\,.\,2\sin\frac{1}{2}A\,.\,\cos\frac{1}{2}B.$$

20. $2\sin\frac{1}{2}A\,.\,\cos\frac{1}{2}A+2\sin\frac{1}{2}B\,.\,\cos\frac{1}{2}B+2\sin\frac{1}{2}C\cos\frac{1}{2}C$
$=\sin A+\sin B+\sin C=4\cos\frac{1}{2}A\,.\,\cos\frac{1}{2}B\,.\,\cos\frac{1}{2}C.$ [p. 192, *Example* 3.]

21. $$\therefore\ (\cos A+\cos B)+(\cos C-1)=2\sin\frac{1}{2}C\{\cos\frac{1}{2}(A-B)-\sin\frac{1}{2}C\}$$
$$=2\sin\frac{1}{2}C\{\cos\frac{1}{2}(A-B)-\cos\frac{1}{2}(A+B)\}$$
$$=2\sin\frac{1}{2}C\,.\,2\sin\frac{1}{2}A\,.\,\sin\frac{1}{2}B.$$

22. $\cos^2 \frac{1}{2} A = \frac{1}{2}(1+\cos A)$;

$\therefore \cos^2 \frac{1}{2} A + \cos^2 \frac{1}{2} B - \cos^2 \frac{1}{2} C = \frac{1}{2}(1+\cos A + \cos B - \cos C)$.

Now $\cos A + \cos B = 2\cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) = 2 \sin \frac{1}{2} C \cos \frac{1}{2}(A-B)$

$\cos C = 1 - 2\sin^2 \frac{1}{2} C$;

$$\begin{aligned}\therefore 1+\cos A+\cos B-\cos C &= 2\sin\tfrac{1}{2}C\,\{\sin\tfrac{1}{2}C+\cos\tfrac{1}{2}(A-B)\}\\ &= 2\sin\tfrac{1}{2}C\,\{\cos\tfrac{1}{2}(A+B)+\cos\tfrac{1}{2}(A-B)\}\\ &= 2\sin\tfrac{1}{2}C\,.\,2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B.\end{aligned}$$

23. $\sin^2 \frac{1}{2} A = \frac{1}{2}(1-\cos A)$;

$$\begin{aligned}\therefore \sin^2\tfrac{1}{2}A-\sin^2\tfrac{1}{2}B+\sin^2\tfrac{1}{2}C &= \tfrac{1}{2}\{1-(\cos A-\cos B+\cos C)\}\\ &= \tfrac{1}{2}[1-2\cos\tfrac{1}{2}C\,\{\sin\tfrac{1}{2}(B-A)+\cos\tfrac{1}{2}C\}+1]\\ &= \tfrac{1}{2}[2-2\cos\tfrac{1}{2}C\,\{\sin\tfrac{1}{2}(B-A)+\sin\tfrac{1}{2}(B+A)\}]\\ &= \tfrac{1}{2}\{2-2\cos\tfrac{1}{2}C\,.\,2\sin\tfrac{1}{2}B\cos\tfrac{1}{2}A\}.\end{aligned}$$

24. The expression $= \sin(90° - A) + \sin(90° - B) + \sin(90° - C) - 1$

$= \cos A + \cos B + \cos C - 1$

$= 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C$. [See **21.**]

25. $(\sin 2A + \sin 2B) + \sin 2C = 2 \sin C\,\{\cos(A-B) - \cos(A+B)\}$

$= 2\sin C\,.\,2\sin A\,.\,\sin B$.

26. $(\sin 2A - \sin 2B) + \sin 2C = \cos C\,\{\sin(A+B) - \sin(A-B)\}$

$= 2\cos A \sin B \cos C$.

27. The expression $= \sin(180° - 2A) - \sin(180° - 2B) + \sin(180° - 2C)$

$= \sin 2A - \sin 2B + \sin 2C$

$= 4\cos A \sin B \cos C$. [Vid. **26.**]

28. $\therefore \cos 2A + \cos 2B + \cos 2C = -2\cos C\,\{\cos(A-B) - \cos C\} - 1$

$= -2\cos C\,\{\cos(A-B) + \cos(A+B)\} - 1$

$= -2\cos C\,.\,2\cos A\cos B - 1$.

29. $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$, etc.

$$\begin{aligned}\therefore \sin^2 A-\sin^2 B+\sin^2 C &= \tfrac{1}{2}\{1-(\cos 2A-\cos 2B+\cos 2C)\}\\ &= \tfrac{1}{2}[1-2\sin C\,\{\sin(B-A)-\sin C\}-1]\\ &= -\sin C\,\{\sin(B-A)-\sin(B+A)\}\\ &= -\sin C\,.\,2\cos B\sin(-A).\end{aligned}$$

30. The expression

$$\begin{aligned}&= \cos(180°-2A)+\cos(180°-2B)-\cos(180°-2C)+1\\ &= 1-(\cos 2A+\cos 2B-\cos 2C)\\ &= 1+2\cos C\cos(A-B)+2\cos^2 C-1\\ &= 2\cos C\,\{\cos(A-B)-\cos(A+B)\}\\ &= 2\cos C\,.\,2\sin A\sin B.\end{aligned}$$

31. This is merely the expression of the fact that

$\sin \frac{1}{2}(A+B+C) = \sin 90° = 1$.

32. $\tan(A+B+C)=\dfrac{\sin(A+B+C)}{\cos(A+B+C)}$ = expanding numerator and denominator,

$$\frac{\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B-\sin A\sin B\sin C}{\cos A\cos B\cos C-\cos A\sin B\sin C-\cos B\sin A\sin C-\cos C\sin A\sin B},$$

dividing both numerator and denominator by $\cos A\cos B\cos C$, we have

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}$$

$$=0 \text{ because } (A+B+C)=180°.$$

33. Proceeding as in **32** we have

$$\tan\tfrac{1}{2}(A+B+C)=\frac{\tan\frac{1}{2}A+\tan\frac{1}{2}B+\tan\frac{1}{2}C-\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C}{1-\tan\frac{1}{2}A\tan\frac{1}{2}B-\tan\frac{1}{2}B\tan\frac{1}{2}C-\tan\frac{1}{2}C\tan\frac{1}{2}A};$$

$\therefore$ the denominator is zero.

EXAMPLES. LXIII. Page 195.

1. $\tan A=\tan(90°-B)=\cot B$.

2. $\tan B=\tan(90°-A)=\cot A$; $\cos C=\cos 90°=0$.

3. $\sin 2A=2\sin A\cos A=\sin(180°-2B)=\sin 2B$.

4. $\cos 2A+\cos 2B=2\cos(A+B)\cos(A-B)=2\cos 90°\cos(A-B)=0$.

5. $\sin 2A=2\sin A\cos A=2\cdot\dfrac{b}{c}\cdot\dfrac{a}{c}=\dfrac{2ab}{c^2}$.

6. $\operatorname{cosec} 2B=\dfrac{1}{\sin 2B}=\dfrac{1}{2}\cdot\dfrac{c^2}{ab}\cdot=\dfrac{a^2+b^2}{2ab}$.

7. $\cos 2A=1-2\sin^2 A$ (See E. T. Art. 164.)

$$=1-\frac{2a^2}{c^2}=\frac{b^2+a^2-2a^2}{c^2}.$$

8. $\cos 2B=\cos^2 B-\sin^2 B$

$$=\frac{\cos^2 B-\sin^2 B}{\cos^2 B+\sin^2 B}=\frac{\sin^2 A-\sin^2 B}{\sin^2 A+\sin^2 B}.$$

9. $\sin^2\frac{1}{2}B=\frac{1}{2}\{1-\cos B\}=\frac{1}{2}\left(1-\dfrac{a}{c}\right)$.

10. $\cos^2\frac{1}{2}A=\frac{1}{2}(1+\cos A)$, as in **9**.

11. $(\cos\frac{1}{2}A+\sin\frac{1}{2}A)^2=1+\sin A=1+\dfrac{a}{c}$.

12. Divide both numerator and denominator by c.

Then $\dfrac{a-b}{a+b}=\left(\dfrac{a}{c}-\dfrac{b}{c}\right)\div\left(\dfrac{a}{c}+\dfrac{b}{c}\right)=\dfrac{\sin A-\sin B}{\sin A+\sin B}$

$$=\frac{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{\tan\frac{1}{2}(A-B)}{\tan 45^\circ}.$$

13. $\sin(A-B)+\cos 2A=\sin A\cos B-\cos A\sin B+\cos 2A$

$$=\sin A\,.\,\sin A-\cos A\,.\,\cos A+\cos^2 A-\sin^2 A.$$

14. $\sin(A-B)+\sin(2A+C)=\sin(A-B)+\cos 2A.$

15. $(\sin A-\sin B)^2+(\cos A+\cos B)^2=2-2\sin A\sin B+2\cos A\cos B$

$$=2-2\sin A\sin B+2\sin A\sin B.$$

16. $\sqrt{\dfrac{a+b}{a-b}}+\sqrt{\dfrac{a-b}{a+b}}$

$$=\sqrt{\frac{\sin A+\sin B}{\sin A-\sin B}}+\sqrt{\frac{\sin A-\sin B}{\sin A+\sin B}} \qquad \text{[See 12.]}$$

$$=\frac{(\sin A+\sin B)+(\sin A-\sin B)}{\sqrt{\sin^2 A-\sin^2 B}}=\frac{2\sin A}{\sqrt{(\cos^2 B-\sin^2 B)}}.$$

EXAMPLES. LXIV. PAGE 203.

1. By iii. p. 203, $\sin A=ak$, $\sin B=bk$, $\sin C=ck$;

$$\therefore\ \frac{\sin A+2\sin B}{a+2b}=\frac{ak+2bk}{a+2b}=k=\frac{\sin C}{c}.$$

2. $\dfrac{\sin^2 A-m\sin^2 B}{a^2-m\,.\,b^2}=\dfrac{k^2(a^2-mb^2)}{a^2-mb^2}=k^2=\dfrac{\sin^2 C}{c^2}.$

3. By iii. p. 203, $a=d\sin A$, $b=d\sin B$, $c=d\sin C$;

$\therefore$ $a\cos A+b\cos B-c\cos C$ may be written

$$d\sin A\cos A+d\sin B\,.\,\cos B-d\sin C\,.\,\cos C$$

$=\frac{1}{2}d(\sin 2A+\sin 2B-\sin 2C)=2d\sin C\cos A\cos B=2c\cos A\,.\,\cos B.$

See Examples LXII. **25.**

4. $(a+b)\sin\frac{1}{2}C=d(\sin A+\sin B)\sin\frac{1}{2}C$

$$=2d\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}C$$

$$=2d\cos\tfrac{1}{2}C\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}C=d\sin C\cos\tfrac{1}{2}(A-B).$$

5. $(b-c)\cos\frac{1}{2}A=d(\sin B-\sin C)\cos\frac{1}{2}A$

$=d\sin A\sin\frac{1}{2}(B-C)$, as in **4.**

6. The expression $=\dfrac{\sin A\cos A+\sin B\cos B+\sin C\cos C}{\sin A\sin B\sin C}$

$$=\frac{\frac{1}{2}(\sin 2A+\sin 2B+\sin 2C)}{\sin A\sin B\sin C}$$

$$=\frac{2\sin A\sin B\sin C}{\sin A\sin B\sin C} \qquad \text{Examples LXII. 25.}$$

7. $a \sin(B-C) + b\sin(C-A) + c\sin(A-B)$
$= d\sin A \sin(B-C) + d\sin B\sin(C-A) + d\sin C\sin(A-B)$
$=0.$ [If $\sin(B-C)$ be written out in full.]

8. $$\frac{a-b}{c} = \frac{d(\sin A - \sin B)}{d\sin C} = \frac{\sin A - \sin B}{\sin(A+B)}$$
$$= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)} = \frac{2\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B)}{2\sin^2\frac{1}{2}(A+B)}$$
$$= \frac{\cos B - \cos A}{2\cos^2\frac{1}{2}C}.$$

9. $$\frac{b+c}{a} = \frac{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B+C)} = \frac{\cos B + \cos C}{2\sin^2\frac{1}{2}A}.$$ [As in **8.**]

10. $\sqrt{(bc\sin B\sin C)} = d\sin B\sin C$
$$= \frac{d^2\sin B\sin C(\sin B + \sin C)}{b+c}$$
$$= \frac{b^2\sin C + c^2\sin B}{b+c}.$$

11. From p. 237 E. T., $a = b\cos C + c\cos B,$
$b = c\cos A + a\cos C,$
$c = a\cos B + b\cos A;$
$\therefore\ a+b+c = (b+c)\cos A + (c+a)\cos B + (a+b)\cos C.$

12. As in **11**, $b+c-a = (b+c)\cos A - (c-a)\cos B + (a-b)\cos C.$

13. $$\frac{a\sin C}{b - a\cos C} = \frac{\sin A\sin C}{\sin B - \sin A\cos C} = \frac{\sin A\sin C}{\sin(A+C) - \sin A\cos C}$$
$$= \frac{\sin A\sin C}{\cos A\sin C}.$$

14. The expression $= bc(b\cos C + c\cos B) + ca(c\cos A + a\cos C)$
$+ ab(a\cos B + b\cos A) = bca + cab + abc.$

15. $a\cos(A+B+C) - b\cos(B+A) - c\cos(A+C)$
$= -a + b\cos C + c\cos B.$ [Art. 237.]

16. $$= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}.$$

17. $$\frac{\tan B}{\tan C} = \frac{\sin B\cos C}{\sin C\cos B} = \frac{b\cos C}{c\cos B} = \frac{a^2+b^2-c^2}{2a} \div \frac{a^2-b^2+c^2}{2a}.$$

18. $$= \frac{s(s-c)}{a} + \frac{s(s-b)}{a} = \frac{s(2s-b-c)}{a} = \frac{as}{a} = s.$$

19. $$= \sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}} \times \sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}} = \frac{s-a}{s}.$$

20. $\tan\frac{1}{2}A \div \tan\frac{1}{2}B = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)} \times \frac{s(s-b)}{(s-c)(s-a)}\right\}} = \frac{s-b}{s-a}.$

21. $c^2 = a^2 + b^2 - 2ab\cos C$

$$= a^2(\sin^2\tfrac{1}{2}C + \cos^2\tfrac{1}{2}C) + b^2(\sin^2\tfrac{1}{2}C + \cos^2\tfrac{1}{2}C) - 2ab(\cos^2\tfrac{1}{2}C - \sin^2\tfrac{1}{2}C)$$

$$= \sin^2\tfrac{1}{2}C(a^2 + 2ab + b^2) + \cos^2\tfrac{1}{2}C(a^2 - 2ab + b^2).$$

MISCELLANEOUS EXAMPLES. LXV. Pages 204, 205.

1. Let p_1 be the length of the perpendicular from B on AC.

Then $$ap = bp_1 \text{ by areas.}$$

$$\sin A = \frac{p_1}{c} = \frac{bp_1}{bc} = \frac{ap}{bc}.$$

2. If $2\cos B\sin C = \sin A$, $\therefore 2\sin A\cos B\sin C = \sin^2 A$;

$\therefore \sin^2 A - \sin^2 B + \sin^2 C = \sin^2 A$; [Examples LXII. 29.]

$\therefore \sin^2 B = \sin^2 C$; $\therefore \sin B = \sin C$; $\therefore B = C$.

3. $\frac{\sin B}{\sin A} = \frac{b}{a} = \frac{\sin B}{\sin 3B} = \frac{\sin B}{3\sin B - 4\sin^3 B} = \frac{1}{3 - 4\sin^2 B}$;

$\therefore \frac{b}{a} = \frac{1}{3 - 4\sin^2 B}$. $\therefore \sin B = \frac{1}{2}\sqrt{\frac{3b-a}{b}}$.

4. $\frac{\sin B}{b} = \frac{\sin C}{c} = k$, suppose;

$\therefore \sqrt{\{bc\sin B\sin C\}} = \frac{b^2\sin B + c^2\sin C}{b+c}$ may be written

$kbc = \frac{k(b^3+c^3)}{b+c}$. i.e. $bc = b^2 + c^2 - bc$,

$\therefore (b-c)^2 = 0$, i.e. $b = c$.

5. $a\cos\frac{1}{2}B \,.\, \cos\frac{1}{2}C \operatorname{cosec}\frac{1}{2}A = a\sqrt{\left\{\frac{s(s-b)}{ac} \,.\, \frac{s(s-c)}{ab} \,.\, \frac{bc}{(s-b)(s-c)}\right\}} = s$,
similarly for the other expressions.

6. $\sin A = \frac{4}{5}$, $\sin B = \frac{5}{13}$

$-\cos C = \cos(A+B) = \cos A\cos B - \sin A\sin B = \frac{3}{5} \,.\, \frac{12}{13} - \frac{4}{5} \,.\, \frac{5}{13} = \frac{16}{65}$.

7. $\therefore b^2 + c^2 = a^2$; $\therefore A = 90^\circ$.

8. $\sin 2B - \sin 2A + \sin 2C = 4 \sin A \cos B \cos C$; [Examples LXII. 27.]

$$\therefore\ 4 \sin A \cos B \cos C = 0,$$

$$\text{i.e. } \sin A = 0, \text{ or } \cos B = 0, \text{ or } \cos C = 0.$$

9. $A = \frac{1}{8}$ of $180°$, $B = \frac{2}{8}$ of $180°$, $C = \frac{5}{8}$ of $180°$,

$$1 + 4 \cos A \cos B \cos C = -\cos 2A - \cos 2B - \cos 2C$$
$$= -\cos 45° - \cos 90° - \cos 225° = 0;$$
$$2 \sin^2 A + 2 \sin^2 C - 4 \sin^2 B = \cos 2A + \cos 2C - 2 \cos 2B$$
$$= \cos 45° + \cos 225° = 0.$$

10. The exp. $= a \cos \frac{1}{2}(B + C) \sin \frac{1}{2}(B - C) + \text{etc.}$

$$= \tfrac{1}{2} a (\sin B - \sin C) + \tfrac{1}{2} b (\sin C - \sin A) + \tfrac{1}{2} c (\sin A - \sin B)$$
$$= 0; \text{ for } a \sin B = b \sin A.$$

11. In fig. 1 of p. 196 let D be the *middle* point of BC,

then $AC^2 = AD^2 + CD^2 - 2AD \,.\, CD \cos ADC$,

also $AB^2 = AD^2 + BD^2 - 2AD \,.\, BD \cos ADB$.

Now $\cos ADC = \cos (180° - ADB) = -\cos ADB$,

also $CD = BD$;

$\therefore$ by addition $AC^2 + AB^2 = 2AD^2 + CD^2 + BD^2$;

$$\therefore\ b^2 + c^2 = 2AD^2 + \tfrac{1}{4} a^2 + \tfrac{1}{4} a^2,$$

and the result follows.

12.

$$b \sin A = a \sin B;$$
$$\therefore\ \sin 3B = 2 \sin B;$$
$$\therefore\ 3 \sin B - 4 \sin^3 B = 2 \sin B;$$
$$\therefore\ 3 - 4 \sin^2 B = 2; \ \therefore\ \sin B = \tfrac{1}{2}.$$

Hence $B = 30°$, $A = 3B = 90°$ and $C = 180° - A - B = 60°$.

13. $abc (a \cos A + b \cos B + c \cos C)$

$$= \tfrac{1}{2} abcd (\sin 2A + \sin 2B + \sin 2C)$$
$$= 2abcd \sin A \sin B \sin C \qquad \text{[Ex. LXII. 25.]}$$
$$= 2a^2bc \sin B \sin C = 8S^2. \qquad \text{[iii. p. 203.]}$$

14. $2b \cos^2 \frac{1}{2} C + 2c \cos^2 \frac{1}{2} B = b + b \cos C + c + c \cos B$

$$= b + c + \frac{a^2 + b^2 - c^2}{2a} + \frac{c^2 + a^2 - b^2}{2a} = \frac{1}{a}(ab + ac + a^2) = a + b + c.$$

If then this $= 3a$ we have $b + c = 2a$. Q.E.D.

15. By **11** above we have

$$4AD^2 = (2b^2 + 2c^2 - a^2),$$
$$4BE^2 = (2c^2 + 2a^2 - b^2),$$
$$4CF^2 = (2a^2 + 2b^2 - c^2);$$

$\therefore$ by addition the result follows.

16. In the figure of p. 230 let AD be the *perpendicular* from A on CB; draw AD' to bisect CB in D'; then

$$\cot ADB = \frac{DD'}{AD} = \frac{CD - BD}{2AD} = \frac{b\cos C - c\cos B}{2AD}$$

$$= \frac{a^2+b^2-c^2-c^2-a^2+b^2}{4a \, . \, AD} = \frac{b^2-c^2}{2aAD} = \frac{b^2-c^2}{2ac\sin B} = \frac{b^2-c^2}{4S}.$$

17. We have $d = c\sin B = b\sin C$, $e = c\sin A$, $f = a\sin B$.

Also $a = k\sin A$, $b = k\sin B$, $c = k\sin C$;

$$\therefore\ d = \frac{bc}{k},\quad e = \frac{ca}{k},\quad f = \frac{ab}{k};$$

$$\therefore\ 2d\cos A = 2d\,\frac{b^2+c^2-a^2}{2bc} = \frac{b^2+c^2-a^2}{k};$$

$$\therefore\ 2(d\cos A + e\cos B + f\cos C) = \frac{1}{k}\{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2\}$$

$$= \frac{1}{k}\{a^2+b^2+c^2\} = a \, . \, \frac{a}{k} + b \, . \, \frac{b}{k} + c \, . \, \frac{c}{k}$$

$$= a\sin A + b\sin B + c\sin C.$$

EXAMPLES. LXVI. Pages 208, 209.

1. $s = 674{\cdot}10$, $s-a = 321{\cdot}85$, $s-b = 160{\cdot}83$, $s-c = 191{\cdot}42$,

$$L\tan\tfrac{1}{2}A - 10 = \tfrac{1}{2}\{\log(s-b) + \log(s-c) - \log s - \log(s-a)\};$$

$$\therefore\ L\tan\tfrac{1}{2}A = 10 + \tfrac{1}{2}\{\log 160{\cdot}83 + \log 191{\cdot}42 - \log 674{\cdot}10 - \log 321{\cdot}85\}$$

$$= 10 + \tfrac{1}{2}\{2{\cdot}2063401 + 2{\cdot}2819873 - 2{\cdot}8287248 - 2{\cdot}5076535\}$$

$$= 9{\cdot}5759748$$

9·5761934	9·5759748	$\therefore\ D = \frac{1644}{3830}$ of $60''$
9·5758104	9·5758104	$= 25{\cdot}75''$;
·0003830	·0001644	

$$\therefore\ \tfrac{1}{2}A = 20^\circ\, 38'\, 25{\cdot}75'',\quad A = 41^\circ\, 16'\, 51{\cdot}5''.$$

2. $a = 484$, $b = 376$, $c = 522$, $s = 691$, $(s-a) = 207$, $s-b = 315$, $s-c = 169$. The largest angles are opposite to the greatest sides and are therefore A and C.

$$L\tan\tfrac{1}{2}C - 10 = \tfrac{1}{2}\{\log(s-a) + \log(s-b) - \log s - \log(s-c)\},$$

$$L\tan\tfrac{1}{2}C = 10 + \tfrac{1}{2}\{\log 207 + \log 315 - \log 691 - \log 169\}$$

$$= 9{\cdot}8734581 = L\tan 36^\circ\, 46'\, 6'';$$

$$\therefore\ \tfrac{1}{2}C = 36^\circ\, 46'\, 6'';\quad \therefore\ C = 73^\circ\, 32'\, 12''.$$

$$L\tan\tfrac{1}{2}A - 10 = \tfrac{1}{2}\{\log(s-c) + \log(s-b) - \log s - \log(s-a)\},$$

$$L\tan\tfrac{1}{2}A = 10 + \tfrac{1}{2}\{2{\cdot}2278867 + 2{\cdot}4983106 - 2{\cdot}8394780 - 2{\cdot}3159703\}$$

$$= 9{\cdot}7853745 = L\tan 31^\circ\, 23'\, 9'';$$

$$\therefore\ \tfrac{1}{2}A = 31^\circ\, 23'\, 9'';\quad \therefore\ A = 62^\circ\, 46'\, 18''.$$

3. $s=10142,\ s-a=4904,\ s-b=4480,\ s-c=758,$

$$L\tan\tfrac{1}{2}A=10+\tfrac{1}{2}\{\log 4480+\log 758-\log 10142-\log 4904\}$$
$$=10+\tfrac{1}{2}\{3{\cdot}651278+2{\cdot}8796692-4{\cdot}0061236-3{\cdot}6905505\}$$
$$=9{\cdot}4171366$$

$$\begin{array}{ll} 9{\cdot}4173265 & 9{\cdot}4171366 \\ 9{\cdot}4168099 & 9{\cdot}4168099 \\ \hline {\cdot}0005166 & {\cdot}0003267 \end{array} \qquad \therefore D=\tfrac{3267}{5166}\text{ of }60''=38'';$$

$$\therefore \tfrac{1}{2}A=14^\circ\,38'\,38'';\ \therefore A=29^\circ\,17'\,16''.$$

$$L\tan\tfrac{1}{2}B-10=\tfrac{1}{2}\{\log(s-c)+\log(s-a)-\log s-\log(s-b)\};$$
$$\therefore L\tan\tfrac{1}{2}B=10+\tfrac{1}{2}\{\log 758+\log 4904-\log 10142-\log 4480\}$$
$$=10+\tfrac{1}{2}\{2{\cdot}8796692+3{\cdot}6905505-4{\cdot}0061236-3{\cdot}6512780\}$$
$$=9{\cdot}4564091$$

$$\begin{array}{ll} 9{\cdot}4565420 & 9{\cdot}4564091 \\ 9{\cdot}4560641 & 9{\cdot}4560641 \\ \hline {\cdot}0004779 & {\cdot}0003450 \end{array} \qquad D=\tfrac{3450}{4779}\text{ of }60''=43'';$$

$$\therefore 9{\cdot}4564091=L\tan 15^\circ\,57'\,43''=L\tan\tfrac{1}{2}B;$$
$$\therefore \tfrac{1}{2}B=15^\circ\,57'\,43'';\ \therefore B=31^\circ\,55'\,26''.$$

4. $s=5875{\cdot}5,\ s-a=1785{\cdot}5,\ b=3850,\ c=3811,$

$$L\cos\tfrac{1}{2}A=10+\tfrac{1}{2}\{\log s+\log(s-a)-\log b-\log c\}$$
$$=10+\tfrac{1}{2}\{\log 5875{\cdot}5+\log 1785{\cdot}5-\log 3850-\log 3811\}$$
$$=10+\tfrac{1}{2}\{3{\cdot}7690448+3{\cdot}2517599-3{\cdot}5854607-3{\cdot}5810389\}$$
$$=9{\cdot}9271526$$

$$\begin{array}{ll} 9{\cdot}9272306 & 9{\cdot}9272306 \\ 9{\cdot}9271509 & 9{\cdot}9271526 \\ \hline {\cdot}0000797 & {\cdot}0000780 \end{array} \qquad D=\tfrac{780}{797}\text{ of }60''=59'';$$

$$\therefore \tfrac{1}{2}A=32^\circ\,15'\,59'';\ \therefore A=64^\circ\,31'\,58''.$$

5. $a=7$, $b=8$, $c=9$, the greatest angle is opposite to the greatest side and is C, $s=12$, $s-c=3$,

$$\cos\tfrac{1}{2}C=\sqrt{\left\{\frac{s(s-c)}{ab}\right\}}=\sqrt{\left\{\frac{12\,.\,3}{8\,.\,7}\right\}}=\sqrt{\frac{9}{14}},$$

$$L\cos\tfrac{1}{2}C-10=\tfrac{1}{2}(\log 9-\log 14)=\tfrac{1}{2}(2\log 3-\log 14),$$
$$L\cos\tfrac{1}{2}C=10+\tfrac{1}{2}({\cdot}9542426-1{\cdot}146128)$$
$$=9{\cdot}9040573$$

$$\begin{array}{l} 9{\cdot}9040573 \\ 9{\cdot}9040529 \\ \hline {\cdot}0000044 \end{array} \qquad \therefore D=\tfrac{44}{942}\text{ of }60''=2{\cdot}8'';$$

$$\therefore \tfrac{1}{2}C=36^\circ\,42'-2{\cdot}8''=36^\circ\,41'\,57{\cdot}2'';$$
$$\therefore C=73^\circ\,23'\,54{\cdot}4''.$$

6. The smallest angle is opposite to the least side and is A, $s-b=5$, $s-c=3$,

$$\sin\tfrac{1}{2}A=\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}=\sqrt{\left\{\frac{5.3}{10.12}\right\}}=\sqrt{\frac{1}{8}},$$

$$L\sin\tfrac{1}{2}A-10=\tfrac{1}{2}\{\log 1-\log 8\}=-\tfrac{3}{2}\log 2,$$

$$L\sin\tfrac{1}{2}A=10-\cdot 451545=9\cdot 548455.$$

Here $D=\frac{965}{3342}$ of $60''=17\cdot 3''$;

$$\tfrac{1}{2}A=20^\circ\,42'\,17\cdot 3'';\quad\therefore\ A=41^\circ\,24'\,34\cdot 6''.$$

7. Let $a=4$, $b=5$, $c=6$, $s=7\cdot 5$, $(s-c)=1\cdot 5$,

$$\cos\tfrac{1}{2}C=\sqrt{\left\{\frac{s(s-c)}{ab}\right\}}=\sqrt{\left\{\frac{5\times 3\times 3}{4\times 4\times 5}\right\}}=\sqrt{\frac{9}{16}}=\frac{3}{4};$$

$$L\cos\tfrac{1}{2}C-10=\log 3-\log 4=\log 3-2\log 2,$$

$$L\cos\tfrac{1}{2}C=10+\cdot 4771213-\cdot 6020600=9\cdot 8750613;$$

$$\begin{array}{r}9\cdot 8750613\\ 9\cdot 8750142\\ \hline \cdot 0000471\end{array}\qquad\therefore\ D=\frac{471}{1115}\text{ of }60''=25\cdot 35'';$$

$$\therefore\ \tfrac{1}{2}C=41^\circ\,25'-25\cdot 35''=41^\circ\,24'\,34\cdot 65'';$$

$$\therefore\ C=82^\circ\,49'\,9\cdot 3''.$$

8. $$a=2,\ b=\sqrt{6},\ c=1+\sqrt{3},$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{6+(\sqrt{3}+1)^2-4}{2(1+\sqrt{3})\sqrt{6}}=\frac{6+2\sqrt{3}}{2(1+\sqrt{3})\sqrt{6}}$$

$$=\frac{\sqrt{3}(1+\sqrt{3})}{(1+\sqrt{3})\sqrt{6}}=\frac{1}{\sqrt{2}};\quad\therefore\ A=45^\circ,$$

$$\cos B=\frac{(1+\sqrt{3})^2+4-6}{4(1+\sqrt{3})}=\frac{2+2\sqrt{3}}{4(1+\sqrt{3})}=\frac{1}{2};\quad\therefore\ B=60^\circ,$$

$$C=180^\circ-60^\circ-45^\circ=75^\circ.$$

9. $$a=2,\ b=\sqrt{2},\ c=\sqrt{3}-1,$$

$$\cos A=\frac{2+(\sqrt{3}-1)^2-4}{2\sqrt{2}(\sqrt{3}-1)}=\frac{2-2\sqrt{3}}{2\sqrt{2}(\sqrt{3}-1)}=-\frac{1}{\sqrt{2}};\quad\therefore\ A=135^\circ,$$

$$\cos B=\frac{(\sqrt{3}-1)^2+4-4}{4(\sqrt{3}-1)}=\frac{6-2\sqrt{3}}{4(\sqrt{3}-1)}=\frac{\sqrt{3}}{2};\quad\therefore\ B=30^\circ;$$

$$C=180^\circ-(A+B)=180^\circ-(135^\circ+30^\circ)=15^\circ.$$

EXAMPLES. LXVII. Page 211.

1. $$C=180^\circ-A-B=180^\circ-53^\circ\,24'-66^\circ\,27'=60^\circ\,9',$$

$$a=\frac{c\sin A}{\sin C}=\frac{338\cdot 65\times\sin 53^\circ\,24'}{\sin 60^\circ\,9'},$$

$$\log a=\log 338\cdot 65+L\sin 53^\circ\,24'-L\sin 60^\circ\,9'$$

$$=2\cdot 5297511+9\cdot 9046168-9\cdot 9381851$$

$$=2\cdot 4961828=\log 313\cdot 46;$$

$$\therefore\ a=313\cdot 46\text{ yds.}$$

2.
$$C=180^\circ-A-B=180^\circ-48^\circ-54^\circ=78^\circ,$$
$$a=\frac{c\sin A}{\sin C}=\frac{38\times\sin 48^\circ}{\sin 78^\circ},$$
$$\log a=\log 38^\circ+L\sin 48^\circ-L\sin 78^\circ$$
$$=1{\cdot}5797836+9{\cdot}8710735-9{\cdot}9904044$$
$$=1{\cdot}4604527=\log 28{\cdot}8704;$$
$$\therefore\ a=28{\cdot}8704,$$
$$b=\frac{c\sin B}{\sin C}=\frac{38\times\sin 54^\circ}{\sin 78^\circ},$$
$$\log b=\log 38+L\sin 54^\circ-L\sin 78^\circ$$
$$=1{\cdot}5797836+9{\cdot}9079576-9{\cdot}9904044$$
$$=1{\cdot}4973368=\log 31{\cdot}4295;$$
$$\therefore\ b=31{\cdot}43.$$

3.
$$c=\frac{a\sin C}{\sin A}=\frac{1000\times\sin 66^\circ}{\sin 50^\circ};$$
$$\therefore\ \log c=\log 1000+L\sin 66^\circ-L\sin 50^\circ$$
$$=3+9{\cdot}9607302-9{\cdot}8842540$$
$$=3{\cdot}0764762=\log 1192{\cdot}55;$$
$$\therefore\ c=1192{\cdot}55.$$

4. $A=180^\circ-B-C$
$$=180^\circ-(32^\circ\,15'+21^\circ\,47'\,20'')=180^\circ-(54^\circ\,2'\,20'');$$
$$\therefore\ \sin A=\sin(180^\circ-54^\circ\,2'\,20'')=\sin 54^\circ\,2'\,20'';$$
$$\therefore\ d=\frac{20''}{60''}\text{ of }{\cdot}000092={\cdot}000031;$$
$$\therefore\ L\sin A=L\sin 54^\circ\,2'\,20''=9{\cdot}908141+{\cdot}000031=9{\cdot}908172.$$
$$b=\frac{a\sin B}{\sin A}=\frac{34^\circ\times\sin 32^\circ\,15'}{\sin(180^\circ-54^\circ\,2'\,20'')},$$
$$\therefore\ \log b=\log 34+L\sin 32^\circ\,15'-L\sin(180^\circ-54^\circ\,2'\,20'')$$
$$=1{\cdot}531479+9{\cdot}9727228-9{\cdot}908172=1{\cdot}350535.$$
$$\text{Here } d=\tfrac{93}{194}\text{ of }{\cdot}001={\cdot}0005;$$
$$\therefore\ 1{\cdot}350535=\log(22{\cdot}41+{\cdot}0005)=\log 22{\cdot}415;$$
$$\therefore\ b=22{\cdot}415.$$

5. $C=180^\circ-A-B$
$$=180^\circ-114^\circ\,18''=65^\circ\,59'\,42''.$$
$$\text{Here } d=\frac{42''}{60''}\text{ of }{\cdot}0000563={\cdot}0000394$$
$$\therefore\ L\sin 65^\circ\,59'\,42''=9{\cdot}9606739+{\cdot}0000394=9{\cdot}9607133.$$
$$a=\frac{c\sin A}{\sin C},$$

$$\log a = \log c + L\sin A - L\sin C$$
$$= \log 24 + L\sin 72^\circ 4' - L\sin 65^\circ 59' 42''$$
$$= 1{\cdot}3802112 + 9{\cdot}9783702 - 9{\cdot}9607133 = 1{\cdot}3978681.$$

$$\begin{array}{rr} \cdot 3978705 & \cdot 3978681 \\ \cdot 3978531 & \cdot 3978531 \\ \hline \cdot 0000174 & \cdot 0000150 \end{array} \quad \therefore d = \tfrac{150}{174} \text{ of } \cdot 0001 = \cdot 0000862;$$

$$\therefore \cdot 3978681 = \log(2{\cdot}4995 + \cdot 0000862) = \log 2{\cdot}499586;$$
$$\therefore 1{\cdot}3978681 = \log 24{\cdot}996; \quad \therefore a = 25 \text{ feet nearly.}$$

$$\begin{array}{r} 9{\cdot}8249959 \\ 9{\cdot}8249725 \\ \hline \cdot 0000234 \end{array} \quad \therefore d = \tfrac{8}{10} \text{ of } \cdot 0000234 = \cdot 00001872;$$

$$\therefore L\sin 41^\circ 56' 18'' = 9{\cdot}8249725 + \cdot 00001872 = 9{\cdot}82499122.$$

$$b = \frac{c\sin B}{\sin C};$$

$$\therefore \log b = \log c + L\sin B - L\sin C$$
$$= \log 24 + L\sin 41^\circ 56' 18'' - L\sin 65^\circ 59' 42''$$
$$= 1{\cdot}3802112 + 9{\cdot}8249912 - 9{\cdot}96077133 = 1{\cdot}2444891.$$

Here $d = \tfrac{2120}{2474}$ of $\cdot 001 = \cdot 00085$;

$$\therefore 1{\cdot}2444891 = \log(17{\cdot}55 + \cdot 0085) = \log 17{\cdot}5585;$$
$$\therefore b = 17{\cdot}559 \text{ feet.}$$

EXAMPLES. LXVIII. Pages 213, 214.

