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NEW YORK UNIVERSITY

[LECTURE NOTES

VOLUME 5]

Linear operators and their spectra

by

Kurt Friedrichs

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LINEAR OPERATORS AND THEIR SPECTRA

by Kurt Friedrichs

The theory of linear operators and their spectra has emerged from generalizing the problem of principal axes of quadratic forms. Let

$$xQx = q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2$$

be a quadratic form, $xQx = 1$ a quadric in the (x_1, x_2) -plane, then the problem is to find a new system of rectangular coordinates (a_1, a_2) such that

$$xQx = a_1^2/\alpha_1^2 + a_2^2/\alpha_2^2$$

$$x_1^2 + x_2^2 = a_1^2 + a_2^2;$$

α_1, α_2 are then the semi-axes of the quadric.

A first generalization of this problem was Hilbert's theory of quadratic forms in infinitely many variables (1906), with application to integral equations. Hilbert's theory was also formulated in terms of infinite matrices. Infinite matrices were later on basic in the quantum theory of Heisenberg (1925); they were, however, of an essentially more general character than those that could be treated by Hilbert's theory.

Quite a different class of problems in analysis present strong analogies to the problem of principal axes of quadratic forms: namely, the problems concerning characteristic values of differential equations and the connected expansions of the type of Fourier's series and Fourier's integral. A typical problem of this class can be formulated as follows: We take the differential equation $-\frac{d}{ds}x(s) = \lambda x(s)$,

where λ is a value to be determined, and consider two classes of admitted functions.

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Mathematical Induction

Let $P(n)$ be a statement involving n . To prove $P(n)$ is true for all $n \in \mathbb{N}$, we use the principle of mathematical induction. The first step is to verify that $P(1)$ is true. The second step is to assume $P(k)$ is true for some $k \in \mathbb{N}$ and prove that $P(k+1)$ is true.

For example, let $P(n)$ be the statement that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$. We first verify $P(1)$ is true. Then we assume $P(k)$ is true and show $P(k+1)$ is true.

$$1 + 2 + \dots + k = \frac{k(k+1)}{2}$$

Assuming $P(k)$ is true, we have $1 + 2 + \dots + k = \frac{k(k+1)}{2}$. To prove $P(k+1)$ is true, we need to show $1 + 2 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$. Adding $(k+1)$ to both sides of the equation for $P(k)$ gives $1 + 2 + \dots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$. Thus, $P(k+1)$ is true.

Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. This method is used to prove many mathematical statements involving natural numbers.

Q.E.D.

1. We restrict s to the range $0 \leq s \leq 1$ and impose the boundary condition $x(0) = x(1) = 0$. Then the solutions are

$$x = u_n(s) = 2^{-1/2} \sin n\pi s, \quad n = 1, 2, 3, \dots$$

$$\lambda = \lambda_n = n^2 \pi^2 \quad \text{"eigen-values"}$$

With them we have the expansions.

$$x(s) = \sum_n a_n u_n(s); \quad \int_0^1 x(s)^2 ds = \sum_n a_n^2$$

$$-\frac{d^2}{ds^2} x(s) = \sum_n a_n u_n(s) \quad \int_0^1 \left(\frac{d}{ds} x(s)\right)^2 ds = \sum_n \lambda_n a_n^2$$

The first expansion is simply a Fourier series. The latter two formulas present an obvious analogy to the reduction of quadratic forms to principal axes. The set of values λ_n is called the spectrum of $-\frac{d}{ds^2}$; in particular it is a point spectrum

since it consists of discrete points on the λ -axis.

2. Secondly we restrict s to the range $0 \leq s < \infty$ and impose as boundary condition only $x(0) = 0$. Then the solutions are

$$x(s) = u_\mu(s) = (\pi/2)^{1/2} \sin \mu s,$$

where μ runs over the entire range $0 \leq \mu < \infty$, $\lambda = \lambda_\mu = \mu^2$, and we have the expansions

$$x(s) = \int_0^\infty a_\mu u_\mu(s) ds; \quad \int_0^\infty x(s)^2 ds = \int_0^\infty a_\mu^2 d\mu$$

$$-\frac{d^2}{ds^2} x(s) = \int_0^\infty \lambda_\mu a_\mu u_\mu(s) ds; \quad \int_0^\infty \left(\frac{d}{ds} x(s)\right)^2 ds = \int_0^\infty \lambda_\mu a_\mu^2 d\mu.$$

The first expansion is nothing but a Fourier-Integral.

In this case where the characteristic values run over a continuous range $0 \leq \lambda < \infty$, the spectrum is termed "continuous". It is interesting to note that here the characteristic functions themselves are not representable because the integral $\int_0^\infty u_\mu^2(s) d\mu$ is infinite - a point which was left in obscurity for some time.

Let $f(x) = x^2 + 2x + 1$. Then $f(x) = (x+1)^2$. The function $f(x)$ is always non-negative, and it is zero only when $x = -1$.

$$f(x) = x^2 + 2x + 1 = (x+1)^2$$

$$f(x) = (x+1)^2$$

$$f(x) = (x+1)^2 \geq 0$$

The function $f(x) = (x+1)^2$ is a parabola opening upwards with its vertex at $(-1, 0)$. Since the square of any real number is non-negative, $f(x) \geq 0$ for all $x \in \mathbb{R}$. The only root of the equation $f(x) = 0$ is $x = -1$.

$$f(x) = (x+1)^2 = 0 \implies x = -1$$

Therefore, the function $f(x) = x^2 + 2x + 1$ is non-negative for all real numbers x , and it is zero only at $x = -1$.

$$f(x) = (x+1)^2 \geq 0$$

$$f(x) = (x+1)^2 \geq 0$$

The function $f(x) = (x+1)^2$ is a parabola opening upwards with its vertex at $(-1, 0)$. Since the square of any real number is non-negative, $f(x) \geq 0$ for all $x \in \mathbb{R}$. The only root of the equation $f(x) = 0$ is $x = -1$.

The classical application of problems of the preceding type is that to natural vibrations of elastic media. In 1926 Schrödinger caused quite a stir when he presented his quantum theory based on differential equations of a similar though more complicated type. Although Schrödinger himself showed the formal identity of his and Heisenberg's theory, a complete mathematical theory of which all the mentioned theories are special cases, was lacking.

It was the decisive merit of v. Neumann to have broken the mathematical deadlock. He discovered that this deadlock was due to the use of a wrong system of concepts and he developed an appropriate new system of concepts. Using it, he and Stone (1929) derived a general theory of spectral resolution. It is true, von Neumann and Stone did not show that the differential equation problems of quantum mechanics fall under their general theory; to show this has been one of my personal interests. The decisive step, however, was the discovery of the right system of concepts by v. Neumann. Strongly enough, physicists have apparently failed to appreciate von Neumann's work sufficiently. This lack of appreciation may partly be due to the fact that v. Neumann's system is the right one for formulating the principles of quantum theory, not, however, the right one for actually calculating solutions of special problems.

Quadratic forms	Differential equations
Infinite matrices Hilbert	Fourier series
Integral equations 1906	Fourier integral
Quantum theory Heisenberg 1925	Quantum theory Schrödinger 1927
Operators in Hilbert spaces	
v. Neumann 1927/29	
Stone 1929	

In this course I do not want to derive methods for actually calculating solutions of special characteristic value problems. On the other hand I do not want to derive the abstract concepts of von Neumann in their full generality. Rather I wish to derive them gradually in immediate connection with special problems, in particular, differential equation problems; and I want to show how naturally all existence, completeness and expansion theorems result from a general theory.

Literature von Neumann, Mathematische Grundlagen der Quantenmechanik. 1932

Stone, Linear Transformations in Hilbert Space, 1932

Chapter I. Space of finite dimension.

Space of vectors $x = \{x_1, x_2, \dots, x_m\}$;

length $|x| = (x, x)^{1/2}$, where

$x, x = x_1^2 + x_2^2 + \dots$ is the unit-form.

Quadratic form

$xQx = q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2 + \dots$, $xQx = 1$ quadric.

Eigen-vector or E-vector u : normal to $xQx = \text{const.}$ at

$x = u$, has direction of u : $(1/2) \partial xQx / \partial x =$

$2x = \{q_{11}x_1 + q_{12}x_2 + \dots, q_{21}x_1 + q_{22}x_2 + \dots, \dots\}$, $q_{12} = q_{21}$.

Q operator: transforms vector x into Qx ;

Q is linear: $Q(\alpha x + \beta y) = \alpha Qx + \beta Qy$.

Q given by matrix $q_{\sigma\tau}$.

u is E-vector if

$Qu = Ku$; K Eigen-value or E-value.

Example: Ellipse $xQx = x_1^2/\alpha_1^2 + x_2^2/\alpha_2^2 = 1$.

E-vectors $u^1 = \{1, 0\}$, $u^2 = \{0, 1\}$;

E-values $K_1 = \alpha_1^{-2}$, $K_2 = \alpha_2^{-2}$.

Circle $xQx = (x_1^2 + x_2^2)/R^2 = 1$. Every vector is

E-vector since $Qx = \{R^{-2}x_1, R^{-2}x_2\} = R^{-2}x$.

Relation $Qx - Kx = 0$ is

$(q_{11} - K)x_1 + q_{12}x_2 + \dots = 0$, $q_{21}x_1 + (q_{22} - K)x_2 + \dots = 0$.

Condition for solvability is secular equation, (for $m=2$):

$$\begin{vmatrix} q_{11} - K & q_{12} \\ q_{21} & q_{22} - K \end{vmatrix} = 0.$$

quadratic equation (of order- m in general) for K .

Problem 1. Prove that solutions of secular equation are real.

Example: $2x_1^2 + 12x_1x_2 - 7x_2^2 = 1$.

$(2 - K)x_1 + 6x_2 = 0$, $6x_1 - (7 + K)x_2 = 0$.

$(2 - K)(7 + K) + 36 = 0$. $K^2 + 5K - 50 = 0$

$K_1 = 5$, $K_2 = 10$. $u^1 = \{2, 1\}$, $u^2 = \{1, -2\}$.

The following table shows the results of the
 experiments conducted during the year 1911.
 The first column gives the number of the
 experiment, the second column the date,
 the third column the time of day, and the
 fourth column the result.

Exp. No.	Date	Time	Result
1	Jan 15	10:00	...
2	Jan 22	11:30	...
3	Jan 29	12:45	...
4	Feb 5	14:00	...
5	Feb 12	15:15	...
6	Feb 19	16:30	...
7	Feb 26	17:45	...
8	Mar 5	19:00	...
9	Mar 12	20:15	...
10	Mar 19	21:30	...
11	Mar 26	22:45	...
12	Apr 2	24:00	...
13	Apr 9	25:15	...
14	Apr 16	26:30	...
15	Apr 23	27:45	...
16	Apr 30	29:00	...
17	May 7	30:15	...
18	May 14	31:30	...
19	May 21	32:45	...
20	May 28	34:00	...
21	Jun 4	35:15	...
22	Jun 11	36:30	...
23	Jun 18	37:45	...
24	Jun 25	39:00	...
25	Jul 2	40:15	...
26	Jul 9	41:30	...
27	Jul 16	42:45	...
28	Jul 23	44:00	...
29	Jul 30	45:15	...
30	Aug 6	46:30	...
31	Aug 13	47:45	...
32	Aug 20	49:00	...
33	Aug 27	50:15	...
34	Sep 3	51:30	...
35	Sep 10	52:45	...
36	Sep 17	54:00	...
37	Sep 24	55:15	...
38	Oct 1	56:30	...
39	Oct 8	57:45	...
40	Oct 15	59:00	...
41	Oct 22	60:15	...
42	Oct 29	61:30	...
43	Nov 5	62:45	...
44	Nov 12	64:00	...
45	Nov 19	65:15	...
46	Nov 26	66:30	...
47	Dec 3	67:45	...
48	Dec 10	69:00	...
49	Dec 17	70:15	...
50	Dec 24	71:30	...

The results of these experiments show that
 the time of day has a marked influence
 on the results obtained. The results
 are generally better during the day
 than during the night. This is
 probably due to the fact that the
 temperature is higher during the day
 and the rate of reaction is therefore
 increased.

Proceeding procedure best for numerical determination; to be abandoned as theoretical approach since it can be generalized only to restricted class of quadratic forms in spaces of infinite dimension.

Maximum problem. Consider quotient

$xQx/x, x$; for $x \neq 0$. Unchanged when x replaced by cx ;

hence $x, x = 1$ no restriction

Geometric meaning: since no restriction we assume $xQx = \text{const}$; i.e. x to be on quadric, quotient is reciprocal square of distance between point on quadric and origin.

For minimum distance E-vector expected.

Quotient is bounded; assume $x, x = x_1^2 + x_2^2 + \dots = 1$,

then $|x_n| \leq 1$. $xQx \leq |q_{11}| + 2|q_{12}| + \dots$. Hence:

$xQx/x, x$ has maximum, let its value be K , attained for $x = u$,

no restriction: $|u| = 1$. Condition:

$$0 = 1/2 \partial [xQx/x, x] / \partial x^{x=u} = (u, u)^{-2} [qu(u, u) - (uQu)u]$$

$= Qu - Ku$. Hence we have proved.

Theorem 1.1 There exists one E-vector $\neq 0$.

Spectral resolution: To give all E-vectors.

Completeness: Manifold of all E-vectors q_{11}, q_{12}, \dots

Orthogonality: E-vectors to different E-values are \perp .

Both properties to be established.

Inner product:

$$x, y = x_1 y_1 + x_2 y_2 + \dots, \quad x \perp y \text{ means } x, y = 0.$$

Bilinear form:

$$xQy = x_1 q_{11} y_1 + x_1 q_{12} y_2 + x_2 q_{11} y_1 + x_2 q_{12} y_2 + \dots,$$

$$q_{12} = q_{21}; \dots$$

Symmetry of form: $xQy = yQx$; due to $q_{12} = q_{21}, \dots$

Green's formula: $x, Qy = (x, Qy)$

Symmetry of operator: $x, Qy = Qx, y$.

The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This ensures transparency and allows for easy verification of the data.

In the second section, the author outlines the various methods used to collect and analyze the data. This includes both manual and automated processes. The goal is to identify trends and anomalies that might not be immediately apparent from a simple review of the raw data.

The third part of the document provides a detailed breakdown of the results. It shows that there is a significant correlation between the variables being studied. This finding is supported by statistical analysis and is consistent with previous research in the field.

Finally, the document concludes with a series of recommendations for future work. It suggests that further research should be conducted to explore the underlying causes of the observed trends. Additionally, it recommends that the current findings be applied to improve existing processes and systems.

Theorem 1.2 E-vectors to different E-values are \perp .

Proof: Let $Qu^1 = \kappa_1 u^1$, $Qu^2 = \kappa_2 u^2$, then

$$u^1 Qu^2 = \kappa_2 (u^1 u^2); \quad u^2 Qu^1 = \kappa_1 (u^2 u^1);$$

difference: $(\kappa_1 - \kappa_2) (u^2 u^1) = 0$; hence $u^2 \perp u^1$ if $\kappa_1 \neq \kappa_2$

Lemma 1.1 If $x \perp$ E-vector u then $Qx \perp u$.

Proof $u, Qx = u Q x = x Qu = x, Qu = \kappa (x, u) = 0$.

