LESSONS IN GEOMETRY

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LESSONS

IN

GEOMETRY.

FOR THE USE OF BEGINNERS.

BY

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BOSTON:

PUBLISHED BY GINN & COMPANY.

1888.

ELECTROTYPED BY J. S. CUSHING & CO., BOSTON.

PRESSWORK BY GINN & CO., BOSTON.

PREFACE.

This book has been prepared to meet the wishes of those who prefer a shorter and easier introductory course in Geometry than that given in the "Geometry for Beginners."

The metric system of units is explained; but all exercises in metric units are confined to lessons placed at the ends of the chapters, and may be omitted if desired.

The method of instruction is that best adapted to the mental condition of pupils between the ages of twelve and sixteen. The training in consecutive reasoning is introduced very gradually, and is confined mainly to the laws of equal triangles and a few of their simple applications.

To one feature of the method the author desires to call special attention; namely, the numerous exercises which involve the use of instruments and drawing to scale. It is assumed that every pupil is provided with ruler, divided scale, pencil compasses, triangle, and protractor. Any objection on the ground of expense has been met by the publishers, who are prepared to supply, at a very low price, these instruments enclosed in a strong wooden box.

Experience shows that for the beginner of Geometry the careful execution of easy constructions is the most useful as well as the most interesting part of the daily lesson. This work calls into action the eye, the hand, and the judgment. It holds the attention. It is attended with the pleasing sense of the successful exercise of new-found power. Under these conditions progress in knowledge is sure to be rapid. The precept, "Do that you may know," finds here a pertinent application.

No teacher, however, can expect to obtain the best results from this kind of work, unless he insists strenuously upon neatness and a reasonable degree of accuracy. Every teacher should lay down a standard of neatness and accuracy, — remembering that his pupils are only beginners working with cheap instruments, — and then should criticise unsparingly every drawing which, tried by his standard, is a slovenly or inaccurate piece of work.

The contents of the book are so arranged that the course may be considerably abridged, if so desired. There are, in all, ninety-six lessons and fifteen drawing exercises. This makes abundant material for a course of three hours per week for a year, or, what is better for the pupil, a course of one hour per week for a year, and a course of two hours per week for the year following. The last two chapters, however, may be omitted, and likewise the lessons in which metric units are used; there will then be left sixty-five lessons and the drawing exercises, or a course of two hours per week for one year.

Geometry, as here presented, should be studied before Algebra. If this is done, pupils, while learning the properties of figures and the measurement of areas and volumes, will see for themselves the great advantage of using letters to represent quantities. Thus the chief stumbling-block to every beginner of Algebra will be removed.

The author takes this opportunity to express his warm thanks to the teachers who have had the kindness to read and correct the proof-sheets. A special acknowledgment of obligation for this assistance is due to Prof. G. A. Wentworth, of Exeter, N.H.; Prof. H. D. Wood, of Trenton, Ga.; Mr. J. E. Clarke, of Chelsea, Mass.; and Mr. E. H. Nichols, of Cambridge, Mass.

G. A. HILL.

Cambridge, Feb. 29, 1888.

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SIGNS AND ABBREVIATIONS.

$+$, increased by.

$-$, diminished by.

$\times$, multiplied by.

$\div$, divided by.

$=$, is (or are) equal to.

$>$, is (or are) greater than.

$<$, is (or are) less than.

$\parallel$, parallel.

$\perp$, perpendicular.

$\sqrt{}$, the square root of.

$\angle$, angle.

$\angle$s, angles.

$\triangle$, triangle.

$\triangle$s, triangles.

Ax., axiom.

Def., definition.

Hyp., hypothesis.

Con., conclusion.

Const., construction.

rt., right.

LESSONS IN GEOMETRY.

CHAPTER I.

INTRODUCTION.

Lesson 1.

1. *Why is this block of wood (Fig. 1) called a body? Why is it also called a cube?*

The block is called a body because it occupies space, or, in other words, "takes up room." Everything which occupies a portion of space is called a **body**.

The block is called a cube on account of its shape. Every body having the shape of this block is called a **cube**.

2. *Point out and name its dimensions.*

The block can be measured in three main directions:

(1) From left to right;
(2) From front to back;
(3) From top to bottom.

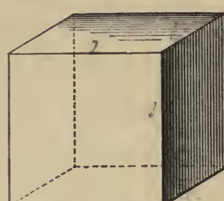


FIG. 1.

These three measurements are called the **dimensions** of the block; one is called the **length**, another the **breadth**, the other the **thickness**.

The dimension which is measured straight upwards or downwards is also called the **height** of the block.

3. *Are the dimensions of a cube equal or unequal?*

The three dimensions of this cube (Fig. 2) seem to be equal. If we measure them with a foot rule, we shall find that they are equal. The dimensions of a cube are equal.

4. *Describe the surface of the cube.*

The cube is limited or bounded by its surface. When we look at the cube, it is its surface only which we see. When we handle the cube, it is its surface only which we touch. The surface of the cube consists of six distinct parts, called *faces:*

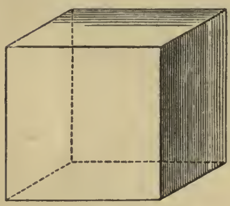


Fig. 2.

The front and the back faces;
The right and the left faces;
The upper and the lower faces.

The faces have no thickness; each face has only *two* dimensions, length and breadth, or length and height. The faces of the cube are *flat* or *plane* surfaces.

5. *Describe the edges of the cube.*

Each face of the cube is bounded by four edges.

Each edge is the place where two of the faces meet.

There are in all twelve edges.

The edges of the cube are *lines;* they have neither breadth nor thickness; they have only *one* dimension, length.

The edges of the cube are *straight* lines.

6. *Describe the corners of the cube.*

Each edge of the cube is limited by two corners.

Each corner is the place of meeting of three edges.

There are in all eight corners.

The corners of the cube are *points;* they have no dimensions, neither length, breadth, nor thickness.

They have only *place* or *position.*

7. *The body represented in Figure 3 is called a cylinder. Point out and name its dimensions.*

FIG. 3.

The cylinder has three dimensions. The dimension measured straight upwards or downwards is called the length or height. The other dimensions are called the breadth and thickness; they are measured, one of them from left to right, the other from front to back.

The breadth is not everywhere the same; neither is the thickness. But in such cases we always choose the *greatest* breadth and the *greatest* thickness. The greatest breadth and the greatest thickness are equal, as we can see by measuring them with a foot rule.

8. *Describe the surface of the cylinder.*

The cylinder is bounded on all sides by its surface. The surface has no thickness; it has only two dimensions, length and breadth.

The surface consists of three parts: two flat or plane surfaces, called the **bases**; and a surface extending all around the cylinder between the bases, called the **convex** surface.

The convex surface is not flat; it is a *curved* surface.

9. *Describe the edges of the cylinder.*

The cylinder has two edges. These edges are the boundaries of the bases. They are also the places where the lateral surface meets the bases.

The edges are lines, and have one dimension, length.

These edges are *curved* lines.

The cylinder has no corners.

10. Measure with a foot rule the dimensions of the cylinder which is given you.

Lesson 2.

1. The body represented in Figure 4 is called a **prism**. Point out and name its dimensions. Are they equal or unequal? Which is the longest? Which the shortest? Describe carefully the surface, the edges, and the corners, as those of the cube were described in Lesson 1.

2. Examine in like manner the four-sided prism (Fig. 5).

3. Examine in like manner the six-sided prism (Fig. 6).

4. Examine in like manner the four-sided **pyramid** (Fig. 7).

5. Examine in like manner the **cone** (Fig. 8).

6. Examine in like manner the **frustum** of a cone (Fig. 9).

7. Examine in like manner the **sphere** (Fig. 10).

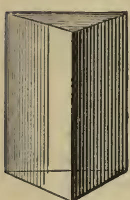


FIG. 4.

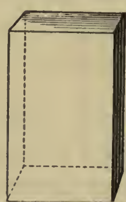


FIG. 5.

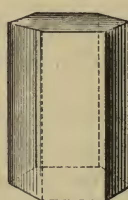


FIG. 6.

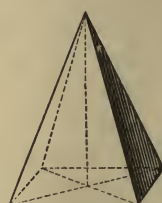


FIG. 7.

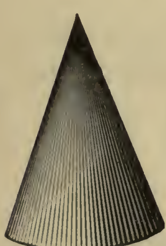


FIG. 8.

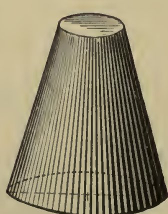


FIG. 9.

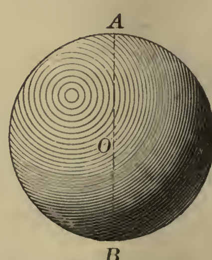


FIG. 10.

figure in matter.

Lesson 3.

1. *What is a body, as understood in Geometry?*

A body, in common language, is a portion of space filled with something which we can see, touch, handle, etc.; for example: a pencil, a book, a stone, the sun. But in Geometry we pay no attention to the matter of which a body is composed; we study simply its size and its shape. In Geometry, a **body** means simply a limited portion of space.

Geometrical bodies are sometimes called **solids.**

2. *How many dimensions has a body? How are they named?*

Every body has three dimensions. The words used for naming dimensions are *length*, *breadth* (or *width*), *thickness*, *height*, and *depth.* Usually the longest dimension is called the length, and the shortest the thickness. The terms height and depth are used only for dimensions measured straight upwards or downwards.

3. *Define the terms surface, line, and point.*

A **surface** is a space magnitude having only *two* dimensions.

A **line** is a space magnitude having only *one* dimension.

A **point** is a position in space without magnitude.

The boundaries of a body are surfaces.

The boundaries of a surface are lines.

The limits of a line are points.

4. *Define the intersection of two surfaces.*

When two surfaces meet or cut each other, the place where they meet is called their **intersection.**

The intersection of two surfaces is always a line.

For example: the faces of the cube intersect in the edges.

5. *Define the intersection of two lines.*

The place where two lines meet or cut each other is also called their intersection.

The intersection of two lines is always a point.

For example: the edges of the cube intersect at the corners; two lines on paper which meet always meet in a point.

6. *What are the parts of bodies, surfaces, and lines?*

If we divide a body into parts, each part, however small, will also be a body having length, breadth, and thickness. In this way we never could obtain a part so small that it was a surface, a line, or a point.

The parts of a body are bodies; the parts of a surface are surfaces; and the parts of a line are lines.

Therefore the surface of a body is not a part of the body. If any number of flat surfaces were placed one upon another, they would have no thickness, and would form but a single surface. The common boundary which separates water from oil resting on it is neither water nor oil nor any other substance; it is not a body, but a surface.

Likewise a line is not a part of a surface, and a point is not a part of a line.

7. *How are points and lines represented to the eye?*

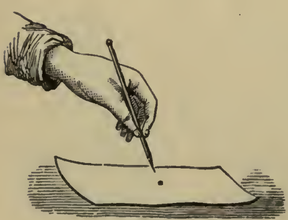


FIG. 11.

Strictly speaking, a point cannot be seen, because it has no magnitude. On paper we *represent* a point by a small dot, and *name* it by a letter, as *A*. The dot is really a very small body, but we do not regard its dimensions, and think of it as having position only.

A line is represented by a narrow mark, and named generally by two letters, one placed at each end of the line.

Surfaces are represented by their bounding lines, and solids by their bounding surfaces.

8. *What is the path of a point?*

When we move the point of a pencil on paper, it leaves behind a trace which we call a line; imagine the point of the pencil to be a point as understood in Geometry, and then the trace or path of the pencil would be a geometrical line. Who has not observed that if the red-hot point of an iron rod be moved rapidly in the dark, the path of the point appears as a luminous line?

The path of a moving point is a line.

9. Can you think of a body shaped like a cube? And one shaped like a cylinder? And one shaped like a sphere?

10. Is the space within the room a body? Why?

11. Name the dimensions of this room; the dimensions of the floor; the dimensions of one of the walls.

12. Name the dimensions of a pencil; a cistern; a brick wall.

13. Choose among the bodies before you one which has two of its dimensions equal. Which are they?

14. Point out upon a cube the intersection of two surfaces. Also the intersection of two lines.

15. How many edges and how many corners has a square box?

16. In common language, a telegraph wire is called a line. Is it a geometrical line? What is it? Why?

17. We call the trace of a pencil-point on paper a line. What is it in reality? Why?

Lesson 4.

1. *Divide lines into two classes. Define each class.*

There are two kinds of lines, straight and curved.

A **straight** line has the same direction from end to end.

A **curved** line is a line whose direction is continually changing. Curved lines are often called **curves**. A moving point describes a straight line when the direction of its motion does not change, and a curve when the direction of the motion continually changes. In Figure 12,

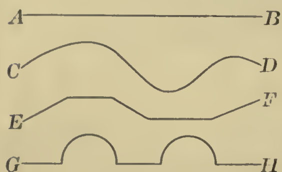


Fig. 12.

AB is a straight line;
CD is a curve;
EF is a broken line;
GH is a composite line.

The figures on carpets and on wall-papers furnish numberless examples of broken and composite lines.

The word "line" used alone means *straight line.*

2. Draw and name with letters two straight lines which intersect. Also two straight lines which do not intersect.

3. The same exercise with two curved lines.

4. Draw three broken lines. Define a broken line.

5. Draw three composite lines. Define a composite line.

6. What kind of lines are the edges of a cube? the edges of a prism? the edges of a pyramid? the edges of a cylinder? the edges of the room? the edges of a ruler?

7. What kind of lines are represented by the spokes of a wheel? the tire of the wheel? a telegraph wire? a watch-chain? the letters *M*, *O*, *S*, *T*, *V*, *Z*? the digits 2, 4, 6, 8?

8. What is the path of a falling apple? a ball thrown straight upwards? a ball thrown sideways into the air? the end of a clock hand? a raindrop? a flying bird?

9. *Classify surfaces, and define each class.*

There are two kinds of surfaces, plane and curved.

A **plane** surface is a surface on which straight lines can be drawn in as many different directions as you please.

Plane surfaces are also called **planes**.

A **curved** surface is a surface which is not plane.

10. What kind of surfaces are the faces of a cube? the faces of a prism? the faces of a pyramid? the base of a cone? the lateral surface of a cone? the surface of a sphere?

11. Classify the surfaces of the following bodies: a table, a room, a hat, a lamp-shade, a glass chimney, a mirror, a basin, the water in the basin, the ocean.

12. Hold a sheet of paper so that its surface is at first a plane, and then a curved surface.

13. *How can you test whether a surface is a plane?*

Apply to the surface the straight edge of a ruler in several different directions; if the edge always touches the surface at all points, the surface is a plane.

Test if the surface given you is a plane or not.

14. *Classify bodies, and define each class.*

Bodies are of two kinds, polyhedrons and curved bodies.

A **polyhedron** is a body bounded wholly by planes.

A **curved** body is a body bounded partly or wholly by curved surfaces.

15. What kind of body is a cube? a cylinder? a sphere?

16. *What does the word figure mean in Geometry?*

In Geometry every line or group of lines — in short, everything having shape or form — is called a **figure**.

17. *What is Geometry?*

Geometry is the study of geometrical figures, their properties, their construction, and their measurement.

CHAPTER II.

STRAIGHT LINES.

Lesson 5.

1. *How many directions has a straight line?*

A straight line AB (Fig. 13) has two directions, — the direction from A towards B, and the opposite direction from B towards A. If either direction be known or *given*, the other direction is of course also known.

NOTE. — In Geometry "given" is used very nearly in the sense of "known."

2. *How is the position of a straight line determined?*

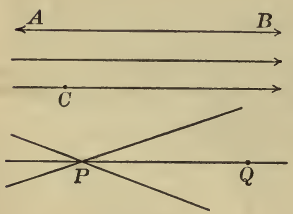


FIG. 13.

Different straight lines may be drawn, all having the same direction; also different straight lines may be drawn through a given point P.

But through a given point C in a given direction (as that of the arrow) only *one* straight line can be drawn.

Also through two given points P and Q only *one* straight line can be drawn.

Hence if we know either *one point* and the *direction* of a straight line, or *two points* of the line, we know enough about the line to be able to construct it, or to represent it by drawing it on paper.

In other words, the line is *determined in position.*

3. *How are straight lines drawn on paper?*

Straight lines are drawn on paper by means of a **ruler** with a straight edge. To draw a line through two given points, lay the ruler on the paper so that its edge just touches the two points; then draw the line with a well-sharpened pencil, holding the pencil nearly vertical, and always touching the edge of the ruler.

FIG. 14.

4. *What is freehand drawing?*

If we draw a straight line without the aid of a ruler to guide the hand, we are said to draw it *freehand.*

In general, **freehand drawing** is drawing without the aid of instruments, except the pencil or pen.

5. Draw four lines through a point, and name them.

6. Make two points, and join them by a straight line.

NOTE. — To "join *AB*" means to "draw a straight line from *A* to *B*."

7. Make three points, and join them by drawing as many straight lines through them as possible.

8. The same exercise with four points.

9. The same exercise with five points.

10. Draw freehand a straight line; correct or rectify it with the aid of a ruler. Repeat several times.

11. Draw one of the faces of the body given you in three different ways, as follows:

(1) Lay the face on paper and trace its boundary.

(2) Lay the face on paper; mark the corners; join them.

(3) Draw the face as well as you can freehand.

12. Test whether the edge of your ruler is straight.

To do this draw a line through two points, turn the ruler end for end, draw a line through the same points along the same edge as before. The lines ought to coincide.

Lesson 6.

1. *How are straight lines and plane surfaces classified with respect to the surface of the earth?*

With respect to the surface of the earth straight lines and planes are either vertical, horizontal, or inclined.

A **vertical** line is a line having the direction of a *plumb* line, or string held at rest in the hand, and supporting at its lower end a small weight (Fig. 15).

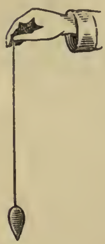


FIG. 15.

Every plane in which a vertical line can be drawn is a *vertical* plane.

A **horizontal** line is a line having the direction of a pencil, stick, or other object which is floating on the surface of still water.

The surface of still water, and every plane similarly placed with respect to the earth's surface, is a *horizontal* plane.

A line or a plane which is neither vertical nor horizontal is said to be **inclined**.

2. Give examples of lines and planes vertical, horizontal, and inclined.

3. Hold a pencil vertical, horizontal, inclined.

4. Hold a book vertical, horizontal, inclined.

5. Draw on the blackboard a line of each kind.

6. What direction has the mast of a ship? the path of a falling apple? a rail on a railway track? the side of a house? the roof of the house? the trunk of a growing tree?

7. Place the body given you on the table, and then examine the position of its edges and its faces with respect to the surface of the earth.

8. At what time of day are the sun's rays most nearly vertical? When are they horizontal?

9. When is the hour-hand of a clock vertical? horizontal?

10. When two vertical planes intersect, what kind of a line is the intersection? Example: the walls of a room.

11. When a vertical plane intersects a horizontal plane, what kind of a line is the intersection? Give an example.

12. *How are vertical and horizontal lines represented on paper, when they are situated in a vertical plane which stands directly in front of the eye?*

Vertical lines are represented by lines drawn on paper straight *towards* us or straight *from* us; horizontal lines by lines drawn from *left to right* or from *right to left.*

13. Draw a vertical line, mark on it four points, and draw through each point a horizontal line. Draw freehand.

14. Draw a horizontal line, mark on it four points, and draw through each point a vertical line. Draw freehand.

15. Draw an inclined line, mark on it four points, and draw through each point, freehand, a horizontal and a vertical.

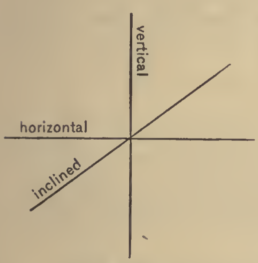


FIG. 16.

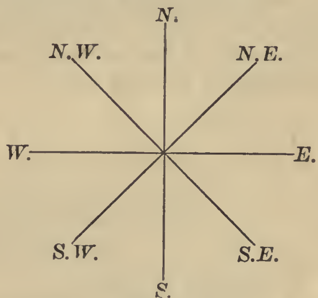


FIG. 17.

16. *How are the cardinal directions of a horizontal plane represented on paper (Fig. 17)?*

The cardinal directions are **North, South, East,** and **West.** A line from north to south is drawn straight *towards* us. A line from east to west is drawn from *right to left.*

17. Draw, freehand, through a point, the four cardinal directions, and also the four directions which lie midway between them and are termed, *northeast*, *northwest*, *southeast*, *southwest*. Name the directions (as in Fig. 17), by affixing the letters

N., S., E., W., N.E., N.W., S.E., S.W.

18. What direction is opposite to N.E.? opposite to S.E.?

19. How many horizontal lines can be drawn through a point? How many vertical lines? How many inclined lines?

20. How many horizontal lines can be drawn in a horizontal plane? How many in a vertical plane?

21. How many vertical lines can be drawn in a vertical plane? How many in a horizontal plane?

NOTE.—The *plumb line* is used for testing vertical walls, and the *spirit level* for testing horizontal planes. On the ground the difference in level of two places is often found by means of a *water level* (Fig. 18). It is a metal tube mounted horizontally on a tripod stand with glass vials cemented vertically at its ends. It contains water colored red to distinguish it from the glass.

FIG. 18.

Lesson 7.

1. *Explain how a straight line may change in direction.*

The direction of a straight line may be conceived to change. The hands of a clock represent straight lines which are continually changing in direction.

Suppose a line OA (Fig. 19) to turn about the point O from the position OA to the position OB; then the line is said to **rotate**, or **revolve**, about the point O. If the motion continue, the line will, after a time, return to its first position OA. It is then said to have made one **revolution.**

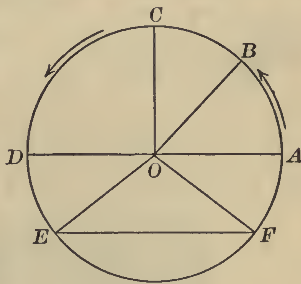


FIG. 19.

If, during the motion, the length of the line does not change, the point A will describe a curve which returns into itself, and every point of which is equally distant from O.

2. *Define the terms circumference and circle.*

A **circumference** is a curve, every point of which is equally distant from a fixed point called the *centre.*

A **circle** is a portion of a plane surface bounded by a circumference.

3. *Define the terms radius, arc, chord, diameter.*

A straight line drawn from the centre of a circle to the circumference is called a **radius** (plural *radii*).

A part of a circumference is called an **arc.**

A straight line joining the ends of an arc is called a **chord.**

A chord passing through the centre is called a **diameter.**

4. *How are circles drawn or described on paper?*

Circles, or strictly speaking, circumferences, are described on paper with the aid of the **compasses, or dividers.**

Fig. 20.

This instrument has two legs joined together by a pivot, about which they can turn. In the form best adapted for school work, one leg is provided with a metal point, and the other leg with a pencil point. In order to describe a circle on paper, we open the legs and proceed as illustrated in Fig. 20.

On the ground circles are made as illustrated in Fig. 21.

Note.—The shorter word, "circle," is often used instead of the word "circumference," in cases where a misunderstanding is not possible. For example: "describe a circle"; "an arc of a circle."

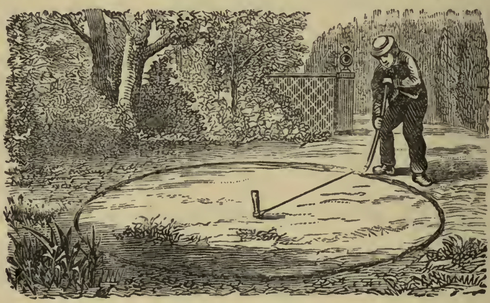


Fig. 21.

5. Name in Fig. 19 a radius, an arc, a chord, a diameter.

6. Give an example of a circle or a circumference.

7. What does the tire of a carriage wheel represent? the spokes? the axle? the part of the tire between two spokes?

8. What is true of all radii of the same circle?

9. What is true of all diameters of the same circle?

10. Compare a radius and a diameter of the same circle.

11. Draw a circle, a radius, a diameter, and a chord.

12. Describe an arc; draw its chord and radii to its ends.

13. Describe a circle; then draw (1) a chord equal to the radius, (2) the longest possible chord.

14. Describe several circles freehand.

15. Describe three *concentric* circles, or circles having the same point for centre.

16. Describe two circles so that their circumferences shall touch each other, and one shall lie wholly outside the other.

17. Describe two circles so that their circumferences shall touch each other, and one shall be wholly within the other.

18. Describe two equal circles so that the circumference of each shall pass through the centre of the other.

19. Describe three unequal circles so that the centre of each shall lie on the circumference of one of the others.

20. Construct Figure 22. First describe the circle; then the arcs whose centres are marked 1, 2, 3, 4, 5, 6; then find the centres of the remaining arcs by repeated trial.

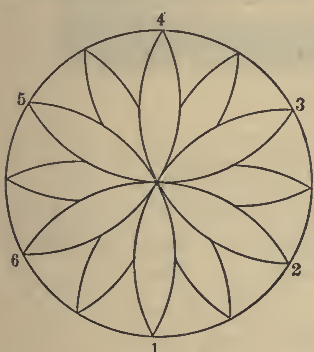


FIG. 22.

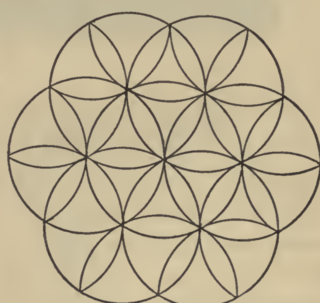


FIG. 23.

21. Construct Figure 23. The circles have the same radius. Begin by describing the circle in the middle of the figure.

Lesson 8.

1. *Define parallel lines, and give examples.*

Parallel lines are lines having the same direction.

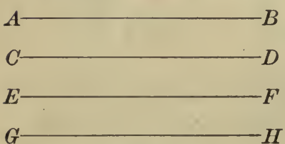


Fig. 24.

Examples. — The lines AB, CD, EF, and GH (Fig. 24); the legs of a table; the rails of a railroad (Fig. 25).

Abbreviation. — The sign ∥ is used for the word "parallel"; "AB is ∥ to CD," is read "AB is parallel to CD."

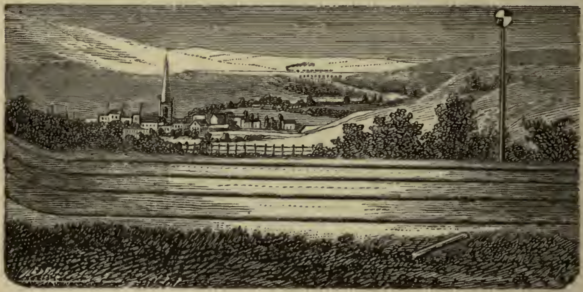


Fig. 25.

2. *Parallel lines cannot meet however far produced. Show why this follows from the definition.*

Suppose that two parallel lines, for example, AB and CD (Fig. 24), when produced (prolonged), should at length meet, and call the point of meeting P. Then we should have two lines, AB and CD, drawn through P *in the same direction.* Now, two lines so drawn *must* coincide (p. 10, No. **2**). But AB and CD plainly do not coincide, therefore we know that they could never meet, however far produced.

3. Are two lines, which would never meet if produced, necessarily parallel? Illustrate by an example.

4. Problem.—*To draw a line parallel to a given line AB through a given point C.*

METHOD I. (Fig. 26).—Instruments: ruler and compasses.

Join C to any point D in AB.

With centre D and radius DC describe an arc, cutting AB in E.

With centre C and radius DC describe the arc DF.

With centre D and radius CE cut the arc DF in F.

Through C and F draw a straight line.

The straight line CF is the parallel required.

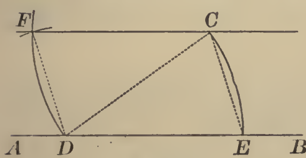


FIG. 26.

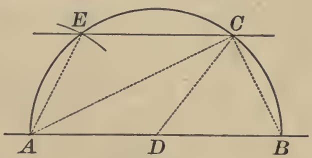


FIG. 27.

METHOD II. (Fig. 27).—Instruments: ruler and compasses.

With any point D in AB as centre and DC as radius, describe a semicircle $AECB$.

With centre A and radius BC cut the semicircle in E.

Draw CE. The line CE is the parallel required.

METHOD III. (Fig. 28).—Instruments: ruler, and a piece of wood with three straight edges, called a **triangle**.

Place the ruler and triangle as shown in the figure. Slide the triangle along till its edge touches the point C; then draw a straight line along the edge of the triangle. This line will be parallel to AB because the edge of the triangle during the motion remains parallel to AB.

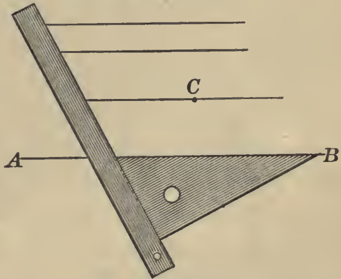


FIG. 28.

5. Draw a straight line; then draw a parallel line through a point not in the line by Method I.

6. The same exercise, using Method II.

7. The same exercise, using Method III.

8. Draw a parallel to a given line through a given point by Method I. Test the accuracy of your result by Method III.

9. Draw by Method III. six parallels as nearly equidistant as you can.

10. Draw a vertical line, mark five points on it, and draw through the points parallel lines by Method III.

11. Draw six parallel lines, two vertical, two horizontal, and two inclined.

12. Draw a three-sided figure, and then draw through each of its corners a line $\parallel$ to the opposite side.

13. Describe a circle, and draw two parallel chords.

14. Draw a straight line; then try to draw freehand a parallel to it. Begin by marking points which, as well as you can judge, must be in the required line.