1. $b = 131$, $c = 72$, $b - c = 59$, $b + c = 203$;

$$B + C = 180^\circ - A = 180^\circ - 40^\circ = 140^\circ; \quad \tfrac{1}{2}A = 20^\circ;$$
$$L\tan\tfrac{1}{2}(B - C) = \log(b - c) - \log(b + c) + L\cot\tfrac{1}{2}A$$
$$= \log 59 - \log 203 + L\cot 20^\circ$$
$$= 1{\cdot}7708520 - 2{\cdot}3074960 + 10{\cdot}4389341$$
$$= 9{\cdot}9022901.$$

$$\begin{array}{rr} 9{\cdot}9024195 & 9{\cdot}9022901 \\ 9{\cdot}9021604 & 9{\cdot}9021604 \\ \hline \cdot 0002591 & \cdot 0001297 \end{array} \quad \therefore D = \tfrac{1297}{2591} \text{ of } 60'' = 30'';$$

$$\therefore \tfrac{1}{2}(B - C) = 38^\circ 36' 30''. \quad \therefore B - C = 77^\circ 13'. \quad \text{Also } B + C = 140^\circ;$$
$$\therefore 2B = 217^\circ 13'; \quad \therefore B = 108^\circ 36' 30'';$$
$$2C = 62^\circ 47'; \quad \therefore C = 31^\circ 23' 30''.$$

2. $a - b = 14$, $a + b = 56$; $A + B = 180^\circ - 50^\circ = 130^\circ$;

$$\tan\tfrac{1}{2}(A - B) = \tfrac{14}{56}\cot\tfrac{1}{2}C = \tfrac{1}{4}\cot\tfrac{1}{2}C;$$
$$\therefore L\tan\tfrac{1}{2}(A - B) = \log 1 - \log 4 + L\cot\tfrac{1}{2}C.$$
But $\cot\tfrac{1}{2}C = \tan\tfrac{1}{2}(A + B) = \tan 65^\circ$;
$$\therefore L\tan\tfrac{1}{2}(A - B) = L\tan 65^\circ - 2\log 2$$
$$= 10{\cdot}331327 - \cdot 602060 = 9{\cdot}729267.$$

$$\text{Here } D = \tfrac{247}{303} \text{ of } 60'' = 49'';$$
$$\therefore \tfrac{1}{2}(A - B) = 28^\circ\, 11'\, 49'';$$
$$\therefore A - B = 56^\circ\, 23'\, 38'', \text{ also } A + B = 130^\circ;$$
$$\therefore A = 93^\circ\, 11'\, 49'' \text{ and } B = 36^\circ\, 48'\, 11''.$$

3. $c - b = 1,\ c + b = 39,\ \tfrac{1}{2}A = \tfrac{1}{2}(60^\circ) = 30^\circ;\ C + B = 180^\circ - 60^\circ = 120^\circ.$

$$\begin{aligned} L \tan \tfrac{1}{2}(C - B) &= \log(c - b) - \log(c + b) + L \cot \tfrac{1}{2} A \\ &= \log 1 - \log 39 + L \cot 30^\circ \\ &= -1{\cdot}591065 + 10{\cdot}238561 = 8{\cdot}647496. \end{aligned}$$

$$\text{Here } D = \tfrac{1643}{2851} \text{ of } 60'' = 34{\cdot}6'';$$
$$\therefore \tfrac{1}{2}(C - B) = 2^\circ\, 32'\, 34{\cdot}6'';$$
$$\therefore C - B = 5^\circ\, 5'\, 9{\cdot}2'', \text{ also } C + B = 120^\circ.$$
$$\therefore C = 62^\circ\, 22'\, 34{\cdot}6'' \text{ and } B = 57^\circ\, 27'\, 25{\cdot}4''.$$

4. $a - b = 124{\cdot}610,\ a + b = 628{\cdot}140,\ \tfrac{1}{2}C = 39^\circ\, 13';\ A + B = 101^\circ\, 34'.$

$$\begin{aligned} L \tan \tfrac{1}{2}(A - B) &= \log 124{\cdot}610 - \log 628{\cdot}140 + L \cot 39^\circ\, 13' \\ &= 2{\cdot}0955529 - 2{\cdot}7980565 + 10{\cdot}0882755 \\ &= 9{\cdot}3857719. \end{aligned}$$

Here $$D = \tfrac{4349}{5506} \text{ of } 60'' = 47'';$$
$$\therefore \tfrac{1}{2}(A - B) = 13^\circ\, 39'\, 47'';$$
$$\therefore A - B = 27^\circ\, 19'\, 34'' \text{ and } A + B + 101^\circ\, 34';$$
$$\therefore A = 64^\circ\, 26'\, 47'' \text{ and } B = 37^\circ\, 7'\, 13''.$$

5. $a - b = 30,\ a + b = 240,\ \tfrac{1}{2}C = 30^\circ,\ A + B = 180^\circ - 60^\circ = 120^\circ,$

$$\tan \tfrac{1}{2}(A - B) = \tfrac{30}{240} \cot 30^\circ = \tfrac{1}{8}\sqrt{3},$$
$$L \tan \tfrac{1}{2}(A - B) - 10 = \log \sqrt{3} - \log 8 = \tfrac{1}{2}\log 3 - 3 \log 2;$$
$$\begin{aligned} \therefore L \tan \tfrac{1}{2}(A - B) &= 10 + \tfrac{1}{2}\log 3 - 3 \log 2 \\ &= 10 + {\cdot}23856065 - {\cdot}9030900 = 9{\cdot}3354707. \end{aligned}$$

Here $$D = \tfrac{5996}{6112} \text{ of } 60'' = 59'';$$
$$\therefore \tfrac{1}{2}(A - B) = 12^\circ\, 12'\, 59'';$$
$$\therefore A - B = 24^\circ\, 25'\, 58'', \text{ also } A + B = 120^\circ;$$
$$\therefore A = 72^\circ\, 12'\, 59''.$$

6.
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ &= 21^2 + 20^2 - 2 \times 21 \times 20 \times \cos 60^\circ \\ &= 441 + 400 - 2 \times 21 \times 20 \times \tfrac{1}{2} = 421; \end{aligned}$$
$$\therefore c = \sqrt{421} = 20{\cdot}5.$$

7.
$$\begin{aligned} c^2 &= (135)^2 + (105)^2 - 2 \times 135 \times 105 \times \cos 60^\circ \\ &= 18225 + 11025 - 2 \times 135 \times 105 \times \tfrac{1}{2} = 15075; \end{aligned}$$
$$\therefore c = \sqrt{15075} = 122{\cdot}7.$$

8. Let $a=5$, $b=3$, $C=70°\,30'$, $A+B=180°-70°\,30'=109°\,30'$,

$$\tan\tfrac{1}{2}(A-B)=\frac{5-3}{5+3}\cot\tfrac{1}{2}C=\tfrac{1}{4}\cot\tfrac{1}{2}C;$$

$$\therefore L\tan\tfrac{1}{2}(A-B)=\log 1-\log 4+L\cot 35°\,15'$$
$$=-2\log 2+L\cot 35°\,15'$$
$$=-\cdot 6020600+10\cdot 1507464$$
$$=9\cdot 5486864=L\tan 19°\,28'\,50'';$$
$$\therefore \tfrac{1}{2}(A-B)=19°\,28'\,50'';$$
$$A-B=38°\,57'\,40'', \text{ also } A+B=109°\,30';$$
$$\therefore A=74°\,13'\,50'' \text{ and } B=35°\,16'\,10''.$$

EXAMPLES. LXIX. Pages 218, 219.

1.
$$L\sin A=\log a+L\sin B-\log b$$
$$=\log 170\cdot 6+L\sin 40°-\log 140\cdot 5$$
$$=2\cdot 2319790+9\cdot 8080675-2\cdot 1476763$$
$$=9\cdot 8923702.$$

9·8924354	9·8923702
9·8923342	9·8923342
·0001012	·0000360

$\therefore D=\frac{360}{1012}$ of $60''=21''$;

$$\therefore 9\cdot 8923702=L\sin 51°\,18'\,21'';$$
$$\therefore A=51°\,18'\,21'' \text{ or } (180°-51°\,18'\,21''), \text{ i. e. or } 128°\,41'\,39''.$$

Since b is less than a each of these values is admissible.

When $A=51°\,18''\,21''$, $C=180°-A-B$;

$$\therefore C=180°-91°\,18'\,21''=88°\,41'\,39''.$$

When $A=128°\,41'\,39''$, $C=180°-168°\,41'\,39''=11°\,18'\,21''$.

2.
$$L\sin B=\log b+L\sin A-\log a$$
$$=\log 119+L\sin 50°-\log 97$$
$$=2\cdot 075547+9\cdot 884254-1\cdot 986772=9\cdot 973029.$$

Here, $D=\frac{43}{46}$ of $60''=56''$;

$$\therefore 9\cdot 973029=L\sin 70°\,0'\,56'';$$
$$\therefore B=70°\,0'\,56'' \text{ or } 180°-70°\,0'\,56'' \text{ or } 109°\,54'\,4''.$$

Since a is less than b each of these values is admissible.

When $B=70°\,0'\,56''$, $C=180°-A-B=180°-120°\,0'\,56''=59°\,59'\,4''$.

When $B=109°\,54'\,4''$, $C=180°-159°\,59'\,4''=20°\,0'\,56''$.

3.
$$L\sin B=\log 97+L\sin 50°-\log 119$$
$$=1\cdot 986772+9\cdot 884254-2\cdot 075547$$
$$=9\cdot 795479=L\sin 38°\,38'\,24'';$$
$$\therefore B=38°\,38'\,24'' \text{ or } 180°-38°\,38'\,24''=141°\,21'\,36''.$$

Since a is greater than b, angle A is greater than angle B, and only the less value of B is admissible;

$$\therefore\ B = 38^\circ\, 38'\, 24'',$$
$$C = 180^\circ - A - B = 180^\circ - 88^\circ\, 38'\, 24'' = 91^\circ\, 21'\, 36'',$$
$$\log c = \log b + L \sin C - L \sin B$$
$$= \log 97 + L \sin 91^\circ\, 21'\, 36'' - L \sin 38^\circ\, 38'\, 24''.$$

Now $L \sin 91^\circ\, 21'\, 36'' = L \sin (180^\circ - 91^\circ\, 21'\, 36'') = L \sin 88^\circ\, 38'\, 24''$;

$$\therefore\ \log c = \log 97 + L \sin 88^\circ\, 38'\, 24'' - L \sin 38^\circ\, 38'\, 24''$$
$$= 1{\cdot}986772 + 9{\cdot}999876 - 9{\cdot}795479$$
$$= 2{\cdot}191169 = \log 155{\cdot}3.$$

4. $$L \sin A = \log a + L \sin C - \log c$$
$$= \log 24 + L \sin 65^\circ\, 59' - \log 25$$
$$= 1{\cdot}3802112 + 9{\cdot}9606739 - 1{\cdot}3979400 = 9{\cdot}9429451.$$

Here $D = \frac{116}{693}$ of $60'' = 10''$;

$\therefore\ 9{\cdot}9429451 = L \sin 61^\circ\, 16'\, 10''$ or $L \sin (180^\circ - 61^\circ\, 16'\, 10'')$
or $L \sin 118^\circ\, 43'\, 50''$;

$$\therefore\ A = 61^\circ\, 16'\, 10'' \text{ or } 118^\circ\, 43'\, 50''.$$

Since $c > a$, $\therefore\ C > A$, and only the less value of A is admissible,

$$\therefore\ A = 61^\circ\, 16'\, 60''.$$

5. $$L \sin A = \log a + L \sin C - \log c$$
$$= \log 25 + L \sin 65^\circ\, 59' - \log 24$$
$$= 1{\cdot}3979400 + 9{\cdot}9606739 - 1{\cdot}3802112 = 9{\cdot}9784027.$$

Here $D = \frac{325}{409}$ of $60'' = 48''$;

$\therefore\ 9{\cdot}9784027 = L \sin 72^\circ\, 4'\, 48''$ or $L \sin (180^\circ - 72^\circ\, 4'\, 48'') = L \sin 107^\circ\, 55'\, 12''$;

$$\therefore\ A = 72^\circ\, 4'\, 48'' \text{ or } 107^\circ\, 55'\, 12''.$$

Since c is less than a, both values of A are admissible.

When $A = 72^\circ\, 4'\, 48''$, $B = 180^\circ - A - C = 180^\circ - 138^\circ\, 3'\, 48'' = 41^\circ\, 56'\, 12''$.

When $A = 107^\circ\, 55'\, 12''$, $B = 180^\circ - 173^\circ\, 54'\, 12'' = 6^\circ\, 5'\, 48''$.

We shall have the greater value of b when we take the greater value of B, i.e. when we take the less value of A. When $A = 72^\circ\, 4'\, 48''$, $B = 41^\circ\, 56'\, 12''$,

$$b = \frac{a \sin B}{\sin A};$$

$$\therefore\ \log b = \log a + L \sin B - L \sin A$$
$$= \log 25 + L \sin 41^\circ\, 56'\, 12'' - L \sin 72^\circ\, 4'\, 48''.$$

Here $d = \dfrac{2''}{16''}$ of $\cdot0000234 = \cdot0000030$.

$$\therefore\ L \sin 41^\circ\, 56'\, 12'' = 9{\cdot}8249725 + \cdot0000030 = 9{\cdot}8249755;$$
$$\therefore\ \log b = 1{\cdot}3979400 + 9{\cdot}8249755 - 9{\cdot}9784027 = 1{\cdot}2445123.$$

Here $d = \frac{2352}{2474}$ of $\cdot001 = \cdot00095$;

$$\therefore\ 2445123 = \log (1{\cdot}755 + \cdot00095) = \log 1{\cdot}756;$$
$$\therefore\ 1{\cdot}2445123 = \log 17{\cdot}56 = \log b,\ \therefore\ b = 17{\cdot}56.$$

6.
$$\sin C = \frac{c \sin A}{a}.$$

(α)
$$\sin C = \tfrac{250}{125} \times \sin 30° = \tfrac{125}{125} = 1;$$
$\therefore$ $C = 90°$ and the triangle is *not* ambiguous.

(β)
$$\sin C = \tfrac{250}{200} \times \sin 30° = \tfrac{125}{200} = \tfrac{5}{8}.$$

The triangle is possible, and there are two admissible values for C; since $c > a$, $\therefore$ $C > A$, and C may therefore be acute or obtuse; the triangle *is* therefore ambiguous.

(γ)
$$\sin C = \tfrac{125}{200} \times \sin 30° = \tfrac{125}{400} = \tfrac{5}{16};$$
$\therefore$ the triangle is possible; but since $a > c$, $A > C$, and C can only be acute. The triangle is *not* ambiguous.

In the ambiguous case (β)

$$\begin{aligned} L \sin C &= 10 + \log 250 + \log \sin 30° - \log 200 \\ &= 10 + \log 1000 - \log 4 + \log 1 - \log 2 - \log 100 - \log 2 \\ &= 10 + 3 - 2 \log 2 - \log 2 - 2 - \log 2 \\ &= 11 - 4 \log 2 = 11 - 1{\cdot}2041200 \end{aligned}$$

$= 9{\cdot}7958800 = L \sin 38° \, 41'$ or $L \sin (180° - 38° \, 41')$, i.e. $L \sin 141° \, 19'$;

$\therefore$ the angles of the obtuse-angled triangle are

$$A = 30°, \; C = 141° \, 19', \; B = 180° - A - C = 8° \, 41'.$$

Now
$$\begin{aligned} \log b &= \log c + L \sin B - L \sin C \\ &= \log 250 + L \sin 8° \, 41' - L \sin 141° \, 19' \\ &= \log 1000 - \log 4 + L \sin 8° \, 41' - L \sin 141° \, 19' \\ &= 3 - {\cdot}6020600 + 9{\cdot}1789001 - 9{\cdot}7958800 = 1{\cdot}7809601. \end{aligned}$$

Here
$$d = \tfrac{23}{72} \text{ of } {\cdot}001 = {\cdot}00003;$$

$$\therefore \; {\cdot}7809601 = \log (6{\cdot}0389 + {\cdot}00003) = \log 6{\cdot}03893,$$
$$1{\cdot}7809601 = \log 60{\cdot}3893, \; \therefore \; b = 60{\cdot}3893.$$

MISCELLANEOUS EXAMPLES. LXX. PAGE 220.

1. A is obviously the smallest angle of the triangle, since it is opposite to the least side. If we therefore apply the formula

$$\sin \frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}},$$

only the smaller value of A is admissible,

$$b = 576{\cdot}2, \; c = 759{\cdot}3, \; s - b = 278{\cdot}8, \; s - c = 95{\cdot}7.$$

Now
$$\begin{aligned} L \sin \tfrac{1}{2} A &= 10 + \tfrac{1}{2} \{\log (s-b) + \log (s-c) - \log b - \log c\} \\ &= 10 + \tfrac{1}{2} \{\log 278{\cdot}8 + \log 95{\cdot}7 - \log 576{\cdot}2 - \log 759{\cdot}3\} \\ &= 10 + \tfrac{1}{2} \{2{\cdot}4452928 + 1{\cdot}9809119 - 2{\cdot}7605733 - 2{\cdot}8804134\} \\ &= 9{\cdot}3926090. \end{aligned}$$

$L \sin 14° \, 17' = 9{\cdot}3921993$, difference for $60'' = 4959$;

$$\therefore \; D = \tfrac{4097}{4959} \text{ of } 60'' = 49{\cdot}6'';$$

$$\therefore \; \tfrac{1}{2} A = 14° \, 17' \, 49{\cdot}6'', \; \therefore \; A = 28° \, 35' \, 39''.$$

2. Here B is the greatest angle and it is easy to see that $b^2 > a^2 + c^2$, $\therefore$ b is obtuse and therefore if we find B from the formula for $\sin \frac{1}{2}B$ we must take the obtuse angle. It is however best to avoid this investigation by using either the $\tan \frac{1}{2}B$ formula or the $\cos \frac{1}{2}B$ formula.

Here $s = 10851\cdot5$, $s - a = 6850\cdot5$, $s - b = 3208\cdot5$, $s - c = 2909\cdot5$,

$$\therefore\ L \tan \tfrac{1}{2}B = \tfrac{1}{2}\{\log(s-a) + \log(s-c) - \log(s-b) - \log s\} + 10$$
$$= \tfrac{1}{2}\{3\cdot8357223 + 3\cdot4638184 - 3\cdot0380237 - 4\cdot0354898\} + 10$$
$$= 9\cdot1130136.$$

$$L \tan 52^\circ\, 22' = 10\cdot1129282, \text{ difference for } 1' = 2613.$$

Here $$D = \tfrac{854}{2613} \text{ of } 60'' = 19\cdot6'';$$
$$\therefore\ \tfrac{1}{2}B = 52^\circ\, 22'\, 19\cdot6'',\ \therefore\ B = 104^\circ\, 44'\, 39''.$$

3. Here $s = 10549$, $s - a = 1787\cdot8$, $s - b = 2906$, $s - c = 5855\cdot2$;

$$\therefore\ L \tan \tfrac{1}{2}C = 10 + \tfrac{1}{2}\{3\cdot2523189 + 3\cdot4632956 - 3\cdot7675417 - 4\cdot0232113\}$$
$$= 9\cdot4624307.$$

$$L \tan 16^\circ\, 10' = 9\cdot4622423, \text{ difference for } 60'' = 4722.$$

Here $$D = \tfrac{1884}{4722} \text{ of } 60'' = 24'';$$
$$\therefore\ \tfrac{1}{2}C = 16^\circ\, 10'\, 24'',\ \therefore\ C = 32^\circ\, 20'\, 48''.$$

4. $$\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{541}{10401} \cot 43^\circ\, 9'\, 30'',$$

$$L \cot 43^\circ\, 9' = 10\cdot0280650, \text{ difference for } 60'' = \cdot0002532;$$
$$\therefore\ d = \frac{30''}{60''} \text{ of } \cdot0002532 = \cdot00012666;$$
$$\therefore\ L \cot 43^\circ\, 9'\, 30'' = 10\cdot0280650 - \cdot00012666 = 10\cdot0279384.$$

Now $$L \tan \tfrac{1}{2}(C-B) = \log 541 - \log 10401 + L \cot 43^\circ\, 9'\, 30''$$
$$= 2\cdot7331973 - 4\cdot0170751 + 10\cdot0279384$$
$$= 8\cdot7440606.$$

$$8\cdot7429222 = L \tan 3^\circ\, 10', \text{ difference for } 60'' = \cdot0022845;$$
$$\therefore\ D = \tfrac{11384}{22845} \text{ of } 60'' = 30'';$$
$$\therefore\ \tfrac{1}{2}(C-B) = 3^\circ\, 10'\, 30'',\ \therefore\ C - B = 6^\circ\, 21'.$$

Also $$C + B = 180^\circ - A = 180^\circ - 86^\circ\, 19' = 93^\circ\, 41';$$
$$\therefore\ 2B = 87^\circ\, 20',\ \therefore\ B = 43^\circ\, 40'.$$

5. $$L \tan \tfrac{1}{2}C - A = \log(c-a) - \log(c+a) + L \cot \tfrac{1}{2}B$$
$$= \log 1109\cdot3 - \log 2637\cdot7 + L \cot 16^\circ\, 29'$$
$$= 3\cdot0440490 - 3\cdot4212254 + 10\cdot5288593$$
$$= 10\cdot1516829.$$

$$10\cdot1515508 = L \tan 54^\circ\, 48', \text{ difference for } 60'' = \cdot0002682;$$
$$\therefore\ D = \tfrac{1321}{2682} \text{ of } 60'' = 30'';$$
$$\tfrac{1}{2}(C-A) = 54^\circ\, 48'\, 30'';\ \therefore\ C - A = 109^\circ\, 37',$$
$$C + A = 147^\circ\, 2';\ \therefore\ 2C = 256^\circ\, 39'.$$

6. $L \tan \frac{1}{2}(B-A) = \log 52629 - \log 125711 + L \cot 54° 13' 30''$;

$\therefore \log 125711 = 5{\cdot}0993698 + {\cdot}0000035 = 5{\cdot}0993733$;

$\therefore L \cot 54° 13' 30'' = 9{\cdot}8578031 - {\cdot}0001332 = 9{\cdot}8576699$.

Now $L \tan \frac{1}{2}(B-A) = \log 52629 - \log 125711 + L \cot 54° 13' 30''$

$= 4{\cdot}7212251 - 5{\cdot}0993733 + 9{\cdot}8576699$

$= 9{\cdot}4795217$.

$9{\cdot}4794319 = L \tan 16° 47'$, difference for $60''$ is 4568;

$\therefore D = \frac{898}{4568}$ of $50'' = 12''$;

$\therefore \frac{1}{2}(B-A) = 16° 47' 12''$ and $B - A = 33° 34' 24''$.

$B + A = 71° 33'$; $\therefore B = 52° 33' 42''$.

$\sin C = \sin 108° 27' = \sin 71° 33'$,

$\log c = \log 89170 + L \sin 71° 33' - L \sin 52° 33' 42''$.

$L \sin 52° 33' = 9{\cdot}8997572$, difference for $60'' = {\cdot}0000967$;

$\therefore d = \frac{42''}{60''}$ of ${\cdot}0000967 = {\cdot}0000676$;

$\therefore \log c = 4{\cdot}9502188 + 9{\cdot}9770832 - 9{\cdot}8998248 = 5{\cdot}0274772$.

$5{\cdot}0274719 = \log 106530$, difference for 10 is 408;

$\therefore d = \frac{53}{408}$ of $10 = 1{\cdot}3$;

$\therefore 5{\cdot}0274772 = \log(106530 + 1{\cdot}3) = \log 106531{\cdot}3$.

7. $\log c = \log 3720 + L \sin 62° 45' - L \sin 74° 10'$

$= 3{\cdot}5705429 + 9{\cdot}9489101 - 9{\cdot}9832019 = 3{\cdot}5362511$.

$3{\cdot}5362427 = \log 3437{\cdot}5$, difference for ${\cdot}1 = {\cdot}0000126$;

$\therefore d = \frac{84}{126} \times {\cdot}1 = {\cdot}0\dot{6}$.

$\therefore 3{\cdot}5362511 = \log 3437{\cdot}5\dot{6}$; $\therefore c = 3437{\cdot}6$ yards.

8. $A = 180 - 145° 18' = 34° 42'$.

$\log b = \log 1000 + L \sin(180° - 100° 19') - L \sin 34° 42'$

$= 3 + 9{\cdot}9929214 - 9{\cdot}7553256 = \log 1728{\cdot}2$.

9. $A = 180° - 138° 16' 20'' = 41° 43' 40''$

$= \log 9964 + L \sin 41° 43' 40'' = L \sin 15° 9'$.

$L \sin 41° 43' = 9{\cdot}8231138$, difference for $60'' = {\cdot}0001417$;

$\therefore d = \frac{40''}{60''}$ of ${\cdot}0001417 = {\cdot}0000945$;

$\therefore \log a = 3{\cdot}9984337 + 9{\cdot}8232083 - 9{\cdot}4172174$

$= 4{\cdot}4044246 = \log 25376$;

$\therefore a = 25376$.

10. $L\sin B = \log 1450 + L\sin(180° - 100° 37') - \log 6374$

$= 3{\cdot}1613680 + 9{\cdot}9925013 - 3{\cdot}8044121 = 9{\cdot}3494572.$

$9{\cdot}3493429 = L\sin 12° 55'$, difference for $60'' = {\cdot}0005505$;

$\therefore D = \frac{1143}{5505}$ of $60'' = 12''$;

$\therefore 9{\cdot}3494572 = L\sin 12° 55' 12''$.

$\sin B = \sin 12° 55' 12''$ or $\sin(180° - 12° 55' 12'')$, i.e. $167° 4' 48''$.

Since $c > b$, $\therefore C > B$, and only the less value of B is admissible;

$\therefore B = 12° 55' 12''$.

$A = 180° - 113° 32' 12'' = 66° 27' 48''$.

11. $L\sin B = \log 643 + L\sin 52° 10' - \log 872$

$= 2{\cdot}8082110 + 9{\cdot}8975162 - 2{\cdot}9405165 = 9{\cdot}7652107.$

$9{\cdot}7651911 = L\sin 35° 37'$, difference for $60'' = {\cdot}0001763$;

$\therefore D = \frac{196}{1763}$ of $60'' = 7''$;

$\therefore 9{\cdot}7652107 = L\sin 35° 37' 7'' = L\sin B$,

$\sin B = \sin 35° 37' 7''$ or $\sin(180° - 35° 37' 7'') = \sin 144° 22' 53''$.

Since $c > b$, $\therefore C > B$, and only the less value is admissible;

$\therefore B = 35° 37' 7''$; $A = 180° - B - C = 180° - 87° 47' 7'' = 92° 12' 53''$.

12. $L\sin B = \log 1000 + L\sin 76° 2' 30'' - \log 2000.$

$L\sin 76° 2' 30''$, $9{\cdot}9869670 + {\cdot}0000157 = 9{\cdot}869827$;

$\therefore L\sin B = 3 + 9{\cdot}969827 - 3{\cdot}3010300 = 9{\cdot}6859527.$

$9{\cdot}6857991 = L\sin 29° 1$, difference for $60'' = {\cdot}0002276$;

$\therefore D = \frac{1536}{2276}$ of $60'' = 40''$;

$\therefore 9{\cdot}685927 = L\sin 29° 1' 40'' = L\sin B$;

$\therefore \sin B = 29° 1' 40''$ or $\sin(180° - 29° 1' 40'') = \sin 150° 58' 20''$.

$a > b$, $\therefore A > B$, and only the less value of B is admissible,

$\therefore B = 29° 1' 40''$, $C = 180° - 105° 4' 10'' = 74° 55' 50''$.

13. $L\sin B = \log 873{\cdot}4 + L\sin 54° 23' - \log 752{\cdot}8$

$= 2{\cdot}9412132 + 9{\cdot}9100529 - 2{\cdot}8766796$

$= 9{\cdot}9745875.$

$9{\cdot}9745697 = L\sin 70° 35'$, difference for $60'' = {\cdot}0000445$;

$\therefore D = \frac{178}{445}$ of $60'' = 24''$;

$\therefore 9{\cdot}9745875 = L\sin 70° 35' 24''$ or $L\sin(180° - 70° 35' 24'')$;

i.e. $L\sin 109° 24' 36''$; $\therefore B = 70° 35' 24''$ or $109° 24' 36''$.

Both values of B are admissible; for $b > c$, i.e. $B > C$; $\therefore B$ may be obtuse or acute.

$A = 180° - 124° 58' 24'' = 55° 1' 36''$ or $180° - 163° 47' 36'' = 16° 12' 24''$.

14. $L\sin B = \log 674{\cdot}5 + L\sin 18° 21' - \log 269{\cdot}7$

$$= 2{\cdot}8289820 + 9{\cdot}4980635 - 2{\cdot}4308809 = 9{\cdot}8961646.$$

$9{\cdot}8961369 = L\sin 51° 56'$, difference for $60'' = {\cdot}0000989$;

$$\therefore D = \tfrac{227}{989} \text{ of } 60'' = 17''.$$

$$9{\cdot}8961646 = L\sin 51° 56' 17'' \text{ or } L\sin(180° - 51° 56' 17'') = L\sin 128° 3' 43'';$$

$$\therefore B = 51° 56' 17'' \text{ or } 128° 3' 43''.$$

Since $b > c$ both values of B are admissible.

15. $L\sin B = \log 7934 + L\sin 29° 11' 43'' - \log 4379$,

$$\therefore L\sin 29° 11' 43'' = 9{\cdot}6880688 + {\cdot}0001620 = 9{\cdot}6882308;$$

$$\therefore L\sin B = 3{\cdot}8994922 + 9{\cdot}6882308 - 3{\cdot}6413749 = 9{\cdot}9463481.$$

$9{\cdot}9463371 = L\sin 62° 6'$, difference for $60'' = {\cdot}0000669$;

$$\therefore D = \tfrac{110}{669} \text{ of } 60'' = 10'';$$

$$\therefore 9{\cdot}9463481 = L\sin 62° 6' 10'' \text{ or } L\sin(180° - 62° 6' 10'');$$

$$\therefore B = 62° 6' 10'' \text{ or } 117° 53' 50''.$$

Since $b > a$ both values of B are admissible.

16. Let B, C be the angles at the base of the triangle, the sides subtending them being b and c; A being the third angle,

$$\therefore \tfrac{1}{2}(B - C) = \tfrac{1}{2}(17° 48') = 8° 54'. \quad b - c = 28{\cdot}5; \; b + c = 182;$$

$$\tan 8° 54' = \tfrac{285}{1820} \cot \tfrac{1}{2}A;$$

$$\therefore L\tan 8° 54' = \log 28{\cdot}5 - \log 182 + L\cot\tfrac{1}{2}A;$$

$$\therefore L\cot\tfrac{1}{2}A = L\tan 8° 54' - \log 28{\cdot}5 + \log 182$$

$$= 9{\cdot}1947802 - 1{\cdot}4548449 + 2{\cdot}2600714$$

$$= 10{\cdot}0000077 = L\cot 45° \text{ nearly};$$

$$\therefore \tfrac{1}{2}A = 45° \text{ nearly}; \; A = 90° \text{ nearly}.$$

17. Here $$\frac{c-b}{c+b} = \frac{1}{9};$$

$$\therefore L\tan\tfrac{1}{2}(C - B) = \log 1 - \log 9 + L\cot 18° 39' 30'';$$

$$\therefore L\cot 18° 39' 30'' = 10{\cdot}4717147 - {\cdot}0002084 = 10{\cdot}4715063;$$

$$\therefore L\tan\tfrac{1}{2}(C - B) = -{\cdot}9542425 + 10{\cdot}4715063 = 9{\cdot}5172638.$$

$9{\cdot}5169097 = L\tan 18° 12'$, difference for $60'' = {\cdot}0004256$;

$$\therefore D = \tfrac{3541}{4256} \text{ of } 60'' = 50'';$$

$$\therefore \tfrac{1}{2}(C - B) = 18° 12' 50''; \quad \therefore C - B = 36° 25' 40''.$$

$$C + B = 180° - A = 180° - 37° 19' = 142° 41';$$

$$\therefore 2B = 106° 15' 20''; \quad \therefore B = 53° 7' 40'';$$

$$\therefore \log b = \log 1000 + L\sin 53° 7' 40'' - L\sin 37° 19';$$

$$\therefore L\sin 53° 8' 40'' = 9{\cdot}9030136 + {\cdot}0000672 = 9{\cdot}9030768;$$

$$\therefore \log b = 3 + 9{\cdot}9030768 - 9{\cdot}7826301$$

$$= 3{\cdot}1204467 = \log 1319{\cdot}6 \text{ nearly};$$

$$\therefore b = 1319{\cdot}6.$$

EXAMPLES. LXX. b. PAGES 220 (i), (ii).

1. When $a=b=c$; $\cos A=\frac{1}{2}$; $\cos\frac{1}{2}A=\sqrt{\left\{\frac{\frac{3}{2}\times\frac{1}{2}}{1\times 1}\right\}}=\frac{1}{2}\sqrt{3}$.

2. $\cos A=\frac{4+6-(1+\sqrt{3})^2}{4\sqrt{6}}=\frac{6-2\sqrt{3}}{4\sqrt{6}}=\frac{\sqrt{6}-\sqrt{2}}{4}=\frac{(\sqrt{3}-1)}{2\sqrt{2}}$.

$\cos B=\frac{6+4+2\sqrt{3}-4}{2\sqrt{6}(1+\sqrt{3})}=\frac{2\sqrt{3}(\sqrt{3}+1)}{2\sqrt{6}(\sqrt{3}+1)}=\frac{1}{\sqrt{2}}$.

$\cos C=\frac{4+2\sqrt{3}+4-6}{4(1+\sqrt{3})}=\frac{2(1+\sqrt{3})}{4(1+\sqrt{3})}=\frac{1}{2}$;

$$\therefore C=60^\circ,\ B=45^\circ,\ A=75^\circ.$$

3. $\cos A=\frac{4+8-4(4-2\sqrt{3})}{16\sqrt{2}}=\frac{8\sqrt{3}-4}{16\sqrt{2}}=\frac{2\sqrt{3}-1}{4\sqrt{2}}$.

$\cos B=\frac{8+16-8\sqrt{3}-16}{8\sqrt{2}(\sqrt{3}-1)}=\frac{8(1-\sqrt{3})}{8\sqrt{2}(\sqrt{3}-1)}=-\frac{1}{\sqrt{2}}$.

$\cos C=\frac{16-8\sqrt{3}+16-8}{16(\sqrt{3}-1)}=\frac{24-8\sqrt{3}}{16(\sqrt{3}-1)}=\frac{8\sqrt{3}(\sqrt{3}-1)}{16(\sqrt{3}-1)}=\frac{\sqrt{3}}{2}$;

$$\therefore C=30^\circ,\ B=135^\circ,\ A=15^\circ.$$

4. $$c^2=a^2+b^2-2ab\cos 120^\circ,$$

$\therefore 19=4+b^2+2b$; $\therefore b^2+2b+1=16$; $b+1=\pm 4$, $\therefore b=3$.

5. $a^2=16\times 7+36\times 7-24\times 7=28\times 7=(2\times 7)^2$.

6. $c=\frac{a\sin C}{\sin A}=\frac{2\sin 75^\circ}{\sin 45}=2\sqrt{2}\times\frac{1}{4}\sqrt{2}(\sqrt{3}+1)=\sqrt{3}+1$.

7. $\cos A=\frac{49+64-169}{2\times 56}=-\frac{56}{2\times 56}=-\frac{1}{2}=120^\circ$.

8. $\cos A=\frac{4+1-7}{4}=-\frac{1}{2}$, $\therefore A=120^\circ$.

9. $\cos A=\frac{a^2+b^2-a^2-ab-b^2}{2ab}=-\frac{1}{2}$; $\therefore A=120^\circ$.

10. $\cos A=\frac{9+16-25}{2\times 3\times 4}=0$; $\therefore A=90^\circ$.

$\cos C=\frac{16+25-9}{40}=\frac{32}{40}=\cdot 8$; $\therefore C=36^\circ 52'$.

11. $\cos A = \dfrac{25+36-100}{2\times5\times6} = -\tfrac{39}{60} = -\cdot65$; $\therefore A = 180^\circ - 49^\circ 33' = 130^\circ 27'$.

12. $\cos C = \dfrac{16+25-64}{2\times4\times5} = -\tfrac{23}{40} = -\cdot575 = \cos(180^\circ - 54^\circ 54')$.

13. $\tan\tfrac{1}{2}(A-B) = \dfrac{a-b}{a+b}\cot\tfrac{1}{2}C = \dfrac{\sqrt{3}-1}{\sqrt{3}(1+\sqrt{3})}\cot 15^\circ = \dfrac{1}{\sqrt{3}} = \tan 30^\circ$,

$$\left.\begin{aligned}\therefore A-B &= 60^\circ\\ A+B &= 180^\circ - 30^\circ = 150^\circ\end{aligned}\right\}; \quad \therefore A = 105^\circ;\ B = 45^\circ.$$

14. $c\sin A = a\sin C$; $\therefore (\sqrt{5}-1)\sin A = \tfrac{1}{4}(\sqrt{5}+1)(\sqrt{5}-1)$;

$$\therefore \sin A = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}.$$

$$\sin 54^\circ = \sin(3\times18^\circ) = 3\sin18^\circ - 4\sin^3 18^\circ$$
$$= \tfrac{1}{4}(\sqrt{5}-1)\{3 - \tfrac{1}{4}(6-2\sqrt{5})\} = \tfrac{1}{4}(\sqrt{5}-1)\tfrac{1}{2}(\sqrt{5}+3) = \tfrac{1}{4}(\sqrt{5}+1).$$
$$\therefore \sin A = \sin 54^\circ; \quad \therefore A = 54^\circ \text{ or } 180^\circ - 54^\circ.$$

15.
$$c^2 = a^2 + b^2 - 2ab\cos C,$$
$$\therefore 13 = a^2 + 9 + 3a;$$
$$\therefore a^2 + 3a - 4 = 0 \text{ or } (a-1)(a+4) = 0; \quad \therefore a = 1.$$
$$\sin A = \frac{a}{c}\sin C, \quad \sin B = \frac{b}{c}\sin C.$$

16.
$$C = 180^\circ - 105^\circ - 45^\circ = 30^\circ,$$
$$a = \frac{c\sin A}{\sin C} = \frac{\sqrt{2}\cos 15^\circ}{\tfrac{1}{2}} = \sqrt{3}+1,$$
$$b = 2\sqrt{2}\sin 45^\circ = 2.$$

17.
$$A = 180^\circ - 75^\circ - 30^\circ = 75^\circ = B.$$

$\therefore a = b$. Also since the triangle is isosceles $\tfrac{1}{2}c = a\cos 75^\circ$;

$$\therefore \sqrt{2} = a\frac{\sqrt{3}-1}{2\sqrt{2}}; \quad \therefore a = 2(\sqrt{3}+1).$$

18. $\sin C = \dfrac{\sqrt{75}}{\sqrt{50}}\sin 45^\circ = \dfrac{\sqrt{3}}{\sqrt{2}}\times\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{2} = \sin 60^\circ$;

$$\therefore C = 60^\circ \text{ or } 120^\circ,\ A = 75^\circ \text{ or } 15^\circ;$$
$$a = \frac{\sin 75^\circ}{\sin 60^\circ}\times\sqrt{75} = \frac{\sqrt{3}+1}{\sqrt{3}\sqrt{2}}\sqrt{75} = 5\times\frac{\sqrt{3}+1}{\sqrt{2}},$$
$$\text{or } a = \frac{\sin 15^\circ}{\sin 120^\circ}\sqrt{75} = \frac{\sqrt{3}-1}{\sqrt{3}\sqrt{2}}\sqrt{75} = 5\times\frac{\sqrt{3}-1}{\sqrt{2}}.$$

19. $\sin C = \frac{150}{50\sqrt{3}} \sin 30° = \frac{\sqrt{3}}{2} = \sin 60°$;

$$\therefore C = 60° \text{ or } 120°; \ \therefore A = 90° \text{ or } 30°.$$

When the triangle is right-angled $a^2 = b^2 + c^2 = 50^2(9+3) = (100\sqrt{3})^2$.

20. $\sin C = 1$, $\therefore C = 90°$ and the triangle is not ambiguous; see Fig. II. p. 216.

23. $C = 180° - 54°$, $\sin C = \sin 54° = \cos 36°$

$$= 1 - 2\sin^2 18° = 1 - \tfrac{1}{8}(6 - 2\sqrt{5}) = \tfrac{1}{4}(\sqrt{5}+1);$$

$$\therefore c = \tfrac{1}{4}(\sqrt{5}+1)\sqrt{6}\sqrt{2} = \tfrac{1}{2}\sqrt{3}(\sqrt{5}+1).$$

24. $$b^2 = a^2 + c^2 - 2ac\cos B;$$

$$\therefore 4 - 2\sqrt{3} = a^2 + 4 + 2\sqrt{3} - 2a(\sqrt{3}+1)(\sqrt{3}+1)\tfrac{1}{4}\sqrt{2};$$

$$\therefore a^2 - a(2+\sqrt{3})\sqrt{2} + 4\sqrt{3} = 0;$$

$$\therefore (a - 2\sqrt{2})(a - \sqrt{6}) = 0;$$

$$\therefore a = 2\sqrt{2} \text{ or } \sqrt{6};$$

$$\therefore \sin A = \frac{2\sqrt{2}}{\sqrt{3}-1} \cdot \frac{\sqrt{3}-1}{2\sqrt{2}} = 1; \ \therefore A = 90°,$$

or $$\sin A = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}; \ \therefore A = 60°.$$

When $A = 90°$, $C = 75°$; when $A = 60°$, $C = 105°$.

25. $\sin A = \tfrac{1}{4} \cdot 5 \times \tfrac{2}{5} = \tfrac{1}{2}$; $\therefore A = 30°$ or $150°$.
See Figure III. of p. 216.

26. $\sin A = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{4(1+\sqrt{3})}{4} = \frac{1}{\sqrt{2}}$; $\therefore A = 45°$ or $135°$.

When $A = 45°$, $B = 120°$; when $A = 135°$, $B = 30°$,

$$\therefore b = \frac{\sin 120°}{\sin 45°} 4(1+\sqrt{3}) = 2\sqrt{6}(1+\sqrt{3}),$$

or, $$b = \frac{\sin 30°}{\sin 45°} 4(1+\sqrt{3}) = 2\sqrt{2}(1+\sqrt{3}).$$

27. $a^2 = b^2 + c^2 - 2bc\cos A = 9\{6 + 4 + 2\sqrt{3} - \sqrt{6}(\sqrt{3}+1)\sqrt{2}\}$

$$= 9\{10 + 2\sqrt{3} - 6 - 2\sqrt{3}\} = 36;$$

$$\therefore a = 6.$$

$$\sin B = \frac{3\sqrt{6}}{6} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}; \ \therefore B = 60°;$$

$$\therefore C = 180° - 60° - 45° = 75°.$$

28. $\sin B = \frac{100}{45} \times \frac{1}{2}$ which is greater than 1.