Theorem 1.3 E-vectors span whole space.

Proof. - If they did not, subspace \mathcal{G}^* of all x^* which are \perp to all E-vectors would have dimension ≥ 1 .

Lemma 1.1 shows that Qx is also in \mathcal{G}^* . Apply Theorem

1.1 to Q considered an operator in \mathcal{G}^* ; it follows that there would be an E-vector $u \neq 0$ in \mathcal{G}^* . However,

\mathcal{G}^* cannot contain such E-vectors since it is \perp to them. Contradiction.

Problem 2: Analyze the steps hidden in this reasoning.

Herewith the spectral resolution is completed.

Spectral representation:

Lemma 1.2 Let u, v be E-vectors to same E-value κ then linear combination $\alpha u + \beta v$ is E-vector, too.

Proof: Linearity of Q .

Take all E-vectors to same E-value κ ; they form "E-subspace" \mathcal{G}_κ to Q ; \mathcal{G}_κ can be spanned by normed orthogonal E-vectors u .

All normed E-vectors obtained this way are \perp in view of Th. 1.2; they form a coordinate system: u^1, u^2, \dots . Let $\kappa_1, \kappa_2, \dots$ be the corresponding E-values (not necessarily different). Then each vector x can be written:

$$x = a_1 u^1 + a_2 u^2 + \dots + a_m u^m; \quad \text{or } x = \{a_1, a_2, \dots\};$$

the operation Qx may be written, since Q is linear,

$$Qx = \kappa_1 a_1 u^1 + \kappa_2 a_2 u^2 + \dots \text{ or } Qx = \{\kappa_1 a_1, \kappa_2 a_2, \dots\}.$$

The unit-form and the quadratic form become

$$x, x = a_1^2 + a_2^2 + \dots$$

$$x, Qx = \kappa_1 a_1^2 + \kappa_2 a_2^2 + \dots$$

Thus the spectral representation is completed.

Chapter II Hilbert space,

Original Hilbert space H_0 : all vectors with infinitely many components.

$$x = \{x_1, x_2, \dots\}, \quad \text{for which}$$

$$(x, x) = x_1^2 + x_2^2 + \dots < \infty.$$

Space of infinitely many dimensions.

Not suitable for differential operators and quantum theory.

Von Neumann's "abstract" Hilbert space; to be developed.

Linear space \mathcal{Q} of elements or vectors x .

Multiplication by real numbers and addition defined.

$$\alpha x + \beta y = z \quad \text{in } \mathcal{Q}, \quad \text{if } x, y \text{ in } \mathcal{Q}; \alpha, \beta \text{ real.}$$

Vector "0". Axioms.

In what follows space is supposed to imply linearity.

Examples. - 1. Space of finite dimension.

2. Space of all $\{x_1, x_2, \dots\}$, with or without restriction $(x, x) < \infty$.

3. Function spaces: x is function $x(s)$ of real variable s , defined in interval S , say $-\infty < s < \infty$ or else.

Space of continuous functions \mathcal{C} , of continuously differentiable functions \mathcal{C}^1 . Notation: $D = d/ds$.

Form (bilinear)

$$x \underline{B} y$$

Symmetry: $x \underline{B} y = y \underline{B} x$.

Linearity: $x \underline{B} (\beta y + \gamma z) = \beta (x \underline{B} y) + \gamma (x \underline{B} z)$;

the same for left vector due to symmetry.

Non-negative form \underline{P} :

$$x \underline{P} x \geq 0$$

Form $0 \leq (\alpha x + \beta y) \underline{P} (\alpha x + \beta y) = \alpha^2 (x \underline{P} x) + 2\alpha\beta (x \underline{P} y) + \beta^2 (y \underline{P} y)$

Schwarz inequality (SI): $|x \underline{P} y|^2 \leq (x \underline{P} x)(y \underline{P} y)$, and

Triangular inequality (TI): $|(x + y) \underline{P} (x + y)|^{1/2} \leq |x \underline{P} x|^{1/2} + |y \underline{P} y|^{1/2}$.

Form is positive-definite if

$$x \underline{P} x > 0 \quad \text{for } x \neq 0.$$

Director of the
Federal Bureau of Investigation
Washington, D. C.

Dear Sir:

I am writing to you regarding the
information received from the
New York office on the subject of

James Earl Ray, who is
currently in the custody of the
Federal Bureau of Investigation.

It is noted that you have
been advised that Ray is
currently in the custody of the
New York office.

It is also noted that you
have been advised that Ray
is currently in the custody of
the New York office.

Very truly yours,
Special Agent in Charge

James Earl Ray
New York, New York

Enclosed for you are
two copies of the report
dated and captioned as above.

Very truly yours,
Special Agent in Charge

James Earl Ray
New York, New York

Examples: In finite space $\sum_{\sigma, \tau} a_{\sigma\tau} x_{\sigma} x_{\tau}$ is bilinear form, symmetric if $a_{\sigma\tau} = a_{\tau\sigma}$; $x, y = x_1 y_1 + \dots + x_m y_m$ is

positive-definite form.

In \mathcal{H}_0 we can define inner product by

$$x, y = x_1 y_1 + x_2 y_2 + \dots$$

Problem: Prove absolute convergence of (x, y) from $(x, x) < \infty$, $(y, y) < \infty$. $(x, y) \leq |x| |y|$

In space of functions $x(s)$ we may define bilinear form as follows

1. $\int x(s) r(s) y(s) ds$, e.g. $r = 1$.
2. $\int \int x(s) k(s, t) y(t) ds dt$, $k(s, t) = k(t, s)$, e.g. $k = e^{-|s-t|}$.
3. $\int Dx(s) p(s) Dy(s) ds$ e.g. $p = 1$.

Metriization Take one positive-definite form and define it as inner product x, y ; unit-form:

$$x, x > 0 \quad \text{if} \quad x \neq 0,$$

Norm: $\|x\| = (x, x)^{1/2}$, distance $\|x - y\|$.

- SI $|x, y| \leq \|x\| \|y\|$
 TI $\|x + y\| \leq \|x\| + \|y\|$ } except if $\alpha x + \beta y = 0$.

Examples in function spaces

$$x, x = \int r(s) x^2(s) ds, \quad r > 0 \quad \text{used for } \mathcal{L}_2,$$

$$x, x = \int \{ p(s) Dx^2(s) ds + q(s) x^2(s) \} ds, \quad p(s) > 0, q(s) > 0, \text{ used for } \mathcal{L}_2. \text{ (may require restriction).}$$

Limit vector x of sequence x^σ : $\|x^\sigma - x\| \rightarrow 0$ as $\sigma \rightarrow \infty$. Also

$x^\sigma \rightarrow x$; convergence (strong); implies

$$\|x^\sigma\| \rightarrow \|x\| \text{ (from TI),}$$

$$y, x^\sigma \rightarrow y, x \text{ (from SI).}$$

Examples. In \mathcal{H}_0 , convergence of each component not sufficient.

In \mathcal{L}_2 (with above norm) $\int r [x^\sigma - x]^2 ds \rightarrow 0$ (i.e. in the mean).

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Cauchy sequence x^σ ; $\|x^\sigma - x^\tau\| \rightarrow 0$ as $\sigma, \tau \rightarrow \infty$.

If $x^\sigma \rightarrow x$ then x^σ is Cauchy sequence.

Space is complete if each Cauchy sequence has limit;

Complete space is termed Euclidean or general Hilbert space (with reference to norm of the type $\|x\| = (x, x)^{1/2}$).

L_2 is complete, hence Euclidean.

Above function spaces are not complete. Three approaches:

1. Function spaces can be completed by adjoining functions with squares integrable in Lebesgue's sense.
2. Theory can be worked out in incomplete spaces, cf. Courant-Hilbert Vol. II Ch. VII.
3. Completing by adjoining ideal elements.

Consider space \mathcal{R} with unit-form (x, x) . Let x^σ be a Cauchy sequence. Either it has limit vector in \mathcal{R} or else assign to it ideal limit vector \bar{x} . If for two sequences $x^\sigma, x^\tau : \|x^\sigma - x^\tau\| \rightarrow 0$, assign the same limit vector to them.

Define $\alpha x + \beta y = \lim (\alpha x^\sigma + \beta y^\sigma)$. In this way space \mathcal{R} can be extended to a space $\bar{\mathcal{R}}$.

(x, y) and $\|x\|$ can be defined in $\bar{\mathcal{R}}$:

From II one derives:

$|\|x^\sigma\| - \|x^\tau\|| \leq \|x^\sigma - x^\tau\| \rightarrow 0$, hence $\|x^\sigma\|$ is Cauchy sequence and therefore bounded. From III one derives for two Cauchy sequences $x^\sigma, y^\sigma : \|x^\sigma - x^\tau\| \leq \|x^\sigma - x^\tau\| \|y^\sigma\| + \|x^\tau\| \|y^\sigma - y^\tau\| \rightarrow 0$, hence (x^σ, y^σ) is Cauchy sequence and therefore converges to limit number. Let $x = \lim x^\sigma, y = \lim y^\sigma$. If x and y are in \mathcal{R} , $(x, y) \rightarrow (x, y)$; otherwise define $(x, y) = \lim (x^\sigma, y^\sigma)$. Prove that (x, y) is symmetric and bilinear, further $\|x\| \geq 0$. Moreover $\|x\| > 0$ for $x \neq 0$.

$\|x\| = 0$ would mean $\|x^\sigma\| \rightarrow 0$, and x^σ would have limit vector 0 in \mathcal{R} .

$\bar{\mathcal{R}}$ is complete. Let \bar{x}^σ be Cauchy sequence in $\bar{\mathcal{R}}$, $\|\bar{x}^\sigma - \bar{x}^\tau\| \leq \epsilon(\sigma) \rightarrow 0$ for $\sigma \geq \tau$; choose x^σ in \mathcal{R} such that $\|x^\sigma - \bar{x}^\sigma\| \leq \epsilon(\sigma)$, then

$\|x^\sigma - x^\tau\| \leq 2\epsilon(\sigma) + \epsilon(\tau) \rightarrow 0$; Let $x = \lim x^\sigma$, then

$\|\bar{x}^\sigma - x\| \leq \|x^\sigma - x\| + \epsilon(\sigma) \rightarrow 0$.

Reference: F. Hausdorff Mengenlehre 2nd ed. 1927. § 21.3.

Handwritten notes at the top left of the page.

Main body of handwritten text, appearing to be a list or series of notes.

Bottom section of handwritten text, possibly a conclusion or summary.

Example The space \mathcal{L}_3 with unit-form $(x,x) = \int_0^1 x^2 ds$ may be extended to complete space \mathcal{L}_3 , the space \mathcal{D}_3 with unit-form $\int_0^1 \{pDx^2 + qx^2\} ds$ to complete space \mathcal{L}_3 . We do not investigate whether the ideal adjoined vector can be realized by functions.

Subspace of \mathcal{H}_3 is closed if it contains all of its limit vectors. Subspace is dense in \mathcal{H}_3 if each vector of space \mathcal{H}_3 is limit vector of subspace.

Examples. In \mathcal{H}_3 : vectors for which $x_1 = 0$ form closed subspace, vectors which have only finite number of non-vanishing coefficients form dense subspace.

In \mathcal{L}_3 : Dense subspaces are: \mathcal{L}_3^- and all piecewise linear functions; further: analytic functions, polynomials, (if \mathcal{S} is bounded).

Problem: Prove that subspace in \mathcal{L}_3 or \mathcal{D}_3 of functions vanishing at a fixed point is dense in \mathcal{L}_3 , not dense in \mathcal{D}_3 respectively.

Proof for case $\mathcal{S} = (0 \leq s \leq 1)$, $\|x\|^2 = \int_0^1 x^2 ds$ or $\int_0^1 \{Dx^2 + x^2\} ds$ respectively; fixed point: $x = 0$.

1. Let $\eta_\tau(s) = 1 - \tau^{-1}s$ for $0 \leq s \leq \tau$, $= 0$ for $s > \tau$.

Let \bar{x} be any vector in \mathcal{L}_3 , x in \mathcal{L}_3 such that $\|x - \bar{x}\| < \epsilon/2$.

Set $x^\tau = (1 - \eta_\tau)x$; $x^\tau(0) = 0$; $\|x - x^\tau\|^2 = \int_0^\tau \eta_\tau^2 x^2 ds \leq \int_0^\tau x^2 ds \leq \frac{\epsilon}{2}$

if $\tau = \tau_\epsilon$ sufficiently small. Then $\|x_{\tau_\epsilon} - \bar{x}\| \leq \|x - \bar{x}\| + \|x_{\tau_\epsilon} - x\| \leq \epsilon$.

2. First prove (*) $|x(0)| \leq 2\|x\|$ for x in \mathcal{D}_3 .

$$|x(0)|^2 = [x(s) + \int_0^s Ds(s') ds']^2 \leq 2x^2(s) + 2\int_0^s Dx^2(s) ds \quad \text{see TV 10}$$

$$|x(0)|^2 \leq 2\int_0^1 x^2(s) ds + 2\int_0^1 Dx^2(s) ds = 2\|x\|^2.$$

Now let y be in \mathcal{D}_3 with $y(0) \neq 0$. If it could be approximated by x^ϵ in \mathcal{D}_3 with $x^\epsilon(0) = 0$, contradiction $|y(0)| \leq 2\|x^\epsilon - y\| \rightarrow 0$ would result from (*).

Subset spans (determines) space, if linear combinations are dense.

Examples: \mathcal{H}_3 is spanned by coordinate system; i.e. by set of vectors $\{0, 0, \dots, 0, 1, 0, \dots\}$.

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\mathcal{L}_S is spanned by piecewise linear functions with peak values 1 and rational-salient points, or, for bounded S , by powers $1, x, x^2, \dots$, or, for $S: 0 \leq x \leq 1$, by $\sin n\pi x$.

Space is denumerable if it is spanned by denumerable set. (Proper Hilbert space).

Examples: \mathcal{L}_0 and \mathcal{L}_S are denumerable.

Problem - Prove that \mathcal{L}_S is denumerable. ~~see above~~

Projection. Let \mathcal{Y} be closed subspace of vectors y . Projection Px of x in \mathcal{Y} is vector in \mathcal{Y} such that

$$x - Px \perp \mathcal{Y}.$$

Minimum Property:

$$\|x - Px\| < \|x - y\| \quad \text{for all } y \neq Px \text{ in } \mathcal{Y}.$$

Proof: $\|x - y\|^2 = \|x - Px\|^2 + (x - Px, y - Px) + \|y - Px\|^2$
 $= \|x - Px\|^2 + \|y - Px\|^2,$

because of $x - Px \perp y - Px$.

There is only one projection Px . Otherwise, from $x - P_1x \perp \mathcal{Y}$, $x - P_2x \perp \mathcal{Y}$, contradiction $P_1x - P_2x \perp \mathcal{Y}$ would result.

Projection theorem 2.1 Let \mathcal{Y} be a closed subspace in general-Hilbert space \mathcal{H} , x a vector in \mathcal{H} . Then there is a vector Px in \mathcal{Y} such that $x - Px \perp \mathcal{Y}$.

