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N E W   Y O R K   U N I V E R S I T Y

[LECTURE NOTES

VOLUME 5]

Linear operators and their spectra

by

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LINEAR OPERATORS AND THEIR SPECTRA

by Kurt Friedrichs

The theory of linear operators and their spectra has emerged from generalizing the problem of principal axes of quadratic forms. Let

$$x\underset{-}{Q}x = q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2$$

be a quadratic form, $x\underset{-}{Q}x = 1$ a quadric in the (x_1, x_2)-plane, then the problem is to find a new system of rectangular coordinates (a_1, a_2) such that

$$x\underset{-}{Q}x = a_1^2/\alpha_1^2 + a_2^2/\alpha_2^2$$

$$x_1^2 + x_2^2 = a_1^2 + a_2^2;$$

α_1, α_2 are then the semi-axes of the quadric.

A first generalization of this problem was Hilbert's theory of quadratic forms in infinitely many variables (1906), with application to integral-equations. Hilbert's theory was also formulated in terms of infinite matrices. Infinite matrices were later on basic in the quantum theory of Heisenberg (1925); they were, however, of an essentially more general charcter than those that could be treated by Hilbert's theory.

Quite a different class of problems in analysis present strong analogies to the problem of principal axes of quadratic forms: namely, the problems concerning characteristic values of differential equations and the connected expansions of the type of Fourier's series and Fourier's integral. A typical problem of that class can be formulated as follows: We take the differential equation $-\frac{d^2}{ds^2}x(s) = \lambda x(s)$, where λ is a value to be determined, and consider two classes of admitted functions.

1. We restrict s to the range $0 \leq s \leq 1$ and impose the boundary condition $x(0) = x(1) = 0$. Then the solutions are

$$x = u_n(s) = 2^{-1/2} \sin n\pi s, \qquad n = 1,2,3,\ldots$$

$$\lambda = \lambda_n = n^2\pi^2 \qquad \text{"eigen-values"}$$

With hem we have the expansions.

$$x(s) = \sum_n a_n u_n(s); \qquad \int_0^1 x(s)^2 ds = \sum_n a_n^2$$

$$-\frac{d^2}{ds^2} x(s) = \sum_n a_n u_n(s) \qquad \int_0^1 \left(\frac{d}{ds} x(s)\right)^2 ds = \sum_n \lambda_n a_n^2$$

The first expansion is simply a Fourier series. The latter two formulas present an obvious analogy to the reduction of quadratic forms to principal axes. The set of values λ_n is called the spectrum of $-\frac{d}{ds^2}$; in particular it is a point spectrum since it consists of discrete points on the λ -axis.

2. Secondly we restrict s to the range $0 \leq s < \infty$ and impose as boundary condition only $x(0) = 0$. Then the solutions are

$$x(s) = u_\mu(s) = (\pi/2)^{1/2} \sin \mu s,$$

where μ runs over the entire range $0 \leq \mu < \infty$, $\lambda = \lambda_\mu = \mu^2$, and we have the expansions

$$x(s) = \int_0^\infty a_\mu u_\mu(s) ds; \qquad \int_0^\infty x(s)^2 ds = \int_0^\infty a_\mu{}^2 d\mu$$

$$-\frac{d^2}{ds^2} x(s) = \int_0^\infty \lambda_\mu a_\mu u_\mu(s) ds; \int_0^\infty \left(\frac{d}{ds} x(s)\right)^2 ds = \int_0^\infty \lambda_\mu a_\mu{}^2 d\mu.$$

The first expansion is nothing but a Fourier-Integral.

In this case where the characteristic values run over a continuous range $0 \leq \lambda < \infty$, the spectrum is termed "continuous". It is interesting to note that here the characteristic functions themselves are not representable because the integral $\int_0^\infty u_\mu^2(s) ds$ is infinite- a point which was left in obscurity for some time.

The classical application of problems of the preceding type is that to natural vibrations of elastic media. In 1926 Schrödinger caused quite a stir when he presented his quantum theory based on differential equations of a similar though more complicated type. Although Schrödinger himself showed the formal identity of hes and Heisenberg's theory, a complete mathematical theory of which all the mentioned theories are special cases, was lacking.

It was the decisive merit of v.Neumann to have broken the mathematical deadlock. He discovered that this deadlock was due to the use of a wrong system of concepts and he developed an appropriate new system of concepts. Using it, he and Stone (1929) derived a general theory of spectral resolution. It is true, von Neumann and Stone did not show that the differential equation problems of quantum mechanics fall under their general theory; to show this has been one of my personal interests. The decisive step, however, was the discovery of the right system of concepts by v.Neumann. Strangely enough, physicists have apparently failed to appreciate von Neumann's work sufficiently. This lack of appreciation may partly be due to the fact that v.Neumann's system is the right one for formulating the principles of quantum theory, not, however, the right one for actually calculating solutions of special problems.

Quadratic forms	Differential equations
Infinite matrices Hilbert	Fourier series
Integral equations 1906	Fourier integral
Quantum theory Heisenberg 1925	Quantum theory Schrödinger 1927

Operators in Hilbert spaces
v. Neumann 1927/29
Stone 1929

In this course I do not want to derive methods for actually calculating solutions of special characteristic value problems. On the other hand I do not want to derive the abstract concepts of von Neumann in their full generality. Rather I wish to derive them gradually in immediate connection with special problems, in particular, differential equation problems; and I want to show how naturally all existence, completeness and expansion theorems result from a general theory.

[illegible]

system is [illegible] for [illegible]

theory, [illegible], the [illegible]

solutions of [illegible]

[illegible] field [illegible]

Infinite [illegible]

Integral equations [illegible]

Quantum theory [illegible]

[illegible]

Literature von Neumann, Mathematische Grundlagen der Quantenmechanik. 1932
Stone, Linear Transformations in Hilbert Space, 1932

Chapter I. Space of finite dimension.

Space of vectors $x = \{x_1, x_2, \ldots, x_m\}$;
length $|x| = (x,x)^{1/2}$, where
$x,x = x_1^2 + x_2^2 + \ldots$ is the unit-form.

Quadratic form
$x\underline{Q}x = q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2 + \ldots$, $x\underline{Q}x = 1$ quadric.

Eigen-vector or E-vector u: normal to $x\underline{Q}x$ = const. at $x = u$, has direction of u: $(1/2)\,\partial x\underline{Q}x/\partial x =$
$Qx = \{q_{11}x_1 + q_{12}x_2 + \ldots,\ q_{21}x_1 + q_{22}x_2 + \ldots, \ldots\}$, $q_{12} = q_{21}$.

Q operator: transforms vector x into Qx;
Q is linear: $Q(\alpha x + \beta y) = \alpha Qx + \beta Qy$.
Q given by matrix $q_{\sigma\tau}$.
u is E-vector if
$Qu = \kappa u$; κ Eigen-value or E-value.

Example: Ellipse $x\underline{Q}x = x_1^2/\alpha_1^2 + x_2^2/\alpha_2^2 = 1$.
E-vectors $u^1 = \{1.0\}$, $u^2 = \{0,1\}$;
E-values $\kappa_1 = \alpha_1^{-2}$, $\kappa_2 = \alpha_2^{-2}$.

Circle $x\underline{Q}x = (x_1^2 + x_2^2)/R^2 = 1$. Every vector is E-vector since $Qx = \{R^{-2}x_1, R^{-2}x_2\} = R^{-2}x$.

Relation $Qx - \kappa x = 0$ is
$(q_{11} - \kappa)x_1 + q_{12}x_2 + \ldots = 0$, $q_{21}x_1 + (q_{22} - \ )x_2 + \ldots = 0$.
Condition for solvability is secular equation, (for m=2):

$$\begin{vmatrix} q_{11} - \kappa & q_{12} \\ q_{21} & q_{22} - \kappa \end{vmatrix} = 0,$$

quadratic equation (of order m in general) for κ.

Problem 1. Prove that solutions of secular equation are real.

Example: $2x_1^2 + 12\,x_1x_2 - 7\,x_2^2 = 1$.
$(2 - \kappa)x_1 + 6\,x_2 = 0$, $6\,x_1 - (7 + \kappa)x_2 = 0$.
$(2 - \kappa)(7 + \kappa) + 36 = 0$. $\kappa^2 + 5\kappa - 50 = 0$
$\kappa_1 = 5$, $\kappa_2 = 10$. $u^1 = \{2,1\}$, $u^2 = \{1,-2\}$.

Preceding prodedure best for numerical determination; to be abandoned as theoretical approach since it can be generalized only to restricted class of quadratic forms in spaces of infinite dimension.

Maximum problem. Consider quotient $xQx/x,x$; for $x \neq 0$. Unchanged when x replaced by cx; hence $x,x = 1$ no restriction

Geometric meaning: since no restriction to assume xQx = const; i.e. x to be on quadric, quotient is reciprocal square of distance between point on quadric and origin. For minimum distance E-vector expected.

Quotient is bounded; assume $x,x = x_1^2 + x_2^2 + \ldots = 1$,

then $|x_\alpha| \leq 1$. $xQx \leq |q_{11}| + 2|q_{12}| + \ldots$. Hence:

$xQx/x,x$ has maximum, let its value be κ, attained for $x = u$, no restriction: $|u| = 1$. Condition:

$0 = 1/2\,\partial[xQx/x,x]/\partial x^{x=u} = (u,u)^{-2}[Qu(u,u)-(uQu)u]$

$= Qu - \kappa u$. Hence we have proved.

Theorem 1.1 There exists one E-vector $\neq 0$.

Spectral resolution: To give all E-vectors.

Completeness: Manifold of all E-vectors spans space.

Orthogonality: E-vectors to different E-values are $\perp$.

Both properties to be established.

Inner product:

$x,y = x_1y_1 + x_2y_2 + \ldots$, $x \perp y$ means $x,y = 0$.

Bilinear form:

$xQy = x_1q_{11}y_1 + x_1q_{12}y_2 + x_2q_{11}y_1 + x_2q_{22}y_2 + \ldots$,

$q_{12} = q_{21}, \ldots$.

Symmetry of form: $xQy = yQx$; due to $q_{12} = q_{21}, \ldots$.

Green's formula: $x,Qy = xQy$. $(x, Qy) = (xQy)$

Symmetry of operator: $x,Qy = Qx,y$.

Theorem 1.2 E-vectors to different E-values are $\perp$.
Proof: Let $Qu^1 = \kappa_1 u^1$, $Qu^2 = \kappa_2 u^2$; then
$u^2, Qu^1 = \kappa_1 (u^2, u^1)$; $u^1, Qu^2 = \kappa_2 (u^1, u^2)$;
difference: $(\kappa_1 - \kappa_2)(u^2, u^1) = 0$; hence $u^2 \perp u^1$ if $\kappa_1 \neq \kappa_2$
Lemma 1.1 If $x \perp$ E-vector u then $Qx \perp u$.
Proof $u, Qx = uQx = xQu = x, Qu = \kappa (x,u) = 0$.
Theorem 1.3 E-vectors span whole space.
Proof. If they did not, subspace $\mathcal{J}^*$ of all x^* which are $\perp$ to all E-vectors would have dimension ≥ 1.
Lemma 1.1 shows that Qx is also in $\mathcal{J}^*$. Apply Theorem 1.1 to Q considered an operator in $\mathcal{J}^*$; it follows that there would be an E-vector $u^* \neq 0$ in $\mathcal{J}^*$. However, $\mathcal{J}^*$ cannot contain such E-vectors since it is $\perp$ to them. Contradiction,
Problem2: Analyze the steps hidden in this reasoning.
Herewith the spectral resolution is completed.
Spectral representation:
Lemma 1.2 Let u,v be E-vectors to same E-value κ then linear combination $\alpha u + \beta v$ is E-vector, too.
Proof: Linearity of Q.
Take all E-vectors to same E-value κ; they form "E-subspace" $\mathcal{J}_\kappa$ to κ; $\mathcal{J}_\kappa$ can be spanned by normed orthogonal E-vectors u.
All normed E-vectors obtained this way are $\perp$ in view of Th. 1.2; they form a coordinate system: $u^1, u^2, \ldots$. Let $\kappa_1, \kappa_2, \ldots$ be the corresponding E-values (not necessarily different). Then each vector x can be written:

$$x = a_1 u^1 + a_2 u^2 + \ldots + a_m u^m; \quad \text{or } x = \{a_1, a_2, \ldots\};$$

the operation Qx may be written, since Q is linear,

$$Qx = \kappa_1 a_1 u^1 + \kappa_2 a_2 u^2 + \ldots \text{ or } Qx = \{\kappa_1 a_1, \kappa_2 a_2, \ldots\}.$$

The unit-form and the quadratic form become

$$x, x = a_1^2 + a_2^2 + \ldots$$

$$xQx = \kappa_1 a_1^2 + \kappa_2 a_2^2 + \ldots$$

Thus the spectral representation is completed.