29. $C = 75°,\ 2c = 2a \sin 75° \div \sin 45° = a(1+\sqrt{3})$.

30. $\sin A = \frac{1}{2}\sqrt{3},\ \sin B = \frac{4}{5},$

$\sin C = \sin(A+B) = \sin A \cos B + \cos A \sin B = \frac{1}{2}\sqrt{3} \times \frac{3}{5} + \frac{1}{2} \times \frac{4}{5} = \frac{1}{10}(3\sqrt{3}+4);$

$\therefore$ the required ratio which is equal to the ratio of the sines is

$$\tfrac{1}{2}\sqrt{3} \times 10\sqrt{3} : \tfrac{4}{5} \times 10\sqrt{3} : \tfrac{1}{10}(3\sqrt{3}+4) \times 10\sqrt{3} = 15 : 8\sqrt{3} : 9+4\sqrt{3}.$$

EXAMPLES. LXXI. Pages 224—226.

1. ABC is a triangle in which $A = 60°$, $C = 30$ miles, $b = 15$ miles,

$$a^2 = b^2 + c^2 - 2bc \cos A = 15^2 + 30^2 - 2 \times 15 \times 30 \cos 60°$$
$$= 225 + 900 - 450 = 675;$$
$$\therefore\ a = \sqrt{675} = 25{\cdot}98\ldots$$

2. Let A be the mouth of the harbour; let B, C be the position of the ships respectively after $1\frac{1}{2}$ hours; then $AB = \frac{3}{2}$ of $\frac{15}{2}$ miles, $AC = \frac{3}{2}$ of 10 miles and $BAC = 45°$;

$$\therefore\ BC^2 = (\tfrac{45}{4})^2 + (15)^2 - 2 \times \tfrac{45}{4} \times 15 \times \tfrac{1}{2}\sqrt{2} = 15^2\{\tfrac{9}{16} + 1 - \tfrac{3}{4}\sqrt{2}\}$$
$$= (\tfrac{15}{4})^2\{25 - 12\sqrt{2}\} = (\tfrac{15}{4})^2\{25 - 12 \times 1{\cdot}4142\ldots\}$$
$$= (\tfrac{15}{4})^2 \times (2{\cdot}84\ldots)^2;$$
$$\therefore\ BC = 10{\cdot}6 \text{ nearly.}$$

3. $AB = c,\ BC = a,\ CA = b,$

$$b = \frac{c \sin B}{\sin C} = \frac{\sin 120°}{\sin 15°}$$
$$= \frac{\sqrt{3}}{2} \div \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{3}-1}$$ Vid. E. T. Examples XXXIV.
$$= \frac{\sqrt{18}+\sqrt{6}}{3-1} = \frac{\sqrt{2}(3+\sqrt{3})}{2} = \frac{1{\cdot}4142 \times 4{\cdot}732}{2}$$
$$= {\cdot}7071 \times 4{\cdot}732 = 3{\cdot}346 \text{ miles.}$$

4. From (3) $AC = \frac{1}{2}\{\sqrt{2}(3+\sqrt{3})\} = \frac{1}{2}\{\sqrt{2}\sqrt{3}(1+\sqrt{3})\}$.

The distance of the spire from the plane of A is

$$AC \sin 15° = \frac{\sqrt{3}(\sqrt{3}+1)}{\sqrt{2}} \times \frac{(\sqrt{3}-1)}{2\sqrt{2}} = \frac{\sqrt{3}}{2} = \frac{1{\cdot}732}{2} = {\cdot}866 \text{ miles.}$$

5. Draw a quadrilateral $ABCD$, so that AC and BD be the two diagonals and $\angle ABC=120°$, $\angle BAC=45°$, $\angle ACB=15°$, $\angle DAB=90°$, $\angle DBA=45°$, $\angle DAC=45°$. In the right-angled triangle DAB, $DB=AB \sec 45°=\sqrt{2}$ miles In the triangle ABC, $BC=\frac{AB \sin A}{\sin C}=\frac{\sin 45°}{\sin 15°}=\frac{1}{\sqrt{2}} \div \frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{2}{\sqrt{3}-1}$.

Now $CD^2=DB^2+BC^2-2DB \,.\, BC \cos DBC$

$$=2+\frac{4}{(\sqrt{3}-1)^2}-\frac{4\sqrt{2}}{\sqrt{3}-1}\cos 75°$$

$$=2+\frac{4}{(\sqrt{3}-1)^2}-\frac{4\sqrt{2}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{4}{(\sqrt{3}-1)^2}; \quad [\cos 75°=\sin 15°]$$

$$\therefore CD=\frac{2}{\sqrt{3}-1}=\frac{2(\sqrt{3}+1)}{2}=\sqrt{3}+1=2{\cdot}732=2\tfrac{3}{4} \text{ miles nearly.}$$

6. Angle $CAB=36° \, 18'$, $CBA=120° \, 27'$; $\therefore ACB=23° \, 15'$.

Let $AB=c$, $BC=a$, $CAB=A$, $ACB=C$.

Now
$$\log a=\log c+L\sin A+L\operatorname{cosec} C-20$$
$$=\log 1760+L\sin 36° \, 18'+L\operatorname{cosec} 23° \, 15'-20$$
$$=3{\cdot}2455127+9{\cdot}7723314+10{\cdot}4036846-20$$
$$=3{\cdot}4215287=\log 2639{\cdot}5 \text{ nearly};$$
$$\therefore a=2639{\cdot}5.$$

7. This is Case III. of the Solution of Triangles;

$$\therefore L\tan\tfrac{1}{2}(A-B)=\log 1346-\log 4934+L\cot 29° \, 8' \, 30''$$
$$=3{\cdot}1290451-3{\cdot}6931991+10{\cdot}2537194$$
$$=9{\cdot}6895654=L\tan 26° \, 4' \, 19'';$$
$$\therefore \tfrac{1}{2}(A-B)=26° \, 4' \, 19'' \text{ and } A-B=52° \, 8' \, 38'';$$
$$A+B=180°-58° \, 17'; \quad \therefore A+B=121° \, 43';$$
$$\therefore 2A=173° \, 51' \, 38''; \quad \therefore A=86° \, 55' \, 49''.$$

8. Let the distance from A to $C=b$,

" " " A to $B=c$,

" " " C to $B=a$.

Let angle $CAB=A$, $\angle CBA=B$, $\angle ACB=C$;

$$\therefore C=180°-A-B=180°-61° \, 53'-76° \, 49'=41° \, 18',$$
$$b=c\sin B\operatorname{cosec} C;$$
$$\therefore \log b=\log c+L\sin B+L\operatorname{cosec} C-20$$
$$=\log 34920+L\sin 76° \, 49'+L\operatorname{cosec} 41° \, 18'-20$$
$$=4{\cdot}5430742+9{\cdot}9884008+10{\cdot}1804552-20$$
$$=4{\cdot}7119302=\log 51515;$$
$$\therefore b=51515 \text{ feet.}$$

9. Using same notation as in (8),

$$C = 180^\circ - 72^\circ\,34' - 81^\circ\,41' = 25^\circ\,45',$$
$$b = c \sin B \operatorname{cosec} C;$$
$$\therefore \log b = \log c + L \sin B + L \operatorname{cosec} C - 20$$
$$= \log 37412 + L \sin 81^\circ\,41' + L \operatorname{cosec} 25^\circ\,45' - 20$$
$$= 3{\cdot}5731038 + 9{\cdot}9954087 + 10{\cdot}3620649 - 20$$
$$= 3{\cdot}9305774 = \log 8522{\cdot}7;$$
$$\therefore b = 8522{\cdot}7 \text{ yards.}$$

10. The height of the one above the other is $4970 \times \sin 9^\circ\,14'$,

$$\text{i. e. } 4970 \times {\cdot}1604555 = 797{\cdot}5 \text{ yards.}$$

11. In the triangle ABC, let A represent the point of intersection of the railways; at the end of an hour let the first train be at B and the second at C then BC represents their distance apart; $\therefore$ $BC = 35$ mls., $AB = 40$ mls., $AC = x$ mls., $a^2 = b^2 + c^2 - 2bc \cos A$;

$$\therefore (35)^2 = x^2 + (40)^2 + 2 \times x \times 40 \cos 60^\circ;$$
$$\therefore x^2 - 40x - 25 = 0.$$

From this quadratic $x = 25$ mls. or 15 mls.

N.B. This is an instance of the ambiguous case.

12. Using same notation as in (8),

$$c^2 = a^2 + b^2 - 2ab \cos C = 8^2 + 10^2 - 2 \times 8 \times 10 \times \tfrac{1}{2} = 84;$$
$$\therefore c = \sqrt{84} = 9{\cdot}165.$$

13. The height of B above A is the difference of the elevation of C above B and the elevation of C above A;

$$\text{i.e. } 10 \sin 8^\circ - 8 \sin 2^\circ\,48'\,24'' = 10 \times {\cdot}1391731 - 8 \times {\cdot}0489664$$
$$= 1{\cdot}391731 - {\cdot}3915312 = 1 \text{ nearly.}$$

14. The sine of the angle which the tunnel makes with the horizon is

$$\frac{\text{the height of } A \text{ above } B}{\text{the distance between } A \text{ and } B} = \frac{1}{9{\cdot}165} = {\cdot}1091$$
$$= \sin 6^\circ\,16'; \ \therefore \text{ the angle is } 6^\circ\,16';$$
$$\tan \tfrac{1}{2}(B - A) = \frac{10 - 8}{10 + 8} \cot 30^\circ = \frac{1}{9} \times \sqrt{3} = {\cdot}192450;$$
$$\therefore B - A = 2 \times (10^\circ\,53'\,36'') = 21^\circ\,47'\,12'',\ B + A = 120^\circ;$$
$$\therefore A = \tfrac{1}{2}(98^\circ\,12'\,48'') = 49^\circ\,6'\,24''.$$

15. Using the notation of (8) angle $C = 180° - A - B$

$$= 180° - 38° 19' - 132° 42' = 8° 59'.$$

In the triangle ABC $a = c \sin A \operatorname{cosec} C$;

$$\therefore \log a = \log c + L \sin A + L \operatorname{cosec} C - 20$$
$$= \log 1760 + L \sin 38° 19' + L \operatorname{cosec} 8° 59' - 20$$
$$= 3{\cdot}2455127 + 9{\cdot}7923968 + 10{\cdot}8064659 - 20$$
$$= 3{\cdot}844375.$$

If h is height of mountain above B, $h = a \sin 10° 15'$;

$$\therefore \log h = \log a + L \sin 10° 15' - 10 = 3{\cdot}844375 + 9{\cdot}2502822 - 10$$
$$= 3{\cdot}0946576 = \log 1243{\cdot}5;$$
$$\therefore h = 1243{\cdot}5 \text{ yards.}$$

16. Using the same notation as in (8),

$$C = 180° - A - B = 180° - 65° 37' - 53° 4' = 61° 19'.$$

In the triangle ABC, $b = c \sin B \operatorname{cosec} C$;

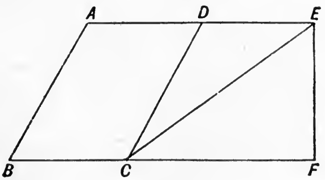
$$\therefore \log b = \log c + L \sin B + L \operatorname{cosec} C - 20$$
$$= \log 1000 + L \sin 53° 4' + L \operatorname{cosec} 61° 19' - 20$$
$$= 3 + 9{\cdot}9027289 + 10{\cdot}0568589 - 20 = 2{\cdot}9595878.$$

If p be the perpendicular breadth of the river,

$$p = b \sin A;$$
$$\therefore \log p = \log b + L \sin A - 10$$
$$= 2{\cdot}9595878 + 9{\cdot}9594248 - 10$$
$$= 2{\cdot}9190126 = \log 829{\cdot}87;$$
$$\therefore p = 829{\cdot}87.$$

EXAMPLES. LXXII. Pages 227, 228, 229.

1. Let A be the position of the balloon when it was first observed, and B the position of the man, so that the angle $ABC = 60°$; produce AE horizontally to represent 1 mile, the distance travelled by the balloon in 10



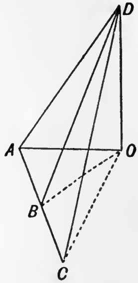
minutes. Let $BC = \frac{1}{2} AE$ drawn parallel to it represent half a mile, the distance travelled by the man in 10 minutes. Produce BC to F; join EC; then, according to the question, angle $ECF = 30°$. Draw CD parallel to BA and EF perpendicular to BF. EF represents the height of the balloon above the road. Now $ABCD$ is a parallelogram; $\therefore$ $AD = BC = \frac{1}{2}$ mile; $\therefore$ DE also $= \frac{1}{2}$ mile; $\angle DCF = \angle ABC = 60°$ and angle $ECF = 30°$; $\therefore$ $\angle DCE = 30°$; $\angle DEC = \angle ECF = 30°$; $\therefore$ $\angle DCE = \angle DEC = 30°$; $\therefore$ $\angle CDE = 120°$ and

$$CD = DE = \tfrac{1}{2} \text{ mile}.$$

$$\frac{CE}{DE} = \frac{\sin 120°}{\sin 30°}; \therefore CE = \frac{\sin 60°}{\sin 30°} \times \tfrac{1}{2} \text{ mile} = \frac{\sqrt{3}}{2} \text{ miles};$$

$EF = CE \sin ECF = \frac{1}{2}\sqrt{3} \sin 30° \text{ mi.} = \frac{1}{4}\sqrt{3} \text{ m.} = \frac{1}{4}\sqrt{3} \times 1760 \text{ yds.} = 440\sqrt{3}$ yds.

2. In the figure let O represent the foot of the tower, and $OD = x$ its height; $\angle DAO = 60°$, $\angle OAB = 90°$,



$$\angle DBO = 45°, \angle DCO = 30°;$$

$$\frac{OD}{AO} = \tan 60° = \sqrt{3}; \therefore OD = AO\sqrt{3} = x; \therefore AO = \frac{x}{\sqrt{3}};$$

$$\frac{OD}{BO} = \tan 45° = 1; \therefore OD = BO = x;$$

$$\frac{OD}{CO} = \tan 30° = \frac{1}{\sqrt{3}}; \therefore CO = OD\sqrt{3} = x\sqrt{3}.$$

Since $OAB = 90°$; $\therefore$ $AB^2 = OB^2 - OA^2$;

$$\therefore AB^2 = x^2 - \frac{x^2}{3} = \frac{2x^2}{3}; \therefore AB = \frac{x\sqrt{6}}{3}.$$

Also $AC^2 = OC^2 - OA^2 = 3x^2 - \frac{x}{3} = \frac{8x^2}{3}$, $\therefore$ $AC = \frac{2x\sqrt{6}}{3}$;

$$\therefore AC = 2AB, \text{ i.e. } AC = AB + BC; \therefore AB = BC.$$

3. Let x be the no. of miles the balloon travels per hour.

Then $\frac{x}{3}$ " " " " " in 20 mins.

Using figure E. T. p. 62, let Q be the position of balloon after travelling 1 mile, and P its position 20 minutes after; $\therefore$ $MQ = 1$ mile, $QP = \frac{x}{3}$ miles.

$$\angle QOM = 35° 20', \angle POM = 55° 40'.$$

$$\frac{OM}{MQ} = \cot QOM = \cot 35° 20'; \therefore OM = \cot 35° 20' \text{ miles},$$

$$MP = MO \tan 55° 40';$$

$$\therefore 1 + \frac{x}{3} = \cot 35° 20' \tan 55° 40', \frac{x}{3} = \cot 35° 20' \tan 55° 40' - 1$$

$$= \frac{\sin 55° 40' \cos 35° 20' - \cos 55° 40' \sin 35° 20'}{\sin 35° 20' \cos 55° 40'}$$

$$= \frac{\sin (55° 40' - 35° 20')}{\sin 35° 20' \cos 55° 40'};$$

$$\therefore x = 3 (\sin 20° 20') (\sec 55° 40') (\operatorname{cosec} 35° 20').$$

4. Draw a figure of the form indicated in (2) and retaining the same letters $\angle DAO=30°$, $\angle OAB=90°$, $\angle DBO=45°$, $AB=a$ feet.

$$\frac{OD}{AO}=\tan 30°=\frac{1}{\sqrt{3}};\quad \therefore\ AO=OD\sqrt{3}=x\sqrt{3},$$

$$OB^2=AB^2+OA^2=a^2+3x^2;\quad \therefore BO=\sqrt{(a^2+3x^2)},$$

$$\frac{OD}{BO}=\frac{x}{\sqrt{(a^2+3x^2)}}=\tan 18°=\frac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}}.$$

$$\therefore\ \frac{x^2}{a^2+3x^2}=\frac{3-\sqrt{5}}{5+\sqrt{5}};$$

$$\therefore\ 4x^2=\frac{a^2(3-\sqrt{5})}{\sqrt{5}-1}=\frac{a^2(3-\sqrt{5})(3+\sqrt{5})}{(\sqrt{5}-1)(\sqrt{5}+3)}=\frac{2a^2}{1+\sqrt{5}}.$$

$$x^2=\frac{a^2}{2(1+\sqrt{5})};\quad \therefore\ x=\frac{a}{\sqrt{\{2(1+\sqrt{5})\}}}.$$

5. Using the figure and letters of (2) $OD=x$ the height of the steeple, $\angle DAO=45°$, $\angle OAB=90°$, $\angle DBO=15°$.

$$AB=a\text{ ft.},\ \frac{OD}{AO}=\tan 45°=1,\quad \therefore\ OD=OA=x,$$

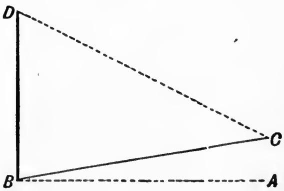
$$OB^2=AB^2+OA^2=a^2+x^2;\quad \therefore\ OB=\sqrt{(a^2+x^2)}.$$

$$\frac{OD}{OB}=\frac{x}{\sqrt{(a^2+x^2)}}=\tan 15°=\frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

$$\therefore\ 2x^2\sqrt{3}=2a^2-a^2\sqrt{3},\quad 4x^2\sqrt{3}=a^2(4-2\sqrt{3});$$

$$\therefore\ 2x3^{\frac{1}{4}}=a(3^{\frac{1}{2}}-1),\quad x=\frac{a}{2}(3^{\frac{1}{4}}-3^{-\frac{1}{4}}).$$

6. Let BC be the inclined plane; BD the tower at its foot; $\angle CBA=9°$, $\therefore \angle CBD=(90°-9°)=81°$; let BC be length of line$=100$ ft., $\angle DCB=54°$; $\therefore\ \angle BDC=45°$.



$$BD=\frac{BC\sin C}{\sin D}=\frac{100\times\sin 54°}{\sin 45°}$$
$$=100\times\tfrac{1}{4}(1+\sqrt{5})\times\sqrt{2}=25(\sqrt{2}+\sqrt{10})$$
$$=25(1{\cdot}4142+3{\cdot}1628)$$
$$=25\times 4{\cdot}577=114{\cdot}4\text{ ft.}$$

7. Let $\angle ABC=47°$; $BD=1000$ ft., $\angle DBC=32°$,

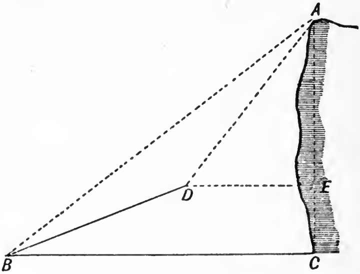
$$\angle ABD=\angle ABC-\angle DBC=15°;\quad \angle ADC=77°.$$

Since ABC is a right-angled triangle and $\angle ABC=47°$;

$$\therefore\ \angle BAC=90°-47°=43°,$$

and since ADE is a right-angled triangle and $\angle ADE = 77°$;

$$\therefore \angle DAE = 90° - 77° = 13°; \quad \therefore \angle BAD = 43° - 13° = 30°,$$

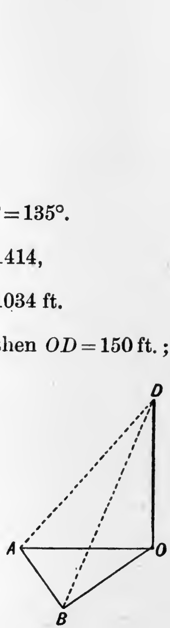


$$\angle ADB = 180° - \angle ABD - \angle BAD = 180° - 15° - 30° = 135°.$$

$$AB = \frac{BD \sin ADB}{\sin BAD} = \frac{1000 \times \sin 135°}{\sin 30°} = 1000\sqrt{2} = 1414,$$

$$AC = AB \sin ABC = AB \sin 47° = 1414 \times \cdot 73135 = 1034 \text{ ft.}$$

8. Let OD be the chimney; OAB the triangular area; then $OD = 150$ ft.; $\angle BDO = 30°$; $\angle ADO = 45°$; $\angle BDA = 30°$.



$$\frac{BO}{OD} = \tan 30° = \frac{1}{\sqrt{3}};$$

$$\therefore BO = \frac{OD}{\sqrt{3}} = \frac{150}{\sqrt{3}} = 50\sqrt{3} = 86 \cdot 6 \text{ ft.}$$

$$\frac{AO}{DO} = \tan 45° = 1; \quad \therefore AO = DO = 150 \text{ ft.}$$

$$AB^2 = AD^2 + BD^2 - 2AD \,.\, BD \cos 30°.$$

Now $AD = OD \operatorname{cosec} OAD = 150 \operatorname{cosec} 45° = 150\sqrt{2}$,

$BD = OD \operatorname{cosec} OBD = 150 \operatorname{cosec} 60° = 100\sqrt{3}$;

$$\therefore AB^2 = (150\sqrt{2})^2 + (100\sqrt{3})^2 - 2 \times 150\sqrt{2} \times 100\sqrt{3} \times \frac{\sqrt{3}}{2}$$

$$= 45000 + 30000 - 45000\sqrt{2} = 75000 - 45000 \times 1 \cdot 414 = 11370;$$

$$\therefore AB = \sqrt{11370} = 106 \cdot 6.$$

9. Figure of E.T. p. 62. Let $PQ = h$ (i.e. height of flagstaff), $QM = x$ (i.e. height of tower), $\angle POM = \alpha$, $\angle QOM = \beta$, $OM = y$, i.e. distance of the point of observation from the foot of the tower.

$$\frac{x+h}{y} = \tan\alpha, \quad \frac{x}{y} = \tan\beta,$$

$$\frac{x+h}{y} - \frac{x}{y} = \tan\alpha - \tan\beta; \quad \therefore y = \frac{h}{\tan\alpha - \tan\beta}.$$

Now $x = y\tan\beta = \dfrac{h\tan\beta}{\tan\alpha - \tan\beta} = \dfrac{h\sin\beta\cos\alpha}{\sin\alpha\cos\beta - \sin\beta\cos\alpha}$

$$= \frac{h\sin\beta\cos\alpha}{\sin(\alpha-\beta)}.$$

10. Figure of E. T. p. 62. Let $OM = h$ height of the cliff, $OPM = \beta$, $OQM = \alpha$, $\therefore PQ$ is the distance between the ships.

$$PQ = PM - QM,$$

$$\frac{PQ}{OM} = \frac{PM}{OM} - \frac{QM}{OM},$$

$$PQ = h(\cot\beta - \cot\alpha) \text{ feet.}$$

11. Draw a figure like that indicated in Ex. (8), putting the A, O, B in similar positions.

Let B be due S. of O and A due W. of B.

Then $$OB = h\cot\alpha,$$

$$OA^2 = BO^2 + AB^2 = h^2\cot^2\alpha + d^2;$$

$$\frac{OD^2}{OA^2} = \frac{h^2}{h^2\cot^2\alpha + d^2} = \tan^2\beta.$$

$$\therefore h^2(1 - \cot^2\alpha\tan^2\beta) = d^2\tan^2\beta,$$

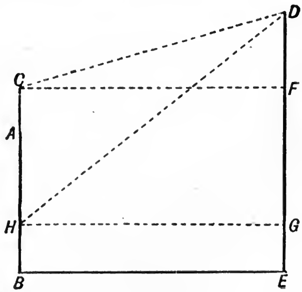
$$h^2 = \frac{d^2}{\cot^2\beta - \cot^2\alpha}.$$

$$\therefore h = \frac{d}{\sqrt{(\cot^2\beta - \cot^2\alpha)}} = \frac{d\sin\alpha\sin\beta}{\sqrt{\{\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta\}}}$$

$$= \frac{d\sin\alpha\sin\beta}{\sqrt{\{(\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)\}}}$$

$$= \frac{d\sin\alpha\sin\beta}{\sqrt{\{\sin(\alpha+\beta)\,.\,\sin(\alpha-\beta)\}}}.$$

12. Let AB be the height of the wall $= h$; AC the height of the man $= BH$;
$\angle DCF = \alpha$, $\angle DHG = \beta$, $\angle BCD = 90° + \alpha$;
$\angle DHC = 90° - \beta$, $\angle CDH = \beta - \alpha$.



Since $CA = HB$, $CH = AB = h$.

In the parallelogram BG, $GE = HB$;

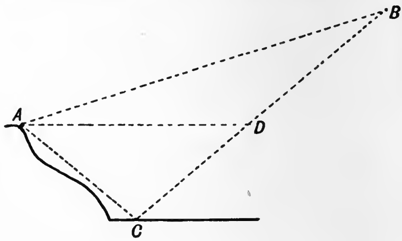
$\therefore ED - HB = ED - GE$

$= DG$ the required difference.

Now $DH = \dfrac{CH\sin(90° + \alpha)}{\sin(\beta - \alpha)} = \dfrac{h\cos\alpha}{\sin(\beta-\alpha)}$,

$$\therefore DG = DH\sin\beta = \frac{h\cos\alpha\sin\beta}{\sin(\beta-\alpha)}.$$

13. Let A be the point of observation, B the cloud, C the point on the surface of the lake from which the light is reflected from the cloud. Since from Optics the angle of incidence = the angle of reflexion, $\therefore$ BC and AC are equally inclined to the surface of the lake and $\therefore$ also to the horizontal line AD drawn through A.



$\angle BAC = \alpha + \beta$, $\angle ACB = 180° - 2\beta$,

$\therefore \angle ABC$
$= 180 - (\alpha + \beta + 180° - 2\beta)$
$= \beta - \alpha$;

$$CB = \frac{AC \sin BAC}{\sin ABC} = \frac{AC \cdot \sin(\beta + \alpha)}{\sin(\beta - \alpha)}.$$

Height of cloud $= CB \sin\beta$

$$= \frac{AC \cdot \sin\beta \cdot \sin(\beta + \alpha)}{\sin(\beta - \alpha)} = \frac{h \sin(\beta + \alpha)}{\sin(\beta - \alpha)}.$$

14. Let A be the position of the spire which is nearer to the road and B that of the spire which is farther from the road.

The height of A above the road can be found as in (13) to be $\dfrac{h \sin(\beta + \alpha)}{\sin(\beta - \alpha)}$; in the same manner the height of B above the road is $\dfrac{h \sin(\gamma + \alpha)}{\sin(\gamma - \alpha)}$. The difference in height between B and A

$$= h \left\{ \frac{\sin(\gamma + \alpha)}{\sin(\gamma - \alpha)} - \frac{\sin(\beta + \alpha)}{\sin(\beta - \alpha)} \right\}$$

$$= h \frac{\sin(\gamma + \alpha)\sin(\beta - \alpha) - \sin(\beta + \alpha)\sin(\gamma - \alpha)}{\sin(\gamma - \alpha)\sin(\beta - \alpha)}$$

$$= h \frac{\cos(\gamma - \beta + 2\alpha) - \cos(\beta - \gamma + 2\alpha)}{2\sin(\gamma - \alpha)\sin(\beta - \alpha)}$$

$$= h \cdot \frac{\sin 2\alpha \sin(\beta - \gamma)}{\sin(\gamma - \alpha)\sin(\beta - \alpha)}.$$

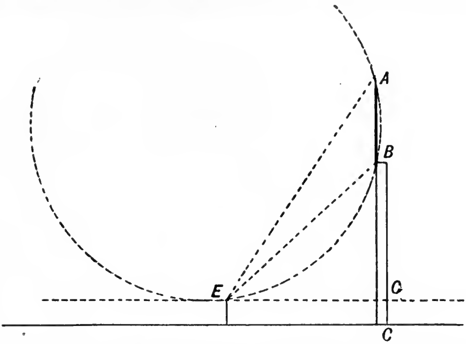
Now $\dfrac{\text{the difference in height of the spires}}{\text{the horizontal distance of the spires}} = \tan\alpha$;

$\therefore$ The horizontal distance = the difference in height $\times \cot\alpha$

$$= \frac{h \sin 2\alpha \sin(\beta - \gamma)}{\sin(\gamma - \alpha)\sin(\beta - \alpha)} \cdot \frac{\cos\alpha}{\sin\alpha} = \frac{2h \sin\alpha\cos\alpha\sin(\beta - \gamma)}{\sin(\gamma - \alpha)\sin(\beta - \alpha)} \cdot \frac{\cos\alpha}{\sin\alpha}$$

$$= 2h\cos^2\alpha \sin(\beta - \gamma) \cdot \operatorname{cosec}(\beta - \alpha) \cdot \operatorname{cosec}(\gamma - \alpha).$$

15. From note E. T. Answers, the point E in an unlimited straight line



GE, at which a finite straight line AB subtends the greatest angle, is on the circumference of the circle of which GE is a tangent and ABG a secant.

Let $\angle BEG=\gamma$ and $GE=c$;

$$\therefore\ AG=c\tan(\alpha+\gamma),\quad BG=c\tan\gamma.$$

From Euclid III. 32, $\angle BAE=\angle BEG=\gamma$; and $\therefore$ in the right-angled triangle ACE $\alpha+2\gamma=\frac{1}{2}\pi$, $\therefore\ \gamma=\frac{1}{4}\pi-\frac{1}{2}\alpha$.

$$\begin{aligned}AB=AG-BG&=c\tan(\alpha+\gamma)-c\tan\gamma\\&=c\left\{\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha\right)-\tan\left(\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha\right)\right\}\\&=c\left\{\frac{1+\tan\frac{1}{2}\alpha}{1-\tan\frac{1}{2}\alpha}-\frac{1-\tan\frac{1}{2}\alpha}{1+\tan\frac{1}{2}\alpha}\right\}=2c\cdot\frac{2\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha}\\&=2c\tan\alpha.\end{aligned}$$

The height of the pillar $=b+BG=b+c\tan\gamma=b+c\tan\left(\frac{1}{4}\pi-\frac{1}{2}\alpha\right)$.

Or, let O be the centre of the circle of which GE is tangent and ABG secant; draw OM perpendicular to AB; $\therefore\ AM=MB$ and $AB=2AM$. Draw the straight line OE and (Euclid III. 18) OEG is a right angle; $\therefore\ MOEC$ is a rectangle and $MO=GE=c$. Also angle $AOM=$ angle $AEB=\alpha$ (Eucl. III. 20); $\therefore\ AM=MO\tan\alpha=c\tan\alpha$; $\therefore\ AB=2c\tan\alpha$.

Also $\quad GB\cdot GA=GE^2$, i.e. $GB(GB+AB)=GE^2$.

Let $\quad GB=a$; $\therefore\ a(a+2c\tan\alpha)=c^2$; $\therefore\ a^2+2ac\tan\alpha-c^2=0$.

From this quadratic $\quad a=c\tan\left(\frac{1}{4}\pi-\frac{1}{2}\alpha\right)$.

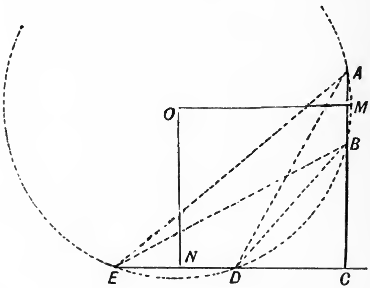
The height of the pillar $\quad =b+a=b+c\tan\left(\frac{1}{4}\pi-\frac{1}{2}\alpha\right)$.

16. In the above fig. let A denote the object which is further removed from the road, B that which is nearer to the road E, the point where AB subtends the greatest angle, G the second point of observation. From note E. T. p. 272 it is known that the point E is on the circumference of a circle of which GE is tangent and ABG secant [cf. (15) above]. Let $\angle BEG=\gamma$ and $GE=a$;

$$\therefore\ AG=c\tan(\alpha+\gamma),\ BG=c\tan\gamma.$$

From Euclid III. 32 $\angle BAE = \angle BEG = \gamma$; $\therefore$ in the right-angled triangle AGE
$$\alpha + 2\gamma = \tfrac{1}{2}\pi; \quad \therefore \gamma = \tfrac{1}{4}\pi - \tfrac{1}{2}\alpha,$$
$$AB = AG - BG = c\tan(\alpha+\gamma) - c\tan\gamma$$
$$= a\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\alpha) - a\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}\alpha) = 2a\tan\alpha \ldots \text{cf. } \mathbf{15}.$$

17. Let AB be the flagstaff and BC the tower, CE being the horizontal plane. E the first point of observation and D the second. Since the angle



AEB is equal to the angle ADB, the points A, B, D, E lie on the circumference of a circle. Euclid III. 21. The centre O of the circle is in the straight line bisecting AB at right angles in M and in the straight line bisecting DE at right angles in N; $\therefore$ $MONC$ is a rectangle and $MO = CN$. The angle AOM is half of the angle AOB; hence by Euclid III. 20 the angle AOM is equal to the angle AEB; i.e. $\angle AOM = \alpha$,
$$MO = CN = CD + DN = h - 2k + k = h - k,$$
$$\frac{AM}{MO} = \tan AOM; \quad \therefore AM = MO\tan AOM = (h-k)\tan\alpha;$$
$$\therefore AB = 2AM = 2(h-k)\tan\alpha.$$

18. By comparing (15) and (16) above and note E. T. p. 272, it is known that E the first point of observation is on the circumference of a circle of which GE is tangent and ABG secant. The figure is like that on p. 175 except that AGE is not a right angle. G is second point of observation and the straight line ABG makes with GE the angle $AGE = \beta$.

Let angle $BEG = \gamma$; from Euclid III. 32 $\angle BEG = \angle EAG$; $\therefore$ angle $EAG = \gamma$, and angle $ABE = 180° - \gamma - \alpha$. But angle $EBG = 180° - \beta - \gamma$; $\therefore$ $\angle ABE = \beta + \gamma$; $\therefore$ $180° - \gamma - \alpha = \beta + \gamma$; $\therefore$ $2\gamma = 180° - \alpha - \beta$.

Now $$\frac{BE}{EG} = \frac{\sin\beta}{\sin(\beta+\gamma)}; \quad \therefore BE = \frac{a\sin\beta}{\sin(\beta+\gamma)}; \quad \frac{AB}{BE} = \frac{\sin\alpha}{\sin\gamma};$$
$$\therefore AB = \frac{a\sin\alpha\sin\beta}{\sin\gamma\sin(\beta+\gamma)} = \frac{2a\sin\alpha\sin\beta}{\cos\beta - \cos(2\gamma+\beta)}$$
$$= \frac{2a\sin\alpha\sin\beta}{\cos\beta - \cos(180° - \alpha)} = \frac{2a\sin\alpha\sin\beta}{\cos\alpha + \cos\beta} \text{ yds.}$$

EXAMPLES. LXXIII. Pages 239, 240.

1. $\Delta = \frac{ab}{2} \sin C = \frac{bc}{2} \sin A = \frac{ca}{2} \sin B.$

(i) $\Delta = \frac{10 \times 4}{2} \sin 30° = \frac{10 \times 4}{2 \times 2} = 10$ sq. ft.

(ii) $\Delta = \frac{5 \times 20}{2} \sin 60° = 25\sqrt{3} = 25 \times 1{\cdot}732 = 43{\cdot}3$ sq. in.

(iii) $\Delta = \frac{66\frac{2}{3} \times 15}{2} \sin 17° 14' = \frac{200 \times 15}{3 \times 2} \times {\cdot}29626$

$= 500 \times {\cdot}29626 = 148{\cdot}13$ sq. yds.

(iv) $\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}}$; $s=21$, $s-a=8$, $s-b=7$, $s-c=6$

$= \sqrt{\{21 \times 8 \times 7 \times 6\}} = 84$ sq. chains.

(v) $\Delta = \frac{ap}{2} = \frac{10 \times 20}{2} = 100$ sq. feet.

(vi) $s=1017$, $s-a=392$, $s-b=512$, $s-c=113$;

$\therefore \Delta \sqrt{\{1017 \times 392 \times 512 \times 113\}} = (113 \times 7 \times 8 \times 8 \times 3) = 151872$ sq. yds.

2. $s=21$, $s-a=8$, $s-b=7$, $s-c=6$, $\Delta = \sqrt{\{21 \times 8 \times 7 \times 6\}} = 84.$

Radius of inscribed circle $= r = \frac{\Delta}{s} = \frac{84}{21} = 4,$

$$r_1 = \frac{\Delta}{s-a} = \frac{84}{8} = 10{\cdot}5, \quad r_2 = \frac{\Delta}{s-b} = \frac{84}{7} = 12,$$

$$r_3 = \frac{\Delta}{s-c} = \frac{84}{6} = 14.$$

3. $$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The circle in which the first triangle can be inscribed is of diameter $\frac{2}{\sin 60°} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$. Diameter of circle in which the second triangle can be inscribed is $\frac{\frac{2}{3}\sqrt{3}}{\sin 30°} = \frac{4\sqrt{3}}{3}$; i.e. the diameters of the circles are equal; therefore the circles themselves are equal, and the two triangles can be both inscribed in a circle with diameter $\frac{4}{3}\sqrt{3}$.

4. $$2R = \frac{a}{\sin A} = \frac{abc}{bc \sin A} = \frac{abc}{2S}; \quad \therefore R = \frac{abc}{4S}.$$

The area of triangle of (2) is 84 ((i), (iv)) sq. ft.;

$$\therefore R = \frac{13 \times 14 \times 15}{4 \times 84} = \frac{65}{8} = 8\tfrac{1}{8} \text{ feet}.$$

5. When the circle is the same the radius is constant; and when the perimeters are all equal, $2s$ is constant and therefore s is constant; $\therefore$ r, and s in the formula $r=\frac{S}{s}$, are constant; $\therefore$ S is also constant; i. e. the areas of the triangles are all equal.

6. $$\frac{\sin B}{b}=\frac{\sin A}{a};\ \therefore\ \sin B=\frac{b\sin A}{a}.$$

Since $b>a$; $\therefore$ $B>A$ and only the smaller value of B is admissible;

$$\therefore\ \sin B=\sqrt{2}\times\frac{\sqrt{3}}{2}\div\sqrt{3}=\frac{1}{\sqrt{2}};\ \therefore\ B=45°;\ A=60°;$$

$$\therefore\ C=180°-45°-60°=75°.$$

Area of the triangle $=\frac{ab\sin C}{2}=\frac{1}{2}\times\sqrt{3}\times\sqrt{2}\sin 75°$

$$=\sqrt{\tfrac{3}{2}}\cos 15°=\sqrt{\tfrac{3}{2}}\cdot\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{3}\,(\sqrt{3}+1)}{4}=\tfrac{1}{4}(3+\sqrt{3}).$$

7. (i) $\Delta=\frac{ab\sin C}{2}$. Now $\frac{c}{\sin C}=2R$; $\therefore$ $\sin C=\frac{c}{2R}$;

$$\therefore\ \Delta=\frac{abc}{4R};\ \text{or, from (4)},\ R=\frac{abc}{4S};\ \therefore\ S=\frac{abc}{4R}.$$

(ii) $\frac{a}{\sin A}=\frac{b}{\sin B}=2R$; $\therefore$ $a=2R\sin A$, $b=2R\sin B$.

Now $$\Delta=\frac{ab\sin C}{2}=\frac{4R^2\sin A\sin B\sin C}{2}=2R^2\sin A\sin B\sin C.$$

(iii) $r=\frac{S}{s}$; $\therefore$ $S=rs=\Delta$.