15. Draw freehand a series of six parallel lines.

16. Hold two pencils (1) parallel; (2) so that they would intersect if prolonged; (3) so that they are not parallel, and also would not intersect.

17. Point out on the body given to you edges which are (1) parallel; (2) intersecting; (3) neither parallel nor intersecting.

18. The same exercise with the edges of the room.

19. When is a straight line parallel to a plane?

A straight line is parallel to a plane when the line will never meet the plane however far they are both produced.

20. When are two planes parallel to each other?

21. Give an example of a line $\parallel$ to a plane; also of two parallel planes.

Lesson 9.

1. *What has a straight line besides direction?*

Besides direction, a straight line has *length*. As regards length, two straight lines are either equal or unequal.

2. *Define equal and unequal straight lines.*

Two straight lines are **equal** if they can be so placed, one upon the other, that their ends coincide. If this cannot be done, the lines are **unequal**.

3. *How is the equality of two lines expressed?*

The equality of two lines AB and CD is expressed thus:

$$AB = CD.$$

This is called an **equation**, and is read, "AB is equal to CD."

The sign $=$ is the sign of equality.

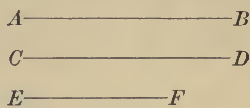


Fig. 29.

4. *How is the inequality of two lines expressed?*

The inequality of the two lines AB, EF is thus expressed:

$$AB > EF, \text{ or } EF < AB.$$

These expressions are read, "AB is greater than EF," and "EF is less than AB."

5. *How is the equality of two lines tested?*

The lines are usually compared with a third line.

Suppose I wish to test whether the lines AB and CD (Fig. 29) are equal, I open the dividers, place one point on A, and the other on B. This is called "taking the distance AB between the points of the dividers." Then, keeping the opening of the dividers unchanged, I place one point on C, and observe whether the other point will fall on D. If it does fall on D, I know that $AB = CD$.

Here the third line with which AB and CD are compared is the distance between the points of the dividers.

6. Draw freehand from a point two lines as nearly equal as you can, then test their equality with the dividers.

7. The same exercise, one line to be horizontal, the other vertical.

8. Draw freehand a three-sided figure, with its sides as nearly equal as you can; test the equality with the dividers.

9. Draw four parallels exactly equal in length.

10. Draw a line equal to an edge of the body given you.

11. Draw lines a, b, c, d so that $a = b$, $c > a$, $d < b$.

12. Read the following: $m = n$, $AB < CD$, $x > y$.

13. Draw four straight lines, a, b, c, d; then compare a with each of the other lines, writing the result with the proper sign.

14. Draw freehand two lines as nearly equal as you can, each longer than the greatest opening of your dividers. Can you test their equality by means of your dividers?

15. *Compare different lines drawn between two points.*

A straight line is the shortest line between two points; hence the length of a straight line joining two points is taken as the **distance** of the points from each other.

FIG. 30.

16. How do sign-painters make use of this truth (Fig. 30)?

They chalk a cord and stretch it tightly between the points through which the line is to pass; then, seizing this cord by the middle, they draw it back a little from the wood, and then let it go. It springs back, strikes the wood a sharp blow, and leaves on it a white trace, which is a straight line.

17. *Define an axiom, and illustrate the meaning.*

There are statements or assertions which are so obvious that they stand in no need of any explanation or proof. Every one *sees* at a glance that they are true.

In Geometry, an assertion which is admitted to be true, without proof, is called an *axiom*.

An instance of an axiom occurs in No. **5**. We assume that if AB and CD are separately equal to the distance between the points of the dividers, they are equal to each other.

In general, if any two magnitudes are each equal to a third magnitude, we assume, as quite obvious, that they are equal to each other.

Another example of an axiom is the assertion that a straight line is the shortest distance between two points.

18. *State the most important axioms which are found useful in the study of Geometry.*

1. *Two magnitudes, each equal to a third, are equal to each other;* or, *for any magnitude, its equal may be substituted.*
2. *If equals are added to equals, the sums are equal.*
3. *If equals are taken from equals, the remainders are equal.*
4. *If equals are multiplied by equals, the products are equal.*
5. *If equals are divided by equals, the quotients are equal.*
6. *A whole is greater than any of its parts.*
7. *Through a given point in a given direction, only one straight line can be drawn.*
8. *Through two points, only one straight line can be drawn.*
9. *A straight line is the shortest ~~distance~~* line *between two points.*
10. *Through a given point only one parallel to a given straight line can be drawn.*

Lesson 10.

1. *Explain the addition and subtraction of lines.*

In order to add two straight lines, let AB and CD (Fig. 31) be two lines, draw a straight line and take on it $EF = AB$, $FG = CD$; then the line EG will be the *sum* of AB and BC; or,

$$EG = AB + CD.$$

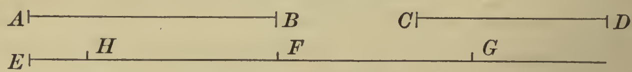


FIG. 31.

If, instead of laying off CD towards the right, we lay it off from F to H towards the left, then we obtain a length EH which is the *difference* between AB and CD; or, using signs,

$$EH = AB - CD.$$

2. *How are lines multiplied or divided by numbers?*

To multiply a line by a number is to add the line repeatedly to itself. Thus: produce AB (Fig. 32) by adding to it lines BC, CD, etc., each equal to itself; then AC is twice as long as AB, AD three times as long, etc.

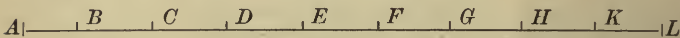


FIG. 32.

These results are usually written as follows:

$$AC = 2\,AB,\ \ AD = 3\,AB,\ \ AE = 4\,AB,\ \ AL = 9\,AB.$$

Division is the inverse of multiplication. To divide a line by a number is to separate the line into as many equal parts as there are units in the number, and take one of the parts as the quotient. The operation of division is usually expressed in the form of a fraction, thus (see Fig. 32):

$$AB = \frac{AC}{2} = \frac{AD}{3} = \frac{AL}{9},\ AD = \frac{AH}{2}, AC = \frac{AG}{3}, \text{ etc.}$$

3. Problem.—*To bisect a given straight line AB.*

With centres A and B (Fig. 33), and a radius greater than half of AB, describe arcs intersecting at C and D.

Join CD, which will cut AB at a point M.

Then M is the middle point of AB.

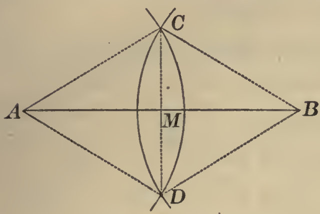


FIG. 33.

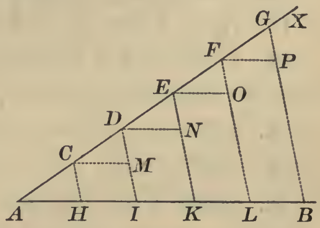


FIG. 34.

4. Problem.—*To divide a given straight line AB into any number (say five) equal parts.*

Draw through A (Fig. 34) any straight line AX.

Upon AX lay off five equal lengths from A to G.

Join GB, and draw through the points C, D, E, F, between A and G, lines $\parallel$ to GB (p. 19, No. 4).

These parallels will divide AB into five equal parts.

NOTE.—The use of the lines CM, DN, etc., will appear hereafter.

5. Draw any two straight lines; then construct a line equal to their sum, and a line equal to their difference.

6. In Fig. 32 name lengths equal to the following:

$AE + BD$, $AC + GL$, $AE - AD$, $AG - EH$, $AD - HL$.

$4\,AC$, $2 \times (AB + BD)$, $3 \times (DH - DE)$, $4 \times (AG - BF)$.

$\frac{1}{4}AE$, $\frac{1}{3}AL$, $\frac{1}{2}BL$, $\frac{1}{2}(AD + AH)$, $\frac{1}{4}(CL - CF)$.

7. Draw a broken line, then develop it; that is, draw a straight line equal to the entire length of the broken line.

8. Draw a straight line equal in length to the sum of the edges of one face of the solid given to you.

9. Draw $AB = CD$, and $MN = PQ$. Then draw lines equal to $AB + MN$, $CD + PQ$, $AB - MN$, $CD - PQ$.

Are the first two of these lines equal? Why? Are the last two of these lines equal? Why?

10. Produce a line AB to P, making $AP = 5\ AB$.

11. Draw a line AB and then bisect it.

12. Divide a line into four equal parts. First bisect the line and then bisect each of the halves.

13. Divide a line into six equal parts.

14. Divide a line into seven equal parts.

15. Bisect a line freehand. Test and correct the result.

16. Trisect a line freehand. Test and correct the result.

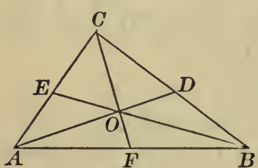


FIG. 35.

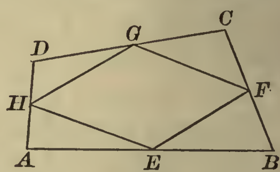


FIG. 36.

17. Bisect a line by *repeated trial* with the dividers.

18. Draw a three-sided figure ABC. Bisect its sides at D, E, F. Join the middle points to the opposite corners. If your work is accurate, the lines AD, BE, CF will all pass through the same point O (Fig. 35).

19. Draw a four-sided figure $ABCD$. Bisect its sides at E, F, G, H. Join the middle points, taken in order around the figure. If your work is accurate, EF will be $\parallel$ to GH, and $FG \parallel$ to EH (Fig. 36).

NOTE. — In performing Exercises 18 and 19 do not try to choose points so that your figures will resemble Figures 35 and 36, but mark their positions on your paper without reference to these figures. Do not choose your points too near together; the farther they are from each other the better.

Lesson 11.

1. *What is a unit of length? Give examples.*

A **unit of length** is a length named and defined by law. Examples: the *foot*, the *yard*, the *mile*, the *meter*.

2. *What is meant by measuring a line?*

To **measure** a line is to find how many times it will contain a unit of length. The number of times a line will contain a unit of length, joined to the name of the unit, is called the **length** of the line. Examples: 6 feet, 9 miles.

3. *What units of length are in common use?*

The units of length in common use in this country and in England are the **inch**, the **foot**, the **yard**, the **rod**, the **mile**.

Abbreviations: *in.* = inch, *ft.* = foot, *yd.* = yard.

4. *How are these units related to one another?*

They are related as shown in the following table:

12 inches = 1 foot.	$5\frac{1}{2}$ yards = 1 rod.
3 feet = 1 yard.	320 rods = 1 mile.

5. *How are lengths less than an inch expressed?*

The inch is usually divided into *halves*, *quarters*, *eighths*, and *sixteenths*. If we wish to take into account very small parts of an inch, it is more convenient to divide it decimally into *tenths*, *hundredths*, etc.

6. How many inches in one rod? in one mile?

7. How many feet are there in one mile?

8. How many inches in 3 feet? 10 feet? $2\frac{1}{2}$ yds.?

9. What part of a foot is 3 inches? 4 inches? 8 inches? 9 inches? 10 inches?

10. Reduce 100 inches to feet. Also to yards.

11. You have a **scale** for measuring lengths. What is the length of the scale? Describe how each inch is subdivided. Name one of the shortest parts. Name two of them taken together. Name three of them taken together, etc.

12. How many half-inches are there in $5\frac{1}{2}$ inches?

13. How many quarter-inches are there in $4\frac{3}{4}$ inches?

14. Reduce $9\frac{1}{4}$ inches to quarters of an inch.

15. Reduce $3\frac{1}{2}$ inches to quarters of an inch.

16. Reduce $2\frac{1}{4}$ inches to eighths of an inch.

17. Reduce $4\frac{1}{4}$ inches to sixteenths of an inch.

18. Reduce 24 quarters of an inch to inches.

19. Reduce 28 eighths of an inch to inches.

20. Find the difference between $6\frac{7}{8}$ in. and $6\frac{13}{16}$ in.

21. How many eighths of an inch are there in one foot?

22. How many quarter-inches are there in one yard?

23. Draw straight lines having lengths as follows:

$4\frac{1}{2}$ in.; $3\frac{3}{4}$ in.; $2\frac{5}{8}$ in.; $1\frac{11}{16}$ in.

24. Through a point A draw three straight lines. Upon one of them, mark off a length $AB = 2\frac{3}{4}$ inches; upon another, $AC = 3\frac{1}{8}$ inches; upon the other, $AD = 4\frac{3}{16}$ inches.

25. Draw four parallel lines (**7**, 4). Then lay off upon the first $AB = 4\frac{1}{2}$ in.; upon the second, $CD = 2\frac{7}{8}$ in.; upon the third, $EF = AB + CD$; upon the fourth, $GH = AB - CD$. Then measure EF and GH. Do not put the scale on the paper. Record the lengths of EF and GH thus:

$EF =$, $GH =$.

26. Draw a broken line, $ABCDEF$, making $AB = 2$ in., $BC = 2\frac{3}{8}$ in., $CD = 1\frac{1}{2}$ in., $DE = 3\frac{1}{16}$ in., $EF = 2\frac{3}{4}$ in. Then develop it. In what part does the middle point of the developed line lie? Find the length of the developed line (1) by adding the parts, (2) by measuring it.

Lesson 12.

DIRECTIONS TO BE FOLLOWED.

Work out Exercises 1 and 2 before the hour of recitation, and bring the results with you. Do each of the other exercises *twice;* first, using no instruments but your pencil, and estimating all lengths as well as you can by your eye; secondly, using rule, scale, and dividers, and measuring lengths correct to a sixteenth of an inch at least. *Record* neatly all your results, both the estimated lengths and the measured lengths.

1. Find the average value of your pace. To do this, take ten steps straight forward as you naturally walk, measure the distance walked, and divide this distance by ten. Repeat, and if the two results differ, take their *mean*, which is equal to half their sum.

2. Measure by pacing the distance from the schoolhouse to your home.

3. Draw a three-sided figure ABC. Then measure and record the lengths of the sides, AB, BC, AC.

FORM OF RECORD.

	AB	BC	AC
Estimated Length. Measured Length.			
Difference.			

4. Measure the length (1) of one of the horizontal bars in the form of record above, (2) of one of the vertical bars. Make a neat form of record for yourself.

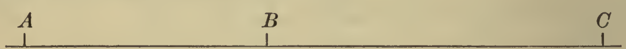
5. Draw $AB = 1\frac{1}{2}$ in., $AC = 2$ in. Join BC. Produce AB to D making $BD = 1\frac{1}{2}$ in. Draw $DE \parallel$ to BC and meeting AC produced at E. Measure BC, DE, CE.

6. Measure and record the dimensions of the body which is given you.

Lesson 13.

1. *Suppose three points A, B, C on the ground are in a straight line, and that by measurement you find AB = 125 feet, AC = 300 feet. How can these lengths be represented on this page by a straight line drawn from left to right?*

Since the width of the printed matter on the page is only about $3\frac{1}{2}$ inches, it is clear that the line which I draw to represent AC cannot be more than $3\frac{1}{2}$ inches long. Suppose I take 1 inch on paper to represent 100 feet or 1200 inches on the ground. Then $\frac{1}{100}$ of an inch will represent 1 foot. Therefore 300 feet will be represented by $\frac{300}{100}$ inches, or 3 inches, and 125 feet by $\frac{125}{100}$ inches, or $1\frac{1}{4}$ inches. Hence, to represent the measured lines I draw on the paper $AC = 3$ inches, and take $AB = 1\frac{1}{4}$ inches.



Scale: 1 in. = 100 ft.

The two measured lengths have now been *drawn to scale*; the *reducing factor* is 1200; the *scale of reduction* is $\frac{1}{1200}$, or as sometimes written 1 : 1200; a common way of expressing it is as follows: 1 inch = 100 feet.

2. *What is meant by "drawing to scale"?*

Drawing to scale means drawing lines on paper so that each line shall be the same fractional part of the line on the ground which it represents.

3. *What is the reducing factor?*

The **Reducing factor** is the number by which each line on the ground must be divided in order to obtain the reduced length to lay off on paper.

4. *What is the scale of reduction?*

The **Scale of reduction** is the fraction having 1 for numerator and the reducing factor for denominator.

In making a drawing, the scale of reduction should be so chosen as to leave a fair margin around the paper after the lines are all drawn, and should be neatly written by the side of the drawing, usually at the lower right-hand corner.

5. Give examples of drawing to scale.

On maps distances of hundreds and even thousands of miles are represented by lines a few inches long.

Plans of estates and buildings are always drawn to a reduced scale. In general, the same is true of all pictures and paintings.

6. Write in the form of a fraction the following scales: 1 in. = 1 yd., $\frac{1}{4}$ in. = 1 ft., 2 in. = 1 mile, $\frac{1}{2}$ in. = 10 miles.

7. If the scale of reduction be $\frac{1}{4800}$, what length on the ground will be represented by 1 inch on the paper?

8. If 4 inches on paper represent a line 200 feet long, express the scale of reduction in a fractional form. What length on paper will represent half a mile?

9. The scale of a map is $\frac{1}{3600}$. The distance between two towns on the map is 8.8 inches; what is the real distance between the towns?

10. Draw 5 parallels and lay off on them to the scale $\frac{1}{200}$ the lengths: 100 feet, 150 feet, 70 feet, 50 feet, 30 feet.

11. Draw a three-sided figure, and find what lengths the sides would represent to the scale 1 inch = 1000 feet.

12. Find from a map of the United States the distance in a straight line from New York to Chicago.

13. Measure the line on the blackboard assigned to you, and record its length. Then represent the line on paper, choosing a suitable scale of reduction for this purpose.

Lesson 14.

1. *What is the nature of the Metric System?*

The Metric System of units is a decimal system.

The fundamental unit of length is the **meter.**

All the other units of length are derived from the meter by decimal multiplication and division.

2. *Write out in tabular form the names, abbreviations, and relative values of the units of length.*

THE METRIC UNITS OF LENGTH.

Name.	*Abbreviation.*	*Relative value.*	
Meter	m.		
Dekameter	dkm.	10	meters.
Hectometer	hm.	100	"
Kilometer	km.	1000	"
Decimeter	dm.	0.1	of a meter.
Centimeter	cm.	0.10	" "
Millimeter	mm.	0.100	" "

The units most largely employed in practice are the **meter,** the **kilometer,** the **centimeter,** the **millimeter.**

3. *How is one unit reduced to another unit?*

Reduction from a *larger* unit to a *smaller* unit is performed by *multiplying* by 10 as many times as may be necessary; that is, by *moving the decimal point towards the right* the proper number of places.

Reduction from a *smaller* unit to a *larger* unit is performed by *dividing* by 10 as many times as may be necessary; that is, by *moving the decimal point towards the left* the proper number of places.

Examples. $32^{m} = 320^{dm} = 3200^{cm} = 32000^{mm}$;

$$0.764^{km} = 7.64^{hm} = 76.4^{dkm} = 764^{m} = 764000^{mm};$$

$$376^{mm} = 37.6^{cm} = 0.376^{m} = 0.000376^{km}.$$

4. In practice, *only one unit* is used in expressing a length. Suppose the length of a line is measured and found to be 7^{dm}, 6^{cm}. Reduce this to meters; decimeters; centimeters.

5. Reduce 12.4^{km} to each of the other units.

6. Reduce 1800^{cm} to decimeters; to meters; to kilometers.

7. What part of a kilometer is 1 millimeter?

8. What part of a meter is 5^{cm}? 2^{dm}? 10^{mm}?

9. $4^{km} + 4^{m} + 4^{cm} + 4^{mm}$ = how many meters?

10. What is the difference between 4^{cm} and 28^{mm}?

11. From 10^{m} take 10^{mm}.

12. How long a piece of wood is required to make 20 rulers, each ruler to be 30^{cm} long?

13. How many rails 7.5^{m} long are required to build a railroad track 300^{km} long?

14. Reduce the following values to English equivalents: height of a barometer, 76^{cm}; height of Mt. Blanc, 4815^{m}.

15. Reduce 236 miles to kilometers.

16. Describe the divisions of your metric scale.

17. Measure to the nearest millimeter (1) the length of this page; (2) the length of a line of print upon the page. Record the results neatly in a tabular form.

18. Construct on paper a scale, 1^{dm} long, precisely like the decimeter scale printed below.

TABLE OF ENGLISH AND METRIC EQUIVALENTS.

Approximately.		*More exactly.*	
8 kilometers	= 5 miles.	1 kilometer	= 0.6214 mile.
1 meter	= $1\frac{1}{11}$ yards.	1 meter	= 39.37 inches.
$2\frac{1}{2}$ centimeters	= 1 inch.	1 centimeter	= 0.3937 inch.

Lesson 15.

1. Express in fractional form the scale $4^{cm} = 1^{m}$.

2. Express in fractional form the scale $2^{mm} = 1^{km}$.

3. A certain map is drawn to the scale $1^{cm} = 100^{m}$. What length on the map will represent a distance of 1.2^{km}?

4. The distance between two towns on a map drawn to the scale 1 : 40000 is represented by a line 1^{dm} in length. What is the actual distance between the towns?

5. A monument is 40^{m} high, and a man standing beside it is 2^{m} high. Draw to the scale 1 : 200 straight lines to represent the monument and the man.

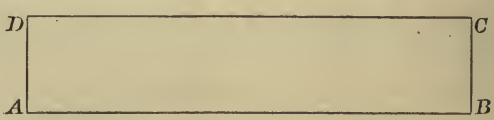
6. Draw three parallel lines of any lengths, and then find what lengths they would represent to the scale 1 : 200. For example: a line 3^{cm} long would represent $3^{cm} \times 200 = 600^{cm} = 6^{m}$.

7. Draw $AB = 25^{mm}$, $AC = 37^{mm}$. Join BC. Produce AB to D, making $BD = 50^{mm}$. Draw $DE \parallel$ to BC, and meeting AC produced at E. Measure BC, DE, CE.

8. Draw a line by your eye 4^{cm} long; measure it; record your error in millimeters. Repeat several times.

Note.—In doing Exercises 9, 10, and 11 follow the directions given in Lesson 12, and measure lengths to the nearest *half*-millimeter.

9. Measure the dimensions of the following figure:



10. Draw a four-sided figure $ABCD$, and measure the lengths of the sides.

11. Measure the dimensions of the body given you.

Lesson 16. Review.

1. Review the italicized exercises in Lessons 5–9.

2. What is the greatest possible number of straight lines that can be drawn through four points? Mark four points, and draw all the straight lines possible through them.

3. Place four points so that *four* and *only four* straight lines can be drawn through them. Draw the lines.

4. In how many points will five lines intersect if four of the lines are parallel? if three are parallel? if two are parallel? if no two of the lines are parallel?

5. Through a vertical line how many vertical planes can be passed (that is, how many different vertical planes can be imagined, all of which shall contain the vertical line)? how many horizontal planes? how many inclined planes?

6. Through a horizontal line how many vertical planes can be passed? how many horizontal planes? how many inclined planes?

7. Through an inclined line how many vertical planes can be passed? how many horizontal planes? how many inclined planes?

8. Describe two concentric circles. Then describe two circles which shall touch, but not cut, the concentric circles.

9. Describe a circle with radius two inches, and draw in it a chord equal to one and one-half times the radius.

10. Describe a circle, and within this circle two smaller circles which shall touch each other and the large circle.

11. Describe a semicircle, and draw in it three chords parallel to the diameter.

12. Give an example of parallel lines; of parallel planes.

13. Describe the situation of all the points in a plane which are one inch from a straight line drawn in the plane.

Lesson 17. Review.

1. Review the italicized exercises in Lessons 10–13.

2. Which is the greater, 3.6 in. or $3\frac{5}{8}$ in.? What is the difference?

3. If a pace $= 2\frac{1}{2}$ ft., how many paces are there in 1 mile?

4. Express as fractions the following scales: 1 in. = 1 ft.; 1 in. = 1 yd.; 1 in. = 1 rod; 1 in. = 1 mile.

5. If 4 in. represent 66 ft., what will represent 44 yds.?

6. Among the scales in use in the engineer service are:

Plans of buildings,	1 in. = 10 ft.
Maps $1\frac{1}{2}$ miles square,	2 ft. = 1 mile.
Maps 3 miles square,	1 ft. = 1 mile.
Maps 4 to 8 miles square,	6 in. = 1 mile.
Maps 9 miles square,	4 in. = 1 mile.
Maps 12 to 24 miles square,	2 in. = 1 mile.
Maps 50 miles square,	1 in. = 1 mile.

Express these scales in the form of fractions.

7. Divide by your eye an inch into tenths.

8. Divide a line into 8 equal parts.

9. Draw a three-sided figure; bisect its sides, and produce (if necessary) the bisecting lines till they meet.

They should all meet in the same point.

10. Draw a line *AB*, and divide it into three parts, such that the second part shall be double the first, and the third part double the second.

11. Draw a four-sided figure, two sides of which are parallel; then describe circles upon the sides taken as diameters.

12. Draw a four-sided figure *ABCD*, having its opposite sides parallel; join *AC* and *BD*, and mark their point of intersection *O*. Measure the lengths *AC*, *BC*, *CD*, *DA*, *AO*, *CO*, *BO*, *DO*.

CHAPTER III.

ANGLES.

Lesson 18.

1. *Define an angle, its sides, and its vertex.*

An **angle** is the difference in direction of two straight lines which meet, or would meet if produced.

The lines are called the **sides** of the angle; their point of meeting is called the **vertex.**

2. Illustrate these definitions.

The lines AB and AC (Fig. 37) form the angle BAC. A is the vertex, AB and AC the sides. The lines DE and FG (Fig. 38) also form an angle. Its vertex is the point H, where the lines would meet if produced.

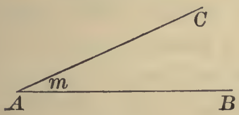


FIG. 37.

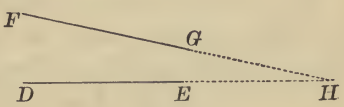


FIG. 38.

3. *How is an angle named or denoted?*

An angle may be named in either of two ways:

(1) By a *small* letter placed just inside the vertex.

(2) By writing the letter which stands outside the vertex between two letters which stand one on each of the sides. Thus, the angle in Fig. 37 is the angle m, or the angle BAC.

Abbreviations: $\angle$ stands for "angle," $\measuredangle s$ for "angles."

4. *Angles differ in magnitude. How can a clear idea of the magnitude of an angle be obtained?*

To get a clear idea of angular magnitude we must conceive an angle to be described by the rotation of a line.

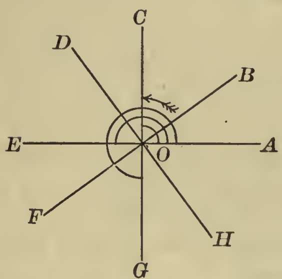


FIG. 39.

Suppose a line to start from the position OA (Fig. 39), and turn about the point O in the direction of the arrow. When the line has reached the position OB, it has described the angle AOB; when it has reached the position OC, it has described the angle AOC. If the motion continue, an angle larger than AOC will be described. It is quite obvious that the magnitude of the angle depends entirely upon the *amount of rotation* of the line, and not at all upon the length of the line.

5. *Define the right angle and the straight angle.*

When a line makes *one-fourth* of a complete revolution it describes an angle which is called a **right angle**.

Thus, the angle AOC (Fig. 39) is a right angle.

When a line makes *one-half* of a complete revolution it describes an angle which is called a **straight angle**. The angle AOE (Fig. 39) is a straight angle; its sides are opposite in direction, and form a straight line; hence its name.

6. *Can an angle be greater than a straight angle?*

Yes. When the rotating line (Fig. 39) has made *three-fourths* of a revolution it takes the position OG, and the angle described, AOG, is equal to *three* right angles. The side OG of this angle is opposite in direction to the line OC.

When the rotating line has made exactly one revolution it arrives at the position OA from which it started, and has described an angle equal to four right angles.

7. *Define an acute angle and an obtuse angle.*

An angle less than a right angle is called an **acute** angle, and an angle greater than a right angle is called an **obtuse** angle.

8. What kind of an angle (Fig. 39) is AOB? AOD? DOE? COE? FOG? EOG? DOG? GOH? HOC?

9. Is the angle a on the blackboard right, acute, or obtuse? the $\angle b$? the $\angle c$? etc.

10. Mark three points A, B, C. Join them by straight lines. How many angles are formed? Name them.

11. Draw two intersecting lines. How many angles are formed? Name them, using as few letters as possible.

12. Draw freehand an acute angle, an obtuse angle, a right angle.

13. What angle does a vertical line make with a horizontal line?

14. What kind of an angle do the hands of a clock make with each other at three o'clock? at six o'clock?

15. Mention times of the day when the hands of a clock make a right angle, an acute angle, an obtuse angle.

16. How long a time does it take the minute-hand of a watch to describe a right angle? How long does it take the hour-hand to do the same?

17. What kind of an angle does the hour-hand describe in one hour? three hours? five hours? six hours?

18. A man is walking due north. If he suddenly change the direction in which he is moving through two right angles, in what direction will he now be going? In what direction if he makes a change from due north through one right angle to his left? through three right angles to his right? through two and one-half right angles to his right?

Lesson 19.

1. *Define perpendicular lines and oblique lines.*

If two lines form a right angle, they are said to be **perpendicular** to each other; if they form any other angle, they are said to be **oblique** to each other.

Abbreviation: The sign ⊥ means "perpendicular."