Proof (Rellich, F. Riesz 1934). Let d be the g.l.b. of all $\|x - y\|$ where y in \mathcal{Y} . Then for all z in \mathcal{Y}

$$0 \leq (\beta z + y - x, \beta z + y - x) - d^2 = \beta^2 z^2 + 2\beta(z, y - x) + \|y - x\|^2 - d^2.$$

i.e. this quadratic function is non-negative; hence

$$(1) \quad |z, y - x| \leq \|z\| \{ \|y - x\|^2 - d^2 \}^{1/2}.$$

Apply to y^1 and y^2 , use $|z, y^1 - y^2| \leq |z, y^1 - x| + |z, y^2 - x|$,

set $z = y^1 - y^2$,

$$(2) \quad \|y^1 - y^2\| \leq \{ \|y^1 - x\|^2 - d^2 \}^{1/2} + \{ \|y^2 - x\|^2 - d^2 \}^{1/2}.$$

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Now take minimizing sequence y^σ , i.e. $\|y^\sigma - x\| \rightarrow d$, (exists!).
 From (2) we have $\|y^\sigma - y^\tau\| \rightarrow 0$, hence y^σ is Cauchy-sequence.
 Since H is complete, there is a limit vector \hat{y} such that
 $y^\sigma \rightarrow \hat{y}$; since \mathcal{L} is closed, \hat{y} is in \mathcal{L} . Evidently $\|\hat{y} - x\| = d$.
 From (1) we have $|z, \hat{y} - x| = 0$, i.e. $\hat{y} - x \perp \mathcal{L}$. Hence $\hat{y} = Px$ is
 projection.

Complementary space \mathcal{L}^* to closed subspace \mathcal{L} consists of all
 vectors $\perp \mathcal{L}$.

Theorem 2.2 $\mathcal{L} \oplus \mathcal{L}^* = H$, i.e. \mathcal{L} and \mathcal{L}^* span H . Or, each
 vector x in H can be split into sum of vector in \mathcal{L} and vector
 in \mathcal{L}^* .

In fact: $x = Px + (x - Px)$.

Problem: Let \mathcal{L} be subspace in \mathcal{V} of functions vanishing at
 fixed point. Construct complement \mathcal{L}^* to closure $\bar{\mathcal{L}}$ of \mathcal{L} in \mathcal{V} .

Coordinate system: $u^1, u^2, u^3, \dots, \|u^\sigma\| = 1; u^\sigma \perp u^\tau, \sigma \neq \tau$.

Complete if it spans space.

Problem. If space is finite or denumerable, a complete coordin-
 ate-system exists.

Theorem 2.3: Let $\{u^\mu\}$ be a complete coordinate system, then
 for any x in

$$x = \alpha_1 u^1 + \alpha_2 u^2 + \dots, \text{ where } \alpha_\mu = (x, u^\mu).$$

$$\|x\|^2 = \alpha_1^2 + \alpha_2^2 + \dots; \text{ i.e.}$$

H is equivalent to H_0 or finite.

Proof: Take subspace \mathcal{U}_m spanned by u^1, \dots, u^m ; $P_m x$ projection
 of x into \mathcal{U}_m . Let $P_m x = \beta_1 u^1 + \dots + \beta_m u^m$; then, for $1 \leq \mu \leq m$,

$$\alpha_\mu = (x, u^\mu) = (P_m x, u^\mu) = \beta_\mu; \text{ hence}$$

$$P_m x = \alpha_1 u^1 + \dots + \alpha_m u^m.$$

Since system $\{u^\mu\}$ is complete there is sequence y^m in \mathcal{U}_m
 such that $\|y^m - x\| \rightarrow 0$. Now $\|P_m x - x\| \leq \|y^m - x\|$, hence
 $\|P_m x - x\| \rightarrow 0$, whence $\|P_m x\| \rightarrow \|x\|$, but $\|P_m x\|^2 = \alpha_1^2 + \dots + \alpha_m^2$,

$$\text{Hence } \|x\|^2 = \alpha_1^2 + \alpha_2^2 + \dots$$

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Chapter III. Completely continuous forms and joint spectra.

Hilbert space \mathcal{H} of vectors x ; unit form (x, x)

Quadratic form $x \underline{Q} x$ resulting from symmetric

Bilinear form $x \underline{Q} y = y \underline{Q} x$. Assume

$0 < x \underline{Q} x < k(x, x)$; i.e. \underline{Q} is positive-definite and bounded.

(Positive-definiteness immaterial).

Lemma 3.1 Let $x^\sigma \rightarrow x$ then $y \underline{Q} x^\sigma \rightarrow y \underline{Q} x$, $x^\sigma \underline{Q} x^\sigma \rightarrow x \underline{Q} x$.

Proof 1. $|y \underline{Q} (x^\sigma - x)| \leq y \underline{Q} y ((x^\sigma - x) \underline{Q} (x^\sigma - x)) \leq y \underline{Q} y k \|x^\sigma - x\|^2 \rightarrow 0$.

2. $|x^\sigma \underline{Q} x^\sigma - x \underline{Q} x| \leq ((x^\sigma - x) \underline{Q} (x^\sigma - x)) \leq k \|x^\sigma - x\|^2 \rightarrow 0$

Definition of \mathbb{E} -values κ and \mathbb{E} -vectors u without reference to operators:

(3.1) $y \underline{Q} u = \kappa(y, u)$ for all y in \mathcal{H} .

Theorem 3.1 Two \mathbb{E} -vectors to different \mathbb{E} -values κ are \perp .

Proof: $\kappa_1(u^1, u^2) = u^1 \underline{Q} u^2 = \kappa_2(u^1, u^2)$.

Theorem 3.2 \mathbb{E} -vectors to same \mathbb{E} -value κ form closed space \mathcal{U}_κ .

Proof 1. \mathcal{U}_κ is linear, from (3.1). 2. \mathcal{U}_κ is closed: let u^σ be sequence with limit v ; then $(y, u^\sigma) \rightarrow (y, v)$ and, from Lemma 3.1, $y \underline{Q} u^\sigma = y \underline{Q} v$; hence $y \underline{Q} v = \kappa(y, v)$, i.e. v in \mathcal{U}_κ .

Lemma 3.2 Let u^1, \dots, u^m be an orthonormal set of \mathbb{E} -functions with \mathbb{E} -values $\kappa_1, \dots, \kappa_m$; then for $x = \alpha_1 u^1 + \dots + \alpha_m u^m$,

$(x, x) = \alpha_1^2 + \dots + \alpha_m^2$, $x \underline{Q} x = \kappa_1 \alpha_1^2 + \dots + \kappa_m \alpha_m^2$.

Proof: $(x, u^\mu) = \alpha_\mu$; $(x, x) = \sum_\mu \alpha_\mu (x, u^\mu) = \sum_\mu \alpha_\mu^2$;

$x \underline{Q} x = \sum_\mu \alpha_\mu (x \underline{Q} u^\mu) = \sum_\mu \alpha_\mu \kappa_\mu \alpha_\mu^2$

Theorem 3.3 Let \mathcal{U}_ω be space spanned by \mathbb{E} -vectors to \mathbb{E} -values $\geq \omega$; then $x \underline{Q} x \geq \omega(x, x)$ for x in \mathcal{U}_ω .

Proof: 1. Let $x = \alpha_1 u^1 + \dots + \alpha_m u^m$; in view of Th. 3.2 $\kappa_1, \dots, \kappa_m$ may be considered different; in view of Th. 3.1 the u^μ are perpendicular; then the statement follows from Lemma-3.2. 2. For limits of such linear combinations the statement follows from Lemma 3.1.

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Desired type of spectral representation: "positive-discrete" representation:-

Complete orthogonal system of normal \mathbb{E} -vectors u^1, u^2, \dots ;
 $u^\mu \perp u^\nu, \|u^\mu\| = 1$.

Decreasing sequence of positive \mathbb{E} -values $K_1 \geq K_2 \geq K_3 \geq \dots \rightarrow 0$.

Able to represent each vector in \mathcal{H} :

$x = \alpha_1 u^1 + \alpha_2 u^2 + \dots$, where $\alpha_\mu = (u^\mu, x)$; further

$(x, x) = \alpha_1^2 + \alpha_2^2 + \dots$, $(x, x) = K_1 \alpha_1^2 + K_2 \alpha_2^2 + \dots$.

The latter relation follows from $P_m x = \alpha_1 u^1 + \dots + \alpha_m u^m \rightarrow x$ and

$P_m x P_m x = K_1 \alpha_1^2 + \dots + K_m \alpha_m^2$ (cf. Lemma 3.2) in view of Lemma 3.1.

Above-type of spectral representation implies following properties of spectral resolution:

Pure point spectrum:- set of \mathbb{E} -vectors span whole space.

Positive-discrete: point spectrum $K > 0$ and \mathbb{E} -vectors to $K \geq \omega > 0$ span finite space; i.e. $\mathcal{H} \geq \omega$ has finite dimension.

Theorem-3.4 If \mathbb{E} has pure positive-discrete point spectrum then "positive-discrete" representation holds.

Proof Since \mathbb{E} -vectors to each \mathbb{E} -value form finite space, they can be spanned by finite coordinate system. All \mathbb{E} -vectors thus obtained can be ordered such that $K_1 \geq K_2 \geq K_3 \geq \dots \rightarrow 0$. The resulting system is complete; hence representation follows from Th. 2.3.

Hilbert's discovery: Forms with pure positive-discrete point spectrum can be simply characterized.

Property "v": To each $\varepsilon > 0$ there are $r = r(\varepsilon)$ vectors

y^1, \dots, y^r in \mathcal{H}_ε such that for all x in \mathcal{H}_ε

$$(w) - |(x, x)| \leq \sum_{g=1}^r (y^g, x)^2 + \varepsilon (x, x).$$

Theorem 3.5 "Positive-discrete" representation implies "v".

Proof: Set $y^g = K_g^{-1/2} u^g$ for $K_g \geq \varepsilon$.

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DEPARTMENT OF CHEMISTRY

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BOARD OF TRUSTEES

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We shall show that positive-definite form \underline{Q} enjoying "W"-has positive-discrete point spectrum and, consequently, a "positive-discrete" representation. Before doing so we mention:

Hilbert had given different-condition instead of "W".

Property "CC" of complete continuity: Let x^σ be a sequence with bounded $\|x^\sigma\|$ such that $(y, x^\sigma) \rightarrow 0$ for all y in \mathcal{H}_y (weak convergence), then $x^\sigma \underline{Q} x^\sigma \rightarrow 0$.

Theorem 3.6 - "W" implies "CC". Evident from "W".

Problem: Prove: "CC" implies "W".

Theorem 3.7 If positive-definite form \underline{Q} enjoys "W" its point spectrum is positive-discrete.

Proof: Let $\mathcal{U} \supseteq \omega$ (cf. Th.3.3) have dimension $> r(\varepsilon)$ for $\omega > \varepsilon$: then it contains vector x such that $x y^p = 0$, $p = 1, \dots, r$.

For this x , (W) gives contradiction to Th.3.3.

Theorem 3.8 - If positive-definite form \underline{Q} enjoys "W" it possesses \mathbb{E} -vector, provided the space \mathcal{H}_y possesses one vector $x \neq 0$.

Proof based on Maximum-Problem for $x \underline{Q} x / x, x$ for $x \neq 0$.

Let $K > 0$ be least upper bound; then $K - \underline{Q}$ non-negative

form. Consider normed maximizing sequence x^σ , $\|x^\sigma\| = 1$,

$x^\sigma (K - \underline{Q}) x^\sigma = \varepsilon_\sigma^2 \rightarrow 0$. Set $x^\sigma - x^\tau = x^{\sigma\tau}$. If applied to $x^{\sigma\tau}$ for $K - \underline{Q}$ yields

$$x^{\sigma\tau} (K - \underline{Q}) x^{\sigma\tau} \leq (\varepsilon_\sigma + \varepsilon_\tau)^2.$$

Upon adding (W) for $x^{\sigma\tau}$,

$$(*) \cdot (K - \varepsilon)(x^{\sigma\tau}, x^{\sigma\tau}) \leq \sum_{p=1}^r (y^p \underline{Q} x^{\sigma\tau})^2 + (\varepsilon_\sigma + \varepsilon_\tau)^2.$$

Choose $\varepsilon < K$, then r and y^1, \dots, y^r ; further subsequence x^σ such that the r bounded sequences $(y^p \underline{Q} x^\sigma)$ converge, which implies $(y^p \underline{Q} x^{\sigma\tau}) \rightarrow 0$. For this subsequence, (*) yields $\|x^{\sigma\tau}\| \rightarrow 0$;

i.e. - it is Cauchy sequence and converges to limit vector u due to completeness of space. $\|x^\sigma\| = 1$ yields $\|u\| = 1$, $x^\sigma (K - \underline{Q}) x^\sigma \rightarrow 0$

yields $u(K - \underline{Q})u = 0$ (Lemma 3.1). Further from 3I,

$$|y(K - \underline{Q})u|^2 \leq |y(K - \underline{Q})y| |u(K - \underline{Q})u| = 0;$$

hence $y \underline{Q} u = K(y, u)$ for all y in \mathcal{H}_y ; i.e. u is \mathbb{E} -vector with \mathbb{E} -value K .

1. The first part of the document is a list of names and addresses of the members of the committee. The names are listed in alphabetical order, and the addresses are given in full. The list includes names such as Mr. J. H. Smith, Mr. W. B. Jones, and Mr. C. D. Brown, among others.

2. The second part of the document is a list of the names and addresses of the members of the committee who have been elected to the office of Chairman. The names are listed in alphabetical order, and the addresses are given in full. The list includes names such as Mr. J. H. Smith, Mr. W. B. Jones, and Mr. C. D. Brown, among others.

3. The third part of the document is a list of the names and addresses of the members of the committee who have been elected to the office of Secretary. The names are listed in alphabetical order, and the addresses are given in full. The list includes names such as Mr. J. H. Smith, Mr. W. B. Jones, and Mr. C. D. Brown, among others.

4. The fourth part of the document is a list of the names and addresses of the members of the committee who have been elected to the office of Treasurer. The names are listed in alphabetical order, and the addresses are given in full. The list includes names such as Mr. J. H. Smith, Mr. W. B. Jones, and Mr. C. D. Brown, among others.

5. The fifth part of the document is a list of the names and addresses of the members of the committee who have been elected to the office of Auditor. The names are listed in alphabetical order, and the addresses are given in full. The list includes names such as Mr. J. H. Smith, Mr. W. B. Jones, and Mr. C. D. Brown, among others.

Theorem 3.9 If positive-definite form \underline{Q} enjoys property "W" its \mathbb{E} -vectors are complete.

Proof Consider closed space \mathcal{Z} spanned by all \mathbb{E} -vectors and complementary space \mathcal{Z}^* of all $x \perp \mathcal{Z}$. Since \mathcal{Z} and \mathcal{Z}^* span \mathcal{H} , (cf. Theorem 2.2), we have to prove $\mathcal{Z}^* = 0$, i.e. consisting of $x = 0$ only. First we observe

(*) $t \underline{Q}x$ for any t in \mathcal{Z} , x in \mathcal{Z}^* .

