RICE UNIVERSITY

AN ELEMENTARY APPROACH TO

BOUNDED SYMMETRIC DOMAINS

Max Koecher

HOUSTON, TEXAS
1969

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	-11	$\mathfrak{P}$	s
6	15	obtain	o-
			tions
3	- 2	$[v, g_p^2]$	ne
8	- 3	$\Xi(\mathfrak{D})$	
6	- 1	$W^{\#} \in \Gamma(\mathfrak{D})$	ven.
	- 2	$\Theta\nabla_W\Theta$	
8	- 1	σ	
7	- 2	$\Theta b \in \mathfrak{J}$	ailed
7	1	$(T_{u,v})^* = T_{u^t,v^t}$.	ned.
7	5	maximal compact	
5	- 1	$\Xi(\mathfrak{D})$	l
	- 2	$a \in V$	mains
7	- 1	σ	ntary.
3	- 1	$\sigma(c,\bar{c})$	
1	11	[2]	
23	-9 thru -5	det $Q_c \neq 0$ for $c \in Z$.	le

But Q_c is hermitian and $Q_o = I > 0$. Since Z is connected we end up with $Q_c > 0$ for $c \in Z$. So the first inclusion is proved.

PREFACE

These notes contain the material of lectures given at Rice University, Houston, Texas, during two months in the spring of 1969.

In the first two chapters finite dimensional subalgebras $\mathfrak{Q}$ of the Lie algebra Rat V of rational functions on a vector space V are considered. In particular the group of automorphisms of $\mathfrak{Q}$ is investigated and a connection with groups of birational functions is given. The algebraic construction generalizes to arbitrary fields (of characteristic $\neq 2$ and 3) the groups of biholomorphic mappings of bounded symmetric domains, and thereby generalizes the domains themselves. Detailed information about these algebraic groups, their Lie algebras, and the associated Killing forms, is obtained.

The reader should compare the results with the examples given in chapter I, §5, and in chapter III.

In chapter IV and V the algebraic method is used for an explicit construction of bounded symmetric domains which covers all domains of this type.

The methods used in these notes are quite elementary. For completeness the proofs of well-known results on linear Lie groups are included.

I wish to express my thanks to my friend H. L. Resnikoff for his continuous interest and his valuable discussions and suggestions. I am also grateful for Nancy Singleton's excellent preparation of the notes.

München, June 15, 1969
M. KOECHER

CONTENTS

Chapter I

LIE ALGEBRAS OF RATIONAL FUNCTIONS

Let K be an infinite field of characteristic different from 2 and 3 and let V be a vector space over K of finite dimension $n > 0$. If R is an extension ring of K, the tensor product $R \otimes V$ (over K) is called the <u>scalar extension</u> of V by R.

§1. The Lie algebra Rat V.

1. Let $\tau_1, \ldots, \tau_n$ be algebraically independent elements of an extension field of K and let $K' := K(\tau_1, \ldots, \tau_n)$ be the field of rational functions in $\tau_1, \ldots, \tau_n$ with coefficients in K. For an arbitrary vector space E over K, denote by E' the scalar extension of E by K'. Choosing a basis $b_1, \ldots, b_n$ of V we obtain the element

$$x = \tau_1 b_1 + \cdots + \tau_n b_n$$

of V'. Let $e_1, \ldots, e_m$ be a basis of the vector space E over K; then the elements f of E' have a unique representation as

$$f = f_1 e_1 + \cdots + f_m e_m, \qquad f_j \in K',$$

and we write $f = f(x)$. We call f a rational function of x, moreover, f is called a polynomial or a homogeneous polynomial of degree r if the f_j's have these respective properties. Writing the f_j's as reduced quotients of polynomials, the least common multiple δ_f of their denominators is uniquely determined (up to a constant factor) and it is called the denominator of f.

Let $\varphi = \varphi(x)$ be a polynomial of K' and let $a = \alpha_1 b_1 + \cdots + \alpha_n b_n$ be an element of a scalar extension of V. Then $\varphi(a)$ is defined by replacing the τ_j's in $\varphi(x)$ by the α_j's. More generally if δ_f is the denominator of a rational function $f \in E'$ and if we call $\text{Dom } f = \{a;\ a \in V,\ \delta_f(a) \neq 0\}$ the domain of f, then f(a) is defined for $a \in \text{Dom } f$. One says that f(a) is obtained from f(x) by the specialization $x \to a$ and one writes $f(a) = f(x)\big|_{x \to a}$.

2. An element $f \in V'$ is called a generic element of V, if $\varphi(f(x)) = 0$, $\varphi \in K'$, implies $\varphi = 0$. Hence f is generic if and only if the coefficients of f with respect to a basis of V are algebraically independent over K. In particular, x is a generic element of V. Finitely many elements of V' are called generically independent if all coefficients with respect to a basis of V are algebraically independent.

Let $g \in E'$, $f \in V'$ and let δ_g be the denominator of g. We say that g and f are composable if $\delta_g(f(x)) \neq 0$, i.e., if we can specialize $x \to f(x)$ in g. If g and f are composable then $g(f(x))$ is again an element of E' which is denoted by $g \circ f$. Denote by I the polynomial $Ix = x$.

Let $\mathbf{P}(V)$ be the set of rational functions $f \in V'$ for which there exists a rational function $\tilde{f} \in V'$ such that f and $\tilde{f}$ as well as $\tilde{f}$ and f are composable and such that $f \circ \tilde{f} = \tilde{f} \circ f = I$ holds. Hence any element f of $\mathbf{P}(V)$ is a generic element of V and therefore, we can specialize $x \to f(x)$ in an arbitrary rational function. Moreover, $\mathbf{P}(V)$ turns out to be a group with respect to the product $(f,g) \to f \circ g$. The elements of $\mathbf{P}(V)$ are called birational functions.

For $u \in V$ and a rational function $g \in E'$ the differential operator Δ_x^u is given by

$$\Delta_x^u\, g(x) := \frac{d}{d\tau}\, g(x + \tau u)\Big|_{\tau \to 0} .$$

The map $u \to \Delta_x^u\, g(x)$ of V into E' is linear, hence it can be extended to an arbitrary scalar extension of V.

Furthermore, let $f \in V'$ and suppose that g and f are composable. Then we have the chain rule

$$\Delta_x^u (g \circ f)(x) = \Delta_{f(x)}^w\, g(f(x)) \qquad \text{where } w := \Delta_x^u f(x).$$

Each $f \in V'$ induces an endomorphism $\frac{\partial f(x)}{\partial x}$ of V' via

$$\frac{\partial f(x)}{\partial x}\, u := \Delta_x^u\, f(x).$$

If in addition g belongs to V', then the chain rule becomes

$$\frac{\partial (g \circ f)(x)}{\partial x} = \frac{\partial g(f(x))}{\partial f(x)} \frac{\partial f(x)}{\partial x} .$$

3. For rational functions $h,k \in V'$, we define a product $h \bullet k$ by

$$(h \bullet k)(x) := \Delta_x^{k(x)} h(x) = \frac{\partial h(x)}{\partial x} k(x).$$

The map $(h,k) \to h \bullet k$ of $V' \times V'$ into V' is K-bilinear. Hence V' as a vector space over K together with the product $h \cdot k$ becomes a (non-associative) algebra. Using the associator $(h,k,l) = (h \bullet k) \bullet l - h \bullet (k \bullet l)$, we obtain

$$\begin{aligned}(h,k,l)(x) &= \Delta_x^{l(x)} (h \bullet k)(x) - \Delta_x^{(k \bullet l)(x)} h(x) \\ &= \Delta_x^{l(x)} \frac{\partial h(x)}{\partial x} k(x) - \frac{\partial h(x)}{\partial x} \Delta_x^{l(x)} k(x) \\ &= \Delta_x^{u} \Delta_x^{v} h(x),\end{aligned}$$

where after the differentiation we put $u = l(x)$ and $v = k(x)$. Since the last term is symmetric in u and v, we get

(1.1) $\qquad (h,k,l) = (h,l,k) \quad$ for $h,k,l \in V'$.

Algebras satisfying this condition are called <u>right symmetric</u>.

Denote by Rat V the algebra over K with vector space V' and the product $[h,k] := h \bullet k - k \bullet h$. Obviously Rat V is anti-commutative. The identity

$$[[h,k],l] + [[k,l],h] + [[l,h],k]$$

$$= (h,k,l)+(k,l,h)+(l,h,k)-(k,h,l)-(l,k,h)-(h,l,k)$$

shows _that_ Rat V _is_ _a_ _Lie_ _algebra_ _over_ K.

4. Denote by Pol V the subset of Rat V of all polynomials in x. Hence Pol V is a subalgebra of Rat V. Let $\mathfrak{P}_r$, r = 0,1,2,..., be the subspace of Pol V of all homogeneous polynomials of degree r; then we get a direct sum decomposition

$$\text{Pol } V = \bigoplus_{r\geq o} \mathfrak{P}_r \ .$$

Setting $\mathfrak{P}_{-1} = 0$ we obtain

$$(1.2) \qquad [\mathfrak{P}_r, \mathfrak{P}_s] \subset \mathfrak{P}_{r+s-1} \quad \text{for} \quad r,s = 0,1,2,\ldots \ .$$

From (1.2) it follows that $\mathfrak{P}_o = V$, $\mathfrak{P}_1$ and $\mathfrak{P}_o + \mathfrak{P}_1$ are subalgebras of Pol V. But

$$\mathfrak{P} := \mathfrak{P}_o + \mathfrak{P}_1 + \mathfrak{P}_2$$

is not closed under the Lie product.

$\mathfrak{P}_1$ contains the function Ix = x. For $h \in \mathfrak{P}_r$ the Euler differential equation $\Delta_x^x\, h(x) = r\, h(x)$ shows that $[h,I](x) = \Delta_x^x\, h(x) - h(x) = (r-1)h(x)$. Hence

$$(1.3) \qquad [h,I] = (r-1)h \qquad \text{for } h \in \mathfrak{P}_r.$$

Let $h = h_o + h_1 + \ldots,\ h_r \in \mathfrak{P}_r$, be an element of Pol V that commutes with I and all $a \in \mathfrak{P}_o = V$. We obtain

$$(r-1)h_r = 0 \text{ and } \Delta_x^a\, h_r(x) = 0 \text{ for } a \in V.$$

Because of the linearity we can replace a by x and obtain $(r-1)h_r = rh_r = 0$ and consequently $h_r = 0$ for all r. Hence only 0 commutes with I and all $a \in V$. In particular, any subalgebra of Pol V that contains I and all constant polynomials has center 0.

Denote by End V the ring of endomorphisms of the vector space V. Since an element T of End V can be extended to an endomorphism of any scalar extension of V, the linear function Tx belongs to $\mathfrak{P}_1$. Conversely each element of $\mathfrak{P}_1$ has this form. In keeping with the notation $f = f(x)$ for $f \in$ Rat V, we also write T instead of Tx and I instead of $Ix = x$. From the context it will be clear whether we mean the endomorphism or the corresponding linear function. Calculating $[T,S](x) = (TS-ST)x$ for $S,T \in \mathfrak{P}_1$ we see that the product in the subalgebra $\mathfrak{P}_1$ of Pol V corresponds with the commutator product of the endomorphisms.

Without proof we mention that Pol V is a simple algebra if and only if the ground field has characteristic zero.

Let L be an extension field of K and suppose that x is generic over L. Then for the scalar extension we have

$$L \underset{K}{\otimes} \text{Rat } V \subsetneq \text{Rat}(L \underset{K}{\otimes} V) \text{ and } L \underset{K}{\otimes} \text{Pol } V = \text{Pol}(L \underset{K}{\otimes} V).$$

5. Denote by $\mathbb{P}_o(V)$ the set of $f \in$ Rat V for which the scalar rational function

$$\det \frac{\partial f(x)}{\partial x}$$

is not the zero function. For $h \in$ Rat V and $f \in \mathbb{P}_o(V)$

we define a function h^f by

(1.4) $$h^f(x) := \left(\frac{\partial f(x)}{\partial x}\right)^{-1} h(f(x)),$$

provided h and f are composable. Obviously h^f belongs to Rat V. In the notation of 3 we have

(1.5) $$f\cdot h^f = h\circ f.$$

Suppose that h and f as well as k and f are composable; then [h,k] and f are composable too, and we get

(1.6) $$[h^f,k^f] = [h,k]^f.$$

For the proof we use (1.5) and $(h\circ f)\cdot k^f = (h\cdot k)\circ f$ in the following calculation:

$$\begin{aligned}(f,h^f,k^f) &= (f\cdot h^f)\cdot k^f - f\cdot(h^f\cdot k^f)\\ &= (h\circ f)\cdot k^f - f\cdot(h^f\cdot k^f) = (h\cdot k)\circ f - f\cdot(h^f\cdot k^f)\\ &= f\cdot(h\cdot k)^f - f\cdot(h^f\cdot k^f).\end{aligned}$$

Formula (1.1) shows that the left side is symmetric in h and k, hence we get $f\cdot[h,k]^f = f\cdot[h^f,k^f]$ and (1.6) is proved.

Each $h \in$ Pol V is composable with each $f \in \mathbb{P}_o(V)$. From (1.6) it follows <u>that</u> $h \to h^f$ <u>is a homomorphism of the Lie algebra</u> Pol V <u>into</u> Rat V.

The group $\mathbb{P}(V)$ of birational functions is a subset of $\mathbb{P}_o(V)$. Hence h^f is defined for $h \in$ Rat V and $f \in \mathbb{P}(V)$. The chain rule implies $h^{f\circ g} = (h^f)^g$ for $f,g \in \mathbb{P}(V)$. Hence $h \to h^f$ is a linear bijection of

Rat V onto itself. Moreover $h^f = h$ for all $h \in$ Rat V implies $f = I$, therefore $\mathbb{P}(V)$ acts effectively on Rat V. Again from (1.6) it follows that $h \to h^f$ is an automorphism of the Lie Algebra Rat V for each $f \in \mathbb{P}(V)$.

Setting $\nabla_f(h) := h^{f^{-1}}$ we obtain an automorphism ∇_f of Rat V. Furthermore the map

$$(1.7) \qquad \nabla : \mathbb{P}(V) \to \text{Aut Rat } V, \quad f \to \nabla_f,$$

is a monomorphism of the group $\mathbb{P}(V)$ into the automorphism group of Rat V.

6. We now construct two special types of automorphisms of Rat V. Denote by GL(V) the group of bijective endomorphisms of V. For $W \in GL(V)$ we have the linear function Wx that is birational. Hence GL(V) can be considered as a subgroup of $\mathbb{P}(V)$. We get

$$(1.8) \qquad (\nabla_W h)(x) = (h^{W^{-1}})(x) = W\, h(W^{-1}x), \quad W \in GL(V).$$

For $b \in V$ we define the polynomial t_b by

$$t_b(x) = x + b.$$

From $t_b \circ t_c = t_{b+c}$ it follows that t_b belongs to $\mathbb{P}(V)$ and that $(t_b)^{-1} = t_{-b}$ holds. From the definitions it follows that

$$(1.9) \qquad W \circ t_b = t_{Wb} \circ W \quad \text{for } W \in GL(V),\ b \in V.$$

As an abbreviation set

$$(1.10) \quad \Psi_b := \nabla_{t_b}, \text{ hence } (\Psi_b h)(x) = (h^{t_{-b}})(x) = h(x-b).$$

Again Ψ_b is an automorphism of Rat V. Formula (1.9) yields

$$(1.11) \quad \nabla_W \Psi_b = \Psi_{Wb} \nabla_W, \quad W \in GL(V),\ b \in V,$$

and we have

$$(1.12) \quad \Psi_b \Psi_c = \Psi_{b+c} \quad \text{for } b,c \in V.$$

§2. Binary Lie algebras.

1. Suppose that the ground field K has a characteristic different from 2 and 3. The elements of $\mathfrak{P} = \mathfrak{P}_0 + \mathfrak{P}_1 + \mathfrak{P}_2$ are written as

$$q = a + T + p, \quad a \in V,\ T \in \mathfrak{P}_1,\ p \in \mathfrak{P}_2.$$

Here p is a homogeneous polynomial of degree 2. Hence there exists a bilinear symmetric mapping $p : V\times V \to V$ and a linear map $a \to S_a^p$ of V into End V such that

$$(2.1) \quad p(x) = p(x,x),\ \Delta_x^a\, p(x) = 2p(x,a),\ p(x,a) = S_a^p\, x,$$

holds for $a \in V$.

Let $\mathfrak{T}$ be a subspace of $\mathfrak{P}_1$ and $\tilde{V}$ be a subspace of $\mathfrak{P}_2$ satisfying the following conditions

(B.1) $\mathfrak{T}$ is a subalgebra of $\mathfrak{P}_1$,

(B.2) $[V,\tilde{V}] \subset \mathfrak{T}$,

(B.3) $[\mathfrak{T},\tilde{V}] \subset \tilde{V}$, (B.4) $[\tilde{V},\tilde{V}] = 0$,

(B.5) $I \in \mathfrak{T}$.

Forming the subspace $\mathfrak{Q} = V + \mathfrak{T} + \tilde{V}$ of Pol V and using (1.2) we see, that (B.1) to (B.4) mean that $\mathfrak{Q}$ is a subalgebra of the Lie algebra Pol V. We call $\mathfrak{Q}$ a <u>binary Lie algebra</u> if in addition (B.5) is satisfied. Using (2.1) we get $[p,a](x) = 2\,p(x,a) = 2\,S_a^p\,x$ and hence (B.2) is equivalent to

(B.2′) $S_a^p \in \mathfrak{T}$ for $a \in V$ and $p \in \tilde{V}$.

From §1.4 we know that <u>a binary Lie algebra has center</u> 0. Let L be an extension field of K and suppose that x is generic over L. Then $L\otimes\mathfrak{Q}$ turns out to be a binary Lie algebra of Pol $(L\otimes\mathfrak{Q})$.

2. Let $\mathfrak{Q}$ be a binary Lie algebra and let $\Phi : \mathfrak{Q} \to \text{Rat } V$ be a homomorphism of the Lie algebras. Hence $a \to \Phi a$ is a linear map of V into the vector space Rat V. Therefore there exists a linear transformation $H_\Phi(x)$ of V' that is rational in x such that

(2.2) $$(\Phi a)(x) = H_\Phi(x)a \quad \text{for } a \in V.$$

The homomorphism $\Phi : \mathfrak{Q} \to \text{Rat } V$ is called <u>essential</u> if the determinant of the endomorphism $H_\Phi(x)$ is not the zero function.

<u>THEOREM 2.1</u>. <u>Let</u> $\mathfrak{Q}$ <u>be a binary Lie algebra and let</u> $\Phi : \mathfrak{Q} \to \text{Rat } V$ <u>be a homomorphism of the Lie algebras</u>.

<u>Then Φ is essential if and only if there exists</u> $f \in \mathbf{P}_o(V)$ <u>such that</u> $\Phi q = q^f$ <u>for</u> $q \in \mathfrak{Q}$.

<u>Proof</u>: If there is an $f \in \mathbf{P}_o(V)$ such that $\Phi q = q^f$, then let $q = a \in V$ and in view of (2.2) we have

$$H_\Phi(x) = \left(\frac{\partial f(x)}{\partial x}\right)^{-1} .$$

Conversely, let $\Phi : \mathfrak{Q} \to \text{Rat } V$ be an essential homomorphism of the Lie algebras and let $H_\Phi(x)$ be its associated linear transformation. Set $F = F(x) = [H_\Phi(x)]^{-1}$. Due to the linearity of Φ, we can write

(i) $\quad \Phi q = F^{-1}[a+b_T+c_p]$, where $q = a + T + p$.

$T \to b_T$ and $p \to c_p$ are linear mappings of $\mathfrak{T}$ and $\widetilde{V}$, respectively, into Rat V.

For two elements q_1 and q_2 in $\mathfrak{Q}$ we abbreviate $w_j = \Phi q_j$ and write w_j as in (i). In the notation of §1.2 we obtain

$$(\Phi q_1 \cdot \Phi q_2)(x) = (w_1 \cdot w_2)(x) = \Delta_x^{w_2} w_1(x)$$

$$= -F^{-1}[\Delta_x^{w_2} F(x)]w_1(x) + F^{-1} \Delta_x^{w_2}[b_{T_1}(x) + c_{p_1}(x)],$$

by using the fact that $\Delta_x^u[F(x)]^{-1} = -F^{-1}[\Delta_x^u F(x)]F^{-1}$. It now follows that

$$(\Phi[q_1,q_2])(x) = [\Phi q_1, \Phi q_2](x) = [w_1,w_2](x)$$

$$= F^{-1}\Big(-[\Delta_x^{w_2} F(x)]w_1 + [\Delta_x^{w_1} F(x)]w_2$$

$$+ \Delta_x^{w_2} [b_{T_1}(x)+c_{p_1}(x)] - \Delta_x^{w_1}[b_{T_2}(x)+c_{p_2}(x)]\Big) .$$

Setting $q_j = a_j \in V$, we find that $[q_1,q_2] = 0$ and we then obtain

$$[\Delta_x^{u_2} F(x)]u_1 = [\Delta_x^{u_1} F(x)]u_2 \quad \text{for } u_j = F^{-1}a_j.$$

As this expression is bilinear in u_1 and u_2, this equation is also valid in any scalar extension of V. The above equation simplifies to

(ii) $\quad F\Phi[q_1,q_2] = (b_{T_1} + c_{p_1})\cdot\Phi q_2 - (b_{T_2} + c_{p_2})\cdot\Phi q_1.$

Now let $q_1 = T \in \mathfrak{T}$ and $q_2 = a \in V$. As $[q_1,q_2] = Ta$, $\Phi[q_1,q_2] = F^{-1}Ta$ and $\Phi a = F^{-1}a$, it follows that $Ta = b_T\cdot(F^{-1}a)$. Since both sides of this equation are K'-linear in a, we can replace a by an arbitrary element of Rat V and the equation will remain valid. We thus have

(iii) $\quad b_T\cdot h = T F h \quad$ for $T \in \mathfrak{T}$, $h \in$ Rat V.

As $I \in \mathfrak{T}$ we can substitute $T = I$ and obtain

(iv) $\quad F(x) = \frac{\partial f(x)}{\partial x} \quad$ for $f(x) := b_I(x)$.

As the determinant of $F = F(x)$ is not zero, f is an element of $\mathbb{P}_o(V)$. Now substitute $q_j = T_j \in \mathfrak{T}$ in equation (ii). Since $[q_1,q_2] = [T_1,T_2] \in \mathfrak{T}$, we have that $\Phi[q_1,q_2] = F^{-1}b_{[T_1,T_2]}$, and using (iii) we obtain

$$b_{[T_1,T_2]} = b_{T_1}\cdot(F^{-1}b_{T_2})-b_{T_2}\cdot(F^{-1}b_{T_1}) = T_1 b_{T_2}-T_2 b_{T_1}.$$

Now for $T_1 = I$, $T_2 = T$, we get the relation

(v) $\qquad b_T = Tf \quad$ for all $T \in \mathfrak{T}$.

For $q_1 = p \in \widetilde{V}$, $q_2 = a \in V$, we apply (2.1) to the calculation of $[q_1,q_2] = 2\, S_a^p = 2\, S_a$. Using (v) and the fact from (B.2′) that $S_a \in \mathfrak{T}$ we get

$$\tfrac{1}{2}\Phi[q_1,q_2] = F^{-1}\, b_{S_a} = F^{-1}\, S_a f,$$

so that (ii) yields

$$2\, S_a f = c_p\cdot\Phi a = c_p\cdot(F^{-1}a)\ .$$

Now replace a by Fh in the above equation and obtain

(vi) $\qquad 2\, S_{Fh}f = c_p\cdot h \quad$ for $h \in$ Rat V.

Finally, substitute $q_1 = I$, $q_2 = p \in \widetilde{V}$ in (ii) and in view of (1.3) and (iv), it follows that

$$-F\Phi p = b_I\cdot\Phi p - c_p\cdot\Phi I \text{ and } 2\, c_p = c_p\cdot(F^{-1}f).$$

A comparison with (v) yields

$$c_p = S_f f = p(f,f) = p\circ f.$$

Taking this and (v) together, we get that the image of $q = a + T + p$ under Φ is given by

$$(\Phi q)(x) = [F(x)]^{-1}[a+Tf(x)+p(f(x))] = (q^f)(x).$$

This completes the proof of the theorem.

3. For an essential homomorphism $\Phi : \mathfrak{L} \to \text{Rat } V$ there is an $f \in \mathbb{P}_o(V)$ such that $\Phi q = q^f$ for $q \in \mathfrak{L}$. We define the rational function r_Φ by

$$r_\Phi(x) := [H_\Phi(x)]^{-1} (\Phi I)(x). \tag{2.3}$$

Obviously r_Φ depends only on the images ΦI and Φa, $a \in V$. Writing $\Phi q = q^f$ for $q = I$ and for $q = a \in V$ we obtain

$$(\Phi I)(x) = \left(\frac{\partial f(x)}{\partial x}\right)^{-1} f(x),$$

$$(\Phi a)(x) = H_\Phi(x)a = \left(\frac{\partial f(x)}{\partial x}\right)^{-1} a.$$

Hence $f = r_\Phi$ and

$$\frac{\partial r_\Phi(x)}{\partial x} = [H_\Phi(x)]^{-1}. \tag{2.4}$$

In particular the rational function f is uniquely determined by Φ. We say that $f = r_\Phi$ belongs to the essential homomorphism Φ.

4. Let $W \in GL(V)$ and consider the automorphism ∇_W of Rat V given by (1.8). It follows that $\nabla_W I = I$ and the image $\nabla_W \mathfrak{L}$ is again a binary Lie algebra. Moreover the restriction of ∇_W to $\mathfrak{L}$ is an essential homomorphism and W^{-1} belongs to it.

For $b \in V$ we consider the automorphism $\Psi_b = \nabla_{t_b}$ of Rat V given by (1.10), and we show that the restriction of Ψ_b to $\mathfrak{L}$ is an automorphism of the binary Lie

<u>algebra</u> $\mathfrak{L}$. Because of (1.12) it is enough to prove $\Psi_b\mathfrak{L} \subset \mathfrak{L}$. Writing $q = a+T+p \in \mathfrak{L} = V+\mathfrak{T}+\tilde{V}$ we obtain from (1.10)

$$(\Psi_b q)(x) = q(x-b) = [a-Tb+p(b)]+[Tx-2p(x,b)]+p(x).$$

Hence we have only to show that $2p(x,b) = [p,b](x)$ belongs to $\mathfrak{T}$. But this is a consequence of (B.2).

Furthermore the restriction of Ψ_b to $\mathfrak{L}$ is an essential homomorphism and t_{-b} belongs to it.

5. Later we will see that the essential automorphisms of $\mathfrak{L}$ form a group. As a first step we prove

<u>LEMMA 2.2</u>. <u>Let</u> $\mathfrak{L}$ <u>and</u> $\mathfrak{L}'$ <u>be binary Lie algebras and let</u>

$$\Phi : \mathfrak{L} \to \mathfrak{L}', \quad \Phi' : \mathfrak{L}' \to \text{Rat } V,$$

<u>be essential homomorphisms such that</u> r_Φ <u>and</u> $r_{\Phi'}$ <u>are composable</u>. <u>Then</u> $\Phi'\Phi : \mathfrak{L} \to \text{Rat } V$ <u>is essential and we have</u>

$$r_{\Phi'\Phi} = r_\Phi \circ r_{\Phi'} .$$

<u>Proof</u>: Put $f = r_\Phi$ and $g = r_{\Phi'}$. Since f and g are composable, the chain rule shows that $f\circ g$ belongs to $P_o(V)$ too, and that $h^{f\circ g} = (h^f)^g$ holds for $h \in \text{Pol } V$. From Theorem 2.1 we conclude

$$\Phi'\Phi q = (\Phi q)^g = (q^f)^g = q^{f\circ g} \quad \text{for } q \in \mathfrak{L}.$$

Hence $\Phi'\Phi$ is essential and $f\circ g$ belongs to it.

The assumptions of the Lemma are certainly satisfied if

$$r_\Phi \in \text{Pol } V,\ r_{\Phi'} \in \text{Rat } V \quad \text{or} \quad r_\Phi \in \text{Rat } V,\ r_{\Phi'} \in \mathbb{P}(V).$$

In particular we get the

COROLLARY. *If $\Phi : \mathfrak{D} \to \text{Rat } V$ is essential and if $b,c \in V$ then $\Psi_b\,\Phi\,\Psi_c$ is essential and $t_{-c}\circ r_\Phi \circ t_{-b}$ belongs to it.*

6. Let $v \in \mathfrak{P}_2$. We define a linear transformation $\tilde{\Psi}_v$ of Rat V by

$$\tilde{\Psi}_v = \text{Id} + \text{ad } v + \tfrac{1}{2}(\text{ad } v)^2, \tag{2.5}$$

where as usual the adjoint representation ad v is given by $(\text{ad } v)h = [v,h]$, $h \in \text{Rat } V$. We know that ∇_W, $W \in GL(V)$, is an automorphism of Rat V, hence we

$$\nabla_W \tilde{\Psi}_v = \tilde{\Psi}_u \nabla_W \quad \text{for } W \in GL(V),\ v \in \mathfrak{P}_2,\ u := \nabla_W v. \tag{2.6}$$

The restrictions of ad v, ∇_W, Ψ_b and $\tilde{\Psi}_v$ to $\mathfrak{D}$ will be denoted with the same symbol if there is no possibility of misunderstanding.

Furthermore we define a linear map $B_v(x)$ of V into Pol V by

$$[B_v(x)]a = (a - [v,a] + \tfrac{1}{2}[v,[v,a]])(x) \tag{2.7}$$

$$= a - 2v(x,a) + 2\,v(x,v(x,a)) - v(a,v(x))$$

for $a \in V$. Then $B_v(x)$ can be extended to any scalar extension of V. One sees that $B_v(x)$ is a polynomial of highest degree 2 and one has $B_v(0) = I$. Hence $B_v(x)$ considered as a linear transformation of V' is invertible. The expression

$$(2.8) \qquad \tilde{t}_v(x) := [B_v(x)]^{-1}[x - v(x)], \quad v \in \mathfrak{P}_2$$

is a rational function. In particular $\tilde{t}_v(0)$ is defined and we have

$$\tilde{t}_v(0) = 0, \qquad \left.\frac{\partial \tilde{t}_v(x)}{\partial x}\right|_{x \to o} = I.$$

Hence $\tilde{t}_v$ lies in $\mathbf{P}_o(V)$.

THEOREM 2.3. Let $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$ be a binary Lie algebra and let $v \in \tilde{V}$. Then

(a) $\tilde{t}_v \in \mathbf{P}(V)$ and $\dfrac{\partial \tilde{t}_v(x)}{\partial x} = [B_v(x)]^{-1}$,

(b) $\tilde{\Psi}_v$ is an essential automorphism of $\mathfrak{L}$ and $\tilde{t}_{-v}$ belongs to it,

(c) $\tilde{t}_{u+v} = \tilde{t}_u \circ \tilde{t}_v$ and $\tilde{\Psi}_{u+v} = \tilde{\Psi}_u \tilde{\Psi}_v$ for $u \in \tilde{V}$.

(d) $W \circ \tilde{t}_v = \tilde{t}_u \circ W$, where $u = \nabla_W v$.

Proof: (1) Using (B.1) to (B.4) we see that $(\text{ad } \dot{v})^3 \mathfrak{L} = 0$. Hence from (2.5) it follows that

$$\tilde{\Psi}_v q = (\exp \text{ ad } v) q \quad \text{for } q \in \mathfrak{L}$$

holds. Since ad v is a derivation of $\mathfrak{L}$ the restriction of exp ad v to $\mathfrak{Q}$ turns out to be an automorphism of $\mathfrak{Q}$. Moreover for $u \in \tilde{V}$ we obtain $[u,[v,q]] - [v,[u,q]] = [q,[v,u]] = 0$ from (B.4). Therefore ad u and ad v commute on $\mathfrak{Q}$. Hence

$$\tilde{\Psi}_u\tilde{\Psi}_v = (\exp \operatorname{ad} u)(\exp \operatorname{ad} v) = \exp \operatorname{ad}(u+v) = \tilde{\Psi}_{u+v}.$$

(2) As an abbreviation write $\Phi = \tilde{\Psi}_v$. Then combining (2.2) and (2.7) we get

$$H_\Phi(x)a = (\Phi a)(x) = a+[v,a](x) + \tfrac{1}{2}[v,[v,a]](x)$$
$$= B_{-v}(x)a.$$

In particular $\tilde{\Psi}_v$ is essential. Furthermore from (1.3) we obtain $[v,I] = v$ and $\Phi I = I+v$. Hence $r_\Phi = \tilde{t}_{-v}$. So part (b) together with (2.4) implies the second statement of part (a).