Theorem 1.6 [illegible]

Proof. [illegible]

[illegible]

Difference; [illegible]

Lemma 1.1 [illegible]

Proof [illegible]

Theorem 1.7 [illegible]

[illegible]

[illegible] would have linear [illegible]

Lemma 1.1 [illegible]

[illegible]

would be [illegible]

[illegible]

[illegible]

Hence [illegible]

[illegible]

Lemma 1.2 [illegible]

linear combinations [illegible]

Proof. Linearity [illegible]

[illegible]

[illegible]

vectors [illegible]

All normed [illegible]

of [illegible]

[illegible] is the [illegible]

[illegible] differentiable [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Chapter II Hilbert space,

Original Hilbert space $\mathfrak{h}_0$: all vectors with infinitely many components.

$$x = \{x_1, x_2, \ldots\}, \qquad \text{for which}$$

$$(x,x) = x_1^2 + x_2^2 + \ldots < \infty.$$

Space of infinitely many dimensions.-
Not suitable for differential operators and quantum theory.
Von Neumann's "abstract" Hilbert space; to be developed.
<u>Linear Space</u> $\mathfrak{R}$ of elements or vectors x.
Multiplication by real numbers and addition defined.

$$\alpha x + \beta y = z \quad \text{in } \mathfrak{R}, \quad \text{if } x,y \text{ in } \mathfrak{R}; \ \alpha, \beta \text{ real.}$$

Vector "0". Axioms.
In what follows <u>space</u> is supposed to imply linearity.
Examples.- 1. Space of finite dimension.
2. Space of all $\{x_1, x_2, \ldots\}$, with or without restriction $(x,x) < \infty$.
3. Function spaces: x is function $x(s)$ of real variable s, defined in interval S, say $-\infty < s < \infty$ or else.
Space of continuous functions $\mathfrak{C}$, of continuously differentiable functions $\mathfrak{D}$. Notation: $D = d/ds$.
<u>Form</u> (bilinear)

$$x \underline{B} y$$

<u>Symmetry</u>: $x \underline{B} y = y \underline{B} x$.
<u>Linearity</u>: $x \underline{B}(\beta y + \gamma z) = \beta(x \underline{B} y) + \gamma(x \underline{B} z)$;
the same for left vector due to symmetry.

<u>Non-negative form</u> $\underline{P}$:

$$x \underline{P} x \geq 0$$

Form $0 \leq (\alpha x + \beta y) \underline{P} (\alpha x + \beta y) = \alpha^2 (x \underline{P} x) + 2\alpha\beta (x \underline{P} y) + \beta^2 y \ldots$

Schwarz inequality (SI): $|x \underline{P} y|^2 \leq (x \underline{P} x)(y \underline{P} y)$, and

Triangular inequality (TI): $|(x+y)\underline{P}(x+y)|^{1/2} \leq |x \underline{P} x|^{1/2} + |y \underline{P} y|^{1/2}$.

Form is <u>positive-definite</u> if

$$x \underline{P} x > 0 \qquad \text{for} \quad x \neq 0.$$

Examples: In finite space $\Sigma_{\sigma,\tau} a_{\sigma\tau} x_\sigma x_\tau$ is bilinear form, symmetric if $a_{\sigma\tau} = a_{\tau\sigma}$; $x,y = x_1 y_1 + \dots + x_m y_m$ is positive-definite form.

In $\mathfrak{h}_o$ we can define inner product by

$$x,y = x_1 y_1 + x_2 y_2 + \dots$$

Problem: Prove absolute convergence of (x,y) from $(x,x) < \infty$, $(y,y) < \infty$. $(x,y))^2 \leq |x,x|\,|y,y|$

In space of functions $x(s)$ we may define bilinear form as follows

1. $\int_S x(s)r(s)y(s)ds$, e.g. $r = 1$.
2. $\int_S x(s)k(s,t)y(t)dsdt$, $k(s,t) = k(t,s)$, e.g. $k = e^{-|s-t|}$.
3. $\int_S Dx(s)\, p(s)\, Dy(s)\, ds$ e.g. $p = 1$.

Metrization Take one positive-definite form and define it as inner product x,y; unit-form:

$$x,x > 0 \quad \text{if} \quad x \neq 0,$$

Norm: $\|x\| = (x,x)^{1/2}$, distance $\|x - y\|$.

SI $|x,y| < \|x\|\,\|y\|$

TI $\|x + y\| < \|x\| + \|y\|$ } except if $\alpha x + \beta y = 0$.

Examples in function spaces

$x,x = \int_S r(s)x^2(s)ds$, $r > 0$ used for $\mathcal{L}_S$,

$x,x = \int_S \{p(s)Dx^2(s)ds + q(s)x^2(s)\}\, ds$, $p(s) > 0$, $q(s) > 0$, used for $\mathcal{D}_S$. (may require restriction).

Limit vector x of sequence x^σ : $\|x^\sigma - x\| \longrightarrow 0$ as $\sigma \longrightarrow \infty$. Also $x^\sigma \longrightarrow x$; convergence (strong); implies

$\|x^\sigma\| \longrightarrow \|x\|$ (from TI),

$y,x^\sigma \longrightarrow y,x$ (from SI).

Examples. In $\mathfrak{h}_o$, convergence of each component not sufficient. In $\mathcal{L}_{S'}$ (with above norm) $\int_{S'} r\,[x^\sigma - x]^2 ds \longrightarrow 0$ (i.e. in the mean).

Cauchy sequence x^σ: $\|x^\sigma - x^\tau\| \to 0$ as $\sigma, \tau \to \infty$.
If $x^\sigma \to x$ then x^σ is Cauchy sequence.
Space is <u>complete</u> if each Cauchy sequence has limit;
Complete space is termed <u>Euclidean</u> or general <u>Hilbert</u> space (with reference to norm of the type $\|x\| = (x,x)^{1/2}$).
$\mathfrak{H}_\sigma$ is complete, hence Euclidean.
Above function spaces are not complete. Three approaches:

1. Function spaces can be completed by adjoining functions with squares integrable in Lebesgue's sense.
2. Theory can be worked out in incomplete spaces, cf. Courant-Hilbert Vol. II Ch. VII.
3. Completing by adjoining ideal elements.

Consider space $\breve{\mathcal{R}}$ with unit-form (x,x). Let x^σ be a Cauchy sequence. Either it has limit vector in $\breve{\mathcal{R}}$ or else assign to it ideal limit vector x. If for two sequences x^σ, x_*^σ: $\|x^\sigma - x_*^\sigma\| \to 0$, assign the same limit vector to them. ?

Define $\alpha x + \beta y = \lim (\alpha x^\sigma + \beta y^\sigma)$. In this way space $\breve{\mathcal{R}}$ can be extended to a space $\bar{\mathcal{R}}$.

(x,y) and $\|x\|$ can be defined in $\bar{\mathcal{R}}$:

From TI one derives:
$\big|\, \|x^\sigma\| - \|x^\tau\| \,\big| \leq \|x^\sigma - x^\tau\| \to 0$, hence $\|x^\sigma\|$ is Cauchy sequence and therefore bounded. From SI one derives for two Cauchy sequences x^σ, y^σ: $|(x^\sigma, y^\sigma) - (x^\tau, y^\tau)| \leq \|x^\sigma - x^\tau\| \, \|y^\sigma\| + \|x^\tau\| \, \|y^\sigma - y^\tau\| \to 0$, hence (x^σ, y^σ) is Cauchy sequence and therefore converges to limit number. Let $x = \lim x^\sigma$, $y = \lim y^\sigma$. If x and y are in $\breve{\mathcal{R}}$, $(x,y) \to (x,y)$; otherwise define $(x,y) = \lim (x^\sigma, y^\sigma)$. Prove that (x,y) is symmetric and bilinear, further $\|x\| \geq 0$. Moreover $\|x\| > 0$ for $x \neq 0$: $\|x\| = 0$ would mean $\|x^\sigma\| \to 0$, and x^σ would have limit vector 0 in $\breve{\mathcal{R}}$.

$\bar{\mathcal{R}}$ is complete. Let $\bar{x}^\sigma$ be Cauchy sequence in $\bar{\mathcal{R}}$, $\|\bar{x}^\sigma - \bar{x}^\tau\| \leq \varepsilon(\sigma) \to 0$ for $\sigma \geq \tau$; choose x^σ in $\breve{\mathcal{R}}$ such that $\|x^\sigma - \bar{x}^\sigma\| < \varepsilon(\sigma)$, then $\|x^\sigma - x^\tau\| \leq 2\varepsilon(\sigma) + \varepsilon(\tau) \to 0$; Let $x = \lim x^\sigma$, then $\|\bar{x}^\sigma - x\| \leq \|x^\sigma - x\| + \varepsilon(\sigma) \to 0$.

Reference: F. Hausdorff Mengenlehre 2nd ed. 1927. § 21.3.

Example The space $\mathfrak{L}_S$ with unit-form $(x,x) = \int_S rx^2 ds$ may be extended to complete space $\bar{\mathfrak{L}}_S$, the space $\mathfrak{D}_S$ with unit-form $\int_S \{pDx^2 + qx^2\} ds$ to complete space $\bar{\mathfrak{D}}_S$. We do not investigate whether the ideal adjoined vector can be realized by functions.

Subspace of $\mathfrak{h}$ is <u>closed</u> if it contains all of its limit vectors. Subspace is <u>dense</u> in $\mathfrak{h}$ if each vector of space $\mathfrak{h}$ is limit vector of subspace.

Examples. In $\mathfrak{h}_0$: vectors for which $x_1 = 0$ form closed subspace, vectors which have only finite number of non-vanishing coefficients form dense subspace.
In $\bar{\mathfrak{L}}_S$: Dense subspaces are: $\mathfrak{L}_S$ and all piecewise linear functions; further: analytic functions, polynomials, (if S is bounded).
<u>Problem</u>: Prove that subspace in $\mathfrak{L}_S$ or $\mathfrak{D}_S$ of functions vanishing at a fixed point is dense in $\bar{\mathfrak{L}}_S$, not dense in $\bar{\mathfrak{D}}_S$ respectively.
<u>Proof</u> for case $S = (0 \leq s \leq 1)$, $\|x\|^2 = \int_0^1 x^2 ds$ or $= \int_0^1 \{Dx^2 + x^2\} ds$ respectively; fixed point: $s = 0$.
1. Let $\eta_\tau(s) = 1 - \tau^{-1} s$ for $0 \leq s \leq \tau$, $= 0$ for $s > \tau$.
Let $\bar{x}$ be any vector in $\bar{\mathfrak{L}}$, x in $\mathfrak{L}$ such that $\|x - \bar{x}\| < \varepsilon/2$.
Set $x^\tau = (1 - \eta_\tau)x$; $x^\tau(0) = 0$; $\|x - x^\tau\|^2 = \int_0^\tau \eta_\tau^2 x^2 ds \leq \int_0^\tau x^2 ds \leq \frac{\varepsilon}{2}$

if $\tau = \tau_\varepsilon$ sufficiently small. Then $\|x_{\tau_\varepsilon} - \bar{x}\| \leq \|x - \bar{x}\| + \|x_{\tau_\varepsilon} - x\| \leq \varepsilon$.
2. First prove (*) $|x(0)| \leq 2\|x\|$ for x in $\mathfrak{D}$.
$|x(0)|^2 = [x(s) + \int_0^s Dx(s') ds']^2 \leq 2x^2(s) + 2\int_0^1 Dx^2(s) ds$ see TV I D
$|x(0)|^2 \leq 2\int_0^1 x^2(s) ds + 2\int_0^1 Dx^2(s) ds = 2\|x\|^2$.
Now let y be in $\mathfrak{D}$ with $y(0) \neq 0$. If it could be approximated by x^ε in $\mathfrak{D}$ with $x^\varepsilon(0) = 0$, contradiction $|y(0)| \leq 2\|x^\varepsilon - y\| \longrightarrow 0$ would result from (*).
Subset <u>spans</u> (determines) space, if linear combinations are dense.
Examples: $\mathfrak{h}_0$ is spanned by coordinate system; i.e. by set of vectors $\{0,0,\ldots,0,1,0,\ldots\}$.

$\mathfrak{L}_S$ is spanned by piecewise linear functions with peak values 1 and rational salient points, or, for bounded S, by powers $1, x, x^2, \ldots$, or, for $S: 0 \leq x \leq 1$, by $\sin n\pi x$.
Space is denumerable if it is spanned by denumerable set.
(Proper Hilbert space).
Examples: $\mathfrak{h}_0$ and $\mathfrak{L}_S$ are denumerable.
Problem - Prove that $\mathfrak{I}_S$ is denumerable.
Projection. Let γ be closed subspace of vectors y. Projection Px of x in γ is vector in γ such that

$$x - Px \perp \gamma .$$

Minimum Property:

$$\|x - Px\| < \|x - y\| \quad \text{for all} \quad y \neq Px \text{ in } \gamma .$$

Proof: $\|x - y\|^2 = \|x - Px\|^2 + (x - Px, y - Px) + \|y - Px\|^2$
$= \|x - Px\|^2 + \|y - Px\|^2$,
because of $x - Px \perp y - Px$.
There is only one projection Px. Otherwise, from $x - P_1 x \perp \gamma$, $x - P_2 x \perp \gamma$, contradiction $P_1 x - P_2 x \perp \gamma$ would result.