(iv) $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$.

From (iii) $S=rs=r\tfrac{1}{2}(a+b+c)=rR(\sin A+\sin B+\sin C)$.

(v) $$\Delta=\tfrac{1}{2}\cdot ab\sin C=\tfrac{1}{2}\cdot\frac{a^2b\sin C}{a}=\tfrac{1}{2}\cdot\frac{a^2\,2R\sin B\sin C}{2R\sin A}$$

$$=\tfrac{1}{2}\cdot\frac{a^2\sin B\sin C}{\sin A}=\tfrac{1}{2}a^2\sin B\sin C\operatorname{cosec} A.$$

(vi) $S=rs=r\tfrac{1}{2}(a+b+c)=rR(\sin A+\sin B+\sin C)$
$=4rR\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$. E. T. p. 192.

Now $$2R=\frac{a}{\sin A}=\frac{a}{2\sin\frac{1}{2}A\cos\frac{1}{2}A};$$

$$\therefore\ S=\frac{2ar\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}=ra\operatorname{cosec}\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

(vii) $$(rr_1r_2r_3)^{\frac{1}{2}}=\sqrt{\left\{\frac{S}{s}\cdot\frac{S}{s-a}\cdot\frac{S}{s-b}\cdot\frac{S}{s-c}\right\}}$$

$$=\frac{S^2}{\sqrt{\{s(s-a)(s-b)(s-c)\}}}=\frac{S^2}{S}=S.$$

(viii) $$= \tfrac{1}{2} ab \sin C = \frac{\tfrac{1}{2}(a^2 - b^2) ab \sin C}{a^2 - b^2}$$
$$= \frac{\tfrac{1}{2}(a^2 - b^2) 4R^2 \sin A \sin B \sin C}{4R^2 (\sin^2 A - \sin^2 B)}$$
$$= \frac{\tfrac{1}{2}(a^2 - b^2) \sin A \sin B \sin C}{\sin^2 A - \sin^2 B} = \frac{\tfrac{1}{2}(a^2 - b^2) \sin A \sin B \sin C}{\sin (A+B) \sin (A-B)}$$
$$= \frac{\tfrac{1}{2}(a^2 - b^2) \sin A \sin B \sin C}{\sin C \sin (A-B)}$$
$$= \tfrac{1}{2}(a^2 - b^2) \sin A \sin B \operatorname{cosec} (A-B).$$

8. From (4) $$R = \frac{abc}{4S} \text{ and } \frac{S}{s} = r;$$
$$\therefore Rr = \frac{abc}{4s} \text{ and } 6Rr = \frac{3abc}{2s}.$$

Since a, b, c are in A. P., $a + c = 2b$; $\therefore a + b + c = 3b = 2s$;
$$\therefore \frac{3abc}{2s} = \frac{3abc}{3b} = ac = 6rR.$$

9. $\Delta = \frac{ab}{2} \sin C$; let a and b be constants being equal to 50 and 60 feet respectively; the area will therefore be greatest when $\sin C$ has its greatest value, i.e. when $\sin C = 1$; $\therefore$ greatest area $= \tfrac{1}{2} \times 50 \times 60 = 1500$ sq. ft.

10. In the isosceles triangle ABC let b and c be the equal sides; $2a$ the perpendicular and $2x$ the base, the perpendicular will bisect the base at right angles;
$$\therefore b = c = \sqrt{\{(2x)^2 + x^2\}} = x\sqrt{5}; \quad \therefore \sin B = \frac{2x}{x\sqrt{5}} = \frac{2}{\sqrt{5}};$$
now $$2R = \frac{b}{\sin B} = x\sqrt{5} \div \frac{2}{\sqrt{5}} = \frac{5}{2}x; \quad \therefore R = \frac{5}{4} \cdot x = \frac{5}{8} \cdot 2x = \frac{5}{8} \text{ of the base.}$$

11. $R(\sin A + \sin B + \sin C)$
$$= \tfrac{1}{2}(2R \sin A + 2R \sin B + 2R \sin C) = \tfrac{1}{2}(a + b + c) = s.$$

12. $4R^2 (\cos A + \cos B \cdot \cos C)$
$$= 4R^2 \{-\cos (B + C) + \cos B \cos C\}$$
$$= 4R^2 \sin B \sin C = 2R \sin B \cdot 2R \sin C = bc.$$

13. $$2R = \frac{c}{\sin C} = \frac{c}{\sin 90^\circ} = c,$$
$$r = \frac{\tfrac{1}{2} ab \sin C}{s} = \frac{ab}{2} \div \frac{a+b+c}{2} = \frac{ab}{a+b+c}.$$
$$\therefore 2R + 2r = c + \frac{2ab}{a+b+c} = \frac{ac + bc + c^2 + 2ab}{a+b+c}$$
[since $C = 90^\circ$; $\therefore c^2 = a^2 + b^2$]
$$= \frac{ac + bc + a^2 + b^2 + 2ab}{a+b+c} = \frac{c(a+b) + (a+b)^2}{a+b+c} = \frac{(a+b)(a+b+c)}{a+b+c} = a + b.$$

14. $r_2r_3+r_3r_1+r_1r_2$

$$=\frac{S}{s-b}\cdot\frac{S}{s-c}+\frac{S}{s-c}\cdot\frac{S}{s-a}+\frac{S}{s-a}\cdot\frac{S}{s-b}$$

$$=\frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)}+\frac{s(s-a)(s-b)(s-c)}{(s-c)(s-a)}+\frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)}$$

$$=s(s-c)+s(s-a)+s(s-b)=3s^2-s(a+b+c)=3s^2-2s^2=s^2.$$

15. From (4), $R=\frac{abc}{4S},\quad r=\frac{S}{s};$

$$\therefore\ 2rR=2\,.\frac{abc}{4S}\cdot\frac{S}{s}=\frac{abc}{2s}=\frac{abc}{a+b+c};$$

$$\therefore\ \frac{1}{2rR}=\frac{a+b+c}{abc}=\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}.$$

16. In fig. E. T. p. 234, $r_1=I_1F_1=AF_1\tan F_1AI_1$
$=s\tan\frac{1}{2}A$ [by ii. p. 234].

Similarly $r_2\cot\frac{1}{2}B=s=r_3\cot\frac{1}{2}C.$

Also $r\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C=\frac{S}{s}\sqrt{\left\{\frac{s(s-a)}{(s-b)(s-c)}\cdot\frac{s(s-b)}{(s-c)(s-a)}\cdot\frac{s(s-c)}{(s-a)(s-b)}\right\}}$

$=s.$

17. $$r_1+r_2=S\left(\frac{1}{s-a}+\frac{1}{s-b}\right)=S\left(\frac{2s-(a+b)}{(s-a)(s-b)}\right)$$

$$=S\frac{a+b+c-(a+b)}{(s-a)(s-b)}=\frac{c\sqrt{\{s(s-a)(s-b)(s-c)\}}}{(s-a)(s-b)}$$

$$=\frac{c\sqrt{\{s(s-c)\}}}{\sqrt{\{(s-a)(s-b)\}}}=c\cot\tfrac{1}{2}C.\quad \text{E. T. p. 200.}$$

18. $$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{r_2r_3+r_3r_1+r_1r_2}{r_1r_2r_3}=\frac{r(r_2r_3+r_3r_1+r_1r_2)}{rr_1r_2r_3}.$$

The numerator of this fraction from (14) is $rs^2=Ss$; the denominator is (from (7), viii) S^2;

$$\therefore\ \text{the expression is } \frac{Ss}{S^2}=\frac{s}{S}=\frac{1}{r}.$$

19. By (vii) E. T. p. 235, $II_1=a\sec\frac{1}{2}A$, but $r_1-r=II_1\sin\frac{1}{2}A$;

$$\therefore\ \frac{r_1-r}{a}=\tan\tfrac{1}{2}A,\ \text{similarly}\ \frac{r_2-r}{b}=\tan\tfrac{1}{2}B;$$

$$\therefore\ \text{their sum}=\tan\tfrac{1}{2}A+\tan\tfrac{1}{2}B=\sin\tfrac{1}{2}(A+B)\sec\tfrac{1}{2}A\,.\,\sec\tfrac{1}{2}B$$

$$=\cos\tfrac{1}{2}C\sec\tfrac{1}{2}A\sec\tfrac{1}{2}B=\frac{c}{r_3},\ \text{by (x) p. 235.}$$

20. By (x) p. 235, $r_1 = 4R \sin \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$;

$$\therefore r_1 + r_2 + r_3 - r = 4R \{\sin \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C + \sin \tfrac{1}{2} B \cos \tfrac{1}{2} C \cos \tfrac{1}{2} A + \sin \tfrac{1}{2} C \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B - \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C\}$$
$$= 4R \sin \tfrac{1}{2} (A + B + C) = 4R.$$

21. From (7, i.) we obtain $\dfrac{abc}{4R} = S = \sqrt{\{s(s-a)(s-b)(s-c)\}}$

$$\therefore abc = 4R\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Now $r = S \div s = \sqrt{\{s(s-a)(s-b)(s-c)\}} \div s$; $\therefore abcr = 4R(s-a)(s-b)(s-c)$.

22. By (vii) p. 235, $II_1 = a \sec \frac{1}{2} A = 2R \sin A \sec \frac{1}{2} A = 4R \sin \frac{1}{2} A$. Similarly the other distances are $4R \sin \frac{1}{2} B$, $4R \sin \frac{1}{2} C$.

23. By 17, $r_2 + r_3 = a \cot \frac{1}{2} A = a$; when $\frac{1}{2} A = 45°$.

24. Each of the angles of the equilateral triangle is 60°; let each of the sides be a,

$$R = \frac{a}{2 \sin A} = \frac{a}{2 \sin 60°} = \frac{a}{\sqrt{3}}; \quad \therefore 3R = a\sqrt{3},$$

$$r = \frac{S}{s} = \frac{\frac{1}{2} a^2 \sin 60°}{\frac{1}{2} 3a} = \frac{a^2\sqrt{3}}{4} \times \frac{2}{3a} = \frac{a}{6}\sqrt{3}; \quad \therefore 6r = a\sqrt{3},$$

$$r_1 = \frac{S}{s-a} = \frac{\frac{1}{2} a^2 \sin 60°}{\frac{1}{2} a} = \frac{a^2\sqrt{3}}{4} \times \frac{2}{a} = \frac{a\sqrt{3}}{2}; \quad \therefore 2r_1 = a\sqrt{3};$$

$$\therefore 3R = 6r = 2r_1.$$

25. $$\frac{r_1}{bc} = \frac{a \cos \frac{1}{2} B \cos \frac{1}{2} C \sec \frac{1}{2} A}{bc} = \frac{1}{4R} \cdot \frac{\sin \frac{1}{2} A}{\sin \frac{1}{2} B \sin \frac{1}{2} C};$$

$$\therefore \frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{4R \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C} \{\sin^2 \tfrac{1}{2} A + \sin^2 \tfrac{1}{2} B + \sin^2 \tfrac{1}{2} C\}$$

$$= \frac{1 - 2 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C}{4R \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C} \qquad \text{[Ex. 21, p. 194.]}$$

$$= \frac{1}{4R \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C} - \frac{1}{2R} = \frac{1}{r} - \frac{1}{2R}. \qquad \text{[ix. p. 235.]}$$

EXAMPLES. LXXIV. PAGE 244.

1. The perimeter of a regular polygon inscribed in a circle of radius r is $2nr \sin \dfrac{\pi}{n}$; therefore the ratio of the perimeter of the polygon to the diameter of the circumscribing circle is $\dfrac{2nr}{2r} \sin \dfrac{\pi}{n} = n \sin \dfrac{\pi}{n}$.

When $n = 4$, ratio is $4 \sin \frac{1}{4}\pi = 4 \sin 45° = 2\sqrt{2} = 2 \times 1{\cdot}4142 = 2{\cdot}8284$.

When $n = 6$, ratio is $6 \sin \frac{1}{6}\pi = 6 \sin 30° = 3$.

When $n = 8$, ratio is $8 \sin \frac{1}{8}\pi = 8 \sin 22\frac{1}{2}°$.

Now $\sin 22\tfrac{1}{2}° = \tfrac{1}{2}\sqrt{(1-\cos 45°)} = \tfrac{1}{2}\sqrt{(2-\sqrt{2})}$;

$\therefore$ ratio is $4\sqrt{(2-\sqrt{2})} = 4\sqrt{\cdot 5858\ldots} = 3\cdot 06148\ldots$

When $n=10$, the ratio is $10 \sin \tfrac{1}{10}\pi$; that is, $10 \sin 18°$.

But $\sin 18° = \tfrac{1}{4}(\sqrt{5}-1)$. [Art. 93, Ex. XLII. 6].

Therefore the required ratio is $\tfrac{10}{4}(\sqrt{5}-1) = \tfrac{5}{2} \times 1\cdot 23606 = 3\cdot 09015$.

When $n=12$, ratio is $12 \sin \tfrac{1}{12}\pi = 12 \sin 15° = 12 \times \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$= 3\sqrt{2}(\sqrt{3}-1) = 3 \times 1\cdot 4142 \times \cdot 732 = 3\cdot 10558$.

When $n=20$, the ratio is $20 \sin \tfrac{1}{20}\pi = 20 \sin 9°$, $\sin 9°$ may be thus found.

From E. T. p. 150, $\sin 9° + \cos 9° = \sqrt{(1+\sin 18°)} = \tfrac{1}{2}\sqrt{(3+\sqrt{5})}$,

$\sin 9° - \cos 9° = -\sqrt{(1-\sin 18°)} = -\tfrac{1}{2}\sqrt{(5-\sqrt{5})}$;

$\therefore \sin 9° = \tfrac{1}{4}\{\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}\}$;

$\therefore$ the required ratio is $5\{\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}\}$

$= 5\{\tfrac{1}{2}\sqrt{10} + \sqrt{2} - \sqrt{(5-2\cdot 23608)}\} = 5(2\cdot 28825 - 1\cdot 6625) = 3\cdot 1287\ldots$

When $n=60°$, the ratio is $60 \sin \tfrac{1}{60}\pi = 60 \sin 3°$,

$\sin 3° = \sin(18° - 15°) = \sin 18° \cos 15° - \cos 18° \sin 15°$.

Now $\cos 18° = \sqrt{1-\sin^2 18°} = \tfrac{1}{4}\sqrt{(10+2\sqrt{5})}$;

$$\therefore \sin 3° = \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}} - \frac{\sqrt{(10+2\sqrt{5})}}{4} \cdot \frac{\sqrt{3}-1}{2\sqrt{2}};$$

$\therefore$ the required ratio is $\dfrac{15}{2\sqrt{2}}\{(\sqrt{5}-1)(\sqrt{3}+1) - \sqrt{10+2\sqrt{5}}(\sqrt{3}-1)\}$

$$= \frac{15}{2\sqrt{2}}\{1\cdot 236 \times 2\cdot 732 - 3\cdot 8 \times \cdot 732\} = \tfrac{15}{4}\sqrt{2} \times \cdot 595\ldots$$

$$= \tfrac{1}{4} \times 15 \times \cdot 595 \times \sqrt{2} = 15 \times \cdot 595 \times \cdot 353 = 3\cdot 1405.$$

2. Area of a polygon described about a circle is $nr^2 \tan \tfrac{1}{4}\pi$

$= 12 \times 1 \times \tan 15° = 12 \times \cdot 2679\ldots = 3\cdot 215$ nearly. [E. T. p. 120.]

3. Let r be the radius of the circle.

Then the area of the square described about it $= 4r^2 \tan 45° = 4r^2$.

The area of dodecagon inscribed in it

$$= 12r^2 \sin 15° \cos 15° = 12r^2 \cdot \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{12}{4} r^2 = 3r^2.$$

The ratio of the areas is $\dfrac{4r^2}{3r^2} = \dfrac{4}{3}$.

4. (i) The perimeter of the described polygon is $2nr \tan \dfrac{\pi}{n}$, $n=100$,

$2r=1$; $\therefore$ the perimeter $= 100 \tan \dfrac{18°}{10} = 100 \times \tan 1° 48'$

$= 100 \times \cdot 0314263\ldots = 3\cdot 14263\ldots$.

(ii) The perimeter of the inscribed polygon is $2nr \sin \frac{\pi}{n}$, $n=100$,

$2r=1$; $\therefore$ the perimeter $= 100 \sin \frac{18^\circ}{10} = 100 \times \sin 1^\circ 48'$

$$= 100 \times \cdot 0314108\ldots = 3\cdot 14108\ldots.$$

5. Let r_1 be the radius of the circle inscribed in the equilateral triangle, then the perimeter of the triangle is $6r_1 \tan \frac{1}{3}\pi = 6r_1\sqrt{3}$.

Let r_2 be the radius of the circle inscribed in the regular hexagon, then the perimeter of the hexagon is $12r_2 \tan \frac{1}{6}\pi = 4r_2\sqrt{3}$.

Now by the question $6r_1\sqrt{3} = 4r_2\sqrt{3}$; $\therefore 3r_1 = 2r_2$,

therefore $$r_1 : r_2 :: 2 : 3,$$

i.e. the radii of the circles (and therefore their diameters) are in the ratio of $2 : 3$; but circles are to one another as the squares on their diameters, therefore the given circles are to one another as $2^2 : 3^2$, i.e. as $4 : 9$.

6. In the figure E. T. p. 242, bisect the angle at H by OH and the angle at K by OK; produce HO, KO to meet at O; it may then be shewn that straight lines drawn from O to the other angular points of the polygon bisect those angles. Therefore OH, OK... are all equal, and the polygon is divided into as many isosceles triangles as it has sides. Let n be the number of sides of the polygon, then the angle $HOK = \frac{2\pi}{n}$ and the area of the polygon is n times the area of the triangle HOK. In the isosceles triangle HOK draw the perpendicular OM, then OM bisects the angle HOK and the base HK; $\therefore \angle HOM = \frac{\pi}{n}$ and $OM = HM \cot \frac{\pi}{n} = \frac{a}{2} \cot \frac{\pi}{n}$.

The area of the triangle HOK is $OM \times HM = \frac{a}{2} \cdot \frac{a}{2} \cot \frac{\pi}{n}$.

$$\therefore \text{the area of the polygon is } n \cdot \frac{a}{2} \cdot \frac{a}{2} \cot \frac{\pi}{n} = \frac{na^2}{4} \cdot \cot \frac{180^\circ}{n}.$$

7. Let a be the side of the pentagon, then area

$$= \frac{5a^2}{4} \cdot \cot 36^\circ = \frac{5a^2}{4} \cdot \frac{1+\sqrt{5}}{\sqrt{(10-2\sqrt{5})}}.$$

Let b be the side of the decagon, then area

$$= \frac{10b^2}{4} \cot 18^\circ = \frac{5b^2}{2} \cdot \frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}.$$

Now $$\frac{5a^2}{4} \cdot \frac{1+\sqrt{5}}{\sqrt{(10-2\sqrt{5})}} = \frac{5b^2}{2} \cdot \frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}.$$

$$a^2(\sqrt{5}+1)(\sqrt{5}-1) = 2b^2 \sqrt[2]{(10+2\sqrt{5})} \sqrt[2]{(10-2\sqrt{5})}$$

$$2a^2 = b^2\sqrt{80}, \quad a^2 = b^2\sqrt{20}, \quad \therefore a = b\sqrt[4]{20};$$

$$\therefore \frac{a}{b} = \frac{\sqrt[4]{20}}{1}.$$

8. In fig. E. T. p. 242, $OH=R$ and $OM=r$, $OH=HM\operatorname{cosec}HOM$;

$$\therefore\ R=\frac{a}{2}\operatorname{cosec}\frac{\pi}{n};\ \therefore\ 2R=a\operatorname{cosec}\frac{\pi}{n},\ OM=HM\cot MOH;$$

$$\therefore\ r=\frac{a}{2}\cot\frac{\pi}{n};\ \therefore\ 2r=a\cot\frac{\pi}{n}.$$

9.
$$R+r=\frac{a}{2}\left(\operatorname{cosec}\frac{\pi}{n}+\cot\frac{\pi}{n}\right)=\tfrac{1}{2}a\,.\left(1+\cos\frac{\pi}{n}\right)\div\sin\frac{\pi}{n}$$
$$=\frac{a}{2}\cot\frac{\pi}{2n}.$$ [Art. 165.]

10. Let r be the radius of the circle, then from E. T. p. 283,
Side of Pentagon $=2r\sin\frac{1}{5}\pi=\frac{1}{2}r\sqrt{(10-2\sqrt{5})}$.
Side of Hexagon $=2r\sin\frac{1}{6}\pi=r$.
Side of Decagon $=2r\sin\frac{1}{10}\pi=\frac{1}{2}r(\sqrt{5}-1)$.

If the triangle formed from one side each of these polygons is right-angled, then the sum of the squares on the sides of the hexagon and the decagon is equal to the square on the side of the pentagon.

Now $r^2+\{\frac{1}{2}r(\sqrt{5}-1)\}^2=r^2+\frac{1}{4}r^2(\sqrt{5}-1)^2=\frac{1}{4}r^2(10-2\sqrt{5})$. Q.E.D.

11. Let $a, b, c, d\ldots$ be the n sides of an irregular polygon described about a circle of radius r; then the polygon can be divided into n triangles of which the areas are $\frac{1}{2}ar$, $\frac{1}{2}br$, $\frac{1}{2}cr$, $\frac{1}{2}dr$,... Therefore the area of the polygon is $\frac{1}{2}r(a+b+c+d\ldots)$, i.e. the product of the radius and half the perimeter of the polygon.

12. Let $ABCDEF$ be an irregular polygon of an even number of sides described about the circle $HIKLMN$, so that the sides AB, BC, CD, DE, EF, FA touch the circle on the points H, I, K, L, M, N respectively. Let r be the radius of the circle. Then from (11), the area of the polygon is

$$\tfrac{1}{2}r(AB+BC+CD+DE+EF+FA)$$
$$=\tfrac{1}{2}r\{(AH+HN)+(BH+BI)+(CI+CK)$$
$$+(DK+DL)+(EL+EM)+(FM+FN)\}.$$

It may be proved by means of Euclid III. 37 that

$$AH=HN,\ BH=BI,\ CI=CK,\ DK=DL,\ EL=EM,\ FM=FN;$$

$\therefore$ the area of the polygon

$$=\tfrac{1}{2}r\{2AH+2BH+2CK+2DK+2EM+2FM\}$$
$$=r\{AH+BH+CK+DK+EM+FM\}$$
$$=\{AB+CD+EF\}.$$

In the same way if we begin with BC instead of AB we can prove that the area of the polygon $=r\{BC+DE+FA\}$; and so on for every other side; therefore the area of the polygon is radius $\times$ the sum of every alternate side.

13. Area of a circle $=\pi r^2$;

$\therefore$ area covered by the dome $=\pi(54)^2=3{\cdot}1416\times(54)^2$ sq. feet
$$=3{\cdot}1416\times324=1017{\cdot}8784$$
$$=1018 \text{ sq. yards,}$$

14. Area of the circle $= \pi r^2 = 1$ acre $= 4840$ sq. yards;

$$\therefore r^2 = \frac{4840}{\pi} = \frac{4840 \times 7}{22} = 1540;$$

$$\therefore r = \sqrt{1540} = 39{\cdot}25\ldots \text{yds.}$$

15. The area of the base of the cylinder is the length of the paper × its thickness $= 300 \times 3 \times 12 \times \frac{1}{150} = 72$ sq. inches.

If r = radius of cylinder, πr^2 = area of its base, and $2r$ = its diameter;

$$\therefore \pi r^2 = 72 \text{ sq. inches},$$

$$4\pi r^2 = 288 \text{ sq. inches},$$

$$4r^2 = \frac{288}{\pi} = 91{\cdot}673\ldots \text{sq. in.},$$

$$2r = \sqrt{91{\cdot}673\ldots} \text{ inches} = 9{\cdot}575 \text{ inches}.$$

16. Let l be the length of the carpet in feet.

Then area of the base is $\frac{12}{8} l$ sq. inches; the area is also r^2 where $2r$ is diameter $= 144\pi$ sq. inches;

$$\therefore \tfrac{12}{8} l = 144\pi; \quad \therefore l = 96\pi = 96 \times 3{\cdot}1416 = 301{\cdot}6 \text{ feet}.$$

EXAMPLES. LXXV. Page 252.

1. $$\frac{n}{2} R^2 \sin\frac{2\pi}{n} = \pi \cdot \frac{n}{2\pi} \cdot R^2 \sin\frac{2\pi}{n} = \pi R^2 \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}.$$

When $n = \infty$, $\frac{2\pi}{n} = 0$, and $\dfrac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} = 1$; [E. T. p. 247.]

$$\therefore \text{ the limit when } n = \infty \text{ of } \frac{n}{2} R^2 \sin\frac{2\pi}{n} = \pi R^2 \text{ when } n = \infty.$$

2. $$nr^2 \tan\frac{\pi}{n} = r^2 \cdot \pi \cdot \frac{n}{\pi} \tan\frac{\pi}{n} = \pi r^2 \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}} = \pi r^2 \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \cdot \frac{1}{\cos\frac{\pi}{n}}.$$

When $n = \infty$, $\frac{\pi}{n} = 0$; $\therefore \dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 1$ and $\dfrac{1}{\cos\frac{\pi}{n}} = 1$;

$$\therefore nr^2 \tan\frac{\pi}{n} = \pi r^2 \text{ when } n = \infty.$$

3. The circular measure of $10''$ is $\dfrac{10\pi}{180\times 60\times 60}=\dfrac{\pi}{64800}$.

Now $\sin\theta<\theta$; $\therefore$ $\sin 10''$ is less than $\dfrac{\pi}{64800}$ $\sin\theta>\theta-\dfrac{\theta^3}{4}$;

$$\therefore \sin 10'' \text{ is greater than } \frac{\pi}{64800}-\frac{1}{4}\left(\frac{\pi}{64800}\right)^3.$$

If $\pi=3{\cdot}141592653589793$ then $\dfrac{\pi}{64800}=\cdot00004848136811......$; sine $10''$ is therefore less than this decimal. But $\dfrac{\pi}{64800}$ is less than $\cdot00005$, therefore *a fortiori* sine $10''$ is greater than $\cdot00004848136811......-\frac{1}{4}(\cdot00005)^3$; i.e. $\sin 10''$ is greater than $\cdot000048481368078...$

The two decimals between which $\sin 10''$ lies correspond in their first thirteen figures, therefore we have

$$\sin 10''=\cdot0000484813681......$$

4.
$$\begin{aligned}2\sin(72^\circ+A)-2\sin(72^\circ-A)&=2\{\sin(72^\circ+A)-\sin(72^\circ-A)\}\\&=4\sin A\cos 72^\circ=4\sin A\sin 18^\circ\\&=(\sqrt{5}-1)\sin A,\end{aligned}$$ [E. T. p. 59.]

$$\begin{aligned}2\sin(36^\circ+A)-2\sin(36^\circ-A)&=2\{\sin(36^\circ+A)-\sin(36^\circ-A)\}\\&=4\sin A\cos 36^\circ=4\sin A\sin 54^\circ\\&=(\sqrt{5}+1)\sin A.\end{aligned}$$

5. Fig. E. T. p. 251, $TP^2=2\,.\,RO\,.\,TR$ nearly.

Let $x=TP$, $\frac{150}{5280}$ mile $=TR$,

$$x^2=7914\times\tfrac{150}{5280}=225 \text{ nearly};$$
$$\therefore\ x=15 \text{ miles nearly}.$$

6. From E. T. Art. 300, $TP^2=2TR\,.\,RO$;

$$\therefore\ TR=\frac{TP^2}{2\,.\,RO}.$$

From Euclid VI. 8, $\dfrac{TP}{OR}=\dfrac{TM}{PM}=\cdot025$;

$$\therefore\ \frac{TP^2}{OR^2}=\cdot000625, \text{ and } \frac{TP^2}{2\,.\,OR}=\frac{\cdot000625\times 3957}{2} \text{ miles},$$

$$\text{i.e. } TR=\cdot000625\times 3957\times 2640 \text{ feet}$$
$$=6530 \text{ feet nearly}.$$

EXAMPLES. LXXVI. a. PAGE 253.

1. The proof of Art. 107 is true for each of the four figures on page 96.

$$\tan A = \tfrac{3}{4};\ \therefore \sqrt{(1+\tan^2 A)} = \tfrac{5}{4};\ \therefore \cos A = \tfrac{4}{5};$$
$$\sin A = \sqrt{(1-\cos^2 A)} = \tfrac{3}{5}.$$

2. When $\theta = 0$, $\cos\theta = 1$ and $\sec\theta = 1$, $\therefore \cos\theta - \sec\theta = 0$, let λ represent $\cos\theta - \sec\theta$.

As θ increases from 0 to $\frac{1}{2}\pi$ $\cos\theta$ diminishes to 0 and $\therefore$ $\sec\theta$ increases from 1 to ∞.

$\therefore \lambda$ is negative and increases numerically from 0 to ∞.

As θ increases from $\frac{1}{2}\pi$ to π $\cos\theta$ increases numerically from 0 to 1 and is negative.

$\therefore$ $\sec\theta$ decreases numerically from ∞ to 1 and is negative;

$\therefore \lambda$ is positive and decreases from ∞ to 1.

3. See T. p. 104, $(2-\sec A)\sin A = 0$;

$\therefore \sin A = 0$ and $A = n\times 180°$, or $\cos A = \frac{1}{2}$ and $A = 2n\times 180° \pm 60°$.

4. (1) $\sin(A+B)\,.\,\sin(A-B) = (\sin A\,.\,\cos B + \cos A\,.\,\sin B)$
$\times(\sin a\,.\,\cos B - \cos A\,.\,\sin B)$

$$= \sin^2 A\,.\,\cos^2 B - \cos^2 A\,.\,\sin^2 B$$
$$= \sin^2 A\,.\,(1-\sin^2 B) - (1-\sin^2 A)\,.\,\sin^2 B = \sin^2 A - \sin^2 B.$$

(2) $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\frac{1}{2}(a+B)\,.\,\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(a+B)\,.\,\sin\frac{1}{2}(a-B)} = \frac{\tan\frac{1}{2}(a+B)}{\tan\frac{1}{2}(a-B)}.$$

5. $\cos^2 A - \cos A\,.\,\cos(60°+A) + \sin^2(30°-A)$

$= \cos^2 A - \cos A\,.\,\sin(30°-A) + \sin^2(30°-A)$

[since $\cos(60°+A) = \sin\{90° - (60°+A)\}$]

$= \cos^2 A - \cos A \times (\frac{1}{2}\cos A - \frac{1}{2}\sqrt{3}\sin A)$

$+\frac{1}{4}\cos^2 A - \frac{1}{2}\sqrt{3}\sin A\,.\,\cos A + \frac{3}{4}\sin^2 A$

$= \cos^2 A \times (1 - \frac{1}{2} + \frac{1}{4} - \frac{3}{4}) + \frac{3}{4} = \frac{3}{4}.$

6. Greatest angle $= 78°\,14'$, greatest side $= 2183\,.\,\sin 78°\,14' \div \sin 30°\,22'$;

log (greatest side) $= \log 2183 + L\sin 78°\,14' - L\sin 30°\,22'$

$= 3{\cdot}3390537 + 9{\cdot}9907766 - 9{\cdot}7037486 = 3{\cdot}6260817$:

by the rule of Proportional Parts, $d = {\cdot}0001 \times \frac{84}{103} = {\cdot}0000815$;

$\therefore {\cdot}6260817 = \log(4{\cdot}2274 + {\cdot}0000815) = \log 4{\cdot}2274815.$

Hence greatest side $= 4227{\cdot}4815$.

7. See Examples.XVI. 1.

8. See T. pp. 106, 107.

9. $\sin 30° = \frac{1}{2}$, $\sin 60° = \frac{1}{2}\sqrt{3}$, $\sin 90° = 1$, $\sin 120° = \sin 60° = \frac{1}{2}\sqrt{3}$,
$\sin 150° = \sin 180° - 150° = \sin 30° = \frac{1}{2}$,
$\sin 180° = 0$ $\sin 7 \times 30° = \sin 210° = +\sin(180° + 30°) = -\sin 30° = -\frac{1}{2}$,
$\sin 240° = \sin 180° + 60° = -\sin 60° = -\frac{1}{2}\sqrt{3}$,
$\sin 270° = \sin(180° + 90°) = -\sin 90° = -1$,
$\sin 300° = \sin(360° - 300°) = -\sin 60° = -\frac{1}{2}\sqrt{3}$,
$\sin 330° = \sin(360° - 330°) = -\sin 30° = -\frac{1}{2}$.

10. $\tan^2 A = \dfrac{2\sin^2 A}{2\cos^2 A} = \dfrac{1 - \cos 2A}{1 + \cos 2A}$.

11. $\tan A + \sqrt{(1 + \tan^2 A)} = 2$, $\therefore \tan A = \frac{3}{4}$, see 1 above.

When $\sin A = \frac{4}{5}$, then $\tan A = \frac{4}{3}$ and $\sec A = \frac{5}{3}$, the ratios are all positive when A is less than 90°.

12. Let a, b, 1035·43 be the lengths of the three sides,
$a = 1035{\cdot}43 \,.\, \sin 44° \div \sin 70°$, $b = 1035{\cdot}43 \,.\, \sin 66° \div \sin 70°$.

$$\log a = \log 1035{\cdot}43 + L\sin 44° - L\sin 70°$$
$$= 3{\cdot}0151212 + 9{\cdot}8417713 - 9{\cdot}9729858 = 2{\cdot}8839067$$
$$= \log 765{\cdot}432;\ \therefore a = 765{\cdot}432 \text{ feet.}$$
$$\log b = \log 1035{\cdot}43 + L\sin 66° - L\sin 70°$$
$$= 3{\cdot}0151212 + 9{\cdot}9607302 - 9{\cdot}9729858 = 3{\cdot}0028656$$
$$= \log 1006{\cdot}6;\ \therefore 6 = 1006{\cdot}6 \text{ ft.}$$

13. See T. Ex. XVI. 2.

14. See T. pages 104, 106.

15. $\tan 30° = \dfrac{1}{\sqrt{3}}$, $\tan 60° = \sqrt{3}$, $\tan 90° = \infty$,
$\tan 120° = \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$,
$\tan 150° = \tan(180° - 30°) = -\tan 30° = -\dfrac{1}{\sqrt{3}}$, $\tan 180° = 0$,
$\tan 210° = \tan(180° + 30°) = \tan 30° = \dfrac{1}{\sqrt{3}}$,
$\tan 240° = \tan(180° + 60°) = \tan 60° = \sqrt{3}$,
$\tan 270° = \infty$, $\tan(360° - 60°) = -\tan 60° = -\sqrt{3}$,
$\tan(360° - 30°) = -\tan 30° = -\dfrac{1}{\sqrt{3}}$.

16. $\tan A + \sqrt{(1 + \tan^2 A)} = 3$, $\therefore \tan A = \frac{4}{3}$.

$$\sin A = \frac{\tan A}{\sqrt{(1 + \tan^2 A)}} = \frac{4}{5}.$$

When $\sin A = \frac{3}{5}$ then by 1 $\tan A = \frac{3}{4}$ and $\sec A = \frac{5}{4}$, the ratios are all positive when A is less than 90°.

17. Let $a=193$, $b=194$, $c=195$.

$$\sin A=\frac{2}{194\times 195}\sqrt{\{s(s-a)(s-b)(s-c)\}}$$
$$=\frac{2}{194\times 195}\sqrt{\{291\times 96\times 97\times 98\}}$$
$$=\frac{2}{194\times 195}\sqrt{\{3\times 97^2\times 2^2\times 48\times 7^2\}}$$
$$=\frac{2}{194\times 195}\sqrt{\{3^2\times 97^2\times 2^2\times 4^2\times 7^2\}}$$
$$=\frac{2}{97\times 2\times 39\times 5}\times 3\times 97\times 2\times 4\times 7=\frac{56}{65}=\cdot 861538;$$
$$\therefore A=\sin^{-1}.86154=59^\circ\,29'\,23''.$$

Similarly it may be shewn that $B=59^\circ\,59'\,23''$, $\therefore C=60^\circ\,31'\,14''$.

18. (1) $\cos\frac{3}{2}A=\cos A\cos\frac{1}{2}A-\sin A\sin\frac{1}{2}A$
$$=\cos A\cos\tfrac{1}{2}A-2\cos\tfrac{1}{2}A\sin^2\tfrac{1}{2}A$$
$$=\cos A\cos\tfrac{1}{2}A-\cos\tfrac{1}{2}A(1-\cos A).$$

(2) $\cos\theta-\cos(\theta+\delta)=2\sin(\theta+\frac{1}{2}\delta)\sin\frac{1}{2}\delta$
$$=2\sin\theta\cos\tfrac{1}{2}\delta\sin\tfrac{1}{2}\delta+2\cos\theta\sin^2\tfrac{1}{2}\delta$$
$$=\sin\theta\sin\delta+\cos\theta\sin\delta\tan\tfrac{1}{2}\delta.$$

19. See T. Art. 107, $\sec A=1\div\sqrt{(1-\sin^2 A)}=\pm\sqrt{\frac{9}{8}}$.

20. The expression $=-\dfrac{1+\tan\theta}{1-\tan\theta}=-\tan(\theta+45^\circ)$;

$\therefore$ the values of the expression are the same as those of $\tan\alpha$ as α changes from $\frac{5}{4}\pi$ to $\frac{9}{4}\pi$.

21. Write out Art. 148 putting $\cot\theta$ for $\tan\theta$,
$$\tan^2\theta=1,\quad \tan\theta=\pm 1,\quad \theta=n\pi\pm\tfrac{1}{4}\pi.$$

22. See T. p. 118,

$$\cos 5a=\cos(4a+a)=\cos 4a\,.\cos a-\sin 4a\,.\sin a$$
$$=(2\cos^2 2a-1)\cos a-2\sin 2a\,.\cos 2a\,.\sin a$$
$$=[2(2\cos^2 a-1)^2-1]\cos a-4\sin a\,.\cos a\times(2\cos^2 a-1)\,.\sin a$$
$$=[2\times(4\cos^4 a-4\cos^2 a+1)-1]\cos a-4\sin^2 a\,.2\cos^3 a+4\sin^2 a\cos a$$
$$=8\cos^5 a-8\cos^3 a+\cos a-4(1-\cos^2 a)\,2\cos^3 a+4(1-\cos^2 a)\cos a$$
$$=8\cos^5 a-8\cos^3 a+\cos a-12\cos^3 a+8\cos^5 a+2\cos a-4\cos^3 a$$
$$=16\cos^5 a-20\cos^3 a+5\cos a.$$

23. See Art. 179. If A lies between 540° and 630°, $\frac{1}{2}a$ lies between 270° and 315°,

$\sin\frac{1}{2}a$ is negative and is greater in magnitude than $\cos\frac{1}{2}a$ which is positive;

$$\therefore \sin\tfrac{1}{2}a+\cos\tfrac{1}{2}a=-\sqrt{(1+\sin A)}$$
$$\sin\tfrac{1}{2}a-\cos\tfrac{1}{2}a=-\sqrt{(1-\sin a)}$$
$$=2\sin\tfrac{1}{2}a=-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}.$$

24. Length of string $=\sqrt{(144+25)}$ in. $=13$ in.,
the points of suspension of the ring are the angular points of a regular hexagon inscribed in the ring;

$\therefore$ the side $=5$ inches,

$$\text{cos of angle required}=\frac{(13)^2+(13)^2-5^2}{2\times 13\times 13}=\frac{313}{338}.$$

25. $1^\circ=$ one-ninetieth part of a right angle, $180^\circ=\pi$ radians;

$$\therefore A^\circ=\frac{A}{180}\times\pi \text{ radians}=\frac{22}{7\times 180}\times A \text{ radians},$$

$$\cdot 1 \text{ radian}=\frac{180^\circ}{10\pi}=\frac{18\times 7^\circ}{22}=5\tfrac{8}{11}{}^\circ.$$

26. $\sin A=\sqrt{(1-\frac{9}{25})}=\frac{4}{5}$; $\tan A=\frac{4}{5}\div\frac{3}{5}=\frac{4}{3}$.