(3) The determinant of $B_{-v}(x)$ is a denominator of $\tilde{t}_{-v}(x)$ and $\tilde{t}_{-u}(0) = 0$, $B_{-v}(0) = I$. Hence $\tilde{t}_{-v}$ and $\tilde{t}_{-u}$ are composable. Applying Lemma 2.2 we see that $\tilde{\Psi}_u\,\tilde{\Psi}_v$ is again essential and $\tilde{t}_{-v}\circ\tilde{t}_{-u}$ belongs to it. From (1) and (2) we obtain $\tilde{\Psi}_u\,\tilde{\Psi}_v = \tilde{\Psi}_{u+v}$ and the function belonging to it equals $\tilde{t}_{-u-v}$. Hence $\tilde{t}_{-v}\circ\tilde{t}_{-u} = \tilde{t}_{-u-v}$ and part (c) is proved. In addition we see that $\tilde{t}_v$ is a birational function.

(4) Part (d) follows from (2.6).

In particular $\tilde{t}_{-v}\circ\tilde{t}_v = I$ for $v \in \tilde{V}$. Using the chain rule together with part (a) of the theorem we end up with

$$(2.9)\qquad B_v(x)\ B_{-v}(\tilde{t}_v(x)) = I, \quad v \in \tilde{V}.$$

Finally from the definitions (2.7) and (2.8) we obtain

$$(2.10)\qquad B_{-v}(-x) = B_v(x), \quad \tilde{t}_{-v}(-x) = -\tilde{t}_v(x), \quad v \in \tilde{V}.$$

Parts (a) and (d) of Theorem 2.3 yields

$$(2.11)\qquad W\ B_v(x) = B_u(Wx)\ W, \quad v \in \tilde{V},\ W \in GL(V),\ u = \nabla_W v.$$

7. Any binary Lie algebra $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$ gives rise to a family of Jordan algebras defined on the vector space V. We are going to prove

MEYBERG'S THEOREM. *Let* $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$ *be a binary Lie algebra and let* $v \in \tilde{V}$. *Then* V *together with the bilinear product* $ab = [[a,v],b]$ *turns out to be a Jordan algebra*.

Proof: As an abbreviation we write

$$\{a,u,b\} = [[a,u],b], \quad \{u,a,v\} = [[u,a],v] \text{ for } a,b \in V,\ u,v \in \tilde{V}.$$

Since V and $\tilde{V}$ are abelian subalgebras of $\mathfrak{L}$, both triples are symmetric in the first and last entries. In order to prove

$$(2.12)\qquad \{a,u,\{b,v,c\}\} - \{b,v,\{a,u,c\}\}$$
$$= \{\{a,u,b\},v,c\} - \{b,\{u,a,v\},c\},$$
$$a,b,c \in V, \quad u,v \in \tilde{V},$$

one puts $T = [a,u]$ and uses the Jacobi identity. Analogously we get

$$(2.12') \quad \{u,a,\{v,b,w\}\} - \{v,b,\{u,a,w\}\} = \{\{u,a,v\},b,w\} - \{v,\{a,u,b\},w\},$$
$$a,b \in V, \quad u,v,w \in \tilde{V}.$$

The left side of (2.12) is skew-symmetric in (a,u) and (b,v); hence

$$(2.13) \quad \{\{a,u,b\},v,c\} - \{b,\{u,a,v\},c\} + \{\{b,v,a\},u,c\} - \{a,\{v,b,u\},c\} = 0.$$

Choosing $a = b$ and $u = v$ we get

$$(2.14) \quad \{b_v,v,b\} = \{b,v_b,b\} \quad \text{where } b_v = \{b,v,b\} \text{ and } v_b = \{v,b,v\}.$$

In the same way from (2.12′) it follows that

$$(2.14') \quad \{v_b,b,v\} = \{v,b_v,v\}.$$

Using the product $ab = \{a,v,b\} = [[a,v],b]$, we get $b_v = b^2$. Now in (2.13) we choose $a = b$ and replace u by v, and v by v_b. We then obtain

$$(2.15) \quad 2\{b,\{v_b,b,v\},c\} = c\{b,v_b,b\} + \{c,v_b,b^2\}.$$

Next we set $u = v$ and replace a by b^2 in (2.12):

$$2b^2(bc)-2b(b^2c) = 2b^3c-2\{b,\{v,b^2,v\},c\}.$$

Using (2.14′), we put (2.15) in this equation:

$$2b^2(bc)-2b(b^2c) = b^3c-\{c,v_b,b^2\}.$$

Finally put u = v and replace a by b and b by b^2 in (2.12):

$$b(b^2c)-b^2(bc) = b^3c-\{b^2,v_b,c\}.$$

This means $3[b^2(bc)-b(b^2c)] = 0$ and the theorem is proved.

§3. A description of the essential homomorphisms.

1. Again let $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$ be a binary Lie algebra in Pol V and let $\Phi : \mathfrak{L} \to$ Pol V be a linear map. Hence we obtain a representation

$$\Phi q = \sum_{\nu\geq o} g_q^\nu \quad \text{where } g_q^\nu \in \mathfrak{P}_\nu \tag{3.1}$$

as a finite sum. Here $q \to g_q^\nu$ is a linear map of $\mathfrak{L}$ into $\mathfrak{P}_\nu$. We write $g_q^\nu = g_a^\nu + g_T^\nu + g_p^\nu$ whenever $q = a + T + p \in \mathfrak{L}$. If $\Phi : \mathfrak{L} \to$ Pol V is a homomorphism of the Lie algebras then (1.2) implies

$$g_{[q,q']}^\nu = \sum_{\mu=o}^{\nu+1} [g_q^\mu, g_{q'}^{\nu+1-\mu}] \text{ for } q,q' \in \mathfrak{L}. \tag{3.2}$$

We obtain our first information about the homomorphisms of binary Lie algebras in

LEMMA 3.1. *Let $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$ and $\mathfrak{L}' = V + \mathfrak{T}' + \tilde{V}'$ be binary Lie algebras in Pol V. Suppose that*

$\Phi : \mathfrak{L} \to \mathfrak{L}'$ is an epimorphism of the Lie algebras with $g_I^o = 0$ and such that $a \to g_a^o$ is a bijection of V. Then there exist $W \in GL(V)$ and $v \in \tilde{V}$ such that

$$\Phi q = \nabla_W \Psi_v q \qquad \text{for } q \in \mathfrak{L}$$

and $\mathfrak{L}' = \nabla_W \mathfrak{L}$.

Proof: We define the linear transformation W of V by $Wa = g_a^o$. Hence W is bijective and consequently $W \in GL(V)$. We know from §1.6 that ∇_W is a homomorphism of Pol V that maps binary Lie algebras onto binary Lie algebras. Setting $\tilde{\Phi} := \nabla_W^{-1} \Phi$, $\mathfrak{L}'' := \nabla_W^{-1}\mathfrak{L}'$, we obtain a homomorphism $\tilde{\Phi} : \mathfrak{L} \to \mathfrak{L}''$ satisfying $\tilde{g}_I^o = 0$ and $\tilde{g}_a^o = a$, where $\tilde{g}_q^\nu = \nabla_W^{-1} g_q^\nu$.

Hence we may assume that $\Phi : \mathfrak{L} \to \mathfrak{L}'$ is a homomorphism satisfying $g_I^o = 0$ and $g_a^o = a$, and we have to prove $\mathfrak{L}' = \mathfrak{L}$ and $\Phi = \Psi_v$ for some $v \in \tilde{V}$.

Substituting $q = T \in \mathfrak{T}$, $q' = a \in V$ in (3.2) we get $[q,q'] = Ta$ and

$$(3.3) \qquad g_{Ta}^o = g_T^1(g_a^o) - g_a^1(g_T^o).$$

$T = I$ yields $a = g_a^o = g_I^1(a)$, hence $g_I^1 = I$ and

$$\Phi I = I + v, \qquad v \in \tilde{V}'.$$

For $q = a + T + p \in \mathfrak{L}$ we get $[I,q] = a - p$ from (1.3). Consequently $\Phi(a-p) = \Phi[I,q] = [\Phi I, \Phi q]$ implies

$$(3.4) \quad \Phi a = [\Phi I, \Phi a], \quad 0 = [\Phi I, \Phi T], \quad -\Phi p = [\Phi I, \Phi p].$$

Together with (3.2) the first condition leads to

$$\sum_{\nu=0}^{2} \nu \; g_a^{\nu} = \sum_{\nu=0}^{2} [v, g_a^{\nu}].$$

We compare the homogeneous terms and get

$$g_a^1 = [v,a], \; g_a^2 = \tfrac{1}{2}[v,[v,a]] \text{ resp. } [v,[v,[v,a]]] = 0.$$

That means

$$\Phi a = a + [v,a] + \tfrac{1}{2}[v,[v,a]], \; (\mathrm{ad}\; v)^3 a = 0 \text{ for } a \in V.$$

In the same way the second condition of (3.4) yields

$$\sum_{\nu=0}^{2} (\nu-1) \; g_T^{\nu} = \sum_{\nu=0}^{2} [v, g_T^{\nu}].$$

Again we compare the homogeneous terms and obtain $g_T^{\nu} = 0$, $g_T^2 = [v, g_T^1]$ and $[v, g_T^2] = 0$. Formula (3.3) leads to $g_T^1 = T$ and hence $g_T^2 = [v,T]$, $[v,[v,T]] = 0$. This means

$$\Phi T = T + [v,T], \quad (\mathrm{ad}\; v)^2 T = 0, \text{ for } T \in \mathfrak{T}.$$

Finally the third condition of (3.4) leads to

$$\sum_{\nu=0}^{2} (\nu-2) \; g_p^{\nu} = \sum_{\nu=0}^{2} [v, g_p^{\nu}].$$

Hence $g_p^o = g_p^1 = 0$ and $[v, g_p^2] = 0$. Substituting $q = p \in \tilde{V}$, $q' = a \in V$ and $\nu = 1$ in (3.2) we get

$$g^1_{[p,a]} = [g^2_p, g^o_a] = [g^2_p, a].$$

From (B.2) we obtain $[p,a] \in \mathfrak{T}$ and therefore $g^1_T = T$ yields $[p,a] = [g^2_p, a]$. This means $p(x,a) = g^2_p(x,a)$ and consequently $g^2_p = p$ as well as $[v,p] = 0$. Hence

$$\Phi p = \tilde{\Psi}_v p, \quad [v,p] = 0, \text{ for } p \in \tilde{V}.$$

Summing up we have

(3.5) $\Phi q = [I + \text{ad } v + \frac{1}{2}(\text{ad } v)^2]q, \quad (\text{ad } v)^3 q = 0, \; q \in \mathfrak{L}.$

Because of $v \in \tilde{V}'$ we know from Theorem 2.3b that $\tilde{\Psi}_{-v}$ is an automorphism of $\mathfrak{L}'$. Hence $\tilde{\Psi}_{-v}\Phi : \mathfrak{L} \to \mathfrak{L}'$ turns out to be a homomorphism, but (3.5) leads to $\tilde{\Psi}_{-v}\Phi q = q$. Therefore $\mathfrak{L}' = \mathfrak{L}$ and $\Phi_q = \tilde{\Psi}_v q$ for $q \in \mathfrak{L}$.

2. As a first application we prove

THEOREM 3.2. *Let $\Phi : \mathfrak{L} \to \mathfrak{L}'$ be an isomorphism of the binary Lie algebras satisfying $\Phi I = I$. Then there exists a W in GL(V) such that $\Phi q = \nabla_W q$ for $q \in \mathfrak{L}$.*

Proof: In the notation of (3.1) we have $g^o_I = 0$. Moreover from $\Phi I = I$ it follows that $\Phi a = \Phi[I,a] = [I,\Phi a]$ and (1.2) implies $g^1_a = g^2_a = 0$. Hence $a \to g^o_a$ is a bijection and we can apply Lemma 3.1. There is a $W \in GL(V)$ and $v \in \tilde{V}'$ such that $\Phi q = \nabla_W \tilde{\Psi}_v q$ for $q \in \mathfrak{L}$. Substituting $q = I$ we get $v = 0$ and the Theorem is proved.

A second application leads to the following main result on the automorphisms:

THEOREM 3.3. *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be binary Lie algebras in* Pol V. *Then*:

a) (i) *If* $\Phi : \mathfrak{L} \to \mathfrak{L}'$ *is an essential isomorphism, then* Φ *can be written as*

(3.6) $$\Phi = \nabla_W \Psi_b \tilde{\Psi}_v \Psi_c, \quad \text{where } W \in GL(V),\ b,c \in V,\ v \in \tilde{V},$$

and $\mathfrak{L}' = \nabla_W \mathfrak{L}$.

(ii) *The rational function* r_Φ *belonging to* Φ *is birational and one has*

(3.7) $$r_\Phi = (W \circ t_b \circ \tilde{t}_v \circ t_c)^{-1}.$$

(iii) *If there is a* $d \in V$ *in the domain of definition of* r_Φ *such that*

(3.8) $$r_\Phi(d) = 0 \quad \text{and} \quad \det \left.\frac{\partial r_\Phi(x)}{\partial x}\right|_{x \to d} \neq 0,$$

then the statements are true for $c = 0$.

b) *Each map of the form* (3.6) *turns out to be an essential isomorphism of* $\mathfrak{L}$ *onto* $\mathfrak{L}' = \nabla_W \mathfrak{L}$.

Proof: 1) Suppose first that the essential isomorphism Φ satisfies the condition of (iii). From §2.4 we know that Ψ_d is an essential automorphism of $\mathfrak{L}'$, hence $\tilde{\Phi} = \Psi_{-d} \Phi : \mathfrak{L} \to \mathfrak{L}'$ is again an isomorphism of the Lie algebras which is essential because of the Corollary of Lemma 2.2 and $f = r_{\tilde{\Phi}} \circ t_d$ belongs to $\tilde{\Phi}$. From (3.8) we get

$$f(0) = 0 \quad \text{and} \quad Q = \left.\frac{\partial f(x)}{\partial x}\right|_{x \to o} \quad \text{is invertible.}$$

We write $\tilde{\Phi}q$ in the form (3.1) and specialize $x \to 0$ in $\tilde{\Phi}q = q^f$, $q = a + T + p \in \Omega$. Hence

$$g_a^o + g_T^o + g_p^o = (q^f)(0) = Q^{-1}a.$$

In particular $g_T^o = 0$ and $a \to g_a^o$ is a bijection. We apply Lemma 3.1 and obtain

$$\tilde{\Phi}q = \nabla_W \tilde{\Psi}_v q \quad \text{for} \quad q \in \Omega$$

and $W \in GL(V)$, $v \in \tilde{V}$. Hence $\Phi q = \nabla_q \Psi_b \tilde{\Psi}_v q$ and $r_{\Phi} = (W \circ t_b \circ \tilde{t}_v)^{-1}$, where $b = W^{-1}d$ according to (1.11). So (iii) is proved.

2) Now let $\Phi : \Omega \to \Omega'$ be an arbitrary essential isomorphism. The Corollary of Lemma 2.2 shows again that $\tilde{\Phi} = \Phi \Psi_{-c}$ is essential and $r_{\tilde{\Phi}} = t_c \circ r_{\Phi}$ belongs to it. We choose $c, d \in V$ such that d is in the domain of definition of $r_{\tilde{\Phi}}$ and that (3.8) is satisfied for $r_{\tilde{\Phi}}$. Part (i) of the proof yields (3.6) and (3.7).

3) Since the functions W, t_b and $\tilde{t}_v$ are birational we obtain the statement of part b) from Lemma 2.2 and Theorem 2.1.

Using the definition of ∇_f in §1.5 we obtain the

COROLLARY. *If $\Phi : \mathfrak{L} \to \mathfrak{L}'$ is an essential isomorphism, then there exists a (uniquely determined) $f \in \mathbb{P}(V)$ such that $\Phi = \nabla_f$. Moreover $f = W \circ t_b \circ \tilde{t}_v \circ t_c$ in the notation of (3.6).*

§4. The group of essential automorphisms.

1. Let $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$ be a binary Lie algebra and denote by $\Gamma(\mathfrak{L})$ the group of $W \in GL(V)$ satisfying $\nabla_W \mathfrak{L} = \mathfrak{L}$. Obviously, $W \in \Gamma(\mathfrak{L})$ is equivalent to

$$\nabla_W \mathfrak{T} = W \mathfrak{T} W^{-1} = \mathfrak{T} \quad \text{and} \quad \nabla_W \tilde{V} = \tilde{V}.$$

In particular, $\alpha \cdot I$, $0 \neq \alpha \in K$, belongs to $\Gamma(\mathfrak{L})$. One can show that $\Gamma(\mathfrak{L})$ is a linear algebraic group defined over K.

Denote by $\mathrm{Aut}^*\mathfrak{L}$ the subgroup of the automorphism group $\mathrm{Aut}\,\mathfrak{L}$ of $\mathfrak{L}$ that is generated by the automorphsims

$$(4.1) \qquad \nabla_W \text{ for } W \in \Gamma(\mathfrak{L}), \quad \Psi_b \text{ for } b \in V, \quad \tilde{\Psi}_v \text{ for } v \in \tilde{V}$$

(see §1.6 and Theorem 2.3).

THEOREM 4.1. *Let $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$ be a binary Lie algebra. Then:*

a) *The set of essential automorphisms of $\mathfrak{L}$ coincides with the group $\mathrm{Aut}^*\mathfrak{L}$, which is Zariski-open in $\mathrm{Aut}\,\mathfrak{L}$.*

b) *Each Φ in $\mathrm{Aut}^*\mathfrak{L}$ can be written as*

$$\Phi = \nabla_W \Psi_b \tilde{\Psi}_v \Psi_c, \text{ where } W \in \Gamma(\mathfrak{D}), \quad b,c \in V, \quad v \in \tilde{V},$$

<u>and the rational function belonging to</u> Φ <u>is given by</u>

$$r_\Phi = (W \circ t_b \circ \tilde{t}_v \circ t_c)^{-1}.$$

c) $\Phi \to r_{\Phi^{-1}}$ <u>gives a monomorphism of</u> $\mathrm{Aut}^*\mathfrak{D}$ <u>into</u> $\mathbb{P}(V)$.

<u>Proof</u>: The generators (4.1) are essential automorphisms of $\mathfrak{D}$ and the rational functions belonging to them are birational. Using Lemma 2.2 we see that $\mathrm{Aut}^*\mathfrak{D}$ consists only of essential automorphisms. Conversely, each essential automorphism of $\mathfrak{D}$ belongs to $\mathrm{Aut}^*\mathfrak{D}$ because of Theorem 3.3. Since an automorphism Φ is essential if and only if $\det H_\Phi(x) \neq 0$ (see §2.2), the set of essential automorphisms turns out to be a Zariski open subset of Aut $\mathfrak{D}$. So parts a) and b) are proved.

From Lemma 2.2 we get $r_{\Phi'\Phi} = r_\Phi \circ r_{\Phi'}$. Hence we need only prove that $\Phi \to r_\Phi$ is an injection. Consider $\Phi \in \mathrm{Aut}^*\mathfrak{D}$ such that $r_\Phi = I$. From (2.4) it follows that $H_\Phi(x) = I$ and consequently $\Phi I = I$. Applying Theorem 3.2 we obtain $\Phi = \nabla_W$ for some $W \in GL(V)$. Hence $r_\Phi = W^{-1}$ and $W = I$.

2. Sometimes it is useful to consider the image $\mathfrak{I}(\mathfrak{D})$ of $\mathrm{Aut}^*\mathfrak{D}$ under the injection

(4.2) $$\Phi \to r_{\Phi^{-1}}, \quad \Phi \in \mathrm{Aut}^*\mathfrak{D},$$

i.e., the set

$$\Xi(\mathfrak{D}) = \{r_\Phi \ ; \ \Phi \in \mathrm{Aut}^*\mathfrak{D}\}.$$

Part c) of Theorem 4.1 shows that $\Xi(\mathfrak{D})$ is a subgroup of $\mathbf{P}(V)$ and (4.2) turns out to be an isomorphism of $\mathrm{Aut}^*\mathfrak{D}$ onto $\Xi(\mathfrak{D})$. Comparing $\Phi q = q^f$ for $f = r_\Phi$ and the definition of ∇_f (see §1.5) we see

(4.3) $$\nabla_f = \Phi^{-1} \Leftrightarrow f = r_\Phi.$$

Hence

$$\nabla : \Xi(\mathfrak{D}) \to \mathrm{Aut}^*\mathfrak{D}, \quad f \to \nabla_f,$$

is the inverse map to (4.2).

In case Φ equals ∇_W, Ψ_b, or $\tilde{\Psi}_v$, the function r_Φ belonging to Φ equals W^{-1}, $(t_b)^{-1} = t_{-b}$, or $(\tilde{t}_v)^{-1} = \tilde{t}_{-v}$, respectively. <u>Hence the group</u> $\Xi(\mathfrak{D})$ <u>is generated by the birational mappings</u>

(4.4) $$W \in \Gamma(\mathfrak{D}), \quad t_b \text{ for } b \in V, \quad \tilde{t}_v \text{ for } v \in \tilde{V}.$$

Moreover from part b) of Theorem 4.1, we know that each f in $\Xi(\mathfrak{D})$ can be written as

(4.5) $$f = W \circ t_b \circ \tilde{t}_v \circ t_c \quad \text{where } W \in \Gamma(\mathfrak{D}), \quad b,c \in V, \quad v \in \tilde{V}.$$

Using the chain rule together with part a) of Theorem 2.3 we get

(4.6) $$\frac{\partial f(x)}{\partial x} = W[B_v(x+c)]^{-1}.$$

As a first application we prove a lemma that is trivial in the case of characteristic zero.

LEMMA 4.2. *Let* f *and* g *be in* $\Xi(\mathfrak{D})$. *Then* $\frac{\partial f(x)}{\partial x} = \frac{\partial g(x)}{\partial x}$ *if and only if there exists an element* $a \in V$ *such that* $f = t_a \circ g$.

Proof: We set $h = f \circ g^{-1}$ and we obtain $\frac{\partial h(x)}{\partial x} = I$ using the chain rule. Therefore it is enough to prove the statement in the case $g = I$. Writing $f = W \circ t_b \circ \tilde{t}_v \circ t_c$ we obtain $W = B_v(x+c)$ from (4.6). So $x = -c$ leads to $W = I$ and to $B_v(x) = I$. From (2.7) we get $v(x,a) = 0$ for $a \in V$ and hence $v = 0$ and $f = t_{b+c}$.

Clearly $f = t_a \circ g$ implies $\frac{\partial f(x)}{\partial x} = \frac{\partial g(x)}{\partial x}$.

Remark. From (4.5) we obtain a decomposition

$$\Xi(\mathfrak{D}) = \Gamma(\mathfrak{D}) \circ \Sigma \circ \tilde{\Sigma} \circ \Sigma$$

where $\Sigma = \{t_a; a \in V\}$ and $\tilde{\Sigma} = \{t_v; v \in \tilde{V}\}$ are abelian subgroups of $\Xi(\mathfrak{D})$. From (1.9) and part d) of Theorem 2.3 we get $\Gamma(\mathfrak{D}) \circ \Sigma = \Sigma \circ \Gamma(\mathfrak{D})$ as well as $\Gamma(\mathfrak{D}) \circ \tilde{\Sigma} = \tilde{\Sigma} \circ \Gamma(\mathfrak{D})$. This decomposition induces an equivalence relation on the set $\tilde{V}$: For $u, v \in \tilde{V}$ we define $u \sim v$ whenever $\tilde{t}_u \in \Gamma(\mathfrak{D}) \circ \Sigma \circ \tilde{t}_v \circ \Sigma$. In particular $u \sim v$ and $W \in \Gamma(\mathfrak{D})$ implies $\nabla_W u \sim \nabla_W v$.

Moreover from (4.6)

$$\omega_f(x) := \det\left(\frac{\partial f(x)}{\partial x}\right)^{-1}$$

turns out to be a polynomial whenever $f \in \Xi(\mathfrak{L})$. Denote by D_f the set of $a \in V$ such that $\omega_f(a) \neq 0$. Furthermore the chain rule yields

$$(4.7) \qquad \omega_{g\circ f}(x) = \omega_g(f(x))\,\omega_f(x) \qquad \text{for } f,g \in \Xi(\mathfrak{L}).$$

Writing $f = r_\Phi \in \Xi(\mathfrak{L})$ the formulas (2.3) and (2.4) lead to

$$f(x) = \frac{\partial f(x)}{\partial x}\,(\Phi I)(x).$$

ΦI belongs to $\mathfrak{L}$ and consequently ΦI is a polynomial. Therefore we get: <u>For</u> $f \in \Xi(\mathfrak{L})$ <u>we have</u> $D_f \subset \mathrm{Dom}\, f$.

3. Note that a representation (4.5) is <u>not</u> unique. But setting

$$\Xi^o = \Xi^o(\mathfrak{L}) = \{f \; ; \; f \in \Xi(\mathfrak{L}),\ \omega_f(0) \neq 0\},$$

we get the

<u>THEOREM 4.3</u>. <u>Let</u> $\mathfrak{L}$ <u>be a binary Lie algebra</u>. <u>Then</u>

a) <u>The elements of</u> $\Xi^o(\mathfrak{L})$ <u>are exactly the functions</u>

$$(4.8) \qquad f = W\circ t_b\circ \tilde{t}_v, \quad \underline{\text{where}}\ W \in \Gamma(\mathfrak{L}),\ b \in V,\ v \in \tilde{V},$$

<u>and this representation is unique</u>.

b) <u>The image of</u> $\Xi^o(\mathfrak{L})$ <u>under the map</u> $f \to \nabla_f$ <u>is Zariski-open in</u> $\mathrm{Aut}^*\mathfrak{L}$.

<u>Proof</u>: a) From $B_v(0) = I$ and from (4.6) we conclude that any f given in (4.8) belongs to $\Xi^o(\mathfrak{L})$. Conversely,

let $f \in \Xi^o(\mathfrak{D})$. Hence 0 is in Dom f. Put $d = f(0)$ and use (4.7) for $g = f^{-1}$ in order to get $\omega_g(d) \neq 0$. Hence d is in Dom g. Choose $\Phi \in \mathrm{Aut}^*\mathfrak{D}$ such that $f = (r_\Phi)^{-1} = r_{\Phi^{-1}}$, i.e., $g = r_\Phi$. Hence d is in Dom r_Φ and we get $r_\Phi(d) = f^{-1}(d) = 0$ as well as $\omega_{r_\Phi}(d) \neq 0$. This is exactly the condition (3.8), so Theorem 3.3 implies $f = (r_\Phi)^{-1} = W \circ t_b \circ \tilde{t}_v$, where $W \in GL(V)$, $b \in V$ and $v \in \tilde{V}$. Consequently $W \in \Gamma(\mathfrak{D})$ and (4.8) is proved.

Because of (1.9) and part c) of Theorem 2.3, to prove uniqueness we need only consider the equation $W \circ t_b \circ \tilde{t}_v = I$. From $\tilde{t}_v(0) = 0$ we get $b = 0$ and then $B_v(x) = W$ using (4.6). The definition (2.8) yields $x = Ix = x - v(x)$. Hence $v = 0$ and $W = I$.

b) For $f = r_\Phi$, $\Phi \in \mathrm{Aut}^*\mathfrak{D}$, the equation (2.4) leads to $\omega_f(x) = \det H_\Phi(x)$ and the proof is complete.

<u>COROLLARY 1</u>. <u>Let</u> $a \in V$ <u>and</u> $v \in \tilde{V}$. <u>If</u> $\det B_v(a) \neq 0$ <u>then</u> $B_v(a)$ <u>belongs to</u> $\Gamma(\mathfrak{D})$.

<u>Proof:</u> We put $f = \tilde{t}_v \circ t_a$ and obtain $\omega_f(x) = \det B_v(x+a)$ from part a) of Theorem 2.3. Hence $\omega_f(0) \neq 0$ and consequently $f \in \Xi^o(\mathfrak{D})$. Part a) of the theorem yields $\tilde{t}_v \circ t_a = f = W \circ t_b \circ \tilde{t}_u$ for some $W \in \Gamma(\mathfrak{D})$, $b \in V$ and $u \in \tilde{V}$. The equation (4.6) yields $B_v(x+a) = B_u(x)W^{-1}$ and $x = 0$ leads to $B_v(a) = W^{-1} \in \Gamma(\mathfrak{D})$. In view of (4.6) we obtain the

<u>COROLLARY 2</u>. <u>Suppose that</u> a <u>is in</u> D_f <u>for some</u> $f \in \Xi(\mathfrak{D})$. <u>Then</u> $\left.\frac{\partial f(x)}{\partial x}\right|_{x \to a}$ <u>belongs to</u> $\Gamma(\mathfrak{D})$.

§5. The case n = 1.

As an illustration we consider the case that $V = K$ is the one dimensional vector space over K. Hence the generic element x is an indeterminate over K. Denote by $\mathfrak{M}_2$ the group of invertible two-by-two matrices over K and set

$$f_M(x) = \frac{\alpha x+\beta}{\gamma x+\delta} \quad \text{where} \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{M}_2.$$

Then $M \to f_M$ defines an epimorphism of $\mathfrak{M}_2$ onto $\mathbb{P}(K)$ having the kernel $\{\alpha I;\ 0 \neq \alpha \in K\}$ where I denotes the unit matrix of $\mathfrak{M}_2$.

The Lie algebra Rat K is given by the vector space K(x) of rational functions together with the product $[h,k] = h'k-hk'$, where h' denotes the derivative of h. The only subalgebras of Pol K that contain $V = K$ are K, $\{a+bx; a,b\in K\}$, and the binary Lie algebra

$$\mathfrak{Q} = \mathfrak{P} = \{q;\ q(x) = a+bx+cx^2;\ a,b,c \in K\}.$$

In fact, $\mathfrak{Q}$ is (up to isomorphisms) the split three-dimensional simple Lie algebra. We see that $\Gamma(\mathfrak{Q})$ coincides with the mulitplicative group of non-zero elements of K.

A verification shows $h^f \in \mathfrak{Q}$ whenever $q \in \mathfrak{Q}$ and $f \in \mathbb{P}(K)$. Hence $f \to \nabla_f$ maps $\mathbb{P}(K)$ into Aut $\mathfrak{Q}$. We observe further that each automorphism of $\mathfrak{Q}$ is essential. According to Theorem 3.3 it follows that $\nabla : \mathbb{P}(K) \to \text{Aut}\ \mathfrak{Q}$ <u>is an isomorphism of the groups</u>. In particular we get $\text{Aut}^*\mathfrak{D} = \text{Aut }\mathfrak{D}$ and $\Xi(\mathfrak{D}) = \mathbb{P}(K)$ (see §4.$\underline{\underline{2}}$).

Finally let us consider the generators of the group $\Xi(\mathfrak{D}) = \mathbb{P}(K)$ according to §4.$\underline{\underline{2}}$. At first we have $Wx = wx$, $0 \neq w \in K$, and $t_a(x) = x+a$, $a \in K$. In order to describe $\tilde{t}_v$ where $v(x) = bx^2$, $b \in K$, we observe according to (2.7) and to (2.8)

$$B_v(x) = (1-bx)^2 \quad \text{and} \quad \tilde{t}_v(x) = x(1-bx)^{-1}.$$

Indeed we obtain the usual set of generators of $\mathbb{P}(K)$.

Chapter II

THE CONCEPT OF SYMMETRIC LIE ALGEBRAS

§1. Symmetric Lie algebras.

1. A pair $(\mathfrak{L}, \Theta)$ is called a symmetric Lie algebra if

(i) $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$ is a binary Lie algebra,

(ii) Θ is an automorphism of $\mathfrak{L}$ of period 2 such that $\Theta V = \tilde{V}$.

From $\Theta^2 = \mathrm{Id}$ we get $\Theta\tilde{V} = V$. In order to prove

$$(1.1) \qquad \Theta\mathfrak{T} = \mathfrak{T} \quad \text{and} \quad \Theta I = -I$$

we put $\Theta I = a + S + p$. But $[I,b] = b$ implies $[a,\Theta b] = 0$ for $b \in V$ and hence $a = 0$. From (I;1.3) we get $[I,\Theta I] = -p$ and $-\Theta p = \Theta[I,\Theta I] = [\Theta I, I] = p$ yields $p = 0$. Next $\Theta v = \Theta[v,I] = [\Theta v, S]$ for $v \in \tilde{V}$ leads to $[S,b] = -b$ for $b \in V$ and hence to $S = -I$. Since I is in the center of $\mathfrak{T}$ we obtain $\Theta\mathfrak{T} \subset \mathfrak{T}$ from (I;1.3). The elements of a symmetric Lie algebra $(\mathfrak{L}, \Theta)$ we write as $q = a + T + \Theta b$, where $a, b \in V$ and $T \in \mathfrak{T}$.

For a symmetric Lie algebra we are able to express the automorphisms $\tilde{\Psi}_{\Theta b}$ of $\mathfrak{L}$ by Θ and the automorphisms Ψ_b.

LEMMA 1.1. *For* $b \in V$ *we have* $\tilde{\Psi}_{\Theta b} = \Theta\Psi_b\Theta$.