Projection theorem 2.1 Let γ be a closed subspace in general Hilbert space $\mathfrak{h}$, x a vector in $\mathfrak{h}$. Then there is a vector Px in γ such that $x - Px \perp \gamma$.
Proof (Rellich, F. Riesz 1934). Let d be the g.l.b. of all $\|x - y\|$ where y in γ. Then for all z in γ

$$0 \leq (\beta z + y - x, \beta z + y - x) - d^2 = \beta^2 z^2 + 2\beta(z, y - x) + \|y - x\|^2 - d^2.$$

i.e. this quadratic function is non-negative; hence

(1) $\quad |z, y - x| \leq \|z\| \left\{\|y - x\|^2 - d^2\right\}^{1/2}$.

Apply to y^1 and y^2, use $|z, y^1 - y^2| \leq |z, y^1 - x| + |z, y^2 - x|$, set $z = y^1 - y^2$,

(2) $\quad \|y^1 - y^2\| \leq \left\{\|y^1 - x\|^2 - d^2\right\}^{1/2} + \left\{\|y^2 - x\|^2 - d^2\right\}^{1/2}$.

Now take minimizing sequence y^{σ}, i.e. $\|y^{\sigma} - x\| \longrightarrow d$, (exists!).
From (2) we have $\|y^{\sigma} - y^{\tau}\| \longrightarrow 0$, hence y^{σ} is Cauchy-sequence.
Since $\mathfrak{h}$ is complete, there is a limit vector $\mathring{y}$ such that $y^{\sigma} \longrightarrow \mathring{y}$; since γ is closed, $\mathring{y}$ is in γ. Evidently $\|\mathring{y} - x\| = d$.
From (1) we have $|z, \mathring{y} - x| = 0$, i.e. $\mathring{y} - x \perp \gamma$. Hence $\mathring{y} = Px$ is projection.

Complementary space γ^* to closed subspace γ consists of all vectors $\perp \gamma$.

Theorem 2.2 $\gamma \oplus \gamma^* = \mathfrak{h}$, i.e. γ and γ^* span $\mathfrak{h}$. Or, each vector x in $\mathfrak{h}$ can be split into sum of vector in γ and vector in γ^*.

In fact: $x = Px + (x - Px)$.

Problem: Let $\mathcal{L}$ be subspace in $\mathfrak{J}$ of functions vanishing at fixed point. Construct complement $\overline{\mathcal{J}}^*$ to closure $\overline{\mathcal{J}}$ of $\mathcal{J}$ in $\mathfrak{J}$.

Coordinate system: $u^1, u^2, u^3, \ldots, \|u^{\sigma}\| = 1$; $u^{\sigma} \perp u^{\tau}$, $\sigma \neq \tau$.
Complete if it spans space.

Problem. If space is finite or denumerable, a complete coordinate-system exists.

Theorem 2.3: Let $\{u^{\mu}\}$ be a complete coordinate system, then for any x in

$$x = \alpha_1 u^1 + \alpha_2 u^2 + \ldots\ldots, \quad \text{where} \quad \alpha_{\mu} = (x, u^{\mu}).$$

$$x, x = \alpha_1^{\,2} + \alpha_2^{\,2} + \ldots\ldots; \text{ i.e.}$$

$\mathfrak{h}$ is equivalent to $\mathfrak{h}_0$ or finite.

Proof: Take subspace $\mathcal{U}_m$ spanned by $u^1, \ldots, u^m$; $P_m x$ projection of x into $\mathcal{U}_m$. Let $P_m x = \beta_1 u^1 + \ldots + \beta_m u^m$; then, for $1 \leq \mu \leq m$,

$$\alpha_{\mu} = (x, u^{\mu}) = (P_m x, u^{\mu}) = \beta_{\mu}; \text{ hence}$$

$$P_m x = \alpha_1 u^1 + \ldots + \alpha_m u.$$

Since system $\{u^{\mu}\}$ is complete there is sequence y^m in $\mathcal{U}_m$ such that $\|y^m - x\| \longrightarrow 0$. Now $\|P_m x - x\| \leq \|y^m - x\|$, hence $\|P_m x - x\| \longrightarrow 0$, whence $\|P_m x\| \longrightarrow \|x\|$, but $\|P_m x\|^2 = \alpha_1 + \ldots + \alpha_m^{\,2}$,
Hence $\|x\|^2 = \alpha_1^{\,2} + \alpha_2^{\,2} + \ldots$.

Chapter III. Completely continuous forms and point spectra.
Hilbert space $\mathfrak{H}$ of vectors x; unit form (x,x)
Quadratic form xQx resulting from symmetric
Bilinear form $xQy = yQx$. Assume
$0 < xQx < k(x,x)$; i.e. Q is positive-definite and bounded.
(Positive- definiteness immaterial).

<u>Lemma 3.1</u> Let $x^\sigma \to x$ then $yQx^\sigma \to yQx$, $x^\sigma Q x^\sigma \to xQx$.

Proof 1. $|yQ(x^\sigma - x)| \le yQy((x^\sigma - x)Q(x^\sigma - x)) \le yQy\, k\|x^\sigma - x\|^2 \to 0$.
2. $|(x^\sigma Q x^\sigma)^{1/2} - (xQx)^{\frac{1}{2}}| \le ((x^\sigma - x)Q(x^\sigma - x))^{1/2} \le k^{1/2}\|x^\sigma - x\| \to 0$.

Definition of E-values κ and E-vectors u without reference to operators:

(3.1) $yQu = \kappa(y,u)$ for all y in $\mathfrak{H}$.

<u>Theorem 3.1</u> Two E-vectors to different E-values κ are $\perp$.

Proof: $\kappa_1(u^1, u^2) = u^1Qu^2 = \kappa_2(u^1, u^2)$.

<u>Theorem 3.2</u> E-vectors to same E-value κ form closed space $\mathfrak{M}_\kappa$.

Proof 1. $\mathfrak{M}_\kappa$ is linear, from (3.1). 2. $\mathfrak{M}_\kappa$ is closed: let u^σ be sequence with limit v; then $(y,u^\sigma) \to (y,v)$ and, from Lemma 3.1, yQu^σ yQv; hence $yQv = \kappa(y,v)$, i.e. v in $\mathfrak{M}_\kappa$.

<u>Lemma 3.2</u> Let $u^1, \ldots, u^m$ be an orthonormed set of E-functions with E-values $\kappa_1, \ldots, \kappa_m$; then for $x = \alpha_1 u^1 + \ldots + \alpha_m u^m$,

$(x,x) = \alpha_1^2 + \ldots + \alpha_m^2$, $xQx = \kappa_1\alpha_1^2 + \ldots + \kappa_m\alpha_m^2$.

Proof: $(x,u^\mu) = \alpha_\mu$; $x,x = \sum_\mu \alpha_\mu(x,u^\mu) = \sum_\mu \alpha_\mu^2$;
$xQu^\mu = \kappa_\mu\alpha_\mu$; $xQx = \sum_\mu \alpha_\mu(xQu^\mu) = \sum_\mu \kappa_\mu\alpha_\mu^2$

<u>Theorem 3.3</u> Let $\mathfrak{M}_{\ge\omega}$ be space spanned by E-vectors to E-values $\ge \omega$; then $xQx \ge \omega(x,x)$ for x in $\mathfrak{M}_{\ge\omega}$.

<u>Proof: 1.</u> Let $x = \alpha_1 u^1 + \ldots + \alpha_m u^m$; in view of Th. 3.2 $\kappa_1, \ldots, \kappa_m$ may be considered different; in view of Th. 3.1 the u^μ are perpendicular; then the statement follows from Lemma 3.2. 2. For limits of such linear combinations the statment follows from Lemma 3.1.

Desired type of <u>spectral representation</u> : "positive-discrete" representation:-

Complete orthogonal system of normal E-vectors $u^1, u^2, \ldots,$; $u^\mu \perp u^\nu$, $\|u^\mu\| = 1$.

Decreasing sequence of positive E-vectors $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq \ldots \longrightarrow 0$.

Able to represent each vector in :

$x = \alpha_1 u^1 + \alpha_2 u^2 + \ldots$, where $\alpha_\mu = (u^\mu, x)$; further

$x, x = \alpha_1^2 + \alpha_2^2 + \ldots$, $x\mathfrak{Q}x = \kappa_1\alpha_1^2 + \kappa_2\alpha_2^2 + \ldots$.

The latter relation follows from $P_m x = \alpha_1 u^1 + \ldots + \alpha_m u^m \longrightarrow x$ and $P_m x \mathfrak{Q} P_m x = \kappa_1\alpha_1^2 + \ldots + \kappa_m \alpha_m^2$ (cf. Lemma 3.2) in view of Lemma 3.1.

Above type of spectral representation implies following properties of <u>spectral resolution:</u>

Pure point spectrum:- set of E-vectors span whole space.-

Positive-discrete: point spectrum $\kappa > 0$ and E-vectors to $\kappa \geq \omega > 0$ span finite space; i.e. $\mathfrak{M} \geq \omega$ has finite dimension.

<u>Theorem 3.4</u> If $\mathfrak{Q}$ has pure positive-discrete point spectrum then "positive-discrete" representation holds.

<u>Proof</u> Since E-vectors to each E-value form finite space, they can be spanned by finite coordinate system. All E-vectors thus obtained can be ordered such that $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq \ldots \longrightarrow 0$. The resulting system is complete; hence representation follows from Th. 2.3.

Hilbert's discovery: Forms with pure positive-discrete point spectrum can be simply characterized.

Property "W": To each $\varepsilon > 0$ there are $r = r(\varepsilon)$ vectors $y^1, \ldots, y^r$ in $\mathfrak{H}$ such that for all x in $\mathfrak{H}$

(W) $|x\mathfrak{Q}x| \leq \Sigma_{\rho=1}^{r} (y^\rho \mathfrak{Q} x)^2 + \varepsilon(x, x)$.

<u>Theorem 3.5</u> "Positive-discrete" representation implies "W".

Proof: Set $y^\rho = \kappa_\rho^{-1/2} u^\rho$ for $\kappa_\rho \geq \varepsilon$.

We shall show that positive-definite form $\underline{Q}$ enjoying "W" has positive-discrete point spectrum and, consequently, a "positive-discrete" representation. Before doing so we mention:
Hilbert had given different condition instead of "W".
Property "CC" of complete continuity: Let x^σ be a sequence with bounded $\|x^\sigma\|$ such that $(y\underline{Q}x^\sigma) \longrightarrow 0$ for all y in $\mathfrak{h}$ (weak convergence), then $x^\sigma\underline{Q}x^\sigma \longrightarrow 0$.

<u>Theorem 3.6</u> "W" implies "CC". Evident from "W".

<u>Problem:</u> Prove: "CC" implies "W".

<u>Theorem 3.7</u> If positive-definite form $\underline{Q}$ enjoys "W" its point spectrum is positive-discrete.

Proof: Let $\mathfrak{M} \geq \omega$ (cf. Th.3.3) have dimension $> r(\varepsilon)$ for $\omega > \varepsilon$: then it contains vector x such that $x\underline{Q}y^\rho = 0$, $\rho = 1,\ldots,r$. For this x, (W) gives contradiction to Th.3.3.

<u>Theorem 3.8.</u> If positive-definite form $\underline{Q}$ enjoys "W" it possesses E-vector, provided the space $\mathfrak{h}$ possesses one vector $x \neq 0$.

<u>Proof</u> based on Maximum-problem for $x\underline{Q}x/x,x$ for $x \neq 0$.
Let $K > 0$ be least upper bound; then $K - \underline{Q}$ non-negative form. Cinsider normed maximizing sequence x^σ, $\|x^\sigma\| = 1$, $x^\sigma(K - \underline{Q})x^\sigma = \varepsilon_\sigma^2 \longrightarrow 0$. Set $x^\sigma - x^\tau = x^{\sigma\tau}$. TI applied to $x^{\sigma\tau}$ for $K - \underline{Q}$ yields

$$x^{\sigma\tau}(K - \underline{Q})x^{\sigma\tau} \leq (\varepsilon_\sigma + \varepsilon_\tau)^2.$$

Upon adding (W) for $x^{\sigma\tau}$,

$$(*)\quad (K - \varepsilon)(x^{\sigma\tau}, x^{\sigma\tau}) \leq \Sigma_{\rho=1}^{r} (y^\rho\underline{Q}x^{\sigma\tau})^2 + (\varepsilon_\sigma + \varepsilon_\tau)^2.$$

Choose $\varepsilon < K$, then r and $y^1,\ldots,y^r$; further subsequence x^σ such that the r bounded sequences $(y^\rho\underline{Q}x^\sigma)$ converge, which implies $(y^\rho\underline{Q}x^{\sigma\tau}) \longrightarrow 0$. For this subsequence, (*) yields $\|x^{\sigma\tau}\| \to 0$; i.e. it is Cauchy sequence and converges to limit vector u due to completeness of space. $\|x^\sigma\| = 1$ yields $\|u\| = 1$, $x^\sigma(K-\underline{Q})x^\sigma \to 0$ yields $u(K - \underline{Q})u = 0$ (Lemma 3.1). Further from SI,

$$|y(K - \underline{Q})u|^2 \leq |y(K - \underline{Q})y|\,|u(K - \underline{Q})u| = 0;$$

hence $y\underline{Q}u = K(y,u)$ for all y in $\mathfrak{h}$; i.e. u is E-vector with E-value K.