2. Problem. — *To erect a perpendicular at any point C of a given straight line AB.*

Method I. (Fig. 40). — Instruments: ruler and compasses.

With centre C, and any radius, cut AB in D and E.

With centres D and E, and a greater radius than before, describe arcs intersecting at F.

Join CF; CF is the perpendicular required.

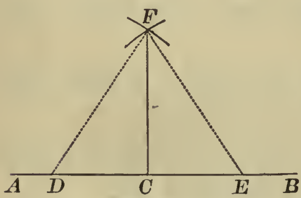


Fig. 40.

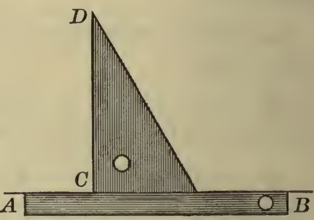


Fig. 41.

Method II. (Fig. 41). — Instruments: ruler and triangle.

Place the ruler and the triangle as seen in the figure.

Draw a line along the edge CD.

This line will be the perpendicular required.

3. Give examples of perpendicular lines.

4. Draw a line AB; erect a perpendicular by Method I.

5. Draw a horizontal line, and erect (by Method II.) perpendiculars at six equidistant points along the line.

6. How many right angles are formed when two perpendicular lines cut each other?

7. Problem. — *To erect a perpendicular at the end of a line AB without producing the line (Fig. 42).*

Mark any point C not in the line.

With centre C and radius CB describe an arc greater than a semicircumference, cutting AB at D.

Join DC, and produce DC to meet the arc at E.

Join EB; EB is the perpendicular required.

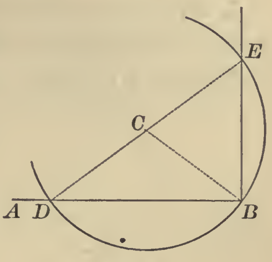


FIG. 42.

8. Draw a line $AB = 4$ inches; erect a perpendicular at each end, making each perpendicular, as nearly as you can judge by the eye, equal to AB. Test the result with dividers.

9. Draw a line AB; erect at A a perpendicular equal to $4\,AB$; at B a perpendicular equal to $\frac{1}{2}AB$.

Work out the exercise (1) freehand; (2) with instruments.

10. Draw a horizontal line AB four inches long; at A erect a perpendicular $AC = 1\frac{1}{2}$ in.; at B erect a perpendicular $BD = 4\frac{1}{2}$ in. Join CD; measure CD, and write its length along its side.

Work the exercise (1) freehand; (2) with instruments.

11. Draw $AB = 5$ in., $BC \perp$ to AB and $= 3$ in., $CD \parallel$ to AB and $= 1$ in. Join AD, and measure its length.

Work (1) freehand; (2) with instruments.

12. Draw $AB = 6$ in. Upon AB as diameter, describe a semicircle. At a point C, two inches from A, draw $CD \perp$ to AB, and meeting the curve at D. Measure CD, and multiply its length by itself. Record the result thus: $CD \times CD =$

If your work were perfectly done, the product would be 8. The nearer your result is to 8, the better your work.

13. Erect a perpendicular freehand, and test the result with ruler and triangle.

Lesson 20.

1. Problem. — *To drop a perpendicular from a given point C to a given line AB.*

METHOD I. (Fig. 43). — Instruments: ruler and compasses. With centre C and radius greater than the distance from C to AB, cut AB at D and E. With centres D and E and the same radius as before, describe arcs intersecting at F on the other side of AB. Join CF; CF is $\perp$ to AB.

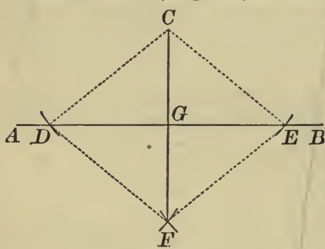


FIG. 43.

METHOD II. — Instruments: ruler and triangle. This method will be obvious from Fig. 41.

2. Drop by Method I. a perpendicular from a point C to a line AB.

3. The same exercise, using Method II.

4. Draw a four-sided figure $ABCD$, and drop perpendiculars from C and D to the side AB.

5. Draw a three-sided figure and drop perpendiculars from the corners to the opposite sides (produced when necessary). If your work is accurate, the three perpendiculars will all pass through the same point.

6. Draw an angle BAC. Upon AC take $AM = 1\frac{1}{4}$ in.; $AN = 2\frac{1}{2}$ in.; $AO = 3\frac{3}{4}$ in.; $AP = 5$ in. Drop perpendiculars MR, NS, OT, PV to AB. Measure their lengths, and also AR, AS, AT, AV. Record your results in a table.

7. A path goes straight for 15 yards, then turns through a right angle and goes straight for 40 yards, then turns and runs straight to its starting-point. Find its entire length by drawing to scale (1 in. = 10 yds.).

Lesson 21.

1. *Define equal angles.*

Two angles are **equal** if they can be placed so that their vertices coincide in position, and their sides coincide in direction. For example, the angles AOB, BOC, COD, etc. (Fig. 44), are equal angles.

2. *What consequences follow from this definition?*

(1) *All straight angles are equal;* for the sides of a straight angle form a straight line, and two straight lines can always be so placed that they coincide.

(2) *All right angles are equal;* for a right angle is half of a straight angle, and the halves of equal things are also equal (see Axiom VIII.).

(3) *Angles, like lines, can be added, subtracted, etc.*

For example: in Fig. 44,

$$AOC = AOB + BOC$$

and $AOB = AOC - BOC.$

Also, the angles being equal,

$$AOC = 2\,AOB,$$
$$AOD = 3\,AOB, \text{ etc.}$$

And it is also clear that

$$AOB = \tfrac{1}{2}AOC,$$
$$= \tfrac{1}{3}AOD, \text{ etc.}$$

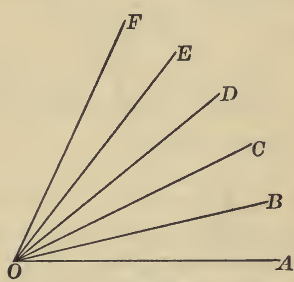


FIG. 44.

3. *Describe the units used for measuring angles.*

The right angle is divided into 90 equal parts called **degrees**, the degree into 60 equal parts called **minutes**, the minute into 60 equal parts called **seconds**. Abbreviations: the sign ° stands for "degree," ′ for "minute," ″ for "second."

4. Reduce to degrees 4 rt. ∠s, $\frac{1}{2}$ rt. ∠, $\frac{1}{3}$ rt. ∠, $\frac{1}{8}$ rt. ∠.

5. Reduce 48° 54′ 36″ to seconds; 120,000″ to degrees.

6. *Define equal arcs.*

Two arcs are **equal** if they can be placed, one upon the other, so that they coincide. Example: the arcs AB and CD (Fig. 45).

7. *Suppose a circumference (Fig. 45) to be divided into any number of equal arcs by drawing radii, and then compare the corresponding angles at the centre.*

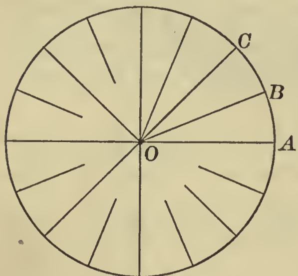


FIG. 45.

Compare any two adjacent angles, as AOB and BOC.

Imagine AOB folded upon the line OB till A falls on C.

Why *must* A fall on C?

Then OA will coincide with OC.

Why *must* OA and OC coincide?

Therefore, $\angle AOB = \angle BOC$.

In the same way all the angles at the centre can be proved equal.

Therefore, *if a circumference be divided by radii into any number of equal arcs, the corresponding angles at the centre are also equal.*

8. *How is this relation between arcs and the corresponding angles at the centre applied in order to measure angles?*

The circumferences of circles are supposed to be divided into degrees, minutes, and seconds precisely in the same way as angles. A circumference, therefore, contains 360 degrees of arc, each degree contains 60 minutes, and each minute contains 60 seconds. It follows that the arc, described from the vertex of an angle as centre, and with *any* radius, will contain the same number of degrees, etc., as the angle. And the angle is measured by finding how many degrees, etc., there are in this arc.

9. *Why may the arc employed to measure an angle be described with* **any** *radius?*

It is true that the greater the radius the greater will be the arc. Thus: the arc AB (Fig. 46) is larger than the arc CD; but if the arc AB be divided into degrees, etc., and radii are drawn to all the points of division, these radii will divide the arc CD into exactly as many equal parts as there are in AB, only the parts will be smaller. Therefore, we shall get the same measure for the angle whichever arc we use.

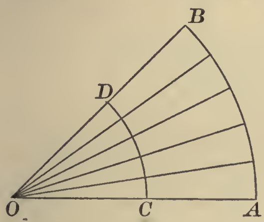


Fig. 46.

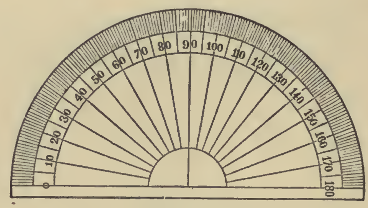


Fig. 47.

10. *How are angles drawn on paper measured?*

Angles on paper are measured with an instrument called a **protractor** (Fig. 47). It is a semicircle, the circular edge of which is divided into 180 degrees.

To measure an angle, place the centre of the protractor over the vertex and the zero line on one side of the angle; then read on the divided edge the division through which the other side of the angle passes.

11. Make an angle BAC, and measure it with the protractor.

12. Measure with your protractor the angle given you.

13. Make with your protractor the angles: 30°, 60°, 75°, 120°, 150°.

14. Make with your protractor the angles: 45°, 63°, 107°.

Lesson 22.

1. Problem. — *At a point C in a line AB to construct an angle equal to a given angle DEF (Fig. 48).*

METHOD I. — Instruments: ruler and compasses.

With centre E and any radius describe the arc GH.

With centre C and the same radius describe an arc LM, cutting AB at L.

With centre L and radius GH, cut the arc LN at N.

Join CN; $\angle BCN = \angle DEF$.

METHOD II. — Instruments: ruler and protractor.

This method is obvious without explanation.

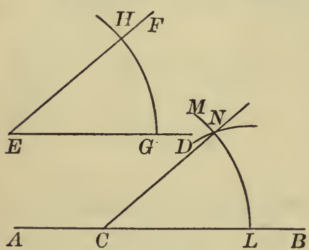


FIG. 48.

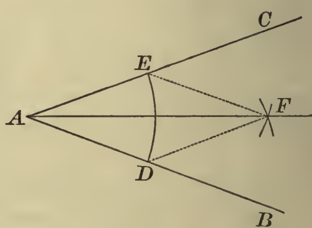


FIG. 49.

2. Problem. — *To bisect a given angle BAC (Fig. 49).*

With centre A describe the arc DE.

With centres D and E, and the same radius as before, describe arcs intersecting at F.

Draw AF; the line AF bisects the $\angle BAC$.

3. Make an angle BAC, and then construct with ruler and compasses an angle DEF equal to BAC.

4. Make with your protractor an angle of 40°; then construct an equal angle with ruler and compasses.

5. Construct a right angle, and then bisect it. What is the value of each of the equal parts?

6. Problem. — *To construct an angle of 60° (Fig. 50).*

Draw a line AB.

With centre A and any radius describe the arc CD.

With centre C, and the same radius, cut the arc CD in E.

Join AE; $\angle BAE = 60°$.

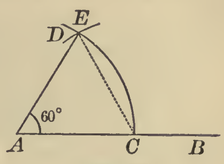


FIG. 50.

7. Construct an angle of 60°, and then bisect it.

8. Construct an angle $BAC = 60°$; from C drop $CD \perp$ to AB; and find the value of the angle ACD.

9. Construct an angle $BAC = 60°$; join BC; and find the values of the angles ABC and ACB.

10. Construct with ruler and compasses an angle of 30°. Verify your result with your protractor.

11. Construct with ruler and compasses an angle of $22\frac{1}{2}°$. Verify with your protractor.

12. Draw any acute angle BAC; then construct an angle three times as large.

13. Draw freehand the angles 30°, 45°, 60°. Correct your results with your protractor. Record your errors in degrees.

14. Make an angle, and bisect it freehand. Correct your result with your protractor. Record your error.

15. Draw a three-sided figure ABC. Find the values of its angles, (1) by estimating them by the eye, (2) by measuring them with your protractor. Record your results in tabular form, thus: —

	BAC	*ABC*	*ACB*	*Sum.*
Estimated Value. Measured Value.				
Difference.				

Lesson 23.

1. *Define complementary and supplementary angles.*

Two angles are said to be **complementary** if their sum is equal to a right angle, or 90°, and each is called the **complement** of the other. Example: 67° and 23°.

Two angles are said to be **supplementary** if their sum is equal to a straight angle, or 180°, and each is called the **supplement** of the other. Example: 150° and 30°.

2. *Define adjacent angles. When are they supplementary?*

Two angles which have the same vertex and a common side between them are called **adjacent** angles.

If a line OC (Fig. 51) is drawn from any point O in AB, two adjacent angles AOC, BOC are formed; and whatever be the direction of OC it is obvious that

$$AOC + COB = 180°.$$

The two adjacent angles formed when one straight line meets another are supplementary.

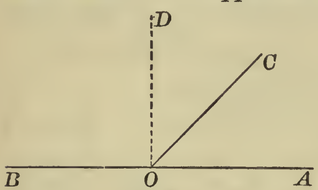


Fig. 51.

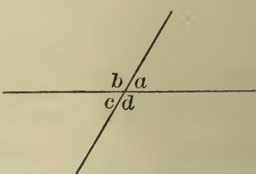


Fig. 52.

3. What is the complement of 35°? 60°? 1°? 89°? 45°?

4. What is the supplement of 60°? 90°? 1°? 135°? 62° 12′?

5. Draw two angles adjacent but not supplementary.

6. If OD (Fig. 51) is ⊥ to AB, what is the complement of ∠ AOC?

7. If ∠ $AOC = 48°$, find ∠ COD and ∠ COB.

8. *Define vertical angles.*

Two angles are called **vertical** angles if one of them is formed by producing the sides of the other from the vertex.

When two straight lines intersect (Fig. 52) two pairs of vertical angles, a and c, b and d, are formed.

9. *Two vertical angles are equal. How can this be shown to be true?*

We may use three different methods:

(1) Measure them directly with a protractor.

(2) Cut them out, and place one upon the other.

(3) Observe that the vertical angles a and c (Fig. 52) both have the angle b for their supplement, and then reason as follows:

Since $a = 180° - b$,

and $c = 180° - b$,

therefore $a = c$. (Axiom 1).

10. *What advantages has the third method?*

The third method requires no instruments; it employs only *reasoning*. Moreover, it is easy to see that this reasoning applies equally well to the vertical angles b and d, or to any pair of vertical angles whatever. Therefore it convinces us immediately that *vertical angles are always equal.*

11. If $\angle a = 37°$ (Fig. 52), find the $\measuredangle$s b, c, and d.

12. Two lines intersect so that one of the angles formed is 90°. What are the values of the other angles?

13. Draw any two intersecting lines, and find the values of the four angles which are formed.

14. Write out the proof by the third method employed in No. **9** that $\angle b = \angle d$ (Fig. 52).

Lesson 24.

1. *Name certain pairs of angles formed when a straight line EF cuts two parallel lines AB and CD (Fig. 53).*

The line EF forms with AB four angles, a, b, c, d, and with CD four angles, m, n, o, p.

The pair of angles, a and m, and the other pairs similarly placed with respect to AB and CD, are called **exterior-interior** angles. The pairs, c and m, d and n, are called **alternate-interior** angles.

2. Draw two parallels. Cut them by a third line. Mark the angles about the points of intersection 1, 2, 3, 4 and 5, 6, 7, 8. Name the exterior-interior angles and the alternate-interior angles.

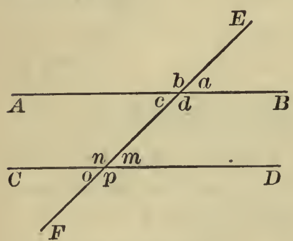


FIG. 53.

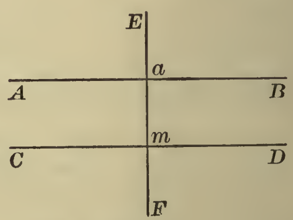


FIG. 54.

3. *The exterior-interior angles are equal. How can this be shown without use of instruments?*

If (Fig. 53) AB be moved towards CD, keeping all the time parallel to AB, the angles m, n, o, p will not change, because their sides do not change in direction. Let the motion continue till AB and CD cut EF at the same point; then AB and CD will coincide (see Axiom 10 for the reason). Therefore a will coincide with m, b with n, c with o, d with p. In other words, $a = m$, $b = n$, $c = o$, $d = p$.

4. *The alternate-interior angles are equal. How can this be shown to be true?*

Compare the alternate-interior angles c and m (Fig. 53).

We have just shown that $m = a$.
We know also that $c = a$ (p. 49, No. **9**).
Therefore, by Axiom 1, $c = m$.

In the same way we can show that $d = n$.

5. *If two straight lines AB, CD, cut a third line EF so that the exterior-interior angles, or the alternate-interior angles are equal, the two lines are parallel. How can this be shown to be true?*

(1) Suppose the exterior-interior angles, a and m, equal (Fig. 53). If AB be moved towards CD without change of direction till both lines cut EF at the same point, the two lines will then coincide, since they will pass through the same point and make equal angles, a and m, with the same portion of the line EF.

Therefore AB and CD must have the same direction.

Therefore AB and CD are parallel (p. 18, No. **1**).

(2) Suppose the alternate-interior angles, c and m, equal. The angles c and a are equal. Why? Therefore the angles a and m are equal, and the reasoning in (1) holds good.

6. If (Fig. 53) $a = 40°$, find the values of the other angles.

7. If (Fig. 54) AB is ∥ to CD, and EF is ⊥ to AB, is EF also ⊥ to CD? What reason can you give?

8. If (Fig. 54) AB and CD are each ⊥ to EF, does it follow that AB is ∥ to CD? Why?

9. Draw two parallels. Cut them by a third line. Then find the values of all the angles which are formed, using your protractor as little as possible.

Lesson 25. Review.

1. Review the italicized exercises in Lessons 18–20.

2. Can two horizontal lines be $\perp$ to each other?

3. A horse running due northeast turns to the left through $1\frac{1}{2}$ right angles. In what direction is he now running?

4. Can you find a way to test whether the edges of your triangle are exactly $\perp$ to each other?

5. Draw a line, and erect a $\perp$ by folding the paper.

6. Erect a $\perp$ freehand. Find, with protractor, your error.

7. Draw a three-sided figure, and erect perpendiculars at the middle points of the sides.

8. Describe a circle, draw a diameter, and join any three points in the circumference to the ends of the diameter. Measure the three angles formed at the three points. Do your results seem to indicate any general truth?

9. A garden has the shape of the annexed figure. Its dimensions are as follows:

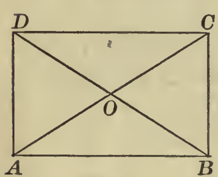


FIG. 55.

Length, $AB = 300$ ft.

Breadth, $BC = 200$ ft.

Draw a plan to the scale 1 : 1,200. Join AC, BD, and find their lengths.

Do AC and BD bisect each other?

10. A man walks 2 miles; then turns to his right, through a right angle, and walks 3 miles; then turns to his left, through a right angle, and walks a mile. Draw a plan, and find his distance from his starting-point (scale, 1 in. = 1 mile).

11. A path goes straight for 3 yds.; then goes at right angles for 8 yds.; then goes straight to the middle point between where it started and where it first turned off. Find its length by drawing to scale.

Lesson 26. Review.

1. Review the italicized exercises in Lessons 21–24.

2. Add together $28° \ 39'$, $37° \ 48' \ 35''$, and $78° \ 9' \ 55''$.

3. Find the difference between the complements of $28° \ 5'$ and $37° \ 27'$.

4. What is the supplement of five times $17° \ 21'$?

5. If seven equal angles are constructed about a point, find the value of each angle correct to the nearest minute.

6. Describe a circle with centre O. Then construct with protractor the angles $AOB = 15°$, $BOC = 37°$, $COD = 50°$, $DOE = 90°$, $EOF = 110°$. Find the value of $\angle FOA$ (1) by the eye; (2) by the protractor; (3) by calculation.

7. Construct with ruler and compasses an angle of 120°.

8. Make two supplementary adjacent angles, and bisect each of them. Then measure the angle formed by the two bisectors.

9. Trisect a right angle (see p. 47, No. **6**).

10. Make any angle BAC, a point P, and a line DE. Then draw through P a line making with DE an angle equal to BAC. (Apply No. **1**, p. 46; No. **4**, p. 19; No. **3**, p. 50.)

11. Construct an angle $BAC = 30°$. From any point D in AC drop $DE \perp$ to AC and meeting AC at E. Measure AD and DE.

12. A man walks 2 miles. He then turns to the right through an angle of 60°, and walks 2 miles. Again he turns to the right through 60°, and walks 2 miles. How far is he now from his starting-point? Construct to scale.

13. A man walks 6 miles, turns through 45° to his left, walks 1 mile, turns through 90° to his left, and walks 2 miles further. How far is he now from home? Construct to scale, and find the answer correct to eighths of a mile.

Lesson 27. Review.

1. Review all the italicized exercises in Chapter III.

2. Draw $AB = 50^{mm}$; erect at a point C, 1^{cm} from B, a perpendicular $CD = 40^{mm}$; join DB and DA, and find in millimeters the difference between DB and DA.

3. Describe a circle with centre O and radius $= 60^{mm}$. Draw any radius OA, bisect it at C, and let D, E be the points where the line of bisection meets the circumference. Join OD, OE. Measure the length DE, and also measure with the protractor the angles ODE, OED, DOE.

4. Draw $AB = 80^{mm}$. Bisect it at C. Erect perpendiculars at A, B, C. Draw any line cutting these perpendiculars at D, E, F, respectively. Measure AD, BE, CF. Find with protractor the sum of the angles ADE and BED.

5. Draw any line AB. Erect the bisecting perpendicular. Join any two points D, E in this perpendicular to A and B. Measure AD, BD, AE, BE. Is any general truth suggested by your results?

6. Three ships start from the same port, sailing N., E., S., at the rates 6, 8, 10 kilometers an hour, respectively. How far are they from one another at the end of 4 hours? Find the distances correct to half-kilometers (scale, $2^{mm} = 1^{km}$).

7. Two cities are connected by a road which forms a broken line $ABCDE$ (Fig. 56). $AB = 5^{km}$, $BC = 3.75^{km}$, $CD = 3^{km}$, $DE = 4^{km}$; $\angle ABC = 90°$, $\angle BCD = 90°$, $\angle CDE = 135°$. A new road is made, going straight from A to C. Find by drawing to scale the distance from A to E by the new road.

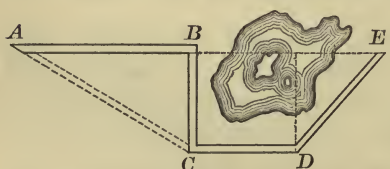


FIG. 56.

CHAPTER IV.

TRIANGLES.

Lesson 28.

1. *Define a triangle, its sides, perimeter, vertices.*

A **triangle** is a plane figure bounded by three straight lines. The straight lines are called its **sides**; their sum, its **perimeter**; their points of intersection, its **vertices**.

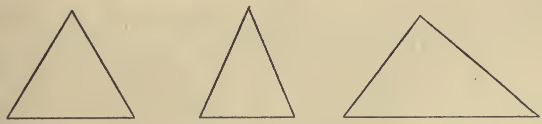
The sides form three angles; the three sides and the three angles are called the **parts** of the triangle.

The sign $\triangle$ stands for "triangle," $\triangle\!\!\!\triangle$ for "triangles."

2. *How are triangles divided with respect to their sides?*

There are three classes; namely:

Equilateral triangles, having all the sides equal.
Isosceles triangles, having two sides equal.
Scalene triangles, having no two sides equal.



EQUILATERAL. ISOSCELES. SCALENE.

3. Draw, freehand, a triangle, and name its six parts, using only the letters placed at the vertices.

4. Draw, freehand, two equilateral triangles, two isosceles triangles, and two scalene triangles.

5. *How are triangles divided with respect to their angles?*

As regards angles there are three classes; namely:

Acute triangles, having all the angles acute.

Obtuse triangles, having *one* obtuse angle.

Right triangles, having *one* right angle.

In a right triangle the side opposite the right angle is called the **hypotenuse,** and the other sides are called the **legs.**

ACUTE. OBTUSE. RIGHT.

6. Draw, freehand, two acute triangles, two obtuse triangles, and two right triangles.

7. *How are the dimensions of a triangle named?*

The two dimensions are named **base** and **altitude.**

In the isosceles triangle the side not equal to the others is called the base, and the equal sides are called the **legs.**

In other kinds of triangles any side may be taken as base.

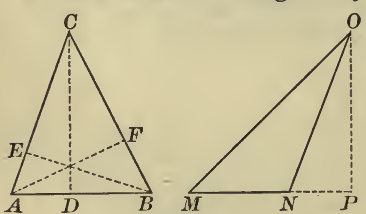


FIG. 57.

In all triangles the altitude is the perpendicular dropped to the base from the opposite vertex.

In Fig. 57 the three altitudes are all drawn: notice that they meet in one point.

In an obtuse triangle two of the altitudes lie outside the figure, and meet, not the base, but the base produced.

Thus, in the obtuse triangle *MNO* (Fig. 57) the altitude *OP* meets the base *MN* produced at *P*.

8. Draw an acute triangle, and construct the altitudes.

9. Draw an obtuse triangle, and construct the altitudes.

10. Draw a right triangle, take the hypotenuse as base, and construct the altitude. What is the altitude if one of the legs is taken as the base?

11. Draw any two lines a and b; then make a triangle having a base equal to a and an altitude equal to b. Can you make more than one such triangle?

12. The perimeter of an isoceles $\triangle = 17$ in. Find (1) the base if one leg $= 5$ in.; (2) each leg if the base $= 5$ in.

13. Draw a $\triangle\ ABC$. Which is the greater, AB or $AC + CB$? Why? BC or $AB + AC$? AC or $AB + BC$? What is true of the sum of two sides of any triangle?

14. Problem.—*To construct a right triangle having given the lengths of the hypotenuse and one leg.*

Let a and b be the given lengths.

Draw AB equal to b.

At B erect a $\perp\ BD$.

With centre A and radius a cut BD at C.

Join AC; then the $\triangle\ ABC$ is the $\triangle$ required.

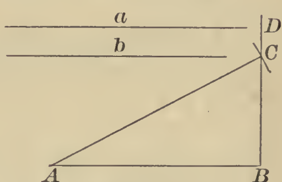


FIG. 58.

15. Construct a right triangle having given the hypotenuse $3\frac{3}{4}$ in., one leg $2\frac{1}{4}$ in. Measure the other leg.

16. Construct right triangles having for legs (1) 3 in. and 4 in.; (2) $1\frac{1}{2}$ in. and 2 in. Measure the hypotenuses. Compare your results.

17. On the ground in front of a wall is a flower-bed 15 ft. wide. It requires a ladder 25 ft. long to reach from the outer edge of the bed to the top of the wall. How high is the wall? Solve by drawing to scale.

Lesson 29.

1. Problem. — *To construct a triangle, having given the lengths of the three sides (Fig. 59).*

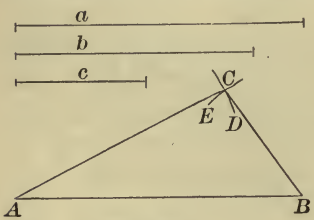


Fig. 59.

Let a, b, c be the lengths.

Draw AB equal to a.

With centre A and radius b describe an arc DC.

With centre B and radius c describe an arc EC, cutting the arc DC at C. Join AC and BC. $\triangle ABC$ is the $\triangle$ required.

2. Construct a triangle, with sides 4 in., $3\frac{1}{4}$ in., $2\frac{1}{8}$ in.

3. Construct a triangle, with sides 4 in., 3 in., 3 in.

4. Construct an isosceles triangle, given: base = 3 in.; one leg = 5 in.

5. Construct an equilateral triangle, having given one side.

6. Construct an equilateral triangle with a perimeter equal to 8 in.

7. Problem. — *To construct a triangle, having given two sides and the included angle (Fig. 60).*

Let a, b, be the sides, and $\angle m$ the given angle (Fig. 60).

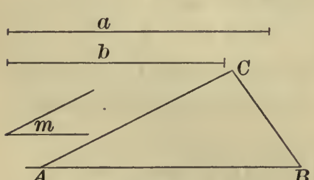


Fig. 60.

Draw a straight line, and at any point A of the line construct an angle equal to m.

Upon its sides lay off $AB = a$, $AC = b$. Join BC. $\triangle ABC$ is the $\triangle$ required.

Note. — In Fig. 60 the auxiliary lines used for constructing the angle are not given. Draw auxiliary lines *dotted* and *very fine.*

8. Construct a triangle, given $AB = 3$ in., $AC = 3\frac{1}{4}$ in., $\angle A = 45°$.

9. Construct an isosceles right triangle, having given one leg.

10. *Explain how the distance between two points, situated on opposite sides of a pond, can be measured.*

Let A and B (Fig. 61) be the points.

First choose a point C from which A and B are both visible; then measure AC, BC, and $\angle ACB$; then construct to scale on paper a triangle acb, making $\angle acb = \angle ACB$, and taking ac of the proper length to represent AC, and bc to represent BC. Measure ab, and multiply it by the reducing factor you have used. The product will be the distance from A to B.