Proof: From (I;1.10) we observe that

$$\Psi_b a = a, \quad \Psi_b I = -b + I, \quad \text{where } a,b \in V.$$

Consider the automorphism $\Phi := \Theta\widetilde{\Psi}_{\Theta b}\Theta$ of $\mathfrak{L}$. Since $\widetilde{\Psi}_{\Theta b}$ is the identity on $\widetilde{V}$ and since $\widetilde{\Psi}_{\Theta b} I = I + \Theta b$ holds, we observe

$$\Phi a = a, \quad \Phi I = -b + I.$$

In particular Φ is essential. In I, §2.3 we have seen that an essential automorphism is uniquely determined by the images ΦI and Φa for $a \in V$. Hence $\Phi = \Psi_b$.

2. The automorphism Θ induces involutions, i.e., involutorial anti-automorphisms, of the groups $\Gamma(\mathfrak{L})$ and $\Xi(\mathfrak{L})$. First we show

LEMMA 1.2. *Let $(\mathfrak{L}, \Theta)$ be a symmetric Lie algebra. Then there exists an involution $W \to W^{\#}$ of $\Gamma(\mathfrak{L})$ such that*

a) $\nabla_W \Theta \nabla_{W^{\#}} = \Theta$ *for all* $W \in \Gamma(\mathfrak{L})$,

b) $\Gamma(\mathfrak{L})$ *acts as a group of inner automorphisms on the Lie algebra* $\mathfrak{T}$, *moreover*

$$(WTW^{-1}) = W^{\#-1}(\Theta T)W^{\#} \quad \textit{for } T \in \mathfrak{T} \textit{ and } W \in \Gamma(\mathfrak{L}).$$

Proof: Consider the automorphism $\Phi = \Theta\nabla_W\Theta$ of $\mathfrak{L}$. Using $\nabla_W I = I$ we get $\Phi I = I$. Hence I. Theorem 3.2. can be applied. There is a $W^{\#} \in GL(V)$ such that $\nabla_W = \Phi = \nabla_{W^{\#-1}}$. Since Φ is an automorphism of $\mathfrak{L}$ we get $W^{\#} \in \Gamma(\mathfrak{L})$. Passing to the inverse we see that

$\nabla_{W^\#} \Theta \nabla_W = \Theta$ and consequently $(W^\#)^\# = W$. The statement b) follows by applying a) on the elements of $\mathfrak{T}$.

Now we can extend the map $W \to W^\#$ to the group $\Xi(\mathfrak{L})$ of birational mappings.

__THEOREM 1.3.__ __Let__ $(\mathfrak{L},\Theta)$ __be a symmetric Lie algebra. Then there exists an involution__ $\Phi \to \Phi^\#$ __of__ $\mathrm{Aut}^*\mathfrak{L}$ __and__ $f \to f^\#$ __of__ $\Xi(\mathfrak{L})$, __respectively, such that__

a) $\Phi\Theta\Phi^\# = \Theta$ __for__ $\Phi \in \mathrm{Aut}^*\mathfrak{L}$,

b) $\nabla_f\Theta\nabla_{f^\#} = \Theta$ __and__ $(\nabla_f)^\# = \nabla_{f^\#}$ __for__ $f \in \Xi(\mathfrak{L})$.

__Proof__: Since $\nabla : \Xi(\mathfrak{L}) \to \mathrm{Aut}^*\mathfrak{L}$ is an isomorphism, it is enough to prove part a). Let Φ be in $\mathrm{Aut}^*\mathfrak{L}$, then $\Phi = \nabla_W \Psi_a \tilde{\Psi}_{\Theta b} \Psi_c$, where $W \in \Gamma(\mathfrak{L})$, $a,b,c \in V$, because of part b) in I, Theorem 4.1. Using Lemma 1.1 we observe that

$$\Theta\Phi\Theta = \Theta\nabla_W \Psi_a{}^\Theta \Psi_b{}^\Theta \Psi_c{}^\Theta$$

$$= \nabla^{-1}_{W^\#}(\Theta\Psi_a\Theta)\Psi_b(\Theta\Psi_c\Theta) \in \mathrm{Aut}^*\mathfrak{L}.$$

Hence $\Phi \to (\Theta\Phi\Theta)^{-1}$ turns out to be an involution of $\mathfrak{L}$. Now the statement follows by setting $\Phi^\# = (\Theta\Phi\Theta)^{-1}$.

Using the theorem we calculate

(1.2) $\quad (\Psi_b)^\# = \tilde{\Psi}_{-\Theta b}$ and $(t_b)^\# = \tilde{t}_{-\Theta b}$ for $b \in V$.

3. Now we are going to prove some basic relations and identities. Let $(\mathfrak{L},\Theta)$ be a symmetric Lie algebra,

$\mathfrak{D} = V + \mathfrak{T} + \tilde{V}$. For a,b $\in$ V we put

$$B(a,b) = B_{\Theta}(a,b) = B_{\Theta b}(a).$$

Hence B(a,b) is a polynomial in a and b which is defined for all a,b in any scalar extension of V, because it is a polynomial of degree ≤ 2 in a and in b and because the characteristic of K is not 2. From (I;2.10) we get $B(-a,-b) = B(a,b)$.

From (I;2.8) it follows then that

$$\tilde{t}_{\Theta b}(x) = [B(x,b)]^{-1}[x-(\Theta b)(x)], \quad b \in V;$$

and part d) of I, Theorem 2.3, implies

$$W \circ \tilde{t}_{\Theta b} = \tilde{t}_{\Theta c} \circ W, \quad \text{where } W \in \Gamma(\mathfrak{D}) \text{ and } c = W^{\#-1}b.$$

From (I;4.5) we see that $f \in \Xi(\mathfrak{D})$ can be written as

(1.3) $\quad f = W \circ t_a \circ \tilde{t}_{\Theta b} \circ t_c$, where $W \in \Gamma(\mathfrak{D})$ and a,b,c $\in$ V.

Moreover (I;4.6) yields

(1.4) $$\frac{\partial f(x)}{\partial x} = W[B(x+c,b)]^{-1}.$$

This formula remains valid in any scalar extension for which x is generic.

<u>LEMMA 1.4</u>. <u>Let</u> a <u>and</u> b <u>be in some scalar extension</u> $L \otimes V$ <u>of</u> V, <u>such that</u> det $B(a,b) \neq 0$. <u>Then</u>

a) $B(a,b) \in \Gamma(L \otimes \mathfrak{D})$ <u>and</u> $[B(a,b)]^{\#} = B(b,a)$.

b) $\tilde{t}_{\Theta b} \circ t_a = W \circ t_c \circ \tilde{t}_{\Theta d}$, <u>where</u> $W = [B(a,b)]^{-1}$,

$c = a-(\ominus b)(a)$, and $d = \tilde{t}_{\ominus a}(b)$.

c) $B(x+a,b) = B(x,\tilde{t}_{\ominus a}(b))\, B(a,b)$.

d) $B(Wa,b) = WB(a,W^{\#}b)W^{-1}$ for $W \in \Gamma(L\otimes\mathfrak{D})$.

Proof: It suffices to prove the lemma in the case that a,b are generically independent elements of some scalar extension $L\otimes V$ and that x is still generic over L.

We put $f = \tilde{t}_{\ominus b}\, t_a$ and obtain $\omega_f(x) = \det B(x+a,b)$ from (1.4) and part a) of I, Theorem 2.3. Hence $\omega_f(0) \neq 0$ and consequently $f \in \Xi^o(L\otimes\mathfrak{D})$. Part a) of I, Theorem 4.3, yields

$$(1.5) \qquad \tilde{t}_{\ominus b}\circ t_a = W\circ t_c \circ \tilde{t}_{\ominus d}$$

for some $W \in \Gamma(L\otimes\mathfrak{D})$ and $c,d \in L\otimes V$. Passing from f to $f^{\#}$ we obtain

$$(1.6) \qquad \tilde{t}_{-\ominus a}\circ t_{-b} = t_{-d}\circ \tilde{t}_{-\ominus c}\circ W^{\#}.$$

Using (1.4) twice we conclude

$$(1.7) \qquad B(x+a,b)W = B(x,d), \quad W^{\#}B(x-b,-a) = B(W^{\#}x,-c).$$

Choosing $x = 0$ we obtain

$$W = [B(a,b)]^{-1} \text{ and } W^{\#} = [B(b,a)]^{-1}.$$

So part a) is proved. Substituting $x = 0$ in (1.6) and using (I;2.10) we get $d = \tilde{t}_{\ominus a}(b)$. Moreover $x = 0$ in (1.5) leads to $Wc = \tilde{t}_{\ominus b}(a)$, i.e., $c = a-(\ominus b)(a)$ according to (I;2.8). So part b) is proved. Now part

c) follows from (1.7).

In order to prove part d) we apply (I;2.11) and get $B(Wa,b)W = WB(a,c)$ where $u = \Theta b$, $v = \Theta c$ and $u = \nabla_W v$. Hence $c = \Theta\nabla_W^{-1}\Theta b = \nabla_{W^\#} b = W^\# b$.

According to (1.4) and part a) of the lemma, for $f \in \Xi(\mathfrak{L})$ the endomorphism $\frac{\partial f(x)}{\partial x}$ belongs to $\Gamma(K'\otimes\mathfrak{L})$ and we have

$$\left(\frac{\partial f(x)}{\partial x}\right)^\# \Bigg|_{x\to a} = \left(\frac{\partial f(x)}{\partial x}\Bigg|_{x\to a}\right)^\# \quad \text{for } a \in \operatorname{Dom} f.$$

§2. The group $\Xi(\mathfrak{L},\Theta)$.

1. Again let $(\mathfrak{L},\Theta)$ be a symmetric Lie algebra, $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$, and let $f \to f^\#$ be the involution of $\Xi(\mathfrak{L})$ induced by Θ according to Theorem 1.3. We consider the group

$$\Xi(\mathfrak{L},\Theta) = \{f; f\in\Xi(\mathfrak{L}),\ f^\#\circ f=I\} = \{f. f\in\Xi(\mathfrak{L}),\ \nabla_f\Theta=\Theta\nabla_f\}.$$

Clearly $f \to f^\#$ maps this group onto itself. Furthermore, let

$$\Gamma(\mathfrak{L},\Theta) = \Xi(\mathfrak{L},\Theta)\cap\Gamma(\mathfrak{L}) = \{W; W\in\Gamma(\mathfrak{L}),\ W^\# W=I\}.$$

Next we define the subset

$$D(\mathfrak{L},\Theta) = \{a; a\in V,\ \text{there exists } W\in\Gamma(\mathfrak{L}) \text{ such that } B(a,-a)=W^\# W\}$$

of V. Clearly 0 belongs to it and $a \in D(\mathfrak{L},\Theta)$ implies $Wa \in D(\mathfrak{L},\Theta)$ for $W \in \Gamma(\mathfrak{L},\Theta)$ because of part d) of

Lemma 1.4. For $a \in D(\mathfrak{D},\Theta)$ we choose $W_a \in \Gamma(\mathfrak{D})$ (not canonically) such that

$$B(a,-a) = W_a^{\#} W_a.$$

Obviously W_a is uniquely determined up to a left-factor out of $\Gamma(\mathfrak{D},\Theta)$. We may choose $W_o = I$, $W_{-a} = W_a$.

Moreover, $a \in \mathrm{Dom}\ \tilde{t}_{-\Theta a}$ for $a \in D(\mathfrak{D},\Theta)$. Hence the element

$$\tilde{a} = W_a\ \tilde{t}_{-\Theta a}(a), \quad a \in D(\mathfrak{D},\Theta),$$

of V can be defined. Finally we put

$$s_a = t_{\tilde{a}} \circ W_a \circ \tilde{t}_{\Theta a}, \quad a \in D(\mathfrak{D},\Theta),$$

and we obtain an element of the subset $\Xi^o(\mathfrak{D})$ of $\Xi(\mathfrak{D})$. Clearly, $-a \in \mathrm{Dom}\ s_a$ so $s_a(-a) = 0$ and $s_a(0) = \tilde{a}$. Moreover, $-s_a(x) = s_{-a}(-x)$.

2. We show that $f \in \Xi(\mathfrak{D},\Theta) \cap \Xi^o(\mathfrak{D})$ is equivalent to $f = W \circ s_a$ where $W \in \Gamma(\mathfrak{D},\Theta)$ and $a \in D(\mathfrak{D},\Theta)$.

According to I, Theorem 4.3, part a), the elements of $\Xi^o(\mathfrak{D})$ are exactly the functions

$$f = W \circ t_b \circ \tilde{t}_{\Theta a} \quad \text{for } a,b \in V \text{ and } W \in \Gamma(\mathfrak{D}).$$

Using (1.2) we observe

$$f^{\#-1} = W^{\#-1} \circ \tilde{t}_{\Theta b} \circ t_a.$$

Hence $f^{\#} \circ f = I$ is equivalent to

$$(2.1) \qquad W^{\#}W\circ t_b\circ \tilde{t}_{\Theta a} = \tilde{t}_{\Theta b}\circ t_a .$$

Using (1.4) we get

$$(2.2) \qquad B(x,a) = B(x+a,b)W^{\#}W.$$

Conversely suppose that (2.2) is satisfied. Hence

$$\frac{\partial f^{\#-1}(x)}{\partial x} = W^{\#-1}[B(x+a,b)]^{-1} = W[B(x,a)]^{-1} = \frac{\partial f(x)}{\partial x}$$

and I, Lemma 4.2, yields $f^{\#-1} = t_c\circ f$ for some $c \in V$. Therefore $f = \tilde{t}_{\Theta c}\circ f^{\#-1} = \tilde{t}_{\Theta c}\circ t_c\circ f$ and we get $\tilde{t}_{\Theta c} = t_{-c}$ from which we obtain $c = 0$. <u>Hence</u> (2.2) <u>is equivalent to</u> $f^{\#}\circ f = I$, $f = W\circ t_b\circ \tilde{t}_{\Theta a}$.

In particular,

$$s_a = t_{\tilde{a}}\circ W_a\circ \tilde{t}_{\Theta a} = W_a\circ t_b\circ \tilde{t}_{\Theta a}, \quad b = W_a^{-1}\tilde{a} = \tilde{t}_{-\Theta a}(a),$$

belongs to $\Xi(\mathfrak{D},\Theta)$ whenever

$$B(x,a) = B(x+a,\tilde{t}_{-\Theta a}(a))B(a,-a)$$

holds. But we get this identity by replacing a by -a, b by a and x by x+a in part c) of Lemma 1.4. So we proved $s_a \in \Xi(\mathfrak{D},\Theta)\cap\Xi^{o}(\mathfrak{D})$.

Now in the notation above let f be in $\Xi(\mathfrak{D},\Theta)$. Substituting $x = -a$ in (2.2) we get $a \in D(\mathfrak{D},\Theta)$ and doing the same in (2.1) we get $W^{\#}W(b+\tilde{t}_{\Theta a}(-a)) = 0$, i.e., $b = \tilde{t}_{-\Theta a}(a) = W_a^{-1}\tilde{a}$. Using (I;1.9) we observe that $f = W_1\circ s_a$, $W_1 \in \Gamma(\mathfrak{D})$. But we know already that s_a satisfies $s_a^{\#}\circ s_a = I$, so $W_1 \in \Gamma(\mathfrak{D},\Theta)$.

3. Suppose that x and y are generically independent elements of a scalar extension of V. Let G be the group of $f \in \Xi(\mathfrak{D})$ satisfying the differential equation

$$B(f(x),-f(y)) = \frac{\partial f(x)}{\partial x} B(x,-y) \left(\frac{\partial f(y)}{\partial y}\right)^{\#} . \tag{2.3}$$

<u>We prove first</u>, <u>that</u> $s_a \in G$ <u>for</u> $a \in D(\mathfrak{D},\Theta)$.

For the proof we rewrite part c) of Lemma 1.4 as follows

$$B(t_a(x),y) = B(x,\tilde{t}_{\Theta a}(y))B(a,y). \tag{2.4}$$

Applying the involution # and interchanging x and y we get

$$B(\tilde{t}_{\Theta a}(x),y) = [B(x,a)]^{-1} B(x,t_a(y)). \tag{2.4'}$$

Because of 2 we have

$$s_a = t_{\tilde{a}} \circ W_a \circ \tilde{t}_{\Theta a} = s_a^{\#-1} = \tilde{t}_{\Theta\tilde{a}} \circ W_a^{\#-1} \circ t_a$$

and hence using (2.4)

$$\begin{aligned} B(s_a(x),-s_a(y)) &= B(s_a(x),-s_a^{\#-1}(y)) \\ &= B(W_a \circ \tilde{t}_{\Theta a}(x), W_a^{\#-1} t_{-a}(-y))W_a Q(y) \end{aligned}$$

where Q(y) does not depend on x. Now part d) of Lemma 1.4 and (2.4′) yields

$$B(s_a(x),-s_a(y)) = W_a B(\tilde{t}_{\Theta a}(x),t_{-a}(-y))Q(y)$$

$$= W_a[B(x,a)]^{-1} B(x,-y)Q(y).$$

Using $f = s_a$ we have

$$B(f(x),-f(y)) = \frac{\partial f(x)}{\partial x} B(x,-y)Q(y).$$

Applying # and interchanging x and y we get

$$B(f(x),-f(y)) = [Q(x)]^{\#} B(x,-y)\left(\frac{\partial f(y)}{\partial y}\right)^{\#}.$$

Specializing $x \to -a$ we have $f(0) = 0$,

$$\left(\frac{\partial f(x)}{\partial x}\right)_{x\to -a} = W_a[B(-a,a)]^{-1} = W^{\#-1}$$

and therefore we obtain $I = [Q(y)]^{\#} B(a,y)W_a^{\#-1}$ and this means that $Q(y) = \left(\frac{\partial f(y)}{\partial y}\right)^{\#}$. Hence $s_a \in G$.

4. As in I, §4.2, we denote by D_f the set of $a \in V$ such that $\omega_f(a) \neq 0$; we know $D_f \subset \mathrm{Dom}\, f$. We introduce now the condition

(A) $\qquad D(\mathfrak{L},\Theta) \cap D_f \neq \emptyset \quad$ for all $f \in \Xi(\mathfrak{L})$,

which is certainly satisfied, if $D(\mathfrak{L},\Theta)$ meets all the Zarski-open subsets of V.

THEOREM 2.1. *Let $(\mathfrak{L},\Theta)$ be a symmetric Lie algebra, $\mathfrak{L} = V + \mathfrak{T} + \tilde{V}$, and suppose that* (A) *is satisfied. Then for $f \in \Xi(\mathfrak{L})$ the following conditions are equivalent*:

a) $f^{\#}\circ f = I$, i.e., $f \in \Xi(\mathfrak{L},\Theta)$.

b) $B(f(x),-f(y)) = \frac{\partial f(x)}{\partial x} B(x,-y) \left(\frac{\partial f(y)}{\partial y}\right)^{\#}$,

<u>where</u> x <u>and</u> y <u>are generically independent</u>.

c) $f = W \circ s_a \circ s_b$, <u>where</u> $W \in \Gamma(\mathfrak{D},\Theta)$ <u>and</u> $a,b \in D(\mathfrak{D},\Theta)$.

<u>In this case</u> $f = W \in \Gamma(\mathfrak{D},\Theta)$ <u>if and only if</u>

$0 \in D_f$ <u>and</u> $f(0) = 0$.

<u>COROLLARY</u>. <u>If</u> $f \in \Xi(\mathfrak{D},\Theta) \cap \Xi^{o}(\mathfrak{D})$ <u>then</u> $f = W \circ s_a$ <u>where</u> $W \in \Gamma(\mathfrak{D},\Theta)$ <u>and</u> $a \in D(\mathfrak{D},\Theta)$ <u>and this representation is unique</u>.

<u>Proof</u>: a) $\Rightarrow$ c): Let $f^{\#} \circ f = I$, then by (A) there exists $a \in D(\mathfrak{D},\Theta) \cap D_f$. Forming $g = f \circ s_{-a}^{-1}$ we get $g^{\#} \circ g = I$ because of $\underline{\underline{2}}$ and $0 \in D_g$. Applying $\underline{\underline{2}}$ we have $g = W \circ s_b$, where $W \in \Gamma(\mathfrak{D},\Theta)$ and $b \in D(\mathfrak{D},\Theta)$. So c) is proved.

b) $\Rightarrow$ c): Again by (A) there exists $a \in D(\mathfrak{D},\Theta) \cap D_f$. Specializing $x \to a$ and $y \to a$ in (2.3) we get

$$B(f(a),-f(a)) = W\, B(a,-a) W^{\#}, \quad W = \left.\frac{\partial f(x)}{\partial x}\right|_{x \to a} .$$

Hence $b = -f(a) \in D(\mathfrak{D},\Theta)$. Forming $g = s_b \circ f$ we see that g satisfies (2.3) because of $\underline{\underline{3}}$ and $g(a) = 0$. Specializing $x \to a$ and $y \to a$ in (2.3) we get $a \in D(\mathfrak{D},\Theta)$ and $h = g \circ s_{-a}^{-1} = s_b \circ f \circ s_{-a}^{-1}$ satisfies $h(0) = 0$ and (2.3). Specializing $y \to 0$ in (2.3) we see that $\frac{\partial h(x)}{\partial x}$ does not depend on x. Hence $h = t_c \circ W$ by I, Lemma 4.2, and $h(0) = 0$ yields $h = W$. Again (2.3) leads to $W \in \Gamma(\mathfrak{D},\Theta)$.

So c) and in addition the last statement of the theorem are proved.

The conclusions c) ⇒ b) and b) ⇒ a) follow from 2 and 3.

§3. Constructions of symmetric Lie algebras.

1. A bilinear map $(a,b) \to a\square b$ of $V\times V$ into End V is called a pairing of V. Let $\mathfrak{T} = \mathfrak{T}_\square$ be the subspace of End V spanned by $a\square b$ for all $a,b \in V$. The symmetric bilinear form $\sigma = \sigma_\square$ of V is given by

$$\sigma(a,b) = \text{trace } (a\square b+b\square a).$$

We call σ the trace form of the pairing $\square$. Suppose that

(P.1) σ is non degenerate.

Then by T^* we denote the adjoint endomorphism of $T \in$ End V with respect to σ. Denote by $[T,S] = TS - ST$ the commutator product in End V. We assume that in addition the following conditions hold:

(P.2) $(a\square b)c = (c\square b)a$ for $a,b,c \in V$.

(P.3) $[T,a\square b] = Ta\square b - a\square T^*b$ for $a,b \in V$ and $T \in \mathfrak{T}$

(P.4) $(a\square b)^* = b\square a$ for $a,b \in V$.

From (P.3) we observe that <u>$\mathfrak{T}$ turns out to be a Lie algebra of endomorphisms of</u> V, for which $T \to -T^*$ is an automorphism of period 2. According to (P.4) we get $\sigma(a,b) = 2$ trace $a\square b$.

Using (P.2) and (P.4) we observe that $\sigma((a\square b)c,d) = \sigma((c\square b)a,d) = \sigma(a,(b\square c)d) = \sigma((c\square d)a,b)$. Hence by linear extension of

$$\sigma_{\mathfrak{T}}(T,S) = \sigma(Tc,d) = \sigma(Sa,b) \quad \text{for } T=a\square b,\ S=c\square d,$$

we may define the symmetric bilinear form $\sigma_{\mathfrak{T}}$ of $\mathfrak{T}$, which is also non degenerate. A verification shows <u>that $\sigma_{\mathfrak{T}}$ is an associative bilinear form of $\mathfrak{T}$ satisfying</u>

(3.1) $$\sigma_{\mathfrak{T}}(T^*,S) = \sigma_{\mathfrak{T}}(T,S^*) \quad \text{for } T,S \in \mathfrak{T}.$$

<u>LEMMA 3.1</u>. <u>The identity</u> I <u>belongs to</u> $\mathfrak{T}$ <u>and</u> trace $T = \frac{1}{2}\sigma_{\mathfrak{T}}(T,I)$ <u>for</u> $T \in \mathfrak{T}$.

<u>Proof</u>: Since $\sigma_{\mathfrak{T}}$ is a non degenerate bilinear form of the vector space $\mathfrak{T}$, to the linear form trace T there corresponds $J \in \mathfrak{T}$ such that trace $T = \sigma_{\mathfrak{T}}(T,J)$. From trace T^* = trace T together with (3.1) we observe $J^* = J$. For $T = a\square b$ it follows that

$$\sigma(a,b) = \text{trace }(a\square b+b\square a) = \sigma_{\mathfrak{T}}(J,a\square b+b\square a) = 2\sigma(Ja,b).$$

Hence $2J = I \in \mathfrak{T}$.

<u>2</u>. Let x,y,z be generically independent elements of some scalar extension of V. We define endomorphisms

$P(x,y)$ and $P(x)$ of a suitable scalar extension by

(3.2) $P(x,y)z = \frac{1}{2}(x\square z)y$ and $P(x) = P(x,x)$, respectively.

Because of (P.2) the endomorphism $P(x,y)$ is symmetric and linear in x and y. Moreover

$$P(x+y) = P(x) + 2P(x,y) + P(y).$$

Let $a \in V$ be such that $P(x)a = 0$, hence $(x\square a)x = 0$ and by linearizing we get $(x\square a)y = 0$ and hence $b\square a = 0$ for all $b \in V$. By (P.4) we observe $a\square b = 0$ and consequently $\sigma(a,b) = 0$ for all $b \in V$. So we proved that

(3.3) $$P(x)a = 0,\ a \in V, \text{ implies } a = 0.$$

Note that nevertheless the determinant of $P(x)$ can be the zero function.

Replacing a by x and b by y in (P.3) and applying the result on x we get $T(x\square y)x-(x\square y)Tx = (Tx\square y)x-(x\square T^{*}y)x$ and consequently

(3.4) $$2P(x,Tx) = TP(x)+P(x)T^{*} \quad \text{for } T \in \mathfrak{T}.$$

By using (P.2) and (P.4) in scalar extensions of V we observe that

$$\sigma(P(x)a,b) = \sigma((x\square a)x,b) = \sigma(x,(a\square x)b) = \sigma(x,(b\square x)a)$$
$$= \sigma((x\square b)x,a) = \sigma(a,P(x)b)$$

and consequently $P(x)$ <u>is self-adjoint with respect to</u> σ.

3. Next we consider for a given pairing $\square : V \times V \to \mathrm{End}\, V$ the direct sum of vector spaces

$$\mathfrak{L} = \mathfrak{L}_{\square} = \mathfrak{T} \oplus V \oplus V$$

and we write the elements of $\mathfrak{L}$ as $u = T \oplus a \oplus b$ where $T \in \mathfrak{T}$ and $a, b \in V$. By

(3.5) $$q_u(x) = -a + Tx + P(x)b$$

we obtain a linear injection $u \to q_u$ of $\mathfrak{L}$ into the subspace $\mathfrak{P} = \mathfrak{P}_o + \mathfrak{P}_1 + \mathfrak{P}_2$ of Pol V. We again identify $\mathfrak{T}$ with the space $\{Tx; T \in \mathfrak{T}\}$ of linear functions and put

$$\tilde{V} = \tilde{V}_{\square} = \{P(x)b; b \in V\}.$$

Hence the image of $\mathfrak{L}$ under the map $u \to q_u$ is given by

$$\mathfrak{Q} = \mathfrak{Q}_{\square} = V + \mathfrak{T} + \tilde{V}.$$

It follows from (3.3) that $b \to P(x)b$ is a linear bijection of V onto $\tilde{V}$.

THEOREM 3.2. *Suppose that the pairing* $\square : V \times V \to \mathrm{End}\, V$ *satisfies the conditions* (P.1) *to* (P.4). *Then* $\mathfrak{Q} = \mathfrak{Q}_{\square}$ *turns out to be a binary Lie algebra, for which in addition* $\mathfrak{T} = [V, \tilde{V}]$ *holds.*

Proof: We have to prove that the conditions (B.1) to (B.5) of I, §2.1, are fulfilled. (B.1) is clear because $\mathfrak{T}$ is a Lie algebra of endomorphisms of V. (B.5) follows from Lemma 3.1. Let $p(x) = P(x)b$ and

$q(x) = P(x)c$ be elements of $\tilde{V}$. For $a \in V$ we observe that

$$[p,a](x) = 2p(x,a) = 2P(x,a)b = 2(a\Box b)x,$$

hence $[p,a] = a\Box b \in \mathfrak{T}$ and (B.2) is proved.

For $T \in \mathfrak{T}$ we obtain

$$[p,T](x) = 2p(x,Tx)-Tp(x) = [2P(x,Tx)-TP(x)]b = P(x)T^{*}b$$

by using (3.4). Hence $[p,T] \in \tilde{V}$ and (B.3) is proved.

Finally we have

$$\tfrac{1}{2}[p,q](x) = p(x,q(x))-q(x,p(x)) = P(x,P(x)c)b-P(x,P(x)b)c.$$

We apply (P.3) for $T = a\Box d$ on a and obtain

$$2P(a,P(a)b)d-P(a,P(a)d)b = P(a)P(b,d)a.$$

Since the right side is symmetric in b,d we observe $3P(a,P(a)b)d = 3P(a,P(a)d)b$. Hence $[p,q] = 0$ and the theorem is proved.

We apply Meyberg's Theorem (I,§2.7) to this case. For $v(x) = -P(x)c$ we obtain $[[a,v],b] = 2[P(x,a)c,b]$ $= 2P(a,b)c = (a\Box c)b$ and hence we have

THEOREM 3.3. *Suppose that the pairing* $\Box : V\times V \to \operatorname{End} V$ *satisfies the conditions* (P.1) *to* (P.4). *Then for any given* $c \in V$ *the product* $(a,b) \to P(a,b)c$ *defines a Jordan algebra in* V.

Using the bijection $u \to q_u$ of $\mathfrak{B}$ onto $\mathfrak{Q}$ we lift the product of $\mathfrak{Q}$ to $\mathfrak{B}$. Hence $\mathfrak{B}$ turns out to be a Lie algebra with respect to product $u = [u_1,u_2] = T\oplus a\oplus b$, $u_i = T_i\oplus a_i\oplus b_i$, that is given by

$$T = [T_1,T_2] + a_1\Box b_2 - a_2\Box b_1,$$

(3.6)

$$a = T_1a_2 - T_2a_1, \quad b = T_2^*b_1 - T_1^*b_2 .$$

Algebras of this type are considered in [8] in a more general set-up.

4. For the given pairing $\Box : V\times V \to \operatorname{End} V$ we write $\Gamma = \Gamma_\Box = \Gamma(\mathfrak{L}_\Box)$. Remember that $W \in GL(V)$ belongs to Γ if and only if

$$W\mathfrak{T}W^{-1} = \mathfrak{T} \text{ and } \nabla_W\tilde{V} = \tilde{V}$$

(see I, §4.1).

LEMMA 3.4. Let W be in GL(V). Then the following conditions are equivalent:

a) $W \in \Gamma$,

b) $P(Wx) = WP(x)W^*$,

c) $W(a\Box b)W^{-1} = Wa\Box W^{*-1}b$.

Proof: First of all, $W \in \Gamma$ is equivalent to

$$W(a\Box b)W^{-1} \in \mathfrak{T} \text{ and } WP(W^{-1}x)b = P(x)\tilde{W}b$$

for $a,b \in V$, when $\tilde{W}$ is some endomorphism of V. The second condition means $WP(x) = P(Wx)\tilde{W}$ and this is equivalent to

$$(3.7) \qquad W(a\square b)W^{-1} = Wa\square\widetilde{W}b \quad \text{for } a,b \in V.$$

Hence (3.7) is equivalent to $W \in \Gamma$. Going over to the trace in (3.7) we get $\sigma(a,b) = \sigma(Wa,\widetilde{W}b)$ and consequently $\widetilde{W} = W^{*-1}$.

5. We define a bijection Θ of $\mathfrak{Q} = \mathfrak{Q}_\square$ by setting

$$(\Theta q)(x) = -b-T^*x-P(x)a \quad \text{where } q(x) = a+Tx+P(x)b.$$

Hence $\Theta^2 = \mathrm{Id}$ and $\Theta V = \widetilde{V}$, $\Theta\mathfrak{T} = \mathfrak{T}$. A verification shows that <u>that Θ is an automorphism of the Lie algebra $\mathfrak{Q}$ and $(\mathfrak{Q},\Theta)$ turns out to be a symmetric Lie algebra</u>. In particular one has

$$\Theta T = -T^*, \quad [a,\Theta b] = a\square b.$$

Again we write $q = a + T + \Theta b$ for the elements in $\mathfrak{Q}$.

The symmetric Lie algebra $(\mathfrak{Q},\Theta)$ induces an involution $W \to W^\#$ of Γ according to Lemma 1.2. In order to prove

$$W^\# = W^*$$

we apply $\nabla_W \Theta \nabla_{W^\#} = \Theta$ to $a \in V$ and observe that $WP(W^{-1}x)W^\# a = P(x)a$. Hence the statement follows from Lemma 3.4.

Using the abbreviation given in §1.3 we are going to prove

$$(3.8) \qquad B(a,b) = B_{\Theta b}(a) = I + a\square b + P(a)P(b).$$

We put $T = [\Theta b, a] = -a\square b$ and get

$$[\Theta b, [\Theta b, a]] = [\Theta b, T] = \Theta[b, \Theta T] = \Theta[T^*, b] = \Theta(T^* b)$$

where $T^* b = -(b\square a)b = -2P(b)a$. Using the definition (I;2.7) we observe that $[B_{\Theta b}(x)]a = a+(a\square b)x+P(x)P(b)a = [I+x\square b+P(x)P(b)]a$, hence (3.8) is proved.