<u>Theorem 3.9</u> If positive-definite form $\underline{Q}$ enjoys property "W" its E-vectors are complete.

<u>Proof</u> Consider closed space $\mathfrak{T}$ spanned by all E-vectors and complementary space $\mathfrak{T}^*$ of all $x \perp \mathfrak{T}$. Since $\mathfrak{T}$ and $\mathfrak{T}^*$ span $\mathfrak{H}$, (cf. Theorem 2.2), we have to prove $\mathfrak{T}^* = 0$, i.e. consisting of $x = 0$ only. First we observe

(*) $\quad t\underline{Q}x_*$ for any t in $\mathfrak{T}$, x_* in $\mathfrak{T}_*$.

This relation holds if $t = u$ is an E-vector because of $u\underline{Q}x_* = K(u,x_*) = 0$. It therefore holds for every linear combination of E-vectors and in view of Lemma 3.1 also for every t in $\mathfrak{T}$. Hence

($\overset{*}{*}$) $\quad y\underline{Q}x_* = y_*\underline{Q}x_*$ where y is any vector in $\mathfrak{H}$, y_* its projection in $\mathfrak{T}_*$ and x_* any vector in $\mathfrak{T}_*$.

We now consider $\underline{Q}$ a positive-definite form in $\mathfrak{T}_*$. An E-vector u_* of $\underline{Q}$ in $\mathfrak{T}_*$ is also E-vector of $\underline{Q}$ in $\mathfrak{T}$ because $y_*\underline{Q}u_* = K_*(y_*,u_*)$ for all y_* in $\mathfrak{T}_*$ implies $y\underline{Q}u_* = K_*(y,u_*)$ for all y in $\mathfrak{H}$, according to ($\overset{*}{*}$). Since every E-vector of $\underline{Q}$ in $\mathfrak{H}$ is $\perp$ to $\mathfrak{T}_*$, there is no E-vector of $\underline{Q}$ in $\mathfrak{T}_*$. However, when relation (W) is applied to $x = x_*$ in $\mathfrak{T}_*$, one may replace $y^\rho\underline{Q}x_*$ by $y^\rho_*\underline{Q}x_*$ where y^ρ_* is the projection of y^ρ in $\mathfrak{T}_*$; This is obvious from ($\overset{*}{*}$). Thus one obtains

$$x_*\underline{Q}x_* \leq \sum_{\rho=1}^{n} (y^\rho_*\underline{Q}x_*)^2 + \varepsilon(x_*,x_*),$$

which expresses that $\underline{Q}$ in $\mathfrak{T}_*$ enjoys "W". Therefore, Th.3.8 would be applicable and would yield the existence of an E-vector in $\mathfrak{T}_*$ unless $\mathfrak{T}_*$ contained only the vector "0". The latter must be the case since there is no E-vector in $\mathfrak{T}_*$.

Differentialform. Open interval S: $s^- < |s| < s^+$ (finite or infinite). Space $\dot{\vartheta} = \dot{\vartheta}_S$ of functions $x(s)$ with continuous derivative vanishing in neighborhood of endpoints of S. Unit-form

$$(x,x) = \int_{S'} \{p(s)Dx^2(s) + q(s)x^2(s)\} ds,$$

Quadratic form

$$x\underline{Q}x = \int_S r(s)x^2(s)ds$$

where $p(s) > 0$, $q(s) \geq r(s)$, $r(s) > 0$ and continuous in open S.

Complete this space $\dot{\vartheta}$ to a space $\hat{\vartheta} = \hat{\vartheta}_{S'}$ by adjoining ideal vectors. $\hat{\vartheta}$ implies a boundary condition, the generalization of "$x = 0$ at the endpoints of S".

Obviously $\underline{Q}$ in $\dot{\vartheta}$ is positive-definite and bounded:

(*) $0 < x\underline{Q}x \leq k(x,x)$, $x \neq 0$.

It can be extended to $\hat{\vartheta}$:

Let x^σ, y^σ be Cauchy sequences in $\dot{\vartheta}$ approaching x,y in $\hat{\vartheta}$. Then $x^{\sigma\tau}\underline{Q}y^{\sigma\tau} \leq (x^{\sigma\tau}\underline{Q}x^{\sigma\tau})^{1/2}(y^{\sigma\tau}\underline{Q}y^{\sigma\tau})^{1/2} \leq k\|x^{\sigma\tau}\|\|y^{\sigma\tau}\| \to 0$. Therefore $x^\sigma\underline{Q}y^\sigma$ converges, and we may define $x\underline{Q}y = \lim x^\sigma\underline{Q}y^\sigma$. It is easily seen that $x\underline{Q}y$ is a bilinear form in $\hat{\vartheta}$ with the bound k. $\underline{Q}$ in $\hat{\vartheta}$ is obviously non-negative; it is, however, not obvious that $\underline{Q}$ is positive-definite; i.e. that never $x\underline{Q}x = 0$ for $x \neq 0$ in $\hat{\vartheta}$. We postpone the proof of this fact to the next chapter.

It may be noted that it is not true that every bounded form which is positive-definite in a dense subspace is also positive-definite in the complete space.

Counter-example. $(x,x) = \int_0^1 \{Dx^2 + x^2\} ds + (Dx(1))^2$, $x\underline{Q}x = \int_0^1 x^2 ds$. In the space ϑ of all $x(s)$ with continuous derivatives Dx the form Q is positive-definite; when this space is completed, this no longer holds. To show this we take the sequence

$x^\varepsilon(s) = 0$ for $s \leq 1 - \varepsilon$, $= s - 1 + \varepsilon$ for $1 - \varepsilon \leq |s| \leq 1$.
Then $(x^\varepsilon, x^\varepsilon) = \varepsilon + \frac{1}{2}\varepsilon^2 + 1 \to 1$, $x^\varepsilon \underline{Q} x^\varepsilon = \frac{1}{2}\varepsilon^2 \to 0$.

$$(x^{\varepsilon_1\varepsilon_2}, x^{\varepsilon_1\varepsilon_2}) = (\varepsilon_2 - \varepsilon_1) + \frac{1}{2}(\varepsilon_2 - \varepsilon_1)^2 + \varepsilon_1(\varepsilon_2 - \varepsilon_1)^2 \to 0,$$

as $\varepsilon_1 < \varepsilon_2 \to 0$. Hence x is a Cauchy sequence; and $(x,x) = 1$, $x\underline{Q}x = 0$ for the limit vector.

<u>Examples.</u>

1. $x\underline{L}x = \int_0^1 Dx^2 ds$, $x\underline{Q}x = \int_0^1 x^2 ds$; $(x,x) = x\underline{L}x + x\underline{Q}x$; or, $= \underline{L} + \underline{Q}$.

E-vectors: $\sin n\pi s$, E-values $K_n = [1 + n^2\pi^2]^{-1}$, Fourier expansion.

2. $x\underline{L}x = \int_{-1}^1 (1 - s^2) Dx^2 ds$, $x\underline{Q}x = \int_{-1}^1 x^2 ds$; $\quad , = \underline{L} + \underline{Q}$.

E-vectors; Legendre polynomials, $K_n = [1 + n(n+1)]^{-1}$.

3. $x\underline{L}x = \int_0^1 \{sDx^2 + n^2 s^{-1} x^2\} ds$, $x\underline{Q}x = \int_0^1 x^2 ds$; $\quad , = \underline{L} + \underline{Q}$.

Bessel functions $J_m(\rho_{mn} s)$, $K_n = [1 + \rho_{mn}^2]^{-1}$, where $J_m(\rho_{mn}) = 0$.

4. $x\underline{L}x = \int_{-\infty}^{\infty} \{Dx^2 + \frac{1}{4}s^2x^2\} ds$, $x\underline{Q}x = \int_{-\infty}^{\infty} x^2 ds$; $\quad , = \underline{L} + \underline{Q}$.

$H_n(s)e^{-s^2/4}$, H_n Hermite polynomials, $K_n = [1 + n - \frac{1}{2}]^{-1}$.

5. $x\underline{L}x = \int_0^\infty \{Dx^2 + \frac{1}{4}x^2\} s^2 ds$, $x\underline{Q}x = \int_0^\infty x^2 s ds$; $\quad , = \underline{L} + \underline{Q}$.

$L_n'(s)e^{-s/2}$, L_n Laguerre polynomials, $K_n = [1 + n]^{-1}$.

6. $x\underline{Q}x = \int_0^\infty \{aDx^2 + 2bxDx\} s^2 ds$, $x\underline{H}x = \int_0^\infty x^2 s^2 ds$; $\quad , = \underline{Q} + c\underline{H}$,

where $a > 0$, $c > b^2/a$.

Negative discrete spectrum $K_n = \lambda_n / c + \lambda_n$, $\lambda_n = -n^{-2} b^2/a$.

$L_n'(2sb/na)e^{-sb/na}$; Schrodinger functions for $a = \hbar/2m$, $2b = e^2$

7. $x\underline{L}x = \int_{-\infty}^{\infty} Dx^2 ds$, $x\underline{Q}x = \int_{-\infty}^{\infty} x^2 ds$; $\quad , = \underline{L} + \underline{Q}$.

No discrete spectrum.

We shall derive three criteria that the above differential form $\underline{Q}$ possess property "W" with respect to the above unit-form (x,x). Before doing so we observe that the coefficient p can be made 1 by a transformation. Since $p > 0$ we may introduce

$$t = \int_{s_0}^{s} ds'/p(s')$$

as new variable; then

$$(x,x) = \int_{t-}^{t+} \left\{ \left(\frac{dx}{dt}\right)^2 + qp\, x^2 \right\} dt, \qquad x\underline{Q}x = \int_{t-}^{t+} x^2 rp\, dt;$$

qp and rp take the place of q and r. Note that $\int qp\, dt = \int q\, ds$, $qp/rp = q/r$ remain unchanged.

Criterion 1. $\mathfrak{Q}$ enjoys "W" if $\int_S r\,ds < \infty$, $\int_S |t|\,r\,ds < \infty$.

Proof: We introduce $\int_{s_0}^{s} r(s')ds' = l(s)$, $\int_{s_0}^{s} |t(s')|\; r(s')ds' = j(s)$

Let Δs be a subinterval of S (finite or infinite), then we set

$\int_{\Delta s} r\,ds = \delta l$, $\int_{\Delta s} |t|\,r\,ds = \delta j$. We then proceed to prove first

Poincare's generalized inequality:

$$\int_{\Delta s} rx^2 ds \leq (\delta l)^{-1}\left(\int_{\Delta s} r x\,ds\right)^2 + \delta j \int_{\Delta s} pDx^2 ds$$

for x in $\dot{\vartheta}$. (even x in ϑ) (t, r & p bounded)

Proof: In view of the preceding remarks we may assume p = 1, t = s. Let s_1, s_2 be in Δs; $|x(s_1) - x(s_2)|^2 = \left|\int_{s_1}^{s_2} Dx\,ds\right|^2$

$\leq |s_2 - s_1| \int_{s_1}^{s_2} Dx^2 ds \leq \left[|s_2| + |s_1|\right] \int_{\Delta s} Dx^2 ds$. By integratingwith respect to s_1 and s_2 over Δs,

$$\int_{\Delta s}\int_{\Delta s} r(s_1)r(s_2)\left|x(s_1) - x(s_2)\right|^2 ds_1 ds_2 \leq 2\,\delta l\,\delta j \int_{\Delta s} Dx^2 ds.$$

The left member is $2\,\delta l \int_{\Delta s} rx^2 ds - 2\left(\int_{\Delta s} rx\,ds\right)^2$. Hence division by $2\,\delta l$ yields the inequality.

Divide S into R intervals $\Delta_\rho s$ such that all $\delta_\rho j \leq \varepsilon$. By adding Poincare-inequalities:

$$\int_S rx^2 ds \leq \sum_{\rho=1}^{R} (\delta_\rho l)^{-1}\left(\int_{\Delta_\rho s} r x\,ds\right)^2 + \varepsilon \int_S Dx^2 ds.$$

Set $z^\rho = (\delta_\rho l)^{-1/2}$ in $\Delta_\rho s$, $= 0$ outside; then preceding formula reads

$$\int_S rx^2 ds \le \sum_{\varrho=1}^{R} \left(\int_S rz^\varrho x ds\right)^2 + \varepsilon \int_S (Dx)^2 ds.$$

The functions z^ϱ are not in $\dot{\mathcal{D}}$; approximate them by functions y^ϱ in $\dot{\mathcal{D}}$ such that $\int_S r(z^\varrho - y^\varrho)^2 ds \le \varepsilon^2/4R^2 \int_S r(z^\varrho)^2 ds$.