This relation holds if $t = u$ is an \mathbb{E} -vector because of $u \underline{Q}x = K(u, x) = 0$. It therefore holds for every linear combination of \mathbb{E} -vectors and in view of Lemma 3.1 also for every t in \mathcal{Z} . Hence

(*) $y \underline{Q}x = y_* \underline{Q}x_*$ where y is any vector in \mathcal{H} , y_* its projection in \mathcal{Z}_* and x_* any vector in \mathcal{Z}_* .

We now consider \underline{Q} a positive-definite form in \mathcal{Z}_* . An \mathbb{E} -vector u_* of \underline{Q} in \mathcal{Z}_* is also \mathbb{E} -vector of \underline{Q} in \mathcal{Z} because $y_* \underline{Q}u_* = K_*(y_*, u_*)$ for all y_* in \mathcal{Z}_* implies $y \underline{Q}u_* = K_*(y, u_*)$ for all y in \mathcal{H} , according to (*). Since every \mathbb{E} -vector of \underline{Q} in \mathcal{H} is \perp to \mathcal{Z}_* , there is no \mathbb{E} -vector of \underline{Q} in \mathcal{Z}_* . However, when relation (*) is applied to $x = x_*$ in \mathcal{Z}_* , one may replace $y \underline{Q}x_*$ by $y_* \underline{Q}x_*$ where y_* is the projection of y in \mathcal{Z}_* ; This is obvious from (*). Thus one obtains

$$x_* \underline{Q}x_* \leq \sum_{p=1}^{\infty} (y_*^p \underline{Q}x_*)^2 + \varepsilon(x_*, x_*),$$

which expresses that \underline{Q} in \mathcal{Z}_* enjoys "W". Therefore, Th. 3.8 would be applicable and would yield the existence of an \mathbb{E} -vector in \mathcal{Z}_* unless \mathcal{Z}_* contained only the vector "0". The latter must be the case since there is no \mathbb{E} -vector in \mathcal{Z}_* .

The first of these is the fact that the
 government has been unable to
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Differential form. Open interval \mathcal{J} : $s^- < |s| < s^+$ (finite or infinite).

space $\mathcal{J} = \mathcal{J}_S$ of functions $x(s)$ with continuous derivative vanishing in neighborhood of endpoints of \mathcal{J} . Unit-form

$$(x, x) = \int_{\mathcal{J}} \{p(s) Dx^2(s) + q(s)x^2(s)\} ds,$$

Quadratic form

$$xQx = \int_{\mathcal{J}} r(s)x^2(s) ds$$

where $p(s) > 0$, $q(s) \geq r(s)$, $r(s) > 0$ and continuous in open \mathcal{J} .

Complete this space \mathcal{J} to a space $\hat{\mathcal{J}} = \hat{\mathcal{J}}_S$ by adjoining ideal vectors. $\hat{\mathcal{J}}$ implies a boundary condition, the generalization of " $x = 0$ at the endpoints of \mathcal{J} ".

Obviously Q in $\hat{\mathcal{J}}$ is positive-definite and bounded:

$$(*) \quad 0 < xQx \leq k(x, x), \quad x \neq 0.$$

It can be extended to $\hat{\mathcal{J}}$:

Let x^σ, y^σ be Cauchy sequences in \mathcal{J} approaching x, y in $\hat{\mathcal{J}}$. Then $x^\sigma Q y^\sigma \leq (x^\sigma Q x^\sigma)^{1/2} (y^\sigma Q y^\sigma)^{1/2} \leq k \|x^\sigma\| \|y^\sigma\| \rightarrow 0$. Therefore $x^\sigma Q y^\sigma$ converges, and we may define $xQy = \lim x^\sigma Q y^\sigma$. It is easily seen that xQy is a bilinear form in $\hat{\mathcal{J}}$ with the bound k . Q in $\hat{\mathcal{J}}$ is obviously non-negative; it is, however, not obvious that Q is positive-definite; i.e. that never $xQx = 0$ for $x \neq 0$ in $\hat{\mathcal{J}}$. We postpone the proof of this fact to the next chapter.

It may be noted that it is not true that every bounded-form which is positive-definite in a dense subspace is also positive-definite in the complete space.

Counter-example. $(x, x) = \int_0^1 \{Dx^2 + x^2\} ds + (Dx(1))^2$, $xQx = \int_0^1 x^2 ds$. In the space \mathcal{J} of all $x(s)$ with continuous derivatives Dx the form Q is positive-definite; when this space is completed, this no longer holds. To show this we take the sequence

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$x^\varepsilon(s) = 0$ for $s \leq 1 - \varepsilon$, $= s - 1 + \varepsilon$ for $1 - \varepsilon \leq |s| \leq 1$.
Then $(x^\varepsilon, x^\varepsilon) = \varepsilon + \frac{1}{2}\varepsilon^2 \rightarrow 1$, $x^\varepsilon \underline{Q} x^\varepsilon = \frac{1}{2}\varepsilon^2 \rightarrow 0$.

$$(x^{\varepsilon_1}, x^{\varepsilon_2}) = (\varepsilon_2 - \varepsilon_1) + \frac{1}{2}(\varepsilon_2 - \varepsilon_1)^2 + \varepsilon_1(\varepsilon_2 - \varepsilon_1)^2 \rightarrow 0,$$

as $\varepsilon_i \rightarrow 0$. Hence x is a Cauchy sequence; and $(x, x) = 1$,
 $x \underline{Q} x = 0$ for the limit vector.

Examples.

$$1. \quad x \underline{L} x = \int_0^1 D x^2 ds, \quad x \underline{Q} x = \int_0^1 x^2 ds; \quad (x, x) = x \underline{L} x + x \underline{Q} x; \quad \text{or, } = \underline{L} + \underline{Q}.$$

\mathbb{E} -vectors; $\sin n\pi s$, \mathbb{E} -values $K_n = [1 + n^2\pi^2]^{-1}$, Fourier expansion.

$$2. \quad x \underline{L} x = \int_{-1}^1 (1 - s^2) D x^2 ds, \quad x \underline{Q} x = \int_{-1}^1 x^2 ds; \quad \text{or, } = \underline{L} + \underline{Q}.$$

\mathbb{E} -vectors; Legendre polynomials, $K_n = [1 + n(n+1)]^{-1}$.

$$3. \quad x \underline{L} x = \int_0^1 \{s D x^2 + n^2 s^{-1} x^2\} ds, \quad x \underline{Q} x = \int_0^1 x^2 ds; \quad \text{or, } = \underline{L} + \underline{Q}.$$

Bessel functions $J_n(\rho_{mn} s)$, $K_n = [1 + \rho_{mn}^2]^{-1}$, where $J_n(\rho_{mn}) = 0$.

$$4. \quad x \underline{L} x = \int_{-\infty}^{\infty} \left\{ D x^2 + \frac{1}{4} s^2 x^2 \right\} ds, \quad x \underline{Q} x = \int_{-\infty}^{\infty} x^2 ds; \quad \text{or, } = \underline{L} + \underline{Q}.$$

$H_n(s) e^{-s^2/4}$, H_n Hermite polynomials, $K_n = \left[1 + n \frac{1}{2}\right]^{-1}$.

$$5. \quad x \underline{L} x = \int_0^{\infty} \left\{ D x^2 + \frac{1}{4} x^2 \right\} s^2 ds, \quad x \underline{Q} x = \int_0^{\infty} x^2 s ds; \quad \text{or, } = \underline{L} + \underline{Q}.$$

$L'_n(s) e^{-s/2}$, L_n Laguerre polynomials, $K_n = [1 + n]^{-1}$.

$$6. \quad x \underline{Q} x = \int_0^{\infty} \left\{ a D x^2 + 2bx D x \right\} s^2 ds, \quad x \underline{L} x = \int_0^{\infty} x^2 s^2 ds; \quad \text{or, } = \underline{Q} + c \underline{L},$$

where $a > 0$, $c > b^2/a$.

Negative discrete spectrum $K_n = \lambda_n / c + \lambda_n$, $\lambda_n = -n^{-2} b^2/a$.

$L'_n(2sb/na) e^{-sb/na}$; Schrodinger functions for $a = \hbar^2/2m$, $2b = e^2$.

$$7. \quad x \underline{L} x = \int_{-\infty}^{\infty} D x^2 ds, \quad x \underline{Q} x = \int_{-\infty}^{\infty} x^2 ds; \quad \text{or, } = \underline{L} + \underline{Q}.$$

No discrete spectrum.

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We shall derive three criteria such that the above differential form Q possess property "W" with respect to the above unit-form (x, x) . Before doing so we observe that the coefficient p can be made 1 by a transformation. Since $p > 0$ we may introduce

$$t = \int_{s_0}^s ds' / p(s')$$

as new variable; then

$$(x, x) = \int_{t_0}^{t_1} \left\{ \left(\frac{dx}{dt} \right)^2 + q_p x^2 \right\} dt, \quad \underline{Q} = \int_{t_0}^{t_1} x^2 r_p dt;$$

q_p and r_p take the place of q and r . Note that

$$\int q_p dt = \int q ds, \quad q_p / r_p = q / r \text{ remain unchanged.}$$

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Criterion 1. Q enjoys "W" if $\int_{\mathcal{S}} r ds < \infty$, $\int_{\mathcal{S}} |t| r ds < \infty$.

Proof: We introduce $\int_{s_0}^s r(s') ds' = l(s)$, $\int_{s_0}^s |t(s')| r(s') ds' = j(s)$

Let Δs be a subinterval of \mathcal{S} (finite or infinite), then we set

$\int_{\Delta s} r ds = \delta \ell$, $\int_{\Delta s} |t| r ds = \delta j$. We then proceed to prove first

Poincaré's generalized inequality:

$$\int_{\Delta s} r x^2 ds \leq (\delta \ell)^{-1} \left(\int_{\Delta s} r ds \right)^2 + \delta j \int_{\Delta s} p D x^2 ds$$

for x in \mathcal{S} . (even in \mathcal{I})

cf. 2 p bound

Proof: In view of the preceding remarks we may assume $p = 1$,

$t = s$. Let s_1, s_2 be in Δs ; $|x(s_1) - x(s_2)|^2 = \left| \int_{s_1}^{s_2} D x ds \right|^2$

$\leq |s_2 - s_1| \int_{s_1}^{s_2} D x^2 ds \leq [|s_2| + |s_1|] \int_{\Delta s} D x^2 ds$. By integrating with

respect to s_1 and s_2 over Δs ,

$$\int_{\Delta s} \int_{\Delta s} r(s_1) r(s_2) |x(s_1) - x(s_2)|^2 ds_1 ds_2 \leq 2 \delta \ell \delta j \int_{\Delta s} D x^2 ds.$$

The left member is $2 \delta \ell \int_{\Delta s} r x^2 ds - 2 \left(\int_{\Delta s} r ds \right)^2$. Hence division

by $2 \delta \ell$ yields the inequality.

Divide \mathcal{S} into R intervals $\Delta \mathcal{S}_q$ such that all $\delta \ell_j \leq \varepsilon$. By adding

Poincaré-inequalities:

$$\int_{\mathcal{S}} r x^2 ds \leq \sum_{q=1}^R (\delta \ell_q)^{-1} \left(\int_{\Delta \mathcal{S}_q} r ds \right)^2 + \varepsilon \int_{\mathcal{S}} D x^2 ds.$$

Set $z^q = (\delta \ell_q)^{-1/2}$ in $\Delta \mathcal{S}_q$, = 0 outside; then preceding formula

reads

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$$\int_S r x^2 ds \leq \sum_{s=1}^R (\int_S r z^k ds)^2 + \sum_{s=1}^R (Dx)^2 ds.$$

The functions z^k are not in \mathcal{J} ; approximate them by functions

$$y^k \text{ in } \mathcal{J} \text{ such that } \int_S r (z^k - y^k)^2 ds \leq \epsilon^2 / 4R^2 \int_S r (z^k)^2 ds.$$

$$\begin{aligned} \text{Then } (\int_S r z^k ds)^2 - (\int_S r y^k ds)^2 &\leq 2 \int_S r (z^k - y^k) ds \int_S r z^k ds \\ &\leq 2 \left(\int_S r (z^k - y^k)^2 ds \right)^{1/2} \left(\int_S r (z^k)^2 ds \right)^{1/2} \int_S r x^2 ds \leq 3R^{-1} \epsilon \int_S r x^2 ds. \end{aligned}$$

Inserting we obtain

$$\begin{aligned} (*) \quad \int_S r x^2 ds &\leq \sum_{s=1}^R (\int_S r y^k ds)^2 + \epsilon \int_S \{ Dk^2 + r x^2 \} ds \quad \text{or} \\ \underline{Q} &\leq \sum_{s=1}^R (\int_S r \underline{Q} ds)^2 + \epsilon(x, k) \end{aligned}$$

This inequality is proved for all x in \mathcal{J} . In view of Lemma 3.1

it holds for all x in $\hat{\mathcal{J}}$. Thus we have proved (W) and establish

Criterion 1.

Application: Examples 1.2.3. The forms \underline{Q} in these examples

may now be seen to enjoy property "W" with respect to the unit-form $\underline{L} + \underline{Q}$ and, therefore, possess a positive discrete spectrum.

We note that the formula (*) derived for the proof of criterion 1 also holds if all integrations are extended over a subinterval of \mathcal{J} ; the conditions $\int r ds < \infty$, $\int |k| r ds < \infty$ need only be satisfied for this sub-interval.

Criterion 2: \underline{Q} enjoys "W" if $q/r \rightarrow \infty$ as s approaches the endpoints.

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Proof: To $\epsilon > 0$ find an inner subinterval S_ϵ outside of which

$$r/q \leq \epsilon. \quad \text{Then } \int_{S-S_\epsilon} r x^2 ds \leq \epsilon \int_{S-S_\epsilon} q x^2 dx \leq \epsilon \int_{S-S_\epsilon} \{Dx^2 + qx^2\} ds.$$

Since formula (*) of criterion 2 holds for the subinterval S_ϵ , addition yields (W).

A similar reasoning leads to

Criterion 3: Q enjoys "W" if at each endpoint either $\int r ds$ and $\int |t| r ds$ are finite or if there $q/r \rightarrow \infty$.

Application of Criterion 3: Example 4.

Application of Criterion 3: Example 5, where $r = s$, $q = s^2$,

hence $q/r \rightarrow \infty$ as $s \rightarrow \infty$; $t = s^2$, $t = -s^{-1}$; $\int |t| r ds = s$ is finite for $s \rightarrow 0$.

For the discussion of example 6 we observe first that for x in \mathcal{J}

$$\int_0^\infty -Dx s^2 ds = - \int_0^\infty x^2 s ds < 0; \text{ hence}$$

$$(x, x) \geq -2b \int_0^\infty x^2 s ds. \quad \text{Further we note that a positive } \alpha \text{ can}$$

be found such that

$$(x, x) \geq \alpha \int_0^\infty \left\{ Dx^2 - xDx + \frac{1}{4} x^2 \right\} s^2 ds.$$

Applying formula (W) of Example 5 we obtain

$$-(x, x) \leq \sum_{q=1}^{\infty} \left(\int_0^\infty y^q r s ds \right)^2 + \xi (x, x).$$

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This formula expresses that Q enjoys property "W" except that $\int_0^\infty y^p x^p ds$ appears instead of $y^p Qx$. As we shall see in the next chapter this modified "W" is equivalent to "W-u". From this then it can be deduced that the negative part of the spectrum of Q is discrete.