From Lemma 1.4 we obtain

$$B(a,b) \in \Gamma \quad \text{if} \quad \det B(a,b) \neq 0$$

and hence Lemma 3.4 yields

$$P(B(a,b)x) = B(a,b)P(x)B(b,a), \quad a,b \in V. \tag{3.9}$$

In part c) of Lemma 1.4 we compare the terms that are of degree two in x and observe that $P(x)P(b) = P(x)P(\tilde{t}_{\Theta a}(b))B(a,b)$ whenever $\det B(a,b) \neq 0$. Now (3.3) leads to

$$P(y) = P(\tilde{t}_{\Theta a}(y))B(a,y).$$

But from the definition (I;2.8) and (3.9) it follows that

$$P(y+P(y)a) = B(y,a)P(y)$$

and again comparing the terms of highest degree in y we end up with

$$P(P(a)y) = P(a)P(y)P(a). \tag{3.10}$$

Hence our method is powerful enough to prove non-trivial identities about the pairings.

6. We generalize Θ by setting

$$\Theta_J = \Theta\nabla_J = \nabla_J^{-1}\Theta, \text{ whenever } J \in \Gamma \text{ and } J^* = J.$$

Using the abbreviation

$$T^J = J^{-1}T^*J, \quad T \in \text{End } V,$$

we have more explicitely

$$(3.11) \qquad \Theta_J q = J^{-1}b - T^J + \Theta(Ja), \text{ where } q = a + T + \Theta b \in \mathfrak{Q}.$$

Again Θ_J is an automorphism of period two satisfying $\Theta_J V = \tilde{V}$. <u>Hence for any</u> $J \in \Gamma$, $J^* = J$, <u>the pair</u> $(\mathfrak{Q}, \Theta_J)$ <u>is a symmetric Lie algebra</u>. Using I, Theorem 3.2, one can easily show that these are the only automorphisms of $\mathfrak{Q}$ which lead to a symmetric Lie algebra.

Note that $T \to -T^J$ is an automorphism of the Lie algebra $\mathfrak{T}$. By the same argument that we used in 5, one shows <u>that the involution of</u> Γ <u>induced by</u> Θ_J <u>is given by</u> $W^\# = W^J$.

§4. Killing forms.

1. Let $\mathfrak{G}$ be a Lie algebra over K. Denote by ad u the adjoint representation. For a linear transformation A of $\mathfrak{G}$ mapping a subspace $\mathfrak{b}$ of $\mathfrak{G}$ into itself we denote by $A_{\mathfrak{b}}$ the restriction of A to $\mathfrak{b}$. Let

$$\langle u,v \rangle_{\mathfrak{G}} = \text{trace (ad u)(ad v)}$$

denote the Killing form of $\mathfrak{G}$.

Suppose that there is a direct sum decomposition

$$\mathfrak{G} = \mathfrak{a} + \mathfrak{b}, \ [\mathfrak{a},\mathfrak{a}] \subset \mathfrak{a}, \ [\mathfrak{a},\mathfrak{b}] \subset \mathfrak{b}, \ [\mathfrak{b},\mathfrak{b}] \subset \mathfrak{a}.$$

Then we prove first

LEMMA 4.1. a) The subspaces $\mathfrak{a}$ and $\mathfrak{b}$ are orthogonal with respect to the Killing form of $\mathfrak{G}$.

b) For $g = a + b$, $a \in \mathfrak{a}$, $b \in \mathfrak{b}$, one has

$$\langle g,g\rangle_{\mathfrak{G}} = \langle a,a\rangle_{\mathfrak{a}} + \operatorname{trace} (\operatorname{ad} a)^2_{\mathfrak{b}} + 2 \operatorname{trace} (\operatorname{ad} b)^2_{\mathfrak{b}}.$$

Proof: a) From $(\operatorname{ad} a)(\operatorname{ad} b)\mathfrak{a} \subset \mathfrak{b}$ and $(\operatorname{ad} a)(\operatorname{ad} b)\mathfrak{b} \subset \mathfrak{a}$ it follows that $\langle a,b\rangle_{\mathfrak{G}} = \operatorname{trace} (\operatorname{ad} a)(\operatorname{ad} b) = 0$.

b) For $g \in \mathfrak{G}$ put $(\operatorname{ad}_+ g)(a+b) = [g,a]$ and $(\operatorname{ad}_- g)(a+b) = [g,b]$. Hence $\operatorname{ad} g = \operatorname{ad}_+ g + \operatorname{ad}_- g$. A verification yields

$$(4.1) \quad (\operatorname{ad} a)^2 = (\operatorname{ad}_+ a)^2 + (\operatorname{ad}_- a)^2,$$

$$(\operatorname{ad} b)^2 = (\operatorname{ad}_+ b)(\operatorname{ad}_- b) + (\operatorname{ad}_- b)(\operatorname{ad}_+ b).$$

Since $(\operatorname{ad}_+ a)^2$ is zero on $\mathfrak{b}$ and equals the square of the adjoint representation on $\mathfrak{a}$ we get $\operatorname{trace} (\operatorname{ad}_+ a)^2 = \langle a,a\rangle_{\mathfrak{a}}$. Moreover, $(\operatorname{ad}_- a)^2$ is zero on $\mathfrak{a}$ and equals $(\operatorname{ad} a)^2_{\mathfrak{b}}$ on $\mathfrak{b}$, hence $\operatorname{trace} (\operatorname{ad}_- a)^2 = \operatorname{trace} (\operatorname{ad} a)^2_{\mathfrak{b}}$. So we obtain the statement for $g = a$ using (4.1). Again from (4.1) we obtain $\operatorname{trace} (\operatorname{ad} b)^2 = 2 \operatorname{trace} (\operatorname{ad}_+ b)(\operatorname{ad}_- b)$. But $(\operatorname{ad}_+ b)(\operatorname{ad}_- b)$ is zero on $\mathfrak{a}$ and equals $(\operatorname{ad} b)^2_{\mathfrak{b}}$ on $\mathfrak{b}$. Hence the lemma is proved.

2. Now let $\square : V\times V \to \mathrm{End}\, V$ be again a pairing satisfying the conditions (P.1) to (P.4) and let σ be its trace form. We consider the Lie algebra $\mathfrak{L} = \mathfrak{L}_\square = V + \mathfrak{T} + \tilde{V}$ together with the involution Θ. Let $\sigma_\mathfrak{T}$ be the associative bilinear form of $\mathfrak{T}$ given in §3.1. For $q_\nu = a_\nu + T_\nu + \Theta b_\nu \in \mathfrak{L}$ we put

$$\sigma_\mathfrak{L}(q_1,q_2) = \sigma_\mathfrak{T}(T_1,T_2) + \sigma(a_1,b_2) + \sigma(a_2,b_1).$$

Clearly $\sigma_\mathfrak{L}$ is a symmetric non degenerate bilinear form of $\mathfrak{L}$ and a verification shows that $\sigma_\mathfrak{L}$ is an associative bilinear form for $\mathfrak{L}$.

LEMMA 4.2. *The Killing form of the Lie algebra $\mathfrak{L}$ is non degenerate and coincides with $\sigma_\mathfrak{L}$. In addition we have*

$$\langle T,T\rangle_\mathfrak{T} + 2 \text{ trace } T^2 = \sigma_\mathfrak{T}(T,T) \qquad \textit{where } T\in\mathfrak{T}\ .$$

Proof: We apply Lemma 4.1 to the case $\mathfrak{G} = \mathfrak{L}$, $\mathfrak{a} = \mathfrak{T}, \mathfrak{b} = V+\tilde{V}$. Applying

$$[T,\Theta b] = \Theta[\Theta T,b] = \Theta[-T^*,b] = -\Theta(T^*b)$$

we calculate for $c,d \in V$

$$(\text{ad } T)^2(c+\Theta d) = [T,Tc-\Theta(T^*b)] = T^2c + \Theta(T^{*2}b).$$

Hence

$$\text{trace } (\text{ad } T)^2_{V+\tilde{V}} = \text{trace } (T^2+T^{*2}) = 2 \text{ trace } T^2.$$

Furthermore we have for $T = a\square d - c\square b$

$$(\mathrm{ad}(a+\Theta b))^2(c+\Theta d) = [a+\Theta b,[a,\Theta d]+[\Theta b,c]] = [a+\Theta b,T]$$

$$= -Ta+\Theta T^* b = +(a\Box b)c-(a\Box d)a+\Theta(-(b\Box c)b+(b\Box a)d)$$

and consequently

$$\mathrm{trace}\,[\mathrm{ad}(a+\Theta b)]^2_{V+\tilde{V}} = \mathrm{trace}(a\Box b+b\Box a) = \sigma(a,b).$$

Summing up we get

$$\langle q,q\rangle_{\mathfrak{L}} = \langle T,T\rangle_{\mathfrak{T}} + 2\ \mathrm{trace}\ T^2 + 2\sigma(a,b) \text{ where } q = a+T+\Theta b.$$

Since the Killing form of a Lie algebra is associative, we obtain an associative bilinear form λ by setting $\lambda(q_1,q_2) = \langle q_1,q_2\rangle_{\mathfrak{L}} - \sigma_{\mathfrak{L}}(q_1,q_2)$. But $\lambda(q_1,q_2) = \lambda(T_1,T_2)$ implies $\lambda(T,a\Box b) = \lambda(T,[a,\Theta b]) = \lambda(Ta,\Theta b) = 0$. So $\lambda = 0$ and the lemma is proved.

3. In order to give a sufficient condition for $\mathfrak{L} = \mathfrak{L}_\Box$ to be simple we prove first

LEMMA 4.3. *A subset $\mathfrak{J}$ of $\mathfrak{L}$ is an ideal of $\mathfrak{L}$ if and only if*

$$\mathfrak{J} = V_o + \mathfrak{T}_o + \Theta V_1,$$

where V_o and V_1 are subspaces of V, $\mathfrak{T}_o$ an ideal of $\mathfrak{T}$ such that

$$\mathfrak{T}V_\nu \subset V_\nu,\ \mathfrak{T}_o V \subset V_o,\ \mathfrak{T}_o^* V \subset V_1,\ [V,\Theta V_\nu] \subset \mathfrak{T}_o$$

holds for $\nu = 1,2$.

Proof: Let $\mathfrak{J}$ be an ideal of $\mathfrak{L}$ and let $q = a + T + \Theta b \in \mathfrak{J}$. We observe $[I,q] = a-\Theta b \in \mathfrak{J}$ and $[I,[I,q]] = a+\Theta b \in \mathfrak{J}$. Hence T, a and Θb belong to $\mathfrak{J}$ and we have $\mathfrak{J} = V_o + \mathfrak{T}_o + \Theta V_1$. A verification leads now to the conditions listed in the lemma.

<u>THEOREM 4.4.</u> <u>If</u> $\mathfrak{T}$ <u>acts irreducibly on</u> V <u>then</u> $\mathfrak{L}$ <u>is a simple Lie algebra</u>.

<u>Proof</u>: Let $\mathfrak{J}$ be an ideal of $\mathfrak{L}$. Then $\mathfrak{J} = V_o + \mathfrak{T}_o + \Theta V_1$ according to Lemma 4.3 and we have $\mathfrak{T}V_\nu \subset V_\nu$. Hence the V_ν's are invariant under $\mathfrak{T}$. By assumption the only invariant subspaces of V are 0 and V itself.

The case $V_o = 0$ or $V_1 = 0$ yields $\mathfrak{T}_o = 0$ and hence $[V, \Theta V_\nu] = 0$. For $a \in V$ and $b \in V_\nu$ we obtain $a \square b = 0$ and hence $\sigma(a,b) = 0$. That means that $V_o = 0$ or $V_1 = 0$ implies $V_o = V_1 = 0$ and hence $\mathfrak{J} = 0$.

In the case $V_o = V_1 = V$ we get $[V, \Theta V] \subset \mathfrak{T}_o$ and hence $\mathfrak{J} = \mathfrak{L}$.

4. According to the criterion of Killing-Cartan in case of characteristic zero, any pairing gives rise to a <u>semi-simple</u> Lie algebra.

As a further application of the lemma we prove

<u>LEMMA 4.5.</u> <u>An endomorphism</u> T <u>of</u> V <u>belongs to</u> $\mathfrak{T}$ <u>if and only if</u>

$$2P(Tx,x) = TP(x) + P(x)T^*.$$

Comparing this result with Lemma 3.4 we see that $\mathfrak{T}$ coincides with the Lie algebra belonging to the linear algebraic group $\Gamma = \Gamma_\square$.

<u>Proof</u>: Because of (3.4) it is enough to consider a T satisfying the condition above. By linearization we get $[T, a\square b] = Ta\square b - a\square T^* b$ and hence $[T, \mathfrak{T}] \subset \mathfrak{T}$. On the other hand for $v(x) = P(x)a$ we get $[v, T](x) = 2P(Tx, x)a - TP(x)a = P(x)T^* a \in \tilde{V}$. Therefore $[T, \mathfrak{Q}] \subset \mathfrak{Q}$ and hence $q \to [T, q]$ turns out to be a derivation of $\mathfrak{Q}$. But a Lie algebra with non degenerate Killing form has only inner derivations (see N. Jacobson, Lie algebras, page 74). Hence $[T, q] = [q_o, q]$ for some $q_o \in \mathfrak{Q}$. Since a binary Lie algebra has center 0 we end up with $T = q_o$ and hence $T \in \mathfrak{T}$.

§5. A characterization of symmetric Lie algebras.

Essential parts of the following results are due to K. Meyberg and U. Hirzebruch.

1. Let $\mathfrak{G}$ be a finite dimensional Lie algebra over a field K of characteristic different from 2 and 3. Suppose there exists a direct sum decomposition

$$\mathfrak{G} = \mathfrak{h} + \mathfrak{a} + \mathfrak{b}$$

as vector spaces having the composition rules

(1) $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} = [\mathfrak{a}, \mathfrak{b}], \ [\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}, \ [\mathfrak{h}, \mathfrak{b}] \subset \mathfrak{b},$

$$[\mathfrak{a}, \mathfrak{a}] = [\mathfrak{b}, \mathfrak{b}] = 0.$$

The elements of $\mathfrak{G}$ are in an obvious notation written as $u = h+a+b$. Suppose further

(2) The Killing form $\langle u,v\rangle$ of $\mathfrak{G}$ is non degenerate.

(3) There exists an automorphism τ of $\mathfrak{G}$ of period 2 satisfying

$$\tau\mathfrak{h} = \mathfrak{h}, \quad \tau\mathfrak{a} = \mathfrak{b}, \quad \tau\mathfrak{b} = \mathfrak{a}.$$

If a pairing $\square : V\times V \to \mathrm{End}\, V$ *satisfies the conditions* (P.1) *to* (P.4) *of* §3.1, *then* $\mathfrak{D} = \mathfrak{D}_\square = V+\mathfrak{T}+\tilde{V}$ *together with the automorphism* $\tau = \Theta$ *defined in* §3.5 *satisfies the conditions* (1) *to* (3) *for* $\mathfrak{h} = \mathfrak{T}$, $\mathfrak{a} = V$ *and* $\mathfrak{b} = \tilde{V}$ (see Theorem 3.2 and Lemma 4.2).

2. Suppose now that $\mathfrak{G}$ satisfies the conditions (1) to (3). We are going to prove some propositions:

PROPOSITION 1. *There exists* h_o *in the center of* $\mathfrak{h}$ *such that*

$$[h_o,a] = a \quad \textit{and} \quad [h_o,b] = -b$$

for $a\in\mathfrak{a}$, $b\in\mathfrak{b}$.

Proof: Using (1), a verification shows that the map $h+a+b \to a-b$ is a derivation of $\mathfrak{G}$. Because of (2) any derivation of $\mathfrak{G}$ is inner (see N. Jacobsen [6], page 74), hence there exists $u_o\in \mathfrak{G}$ such that $[u_o,u] = a-b$ for $u = h+a+b \in \mathfrak{G}$. Hence

$[h_o,h] = 0, \quad a = [h_o,a]+[a_o,h], \quad -b = [h_o,b]+[b_o,h].$

We observe $a_o = b_o = 0$ and the proposition is proved.

PROPOSITION 2. *Let* $u_i = h_i+a_i+b_i \in \mathfrak{G}$. *Then one has*

$$\langle u_1,u_2\rangle = \langle h_1,h_2\rangle+\langle a_1,b_2\rangle+\langle a_2,b_1\rangle$$

and the following implications

$$\langle b,\mathfrak{a}\rangle = 0 \Rightarrow b = 0 \qquad \langle a,\mathfrak{b}\rangle = 0 \Rightarrow a = 0,$$

$$[h,\mathfrak{a}] = 0 \Rightarrow h = 0, \qquad [h,\mathfrak{b}] = 0 \Rightarrow h = 0,$$

where $h\in\mathfrak{h}$, $a\in\mathfrak{a}$, $b\in\mathfrak{b}$.

Proof: Using Proposition 1, we observe that $\langle h,a\rangle = \langle h,[h_o,a]\rangle = \langle [h,h_o],a\rangle = 0$ and similarly $\langle h,b\rangle = 0$. Furthermore $\langle a_1,a_2\rangle = \langle [h_o,a_1],a_2\rangle$ $= -\langle a_1,[h_o,a_2]\rangle = -\langle a_1,a_2\rangle$, hence $\langle a_1,a_2\rangle = 0$ and similarly we obtain $\langle b_1,b_2\rangle = 0$. So the Killing form of $\mathfrak{G}$ has the form indicated in the statement. From (2) we obtain the first two implications. Finally suppose $[h,\mathfrak{a}] = 0$. Then $0 = \langle [h,\mathfrak{a}],\mathfrak{b}\rangle = \langle \mathfrak{a},[h,\mathfrak{b}]\rangle$ and $[h,\mathfrak{b}] = 0$. Similarly, $[h,\mathfrak{b}] = 0$ implies $[h,\mathfrak{a}] = 0$. Next $[h,\mathfrak{h}] = [h,[\mathfrak{a},\mathfrak{b}]] = [[h,\mathfrak{a}],\mathfrak{b}] + [\mathfrak{a},[h,\mathfrak{b}]] = 0$ and h is in the center of $\mathfrak{G}$. But $\mathfrak{G}$ is centerless because of (2).

3. Next we put $V = \mathfrak{a}$ and we write now the elements of $\mathfrak{G}$ as $h+a+\tau b$, where $h\in\mathfrak{h}$ and $a,b \in V$. Setting

$$\sigma(a,b) = \langle a, \tau b\rangle, \qquad a,b \in V$$

we obtain a symmetric bilinear form of V because of $\langle \Theta u_1, \Theta u_2\rangle = \langle u_1, u_2\rangle$. By Proposition 2, σ <u>is non degenerate</u>.

For $h \in \mathfrak{h}$ we define an endomorphism T_h of V by

$$T_h a = [h,a], \qquad a \in V.$$

By Proposition 2, $h \to T_h$ is a linear injection of $\mathfrak{h}$ into End V. Moreover, a pairing $\square : V\times V \to \text{End } V$ is given by

$$a\square b = T_{[a,\tau b]} \quad \text{where } a,b \in V.$$

<u>PROPOSITION 3</u>. <u>The pairing</u> $\square : V\times V \to \text{End } V$ <u>satisfies the conditions</u> (P.1) <u>to</u> (P.4) <u>of</u> §3.1 <u>and</u> σ <u>is its trace form</u>. <u>Furthermore</u>, $h \to T_h$ <u>defines an isomorphism of the Lie algebra</u> $\mathfrak{h}$ <u>onto the Lie algebra</u> $\mathfrak{T}$ <u>associated with the pairing</u>.

<u>Proof</u>: Since $V = \mathfrak{a}$ is an abelian subalgebra of $\mathfrak{G}$ we observe (P.2). Then using Lemma 4.1 we have for $a,b \in V$

$$\langle a, \tau b\rangle = 2 \text{ trace } [\text{ad}(a+\Theta b)]^2_{V+\tilde{V}} .$$

By a verification, the right side equals trace $(a\square b + b\square a)$. Hence σ is the trace form of the pairing and (P.1) is proved. For (P.4) we have

$$\sigma(T_h a, \tau b) = \langle [h,a], \tau b\rangle = -\langle a, [h, \tau b]\rangle$$

$$= -\langle a, \tau[\tau h, b]\rangle = -\sigma(a, T_{\tau h}, b)$$

and consequently

$$T_h^* = - T_{\tau h}.$$

Hence (P.4) is satisfied, too. From the definition of T_h we observe

$$T_{[h,k]} = [T_h, T_k], \quad h,k \in \mathfrak{h}.$$

Finally, using the Jacobi identity we get

$$[T_h, a\Box b] = [T_h, T_{[a,\tau b]}] = T_{[h,[a,\tau b]]}$$

$$= [h,a]\Box b + a\Box[\tau h, b]$$

$$= T_h a\Box b - a\Box T_h^* b.$$

Hence (P.3) is valid, too.

4. We construct the binary Lie algebra $\mathfrak{L}_\Box = V + \mathfrak{T} + \tilde{V}$ associated with the pairing $\Box : V\times V \to \operatorname{End} V$, where $\tilde{V} = \Theta V$ and where Θ is the automorphism of $\mathfrak{L}_\Box$ defined in §3.5. Hence

$$h + a + \tau b \to a + T_h + \Theta b$$

turns out to be a linear bijection of $\mathfrak{G}$ onto $\mathfrak{L}_\Box$. Now a verification shows that this map is a homomorphism of the Lie algebras. Summing up we have

THEOREM 5.1. *If the Lie algebra* $\mathfrak{G}$ *satisfies the conditions* (1) *to* (3), *then* $\mathfrak{G}$ *is isomorphic to a symmetric Lie algebra* $\mathfrak{L}_{\square}$, *where the pairing* $\square$ *satisfies the conditions* (P.1) *to* (P.4), *and vice versa*.

5. For example let Z be a bounded symmetric domain in a complex vector space. (See S. Helgason [3], Chapter VIII, §7.)

Denote by G the group of biholomorphic mappings of Z onto itself and denote by $\mathfrak{G}$ the complexification of the Lie algebra of G. Then $\mathfrak{G}$ considered as a Lie algebra over $\mathbb{R}$ satisfies the conditions (1) to (3). Hence we get

THEOREM 5.2. *If* Z *is a bounded symmetric domain then* $\mathfrak{G}$ *considered as a real Lie algebra is isomorphic to a symmetric Lie algebra* $\mathfrak{L} = \mathfrak{L}_{\square}$.

Chapter III

EXAMPLES

§1. Symmetric and skew-symmetric matrices.

1. Let $\mathfrak{M}_r$ be the vector space over K that consists of all r by r matrices with entries in K. For $\epsilon = \pm 1$ denote by $V = V^{\epsilon}$ the subspace of $a \in \mathfrak{M}_r$ such that $a^t = \epsilon a$, where a^t stands for the transpose of a. Hence the dimension of V equals $\frac{1}{2}r(r+\epsilon)$. Furthermore let GL(r,K) be the group of invertible matrices of $\mathfrak{M}_r$ and let e be the unit matrix of $\mathfrak{M}_r$.

For $u \in \mathfrak{M}_r$ we define an endomorphism T_u of V by $T_u x = u^t x + xu$. Hence $u \to T_u$ is a linear injection of $\mathfrak{M}_r$ into End V. Note that this is not true in the case $\epsilon = -1$ and $r = 2$. A verification shows

$$(1.1) \qquad [T_u, T_v] = T_{[u,v]} \quad \text{for } u,v \in \mathfrak{M}_r.$$

PROPOSITION 1. For $u \in \mathfrak{M}_r$ one has trace $T_u = (r+\epsilon)$ trace u.

Proof: Define a linear form λ of $\mathfrak{M}_r$ by $\lambda(u) =$ trace T_u. Hence $\lambda(uv) = \lambda(vu)$ because of (1.1). Since the bilinear form of $\mathfrak{M}_r$ that is given by $(u,v) \to$ trace(uv) is non degenerate, there exists an element $a \in \mathfrak{M}_r$ such that $\lambda(u) =$ trace(au) and we get trace(auv) = trace(avu) = trace(uav). Hence au = ua for $u \in \mathfrak{M}_r$ and consequently $a = \alpha e$ where $\alpha \in K$. So we get trace $T_u = \alpha$ trace u. For u = e we find trace T_u = $2\cdot\dim V$ and trace e = r, hence $\alpha = r+\epsilon$.

PROPOSITION 2. The set $\{T_u; u\in\mathfrak{M}_r\}$ of linear transformations of V acts irreducibly on V.

Proof: We have to show that 0 and V are the only subspaces of V that are mapped into itself under the maps $x \to u^t x + xu$ for $u\in\mathfrak{M}_r$. Let u be the matrix with 1 at the first entry of the diagonal and zero elsewhere. Then $x-(u^t x+xu)$ is obtained from x by replacing the first row and first column (except the first diagonal element) by zeros. Now an induction argument completes the proof.

PROPOSITION 3. The vector space $\mathfrak{M}_r$ is spanned by elements of the form ab where $a,b \in V$.

Proof: Let $\epsilon = 1$. Since $\mathfrak{M}_r$ is spanned by the matrices that have non-zero entries only at the intersections of two rows and the corresponding two columns it suffices to prove the proposition for 2 by 2 matrices. But in this case the statement follows from

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \gamma\\ \gamma & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \beta\\ \beta & \delta\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} .$$

In the case $\epsilon = -1$ a similar argument shows that it is enough to prove the statement for three by three matrices, for which again one uses a verification.

There is another type of endomorphisms of V given by elements of $\mathfrak{M}_r$. For $u \in \mathfrak{M}_r$ define W_u by

$$W_u x = u^t x u, \quad x \in V.$$

A verification yields

(1.2) $$W_u W_v = W_{uv} \quad \text{for} \quad u,v \in \mathfrak{M}_r.$$

PROPOSITION 4. For $u \in \mathfrak{M}_r$ one has $\det W_u = \pm(\det u)^{r+\epsilon}$.

Proof: Since the field K is infinite it suffices to prove the statement for $u \in GL(r,K)$. But both sides are (up to a sign) multiplicative, so it is enough to prove it for a set of generators of the group $GL(r,K)$. Using the standard generators of $GL(r,K)$ the proof can be completed.

2. Suppose now that the characteristic of K does not divide $r+\epsilon$. We define a pairing $\square : V \times V \to \text{End } V$ by

$$(a \square b)c = ab^t c + cb^t a, \quad a,b,c \in V.$$

Let $\mathfrak{T}$ be the subspace of End V spanned by $a \square b$ for $a,b \in V$. Clearly the elements of $\mathfrak{T}$ are the endomorphism T_u where u is in the vector space spanned by $ab^t = \epsilon ab$ for $a,b \in V$. Hence by Proposition 3 we get

$$\mathfrak{T} = \{T_u ; u \in \mathfrak{M}_r\}.$$

By Proposition 1 the trace form of the pairing $\square$ is given by

$$\sigma(a,b) = (r+\epsilon)\mathrm{trace}(ab^t) \text{ for } a,b \in V.$$

Hence σ is non degenerate and $T_u^* = T_{u^t}$.

Because of the associativity of the matrix product and the commutativity of the trace of a matrix, one verifies then that the pairing $\square : V\times V \to \mathrm{End}\, V$ satisfies our conditions of II,§3.1. We obtain

(1.3) $$P(a)b = ab^ta \quad \text{for} \quad a,b \in V.$$

and the associated binary Lie algebra $\mathfrak{L} = \mathfrak{L}^\epsilon$ consists of the elements

(1.4) $$q(x) = a+u^tx+xu+xb^tx \quad \text{where } a,b \in V \text{ and } u\in\mathfrak{M}_r.$$

Here the generic element x of V can be chosen as a matrix $x = (\tau_{ij})$, $\tau_{ji} = \epsilon\tau_{ij}$, where the τ_{ij}'s ($i\leq j$ if $\epsilon = 1$ and $i<j$ if $\epsilon = -1$) are algebraically independent over K. According to II, Theorem 4.4, and to Proposition 2 the Lie algebra $\mathfrak{L}$ is simple. Clearly the dimension of $\mathfrak{L}$ over K equals $r(2r+\epsilon)$.

P(a) is an endomorphism of V provided $a\in V$. Comparing (1.3) and the definition of W_a we observe $P(a) = \epsilon W_a$, $a\in V$. Hence using Proposition 4 we obtain

(1.5) $$\det P(a) = \pm(\det a)^{r+\epsilon}.$$

Let the automorphism Θ of $\mathfrak{L}$ be defined as in II,§3.5. Hence we get

$$H_\Theta(x) = -P(x)$$

(see I, §2.2) and from (1.5) it follows that Θ is essential if and only if $\epsilon = 1$ ($r\geq 1$) or $\epsilon = -1$ ($r\geq 3$ even).

For $u \in GL(r,K)$ we consider the endomorphism W_u of V (see 1). It follows from Proposition 4 that W_u belongs to GL(V). A verification shows that the adjoint of W_u with respect to the trace form σ equals W_{u^t}. Hence we obtain $P(W_u x) = W_u P(x) W_u^*$ and according to II, Lemma 3.4, we get

$$W_u \in \Gamma(\mathfrak{D}) \quad \text{for } u \in GL(r,K).$$

3. We consider now the group $\Xi(\mathfrak{D})$ of birational functions. From (II;3.8) we know $B(a,b) = I+a\Box b+P(a)P(b)$ and hence we have

$$B(a,b)c = (e+ab^t)c(e+b^t a) \quad \text{where } a,b,c \in V. \tag{1.6}$$

and $B(a,b)$ equals W_u for $u = e+b^t a$. Furthermore using II, §1.3, we observe

$$\tilde{t}_{\Theta b}(x) = x(e+\epsilon bx)^{-1} = (e+\epsilon xb)^{-1}x. \tag{1.7}$$

In order to describe the group $\Xi(\mathfrak{D})$ we define a 2r by 2r matrix Q by

$$Q = \begin{pmatrix} 0 & e \\ -\epsilon e & 0 \end{pmatrix}$$

and we denote by $\mathcal{G} = \mathcal{G}^\epsilon$ the group of 2r by 2r matrixes M satisfying the condition

$$M^t Q M = Q. \tag{1.8}$$

Note that $M^t \in \mathcal{G}$ whenever $M \in \mathcal{G}$. Writing

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where } a,b,c,d \text{ are } r\times r \text{ matrices,}$$

a verification shows that $M \in \mathcal{G}$ is equivalent to

$$(1.9) \quad a^t c = \epsilon c^t a, \quad b^t d = \epsilon d^t b, \quad a^t d - \epsilon c^t b = e.$$

From (1.8) it follows that the inverse of M is given by $M^{-1} = -\epsilon Q M^t Q$, hence

$$M^{-1} = \begin{pmatrix} d^t & -\epsilon b^t \\ -\epsilon c^t & a^t \end{pmatrix}.$$

Next let $\mathcal{G}^*$ denote the set of $M \in \mathcal{G}$ such that the determinants of $cx+d$ and of $-\epsilon c^t x + a^t$ are not the zero polynomials in x. Hence we can define

$$f_M(x) = (ax+b)(cx+d)^{-1} \text{ where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}^*.$$

From (1.9) we observe

$$(1.10) \quad [f_M(x)]^t = \epsilon . f_M(x) \text{ and } \frac{\partial f_M(x)}{\partial x} = W_u \text{ where } u=(cx+d)^{-1}.$$

Moreover, for $N = M^{-1}$ the function f_N is also defined and one verifies $f_M \circ f_N = f_N \circ f_M = I$. <u>Hence f_M belongs to the group</u> $\mathbb{P}(V)$ <u>of birational functions</u>. A verification yields now $f_{M_1} \circ f_{M_2} = f_{M_1 M_2}$ where M_1 and M_2 are in $\mathcal{G}^*$.

According to (1.4) we write the elements of $\mathfrak{Q}$ as

$$q(x) = q_1 + q_2^t x + x q_2 + x q_3^t x \text{ where } q_1, q_3 \in V \text{ and } q_2 \in \mathfrak{M}_r.$$

In the notation of (I;1.4) we obtain

(1.11) $q^f = u^t q_1 u + u^t q_2^t v + \epsilon v^t q_2 u + \epsilon v^t q_3^t v, \quad f = f_M,$

where $u = cx+d$ and $v = ax+b$. From (1.10) one concludes that q^f is in $\mathfrak{D}$ whenever $q \in \mathfrak{D}$. Since the same is true for f^{-1} instead of f, the map $q \to q^f$ is a bijection of $\mathfrak{D}$. According to I,§1.5, <u>we obtain an automorphism</u> ∇_f <u>of</u> $\mathfrak{D}$ <u>whenever</u> $f = f_M$, $M \in \mathfrak{G}^*$. One concludes from (1.11) or from I, Theorem 2.1, that ∇_f is <u>essential</u>. Then from I,§4.2, it follows that f belongs to the group $\Xi(\mathfrak{D})$. So we proved $f_M \in \Xi(\mathfrak{D})$ whenever $M \in \mathfrak{G}^*$.