Then $\left(\int_S rz^\varrho x ds\right)^2 - \left(\int_S ry^\varrho x ds\right)^2 \le 2\int_S r(z^\varrho - y^\varrho)x ds \int_S rz^\varrho x ds$

$\le 2\left(\int_S r(z^\varrho - y^\varrho)^2 ds\right)^{1/2}\left(\int_S r(z^\varrho)^2 ds\right)^{1/2}\int_S rx^2 ds \le 3R^{-1}\varepsilon\int_S rx^2 ds.$

Inserting we obtain

(*) $\int_S rx^2 ds \le \sum_{\varrho=1}^{R} \left(\int_S ry^\varrho x ds\right)^2 + \varepsilon \int_S \{Dx^2 + rx^2\} ds$ or

$$x\underline{Q}x \le \sum_{\varrho=1}^{R} (y^\varrho \underline{Q} x)^2 + \varepsilon(x,x)$$

This inequality is proved for all x in $\dot{\mathcal{D}}$. In view of Lemma 3.1 it holds for all x in $\hat{\mathcal{D}}$. Thus we have proved (W) and establish Criterion 1.

<u>Application:</u> Examples 1.2.3. The forms $\underline{Q}$ in these examples may now be seen to enjoy property "W" with respect to the unit-form $\underline{L} + \underline{Q}$ and, therefore, possess a positive discrete spectrum.

We note that the formula (*) derived for the proof of criterion 1 also holds if all integrations are extended over a subinterval of S; the conditions $\int r ds < \infty$, $\int |t| r ds < \infty$ need only be satisfied for this sub-interval.

<u>Criterion 2:</u> $\underline{Q}$ enjoys "W" if $q/r \to \infty$ as s approaches the endpoints.

Proof: To $\varepsilon > 0$ find an inner subinterval S_ε outside of which $r/q < \varepsilon$. Then $\int_{S-S_\varepsilon} rx^2 ds \leq \varepsilon \int_{S-S_\varepsilon} qx^2 dx \leq \varepsilon \int_{S-S_\varepsilon} \{Dx^2 + qx^2\} ds$. Since formula (*) of criterion 2 holds for the subinterval S_ε, addition yields (W).

A similar reasoning leads to

<u>Criterion 3</u>: $\underline{Q}$ enjoys "W" if at each endpoint either $\int rds$ and $\int |t| rds$ are finite or if there $q/r \to \infty$.

Application of Criterion 2: Example 4.

Application of Criterion 3: Example 5, where $r = s$, $q = s^2$, hence $q/r \to \infty$ as $s \to \infty$; $p = s^2$, $t = -s^{-1}$; $\int |t| rds = s$ is finite for $s \to 0$.

For the discussion of example 6 we observe first that for x in ϑ

$\int_0^\infty xDxs^2 ds = -\int_0^\infty x^2 sds < 0$; hence

$(x\underline{Q}x) \geq -2b \int_0^\infty x^2 sds$. Further we note that a positive α can be found such that

$$(x,x) \geq \alpha \int_0^\infty \left\{Dx^2 - xDx + \frac{1}{4} x^2\right\} s^2 ds.$$

Applying formula (W) of Example 5 we obtain

$$-(x\underline{Q}x) \leq \sum_{\varrho=1}^{\lambda} \left(\int_0^\infty y^\varrho xsds\right)^2 + \varepsilon (x,x).$$

This formula expresses that $\underline{Q}$ enjoys property "W^-" except that $\int_0^\infty y^\rho x r ds$ appears instead of $y^\rho \underline{Q} x$. As we shall see in the next chapter this modified "W^-" is equivalent to "W^-". From this then it can be deduced that the negative part of the spectrum of $\underline{Q}$ is discrete.

In Example 7 $\underline{Q}$ posseses no discrete spectrum. To show this we prove that property "CC" is not satisfied. To this effect we need only construct a sequence $x^\sigma(s)$ such that

$x^\sigma, x^\sigma \longrightarrow a \neq 0$, $x^\sigma \underline{Q} x^\sigma \longrightarrow b \neq 0$, $y \underline{Q} x^\sigma \longrightarrow 0$ for all y in $\mathcal{D}$.

We take $z(s) = (1 - s^2)^2$ and set

$$x^\sigma(s) = \sigma^{-1/2} z(\sigma^{-1} s - z).$$

We then have

$$x^\sigma \underline{Q} x^\sigma = \int_{-\infty}^{\infty} (x^\sigma)^2 ds = \int_{-\infty}^{\infty} z^2 ds = z\underline{Q}z \neq 0.$$

$$x^\sigma, x^\sigma = \int_{-\infty}^{\infty} \left\{ (Dx^\sigma)^2 + (x^\sigma)^2 \right\} ds = \int_{-\infty}^{\infty} \left\{ \sigma^{-2} (z')^2 + z^2 \right\} ds \longrightarrow z\underline{Q}z \neq 0.$$

To show that $y\underline{Q}x \longrightarrow 0$, approximate y by $\dot{y}$ in $\mathcal{D}$; then

$$\left| y\underline{Q}x^\sigma \right| \leq \left| \dot{y}\underline{Q}x^\sigma \right| + \| y - \dot{y} \| \| x^\sigma \|.$$

$\|y - \dot{y}\|$ can be made arbitrarily small; $\|x^\sigma\|$ is bounded and $\dot{y}\underline{Q}x^\sigma$ equals zero for sufficiently large σ. Hence the sequence x^σ violates (CC).

Example. $(x,x) = \int_0^1 x^2(s) ds$, $x\underline{Q}x = \int_0^1\int_0^1 x(s)k(s,t)x(t) ds dt$, $k(s,t) = k(t,s) \geq 0$ and continuous.

Vibrations of string extended over S. Let p be the axial tension, r mass p.u.l., q distributed spring constant. $k^{-1} = \mu^2$, μ circular frequency. Discrete spectrum if total mass $\int r ds$ and its moment $\int |s| r ds$ finite or spring constant q becomes infinite.

<u>Problem.</u> Prove that $\underline{Q}$ possesses property "V".

<u>Example 8</u>. Buckling of a tapered column. S: $0 < s < b$. $p = 1$, $q = 0$, $r = (EI)^{-1}$; E modulus of elasticity, $I = I_b x^4 b^{-4}$ moment of inertia of cross section. P-applied axial force. P^{-1} E-value. Prove that here form $\underline{Q}$ does not enjoy "W".

<u>Problem</u>: Investigate whether or not "W" holds for $S: 0 < s < \infty$. p and $r = q$ behave like $s^{\alpha_0}, s^{\beta_0}$ or s^{α_∞}, s^{β_∞} at $s = 0$, $s = \infty$ respectively.

Chapter IV. Operators.

Hilbert-space $\mathfrak{H}$ of vectors x. Bounded linear operator assigns to every x a vector Qx such that

$$Q(\alpha x+\beta y) = \alpha Qx + \beta Qy$$

and that there is a number K such that

$$\|Qx\| \leq K\|x\| \text{ for all } x.$$

Q is symetric if $(x,Qy)=(Qx,y)$.

Bilinear form $\underline{Q}$ of bounded lenear operator Q defined by

$$x\underline{Q}y=(x,Qy) = (Qx,y); \text{ it is bounded}$$

$$|x\underline{Q}y| \leq K\|x\|\,\|y\|.$$

Theorem 4.1 To every bounded bilinear form $\underline{Q}$ there is exactly one bounded linear Operator Q such that

(1) $x\underline{Q}y = x,Qy$ holds and

(2) $x\underline{Q}y = x,z$ inplies $z = Qy$.

The second statement follows from the first one by setting $x = z-Qy$. The proof is based on a lemma concerning bounded linear forms, i.e. forms $\underline{E}x$ with properties

$\underline{E}(\alpha x+\beta y) = \alpha\underline{E}x+\beta\underline{E}y$ and there is a number η such that $|\underline{E}x| \leq \eta\|x\|$ for all x.

<u>Lemma</u> 4.1 To every bounded linear form $\underline{E}$ there is exactly one vector e in $\mathfrak{H}$ such that

(*) $\underline{E}x = (e,x)$ for all x; i.e. every lenear form can be represented as an inner product with a fixed vextor.

The <u>proof</u> is based on the projection theorem: 1. The space $\mathfrak{Y}$ of all t for which $\underline{E}t=0$ is closed, because $t^{\sigma}\to z$ implies $|\underline{E}(t^{\sigma}-z)| \leq \eta\|t^{\sigma}-z\|\to 0$, $\underline{E}z = \underline{E}(z-t^{\sigma})\to 0$, $\underline{E}z=0$.

2. Consider the complementary space $\mathfrak{T}^*$. If $\mathfrak{T}^* = 0$ then $\mathfrak{T} = \mathfrak{H}$ and $e=0$ satisfies (*). If $\mathfrak{T}^* \neq 0$ it is spanned by just one vector t^*: If there were two independent vectors t_1^*, t_2^* in $\mathfrak{T}^*$, a linear combination $t = \alpha_1 t_1^* + \alpha_2 t_2^*$ could be found such that $\underline{E}t = \alpha_1 \underline{E} t_2^* + \alpha_2 \underline{E} t_2^* = 0$; i.e. which would be in $\mathfrak{T}$. Set $e = \alpha t^*$ where α is so chosen that $\underline{E}e = (e,e)$; namely $\alpha = \underline{E}t^* / (t^*, t^*)$. Then the projection of any x into $\mathfrak{T}^*$ is $(e,e)^{-1}(e,x)e$ and $x = (e,e)^{-1}(e,x)e + t$ where t is in $\mathfrak{T}$. Hence

$\underline{E}x = (e,e)^{-1}\ (e,x)\ \underline{E}e = (e,x)$.

To prove theorem 4.1 we observe that the bounded form $x\underline{Q}y$ is a bounded linear form in y when x is considered fixed. Hence there is exactly one vector z such that $x\underline{Q}y = (z,y)$; z depends on x; we set $z = Qx$. We have $(\alpha_1 x^1 + \alpha_2 x^2)\ \underline{Q}y = (\alpha_1\ Qx^1 + \alpha_2 Qx^2), y$; since Qx is uniquely determined we have $Q(\alpha_1 x^1 + \alpha_2 x^2) = \alpha_1 Qx^1 + \alpha_2 Qx^2$; i.e. Q is linear. Further we have $|Qx, y| \leq K\ \|x\|\ \|y\|$; hence $\|Qx\|^2 = |Qx, Qx| \leq K \|x\| \|Qx\|$ or $\|Qx\| \leq K\|x\|$; i.e. Q is bounded. Thus Theorem 4.1 is proved.

a. Consider [illegible]

the [illegible]

[illegible]

a linear [illegible]

[illegible]

where α is [illegible]

then the projection [illegible]

the $x=(c,c)^{-1}$ [illegible]

$\underline{B}x = (c,c)^{-1}$ [illegible]

To prove [illegible]

[illegible] is [illegible]

[illegible]

[illegible]

[illegible]

quoting [illegible]

i.e. Q is [illegible]

hence [illegible]

Q is [illegible]

As a consequence of Theorem 4.1 we remark that property "V" ensues from "W" because $y\rho\underline{Q}x$ can now be written $(Qy\rho, x)$. That "W" follows from "V" is not so immediate; these properties are, however, equivalent since both are equivalent to the purely discrete spectral representation.

As an important consequence of Theorem 4.1 the definition of E-vector and E-value $z\underline{Q}u = \kappa(z,u)$ can be replace by

$Qu = \kappa u$,

as is seen from the second statement in Theorem 4.1 i.e. E-vectors and E-values can be characterized by an operator-equation.

To reveal the nature of the operator Q we take Example 1, for the interval $a<s<b$; we take, however, the form $\underline{L}$ for unit-form. That $\underline{Q}$ remains bounded in $\hat{\mathfrak{D}}$ is easily seen. $(x,x)= \int_a^b Dx^2 ds$, $x\underline{Q}x = \int_a^b x^2 ds$. We show that $Qx = \int_a^b k(s,r)x(t)dt$ where the Green's function $k(s,t)=k(t,s)$ is given by

$$k(s,t) = \begin{cases} (b-a)^{-1}(b-s)(t-a) & \text{for } t\leq s \\ (b-a)^{-1}(s-a)(b-t) & \text{for } t\leq s. \end{cases}$$

Proof: We first prove it for x in $\mathfrak{D}'$. We have

$$Qx = (b-a)^{-1}(b-s)\int_a^s (t-a)x(t)dt + (b-a)^{-1}(s-a)\int_s^b (b-t)x(t)dt$$

$D^2Qx = -x$. For y in $\mathfrak{D}'$ we deduce

$$\int_a^b Dy\, DQx\, dx = -\int_a^b yD^2Qx ds = \int_a^b yx ds \quad \text{or} \quad y\underline{L}Qx=yQx.$$

This relation can immediately be extended to x and y in $\hat{\mathfrak{D}}$.