In Example 7 Q possesses no discrete spectrum. To show this we prove that property "CC" is not satisfied. To this effect we need only construct a sequence $x^\sigma(s)$ such that $x^\sigma, x^\sigma \rightarrow a \neq 0$, $x^\sigma Qx^\sigma \rightarrow b \neq 0$, $y Qx^\sigma \rightarrow 0$ for all y in \mathcal{J} .

We take $z(s) = (1 - s^2)^2$ and set

$$x^\sigma(s) = \sigma^{-1/2} z(\sigma^{-1}s - z).$$

We then have

$$x^\sigma Qx^\sigma = \int_{-\infty}^{\infty} (x^\sigma)^2 ds = \int_{-\infty}^{\infty} z^2 ds = z Qz \neq 0.$$

$$x^\sigma, x^\sigma = \int_{-\infty}^{\infty} \left\{ (Dx^\sigma)^2 + (x^\sigma)^2 \right\} ds = \int_{-\infty}^{\infty} \left\{ \sigma^{-2} (z')^2 + z^2 \right\} ds \rightarrow z Qz \neq 0.$$

To show that $y Qx \rightarrow 0$, approximate y by \hat{y} in \mathcal{J} ; then

$$|y Qx^\sigma| \leq |\hat{y} Qx^\sigma| + \|y - \hat{y}\| \|x^\sigma\|.$$

$\|y - \hat{y}\|$ can be made arbitrarily small; $\|x^\sigma\|$ is bounded and $\hat{y} Qx^\sigma$ equals zero for sufficiently large σ . Hence the sequence x^σ violates (CC).

Example. $(z, x) = \int_0^1 x^2(s) ds$, $x Qx = \int_0^1 \int_0^1 x(s) k(s, t) x(t) ds dt$, $k(s, t) = k(t, s) \geq 0$ and continuous.

Vibrations of string extended over \mathcal{J} . Let p be the axial tension, r mass p.u.l., q distributed spring constant. $k^{-1} = \mu^2$, μ circular-frequency. Discrete spectrum if total mass $\int r ds$ and its moment $\int |s| r ds$ finite or spring constant q becomes infinite.

Problem. Prove that Q possesses property "V".

Example 8. Buckling of a tapered column. $\mathcal{S}: 0 < s < b$. $p = 1$, $q = 0$, $r = (EI)^{-1}$; E modulus of elasticity, $I = I_b x^4 b^{-4}$ moment of inertia of cross section. P -applied axial force. P^{-1} E -value. Prove that here form Q does not enjoy "W".

Problem: Investigate whether or not "W" holds for $\mathcal{S}: 0 < s < \infty$. p and $r = q$ behave like s^α , s^β or s^α , s^β at $s = 0$, $s = \infty$ respectively.

This is a very difficult problem. It is a problem that has been studied for many years. The solution is not obvious, but it is possible. We will try to find a way to solve it. We will start by assuming that the solution is of the form $y = e^{rx}$. This is a reasonable assumption because the equation is linear and homogeneous. We will then substitute this into the equation and solve for r .

In order to solve the equation, we need to find the roots of the characteristic equation. This is a quadratic equation, and we can solve it using the quadratic formula. The roots are $r_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Once we have the roots, we can write the general solution of the equation as $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.

$$y'' + 2y' + 2y = 0$$
$$r^2 + 2r + 2 = 0$$
$$r = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$$

The general solution is $y = e^{-x}(C_1 \cos(x) + C_2 \sin(x))$. We can determine the constants C_1 and C_2 by using the initial conditions. If $y(0) = 1$ and $y'(0) = 0$, then $C_1 = 1$ and $C_2 = 0$. The particular solution is $y = e^{-x} \cos(x)$.

Another way to solve this equation is by using the method of variation of parameters. We assume a particular solution of the form $y = u(x)e^{-x} \cos(x)$. We then find $u(x)$ by substituting this into the equation and solving for u . This method is more complicated than the characteristic equation method, but it works for non-homogeneous equations as well.

Chapter IV. Operators.

Hilbert-space \mathcal{H} of vectors x . Bounded linear operator assigns to every x a vector Qx such that

$$Q(\alpha x + \beta y) = \alpha Qx + \beta Qy$$

and that there is a number K such that

$$\|Qx\| \leq K\|x\| \text{ for all } x.$$

Q is symmetric if $(x, Qy) = (Qx, y)$.

Bilinear form Q of bounded linear operator Q defined by

$$xQy = (x, Qy) = (Qx, y); \text{ it is bounded}$$

$$|xQy| \leq K\|x\|\|y\|.$$

Theorem 4.1 To every bounded bilinear form Q there is exactly one bounded linear operator Q such that

- (1) $xQy = (x, Qy)$ holds and
- (2) $xQy = (x, z)$ implies $z = Qy$.

The second statement follows from the first one by setting $x = z - Qy$. The proof is based on a lemma concerning bounded linear forms, i.e. forms $\underline{E}x$ with properties

$$\underline{E}(\alpha x + \beta y) = \alpha \underline{E}x + \beta \underline{E}y \text{ and there is a number } \eta \text{ such that}$$

$$|\underline{E}x| \leq \eta\|x\| \text{ for all } x.$$

Lemma 4.1 To every bounded linear form \underline{E} there is exactly one vector e in \mathcal{H} such that

(*) $\underline{E}x = (e, x)$ for all x ; i.e. every linear form can be represented as an inner product with a fixed vector.

The proof is based on the projection theorem: 1. The space of all t for which $\underline{E}t = 0$ is closed, because $t \rightarrow z$ implies

$$|\underline{E}(t^\epsilon - z)| \leq \eta\|t^\epsilon - z\| \rightarrow 0, \underline{E}z = \underline{E}(z - t^\epsilon) \rightarrow 0, \underline{E}z = 0.$$

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2. Consider the complementary space \mathcal{F}^* . If $\mathcal{F}^* = \emptyset$ then $\mathcal{F} = \mathcal{H}$ and $e=0$ satisfies (*). If $\mathcal{F}^* \neq \emptyset$ it is spanned by just one vector t^* : If there were two independent vectors t_1^*, t_2^* in \mathcal{F}^* , a linear combination $e = \alpha_1 t_1^* + \alpha_2 t_2^*$ could be found such that $\underline{E}t = \alpha_1 \underline{E}t_1^* + \alpha_2 \underline{E}t_2^* = 0$; i.e. which would be in \mathcal{F} . Set $e = \alpha t^*$ where α is so chosen that $\underline{E}e = (e, e)$; namely $\alpha = \underline{E}t^* / (t^*, t^*)$. Then the projection of any x into \mathcal{F}^* is $(e, e)^{-1} (e, x) e$ and $x = (e, e)^{-1} (e, x) e + t$ where t is in \mathcal{F} . Hence $\underline{E}x = (e, e)^{-1} (e, x) \underline{E}e = (e, x) e$.

To prove theorem 4.1 we observe that the bounded form xQy is a bounded linear form in y when x is considered fixed. Hence there is exactly one vector z such that $xQy = (z, y)$; z depends on x ; we set $z = Qx$. We have $(\alpha_1 x^1 + \alpha_2 x^2) Qy = (\alpha_1 Qx^1 + \alpha_2 Qx^2), y$; since Qx is uniquely determined we have $Q(\alpha_1 x^1 + \alpha_2 x^2) = \alpha_1 Qx^1 + \alpha_2 Qx^2$; i.e. Q is linear. Further we have $|Qx, y| \leq K \|x\| \|y\|$; hence $\|Qx\|^2 = |Qx, Qx| \leq K \|x\| \|Qx\|$ or $\|Qx\| \leq K \|x\|$; i.e. Q is bounded. Thus Theorem 4.1 is proved.

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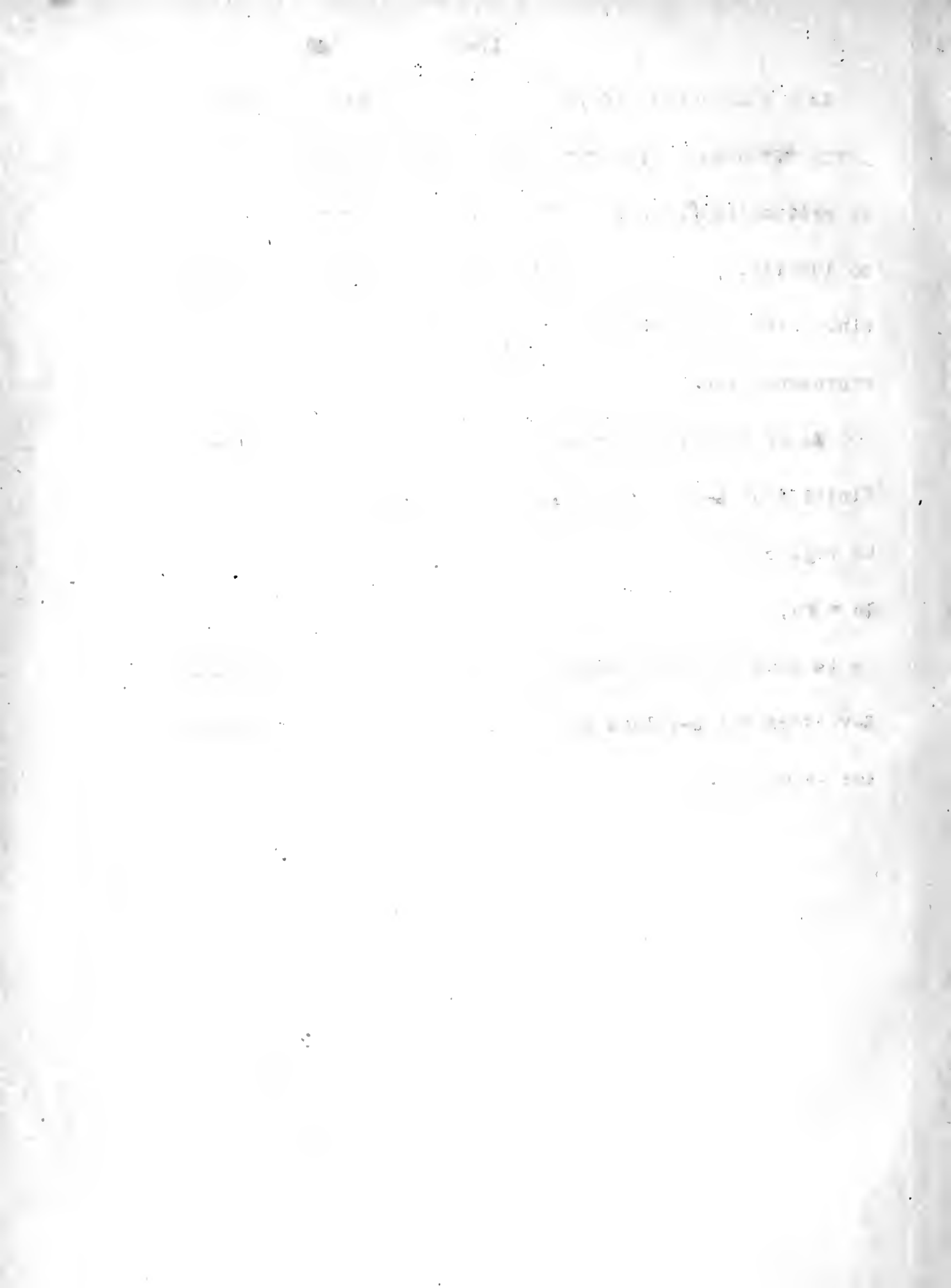
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As a consequence of Theorem 4.1 we remark that property "V" ensues from "W" because $\int Qx$ can now be written (Qy, x) . That "W" follows from "V" is not so immediate; these properties are, however, equivalent since both are equivalent to the purely discrete spectral representation.

As an important consequence of Theorem 4.1 the definition of E-vector and E-value $zQu = K(z, u)$ can be replaced by

$$Qu = \lambda u,$$

as is seen from the second statement in Theorem 4.1 i.e. E-vectors and E-values can be characterized by an operator equation.



To reveal the nature of the operator Q we take ex-
ample 1, for the interval $a < s < b$; we take, however, the
 form \underline{L} for unit-form. That Q remains bounded in \mathcal{J} is
 easily seen. $(x, x) = \int_a^b D^2 x^2 ds$, $x Q x = \int_a^b x^2 ds$. We show
 that $Qx = \int_a^b k(s, r)x(t)dt$ where the Green's function
 $k(s, t) = k(t, s)$ is given by

$$k(s, t) = \begin{cases} (b-a)^{-1}(b-s)(t-a) & \text{for } t \leq s \\ (b-a)^{-1}(s-a)(b-t) & \text{for } t \geq s. \end{cases}$$

Proof: We first prove it for x in \mathcal{J} . We have

$$Qx = (b-a)^{-1}(b-s) \int_a^s (t-a)x(t)dt + (b-a)^{-1}(s-a) \int_s^b (b-t)x(t)dt$$

$D^2 Qx = -x$. For y in \mathcal{J} we deduce

$$\int_a^b Dy DQx dx = - \int_a^b y D^2 Qx ds = \int_a^b y x ds \quad \text{or} \quad y \underline{L} Qx = y \underline{L} x.$$

This relation can immediately be extended to x and y
 in \mathcal{J} .

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Inversion of the operator Q. The inverse of the operator Q in \mathcal{H}_y can be defined as follows. Let $\mathcal{Q}\mathcal{H}_y$ be the subspace of all vectors x in \mathcal{H}_y which are of the form $x = Qy$ where y is in \mathcal{H}_y . The vector y is uniquely determined; for $Qy = 0$ implies $y = 0$ since Q is positive definite. The operator assigning y to x is denoted by M : $y = Mx$. This operator is linear since $Q(\alpha_1 y^1 + \alpha_2 y^2) = \alpha_1 x^1 + \alpha_2 x^2$ implies $M(\alpha_1 x^1 + \alpha_2 x^2) = \alpha_1 Mx^1 + \alpha_2 Mx^2$. Statements (1) and (2) of theorem 4.1 can be written

- (1) $zQMx = z, x$ holds for x in $\mathcal{Q}\mathcal{H}_y, z$ in \mathcal{H}_y
 (2) $zQy = z, x$ for all z in \mathcal{H}_y implies that x is in $\mathcal{Q}\mathcal{H}_y$ and $Mx = y$.

When we apply the latter statement to E -vectors u , we see that the relation $Qu = Ku$ is equivalent to

u is in $\mathcal{Q}\mathcal{H}_y$ and

$Mu = \mu u$ where $\mu = K^{-1}$ is an " E -value of the operator M ".

We turn now to the case that \mathcal{H}_y is the function space \mathcal{V} and Q the operator connected with the differential form Q . We shall see that the inverse M is a differential operator. For the space $\mathcal{Q}\mathcal{H}_y$ we then use the notation $\hat{\mathcal{F}}$.