27. See the Figures in T. p. 110. Let $P_1OM_1=A$; then in Fig. I. $P_2OR=180^\circ-A$; in Fig. II. $P_2OR=180^\circ+A$,
and the two triangles P_2OM_2 are equal in all respects and OM_2 is of the same sign in each;

$$\therefore \cos(180^\circ-A)=\cos(180^\circ+A).$$

In the figures in T. pages 104, 107 ROP' on p. $104=180^\circ-A$, ROP' on p. $107=90^\circ+A$ and $OM'=ON'$, $P'M'=P'N'$.

In figure on p. 104 OP' starts from OL and revolves in the negative direction.

In the figure on p. 107 OP' starts from OU and revolves in the positive direction.

When OP on p. 104 crosses OU on p. 107 it crosses OL; hence

$M'P'$ on p. 104 always equal $-N'P'$ on p. 107,

and $-OM'$ on p. 104 always equals ON' on p. 107;

$$\therefore \frac{OM'}{M'P'}=\frac{ON'}{N'P'}; \therefore \cot(180^\circ-A)=\tan(90^\circ-A).$$

28. $\sin x(2\cos x-1)=2\sin\frac{1}{2}x\cos\frac{1}{2}x(4\cos^2\frac{1}{2}x-3)$
$=2\sin\frac{1}{2}x(4\cos^3\frac{1}{2}x-3\cos\frac{1}{2}x)$.

29. $$\frac{2\sin\theta-\sin 2\theta}{2\sin\theta+\sin 2\theta}=\frac{2\sin\theta(1-\cos\theta)}{2\sin\theta(1+\cos\theta)}=\frac{2\sin^2\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}=\tan^2\tfrac{1}{2}\theta,$$

$\therefore \dfrac{2\sin\theta-\sin 2\theta}{2\sin\theta+\sin 2\theta}$ has the same changes as $\tan^2\frac{1}{2}\theta$ and is always positive.

As θ changes from 0 to π $\tan^2\frac{1}{2}\theta$ changes from 0 to ∞,
.... π to 2π .... ∞ to 0.

30. $2bc\cos 60^\circ=b^2+c^2-a^2=(b+c+a)(b+c-a)-2bc$,

$\therefore bc=(b+c+a)(b+c-a)-2bc$, or $(b+c+a)(b+c-a)=3bc$.

31. π radians $=2$ right angles, θ radians $=\frac{\theta}{\pi}\times 2$ right angles,

$\frac{1}{3}$ radian $=\frac{1}{3\pi}\times 2$ right angles $=\frac{1}{3}\times\frac{7}{22}\times 180^\circ=\frac{1}{11}$ of $180^\circ=19\frac{1}{11}{}^\circ$.

32. Let $\tan\alpha=x$, $\tan\beta=y$,

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\,.\,\tan\beta},\qquad \alpha+\beta=\tan^{-1}\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\,.\,\tan\beta},$$

or
$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}.$$

33. $\sin x\,(2\cos x+1)=2\sin\frac{1}{2}x\cos\frac{1}{2}x\,\{2\,(1-2\sin^2\frac{1}{2}x)+1\}$
$$=2\cos\tfrac{1}{2}x\,(3\sin\tfrac{1}{2}x-4\sin^3\tfrac{1}{2}x).$$

34. $$\frac{\sin\theta+2\sin\frac{1}{2}\theta}{\sin\theta-2\sin\frac{1}{2}\theta}=\frac{\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta-\sin\frac{1}{2}\theta}=\frac{\cos\frac{1}{2}\theta+1}{\cos\frac{1}{2}\theta-1}$$
$$=-\frac{\cos^2\frac{1}{4}\theta}{\sin^2\frac{1}{4}\theta}=-\cot^2\tfrac{1}{4}\theta.$$

Hence the expression is always negative and the numerical changes are the squares of the changes of the cot of $\frac{1}{4}\theta$. As θ changes from θ to 2π $\frac{1}{4}\theta$ changes from 0 to $\frac{1}{2}\pi$ and $\cot\theta$ changes from ∞ to 0.

35. See T. Ex. 3, p. 192, and Ex. 19, p. 194. The equation may be written

$$(4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C)(4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C)$$
$$=12\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\cos\tfrac{1}{2}B;$$
$$\therefore\cos^2\tfrac{1}{2}C=\tfrac{3}{4},\quad\therefore\cos\tfrac{1}{2}C=\tfrac{1}{2}\sqrt{3},\quad\therefore\tfrac{1}{2}C=30^\circ.$$

36. $$c=b\frac{\sin 18^\circ}{\sin 144^\circ}=a,$$
$$\therefore c=a=\frac{\sin 18^\circ}{\sin 36^\circ}=\frac{1}{2\cos 18^\circ}=\cdot 052573.$$

37. See Art. 39. The minute hand has described $(3\times 4+2)$ right angles $+\frac{2}{3}$ right angle between twelve o'clock and 20 minutes to four or $14\frac{2}{3}$ right angles.

38. If A is greater than 90° and less than 180°, $\cos A$ is negative,
$$\cos A=-\sqrt{(1-\sin^2 A)}=-\sqrt{(1-\tfrac{1}{9})}=-\tfrac{2}{3}\sqrt{2}.$$

39. $$\cos\theta+\cos 2\theta=0,$$
$$\therefore 2\cos^2\theta+\cos\theta-1=0;\quad\therefore(2\cos\theta-1)(\cos\theta+1)=0;$$
$$\therefore\text{ either }\cos\theta=-1;\text{ whence }\theta=(2n+1)\pi,$$
or
$$\cos\theta=+\tfrac{1}{2};\text{ whence }\theta=2n\pi\pm\tfrac{1}{3}\pi.$$

40. When $a\cos A=b\cos B$, then
$$\frac{a}{bc}(b^2+c^2-a^2)=\frac{b}{ca}(c^2+a^2-b^2);$$
$$\therefore a^2(b^2+c^2-a^2)=b^2(c^2+a^2-b^2),\quad\therefore a^2c^2-a^4=b^2c^2-b^4.$$
$$c^2(a^2-b^2)-(a^2-b^2)(a^2+b^2)=0;\quad\therefore(a^2-b^2)(c^2-a^2-b^2)=0;$$
$$\therefore\text{ either }a^2=b^2\text{ or }c^2=a^2+b^2.$$ Q. E. D.

41. Let A and B be the sides opposite 1 ft. and $\sqrt{3}$ ft. respectively, then

$$\sin B = \sin 30° \times \sqrt{3} = \tfrac{1}{2}\sqrt{3}; \quad \therefore B = 60° \text{ and } C = 90°;$$

$$\therefore \text{ the sides are } 1, \sqrt{3} \text{ and } 2.$$

42. Let A = the greatest angle,

$$\tan \tfrac{1}{2}A = \sqrt{\left\{\frac{\frac{3}{2}\times\frac{5}{2}}{\frac{9}{2}\times\frac{1}{2}}\right\}} = \sqrt{\left\{\frac{10}{3\times 2}\right\}},$$

$$\begin{aligned} \text{Log}\tan\tfrac{1}{2}A &= 10 + \tfrac{1}{2}\{\log 10 - \log 3 - \log 2\} \\ &= 10 + \tfrac{1}{2}\{1{\cdot}4771213 - {\cdot}3010300\} \\ &= 10 + {\cdot}1109243 = L\,52°\,14'\,19{\cdot}5''. \end{aligned}$$

$$A = 104°\,28'\,39''.$$

43. See Arts. 38, 39. The minute hand describes an angle of $(4+3)$ right angles or 630° between half past four and a quarter past six.

44. If A is between 180° and 270°, $\sin A$ is negative,

$$\therefore \sin A = -\frac{\frac{1}{2}}{\sqrt{1+\frac{1}{4}}} = -\frac{1}{\sqrt{5}}.$$

45. (i) $\sin 2A = 2\sin A \,.\, \cos A$

$$= \frac{2\cos A}{\sin A}\,.\,\sin^2 A = \frac{2\cot A}{\text{cosec}^2 A} = \frac{2\cot A}{1+\cot^2 A}.$$

(ii) This is the same as Example 3, p. 192, for $2A + 2B + 2C = 180°$.

46.

$$\sin\theta + \sin 2\theta = 0, \quad \sin\theta\,(1 + 2\cos\theta) = 0;$$

$$\therefore \sin\theta = 0; \text{ whence, } \theta = n\pi,$$

or

$$\cos\theta = -\tfrac{1}{2}; \text{ whence, } \theta = (2n+1)\pi \pm \tfrac{1}{3}\pi.$$

47. When $b\cos A = a\cos B$, then

$$\frac{b}{bc}(b^2 + c^2 - a^2) = \frac{a}{ac}(c^2 + a^2 - b^2),$$

$$\therefore b^2 + c^2 - a^2 = c^2 + a^2 - b^2 \text{ or } c^2 - a^2 = 0; \quad \therefore c = a. \qquad \text{Q.E.D.}$$

48. Let A be the angle opposite the side $\sqrt{2}$ ft., then

$$\sin A = \sqrt{2}\sin 30° = \tfrac{1}{2}\sqrt{2}; \quad \therefore A = 45°, \text{ or } 135°.$$

$$B = 30° \text{ and } C = 105° \text{ or } 15°,$$

also

$$c = a\cos B + b\cos A = \tfrac{1}{2}\times\sqrt{2}\times\sqrt{3} + \tfrac{1}{2}\times\sqrt{2} = \tfrac{1}{2}(\sqrt{2}+\sqrt{6}).$$

49. (1) $\frac{1}{20}\pi$ radians $= \frac{1}{20}\pi \times \dfrac{180°}{\pi} = 90°.$

(2) 5 radians $= 5 \times \dfrac{180°}{\pi} = \dfrac{7}{11} \times 450° = 286°\,21'\,49''\ldots$

In the third case the unit is $\frac{1}{7}$ of 360°,

$$45° = 45 \div (\tfrac{1}{7} \text{ of } 360) \text{ of } \tfrac{1}{7} \text{ of } 360° = \tfrac{7}{8} \text{ of the unit.}$$

50. $(\sin 30° + \cos 30°)(\sin 60° - \cos 60°)$

$$= (\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3})(\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}) = \tfrac{3}{4} - \tfrac{1}{4} = \tfrac{1}{2} = \sin 30°.$$

51. (1) $\cos^2(\alpha+\beta)-\sin^2\alpha=\cos^2\alpha\,.\,\cos^2\beta+\sin^2\alpha\,.\,\sin^2\beta$
$-\,2\sin\alpha\,.\,\cos\alpha\,.\,\sin\beta\,.\,\cos\beta-\sin^2\alpha$
$=(1-\sin^2\alpha)\cos^2\beta+\sin^2\alpha\,(1-\cos^2\beta)-2\sin\alpha\,.\,\cos\alpha\,.\,\sin\beta\,.\,\cos\beta$
$=\cos^2\beta-2\sin^2\alpha\,.\,\cos^2\beta-2\sin\alpha\,.\,\cos\alpha\,.\,\sin\beta\,.\,\cos\beta$
$=\cos\beta\,(\cos\beta-2\sin^2\alpha\,.\,\cos\beta-\sin 2\alpha\,.\,\sin\beta)$
$=\cos\beta\,(\cos\beta\cos 2\alpha-\sin 2\alpha\,.\,\sin\beta)$
$=\cos\beta\,.\,\cos(2\alpha+\beta)$.

(2) $1+\cot\alpha\,.\,\cot\tfrac{1}{2}\alpha=1+\dfrac{\cos\alpha}{\sin\alpha}\,.\,\dfrac{\cos\frac{1}{2}\alpha}{\sin\frac{1}{2}\alpha}$

$$=\frac{1}{\sin\alpha}\left\{\sin\alpha+\frac{\cos\alpha\,.\,\cos\frac{1}{2}\alpha}{\sin\frac{1}{2}\alpha}\right\}=\frac{\cos\frac{1}{2}\alpha}{\sin\alpha}\left\{2\sin\tfrac{1}{2}\alpha+\frac{\cos\alpha}{\sin\frac{1}{2}\alpha}\right\}$$

$$=\frac{\cos\frac{1}{2}\alpha}{\sin\alpha}\left\{\frac{2\sin^2\frac{1}{2}\alpha+1-2\sin^2\frac{1}{2}\alpha}{\sin\frac{1}{2}\alpha}\right\}=\frac{1}{\sin\alpha}\,.\,\frac{\cos\frac{1}{2}\alpha}{\sin\frac{1}{2}\alpha}.$$

52. (1) $5\tan^2x-\sec^2x=11$,
$\therefore\ 5\tan^2x-(1+\tan^2x)=11$; $\therefore\ 4\tan^2x=12$.
$\tan x=\pm\sqrt{3}$; whence, $x=n\pi\pm\tfrac{1}{3}\pi$.

(2) $\sin 5\theta-\sin 3\theta=\sqrt{2}\,.\,\cos 4\theta$;
$\therefore\ 2\cos 4\theta\,.\,\sin\theta=\sqrt{2}\,.\,\cos 4\theta$. $\cos 4\theta\,(2\sin\theta-\sqrt{2})=0$;
$\therefore\ \cos 4\theta=0$; whence, $4\theta=(2n\pm1)\tfrac{1}{2}\pi$,
or $\sin\theta=\tfrac{1}{2}\sqrt{2}$; whence, $\theta=n\pi+(-1)^n\tfrac{1}{4}\pi$.

53. Area $=\tfrac{1}{2}\times10\times15\times\sin 30^\circ$ sq. ft. $=\tfrac{1}{4}\times150$ sq. ft. $=37\tfrac{1}{2}$ sq. ft.

54. In this case (by Proportional parts) $d=\tfrac{782}{2212}$ of $60''=21{\cdot}2''$,
$\sin^{-1}(0{\cdot}649300)=40^\circ\,29'\,21{\cdot}2''$.

55. (1) $10'=\tfrac{1}{90}\times\tfrac{1}{6}$ right angle $=\tfrac{1}{540}$ of $\tfrac{1}{2}\pi$ radians
$=\tfrac{1}{540}$ of $\tfrac{11}{7}$ radians $=\cdot0026\ldots$ radians.

(2) $\tfrac{1}{5}$ of a right angle $=\tfrac{1}{10}$ of π radians $=\cdot314\ldots$ radians.
In the third case the unit is $\tfrac{1}{5}(5\times180^\circ-360^\circ)=108^\circ$,
a right angle $=\tfrac{90}{108}$ of $108^\circ=\tfrac{5}{6}$ of the unit.

56. $\cos\alpha=\tfrac{1}{7}$; $\therefore\ \sin\alpha=\pm\tfrac{1}{7}\sqrt{48}$; $\therefore\ \tan\alpha=\pm\sqrt{48}=\pm4\sqrt{3}$,
$\operatorname{cosec}\alpha=\pm7\div\sqrt{48}=\pm\tfrac{7}{12}\sqrt{3}$.

57. (1) $\cos^2(\alpha-\beta)-\sin^2(\alpha+\beta)$
$=\cos^2\alpha\,.\,\cos^2\beta+\sin^2\alpha\,.\,\sin^2\beta+2\cos\alpha\,.\,\cos\beta\,.\,\sin\alpha\,.\,\sin\beta$
$-\sin^2\alpha\,.\,\cos^2\beta-\cos^2\alpha\,.\,\sin^2\beta-2\cos\alpha\,.\,\cos\beta\,.\,\sin\alpha\,.\,\sin\beta$
$=\cos^2\alpha\,.\,\cos^2\beta+(1-\cos^2\alpha)(1-\cos^2\beta)-(1-\cos^2\alpha)\cos^2\beta-\cos^2\alpha\,(1-\cos^2\beta)$
$=4\cos^2\alpha\,.\,\cos^2\beta+1-2\cos^2\alpha-2\cos^2\beta$
$=(2\cos^2\alpha-1)(2\cos^2\beta-1)=\cos 2\alpha\,.\,\cos 2\beta$.

(2) $1-\tan\alpha\tan\tfrac{1}{2}\alpha=(\cos\alpha-\sin\alpha\tan\tfrac{1}{2}\alpha)\sec\alpha$
$=\{1-2\sin^2\tfrac{1}{2}\alpha+2\sin^2\tfrac{1}{2}\alpha)\sec\alpha=\sec\alpha$.

58. (1) $5\tan^2 x + 1 + \tan^2 x = 7, \quad \therefore 6\tan^2 x = 6;$

$$\therefore \tan x = \pm 1, \text{ whence } x = n\pi \pm \tfrac{1}{4}\pi.$$

(2) $2\cos 4\theta \cos\theta = \sqrt{2}\cos 4\theta, \quad \therefore \cos\theta = \tfrac{1}{2}\sqrt{2}$; or, $\cos 4\theta = 0$,

whence $\theta = 2n\pi \pm \tfrac{1}{4}\pi$; or, $4\theta = 2n\pi \pm \tfrac{1}{2}\pi$.

59. Area $= \sqrt{(7 \times 4 \times 2 \times 1)} = 2\sqrt{14} = 7{\cdot}478\ldots$ sq. ft.

60. Here $D = \frac{1243}{2279}$ of $60'' = 32{\cdot}2''$,

$$\therefore \sin^{-1}(0{\cdot}621500) = 38^\circ\, 25'\, 32{\cdot}2''.$$

61.
$$76^g = {\cdot}76 \text{ right angles},$$
$$1{\cdot}2^c = (1{\cdot}2 \times 2 \div \pi) \text{ right angles} = 1{\cdot}2 \times \tfrac{7}{11} \text{ right angles}$$
$$= {\cdot}763\ldots \text{ right angles};$$
$$\therefore 1{\cdot}2^c \text{ is greater than } 76^g.$$

62. See Arts. 92, 90. We have that $2\sin A = \sin B + \cos B$,

$$\therefore 1 - 2\sin^2 A = 1 - \tfrac{1}{2}(\sin B + \cos B)^2 = \tfrac{1}{2}(\sin B - \cos B)^2;$$
$$\therefore \cos 2A = \cos^2(B + 45^\circ).$$

63. (1) $\tan^2 A - \sin^2 A = \tan^2 A\,(1 - \sin^2 A \div \tan^2 A)$

$$= \tan^2 A\,(1 - \cos^2 A) = \tan^2 A \,.\, \sin^2 A.$$

(2) $\cot A - \cot 2A = \dfrac{\cos A}{\sin A} - \dfrac{\cos 2A}{\sin 2A}$

$$= \frac{2\cos^2 A - \cos 2A}{\sin 2A} = \frac{1}{\sin 2A} = \operatorname{cosec} 2A.$$

(3) $$\frac{\sin(x+3y) + \sin(3x+y)}{\sin 2x + \sin 2y} = \frac{2\sin 2(x+y) \,.\, \cos(x-y)}{2\sin(x+y)\cos(x-y)}$$
$$= \frac{2\sin(x+y)\cos(x+y)}{\sin(x+y)} = 2\cos(x+y).$$

64. See Art. 178. When $A = 240^\circ$, $\tfrac{1}{2}A = 120^\circ$ and then $\sin\tfrac{1}{2}A$ is greater than $\cos\tfrac{1}{2}A$ and is positive;

$$\therefore \sin\tfrac{1}{2}A + \cos\tfrac{1}{2}A = +\sqrt{(1+\sin A)},\ \sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A = +\sqrt{(1-\sin A)}.$$

Hence the formula is true.

65. See Art. 154.

$$\cos(A+B) = \sin\{90 - (A+B)\} = \sin\{(90 - A) + (-B)\}$$
$$= \sin(90^\circ - A) \,.\, \cos(-B) + \cos(90 - A) \,.\, \sin(-B)$$
$$= \cos A \,.\, \cos B - \sin A \,.\, \sin B.$$

[Since $\sin(-B) = -\sin B$ and $\cos(-B) = \cos B$.

$\sin A \,.\, \cos(B + C) - \sin B \,.\, \cos(A + C)$

$$= \sin A \,.\, \cos B \,.\, \cos C - \sin A \,.\, \sin B \,.\, \sin C - \sin B \,.\, \cos A \,.\, \cos C + \sin B \,.\, \sin A \,.\, \sin C$$
$$= \cos C\,(\sin A \,.\, \cos B - \sin B \,.\, \cos A) = \cos C \,.\, \sin(A - B).$$

66. Let a, b be the two sides,

$$a=102\times\frac{\sin 70^\circ 30'}{\sin 31^\circ 20'},\qquad b=102\times\frac{\sin 78^\circ 10'}{\sin 31^\circ 20'}.$$

$$\log a=\log 102+L\sin 70^\circ 30'-L\sin 31^\circ 20'.$$

$$\log c=2\cdot009+9\cdot974-9\cdot716=2\cdot267=\log 185.$$

$$\log b=\log 102+L\sin 78^\circ 10'-L\sin 31^\circ 20'$$

$$=2\cdot009+9\cdot990-9\cdot716=2\cdot283=\log 192.$$

Area of triangle $=\frac{1}{2}(192\times 185)\,.\,\sin 31^\circ 20'$ sq. ft.

$=17760\times\cdot520016$ sq. ft. $=9235\cdot48416$ sq. ft.

$=1026\cdot1649$ sq. yds.

67. $2\cdot3^c=(2\cdot3\div\pi)$ of $180^\circ=2\cdot3\times\frac{7}{22}$ of $180^\circ=131\frac{8}{11}{}^\circ$,

$\therefore$ $2\cdot3^c$ is the greater.

68. (1) $\cot^2 A-\cos^2 A=\cot^2 A\,(1-\cos^2 A\tan^2 A)=\cot^2 A\,(1-\sin^2 A)$.

(2) $\tan A+\cot 2A=\dfrac{2\sin^2 A}{\sin 2A}+\dfrac{\cos 2A}{\sin 2A}=\operatorname{cosec} 2A\,(2\sin^2 A+\cos 2A)$

$=\operatorname{cosec} 2A$.

(3) $=\dfrac{2\sin 2(x-y)\sin(x+y)}{2\sin(x+y)\cos(x-y)}=2\sin(x-y)$.

69. See Art. 178. When $\frac{1}{2}A=150^\circ$, $\cos\frac{1}{2}A$ is greater than $\sin\frac{1}{2}A$ and is negative;

$\therefore$ $\sin\frac{1}{2}A+\cos\frac{1}{2}A=-\sqrt{(1+\sin A)}$, $\sin\frac{1}{2}A-\cos\frac{1}{2}A=+\sqrt{(1-\sin A)}$.

70. $\sin 30^\circ+\sin 120^\circ=\frac{1}{2}+\frac{1}{2}\sqrt{3}=\sqrt{2}\,.\cos 15^\circ$.

71. (1) $1+\cos A+\sin A=2\cos^2\frac{1}{2}A+2\cos\frac{1}{2}A\sin\frac{1}{2}A$

$=2\cos\frac{1}{2}A\,(\cos\frac{1}{2}A+\sin\frac{1}{2}A)$

$=\sqrt{\{4\cos^2\frac{1}{2}A\,(1+2\cos\frac{1}{2}A\sin\frac{1}{2}A)\}}$

$=\sqrt{\{2\,(1+\cos A)(1+\sin A)\}}$.

(2) $\operatorname{cosec} 2A=\dfrac{1}{2\sin A\cos A}=\dfrac{\sin A}{2\sin^2 A\cos A}$

$=\dfrac{\operatorname{cosec}^2 A}{2\cot A}=\dfrac{\operatorname{cosec}^2 A}{2\sqrt{(\operatorname{cosec}^2 A-1)}}$.

(3) $\sin\frac{2}{7}\pi+\sin\frac{4}{7}\pi-\sin\frac{6}{7}\pi=2\sin\frac{3}{7}\pi\cos\frac{1}{7}\pi-2\sin\frac{3}{7}\pi\cos\frac{3}{7}\pi$

$=2\sin\frac{3}{7}\pi\,(\cos\frac{1}{7}\pi-\cos\frac{3}{7}\pi)=2\sin\frac{3}{7}\pi\sin\frac{2}{7}\pi\sin\frac{1}{7}\pi$.

and $\sin\frac{2}{7}\pi=\sin(\pi-\frac{2}{7}\pi)=\sin\frac{5}{7}\pi$.

72. Let A, B be the angles opposite the sides whose lengths are 185, 192 feet respectively.

$$\sin A=\sin 31^\circ 20'\times\tfrac{185}{102};$$

$$\therefore\ L\sin A=L\sin 31^\circ 20'+\log 185-\log 102$$

$$=9\cdot716+2\cdot267-2\cdot009=9\cdot974$$

$$=L\sin 70^\circ 30',\ A=70^\circ 30'.$$

$$L \sin B = L \sin 31^\circ 20' + \log 192 - \log 102$$
$$= 9{\cdot}716 + 2{\cdot}283 - 2{\cdot}009 = 9{\cdot}990$$
$$= L \sin 78^\circ 10'; \;\therefore\; B = 78^\circ 10'.$$

The area $= \frac{1}{2}(192 \times 185) \sin 31^\circ 21'$ sq. ft.
$= 17760 \times {\cdot}520016$ sq. ft. $= 9235{\cdot}48416\ldots$ sq. ft.

73. Let the angles be $\alpha - 2\beta$, $\alpha - \beta$, α, $\alpha + \beta$, $\alpha + 2\beta$.
Their sum namely, $5\alpha = 2\pi$; $\therefore\; \alpha = \frac{2}{5}\pi$, also $\alpha + 2\beta = 6$ times $(\alpha - 2\beta)$.
$\therefore\; 14\beta = 5\alpha$; $\therefore\; \beta = \frac{5}{14}$ of $\frac{2}{5}$ of $\pi = \frac{1}{7}\pi$; $\therefore\; \alpha + 2\beta = (\frac{2}{5} + \frac{2}{7})\pi$; etc.

74. See Art. 75 and Art. 141, Ex. 4,
$$\sin(180^\circ + A) = \sin(90^\circ + 90^\circ + A) = \cos(90^\circ + A) = -\sin A,$$
$$\cos(180^\circ + A) = \cos(90^\circ + 90^\circ + A) = -\sin(90^\circ + A) = -\cos A.$$

75. (1) See Art. 107. $\cot^2 A = \dfrac{OM^2}{MP^2} = \dfrac{OP^2 - MP^2}{MP^2} = \dfrac{OP^2}{MP^2} - 1.$

(2) $\cot^4 A + \cot^2 A = (\operatorname{cosec}^2 A - 1)^2 + \operatorname{cosec}^2 A - 1$
$= \operatorname{cosec}^4 A - 2\operatorname{cosec}^2 A + 1 + \operatorname{cosec}^2 A - 1.$

When $A = 30^\circ$ the above statement becomes $(\sqrt{3})^4 + (\sqrt{3})^2 = 2^4 - 2^2$,
that is $9 + 3 = 16 - 4$; which is true.

76. See Art. 147. $2\cos^3\theta - \cos^2\theta = 0$; $\therefore\; \cos^2\theta(2\cos\theta - 1) = 0$;
$\therefore$ either, $\cos\theta = 0$; whence, $\theta = n\pi + \frac{1}{2}\pi$,
or $\cos\theta = \frac{1}{2}$; whence, $\theta = 2n\pi \pm \frac{1}{3}\pi$.

77. $\sin^2 B = \sin A \cos A$,
$\therefore\; 1 - 2\sin^2 B = \sin^2 A - 2\sin A\cos A + \cos^2 A = (\cos A - \sin A)^2$:
$\therefore\; \cos 2B = 2\cos^2(A + 45^\circ)$.

78. See 48 above. Let the two triangles be ABC_1, ABC_2 in Fig. III. on p. 216. Then $c = \sqrt{3}$, $b = 1$, $b^2 = c^2 + a^2 - 2ca\cos 30^\circ$;
$\therefore\; 1 = 3 + a^2 - 3a$; $\therefore\; a = \frac{1}{2}(\sqrt{3} + 1)$ or $\frac{1}{2}(\sqrt{3} - 1)$;
$\therefore\; BC_1 = \frac{1}{2}(\sqrt{3} + 1)$, $BC_2 = \frac{1}{2}(\sqrt{3} - 1)$,
and the areas of the triangles are in the ratio $BC_1 : BC_2$.

79. Let the angles subtended by each be
α, $\alpha + \beta$, $\alpha + 2\beta$, $\alpha + 3\beta$, $\alpha + 4\beta$, $\alpha + 5\beta$ respectively,
then $\alpha + 5\beta = 6\alpha$; whence, $\alpha = \beta$; also $6\alpha + 15\beta = 2\pi$;
$\therefore\; \alpha = \beta = \frac{2}{21}\pi = \frac{44}{147}$ radians.

80. Art. 75. See Ex. 4, p. 107.
$$\tan(90^\circ + 90^\circ + A) = -\cot(90^\circ + A) = \tan A.$$

81. See Art. 148. $\sec^3\theta - 2(\tan^2\theta + 1) = 0$; $\therefore\; \sec^3\theta - 2\sec^2\theta = 0$;
$\therefore$ either $\sec\theta = 0$; which gives no solution,
or, $\sec\theta = 2$; whence $\theta = 2n\pi \pm \frac{1}{3}\pi$.

82. Art. 154.

$\sin(A+B) = \cos[90-(A+B)] = \cos[90-A+(-B)]$
$= \cos(90-A) \,.\, \cos(-B) - \sin(90-A) \,.\, \sin(-B)$
$= \sin A \,.\, \cos B + \cos A \,.\, \sin B,$

since $\sin(-B) = -\sin B$ and $\cos(-B) = \cos B$.

$\cos A \,.\, \cos(B+C) - \cos B \,.\, \cos(A+C)$
$= \cos A \,.\, \cos B \,.\, \cos C - \cos A \,.\, \sin B \,.\, \sin C$
$- \cos B \,.\, \cos A \,.\, \cos C + \cos B \,.\, \sin A \,.\, \sin C$
$= \sin C(\cos B \,.\, \sin A - \cos A \,.\, \sin B) = \sin(A-B) \,.\, \sin C.$

83. (1) $1 + \cos A - \sin A = 2\cos^2 \frac{1}{2}A - 2\sin \frac{1}{2}A \cos \frac{1}{2}A$
$= \sqrt{\{4\cos^2 \frac{1}{2}A (\cos \frac{1}{2}A - \sin \frac{1}{2}A)^2\}}$
$= \sqrt{\{2(1+\cos A)(1-\sin A)\}}.$

(2) $\sec 2A = \dfrac{1}{2\cos^2 A - 1} = \dfrac{\sec^2 A}{2 - \sec^2 A}.$

(3) $\cos \frac{2}{7}\pi + \cos \frac{4}{7}\pi + \cos \frac{6}{7}\pi + 1 = 2\cos \frac{3}{7}\pi \cos \frac{1}{7}\pi + 2\cos^2 \frac{3}{7}\pi$
$= 2\cos \frac{3}{7}\pi(\cos \frac{1}{7}\pi + \cos \frac{3}{7}\pi) = 4\cos \frac{3}{7}\pi \cos \frac{2}{7}\pi \cos \frac{1}{7}\pi,$

and $\cos \frac{2}{7}\pi = \cos(\pi - \frac{5}{7}\pi) = -\cos \frac{5}{7}\pi.$ Q. E. D.

84. The shorter diagonal $= \sqrt{(5^2 + 3^2 - 2 \,.\, 5 \,.\, 3 \,.\, \cos 60)}$ in.
$= \sqrt{(25 + 9 - 30 \times \frac{1}{2})}$ in. $= \sqrt{(19)}$ in. $= 4{\cdot}35\ldots$ in.,

the longer diagonal $= \sqrt{(5^2 + 3^2 + 2 \,.\, 5 \,.\, 3 \,.\, \cos 60^\circ)}$ in.
$= \sqrt{(25 + 9 + 30 \times \frac{1}{2})}$ in. $= \sqrt{49}$ in. $= 7$ in.,

area of parallelogram $= 5 \times 3 \times \sin 60^\circ$ sq. in.
$= \frac{15}{2} \times \sqrt{3}$ sq. in. $= \frac{15}{2} \times 1{\cdot}732\ldots$ sq. in. $= 13$ sq. in. nearly.

85. (1) $\sin \frac{3}{2}A = \sin A \cos \frac{1}{2}A + \cos A \sin \frac{1}{2}A$
$= 2\sin \frac{1}{2}A \cos^2 \frac{1}{2}A + \cos A \sin \frac{1}{2}A$
$= \sin \frac{1}{2}A(1 + \cos A + \cos A).$

(2) $\sin(\theta + \delta) - \sin\theta = 2\sin \frac{1}{2}\delta \cos(\theta + \frac{1}{2}\delta)$
$= 2\sin \frac{1}{2}\delta(\cos\theta \cos \frac{1}{2}\delta - \sin\theta \sin \frac{1}{2}\delta)$
$= \cos\theta \sin\delta - \sin\theta \, 2\sin^2 \frac{1}{2}\delta$
$= \cos\theta \sin\delta\{1 - \tan\theta \, 2\sin^2 \frac{1}{2}\delta \div (2\sin \frac{1}{2}\delta \cos \frac{1}{2}\delta)\}.$

86. (1) $\sin 10^\circ + \sin 50^\circ = 2\sin 30^\circ \cos 20^\circ = \cos 20^\circ = \sin 70^\circ.$

(2) $\sqrt{3} = \tan 60^\circ$; $\therefore$ we have to prove that
$\tan 60^\circ + \tan 40^\circ + \tan 80^\circ = \tan 60^\circ \tan 40^\circ \tan 80^\circ,$

which is true by Ex. LXII. 32, since $60^\circ + 40^\circ + 80^\circ = 180^\circ$.

(3) $\sin A - \sin B \cos C = \sin(B+C) - \sin B \cos C = \sin C \cos B.$

Also $\sin B - \sin A \cos C = \sin(A+C) - \sin A \cos C = \sin C \cos A,$

and the result follows.

87. See T. p. 56. $\sin 18° = \frac{1}{4}(\sqrt{5}-1)$.

$$4 \sin 18° \cos 36° = 4 \sin 18° (1 - 2\sin^2 18°) = (\sqrt{5}-1)\{1 - \tfrac{1}{8}(\sqrt{5}-1)^2\}$$
$$= \tfrac{1}{8}(\sqrt{5}-1)(2+2\sqrt{5}) = \tfrac{1}{4}(\sqrt{5}-1)(\sqrt{5}+1) = 1.$$

88. Let a, b be the other sides of the triangle,

$$a = 1006{\cdot}62 \times \sin 70° \div \sin 66°.$$

$$\log a = \log 1006{\cdot}62 + L \sin 70° - L \sin 66°$$
$$= 3{\cdot}0028656 + 9{\cdot}9729858 - 9{\cdot}9607302 = 3{\cdot}0151212 = \log 1035{\cdot}43.$$

$$\log b = \log 1006{\cdot}62 + L \sin 44° - L \sin 66°$$
$$= 3{\cdot}0028656 + 9{\cdot}8417713 - 9{\cdot}9607302 = 2{\cdot}8839067 = \log 765{\cdot}4321.$$

89. Whole circumference $= 2\pi r = 16\pi$ ft.,

$\therefore$ length of arc which subtends at the centre an angle of 50°

$$= \tfrac{50}{360} \times 16\pi \text{ ft.} = \tfrac{20}{9} \times 3{\cdot}1416 \text{ ft.} = 6{\cdot}9813 \text{ ft.}$$

90. Fig. p. 106. Let $POM = A$; $ROP' = A - 180°$.

$-\sin ROP' = -\sin(A - 180°) = \sin A$, $\sin 30° = \frac{1}{2}$, [Art. 92]

$\sin 2010° = \sin(5 \times 360 + 180 + 30) = -\sin 30° = -\frac{1}{2}$.

91. Art. 144. The general value of $\operatorname{cosec}^{-1}(\sqrt{2})$ is $2n\pi \pm \frac{1}{4}\pi$.

92. (1) $\cos^2 A + \cos^2 B - 2 \cos A \,.\, \cos B\, (\cos A \,.\, \cos B - \sin A \,.\, \sin B)$

$= \cos^2 A + \cos^2 B - 2\cos^2 A \,.\, \cos^2 B + 2 \cos A \,.\, \sin B \,.\, \sin A \,.\, \cos B$

$= \cos^2 A\,(1 - \cos^2 B) + \cos^2 B\,(1 - \cos^2 A) + 2 \cos A \,.\, \sin B \,.\, \sin A \,.\, \cos B$

$= \cos^2 A \,.\, \sin^2 B + \cos^2 B \,.\, \sin^2 A + 2\cos A \,.\, \sin B \,.\, \sin A \,.\, \cos B$

$= \sin^2(A+B)$.

(2) $\cos^2 A + \sin^2 A \,.\, \cos 2B = \cos^2 A + (1 - \cos^2 A)(2\cos^2 B - 1)$

$= \cos^2 A + 2\cos^2 B - 2\cos^2 A \,.\, \cos^2 B - 1 + \cos^2 A$

$= \cos^2 B + 2\cos^2 A - 2\cos^2 A \,.\, \cos^2 B - 1 + \cos^2 B$

$= \cos^2 B + (1 - \cos^2 B)(2\cos^2 A - 1)$

$= \cos^2 B + \sin^2 B \cos 2A$.

93. $a^2 \cos 2B + b^2 \cos 2A = a^2(1 - 2\sin^2 B) + b^2(1 - 2\sin^2 A)$

$= a^2 + b^2 - 2a \sin Ba \sin B - 2b \sin Ab \sin A$

$= a^2 + b^2 - 4ab \sin A \sin B$ [for $a \sin B = b \sin A$].

94. $c = a \sin 135° \div \sin 15° 43'$;

$\therefore \log c = \log a + L \sin 45° - L \sin 15° 43'$

$= 2{\cdot}0899051 - \frac{1}{2}\log 2 - 9{\cdot}4327777 + 10$

$= 12{\cdot}0899051 - \frac{1}{2} \times {\cdot}30103 - 9{\cdot}4327777$

$= 2{\cdot}5066124 = \log 321{\cdot}1207$.

95. Art. 60. Let r = the radius of the circle,

$$\frac{2\pi r}{12} = \frac{360°}{50°}, \quad r = \frac{36 \times 12}{5 \times 2\pi} \text{ ft.} = \frac{216}{5 \times 3{\cdot}1416} \text{ ft.} = 13{\cdot}75 \text{ ft.}$$

96. See Ex. XVI. 4. $\cos A + 1 = 5 \sin A$;

$\therefore \cos^2 A + 2\cos A + 1 = 25 - 25\cos^2 A$; $\therefore 26\cos^2 A + 2\cos A = 24$;

$\therefore \cos^2 A + \frac{1}{13}\cos A = \frac{12}{13}$; $\therefore (\cos A - \frac{12}{13})(\cos A + 1) = 0$;

$\therefore$ either $\cos A = \frac{12}{13}$ or $\cos A = -1$.

This last value makes $\sin A = 0$, which satisfies the equation in a partial sense only; so that $A = (2n+1)\pi$.

97. See Art. 147. The angle whose secant is -2 is $(2n+1)\pi \pm \frac{1}{3}\pi$.

98. (1) $\sin^2 A + \sin^2 B + 2\sin A \,.\, \sin B \cos(A+B)$

$= \sin^2 A + \sin^2 B + 2\sin A \,.\, \sin B \,.\, (\cos A \,.\, \cos B - \sin A \,.\, \sin B)$

$= \sin^2 A + \sin^2 B + 2\sin A \,.\, \sin B \,.\, \cos A \,.\, \cos B - 2\sin^2 A \,.\, \sin^2 B$

$= \sin^2 A(1 - \sin^2 B) + \sin^2 B(1 - \sin^2 A) + 2\sin A \,.\, \sin B \cos A \,.\, \cos B$

$= \sin^2 A \,.\, \cos^2 B + \sin^2 B \,.\, \cos^2 A + 2\sin A \,.\, \cos A \,.\, \sin B \,.\, \cos B$

$= \sin^2(A+B)$.

(2) $\sin^2 A - \cos^2 A \cos 2B = \sin^2 A - (1 - \sin^2 A)(2\cos^2 B - 1)$

$= \sin^2 A - 2\cos^2 B + 2\sin^2 A \cos^2 B + 1 - \sin^2 A$

$= \sin^2 B - \cos^2 B(1 - 2\sin^2 A)$.

99. See Art. 178. $1200° = 3 \times 363° + 120°$.

$\sin 120°$ is greater than $\cos 120°$ and is positive,

$\therefore \sin 1200° + \cos 1200° = +\sqrt{(1 + \sin A)}$.

$\sin 1200° - \cos 1200° = -\sqrt{(1 - \sin A)}$.