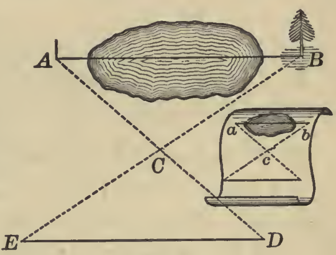


FIG. 61.

Instead of constructing to scale on paper, if the ground is level and clear, we may produce AC to D, making $CD = CA$, and BC to E, making $CE = CB$. Then measure ED. The distance from E to D will be the same as the distance from A to B.

11. Suppose your notes of measurement made to find AB (Fig. 61) to be: $AC = 1200$ yds., $BC = 1500$ yds., $\angle ACB = 110°$. Find the distance from A to B.

12. Two villages A and B are separated by a river. There is a straight road going from each village to a bridge over the river at C. If $AC = 5$ miles, $BC = 6$ miles, $\angle ACB = 30°$, find how far A is from B.

13. Newton is 7 miles west of Boston; Arlington is 6 miles northwest of Boston. How far is Newton from Arlington?

Lesson 30.

1. Problem.—*To construct a triangle, having given one side and the two adjacent angles (Fig. 62).*

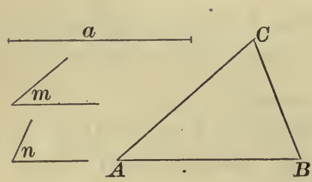


Fig. 62.

Let a be the given side; m and n the given angles.

Draw AB equal to a.

At A construct $\angle BAC = m$.

At B construct $\angle ABC = n$.

Produce the lines AC and BC till they meet at C.

$\triangle ABC$ is the $\triangle$ required.

2. Construct a triangle, given $AB = 3$ in., $BAC = 40°$, $ABC = 80°$.

3. *Define angle of elevation and angle of depression.* If a line is drawn from a point to the eye of an observer, the angle which this line makes with the horizontal plane passing through the eye is the **angle of elevation** or the **angle of depression** of the point, according as the point is *above* or *below* the horizontal plane.

For example (Fig. 63): if the eye is at C, the angle of elevation of A is the angle ACB; if the eye is at A, the angle of depression of C is the angle DAC.

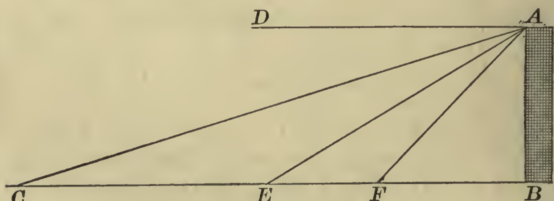


Fig. 63.

4. What is the angle of depression of E as seen from A?

5. What pairs of angles in Fig. 63 are equal? Why?

6. Construct a right △, given one leg $= 2\frac{1}{4}$ in., one angle $= 50°$.

7. Construct a right △, given hypotenuse $= 5$ in., one angle $= 50°$.

8. Construct an isoceles △, given the base, and angle at the base.

9. Find the height of a church spire, if the angle of elevation of the top, 100 yds. from the bottom, is 45°.

10. At a distance of 150 ft. from the foot of a tree, I find the angle of elevation of the top to be 30°. How high is the tree?

11. Wishing to find the height AB of a tower (Fig. 63), I observe the angle of elevation of the top at a point E, and find it to be 30°. At a point F, 120 ft. nearer the tower, I find that the angle of elevation of A is 45°. What is the height of the tower?

12. Wishing to find the breadth AB of a river, I measure on a line ⊥ to AB along the bank a distance of 400 ft., and find that $\angle ACB = 70°$. What is the breadth of the river?

13. A lighthouse A stands on an island. To find its distance from the shore a man runs a base line BC equal to 1000 yds. along the shore, and measures the angles ABC, ACB. He finds $\angle ABC = 80°$, $\angle ACB = 70°$. Find the distance from B to the lighthouse.

NOTE.—The learner who has carefully worked out the exercises in Lesson 29 and in the present lesson will be able to understand what is meant by the *indirect* measurement of a line. To measure any considerable distance by yard-stick, chain, or other direct means, would be a very laborious process. In many cases, also, direct measurement is practically impossible on account of obstacles, such as water, swamps, etc. In such cases, Geometry teaches us how to measure the line *indirectly*, by treating it as a part of a triangle which we can construct after we have found the values of three other parts by direct measurement.

Lesson 31.

1. *How can it be shown to the eye that the sum of the angles of a triangle is equal to a straight angle, or 180°?*

Draw a triangle ABC (Fig. 64) and the altitude CD. Cut out the triangle, and fold the corners over the dotted lines till they come together at D. The three angles of the triangle will just make a straight angle at D.

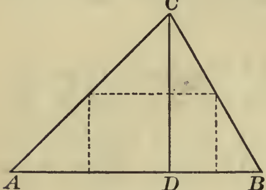


FIG. 64.

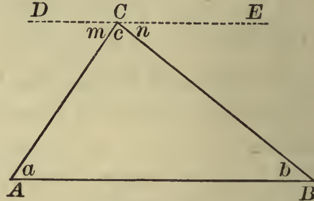


FIG. 65.

2. Prove *that the sum of the angles of every triangle is equal to 180°.*

Draw any triangle ABC (Fig. 65); let a, b, c, denote its angles.

We wish to prove that $a + b + c = 180°$.

Through C draw $DE \parallel$ to AB. Let m denote the angle DCA, and n the angle ECB.

The $\angle$s a and m are equal, because they are alt.-int. angles.

The $\angle$s b and n are equal for the same reason.

Therefore, $a + b + c = m + n + c$. (Ax. 2.)

But $m + n + c =$ a straight angle, or 180°.

Therefore, $a + b + c = 180°$. (Ax. 1.)

This reasoning applies equally well to any triangle; therefore the sum of the angles of every triangle is equal to 180°.

3. *State three truths which follow immediately from what has just been proved.*

(1) *If two angles of a triangle are equal, respectively, to two angles of another triangle, then the third angles must also be equal.*

For the third angle in each case is found by subtracting the sum of the other two from 180°; and the sum of the other two in each case is the same (Ax. 2); and if equals be taken from equals, the remainders will be equal (Ax. 3).

(2) *A triangle can have only one right angle or one obtuse angle.*

For two right angles would make 180°, and two obtuse angles would make more than 180°.

(3) *The two acute angles of a right triangle are complementary; that is, their sum is always* 90°.

For their sum is found by subtracting the right angle, or 90°, from 180°, and $180° - 90° = 90°$.

4. If one angle of a triangle is 45°, what is the sum of the other two angles?

5. If two angles of a triangle are 37° and 75°, find the third angle.

6. If two angles of a triangle are $64° 47' 33''$ and $77° 18' 41''$, find the third angle.

7. If one of the acute angles of a right triangle is 30°, what is the other? If it is $27° 5'$, what is the other?

8. If the acute angles of a right triangle are equal, what is the value of each?

9. Find the value of each angle in an equiangular triangle.

10. Draw a triangle (the larger the better), measure its angles, and find their sum. If the sum is not exactly 180°, what is the reason?

NOTE TO THE TEACHER.—The pupils should work No. 10 together, and the results should be put on the board and compared with 180°.

Lesson 32.

1. *Compare two triangles with respect to size and shape.*

There are four cases, as follows:

(1) Two triangles may have the same size (magnitude).

(2) They may have the same shape (form).

(3) They may agree both in size and shape.

(4) They may differ both in size and shape.

For example (Fig. 66): the triangles ABC and DEF have the same size; ABC and MNO have the same shape; ABC and PQR agree both in size and shape; and DEF and MNO differ both in size and shape.

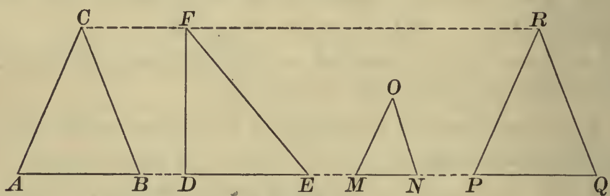


Fig. 66.

2. *Define equivalent triangles; similar triangles; equal triangles.*

Two triangles are **equivalent,** if they have the same size.

Two triangles are **similar,** if they have the same shape.

Two triangles are **equal,** if they agree in size and shape.

3. *To what extent do these definitions apply to other magnitudes than triangles?*

They apply, in general, to lines, to surfaces, and to solids.

Thus, a curved line may have the same length as a straight line; a field bounded by curved lines may enclose the same extent as a field bounded by straight lines; a cubical vessel may hold the same quantity of water as a vessel shaped like a cylinder. In all these cases, the *size* is the same, the *shape* different. They are examples of *equivalent* magnitudes.

Again, two straight lines have the same shape whatever be their lengths; so have two circles, two cubes, or two spheres, however much they differ in size. They are *similar* magnitudes.

If two straight lines have the same length they are *equal* magnitudes; so also are two circles, or two spheres, which have the same size.

Of these three kinds of agreement, — equivalence, similarity, and equality, — the last is the simplest, and should be studied first.

4. *The study of equal triangles raises two questions. What is the first question, and what is the answer?*

If we know that two triangles are equal, what can we infer respecting their sides and their angles?

The answer is obvious. Since two equal triangles agree in size and in shape, they can differ only in *position;* and if placed one upon the other, they must coincide in all their six parts. In other words, their sides and angles must be equal, each to each. In the equal $\triangle$ ABC and PQR (Fig. 66),

$$AB = PQ, \qquad AC = PR, \qquad BC = QR;$$
$$\angle ACB = \angle PRQ, \quad \angle ABC = \angle PQR, \quad \angle BAC = \angle QPR.$$

In any two equal triangles, *the sides opposite two equal angles are equal;* two such sides are called **homologous** sides.

5. Draw, freehand, two equal triangles. Name the equal angles, pair by pair. Name the equal or homologous sides.

6. *What is the other question about equal triangles?*

How many, and what parts, of two triangles must be known to be equal, in order that we may conclude, without error, that the triangles themselves are equal?

To answer this question, special cases must be examined.

Make two *unequal* triangles:

7. Each having a given length for one side.

8. Each having two given lengths for two sides.

9. Each having a given angle.

10. Each having two given angles. What is true of the remaining pair of angles? Why?

11. Each having a given length for one side, and also a given angle adjacent to this side.

12. Each having a given length for one side, and a given angle opposite to this side. Make the angle first.

13. Is a triangle determined if one part is given? If two parts are given? If the three angles are given?

14. Theorem.—*Two triangles are equal, if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.*

HYPOTHESIS.

$\triangle$s ABC and DEF,

$AB = DE$, $AC = DF$,

$\angle BAC = \angle EDF$.

CONCLUSION.

$\triangle ABC = \triangle DEF$.

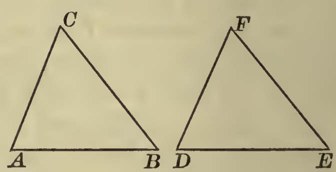


FIG. 67.

PROOF. We may divide the proof into five steps.

(1) Imagine $\triangle ABC$ placed on $\triangle DEF$ so that A falls on D, and the equal angles coincide.

(2) Then E will fall on B, and F on C. Why?

(3) Therefore EF will coincide with BC. Why?

(4) Therefore the two triangles will coincide. Why?

(5) Therefore $\triangle ABC = \triangle DEF$.

Since this reasoning applies equally well to any two triangles, the theorem is proved.

15. Name all the equal parts in the $\triangle$s ABC and DEF.

Lesson 33.

1. *What is a theorem, and what are its parts?*

A **theorem** is a statement to be proved.

A theorem may be separated into two parts:

(1) The **hypothesis,** or what is assumed as true.

(2) The **conclusion,** or what is to be proved.

The **proof** of a theorem is the chain of reasoning by which the conclusion is drawn from the hypothesis.

2. Theorem.—*Two triangles are equal if a side and the two adjacent angles of one are equal, respectively, to a side and the two adjacent angles of the other.*

HYPOTHESIS.

Two $\triangle$s ABC, DEF.

$AB = DE$.

$\angle BAC = \angle EDF$.

$\angle ABC = \angle DEF$.

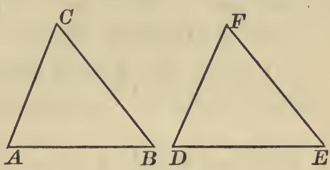


FIG. 68.

CONCLUSION. $\triangle ABC = \triangle DEF$.

PROOF. Use the method of superposition.

(1) Place $\triangle ABC$ on DEF so that the equal parts coincide. Mention these parts, pair by pair.

(2) AC will coincide *in direction* with DF. Why?

(3) BC will coincide in direction with EF. Why?

(4) Therefore C, which is common to AC and BC, must fall on F, which is common to DF and EF.

(5) Therefore the triangles will coincide.

(6) Therefore $\triangle ABC = \triangle DEF$.

The theorem is now proved, because the same reasoning will hold good for any two triangles that can be drawn.

3. Name all the equal parts in the $\triangle$s ABC and DEF.

4. Theorem.—*In an isosceles triangle the angles opposite the equal sides are equal.*

HYPOTHESIS.

ABC an isosceles triangle.

AC and BC the equal sides.

CONCLUSION.

$\angle ABC = \angle BAC.$

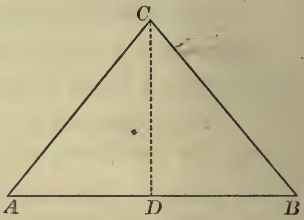


FIG. 69.

PROOF. Successive steps:

(1) Draw CD so as to bisect $\angle ACB$.

(2) Prove that $\triangle DBC = \triangle DAC$. They have three parts equal, each to each. CD is common. $BC = AC$ by hypothesis; $\angle BCD = \angle ACD$ by construction. Why, then, are the two triangles equal?

(3) The two triangles being equal, their six parts are equal, pair by pair. Therefore $\angle ABC = \angle BAC$.

5. *What method of proof is made use of in proving the above theorem?*

The method of equal triangles. This method may often be employed for the purpose of proving two lines equal, or two angles equal. We begin by finding or making two triangles in which the two lines, or the two angles, appear as corresponding parts. We then prove these triangles to be equal. Then it follows immediately that the two lines, or the two angles, must also be equal.

6. *Apply this theorem to an equilateral triangle.*

In an equilateral triangle the angles, taken in pairs, must be equal, because in each case they are opposite equal sides. Therefore all three angles are equal.

An equilateral triangle is also equiangular.

7. Theorem.—*If two angles of a triangle are equal, the opposite sides are also equal.*

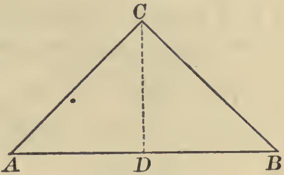


FIG. 70.

HYPOTHESIS.

ABC a triangle.

$\angle ABC = \angle BAC$.

CONCLUSION. $AC = BC$.

PROOF. Successive steps:

(1) Draw CD so as to bisect $\angle ACB$.

(2) Prove that $\angle ADC = \angle BDC$ (see p. 63, No. **3**).

(3) Prove that $\triangle ADC = \triangle BDC$. Give reasons in full.

(4) Therefore $AC = BC$, and the triangle is isosceles.

What method of proof is here employed?

8. *Compare the theorems in Nos. 4 and 7.*

The hypothesis of one is the conclusion of the other.

When two theorems are thus related, either one is called the **converse** of the other.

9. *How is the converse of a theorem formed?*

By changing the hypothesis to the conclusion, and the conclusion to the hypothesis.

10. *Apply the theorem of No. 7 to a triangle, all of whose angles are equal; that is to say, an equiangular triangle.*

If all the angles are equal, then the sides taken in pairs must be equal, because each pair stands opposite equal angles. Therefore all the sides are equal.

An equiangular triangle is also equilateral.

11. What is the value in degrees of each angle of an equilateral triangle?

Lesson 34.

1. Theorem.—*Two triangles are equal if the sides of one are equal, respectively, to those of the other.*

HYPOTHESIS.

In the $\triangle\triangle$ ABC, DEF,

$AB = DE$,

$AC = DF$,

$BC = EF$.

CONCLUSION.

$\triangle ABC = \triangle DEF$.

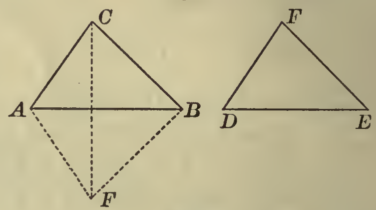


FIG. 71.

PROOF. Successive steps:

(1) Place $\triangle DEF$ so that the longest side DE coincides with AB, and F and C lie on opposite sides of AB.

(2) Join CF. What two isosceles $\triangle\triangle$ are thus formed?

(3) Prove that $\angle ACF = \angle AFC$ (p. 68, No. **4**).

(4) Prove that $\angle BCF = \angle BFC$.

(5) Prove that $\angle ACB = \angle AFB$.

(6) Prove that $\triangle ACB = \triangle AFB$ (p. 66, No. **14**).

(7) Therefore $\triangle ACB = \triangle DEF$.

2. Theorem.—*Two right triangles are equal if the hypotenuse and a leg of one are equal, respectively, to the hypotenuse and a leg of the other.*

HYPOTHESIS.

Two right $\triangle\triangle$ ABC, DEF;

$AC = EF$,

$BC = DF$.

CONCLUSION.

$ABC = \triangle DEF$.

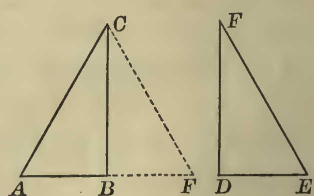


FIG. 72.

PROOF. Successive steps:

(1) Place $\triangle DEF$ so that the leg DF coincides with BC, and A and F are on opposite sides of BC.

What isosceles $\triangle$ is thus formed? Why isosceles?

(2) Show that $\angle ACB = \angle FCB$ (p. 63, No. **3**).

(3) And that $\triangle ABC = \triangle FBC$;

(4) That is, $\triangle ABC = \triangle DEF$.

3. Theorem.—*A perpendicular is the shortest distance from a point to a straight line.*

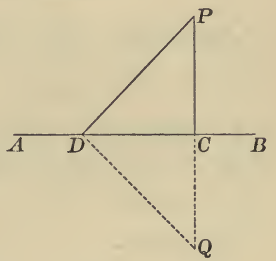


FIG. 73.

HYPOTHESIS.

A straight line AB.

A point P, not in AB.

$PC \perp$ to AB.

PD *any* other line drawn from P to AB.

CONCLUSION. $PC < PD$.

PROOF. Successive steps:

(1) Produce PC, making $CQ = PC$, and join DQ.

(2) Prove that $\triangle PDC = \triangle QDC$. Give all the reasons. What two parts are equal by construction?

(3) Then show that $PQ < PD + DQ$ (p. 23, Ax. 9).

(4) Hence show that $2\,PC < 2\,PD$.

(5) And therefore $PC < PD$.

4. *Define the distance from a point to a straight line.*

By the **distance** from a point to a line is meant the *shortest* distance; therefore the perpendicular dropped from the point to the line.

Lesson 35.

1. Theorem. — *Every point in the perpendicular which bisects a straight line is equidistant from the ends of the line.*

Hypothesis.

D the middle point of AB.
$CD \perp$ to AB.
P any point in CD.

Conclusion. $PA = PB$.

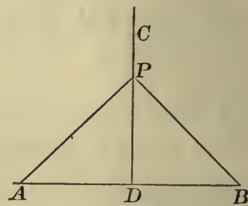


Fig. 74.

Proof. Prove that $\triangle ADP = \triangle BDP$. What follows?

2. Theorem. — *Every point equidistant from the ends of a straight line lies in the perpendicular which bisects the line.*

Hypothesis.

D the middle point of AB.
P any point such that $PA = PB$.

Conclusion. PD is $\perp$ to AB.

Proof. Prove that $\triangle ADP = \triangle BDP$.

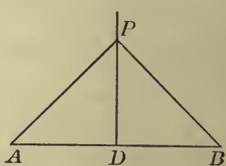


Fig. 75.

What follows with respect to the angles ADP and BDP? Then show that $\angle ADP = 90°$ (see p. 48, No. **2**).

3. *What truth follows from the last theorem?*

Since two points determine a straight line, *any two points equidistant from the ends of a straight line determine the perpendicular which bisects the line.*

This perpendicular is called the **bisecting perpendicular.**

4. How are the two theorems on this page related?

5. By what method are both theorems proved?

6. Theorem.—*Every point in the bisector of an angle is equidistant from the sides of the angle.*

HYPOTHESIS.

AD the bisector of $\angle BAC$.

P any point in AD.

CONCLUSION.

P is equally distant from AB and from AC.

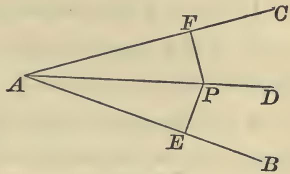


FIG. 76.

PROOF. Successive steps:

(1) Draw $PE \perp$ to AB, $PF \perp$ to AC.

(2) Prove that $\triangle APE = \triangle APF$ (see p. 63, No. 3).

(3) What follows?

7. Theorem.—*Every point within an angle, and equidistant from its sides, lies in the bisector of the angle.*

HYPOTHESIS.

P any point within $\angle BAC$ equidistant from AB and AC.

CONCLUSION.

PA bisects $\angle BAC$.

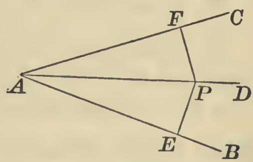


FIG. 77.

PROOF. Successive steps:

(1) Draw $PE \perp$ to AB, $PF \perp$ to AC. Join PA.

(2) Prove that $\triangle APE = \triangle APF$.

(3) What follows?

8. *What truth follows from the last theorem?*

One point equidistant from the sides of an angle determines the bisector of the angle.

Why is *one* point in this case sufficient?

9. What relation is there between the theorems given on this page? Why?

Lesson 36. Review.

1. Review the italicized exercises in Lessons 28–31.

2. Upon a given length as base, how many isosceles triangles can be constructed? How many equilateral triangles?

3. If the acute angles of a right triangle are 30° and 60°, what angle is formed by their bisectors?

4. If one angle of a triangle is 40°, and the other two angles are equal, what are their values?

5. Find the angles of a triangle, if the second angle is twice the first, and the third three times the second.

6. Draw a $\triangle ABC$, produce AB to any point D, and show that $\angle DBC = \angle BAC + \angle ACB$.

7. Show that a $\triangle$ with 1, 2, 3 for sides cannot be made.

8. Show that a $\triangle$ with 1, 2, 4 for sides cannot be made.

9. Two towns, A and B, are 10 miles apart. A place is described as 7 miles from A, and 5 miles from B. Draw a plan showing the situation of C. Is its position determined?

10. Construct a right triangle having one acute angle 20° larger than the other, and hypotenuse $= 4\frac{1}{4}$ in.

11. Construct a $\triangle ABC$; given: $AB = 2$ in., $\angle A = 30°$, $\angle C = 45°$.

12. Construct a $\triangle ABC$; given: $AB = 5$ in., $BC = 3\frac{1}{2}$ in., $\angle BAC = 30°$, and show that there are two solutions.

13. How high is a tower if the angle of elevation of the top, 300 yds. from the base, is 20°?

14. A, B, C, are the corners of a triangular field. A is 80 ft. west of B, and 800 ft. southwest of C. Find the cost of fencing the field at 50 cents a foot.

Lesson 37. Review.

1. Review the italicized exercises in Lessons 32–35.

2. If one side of an equilateral triangle is produced, what is the value of the exterior angle thereby formed?

3. Find the values of the angles at the base of an isosceles triangle if the angle at the vertex is 24°.

4. Find the angles of an isosceles triangle if each angle at the base is double the angle at the vertex.

5. What are the angles of an isosceles right triangle?

6. Can you devise an easy way to find the height of a tree by means of an isosceles right triangle?

7. Construct an isosceles triangle, having given the base and the angle at the vertex. First find, by construction, the base angles.

8. Construct the $\triangle ABC$, given $AB = 2$ in., $AC = 3$ in., $BC = 4$ in. Then find a point D which is 2 in. from C and equidistant from A and B.

9. Prove that the lines joining the middle points of the sides of an equilateral triangle divide the triangle into four equal equilateral triangles.

10. Prove that the altitude of an isosceles triangle bisects the base and the angle at the vertex.

11. A monument AB on a horizontal plane subtends the angle 30° when seen from C (that is, $\angle ACB = 30°$); 150 ft. nearer, the angle it subtends is doubled. Find the height of the monument.

12. A man 75 ft. from the bank of a stream finds that a tree on the opposite bank subtends an angle of 45°. At the bank, this angle is 60°. How high is the tree?

Lesson 38. Review.

1. Prove that the construction in Fig. 26, on page 19, is correct; that is, prove that it follows from the construction that CF is $\parallel$ to AB. For this purpose, show that $\triangle DCF = \triangle DCE$, and then apply No. **5**, p. 51.

Prove the following constructions correct. In each case first state the problem; then describe how the construction was made; then state what must be proved; then prove it.

2. Fig. 33, page 25.

3. Fig. 40, page 40.

4. Fig. 43, page 42.

5. Fig. 48, page 46.

6. Fig. 49, page 46.

7. Fig. 50, page 47.

8. Draw a $\triangle ABC$ and produce each side in both directions. Name with letters all the angles that are formed, and find their values, if $\angle ABC = 70°$ and $\angle BAC = 35°$.

9. *In general, three parts determine a triangle.* Can you think of an exception to this rule?

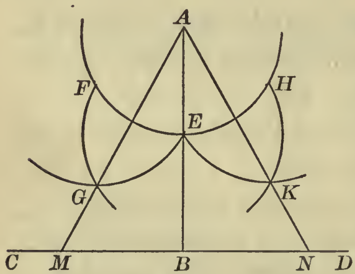


FIG. 78.

10. Find a point in a given straight line AB which is equidistant from two given points C and D. Prove that your construction is correct.

11. In Fig. 78, AB is given, and an equilateral triangle is constructed, having AB for altitude. The construction is quite obvious from the figure. CD is drawn $\perp$ to AB; all the arcs have the same radius.

Make the construction, and prove it to be correct.

Lesson 39. Review.

1. Through each vertex of a triangle a line parallel to the opposite side is drawn. Prove that the triangle formed by these lines is equal to four times the given triangle.

2. Find the values of all the angles of an isosceles triangle, if the angle formed by producing the base is 130°.

3. In a $\triangle ABC$ the side AB is produced to any point D. Prove that the $\angle CBD = \angle BAC + \angle ACB$.

4. Construct an isosceles right triangle whose hypotenuse shall be 6^{cm} long.

5. Describe the position of all points which are situated 2^{cm} from a given point A; 3^{cm} from a given point B; 2^{cm} from A, *and* 3^{cm} from B.

6. A surveyor erects two flagstaffs, A, B, on opposite sides of a pond, chooses a favorable point C, and makes the following measurements: $AC = 660^{\text{m}}$, $BC = 1380^{\text{m}}$, $\angle C = 45°$. Find the distance AB.

7. In a triangular field ABC a man measures $AB = 600^{\text{m}}$, $\angle B = 70°$. He then walks over BC without measuring it. He finds, however, that $\angle C = 45°$. Draw a plan, and find BC.

8. Two roads meet at A, forming the angle 60°. How far apart are two houses, one of which is situated 700^{m} from A along one road, and the other 900^{m} from A along the other road?

9. The distances between three towns A, B, C are: $AB = 9^{\text{km}}$, $AC = 7^{\text{km}}$, $BC = 6^{\text{km}}$. A straight railroad runs from A to B. It is desired to have a station placed as near as possible to C. Draw a plan, and find at what distance from A the station should be built.

Lesson 40. Review.

1. Review all the italicized exercises in this chapter.

2. Find in one side of a triangle a point equidistant from the other sides. Prove your construction correct.

3. Find a point within a triangle equidistant from the three vertices. Prove your construction correct.

4. Find a point within a triangle equidistant from the three sides. Prove your construction correct.

5. Three towns, A, B, C, are situated at the vertices of an isosceles triangle, whose base is the line BC. $\angle ABC = 30°$. The distance from A to the base BC is 6^{km}. Find the distances between the towns.

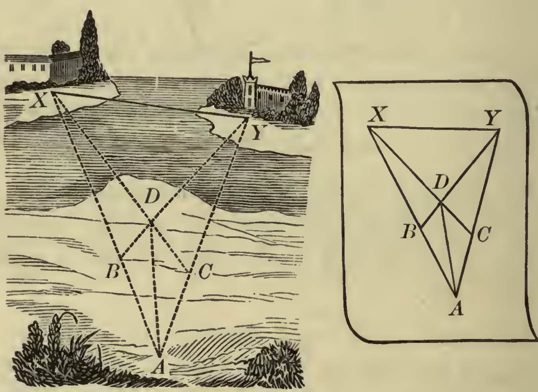


Fig. 79.

6. In order to find the distance XY (Fig. 79), without any means for measuring angles, I chose four points, A, B, C, D, and measured their distances with the following results:

$AB = 140^m$, $AD = 180^m$, $AC = 115^m$, $BD = 65^m$, $CD = 85^m$. Draw a plan, and find XY.

CHAPTER V.

POLYGONS.

Lesson 41.