Inversion of the operator Q. The inverse of the operator Q in $\mathfrak{h}$ can be defined as follows. Let $Q\mathfrak{h}$ be the subspace of all vectors x in $\mathfrak{h}$ which are of the form $x = Qy$ where y is in $\mathfrak{h}$. The vector y is uniquely determined; for $Qy = 0$ implies $y = 0$ since Q is positive definite. The operator assigning y to x is denoted by M: $y = Mx$. This operator is linear since $Q(\alpha_1 y^1 + \alpha_2 y^2) = \alpha_1 x^1 + \alpha_2 x^2$ implies $M(\alpha_1 x^1 + \alpha_2 x^2) = \alpha_1 Mx^1 + \alpha_2 Mx^2$. Statements (1) and (2) of theorem 4.1 can be written

(1) $z\underline{Q}Mx = z,x$ holds for x in $Q\mathfrak{h}$, z in $\mathfrak{h}$

(2) $z\underline{Q}y = z,x$ for all z in $\mathfrak{h}$ implies that x is in $Q\mathfrak{h}$ and $Mx=y$.

When we apply the latter statement to E-vectors u, we see that the relation $Qu = \kappa u$ is equivalent to

u is in $Q\mathfrak{h}$ and

$Mu = \mu u$ where $\mu = \kappa^{-1}$ is an "E-value of the operator M".

We turn now to the case that $\mathfrak{h}$ is the function space $\hat{\vartheta}$ and Q the operator connected with the differential form $\underline{Q}$. We shall see that the inverse M is a differential operator. For the space $Q\mathfrak{h}$ we then use the notation $\hat{f}$.

In what follows we assume that r, q, p, $\frac{dp}{ds}$ have continuous derivatives. We introduce the space f of all functions x(s) with continuous derivatives up to the third order and the subspace $\dot{f}$ of those functions in f that vanish identically in a neighborhood of the end points. In f we define the differential operator

$M = -r^{-1}[DpD-q]$, where the coincidence with the inverse of Q is anticipated in the notation. Then Green's formula

$$z\underline{Q}Mx = z,x \quad \text{for } x \text{ in } \dot{f}, \ z \text{ in } \hat{\vartheta}$$

holds. For z in $\dot{\vartheta}$ this reduces to

$$\int_S z\left[-DpDx + qx\right]ds = \int_S \left\{pDz.Dx + qzx\right\}ds$$

and is proved through integration by parts since the boundary terms vanish; it then extends to z in $\hat{\vartheta}$. (cf. Lemma 3.1.)

The space $\dot{f}$ is dense in $\hat{\vartheta}$; this follows from the fact that every function x(s) in $\hat{\vartheta}$ and its derivative can be uniformly approximated by functions in $\dot{f}$; (prove!).

<u>Inversion</u> of the operator [illegible] inverse [illegible] in $\mathfrak{H}_1$ can be defined as follows: [illegible] the collection of all vectors x in $\mathfrak{H}_1$ [illegible] in $\mathfrak{H}_1$. The vector y is [illegible] since q is positive [illegible] the operator [illegible] denoted by [illegible] = [illegible] operator [illegible] [illegible] implies [illegible] Statements (1) and (2) of theorem [illegible] can be written

(1) [illegible] holds for [illegible] in $\mathfrak{H}_1$

(2) [illegible] for all [illegible] in $\mathfrak{H}_1$ implies [illegible]

When we apply [illegible] that the relation [illegible] is equivalent to

u is in [illegible] and

Mu = [illegible] is [illegible]

We [illegible] to the case [illegible] $\mathfrak{H}_1$ [illegible] and q the operator connected with [illegible] shall say that the inverse M is [illegible] space [illegible]

In what follows we denote [illegible] derivatives. We introduce the space [illegible] functions [illegible] with continuous derivatives [illegible] space [illegible] of these functions [illegible] neighborhood of the end points [illegible] operator

M = [illegible] the inverse of q is [illegible] in the [illegible]

[illegible]

holds. For [illegible] this reduces to

$$[illegible]$$

easily proved through [illegible] terms vanish; it then [illegible] in [illegible].

The space [illegible] that every function [illegible] and its [illegible] formly approximated by [illegible].

We are now in a position to prove the

Theorem 4.2 The form Q is positive definite, which was used in Ch. III. Let y be a vector in $\hat{\mathcal{D}}$ with yQy=0; then the above Green's formula gives (y,x)=0 for all x in $\mathcal{F}$; since $\mathcal{F}$ is dense in $\hat{\mathcal{D}}$ this holds for all x in $\hat{\mathcal{D}}$; in particular for x=y; i.e. (y,y)=0 or y=0. This proves Theorem 4.2.

From Theorem 4.1 (2) we see that zQMx=z,x for all z in $\hat{\mathcal{D}}$ implies x = QMx or QM=1 for x in $\mathcal{F}$. This shows that every function x in $\mathcal{F}$ is of the form x=Qy. Therefore $\mathcal{F}$ is a subspace of $\hat{\mathcal{F}}=Q\hat{\mathcal{D}}$ and the differential operator M in $\mathcal{F}$ coincides with the inverse of Q.

The nature of the space $\hat{\mathcal{F}}$ is revealed by the Theorem 4.31. If x in $\hat{\mathcal{D}}$ then x can be represented by function in $\mathcal{L}$.

4.32. If x in $\hat{\mathcal{F}}$ then x in $\mathcal{D}$, Dx in $\mathcal{D}$, Mx in $\mathcal{L}$.

4.33 If x in $\hat{\mathcal{F}}$, Mx in $\hat{\mathcal{F}}$ thn x, Dx, Mx, DMx in $\mathcal{D}$, M^2x in $\mathcal{L}$.

which could be continued.

An E-vector u admits application of M indefinitely since $Mu=\mu u$, $M^2u=\mu^2u$,.... Hence u is a function in $\mathcal{D}$ with derivative Du in $\mathcal{D}$ while the second derivative is in $\mathcal{D}$ due to $Mu=\mu u$. This implies the important result that E-vectors are functions in the original sense and not merely ideal vectors.

For the proof of Theorem 4.31 we may assume p=1 (as in Ch. III). We first observe that continuous functions z which vanish in the neighborhood of the end points and have piecewise continuous derivatives, belong to $\hat{\mathcal{D}}$ because they can be approximated by functions in $\mathcal{D}$.

Let S' be an inner subinterval of S, let s' represent points in S'. Let $\eta(s)$ be a function with continuous derivatives of any order which equals 1 in a neighborhood of S' and equals zero in a neighborhood of the endpoints. Then

$$k(s) = k(s,s') = \frac{1}{2}\eta(s)\,|s-s'|$$

is a function in $\mathfrak{D}$ of the above mentionned kind. The function

$$j = j(s,s') = -r^{-1}(s)\frac{d^2\,k(s,s')}{ds^2}$$

has then continuous derivatives with respect to s' of any order. For x in $\mathfrak{D}$ we obtain

$$(k,x) = \int_S \{Dk\,Dx + qkx\}\,ds = \int_S \{j + r^{-1}qk\}\,xr\,ds + x(s'),\ \text{or}$$

A: $\quad x(s') = (k,x) - (j + r^{-1}qk)Qx$, whence

$|A|$ $\quad |x(s')| \leq \alpha'\|x\|$, where α' depends on S'.

Let now x in $\mathfrak{D}$ be defined by a Cauchy sequence x^σ in $\mathfrak{D}$. The inequality $|A|$ applied to $x^\sigma - x^\tau$ shows that x^σ converges, uniformly in every subinterval S', and hence possesses a continuous limit function $x(s)$. This limit function represents the vector x because 1. it is evidently independent of the defining Cauchy sequence, 2. no two vectors x possess the same function $x(s)$. The latter statement is equivalent to this: no vector $x \neq 0$ has the limit function $x(s) \equiv 0$. This follows from Theorem 4.2 and the identity $xQx = \int_S r(s)x^2(s)\,ds$ for the limit function $x(s)$ of x. To prove this identity, choose to given $\varepsilon > 0$ a σ such that $x^{\sigma\tau}Qx^{\sigma\tau} \leq \varepsilon^2$ for $\tau > \sigma$. From this $\int_{S'} (x^{\sigma\tau})^2 r\,ds \leq \varepsilon^2$ whence $\int_{S'} (x^\sigma - x)^2 r\,ds \leq \varepsilon^2$ or $\left| \left(\int_{S'} rx^2 ds\right)^{1/2} - \left(\int_{S'} r(x^\sigma)^2 ds\right)^{1/2} \right| \leq \varepsilon$. This holds for every subinterval S'; hence $\left| \left(\int_S rx^2 ds\right)^{1/2} - (x^\sigma Qx^\sigma)^{1/2} \right| \leq \varepsilon$. On the other hand $\left| (xQx)^{1/2} - (x^\sigma Qx^\sigma)^{1/2} \right| \leq \varepsilon$; hence $\left| \left(\int_S rx^2 ds\right)^{1/2} - (xQx)^{1/2} \right| \leq 2\varepsilon$. This proves the identity.

We further observe that formula A and inequality $|A|$ remain valid for x in $\mathcal{D}$.

In order to prove 4.32 we first assume that y is in $\mathcal{D}$, ... we derive from A the formula

$$x(s^1) = \int_S kyrds - \int_S [j + r^{-1}qk]\, xrds$$
$$= \int_S k[y - r^{-1}qx]rds - \int_S jxrds.$$

This formula can be differentiated with respect to s^1 and gives

B: $$Dx(s^1) = \frac{1}{2}\int_S \eta(s)\ \mathrm{sgn}(s^1 - s)[y - r^{-1}qx]rds - \int_S \frac{dj}{ds^1}\, xrds.$$ $\eta(s) = 0$ ...

from which

$|B|$: $|D(s^1)| \leq \beta'\{\|y\| + \|x\|\}$, where β' depends on S'.

(Observe that $\left|\frac{dj}{ds^1}\right| = \frac{1}{2}\left|\frac{d^2\eta}{ds^2}\right|$ is independent of s^1).

Now let x be in $\mathcal{F}$; approximate y=Mx by functions y^σ in $\mathcal{D}$, then $x^\sigma = Qy^\sigma$ approximates x=Qy. Upon applying $|B|$ and $|A|$ to $y^{\sigma\tau}$ and $x^{\sigma\tau}$ we have

$$|Dx^{\sigma\tau}(s^1)| \leq \beta'\{\|y^{\sigma\tau}\| + \|x^{\sigma\tau}\|\} \to 0.$$
$$|x^{\sigma\tau}(s^1)| \leq \alpha'\|x^{\sigma\tau}\| \to 0.$$

Hence the sequence $x^\sigma(s)$ possesses a differentiable limit function x(s) in every subinterval and, therfore, in S. That is, the function x(s) representing x in $\mathcal{F}$ is in $\mathcal{D}$. Formula B, which was derived under the assumption Mx in $\mathcal{D}$ then will hold also for x in $\mathcal{F}$. This formula can be differentiated another time with respect to s' and gives

C: $$D^2x(s^1) = [ry - qx]^{s=s^1}, \text{ since } \frac{d^2j}{(ds^1)^2} = 0; \text{ hence}$$

$r^{-1}[-D^2 + q]x = y = Mx$; i.e. x and Dx are in $\mathcal{D}$ and M is the differential operator. That proves Theorem 4.32.

We further observe that ... the inequality [4] ...

... Hence

... This formula ...

Hence the ...

Supplement to Chapter IV p8 and p,8

Assume that form $\underline{Q}$ possess a positive discrete spectrum. Expansion, with respect to E-vectors u^n to E-values κ_n,

$$x = \Sigma_n \alpha_n u^n \quad , \quad \alpha_n = (u^n, x),$$

implies

$$Qx = \Sigma_n \kappa_n \alpha_n u^n.$$

Proof. Let $x_m = \Sigma_{n=1}^m \alpha_n u^n$, then $Qx_m = \Sigma_{n=1}^m \kappa_n \alpha_n u^n$ in view of $Qu^n = \kappa u^n$. Since $\|Qx_m - Qx\| \leq k\|x_m - x\| \to 0$ we have $\Sigma_{n=1}^m \kappa_n \alpha_n u^n \longrightarrow Qx.$

For x in $Q\mathfrak{H}$ we have

$$Mx = \Sigma_n \mu_n \alpha_n u^n \ , \mu_n = \kappa_n^{-1}.$$

Proof. Let $x = Qy$, $y = \Sigma_n \beta_n u^n$, $\beta_n = (u^n, y)$; then, from preceding, $x = Qy = \Sigma_n \kappa_n \beta_n u^n$.

Now $\kappa_n \beta_n = \kappa_n (u^n, y) = (Qu^n, y) = (u^n, Qy) = (u^n, x) = \alpha_n$.

Hence $\beta_n = \mu_n \alpha_n$.

Discussion of differential operator M for examples.

<u>Example 1.</u> $0 \leq s \leq 1$, $p=1, q=1, r=1$.

$M = -D^2 + 1$,

E-equation $D^2u + (\mu - 1)u = 0$ has solutions

$u = c \sin (\tau s + \varphi)$, where $\mu = 1 + \tau^2$.

For which values of τ and φ will u be in $\hat{\mathcal{D}}$?

<u>Lemma 1.</u> If $x(s)$ in ϑ is in $\hat{\vartheta}$ then $x(0) = x(1) = 0$.