In what follows we assume that $r, q, p, \frac{dp}{ds}$ have continuous derivatives. We introduce the space \mathcal{F} of all functions $x(s)$ with continuous derivatives up to the third order and the subspace $\hat{\mathcal{F}}$ of those functions in \mathcal{F} that vanish identically in a neighborhood of the end points. In \mathcal{F} we define the differential operator

$M = -r^{-1}[DpD - q]$, where the coincidence with the inverse of Q is anticipated in the notation. Then Green's formula

$$zQMx = z, x \quad \text{for } x \text{ in } \hat{\mathcal{F}}, z \text{ in } \hat{\mathcal{V}}$$

holds. For z in $\hat{\mathcal{V}}$ this reduces to

$$\int_S z[-DpDx + qx] ds = \int_S \{pDz \cdot Dx + qzx\} ds$$

and is proved through integration by parts since the boundary terms vanish; it then extends to z in \mathcal{V} . (Lemma 3.1.)

The space $\hat{\mathcal{F}}$ is dense in \mathcal{V} ; this follows from the fact that every function $x(s)$ in \mathcal{V} and its derivative can be uniformly approximated by functions in $\hat{\mathcal{F}}$; (proved).

Let \mathbf{u} and \mathbf{v} be vectors in \mathbb{R}^3 . The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both \mathbf{u} and \mathbf{v} . The magnitude of $\mathbf{u} \times \mathbf{v}$ is the area of the parallelogram determined by \mathbf{u} and \mathbf{v} . The direction of $\mathbf{u} \times \mathbf{v}$ is given by the right-hand rule.

(S) Let \mathbf{u} and \mathbf{v} be vectors in \mathbb{R}^3 . Then the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane containing \mathbf{u} and \mathbf{v} .

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Let \mathbf{u} and \mathbf{v} be vectors in \mathbb{R}^3 . The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane containing \mathbf{u} and \mathbf{v} .

We are now in a position to prove the
 Theorem 4.2 The form Q is positive definite,
 which was used in Ch. III. Let y be a vector in \mathcal{D}
 with $yQy=0$; then the above Green's formula gives
 $(y,x)=0$ for all x in \mathcal{F} ; since \mathcal{F} is dense in \mathcal{D} this holds
 for all x in \mathcal{D} ; in particular for $x=y$; i.e. $(y,y)=0$ or
 $y=0$. This proves Theorem 4.2.

From Theorem 4.1 (2) we see that $zQMx=z, x$
 for all z in \mathcal{D} implies $x = QMx$ or $QM=1$ for x in \mathcal{F} .
 This shows that every function x in \mathcal{F} is of the form
 $x=Qy$. Therefore \mathcal{F} is a subspace of $\mathcal{F} = Q\mathcal{D}$
 and the differential operator M in \mathcal{F} coincides with
 the inverse of Q .

The nature of the space \mathcal{F} is revealed by the
 Theorem 4.31. If x in \mathcal{D} then x can be represented by func-
 tion in \mathcal{L} .

4.32. If x in \mathcal{F} then x in \mathcal{D} , Dx in \mathcal{D} , Mx in \mathcal{L} .

4.33 If x in \mathcal{F} , Mx in \mathcal{F} then x, Dx, Mx, DMx in \mathcal{D} ,
 M^2x in \mathcal{L} .

which could be continued.

An \mathcal{E} -vector u admits application of M indefinitely
 since $Mu=\mu u$, $M^2u=\mu^2u$,.... Hence u is a function in \mathcal{D}
 with derivative Du in \mathcal{D} while the second derivative is in \mathcal{D}
 due to $Mu=\mu u$. This implies the important result that
 \mathcal{E} -vectors are functions in the original sense and not
 merely ideal vectors.

For the proof of Theorem 4.31 we may assume $\mu=1$
 (as in Ch. III). We first observe that continuous functions
 z which vanish in the neighborhood of the end points and
 have piecewise continuous derivatives, belong to \mathcal{D} because
 they can be approximated by functions in \mathcal{D} .

The first part of the report deals with the general situation in the country. It is noted that the economy is still in a state of depression, and that the government is facing a serious financial crisis. The report also mentions the need for a more active role for the state in the economy, and the importance of maintaining social order.

The second part of the report discusses the political situation. It is noted that the government is still in a state of transition, and that there is a need for a more stable and effective government. The report also mentions the need for a more active role for the state in the economy, and the importance of maintaining social order.

The third part of the report discusses the social situation. It is noted that the population is still in a state of poverty, and that there is a need for a more active role for the state in the economy, and the importance of maintaining social order.

The fourth part of the report discusses the international situation. It is noted that the country is still in a state of isolation, and that there is a need for a more active role for the state in the economy, and the importance of maintaining social order.

The fifth part of the report discusses the future of the country. It is noted that the country is still in a state of depression, and that there is a need for a more active role for the state in the economy, and the importance of maintaining social order.

Let S^1 be an inner subinterval of S , let s^1 represent points in S^1 . Let $\eta(s)$ be a function with continuous derivatives of any order which equals 1 in a neighborhood of S^1 and equals zero in a neighborhood of the endpoints. Then

$$k(s) = k(s, s^1) = \frac{1}{2} \eta(s) |s - s^1|$$

is a function in \mathcal{J} of the above mentioned kind. The function

$$j = j(s, s^1) = -r^{-1}(s) \frac{d^2 k(s, s^1)}{ds^2}$$

has then continuous derivatives with respect to s^1 of any order. For x in \mathcal{J} we obtain

$$(k, x) = \int_S \{Dk Dx + qkx\} ds = \int_S \{j + r^{-1}qk\} x r ds + x(s^1), \text{ or}$$

A: $x(s^1) = (k, x) - (j + r^{-1}qk) x$, whence

|A|: $|x(s^1)| \leq \alpha \|k\|$, where α depends on S^1 .

Let now x in \mathcal{J} be defined by a Cauchy sequence x^σ in \mathcal{J} . The inequality |A| applied to $x^\sigma - x^\tau$ shows that x^σ converges, uniformly in every subinterval S^1 , and hence possesses a continuous limit function $x(s)$. This limit function represents the vector x because 1. it is evidently independent of the defining Cauchy sequence, 2. no two vectors x possess the same function $x(s)$. The latter statement is equivalent to this: no vector $x \neq 0$ has the limit function $x(s) = 0$. This follows from Theorem 4.2 and the identity

$xQx = \int_S r(s)x^2(s)ds$ for the limit function $x(s)$ of x . To prove this identity, choose to given $\epsilon > 0$ a σ such that

$x^\sigma Q x^\tau \leq \epsilon^2$ for $\tau > \sigma$. From this $\int_{S^1} (x^\tau)^2 r ds \leq \epsilon^2$ whence

$$\int_{S^1} (x^\sigma - x)^2 r ds \leq \epsilon^2 \text{ or } \left| \left(\int_{S^1} r x^2 ds \right)^{1/2} - \left(\int_{S^1} r (x^\sigma)^2 ds \right)^{1/2} \right| \leq \epsilon.$$

This holds for every subinterval S^1 ; hence

7-10 $\left| \left(\int_S r x^2 ds \right)^{1/2} - \left(\int_S r (x^\sigma)^2 ds \right)^{1/2} \right| \leq \epsilon$. on the other hand

7-10 $\left| \left(\int_S r x^2 ds \right)^{1/2} - \left(\int_S r (x^\sigma)^2 ds \right)^{1/2} \right| \leq \epsilon$; hence $\left| \left(\int_S r x^2 ds \right)^{1/2} - \left(\int_S r (x^\sigma)^2 ds \right)^{1/2} \right| \leq 2\epsilon$.

This proves the identity.

Section 111 of the Criminal Code of Canada
provides that a person who has been
convicted of a crime and is sentenced to
imprisonment for a term of two years or more
shall be liable to deportation.

Section 112 of the Criminal Code of Canada

provides that a person who has been
convicted of a crime and is sentenced to
imprisonment for a term of two years or more
shall be liable to deportation.

Section 113 of the Criminal Code of Canada

provides that a person who has been
convicted of a crime and is sentenced to
imprisonment for a term of two years or more
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Section 114 of the Criminal Code of Canada

provides that a person who has been
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Section 115 of the Criminal Code of Canada

provides that a person who has been
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Section 116 of the Criminal Code of Canada

provides that a person who has been
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Section 117 of the Criminal Code of Canada

provides that a person who has been
convicted of a crime and is sentenced to
imprisonment for a term of two years or more
shall be liable to deportation.

Section 118 of the Criminal Code of Canada

provides that a person who has been
convicted of a crime and is sentenced to
imprisonment for a term of two years or more
shall be liable to deportation.

Section 119 of the Criminal Code of Canada

provides that a person who has been
convicted of a crime and is sentenced to
imprisonment for a term of two years or more
shall be liable to deportation.

Section 120 of the Criminal Code of Canada

provides that a person who has been
convicted of a crime and is sentenced to
imprisonment for a term of two years or more
shall be liable to deportation.

We further observe that formula A and inequality |A| remain valid for x in \mathcal{D} .

In order to prove 4.32 we first assume that y is in \mathcal{D} .
 We derive from A the formula

$$x(s^1) = \int k y r ds - \int [j+r^{-1}qk] x r ds$$

$$= \int k [y-r^{-1}qx] r ds - \int j x r ds.$$

This formula can be differentiated with respect to s^1 and gives

B: $Dx(s^1) = \frac{1}{2} \int \eta(s) \operatorname{sgn}(s^1-s) [y-r^{-1}qx] r ds - \int \frac{dj}{ds^1} x r ds.$

$\eta(s) = 0$
 outside \mathcal{D}
 inside \mathcal{D}

from which

|B|: $|D(s^1)| \leq \beta' \{ \|y\| + \|x\| \}$, where β' depends on s^1 .
 (Observe that $|\frac{dj}{ds^1}| = \frac{1}{2} |\frac{d^2\eta}{ds^2}|$ is independent of s^1).

Now let x be in \mathcal{F} ; approximate $y=Mx$ by functions y^ϵ in \mathcal{D} , then $x^\epsilon = Qy^\epsilon$ approximates $x=Qy$. Upon applying |B| and |A| to y^ϵ and x^ϵ we have

$$|Dx^\epsilon(s^1)| \leq \beta' \{ \|y^\epsilon\| + \|x^\epsilon\| \} \rightarrow 0.$$

$$|x^\epsilon(s^1)| \leq \alpha' \|y^\epsilon\| \rightarrow 0.$$

Hence the sequence $x^\epsilon(s)$ possesses a differentiable limit function $x(s)$ in every subinterval and, therefore, in \mathcal{D} . That is, the function $x(s)$ representing x in \mathcal{F} is in \mathcal{D} . Formula B, which was derived under the assumption Mx in \mathcal{D} then will hold also for x in \mathcal{F} . This formula can be differentiated another time with respect to s^1 and gives

C: $D^2x(s^1) = [ry-qx]_{s=s^1}$, since $\frac{d^2j}{(ds^1)^2} = 0$; hence

$r^{-1} [-D^2 + q]x = y=Mx$; i.e. x and Dx are in \mathcal{D} and M is the differential operator. That proves Theorem 4.32.

At present, the only way to get the best results is to use the best quality of material and to use it in the best way possible.

The following table shows the results of the tests conducted on the different samples of material.

The results of the tests show that the material used in the tests is of a high quality and that it is well suited for the purpose for which it is used.

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Supplement to Chapter IV p5 and p, 5

Assume that form Q possess a positive discrete spectrum.
Expansion, with respect to E -vectors u^n to E -values K_n ,

$$x = \sum_n \alpha_n u^n, \quad \alpha_n = (u^n, x),$$

implies

$$Qx = \sum_n K_n \alpha_n u^n.$$

Proof. Let $x_m = \sum_{n=1}^m \alpha_n u^n$, then $Qx_m = \sum_{n=1}^m K_n \alpha_n u^n$

in view of $Qu^n = Ku^n$. since $\|Qx_m - Qx\| \leq \|x\| \|x_m - x\| \rightarrow 0$

We have $\sum_{n=1}^m K_n \alpha_n u^n \rightarrow Qx$.

For x in $Q\mathcal{D}$ we have

$$Mx = \sum_n \mu_n \alpha_n u^n, \quad \mu_n = K_n^{-1}.$$

Proof. Let $x = Qy$, $y = \sum_n \beta_n u^n$, $\beta_n = (u^n, y)$;

then, from preceding, $x = Qy = \sum_n K_n \beta_n u^n$.

Now $K_n \beta_n = K_n (u^n, y) = (Qu^n, y) = (u^n, Qy) = (u^n, x) = \alpha_n$.

Hence $\beta_n = \mu_n \alpha_n$.

Discussion of differential operator M for examples.

Example 1. $0 < s < 1$, $p=1, q=1, r=1$.

$$M = -D^2 + 1,$$

E -equation $D^2 u + (\mu - 1)u = 0$ has solutions

$u = c \sin(\tau s + \varphi)$, where $\mu = 1 + \tau^2$.

For which values of τ and φ will u be in \mathcal{D} ?

The first part of the paper is devoted to a general
 introduction of the subject. It is shown that the
 theory of the present paper is a special case of
 a more general theory which has been developed
 in the preceding papers.

In the second part of the paper, the author
 discusses the special case of the theory which
 is the subject of the present paper. It is shown
 that the theory of the present paper is a special
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 discusses the special case of the theory which
 is the subject of the present paper. It is shown
 that the theory of the present paper is a special
 case of a more general theory which has been
 developed in the preceding papers.

Lemma 1. If $x(s)$ in \mathcal{J} is in $\hat{\mathcal{J}}$ then $x(0) = x(1) = 0$.

Proof. For $x(s)$ in \mathcal{J} we have $|x(s)|^2 = |x(s_2) + \int_{s_2}^{s_1} Dx ds|^2$,

$$(*) \quad |x(s_1)|^2 \leq 2|x(s_2)|^2 + 2|s_2 - s_1| \int_{s_1}^{s_2} Dx^2 ds.$$

Set $s_1 = 0$, $|s_2 - s_1| \leq 1$ and integrate with respect to s_2 .

$$|x(0)|^2 \leq 2 \int_0^1 x^2 ds + 2 \int_0^1 Dx^2 ds \quad \text{or} \quad |x(0)| \leq \|x\|,$$

in the same way $|x(1)| \leq \|x\|$. Application to $x - \hat{x}$, where x in

x in $\hat{\mathcal{J}}$, gives $|x(0)| \leq \|x - \hat{x}\|$, $|x(1)| \leq \|x - \hat{x}\|$.

Since $\|x - \hat{x}\|$ can be made arbitrarily small, lemma 1 is proved.

Lemma 2. If $x(0) = x(1) = 0$ for x in \mathcal{J} then x in $\hat{\mathcal{J}}$.