1. *Define a polygon: its sides; perimeter; vertices.*

A **polygon** is a plane figure bounded by straight lines. The bounding lines are called its **sides**; their sum, its **perimeter**; their intersections, its **vertices.**

2. *Name the most important polygons.*

Polygons are named according to the number of sides. A **triangle** has three sides; a **quadrilateral,** four sides; a **pentagon,** five sides; a **hexagon,** six sides; an **octagon,** eight sides; a **decagon,** ten sides; a **dodecagon,** twelve sides.

3. *Define a diagonal, and explain how a polygon may be divided into triangles by drawing diagonals.*

A **diagonal** is a straight line joining two vertices not on the same side. A polygon may be divided into triangles by drawing all the diagonals possible from one vertex. The number of diagonals that can be drawn from a vertex is always *three* less than the number of sides; and the number of triangles into which the polygon is divided is always *two* less than the number of sides.

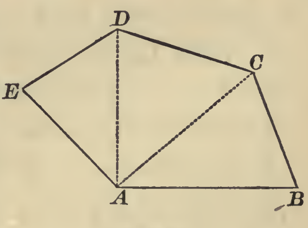


FIG. 80.

4. Theorem.—*The sum of the angles of a polygon is equal to 180° × the number of sides less two.*

We prove this theorem for a hexagon as follows:

HYPOTHESIS.

A hexagon *ABCDEF*.

CONCLUSION.

The sum of the ∠s of the hexagon is equal to 180° × 4, or 720°.

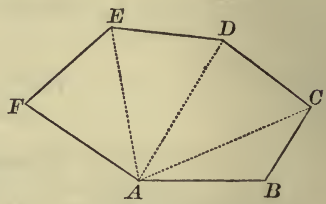


FIG. 81.

PROOF. Divide the hexagon into 6 − 2, or 4, triangles, by drawing the diagonals *AC*, *AD*, *AE*.

Obviously, the sum of the ∠s of the hexagon is equal to the sum of the ∠s of these triangles.

The sum of the ∠s of the △s = 180° × 4. Why?

Therefore the sum of the ∠s of the hexagon = 180° × 4.

Now suppose that in place of 6 sides the polygon has any number n of sides; then there would be $n - 2$ triangles in place of 4; and, therefore, the sum of the ∠s of the polygon would be $180° \times (n - 2)$.

5. Prove, without assuming the above theorem to be true, that the sum of the angles of a quadrilateral is equal to 360°.

6. If the angles of a quadrilateral are all equal, what is the value of each one?

7. Three angles of a quadrilateral are 40°, 70°, and 120°. What is the value of the other angle?

8. Find the sum of the angles of a pentagon.

9. Find the sum of the angles of an octagon.

10. Find the sum of the angles of a decagon.

11. Find the sum of the angles of a dodecagon.

11. *Classify quadrilaterals.*

Quadrilaterals are divided into three classes: parallelograms, trapezoids, and trapeziums.

Parallelograms have their opposite sides parallel.

Trapezoids have only two sides parallel.

Trapeziums have no parallel sides.

PARALLELOGRAM. TRAPEZOID. TRAPEZIUM.

12. *Classify parallelograms.*

There are four kinds of parallelograms: the square, the rectangle, the rhombus, and the rhomboid.

A **square** has four equal sides and four right angles.

A **rectangle** has unequal adjacent sides and four right angles.

A **rhombus** has four equal sides and no right angles.

A **rhomboid** has unequal adjacent sides, and no right angles.

SQUARE. RECTANGLE. RHOMBUS. RHOMBOID.

13. Make, freehand, a parallelogram; a trapezoid; a trapezium.

14. Make, freehand, a trapezium having two right angles.

15. Give examples of parallelograms. What kind is each?

16. Make, freehand, a square; a rhombus; a rectangle; and a rhomboid.

Lesson 42.

NOTE.—In working the exercises in this lesson, construct perpendiculars and parallels with the ruler and the triangle, and angles with the protractor.

1. *How are the dimensions of a parallelogram named?*

They are called **base** and **altitude**. Any side may be taken as the base, and then the altitude will be the perpendicular dropped to the base from any point of the opposite side. Thus, in Fig. 82, AB is the base, DE the altitude.

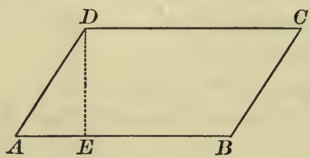


FIG. 82.

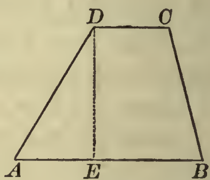


FIG. 83.

2. *How are the dimensions of a trapezoid named?*

They are also named base and altitude. One of the parallel sides is always taken as the base, and the altitude is the perpendicular drawn from one parallel side to the other. Thus, in the trapezoid $ABCD$ (Fig. 83) DE is the altitude.

3. *How are the sides of a trapezoid named?*

The parallel sides are called the **bases**, and the other two sides are called the **legs**.

4. Construct a parallelogram with a base of 3 in. and an altitude of 2 in.; also a trapezoid with the same dimensions.

5. Draw a triangle ABC. Through D, any point in AC, draw $DE \parallel$ to AB, and $DF \parallel$ to BC. What kind of a figure is $ABED$? Why? What kind of a figure is $BEDF$? Why?

6. Construct a square having a side 2 in. long. Draw its diagonals. Measure their lengths and the angle between them.

7. Construct a rhombus with each side 2 in. long, and one angle equal to 60°. Draw its diagonals. Measure their lengths and the angle between them.

8. Construct a rectangle having for two adjacent sides $1\frac{1}{2}$ in. and 2 in. Draw its diagonals. Measure their lengths and the angle between them.

9. Construct a rhomboid $ABCD$, having given $AB = 3\frac{1}{4}$ in., $BC = 1\frac{7}{8}$ in., $\angle ABC = 55°$. Find the values of the other three angles.

10. Draw a line AB; upon one side construct an equilateral $\triangle ABC$, upon the other side a square $ABDE$. Join CD, CE. Prove that $\triangle CDE$ is isosceles.

11. Construct two rectangles, one within the other, and make their adjacent sides everywhere $\frac{1}{8}$ of an inch apart.

12. Construct two squares. Divide one into four equal squares, and the other into four equal rectangles.

13. Construct Fig. 84 (scale, 2 : 1). Then erase the dotted lines, and shade the portions of surface at the corners and at the middle of the sides.

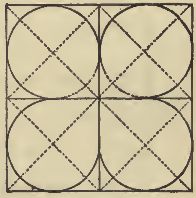


Fig. 84.

14. Through a point O draw two perpendicular lines. With centre O and any radius, describe a circle, cutting the lines in A, B, C, D. Join AB, BC, CD, DA.

What kind of a figure is $ABCD$? Can you prove your answer to be correct?

15. A man walks 5 miles due north, then 3 miles northwest, then 4 miles due south, and then he goes straight home. What figure does his path enclose?

Find, by drawing to scale, how far he has walked.

Lesson 43.

1. Theorem.—*The opposite angles of a parallelogram are equal.*

Hypothesis.

A parallelogram $ABCD$.

Also let its angles be denoted by a, b, c, d, as shown in the figure.

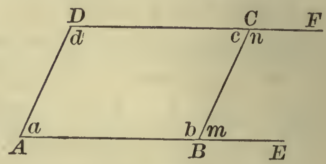


Fig. 85.

Conclusion. $a = c$, $b = d$.

Proof. Produce AB to E, DC to F, and let

$$\angle CBE = m, \ \angle BCF = n.$$

We know that $a = m$ (p. 50, No. **3**).

And that $c = m$ (p. 51, No. **4**).

Therefore $a = c$ (p. 23, Ax. 1).

Also, we know that $d = n$. Why?

And that $b = n$. Why?

Therefore $b = d$. Why?

2. Theorem.—*The opposite sides of a parallelogram are equal.*

What is the hypothesis?

What is the conclusion?

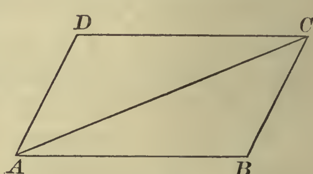


Fig. 86.

Proof. Successive steps:

(1) Draw the diagonal AC.

(2) Prove that $\triangle ABC = \triangle ADC$. What follows?

3. *What truths follow immediately from No. 2?*

(1) A diagonal divides a parallelogram into two equal $\triangle$s.

(2) Parallels between parallels are equal.

4. Theorem. — *Two parallels are everywhere equidistant.*

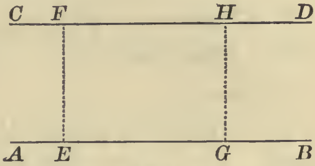


FIG. 87.

HYPOTHESIS.
AB is $\parallel$ to CD.
E, G, any points in AB.
EF is $\perp$ to AB.
GH is $\perp$ to AB.

CONCLUSION. $EF = GH$.

PROOF. $\angle FEG = \angle HGB$. Why?
Then EF is $\parallel$ to GH (p. 51, No. 5).
Hence $EF = GH$. Why?

5. One angle of a parallelogram $= 50°$. Find the others.

6. What is the angle which the diagonal of a square forms with one of the sides?

7. In what kinds of parallelograms is the altitude equal to one of the sides?

8. How many trees 10 yds. apart can be set out around a rectangular park 800 yds. long and 480 yds. wide?

9. Upon a line AB as side, construct with ruler and compasses a square. Draw $BC \perp$ to AB; take $BC = AB$; and with centres A and C, and radius AB, describe arcs meeting at D. $ABCD$ is the required square. Prove that it is a square.

10. Construct a square having its perimeter equal to a line MN.

11. Construct a square having its diagonal equal to a line MN.

12. Make a plan of a wall 20 ft. high and 64 ft. long, with a passage-way in the middle 10 ft. high and 16 ft. wide.

13. Make a plan of two vertical posts 18 ft. high and 24 ft. apart, with a cross-bar 10 ft. from the ground.

Lesson 44.

1. Theorem. — *The diagonals of a parallelogram bisect each other.*

HYPOTHESIS.

Let $ABCD$ be a parallelogram, and let the diagonals AC, BD, intersect at O.

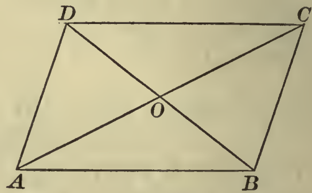


FIG. 88.

CONCLUSION.

$AO = CO$, $BO = DO$.

PROOF. Prove that $\triangle ABO = \triangle DCO$, and keep in mind that in equal triangles equal sides are opposite equal angles.

2. Prove that the diagonals of a rectangle are equal. Begin by stating the hypothesis (Fig. 89); then state the conclusion; then prove by showing that the triangles ABD, ACD, are equal.

3. Prove that the diagonals of a rhombus are perpendicular to each other (Fig. 90). State hypothesis and conclusion, and prove by showing that $\triangle AOB = \triangle AOD$, etc.

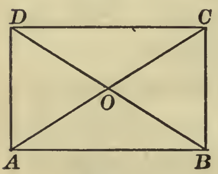


FIG. 89.

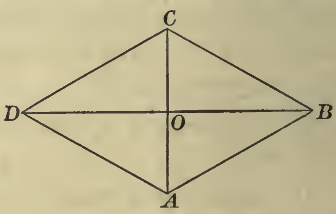


FIG. 90.

4. Prove that the construction, on page 25, for dividing a line into equal parts, is a correct one.

Draw CM, DN, etc., ∥ to AB, prove the ⨻ AHC, CMD, etc., equal, and apply No. **3**, p. 84, and Ax. 1, p. 23.

5. Problem.—*To construct a parallelogram, having given two adjacent sides and the included angle.*

CONSTRUCTION. Instruments: ruler and compasses.

Let a and b be the given sides; m the given angle.

Make $\angle BAD = m$.

Lay off $AB = a$, $AD = b$.

With centre D and radius a describe an arc EF.

With centre B and radius b describe an arc cutting EF at C.

Join BC and DC.

$ABCD$ is the figure required.

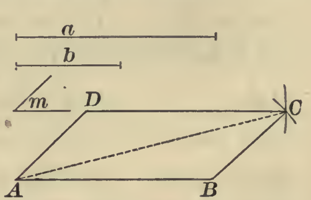


FIG. 91.

PROOF. (1) The figure obviously has the three given parts.

(2) Join AC. $\triangle ABC = \triangle ADC$. Why?

(3) Therefore $\angle BAC = \angle DCA$, and $\angle ACB = \angle CAD$.

(4) Therefore AB is $\parallel$ to DC, and BC is $\parallel$ to AD. Why?

(5) Therefore $ABCD$ is a parallelogram. Why?

NOTE.—In the following exercises, use ruler and compasses only. First *make* the construction; then *describe* it; then *prove* it to be correct.

6. Construct a square, having given a side.

7. Construct a square, having given the diagonal.

8. Construct a rectangle, given two adjacent sides.

9. Construct a rectangle, given a side and a diagonal.

10. Construct a rhombus, given the two diagonals.

11. Construct a rhombus, given a side and an angle.

12. Construct a parallelogram, given two sides and a diagonal.

13. Construct a parallelogram, given one side, one angle, and one diagonal.

14. Construct a trapezoid, given the bases, the altitude, and one of the legs.

Lesson 45.

1. Problem. — *To construct a polygon equal to a given polygon ABCDEF (Fig. 92).*

Construction. Instruments: ruler and compasses.

Draw the diagonals AC, AD, AE.

Make	$\triangle MNO = \triangle ABC$ (explain fully how).
Then make	$\triangle MOP = \triangle ACD$.
Also	$\triangle MPQ = \triangle ADE$.
And	$\triangle MQR = \triangle AEF$.

Then the polygon $MNOPQR$ is the polygon required.

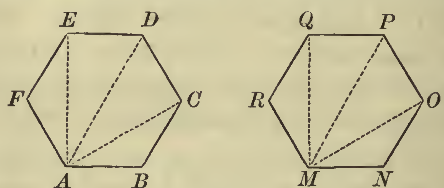


Fig. 92.

Proof. By construction, the $\triangle$s, pair by pair, are equal. They are also, pair by pair, similarly placed.

Therefore the polygon $MNOPQR$ = the polygon $ABCDEF$.

Note. — Use only ruler and compasses for the following exercises. First *make* the construction; then *describe* it; then *prove* it to be correct.

2. Construct a square, given the perimeter.

3. Construct a rectangle, given one side and the angle between the diagonals.

4. Construct a rectangle, given a diagonal and the angle between the diagonals.

5. Construct a rectangle, given one side and the perimeter.

6. Construct a rhombus, given one side and the altitude.

7. Construct a rhombus, given a side and a diagonal.

8. Construct a parallelogram, given the base, the altitude, and one angle.

9. Construct a parallelogram, given the two diagonals and the angle between them.

10. Construct a parallelogram, given the sides, and knowing that one angle is double the other.

11. Construct a trapezoid, given the legs, one base, and the altitude.

12. Construct a trapezoid, given one leg, one of the bases, and the angles at the base.

13. Construct a trapezium, given the four sides and one of the angles.

14. Construct a trapezium, given the four sides and one of the diagonals.

15. Make a pentagon, and then construct an equal pentagon.

16. Make a hexagon, and then construct an equal hexagon.

17. Make an octagon, and then construct an equal octagon.

18. Draw any triangle ABC. Upon AB describe the square $ABEF$, and upon AC the square $ACGH$. Drop a perpendicular from A to BC, and join EC, GB. If the figure is correctly drawn, EC and GB will intersect on the perpendicular.

19. Draw a cross vertical section of a ditch whose depth is 6 ft., breadth at bottom 4 ft., at top 8 ft., and whose sides are equal. Find the length of the side. In this exercise, besides ruler and compasses, the scale must be used.

20. Construct a square, and on the four sides construct equilateral triangles; join their vertices, and show by measurement that the figure so formed is also a square. Can you prove that it is a square?

Lesson 46.

1. *Define an equilateral polygon.*

An **equilateral** polygon is a polygon whose sides are equal.

2. *Define an equiangular polygon.*

An **equiangular** polygon is a polygon whose angles are equal.

3. *Define a regular polygon.*

A **regular** polygon is a polygon which is both equilateral and equiangular.

4. *How is an angle of a regular polygon found?*

First find the sum of all the angles (p. 80, No. 4); then divide this sum by the number of angles; in other words, by the number of sides of the polygon.

5. What are the usual names of the regular triangle and the regular quadrilateral?

6. What is the perimeter of a regular octagon, if one side is 6 in. long?

7. What is the length of one side of a regular decagon, if the perimeter is 35 in.?

8. What quadrilateral is equilateral, but not equiangular? also what quadrilateral is equiangular, but not equilateral?

9. Make a pentagon which shall be equilateral, but not equiangular.

10. Make a hexagon which shall be equiangular, but not equilateral.

11. Find the angle of a regular dodecagon.

12. Find the angles of the regular polygons from three sides to ten sides, and arrange the results in tabular form.

As the number of sides increases, in what way does the angle change?

13. Stone pavements furnish examples of the use of regular polygons. Only those polygons can be used which will fill up all the space about a point.

Show that equilateral triangles may be used, and make a pattern like that in Fig. 93.

14. Make a pattern arranged in squares, like that shown in Fig. 94.

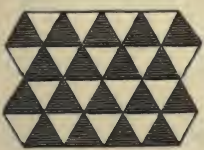


Fig. 93.

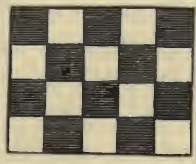


Fig. 94.

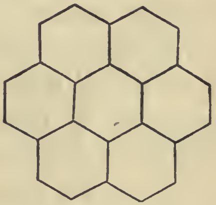


Fig. 95.

15. Show that a pattern can be made of regular hexagons arranged in groups of three about a point, and make a pattern like that in Fig. 95.

16. Make a pattern consisting of a combination of equilateral triangles and hexagons (Fig. 96).

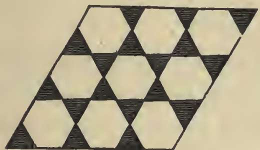


Fig. 96.

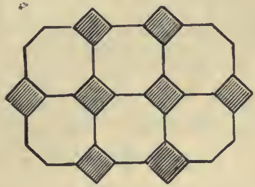


Fig. 97.

17. Show that a pattern cannot be made of regular octagons alone, but that one can be made of regular octagons combined with squares, as shown in Fig. 97.

Lesson 47. Review.

1. Review the italicized exercises in Lessons 41–43.

2. A diagonal divides a parallelogram into two equal triangles. What kind of triangles are they if the parallelogram is a rectangle? a rhombus? a square?

3. The perimeter of a rectangular lot is 360 ft., and the street side is 80 ft. Find the depth of the lot.

4. Make, freehand, a hexagon having as many right angles as possible.

5. Find the sum of the angles of a polygon of 15 sides.

6. Prove that the figure formed by joining the middle points of the sides of a square, taken in order, is also a square.

7. Draw a triangle ABC. Through B and C draw parallels to the opposite sides, meeting at D. What kind of a figure is $ABCD$? Why?

8. A man walks 8 miles east; then $4\frac{1}{2}$ miles southwest; then 8 miles west; then straight home. Draw a plan, and find how far he walked.

9. Construct a trapezoid having an altitude of 1 in., and equal legs each 2 in. long.

10. Construct a quadrilateral $ABCD$, given $AB = AD = 2$ in., $BC = CD = 1$ in., $\angle BAD = 35°$.

11. A man walks 3 miles; turns to his right through 60°, and goes 2 miles; turns again to his right, through 60°, and goes 2 miles; and then goes straight home. Draw a plan, and find how far he has walked.

12. Construct a square $ABCD$, and upon its sides equilateral triangles ABE, BCF, CDG. Join E, F, G, H, taken in order. What kind of a triangle is AEH? What kind of a figure is $EFGH$? Prove your answers.

Lesson 48. Review.

1. Review the italicized exercises in Lessons 44–46.

2. Into what figures is a rhombus divided by drawing both its diagonals?

3. Construct a rhombus, one side of which shall be 2 in. long, and one angle of which shall be double the other.

4. The intersection of the diagonals of a parallelogram is often called its **centre**. Prove that any straight line which can be drawn through the centre will divide the perimeter into two equal parts.

5. Construct a square, given one side and the position of its centre.

6. Construct a trapezoid, given one base, the altitude, and the two diagonals.

7. A stone dam is 20 ft. high, 34 ft. wide at the bottom, and 4 ft. wide at the top. The slant height of each side is the same. Draw to scale a cross section of the dam, and find the slant height.

8. Construct a parallelogram, given one side and the two diagonals.

9. A man walks 5 miles straight; then turns through 30°, and walks 2 miles; then takes a direction *parallel*, but *opposite*, to his first direction. Draw a plan of his route. How far is he from his starting-point?

10. Draw two lines making an angle of 60°, and find a point which shall be 1 in. distant from one line, and 2 in. distant from the other.

11. A certain town is 3 miles from a straight railroad. A man lives 5 miles from the town, and 1 mile from the railroad. Find, by construction, how many locations satisfy these conditions.

Lesson 49. Review.

1. Construct a square, given the diagonal $= 7^{cm}$.

2. Construct a rhombus $ABCD$, given $AB = 5^{cm}$, $\angle ABC = 45°$.

3. Construct a rectangle $ABCD$, given $AB = 4^{cm}$, and the diagonal $AC = 9^{cm}$.

Construct a parallelogram $ABCD$, having given:

4. $AB = 3^{cm}$, $AD = 5^{cm}$, $\angle ABC = 70°$.

5. $AB = 4^{cm}$, $AD = 7^{cm}$, $AC = 9^{cm}$.

6. $AC = 7^{cm}$, $\angle ACD = 30°$, $\angle DAC = 50°$.

7. $AB = 6^{cm}$, $BD = 8^{cm}$, $\angle ABD = 30°$.

8. $AB = 4^{cm}$, $\angle CAB = 80°$, $\angle DAB = 100°$.

Construct a trapezoid $ABCD$ ($AB \parallel$ to CD), having given:

9. $AB = 3^{cm}$, $CD = 5^{cm}$, $AD = 4^{cm}$, $\angle ADC = 70°$.

10. $AB = 5^{cm}$, $AD = 6^{cm}$, $\angle ADC = 130°$, $\angle ABC = 110°$.

11. $BD = 6^{cm}$, $\angle DBC = 60°$, $\angle BDC = 30°$, $\angle ADB = 30°$.

12. $AB = 4^{cm}$, $BD = 5^{cm}$, $BC = 3^{cm}$, $\angle CBD = 80°$.

Construct a quadrilateral $ABCD$, having given:

13. $AB = 3^{cm}$, $AC = 5^{cm}$, $BD = 4^{cm}$, $\angle BAC = \angle ABD = 30°$.

14. $AB = 3^{cm}$, $AC = 4^{cm}$, $BD = 5^{cm}$, $\angle BAC = \angle CBD = 45°$.

15. Draw a plan of a rectangular floor 15^{m} long and 8^{m} wide, and find the distance between the opposite corners.

16. $ABCD$ is a rectangle. $AB = 1200^{m}$, $BC = 600^{m}$. What is the distance from A to the middle point M of the side CD? What kind of a triangle is ABM? Prove your answer.

17. A barn is 12^{m} wide, 8^{m} high to the eaves, and 14^{m} high to the ridgepole. Draw a plan of a vertical cross section. How long are the rafters?

CHAPTER VI.

THE CIRCLE.

Lesson 50.

1. Review the italicized exercises in Lesson 7.

2. *Define an angle at the centre, a sector, a segment, a quadrant, a semicircle.*

An **angle at the centre** is an angle whose vertex is at the centre of a circle. (For example: $\angle AOB$, Fig. 98.)

A **sector** is a portion of a circle bounded by two radii and the intercepted arc.

A **segment** is a portion of a circle bounded by an arc and its chord.

A **quadrant** is a sector whose radii form a right angle.

A **semicircle** is a sector whose radii form a straight angle, or segment whose chord is a diameter of the circle.

3. Define a semi-circumference.

4. Compare, as regards size, a circle, a semicircle, and a quadrant.

5. Draw a circle, and make in it an angle at the centre; a sector; a segment; a quadrant. Also name them with letters.

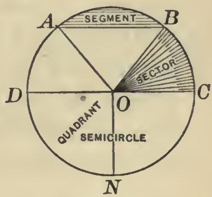


FIG. 98.

6. Into what figures is a sector divided by the chord of its arc?

7. Construct a sector whose angle is 60°.

8. Construct a segment whose chord = radius of the circle.

9. *Define equal circles and equal circumferences.*

Two circles, or two circumferences, are equal if they can be so placed that they coincide.

10. *What truth follows immediately from the definitions which have been given?*

If two circles have equal radii, or equal diameters, they are equal; and their circumferences are also equal.

11. *Define equal arcs (p. 44, No.* **6***).*

12. *What conditions must two equal arcs satisfy?*

(1) They must have equal radii.

(2) They must correspond to equal angles at the centre.

13. Draw two unequal arcs having equal radii.

14. Draw two unequal arcs that correspond to equal angles at the centre. (The radii must be unequal.)

15. Theorem.—*In equal circles, equal angles at the centre intercept equal arcs (Fig. 99).*

What is the hypothesis?
What is the conclusion?

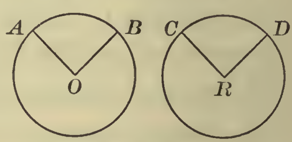


FIG. 99.

PROOF. Place $\angle AOB$ on $\angle CRD$ so that they coincide.
A will fall on C. Why?
B will fall on D. Why?
Arc AB will coincide with arc CD because all their points are at equal distances from R.

16. Theorem.—*In equal circles, equal arcs correspond to equal angles at the centre (Fig. 99).*

What is the hypothesis? What is the conclusion?
Prove by the method of superposition (as in No. **15**).

17. In what relation does No. **16** stand to No. **15**?

18. In what way is the term "subtend" used?

A chord AB (Fig. 100) is said to **subtend** the arc AB, and the arc AB is said "to be subtended by" the chord AB. To subtend means to stretch across.

19. Theorem.—*In equal circles, equal chords subtend equal arcs (Fig. 100).*

What is the hypothesis?

What is the conclusion?

PROOF. Apply No. **1**, p. 70, and No. **15**, p. 96.

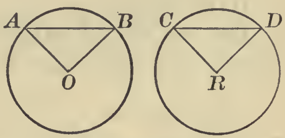


FIG. 100.

20. Theorem.—*In equal circles, equal arcs are subtended by equal chords (Fig. 100).*

What is the hypothesis?

What is the conclusion?

PROOF. Apply No. **16**, p. 96, and No. **14**, p. 66.

21. What relation is there between Nos. **19** and **20**?

22. Problem.—*To make an arc equal to a given arc AB whose centre is at a given point O (Fig. 101).*

CONSTRUCTION. Draw OA, OB, AB.

With radius OA, and any point R as centre, describe an arc CE.

With centre C, and radius equal to AB, cut CE at D.

Then arc $CD =$ arc AB.

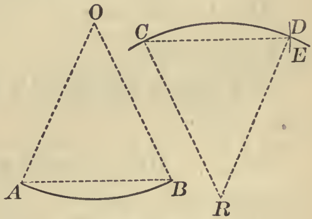


FIG. 101.

PROOF. Apply No. **19**.

23. Make an arc equal to three times a given arc.

Lesson 51.

1. Theorem.—*The radius perpendicular to a chord bisects the chord, the arc subtended by the chord, and the corresponding angle at the centre.*

HYPOTHESIS.

Let the radius OC be $\perp$ to the chord AB, and meet AB at C.

CONCLUSIONS.

$$AD = BD,$$
$$\text{arc } AC = \text{arc } BC,$$
$$\angle AOC = \angle BOC.$$

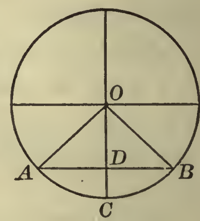


FIG. 102.

PROOF. Compare $\triangle$ AOD, BOD, and apply No. **16**, p. 96.

2. Theorem.—*The perpendicular erected at the middle point of a chord passes through the centre of the circle.*

HYPOTHESIS.

Let AB be a chord.
Let D be its middle point.
Also let DE be $\perp$ to AB.

CONCLUSION.

DE passes through O, the centre of the circle.

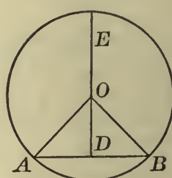


FIG. 103.

PROOF. DE passes through all points which are equally distant from A and B. (No. **1**, p. 72.)
O is a point equally distant from A and B. Why?
Therefore DE passes through O.

3. Bisect an arc, and prove your construction correct.

4. Construct an arc containing 22° 30′.

5. Problem. — *To describe a circumference through three points, A, B, C, not in a straight line.*

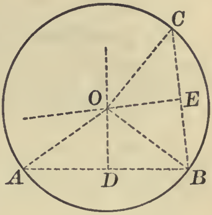


FIG. 104.

CONSTRUCTION.

Draw AB, BC.

Draw the bisecting perpendiculars to AB and BC.

Let these $\perp$s meet at O.

With centre O and radius OA describe a circle.

This circle will pass through the points A, B, and C.

PROOF. Apply No. **1**, p. 72, or No. **2**, p. 98.

6. Can a circle be described through three points which are in the same straight line?

7. Describe a circle and erase the centre. Then find the centre by a construction, and prove that your construction is correct. (Apply No. **2**, p. 98.)

8. Draw a triangle, and then describe a circle which shall pass through its vertices.

9. Make a square of 2 in. side, and describe a circle which shall pass through its vertices.

10. Describe three circles, all passing through two points A and B. How many circles can be described through two given points? In what line are all the centres located?