100.
$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1 - 2\sin^2 A}{a^2} - \frac{1 - 2\sin^2 B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2},$$
since $b^2 \sin^2 A = a^2 \sin^2 B$.

101. (i) $2\cot 2A = \dfrac{\cot^2 A - 1}{\cot A} = \cot A - \tan A$.

(ii) $\sin(\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13}) = \sin(\sin^{-1}\frac{3}{5})\cos(\sin^{-1}\frac{5}{13})$
$- \cos(\sin^{-1}\frac{3}{5})\sin(\sin^{-1}\frac{5}{13})$
$= \frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13} = \frac{16}{65}$. Q. E. D. [See 106 (iii)].

(iii) $\cot(A + 15°) - \tan(A - 15°)$
$$= \frac{\cos(A+15°)\cos(A-15°) - \sin(A-15°)\sin(A+15°)}{\sin(A+15°)\cos(A-15°)} = \frac{2\cos 2A}{\sin 2A + \sin 30°}.$$

102. When $\cos(2x+3y) = \frac{1}{2}$ and $\cos(3x+2y) = \frac{1}{2}\sqrt{3}$,

we have $2x + 3y = 2x\pi \pm \frac{1}{3}\pi$ and $3x + 2y = 2x\pi \pm \frac{1}{6}\pi$;

$\therefore$ solving these equations we obtain the required values.

103. $\operatorname{Sin} x$ increases from 0 to 1 while x changes from 0 to $\frac{1}{2}\pi$;

$\therefore \sin(\pi \sin x)$ changes from 0 to 1 and from 1 to 0 as $\pi \sin x$ goes from 0 to π.

$\sin(\pi \sin x)$ goes from 0 to 1 and from 1 to 0 again as x goes from $\frac{1}{2}\pi$ to π.

$\sin(\pi \sin x)$ goes from 0 to -1 and from -1 to 0 as x goes from π to $\frac{3}{2}\pi$, and repeats as x goes from $\frac{3}{2}\pi$ to 2π.

104.
$$\tan(\tan^{-1}a+\tan^{-1}b)=\frac{a+b}{1-ab},$$
$$\tan(2\tan^{-1}a)=\frac{2a}{1-a^2};$$
$$\therefore \frac{x+1-x}{1-x(1-x)}=\frac{2\sqrt{(x-x^2)}}{1-x+x^2};$$
$$\therefore 4x-4x^2=1;\ \therefore x=\tfrac{1}{2}.$$

105. Let A and B be the two positions of the ship C, D that of the lighthouses.

$$AB=20,\ BAC=22°\,30',\ BAD=45°;\ \therefore BD=20;$$
$$BC=20\,.\,\tan 22°\,30',\ \therefore \log BC=\log 20+L\tan 22°\,30'-10$$
$$=1+\cdot30103+9\cdot6172243-10$$
$$=\cdot9182543=\log 8\cdot2842;$$
$$CD=20-BC=11\cdot7157.$$

106. (i)
$$\tan\tfrac{1}{2}\theta=\sqrt{\frac{2\sin^2\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.$$

(ii)
$$\frac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=\frac{\sin 5\theta\,.\,\cos 3\theta+\sin 3\theta\,.\,\cos 5\theta}{\sin 5\theta\,.\,\cos 3\theta-\sin 3\theta\,.\,\cos 5\theta}$$
$$=\frac{\sin 8\theta}{\sin 2\theta}=\frac{2\sin 4\theta\,.\,\cos 4\theta}{\sin 2\theta}=2\cos 2\theta\,.\,\cos 4\theta.$$

(iii)
$$\sin^{-1}\tfrac{3}{5}=\alpha,\ \therefore \sin\alpha=\tfrac{3}{5};\ \cos\alpha=\tfrac{4}{5},$$
$$\sin^{-1}\tfrac{8}{17}=\beta,\ \therefore \sin\beta=\tfrac{8}{17};\ \cos\beta=\tfrac{15}{17},$$
$$\sin(\alpha+\beta)=\sin\alpha\,.\,\cos\beta+\cos\alpha\,.\,\sin\beta$$
$$=\frac{3}{5}\times\frac{15}{17}+\frac{4}{5}\times\frac{8}{17}=\frac{45+32}{85}=\frac{77}{85},$$
$$\alpha+\beta=\sin^{-1}\tfrac{77}{85},\ \text{or}\ \sin^{-1}\tfrac{3}{5}+\sin^{-1}\tfrac{8}{17}=\sin^{-1}\tfrac{77}{85}.$$

107.
$$2\sin^2\theta-(1+\sqrt{3})\sin 2\theta+2\sqrt{3}\cos^2\theta=0,$$
$$(\sin\theta-\sqrt{3}\cos\theta)(2\sin\theta-2\cos\theta)=0,$$
$$\sin\theta-\sqrt{3}\cos\theta=0;\ \tan\theta=\sqrt{3};\ \therefore \theta=n\pi+\tfrac{1}{3}\pi,$$
or
$$2\sin\theta-2\cos\theta=0;\ \tan\theta=1;\ \therefore \theta=n\pi+\tfrac{1}{4}\pi.$$

108. (i) See Ex. LXIV. 7.

(ii)
$$c(\cos A+\cos B)=c\left\{\frac{b^2+c^2-a^2}{2bc}+\frac{a^2+c^2-b^2}{2ac}\right\}$$
$$=\frac{1}{2ab}\{a(b^2+c^2-a^2)+b(a^2+c^2-b^2)\}$$
$$=\frac{1}{2ab}\{(a+b)(c^2+ab)-(a^3+b^3)\}$$

$$=\frac{1}{2ab}\{(a+b)(c^2-a^2+2ab-b^2)\}$$

$$=\frac{1}{2ab}(a+b)\,4\,(s-a)(s-b)=2(a+b)\sin^2\tfrac{1}{2}C.$$

109. The diameter of the circle circumscribing AEF is $(FE\div\sin A)$ and the angle $FEB=FCB$; $\therefore\ \triangle ABC$ is similar to AFE;

$$\therefore\ FE : AE = a : b;$$

$$\therefore\ \text{diameter of the circle}=\frac{a}{\sin A}\times\frac{AE}{b}=\frac{a}{\sin A}\cos A=a\cot A.$$

Similarly for the other diameter.

110. Let x be the distance from the second point of observation to the top of the tower,

then $x=50\times\sin 10\div\sin 5°$

$=100\times\sin 5°\cos 5°\div\sin 5°=100\cos 5°,$

$\log x=2+L\cos 5°-10$

$=11\cdot9983442-10=1\cdot998342;$

$\therefore$ height of tower $=x\,.\,\sin 15,$

$\log(\text{height})=\log x+L\sin 15°-10$

$=1\cdot9983442+9\cdot4129962-10$

$=1\cdot4113404=\log 25\cdot7834.$

111. $\sin^2 A=\sin B\,.\,\cos B,$

or $1-2\sin^2 A=1-2\sin B\,.\,\cos B$

$=\{\cos^2 B-2\sin B\,.\,\cos B+\sin^2 B\}$

$=2\,\{(\cos B-\sin B)\div\sqrt{2}\}\,\{(\cos B-\sin B)\div\sqrt{2}\}$

$=2\,.\,\sin(45°-B)\cos(45°+B).$

112. (i) $\sin A(\cos 2A+\cos 4A+\cos 6A)$

$=\sin A(2\cos 4A\,.\,\cos 2A+\cos 4A)=\sin A\,.\,\cos 4a(2\cos 2A+1)$

$=\sin A\,.\,\cos 4a(2-4\sin^2 A+1)$

$=\cos 4a(3\sin A-4\sin^3 A)$

$=\sin 3A\,.\,\cos 4a.$

(ii) $\sin^{-1}\tfrac{1}{2}=30°;\ \therefore\ 2\sin^{-1}\tfrac{1}{2}=60°,$

and $\cos^{-1}\tfrac{1}{2}=60°;\ \therefore\ 2\sin^{-1}\tfrac{1}{2}=\cos^{-1}\tfrac{1}{2}.$

113. Let AB and BC produced meet in E.

Draw BF, CG perpendiculars to AE; then,

$$\cos(A+B)=\cos(180°-E)=-\cos E,\ \text{and}$$

$$AB\cos A-BC\cos(A+B)+CD\cos D=AB\times AF\div AB+BC\times\cos E+DG$$

$$=AF+FD+DG=AD.$$

[Notice that $AF=-FA$.]

114. $360° - (2A + 2B) = 2C$; $\therefore$ $\tan 2C = -\tan(2A + 2B)$, and proceed as in LXII. 32.

Let $\tan 2A = x$, $\tan 2B = y$, $\tan 2C = z$, then, since $x + y + z = xyz$, we have $2C = 360° - 2A - 2B$. We have to prove that

$$\frac{(1-y^2)(1-z^2)}{2y2z} + \frac{(1-z^2)(1-x^2)}{2z2x} + \frac{(1-x^2)(1-y^2)}{2x2y} = 1,$$

that is, that, $\cot 4B \cot 4C + \cot 4C \cot 4A + \cot 4A \cot 4B = 1$,
that is, then $\tan 4A + \tan 4B + \tan 4C = \tan 4A \tan 4B \tan 4C$,
and this is true since $4A + 4B + 4C = 2 \times 360°$.

115. Let a and b be the distances the two ships are respectively from the beacon, $b = 1$ mile since the triangle formed is isosceles,

then $$a = \sin 75° \, 9' \, 30'' \div \sin 52° \, 25' 15'',$$

$$\log a = 9\cdot9852635 - 9\cdot8990055 = \cdot0862580 = \log 1\cdot219714.$$

116. Let A, B, C be the three angles

$$\cos a = \tfrac{3}{5}, \; \therefore \; \sin a = \tfrac{4}{5}, \; \cos B = \tfrac{12}{13}, \; \therefore \; \sin B = \tfrac{5}{13},$$

$$\cos C = -\cos(A + B) = -(\cos A \, . \, \cos B - \sin A \, . \, \sin B)$$

$$= -\left(\tfrac{3}{5} \times \tfrac{12}{13} - \tfrac{4}{5} \times \tfrac{5}{13}\right) = -\left(\tfrac{36}{65} - \tfrac{20}{65}\right) = -\tfrac{16}{35};$$

$$\sin C = \sqrt{\{1 - (\tfrac{16}{35})^2\}} = \sqrt{960} \div 35, \quad \tan C = -\sqrt{960} \div 35.$$

117. (i) $$\frac{\sin A + 2\sin 3A + \sin 5A}{\cos A - 2\cos 3A + \cos 5A} = \frac{2\sin 3A \, . \, \cos 2a + 2\sin 3A}{2\cos 3a \, . \, \cos 2a - 2\cos 3a}$$

$$= \frac{\sin 3A\,(1 + \cos 2a)}{\cos 3A\,(\cos 2a - 1)} = -\frac{\sin 3A \times 2\cos^2 A}{\cos 3A \times 2\sin^2 A}$$

$$= \frac{(4\sin^3 A - 3\sin A) \times \cos^2 A}{(4\cos^3 A - 3\cos A) \times \sin^2 A} = \frac{4\sin A - 3\operatorname{cosec} A}{4\cos A - 3\sec A}.$$

(ii) Let $\cot^{-1} 3 = \alpha$, $\cot^{-1} \tfrac{3}{4} = \beta$,

$$\cot \alpha = 3, \qquad \cot \beta = \tfrac{3}{4},$$

$$\cot(\alpha + \beta) = \frac{\cot \alpha \, . \, \cot \beta - 1}{\cot \alpha + \cot \beta} = \frac{3 \times \tfrac{3}{4} - 1}{3 + \tfrac{3}{4}} = \tfrac{1}{3},$$

$$\alpha + \beta = \cot^{-1} \tfrac{1}{3}; \text{ or, } \cot^{-1} 3 + \cot^{-1} \tfrac{3}{4} = \cot^{-1} \tfrac{1}{3}.$$

118. Area $= \sqrt{(1452 \times 1210 \times 240 \times 2)}$ sq. yds.
$= \sqrt{(11^2 \times 2^2 \times 3 \times 11^2 \times 10 \times 2^3 \times 3 \times 10 \times 2)}$ sq. yds.
$= \sqrt{(11^4 \times 2^6 \times 3^2 \times 10^2)}$ sq. yds. $= 11^2 \times 2^3 \times 3 \times 10$ sq. yds.
$= 121 \times 4 \times 10 \times 6$ sq. yds. $= 4840 \times 6$ sq. yds. $= 6$ acres.

119. Fig. p. 232.

$$ad_1^2 + bd_2^2 + cd_3^3 = ar^2 \operatorname{cosec}^2 \tfrac{1}{2} A + br^2 \operatorname{cosec}^2 \tfrac{1}{2} B + cr^2 \operatorname{cosec}^2 \tfrac{1}{2} C.$$

Since $$ar^2 \operatorname{cosec}^2 \tfrac{1}{2} A = \frac{S\,(s-a)\,(s-b)\,(s-c)}{s^2\,(s-b)\,(s-c)} \times abc = abc \times (s-a)\,s;$$

$$\therefore \; ad_1^2 + bd_2^2 + cd_3^2 = abc \times (s - a + s - b + s - c) \div s = abc.$$

120. Let A be the foot of the slope, B, C the two points of observation, D the object, then $BDA=\alpha$, $CDA=\beta$, $\gamma=$angle.

$$\frac{BD}{c}=\frac{\sin(\gamma-\beta)}{\sin(\beta-\alpha)} \text{ also } \frac{BD}{c}=\frac{\sin\gamma}{\sin\alpha};$$

$$\therefore \frac{\sin\gamma}{\sin\alpha}=\frac{\sin\gamma\cos\beta-\cos\gamma\sin\beta}{\sin(\beta-\alpha)};$$

$$\therefore \frac{\sin\gamma}{\cos\gamma}=\frac{\sin\alpha\tan\gamma\cos\beta-\sin\alpha\sin\beta}{\sin(\beta-\alpha)};$$

$$\therefore \tan\gamma\left(1-\frac{\sin\alpha\cos\beta}{\sin(\beta-\alpha)}\right)=-\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)};$$

$$\therefore \cot\gamma=\frac{2\sin\alpha\cos\beta-\sin\beta\cos\alpha}{\sin\alpha\sin\beta}=2\cot\beta-\cot\alpha.$$

121. $\tan^{-1}(\frac{1}{2}\tan 2A)+\tan^{-1}\cot A$

$$=\tan^{-1}\frac{\frac{1}{2}\tan 2A+\cot A}{1-\frac{1}{2}\tan 2A\cot A}=\tan^{-1}\frac{\tan A\div(1-\tan^2 A)+\cot A}{1-1\div(1-\tan^2 A)}$$

$$=\tan^{-1}\frac{\tan^2 A+1-\tan^2 A}{\tan A-\tan^3 A-\tan A}=-\tan^{-1}\cot^3 A. \quad \text{Q.E.D.}$$

122.
$$\frac{1-\tan\frac{1}{2}x}{1+\tan\frac{1}{2}x}+\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}=4.$$

$$\therefore 6\tan^2\tfrac{1}{2}x=2; \quad \therefore \tfrac{1}{2}x=\tan^{-1}\pm\tfrac{1}{3}\sqrt{3}=n\pi\pm\tfrac{1}{6}\pi.$$

123. When A lies between 90° and 180°, then $\frac{1}{2}A$ is between 45° and 90°; in which case $\sin\frac{1}{2}A$ is greater than $\cos\frac{1}{2}A$ and positive;

$$\therefore \sqrt{(1+\sin A)}=\cos\tfrac{1}{2}A+\sin\tfrac{1}{2}A=1-2\sin^2\tfrac{1}{4}A+2\sin\tfrac{1}{4}A\cos\tfrac{1}{4}A$$
$$=1+2\sin\tfrac{1}{4}A(\cos\tfrac{1}{4}A-\sin\tfrac{1}{4}A)$$
$$=1+2\sin\tfrac{1}{4}A\sqrt{(1-\sin\tfrac{1}{2}A)},$$

for $\frac{1}{4}A$ lies between $22\frac{1}{2}$° and 45° and in that case $\cos\frac{1}{4}A$ is greater than $\sin\frac{1}{4}A$ and is positive, and $\therefore \cos\frac{1}{4}A-\sin\frac{1}{4}A=+\sqrt{(1-\sin\frac{1}{2}A)}$.

124. $a\sin A+b\sin B+c\sin C=0, \quad a\cos A+b\cos B+c\cos C=0.$

$$a\sin A\cos C+b\sin B\cos C+c\sin C\cos C=0,$$
$$a\cos A\sin C+b\cos B\sin C+c\sin C\cos C=0;$$
$$\therefore a\sin(C-A)=b\sin(B-C); \quad \therefore a\div\sin(B-C)=b\div\sin(C-A).$$

Similarly $\quad a\div\sin(B-C)=c\div\sin(A-B);$

$$\therefore a:b:c=\sin(B-C):\sin(C-A):\sin(A-B).$$

125. Let AB, BC be the two parts of the ascent, $CD=a$ the height of mountain,

then $\quad AB=AC\cdot\frac{\sin(\beta-\gamma)}{\sin(\beta-\alpha)}; \; BC=AC\cdot\frac{\sin(\gamma-\alpha)}{\sin(\beta-\alpha)}; \; AC=\frac{a}{\sin\gamma}.$

$$\therefore AB+BC=\frac{a}{\sin\gamma}\times\frac{\sin(\beta-\gamma)+\sin(\gamma-\alpha)}{\sin(\beta-\alpha)}$$

$$=a\frac{2\sin\frac{1}{2}(\beta-\alpha)\cos\{\frac{1}{2}(\alpha+\beta)-\gamma\}}{2\sin\gamma\cos\frac{1}{2}(\beta-\alpha)\cdot\sin\frac{1}{2}(\beta-\alpha)}=a\cdot\frac{\cos\{\frac{1}{2}(\alpha+\beta)-\gamma\}}{\sin\gamma\cos\frac{1}{2}(\beta-\alpha)}.$$

126. $\operatorname{cosec} 2A\ (\operatorname{cosec} A + \operatorname{cosec} 3A) = \operatorname{cosec} 2A \left(\dfrac{\sin 3A + \sin A}{\sin A \sin 3A}\right)$

$$= \frac{2 \sin 2A \cos A}{2 \sin A \cos A \sin A \sin 3A} = \frac{\sin(3A - A)}{\sin^2 A \sin 3A}$$

$$= \frac{\sin A \cos 3A - \cos 3A \sin A}{\sin^2 A \sin 3A}.$$

127. $$2 \cos 4\theta \cos \theta + \sqrt{2} (\sin \theta + \cos \theta) \cos \theta = 0,$$

$$\therefore \text{ either } \cos \theta = 0 \text{ and } \theta = n\pi + \tfrac{1}{2}\pi,$$

or $\cos 4\theta + \cos(\theta - \frac{1}{4}\pi) = 0$, that is $2 \cos(\frac{5}{2}\theta - \frac{1}{8}\pi) \cos(\frac{3}{2}\theta + \frac{1}{8}\pi) = 0$,

whence, either $\cos(\frac{5}{2}\theta - \frac{1}{8}\pi) = 0$ and $\frac{5}{2}\theta - \frac{1}{8}\pi = n\pi + \frac{1}{2}\pi$,

or $\cos(\frac{3}{2}\theta + \frac{1}{8}\pi) = 0$ and $\frac{3}{2}\theta + \frac{1}{8}\pi = n\pi + \frac{1}{2}\pi$.

128. See Ex. LXII. 19, we have

$$\sin B + \sin C - \sin A = 4 \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C,$$

$$\sin C + \sin A - \sin B = 4 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C,$$

and the result follows by multiplication.

129. $$10^{1\cdot 5} = 10^{\frac{3}{2}} = \sqrt{(1000)} = 31\cdot 622,$$

$$10^{\cdot\dot{3}} = 10^{\frac{1}{3}} = \sqrt[3]{10} = 2\cdot 154, \quad 10^{1\cdot\dot{3}} = 10^{\frac{4}{3}} = \sqrt[3]{10000} = 21\cdot 534.$$

130. Let AB, AC be the two chords,
α, β the angles which the chords make with the tangent. Join BC,

then $$\frac{AB}{AC} = \frac{\sin ACB}{\sin ABC} = \frac{\sin \alpha}{\sin \beta}.$$

131. Let n be the number of degrees in a polygon of x sides; and let x be the number of grades in a polygon of y sides; then

$$x \times n = 90 \times (2x - 4), \quad y \times n = 100 \times (2y - 4);$$

$$\therefore \frac{10x}{9y} = \frac{x-2}{y-2} \text{ or } xy - 20y + 18x = 0; \quad \therefore x = 20 - 18\frac{x}{y};$$

$\therefore$ since x and y are positive integers $\dfrac{18x}{y}$ is an integer and less than 20,

let the integer be λ, also $\quad y = \dfrac{20y}{x} - 18;$

$$\therefore \frac{20y}{x} \text{ is an integer, } \mu \text{ say;}$$

$$\therefore \lambda\mu = 20 \times 18 = 2 \times 2 \times 2 \times 3 \times 3 \times 5;$$

$\therefore \lambda$ may be either 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, or 18,

and then x is 19, 18, 17, 16, 15, 14, 12, 11, 10, 8, 5, or 2,

and y is 342, 162, 102, 72, 54, 42, 36, 22, 18, 12, 6, or 2 respectively;

but $x = 2$ and $y = 2$ gives $n = 0$ which is not a 'polygon.'

Therefore there are only eleven solutions; of these solutions only $x = 5$, 8, 10, or 12 give integral values for n.

132. $\cos 11A + 3\cos 9A + 3\cos 7A + \cos 5A$

$= 2\cos 8A \,.\, \cos 3A + 6\cos 8A \,.\, \cos A$

$= 2\cos 8A\,(\cos 3A + 3\cos A) = 2\cos 8A\,(4\cos^3 A)$

$= 8\cos^3 A \times \cos 8A = 16\cos^3 A\,\tfrac{1}{2}\,(\cos^2 4A - \sin^2 4A)$

$= 16\cos^3 A \times \cos(4A + \tfrac{1}{4}\pi)\cos(4A - \tfrac{1}{4}\pi).$

133.
$$\sin^{-1}x + \sin^{-1}\tfrac{1}{2}x = \sin^{-1}\tfrac{1}{2}\sqrt{2},$$
$$\therefore\ x\sqrt{(1-\tfrac{1}{4}x^2)} + \tfrac{1}{2}x\sqrt{(1-x^2)} = \tfrac{1}{2}\sqrt{2};$$
$$\therefore\ x^2(4-x^2) = 2 - 2\sqrt{2}x\sqrt{(1-x^2)} + x^2 - x^4;$$
$$\therefore\ 3x^2 - 2 = -2\sqrt{2}x\sqrt{(1-x^2)};$$
$$\therefore\ 17x^4 - 20x^2 + 4 = 0;\ \ \therefore\ x = \sqrt{\{\tfrac{2}{17}(5-2\sqrt{2})\}}.$$

134. $\tfrac{1}{2}\log 2 - \tfrac{1}{2}(L\cos A - 10) = 1 - \log 2 + \tfrac{1}{3}\{2L\cos B - L\sin C - 10\}.$

135. The first distance $= AB\sin 45° = AB \times \tfrac{1}{2}\sqrt{2}$,

the second distance $= AB + AB \times \tfrac{1}{2}\sqrt{2} = AB \times \tfrac{1}{2}(\sqrt{2}+1)\sqrt{2}$,

the third distance $= AD = \sqrt{2}\,AB\tfrac{1}{2}(\sqrt{2}+1)\sqrt{2}$.

136. The expression $= \tfrac{1}{2}\{\cos 2(\alpha+\beta) - \cos 2(\alpha+\gamma) + \cos 2(\beta+\gamma)$
$- \cos 2(\beta+\alpha) + \cos 2(\alpha+\gamma) - \cos 2(\gamma+\beta)\}.$

137. $a = \log\tfrac{1025}{1024} = 2\log 5 - 10\log 2 + \log 41 = 2 - 2b - 10b + \log 41.$

138. $\sin 81° + \sin 39° - \sin 21° + \sin 99°$

$= 2\sin 90°\cos 9° - 2\sin 30°\cos 9° = 2\cos 9°(\sin 90° - \sin 30°)$

$= 4\cos 9°\cos 60°\sin 30° = \cos 9° = \sin 81° = \sin 90^g.$

139. $\sin\overline{n+1}\,.\,\theta + \sin\overline{n-1}\,.\,\theta = \sin 2\theta,$

$2\sin n\theta\,.\,\cos\theta = 2\sin\theta\,.\,\cos\theta;\ \therefore\ \cos\theta = 0;$ whence, $\theta = n\pi \pm \tfrac{1}{2}\pi.$

$\sin n\theta - \sin\theta = 0,\ 2\sin\tfrac{1}{2}(n-1)\theta\,.\,\sin\tfrac{1}{2}(n+1)\theta = 0;$

$\therefore\ \sin\tfrac{1}{2}(n-1)\theta = 0;$ whence $(n-1)\theta = 2m\pi,$

$\sin\tfrac{1}{2}(n+1)\theta = 0;$ whence $(n+1)\theta = 2m\pi.$

140. A circle can be circumscribed about the quadrilateral
$$BE = BD\,.\,\frac{\sin\beta}{\sin(\alpha+\beta+\gamma)} = BA\,.\,\frac{\sin(\alpha+\beta)\sin\beta}{\sin(\gamma-\beta)\sin(\alpha+\beta+\gamma)}.$$

141. $\cos 2A + \sin 2B = \sin(90° - 2A) + \sin 2B$

$= 2\sin\{45° - (A-B)\}\cos\{45° - (A+B)\}.$

$\cos 2A - \sin 2B = \sin(90° - 2A) - \sin 2B$

$= 2\sin\{45° - (A+B)\}\{\cos 45° - (A-B)\}.$

142. (i) $\cos 55° + \cos 65° + \cos 175°$

$= 2\cos 65°\,.\,\cos 120° + \cos 65° = 2\cos 65° \times (-\tfrac{1}{2}) + \cos 65°.$

(ii) $\sin^2 24° - \sin^2 6° = (\sin 24° + \sin 6°)(\sin 24° - \sin 6°)$

$= 4\sin 15°\,.\cos 9°\,.\cos 15°\,.\,\sin 9° = \sin 30°\,.\,\sin 18° = \tfrac{1}{8}(\sqrt{5}-1).$

143. $\cos 2A - \cos 2B + \cos 2C - \cos 2D$

$$= 2 \sin (A + B) \sin (B - A) + 2 \sin (C + D) \sin (D - C)$$
$$= 2 \sin (A + B) \{\sin (B - A) + \sin (D - C)\}$$
$$= 2 \sin (A + B)\, 2 \sin \tfrac{1}{2}(B - A + D - C) \cos \tfrac{1}{2}(B - A - D + C)$$
$$= 4 \sin (A + B) \sin \tfrac{1}{2}(180^\circ - 2A - 2C) \cos \tfrac{1}{2}(2B + 2C - 180^\circ)$$
$$= 4 \sin (A + B) \cos (A + C) \sin (B + C).$$

144. $\log 5 + \log 7 = a$, $2 \log 5 + \log 13 = b$, $\log 5 + 2 \log 7 = c$, and the result follows by solving these equations.

145. In the figure E. T. p. 216, let B be the train in its first position; A the town; C_2C_1 the positions of the train 18 miles from A; then

$$AB = 20, \quad ABC = 45 \text{ and } b^2 = a^2 + c^2 - 2ac \cos 45^\circ;$$
$$\therefore (18)^2 = (24)^2 + c^2 - \sqrt{2} \times 24c; \quad \therefore c = 6(2\sqrt{2} \pm 1);$$
$$\therefore \text{time required} = \tfrac{1}{24} \text{ of } 6(2\sqrt{2} \pm 1) \text{ hours.}$$

146. $4 \sin A \cos A \sin (A - B) = (4 \sin^2 A - 3) \cos (A - B)$

$$\therefore 4 \sin^2 A \cos A \cos B - 4 \sin A \sin B \cos^2 A$$
$$= (4 \sin^2 A - B)(\cos A \cos B + \sin A \sin B);$$
$$\therefore 4 \sin A \sin B (-\cos^2 A - \sin^2 A) = -3(\cos A \cos B + \sin A \sin B);$$
$$\therefore \sin A \sin B = 3 \cos A \cos B. \quad \text{Q. E. D.}$$

147. $\tan (\pi . \cot \theta) = \cot (\pi . \tan \theta)$. Let $\tan \theta = a$,

$$\tan \frac{\pi}{a} = \frac{1}{\tan \pi . a}, \text{ or } 1 - \tan \frac{\pi}{a} . \tan \pi . a = 0.$$

$$\tan \left(\frac{\pi}{a} + \pi . a\right) = \frac{\tan \frac{\pi}{a} + \tan \pi . a}{1 - \tan \frac{\pi}{a} . \tan \pi a} = \infty;$$

$$\therefore \frac{\pi}{a} + \pi . a = n\pi + \frac{\pi}{2}; \quad \therefore 2a^2 - (2n+1)a = -2,$$

or

$$16a^2 - (2n+1)a + (2n+1)^2 = 4n^2 + 4n - 15;$$
$$\therefore 4a \text{ or } 4 \tan \theta = 2n + 1 \pm \sqrt{4n^2 + 4n - 15}.$$

148.

$$x^2 + y^2 = 2 + 2 \cos a.$$
$$2x = 4 \cos^2 a - 2 + 2 \cos a = (x^2 + y^2 - 2)^2 - 2 + x^2 + y^2 - 2$$
$$= (x^2 + y^2)^2 - 3(x^2 + y^2).$$

149. $2R \sin C = c$ and $(c - b \cos A)^2 = c^2 + b^2 \cos^2 A - 2bc \cos A$

$$= a^2 - b^2 + b^2 \cos^2 A = a^2 - b^2 \sin^2 A. \quad \text{Q. E. D.}$$

150.

$$\tan \left\{\tan^{-1} \frac{a}{b+c} + \tan^{-1} \frac{b}{a+c}\right\} = \frac{\frac{a}{b+c} + \frac{b}{a+c}}{1 - \frac{ab}{(b+c)(a+c)}}$$

$$= \frac{a^2 + ac + b^2 + bc}{ab + ac + bc + c^2 - ab} = \frac{(a^2 + b^2) + ac + bc}{c(a + b + c)} = 1 \text{ (since } a^2 + b^2 = c^2) = \tan \tfrac{1}{4}\pi.$$

151. $60° = \frac{2}{3}$ right angle, $50^g = \frac{1}{2}$ right angle, $\frac{3}{4}\pi = \frac{3}{2}$ right angle; the fourth angle $= (4 - \frac{2}{3} - \frac{1}{2} - \frac{3}{2})$ right angle $= \frac{4}{3}$ right angle;

$\therefore$ the angles are $60°, 45°, 135°, 120°$.

152.
$$\cos(\sin^{-1} m + \sin^{-1} n) = \cos\tfrac{1}{2}\pi = 0;$$
$$\therefore \sqrt{(1-m^2)}\sqrt{(1-n^2)} - mn = 0; \quad \therefore 1 - m^2 - n^2 + m^2n^2 = m^2n^2;$$
$$\therefore m^2 = 1 - n^2; \quad \therefore m = \pm\sqrt{(1-n^2)}, \text{ or } \sin^{-1} m = \pm\cos^{-1} n.$$

153.
$$\tan^2\tfrac{1}{2}\alpha = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{1 - e\cos\beta - \cos\beta + e}{1 - e\cos\beta + \cos\beta - e}$$
$$= \frac{(1+e)(1-\cos\beta)}{(1-e)(1+\cos\beta)} = \frac{1+e}{1-e}\tan^2\tfrac{1}{2}\beta.$$

154. $2\cos(\theta - \frac{1}{4}\pi)\cos(2\theta - \frac{1}{4}\pi) = \cos\theta + \cos(3\theta - \frac{1}{2}\pi)$.

155.
$$a^2b^2c^2 - 4b^2S^2 = a^2b^2c^2(1 - \sin^2 B),$$
$$a^2b^2c^2\cos A - 4bcS^2 = a^2b^2c^2\left(\cos A - \frac{bc}{a^2}\sin^2 A\right).$$

We have to prove that
$$\cos^2 B\cos^2 C = \left(\cos A - \frac{\sin B\sin C\sin^2 A}{\sin^2 A}\right)^2$$
$$= (\sin B\sin C - \cos B\cos C - \sin B\sin C)^2.$$

156. $\cos^2(A+B) + \cos^2(A-B) - \cos 2A\cos 2B$
$$= 2\cos^2 A\cos^2 B + 2\sin^2 A\sin^2 B - (\cos^2 A - \sin^2 A)(\cos^2 B - \sin^2 B)$$
$$= \cos^2 A\cos^2 B + \sin^2 A\sin^2 B + \cos^2 A\sin^2 B + \cos^2 B\sin^2 A$$
$$= (\cos^2 A + \sin^2 A)(\cos^2 B + \sin^2 B) = 1 \times 1.$$

157. Let $x°$ and $x° + 10°$ be the units.

Then
$$4kx° = 3k(x° + 10°); \quad \therefore x° = 30°.$$

158.
$$\cos 7°\,30' = \sqrt{\{\tfrac{1}{2}(1 + \cos 15°)\}} = \sqrt{\left\{\frac{1}{2}\left(1 + \frac{\sqrt{6}+\sqrt{2}}{4}\right)\right\}}$$
$$= \tfrac{1}{4}\sqrt{\{2(4 + \sqrt{6} + \sqrt{2})\}} = \tfrac{1}{4}\sqrt{\{2\sqrt{2}(2\sqrt{2} + \sqrt{3} + 1)\}}.$$

Now $(-1 + \sqrt{2} + \sqrt{3})^2 \times (2 + \sqrt{2}) = 2(3 - \sqrt{2} - \sqrt{3} + \sqrt{6})(1 + \sqrt{2})\sqrt{2}$
$$= 2\sqrt{2}(1 + 2\sqrt{2} + \sqrt{3}),$$

and the result follows.

159.
$$\frac{\sin\frac{1}{2}A + \cos\frac{1}{2}B - \sin\frac{1}{2}C}{\sin\frac{1}{2}A + \cos\frac{1}{2}C - \sin\frac{1}{2}B} = \frac{2\cos\frac{1}{4}(A+C)\,.\,\sin\frac{1}{4}(A-C) + \cos\frac{1}{2}B}{2\cos\frac{1}{4}(A+B)\sin\frac{1}{4}(A-B) + \cos\frac{1}{2}C}$$
$$= \frac{2\cos\frac{1}{4}(A+C)\{\sin\frac{1}{4}(A-C) + \sin\frac{1}{4}(A-C)\}}{2\cos\frac{1}{4}(A+B)\{\sin\frac{1}{4}(A-B) + \sin\frac{1}{4}(A+B)\}}$$
$$= \frac{\cos\frac{1}{4}(A+C)(2\sin\frac{1}{4}A\,.\,\cos\frac{1}{4}C)}{\cos\frac{1}{4}(A+B)\,2\sin\frac{1}{4}A\,.\,\cos\frac{1}{4}B} = \frac{\cos(45° - \frac{1}{4}B)\,.\,\cos\frac{1}{4}C}{\cos(45° - \frac{1}{4}C)\,.\,\cos\frac{1}{4}B}$$
$$= \frac{(\cos\frac{1}{4}B + \sin\frac{1}{4}B)\cos\frac{1}{4}C}{(\cos\frac{1}{4}C + \sin\frac{1}{4}C)\cos\frac{1}{4}B}.$$

160. $a^2\{b^2+c^2-2bc\cos(B-C)\}$

$$=a^2b^2+a^2c^2-2a^2bc\cos B\cos C-2a^2bc\sin B\sin C$$
$$=a^2b^2+a^2c^2-\tfrac{1}{2}(a^2+b^2-c^2)(a^2+c^2-b^2)-2b^2c^2\sin^2 A$$
$$=\tfrac{1}{2}\{2a^2b^2+2a^2c^2-a^4+b^4+c^4-2b^2c^2-4b^2c^2+(b^2+c^2-a^2)^2\}$$
$$[\text{since } 2b^2c^2\sin^2 A=2b^2c^2-2b^2c^2\cos^2 A]$$
$$=\tfrac{1}{2}(2b^4+2c^4-4b^2c^2).$$ Q. E. D.

161. $\cos 3\theta+\sin 3\theta=\cos\theta+\sin\theta,\ \sin 3\theta-\sin\theta=\cos\theta-\cos 3\theta,$

$$2\cos 2\theta\,.\,\sin\theta=2\sin 2\theta\,.\,\sin\theta;$$
$$\therefore \text{ either, } \sin\theta=0;\ \text{whence, } \theta=n\pi,$$
or $$\tan 2\theta=1;\ \text{whence, } 2\theta=n\pi\pm\tfrac{1}{4}\pi.$$

162. $x=3\cos\phi+\cos 3\phi=3\cos\phi+4\cos^3\phi-3\cos\phi=4\cos^3\phi,$

$y=3\sin\phi-3\sin\phi+4\sin^3\phi=4\sin^3\phi;$

$$\therefore (\tfrac{1}{4}x)^{\frac{2}{3}}=\cos^2\phi \text{ and } (\tfrac{1}{4}y)^{\frac{2}{3}}=\sin^2\phi;$$
$$\therefore (\tfrac{1}{4}x)^{\frac{2}{3}}+(\tfrac{1}{4}y)^{\frac{2}{3}}=\cos^2\phi+\sin^2\phi=1;\ \text{or, } x^{\frac{2}{3}}+y^{\frac{2}{3}}=4^{\frac{2}{3}}.$$

163. $\cos^2 18^\circ\sin^2 36^\circ-\cos 36^\circ\sin 180^\circ$

$$=\tfrac{1}{16}(10+2\sqrt{5})\times\tfrac{1}{16}(10-2\sqrt{5})-\tfrac{1}{4}(1+\sqrt{5})\tfrac{1}{4}(1-\sqrt{5})=\tfrac{1}{16}\times\tfrac{80}{16}-\tfrac{4}{16}=\tfrac{1}{16}.$$

164. $$2(4\cos^3\theta-3\cos\theta)+2\cos\theta-1=0,$$
$$2\cos\theta(4\cos^2\theta-1)-(2\cos\theta+1)=0;$$
$$\therefore (2\cos\theta+1)(4\cos^2\theta-2\cos\theta-1)=0;$$
$$\therefore \text{ either } \cos\theta=-\tfrac{1}{2};\ \text{whence, } \theta=2n\pi\pm\tfrac{2}{3}\pi,$$
or $$4\cos^2\theta-2\cos\theta-1=0,$$
that is, $$(4\cos\theta-\sqrt{5}-1)(4\cos\theta+\sqrt{5}-1=0,$$
whence $$\theta=2n\pi\pm\tfrac{1}{5}\pi \text{ or } \theta=2n\pi\pm\tfrac{3}{5}\pi.$$

165. In this case the angles at A and C are right angles

$$\therefore BD^2=a^2+d^2=b^2+c^2;$$
$$\therefore (s-a)(s-d)=s^2-(a+d)s+ad$$
$$=s^2-(a+d)\tfrac{1}{2}(a+b+c+d)+ad$$
$$=s^2-\tfrac{1}{2}(a^2+d^2+ab+ac+db+dc+2ad)-ad$$
$$=s^2-\tfrac{1}{2}(b^2+c^2+ab+ac+bd+dc+2bc)-bc$$
$$=s^2-\tfrac{1}{2}(b+c)(a+b+c+d)-bc=(s-b)(s-c).$$

The area $=\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}=\sqrt{\{(s-a)^2(s-d)^2\}}.$

166. $$\sin(A+B)\cos A=3\cos(A+B)\sin A,$$
$$\therefore \sin A\cos A\cos B+\sin B\cos^2 A=3\sin A\cos A\cos B-3\sin^2 A\sin B;$$
$$\therefore \sin 2A\cos B=\sin B(1+2\sin^2 A)=\sin B(2-\cos 2A);$$
$$\therefore \sin(2A+B)=2\sin B;$$
$$\therefore \sin(2A+B)\cos B=2\sin B\cos B;$$
$$\therefore \sin(2A+2B)+\sin 2A=2\sin 2B.$$ Q.E.D.

167. $\sin 18° + \cos 18° = \sqrt{(1+\sin 36°)} = \sqrt{(1+\cos 54°)} = \sqrt{(2\cos^2 27°)}$.