<u>Proof</u>. For $x(s)$ in ϑ we have $|x(s)|^2 = \left| x(s_2) + \int_{s_2}^{s_1} Dx\,ds \right|^2$,

(✱) $|x(s_1)|^2 \le 2|x(s_2)|^2 + 2|s_2 - s_1| \int_{s_1}^{s_2} Dx^2\,ds.$

Set $s_1 = 0$, $|s_2 - s_1| \le 1$ and integrate with respect to s_2.

$|x(0)|^2 \le 2\int_0^1 x^2 ds + 2\int_0^1 Dx^2 ds$ or $|x(0)| \le \|x\|$,

in the same way $|x(1)| \le \|x\|$. Application to $x - \dot{x}$, where x in $\hat{\vartheta}$, $\dot{x}$ in ϑ gives $|x(0)| \le \|x - \dot{x}\|$, $|x(1)| \le \|x - \dot{x}\|$.

Since $\|x - \dot{x}\|$ can be made arbitrarily small, lemma 1 is proved.

<u>Lemma 2.</u> If $x(0) = x(1) = 0$ for x in ϑ then x in $\hat{\vartheta}$.

<u>Proof.</u> From (✱) in proof of lemma 1 for $s_1 = \varepsilon$, $s_2 = 0$ and $s_1 = 1-\varepsilon$, $s_2 = 1$ we have $\left\{x^2(\varepsilon) + x^2(1-\varepsilon)\right\} \le 2\varepsilon \left(\int_0^{\varepsilon} + \int_{1-\varepsilon}^1\right) Dx^2 ds.$

Now replace $x(s)$ by a function $\dot{x}_\varepsilon(s)$ in $\dot{\vartheta}$ obtained from rounding off the function:

$$
\begin{aligned}
x_\varepsilon(s) &= x(s) && \text{for } \varepsilon \le s \le 1-\varepsilon \\
&= (2\varepsilon^{-1}s - 1)x(\varepsilon) && \text{for } \varepsilon/2 \le s \le \varepsilon \\
&= (2\varepsilon^{-1}(1-s) - 1)x(1-\varepsilon) && \text{for } 1-\varepsilon \le s \le 1-\varepsilon/2 \\
&= 0 && \text{for } 0 \le s \le \varepsilon/2,\ 1-\varepsilon/2 \le s \le 1.
\end{aligned}
$$

$$
\begin{aligned}
\text{Then } \|x - x_\varepsilon\|^2 &\le 2\left(\int_0^{\varepsilon} + \int_{1-\varepsilon}^1\right)\left\{Dx^2 + x^2\right\}ds + 2\varepsilon^{-1}(1 + \varepsilon^2/6)\left\{x^2(\varepsilon) + x^2(1-\varepsilon)\right\} \\
&\le (6 + \varepsilon^2/3)\left(\int_0^{\varepsilon} + \int_{1-\varepsilon}^1\right)\left\{Dx^2 + x^2\right\}ds \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.
\end{aligned}
$$

Lemmas 1 and 2 show that those and only those functions $u = c\sin(\tau s + \varphi)$ are E-functions which vanish for $s = 0$,

$s = 1$; i.e., $u_n(s) = c_n \sin n\pi s$ are E-functions, where c_n is determined such that $\|u_n\| = 1$; $\mu_n = 1 + n^2\pi^2$ are the E-values of M. Since the form Q of this example possesses a positive-discrete spectrum, the expansion theorems of Ch. III and Ch. IV supplement yield for every $x(s)$ in $\hat{\vartheta}$ $x(s) = \Sigma_n \alpha_n c_n \sin n\pi s$,

$$\int_0^1 x^2 ds = \Sigma_n K_n \alpha_n^2 , \qquad \int_0^1 \{Dx^2 + x^2\} ds = \Sigma_n \alpha_n^2,$$

and, if x is in $\hat{f}$,

$Mx = \Sigma_n \alpha_n c_n \sin n^2\pi^2 s$, or

$-D^2 x = \Sigma_n n^2\pi^2 \alpha_n c_n \sin n^2\pi^2 s.$

<u>Example 3.</u> $0 \leq s \leq 1$, $p = s$, $q = m^2 s^{-1} + s$, $r = s$.

$M = -s^{-1}DsD + m^2 s^{-2} + 1$. The E-equation

$$s^{-1}DsDu - m^2 s^{-2} u + (\mu - 1)u = 0$$

has for solutions the Bessel functions $J_m(\tau s)$, $Y_m(\tau s)$ and their linear combinations. The functions Y_m behave like s^{-m} for $m \neq 0$, like $\log s$ for $m = 0$; hence $\|Y_m\| = \infty$. Therefore, they must be discarded. The argument of Lemma 1 (preceding example) shows that $J_m(\tau s)$ is also to be discarded unless $J_m(\tau) = 0$. i.e., unless $\tau = \tau_{im}$ is a root of the Bessel function J_m. It then must be shown that $J_m(\tau_{im} s)$ is in $\hat{\vartheta}$. For the endpoint $s = 1$ the argument of Lemma 2 is to be employed. For $s = 0$ an even simpler argument is possible since $J_m(\tau s)$ behaves like s^m for $s = 0$.

The case $n=0$, however, needs a special treatment.
We replace $u=J_0(\tau s)$ by

$$u_\varepsilon(s) = u(s) \qquad \text{for } s\geq\varepsilon$$
$$= (\log\varepsilon^{-1})^{-1}(\log\varepsilon^{-2}s)u(\varepsilon) \qquad \text{for } \varepsilon^2\leq s\leq\varepsilon$$
$$=0 \qquad \text{for } 0<s\leq\varepsilon^2.$$

Then $\int_0^n sD(u-u_\varepsilon)^2 ds \leq 2\int_0^\varepsilon sDu^2 ds + 2\int_{\varepsilon^2}^\varepsilon sDu_\varepsilon^2 ds$

$= 2\int_0^\varepsilon sDu^2 ds + (\log\varepsilon^{-1})^{-1}u^2(\varepsilon) \longrightarrow 0$ since $u(0)=1$.

$\int_0^1 s(u-u_\varepsilon)^2 ds \leq 2\int_0^\varepsilon su^2 ds + 2\int_0^\varepsilon u_\varepsilon^2 ds \rightarrow 0.$

This indicates that it is possible to approximate $J_0(\tau s)$ by functions in $\mathring{\mathcal{D}}$ if $J_0(\tau)=0$.

Since in this example the spectrum is positive discrete, expansions with respect to Bessel-functions are established.

In example 2 we have, $-1\leq s\leq 1$, $p=(1-s^2)$, $q=r=1$.
$M=-D(1-s^2)D+1$; the E-value equation is
$D(1-s^2)Du + (\mu-1)u=0$. It can be shown (cf. e.g. Courant-Hilbert Vol. I Ch. V§ 10) that the solutions of this equation are either Legendre polynomials or behave like $\log(1-s)$ at $s=1$, or like $\log(1+s)$ at $s=1$. For the latter functions $\int_{-1}^1 (1-s^2)Du^2 ds = \infty$; hence they are excluded. That the Legendre polynomials are in $\mathring{\mathcal{D}}$ can be shown by the same reasoning that was applied to the Bessel function J_0 in connection with example 3. The legendre functions thus constitute the E-functions. Since the spectrum is positive discrete in this case, expansion theorems are proved.

CHAPTER V. Spectral Resolution of Operators.

$\mathfrak{H}$: Hilbert space, Q bounded symmetric linear operator.

$\|Qx\| \le k\|x\|$, $(x,Qy) = (Qx,y)$.

We are seeking a generalization of the concept of E-vector. Such a generalization is that of E-vectors belonging to an E-interval. Let Δk be a κ-interval, we consider intervals with and without end points, also a single point is an interval.

In case the discrete spectrum is complete, we say, the E-space $\Delta\mathfrak{H}$ to the E-interval Δk is the subspace spanned by the E-vectors to E-value k in Δk; E vectors to Δk are vectors in $\Delta\mathfrak{H}$. We observe that $\Delta_1\mathfrak{H} \perp \Delta_2\mathfrak{H}$ if $\Delta_1 k \cdot \Delta_2 k$ is void. Further $(u,Qu)(u,u)$ is in Δk for u in $\Delta\mathfrak{H}$. Our aim is to obtain a characterization of E-spaces $\Delta\mathfrak{H}$ without reference to E-values. This can be done by means of the concept of <u>functions of an operator</u>.

First the operator Q^n can be formed from Q , $n=0,1,2,\ldots$. Further, if $p(\kappa)$ is a polynomial $p(\kappa)=a+a_1\kappa+ \ldots +a_n\kappa^n$, we can define $p(Q)=a+a_1Q+ \ldots +a_nQ^n$. We shall see that such functions of the operator Q can also be defined for larger classes of functions $f(\kappa)$.

In <u>case the complete spectrum is complete</u>, the definition of $f(Q)$ can be based on the spectral representation. $x=\sum_n \alpha_n u^n$ and $Qx = \sum_n \kappa_n \alpha_n u^n$. We see immediately that for polynomial $p(\kappa)$, we have $p(Q)x = \sum_n p(\kappa_n)\alpha_n u^n$. Hence we may define for any bounded function $f(\kappa)$: $f(Q)x = \sum_n f(\kappa_n)\alpha_n u^n$.

We have a mapping of the functions $f(K)$ to operators, retaining all functional relations; in particular

$$f(Q)g(Q) = fg(Q) \quad \text{if } f(K)g(K) = fg(K).$$

i.e. multiplication corresponds to successive application.

Consider the function $f_*(K)=1$ for $K = K_*$, $= 0$ for $K \neq K_*$; then $f_*(Q)x = \Sigma_{K_n = K_*} \alpha_n u^n$; i.e. $f_*(Q)x$ is the projection of x into the E-subspace $\mathfrak{H}_*$ to K_*. E-vectors to K_* thus can be characterized as the vectors of the form $f_*(Q)x$.

Let ΔK be an E-interval; consider the function $f_\Delta(K)=1$ in ΔK, $=0$ outside. Then $f_\Delta(Q)x = \Sigma_{K_n \text{ in } \Delta K} \alpha_n u^n$; i.e. $f_\Delta(Q)x$ is the projection of x into the E-subspace $\Delta\mathfrak{H}$ to ΔK, and the vectors in $\Delta\mathfrak{H}$ can be characterized as those of the form $f_\Delta(Q)x$.

By "spectral resolution" we mean from now on, to obtain these operators $f_\Delta(Q)$.

It was the important discovery of F. Riesz that for <u>any</u> symmetric bounded operator Q, a definition of f(Q) can be given without reference to spectral representation, and that a spectral resolution can be obtained from these operator functions

We have a mapping of the [illegible]

operators, [illegible] In par-

ticular

[illegible]

i.e. multiplicative [illegible]

[illegible]

then [illegible]

[illegible]

In what follows functions $f(\kappa)$ are supposed to be defined in $|\kappa| \leq k$ and all statements refer to this interval. Let $f(\kappa)$ represent functions of a class containing sum and product. We say, the linear symmetric operator $f(\eta)$ is "properly" defined for this class if the following statements hold.

A If $f(\kappa) = f_1(\kappa) + f_2(\kappa)$ then $f(\eta) = f_1(\eta) + f_2(\eta)$

B If $f(\kappa) = f_1(\kappa) \cdot f_2(\kappa)$ then $f(\eta) = f_1(\eta) \cdot f_2(\eta)$

where in the last formula the $\cdot$ represents successive application.

C If $f_1(\kappa) \geq f_2(\kappa)$ then $(x, f_1(\eta)x) \geq (x, f_2(\eta)x)$.

For the class of polynomials $p(\kappa)$ the operator $p(\eta)$ was defined; $p(\eta)$ is obviously linear and symmetric.

We have

Theorem 5.1 The definition of the operators $p(\eta)$ for polynomials $p(\eta)$ is proper.

Properties A, B are obvious. It is property C on which the theory of F. Riesz is essentially based. To prove C we refer to a purely algebraic

Lemma 5.1 If the polynomial $p(\kappa) \geq 0$ for $|\kappa| \leq k$, it can be written as the sum of polynomials of the form $g(\kappa)j^2(\kappa)$ where $j(\kappa)$ is

any polynomial and $g(\kappa)$ is either 1, or $k-\kappa$, or $k+\kappa$, or $k^2-\kappa^2$.

Proof: Problem.

From this lemma we derive property C for polynomials $p(\kappa)$ immediately. Since $p_1(\kappa)-p_2(\kappa)$ is a polynomial it is sufficient to assume $p_1(\kappa) = p(\kappa)$, $p_2(\kappa) = 0$. We set $j(Q)x=y$.

Then 1. $(x, j^2(Q)x) = (y,y) \geq 0$

2. $(x,(k\pm Q)j^2(Q)x) = (y,(k\pm Q)y) = k(y,y) \pm (y,Qy) \geq 0$

because of $|y,Qy| \leq \|y\| \, \|Qy\| \leq k\|y\|^2$.