11. Describe a circle with a radius of 2 in. which shall pass through two points A and B, 3 in. apart.

12. Mark two points A and B, and draw a line CD. Then describe a circle whose centre shall be in the line CD, and whose circumference shall pass through both A and B. Can you draw CD so that the problem cannot be solved?

Lesson 52.

1. *Define an inscribed angle.*

An **inscribed** angle is an angle whose vertex is in the circumference of a circle, and whose sides are chords.

For example: $\angle BAC$ (Fig. 105) is an inscribed angle.

2. **Theorem.**—*An inscribed angle is equal to half the angle at the centre, which intercepts the same arc.*

HYPOTHESIS. Circle with centre O. An inscribed $\angle BAC$.

CONCLUSION. $BAC = \frac{1}{2} BOC$.

Case 1 (Fig. 105). The centre O *in* the line AB.

PROOF. $\angle BAC + \angle OCA + \angle AOC = 180°$. Why?

Also $\angle BAC = \angle OCA$. Why?

Therefore $2\angle BAC + \angle AOC = 180°$. Why?

But $\angle BOC + \angle AOC = 180°$.

Hence $2\angle BAC + \angle AOC = \angle BOC + \angle AOC$. Why?

Therefore $2\angle BAC = \angle BOC$. (Ax. 3.)

Therefore $\angle BAC = \frac{1}{2}\angle BOC$. (Ax. 5.)

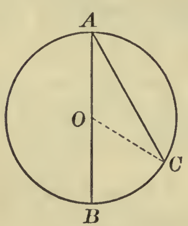


FIG. 105.

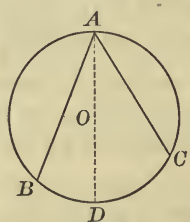


FIG. 106.

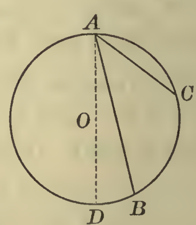


FIG. 107.

Case 2 (Fig. 106). The centre O *between* AB and AC. Draw the diameter AD. Apply Case 1. Add the results.

Case 3 (Fig. 107). The centre O *outside* the angle BAC. Draw AD. Apply Case 1. Subtract the results.

3. *When is an angle inscribed in a segment?*

When its vertex is in the arc of the segment, and its sides pass through the ends of the arc.

4. Make a segment, and inscribe in it three angles.

5. *What truths follow immediately from No.* **2**?

(1) *Inscribed angles that intercept equal arcs are equal.*

(2) *All angles inscribed in the same segment are equal.* For they all intercept the same arc. Thus, the ∠s ACB, ADB, AEB (Fig. 108), all intercept the arc AB.

(3) *Every angle inscribed in a semicircle is a right angle.* For the intercepted arc is a semi-circumference.

Thus, the ∠s ACB, ADB, AEB (Fig. 109) are right angles.

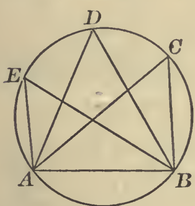


FIG. 108.

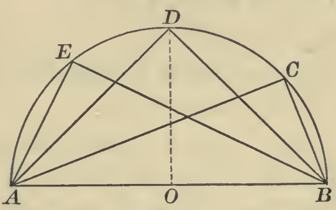


FIG. 109.

6. What is the value of an inscribed angle if the intercepted arc is 60°? 153°? 320°? 118° 35′?

7. What value has an angle inscribed in a segment whose arc contains 60°? 90°? 135°? 180°? 270°? 300°?

8. Prove the construction in Fig. 27, p. 19, to be correct.

Construct a right triangle, having given:

9. Hypotenuse = 3 in., one leg = $1\frac{5}{8}$ in.

10. Hypotenuse = 3 in., altitude upon hypotenuse = 1 in.

11. Hypotenuse = 3 in., one acute angle = 60°.

12. A point A is 2 miles from a line BC; $BC = 9$ miles, $\angle BAC = 90°$. Find AB and AC (by drawing to scale).

Lesson 53.

1. *Define a tangent to a circle.*

A **tangent** to a circle is a straight line which touches the circumference at a point, but does not cut it.

A tangent is said to *touch the circle*. The point where it touches the circle is called the **point of contact.**

2. Theorem. — *A perpendicular erected at the end of a radius is a tangent to the circle.*

Hypothesis.

A circle with centre O.

A radius OA.

The line $MAN \perp$ to OA.

Conclusion.

The line MN is a tangent to the circle at the point A.

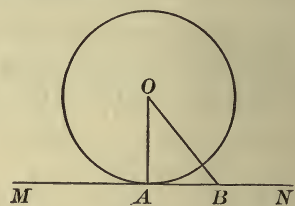


Fig. 110.

Proof. Join O to any other point in MN, as B.

Then we know that $OA < OB$ (No. **3**, p. 71).

Therefore B must lie outside the circle. In the same way we can show that every point in MN except A must lie outside the circle.

Therefore, by definition, the line MN is a tangent, and A is the point of contact.

3. Mark a point A in a line MN, and then describe three circles which shall touch MN at the point A. Take for the radii of the circles 1 in., 2 in., and 3 in. How many circles touching MN at A can be described? In what line do their centres lie?

4. Draw a line MN, and then describe three circles with the same radius, all of which shall touch MN. How many circles, all having the same radius and all touching MN, can be described? In what lines do their centres lie?

5. Problem.—*To draw a tangent to a given circle through a given point A.*

Case 1 (Fig. 111). Let A be in the given circumference.

CONSTRUCTION. Draw the radius OA, and through A draw the line $BAC \perp$ to OA. BAC is the required tangent.

PROOF. See No. 2.

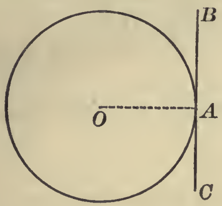


FIG. 111.

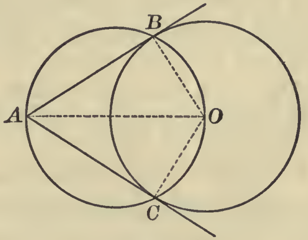


FIG. 112.

Case 2 (Fig. 112). Let A be outside the given circle.

CONSTRUCTION. Join AO.

Upon AO as diameter describe a circle.

This circle will cut the given circle at two points, B and C.

Draw the lines AB and AC.

Then AB and AC are tangents to the given circle.

PROOF. The angles ABO, ACO, are right angles. Why?

Also B and C are the ends of radii.

Therefore AB and AC are tangents. Why?

6. Prove that the tangents AB, AC (Fig. 112), are equal.

7. Describe a circle with a radius of 2 in., and draw tangents to it from a point $3\frac{1}{2}$ in. from the centre.

8. Describe a circle passing through a given point, and touching a given line at a given point in the line.

Describe a circle, and then draw a tangent:

9. Parallel to a given straight line.

10. Perpendicular to a given straight line.

Lesson 54.

1. *When are two circles said to touch each other?*

Two circles are said to **touch** each other, or to be **tangent** to each other, when they both touch the same straight line at the same point.

If the circles lie on opposite sides of the straight line, they are said to touch each other **externally** (Fig. 113).

If they lie on the same side of the straight line, they are said to touch each other **internally** (Fig. 114).

In both cases, the line is a **common tangent** to the circles.

2. *What truth easily follows from No. 7?*

If two circles touch each other, the radii drawn to the point of contact are both perpendicular to the common tangent (No. **2**, p. 102). Therefore they must lie in the same straight line. In other words, *the centres and the point of contact lie in the same straight line.*

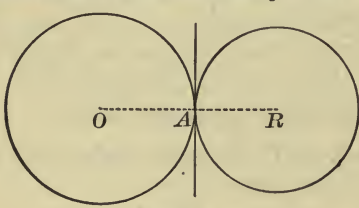


FIG. 113.

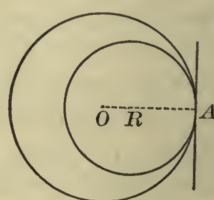


FIG. 114.

3. Construct an equilateral triangle, side 2 in., and by means of it describe three circles, so that each circle shall touch the other two circles.

4. Describe the smallest circle which can touch a given circle and pass through a given point exterior to the given circle.

5. Describe a circle touching two given parallel lines, and passing through a given point between the lines.

6. Draw an angle, and describe three different circles, each touching the sides of the angle. How many circles touching the sides of a given angle can be described? In what line do their centres lie?

7. Draw a triangle, and describe a circle having its centre in one side and touching the other two sides.

8. How many common tangents can be drawn in Fig. 113?

9. *Define circumscribed figures and inscribed figures.*

A circle is **circumscribed** about a polygon if its circumference passes through all the vertices of the polygon; and the polygon is, in this case, **inscribed** in the circle.

A circle is **inscribed** in a polygon if it touches all the sides of a polygon; and in this case the polygon is **circumscribed** about the circle.

10. Illustrate by figures the definitions in No. **9**.

11. Draw a triangle, and then circumscribe a circle about the triangle (see p. 99, No. **5**).

12. Inscribe a circle in a given triangle. (Bisect any two angles. Prove your construction correct. See No. **6**, p. 73.)

13. Circumscribe a circle about an equilateral triangle, and also inscribe a circle in the same triangle.

14. Inscribe a circle in a square whose side is 2 in.

15. Construct a square of side 3 in., inscribe a circle in the square, and form an octagon by cutting off the corners with tangents to the circle.

16. Construct a square of side 4 in., and then describe four equal circles, so that each circle shall touch two of the other circles, and also two sides of the square.

17. Draw a rhombus, and inscribe in it a circle.

Lesson 55.

1. *Define the centre of a regular polygon.*

The **centre** of a regular polygon is a point equidistant from all the vertices of the polygon.

2. Theorem.—*The point where the bisectors of any two angles of a regular polygon meet, is the centre of the polygon.*

Hypothesis.

Let $ABCDE$ be a regular polygon.

Let the bisectors of the $\angle$s A and B meet at O.

Conclusion. AO, BO, CO, DO, EO, are all equal.

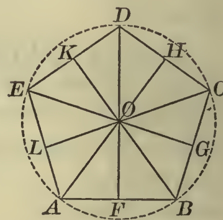


Fig. 115.

Proof. Successive steps. Supply all the reasons.

(1) $\triangle\ OAB$ is isosceles, and $OA = OB$.

(2) $\triangle\ OBC = \triangle\ OAB$ (No. **14**, p. 66). What follows?

(3) $\triangle\ OCD = \triangle\ OBC$ (No. **14**, p. 66). What follows?

(4) $\triangle\ ODE = \triangle\ OCD$ (No. **14**, p. 66). What follows?

3. *What truths follow immediately from No.* **2**?

(1) *The lines drawn from the centre of a regular polygon to the vertices bisect all the angles.*

(2) *A circle can be circumscribed about every regular polygon.*

4. Theorem.—*The centre of a regular polygon is equidistant from all the sides.*

What is the hypothesis (use Fig. 115)?

What is the conclusion?

Proof. Compare the right triangles OFB, OGB; and then OGC, OHC, etc. (see p. 63, No. **3**; p. 67, No. **2**).

5. *What truths follow immediately from No.* 4*?*

(1) *Perpendiculars dropped from the centre of a regular polygon to the sides bisect the sides.*

(2) *A circle can be inscribed in every regular polygon.*

6. *Define the radius and the apothem of a regular polygon.*

The **radius** of a regular polygon is a line drawn from the centre to any one of the vertices.

The **apothem** of a regular polygon is a perpendicular dropped from the centre to any one of the sides.

7. Show how to find the centre of a regular polygon.

8. Inscribe a square in a given circle.

To do this, draw two perpendicular diameters, and join their ends, taken in order (Fig. 116). Prove that the figure thus obtained is a square.

9. Inscribe in a given circle a regular octagon.

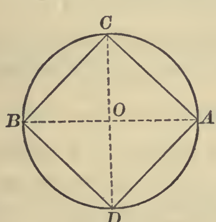


FIG. 116.

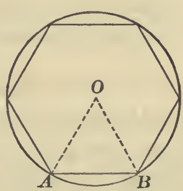


FIG. 117.

10. Inscribe in a given circle a regular hexagon.

To do this, apply the radius as a chord six times (Fig. 117). Then prove that the figure thus formed is a regular hexagon (show that $\angle AOB = 60°$).

11. Inscribe in a given circle an equilateral triangle.

12. Inscribe in a given circle a regular dodecagon.

13. Construct a regular hexagon with a side equal to 3 in.

Lesson 56.

1. *What is the relation between the circumference of a circle and the diameter?*

The diameter of a circle is contained in the circumference a little more than three times. The exact value of the quotient cannot be expressed by a number, but *it is known to be the same for all circles*, and, for convenience, is represented by the Greek letter π (pronounced like *pie*).

The value of π, accurate enough for most purposes, is

$$3\tfrac{1}{7} \text{ or } \frac{22}{7}.$$

The value of π, correct to five decimal places, is 3.14159.

If we take $\pi = \frac{22}{7}$, the relation between circumference and diameter is expressed by either of the following equations:

$$\textit{Circumference} = \textit{diameter} \times \tfrac{22}{7}.$$
$$\textit{Diameter} = \textit{circumference} \times \tfrac{7}{22}.$$

The relation is often expressed in the following general form, in which c denotes circumference, and r radius:

$$c = 2\pi r.$$

NOTE.—The method by which the value of π is shown to be the same for all circles, and the process of computing its approximate values, require the aid of theorems which are not given in this book. The value has been computed to over 700 decimal places.

Find the circumference of a circle, having given:

2. Radius 7 in.

3. Radius 21 in.

4. Radius 40 ft. 10 in.

5. Radius 6 yds. 1 ft. 3 in.

6. Diameter 77 ft.

7. Diameter 49 ft.

8. Diameter 18 yds. 2 ft.

9. Diameter $2\frac{3}{16}$ in.

10. The circumferences of two concentric circles are $16\frac{1}{2}$ feet and $18\frac{1}{3}$ ft. Find the width of the ring between them.

11. If the radius of a circle is $3\frac{1}{2}$ in., what is the length of an arc of $30°$?

The circumference $= 2 \times \frac{7}{2} \times \frac{22}{7}$ in. $= 22$ in.
The circumference is divided into $360°$.
Therefore an arc of $1° = \frac{22}{360}$ in.
And an arc of $30° = \frac{22 \times 30}{360}$ in. $= 1\frac{5}{6}$ in.

Find the length of an arc, having given:

12. Circumference 100 in., angle at centre $30°$.

13. Radius 14 in., angle at centre $45°$.

14. Radius 42 ft., angle at centre $11° 15'$.

15. Diameter 70 in., angle at centre $36°$.

Find the radius of a circle, having given:

16. Circumference 132 ft.

17. Circumference 198 ft.

18. Arc of $60° = 66$ ft.

19. Arc of $15° = 3$ ft. 8 in.

20. A wheel makes 220 revolutions in going half a mile. What is its diameter?

21. A wheel whose diameter is $3\frac{1}{2}$ ft. made 1200 revolutions in going a certain distance. Find the distance.

22. How deep is a well if the wheel, whose diameter is 2 ft. 4 in., makes 30 revolutions in raising the bucket?

23. What angle at the centre will intercept an arc of 6 ft. 5 in. if the radius of the circle is 8 ft. 2 in.?

Circumference $= 2 \times \frac{22}{7} \times 98$ in. $= 616$ in.
Since 616 in. contain $360°$, 1 in. of arc will contain $\frac{360°}{616}$, and 77 in. will contain $\frac{77 \times 360°}{616}$, or $45°$.

24. If the radius of a circle is 7 in., what angle at the centre will intercept an arc 7 in. long?

Lesson 57. Review.

1. Review all the italicized exercises in Lessons 50–53.

2. Make an arc of 225°.

3. Make an arc equal to twice a given arc.

4. Make a sector equal to twice a given sector.

5. Describe a circle with a radius of 2 in. and passing through two given points A and B. When is the solution of the problem impossible?

6. If any number of parallel chords are drawn in a circle, upon what line are all their middle points located?

7. Through a given point within a circle draw that chord which is bisected at the given point.

Prove that your construction is correct.

8. A segment is made by joining the ends of the arc of a quadrant. What is the value of any angle inscribed in this segment?

9. If a series of circles are made, all of them touching two parallel lines, upon what line will all their centres lie?

10. Describe a circle touching two given parallel lines, one of them at a given point A.

11. A man wishes to locate his house so that it shall be 2 miles from a church, and equidistant from the homes of two of his friends. Show by a construction how the proper location may be found.

12. A man walks 3 miles in a straight line; and then walks in the arc of a circle through whose centre he has passed, and whose radius is 1 mile, till he has made the chord of the arc equal to the radius. Find how far he is from home.

Lesson 58. Review.

1. Review all the italicized exercises in Lessons 54–56.

2. Describe a circle, and upon one of its radii OA as a diameter describe another circle. Through A draw any straight line, meeting the circumferences again at B and C. Prove that $AB = BC$.

3. Make two equal circles touching each other externally, and draw all the common tangents.

4. Make two equal circles, exterior to each other and not touching. Then draw all the common tangents.

5. Inscribe a regular hexagon in a circle, and then circumscribe a regular hexagon about the same circle. This is done by drawing tangents through the vertices of the inscribed hexagon. Can you prove that the figure formed by these tangents is a regular hexagon?

6. Inscribe four equal circles in a given square so that each circle shall touch two other circles and one side only of the square.

7. A circular park is laid out having a radius of 350 ft. What will it cost to build a stone wall around the park at $8.00 per yard?

8. The latitude of Leipsic is $51° 21'$, that of Venice is $45° 26'$. Venice is due south of Leipsic. How many miles are they apart? (Take as the radius of the earth 4000 miles.)

9. If the radius of a circle is 4 ft. 8 in., what is the perimeter of a sector whose angle is $45°$?

10. Draw four intersecting lines so as to make four triangles, and circumscribe a circle about each of these triangles. If your work is correct, the four circumferences will be found to pass through the same point.

Lesson 59. Review.

NOTE. — In the exercises of this lesson take $\pi = 3.1416$.

1. Review all the italicized exercises in Chapter VI.

2. Find the circumference of a circle if the radius $= 10^{m}$.

3. Find the radius of a circle if the circumference $= 10^{m}$.

4. What must be the diameter of a round dining-table for 12 persons, if 60^{cm} is allowed to each person?

5. A wheel 75^{cm} in diameter makes 3000 revolutions in going a certain distance. What is the distance?

6. How many trees 10^{m} apart can be set out around the edge of a circular pond 500^{m} in diameter?

7. The radii of two concentric circumferences are 8^{cm} and 10^{cm}. Find the length of the circumference situated just half way between them.

8. Two toothed wheels work together. One has 24 teeth, the other has 144 teeth. The distance between the centres of the teeth is the same on each wheel. How many revolutions will the second wheel make while the first is making 600? If the radius of the first wheel is 4^{cm}, what is the radius of the other?

9. Describe a circle with a radius of 2^{cm}, and then circumscribe about the circle a square. Find (to the nearest millimeter) the difference in length between the circumference of the circle and the perimeter of the square.

Diameter of a circle $= 10^{cm}$. Find the length of the arc:

10. Intercepted by a side of the inscribed square.

11. Intercepted by a side of the inscribed regular pentagon.

12. Intercepted by a side of the inscribed regular hexagon.

13. Intercepted by a side of the inscribed regular octagon.

CHAPTER VII.

AREAS.

Lesson 60.

1. *How are surfaces measured?*

Surfaces, like lines, are measured by choosing a unit, and then finding how many times this unit is contained in the surface which we wish to measure.

The number of times the unit is contained in the surface to be measured, followed by the name of the unit, is called the **area** of the surface.

2. *What units of area are in common use?*

The **square inch**, the **square foot**, the **square yard**, the **acre**, and the **square mile**. All these units, except the acre, are squares whose sides are equal to the units of length.

Their abbreviations are formed by prefixing to those of the corresponding units of length the letters "sq." (standing for "square").

3. *How are those units related?*

These units are related as shown in the following table:

144 sq. in. = 1 sq. ft.	43,560 sq. ft. = 1 acre.
9 sq. ft. = 1 sq. yd.	640 acres = 1 sq. mile

4. How many square feet make 1 square mile?

5. How many square feet in $\frac{1}{4}$ of an acre?

6. Reduce 16 sq. in. to the fraction of a square foot.

7. Reduce $\frac{1}{8}$ of a square foot to square inches.

8. A man bought an acre of land for $5000, and sold it for 20 cents per square foot. How much money did he make?

9. *How can a square be divided into smaller squares?*

Let $ABCD$ (Fig. 118) be a square.

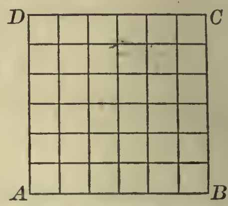


FIG. 118.

Divide AB into any number of equal parts (here, six).

Draw through the points of division lines parallel to AD.

These lines will divide the square into six equal rectangles.

Divide also AD into six equal parts, and draw through the points of division lines parallel to AB. These lines will subdivide each rectangle into six equal squares. Therefore the entire square is now divided into 6×6 or 36 equal squares.

In general, the number of equal squares into which a square is decomposed by this process is found by *multiplying the number of equal parts in one of the sides by itself.*

10. Problem. — *To find the area of a square.*

Let $ABCD$ (Fig. 119) be a square.

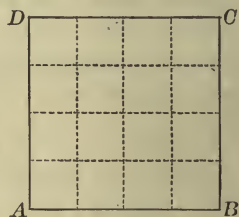


FIG. 119.

Measure one side AB.

Let, for example, $AB = 4$ in.

By proceeding as in No. **9**, we see that the area of the square will be 4×4 or 16 sq. in.

Next, let $AB = 4.25$ in. The same reasoning will still apply, only we shall now have in AB 425 equal parts, each a hundredth of an inch long, and in $ABCD$, 425×425 or 180,625 equal squares. Now one inch contains 100 of these equal parts. Therefore one square inch will contain 100×100 or 10,000 of the equal squares. Therefore the area of the square $ABCD$, *in square inches*, will be $\frac{180625}{10000}$ or 18.0625, a result at once found by multiplying 4.25 by itself.

In general, by multiplying by itself the number of units of length in a line AB, we obtain the number of units of area in the square constructed upon AB as a side; the unit of area being always the square whose side is equal to the unit of length.

Multiplying a number by itself is called **squaring** it, and the square of the number of units of length in a line AB is called, for brevity, "the square of AB," and written $\overline{AB}^2$. Hence the result may be briefly expressed as follows:

Area of a square = square of one side.

Find the area of a square if one side is equal to:

11. 9 in.

12. 14 ft.

13. 18 yds.

14. $7\frac{1}{2}$ in

15. $2\frac{1}{8}$ in.

16. 1 ft. 8 in.

17. 2 ft. 4 in.

18. 4 yds. 1 ft. 2 in.

19. 98 ft. 6 in.

20. 56 ft. 4 in.

21. Find the area in acres and square feet of a square field each side of which is 178 yds. long.

22. How many tiles 8 in. square are required to cover a square court one side of which is 33 ft. 4 in. long?

23. What will it cost, at 20 cents per square foot, to paint a square floor the side of which measures $18\frac{1}{2}$ ft.

24. Find the side of a square whose area is 2000 sq. ft.

SOLUTION.—We must take the square root of 2000. $\sqrt{2000} = 44.72+$. Hence the side (correct to inches) $= 44.72$ ft. $= 44$ ft. 9 in.

25. Find the side of a square whose area is $2\frac{1}{4}$ sq. yds.

26. Find the side of a square whose area is 6000 sq. ft.

27. What will it cost, at \$8 per yard, to enclose with an iron fence a square park containing 1000 acres?

28. Explain why it requires 144 sq. in. to make 1 sq. ft.

Lesson 61.

1. **Problem.**—*To find the area of a rectangle.*

Let $ABCD$ (Fig. 120) be a rectangle.

Measure AB and AD.

Let $AB = 7$ in., $AD = 4$ in.

We can show by reasoning as in No. **9**, p. 114, that the area of the rectangle is equal to 7×4 or 28 sq. in.

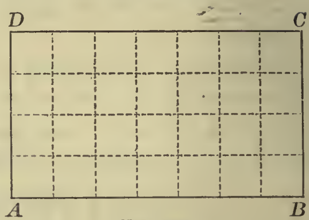


FIG. 120.

In general, the number of units of area in a rectangle is found by multiplying the number of units of length in one side by the number of units of length in the adjacent side.

The adjacent sides are equal to the *dimensions* of the rectangle, and are usually called length and breadth; hence we have the following condensed formula:

$$\textit{Area of a rectangle} = \textit{length} \times \textit{breadth}.$$

Find the area of a rectangle, having given:

2. Length 12 in., breadth 9 in.

3. Length $15\frac{1}{2}$ in., breadth 6 in.

4. Length $3\frac{1}{2}$ in., breadth $2\frac{1}{4}$ in.

5. Length 200 ft., breadth 60 ft.

6. Length 5 ft. 6 in., breadth 1 ft. 6 in.

7. Length 4 yds. 1 ft. 6 in., breadth 6 ft.

Find the other dimension of a rectangle, having given:

8. Area 100 sq. in., one dimension 8 in.

9. Area 288 sq. in., one dimension 6 in.

10. Area $22\frac{1}{2}$ sq. in., one dimension $2\frac{1}{2}$ in.

11. Area 180 sq. ft., one dimension 4 yds.

12. How many bricks 9 in. by 4 in. are required to cover a floor 34 ft. long and 17 ft. wide?

13. A street $\frac{1}{2}$ a mile long has on each side a sidewalk $7\frac{1}{2}$ ft. wide. Find the cost of paving the sidewalk with stones 2 ft. 9 in. by 1 ft. 8 in., and costing, laid down, $1 each.

14. A lawn measures 144 yds. by 98 yds.; the turfs which are used measure 18 in. by 14 in., and cost, laid down, 75 cents per dozen. Find the cost of turfing the lawn.

15. A floor is 27 ft. 6 in. long and 8 yds. wide. If the planks to cover it measure 11 ft. by 9 in., how many are needed, and what will they cost at 10 cents per square foot?

16. How many yards of paper 27 in. wide are required to paper a room 18 ft. long, 12 ft. wide, and 11 ft. high?

17. What will it cost to line with lead, at 2 cents per square inch, the inside of an open cistern 10 ft. long, 6 ft. wide, and 4 ft. deep?

18. A man wants 4 sq. ft. of wood, and has only a plank 18 in. wide from which to cut it off. Find the length of the piece he must cut off.

19. The dimensions of a rectangle are 36 ft. and 20 ft. If the length be diminished by 6 ft., how much must be added to the breadth in order that the area may remain the same?

20. A rectangular field, half a mile long, contains 100 acres. Find the width of the field in feet.

21. A room is 39 ft. by 23 ft. and 17 ft. high, with three windows each $9\frac{1}{2}$ ft. by 8 ft.; two doors each $10\frac{1}{2}$ ft. by 6 ft.; and two fireplaces each $6\frac{1}{2}$ ft. by 4 ft. The carpet is 2 ft. wide, and costs $1.50 per yd.; the paper is 2 ft. wide, and costs 75 cents per roll of 10 yds. Find the cost of carpeting and papering this room, the carpet running lengthwise, and the labor costing $12.

Lesson 62.

1. Problem.—*To find the area of a parallelogram.*

Let $ABCD$ (Fig. 121) be a parallelogram, having the base AB and the altitude BE.

Draw $AF \perp$ to AB, and meeting AB produced at the point F.

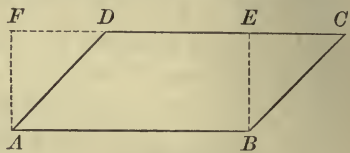


FIG. 121.

The figure $ABEF$ is a rectangle having the same base AB and altitude BE as the parallelogram.

$\triangle BCE = \triangle ADF$. Why?

Entire figure $ABCF - \triangle ADF =$ parallelogram $ABCD$.

Entire figure $ABCF - \triangle BCE =$ rectangle $ABEF$.

Therefore, parallelogram $ABCD =$ rectangle $ABEF$.

But rectangle $ABEF = AB \times BE$ (p.116, No.**1**).

Therefore, parallelogram $ABCD = AB \times BE$.

Hence, the area of a parallelogram is found by measuring its dimensions and multiplying their values together.

Area of a parallelogram = base × altitude.

2. Does this formula also hold true for the square and for the rectangle?

Find the area of a parallelogram, having given:

3. Base 16 in., altitude 5 in.

4. Base $2\frac{1}{2}$ in., altitude 14 in.

5. Base 1 ft. 9 in., altitude 4 ft.

6. Base 15 ft., altitude equal to half the base.

7. The base of a parallelogram = 18 in. What must be its altitude in order that it may contain exactly 1 sq. ft.?

8. *What conclusion may be drawn from No.* 1 *respecting all parallelograms that have equal bases and equal altitudes?*

Parallelograms having equal bases and equal altitudes have also equal areas; in other words, they are equivalent.

For by No. **1** each one of the parallelograms is equivalent to a rectangle having an equal base and an equal altitude. For example (Fig. 122): the parallelograms *ABCD*, *ABEF*, and *ABGH* are equivalent, each being equivalent to the rectangle *ABMN*.

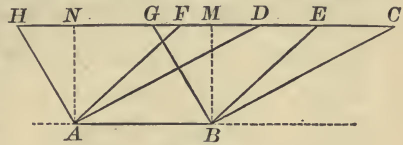


FIG. 122.