168. $$\frac{\tan 2\theta + \tan\theta}{1 - 2\tan 2\theta \tan\theta} + \tan 2\theta + \tan\theta = 0,$$

$\therefore$ **either,** $\tan 2\theta + \tan\theta = 0$,

whence $\tan\theta = 0$; and, $\theta = n\pi$,

or $\tan\theta = \pm\sqrt{3}$; and $\theta = n\pi \pm \frac{1}{3}\pi$,

or, $2 - \tan 2\theta \tan\theta = 0$, whence, $\theta = n\pi \pm \tan^{-1}\frac{1}{2}\sqrt{2}$.

169. $$\begin{aligned} &4\cos^3 2A + 4\cos^3 2B + 4\cos^3 2C \\ &= \cos 6A + \cos 6B + \cos 6C + 3(\cos 2A + \cos 2B + \cos 2C) \\ &= 2\cos 3(A+B)\{\cos 3(A-B) - \cos 3(A+B)\} + 1 \\ &\qquad + 6\cos(A+B)\{\cos(A-B) - \cos(A+B)\} + 3 \\ &= 4\sin 3C \sin 3A \sin 3B + 12\sin 2C \sin 2A \sin 2B + 4. \end{aligned}$$

170. $$\begin{aligned} &bc\cos^2\tfrac{1}{2}A + ca\cos^2\tfrac{1}{2}B + ab\cos^2\tfrac{1}{2}C \\ &= s(s-a) + s(s-b) + s(s-c) = 3s^2 - 2s^2 = s^2. \end{aligned}$$

171. $$\begin{aligned} \sin 7\theta &\equiv \sin 6\theta \cos\theta + \cos 6\theta \sin\theta \\ &\equiv 2\sin 3\theta \cos 3\theta \cos\theta + (4\cos^3 2\theta - 3\cos 2\theta)\sin\theta \\ &\equiv 2(3\sin\theta - 4\sin^3\theta)(4\cos^3\theta - 3\cos\theta)\cos\theta + \ldots \\ &\equiv \sin\theta\{3 - 2(1-\cos 2\theta)\}\{2(1+\cos 2\theta) - 3\}(1+\cos 2\theta) + \ldots \\ &\equiv \sin\theta\{1 + 2\cos 2\theta\}\{2\cos 2\theta - 1\}(1+\cos 2\theta) + \ldots \\ &\equiv \sin\theta\{8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1\}. \end{aligned}$$

Now $7\theta = \pi$ is satisfied if either of the above factors is zero.

But $$\begin{aligned} 8\cos\theta\cos 2\theta\cos 3\theta - 1 &\equiv 4\cos 2\theta\{\cos 4\theta + \cos 2\theta\} - 1 \\ &\equiv 4\cos 2\theta\{2\cos^2 2\theta - 1 + \cos 2\theta\} - 1 \\ &\equiv 8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1. \end{aligned}$$

Hence the statement of the question is true.

172. $$\begin{aligned} &\sin^2\tfrac{1}{2}(A+B) - \sin^2\tfrac{1}{2}(A-B) \\ &= \{\sin\tfrac{1}{2}(A+B) + \sin\tfrac{1}{2}(A-B)\}\{\sin\tfrac{1}{2}(A+B) - \sin\tfrac{1}{2}(A-B)\} \\ &= 2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B \times 2\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B = \sin A \sin B. \end{aligned}$$

Hence when $A+B$ is given $\sin A \sin B$ has its greatest value when $A = B$.

Now suppose $(A+B+C)$ given, then

$\sin A \sin B \sin C$ has its greatest value when $A = B = C$.

For suppose we keep C [and $\therefore$ $(A+B)$] unaltered, then the value of the expression is increased by making $A = B$.

Similarly by keeping $(B+C)$ unaltered the value is increased by making $B = C$.

Hence the greatest value is when $A = B = C$.

Now $\cos A \cos B \cos C = \sin(90° - A)\sin(90° - B)\sin(90° - C)$, and since $(90° - A + 90° - B + 90° - C)$ is constant this expression has its greatest value when $A = B = C$ and then $A = B = C = 30°$.

173. $\cot^{-1}3=\sin^{-1}\sqrt{\frac{1}{10}}=\cos^{-1}\sqrt{\frac{9}{10}},\ \sin^{-1}\frac{1}{5}\sqrt{5}=\cos^{-1}\frac{2}{5}\sqrt{5};$

$$\therefore \sin\{\sin^{-1}\tfrac{1}{5}\sqrt{5}+\cot^{-1}3\}=\tfrac{1}{5}\sqrt{5}\sqrt{\tfrac{9}{10}}+\tfrac{2}{5}\sqrt{5}\sqrt{\tfrac{1}{10}}$$
$$=(\tfrac{3}{5}+\tfrac{2}{5})\tfrac{1}{2}\sqrt{2}=\tfrac{1}{2}\sqrt{2}=\sin\tfrac{1}{4}\pi.$$

174. $\sin 3A+\sin 3B+\sin 3C$

$$=2\sin\tfrac{3}{2}(A+B)\cos\tfrac{3}{2}(A-B)+2\sin\tfrac{3}{2}C\cos\tfrac{3}{2}C$$
$$=2\sin\tfrac{3}{2}(A+B)\{\cos\tfrac{3}{2}(A-B)+\cos\tfrac{3}{2}(A+B)\}$$

[For $\frac{3}{2}C=270^\circ-\frac{3}{2}(A+B)$ and $\sin(270^\circ-\theta)=-\cos\theta$ and $\cos(270^\circ-\theta)=-\sin\theta$]

$$=-4\cos\tfrac{3}{2}C\cos\tfrac{3}{2}A\cos\tfrac{3}{2}B.$$ Q.E.D.

175. We have $\dfrac{a}{2\sin A}=\dfrac{S}{s-a}$; and $a=c$, suppose; $\therefore 4S^2=a^2b(s-a)$;

$\therefore \frac{1}{2}\{(2a+b)b(2a-b)\}=a^2b$; $\therefore b^2=2a^2.$ Q. E. D.

176. $$\sin x=1-\sin^2 x=\cos^2 x.$$
$$\therefore \cos^4 x=\sin^2 x=1-\cos^2 x.$$ Q.E.D.

177. $\sin 10^\circ\sin 50^\circ\sin 70^\circ=\frac{1}{2}(\cos 40^\circ-\cos 60^\circ)\sin 70^\circ$

$$=\tfrac{1}{2}\cos 40^\circ\sin 70^\circ-\tfrac{1}{4}\sin 70^\circ$$
$$=\tfrac{1}{4}\sin 110^\circ+\tfrac{1}{4}\sin 30^\circ-\tfrac{1}{4}\sin 70^\circ=\tfrac{1}{4}\sin 30^\circ=\tfrac{1}{8},$$

for $\sin 110^\circ=\sin 70^\circ.$

178. $\tan\left\{\tan^{-1}\dfrac{x\cos\theta}{1-x\sin\theta}-\tan^{-1}\dfrac{x-\sin\theta}{\cos\theta}\right\}$

$$=\left\{\frac{x\cos\theta}{1-x\sin\theta}-\frac{x-\sin\theta}{\cos\theta}\right\}\div\left\{1-\frac{x\cos\theta}{1-x\sin\theta}\times\frac{x-\sin\theta}{\cos\theta}\right\}$$
$$=\frac{\sin\theta(1-2x\sin\theta+x^2)}{\cos\theta(1-2x\sin\theta+x^2)}=\tan\theta.$$

179. Since $A+B+C+D=360^\circ$,

$\therefore \frac{1}{2}(A+B)=180^\circ-\frac{1}{2}(C+D)$, $\therefore \cos\frac{1}{2}(A+B)+\cos\frac{1}{2}(C+D)=0$;

$$\therefore \cos\tfrac{1}{2}(A+B)+\cos\tfrac{1}{2}(C+D)+\cos\tfrac{1}{2}(A+C)$$
$$+\cos\tfrac{1}{2}(B+D)+\cos\tfrac{1}{2}(B+C)+\cos\tfrac{1}{2}(A+D)=0,$$

expanding, the statement is seen to be true.

180. Let AB be the line, C its middle point; let D be the foot of the tower and h its height,

then $AD=BD$ and DC is perpendicular to AB.

Also $AD=h\cot\alpha,\ CD=h\cot\beta,$

and $a^2=AD^2-CD^2;\ \therefore a^2=h^2(\cot^2\alpha-\cot^2\beta);$

$$\therefore h=a\sin\alpha\sin\beta\div\sqrt{\{\sin^2\beta\cos^2\alpha-\sin^2\alpha\cos^2\beta\}}$$
$$=a\sin\alpha\sin\beta\div\sqrt{\{\sin(\beta+\alpha)\sin(\beta-\alpha)\}}.$$

181. $\sin(\alpha-\beta)\cos 2\beta+\cos(\alpha-\beta)\sin 2\beta$

$$=\sin\{(\alpha-\beta+2\beta)\}=\sin(\beta-\alpha+2\alpha)$$
$$=\sin(\beta-\alpha)\cos 2\alpha+\cos(\beta-\alpha)\sin 2\alpha.$$

182. $\sin\theta=\frac{5}{13}$; $\therefore \cos\theta=\frac{12}{13}$.

$\sin 2\theta=2\times\frac{5}{13}\times\frac{12}{13}=\frac{120}{169}$; $2\cos^2\frac{1}{2}\theta-1=\frac{12}{13}$; $\therefore 2\cos^2\frac{1}{2}\theta=\frac{25}{13}$;

$$\therefore \sin 2\theta \,.\, \cos\tfrac{1}{2}\theta=\tfrac{120}{169}\times\sqrt{\tfrac{25}{26}},$$

and $$\tan\theta=\tfrac{5}{13}\times\tfrac{13}{12}=\tfrac{5}{12}.$$

183. If the angle 2θ is in the first quadrant $\sin 2\theta$ is positive and so are $\cos\theta$ and $\sin\theta$.

If 2θ is in the second quadrant $\sin 2\theta$ is positive and θ would be in the first quadrant and both $\sin\theta$ and $\cos\theta$ are positive.

If 2θ is in the third quadrant $\sin 2\theta$ is negative, $\cos\theta$ would be in the second quadrant and is negative $\sin\theta$ would be positive, $\therefore 2\sin\theta\,.\cos\theta$ is negative.

If 2θ is in the fourth quadrant $\sin 2\theta$ is negative and $\cos\theta$ would be in the second quadrant and is negative $\sin\theta$ would be positive, $\therefore 2\sin\theta\,.\cos\theta$ would be negative; $\therefore \sin 2\theta$ has always the same sign as $2\sin\theta\,.\cos\theta$.

184. $$2\times\tfrac{1}{2}a\times c\cos B=c^2+\tfrac{1}{4}a^2-(AD)^2.$$

Also $$ab\cos C=\tfrac{1}{4}a^2+b^2-(AD)^2;$$

$$\therefore a(b\cos C+c\cos B)=c^2+b^2+\tfrac{1}{2}a^2-2AD^2;$$

$$\therefore a^2=c^2+b^2+\tfrac{1}{2}a^2-2AD^2;$$

$$\therefore 4AD^2=2c^2+2b^2-a^2=c^2+b^2+2bc\cos A.$$

185. $$r=l\sin\tfrac{1}{2}A,$$

$$\therefore \frac{lmn}{r^3}=\operatorname{cosec}\tfrac{1}{2}A\operatorname{cosec}\tfrac{1}{2}B\operatorname{cosec}\tfrac{1}{2}C$$

$$=\frac{abc}{(s-a)(s-b)(s-c)} \text{ and } rs=S;$$ [Ex. LXXIII. (7) iii.]

$$\therefore \frac{lmn}{r}=\frac{abc}{s}.$$

186. Let $\frac{a}{\sin\phi}=k$, then $a\cos\theta+b\cos\phi=k\sin(\theta+\phi)=c$,

$$\therefore \sin(\theta+\phi)=\frac{c}{a}\sin\phi.$$

Again, $c^2-2bc\cos\phi+b^2\cos^2\phi=a^2\cos^2\theta=a^2-a^2\sin^2\phi=a^2-b^2\sin^2\phi$;

$$\therefore 2bc\cos\phi=b^2+c^2-a^2;$$

$$\therefore \sin\phi=\frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}}\equiv\frac{2}{bc}S,$$

where $$s=\tfrac{1}{2}(a+b+c).$$

Similarly $$\sin\theta=\frac{2}{ac}\sqrt{\{s(s-a)(s-b)(s-c)\}}\equiv\frac{2}{ac}S;$$

$$\therefore \cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$
$$=\{(a^2+c^2-b^2)(b^2+c^2-a^2)-16S^2\}\div 4abc^2$$
$$=(2c^4-2a^2c^2-2b^2c^2)\div 4abc^2;$$

$$\therefore \tan(\theta+\phi)=\frac{c}{a}\sin\phi\times 2ab\div(c^2-a^2-b^2),$$

$$\therefore \tan(\theta+\phi)\times(c^2-a^2-b^2)=4bcS.$$ Q.E.D.

187. This is another way of stating that when $A+B+C=180°$;
$\sin A \cos A + \sin B \cos B + \sin C \cos C = 2 \sin A \sin B \sin C$.
[See Ex. LXII. 26.]

188. By Ex. XLIII. 9, we have

$\sin A \sin B \sin C$

$$=\tfrac{1}{4}\{\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)-\sin(A+B+C)\}.$$

Let $A=\beta-\gamma,\ B=\theta-\beta,\ C=\theta-\gamma$,

$\sin(\theta-\beta)\sin(\theta-\gamma)\sin(\beta-\gamma)=\tfrac{1}{4}\{\sin 2(\theta-\beta)+\sin\theta+\sin 2(\beta-\gamma)-\sin 2(\theta-\gamma)$

and two similar statements.

By addition we have

$$\sin(\theta-\beta)\sin(\theta-\gamma)\sin(\beta-\gamma)+\sin(\theta-\gamma)\sin(\theta-\alpha)\sin(\alpha-\gamma)$$
$$+\sin(\theta-\alpha)\sin(\theta-\beta)\sin(\alpha-\beta)=\tfrac{1}{4}\{\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)+\sin 2(\alpha-\beta)\}$$
$$=\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$ [See Ex. XLIII. 7.]

189. $2\sin^2 A+2\sin^2 B+2\sin^2 C=3-\cos 2A-\cos 2B-\cos 2C$
$=2-4\sin A\sin B\sin C$. [Cf. Ex. LXII. 21.]

This expression has its least value when $A=B=C$ by Ex. 1[illegible] and therefore the least value of $\sin^2 A+\sin^2 B+\sin^2 C=3\sin^2 30°=\tfrac{3}{4}$.

190. The perimeter of a regular circumscribing polygon of m sides

$$=2nr\,.\tan\frac{\pi}{n},$$ where r is the radius of circle.

The perimeter of a regular inscribed polygon $=2nr\,.\sin\dfrac{\pi}{n}$,

but here $$2nr\,.\tan\frac{\pi}{n}=4nr\,.\sin\frac{\pi}{n}.$$

$$\therefore 1=2\cos\frac{\pi}{n},\ \text{or}\ \cos\frac{\pi}{n}=\tfrac{1}{2}=\cos\tfrac{1}{3}\pi;\ \therefore n=3.$$

191.
$$m\sin 2\theta=n\sin\theta \quad\ldots\ldots(1),$$
$$p\,.\cos 2\theta=q\,.\cos\theta \quad\ldots\ldots(2).$$

From (1), $2m\,.\cos\theta=n$.

Substituting in (2) for $\cos\theta$, we get

$$2p\,.\frac{n^2}{4m^2}-p=q\,.\frac{n}{2m},\quad 2n^2p-4m^2p=2mnq.$$

192. See Ex. XLIII. 20.

193. $$\tan 5A=\frac{\tan 4A+\tan A}{1-\tan 4A\tan A}=\frac{2\tan 2A+\tan A-\tan^2 2A\tan A}{1-\tan^2 2A-2\tan 2A\tan A}$$
$$=\frac{4\tan A(1-\tan^2 A)+\tan A(1-\tan^2 A)^2-4\tan^3 A}{(1-\tan^2 A)^2-4\tan^2 A-4\tan^2 A(1-\tan^2 A)}$$
$$=\frac{4\tan A-4\tan^3 A+\tan A-2\tan^3 A+\tan^5 A-4\tan^3 A}{1-2\tan^2 A+\tan^4 A-4\tan^2 A-4\tan^2 A+4\tan^4 A}.$$

194. $\cos 36° = \sin 54° = \sin 3 \times 18° = 3 \sin 18° - 4 \sin^3 18°,$

also by XLII. 6, $4 \sin^2 18° = 1 - 2 \sin 18°;$

$$\therefore\ 4 \sin 18° \cos 36° = 12 \sin^2 18° - 16 \sin^4 18°$$
$$= 12 \sin^2 18° - (1 - 2 \sin 18°)^2 = 8 \sin^2 18° - 1 + 4 \sin 18°$$
$$= 2 - 4 \sin 18° - 1 + 4 \sin 18° = 1.$$

Again, $2 \cos 36° - 2 \sin 18° = 2 \sin 3 \times 18° - 2 \sin 18°$
$$= 6 \sin 18° - 8 \sin^3 18° - 2 \sin 18°$$
$$= 4 \sin 18° - 2 \sin 18° (1 - 2 \sin 18°)$$
$$= 2 \sin 18° + 4 \sin^2 18°$$
$$= 2 \sin 18° + 1 - 2 \sin 18° = 1.$$ Q.E.D.

195. $r = \dfrac{S}{s} = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}} = s \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C.$

196. $\cos(\theta + \frac{1}{4}\pi) = \cos(\alpha + \frac{1}{4}\pi), \quad \therefore\ \theta + \frac{1}{4}\pi = 2n\pi \pm (\alpha + \frac{1}{4}\pi).$

This is the complete solution.

197. $\cos(A+B)\cos(A+C)(\cos A+D)$
$$= 2\{\cos(B-C) + \cos(2A+B+C)\}\cos(A+D)$$
$$= \cos(360° - 2C) + \cos(2B - 360°) + \cos(360° + 2A) + \cos(360° - 2D)$$
$$= \cos 2A + \cos 2B + \cos 2C + \cos 2D;$$
$$\therefore\ \cos(x+2A) - \cos 2A + \cos(x+2B) - \cos 2B + \text{etc.} = 0;$$
$$\therefore\ 2 \sin(\tfrac{1}{2}x + 2A) \sin \tfrac{1}{2}x + \text{etc.} = 0;$$

divide by $\sin \frac{1}{2}x$ and multiply by $\cos \frac{1}{2}x$, which may be done provided $\sin \frac{1}{2}x$ is not zero; then

$$2 \sin(\tfrac{1}{2}x + 2A) \cos \tfrac{1}{2}x + \text{etc.} = 0; \quad \therefore\ \sin(x+2A) + \sin 2A + \text{etc.} = 0;$$
$$\therefore\ \sin(x+2A) + \text{etc.} = -(\sin 2A + \sin 2B + \sin 2C + \sin 2D)$$
$$= 4 \sin(A+B) \sin(A+C) \sin(A+D), \text{ as above.}$$

198. $\sin^2 A + \sin^2 B + \sin^2 C - 2 \sin A \,.\, \sin B \,.\, \sin C - 1$
$$= \sin^2 A - 2 \sin A \,.\, \sin B \,.\, \sin C - \tfrac{1}{2}(1 - 2\sin^2 B) - \tfrac{1}{2}(1 - 2 \sin^2 C)$$
$$= \sin^2 A - \sin A\{\cos(B-C) - \cos(B+C)\} - \tfrac{1}{2}(\cos 2B + \cos 2C)$$
$$= \sin^2 A - \sin A\{\cos(B-C) - \cos(B+C)\} - \cos(B+C)\cos(B-C)$$
$$= \{\sin A + \cos(B+C)\}\{\sin A - \cos(B-C)\}$$
$$= \{\cos(90° - A) + \cos(B+C)\}\{\cos(90° - A) - \cos(B-C)\}$$
$$= 2 \cos \tfrac{1}{2}(90° - A + B + C) \,.\, \cos \tfrac{1}{2}(90° - A - B - C)$$
$$\times 2 \sin \tfrac{1}{2}(90° - A - B + C) \,.\, \sin \tfrac{1}{2}(90° - A + B - C).$$

199. $2R\{\cos^2 \frac{1}{2}A + \cos^2 \frac{1}{2}B + \cos^2 \frac{1}{2}C\}$
$$= 2R\{2 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + 2\}$$ [Ex. LXII. 21.]
$$= 4R\,\frac{(s-a)(s-b)(s-c)}{abc} + 4R$$
$$= 4R\frac{S^2}{sabc} + 4R = \frac{S}{s} + 4R = r + 4R.$$ [Ex. LXXIII. 4, 7 (iii.)]

200. Let x be a side of the square $ABCD$; let the diagonals intersect in O.

Let ϕ and ψ be the least angles the diagonals of the quadrilateral make with a side of the square; then one of the angles between the diagonals is $90° - \phi - \psi$.

Twice the area of a quadrilateral whose diagonals AB, CD intersect in O is

$$(OA \times OC + OB \times OC + OB \times OD + OD \times OA) \sin AOB = AB \times CD \sin AOB;$$

$$\therefore 2C = hk \cos(\phi + \psi).$$

Now $$\cos\phi = \frac{x}{h}; \quad \cos\psi = \frac{x}{k};$$

$$\therefore \sin^2\phi = \frac{h^2 - x^2}{h^2}; \quad \sin^2\psi = \frac{k^2 - x^2}{k^2}.$$

$$2C = hk\{\cos\phi\cos\psi - \sin\phi\sin\psi\};$$

$$\therefore 2C = hk \times \frac{x}{h} \times \frac{x}{k} - hk \times \sqrt{\frac{h^2 - x^2}{h^2}} \times \sqrt{\frac{k^2 - x^2}{k^2}};$$

$$\therefore 2C = x^2 - \sqrt{\{h^2k^2 - x^2(h^2 + k^2) + x^4\}};$$

$$\therefore (2C - x^2)^2 = h^2k^2 - x^2(h^2 + k^2) + x^4;$$

$$\therefore x^2 = \frac{h^2k^2 - 4C^2}{h^2 + k^2 - 4C}.$$

MISCELLANEOUS EXAMPLES. LXXVI. b. PAGE 269.

1. $$2\cos\theta - \cos 2\theta = a \ldots\ldots\ldots\ldots(i),$$

$$2\sin\theta - \sin 2\theta = b \ldots\ldots\ldots\ldots(ii).$$

Square (i) and (ii) and add, then we obtain

$$4(\cos^2\theta + \sin^2\theta) + (\cos^2 2\theta + \sin^2 2\theta) - 4(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta) = a^2 + b^2,$$

$$\text{i.e. } 4\cos\theta = 5 - a^2 - b^2,$$

$$\cos 2\theta = 2\cos^2\theta - 1 = \tfrac{2}{16}(5 - a^2 - b^2)^2 - 1 = \tfrac{1}{8}(5 - a^2 - b^2)^2 - 1.$$

Substitute for $\cos\theta$ and $\cos 2\theta$ their values in (i),

then $$\tfrac{1}{4}(5 - a^2 - b^2) - \tfrac{1}{8}(5 - a^2 - b^2)^2 + 1 = a;$$

$$\therefore (a^2 + b^2)^2 - 6(a^2 + b^2) + 3^2 = 12 - 8a; \quad \therefore (a^2 + b^2 - 3)^2 = 12 - 8a.$$

2. $$h\cos\theta + k\sin\theta = 1 \ldots\ldots\ldots\ldots(i),$$

$$l\cos\theta + m\sin\theta = 1 \ldots\ldots\ldots\ldots(ii).$$

Multiply (i) by l and (ii) by h and subtract,

$$\therefore \sin\theta = \frac{l - h}{lk - mh}.$$

Multiply (i) by m and (ii) by k and subtract,

$$\therefore \cos\theta = \frac{m - k}{mh - lk}.$$

Now $$\sin^2\theta + \cos^2\theta = 1; \ \therefore \frac{(l - h)^2}{(lk - mh)^2} + \frac{(m - k)^2}{(mh - lk)^2} = 1.$$

$$\therefore (l - h)^2 + (m - k)^2 = (mh - lk)^2.$$

3. The diagonals of a rhombus bisect each other at right angles; therefore the length of the side of the rhombus with diagonals $2a$, and $2b$, is $\sqrt{a^2+b^2}$. Denote the angle subtended by diagonal of length $2a$, A and that by diagonal of length $2b$, B;

$$\therefore \cos A = \frac{(a^2+b^2)+(a^2+b^2)-4a^2}{2(a^2+b^2)} = \frac{b^2-a^4}{a^2+b^2} \qquad \text{[E. T. Art. 240.]}$$

$$\cos B = \frac{(a^2+b^2)+(a^2+b^2)-4b^2}{2(a^2+b^2)} = \frac{a^2-b^2}{a^2+b^2}.$$

4. $$2\cos^2\tfrac{1}{2}A = 1+\cos A; \quad \therefore 2\cos\tfrac{1}{2}A = \sqrt{2+2\cos A}.$$

Also $$2\cos^2\tfrac{1}{4}A = 1+\cos\tfrac{1}{2}A;$$

$$\therefore 2\cos\tfrac{1}{4}A = \sqrt{2+2\cos\tfrac{1}{2}A} = \sqrt{\{2+\sqrt{(2+2\cos A)}\}}.$$

Similarly, $$2\cos\tfrac{1}{8}A = \sqrt{[2+\sqrt{\{2+\sqrt{(2+2\cos A)}\}}]}.$$

In the same way we may proceed for $2\cos\frac{A}{2^n}$.

5. If $8x = \log_e 3$, $\therefore e^{8x} = 3$ and $e^{2x} = 3^{\frac{1}{4}} = \sqrt[4]{3}$.

Also $$e^{-2x} = 3^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{3}};$$

$$\therefore \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}} = \frac{\sqrt[4]{3}-\frac{1}{\sqrt[4]{3}}}{\sqrt[4]{3}+\frac{1}{\sqrt[4]{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \tan 15^\circ.$$

6. A radians $= \frac{180A}{\pi}$ degrees.

Now we are given that $\tan\frac{A180^\circ}{\pi} = \tan A^\circ$;

$$\therefore \frac{A\,180^\circ}{\pi} = n\times 180^\circ + A^\circ; \quad \therefore A\left(\frac{180^\circ}{\pi}-1\right) = n\times 180^\circ;$$

$$\therefore A^\circ = \frac{n\times 180^\circ\times\pi}{180^\circ-\pi};$$

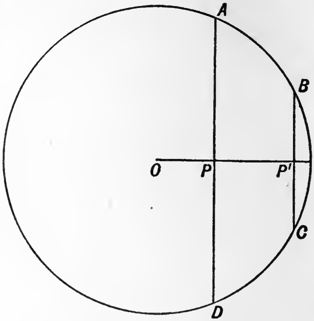
$$\therefore A \text{ is some multiple of } \frac{180\pi}{180^\circ-\pi}.$$

7. Since α, β, γ are in A.P., $\beta-\alpha = \gamma-\beta$ and $\gamma+\alpha = 2\beta$; $\therefore \frac{\gamma+\alpha}{2} = \beta$.

$$\therefore \frac{\gamma-\alpha}{2} = \frac{\gamma+\alpha}{2} - \alpha = \beta-\alpha.$$

Now $$\sin\alpha+\sin\gamma = 2\sin\frac{\gamma+\alpha}{2}\cos\frac{\gamma-\alpha}{2} = 2\sin\beta\cos(\beta-\alpha).$$

8. Let $ABCD$ be the circle of which the centre is O; AD the chord nearer to the centre, and BC the parallel chord further removed from the centre. Join OA, OD, OB, OC; and from O draw the perpendicular OPP' bisecting AD in P and BC in P'; OBC and OAD are isosceles triangles;



$$OBC = \tfrac{1}{2}(180^\circ - 72^\circ) = 54^\circ,$$

and $$OAD = \tfrac{1}{2}(180^\circ - 144^\circ) = 18^\circ.$$

PP' = distance between the chords.

$$PP' = AP' - AP = OP \sin OBC - OA \sin OAD$$
$$= \text{radius} \times \sin 54^\circ - \text{radius} \times \sin 180^\circ$$
$$= \text{radius} \times (\sin 54^\circ - \sin 18^\circ)$$
$$= \text{radius} \times \{\tfrac{1}{4}(\sqrt{5}+1) - \tfrac{1}{4}(\sqrt{5}-1)\}$$
$$= \text{half of the radius.}$$

9. $4 \sin(\theta - \alpha) \sin(m\theta - \alpha) \cos(\theta - m\theta)$
$$= 2\cos(\theta - m\theta)\{\cos(\theta - m\theta) - \cos(\theta + m\theta - 2\alpha)\}$$
$$= 2\cos^2(\theta - m\theta) - 2\cos(\theta - m\theta)\cos(\theta + m\theta - 2\alpha)$$
$$= 1 + \cos 2(\theta - m\theta) - \{\cos(2\theta - 2\alpha) + \cos(2m\theta - 2\alpha)\}$$
$$= 1 + \cos(2\theta - 2m\theta) - \cos(2\theta - 2\alpha) - \cos(2m\theta - 2\alpha).$$

10. $x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2$
$$= -(4x^2y^2 - x^4 - 2x^2y^2 - y^4 + 2y^2z^2 + 2z^2x^2 - z^4)$$
$$= -\{4x^2y^2 - (x^2+y^2)^2 + 2(x^2+y^2)z^2 - z^4\}$$
$$= -[4x^2y^2 - \{(x^2+y^2)^2 - 2(x^2+y^2)z^2 + z^4\}]$$
$$= -\{(2xy)^2 - (x^2+y^2-z^2)^2\}$$
$$= -\{2xy + (x^2+y^2-z^2)\}\{2xy - (x^2+y^2-z^2)\}$$
$$= -\{(x+y)^2 - z\}\{z^2 - (x-y)^2\}$$
$$= -(x+y+z)(x+y-z)(z+x-y)(z-x+y).$$

$\therefore$ log of the first exp. $= \log(x+y+z) + \log(z-x-y)$
$$+ \log(z+x-y) + \log(z+y-x).$$

11. (i) $$\sin 3\theta + \sqrt{3}\cos 3\theta = 1,$$

$$\frac{\sin 3\theta}{2} + \frac{\sqrt{3}}{2} . \cos 3\theta = \tfrac{1}{2}, \ \sin(3\theta + 60^\circ) = \sin 30^\circ;$$

$$\therefore 3\theta + \tfrac{1}{3}\pi = n\pi + (-1)^n \tfrac{1}{6}\pi; \ \therefore 3\theta = n\pi - \tfrac{1}{3}\pi + (-1)^n \tfrac{1}{6}\pi,$$

$$\theta = \frac{n\pi}{3} - \frac{\pi}{9} + (-1)^n \tfrac{1}{18}\pi = \tfrac{1}{18}\{6n\pi - 2\pi + (-1)^n \pi\}.$$

(ii) $$\sin m\theta = \cos n\theta; \ \therefore \cos(\tfrac{1}{2}\pi - m\theta) = \cos n\theta;$$

$$\therefore \tfrac{1}{2}\pi - m\theta = 2r\pi \pm n\theta; \ \therefore \theta(m \pm n) = \tfrac{1}{2}\pi - 2r\pi.$$

(iii) $$\frac{\cos(\beta + x)}{\cos(\alpha - x)} = \frac{m \sin\beta}{n \sin\alpha};$$

$$\therefore \frac{\cos(\beta+x) + \cos(\alpha - x)}{\cos(\alpha - x) - \cos(\beta + x)} = \frac{m\sin\beta + n\sin\alpha}{n\sin\alpha - m\sin\beta};$$

$$\therefore \frac{2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\beta-\alpha+2x)}{2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\beta-\alpha+2x)}=\frac{m\sin\beta+n\sin\alpha}{n\sin\alpha-m\sin\beta};$$

$$\therefore \cot\tfrac{1}{2}(\alpha+\beta)\,.\,\cot\tfrac{1}{2}(\beta-\alpha+2x)=\frac{m\sin\beta+n\sin\alpha}{n\sin\alpha-m\sin\beta};$$

$$\therefore \cot\tfrac{1}{2}(\beta-\alpha+2x)=\tan\tfrac{1}{2}(\alpha+\beta)\,.\,\frac{m\sin\beta+n\sin\alpha}{n\sin\alpha-m\sin\beta};$$

$$\therefore x-\tfrac{1}{2}(\alpha-\beta)=\cot^{-1}\left\{\tan\tfrac{1}{2}(\alpha+\beta)\,.\,\frac{m\sin\beta+n\sin\alpha}{n\sin\alpha-m\sin\beta}\right\}.$$

(iv) $$\tan m\theta=\cot n\theta;$$

$$\therefore \cot(\tfrac{1}{2}\pi-m\theta)=\cot n\theta;\ \therefore\ \tfrac{1}{2}\pi-m\theta=r\pi+n\theta,$$

or, since $\tan m\theta=\cot n\theta$, $\therefore$ $m\theta$ and $n\theta$ are complementary; therefore

$$m\theta+n\theta=r\pi+\tfrac{1}{2}\pi.$$

(v) $$\tan\theta+\tan 2\theta+\tan 3\theta=0;$$

$$\therefore \frac{\sin\theta}{\cos\theta}+\frac{\sin 2\theta}{\cos 2\theta}+\frac{\sin 3\theta}{\cos 3\theta}=0;$$

$$\therefore \frac{\sin\theta\cos 3\theta+\sin 3\theta\cos\theta}{\cos\theta\cos 3\theta}+\frac{\sin 2\theta}{\cos 2\theta}=0,$$

$$\frac{\sin 4\theta}{\cos\theta\cos 3\theta}+\frac{\sin 2\theta}{\cos 2\theta}=0,$$

$$\frac{2\sin 2\theta\cos 2\theta}{\cos\theta\cos 3\theta}+\frac{\sin 2\theta}{\cos 2\theta}=0;$$

$$\therefore \sin 2\theta\,(2\cos^2 2\theta+\cos\theta\cos 3\theta)=0;$$

$$\therefore \sin 2\theta\,(4\cos^2 2\theta+2\cos\theta\cos 3\theta)=0;$$

$$\therefore \sin 2\theta\,(4\cos^2 2\theta+\cos 4\theta+\cos 2\theta)=0;$$

$$\therefore \sin 2\theta\,(6\cos^2 2\theta+\cos 2\theta-1)=0;$$

$$\therefore \sin 2\theta\,(3\cos 2\theta-1)(2\cos 2\theta+1)=0.$$

(vi) $$\cos 8\theta-\cos 5\theta+\cos 3\theta=1,$$

$$\cos 8\theta+2\sin 4\theta\sin\theta=1,$$

$$1-2\sin^2 4\theta+2\sin 4\theta\sin\theta=1,$$

$$2\sin 4\theta\,(\sin\theta-\sin 4\theta)=0,$$

$$4\sin 4\theta\,.\,\cos\tfrac{1}{2}5\theta\sin\tfrac{1}{2}3\theta=0.$$

(vii) $$\cos\theta\,.\,\cos 3\theta=\cos 5\theta\,.\,\cos 7\theta;$$

$$\therefore \cos 4\theta+\cos 2\theta=\cos 12\theta+\cos 2\theta;$$

$$\therefore \cos 4\theta-\cos 12\theta=0;\ \therefore\ 2\sin 4\theta\sin 8\theta=0.$$

12. $$\tan\frac{A-b}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

Now let $\dfrac{a}{b}=\tan\phi$; $\therefore\ \dfrac{a-b}{a+b}=\dfrac{\tan\phi-1}{\tan\phi+1}=\tan\left(\phi-\dfrac{\pi}{4}\right)$;

$\therefore$ if $\phi=\tan^{-1}\dfrac{a}{b}$, $\tan\frac{1}{2}(A-B)=\tan(\phi-\frac{1}{4}\pi)\cot\dfrac{C}{2}$.

13. $b^2+c^2-2bc\cos(60^\circ+A)$

$$=c^2+b^2-2bc(\cos 60^\circ\cos A-\sin 60^\circ\sin A)$$
$$=c^2+b^2-bc\cos A-2bc\sin 60^\circ\sin A$$
$$=c^2+b^2-\tfrac{1}{2}(b^2+c^2-a^2)-2ac\sin 60^\circ\sin B$$
$$=c^2+a^2-\tfrac{1}{2}(a^2+c^2-b^2)-2ac\sin 60^\circ\sin B$$
$$=c^2+a^2-ca\cos B-2ca\sin 60^\circ\sin B$$
$$=c^2+a^2-2ca\cos(60^\circ+B).$$

Let O_1, O_2, O_3 be the centres of the equilateral triangles described on BC, CA and AB respectively.

Then $$O_1O_2^2=O_1C^2+O_2C^2-2O_1C\,.\,O_2C\cos O_1CO_2.$$

Now $$O_1C=\frac{S}{s}=\frac{a^2\sin 60^\circ}{\frac{3}{2}a}=\frac{a}{\sqrt{3}},$$

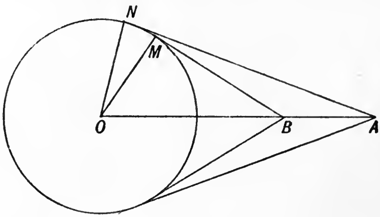
$$O_2C=\frac{S}{s}=\frac{b^2\sin 60^\circ}{\frac{3}{2}b}=\frac{b}{\sqrt{3}};$$

$$\therefore\ 3O_1O_2^2=a^2+b^2-2ab\cos(60^\circ+C)$$
$$=a^2+b^2-\tfrac{1}{2}(a^2+b^2-c^2)+2ab\sin 60^\circ\sin C$$
$$=a^2+b^2-\tfrac{1}{2}(a^2+b^2-c^2)+ab\sin C\sqrt{3}$$
$$=\tfrac{1}{2}(a^2+b^2+c^2)+2S\sqrt{3},$$

$3O_2O_3^2$ and $3O_3O_1^2$ are reducible to this expression, therefore

$$O_1O_2=O_2O_3=O_3O_1.$$

14. Let O be the centre of the base of the tower; from A draw AM and from B draw BN to touch the circular base of the tower, then the angle



$OAM=\alpha$, $OBN=\beta$. Let r be the radius.

Then $$r=OA\sin\alpha=OB\sin\beta;$$

$$\therefore\ AB=OA-OB=\frac{r}{\sin\alpha}-\frac{r}{\sin\beta};\quad\therefore\ r=\frac{a\sin\alpha\sin\beta}{\sin\beta-\sin\alpha};$$

$$\therefore\ \text{the diameter}=2r=\frac{2a\sin\alpha\sin\beta}{\sin\beta-\sin\alpha}.$$

15. Let $OP=h$ be the height of the tower,

$$OBP=\tfrac{1}{4}\pi-a \text{ and } OAP=\tfrac{1}{4}\pi+a.$$

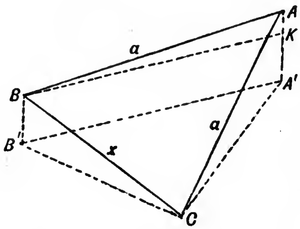
Angle $\qquad ABP=OAP-OBP=\tfrac{1}{4}\pi+a-\tfrac{1}{4}\pi+a=2a,$

$$AB=OB-OA=h\cot(45^\circ-a)-h\cot(45^\circ+a)$$

$$=h\left\{\frac{\cos(45^\circ-a)}{\sin(45^\circ-a)}-\frac{\cos(45^\circ+a)}{\sin(45^\circ+a)}\right\}=h\frac{\sin 2a}{\sin(45^\circ+a)\sin(45^\circ-a)}$$

$$=h\frac{2\sin 2a}{\cos 2a}=2h\tan 2a.$$

16. Through C let the horizontal plane $A'B'C$ be drawn and let AA', BB' be vertical lines.



Then $B'CA'=\theta$, and $ACA'=\lambda$, $BCB'=\mu$, $AB=a=AC$. Let $BC=x$.

Draw BK horizontally to cut AA' in K.