Proof. From (*) in proof of lemma 1 for $s_1 = \varepsilon$, $s_2 = 0$ and

$$s_1 = 1 - \varepsilon, \quad s_2 = 1 \quad \text{we have} \quad \{x^2(\varepsilon) + x^2(1 - \varepsilon)\} \leq 2\varepsilon \int_0^\varepsilon + \int_{1-\varepsilon}^1 Dx^2 ds.$$

Now replace $x(s)$ by a function $\hat{x}_\varepsilon(s)$ in $\hat{\mathcal{J}}$ obtained from

rounding off the function:

$$\begin{aligned} \hat{x}_\varepsilon(s) &= x(s) && \text{for } \varepsilon \leq s \leq 1 - \varepsilon \\ &= (2\varepsilon^{-1}s - 1)x(\varepsilon) && \text{for } \varepsilon/2 \leq s \leq \varepsilon \\ &= (2\varepsilon^{-1}(1-s) - 1)x(1 - \varepsilon) && \text{for } 1 - \varepsilon \leq s \leq 1 - \varepsilon/2 \\ &= 0 && \text{for } 0 \leq s \leq \varepsilon/2, 1 - \varepsilon/2 \leq s \leq 1. \end{aligned}$$

$$\begin{aligned} \text{Then } \|x - \hat{x}_\varepsilon\|^2 &\leq 2 \int_0^\varepsilon + \int_{1-\varepsilon}^1 \{Dx^2 + x^2\} ds + 2\varepsilon^{-1}(1 + \varepsilon^2/6) \{x^2(\varepsilon) + x^2(1 - \varepsilon)\} \\ &\leq (6 + \varepsilon^2/3) \int_0^\varepsilon + \int_{1-\varepsilon}^1 \{Dx^2 + x^2\} ds \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0. \end{aligned}$$

Lemmas 1 and 2 show that those and only those functions

$u = c \sin(\tau s + \varphi)$ are \mathcal{E} -functions which vanish for $s = 0$,

Let $f(x) = \int_0^x (x-t)^2 f''(t) dt$. Then $f(x) = \frac{x^3}{6} f''(0) + \frac{x^4}{24} f'''(0) + \dots$

$$|f(x)| \leq \int_0^x (x-t)^2 |f''(t)| dt \leq \frac{x^3}{6} \max_{t \in [0,x]} |f''(t)|$$

Since $f(0) = 0$, $f'(0) = 0$, and $f''(0) = 6f(0) = 0$.

$$|f(x)| \leq \frac{x^3}{6} \max_{t \in [0,x]} |f''(t)| \leq \frac{x^3}{6} \max_{t \in [0,1]} |f''(t)|$$

Let $M = \max_{t \in [0,1]} |f''(t)|$. Then $|f(x)| \leq \frac{x^3}{6} M$.

$$|f(x)| \leq \frac{x^3}{6} M \leq \frac{1}{6} M$$

Since $f(1) = \int_0^1 (1-t)^2 f''(t) dt = \frac{1}{6} f''(0) + \dots$

$$f(1) = \frac{1}{6} f''(0) + \frac{1}{24} f'''(0) + \dots$$

Since $f(1) = 0$, $f''(0) = -\frac{1}{24} f'''(0) - \dots$

Let $f(x) = \int_0^x (x-t)^2 f''(t) dt$. Then $f(x) = \frac{x^3}{6} f''(0) + \frac{x^4}{24} f'''(0) + \dots$

Since $f(0) = 0$, $f'(0) = 0$, and $f''(0) = 6f(0) = 0$.

Therefore, $|f(x)| \leq \frac{x^3}{6} \max_{t \in [0,x]} |f''(t)|$.

$$|f(x)| \leq \frac{x^3}{6} \max_{t \in [0,x]} |f''(t)| \leq \frac{x^3}{6} \max_{t \in [0,1]} |f''(t)|$$

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Since $f(1) = \int_0^1 (1-t)^2 f''(t) dt = \frac{1}{6} f''(0) + \dots$

Therefore, $|f(x)| \leq \frac{x^3}{6} \max_{t \in [0,x]} |f''(t)|$.

$s = 1$; i.e., $u_n(s) = c_n \sin n\pi s$ are E-functions, where c_n is determined such that $\|u_n\| = 1$; $\mu_n = 1 + n^2\pi^2$ are the E-values of M . Since the form Q of this example possesses a positive-discrete spectrum, the expansion theorems of Ch. III and Ch. IV supplement yield for every $x(s)$ in $\hat{\mathcal{J}}$ $x(s) = \sum_n \alpha_n c_n \sin n\pi s$,

$$\int_0^1 x^2 ds = \sum_n \alpha_n^2, \quad \int_0^1 \{Dx^2 + x^2\} ds = \sum_n \alpha_n^2 \mu_n,$$

and, if x is in $\hat{\mathcal{F}}$,

$$Mx = \sum_n \alpha_n c_n \sin n^2\pi^2 s, \text{ or}$$

$$-D^2 x = \sum_n n^2\pi^2 \alpha_n c_n \sin n^2\pi^2 s.$$

Example 3. $0 \leq s \leq 1$, $p = s$, $q = n^2 s^{-1} + s$, $r = s$.

$M = -s^{-1} D_s D + n^2 s^{-2} + 1$. The E-equation

$$s^{-1} D_s D u - n^2 s^{-2} u + (\mu - 1) u = 0$$

has for solutions the Bessel functions $J_n(\tau s)$, $Y_n(\tau s)$ and their linear combinations. The functions Y_n behave like s^{-n} for $n \neq 0$, like $\log s$ for $n = 0$; hence $\|Y_n\| = \infty$. Therefore, they must be discarded. The argument of Lemma 1 (preceding example) shows that $J_n(\tau s)$ is also to be discarded unless $J_n(\tau) = 0$. i.e., unless $\tau = \tau_{ln}$ is a root of the Bessel function J_n . It then must be shown that $J_n(\tau_{ln} s)$ is in $\hat{\mathcal{J}}$. For the endpoint $s = 1$ the argument of Lemma 2 is to be employed. For $s = 0$ an even simpler argument is possible since $J_n(\tau s)$ behaves like s^n for $s = 0$.

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$$\begin{aligned} & \int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C \\ & \int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C \\ & \int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C \end{aligned}$$

Faint, illegible text at the bottom of the page, possibly a conclusion or additional notes.

The case $n=0$, however, needs a special treatment.

We replace $u=J_0(\tau s)$ by

$$\begin{aligned}
 u_\epsilon(s) &= u(s) && \text{for } s \geq \epsilon \\
 &= (\log \epsilon^{-1})^{-1} (\log \epsilon^{-2} s) u(\epsilon) && \text{for } \epsilon^2 \leq s \leq \epsilon \\
 &= 0 && \text{for } 0 < s < \epsilon^2.
 \end{aligned}$$

Then $\int_0^\epsilon s D(u-u_\epsilon)^2 ds \leq 2 \int_0^\epsilon s D u^2 ds + 2 \int_{\epsilon^2}^\epsilon s D u_\epsilon^2 ds$
 $= 2 \int_0^\epsilon s D u^2 ds + (\log \epsilon^{-1})^{-1} u^2(\epsilon) \rightarrow 0$ since $u(0)=1$.

$$\int_0^\epsilon s (u-u_\epsilon)^2 ds \leq 2 \int_0^\epsilon s u^2 ds + 2 \int_0^\epsilon u_\epsilon^2 ds \rightarrow 0.$$

This indicates that it is possible to approximate $J_0(\tau s)$ by functions in \mathcal{J} if $J_0(\tau)=0$.

Since in this example the spectrum is positive discrete, expansions with respect to Bessel-functions are established.

In example 2 we have, $-1 \leq s \leq 1$, $p=(1-s^2)$, $q=r=1$.

$M=-D(1-s^2)D+1$; the E -value equation is

$D(1-s^2)Du + (\mu-1)u=0$. It can be shown (cf. e.g. Courant-Hilbert Vol. I Ch. V § 10) that the solutions of this equation are either Legendre polynomials or behave like $\log(1-s)$ at $s=1$, or like $\log(1+s)$ at $s=-1$. For the latter functions

$\int_{-1}^1 (1-s^2) D u^2 ds = \infty$; hence they are excluded. That the Legendre polynomials are in \mathcal{J} can be shown by the same reasoning that was applied to the Bessel function J_0 in connection with example 3. The Legendre functions thus constitute the E -functions. Since the spectrum is positive discrete in this case, expansion theorems are proved.

The first part of the paper is devoted to the study of the

of the function $f(x)$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2\pi n x}{1}$$

for $x \in [0, 1]$.

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2\pi n x}{1}$$

It is known that

$$f(x) = \frac{\pi^2}{6} - \frac{\pi^2 x^2}{6} + \frac{\pi^2 x^4}{120} - \frac{\pi^2 x^6}{420} + \dots$$

$$f(x) = \frac{\pi^2}{6} - \frac{\pi^2 x^2}{6} + \frac{\pi^2 x^4}{120} - \frac{\pi^2 x^6}{420} + \dots$$

$$f(x) = \frac{\pi^2}{6} - \frac{\pi^2 x^2}{6} + \frac{\pi^2 x^4}{120} - \frac{\pi^2 x^6}{420} + \dots$$

where the series converges uniformly on the interval $[0, 1]$.

The second part of the paper is devoted to the study of the

of the function $g(x)$ defined by

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2\pi n x}{2}$$

for $x \in [0, 1]$.

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2\pi n x}{2}$$

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CHAPTER V. Spectral Resolution of Operators.

\mathcal{H} : Hilbert space, Q bounded symmetric linear operator.

$$\|Qx\| \leq K\|x\|, \quad (x, Qy) = (Qx, y).$$

We are seeking a generalization of the concept of E-vector.

Such a generalization is that of E-vectors belonging to an E-interval.

Let $\Delta\kappa$ be a κ -interval, we consider intervals with and without end points, also a single point is an interval.

In case the discrete spectrum is complete, we say, the E-space $\Delta\mathcal{H}_\kappa$ to the E-interval $\Delta\kappa$ is the subspace spanned by the E-vectors to E-value κ in $\Delta\kappa$; E vectors to $\Delta\kappa$ are vectors in $\Delta\mathcal{H}_\kappa$. We observe that $\Delta\mathcal{H}_\kappa \perp \Delta\mathcal{H}_\lambda$ if $\Delta\kappa \cdot \Delta\lambda$ is void. Further $(u, Qu)(u, u)$ is in $\Delta\kappa$ for u in $\Delta\mathcal{H}_\kappa$. Our aim is to obtain a characterization of E-spaces $\Delta\mathcal{H}_\kappa$ without reference to E-values. This can be done by means of the concept of functions of an operator.

First the operator Q^n can be formed from Q , $n=0,1,2,\dots$. Further, if $p(\kappa)$ is a polynomial $p(\kappa) = a_0 + a_1\kappa + \dots + a_n\kappa^n$, we can define $p(Q) = a_0 + a_1Q + \dots + a_nQ^n$. We shall see that such functions of the operator Q can also be defined for larger classes of functions $f(\kappa)$.

In case the complete spectrum is complete, the definition of $f(Q)$ can be based on the spectral representation. $x = \sum_n \lambda_n u^n$ and $Qx = \sum_n \lambda_n \lambda_n u^n$. We see immediately that for polynomial $p(\kappa)$, we have $p(Q)x = \sum_n p(\lambda_n) \lambda_n u^n$. Hence we may define for any bounded function $f(\kappa)$: $f(Q)x = \sum_n f(\lambda_n) \lambda_n u^n$.

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We have a mapping of the functions $f(K)$ to operators, retaining all functional relations; in particular

$$f(Q)g(Q) = fg(Q) \quad \text{if } f(K)g(K) = fg(K).$$

i.e. multiplication corresponds to successive application.

Consider the function $f_*(K) = 1$ for $K = K_*$, $= 0$ for $K \neq K_*$;

then $f_*(Q)x = \sum_{K_n = K_*} \alpha_n u^n$; i.e. $f_*(Q)x$ is the pro-

jection of x into the E -subspace \mathcal{E}_* to K_* . E -vectors of the form $f_*(Q)x$ thus can be characterized as the vectors of the form $f_*(Q)x$.

Let Δ_K be an α -interval; consider the function

$f_\Delta(K) = 1$ in Δ_K , $= 0$ outside. Then $f_\Delta(Q)x = \sum_{K_n \in \Delta_K} \alpha_n u^n$;

i.e. $f_\Delta(Q)x$ is the projection of x into the E -subspace

\mathcal{E}_Δ to Δ_K , and the vectors in \mathcal{E}_Δ can be characterized as those of the form $f_\Delta(Q)x$.

By "spectral resolution" we mean from now on, to obtain these operators $f_\Delta(Q)$.

It was the important discovery of F. Riesz that for any symmetric bounded operator Q , a definition of $f(Q)$ can be given without reference to spectral representation, and that a spectral resolution can be obtained from these operator functions

No data available for this year.

Source: Bureau of Economic Analysis, Department of Commerce.

Notes:

(1) Data for 1997 are preliminary.

(2) Data for 1998 are preliminary.

(3) Data for 1999 are preliminary.

(4) Data for 2000 are preliminary.

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(12) Data for 2008 are preliminary.

(13) Data for 2009 are preliminary.

(14) Data for 2010 are preliminary.

(15) Data for 2011 are preliminary.

(16) Data for 2012 are preliminary.

(17) Data for 2013 are preliminary.

(18) Data for 2014 are preliminary.

(19) Data for 2015 are preliminary.

In what follows functions $f(K)$ are supposed to be defined in $|K| \leq k$ and all statements refer to this interval. Let $f(K)$ represent functions of a class containing sum and product. We say, the linear symmetric operator $f(\cdot)$ is "properly" defined for this class if the following statements hold.

A If $f(K) = f_1(K) + f_2(K)$ then $f(\cdot) = f_1(\cdot) + f_2(\cdot)$

B If $f(K) = f_1(K) \cdot f_2(K)$ then $f(\cdot) = f_1(\cdot) \cdot f_2(\cdot)$

where in the last formula the \cdot represents successive application.

C If $f_1(K) \geq f_2(K)$ then $(x, f_1(\cdot)x) \geq (x, f_2(\cdot)x)$.

For the class of polynomials $p(K)$ the operator $p(\cdot)$ was defined; $p(\cdot)$ is obviously linear and symmetric.

We have

Theorem 5.1 The definition of the operators $p(\cdot)$ for polynomials $p(\cdot)$ is proper.

Properties A, B are obvious. It is property C in which the theory of F. Riesz is essentially based. To prove C we refer to a purely algebraic

Lemma 5.1 If the polynomial $p(K) \geq 0$ for $|K| \leq k$, it can be written as the sum of polynomials of the form $g(K)j^2(K)$ where $j(K)$ is

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any polynomial and $g(K)$ is either 1, or $k-K$, or $k+K$, or $k^2 - K^2$.

Proof: Problem.

From this lemma we derive property C for polynomials $p(K)$ immediately. Since $p_1(K) - p_2(K)$ is a polynomial it is sufficient to assume $p_1(K) = p(K)$, $p_2(K) = 0$. We set $j(Q)x = y$.

$$\text{Then } 1. (x, j^2(Q)x) = (y, y) \geq 0$$

$$2. (x, (k+Q)j^2(Q)x) = (y, (k+Q)y) = k(y, y) + (y, Qy) \geq 0$$

because of $|(y, Qy)| \leq \|y\| \|Qy\| \leq k\|y\|^2$.

$$3. (x, (k^2 - Q^2)j^2(Q)x) = k^2(y, y) - (Qy, Qy) \geq 0.$$

Hence, in view of the lemma, $p(K) \geq 0$ implies $(x, p(Q)x) \geq 0$ and theorem 5.1 is proved.

Now let $f(K)$ be a continuous function (for $|K| \leq k$).

Approximate it, according to Weierstrass, by a sequence of polynomials $p_n(K)$, uniformly (in $|K| \leq k$). Set $p_{mn}(K) = p_m(K) - p_n(K)$.

To every $\varepsilon > 0$ there is a $N(\varepsilon)$ such that $|p_{mn}(K)| \leq \varepsilon$ (in $|K| \leq k$).