9. Construct a set of four equivalent parallelograms.

10. Make two parallelograms, differing very much in shape, but having the same area.

11. Construct a parallelogram, and then find its area.

12. The bases of two parallelograms are each $9\frac{1}{2}$ in. long. The altitude of one is 1 in., and the altitude of the other is 3 in. How much greater is the area of one than the area of the other?

13. The bases of two parallelograms are equal; the altitude of one is three times that of the other. Compare their areas. Draw a figure to illustrate your answer.

14. The altitudes of two parallelograms are equal; the base of one is three times that of the other. Compare their areas. Draw a figure to illustrate your answer.

15. Find the area of a rhombus, the perimeter of which is 156 ft., and the altitude 9 ft. 4 in.

16. The area of a field having the shape of a rhombus is exactly 10 acres. The distance from any one side to the opposite side is 400 ft. Required the cost of fencing the field at $2.50 per rod.

Lesson 63.

1. Problem.—*To find the area of a triangle.*

Let ABC (Fig. 123) be a triangle, AB the base, CD the altitude.

Draw $AE \parallel$ to BC, and $CE \parallel$ to AB; the figure $ABCE$ is a parallelogram having AB for base, CD for altitude.

Therefore, $\triangle ABC = \triangle AEC$ (p. 84, No. **3**).

Hence, area of $\triangle ABC = \frac{1}{2}$ area of parallelogram $ABCD$.

Now, area of parallelogram $ABCD = AB \times CD$.

Hence, area of $\triangle ABC = \frac{1}{2} AB \times CD$; or,

$$\textit{Area of a triangle} = \tfrac{1}{2} \times \textit{base} \times \textit{altitude}.$$

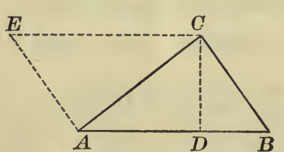


Fig. 123.

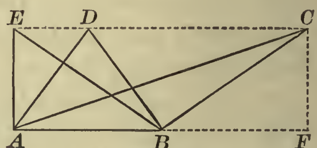


Fig. 124.

2. *What follows immediately from this formula?*

△s having equal bases and equal altitudes are equivalent.

Thus (Fig. 124) the △s ABC, ABD, ABE, are equivalent.

Find the area of a triangle, having given:

3. Base 11 in., altitude 10 in.

4. Base $2\frac{1}{2}$ in., altitude 1 in.

5. Base 3 ft. 6 in., altitude 2 ft.

6. Base 20 ft. 6 in., altitude 10 ft. 4 in.

7. Find the area of a right triangle if the lengths of the two legs are $4\frac{3}{4}$ in. and $3\frac{1}{8}$ in.

8. A garden is laid out in the shape of a right triangle. One leg = 150 yds., the other leg = 200 yds. Find the value of the garden at $12\frac{1}{2}$ cents per square foot.

9. *Apply the formula of No.* 1 *to a rhombus.*

Let $ABCD$ (Fig. 125) be a rhombus. The diagonals AC and BD bisect each other at right angles (p. 86, No. 3). AC divides the rhombus into two equal isosceles triangles, each having AC for base, and half of BD for altitude.

The area of each $\triangle = \frac{1}{2}AC \times \frac{1}{2}BD$; therefore the area of the rhombus $= \frac{1}{2} \times AC \times BD$; or, in general:

Area of a rhombus = half the product of the diagonals.

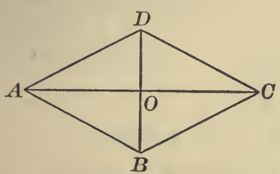


FIG. 125.

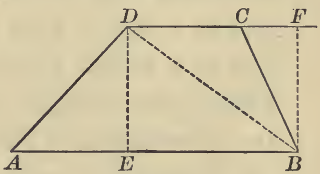


FIG. 126.

10. *Apply the formula of No.* 1 *to a trapezoid.*

Let $ABCD$ (Fig. 126) be a trapezoid, AB and CD the bases, DE the altitude. Draw the diagonal DB, dividing the trapezoid into the two triangles ABD and BDC.

Area of $\triangle\, ABD = \frac{1}{2}AB \times DE$. Why?

Area of $\triangle\, BDC = \frac{1}{2}DC \times DE$.

Hence area of trapezoid $ABCD = \frac{1}{2}(AB + DC) \times DE$.

Putting this result into general terms, we have the formula:

Area of a trapezoid $= \frac{1}{2}$ (*sum of bases*) $\times$ *altitude.*

Find the area of a rhombus, having given:

11. The diagonals $9\frac{3}{8}$ in. and $6\frac{1}{4}$ in.

12. The diagonals 26 yds. and $34\frac{2}{3}$ yds.

13. Find the area of a trapezoid if the two bases are 37 ft. and 25 ft., and the altitude 19 ft.

14. Find in square feet the area of $ABCD$ (Fig. 126) if $AB = \frac{1}{2}$ mile, $CD = \frac{1}{4}$ mile, $DE = \frac{1}{3}$ mile.

Lesson 64.

1. Problem.—*To find the area of a regular polygon.*

Let $ABCDE$ (Fig. 127) be a regular polygon, O the centre, OF the apothem.

Draw the radii OA, OB, etc.

The radii divide the polygon into the equal isosceles $\triangle$ AOB, BOC, etc.

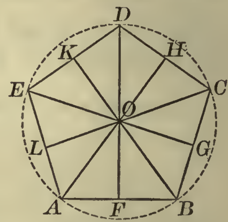


FIG. 127.

Altitude of each $\triangle = OF$.

Area of $\triangle\ AOB = \frac{1}{2} AB \times OF$.

Area of $\triangle\ BOC = \frac{1}{2} BC \times OF$, etc.

Adding these areas, we obtain:

Area of polygon $ABCDE = \frac{1}{2}(AB + BC + \text{etc.}) \times OF$; or,

Area of a regular polygon $= \frac{1}{2} \times$ *perimeter* $\times$ *apothem.*

If one side of a regular polygon and the number of sides are given, the apothem may be found (correct enough for all practical purposes) by multiplying the given value of the side by the proper number in the following table:

No. of sides.	*Apothem.*	*No. of sides.*	*Apothem.*
3	one side $\times$ 0.288	7	one side $\times$ 1.038
4	one side $\times$ 0.500	8	one side $\times$ 1.207
5	one side $\times$ 0.688	10	one side $\times$ 1.539
6	one side $\times$ 0.866	12	one side $\times$ 1.866

Find the area of:

2. An equilateral triangle, if one side = 5 in.

3. A regular pentagon, if one side = 6 in.

4. A regular octagon, if one side = 15 ft.

5. A regular decagon, if one side = 19 ft.

6. A park has the shape of a regular hexagon; each side is 1000 ft. long. Find the value of the park, at 8 cents per square foot.

7. *How may the area of any polygon be found?*

The area of any polygon may be found by dividing it into triangles, or into triangles and trapezoids, computing the areas of these parts, and then adding the results.

8. Find the area of the polygon $ABCD$ (Fig. 128), having given: AC, 400 yds.; BE, 120 yds.; DF, 80 yds.

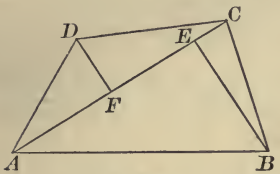


Fig. 128.

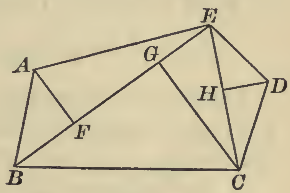


Fig. 129.

9. Find the area of the polygon $ABCDE$ (Fig. 129), having given: BE, 108 yds.; EC, 96 yds.; AF, 49 yds.; HD, 35 yds.; CG, 67 yds.

10. Find the area of the polygon $ABCDE$ (Fig. 130), having given: AC, 475 yds.; BG, 160 yds.; AF, 175 yds.; AG, 320 yds.; AH, 420 yds.; EF, 160 yds.; DH, 90 yds.

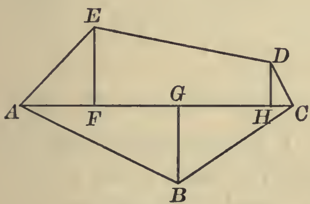


Fig. 130.

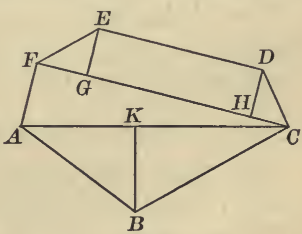


Fig. 131.

11. Find the area of the polygon $ABCDEF$ (Fig. 131), having given: AC, 300 yds.; BK, 115 yds.; FG, 63 yds.; EG, 54 yds.; FH, 283 yds.; DH, 60 yds.; AF, 125 yds.; CF, 325 yds. $\angle AFC = 90°$.

Lesson 65.

1. **Problem.** — *To find the area of a circle.*

Circumscribe about a circle a regular polygon; then

Area of the polygon $= \frac{1}{2} \times$ perimeter $\times$ apothem;

or, since the apothem is equal to the radius of the circle,

Area of the polygon $= \frac{1}{2} \times$ perimeter $\times$ radius.

The greater the number of sides of the polygon, the nearer will its perimeter approach to the circumference of the circle, and its area to the area of the circle.

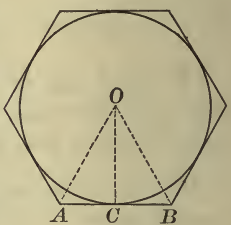


FIG. 132.

If the polygon had 1000 sides, it would not differ sensibly from the circle; if it had 10,000,000 sides, the difference would of course be still less. But however many sides the polygon may have, its area may always be found by the formula above given; and it can be proved that this formula will still hold true if we substitute the area of the circle for the area of the polygon, and the circumference of the circle for the perimeter of the polygon. Making these substitutions, the formula becomes:

Area of a circle $= \frac{1}{2} \times$ *circumference* $\times$ *radius.*

To change this formula into a more practical form, we must substitute for circumference the value which we have previously found (see p. 108); namely, $\frac{22}{7} \times$ diameter, or $\frac{44}{7} \times$ radius. After reducing to the simplest form, we have:

Area of a circle $= \frac{22}{7} \times (\textit{radius})^2$;

or, if one prefers to use the symbols π and r,

Area of a circle $= \pi r^2$.

2. From the formula for the area of a circle find the value of the radius in terms of the area.

Take the equation given in No. **1**:

$$\frac{22}{7} \times (\text{radius})^2 = \text{area}.$$

Multiply both sides by $\frac{7}{22}$, and extract the square root.

Indicating by the radial sign $\sqrt{\ }$ the square root of the second side, we obtain for the result:

$$\text{radius} = \sqrt{\frac{7}{22} \times \text{area}}.$$

Find the area of a circle, having given:

3. Radius 7 in.

4. Radius 14 ft.

5. Radius 40 ft. 10 in.

6. Radius 6 yds. 1 ft. 3 in.

7. Diameter 49 ft.

8. Diameter $18\frac{2}{3}$ yds.

9. Diameter half a mile. (Find the answer in acres.)

10. Find the radius, if the area = 2464 sq. ft.

11. Find the radius, if the area = 6 acres.

12. Find the diameter, if the area = 1 sq. mile.

13. The radii of two concentric circles are $3\frac{1}{2}$ in. and 7 in. Find the area of the ring bounded by their circumferences.

14. Find the area of a circular ring 4 ft. wide, if the radius of the larger circle is 32 ft.

15. Out of a square piece of wood 5 ft. 10 in. long is cut the largest possible circle. Find its area.

16. Find the area of a sector, if the radius of the circle = 14 in. and the angle at the centre = 45°. The sector is the same part of the circle that its angle is of 360°.

17. Find the area of a sector, if the radius = 10 in. and the angle at the centre = 18°.

Lesson 66.

1. Theorem.—*In every right triangle the square of the hypotenuse is equal to the sum of the squares of the legs.*

HYPOTHESIS.

Let ABC be a right triangle, and let $\angle ACB = 90°$.

CONCLUSION.

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2.$$

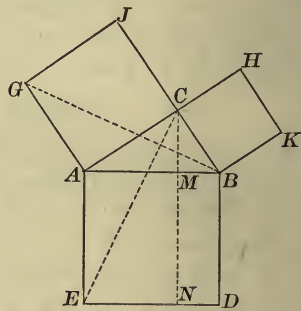


FIG. 133.

PROOF. Construct the squares $ABDE$, $ACFG$, $BCHK$.

Draw $CM \perp$ to AB, and produce it to meet DE at N.

Join BG and CE.

In the $\triangle$s ACE, ABG, $AC = AG$, $AE = AB$. Why?

Also, $\angle CAE = \angle BAG$. Why?

Therefore, $\triangle ACE = \triangle ABG$ (p. 66, No. **14**).

Now, the $\triangle ACE$ and the rectangle $AMNE$ have the same base, AE, and the altitude of each is the distance between the parallels AE and CN.

Therefore, $\triangle ACE = \frac{1}{2}$ rectangle $AMNE$ (p. 120, No. **1**).

Similarly, $\triangle ABG = \frac{1}{2}$ square $ACFG$.

Therefore, $\frac{1}{2}$ rectangle $AMNE = \frac{1}{2}$ square $ACFG$.

Therefore, rectangle $AMNE =$ square $ACFG$.

Similarly (joining CD and AK), we can prove that

rectangle $MBDN =$ square $BCHK$.

Adding the last two equations, we have:

square $ABDE =$ square $ACFG +$ square $BCHK$,

or $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

NOTE.—This very useful theorem is called, from the name of the discoverer, the "Theorem of Pythagoras."

Find the hypotenuse of a right triangle, having given:

3. The two legs 3 in. and 4 in.

4. The two legs 8 in. and 15 in.

5. The two legs 81 ft. and 108 ft.

6. The two legs 237 ft. and 316 ft.

Find one leg of a right triangle, having given:

7. Hypotenuse 13 in., one leg 5 in.

8. Hypotenuse 40 ft., one leg 24 ft.

9. Hypotenuse 51 ft., one leg 24 ft.

10. Hypotenuse 185 ft., one leg 111 ft.

11. Find the area of a right triangle, if the hypotenuse is 25 in. and one leg is 15 in.

12. Find (correct to two decimal places) the hypotenuse of a right triangle, if each leg is 10 in.

13. Find (correct to two decimal places) the legs of an isosceles right triangle, if the hypotenuse is 10 in.

14. The side of a square is 20 ft. Find the diagonal.

15. The diagonal of a square is 20 ft. Find one side.

16. A ladder 34 ft. long just reaches a window when placed with its foot 16 ft. from the side of the house. How high is the window above the ground?

17. Two vessels start at the same time from the same port, and sail one north, the other east. Their rates are 6 and 8 miles per hour. How far apart are they after 3 hours?

18. The perimeter of an isosceles triangle is 64 ft.; the base is 24 ft. Find the area of the triangle.

19. The side of an equilateral triangle is 14 yds. Find the area of the triangle.

20. The radius of a circle is 1 ft. 3 in. Find the distance from the centre to a chord 2 feet long.

Lesson 67.

1. *Define the mean proportional between two lengths.*

The **mean proportional** between two lengths is the length whose square is equal to their product.

Thus: since $6 \times 6 = 4 \times 9$, a line 6 in. long is the mean proportional between a line 4 in. long and a line 9 in. long.

2. Problem.—*To find the mean proportional between two given lines AB and CD (Fig. 134).*

(1) By *calculation.* Measure the two lines, multiply their lengths together, and find the square root of the product.

(2) By *construction.* Draw a straight line, and upon it lay off $AB = a$, and $BC = b$.

Upon AC as a diameter, describe a semicircle.

Erect at B a perpendicular, cutting the semicircle at D.

Then BD will be the mean proportional required.

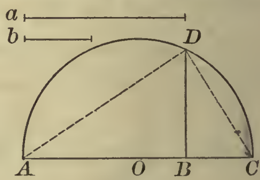


FIG. 134.

NOTE.—This construction can be proved to be correct with the aid of the Theorem of Pythagoras, but the shortest and easiest proof is founded on the properties of similar triangles (see page 151, Ex. 8).

3. Draw a line 2 in. long and a line $3\frac{1}{8}$ in. long, and then find (1) by calculation, (2) by construction, the mean proportional between them.

4. Find the mean proportionals between the following numbers: 1 and 9, 9 and 25, 16 and 25, 1 and $\frac{1}{4}$, $\frac{1}{9}$ and $\frac{4}{49}$.

5. *What is meant by transforming a figure?*

To **transform** a figure is to change its shape without changing its size.

6. Construct a parallelogram, and then transform it by a construction into a rectangle (see p. 118, Ex. 1).

7. Problem. — *To transform a rectangle into a square.*

(1) By *calculation.* Measure the dimensions of the rectangle; find the square root of their product; draw a line equal to this root in length; and upon this line as a side, construct a square.

(2) By *construction.* Denote the two dimensions by a and b.

We must find a length x such that $x^2 = a \times b$.

Hence the side x of the required square is the mean proportional between a and b.

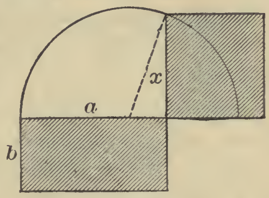


FIG. 135.

Therefore the construction is the same as that in No. 2.

8. Construct a rectangle 4 in. long and $2\frac{1}{4}$ in. wide. Then find (1) by calculation, and (2) by construction, the side of the equivalent square.

9. *How can a triangle be transformed into a square?*

Answer. The area of a triangle $= \frac{1}{2}$ the base $\times$ altitude.

Therefore, the side of an equivalent square is the mean proportional between half of the base and the altitude.

10. Construct a triangle with base 4 in. and altitude $3\frac{1}{8}$ in., and transform it to a square (1) by calculation; (2) by construction.

11. Construct an equilateral triangle, and then transform it by construction into a square.

12. Construct a rhombus, and then transform it by construction into a square (see p. 121, No. 9).

13. Two fields are equal in area. One is a square, and the other a rectangle 625 ft. long and 324 ft. wide. Find the cost of fencing each field at 10 cents a foot.

14. One leg of an isosceles right triangle is 40 ft. long. Find (correct to two decimals) a side of the equivalent square.

Lesson 68.

1. **Problem.**—*To transform any polygon into a triangle.*

Let $ABCDE$ (Fig. 136) be any pentagon.

Draw the diagonal BD.

Draw through C a line parallel to BD, and meeting AB produced at F.

Join DF. The figure $AFDE$ has one side less than the given polygon, and is equivalent to it.

Repeat this construction by joining AD, drawing through E a line parallel to AD, and joining DG.

The $\triangle$ DFG will be equivalent to the given pentagon $ABCDE$.

The repetition of this process will reduce a polygon of any number of sides to an equivalent triangle.

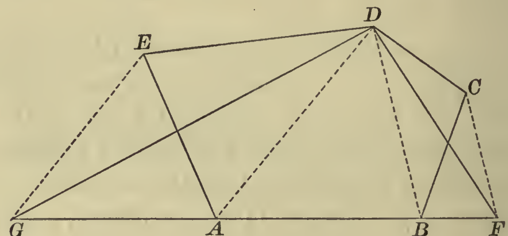


Fig. 136.

Proof. The $\triangle$s BDF, BDC, have the same base BD and equal altitudes (their vertices C, F, being in a line $\parallel$ to BD).

Therefore $\triangle$ BDF is equivalent to $\triangle$ BDC (p. 120, No. 2).

Hence, if $\triangle$ BDC be taken from the given polygon, and $\triangle$ BDF substituted for it, the polygon will suffer no change in area, but it will be transformed into the figure $AFDE$.

For like reasons, $\triangle$ ADG is equivalent to $\triangle$ ADE, and when substituted for it, the figure $AFDE$ will be reduced to the $\triangle$ DFG.

NOTE. — The next seven exercises are to be done by *construction*.

2. Draw a trapezium, and transform it into a triangle.

3. Draw a pentagon, and transform it into a triangle.

4. Draw a hexagon, and transform it into a triangle.

5. Draw a hexagon, and transform it into a square.

6. Draw a triangle, and then upon one side of this triangle as a base, construct an equivalent isosceles triangle.

7. Draw a square, and upon a side of the square as a base, construct an equivalent isosceles triangle.

8. Draw a triangle, and upon one of its sides construct an equivalent rectangle.

NOTE. — The remaining exercises are to be done by *calculation*.

9. The legs of a right triangle are 4 in. and 8 in. Find the side of the equivalent square.

10. Find the side of a square equivalent to a triangle whose base is 18 ft. and altitude 16 ft.

11. Find the side of a square equivalent to a trapezoid whose bases are 30 ft. and 34 ft., and altitude 8 ft.

12. Find, correct to two decimals, the side of a square equivalent to a circle whose radius is 7 in.

13. Find, correct to two decimals, the radius of a circle equivalent to a square whose side is 11 in.

14. A rectangle is 320 ft. long and 100 ft. wide. If the length be reduced by 50 ft., how much must the breadth be increased in order that the area may remain the same as before?

15. The hypotenuse of an isosceles right triangle is 10 in. Find (correct to two decimals) a side of the equivalent square.

Lesson 69. Review.

1. Review the italicized exercises in Lessons 60–65.

2. What will it cost, at $2 per rod, to fence a square field containing $2\frac{1}{2}$ acres?

3. How many five-acre lots can be made out of a field containing 5 square miles?

4. How many planks, each 10 ft. long and 8 in. wide, will cover a floor 14 ft. 3 in. long and 13 ft. 4 in. wide?

5. Find the side of a square having an area equal to that of a regular hexagon whose side is 8 ft.

6. The circumferences of two concentric circles are 440 ft. and 330 ft., respectively. Find the width of the ring.

7. Out of a circular piece of wood whose radius is 3 ft. 4 in. is cut the largest possible square. Find its side.

8. The radius of a circle is 4 ft. Find the area of the inscribed regular hexagon (see p. 107, No. **10**).

9. The side of a square is 42 yds. Find the areas of the inscribed and the circumscribed circles.

10. The diagonals of a diamond-shaped pane of glass are 12 in. and 16 in. How many panes will cover an area of 400 sq. ft.?

11. A square and a rectangle have the same perimeter, 100 yds. The length of the rectangle is four times its breadth. Which has the greater area, and by how much?

12. What will it cost to turf a lawn 35 yds. by 27 yds., with turfs 21 in. by 18 in., and costing 3 cents apiece?

13. A path 8 ft. wide surrounds a rectangular court 60 ft. long and 36 ft. wide. If tiles are 9 in. by 4 in., and cost 10 cents each, find the cost of paving the path with tiles.

Lesson 70. Review.

1. Review the italicized exercises in Lessons 66–68.

2. Show by a figure the difference between half a square foot and half a foot square.

3. Can you make a rectangle, 9 in. by 4 in., and then divide it into two parts of such a form that when placed together in a certain way they will make a square?

4. A man has a lot of land 127 yds. square. What will it cost him, at 30 cents per square yard, to make a gravelled walk 1 yd. wide around his lot?

5. How high is a window from the ground if a ladder 30 ft. long will just reach the window when placed with its foot 18 ft. from the side of the house?

6. The width of a house is 47 ft.; height to ridge, $67\frac{1}{2}$ ft.; height to eaves, 39 ft. Find the cost of painting the side of the house up to the ridge, at 18 cents per square yard, deducting 240 sq. ft. for the windows.

7. If a right triangle has an acute angle of 30°, the leg opposite this angle is always equal to half the hypotenuse. Knowing this truth, find the legs of a right triangle if the hypotenuse = 20 ft. and one angle = 30°.

8. If the distance from the centre of a circle to a chord 10 in. long is 1 ft., find the distance to a chord 2 ft. long.

9. Transform a square into a right triangle.

10. Transform a rectangle into an isosceles triangle.

11. Transform a parallelogram into a square.

12. Construct a square equal to the sum of two given squares.

13. Construct a square equal to the difference of two given squares.

Lesson 71. Review.

1. Show by a figure that the square of $\frac{1}{2}$ is equal to $\frac{1}{4}$.

2. Show by a figure that the square of $1\frac{1}{2}$ is equal to $2\frac{1}{4}$.

3. Find the side of a square equal in area to the sum of the squares constructed upon the lengths 2 in., 3 in., and 4 in.

4. In a railway curve the radius of the circle is 300 ft., and the centre of the circle is located at a point 780 ft. from a station. Draw a plan showing the location of the station, the centre of the circle, and the point where the track begins to curve. Find the distance from this point to the station.

5. How many bricks 9 in. by $4\frac{1}{2}$ in. are required to cover a square yard?

6. Transform a triangle into a rectangle.

7. A circular grass-plot has a diameter of 400 ft., and a walk 3 yds. wide around it. Find the area of the grass-plot, and also the area of the walk.

8. A square, whose side is 4 in., is to be transformed into a rhombus in which one diagonal equals twice the other. Find the lengths of the diagonals and of one side of the rhombus. Can you solve this problem also by construction?

9. Find the radius of a circle equivalent to a square whose side is 11 ft. long.

10. Find the side of a square equivalent to a circle whose radius is 7 ft. long.

11. Find the area of a trapezium, if one diagonal is 54 ft., and the perpendiculars to this diagonal from the opposite vertices are 23 ft. 9 in. and 36 ft. 6 in.

12. Compare the areas of a circle, a square, and an equilateral triangle, if the perimeter of each figure is 132 ft.

Lesson 72. Review.

1. Review all the italicized exercises in this chapter.

2. Find the area of a right triangle, if the hypotenuse is $8\frac{1}{2}$ in. and one of the legs is 4 in.

3. Find the area of an isosceles triangle, if the base is 20 ft. and each of the other sides is 26 ft.

4. Draw any pentagon, and transform it into a square.

5. Construct a square whose diagonal shall be 3 in., and then find its area.

6. Make a square half as large as a given square.

7. Find the side of a square equivalent to a rectangle 400 ft. long and 100 ft. wide.

8. One angle of a rhombus is 60°, and the shorter diagonal is 8 ft. Find the area (see p. 133, No. 7).

9. The angle of a sector is 60°, and the chord of the arc is 14 ft. Find the area of the sector.

10. Two ships, each making 6 miles an hour, sail from the same port. One sails at noon, directly east; the other sails at 3 P.M., directly south. How far apart will they be at 6 P.M.?

11. The top of a centre-table has the shape of a regular octagon. One side measures 2 ft. What will it cost to stain and polish the surface at $2.25 per square foot?

12. A room is $31\frac{1}{2}$ ft. by 17 ft., and $12\frac{1}{2}$ ft. high, with 3 windows each $5\frac{1}{2}$ ft. by 6 ft., 2 doors each $7\frac{1}{2}$ ft. by 4 ft., and a fireplace 7 ft. by $6\frac{1}{2}$ ft. The carpet is 27 in. wide, costs $2.50 per yard, and is laid lengthwise along the floor. The paper is 27 in. wide, and costs $1.80 per roll of 10 yds. Find the cost of the carpet and the paper.

Lesson 73. Review.

1. *Define and name the metric units of area.*

The metric units of area are the squares of the units of length. They are named by placing the word "square" before the name of the corresponding unit of length. Thus: **square meter, square decimeter,** etc.

The square dekameter is also called an **ar,** and the square hectometer a **hectar.**

The abbreviations are formed by placing the letter *q* before those of the corresponding units of length. Thus: *qm* for square meter; *qcm* for square centimeter, etc.

2. *How do they stand related to each other?*

Each unit is 100 times larger than the next smaller one. For example: $1^{qm} = 100^{qdm}$; $1^{qdm} = 100^{qcm}$, etc.

3. Show by a figure that 1 square meter $= 100^{qdm}$.

4. Reduce to square meters 4^{dkm}; $1{,}840{,}000^{qmm}$.

5. How many square meters are there in a hectar?

6. Show that $1^{qm} = 10.764$ sq. ft. (see bottom of p. 33).

7. Show that there are about $15\frac{1}{2}$ sq. in. in 1^{qdm}.

8. A square park contains 64 hectars. How many trees, 20^{m} apart, can be set out around it?

9. If glass is worth $1.40 per square meter, what will be the cost of 1000 panes of glass each 48^{cm} square?

10. Find the area of the largest rectangle which you can make under the condition that its perimeter is 60^{cm}.

11. A table 2.3^{m} by 1.2^{m} is covered with cloth which overlaps 3^{cm} on all sides. How much cloth was required?

12. Find the side of a square equivalent to a triangle whose base is 90^{m} and altitude is 20^{m}.

13. A ladder 13^{m} long leans against the top of a wall, its foot being 5^{m} from the bottom. How high is the wall?

14. Find the area of an equilateral triangle if the length of one side is 20^{m}.

15. Find the cost of covering a round table with cloth 60^{cm} wide, and costing \$2 per meter, if the radius of the table is equal to 75^{cm}.

16. Find the cost of making a stone walk 1^{m} wide around a circular reservoir 64^{m} in diameter, the price paid being \$5 per square meter.

17. At a point E in one side of the rectangular field $ABCD$ there is an artesian well. The owner divides the field into three unequal parts by lines drawn from E to D and C. Find the value of each part, if $AD = AE = 81^{m}$, $EB = 144^{m}$, and a hectar is worth \$3400.

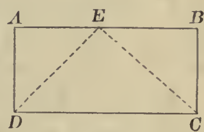


Fig. 137.

18. The radius of a circle is 4^{m}. Find the areas of the circumscribed and the inscribed squares.

19. In order to find the area of a swamp, $ABCDEF$, it was first surrounded by a rectangle, as shown in Fig. 138. The following measurements were then made:

$GB = 42.5^{m}$, $BC = 47^{m}$,

$CH = 75.5^{m}$, $HD = 61.2^{m}$,

$DI = 68.5^{m}$, $IE = 68.5^{m}$,

$KF = 29.5^{m}$, $FA = 58.8^{m}$.