4. Let $M \in \mathfrak{G}^*$ and suppose that f_M equals an element W in $\Gamma(\mathfrak{D})$. We get $ax+b = (Wx)(cx+d)$ and this is equivalent to $b = 0$, $(Wx)d = ax$, $(Wx)(cx) = 0$. Then from (1.9) it follows $a^t d = e$ and hence $Wx = axa^t$ as well as $c = 0$. So for $M \in \mathfrak{G}^*$ we see that $f_M \in \Gamma(\mathfrak{D})$ is equivalent to

$$M = \begin{pmatrix} u^t & 0 \\ 0 & u^{-1} \end{pmatrix} \quad \text{for some } u \in GL(r,K).$$

i.e., to $f_M = W_u$. Denote by $\Gamma_o(\mathfrak{D})$ the subgroup of $\Gamma(\mathfrak{D})$ consisting of the elements W_u where $u \in GL(r,K)$. Hence $f_M \in \Gamma(\mathfrak{D})$ implies $f_M \in \Gamma_o(\mathfrak{D})$. Finally denote by $\Xi^*(\mathfrak{D})$ the subgroup of $\Xi(\mathfrak{D})$ consisting of the functions $f \in \Xi(\mathfrak{D})$ such that

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x \to d} \in \Gamma_o(\mathfrak{D})$$

whenever d is in the domain of definition of f. From (1.10) we obtain that $f_M \in \Xi^*(\mathfrak{D})$ whenever $M \in \mathfrak{G}^*$. Writing

$$f = W \circ t_a \circ \tilde{t}_{\Theta b} \circ t_c \quad \text{where } W \in \Gamma(\mathfrak{D}),\ a,b,c \in V$$

(see I,§4.$\underline{\underline{2}}$) we obtain

$$\frac{\partial f(x)}{\partial x} = W[B(x+c,b)]^{-1}$$

from (I;4.6). In view of (1.6) we see that f belongs to $\Xi^*(\mathfrak{D})$ if and only if $W \in \Gamma_o(\mathfrak{D})$. Using (1.7) we observe

$$f_M = \begin{cases} W_u & \text{if } M = \begin{pmatrix} u^t & 0 \\ 0 & u^{-1} \end{pmatrix}, \\ t_a & \text{if } M = \begin{pmatrix} e & a \\ 0 & e \end{pmatrix}, \\ \tilde{t}_{\Theta b} & \text{if } M = \begin{pmatrix} e & 0 \\ \epsilon b & e \end{pmatrix}, \end{cases}$$

and in each case M belongs to $\mathfrak{G}^*$. Hence for $f \in \Xi^*(\mathfrak{D})$ there exists a $M \in \mathfrak{G}^*$ such that $f = f_M$.

Summing up we proved:

(i) The elements of $\Xi^*(\mathfrak{D})$ are exactly the functions f_M where $M \in \mathfrak{G}^*$.

(ii) $\mathfrak{G}^*$ is a subgroup of $\mathfrak{G}$ and $M \to f_M$ defines an epimorphism of the groups having the kernel $\{\alpha\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix};\ 0 \neq \alpha \in K\}$.

(iii) $\mathfrak{G}^*$ can be generated by the matrices

$$\begin{pmatrix} u^t & 0 \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} e & a \\ 0 & e \end{pmatrix}, \begin{pmatrix} e & 0 \\ b & e \end{pmatrix}$$

where $u \in GL(r,K)$ and $a,b \in V$.

(iv) Each element in $\Xi(\mathfrak{D})$ can be written as $W \circ f_M$ where $W \in \Gamma(\mathfrak{D})$ and $M \in \mathcal{G}^*$. Here W can be chosen in a given set of representatives of $\Gamma(\mathfrak{D})$ modulo $\Gamma_o(\mathfrak{D})$.

5. We consider now the case $\epsilon = 1$. Then the group $\mathcal{G}$ coincides with the <u>symplectic group</u> $Sp(r,K)$. One can show that in this case $\mathcal{G}^*$ equals $\mathcal{G}$. One has only to prove that $\det(cx+d)$ is not the zero polynomial whenever $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}$. Since c can be replaced by ucv^t and d by udv^{-1} where $u,v \in GL(r,K)$ one can choose c in a normal form and then $\det(cx+d) \neq 0$ follows from (1.9). For more details see C. L. Siegel [15].

PROPOSITION 5. *For* $W \in \Gamma(\mathfrak{D})$ *there exists* $\alpha \in K$ *and* $u \in GL(r,K)$ *such that* $W = \alpha \cdot W_u$.

Proof: Assume first that K is algebraically closed. Since any invertible symmetric matrix a can be written as $a = u^t u$, $u \in GL(r,K)$, it suffices to prove the statement for $W \in \Gamma(\mathfrak{D})$ such that $We = e$. Let e_i, $i = 1,2,\dots,r$, be the diagonal matrices having non zero elements only at the i^{th} row. Then given a symmetric matrix a satisfying $a^2 = a \neq 0$ there exists $u \in GL(r,K)$ such that $u^t a u = e_1 + \cdots + e_s$ for some s

and such that $u^t u = e$. Hence we may assume that $We_i = e_i$ for $i = 1,2,...,r$. Now a verification yields $W = I$ and we proved that $W \in \Gamma(\mathfrak{L})$ implies $W = W_u$ for some $u \in GL(r,K)$.

If K is an arbitrary field and if $W \in \Gamma(\mathfrak{L})$ we apply the previous result to the algebraic closure $\overline{K}$ of K and obtain an r by r matrix u with entries in $\overline{K}$ such that $Wa = u^t a u$ for all symmetric matrices a with entries in K. An observation yields $u = \beta v$ where $\beta \in \overline{K}$ and v has entries in K. But W maps V onto itself hence $\alpha = \beta^2 \in K$.

From Proposition 5 and from (iv) it follows now that $\Xi(\mathfrak{L})$ consists of the elements $\alpha \cdot f_M$ where $0 \neq \alpha \in K$ and $M \in Sp(r,K)$. Furthermore from (iii) we obtain the usual set of generators of $Sp(r,K)$.

In particular we see that the Lie algebra $\mathfrak{L}$ is of type C_r.

For $\epsilon = -1$ one can show that $\mathfrak{L}$ is a Lie algebra of type D_r.

§2. The rectangular matrices.

1. Let V be the vector space of r by s matrices with entries in K and suppose $r \geq s$. Hence the dimension of V equals rs. We assume that the characteristic of K does not divide $r+s$. Let $\mathfrak{M}_{rs}$ be the vector space of pairs (u,v) such that $u \in \mathfrak{M}_r$, $v \in \mathfrak{M}_s$ and trace u = trace v.

For $(u,v) \in \mathfrak{M}_{rs}$ we define an endomorphism $T_{u,v}$ of V by $T_{u,v}x = ux+xv$. A verification shows that $T_{u,v}x = 0$ for all x implies $u = \alpha e$ and $v = -\alpha e$ for some $\alpha \in K$. Hence $\alpha \cdot r = -\alpha \cdot s$ and we obtain $\alpha = 0$. The map $x \to T_{u,v}x$ of $\mathfrak{M}_{rs}$ into End V consequently is a linear injection.

PROPOSITION 1. *For* $(u,v) \in \mathfrak{M}_{rs}$ *one has*

$$\operatorname{trace} T_{u,v} = (r+s)\operatorname{trace} u = (r+s)\operatorname{trace} v.$$

Proof: We consider the linear transformation $x \to ux$, $u \in \mathfrak{M}_r$, of V. Writing $x = (x_1,\ldots,x_s)$ where the x_i's are vectors we get $ux = (ux_1,\ldots,ux_s)$ and hence $s \cdot \operatorname{trace} u$ is the trace of this linear transformation. A similar argument shows that $r \cdot \operatorname{trace} v$ is the trace of the transformation $x \to xv$, $v \in \mathfrak{M}_s$.

PROPOSITION 2. *The set* $\{T_{u,v}; (u,v) \in \mathfrak{M}_{rs}\}$ *of linear transformations of* V *acts irreducibly on* V.

Proof: Similar to the proof of Proposition 1 in §1.

PROPOSITION 3. *The vector space* $\mathfrak{M}_{rs}$ *is spanned by elements of the form* $(ab^t, b^t a)$ *where* $a,b \in V$.

Proof: First of all, $\operatorname{trace} ab^t = \operatorname{trace} b^t a$. In the case $s = 1$ we get $\mathfrak{M}_{rs} = \{(u, \operatorname{trace} u); u \in \mathfrak{M}_r\}$ and the proposition follows from the fact that $\mathfrak{M}_r$ is spanned by matrices of the form ab^t where a,b are r by 1 matrices.

In the case $s \geq 2$ again it is enough to show the statement for $r = s = 2$. But one gets this by a verification.

For $u \in \mathfrak{M}_r$, $v \in \mathfrak{M}_s$ we define an endomorphism $W_{u,v}$ of V by

$$W_{u,v}x = uxv, \quad x \in V.$$

A verification yields

(2.1) $W_{u,v}W_{a,b} = W_{ua,bv}$ for $a,u \in \mathfrak{M}_r$ and $b,v \in \mathfrak{M}_s$.

Similarly to the proof of Proposition 4 in §1 we obtain

<u>PROPOSITION 4.</u> <u>For</u> $u \in \mathfrak{M}_r$ <u>and</u> $v \in \mathfrak{M}_s$ <u>one has</u>

$$\det W_{u,v} = \pm(\det u)^s(\det v)^r.$$

2. We define a pairing $\square : V \times V \to \text{End } V$ by

$$(a \square b)c = ab^tc + cb^ta, \quad a,b,c \in V.$$

Let $\mathfrak{T}$ be the subspace of End V spanned by $a \square b$ for $a,b \in V$. The elements of $\mathfrak{T}$ are the endomorphisms $T_{u,v}$ where (u,v) is spanned by (ab^t, b^ta) for $a,b \in V$. Hence by Proposition 3 we get

$$\mathfrak{T} = \{T_{u,v}; (u,v) \in \mathfrak{M}_{rs}\}.$$

By Proposition 1 the trace form of the pairing $\square$ is given by

$$\tau(a,b) = (r+s)\,\text{trace } ab^t \quad \text{for } a,b \in V$$

and consequently σ is non degenerate. Moreover $(T_{u,v})^* = T_{u^t,v^t}$.

A verification shows <u>that the pairing</u> $\square$ <u>satisfies the conditions of</u> II, §3.1. We obtain again

$$(2.2) \qquad P(a)b = ab^ta \quad \text{for } a,b \in V$$

and the associated binary Lie algebra $\mathfrak{L}$ consists of the elements

$$(2.3) \quad q(x) = a + ux + xv + xb^tx \quad \text{where } a,b\in V, \ (u,v)\in\mathfrak{M}_{rs}.$$

The generic element x can be chosen as an r by s matrix having algebraically independent entries. According to II, Theorem 4.4, and to Proposition 2 the Lie algebra $\mathfrak{L}$ is simple. The dimension of $\mathfrak{L}$ equals $(r+s)^2-1$.

Let Θ be the automorphism of $\mathfrak{L}$ defined as in II, §3.5. We get $H_\Theta(x) = -P(x)$ and from (2.2) it follows <u>that</u> Θ <u>is essential if and only if</u> $r = s$.

For $u \in GL(r,K)$ and $v \in GL(s,K)$ we consider the endomorphism $W_{u,v}$ (see 1) of V. From Proposition 4 it follows that $W_{u,v}$ belongs to GL(V). The adjoint of $W_{u,v}$ with respect to the trace form σ equals W_{u^t,v^t} and we obtain

$$W_{u,v} \in \Gamma(\mathfrak{L}) \quad \text{for} \quad u \in GL(r,K) \text{ and } v \in GL(s,K)$$

according to II, Lemma 3.4. Now (2.1) shows that $(u,v) \to W_{u,v}$ is a homomorphism of the group $GL(r,K)\times GL(s,K)$ into $\Gamma(\mathfrak{L})$ having the kernel

$\{(\alpha e, \frac{1}{\alpha} e);\ 0 \neq \alpha \in K\}$.

PROPOSITION 5. *For* $W \in \Gamma(\mathfrak{D})$ *there exists* $u \in GL(r,K)$, $v \in GL(s,K)$ *such that* $W = W_{u,v}$.

Proof: Let a_i be the i^{th} column vector of $a \in V$. Writing

$$[Wa]_i = \sum_j w_{ij} b_j, \quad w_{ij} \in \mathfrak{M}_r,$$

we obtain

$$[W^* b]_i = \sum_j w^t_{ji} b_j$$

and a verification shows that $W \in \Gamma(\mathfrak{D})$ is equivalent to

$$\sum_{k,\ell} w_{jk}\, a_k\, a^t_\ell\, w^t_{i\ell} = \sum_{k,\ell} w_{i\ell}\, a_k\, a^t_\ell\, w_{jk} \tag{2.4}$$

where $a_1, \dots, a_s$ are arbitrary columns. In particular we get

$$w_{jk}\ a\ a^t\ w^t_{ik} = w_{ik}\ a\ a^t\ w_{jk}$$

for arbitrary column a. For a given i there is a k such that $w_{ik} \neq 0$. Hence $w_{ij} = \alpha_{ij}\, u_i$, $\alpha_{ij} \in K$, $u_i \in \mathfrak{M}_r$. Replacing W by $WW_{e,v}$ for suitable $v \in GL(s,K)$ we may assume that $\alpha_{ij} = \delta_{ij}$ holds. Now (2.4) yields $u_i = u$ for all i and the proposition is proved.

3. In order to describe the group $\Xi(\mathfrak{D})$ we obtain from (II;3.8)

(2.5) $B(a,b)c = (e+ab^t)c(e+b^ta)$ where $a,b,c \in V$

and $B(a,b)$ equals $W_{u,v}$ for $u = e+ab^t$, $v = e+b^ta$.
Using II,§1.3, we observe

(2.6) $$\tilde{t}_{\ominus b}(x) = x(e+b^tx)^{-1} = (e+xb^t)^{-1}x.$$

We write the elements of GL(r+s,K) as

$$M = \begin{pmatrix} a & b \\ c^t & d \end{pmatrix} \quad \text{where } a \in \mathfrak{M}_r,\ b,c \in V,\ d \in \mathfrak{M}_s.$$

Since the s by r+s matrix (c^t,d) has maximal rank, the determinant of c^tx+d is not the zero polynomial. Hence for $M \in GL(r+s,K)$ we have the rational function

$$f_M(x) = (ax+b)(c^tx+d)^{-1}.$$

A verification yields $f_M \circ f_N = f_{MN}$ for $M,N \in GL(r+s,K)$ and we obtain a homomorphism $M \to f_M$ of GL(r+s,K) into the group $\mathbb{P}(V)$ of birational functions and its kernel consists of the diagonal matrices. In particular we get

$$f_M = \begin{cases} W_{u,v} & \text{if } M = \begin{pmatrix} u & 0 \\ 0 & v^{-1} \end{pmatrix},\ u \in GL(r,K),\ v \in GL(s,K), \\ t_a & \text{if } M = \begin{pmatrix} e & a \\ 0 & e \end{pmatrix},\ a \in V, \\ \tilde{t}_{\ominus b} & \text{if } M = \begin{pmatrix} e & 0 \\ b^t & e \end{pmatrix},\ b \in V. \end{cases}$$

But the group GL(r+s,K) is generated by the matrices we listed above and we obtain

$$\Xi(\mathfrak{L}) = \{f_M;\ M \in GL(r+s,K)\}.$$

Hence $\mathfrak{L}$ is a Lie algebra of type A_{r+s}.

§3. Jordan pairings.

1. Let V be a vector space over the field K of characteristic different from 2 and 3 and let $\mathfrak{A}$ be a Jordan algebra defined in V with unit element e. Denote the left multiplication by L, i.e., ab = L(a)b, and suppose that its trace form given by

$$(ab) \to \text{trace } L(ab)$$

is non degenerate. Hence $\mathfrak{A}$ is separable and in particular semi-simple (for details about Jordan algebras see [2]).

We define a pairing $\square$: V×V → End V by setting

$$a\square b = 2L(ab) + 2[L(a),L(b)] \text{ where } a,b \in V. \tag{3.1}$$

Then the trace form of this pairing is given by

$$\tau(a,b) = 4 \text{ trace } L(ab) \tag{3.2}$$

and hence it is non degenerate. Moreover τ turns out to be an associative bilinear form of the algebra $\mathfrak{A}$. The adjoint of $T \in$ End V with respect to τ is denoted by T^*. In particular we have $L^*(a) = L(a)$.

It is known and easy to prove (see [8] and [10]) that the pairing (3.1) satisfies besides (P.1) also the conditions (P.2) to (P.4) of II,§3.1. We call such a pairing a Jordan pairing of the first kind. The examples given in §1 are Jordan pairings of the first kind provided $\epsilon = 1$ or $\epsilon = -1$ and $r \geq 3$ is even.

From $(a\Box b)a = 4a(ab) - 2a^2b$ we conclude that the endomorphism $P(a)$ defined by (II;3.2) coincides with the quadratic representation of the Jordan algebra $\mathfrak{A}$. Hence $\Gamma = \Gamma_\Box$ coincides with the structure group $\Gamma(\mathfrak{A})$ of $\mathfrak{A}$ because of II, Lemma 3.4.

The results of II,§3 show that any Jordan pairing of the first kind leads to a binary Lie algebra $\mathfrak{L}_\mathfrak{A} = \mathfrak{L}_\Box$ such that $(\mathfrak{L}_\mathfrak{A}, \Theta)$ is a symmetric Lie algebra. From the definition of Θ in §3.5 we observe

$$H_\Theta(x) = -P(x)$$

(see I,§2.2). Thus $\det H_\Theta(x) \neq 0$ because of $P(e) = I$ and hence Θ is essential.

2. Since Θ is essential, there exists a birational function j in $\Xi(\mathfrak{L}_\mathfrak{A})$ such that $\Theta = \nabla_{-j}$. Here $\Theta^2 = Id$ implies $j \circ j = I$ and because of I,§2.3, the function j is given by

$$j(x) = -[H_\Theta(x)]^{-1}(\Theta I)(x) = -[P(x)]^{-1}x = -x^{-1},$$

where x^{-1} stands for the inverse of x in some scalar extension of the Jordan algebra $\mathfrak{A}$.

We are able to express the birational functions $\tilde{t}_{\Theta b}$ by j and the translations t_b where $b \in V$.

LEMMA 3.1. Suppose that $\square : V \times V \to \operatorname{End} V$ is a Jordan pairing of the first kind. Then for $b \in V$ one has

$$\tilde{t}_{\Theta b} = j \circ t_{-b} \circ j \quad \text{and} \quad B(x,b) = P(x)P(x^{-1}+b).$$

Proof: We know from II, Lemma 1.1, that $\tilde{\Psi}_{\Theta b} = \Theta \Psi_b \Theta$ holds. Moreover the birational functions belonging to Ψ_b or $\tilde{\Psi}_{\Theta b}$ are t_{-b} or $\tilde{t}_{-\Theta b}$, respectively. Hence we get $\tilde{t}_{-\Theta b} = (-j) \circ t_{-b} \circ (-j)$ and this proves the first formula. The second formula now follows from part a) of II, Theorem 2.3, together with $\frac{\partial x^{-1}}{\partial x} = -[P(x)]^{-1}$.

As a conseuqence we see that the group $\Xi(\mathfrak{L}_{\mathfrak{A}})$ is generated by the functions W, t_a and j where $W \in \Gamma(\mathfrak{L}_{\mathfrak{A}}) = \Gamma(\mathfrak{A})$ and $a \in V$. In particular, $\Xi(\mathfrak{L}_{\mathfrak{A}})$ coincides with the group $\Xi(\mathfrak{A})$ considered in [11]. For more results see also H. Braun [1] and [12].

3. Let $\square : V \times V \to \operatorname{End} V$ be an arbitrary pairing satisfying the conditions (P.1) to (P.4) of II,§3.1. Denote by $(\mathfrak{L}_{\square}, \Theta)$ the induced symmetric Lie algebra. Let $d \in V$ and denote by $\mathfrak{A}_d$ the algebra defined on the vector space V by the product $(a,b) \to P(a,b)d$. We know from II, Theorem 3.3, that $\mathfrak{A}_d$ is a Jordan algebra. Denote by L_d and P_d the left multiplication and the quadratic representation of $\mathfrak{A}_d$ respectively. Thus

$$L_d(a) = \tfrac{1}{2}a\Box d.$$

We are going to prove

$$P_d(a) = P(a)P(d). \tag{3.3}$$

Indeed, we apply (P.3) for $T = c\Box b$ on c and obtain

$$(a\Box b)(c\Box b)c - (c\Box b)(a\Box b)c = [(c\Box b)a\Box b]c - [c\Box(b\Box a)b]c.$$

Hence it follows that

$$L_b(L_b(c)c) - L_b^2(c) = L_b^2(c) - P(c)P(b).$$

Since the square of c in $\mathfrak{A}_b$ equals $L_b(c)c$ we get $P_b(c) = P(c)P(b)$ and (3.3) is proved.

4. Two pairings $\Box$ and $\Box'$ of V are said to be *isomorphic* if the associated binary Lie algebras $\mathfrak{L}$ and $\mathfrak{L}'$ are isomorphic under an isomorphism $\Phi : \mathfrak{L} \to \mathfrak{L}'$ that satisfies $\Phi I = I$. According to I, Theorem 3.2, the two pairings are isomorphic if and only if there exists a $W \in GL(V)$ such that $\mathfrak{L}' = \nabla_W \mathfrak{L}$.

THEOREM 3.2. *Let $\Box$ be a pairing of V satisfying the conditions (P.1) to (P.4) of §3.1 and let $(\mathfrak{L}, \Theta)$ be the associated symmetric Lie algebra. Then the following statements are equivalent:*

a) *Θ is essential.*

b) $\det P(x) \neq 0$.

c) *There exists $d \in V$ such that $\mathfrak{A}_d$ has a unit element.*

d) <u>The pairing $\square$ is isomorphic to a Jordan pairing of the first kind</u>.

<u>Proof</u>: From the definition of Θ it follows that

$$H_{\Theta}(x) = -P(x)$$

holds. Hence a) and b) are equivalent (see I, §2.$\underline{\underline{2}}$). But b) implies the existence of $d \in V$ such that $\det P(d) \neq 0$. Hence $\det P_d(d) \neq 0$ from (3.3). Hence the equivalence of b) and c) follows from [2], chapter IV, Theorem 2.7.

It suffices to show that c) implies d). Choose $d \in V$ such that $\mathfrak{A}_d$ has a unit element c. Consider the binary Lie algebra $\mathfrak{L}' = \nabla_W \mathfrak{L}$ where $W = P(c)$. The algebra $\mathfrak{L}'$ can be considered as a binary Lie algebra defined by a pairing $\square'$ of V such that the endomorphism P' is given by

$$P'(x) = WP(W^{-1}x)$$

and $\Theta' = \nabla_W \Theta$ turns out to be the corresponding automorphism of $\mathfrak{L}'$. We obtain $I = P_d(c) = P(c)P(d)$ and using (II;3.10) we observe

$$P'(c) = WP(W^{-1}c) = P(c)[P(c)]^{-1} = I.$$

Hence we may assume that there is $c \in V$ such that $P(c) = I$. Using (3.3) we end up with $P(a) = P_c(a)$. Using [2], chapter IV, Theorem 2.5, we know that the square e of c in $\mathfrak{A}_c$ is the unit element of $\mathfrak{A}_c$ and

$P(e) = P_c(c^2) = [P_c(c)]^2 = I$. We write $\mathfrak{A} = \mathfrak{A}_e$ and obtain a Jordan algebra $\mathfrak{A}$ with unit element such that P is the quadratic representation of $\mathfrak{A}$. Thus the pairing $\square$ is given by (3.1) and the trace form of $\mathfrak{A}$ is non degenerate because of (3.2).

5. Next we are going to define the Jordan pairings of the second kind. We start again with a Jordan algebra $\mathfrak{A}$ in V with unit element and an automorphism $a \to a'$ of $\mathfrak{A}$ of period 2. Suppose again that the trace form of $\mathfrak{A}$ is non degenerate.

The automorphism $a \to a'$ of $\mathfrak{A}$ induces a direct sum decomposition

$$\mathfrak{A} = \mathfrak{A}_+ + \mathfrak{A}_- \text{ where } \mathfrak{A}_\pm = \{a; a \in \mathfrak{A}, a' = \pm a\}.$$

Here $\mathfrak{A}_+$ is a subalgebra of $\mathfrak{A}$, $\mathfrak{A}_- \neq 0$ and one has

$$\mathfrak{A}_+\mathfrak{A}_- \subset \mathfrak{A}_-, \quad \mathfrak{A}_-\mathfrak{A}_- \subset \mathfrak{A}_+.$$

Moreover we conclude $\sigma(a',b') = \sigma(a,b)$ for $a,b \in \mathfrak{A}$. Hence $\mathfrak{A}_+$ turns out to be orthogonal to $\mathfrak{A}_-$ with respect to σ and the restrictions of σ to $\mathfrak{A}_+$ and $\mathfrak{A}_-$ are non degenerate.

The bilinear form of $\mathfrak{A}_+$ given by $u \to \sigma(e,u)$ is <u>normal</u>. Hence $\mathfrak{A}_+$ is <u>non degenerate</u> and consequently semi-simple (see [2], chapter I, §8 and §9).

We define a pairing $\square'$ of $\mathfrak{A}_-$ by

$$(3.4) \quad (a \square' b)c = 2(ab)c + 2a(bc) - 2b(ac) \text{ where } a,b,c \in \mathfrak{A}_-.$$

Clearly $a\Box' b \in \text{End}\, \mathfrak{A}_-$ and $a\Box' b$ is the restriction of $a\Box b$ given by (3.1) to $\mathfrak{A}_-$. Let $\mathfrak{T}'$ be the vector space spanned by $a\Box' b$ where $a,b \in \mathfrak{A}_-$.

If A is a linear transformation of $\mathfrak{A}$ then we denote by $A_\pm$ its restriction to $\mathfrak{A}_\pm$. Hence the trace form σ' of the pairing $\Box'$ is given by

$$(3.5) \quad \sigma'(a,b) = \text{trace}(a\Box' b + b\Box' a) = 4\ \text{trace}\ L_-(ab)$$

where $a,b \in \mathfrak{A}_-$.

<u>PROPOSITION</u>. <u>There exists an element</u> d <u>in the center of</u> $\mathfrak{A}_+$ <u>such that</u> $\sigma'(a,b) = \sigma(da,b)$ <u>where</u> $a,b \in \mathfrak{A}_-$.

<u>Proof</u>: Let $u,v \in \mathfrak{A}_+$ and set $\lambda(u,v) = 4\ \text{trace}\ L_-(uv)$. Then λ is a symmetric bilinear form of $\mathfrak{A}_+$. Using the basic identities about Jordan algebras one observes that λ is associative. But the restriction of σ to $\mathfrak{A}_+$ is associative and non degenerate. Hence there exists d in the center of $\mathfrak{A}_+$ such that $\lambda(u,v) = \sigma(du,v)$ because of Theorem 6.4 in [2], chapter I. Now we obtain from (3.5)

$$\sigma'(a,b) = \lambda(e,ab) = \sigma(d,ab) = \sigma(da,b)$$

where $a,b \in \mathfrak{A}$.

<u>LEMMA 3.3</u>. <u>Suppose that the trace form of</u> $\mathfrak{A}$ <u>and the bilinear form</u> σ' <u>are non degenerate</u>. <u>Then the pairing</u> $\Box'$ <u>given by (3.4) satisfies the conditions (P.1) to (P.4) of</u> II, §3.1.

<u>Proof</u>: Clearly, we have to prove only (P.3) and (P.4). We write the condition (P.3) for the pairing (3.1) where $T = u\Box v$, $T^* = v\Box u$. Choosing all elements in $\mathfrak{A}_-$ we obtain

$$[u\Box' v,\ a\Box' b] = T'a\Box' b - a\Box' Sb \quad \text{where } T' = u\Box' v,\ S = v\Box' u,$$

and in a similar way

$$[b\Box' a,\ v\Box' u] = b\Box' T'a - Sb\Box' a.$$

Taking the trace we get $\sigma'(T'a,b) = \sigma'(a,Sb)$. Hence S equals the adjoint of T' with respect to σ' and consequently (P.3) and (P.4) hold.

We call $\Box'$ a <u>Jordan pairing of the second kind</u>.

The examples given in §1 for $\epsilon = -1$ and in §2 for $r>s$ are Jordan pairings of the second kind.

§4. The two exceptional cases.

<u>1</u>. Let $\mathfrak{C}$ be a Cayley algebra over K and suppose that the characteristic of K is different from 2 and 3. Thus $\mathfrak{C}$ is alternative and there exists a non degenerate bilinear form μ and a linear form λ of $\mathfrak{C}$ such that

$$a^2 = 2\lambda(a)a - \mu(a,a)e \quad \text{for } a \in \mathfrak{C},$$

where e is the unit element of $\mathfrak{C}$ and $\lambda(a) = \mu(a,e)$, $\lambda(e) = 1$, holds (for details see [2], chapter VII, §4). The map $a \to a' = 2\lambda(a)e-a$ defines an involution of $\mathfrak{C}$

and one has

$$a(b'c) + b(a'c) = 2\mu(a,b)e.$$

The dimension of $\mathfrak{C}$ over K equals 8.

Denote by $\mathfrak{H}_3(\mathfrak{C})$ the vector space of 3 by 3 matrices

$$a = \begin{pmatrix} \alpha_1 & a_3 & a_2' \\ a_3' & \alpha_2 & a_1 \\ a_2 & a_1' & \alpha_3 \end{pmatrix}, \quad \alpha_i \in K,\ a_i \in \mathfrak{C}.$$

If ab means the usual matrix product, $\mathfrak{H}_3(\mathfrak{C})$ becomes a central simple Jordan algebra over K with respect to the product $a \circ b = \frac{1}{2}(ab+ba)$, a so-called exceptional algebra (for details see [2], chapter VII,§6).

Associated with the Jordan algebra $\mathfrak{H}_3(\mathfrak{C})$ we obtain a Jordan pairing of the first kind $\square : \mathfrak{H}_3(\mathfrak{C})\times\mathfrak{H}_3(\mathfrak{C}) \to \mathrm{End}\ \mathfrak{H}_3(\mathfrak{C})$. The binary Lie algebra $\mathfrak{L} = \mathfrak{L}_\square$ is an exceptional Lie algebra of type E_7 (see [8]).

2. Let e_1 be the (absolute primitive) idempotent that is given by $\alpha_1 = 1$ and zero elsewhere. Using

$$\mathfrak{A}_\nu = \{a;a\in\mathfrak{H}_3(\mathfrak{C}),\ e_1\circ a = \nu a\}$$

for $\nu = 0,\frac{1}{2},1$ we obtain the Peirce decomposition

$$\mathfrak{A} = \mathfrak{H}_3(\mathfrak{C}) = \mathfrak{A}_0 + \mathfrak{A}_{\frac{1}{2}} + \mathfrak{A}_1$$

which is a direct sum of vector spaces. The map $a \to a^*$ which changes the sign of the component in $\mathfrak{A}_{\frac{1}{2}}$ is an automorphism of the Jordan algebra $\mathfrak{A}$ and the eigenspaces are

$$\mathfrak{A}_+ = \mathfrak{A}_0 + \mathfrak{A}_1, \quad \mathfrak{A}_- = \mathfrak{A}_{\frac{1}{2}} .$$

Set $V = \mathfrak{C}_- \oplus \mathfrak{C}_-$ and write the elements of V as $a = a_1 \oplus a_2$. We define a pairing of V by

$$[(a\Box b)c]_1 = [2\mu(a_1,b_1)+\mu(a_2,b_2)]c_1+[2\mu(c_1,b_1)+\mu(c_2,b_2)]a_1$$
$$- 2\mu(a_1,c_1)b_1+\tfrac{1}{2}[c_2'(b_2a_1)+a_2'(b_2c_1)-b_2'(a_2c_1+c_2a_1)],$$

$$[(a\Box b)c]_2 = [\mu(a_1,b_1)+2\mu(a_2,b_2)]c_2+[\mu(c_1,b_1)+2\mu(c_2,b_2)]a_2$$
$$- 2\mu(a_2,c_2)b_2+\tfrac{1}{2}[(a_2b_1)c_1'+(c_2b_1)a_1'-(a_2c_1+c_2a_1)b_1'].$$

Using the injection

$$\varphi : V \to \mathfrak{H}_3(\mathfrak{C}), \quad \varphi(a_1 \oplus a_2) = \begin{pmatrix} 0 & a_1 & a_2' \\ a_1' & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix} ,$$

we see that $\varphi(V)$ equals $\mathfrak{A}_- = \mathfrak{A}_{\frac{1}{2}}$. Furthermore a verification shows that

$$\tfrac{1}{2}\varphi((a\Box b)c) = [\varphi(a)\varphi(b)]\varphi(c) + \varphi(a)[\varphi(b)\varphi(c)]$$
$$- \varphi(b)[\varphi(a)\varphi(c)]$$

holds. Hence the image of the pairing of V under φ coincides with the pairing of $\mathfrak{A}_-$ given in (3.5).