3. $(x,(k^2-Q^2)j^2(Q)x) = k^2(y,y) - (Qy,Qy) \geq 0$.

Hence, in view of the lemma, $p(\kappa) \geq 0$ implies $(x,p(Q)x) \geq 0$ and theorem 5.1 is proved.

Now let $f(\kappa)$ be a continuous function (for $|\kappa| \leq k$). Approximate it, according to Weierstrass, by a sequence of polynomials $p_n(\kappa)$, uniformly (in $|\kappa| \leq k$). Set $p_{mn}(\kappa)=p_m(\kappa)-p_n(\kappa)$. To every $\varepsilon > 0$ there is a $N(\varepsilon)$ such that $|p_{mn}(\kappa)| \leq \varepsilon$ (in $|\kappa| \leq k$). Hence $p_{mn}^2(\kappa) \leq \varepsilon^2$, and, in view of C, $(x, p_{mn}^2(Q)x) \leq \varepsilon^2(x,x)$ or

$\|p_{mn}(Q)x\| \leq \varepsilon\|x\|$

Consequently, the vectors $p_n(Q)x$ form for every x a Cauchy sequence and, since the space is complete, converge to a limit vector, which we denote by $f(Q)x$.

That $f(Q)$ is independent of the choice of the approximating polynomials follows from the fact that for two such sequences

$p_n^{(1)}$, $p_n^{(2)}$, one has

$$\|(p_n^{(1)}(Q) - p_n^{(2)}(Q))x\| \leq \text{Max}\, |p_n^{(1)}(\kappa) - p_n^{(2)}(\kappa)| \, \|x\| \longrightarrow 0.$$

any polynomial and $r(\lambda)$ be either 1, or [illegible], or [illegible] .

Proof: Problem.

From this lemma we derive [illegible] for polynomials $p(\lambda)$ immediately. Since [illegible] it is sufficient to assume [illegible]

Then [illegible]

[illegible]

[illegible]

[illegible]

Hence, in view of the lemma, [illegible] and theorem [illegible] is proved.

Now let $p(\lambda)$ be a [illegible]

Approximate it, [illegible] by a sequence of polynomials $p_n(\lambda)$, uniformly [illegible]

To every $\varepsilon > 0$ there is a [illegible]

Hence [illegible] and, in [illegible]

$\| p_m(\,) x \| \leq c \| x \|$

Consequently, the vectors [illegible] sequence and, since [illegible] vector, which is denoted by [illegible]

That [illegible] polynomials follows [illegible]

$p_n^{(1)}, p_n^{(2)}$, [illegible]

$\| (p_n^{(1)} \ldots$ [illegible]

One shows easily that $f(Q)x$ depends linearly on x; i.e. that $f(Q)$ is a linear operator, and also that $f(Q)$ is symmetric, since the same holds for the approximating polyninials. Further

<u>Theorem</u> 5.2 The operators $f(Q)$ for continuous $f(\kappa)$ are properly defined.

The properties A and C follow immediately from the corresponding properties for $p_n(Q)$.

However, we want to define operators $f(Q)$ also for functions $f(\kappa)$ which are only piecewise continuous. What we really need is that $f(\kappa)$ is bounded and can be approximated by a non-decreasing sequence of continuous functions, i.e.

$$f_1(\kappa) \leq f_2(\kappa) \leq \cdots \longrightarrow f(\kappa) \leq F = \text{const};$$

we call such functions lower semi-continuous. From theorem 5.2C we have $(x, f_1(Q)x) \leq (x, f_2(Q)x) \leq \cdots \leq F\,(x,x)$. The latter sequence, therefore, converges to a limit value; hence $0 \leq (x, f_{mn}(Q)x) \longrightarrow 0$ for $m \geq n \longrightarrow \infty$. On applying the Schwarz inequality to the non-negative form $\underline{f_{mn}(Q)}$, we obtain

$$(f_{mn}(Q)x, f_{mn}(Q)x)^2 \leq (x, f_{mn}(Q)x)(x, f_{mn}^3(Q)x)$$
$$\leq (x, f_{mn}(Q)x)\; F^3(x,x) \longrightarrow 0\,,$$

since $f_{mn}{}^3(\kappa) \leq f_m^3(\kappa) \leq F^3$.

Hence $f_n(Q)x$ is a Cauchy sequence and converges to a limit function which we denote by $f(Q)x$; $f(Q)x$ depends linearly on x and one proves easily

<u>Theorem</u> 5.3 The operators $f(Q)$ are properly defined for lower-semi-continuous $f(K)$.

Evidently $f(Q)$ is also properly defined for differences of lower-semi-continuous functions, and that is the class of functions we want. For a function that is 1 in ΔK, 0 outside is such a difference.

Hence to every interval ΔK an operator $f_\Delta(Q) = \Delta P$ is defined. It has the following properties:

If $\Delta K = \Delta_1 K + \Delta_2 K$ then $\Delta_1 P + \Delta_2 P = \Delta P$,

$\Delta K = \Delta_1 K \cdot \Delta_2 K$ then $\Delta_1 P \; \Delta_2 P = \Delta P$.

This implies in particular: If $\Delta_1 K \cdot \Delta_2 K$ is void, then $\Delta_1 P \; \Delta_2 P = 0$; further $(\Delta P)^2 = \Delta P$.

If $F_1 \leq f(K) \leq F_2$ for K in ΔK then

$$F_1 \leq (f(Q)\Delta Px, x)/(\Delta Px, x) \leq F_2.$$

This holds in particular for $f(K) = K$ and gives

$$(Q\Delta Px, x)/(\Delta Px, x) \text{ is in } \Delta K.$$

To understand the significance of these properties we introduce the space $\Delta\mathfrak{H}$ of all u of the form $u = \Delta Px$. Since $f_\Delta{}^2 = f_\Delta$ we have $\Delta P^2 = \Delta P$,

$(\Delta Py, (1-\Delta P)x) = (y, (\Delta P - \Delta P^2)x) = 0$; i.e.

$(1-\Delta P)x \perp \Delta \mathfrak{h}$. Hence x can be split up into

$x = \Delta Px + (1-\Delta P)x$, where ΔPx is in $\Delta \mathfrak{h}$ and

$(1-\Delta P)x \perp \Delta \mathfrak{h}$. This shows:

The operator ΔP is the projection into $\Delta \mathfrak{h}$.

Let the interval $|K| \leq k$ be split into a sum of intervals $\Delta_\mu K$ without common points; then we have

$$\Sigma_\mu \Delta_\mu P = 1$$

which means that every vector x can be split up into a sum of E-vectors

$$\Sigma_\mu \Delta_\mu Px = x.$$

Let K_μ be any value of K in $\Delta_\mu K$. Then the operator

$\Sigma_\mu K_\mu \Delta_\mu P$ is approximately Q and

$\Sigma_\mu f(K_\mu)\Delta_\mu P$ is approximately f(Q) for continuous f. The sums on the left hand side really converge to Q or f(Q) respectively if the subdivision is refined.

Consider two subdivisions Δ_μ and Δ'_μ. Then, in view of 1. and 2.,

$$\| \Sigma_\mu f(K_\mu)\Delta_\mu Px - \Sigma_\nu f(K'_\nu)\Delta'_\nu Px\|^2$$
$$= \Sigma_{\mu,\nu} |f(K_\mu) - f(K'_\nu)|^2 \| \Delta_\mu P \Delta'_\nu Px\|^2$$
$$\leq \varepsilon^2 \Sigma_{\mu,\nu} \| \Delta_\mu P \Delta'_\nu Px\|^2 = \varepsilon^2 \|x\|^2,$$

where $\varepsilon = \max |f(K_\mu) - f(K'_\nu)|$ for all μ, ν such that $\Delta_\mu K$ and $\Delta'_\nu K$ have a common point; and ε can be made arbitrarily small.

It is natural to express those facts symbollically by

$$\int dP = 1$$

$$\int K\, dP = Q$$

$$\int f(K)dP = f(Q), \text{ (containing the preceding ones)},$$

which gives a pleasant form of the spectral resolution. In case of a complete discrete spectrum those integrals degenerate into sums.

We are now called to exemplify the preceding results with at least one example. Take the space $\mathcal{L}$ of functions $x(K)$ defined in S, introduce a unit-form

$$(x,x) = \int_S x^2(K)\, r(K)\, dK ,$$

and close it off to a space $\hat{\mathcal{L}}$. In $\mathcal{L}$ introduce the operator Q by

$$Qx(K) = Kx(K).$$

Q is bounded if S is bounded, which we shall assume although that is not essential. Then $\mathcal{L}$ can be extended to $\mathcal{L}$. The functions $x(K)$ in what follows are assumed to be in $\mathcal{L}$. The

results then extend immediately to $x(\kappa)$ in $\mathcal{L}$.

For polynomials $p(\kappa)$ we have

$$p(Q)x(\kappa) = p(\kappa)x(\kappa).$$

For continuous functions $f(\kappa)$ we have

$$f(Q)x(\kappa) = f(\kappa)x(\kappa).$$

For the functions $f_\Delta(\kappa) = 1$ in $\Delta\kappa$, $= 0$ outside, we have $f_\Delta(Q)x(\kappa) = f_\Delta(\kappa)x(\kappa)$, which function may be only piecewise continuous and belongs to $\mathcal{L}$ as such. Hence the projection $f_\Delta(Q) = \Delta P$ is this: from $x(\kappa)$ it cuts off the part outside $\Delta\kappa$. The E-functions of the interval $\Delta\kappa$ are the functions vanishing outside $\Delta\kappa$.

The above example permits various generalizations. Instead of $r(\kappa)d\kappa$ we introduce a non-decreasing function $\rho(\kappa)$ and replace $r(\kappa)d\kappa$ by $d\rho$; i.e., we set

$$(x,x) = \int_S x^2(\kappa)d\rho(\kappa).$$

We then even admit discontinuous functions $\rho(\kappa)$. What this implies may be seen in the case that $\rho(\kappa)$ is piecewise constant with jumps $d\rho_1$, $d\rho_2$,... at $\kappa = \kappa_1, \kappa_2, \ldots$.

We have then

$$(x,x) = \sum_\mu x_\mu^2 d\rho_\mu \quad , \quad \text{where } x_\mu = x(\kappa_\mu),$$

$$(x,Qx) = \sum_\mu \kappa_\mu x_\mu^2 d\rho_\mu .$$

That is, we come back to the case of a pure point spectrum.

If $\rho(K)$ has discontinuities but is not constant in between we have a combination of continuous and point spectrum.

We may generalize in the following way. We consider as element of the Hilbert space a system of functions $x = \{x_1(K), x_2(K), \ldots\}$, and take for unit form

$$(x,x) = \int_S x_1{}^2(K)\,d\rho_1(K) + \int_S x_2{}^2(K)\,d\rho_2(K) + \ldots .$$

Then

$$Qx = \{Kx_1(K), Kx_2(K), \ldots\} \quad \text{and}$$

$$(x,Qx) = \int_S Kx_1{}^2(K)\,dK + \ldots .$$

It can be shown that the latter is the more general case in the following sense:

<u>The Hilbert space can be represented by the space of systems of functions</u> $\{x_\mu(K)\}$ <u>such that</u> Qx <u>is represented by</u> $\{Kx_\mu(K)\}$ <u>and</u> $(x,x) = \sum_\mu \int_S x_\mu{}^2(K)\,d\rho_\mu(K)$. This statement implies the general theory of <u>spectral representation</u>.

It is not difficult to indicate how this representation can be achieved. Take any vector x_1 and all vectors $f_1(Q)x_1$; close this subspace off to a space $\mathfrak{h}_1$; we set

$\Delta\rho_1(K) = (x_1, \Delta P x_1)$ and find

[illegible]

If $\varphi(K)$ are [illegible]

[illegible]

elements of [illegible]

[illegible]

then

[illegible]

[illegible]

$$(x,x) = \Sigma_\mu (x_1, \Delta_\mu P x_1) \quad \text{and}$$

$$(f_1(Q)x_1, f_1(Q)x_1) = \lim \Sigma_\mu f_1^{\,2}(K_\mu)(x, \Delta_\mu P x_1)$$

which can be shown to lead to

$$(f_1(Q)x_1, f_1(Q)x_1) = \int f_1^{\,2}(K)\, d\rho_1(K).$$

Hence for every x in $\mathfrak{h}_1$, represented by $f_1(K)$, we have

$$(x,x) = \int f_1^{\,2}(K)\, d\rho_1(K).$$

We now take the space perpendicular to $\mathfrak{h}_1$, peroform the same operation and obtain a space $\mathfrak{h}_2$. Continuing, we obtain a sequence of spaces $\mathfrak{h}_1$, $\mathfrak{h}_2, \ldots$ which span the whole space $\mathfrak{h}$, as can be shown (if $\mathfrak{h}$ has countable dimension). Since every x in $\mathfrak{h}$ can be split into $x = x_1 + x_2 + \ldots$, with x_1 in $\mathfrak{h}_1$, x_2 in $\mathfrak{h}_2$ and so on, this leads to the above spectral representation with

$$x_\mu(K) = f_\mu(K).$$