Hence $p_{mn}^2(K) \leq \varepsilon^2$, and, in view of C, $(x, p_{mn}^2(Q)x) \leq \varepsilon^2(x, x)$ or

$$\|p_{mn}(Q)x\| \leq \varepsilon \|x\|$$

Consequently, the vectors $p_n(Q)x$ form for every x a Cauchy sequence and, since the space is complete, converge to a limit vector, which we denote by $f(Q)x$.

- That $f(Q)$ is independent of the choice of the approximating polynomials follows from the fact that for two ^{such} sequences

$p_n^{(1)}$, $p_n^{(2)}$, one has

$$\|(p_n^{(1)}(Q) - p_n^{(2)}(Q))x\| \leq \text{Max } |p_n^{(1)}(K) - p_n^{(2)}(K)| \|x\| \rightarrow 0.$$

any polynomial $p(x)$ in $\mathbb{R}[x]$ can be written as $p(x) = q(x)r(x) + r_1(x)$ where $\deg r_1 < \deg r$.
Proof: Problem.

From this lemma we derive the following theorem:
Theorem 1. Let $r(x)$ be a polynomial in $\mathbb{R}[x]$ of degree n . Then $\mathbb{R}[x]$ is a free $\mathbb{R}[x]/(r(x))$ -module with basis $\{1, x, \dots, x^{n-1}\}$.

Proof. Let $M = \mathbb{R}[x]/(r(x))$. Then M is a free $\mathbb{R}[x]/(r(x))$ -module with basis $\{1, x, \dots, x^{n-1}\}$.
Hence, in view of the lemma, $\mathbb{R}[x]$ is a free $\mathbb{R}[x]/(r(x))$ -module with basis $\{1, x, \dots, x^{n-1}\}$.

Now let $r(x) = x^2 + 1$. Then $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.
Theorem 2. Let $r(x) = x^2 + 1$. Then $\mathbb{R}[x]/(r(x)) \cong \mathbb{C}$.

Proof. Let $\phi: \mathbb{R}[x]/(x^2 + 1) \rightarrow \mathbb{C}$ be the map defined by $\phi(x) = i$. Then ϕ is an isomorphism.

Hence $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.
Theorem 3. Let $r(x) = x^2 + 1$. Then $\mathbb{R}[x]/(r(x)) \cong \mathbb{C}$.

Consequently, the vector space $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to \mathbb{C} .
Theorem 4. Let $r(x) = x^2 + 1$. Then $\mathbb{R}[x]/(r(x)) \cong \mathbb{C}$.

$$\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$$

$$\| \mathbb{R}[x]/(x^2 + 1) \| = \mathbb{C}$$

One shows easily that $f(Q)x$ depends linearly on x ;
i.e. that $f(Q)$ is a linear operator, and also that $f(Q)$ is
symmetric, since the same holds for the approximating polynomials.

Further

Theorem 5.2 The operators $f(Q)$ for continuous $f(K)$ are properly
defined.

The properties A and C follow immediately from the
corresponding properties for $p_n(Q)$.

However, we want to define operators $f(Q)$ also for functions
 $f(K)$ which are only piecewise continuous. What we really need
is that $f(K)$ is bounded and can be approximated by a non-
decreasing sequence of continuous functions, i.e.

$$f_1(K) \leq f_2(K) \leq \dots \longrightarrow f(K) \leq F = \text{const};$$

we call such functions lower semi-continuous. From theorem 5.2C

we have $(x, f_1(Q)x) \leq (x, f_2(Q)x) \leq \dots \leq F(x, x)$.

The latter sequence, therefore, converges to a limit value; hence

$0 \leq (x, f_{mn}(Q)x) \rightarrow 0$ for $m \geq n \rightarrow \infty$. On applying the Schwarz

inequality to the non-negative form $f_{mn}(Q)$, we obtain

$$\begin{aligned} (f_{mn}(Q)x, f_{mn}(Q)x)^2 &\leq (x, f_{mn}(Q)x)(x, f_{mn}^3(Q)x) \\ &\leq (x, f_{mn}(Q)x) F^3(x, x) \longrightarrow 0, \end{aligned}$$

since $f_{mn}^3(K) \leq f_{mn}^3(K) \leq F^3$.

1. The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that this is crucial for the company's financial health and for providing reliable information to stakeholders.

2. The second part of the document outlines the specific procedures for recording transactions. It details the steps from initial entry to final review, ensuring that all necessary information is captured and verified.

3. The third part of the document addresses the role of the accounting department in this process. It highlights the need for clear communication and collaboration between different departments to ensure the accuracy of the data.

4. The fourth part of the document discusses the challenges associated with maintaining accurate records. It identifies common pitfalls and provides strategies to avoid them, such as regular audits and the use of standardized formats.

5. The fifth part of the document concludes by reiterating the importance of this process and encourages all employees to take their responsibilities seriously. It expresses confidence in the team's ability to maintain high standards of accuracy and reliability.

Hence $f_n(\gamma)x$ is a Cauchy sequence and converges to a limit function which we denote by $f(\gamma)x$; $f(\gamma)x$ depends linearly on x and one proves easily

Theorem 5.3 The operators $f(\gamma)$ are properly defined for lower-semi-continuous $f(K)$.

Evidently $f(\gamma)$ is also properly defined for differences of lower-semi-continuous functions, and that is the class of functions we want. For a function that is 1 in ΔK , 0 outside is such a difference.

Hence to every interval ΔK an operator $f_{\Delta}(\gamma) = \Delta P$ is defined. It has the following properties:

If $\Delta K = \Delta_1 K + \Delta_2 K$ then $\Delta_1 P + \Delta_2 P = \Delta P$.

If $\Delta K = \Delta_1 K \cdot \Delta_2 K$ then $\Delta_1 P \Delta_2 P = \Delta P$.

This implies in particular: If $\Delta_1 K \cdot \Delta_2 K$ is void, then

$\Delta_1 P \Delta_2 P = 0$; further $(\Delta P)^2 = \Delta P$.

If $F_1 \leq f(K) \leq F_2$ for k in ΔK then

$$F_1 \leq (f(\gamma)\Delta Px, x) / (\Delta Px, x) \leq F_2.$$

This holds in particular for $f(K) = K$ and gives

$$(Q\Delta Px, x) / (\Delta Px, x) \text{ is in } \Delta K.$$

To understand the significance of these properties we introduce

the space $\Delta \mathcal{H}$ of all u of the form $u = \Delta Px$. Since $f_{\Delta}^2 = f_{\Delta}$ we have $\Delta P^2 = \Delta P$.

Faint, illegible text, possibly bleed-through from the reverse side of the page. The text is arranged in several paragraphs and appears to be a formal document or report.

$$(\Delta P y, (1 - \Delta P)x) = (y, (\Delta P - \Delta P^2)x) = 0; \text{ i.e.}$$

$(1 - \Delta P)x \perp \Delta h_y$. Hence x can be split up into

$x = \Delta P x + (1 - \Delta P)x$, where $\Delta P x$ is in Δh_y and

$(1 - \Delta P)x \perp \Delta h_y$. This shows:

The operator ΔP is the projection into Δh_y .

Let the interval $|K| \leq k$ be split into a sum of intervals

$\Delta_\mu K$ without common points; then we have

$$\sum_\mu \Delta_\mu P = 1$$

which means that every vector x can be split up into a sum of

\mathbb{R} -vectors

$$\sum_\mu \Delta_\mu P x = x.$$

Let k_μ be any value of K in $\Delta_\mu K$. Then the operator

$\sum_\mu k_\mu \Delta_\mu P$ is approximately Q and

$\sum_\mu f(k_\mu) \Delta_\mu P$ is approximately $f(Q)$ for continuous f . The

sums on the left hand side really converge to Q or $f(Q)$ respectively if the subdivision is refined.

Consider two subdivisions Δ_μ and Δ'_ν . Then, in view of 1. and

2.,

$$\begin{aligned} & \left\| \sum_\mu f(k_\mu) \Delta_\mu P x - \sum_\nu f(k'_\nu) \Delta'_\nu P x \right\|^2 \\ &= \sum_{\mu, \nu} |f(k_\mu) - f(k'_\nu)|^2 \left\| \Delta_\mu P \Delta'_\nu P x \right\|^2 \\ &\leq \varepsilon^2 \sum_{\mu, \nu} \left\| \Delta_\mu P \Delta'_\nu P x \right\|^2 = \varepsilon^2 \|x\|^2, \end{aligned}$$

1. The first part of the document is a list of names and titles.

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23. The twenty-third part is a list of dates and times.

where $\varepsilon = \max |f(K_\mu) - f(K'_\nu)|$ for all μ, ν such that $\Delta_\mu K$ and $\Delta'_\nu K$ have a common point; and ε can be made arbitrarily small.

It is natural to express these facts symbollically by

$$\begin{aligned} \int dP &= 1 \\ \int K dP &= Q \\ \int f(K) dP &= f(Q), \text{ (containing the preceding ones),} \end{aligned}$$

which gives a pleasant form of the spectral resolution. In case of a complete discrete spectrum these integrals degenerate into sums.

We are now called to exemplify the preceding results with at least one example. Take the space \mathcal{L} of functions $x(K)$ defined in S , introduce a unit-form

$$(x, x) = \int_S x^2(K) r(K) dK,$$

and close it off to a space $\hat{\mathcal{L}}$. In \mathcal{L} introduce the operator Q by

$$Qx(K) = Kx(K).$$

Q is bounded if S is bounded, which we shall assume although that is not essential. Then $\hat{\mathcal{L}}$ can be extended to \mathcal{L} . The functions $x(K)$ in what follows are assumed to be in \mathcal{L} . The

where Δ is a square matrix and X is a vector

matrix

In an attempt to solve the system

$$AX = B$$

$$A^{-1}B = X$$

where A^{-1} is the inverse of A

which gives a unique solution

of a square matrix A

matrix

is given by

the inverse of A

exists if and only if

$$\Delta \neq 0$$

and also is one of the

of

$$A^{-1} = \frac{1}{\Delta} \text{adj}(A)$$

where $\text{adj}(A)$ is the

adjoint of A

and Δ is the determinant

results then extend immediately to $x(K)$ in \mathcal{L} .

For polynomials $p(K)$ we have

$$p(Q)x(K) = p(K)x(K).$$

For continuous functions $f(K)$ we have

$$f(Q)x(K) = f(K)x(K).$$

For the functions $f_{\Delta}(K) = 1$ in ΔK , $= 0$ outside, we have

$$f_{\Delta}(Q)x(K) = f_{\Delta}(K)x(K),$$

which function may be only piecewise continuous and belongs to \mathcal{L} as such. Hence the projection

$f_{\Delta}(Q) = \Delta P$ is this: from $x(K)$ it cuts off the part outside ΔK .

The E -functions of the interval ΔK are the functions vanishing outside ΔK .

The above example permits various generalizations. Instead of $x(K)dK$ we introduce a non-decreasing function $\rho(K)$ and replace $x(K)dK$ by $d\rho$; i. e., we set

$$(x, x) = \int_S x^2(K) d\rho(K).$$

We then even admit discontinuous functions $\rho(K)$. What this

implies may be seen in the case that $\rho(K)$ is piecewise constant with jumps $d\rho_1, d\rho_2, \dots$ at $K = K_1, K_2, \dots$.

We have then

$$(x, x) = \sum_{\mu} x_{\mu}^2 d\rho_{\mu}, \quad \text{where } x_{\mu} = x(K_{\mu}),$$

$$(x, Qx) = \sum_{\mu} K_{\mu} x_{\mu}^2 d\rho_{\mu}.$$

Let $f(x) = x^2 + 2x + 1$ and $g(x) = x^2 - 2x + 1$

Find $(f+g)(x)$ and $(f-g)(x)$

$$(f+g)(x) = (x^2 + 2x + 1) + (x^2 - 2x + 1)$$

$$= x^2 + 2x + 1 + x^2 - 2x + 1$$

$$= 2x^2 + 2$$

Now find $(f-g)(x)$

$$(f-g)(x) = (x^2 + 2x + 1) - (x^2 - 2x + 1)$$

$$= x^2 + 2x + 1 - x^2 + 2x - 1$$

$$= 4x$$

Therefore, $(f+g)(x) = 2x^2 + 2$ and $(f-g)(x) = 4x$

Q.E.D.

That is, we come back to the case of a pure point spectrum.

If $\rho(k)$ has discontinuities but is not constant in between we have a combination of continuous and point spectrum.

We may generalize in the following way. We consider as element of the Hilbert space a system of functions

$x = \{x_1(k), x_2(k), \dots\}$, and take for unit form

$$(x, x) = \int_S x_1^2(k) d\rho_1(k) + \int_S x_2^2(k) d\rho_2(k) + \dots$$

Then

$$Qx = \{kx_1(k), kx_2(k), \dots\} \quad \text{and}$$

$$(x, Qx) = \int_S kx_1^2(k) dk + \dots$$

It can be shown that the latter is the more general case in the following sense:

The Hilbert space can be represented by the space of systems of functions $\{x_\mu(k)$ such that Qx is represented by $\{kx_\mu(k)\}$ and $(x, x) = \sum_\mu \int_S x_\mu^2(k) d\rho_\mu(k)$. This statement implies the general theory of spectral representation.

It is not difficult to indicate how this representation can be achieved. Take any vector x_1 and all vectors $f_1(Q)x_1$; close this subspace off to a space \mathcal{H}_{x_1} ; we set

$$\Delta \rho_1(k) = (x_1, \Delta P x_1) \quad \text{and find}$$

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$$(x, x) = \sum_{\mu} (x_1, \Delta_{\mu} P x_1) \quad \text{and}$$

$$(f_1(Q)x_1, f_1(Q)x_1) = \lim \sum_{\mu} f_1^2(K_{\mu})(x, \Delta_{\mu} P x_1)$$

which can be shown to lead to

$$(f_1(Q)x_1, f_1(Q)x_1) = \int f_1^2(K) d\rho_1(K).$$

Hence for every x in \mathcal{H}_1 , represented by $f_1(K)$, we have

$$(x, x) = \int f_1^2(K) d\rho_1(K).$$

We now take the space perpendicular to \mathcal{H}_1 , perform the same operation and obtain a space \mathcal{H}_2 . Continuing, we obtain a sequence of spaces $\mathcal{H}_1, \mathcal{H}_2, \dots$ which span the whole space \mathcal{H} , as can be shown (if \mathcal{H} has countable dimension). Since every x in \mathcal{H} can be split into $x = x_1 + x_2 + \dots$, with x_1 in \mathcal{H}_1 , x_2 in \mathcal{H}_2 and so on, this leads to the above spectral representation with

$$x_{\mu}(K) = f_{\mu}(K).$$

$$(x^2 + 2x + 1) = (x+1)^2$$

$$10x^2 + 20x + 10 = 10(x+1)^2$$

which can be shown to be

$$10(x+1)^2 = 10x^2 + 20x + 10$$

Since the left side is equal to the right side,

$$(x+1)^2 = x^2 + 2x + 1$$

It has been shown that the square of the sum of two numbers

is equal to the sum of the squares of the two numbers

plus twice the product of the two numbers.

Therefore, the square of the sum of two numbers is

equal to the sum of the squares of the two numbers

plus twice the product of the two numbers.

Q.E.D.

Q.E.D.