An observation shows that the trace form of $\square$ is given by

$$\sigma(a,b) = 48[\mu(a_1,b_1)+\mu(a_2,b_2)].$$

Hence σ is non degenerate. According to Lemma 3.3 we obtain a pairing $\square : V\times V \to \mathrm{End}\, V$ of the second kind. Denote by $\mathfrak{L} = \mathfrak{L}_\square$ the associated binary Lie algebra. One can show (see K. Meyberg [13]) that $\mathfrak{L}$ is a Lie algebra of type E_6.

Chapter IV

APPLICATIONS TO BOUNDED SYMMETRIC DOMAINS

§1. Some elementary results on real linear algebraic groups.

1. For an arbitrary finite dimensional vector space V_o over $\mathbb{R}$ we denote by $V = \mathbb{C} \otimes_{\mathbb{R}} V_o$ its complexification and by $V^{\mathbb{R}}$ the space V considered as vector space over $\mathbb{R}$. Note that V and $V^{\mathbb{R}}$ are the same sets. The elements of V (and of $V^{\mathbb{R}}$) are written as $u = a + ib$ where $a,b \in V_o$. The vector spaces V_o, V and $V^{\mathbb{R}}$ are equipped with the natural topologies. Let $D \neq \emptyset$ be an open subset of V and let $f : D \to V'$ (V' being a vector space over $\mathbb{C}$) be a map. Then f is called holomorphic if in the representation

$$f(z_1b_1 + \cdots + z_nb_n) = \sum_{k=1}^{m} f_k(z_1,\ldots,z_n)b_k'$$

($b_1,\ldots,b_n$ and $b_1',\ldots,b_m'$ being a basis of V and V') the functions f_k are holomorphic in the complex variables $z_1,\ldots,z_n$.

Note that the multiplication by i and the conjugation $u \to \overline{u} = a - ib$ belong to End $V^{\mathbb{R}}$. For $A,B \in$ End V_o the endomorphism $A + iB$ of V is given by $(A+iB)(a+ib) := (Aa-Bb) + i(Ba+Ab)$. Conversely for any

$W \in \text{End}\, V$ there exists $A, B \in \text{End}\, V_o$ such that $W = A + iB$. For $W \in \text{End}\, V$ denote by $W^{\mathbb{R}}$ the induced endomorphism of $V^{\mathbb{R}}$. Moreover, if $\mathfrak{S}$ is a subset of $\text{End}\, V$ we denote $\mathfrak{S}^{\mathbb{R}} = \{W^{\mathbb{R}};\ W \in \mathfrak{S}\}$. Clearly $W \to W^{\mathbb{R}}$ defines a monomorphism of the ring End V into $\text{End}\, V^{\mathbb{R}}$. For $W \in \text{End}\, V^{\mathbb{R}}$ we define $\overline{W}$ by $\overline{W}u = \overline{W\overline{u}}$ where $u \in V^{\mathbb{R}}$. Clearly $\overline{W^{\mathbb{R}}} = \overline{W}^{\mathbb{R}}$.

An arbitrary endomorphism T of $V^{\mathbb{R}}$ can be written as $T(a+ib) = (Aa+Bb) + i(Ca+Db)$ where $A, B, C, D \in \text{End}\, V_o$. We obtain

(1.1) $$\text{trace}\ T = \text{trace}\ A + \text{trace}\ D.$$

The endomorphism T of $V^{\mathbb{R}}$ is $\mathbb{C}$-linear if and only if $D = A$ and $C = -B$, i.e., if $T = W^{\mathbb{R}}$ for some $W \in \text{End}\, V$. Furthermore T commutes with the conjugation $u \to \overline{u}$ if and only if $B = C = 0$. Note that in both cases the conditions are linear equations over $\mathbb{R}$. From (1.1) we observe

(1.2) $$\text{trace}\ (A+iB)^{\mathbb{R}} = 2\ \text{trace}\ A, \text{ where } A, B \in \text{End}\, V_o.$$

For a subset $\mathfrak{S}$ of $\text{End}\, V^{\mathbb{R}}$ denote by $\mathfrak{S}_*$ the set of elements of $\mathfrak{S}$ that are $\mathbb{C}$-linear. Hence there is a subset $\mathfrak{H}$ of End V such that $\mathfrak{S}_* = \mathfrak{H}^{\mathbb{R}}$.

For an endomorphism T of V_o, V or $V^{\mathbb{R}}$ we define the exponential by

$$\exp T := \sum_{m=o}^{\infty} \frac{1}{m!} T^m.$$

Hence we obtain a map exp of the endomorphism space into the corresponding general linear group. For $T \in \mathrm{End}\, V$ we have $\exp(T^{\mathbb{R}}) = (\exp T)^{\mathbb{R}}$. Note that the exponential map is bijective in a neighborhood of zero.

2. Let β be a hermitian positive definite form of V. The adjoint of $T \in \mathrm{End}\, V$ with respect to β is denoted by T^{β}. An element T of End V is called *unitary* (with respect to β) if $T^{\beta}T = I$ and *hermitian* (with respect to β) if $T^{\beta} = T$. Moreover we call T *positive definite* (with respect to β) and we write $T > 0$ if $T^{\beta} = T$ and $\beta(Tu,u) > 0$ for $0 \neq u \in V$. For a subset $\mathfrak{S}$ of End V we write $\mathfrak{S}^{\beta} = \{T^{\beta};\ T \in \mathfrak{S}\}$.

It is well known that the exponential map maps the hermitian endomorphisms of V bijectively onto the positive definite endomorphisms of V.

3. A subgroup $\mathfrak{G}$ of $GL(V^{\mathbb{R}})$ is called a *real linear algebraic* group if there exists a non-empty set $\mathfrak{p}$ of polynomials in an endomorphism variable of $V^{\mathbb{R}}$ having real coefficients such that $W \in GL(V^{\mathbb{R}})$ belongs to $\mathfrak{G}$ if and only if $\pi(W) = 0$ for all $\pi \in \mathfrak{p}$. Note that any real linear algebraic group is closed in $GL(V^{\mathbb{R}})$, hence it is a real Lie group. The subgroup $\mathfrak{G}_{*}$ is again a real linear algebraic group and there is a subgroup $\mathfrak{H}$ of GL(V) such that $\mathfrak{G}_{*} = \mathfrak{H}^{\mathbb{R}}$. A subgroup $\mathfrak{H}$ of GL(V) is called a *real linear algebraic* group if $\mathfrak{H}^{\mathbb{R}}$ has the corresponding property.

Let $\mathcal{G}$ be a closed subgroup of $GL(V^{\mathbb{R}})$. Because of the following lemma we call

(1.3) $\text{Lie}\,\mathcal{G} := \{T;\ T \in \text{End}\,V^{\mathbb{R}},\ \exp \xi T \in \mathcal{G} \text{ for } \xi \in \mathbb{R}\}$

the Lie algebra of $\mathcal{G}$.

LEMMA 1.1. *If $\mathcal{G}$ is a closed subgroup of $GL(V^{\mathbb{R}})$ then $\text{Lie}\,\mathcal{G}$ is a Lie algebra of endomorphisms of $V^{\mathbb{R}}$.*

Proof: We use the formulas

$$(\exp \xi T)(\exp \xi S) = \exp\{\xi(T+S) + O(\xi^2)\},$$

$$(\exp \xi T)^{-1}(\exp \xi S)^{-1}(\exp \xi T)(\exp \xi S) = \exp\{\xi^2[T,S] + O(\xi^3)\}.$$

For a given $\xi \in \mathbb{R}$ and positive integer m we replace ξ by ξm^{-1} and raise the first formula to the power m and the second to the power m^2. Then the limit $m \to \infty$ yields $T + S \in \text{Lie}\,\mathcal{G}$ and $[T,S] \in \text{Lie}\,\mathcal{G}$.

Remark. Suppose that $\mathcal{H}$ is a closed subgroup of $GL(V)$ and let $T \in \text{Lie}\,\mathcal{H}^{\mathbb{R}}$. Hence $\exp \xi T$ and consequently T itself is $\mathbb{C}$-linear. We obtain $\text{Lie}\,\mathcal{H}^{\mathbb{R}} = (\text{Lie}\,\mathcal{H})^{\mathbb{R}}$ where

(1.4) $\text{Lie}\,\mathcal{H} := \{S\ ;\ S \in \text{End}\,V,\ \exp \xi S \in \mathcal{H} \text{ for } \xi \in \mathbb{R}\}.$

Note that $\text{Lie}\,\mathcal{H}$ can be considered as a Lie algebra over $\mathbb{C}$.

4. Let β be a hermitian positive definite form of V. For a closed subgroup $\mathcal{H}$ of $GL(V)$ the condition

$\mathcal{H}^\beta = \mathcal{H}$ implies $(\text{Lie } \mathcal{H})^\beta = \text{Lie } \mathcal{H}$. The group of unitary elements of GL(V) clearly is a real linear algebraic group; its Lie algebra consists of the T in End V such that $T^\beta = -T$. Moreover, this group is compact.

LEMMA 1.2. Let β be a hermitian positive definite form of V and let $\mathcal{H}$ be a real linear algebraic subgroup of GL(V) satisfying $\mathcal{H}^\beta = \mathcal{H}$. Then

a) the unitary elements of $\mathcal{H}$ form a maximal compact subgroup $\mathcal{K}$ of $\mathcal{H}$ that is again a real linear algebraic group.

b) Each element of $\mathcal{H}$ can be uniquely written as UP where $U \in \mathcal{K}$, $P = \exp T > 0$, $T^\beta = T \in \text{Lie } \mathcal{H}$.

c) If $W \in \mathcal{H}$, $W > 0$, then there exists a uniquely determined $W^{\frac{1}{2}} \in \mathcal{H}$ such that $W^{\frac{1}{2}} > 0$ and $(W^{\frac{1}{2}})^2 = W$.

Proof: Each $W \in GL(V)$ can be uniquely written as $W = UP$ where U is unitary and $P > 0$. Hence $P = \exp T$ where $T^\beta = T$ and we obtain $W^\beta W = P^2 = \exp 2T$. From $\mathcal{H}^\beta = \mathcal{H}$ it follows that P^2 belongs to $\mathcal{H}$.

Consider the curve $W(\xi) := \exp 2\xi T$, $\xi \in \mathbb{R}$, in GL(V). For any polynomial π we obtain a finite sum,

$$\pi(W(\xi)) = \sum_m \alpha_m e^{\beta_m \xi},$$

by using the "minimal decomposition" of the (semi-simple) endomorphism T. Now let $\mathfrak{p}$ be a set of polynomials that defines the real linear algebraic group $\mathcal{H}$. For integer k we have $W(k) = P^{2k} \in \mathcal{H}$ and hence $\pi(W(k)) = 0$ for $\pi \in \mathfrak{p}$ and all k. Hence $\alpha_m = 0$ for all m and we obtain $W(\xi) \in \mathcal{H}$. This means $T \in \text{Lie } \mathcal{H}$ and $U \in \mathcal{H}$. So part b) is proved.

In order to prove part c) we write $W = P^2$ where $P > 0$ and $W = \exp T$, $T \in \text{Lie } \mathcal{H}$, according to part b). Since the positive definite square root is unique we get $P = \exp \frac{1}{2}T \in \mathcal{H}$.

For part a) let $\mathcal{K}'$ be a compact subgroup of $\mathcal{H}$ such that $\mathcal{K} \subset \mathcal{K}'$. Let $W \in \mathcal{K}'$; hence part b) implies $W = UP$ where $U \in \mathcal{K}$ and $P > 0$, $P \in \mathcal{H}$. The proof of part a) will be complete if we show that $P \in \mathcal{K}'$, $P > 0$, implies $P = I$.

$\mathcal{K}'$ being a compact subgroup of $\mathcal{H}$ means that $\mathcal{K}'$ is compact in GL(V). By a known result there exists a hermitian positive definite form γ of V such that $\gamma(Wu,Wu) = \gamma(u,u)$ for $u \in V$ and $W \in \mathcal{K}'$. Writing $\gamma(u,v) = \beta(Bu,v)$ we obtain $B > 0$ and $W^\beta BW = B$. Choose $C > 0$ such that $B = C^2$ and put $W = P$, $D = CPC$. Hence $D^2 = B^2$ and therefore $D = B$ which means $P = I$. So the lemma is proved.

5. Again let $\mathcal{H}$ be a real linear algebraic subgroup of GL(V) and suppose that β is a hermitian positive definite form of V such that $\mathcal{H}^\beta = \mathcal{H}$. Hence $(\text{Lie } \mathcal{H})^\beta = \text{Lie } \mathcal{H}$ and $T \to -T^\beta$ is an automorphism of the real Lie algebra Lie $\mathcal{H}$. We set

$$\mathfrak{a} = \{A;\ A \in \text{Lie}\ \mathcal{H},\ A^{\beta} = -A\},$$

$$\mathfrak{b} = \{B;\ B \in \text{Lie}\ \mathcal{H},\ B^{\beta} = B\}$$

and we obtain a direct sum decomposition

$$\text{Lie}\ \mathcal{H} = \mathfrak{a} + \mathfrak{b},\ [\mathfrak{a},\mathfrak{a}] \subset \mathfrak{a},\ [\mathfrak{a},\mathfrak{b}] \subset \mathfrak{b},\ [\mathfrak{b},\mathfrak{b}] \subset \mathfrak{a}.$$

Again let $\mathcal{K}$ be the maximal compact subgroup of $\mathcal{H}$ consisting of the unitary elements of $\mathcal{H}$. Hence we obtain

$$\mathcal{H} = \mathcal{K}\cdot\exp\ \mathfrak{b}, \qquad \text{Lie}\ \mathcal{K} = \mathfrak{a}.$$

Finally we prove

LEMMA 1.3. *The restriction of the Killing form of* Lie $\mathcal{H}$ *to* $\mathfrak{a}$ *or* $\mathfrak{b}$ *is negative semi-definite or positive semi-definite, respectively.*

Proof: For $S, T \in \text{Lie}\ \mathcal{H}$ we put $\gamma(S,T) := \text{trace}\ ST^{\beta}$. From $\text{trace}\ T^{\beta} = \overline{\text{trace}\ T}$ it follows that γ is hermitian positive definite on the vector space Lie $\mathcal{H}$ (considered as a vector space over $\mathbb{C}$). A verification yields $\gamma([\text{ad}\ T]S_1, S_2) = \gamma(S_1, [\text{ad}\ T^{\beta}]S_2)$. Hence for the adjoint with respect to γ we observe $(\text{ad}\ T)^{\gamma} = \text{ad}\ T^{\beta}$ and

$$\langle T, T^{\beta}\rangle_{\text{Lie}\ \mathcal{H}} = \text{trace}\ (\text{ad}\ T)(\text{ad}\ T)^{\gamma} \geq 0,$$

which completes the proof.

6. Let W be in End V_o. Then the extension of W

to V is an endomorphism of V and we will identify End V_o with the sub-ring $\{W;\ W \in \text{End } V,\ \overline{W} = W\}$ of End V. Hence for subgroups $\mathcal{G}$ of $GL(V_o)$ we have the notion of a real linear algebraic group. Suppose that α is a symmetric positive definite bilinear form of V_o. Denote the extension of it to V also by α. Then $(u,v) \to \alpha(u,\overline{v})$ defines a hermitian positive definite form of V. Clearly our results are valid for a real linear algebraic subgroup $\mathcal{G}$ of $GL(V_o)$ and the endomorphisms of Lie $\mathcal{G}$ as well as the decompositions in Lemma 1.2 can be chosen as endomorphisms of V_o.

§2. The group $\Gamma(\mathfrak{D})$.

1. Let V_o be a finite dimensional vector space over $\mathbb{R}$. We suppose that $\square : V_o \times V_o \to \text{End } V_o$ is a pairing satisfying the conditions (P.1) to (P.4) of II, §3.1, such that its trace form σ_o is positive definite.

Denote by $\mathfrak{T}_o$ the Lie algebra spanned by $a \square b$ where $a,b \in V_o$. The adjoint of $T \in \text{End } V_o$ with respect to σ_o is denoted by T^*. Furthermore let $P_o(a)$ be the endomorphism induced by $\square$ according to (II;3.2).

Using the identification of "generic elements" of V_o with "vector variables" we consider the Lie algebra Rat V_o of rational functions in the real variable x of V_o. Denote the binary Lie algebra associated with the pairing by $\mathfrak{Q}_o$, i.e.,

$$\mathfrak{Q}_o = V_o + \mathfrak{T}_o + \tilde{V}_o \quad \text{where } \tilde{V}_o = \{P_o(x)b; b \in V_o\}.$$

2. Now let V be the complexification of V_o. By linearity the pairing $\square$ of V_o extends to a pairing $\square : V \times V \to \operatorname{End} V$ having the trace form σ. A verification shows

(2.1) $\sigma(a+ib, c+id) = \sigma_o(a,c) - \sigma_o(b,d) + i[\sigma_o(a,d) + \sigma_o(b,c)]$,

hence σ coincides with the extension of σ_o to V. In particular, σ is a non degenerate bilinear form of V, and $(u,v) \to \sigma(u,\bar{v})$ <u>defines a hermitian positive definite form of</u> V.

Clearly the vector space (over $\mathbb{C}$) spanned by $u \square v$, where $u,v \in V$, coincides with the complexification $\mathfrak{T}$ of $\mathfrak{T}_o$. Again the adjoint of $T \in \operatorname{End} V$ with respect to σ is denoted by T^*. Hence we have $(A+iB)^* = A^* + iB^*$ where $A, B \in \operatorname{End} V_o$. By linearity <u>the pairing</u> $\square : V \times V \to \operatorname{End} V$ <u>satisfies the conditions</u> (P.1) <u>to</u> (P.4), too. Moreover, let $P(u)$, $u \in V$, be the endomorphism of V induced by the pairing. Then for $a \in V_o$ the restriction of $P(a)$ to V_o equals $P_o(a)$ and we have

$$P(a+ib) = P(a) - P(b) + 2iP(a,b), \quad a,b \in V_o.$$

In particular $\overline{P(u)} = P(\bar{u})$ for $u \in V$.

We choose a complex variable z of V and consider the Lie algebra Rat V of rational functions in z. By construction all functions in Rat V are holomorphic in z. Denote the binary Lie algebra associated with the pairing $\square$ of V by $\mathfrak{L}_*$, i.e.,

$$\mathfrak{L}_* = V + \mathfrak{T} + \tilde{V} \quad \text{where } \tilde{V} = \{P(z)u; u\in V\}.$$

Note that $\mathfrak{L}_*$ is not the complexification of $\mathfrak{L}_o$, but $\mathfrak{L}_*$ can be considered as the holomorphization of $\mathfrak{L}_o$ in the following sense: Let $\mathfrak{L}_o$ be any subspace of Rat V_o. Then the <u>holomorphization</u> $\mathfrak{L}_*$ of $\mathfrak{L}_o$ is obtained by the complexification $\mathfrak{L}$ of $\mathfrak{L}_o$ and by replacing the real variable x by z.

3. There is a third pairing induced by the original one. Let $V^{\mathbb{R}}$ be the space V considered as vector space over $\mathbb{R}$. Then the pairing $\square$ of V can be considered as a pairing of $V^{\mathbb{R}}$ having the trace form $\sigma^{\mathbb{R}}$. Using (1.2) we obtain

$$(2.2) \qquad \sigma^{\mathbb{R}}(a+ib, c+id) = 2[\sigma_o(a,c) - \sigma_o(b,d)]$$

$$= 2 \operatorname{Re} \sigma(a+ib, c+id).$$

Again $\sigma^{\mathbb{R}}$ is non degenerate and $(u,v) \to \sigma^{\mathbb{R}}(u,\bar{v})$ defines <u>a symmetric positive definite bilinear form of</u> $V^{\mathbb{R}}$. The vector space (over $\mathbb{R}$) spanned by $u\square v$ where $u,v \in V^{\mathbb{R}}$ coincides with $\mathfrak{T}^{\mathbb{R}}$. Denote the adjoint of $T \in \operatorname{End} V^{\mathbb{R}}$ with respect to $\sigma^{\mathbb{R}}$ by $T^{\P}$. For $T = A+iB \in \operatorname{End} V$ where $A,B \in \operatorname{End} V_o$ we have the induced endomorphism $T^{\mathbb{R}}$ of $V^{\mathbb{R}}$ (see §1.1) and it follows that $(T^{\mathbb{R}})^{\P} = (T^*)^{\mathbb{R}}$. In the notation we will not distinguish between T and $T^{\mathbb{R}}$ if there is no possibility of misunderstanding.

Hence by linearity the pairing $\square : V^{\mathbb{R}} \times V^{\mathbb{R}} \to \text{End}\, V^{\mathbb{R}}$ satisfies the conditions (P.1) to (P.4). The induced endomorphism $P^{\mathbb{R}}(u)$ coincides with $P(u)$ considered as an endomorphism of $V^{\mathbb{R}}$. Note that the endomorphisms of $\mathfrak{T}^{\mathbb{R}}$ are $\mathbb{C}$-linear.

Let x,y be real variables of V_o. We consider the Lie algebra Rat $V^{\mathbb{R}}$ of rational functions in x and y. Setting $z = x + iy$ we get a complex variable of V and $(\text{Rat } V)^{\mathbb{R}}$ is a subspace of Rat $V^{\mathbb{R}}$. The Cauchy-Riemann differential equations show that it is in fact a subalgebra.

Denote the binary Lie algebra associated with the pairing $\square$ of $V^{\mathbb{R}}$ by $\mathfrak{L}$, i.e.,

$$\mathfrak{L} = V^{\mathbb{R}} + \mathfrak{T}^{\mathbb{R}} + \tilde{V}^{\mathbb{R}} = (\mathfrak{L}_*)^{\mathbb{R}}.$$

Note that $\mathfrak{L}$ is a real Lie algebra, but the product is $\mathbb{C}$-linear, so $\mathfrak{L}$ can be considered as a complex Lie algebra.

4. Next we consider the groups Γ associated with the pairings. According to II, Lemma 3.4, we have first

$$\Gamma(\mathfrak{L}_o) = \{W;\ W \in GL(V_o),\ P(Wx) = W\,P(x)\,W^*\}$$

and second

$$\Gamma(\mathfrak{L}_*) = \{W;\ W \in GL(V),\ P(Wz) = W\,P(z)\,W^*\}.$$

We observe that $\Gamma(\mathfrak{D}_o)$ is the subgroup of $\Gamma(\mathfrak{D}_*)$ consisting of the W's such that $\overline{W} = W$. Finally we get

$$\Gamma(\mathfrak{D}) = \{W;\ W \in GL(V^{\mathbb{R}}),\ P(Wz) = W\,P(z)\,W^{\P}\}.$$

Using the injection $W \to W^{\mathbb{R}}$ of $GL(V)$ into $GL(V^{\mathbb{R}})$ we obtain as the image of $\Gamma(\mathfrak{D}_*)$ the subgroup

$$\Gamma_*(\mathfrak{D}) = \{W^{\mathbb{R}};\ W \in GL(V),\ P(Wz) = W\,P(z)\,W^*\}$$

of $\Gamma(\mathfrak{D})$ of the $\mathbb{C}$-linear elements (see §1.1). Hence $[\Gamma(\mathfrak{D}_*)]^{\mathbb{R}} = \Gamma_*(\mathfrak{D})$. Using the identification mentioned above we also write $\Gamma(\mathfrak{D}_*) = \Gamma_*(\mathfrak{D})$. Furthermore from $\overline{P(u)} = P(\overline{u})$ it follows that the conjugation J given by

$$Jz := \overline{z}$$

belongs to $\Gamma(\mathfrak{D})$ and one has $J^{\P} = J$. For a $\mathbb{C}$-linear endomorphism W the endomorphism $\overline{W} = JWJ$ is $\mathbb{C}$-linear, too. Hence $\overline{W}$ and $\overline{W}^*$ belong to $\Gamma_*(\mathfrak{D})$ whenever $W \in \Gamma_*(\mathfrak{D})$.

LEMMA 2.1. *The Lie algebra of $\Gamma_*(\mathfrak{D})$ coincides with $\mathfrak{T}^{\mathbb{R}}$.*

Proof: An element $S \in \text{End}\,V^{\mathbb{R}}$ belongs to $\mathfrak{T}^{\mathbb{R}}$ if and only if there is a $T \in \mathfrak{T}$ such that $S = T^{\mathbb{R}}$. According to II, Lemma 4.5, it suffices to prove that

$$2\,P(Tz,z) = T\,P(z) + P(z)\,T^* \tag{2.3}$$

for $T \in \text{End}\,V$ is equivalent to $\exp \xi T \in \Gamma(\mathfrak{D}_*)$ for $\xi \in \mathbb{R}$, i.e., to $T \in \text{Lie}\,\Gamma(\mathfrak{D}_*)$. We put

$$W = W(\xi) = \exp \xi T,$$

$$Q = Q(\xi) = P(Wz) - W\, P(z)\, W^*$$

and denote the derivative with respect to ξ by $'$. One gets $W' = WT = TW$ and

$$Q' = 2P(W'z, Wz) - W'\, P(z)\, W^* - W\, P(z)\, W'^*.$$

If $T \in \operatorname{Lie} \Gamma(\Omega_*)$ then $Q(\xi) = 0$ and hence $Q'(0) = 0$, so (2.3) holds. Conversely suppose (2.3). We have $Q^* = Q$ and hence $Q'^* = Q'$. But

$$Q' = 2P(TWz, Wz) - TWP(z)W^* - WP(z)W^*T^* = TQ - Q^*T^*$$

yields $Q'^* = -Q'$ and hence $Q' = 0$. One observes $Q(\xi) = Q(0) = 0$ and consequently $W \in \Gamma(\Omega_*)$, so $T \in \operatorname{Lie} \Gamma(\Omega_*)$.

5. Clearly $\overline{T}^* \in \mathfrak{X}^R$ whenever $T \in \mathfrak{X}^R$. Hence $T \to -\overline{T}^*$ becomes an automorphism of the Lie algebra $\mathfrak{X}^R$ and we have the induced direct sum decomposition as vector spaces over $\mathbb{R}$:

$$(2.4) \qquad \mathfrak{X}^R = \mathfrak{D} + \mathfrak{L}, \qquad \mathfrak{D} = \{D;\ D \in \mathfrak{X}^R,\ \overline{D}^* = -D\},$$
$$\mathfrak{L} = \{L;\ L \in \mathfrak{X}^R,\ \overline{L}^* = L\}.$$

Using II, Lemma 4.1, we see that $\mathfrak{D}$ and $\mathfrak{L}$ are orthogonal with respect to the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{X}^R$. We put

$$K = \{W;\ W \in \Gamma_*(\Omega),\ \overline{W}^*W = I\}$$

and we see that K is a real linear algebraic subgroup of $GL(V^{\mathbb{R}})$.

LEMMA 2.2. The group $\mathcal{K}$ is maximal compact in $\Gamma_*(\mathfrak{Q})$ having $\mathfrak{d}$ as a Lie algebra. Any element of $\Gamma_*(\mathfrak{Q})$ has a unique representation as $U \cdot \exp L$ where $U \in \mathcal{K}$ and $L \in \mathfrak{l}$. Moreover one has

(2.5) $\langle D,D \rangle \leq 0$, $\langle L,L \rangle \geq 0$ for $D \in \mathfrak{d}$ and $L \in \mathfrak{l}$.

Proof: Defining β by $\beta(u,v) = \sigma(u,\bar{v})$ we obtain a hermitian positive definite form of V. For $W \in GL(V)$ the adjoint W^β with respect to β equals $\overline{W}^*$. Let $\mathcal{H} = \{W;\ W \in \Gamma(\mathfrak{Q}_*),\ \overline{W}^*W = I\}$. Then $\mathcal{K} = \mathcal{H}^{\mathbb{R}}$ and $\mathcal{H}^\beta = \mathcal{H}$. So we are able to apply Lemma 1.2 and Lemma 1.3 from which the statements follow.

§3. The group $Aut(\mathfrak{Q},\Theta_-)$.

1. We apply now our results of chapter I and II to the binary Lie algebra

$$\mathfrak{Q} = V^{\mathbb{R}} + \mathfrak{T}^{\mathbb{R}} + \tilde{V}^{\mathbb{R}} \subset (\text{Rat } V)^{\mathbb{R}}$$

(see §2.3). According to II,§3.5, there is an automorphism Θ of $\mathfrak{Q}$ of period two given by

(3.1) $$(\Theta q)(z) := -b - T^*z - P(z)a$$

$$\text{where } q(z) = a + Tz + P(z)b \in \mathfrak{Q},$$

and $(\mathfrak{Q},\Theta)$ is a symmetric Lie algebra in the sense of II,§1.1. Again we write the elements of $\mathfrak{Q}$ as

$q = a + T + \Theta b$ where $a, b \in V^{\mathbb{R}}$ and $T \in \mathfrak{T}^{\mathbb{R}}$. In particular we have

$$\Theta T = -T^* \quad \text{and } [a, \Theta b] = a \square b.$$

Note that Θ is $\mathbb{C}$-linear. From II, §3.5, we know that the involution $W \to W^{\#}$ of $\Gamma(\mathfrak{D})$ induced by Θ (see II, Lemma 1.2) is given by $W^{\P}$. Hence for $W \in \Gamma_*(\mathfrak{D})$ we obtain $W^{\#} = W^* \in \Gamma_*(\mathfrak{D})$. As in II, §1.3, we set

$$B(a,b) = B_{\Theta}(a,b) = B_{\Theta b}(a)$$

and we obtain from (II;3.8)

$$B(a,b) = I + a \square b + P(a)P(b). \tag{3.2}$$

Hence from II, §1.3, we observe

$$\tilde{t}_{\Theta b}(z) = [B(z,b)]^{-1}(z + P(z)b) \quad \text{where } b \in V. \tag{3.3}$$

From II, §1.3, it follows that

$$W \circ \tilde{t}_{\Theta b} = \tilde{t}_{\Theta c} \circ W \quad \text{where } c = W^{*-1}b,\ b \in V,\ W \in \Gamma_*(\mathfrak{D}).$$

Using (3.2) and $[P(a)]^* = P(a)$ we get $[B(a,b)]^* = B(b,a)$ and II, Lemma 1.4, yields

$$B(a,b) \in \Gamma_*(\mathfrak{D}) \quad \text{whenever} \quad \det B(a,b) \neq 0. \tag{3.4}$$

the function $\tilde{t}_{\Theta b}$ is holomorphic in its domain of definition because it is rational in z. Clearly the same is true for t_a, $a \in V$. From (I;4.5) we see that the elements of $\Gamma(\mathfrak{D})$ are exactly the functions

(3.5) $W \circ t_a \circ \tilde{t}_{\ominus b} \circ t_c$ where $W \in \Gamma(\mathfrak{D})$ and $a, b, c \in V$.

Let $\mathfrak{S}$ be a subset of Rat $V^{\mathbb{R}}$. Then in comformity with the previous notations let $\mathfrak{S}_*$ be the set of functions in $\mathfrak{S}$ which are holomorphic in its domain of definition. Hence the subgroup $\Xi_*(\mathfrak{D})$ of $\Xi(\mathfrak{D})$ consists of the elements (3.5) where now $W \in \Gamma_*(\mathfrak{D})$. Applying (I;4.5) to $\mathfrak{D}_*$ instead of $\mathfrak{D}$ we observe

$$\Xi_*(\mathfrak{D}) = \Xi(\mathfrak{D}_*).$$

LEMMA 3.1. Suppose $\Phi = \nabla_f$ for some $f \in \Xi(\mathfrak{D})$. Then f belongs to $\Xi_*(\mathfrak{D})$ if and only if Φ is $\mathbb{C}$-linear.

Proof: If $f \in \Xi_*(\mathfrak{D})$ then $\nabla_f \in \operatorname{Aut} \mathfrak{D}_*$ and Φ is $\mathbb{C}$-linear. Conversely, let Φ be $\mathbb{C}$-linear. We write $f = g^{-1}$, $g = W \circ t_a \circ \tilde{t}_{\ominus b} \circ t_c$ where $W \in \Gamma(\mathfrak{D})$ and $a, b, c \in V$. Hence $\Phi q = q^g$ for $q \in \mathfrak{D}$ and from (I;1.4) it follows that the inverse of the Jacobian of g, $B(z+c, b)\, W^{-1}$, is $\mathbb{C}$-linear. Therefore W is $\mathbb{C}$-linear and $g \in \Xi_*(\mathfrak{D})$.

2. We apply now II, Lemma 4.2, to $\mathfrak{D}$. Since the trace form is given by $\sigma^{\mathbb{R}}$, the Killing form of $\mathfrak{D}$ is given by $(\sigma^{\mathbb{R}})_{\mathfrak{D}}$. But $\sigma^{\mathbb{R}}$ is non degenerate and hence $(\sigma^{\mathbb{R}})_{\mathfrak{D}}$ is non degenerate, too. Hence by the criterion of Killing-Cartan, $\mathfrak{D}$ is a semi-simple Lie algebra. Furthermore, II, Lemma 4.2, yields

$$\langle q_1, q_2 \rangle_{\mathfrak{D}} = \langle T_1, T_2 \rangle_{\mathfrak{T}^{\mathbb{R}}} + 2 \operatorname{trace} T_1 T_2$$
$$+ \sigma^{\mathbb{R}}(a_1, b_2) + \sigma^{\mathbb{R}}(a_2, b_1).$$

where $q_k = a_k + T_k + \Theta b_k$ and where the trace is taken over $\mathfrak{T}^{\mathbb{R}}$.