Then $\quad AB^2=BK^2+AK^2=(B'A')^2+(AA'-BB')^2$

$$=\{B'C^2+CA'^2-2B'C\,.\,CA'\cos\theta\}+(a\sin\lambda-x\sin\mu)^2,$$

or $\qquad a^2=(x^2\cos^2\mu+a^2\cos^2\lambda-2ax\cos\mu\cos\lambda\cos\theta)+(a\sin\lambda-x\sin\mu)^2$

$$=x^2(\cos^2\mu+\sin^2\mu)+a^2(\cos^2\lambda+\sin^2\lambda)-2ax\{\cos\mu\cos\lambda\cos\theta+\sin\lambda\sin\mu\};$$

$$\therefore\ x^2=2ax\{\cos\mu\cos\lambda\cos\theta+\sin\lambda\sin\mu\},$$

or $\qquad x=2a\cos\theta\cos\lambda\cos\mu+2a\sin\lambda\sin\mu$

$$=a\cos\theta\{\cos(\lambda-\mu)+\cos(\lambda+\mu)\}+a\{\cos(\lambda-\mu)-\cos(\lambda+\mu)\}$$

$$=a\cos(\lambda-\mu)\{1+\cos\theta\}+a\sin(\lambda+\mu)\{\cos\theta-1\}$$

$$=2a\{\cos(\lambda-\mu)\cos^2\tfrac{1}{2}\theta-\cos(\lambda+\mu)\sin^2\theta\}.$$

17. $\qquad b^2=c^2+a^2-2ca\cos B;\ \therefore\ a^2-2ca\cos B+c^2-b^2=0.$

If a_1a_2 are the two values of a, we have from the theory of Quadratic Equations in Algebra,

(i) $a_1+a_2=2c\cos B.\qquad$ (ii) $a_1a_2=c^2-b^2.$

(iii) From (i) $\qquad (a_1+a_2)^2=4c^2\cos^2 B.$

From (ii) $\quad 4a_1a_2\cos^2 B=4(c^2-b^2)\cos^2 B;$

$$\therefore\ (a_1+a_2)^2-4a_1a_2\cos^2 B=4b^2\cos^2 B,$$

i.e. $a_1^2+2a_1a_2+a_2^2-4a_1a_2\cos^2 B=4b^2\cos^2 B;$

$$\therefore\ a_1^2-2a_1a_2(2\cos^2 B-1)+a_2^2=4b^2\cos^2 B;$$

$$\therefore\ a_1^2-2a_1a_2\cos 2B+a_2^2=4b^2\cos^2 B.$$

(iv) Vide fig. iii. E. T. p. 216. If a perpendicular be drawn from the middle point of AB, the centres of the two circles lie in this perpendicular; centre of circle circumscribing ABC_2 being the point where the perpendicular from the middle point of BC_2 intersects the perpendicular from the middle point of AB, and the centre of the other circle being the point where the perpendicular from the middle point of BC_1 intersects the perpendicular from the middle point of AB. The length of the distance between the middle points of BC_2 and BC_1 is $\frac{a_1 - a_2}{2}$.

Let x be the distance between the required centres, then

$$\frac{a_1 - a_2}{2x} = \sin B, \quad \therefore\ x = \frac{a_1 - a_2}{2 \sin B}.$$

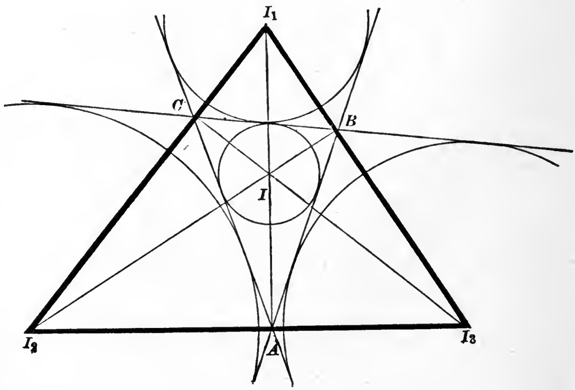
(v) The diameters of the circumscribing circle is $\frac{b}{\sin B}$ or $\frac{c}{\sin C}$, which is the same for both triangles. Therefore the circles are equal.

(vi) E. T. p. 216, fig. 111.

$$\cos C_2AC_1 = \frac{b^2 + b^2 - (a_1 - a_2)^2}{2b^2} = 1 - \frac{(a_1 - a_2)^2}{2b^2},$$

but
$$b^2 = c^2 + a_1{}^2 - \sqrt{2}a_1c = c^2 + a_2{}^2 - \sqrt{2}a_2c;$$
$$\therefore\ 2b^2 = 2c^2 + a_1{}^2 + a_2{}^2 - c\sqrt{2}\,(a_1 + a_2)$$
$$= 2c^2 + a_1{}^2 + a_2{}^2 - 2c^2 \quad \text{..........................see (i)}$$
$$= a_1{}^2 + a_2{}^2;$$
$$\therefore\ \cos C_2AC_1 = 1 - \frac{(a_1 - a_2)^2}{a_1{}^2 + a_2{}^2} = \frac{2a_1a_2}{a_1{}^2 + a_2{}^2}.$$

18. AI_2 bisects the exterior angle at A, so also does AI_3, $\therefore\ I_2AI_3$ is a straight line. I_1A bisects the angle A and the bisectors of the angle A and of the external angle at A are perpendicular.



The angle $CI_1B = 180° - I_1CB - I_1BC = BC'I + C'BI = \frac{1}{2}B + \frac{1}{2}C = 90° - \frac{1}{2}A$; $\therefore$ the angles of $I_1I_2I_3$ are $90° - \frac{1}{2}A$, $90° - \frac{1}{2}B$, $90° - \frac{1}{2}C$, so also are the angles of each of the triangles I_1BC, I_2CA, I_3AB.

Now $\quad I_2I_3 = BI_3 \sec BI_3A = \dfrac{AB \sin BAI_3}{\sin BI_3A \cos BI_3A} = \dfrac{AB \cos \frac{1}{2}A}{\cos \frac{1}{2}C \sin \frac{1}{2}C}$

$$= \frac{2c \cos \frac{1}{2}A}{\sin C} = \frac{2a \cos \frac{1}{2}A}{\sin A} = a \operatorname{cosec} \tfrac{1}{2}A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

(iii) Similarly $I_1I_2 = c \operatorname{cosec} \frac{1}{2}C$ and $I_1I_3 = b \operatorname{cosec} \frac{1}{2}B$.

Area of $I_1I_2I_3 = \frac{1}{2}(I_1I_2 \times I_1I_3 \sin I_2I_1I_3)$

$$= \frac{bc \operatorname{cosec} \frac{1}{2}C \operatorname{cosec} \frac{1}{2}B \sin(\frac{1}{2}\pi - \frac{1}{2}A)}{2} = \frac{bc \cos \frac{1}{2}A}{2 \sin \frac{1}{2}C \sin \frac{1}{2}B}$$

$$= \frac{bc\sqrt{\left\{\dfrac{s(s-a)}{bc}\right\}}}{2\sqrt{\left\{\dfrac{(s-a)(s-b)(s-c)(s-a)}{ab \cdot ca}\right\}}} = \frac{abc\sqrt{s}}{2\sqrt{\{(s-a)(s-b)(s-c)\}}}$$

$$= \frac{abcs}{2S} = \frac{abcs}{bc \sin A} = \frac{sa}{\sin A}.$$

Now $$\frac{abc\sqrt{s}}{2\sqrt{\{(s-a)(s-b)(s-c)\}}} = \frac{abc\sqrt{\{s(s-a)(s-b)(s-c)\}}}{2\sqrt{\{(s-a)(s-b)(s-c)(s-a)(s-b)(s-c)\}}}$$

$$= \frac{\frac{1}{2}S}{\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}.$$

(iv) The radius of the circle circumscribing $I_1I_2I_3$

$$= \frac{I_1I_2}{2 \sin I_1I_3I_2} = \frac{c \operatorname{cosec} \frac{1}{2}C}{2 \sin(\frac{1}{2}\pi - \frac{1}{2}C)} = \frac{c}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C} = \frac{c}{\sin C} = 2R.$$

19. In **18** we proved that ABC was the pedal triangle of $I_1I_2I_3$, and $\therefore$ we have that in the above figure, making the necessary alterations in the letters, $A = 90° - \frac{1}{2}FDE$ or $FDE = \pi - 2A$,

and $\quad BC = EF \sec(90° - \frac{1}{2}FDE)$ or $EF = a \cos A$.

(iii) $\quad AD \times a = 2S = BE \times b = CF \times c$;

$$\therefore \frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{a}{2S} + \frac{b}{2S} + \frac{c}{2S} = \frac{a+b+c}{2}\frac{s}{S} = \frac{1}{S} = \frac{1}{r}.$$

(iv) $$AD = \frac{2S}{a}, \quad BE = \frac{2S}{b}, \quad CF = \frac{2S}{c};$$

$$\therefore \frac{AD^2}{BE \cdot CF} = \frac{4S^2}{a^2} \times \frac{bc}{4S^2} = \frac{bc}{a^2}.$$

(v) The triangle AEF is similar to the triangle ABC, and its sides are respectively $a \cos A$, $b \cos A$, $c \cos A$; therefore the radius of the circle circumscribing AEF is $R \cos A$.

(vi) Radius of the circle circumscribing DEF

$$=\frac{FE}{2\sin FDE}=\frac{FE}{2\sin(\pi-2A)} \text{ ....see (ii)}$$

$$=\frac{a\cos A}{2\sin 2A}=\frac{2R\sin A\cos A}{2\sin 2A} \text{ ....see (i)}$$

$$=\frac{R\sin 2A}{2\sin 2A}=\tfrac{1}{2}R.$$

20. $a^2=b^2+c^2-2bc\cos A=b^2+c^2-2bc(1-2\sin^2\tfrac{1}{2}A)$

$$=b^2+c^2-2bc+4bc\sin^2\frac{A}{2}=(b-c)^2\left\{1+\frac{4bc}{(b-c)^2}\sin^2\frac{A}{2}\right\}.$$

Let $\tan^2\theta=\dfrac{4bc}{(b-c)^2}\sin^2\dfrac{A}{2}$,

then $a^2=(b-c)^2\{1+\tan^2\theta\}=(b-c)^2\sec^2\theta$; $\therefore a=(b-c)\sec\theta$.

21. $a^2=b^2+c^2-2bc\cos A=b^2+c^2-2bc(2\cos^2\tfrac{1}{2}A-1)$

$$=(b+c)^2-4bc\cos^2\frac{A}{2}=(b+c)^2\left\{1-\frac{4bc}{(b+c)^2}\cos^2\frac{A}{2}\right\}.$$

Let β be an angle such that $\sin^2\beta=\dfrac{4bc}{(b+c)^2}\cos^2\dfrac{A}{2}$,

then $a^2=(b+c)^2(1-\sin^2\beta)=(b+c)^2\cos^2\beta$; $\therefore a=(b+c)\cos\beta$.

(i) From 20, let $\tan\theta=\dfrac{2\sqrt{(bc)}}{b-c}\sin\dfrac{A}{2}=\dfrac{2\sqrt{(347\times 293)}\sin 19^\circ 51'}{54}$;

$$\begin{aligned}\therefore L\tan\theta&=\log 2+\tfrac{1}{2}(\log 347+\log 293)+L\sin 19^\circ 51'-\log 54\\&=\cdot 30103+\tfrac{1}{2}(2\cdot 5403295+2\cdot 4668676)+9\cdot 5309151-1\cdot 7323938\\&=10\cdot 6031457=L\tan 75^\circ 59' 51'';\end{aligned}$$

$\therefore \theta=75^\circ 59' 51''$.

Now $a=(b-c)\sec\theta$,

$$\begin{aligned}\therefore \log a&=\log(b-c)+L\sec\theta-10\\&=\log 54+L\sec 75^\circ 59' 51''-10\\&=1\cdot 7323938+10\cdot 6162489-10\\&=2\cdot 3486427=\log 223\cdot 17 \text{ nearly;}\end{aligned}$$

$\therefore a=223\cdot 17$ nearly.

Or, from above, $\sin\beta=\dfrac{2\sqrt{bc}}{b+c}\cos\dfrac{A}{2}=\dfrac{2\sqrt{(347\times 293)}}{640}\cos 19^\circ 51'$;

$$\begin{aligned}\therefore L\sin\beta&=\log 2+\tfrac{1}{2}(\log 347+\log 293)+L\cos 19^\circ 51'-\log 640\\&=\cdot 30103+\tfrac{1}{2}(2\cdot 5403295+2\cdot 4668676)+9\cdot 973398-2\cdot 80618\\&=9\cdot 9718466=L\sin 69^\circ 35' 30'';\end{aligned}$$

$s\,.\,\beta=69^\circ 35' 30''$.

Now $a=(b+c)\cos\beta=640\times\cos 69^\circ\,35'\,30''$;

$$\therefore \log a=\log 640+L\cos 69^\circ\,35'\,30''-10$$
$$=2\cdot80618+9\cdot5424624-10$$
$$=2\cdot3486424=\log 223\cdot17 \text{ nearly};$$
$$\therefore a=223\cdot17 \text{ nearly.}$$

22. Now
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C};$$
$$\therefore \frac{a}{\sin A}=\frac{b-c}{\sin B-\sin C},$$

and
$$\sin B-\sin C=2\cos\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(B-C)$$
$$=2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}(B-C);$$
$$\therefore a=\frac{(b-c)\sin A}{2\sin\frac{1}{2}A\sin\frac{1}{2}(B-C)}=\frac{(b-c)\cos\frac{1}{2}A}{\sin\frac{1}{2}(B-C)}.$$

If $\phi=\tan^{-1}\dfrac{b+c}{b-c}\tan\tfrac{1}{2}A$, then $\tan\phi=\dfrac{b+c}{b-c}\tan\tfrac{1}{2}A$.

But since
$$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\tfrac{1}{2}A,$$
$$\therefore \tan\tfrac{1}{2}A=\frac{b-c}{b+c}\cot\tfrac{1}{2}(B-C);$$
$$\therefore \tan\phi=\frac{b+c}{b-c}\cdot\frac{b-c}{b+c}\cot\tfrac{1}{2}(B-C);$$
$$\therefore \tan\phi=\cot\tfrac{1}{2}(B-C),$$

therefore ϕ and $\frac{1}{2}(B-C)$ are complementary angles and
$$\sin\tfrac{1}{2}(B-C)=\cos\phi;$$
$$\therefore a=\frac{(b-c)\cos\frac{1}{2}A}{\cos\phi};$$
$$\therefore \log a=\log(b-c)+L\cos\tfrac{1}{2}A-L\cos\phi.$$

23. Since $BD=CD$, the triangle ABD = triangle ACD;

therefore
$$2\times\text{area } ABD=\text{area } ABC;$$
$$\therefore 2\times C\times AD\times\sin BAD=bc\sin A;$$
$$\therefore \sin BAD=\frac{b\sin A}{2AD}=\frac{b\sin A}{\sqrt{(2b^2+2c^2-a^2)}} \qquad \text{Ex. 11. p. 205.}$$
$$=\frac{b\sin A}{\sqrt{(b^2+b^2+c^2-a^2+c^2)}}=\frac{b\sin A}{\sqrt{(b^2+2bc\cos A+c^2)}}.$$

24. From E. T. p. 235 (vii.) 11, $=\dfrac{a}{\cos\frac{1}{2}A}=\dfrac{2R\sin A}{\cos\frac{1}{2}A}=4R\sin\frac{1}{2}A=x$,
similarly $4R\sin\frac{1}{2}B=y$; and $4R\sin\frac{1}{2}C=z$,

$$xyz=64R^3\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C,$$

$$d(x^2+y^2+z^2)=32R^3(\sin^2\tfrac{1}{2}A+\sin^2\tfrac{1}{2}B+\sin^2\tfrac{1}{2}C)$$

$$=32R^3(1-2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C) \qquad \text{[LXXIII. 25.]}$$

$$=32R^3-64R^3\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C;$$

$$\therefore\ xyz+d(x^2+y^2+z^2)=32R^3=4d^3.$$

25. The sides $\dfrac{sa}{s-a}$, $\dfrac{sb}{s-a}$, $\dfrac{sc}{s-a}$ are in the proportion of a, b, c; i.e. the two triangles are similar and they are therefore equiangular. The area of the triangle ABC is $\frac{1}{2}ab\sin C$. The area of the second triangle

$$=\tfrac{1}{2}\left\{\frac{sa}{(a-b)}\cdot\frac{sb}{(a-b)}\right\}\sin C=\frac{\frac{1}{2}s^2ab}{(s-a)^2}\sin C.$$

Radius of circle escribed to side BC (i.e. a) of ABC

$$=\frac{S}{s-a}=\frac{ab\sin C}{2(s-a)}.$$

Radius of inscribed circle of the second triangle

$$=\frac{\text{area of the triangle}}{\text{half its perimeter}}$$

$$=\frac{\frac{1}{2}s^2ab}{(s-a)^2}\sin C\div\tfrac{1}{2}\left\{\frac{sa}{s-a}+\frac{sb}{s-a}+\frac{sc}{s-a}\right\}$$

$$=\frac{\frac{1}{2}s^2ab}{(s-a)^2}\div\frac{s^2}{s-a}=\frac{ab\sin C}{2(s-a)}.$$

Therefore the circles are equal.

26. Let $QA=x$, $QB=y$, $QC=z$; angle AQC=angle AQP=angle BQC $=120°$, $\cos 120°=-\frac{1}{2}$, $\sin 120°=\frac{1}{2}\sqrt{3}$,

$$a^2=BQ^2+QC^2+BQ\,.\,QC,$$

$$b^2=AQ^2+QC^2+AQ\,.\,QC,$$

$$c^2=AQ^2+BQ^2+AQ\,.\,QB;$$

$$\therefore\ a^2+b^2+c^2=2x^2+2y^2+2z^2+xy+yz+xz.$$

Let the area of triangle $ABC=\Delta$;

$$\therefore\ 2\Delta=yz\sin 120°+zx\sin 120°+xy\sin 120°,$$

$$\Delta=\tfrac{1}{4}\sqrt{3}(xy+yz+zx),$$

$$3(xy+yz+zx)=4\sqrt{3}\Delta.$$

Now $$2(x^2+y^2+z^2)+(xy+yz+zx)=a^2+b^2+c^2,$$

$$2(x^2+y^2+z^2)+4(xy+yz+zx)=a^2+b^2+c^2+4\sqrt{3}\Delta;$$

$$\therefore\ x+y+z=\sqrt{\{\tfrac{1}{2}(a^2+b^2+c^2)+2\sqrt{3}\Delta\}}=d;$$

$$\therefore\ y+z=d-x,\ y^2+2yz+z^2=(d-x)^2;\ \text{but}\ y^2+z^2+yz=a^2;$$

$$\therefore\ yz=(d-x)^2-a^2$$

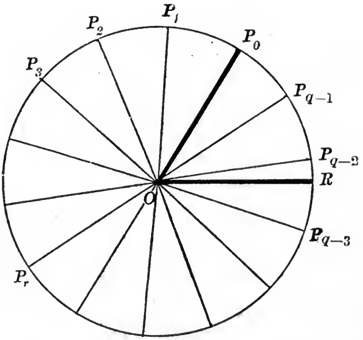
also $$yz+(xy+xz)=\tfrac{4}{3}\sqrt{3}\Delta\,;\ \therefore\ yz=\tfrac{4}{3}\sqrt{3}\Delta-x\,(y+z),$$
but $$x+y+z=d\,;\ \therefore\ yz=\tfrac{4}{3}\Delta\sqrt{3}-x\,(d-x)\,;$$
$$\therefore\ (d-x)^2-a^2=\tfrac{4}{3}\Delta\sqrt{3}-x\,(d-x)\,;$$
$$\therefore\ d^2-dx-b^2=\tfrac{4}{3}\Delta\sqrt{3}\,;$$
$$\therefore\ dx=d^2-a^2-\tfrac{4}{3}\Delta\sqrt{3}=\tfrac{1}{2}(a^2+b^2+c^2)+2\sqrt{3}\Delta-a^2-\tfrac{4}{3}\Delta\sqrt{3}\,;$$
$$\therefore\ dx=\tfrac{1}{2}(b^2+c^2-a^2)+\tfrac{2}{3}\Delta\sqrt{3}\,;$$
$$\therefore\ x=\frac{\sqrt{3}\,(b^2+c^2-a^2)+4\Delta}{2\sqrt{3}}\times\frac{\sqrt{2}}{\sqrt{(a^2+b^2+c^2+4\sqrt{3}\Delta)}}\,;$$
$$\therefore\ QA=\frac{4\sqrt{2}\Delta+\sqrt{6}\,(b^2+c^2-a^2)}{2\,\{12\sqrt{3}\Delta+3\,(a^2+b^2+c^2)\}^{\frac{1}{2}}}.$$

27. When $\sin A$ and $\cos A$ are both known then the different possible values of A differ by 360°.

Hence the different possible values of $\frac{A}{n}$ differ by $\frac{360^\circ}{n}$.

So that if $\frac{a}{n}$ be the least positive value of $\frac{A}{n}$ the different angles are

$$\frac{a}{n}+\frac{m\,360^\circ}{n}.$$



In the figure let P_0OR be $\frac{a}{n}$; let $P_1OP_0=\frac{360^\circ}{n}=P_1OP_2=P_2OP_3$, etc.

Then all possible values of A are given by the lines OP_0, OP_1, OP_2... and these angles in general have different values for their sines; and there are n of them and no more.

28. Let AB be the given arc; R any point in it; O the centre of the circle; OA and OB the bounding radii so that angle $AOB=\alpha$. From R draw RC, RD perpendiculars to the radii; join OR; let angle $AOR=\beta$ and angle $BOR=\alpha-\beta$. Therefore

$$CD\ :\ OC=\sin\alpha\ :\ \cos\beta,$$
and $$OC=OR\cos\beta\,;\ \therefore\ CD\ :\ r\cos\beta=r\sin\alpha\ :\ r\cos\beta.$$

29. Let O be the centre of the smaller circle and B the centre of the larger; E the point in which a tangent touches the smaller circle and D the point in which the same tangent touches the larger; produce BO and DE to meet at A; from the point E draw EC parallel to AB to meet BD in C.

If θ is the angle between the tangents it may be easily shewn that angle $BAD=\frac{1}{2}\theta$. Let $AB=x$, $OB=EC=a+b$, $CD=BD\sim EO=a\sim b$.

Then $$ED^2=EC^2-CD^2=(a+b)^2-(a\sim b)^2=4ab.$$

$$\therefore\ \tan\tfrac{1}{2}\theta=\frac{CD}{ED}=\frac{a\sim b}{2\sqrt{(ab)}}=\frac{1}{2}\left(\sqrt{\frac{a}{b}}\sim\sqrt{\frac{b}{a}}\right);$$

$$\therefore\ \tfrac{1}{2}\theta=\tan^{-1}\frac{1}{2}\left(\sqrt{\frac{a}{b}}\sim\sqrt{\frac{b}{a}}\right).$$

30. E. T. p. 234, fig.

Let AI bisect EF at right angles in K; then $\angle FEI=90^\circ-FEA=\frac{1}{2}A$;

$$\therefore\ EF=2EK=2EI\cos KEI=2r\cos\tfrac{1}{2}A.$$

Similarly, $$FD=2r\cos\tfrac{1}{2}B \text{ and } DE=3r\cos\tfrac{1}{2}C;$$

$$\therefore\ EF:FD:DE::2r\cos\tfrac{1}{2}A:2r\cos\tfrac{1}{2}B:2r\cos\tfrac{1}{2}C$$
$$::\cos\tfrac{1}{2}A:\cos\tfrac{1}{2}B:\cos\tfrac{1}{2}C.$$

31. (i) $$BD:DC=c:b,$$

$$\therefore\ BD:BD+DC=c:b+c;\quad \therefore BD=\frac{ac}{b+c}.$$

$$\frac{AD}{BD}=\frac{\sin B}{\sin\frac{1}{2}A},\quad \therefore\ AD=\frac{ac\sin B}{(b+c)\sin\frac{1}{2}A}=\frac{bc\sin A}{(b+c)\sin\frac{1}{2}A}=\frac{2bc}{b+c}\cos\tfrac{1}{2}A.$$

(ii) $$CD=\frac{ab}{b+c}=\frac{a\sin B}{\sin B+\sin C},\text{ as above.}$$

(iii) $$\triangle DFB:\triangle ADB=BF:BA=a:a+b,$$
$$\triangle ABD:\triangle ABC=BD:AC=c:b+c;$$
$$\therefore\ \triangle BFD:\triangle ABC=ac:(a+b)(b+c).$$

$$\therefore\ \text{Area of triangle } BDF=\frac{Sac}{(b+c)(b+a)}.$$

$$\text{Area of } CDE=\frac{Sab}{(c+a)(c+b)};\ \text{area of } AFE=\frac{Sbc}{(a+b)(a+c)}.$$

$$\triangle DEF=\triangle ABC-(\triangle DBF+\triangle CDE+\triangle AEF)$$

$$=S\left(1-\frac{ac}{(b+c)(b+a)}-\frac{ab}{(c+a)(c+b)}-\frac{bc}{(a+b)(a+c)}\right)$$

$$=\frac{S}{(a+b)(b+c)(c+a)}\left\{(a+b)(b+c)(c+a)-ac(a+c)-ab(a+b)-bc(b+c)\right\}$$

$$=\frac{2abcS}{(a+b)(b+c)(c+a)}.$$

Now $$\frac{abc}{(a+b)(b+c)(c+a)}=\frac{a}{b+c}\cdot\frac{b}{c+a}\cdot\frac{c}{a+b}.$$

$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin\frac{1}{2}A\cos\frac{1}{2}A}{\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}=\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}(B-C)};$$

$$\therefore \frac{abc}{(a+b)(b+c)(c+a)}=\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}(B-C)\cos\frac{1}{2}(C-A)\cos\frac{1}{2}(A-B)};$$

$$\therefore S\frac{2abc}{(a+b)(b+c)(c+a)}=S\frac{2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}(B-C)\cos\frac{1}{2}(C-A)\cos\frac{1}{2}(A-B)}.$$

32. Since $\alpha, \beta, \gamma, \delta$ are the angles of a quadrilateral inscribed in a circle

$$\alpha+\gamma=\pi,\quad \beta+\delta=\pi,\quad \alpha+\gamma+\beta+\delta=2\pi;$$

$$\therefore \gamma+\delta=2\pi-(\alpha+\beta) \text{ and } \delta+\alpha=2\pi-(\beta+\gamma);$$

$$\begin{aligned}&\cos(\alpha+\beta)\,.\cos(\beta+\gamma)\,.\cos(\gamma+\delta)\cos(\delta+\alpha)\\&=\cos(\alpha+\beta)\,.\cos(\beta+\gamma)\,.\cos\{2\pi-(\alpha+\beta)\}\cos\{2\pi-(\beta+\gamma)\}\\&=\cos^2(\alpha+\beta)\,.\cos^2(\beta+\gamma)\\&=\cos^2(\alpha+\beta)\cos^2\{\pi-(\alpha-\beta)\}=\cos^2(\alpha+\beta)\cos^2(\alpha-\beta)\\&=\{\cos(\alpha+\beta)\cos(\alpha-\beta)\}^2=(\sin^2\alpha-\cos^2\beta)^2 \qquad \text{[Ex. xxxv. (25).}\\&=(1-\cos^2\alpha-\cos^2\beta)^2.\end{aligned}$$

33. Since θ_1 and θ_2 are values of θ, we have

$$a\cos\theta_1+b\sin\theta_1=c \text{ ....(i)},$$
$$a\cos\theta_2+b\sin\theta_2=c \text{ ....(ii)}.$$

Multiply (i) by $\sin\theta_1$ and (ii) by $\sin\theta_2$ and subtract;

$$\therefore a(\cos\theta_1\sin\theta_2-\cos\theta_2\sin\theta_1)=c(\sin\theta_2-\sin\theta_1);$$

$$\therefore a\sin(\theta_1-\theta_2)=c(\sin\theta_1-\sin\theta_2);$$

$$\therefore 2a\sin\tfrac{1}{2}(\theta_1-\theta_2)\cos\tfrac{1}{2}(\theta_1-\theta_2)=2c\sin\tfrac{1}{2}(\theta_1-\theta_2)\cos\tfrac{1}{2}(\theta_1+\theta_2);$$

$$\therefore \frac{1}{c}\,.\cos\tfrac{1}{2}(\theta_1-\theta_2)=\frac{1}{a}\cos\tfrac{1}{2}(\theta_1+\theta_2).$$

Now multiply (i) by $\cos\theta_2$ and (ii) by $\cos\theta_1$ and subtract;

$$\therefore b\sin(\theta_1-\theta_2)=c(\cos\theta_2-\cos\theta_1);$$

$$\therefore 2b\sin\tfrac{1}{2}(\theta_1-\theta_2)\cos\tfrac{1}{2}(\theta_1-\theta_2)=2c\sin\tfrac{1}{2}(\theta_1-\theta_2)\sin\tfrac{1}{2}(\theta_1+\theta_2);$$

$$\therefore \frac{1}{c}\cos\tfrac{1}{2}(\theta_1-\theta_2)=\frac{1}{b}\sin\tfrac{1}{2}(\theta_1+\theta_2);$$

$$\therefore \frac{1}{a}\,.\cos\tfrac{1}{2}(\theta_1+\theta_2)=\frac{1}{b}\,.\sin\tfrac{1}{2}(\theta_1+\theta_2)=\frac{1}{c}\cos\tfrac{1}{2}(\theta_1-\theta_2).$$

34. Solving the first two equations, assuming that $\cos\theta$, $\cos\phi$, $\cos\chi$ are none of them zero, we have

$$\frac{a}{\cos\phi+\cos\theta\cos\chi}=\frac{b}{\cos\theta+\cos\phi\cos\chi}=\frac{c}{1-\cos^2\chi}.$$

Substituting in the third equation we have,

$$1-\cos^2\chi=\cos\phi\,(\cos\phi+\cos\theta\cos\chi)+\cos\theta\,(\cos\theta+\cos\phi\cos\chi);$$

$$\therefore\ \cos^2\theta+\cos^2\phi+\cos^2\chi+2\cos\theta\cos\phi\cos\chi-1=0,$$

and the required result follows.

From E. T. Ex. XLIII. (20) the expression may be put into the form

$$8\cos\frac{\theta+\phi+\chi}{2}\,.\,\cos\frac{\phi+\chi-\theta}{2}\,.\,\cos\frac{\chi+\theta-\phi}{2}\,.\,\cos\frac{\theta+\phi-\chi}{2}.$$

In order that this expression may be zero one of the four cosines must be zero; therefore one of the four compound angles must be some odd multiple of a right angle, i.e. $(\theta\pm\phi\pm\chi)=(2n+1)\pi$.

35. E. T. p. 234 fig. The first circle drawn in the manner indicated in the question will have its centre and the point of contact with the inscribed circle of the triangle ABC in the straight line AI. Let r_1 be the radius of this circle, then the length of $AI=r+r_1+r_1\operatorname{cosec}\frac{1}{2}A$; and this whole distance also is $r\operatorname{cosec}\frac{1}{2}A$;

$$\therefore\ r_1(1+\operatorname{cosec}\tfrac{1}{2}A)=r\,(\operatorname{cosec}\tfrac{1}{2}A-1);$$

$$\therefore\ r_1=\frac{r\,(\operatorname{cosec}\frac{1}{2}A-1)}{1+\operatorname{cosec}\frac{1}{2}A}=\frac{r\,(1-\sin\frac{1}{2}A)}{1+\sin\frac{1}{2}A}.$$

Let r_2 be the radius of the second circle drawn in the manner indicated in the question; then, as above, it can be shewn that

$$r_2=\frac{r_1\,(1-\sin\frac{1}{2}A)}{1+\sin\frac{1}{2}A}=r\left(\frac{1-\sin\frac{1}{2}A}{1+\sin\frac{1}{2}A}\right)^2.$$

By repeating the process we have, where r_n is the radius of the nth similarly described circle,

$$r_n=r\left(\frac{1-\sin\frac{1}{2}A}{1+\sin\frac{1}{2}A}\right)^n.$$

Now $r_1+r_2+r_3+\ldots\ldots r_n$, when $n=\infty$

$$=r\left\{\frac{1-\sin\frac{1}{2}A}{1+\sin\frac{1}{2}A}+\left(\frac{1-\sin\frac{1}{2}A}{1+\sin\frac{1}{2}A}\right)^2+\left(\frac{1-\sin\frac{1}{2}A}{1+\sin\frac{1}{2}A}\right)^3+\ldots\ldots\text{ ad infin.}\right\}$$

$$=r\left\{\frac{1-\sin\frac{1}{2}A}{1+\sin\frac{1}{2}A}\div\left(1-\frac{1-\sin\frac{1}{2}A}{1+\sin\frac{1}{2}A}\right)\right\}$$

$$=r\,.\,\frac{1-\sin\frac{1}{2}A}{2\sin\frac{1}{2}A}=\tfrac{1}{2}r\,(\operatorname{cosec}\tfrac{1}{2}A-1).$$

36. From O draw OP perpendicular to BC; then angle $BOP=$ angle A and angle $CBO=90^\circ-A$; angle $ABO=B-(90^\circ-A)=A+B-90^\circ=90^\circ-C$. In the same way it may be shewn that the two angles into which AD divides the angle A are $90^\circ-B$ and $90^\circ-C$; and the two angles into which CF divides C are $90^\circ-A$ and $90^\circ-B$.

$$\frac{BD}{AD}=\frac{\sin(90^\circ-C)}{\sin B}=\frac{\cos C}{\sin B}.$$

Similarly

$$\frac{CD}{AD}=\frac{\sin B}{\sin C}.$$

$$\therefore\ BD+CD=BC=a=AD\left(\frac{\cos C}{\sin B}+\frac{\cos B}{\sin C}\right).$$

$$\therefore\ \frac{1}{AD}=\frac{\sin C\cos C+\sin B\cos B}{a\sin B\sin C}=\frac{\sin B\cos B+\sin C\cos C}{2R\sin A\sin B\sin C}.$$

Similarly
$$\frac{1}{BE}=\frac{\sin C\cos C+\sin A\cos A}{2R\sin A\sin B\sin C},$$

$$\frac{1}{CF}=\frac{\sin A\cos A+\sin B\cos B}{2R\sin A\sin B\sin C};$$

$$\therefore\ \frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}=\frac{\sin 2A+\sin 2B+\sin 2C}{2R\sin A\sin B\sin C}$$

$$=\frac{4\sin A\sin B\sin C}{2R\sin A\sin B\sin C}=\frac{2}{R}.$$ [Ex. LXII. (25).]

37. Draw the figure $ABCD$ and let the diagonals of the quadrilateral intersect at O; let

$$OA=a,\ OC=c,\ \therefore\ a+c=k;\ OB=b,\ OD=d,\ \therefore\ b+d=k.$$

The area of the quadrilateral

$$=\triangle AOD+\triangle BOC+\triangle AOB+\triangle COD$$
$$=\tfrac{1}{2}ad\sin\theta+\tfrac{1}{2}bc\sin\theta+\tfrac{1}{2}ab\sin\theta+\tfrac{1}{2}cd\sin\theta$$
$$=\tfrac{1}{2}(ad+bc+ab+cd)\sin\theta=\tfrac{1}{2}(a+c)(b+d)\sin\theta=\tfrac{1}{2}hk\sin\theta.$$

38. If
$$\frac{x}{\sin X}=\frac{y}{\sin Y}=\frac{z}{\sin Z},$$

$$\therefore\ \frac{x}{\sin X}=\frac{y\cos Z}{\sin Y\cos Z}=\frac{z\cos Y}{\sin Z\cos Y},$$

$$\therefore\ \frac{x}{\sin X}=\frac{y\cos Z+z\cos Y}{\sin Y\cos Z+\sin Z\cos Y}=\frac{y\cos Z+z\cos Y}{\sin(Y+Z)}$$

$$=\frac{y\cos Z+z\cos Y}{\sin(180^\circ-X)}=\frac{y\cos Z+z\cos Y}{\sin X};$$

$$\therefore\ x=y\cos Z+z\cos Y.$$

39. If R and r are respectively the radii of the circumscribed and inscribed circles of a regular polygon of n sides, each of which is a, it may be seen from E. T. Arts. 283, 284 that

$$R=\tfrac{1}{2}a\operatorname{cosec}\frac{\pi}{n}\ \text{ and }\ r=\tfrac{1}{2}a\cot\frac{\pi}{2n}.$$

Let x and y be the length of the sides of the given polygons of n and $2n$ sides respectively.

I. In the first polygon $r=\tfrac{1}{2}x\cot\dfrac{\pi}{n}$.

In the second polygon $R=\tfrac{1}{2}y\operatorname{cosec}\dfrac{\pi}{2n}$.

From the question $R=r$, $\therefore\ 2y\cos\dfrac{\pi}{2n}=x\cos\dfrac{\pi}{n}$.

II. Now $3+\sqrt{3} : 4\sqrt{2} = \frac{1}{2} y \cot\frac{\pi}{2n} : \frac{1}{2} x \operatorname{cosec}\frac{\pi}{n} = y\cos^2\frac{\pi}{2n} : x$

$$= x\cos\frac{\pi}{n}\cos\frac{\pi}{2n} : x \text{ ....from I.}$$

$$\therefore \cos\frac{\pi}{n}\cos\frac{\pi}{2n} = \frac{3+\sqrt{3}}{4\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}\cdot\frac{\sqrt{3}}{2} = \cos 15^\circ . \cos 30^\circ;$$

$$\therefore \frac{\pi}{n} = 30^\circ \text{ and } n = 6.$$

It will be interesting to the student to observe that this is an instance of the Trigonometrical solution of a Cubic Equation.

For assume $\cos\frac{\pi}{2n} = x$, then

$$\cos\frac{\pi}{2n}\cos\frac{\pi}{n} = \cos\frac{\pi}{2n}\left(2\cos^2\frac{\pi}{2n} - 1\right) = x(2x^2-1);$$

and the equation becomes $2x^3 - x = \frac{3+\sqrt{3}}{4\sqrt{2}}$.

40. Since AI_1 bisects the angle A, therefore the points A, I, I_1 lie on the same straight line; produce AI to I_1 cutting the circumscribing circle at V; join I_1O cutting the circumscribing circle at the point K and produce it to meet the circle again at H. Now I_1C bisects the exterior angle at C of the triangle ABC and IC bisects the angle ACB, therefore the angle ICI_1 is a right angle; and a circle with II_1 as diameter can circumscribe the triangle ICI_1. But from E. T. Art. 282, we know that $VC=VI$; therefore V is the centre of the circumscribing circle of the triangle ICI_1, hence $IV=I_1V$.

Now $\qquad I_1O^2 - OK^2 = I_1K . I_1H \qquad$ (Eucl. II. 6)

$\qquad\qquad = I_1A . I_1V. \qquad$ (Eucl. III. 36, *Cor.*)

But $\qquad I_1A = \frac{r_1}{\sin\frac{1}{2}A}, \qquad$ (see fig. E. T. p. 234)

and $\qquad I_1V = IV = VC = 2R\sin\frac{1}{2}A. \qquad$ (E. T. Art. 282.)

Therefore $\qquad I_1O^2 - OP^2 = \frac{r_1}{\sin\frac{1}{2}A} . 2R\sin\frac{1}{2}A = 2Rr_1;$

$$\therefore I_1O^2 - OP^2 = 2Rr_1, \text{ i.e. } I_1O^2 = R^2 + 2Rr_1.$$

41. Assume $x=\tan\frac{1}{2}A$, $y=\tan\frac{1}{2}B$, then $z=\tan\frac{1}{2}C$ from Ex. (33) p. 194 and $\frac{1}{2}A+\frac{1}{2}B+\frac{1}{2}C=90^\circ$; $\therefore A+B+C=180^\circ$; and from Ex. (32) p. 194

$$\tan A + \tan B + \tan C = \tan A . \tan B . \tan C,$$

$$\text{i.e. } \frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A} + \frac{2\tan\frac{1}{2}B}{1-\tan^2\frac{1}{2}B} + \frac{2\tan\frac{1}{2}C}{1-\tan^2\frac{1}{2}C}$$

$$= \frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}\cdot\frac{2\tan\frac{1}{2}B}{1-\tan^2\frac{1}{2}B}\cdot\frac{2\tan\frac{1}{2}C}{1-\tan^2\frac{1}{2}C}.$$

CAMBRIDGE PRINTED BY JOHN CLAY M.A. AT THE UNIVERSITY PRESS.