Next we use the decomposition $\mathfrak{T}^{\mathbb{R}} = \mathfrak{d} + \mathfrak{l}$ introduced in §2.5. For $D \in \mathfrak{d}$, $L \in \mathfrak{l}$ we have, according to (1.2), clearly trace $DL = 2$ trace_o Re DL where trace_o means the trace over V_o. But Re DL = Re $\overline{DL}$ = - Re $D^* L^*$ together with trace_o Re $D^* L^*$ = trace_o Re DL imply trace $DL = 0$. Hence we obtain

(3.6) $$\langle q_1, q_2 \rangle_{\mathfrak{Q}} = \langle D_1, D_2 \rangle_{\mathfrak{d}+\mathfrak{l}} + 2 \text{ trace } D_1 D_2$$
$$+ \langle L_1, L_2 \rangle_{\mathfrak{d}+\mathfrak{l}} + 2 \text{ trace } L_1 L_2 + \sigma^{\mathbb{R}}(a_1, b_2)$$
$$+ \sigma^{\mathbb{R}}(a_2, b_1),$$

where $T_k = D_k + L_k$.

3. Using the conjugation J of $V^{\mathbb{R}}$ given by $Jz = \overline{z}$ we introduce two more automorphisms Θ_+ and Θ_- of $\mathfrak{Q}$ by

$$\Theta_{\pm} = \Theta \, \nabla_{\pm J} = \nabla_{\pm J} \, \Theta$$

(see II, §3.6). According to (II;3.10) we have the explicit definition

(3.7) $$\Theta_{\pm} q = \pm \overline{b} - \overline{T}^* \pm \Theta \overline{a} \quad \text{where } q = a + T + \Theta b \in \mathfrak{Q}.$$

Again $(\mathfrak{Q}, \Theta_+)$ and $(\mathfrak{Q}, \Theta_-)$ are symmetric Lie algebras and one verifies that Θ, Θ_+ and Θ_- commute by pairs. Furthermore one gets

$$\Theta_o := \nabla_{-I} = \Theta_+ \Theta_- .$$

Note that Θ_o is $\mathbb{C}$-linear but $\Theta_\pm$ is not.

The involutions of $\Gamma_*(\mathfrak{Q})$ induced by Θ_+ or Θ_- coincide and are given by $W \to \overline{W}^*$ (see II, §3.6).

LEMMA 3.2. *The symmetric bilinear form β of $\mathfrak{Q}$ given by*

$$\beta(q_1, q_2) := -\langle q_1, \Theta_- q_2 \rangle_{\mathfrak{Q}}$$

is positive definite on $\mathfrak{Q}$.

Proof: From the definition of the Killing form follows $\langle \Phi q_1, \Phi q_2 \rangle_{\mathfrak{Q}} = \langle q_1, q_2 \rangle_{\mathfrak{Q}}$ for each automorphism Φ of $\mathfrak{Q}$. Hence β is symmetric. Using (3.6) we obtain $\Theta_-(D+L) = D-L$ and from (3.7) it follows

$$\langle q, \Theta_- q \rangle_{\mathfrak{Q}} = \langle D, D \rangle_{\mathfrak{d}+\mathfrak{l}} - 2 \text{ trace } D\overline{D}^*$$

$$-\langle L, L \rangle_{\mathfrak{d}+\mathfrak{l}} - 2 \text{ trace } L\overline{L}^* - \sigma^{\mathbb{R}}(a, \overline{a}) - \sigma^{\mathbb{R}}(b, \overline{b}),$$

where $q = a + (D+L) + \Theta b \in \mathfrak{Q}$. Here we have trace $D\overline{D}^* \geq 0$ and trace $L\overline{L}^* \geq 0$. Moreover, the definition (2.2) yields $\sigma^{\mathbb{R}}(a, \overline{a}) \geq 0$ for $a \in V$. Hence (2.5) implies $\langle q, \Theta_- q \rangle_{\mathfrak{Q}} \leq 0$ for $q \in \mathfrak{Q}$. Since the Killing form of $\mathfrak{Q}$ is non degenerate, the same is true for the bilinear form β and we end up with $\beta(q,q) > 0$ for $0 \neq q \in \mathfrak{Q}$.

4. For $\Psi \in \text{Aut } \mathfrak{Q}$ we set

$$\text{Aut}(\mathfrak{Q}, \Psi) := \{\Phi;\ \Phi \in \text{Aut } \mathfrak{Q},\ \Phi\Psi = \Psi\Phi\}.$$

Clearly Aut $\mathfrak{Q}$ and Aut$(\mathfrak{Q}, \Psi)$ are real linear algebraic groups in GL($\mathfrak{L}$). The Lie algebra of Aut $\mathfrak{L}$ coincides with the Lie algebra of all derivations of $\mathfrak{L}$. But $\mathfrak{L}$ is semi-simple and hence

$$\text{Lie Aut } \mathfrak{L} = \{\text{ad } p;\ p \in \mathfrak{L}\}.$$

Moreover, ad $p \in$ Lie Aut$(\mathfrak{L},\Psi)$ is equivalent to $\exp \xi \text{ ad } p = \Psi\cdot\exp \xi \text{ ad } p\cdot\Psi^{-1} = \exp \xi[\Psi\cdot\text{ad } p\cdot\Psi^{-1}]$ and since exp is bijective in a neighborhood of zero, we obtain the equivalent condition $\Psi\cdot\text{ad } p = \text{ad } p\cdot\Psi$, i.e., $\Psi[p,q] = [p,\Psi q]$ for $q \in \mathfrak{L}$. This means $[\Psi p,q] = [p,q]$ for $q \in \mathfrak{L}$ and since $\mathfrak{L}$ is centerless (see II, §2.1) we obtain $\Psi p = p$. Hence

$$\text{(3.8)} \quad \text{Lie Aut}(\mathfrak{L},\Psi) = \{\text{ad } p;\ p \in \mathfrak{L},\ \Psi p = p\}$$

$$\cong \{p;\ p \in \mathfrak{L},\ \Psi p = p\}.$$

The adjoint of $\Phi \in$ Aut $\mathfrak{L}$ with respect to β is given by

$$\text{(3.9)} \qquad \Phi^{\beta} = \Theta_{-}\ \Phi^{-1}\ \Theta_{-}\ .$$

Hence $\Phi \to \Phi^{\beta}$ maps Aut $\mathfrak{L}$ as well as Aut$(\mathfrak{L},\Theta_{+})$ and Aut$(\mathfrak{L},\Theta_{-})$ onto themselves (because Θ_{+} and Θ_{-} commute).

Since the Killing form of a Lie algebra is associative we observe

$$\text{(3.10)} \qquad (\text{ad } p)^{\beta} = -\text{ ad } \Theta_{-}p = -\ \Theta_{-}\cdot\text{ad } p\cdot\Theta_{-}\ .$$

We write $\Phi > 0$ if $\Phi^{\beta} = \Phi$ and if Φ is positive definite with respect to β (see §1.4). As an abbreviation, set

$$\Xi_*^o(\mathfrak{D}) := \Xi^o(\mathfrak{D}) \cap \Xi_*(\mathfrak{D})$$

where $\Xi^o(\mathfrak{D})$ is defined as in I,§4.3. Now we are able to prove

THEOREM 3.3. a) $\mathrm{Aut}(\mathfrak{D},\Theta_-)$ is a maximal compact subgroup of Aut $\mathfrak{D}$.

b) Each element in Aut $\mathfrak{D}$ can be uniquely written as $\Psi\Phi$ where $\Psi \in \mathrm{Aut}(\mathfrak{D},\Theta_-)$ and where

$$\Phi^\beta = \Phi = \exp \mathrm{ad}\, p > 0, \ p\in\mathfrak{D}, \ \Theta_- p = -p.$$

c) If $\Phi \in$ Aut $\mathfrak{D}$ and $\Phi > 0$ then Φ is essential and $\mathbb{C}$-linear. Furthermore there exists $f \in \Xi_*^o(\mathfrak{D})$ such that $\Phi = \nabla_f$.

Proof: We apply Lemma 1.2 to $\mathfrak{H}$ = Aut $\mathfrak{D}$. In view of (3.9) the unitary elements of Aut $\mathfrak{D}$ are exactly the elements of $\mathrm{Aut}(\mathfrak{D},\Theta_-)$, so we already proved part a).

Moreover, we have a unique representation $\Psi\Phi$ of the elements of Aut $\mathfrak{D}$ where $\Psi \in \mathrm{Aut}(\mathfrak{D},\Theta_-)$ and $\Phi = \exp \mathrm{ad}\, p$, $p\in\mathfrak{D}$, such that $(\mathrm{ad}\, p)^\beta = \mathrm{ad}\, p$. Hence $\Theta_- p = -p$ and part b) is proved.

Finally let $\Phi \in$ Aut $\mathfrak{D}$, $\Phi > 0$. Hence $\beta(\Phi q,q) > 0$ for $0 \neq q\in\mathfrak{D}$. Choose $q = a\in V$ and put $\Phi a = b + T + \partial c$ where $b,c \in V$ and $T \in \mathfrak{T}^{\mathbb{R}}$. Clearly $b = (\Phi a)(0) = Aa$ where $A = H_\Phi(0)$ in the notation (I;2.2). We observe $\Theta_- a = -\Theta\bar{a}$ and (3.6) yields

$$\beta(\Phi a,a) = -\langle \Phi a, \Theta_- a\rangle_{\mathfrak{D}} = \langle \Phi a, \Theta \bar{a}\rangle_{\mathfrak{D}}$$

$$= \sigma^{\mathbb{R}}(b,\bar{a}) = \sigma^{\mathbb{R}}(Aa,\bar{a}).$$

From Lemma 3.2 we conclude that $\det H_\Phi(0) \neq 0$. In particular, Φ is essential. According to I, Theorem 2.1, there exists an $f \in \Xi(\mathfrak{D})$ such that $\Phi q = q^f$. But $\Phi = \exp \operatorname{ad} p$ is $\mathbb{C}$-linear and hence $f \in \Xi_*(\mathfrak{D})$ because of Lemma 3.1. From

$$\left(\frac{\partial f(z)}{\partial z}\right)^{-1} = H_\Phi(z)$$

we obtain $\omega_f(0) = \det H_\Phi(0) \neq 0$ (see I, §4.3) and $f \in \Xi^o(\mathfrak{D})$. Using the result for Φ^{-1} instead of Φ we end up with $\Phi = \nabla_f$ where $f \in \Xi^o_*(\mathfrak{D})$.

<u>LEMMA 3.4</u>. <u>Suppose</u> $\Phi = \nabla_f$ <u>for some</u> $f \in \Xi^o_*(\mathfrak{D})$. <u>Then</u> $\Phi^\beta = \Phi$ <u>if and only if</u>

$$f = t_c \circ W \circ \tilde{t}_{\Theta \bar{c}}, \quad \underline{\text{where}}\ c \in V\ \underline{\text{and}}\ \overline{W}^* = W \in \Gamma_*(\mathfrak{D}).$$

<u>Proof</u>: In view of I, Theorem 4.3, we write $f = t_c \circ W \circ \tilde{t}_{\Theta d}$ where $c,d \in V$ and $W \in \Gamma_*(\mathfrak{D})$. Hence

$$\Phi = \Psi_c \nabla_W \Theta \Psi_d \Theta$$

because of II, Lemma 1.1, and I, Theorem 2.3b). Using (3.9) we see that $\Phi^\beta = \Phi$ is equivalent to $\Phi \Theta_- \Phi = \Theta_-$, hence to $\Phi_1^{-1} = \Theta \Phi_1 \Theta$ where $\Phi_1 = \Psi_d \nabla_{-J} \Psi_c \nabla_W = \Psi_{d-\bar{c}} \nabla_{-JW}$. As an equivalent condition we get

$$\nabla^{-1}_{-JW} \Psi_{\bar{c}-d} = \Theta \Psi_{d-\bar{c}} \Theta \nabla^{-1}_{-(JW)^\tau} = \tilde{\Psi}_{\Theta(d-\bar{c})} \nabla^{-1}_{-(JW)^\tau}$$

because of II, Lemma 1.1 and Lemma 1.2. In terms of the rational functions belonging to it we have

$$(JW)^{-1} \circ t_{d-\bar{c}} \circ (JW)^{\P} = \tilde{t}_{\Theta(d-\bar{c})} .$$

But this is equivalent to $d = \bar{c}$ and $JW = (JW)^{\P} = W^*J$, i.e., $\overline{W}^* = W$.

<u>THEOREM 3.5.</u> <u>Let</u> Φ <u>be in</u> Aut $\mathfrak{D}$. <u>Then the following two conditions are equivalent</u>:

a) $\Phi^{\beta} = \Phi > 0.$

b) <u>There exists</u> $W \in \Gamma_*(\mathfrak{D})$ <u>and</u> $c \in V$ <u>such that</u> $\Phi = \nabla_f$ <u>and</u> $f = t_c \circ W \circ \tilde{t}_{\Theta \bar{c}}$ <u>and</u> $\overline{W}^* = W > 0.$

<u>Proof</u>: In view of Lemma 3.4 we know that $\Phi^{\beta} = \Phi$ is equivalent to $\Phi = \nabla_f$ where $f = t_c \circ W \circ \tilde{t}_{\Theta \bar{c}}$ and $\overline{W}^* = W \in \Gamma_*(\mathfrak{D})$. We obtain

$$\nabla_f = \Psi_c \ \nabla_W \ \tilde{\Psi}_{\Theta \bar{c}}$$

and

$$(\Psi_c)^{\beta} = \Theta_- \ \Psi_{-c} \ \Theta_- = \Theta \ \nabla_{-J} \ \Psi_{-c} \ \nabla_{-J} \ \Theta = \Theta \ \Psi_{\bar{c}} \ \Theta = \tilde{\Psi}_{\Theta \bar{c}}$$

because of (3.9). Hence

$$\Phi = \nabla_f = \Psi_c \ \nabla_W \ (\Psi_c)^{\beta}$$

and $\Phi > 0$ means $\nabla_W > 0$. From $\nabla_W > 0$ it follows

$$\beta(\nabla_W a, a) = -\langle Wa, \Theta_- a \rangle_{\mathfrak{D}} = \sigma^{\mathbb{R}}(Wa, \bar{a}) > 0$$

for $0 \neq a \in V$. Hence $W > 0$. Conversely let $W > 0$. Then

$W = U^2$ where $\overline{U}^* = U > 0$ and $\nabla_W = \nabla_U(\nabla_U)^\beta > 0$.

§4. The groups $\mathrm{Aut}(\mathfrak{D}, \Theta_+)$ and $\mathfrak{G}$.

1. Next we consider the group $\mathrm{Aut}(\mathfrak{D}, \Theta_+)$ and its subgroup

$$\mathfrak{M} := \mathrm{Aut}(\mathfrak{D}, \Theta_+) \cap \mathrm{Aut}(\mathfrak{D}, \Theta_-).$$

As in §2.5 we denote by $\mathfrak{K}$ the group of unitary elements of $\Gamma_*(\mathfrak{D})$, i.e., $\mathfrak{K} = \{U;\ U \in \Gamma_*(\mathfrak{D}),\ \overline{U}^*U = I\}$. Moreover let $\mathfrak{M}_o$ and $\mathfrak{K}_o$ be the identity component of $\mathfrak{M}$ and $\mathfrak{K}$. Hence Lie $\mathfrak{K}_o = \mathfrak{d}$.

THEOREM 4.1. a) *$\mathfrak{M}$ is a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{D}, \Theta_+)$ and its Lie algebra is given by $\mathrm{ad}\ \mathfrak{d}$. Moreover, for $D \in \mathfrak{d}$ we have*

$$(\exp \mathrm{ad}\ D)q = \nabla_W q \text{ where } W = \exp D \in \mathfrak{K}_o \text{ and } q \in \mathfrak{D},$$

The map $\nabla : \mathfrak{K}_o \to \mathfrak{M}_o$ is an isomorphism of the groups.

b) *Each element in $\mathrm{Aut}(\mathfrak{D}, \Theta_+)$ can be uniquely written as $\Psi\Phi$ where $\Psi \in \mathfrak{M}$ and $\Phi \in \mathrm{Aut}(\mathfrak{D}, \Theta_+)$ such that*

$$\Phi^\beta = \Phi = \exp \mathrm{ad}\ p > 0,\ p = a + \Theta\overline{a},\ a \in V. \tag{4.1}$$

c) *Each element in the identity component of $\mathrm{Aut}(\mathfrak{D}, \Theta_+)$ is essential, $\mathbb{C}$-linear and it can be uniquely written as $\nabla_U\Phi$ where $U \in \mathfrak{K}_o$ and $\Phi \in \mathrm{Aut}(\mathfrak{D}, \Theta_+)$ satisfying* (4.1).

<u>Proof</u>: We apply Lemma 1.2 to $\mathcal{H} = \mathrm{Aut}(\mathfrak{D}, \Theta_+)$ and we use the bilinear form β of Lemma 3.2. The unique representation together with Theorem 3.3 shows that $\mathfrak{m}$ is a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{D}, \Theta_+)$ and the elements of $\mathrm{Aut}(\mathfrak{D}, \Theta_+)$ have a representation $\Psi\Phi$ where $\Psi \in \mathfrak{m}$ and $\Phi = \exp \mathrm{ad}\, p$, $\Theta_- p = -p$. But $\mathrm{ad}\, p \in \mathrm{Lie}\,\mathrm{Aut}(\mathfrak{D}, \Theta_+)$ yields $\Theta_+ p = p$ because of (3.8). Hence $p = a + \Theta \bar{a}$ where $a \in V$. So part b) is proved.

The Lie algebra of $\mathfrak{m}$ consists of the elements of the form ad p where $\Theta_{\pm} p = p$. Hence $p = D \in \mathfrak{d}$. For $D \in \mathfrak{d}$ and $q = a + T + \Theta b \in \mathfrak{D}$ we observe

$$(\exp \mathrm{ad}\, D) q = \sum_{m=o}^{\infty} \frac{1}{m!} \{D^m a + (\mathrm{ad}\, D)^m T + \Theta(\bar{D}^m b)\}$$

because of $(\mathrm{ad}\, D)(\Theta b) = [D, \Theta b] = \Theta[\Theta D, b] = \Theta(\bar{D} b)$. From

$$(\mathrm{ad}\; D)^m T = \sum_{k=o}^{m} (-1)^k \binom{m}{k} D^{m-k}\, T\, D^k$$

it follows that

$$(\exp \mathrm{ad}\, D) q = Wa + WTW^{-1} + \Theta(\bar{W} b) \text{ where } W = \exp D \in K_o.$$

Using $\bar{W}^* W = I$ we obtain $\bar{W} = W^{*-1}$ and consequently $\exp \mathrm{ad}\, D = \nabla_W$. Hence part a) is proved because $\exp \mathrm{ad}\, D$, $D \in \mathfrak{d}$, generates the identity component $\mathfrak{m}_o$ of $\mathfrak{m}$.

According to part b) the identity component of

$\mathrm{Aut}(\mathfrak{D},\Theta_+)$ consists of the elements $\nabla_U\Phi$ where $U\in K_o$ and Φ satisfies (4.1). In particular ∇_U is essential and $\mathbb{C}$-linear. According to part c) of Theorem 3.3 the same is true for Φ.

2. Let $\mathcal{G} = \mathcal{G}_\square$ be the set of $f \in \Xi(\mathfrak{D})$ such that ∇_f is in the identity component of $\mathrm{Aut}(\mathfrak{D},\Theta_+)$. According to the parts c) of Theorem 3.3 and Theorem 4.1, $\mathcal{G}$ is a group of birational functions contained in $\Xi^o_*(\mathfrak{D})$ and $\mathcal{G}$ is isomorphic to the identity component of $\mathrm{Aut}(\mathfrak{D},\Theta_+)$. Using Theorem 3.5 we see that the elements of $\mathcal{G}$ are exactly the functions $U\circ g$, where $U\in K_o$ and where

(4.2) $$g = t_c\circ W\circ \tilde{t}_{\Theta\bar{c}} \quad \text{where } W \in \Gamma_*(\mathfrak{D}),\ c\in V$$
$$\text{such that } \overline{W}^* = W > 0$$
$$\text{and } \nabla_g\,\Theta_+ = \Theta_+\,\nabla_g.$$

Furthermore, the representation of the elements of $\mathcal{G}$ as $U\circ g$ is unique.

We prove that in (4.2) the condition $\nabla_g\,\Theta_+ = \Theta_+\,\nabla_g$ can be replaced by

(4.3) $$g\circ(-I)\circ g = -I.$$

Indeed, it suffices to show that for $\Phi = \nabla_g$, $\Phi^\beta = \Phi$ (see Theorem 3.5) the condition $\nabla_g\Theta_+ = \Theta_+\nabla_g$ is equivalent to (4.3). But this follows from (3.9) and

$\Theta_- = \Theta_+ \nabla_{-I}$.

3. We define the subalgebras $\mathfrak{D}_+$ and $\mathfrak{D}_-$ of $\mathfrak{D}$ by

$$\mathfrak{D}_\pm = \{p;\ p\in\mathfrak{D},\ \Theta_\pm p = p\}.$$

We know from (3.8) that

$$\text{Lie Aut}(\mathfrak{D},\Theta_\pm) = \mathfrak{D}_\pm. \tag{4.4}$$

Using the isomorphism $f \to \nabla_f$ of $\mathfrak{G}$ onto the identity component of $\text{Aut}(\mathfrak{D},\Theta_+)$ we may consider $\mathfrak{G}$ as a Lie group. Then its Lie algebra will be isomorphic to the Lie algebra of $\text{Aut}(\mathfrak{D},\Theta_+)$ and (4.4) yields

$$\text{Lie}\ \mathfrak{G} \cong \mathfrak{D}_+.$$

Next we prove

THEOREM 4.2. The complexifications of $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are isomorphic to $\mathfrak{D}$, considered as complex Lie algebras.

Proof: We write the complexification of $\mathfrak{D}_\epsilon$, $\epsilon = \pm$, as $\mathfrak{D}_\epsilon + j\mathfrak{D}_\epsilon$ where the sum is direct and where $j^2 = -1$. Define a map

$$\varphi : \mathfrak{D} \longrightarrow \mathfrak{D}_\epsilon + j\mathfrak{D}_\epsilon$$

by

$$\varphi(q) = \tfrac{1}{2}(q+\Theta_\epsilon q) - J[\tfrac{i}{2}(q-\Theta_\epsilon q)] \quad \text{where } q\in\mathfrak{D}.$$

From $\Theta_\epsilon iq = i\Theta_\epsilon q$ we obtain $\varphi(q) \in \mathfrak{D}_\epsilon + j\mathfrak{D}_\epsilon$ and $\varphi(iq) = j\varphi(q)$. Furthermore, φ is injective and $\mathbb{R}$-linear.

For arbitrary $q_1, q_2 \in \mathfrak{Q}$ we set $q = q_1 + iq_2$ and we get $\varphi(q) = q_1 + jq_2$. Hence φ becomes a bijection. A verification shows that φ is a homomorphism of the Lie algebras.

Since $\mathfrak{Q}$ is semi-simple we obtain

COROLLARY 1. $\mathfrak{G}$ *is a (connected) semi-simple Lie group.*

From part a) of Theorem 3.3 together with (4.4) we get the

COROLLARY 2. $\mathrm{Aut}(\mathfrak{Q}, \Theta_-)$ *is a semi-simple compact Lie group.*

§5. The bounded symmetric domain Z.

1. We use now the results of II,§2, about the symmetric Lie algebra $(\mathfrak{Q}, \Theta_+)$. In terms of $B(a,b)$ the endomorphism corresponding to Θ_+ is given by

$$B_+(a,b) = B_{\Theta_+ b}(a) = B(a,\bar{b}) \quad \text{where } a,b \in V$$

because of $\Theta_+ b = \Theta\bar{b}$.

The involution of $\Gamma(\mathfrak{Q})$ induced by Θ is given by the adjoint $W^{\P}$ of W with respect to σ^{P} (see §2.4 and §3.1). Hence the involution $\Gamma(\mathfrak{Q})$ induced by Θ_+ is given by $\overline{W}^{\P}$. For $W \in \Gamma_*(\mathfrak{Q})$ we have $W^{\P} = W^*$, where W^* stands for the adjoint of W with respect to τ.

Rewriting the definitions of II,§2, for Θ_+ instead of Θ we obtain

$$\Xi(\mathfrak{D},\Theta_+) = \{f;\ f\in\Xi(\mathfrak{D}),\ \nabla_f\ \Theta_+ = \Theta_+\ \nabla_f\},$$

$$\Gamma(\mathfrak{D},\Theta_+) = \{W;\ W\in\Gamma(\mathfrak{D}),\ \overline{W}^{\P}W = I\},$$

$$D(\mathfrak{D},\Theta_+) = \{c;\ c\in V,\ \text{there exists } W\in\Gamma(\mathfrak{D}) \text{ such that } B(c,-\overline{c}) = \overline{W}^{\P}W\}.$$

Note that B has to be replaced by B_+. Clearly $\nabla_f \in \mathrm{Aut}(\mathfrak{D},\Theta_+)$ for $f \in \Xi(\mathfrak{D},\Theta_+)$ and the subgroup of $\Gamma(\mathfrak{D},\Theta_+)$ of the $\mathbb{C}$-linear elements equals $\mathcal{K}$ (see §2.5). We know from II,§2.1, that $\mathcal{K}$ maps $D(\mathfrak{D},\Theta_+)$ onto itself.

From (3.2) we see that $B(a,-\overline{a})$ is hermitian with respect to the hermitian positive definite form of V that is given by $(u,v) \to \sigma(u,\overline{v})$. Again we write $A > 0$ if the endomorphism A of V is hermitian positive definite.

<u>PROPOSITION 1</u>. $D(\mathfrak{D},\Theta_+)$ <u>equals</u> $\{c;\ c\in V,\ B(c,-\overline{c}) > 0\}$, <u>being an open subset of</u> V, <u>and the condition</u> (A) <u>of</u> II, §2.5, <u>is satisfied</u>. <u>Moreover</u>, <u>to</u> $c \in D(\mathfrak{D},\Theta_+)$ <u>there exists a unique</u> $B_c > 0$ <u>such that</u> $B(c,-\overline{c}) = (B_c)^2, B_c \in \Gamma_*(\mathfrak{D})$. In particular, II, Theorem 2.1, can be applied.

<u>Proof</u>: Let $c \in D(\mathfrak{D},\Theta_+)$. Hence $B(c,-\overline{c}) = \overline{W}^{\P}W$ for some $W \in \Gamma(\mathfrak{D})$. But $B(c,-\overline{c})$ is $\mathbb{C}$-linear and therefore (2.2) yields

$$2\sigma(B(c,-\bar{c})u,\bar{u}) = \sigma^{R}(\bar{W}^{\P}Wu,\bar{u}) = \sigma^{R}(Wu,\overline{Wu}) > 0$$

of $0 \neq u \in V$. Hence $B(c,-\bar{c}) > 0$.

Conversely let $B(c,-\bar{c}) > 0$ for some $c \in V$. Hence $B(c,-\bar{c}) \in \Gamma_{*}(\mathfrak{D})$ because of (3.4) and part c) of Lemma 1.2 shows that $B_c := [B(c,-\bar{c})]^{\frac{1}{2}}$ belongs to $\Gamma_{*}(\mathfrak{D})$. In particular $B(c,-\bar{c}) = (B_c)^2$ and $c \in D(\mathfrak{D},\Theta_{+})$. Hence $D(\mathfrak{D},\Theta_{+})$ is open in the natural topology of V and the condition (A) is fulfilled.

2. Denote by $Z = Z_{\square}$ the connected component of $D(\mathfrak{D},\Theta_{+})$ that contains zero. Hence Z equals the connected component of $\{z;\ \det B(z,-\bar{z}) \neq 0\}$ that contains zero. In particular, Z is open in the natural topology of V. Clearly, $z \to \bar{z}$ as well as $z \to Uz$, $U \in K_o$, maps Z onto itself.

We define

$$(5.1) \qquad g_c := t_c \circ B_c \circ \tilde{t}_{\Theta\bar{c}} \qquad \text{for } c \in D(\mathfrak{D},\Theta_{+}).$$

Clearly g_c belongs to $\Xi_{*}^{o}(\mathfrak{D})$.

Let $\mathcal{G}$ be the group of birational functions as defined in §4.2. Let D be a non empty open subset of V. A mapping $f : D \to D$ is called biholomorphic if f is bijective and if f as well as the inverse mapping f^{-1} is holomorphic in D. The domain D is called symmetric if

(i) the group of biholomorphic mappings of D onto itself acts transitively on D,

(ii) there exists $d \in D$ and a biholomorphic map f of D such that d is an isolated fixed point of f and $f \circ f = I$.

THEOREM A.

a) Z is a bounded symmetric domain in V.

b) The elements of $\mathcal{G}$ are exactly the birational functions $f = U \circ g_c$ where $U \in \mathcal{K}_o$ and $c \in Z$. Moreover, this representation of f is unique.

c) Each $f \in \mathcal{G}$ is holomorphic in Z and $\mathcal{G}$ acts on Z via $\mathcal{G} \times Z \to Z$, $(f,z) \to f(z)$, as a transitive group of biholomorphic mappings.

d) The isotropic subgroup of $\mathcal{G}$ with respect to zero equals $\mathcal{K}_o$, i.e., $f(0) = 0$ for $f \in \mathcal{G}$ is equivalent to $f = U \in \mathcal{K}_o$.

3. The proof is divided into several propositions. If X is a topological space then we write $\varphi \sim \psi$ for $\varphi, \psi \in X$ provided there is a continuous curve in X connecting φ and ψ.

PROPOSITION 2. Let c be in Z. Then

a) $g_c \in \mathcal{G}$ and $g_c^{-1} = g_{-c}$,

b) $B_c c = c - P(c)\bar{c}$.

Proof: As in II, §2, we define (now in a canonical way)

$$\tilde{c} = B_c \circ \tilde{t}_{-\Theta\bar{c}}(c), \quad s_c = t_{\tilde{c}} \circ B_c \circ \tilde{t}_{\Theta\bar{c}}. \quad c \in D(\mathfrak{D}, \Theta_+).$$

Note $s_c(0) = \tilde{c}$. Let c be in Z. Then ∇_f, $f = s_c$ commutes with Θ_+ because of II, Theorem 2.1 (notice, that B has to be replaced by B_+). But $c \sim 0$ implies $B_c \sim I$, $\tilde{c} \sim 0$ and $s_c \sim I$. Hence $s_c \in \mathfrak{G}$.

In part b) of II, Theorem 2.1, we choose $f = s_c$, $x = y = 0$ and obtain

$$B(\tilde{c}, -\bar{\tilde{c}}) = (B_c)^2 = B(c, -\bar{c}).$$

Hence $\tilde{c} \in Z$ and $B_{\tilde{c}} = B_c$,

We define $f = s_{-\tilde{c}} \circ s_c$ and we obtain an element of $\mathfrak{G}$. From $s_c(0) = c$ and $s_b(-b) = 0$ follows $f(0) = 0$. Using the chain rule and (I;4.6) we see that the Jacobian of f at the point 0 equals I. Hence $\omega_f(0) \neq 0$ and the last statement in II, Theorem 2.1, yields $f \in \Gamma(\mathfrak{D}, \Theta_+)$ and consequently $f = I$. So we proved $s_c^{-1} = s_{-\tilde{c}}$.

Since $s_c \in \mathfrak{G}$ we may apply (4.2). Hence there exists $d \in V$, $U \in K_o$ and $W \in \Gamma_*(\mathfrak{D})$ such that

$$s_c = U \circ t_d \circ W \circ \tilde{t}_{\Theta\bar{d}} \quad \text{where } \bar{W}^* = W > 0.$$

The uniqueness result of I, Theorem 4.3, yields $d = c$ and thus we have $t_{\tilde{c}} \circ B_c = U \circ t_d \circ W = t_{Ud}\, UW$. It follows $\tilde{c} = Ud$ and $B_c = UW$. Here U is hermitian and B_c as well as W is positive definite. Hence the uniqueness of Lemma 1.2 yields $U = I$ and $W = B_c$. So $\tilde{c} = c$ and $s_c = g_c$.

PROPOSITION 3. a) *Each element* f *in* $\mathcal{G}$ *can be uniquely written as* $f = U \circ g_c$ *where* $U \in \mathcal{K}_o$ *and* $c \in Z$. *Moreover* $f(0) = 0$ *is equivalent to* $c = 0$, *i.e., to* $f = U \in \mathcal{K}_o$.

b) *For* $f \in \mathcal{G}$ *we have* $Z \subset D_f$ *(see* I, §4.2) *and* $z \to f(z)$ *maps* Z *biholomorphically onto itself.*

Proof: The corollary of II, Theorem 2.1, shows that $f \in \mathcal{G}$ can be uniquely written as $f = U \circ s_c$ where $U \in \Gamma(\mathfrak{D}, \Theta_+)$ and $c \in D(\mathfrak{D}, \Theta_+)$. But $f \sim I$ yields $f^{-1}(0) \sim 0$ and hence $c \sim 0$. So c belongs to Z and s_c equals g_c in view of the proof of Proposition 2. From $f(0) = 0$ follows $Uc = 0$ and hence $c = 0$. So part a) is proved.

Let $f \in \mathcal{G}$ and $b \in Z$. Then $f \circ g_b$ belongs to $\mathcal{G}$ and part a) yields $h = U \circ g_c$ for some $U \in \mathcal{K}_o$ and $c \in Z$. It follows that

$$\omega_h(z) = \omega_f(g_b(z))\ \omega_{g_c}(z)$$

according to (I;4.7). Since $\mathcal{G}$ is contained in $\Xi_*^o(\mathfrak{D})$ we have $\omega_h(0) \neq 0$. Hence $b = g_b(0) \in D_f$. Thus f is holomorphic in Z and $f(b) = h(0) = Uc$ belongs to Z. So $z \to f(z)$ maps Z into itself. Since f is birational it is biholomorphic.

PROPOSITION 4. Z *is a bounded symmetric domain and*

$$Z \subset \{z;\ z \in V,\ I - P(z)P(\bar{z}) > 0\}$$

$$\subset \{z;\ z \in V,\ 2I - z \square \bar{z} > 0\}.$$

<u>Proof</u>: Let $c \in Z$ and set $g = g_c$. From Theorem 3.5 follows $\nabla_g > 0$ (with respect to the bilinear form β) and hence $\beta(q^g, q) > 0$ for $0 \neq q \in \mathfrak{Q}$. Choose $q = a + \Theta b$ where $a, b \in V$ and set $q^g = a_1 + T_1 + \Theta b_1$. Thus

$$\beta(q^g, q) = \langle a_1 + T_1 + \Theta b_1, \bar{b} + \Theta \bar{a} \rangle_{\mathfrak{Q}}$$

$$= \sigma^{\mathbb{R}}(a_1, \bar{a}) + \sigma^{\mathbb{R}}(b, \bar{b}_1)$$

according to §3.2. A verification leads to

$$a_1 = (q^g)(0) = B_c^{-1}[a - P(c)b],$$

and from $\Theta_+ q^g = \Theta_+ \nabla_g^{-1} q = \nabla_g^{-1} \Theta_+ q = (\Theta_+ q)^g$ it follows that

$$\bar{b}_1 = (\Theta_+ q^g)(0) = (\Theta_+ q)^g(0) = B_c^{-1}(\bar{b} - P(c)\bar{a}).$$

Hence choosing $a = P(c)b$ we get $a_1 = 0$ and

$$0 < \beta(q^g, q) = \sigma^{\mathbb{R}}(b, B_c^{-1} Q_c \bar{b}) \quad \text{if } b \neq 0$$

where $Q_c = I - P(c)P(\bar{c})$. In particular, det $Q_c \neq 0$ f[illegible] $c \in Z$. But Q_c is hermitian and [illegible] is connected we [illegible] with $Q_c > 0$ for $c \in Z$. So the first inclusion is proved.

Next for $c \in Z$ we have

$$0 < B(c, -\bar{c}) = I - c \square \bar{c} + P(c)P(\bar{c}) < 2I - c \square \bar{c}$$

and the second inclusion holds. Taking the trace in $2I - c \square \bar{c} > 0$ we obtain $2 \cdot \dim V >$ [illegible]$(c, \bar{c})$. Thus Z is bounded.

Since f(0) runs through all of Z if $f \in \mathcal{G}$ (see Proposition 3a), $\mathcal{G}$ induces a transitive group of biholomorphic mappings of Z and the symmetry $z \to -z$ is contained in $\mathcal{G}$. Hence Z is a symmetric domain.

Putting the propositions together we complete the proof of Theorem A.

4. As a generalization of the representation of a complex number in polar coordinates we give a theorem, for which the proof is based on an idea of U. Hirzebruch [4]. Introducing the condition

(*) If $x, y \in V_o$ such that $x \square y + y \square x = 0$ and $\sigma(Lx, Lx) \geq \sigma(Ly, Ly)$ for all $L \in \mathfrak{T}_o$, $L^* = L$, then $y = 0$.

we have

HIRZEBRUCH's Theorem. *Suppose that the pairing* $\square$ *of* V_o *satisfies in addition the condition* (*). *Then to each* $w \in V$ *there exists* U *in the identity component* $\mathcal{K}_o$ *of* $\mathcal{K}$ *such that* Uw *belongs to* V_o.

It is not known whether or not the condition (*) is a consequence of our assumptions on the pairing of V_o. We will see later, that (*) holds whenever $\square$ is a Jordan pairing of the first kind.

Proof: Since $\mathcal{K}_o$ is a compact group there exists $z = x + iy$ in the orbit $\mathcal{K}_o w$ such that

$$\sigma(y,y) \le \sigma(\mathrm{Im}\ Uw,\ \mathrm{Im}\ Uw) \quad \text{for all } U \in \mathcal{K}_o .$$

According to Lemma 2.2 the Lie algebra of $\mathcal{K}_o$ equals $\mathfrak{d}$. Hence $U = \exp D$, $D \in \mathfrak{d}$, belongs to $\mathcal{K}_o$. We obtain

$$0 \le 2\sigma(y, \mathrm{Im}\ Dz) + \sigma(y, \mathrm{Im}\ D^2 z) + \sigma(\mathrm{Im}\ Dz, \mathrm{Im}\ Dz) + \cdots .$$

Replacing D by αD, $0 < \alpha \in \mathbb{R}$, we get $\sigma(y, \mathrm{Im}\ Dz) \ge 0$ and hence

$$\sigma(y, \mathrm{Im}\ Dz) = 0 \text{ and } \sigma(y, \mathrm{Im}\ D^2 z) + \sigma(\mathrm{Im}\ Dz, \mathrm{Im}\ Dz) \ge 0$$

for all $D \in \mathfrak{d}$. Choosing $D = iL$ where $L^* = L \in \mathfrak{T}_o$ we obtain

$$\sigma(y, Lx) = 0 \text{ and } \sigma(Lx, Lx) \ge \sigma(y, L^2 y) = \sigma(Ly, Ly).$$

We choose $L = a \square b + b \square a$ where $a, b \in V_o$ and the first conditions imply $x \square y + y \square x = 0$. Hence $y = 0$ follows from (*).

5. Let D be an arbitrary bounded symmetric domain in a complex vector space V and denote by $\mathcal{G}$ the group of biholomorphic mappings of D onto itself. The complexification of the real Lie algebra of $\mathcal{G}$ is denoted by $\mathfrak{G}$. We have seen in II, Theorem 5.2, that there exists a pairing of the vector space V satisfying the conditions (P.1) to (P.4) such that $\mathfrak{G}$ is isomorphic to the binary Lie algebra associated with the pairing.

THEOREM B. *If D is a bounded symmetric domain in a complex vector V space then there exists a real form*

V_o of V and a pairing $\square$ of V_o satisfying the conditions of §2.1 such that D is linearly equivalent to the domain $Z_\square$.

We give a sketch of the proof. From S. Helgason [3], chapter VIII, §7, it follows that there is a real form V_o of V such that the restriction of the pairing $\square$ to V_o satisfies the conditions in §2.1. Furthermore, let $\mathfrak{L}_\square$ be the binary Lie algebra associated with the pairing of V_o then the conjugation τ coincides with Θ_- and the bounded domain $Z_\square$ is linearly equivalent to D.

§6. The Bergman kernel of Z.

1. Let D be a domain in V and put $\overline{D} = \{\overline{z}; z \in D\}$. Denote by Bih D the group of all biholomorphic mappings of D onto itself. A function $\rho : D \times \overline{D} \to \mathbb{C}$ is called a Bergman kernel of D if

(i) $\rho(f(z), \overline{f(w)}) \cdot \det \frac{\partial f(z)}{\partial z} \cdot \overline{\det \frac{\partial f(w)}{\partial w}} = \rho(z, \overline{w})$

holds for $z, w \in D$ and $f \in$ Bih D,

(ii) $\rho(z, \overline{z}) > 0$ for $z \in D$ and $\overline{\rho(z, \overline{w})} = \rho(w, \overline{z})$.

We need the following theorem due to St. Bergman.

THEOREM 6.1. *If D is a bounded domain in V then there exists a Bergman kernel of D.*

For a proof see S. Helgason [3], Chapter VIII, §3.

COROLLARY. Suppose that the function $\xi : D\times\overline{D} \to \mathbb{C}$ satisfies the condition (i) for all f in a transitive subgroup of Bih D as well as (ii). Then each Bergman kernel of D equals $\gamma\xi$ where γ is a positive constant.

Proof: Let ρ be a Bergman kernel of D and put $\eta = \xi/\rho$. Then $\eta(f(z),\overline{f(w)}) = \eta(z,\overline{w})$ for $z,w\in D$ and all f in the given transitive subgroup of Bih D. Hence η does not depend on z. But $\overline{\eta(z,\overline{w})} = \eta(w,\overline{z})$ shows that η is constant.

2. Now let $Z = Z_\square$ be the bounded symmetric domain given by the pairing $\square$ of V_o. Since the subgroup $\mathcal{G}$ of Bih D is contained in $\Xi(\mathfrak{D},\theta_+)$ (see §5.1) we conclude

$$(6.1) \qquad B(f(z),\overline{-f(w)}) = \frac{\partial f(z)}{\partial z} B(z,-\overline{w}) \left(\overline{\frac{\partial f(w)}{\partial w}}\right)^*$$

for $f\in\mathcal{G}$ from II, Theorem 2.1. Notice that B has to be replaced by B_+ (see §5.1). We define the holomorphic function $\zeta : D\times D \to \mathbb{C}$ by

$$(6.2) \qquad \zeta(z,w) = \det B(z,-w) \quad \text{for } z,w\in Z.$$

From $\overline{B(z,w)}^* = B(\overline{w},\overline{z})$ we conclude $\overline{\zeta(z,\overline{w})} = \zeta(w,\overline{z})$. Furthermore, since $B(z,-\overline{z})$, $z\in Z$, is hermitian positive definite we obtain $\zeta(z,\overline{z}) > 0$. Hence the function ζ^{-1} satisfies (ii) and (i) for $f\in G$. Hence the Corollary of

Theorem 6.1 yields

THEOREM 6.2. *Each Bergman kernel of* Z *equals* $\gamma\zeta^{-1}$ *where* γ *is positive constant.*

Since Z is bounded the function $\zeta(z,\bar{w})$ is bounded for $z,w \in Z$. Hence we obtain the

COROLLARY. *Each Bergman kernel of* Z *is bounded away from zero.*

For bounded symmetric domains this result is due to H. L. Resnikoff [14]. We are going to prove

LEMMA 6.3. *Let* $z,w \in Z$ *and* $a,b \in V$. *Then*

$$(6.3) \qquad \Delta_z^a \Delta_{\bar{w}}^{\bar{b}} \log \zeta(z,\bar{w}) = -\sigma([B(z,-\bar{w})]^{-1} a,\bar{b}),$$

and (6.1) *holds for all* f *in* Bih Z.

Proof: Note first that

$$\overline{\Delta_z^a \varphi(z)} = \Delta_{\bar{z}}^{\bar{a}} \overline{\varphi(z)}$$

holds whenever φ is holomorphic in z. Hence the left side of (6.3) defines a hermitian form $\lambda_{z,w}$ of V. Since $(a,b) \to \sigma(a,\bar{b})$ defines a hermitian positive definite form of V there exists an endomorphism $Q(z,\bar{w})$ of V that is hermitian and rational in $z,\bar{w}$ such that

$$\lambda_{z,w}(a,b) = -\sigma(Q(z,\bar{w})a,\bar{b}).$$

Since the condition (i) holds for $\rho = \zeta$ we obtain

$$\overline{\frac{\partial f(w)}{\partial w}}^{*} Q(f(z), \overline{f(w)}) \frac{\partial f(z)}{\partial z} = Q(z, \bar{w})$$

for $f \in \operatorname{Bih} Z$ because of the chain rule. Hence the function

$$R(z, \bar{w}) := B(z, -\bar{w})\, Q(z, \bar{w})$$

satisfies

$$(6.4) \quad R(f(z), \overline{f(w)}) = \frac{\partial f(z)}{\partial z} R(z, \bar{w}) \left(\frac{\partial f(z)}{\partial z}\right)^{-1} \quad \text{for } f \in \mathfrak{G}$$

because of (6.1).

From the definition of Q we observe $-\lambda_{z,0}(a,b) =$ trace $a \square \bar{b} = \sigma(a, \bar{b})$. Hence $Q(z,0) = I$ and (6.4) yields

$$R(f(z), \overline{f(0)}) = I \text{ for } f \in \mathfrak{G}.$$

Since $\mathfrak{G}$ acts transitively on Z we conclude $R(z, \bar{w}) = I$ and the lemma is proved.

3. Denote by $\mathcal{K}'$ the subgroup of $\mathcal{K}$ consisting of the transformations W which map Z onto itself (see §2.5). Clearly, the connected component $\mathcal{K}_o$ of $\mathcal{K}$ is a normal subgroup of $\mathcal{K}'$ of finite index.

THEOREM 6.4. *The group* $\operatorname{Bih} Z$ *of biholomorphic mappings of* Z *onto itself consists exactly of the functions* $U \circ g_c$ *where* $U \in \mathcal{K}'$ *and* $c \in Z$ *and this representation is unique. The index* $[\operatorname{Bih} Z : \mathfrak{G}] = [\mathcal{K}' : \mathcal{K}_o]$ *is finite.*

<u>Proof</u>: Let f be a holomorphic map of Z onto itself. We choose g_c, $c \in Z$, such that the function $h = f \circ g_c^{-1}$ satisfies $h(0) = 0$. By Lemma 6.3 the condition (6.1) holds for h. Substituting $w = 0$ we see that $\frac{\partial h(z)}{\partial z}$ is constant. Hence $h(z) = Uz$ where $U \in GL(V)$. But again (6.1) yields $\bar{U}^* U = I$ and hence $U \in K'$.

As a consequence we see that the Lie algebras of Bih Z and of $\mathfrak{G}$ coincide. Using (4.5) we obtain the

COROLLARY. *The real Lie algebra of* Bih Z *is isomorphic to the subalgebra* $\mathfrak{D}_+ = \{p;\ p \in \mathfrak{D},\ \Theta_+ p = p\}$ *of* $\mathfrak{D}$.

In a similar way we observe

THEOREM 6.5. *Let* $\square$ *and* $\square'$ *be two pairings of* V_o *that satisfy the conditions of* §2.1 *and let* Z *and* Z′ *be the corresponding bounded symmetric domains. Then the following statements are equivalent*:

a) *There exists a biholomorphic map* $f : Z \to Z'$.

b) *There exists a* $W \in GL(V)$ *such that* $Z' = WZ$.

c) *The pairings* $\square$ *and* $\square'$ *are isomorphic* (*in the sense of* III, §3.4).

Chapter V

AN EXPLICIT DESCRIPTION OF THE BOUNDED SYMMETRIC DOMAINS

§1. Formal real Jordan algebras.

1. Let $\mathfrak{A}_o$ be a finite dimensional semi-simple Jordan algebra over R. Hence $\mathfrak{A}_o$ contains a unit element e and its trace form (a,b) → trace L(ab) is non-degenerate (see III, §3, and [2], chapter XI). We obtain a pairing □ of the vector space $\mathfrak{A}_o$ by

$$a\Box b := 2(L(ab) + [L(a),L(b)])$$

that is a Jordan pairing of the first kind (see III, §3). Using [ ], chapter XI, Satz 3.4, we see that the pairing has a positive definite trace form

$$\sigma_o(a,b) = 4 \text{ trace } L(ab)$$

if and only if $\mathfrak{A}_o$ is formal real, i.e., if $a^2 + b^2 = 0$ implies $a = b = 0$.

Suppose now that $\mathfrak{A}_o$ is formal real. We know from III, §3.1, that the endomorphism P(a) associated with the pairing coincides with the quadratic representation of $\mathfrak{A}_o$, i.e.,

$$P(a) := 2L^2(a) - L(a^2).$$

For $a \in \mathfrak{A}_o$ the exponential exp a is given by

$$\exp a := \sum_{m=o}^{\infty} \frac{1}{m!} a^m$$

and one has

$$P(\exp a) = \exp 2 L(a)$$

(see [2], chapter XI, Satz 2.2). Since σ_o is an associative bilinear form of $\mathfrak{A}_o$, the endomorphism L(a) is self adjoint with respect to σ_o. Furthermore the group $\Gamma_\Box$ equals the structure group $\Gamma(\mathfrak{A}_o)$ of $\mathfrak{A}_o$.

2. Let

$$\mathfrak{L}_o = \mathfrak{A}_o + \mathfrak{T}_o + \tilde{\mathfrak{A}}_o, \quad \tilde{\mathfrak{A}}_o = \{P(x)b;\ b \in \mathfrak{A}_o\},$$

be the binary Lie algebra associated with the pairing $\Box$ of the vector space $V_o = \mathfrak{A}_o$ (see IV, §2.1). We know from III, §3.1, that the automorphism Θ of $\mathfrak{L}_o$ is essential and III, Lemma 3.1, shows that $\Xi(\mathfrak{L}_o)$ is generated by the birational functions

$$W,\ t_a \text{ and } j \text{ where } W \in \Gamma(\mathfrak{A}_o),\ a \in \mathfrak{A}_o,$$

and where j is given by $j(x) = -x^{-1}$. As mentioned in III, §3, the group $\Xi(\mathfrak{L}_o)$ coincides with the group $\Xi(\mathfrak{A}_o)$ considered in [11].

In the notation of I, §4.2, we have the

THEOREM 1.1. *Each automorphism of $\mathfrak{L}_o$ is essential and*

$$\nabla : \Xi(\mathfrak{D}_o) \to \mathrm{Aut}\ \mathfrak{D}_o,\ f \to \nabla_f,$$

<u>defines an isomorphism of the groups</u>.

For a <u>proof</u> see [12].

3. Let $Y = Y(\mathfrak{A}_o)$ be the <u>domain of positivity</u> given by the formal real Jordan algebra $\mathfrak{A}_o$. According to [2], chapter XI, Satz 3.6 and Satz 3.7, we have the descriptions

$$Y = \exp \mathfrak{A}_o = \{a;\ a \in \mathfrak{A}_o,\ L(a) > 0\}$$

and the closure of Y equals $\{a^2;\ a \in \mathfrak{A}_o\}$. Furthermore, Y is an open convex cone and equals the connected component of the set $\{z,\ z \in \mathfrak{A},\ \det P(z) \neq 0\}$ containing e. Denote by $\mathfrak{H} = \mathfrak{H}(\mathfrak{A}_o)$ the group of $W \in \Gamma(\mathfrak{A}_o) = \Gamma(\mathfrak{D}_o)$ such that $a \to Wa$ maps Y onto itself. Then $\mathfrak{H}$ acts transitively on Y and the index of $\mathfrak{H}$ in $\Gamma(\mathfrak{A}_o)$ is finite.

4. Denote by $\mathfrak{A}$ the complexification of the formal real Jordan algebra $\mathfrak{A}_o$. Hence $\mathfrak{A}$ is a semi-simple complex Jordan algebra. Let

$$H = H(\mathfrak{A}_o) = \mathfrak{A}_o + iY = \{z;\ z \in \mathfrak{A},\ \mathrm{Im}\ z \in Y\},$$

then H is a domain in the complex vector space $\mathfrak{A}$. It is known (see U. Hirzebruch [4], [7]) <u>that the subgroup of</u> $\Xi(\mathfrak{D}_o)$ <u>generated by</u> W, t_a <u>and</u> j <u>where</u> $W \in \mathfrak{H}$, $a \in \mathfrak{A}_o$, <u>acts as a transitive group of biholomorphic mappings on</u> H. In particular, for f in this subgroup one has $H \subset \mathrm{Dom}\ f$

and $z \to f(z)$ maps H onto itself.

The real pairing $\square$ induces a pairing of $\mathfrak{A}^{\mathbb{R}}$ (see IV, §2.3) and we obtain the binary Lie algebra

$$\mathfrak{L} = \mathfrak{A}^{\mathbb{R}} + \mathfrak{T}^{\mathbb{R}} + \tilde{\mathfrak{A}}^{\mathbb{R}}, \quad \tilde{\mathfrak{A}}^{\mathbb{R}} = \{P(z)b;\ b \in \mathfrak{A}^{\mathbb{R}}\}.$$

Again the group $\Xi(\mathfrak{L})$ is generated by the birational functions

$$W,\ t_a \text{ and } j \text{ where } W \in \Gamma(\mathfrak{L}),\ a \in \mathfrak{A},$$

and $\Xi(\mathfrak{L}_o)$ becomes a subgroup of $\Xi(\mathfrak{L})$. The pairing induces a bounded symmetric domain $Z = Z_\square$ in $\mathfrak{A}$ according to IV, §5.2, and to Theorem A. Using the element p of $\Xi_*(\mathfrak{L})$ given by

$$p(z) = (z-ie)(z+ie)^{-1} = e - 2i(z+ie)^{-1},$$

i.e., $p = t_e \circ 2iI \circ j \circ t_{ie}$, we are going to prove

<u>THEOREM 1.2</u>. <u>The function</u> p <u>maps</u> H <u>biholomorphically onto the bounded symmetric domain</u> Z.

<u>Proof</u>: Let z be in H. Hence $z+ie \in H$ and p is holomorphic in H because j is holomorphic in H. A verification yields

$$p^{-1}(w) = i(e+w)(e-w)^{-1} = -ie + 2i(e-w)^{-1}$$

provided e-w is invertible in $\mathfrak{A}$. Thus the imaginary part is given by

$$\operatorname{Im} p^{-1}(w) = -e + (e-w)^{-1} + (e-\overline{w})^{-1}.$$

We use the well-known formulas

$$L(a^{-1}) = L(a)[P(a)]^{-1} = [P(a)]^{-1}L(a),$$

$$P(a^{-1} + b^{-1}) = [P(a)]^{-1}P(a+b)[P(b)]^{-1}$$

where $a,b \in \mathfrak{A}$ are invertible. Writing $a = e-w$, $b = e-\bar{w}$ we obtain

$$P(e-a^{-1}-b^{-1}) = I - 2L(a^{-1}+b^{-1}) + P(a^{-1}+b^{-1})$$

$$= [P(a)]^{-1}[P(a)P(b) - 2L(a)P(b) - 2P(a)L(b)$$

$$+ P(a+b)][P(b)]^{-1}$$

provided a and b are invertible. A verification yields now

$$P(\operatorname{Im} p^{-1}(w)) = [P(e-w)]^{-1}[I - w\square\bar{w} + P(w)P(\bar{w})][P(e-\bar{w})]^{-1}$$

provided e-w is invertible.

Denote the image of H under p by $\tilde{Z}$. Clearly e-w is invertible whenever $w\in\tilde{Z}$. Thus $w\in\tilde{Z}$ if and only if $\operatorname{Im} p^{-1}(w)$ lies in Y, i.e., lies in the connected component of the set $\{y;\ y\in\mathfrak{N}_o,\ \det P(y) \neq 0\}$ containing e. Hence w is in $\tilde{Z}$ if and only if w is in the connected component of the set $\{w;\ w\in\mathfrak{A},\ \det B(w,-\bar{w}) \neq 0\}$ containing zero which equals Z.

5. We are going to give some more descriptions of the bounded symmetric domain Z that is associated with a Jordan pairing of the first kind induced by a formal real Jordan algebra. First we have

<u>THEOREM 1.3</u> <u>To each</u> $w \in \mathfrak{A}$ <u>there exists</u> U <u>in the identity component</u> $\mathcal{K}_o$ <u>of</u> $\mathcal{K}$ <u>such that</u> Uw <u>belongs to the closure</u> $\overline{Y}$ <u>of</u> Y.

<u>Proof</u>: We apply Hirzebruch's Theorem and we have to show that the condition (*) in IV, § 5.4, holds. From $x \square y + y \square x = 4L(xy)$ we get $xy = 0$. Choosing $L = L(y)$ the second condition in (*) yields $\sigma_o(y^2, y^2) = 0$ and hence $y = 0$ because $\mathfrak{A}_o$ is formal real. Hence there exists $U \in \mathcal{K}_o$ such that Uw belongs to $\mathfrak{A}_o$. Let

$$Uw = \sum_{\nu} \lambda_\nu c_\nu, \quad \lambda_\nu \in \mathbb{R},$$

be the minimal decomposition of Uw (see [2], chapter XI, §3) where the c_ν's form a complete orthogonal system of idempotents of $\mathfrak{A}_o$. We choose $\varphi_\nu \in \mathbb{R}$ such that $e^{i\varphi_\nu} \lambda_\nu \geq 0$ and set

$$q = \sum_{\nu} e^{i\varphi_\nu} c_\nu .$$

Thus q is invertible and $\overline{q} = q^{-1}$. Clearly $P(q) \in \mathcal{K}_o$ and $P(q)Uw$ has a minimal decomposition with non-negative eigenvalues. Hence $P(q)Uw$ belongs to $\overline{Y}$.

In view of Theorem 1.2 we may apply Theorem 12 in [7], chapter VII. We use the orderings ">" and "≥" of $\mathfrak{A}_o$ which are given by

$$a > b \Leftrightarrow a-b \in Y, \qquad a \geq b \Leftrightarrow a-b \in \overline{Y}.$$

THEOREM 1.4. For $z \in \mathfrak{A}$ the following conditions are equivalent:

a) $z \in Z$,

b) $z = Ur$ where $U \in K_o$ and $r \in \mathfrak{A}_o$ such that $e > r \geq 0$,

c) $I - P(z)P(\bar{z}) > 0$,

d) $2I - z\Box\bar{z} > 0$.

Note that part c) and d) state a sharper result than that given in Proposition 4 in IV, §5.

Here $A > 0$ means that the endomorphism A is positive definite with respect to the hermitian form $(u,v) \to \sigma_o(u,\bar{v})$.

Proof: As an abbreviation set

$$Q_1(z) = B(z,-\bar{z}),\ Q_2(z) = I - P(z)P(\bar{z}),\ Q_3(z) = 2I - z\Box\bar{z}.$$

Hence

$$U\,Q_k(z)\,\bar{U}^* = Q_k(Uz) \text{ where } U \in K \text{ and } k = 1,2,3.$$

In view of Theorem 1.3 it suffices to prove the equivalence of the conditions a) to d) for $z = r \in \mathfrak{A}_o$ such that $r \geq 0$. We obtain

$$Q_1(r) = P(e-r)^2,\quad Q_2(r) = I - P(r^2),\quad Q_3(r) = 2L(e-r^2).$$

Let

$$r = \sum_\nu \lambda_\nu c_\nu,\quad 0 \leq \lambda_\nu \in \mathbb{R},$$

be the minimal decomposition of r. From the definition of Z it follows that $r \in Z$ is equivalent to $e - r^2 > 0$ (see 3) and hence to $e > r$. Using [2], chapter VIII, Satz 1.3, we see that $Q_2(r) > 0$ is equivalent to $1 > \lambda_\nu$ for all ν and hence to $e > r$. But $Q_3(r) > 0$ means $e - r^2 > 0$, too.

§2. The classification of the bounded symmetric domains.

1. Let $\mathfrak{M}_{r,s}$ be the space of $r \times s$ complex matrices and denote by e_r the $r \times r$ unit matrix. Cartan's classification shows that each irreducible bounded symmetric domain is linearly equivalent either to a domain in the following list

notation: Cartan	notation: Helgason	domain	$\dim_{\mathbb{C}}$
$I_{r,s}$	A III	$\{z;\ z \in \mathfrak{M}_{r,s},\ \bar{z}^t z < e_s\}$	rs
II_r	D III	$\{z;\ z \in \mathfrak{M}_{r,r},\ \bar{z}^t z < e_r,\ z^t = -z\}$	$\frac{r(r-1)}{2}$
III_r	C I	$\{z;\ z \in \mathfrak{M}_{r,r},\ \bar{z}^t z < e_r,\ z^t = z\}$	$\frac{r(r+1)}{2}$
IV_r	BD I (q=2)	$\{z;\ z \in \mathbb{C}^r,\ \bar{z}^t z < \frac{1}{2}(1 + \lvert z^t z \rvert^2) < 1\}$	r

or to an exceptional domain of type E_6 or E_7 of dimension 16 or 27 respectively.

Each of these domains can be obtained as a domain $Z_\square$ (see IV, §5.$\underline{\underline{2}}$) where $\square$ is some pairing of a real vector space satisfying the conditions of IV, §2.$\underline{\underline{1}}$. For a real vector space V_o let V be its complexification.

<u>Type $I_{r,s}$:</u> Let V_o be the real vector space of $r \times s$ matrices with real entries. As pointed out in III, §2.$\underline{\underline{1}}$, we obtain a pairing $\square$ of V_o by

$$(a\square b)c = ab^t c + cb^t a$$

having

$$\sigma_o(a,b) = (r+s) \text{ trace } ab^t$$

as trace form. Clearly, σ_o is positive definite and therefore the pairing satisfies our conditions.

According to (III;2.5) the endomorphism $B(a,b)$ is given by

$$B(a,b)c = (e+ab^t)c(e+b^t a) \text{ where } a,b,c \in V,$$

and Proposition 4 in III, §2, shows that $\det B(a,-\bar{a}) \neq 0$ is equivalent to

$$\det(e-a\bar{a}^t) \neq 0 \quad \text{and} \quad \det(e-\bar{a}^t a) \neq 0.$$

Using

$$M\begin{pmatrix} e & 0 \\ 0 & e - a\bar{a}^t \end{pmatrix}\bar{M}^t = \bar{M}^t\begin{pmatrix} e - \bar{a}^t a & 0 \\ 0 & e \end{pmatrix}M \text{ where } M = \begin{pmatrix} e & 0 \\ a & e \end{pmatrix}$$

we see that the last two conditions are equivalent. Hence the domain $Z_\square$ associated with our pairing coincides with the set of z's such that $\det(e-\bar{z}^t z) \neq 0$ and hence with the domain listed under $I_{r,s}$.

<u>Type II_r and III_r</u>: For $\epsilon = \pm 1$ denote by V_o^ϵ the vector space of $r \times r$ real matrices a satisfying $a^t = \epsilon a$. According to III, §1, we obtain a pairing $\square$ of V_o^ϵ by

$$(a\square b)c = ab^t c + cb^t a$$

having

$$\sigma_o(a,b) = (r+\epsilon) \text{ trace } ab^t$$

as trace form. Again σ_o is positive definite and the pairing satisfies our conditions. From (III;1.6) we conclude that the domain $Z_\square$ associated with the pairing coincides with the domain listed under II_r provided $\epsilon = -1$ or listed under III_r provided $\epsilon = 1$.

2. We use now our results of §1. Let $\mathfrak{A}_o$ be a formal real Jordan algebra of dimension n and let $\square$ be the induced Jordan pairing of the first kind, i.e.,

$$a\square b = 2(L(ab) + [L(a),L(b)]).$$

We know from §1.1 that its trace form is positive definite. In the following list we write all simple formal real Jordan algebras (in the notation of [2], chapter XI, §5) and the type of the domain $Z_\square$ associated with the pairing:

$\mathfrak{A}_o$	$[X,\mu,e]$	$\mathfrak{H}_r(\mathbb{R})$	$\mathfrak{H}_r(\mathbb{C})$	$\mathfrak{H}_r(\mathfrak{C}_4)$	$\mathfrak{H}_3(\mathfrak{C}_8)$
type	IV_n	III_r	$I_{r,r}$	II_{2r}	E_7

Hence all irreducible bounded symmetric domains except the domain of type E_6 are constructed by a pairing.

3. Finally we show that the domain of type E_6 is also covered by our construction. According to III, §4.2, let $\mathfrak{C} = \mathfrak{C}_8$ be the Cayley division algebra over $\mathbb{R}$ and put $V_o = \mathfrak{C}\oplus\mathfrak{C}$. Then there is a pairing $\square$ of V_o having the trace form

$$\sigma_o(a,b) = 48[\mu(a_1,b_1) + \mu(a_2,b_2), \quad a = a_1\oplus a_2, \ b = b_1\oplus b_2 \in V_o.$$

Since the bilinear form μ of $\mathfrak{C}$ is positive definite we see that σ_o is positive definite, too. Hence the pairing satisfies our conditions. According to a recent result of K. Meyberg [13] the Lie algebra $\mathfrak{L} = \mathfrak{L}_\square$ is of type E_6. The Lie algebra of the group of biholomorphic mappings of the associated domain $Z_\square$ is isomorphic to $\mathfrak{L}_+$ (see the Corollary of IV, Theorem 6.5) and hence of type E_6 (see IV, Theorem 4.2).

Summing up we see that all bounded symmetric domains are linearly equivalent to a domain $Z_\square$ where the pairing $\square$ is a Jordan pairing of first or second kind satisfying the conditions of IV, §2.1.

FINIS

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