Find the value of the swamp, one hectar being worth \$1500.

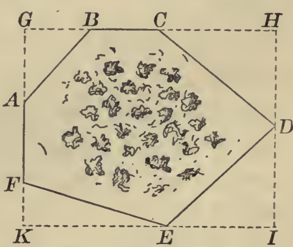


Fig. 138.

20. The diagonals of a rhombus are 4^{m} and 6^{m}. Find its area, its perimeter, and its altitude.

Lesson 74. Review.

1. Review all the italicized exercises in this chapter.

2. Find the area of the figure which is given to you.

3. If a triangle and a parallelogram have equal bases and equal areas, what is true of their altitudes?

4. Find the radius of a circle three times as large as a circle whose radius is 7^m.

5. Show by a figure that the square of $\frac{1}{3}$ is equal to $\frac{1}{9}$.

6. The three dimensions of a room are each 6.4^m. Find the cost of papering the walls at 30 cents per square meter.

7. Through the middle of a rectangular garden, 180^m by 76^m, there are two paths $\perp$ to each other and $\parallel$ to the sides. Find the area of the remainder of the garden.

8. If a rhombus and a square have equal perimeters, which has the greater area? Prove your answer correct.

9. Find the side of a square equivalent to a trapezoid whose bases are 60^m and 32^m, and altitude is 124^m.

10. Find the side of a square equivalent to a regular hexagon, one side of which is 8^m.

11. The perimeter of a regular octagon is 16^m. Find the area.

12. One leg of an isosceles right triangle is 32^m. Find the area.

13. The diagonal of a square is 12^m. Find the area.

14. Find the length of the shortest chord that can be drawn through a point 9^{cm} from the centre of a circle whose radius is 15^{cm}.

15. What is the mean proportional between 9 and 16?

16. The angle of a sector is 120°, and the length of its arc is 66^{cm}. Find its area.

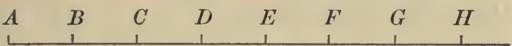
CHAPTER VIII.

RATIOS.

Lesson 75.

1. *Define the numerical measure of a quantity.*

The number of times one quantity contains another quantity of the same kind is called the **numerical measure** of the first quantity referred to the second as a unit.



For example: if $AB = BC = CD = DE = EF = FG = GH$, the numerical measure of AH referred to AB as a unit is 7, and the numerical measure of AB referred to AH as a unit is $\frac{1}{7}$; in other words, $AH = 7\,AB$, $AB = \frac{1}{7}$ of AH.

2. If in the above figure AB is taken as the unit, what is the numerical measure of AG? AF? AE? AD? AC? AB?

3. If AC is taken as the unit, what is the numerical measure of AH? AG? AF? AE? AD? AC? AB?

4. *Define the ratio of two numbers.*

The **ratio** of one number to another number is their relative magnitude, and is found by dividing the first number by the second number.

A ratio is written either as a fraction or by placing a colon (:) between the two numbers. For example:

the ratio of 4 to 7 is $\frac{4}{7}$, or $4:7$;
the ratio of 3 to 1 is $\frac{3}{1}$ (that is, 3), or $3:1$.

The two numbers which form a ratio are called its **terms**.

5. *Define the ratio of two quantities.*

The **ratio** of two quantities of the same kind is the ratio of their numerical measures referred to the same unit.

(1) The two quantities must be *the same in kind.* Two quantities which differ in kind cannot have a ratio to each other. For example, we cannot compare, as regards magnitude, a line with an angle, or dollars with days.

(2) The two quantities must be *referred to the same unit.* We may compare, for example, 6 in. with 2 ft., but their ratio is not 6 : 2, because 1 ft. = 12 in.; it is 6 : 24, or 1 : 4.

6. In the above figure what is the ratio of AC to AD? AD to AC? AC to AH? AD to AG? AE to AG? etc.

7. What is the ratio of 2 in. to 6 in.? 2 in. to 6 ft.?

8. What is the ratio of 15° to a right angle?

9. Problem.—*To divide a straight line AB into two parts having a given ratio; say, 3 : 4.*

Draw through A a line AC, making an acute angle with AB.

Begin at A and lay off upon AC 3 + 4, or 7, equal parts.

Mark the end of the 7th part D, and the end of the 3d part E.

Join BD, and draw $EF \parallel$ to BD, and cutting AB at F.

Then $AF : BF = 3 : 4$.

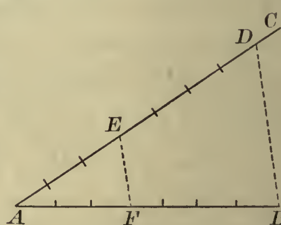


FIG. 139.

PROOF. Draw parallels to BD through all the points of division in AC, and prove that they divide AB into 3 + 4, or 7, equal parts (p. 25, No. **4**; p. 86, No. **4**).

Therefore $AF : FB = 3 : 4$.

10. Divide a line into two parts, having the ratio 2 : 5.

11. *Compare two triangles whose altitudes are equal.*

Let the triangles ABC, ADC (Fig. 140), have the same altitude CE, but let their bases AB, AD, be unequal.

The areas of the two triangles have the values,

$$\triangle ABC = \tfrac{1}{2} AB \times CE, \ \triangle ADC = \tfrac{1}{2} AD \times CE.$$

If we divide one of these equations by the other, we obtain the ratio of the areas written as a fraction; this fraction may then be reduced to its lowest terms by cancelling the common factors $\frac{1}{2}$ and CE. We have as the result,

$$\frac{\triangle ABC}{\triangle ADC} = \frac{\frac{1}{2} AB \times CE}{\frac{1}{2} AD \times CE} = \frac{AB}{AD}.$$

Hence we see that *triangles having equal altitudes are to each other as their bases.*

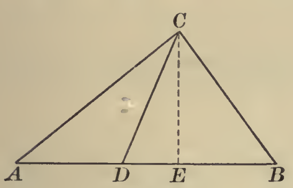


FIG. 140.

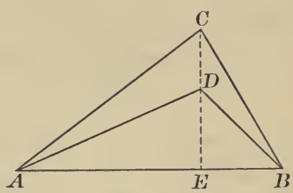


FIG. 141.

12. *Show that two triangles having equal bases are to each other as their altitudes (Fig. 141).*

13. *Compare two parallelograms having equal altitudes.*

14. *Compare two parallelograms having equal bases.*

15. Two parallelograms have equal altitudes, but the base of one is ten times that of the other. Compare their areas.

16. A triangle and a parallelogram have equal bases, but the altitude of the triangle is four times that of the parallelogram. What is the ratio of their areas?

17. Construct a square and a rhombus so that their perimeters shall be equal, and their areas be to each other as 3 : 2.

Lesson 76.

1. *Compare the two parts into which a triangle is divided by drawing a line from one vertex to the opposite side.*

In the triangle ABC (Fig. 142) let CD be drawn from the vertex C to any point D of the base AB. The two triangles thus formed ADC, BDC, have the same altitude CE, but unequal bases AD, BD. Therefore (p. 141, No. **11**),

$$\triangle ADC : \triangle BDC = AD : BD, \text{ or } \frac{\triangle ADC}{\triangle BDC} = \frac{AD}{BD}.$$

Hence *a line drawn from the vertex to the base of a triangle divides the triangle and its base into parts that have the same ratio.*

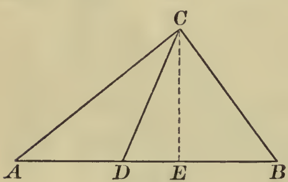


Fig. 142.

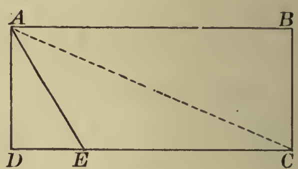


Fig. 143.

2. How must the line CD (Fig. 142) be drawn in order to divide $\triangle ABC$ into two equivalent parts? What would be the ratio of these parts? What would be the ratio of one of them to the whole triangle?

3. Draw a triangle and divide it into two parts which shall be to each other as $2:3$. What part of the whole triangle is each one of the parts?

4. It is desired to run a division line AE across a rectangular field $ABCD$ (Fig. 143) so that the triangle cut off, ADE, may be equal in area to $\frac{1}{8}$ of the whole field. How would you determine the point E? What part of DC is DE? What is the ratio of DE to EC?

5. In Fig. 143, if $DE = \frac{1}{5} DC$, what is the ratio of DE to DC? DC to DE? DE to EC? What part of the rectangle $ABCD$ is the $\triangle ADE$?

How should AE (Fig. 143) be drawn in order that

6. $\triangle ADE$ may be $\frac{1}{4}$ of the rectangle $ABCD$?

7. $\triangle ADE$ may be $\frac{1}{6}$ of the rectangle $ABCD$?

8. $\triangle ADE$: rectangle $ABCD = 1 : 12$?

9. $\triangle ADE$: rectangle $ABCD = 3 : 7$?

10. Draw a triangle and divide it into three parts which shall be to each other as the numbers 1, 2, 3.

11. Draw a rectangle and then divide it into four equivalent parts by lines drawn from one of the vertices.

12. Draw a rectangle and then divide it into three equivalent parts by lines drawn from one of the vertices.

13. Draw a rectangle and then divide it by a line ∥ to one side into two parts having the ratio $2 : 3$. What fraction of the rectangle is each part?

14. In a triangle ABC (Fig. 144) the middle points D, E, of the sides AB, AC, are joined. What is the ratio of the triangle ADE to the triangle ABC?

Join BE. Since $AD = \frac{1}{2} AB$,
$\triangle ADE = \frac{1}{2} \triangle ABE$ (p. 141, No. **11**).
Also, since $AE = \frac{1}{2} AC$,
$\triangle ABE = \frac{1}{2} \triangle ABC$.
Substituting this value of $\triangle ABE$ in the first equation, we have:
$\triangle ADE = \frac{1}{2} \times \frac{1}{2} \triangle ABC = \frac{1}{4} \triangle ABC$;
or, written as ratios, $\triangle ADE : \triangle ABC = 1 : 4$.

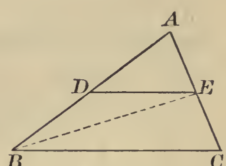


Fig. 144.

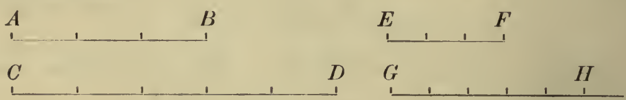
15. In Fig. 144, if $AD = \frac{1}{3} AB$, and $AE = \frac{1}{3} AC$, show that $\triangle ADE : \triangle ABC = 1 : 9$.

Lesson 77.

1. *Define a proportion.*

Four quantities are said to be **proportional,** or to form a **proportion,** if the ratio of the first to the second is equal to the ratio of the third to the fourth.

For example: since $\frac{4}{10} = \frac{2}{5}$, the numbers 4, 10, 2, 5, are proportional.



Again, the lines AB, CD, have the ratio 3 : 5, and the lines EF, GH, have the same ratio 3 : 5. Therefore the four lengths AB, CD, EF, GH, form the proportion

$$AB : CD = EF : GH,$$

which is read "AB is to CD as EF is to GH."

The four quantities that form a proportion are called its **terms**; the first and fourth are called the **extremes**; the second and third the **means**.

2. Give an example of four proportional numbers.

3. Give examples of proportions on pp. 142, 143.

4. *Prove that, if four numbers form a proportion, the product of the extremes is equal to the product of the means.*

If the four numbers a, b, c, d, are proportional, then

$$\frac{a}{b} = \frac{c}{d}.$$

Multiply both sides of this equation by the product bd.

Then $$\frac{abd}{b} = \frac{cbd}{d}.$$

Cancelling common factors, we have left $ad = bc$.

5. *Find the first term of a proportion if the other three terms are 9, 28, 7.*

Let x denote the first term; then $x : 9 = 28 : 7$.

Hence (No. **5**), $7x = 9 \times 28$.

Therefore $$x = \frac{9 \times 28}{7} = 9 \times 4 = 36.$$

In general, *either extreme is equal to the product of the means divided by the other extreme.*

6. Find the first term if the other terms are 3, 7, 84.

7. Find the fourth proportional to 13, 65, 120.

8. *Compare two circumferences and their radii.*

Let c, d, denote the circumferences; r, s, their radii.

Then $c = 2\pi r$, $d = 2\pi s$ (p. 108, No. **1**).

Therefore $$\frac{c}{d} = \frac{2\pi r}{2\pi s} = \frac{r}{s}.$$

Or, *two circumferences are to each other as their radii.*

9. *Compare the areas of two circles and their radii.*

Let a, b, denote the areas of two circles; r, s, their radii.

Then $a = \pi r^2$, $b = \pi s^2$ (p. 124, No. **1**).

Therefore $$\frac{a}{b} = \frac{\pi r^2}{\pi s^2} = \frac{r^2}{s^3}.$$

In other words, *the areas of two circles are to each other as the squares of their radii.*

10. How is the circumference, and also the area, of a circle changed if the radius is doubled? trebled? halved? ?

11. The radii of two circles are as $2 : 5$. The area of the first circle is 100 sq. ft. What is the area of the other circle?

12. The radii of three circular tanks are 6 ft., 9 ft., and 15 ft. If a mason charge $5 for cementing the smallest tank, what ought he to charge for cementing each of the others?

Lesson 78.

1. *Define similar figures (see p. 64, No. 3).*

Similar figures are figures that have the same shape.

For example, all circles are similar figures.

2. *Define similar triangles.*

Two triangles have the same shape if their angles are equal, each to each; hence we may define similar triangles as *triangles that are equiangular with respect to each other.*

3. Explain by reference to Fig. 145 what is meant by two triangles being equiangular with respect to each other.

4. Name by pairs the homologous sides of the triangles in Fig. 145. For the meaning of homologous, see p. 65, No. 4.

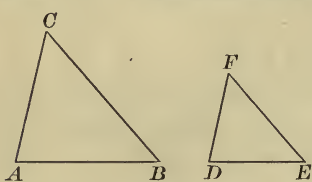


FIG. 145.

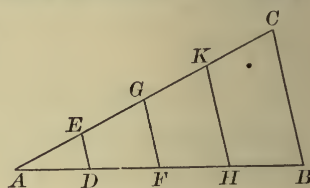


FIG. 146.

5. *What is the most important property which two similar triangles have?*

The homologous sides of two similar triangles are proportional.

NOTE. — This property can be deduced as a necessary consequence of the definition in No. 2, but the proof will not be given here.

6. One proportion formed by the homologous sides in Fig. 145 is $AB : DE = BC : EF$. State the two others.

7. If $AB = 15$ ft., $DE = 9$ ft., $BC = 20$ ft., find EF.

8. If $AC = 10$ ft., $BC = 20$ ft., $DF = 6$ ft., find EF.

9. If (Fig. 145) AB is three times DE, what is true of AC and DF? of BC and EF?

10. *Between the sides AB, AC of the triangle ABC (Fig. 146) a line DE parallel to BC is drawn. Compare the two triangles ABC, ADE.*

The $\angle A$ is common to the two triangles.

$\angle ADE = \angle ABC$, and $\angle AED = \angle ACB$. Why?

Therefore the triangles ABC, ADE are similar (No. **4**).

11. The side AB of $\triangle ABC$ (Fig. 146) is divided into four equal parts, and through the points of division D, F, H, parallels to BC are drawn, meeting AC at E, G, K, respectively. How many similar ⚠ are formed? Why similar? State several proportions formed by their homologous sides.

12. Let (Fig. 146) $AD = 14$ ft., $AE = 18$ ft., $DE = 9$ ft. Find AB, AC, BC.

13. Prove that $AE = EG = GK = KC$. (Two methods.)

14. Draw any line AB, and then construct a length CD so that $CD : AB = 5 : 3$ (see No. **10**, and p. 140, No. **9**).

15. Draw any triangle ABC, and construct a similar triangle so that the ratio of any two homologous sides shall be equal to that of the numbers 4 and 7.

16. *Define ratio of similitude.*

The ratio of similitude of a triangle with respect to a similar triangle is the ratio of any one of its sides to the homologous side of the similar triangle.

If (Fig. 145) $AB = 15$ ft., and $DE = 9$ ft., then the ratio of similitude of $\triangle ABC$ with respect to $\triangle DEF$ is $\frac{5}{3}$, or $5 : 3$, and that of $\triangle DEF$ with respect to $\triangle ABC$ is $\frac{3}{5}$, or $3 : 5$.

17. What is the ratio of similitude of $\triangle ABC$ (Fig. 146) with respect to $\triangle ADE$? $\triangle ABC$ with respect to $\triangle AFG$? etc.

The sides of a triangle are 12, 16, 20. Find the sides of a similar triangle if its ratio of similitude is

18. $2 : 1$.	**20.** $1 : 4$.	**22.** $2 : 3$.	**24.** $1 : 20$.
19. $1 : 2$.	**21.** $5 : 2$.	**23.** $4 : 3$.	**25.** $100 : 1$.

Lesson 79.

1. *How can the height of a vertical object be found by means of its shadow?*

Let the tree AB (Fig. 147) be the object. Fix a rod DE vertically in the ground near the tree. Measure DE and the lengths AC, DF of the shadows cast by the tree and the rod. Since rays of light are parallel to one another, the triangles ABC, DEF are equiangular with respect to each other; hence they are similar; hence *the shadows cast by the two objects are proportional to the heights of the objects.* Therefore if x denote the height of the tree AB,

$$\frac{x}{DE}=\frac{AC}{DF}; \text{ whence } x=\frac{DE\times AC}{DF}.$$

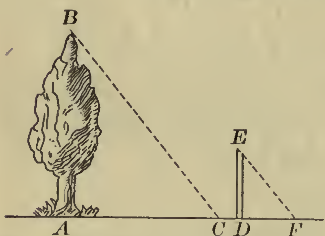


Fig. 147.

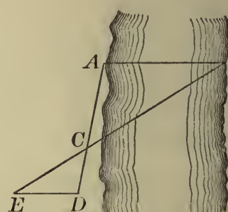


Fig. 148.

2. A tree casts a shadow 78 ft. long at the same time that a rod 8 ft. high casts a shadow $6\frac{1}{2}$ ft. long. How high is the tree?

3. Under what condition would the length of the shadow AC (Fig. 147) be just equal to the height AB of the tree?

4. Explain how you would proceed in order to find the breadth AB of a river (Fig. 148) by means of two similar triangles ABC, EDC.

If $AC=300$ ft., $CD=50$ ft., $ED=150$ ft., find AB.

5. If $ED=500$ ft., and $AC=2\,CD$, find AB.

6. *In what way are the areas of two similar triangles related to their homologous sides?*

The relation is expressed by the following theorem:

The areas of two similar triangles are proportional to the squares of any two homologous sides.

The proof of this theorem will not be given here.

7. *Illustrate the meaning of this theorem.*

Draw a $\triangle ABC$ (Fig. 149), divide each side into 4 equal parts, and draw through the points of division lines ∥ to the sides. Several similar ⊿s will be formed. Why similar? Also the $\triangle ABC$ will be divided into 16 equal ⊿s in such a way that the ratio of the areas of any two similar ⊿s is seen almost at a glance.

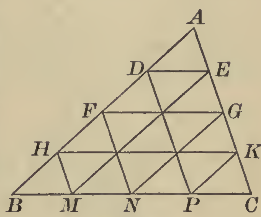


FIG. 149.

Compare, for example, the ⊿s ABC, AHK. The ratio of their areas is seen to be 16 : 9, or equal to the *square* of the ratio 4 : 3 of any two homologous sides.

8. Compare the areas and the homologous sides of the ⊿s ADE, AFG; the ⊿s ADE, AHK; the ⊿s AFG, AHK, etc.

9. The homologous sides of two similar triangles are 3 ft. and 7 ft. What is the ratio of their areas? If the area of the first triangle is 36 sq. ft., find the area of the second triangle.

10. Compare the areas of two similar triangles having a and b for homologous sides, if $a = 2b$; $a = 3b$; $a = 10b$; $a = \frac{1}{6}b$.

11. The side of a $\triangle = 3$ ft. Find the homologous side of a similar $\triangle$ 25 times as large; also of one half as large.

12. It is desired to divide a $\triangle ABC$ into two equivalent parts by drawing a line DE ∥ to BC. If $AB = 400$ ft., $AC = 500$ ft., $BC = 600$ ft., find AD and AE.

Lesson 80.

1. *Show that two triangles are similar if they have two angles equal each to each.*

Apply No. **3**, p. 63, and No. **2**, p. 146.

2. *Prove that in a right triangle the perpendicular dropped from the vertex of the right angle to the hypotenuse divides the triangle into two right triangles which are similar to each other and to the entire triangle (Fig. 150).*

In the $\triangle ABC$ let
$BC = a$, CD be $\perp$ to AB,
$AC = b$, $\angle ACB = 90°$,
$AB = c$, $\angle BAC = r$,
$AD = d$, $\angle ABC = s$,
$DB = e$,
$CD = h$.

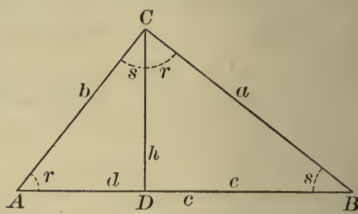


Fig. 150.

Show that $\angle ACD = s$, and $\angle BCD = r$, and then apply No. **2**, p. 146, or No. **1**, above.

3. In Fig. 150 a right triangle is divided into two similar right triangles, as explained in No. **2**, and the sides and angles are named by small letters, as shown in the figure. Complete the following table by writing with each angle the homologous sides of the three triangles.

	$\triangle$ *ABC*	$\triangle$ *BCD*	$\triangle$ *ACD*
90°			
r			
s			

4. Six *different* proportions may be formed from the homologous sides of the similar triangles in Fig. 150, and it is very easy to form them after the table in No. **3** has been completed. Write out all these proportions in a vertical column, thus:

$$c : a = a : e,$$
$$c : a = b : h.$$
$$. \quad . \quad . \quad . \quad .$$
$$. \quad . \quad . \quad . \quad .$$

5. Transform the six proportions obtained in No. **4** into equations by placing in each case the product of the means equal to the product of the extremes.

6. Define a mean proportional (p. 128, No. **1**).

The proportion $c : a = a : e$ gives the equation $a^2 = ce$.

Between what two lines is a a mean proportional?

7. What two other examples of mean proportionals are there among the equations obtained in No. **5**.?

8. Prove that the construction in No. **2**, p. 128, is correct.

HINTS.—First show that the triangle ACD (Fig. 134) is a right triangle; then make use of one of the proportions which can be obtained from the table in No. **3**.

9. Among the equations obtained in No. **5** are the two following: $a^2 = ce$, $b^2 = cd$. Hence show that $a^2 + b^2 = c^2$. State this result as a theorem. Compare with No. **1**, p. 126.

HINT.—Observe that $d + e = c$.

10. If (Fig. 150) $c = 9$ in., $e = 4$ in., find a, h, and b.

11. If (Fig. 150) $a = 6$ in., $b = 8$ in., find c, d, and e.

12. If (Fig. 150) $e = 3$ in., $h = 4$ in., find a, b, and c.

13. One of the equations obtained in No. **5** is the following: $ab = ch$. State this equation as a theorem.

Lesson 81. Review.

1. Review the italicized exercises in Lessons 75–77.

2. What is the ratio of 3 inches to 2 feet?

3. What is the ratio of 6 feet to 9 yards?

4. What is the ratio of $\frac{1}{2}$ of an inch to $\frac{1}{8}$ of an inch?

5. What is the ratio of $\frac{1}{2}$ of an inch to $\frac{1}{4}$ of a foot?

6. Produce a straight line AB to a point C such that $AB : BC = 4 : 5$. What is the ratio of AC to BC?

7. Divide a given line AB into three parts proportional to the numbers 2, 3, 5.

8. Divide a given line AB into three parts proportional to the numbers 1, $\frac{1}{2}$, $\frac{1}{3}$.

9. Construct a triangle whose sides shall be to each other as the numbers 3, 4, 5.

10. Construct an isosceles triangle so that the ratio of the base to one leg shall be as $2 : 3$.

11. Compare the areas of a triangle and a parallelogram if they have equal bases and equal altitudes.

12. A triangle and a parallelogram have equal areas and equal altitudes. What is the ratio of their bases?

13. Divide a triangle into two parts which are as 3 to 7.

14. A triangular lot is worth $6000. Show by a figure how you would run a line from one corner to the opposite side so as to cut off a piece worth $2400.

15. Divide a parallelogram by a line drawn from one vertex into two parts having to each other the ratio $3 : 8$.

16. Find the fourth proportional to the numbers 9, 117, 10.

17. If the radii of two circles are as $1 : 4$, and the area of the first circle is 1 square inch, what is the area of the other circle?

18. Given a circle; construct a concentric circle $\frac{1}{4}$ as large.

Lesson 82. Review.

1. Review the italicized exercises in Lessons 78–80.

2. Prove that two isosceles triangles are similar if an angle of one is equal to the homologous angle of the other.

3. Are two equilateral triangles similar? Why?

4. The sides of a triangle are 5, 6, 7. Find the sides of a similar triangle if its ratio of similitude to the given triangle is 10 : 7.

5. Draw an isosceles triangle, and then construct a similar triangle whose ratio of similitude to the first triangle is 3 : 1. What is the ratio of their areas?

6. Compare the areas of the two parts into which a parallelogram is divided by the line joining the middle points of two adjacent sides. (See p. 84, No. **3**; p. 143, No. **14**.)

7. The ratio of similitude of two similar triangles is 3 : 5. The first triangle has an area of 27 sq. ft. What is the area of the other triangle?

8. A vertical rod 6 ft. high throws a shadow 4 ft. long at the same time that a church spire casts a shadow 120 ft. long. How high is the spire?

9. In order to find the distance from my position A to the enemy's fort B, I run a line $AC \perp$ to AB and equal to 2000 ft. Through a point D in AC 100 ft. from C I run a line $DE \parallel$ to AB, meeting BC at E. If DE is found to be 800 ft., how far am I from the fort?

10. The radius of a circle $= 17$ ft. Through a point upon a diameter 9 ft. from the circumference a chord $\perp$ to this diameter is drawn. Find the length of this chord.

11. How many miles is the light of a lighthouse 200 ft. high visible at sea? (Radius of earth $= 4000$ miles.)

Lesson 83. Review.

1. Review all the italicized exercises in this chapter.

2. What is the ratio of 4^{m} to 1^{km}?

3. What is the ratio of 5^{cm} to 5^{m}?

4. What is the ratio of 8^{cm} to 24^{dm}?

5. What length is to 60^{m} as 5 is to 6?

6. The areas of two triangles are 480^{qm} and 300^{qm}. If their bases are equal, compare their altitudes.

7. If the altitude of the larger triangle in No. **6** is 30^{m}, find its base and the altitude of the other triangle.

8. Compare the areas of two similar triangles if the ratio of similitude is equal to $\frac{1}{3}$.

9. From one corner of a field in the shape of an equilateral triangle I wish to cut off a similar triangle equal in area to $\frac{1}{100}$ of the entire triangle. How must I run the division line?

10. A rectangle 36^{cm} by 20^{cm} is so divided into three parts by lines drawn from one vertex that the parts are as the numbers 2, 3, 4. How great is each part? How are the sides divided by the division lines?

11. Two sides of a field in the shape of an isosceles right triangle are each 400^{m} long. It is desired to cut from the field a lot containing one hectar of land by a straight fence starting from the vertex of the right angle. What portion of the hypotenuse will be cut off by this fence?

12. What is the mean proportional between 49 and 121?

13. How high is a tower if it casts a shadow 28^{m} long at the same time that a vertical staff 4^{m} high casts a shadow 1.45^{m} long?

See also p. 4.

CHAPTER IX.

SOLIDS.

Lesson 84.

1. *Define a plane (see p. 9, No.* 9*).*

A plane is named by *three* or more letters written on its boundary lines, or by a single letter written upon its surface; thus the plane in Fig. 151 is the plane $ABCD$, or the plane ABC, or simply the plane M.

2. *When is a straight line parallel to a plane?*

A straight line and a plane are parallel when they will not meet, however far both are produced.

3. *What is the foot of a line?*

If a straight line meets a plane, the point where it meets the plane is called its **foot**.

4. *When is a straight line perpendicular to a plane?*

When it is perpendicular to every straight line that can be drawn through its foot in the plane.

5. Name in Fig. 151 lines ∥ to the plane M, ⊥ to M, oblique or inclined to M.

6. Are two lines necessarily parallel if they are ⊥ to the same plane? If they are ∥ to the same plane?

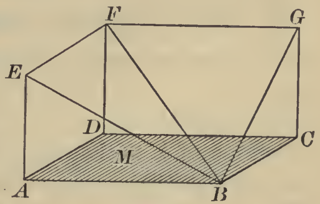


Fig. 151.

Illustrate your answers by reference to Fig. 151.

7. *When are two planes parallel to each other?*

Two planes are parallel if they will not meet, however far both are produced.

Two parallel planes are everywhere equally distant, and their distance apart is equal to the length of a perpendicular dropped from any point in one of the planes to the other plane.

8. *What is the intersection of two planes?*

The intersection of two planes is a straight line.

9. *Define a dihedral angle.*

If two planes intersect, the amount of rotation about the intersection required to make one of the planes coincide with the other is called the **dihedral** angle of the two planes.

The two planes are the **faces** of the angle; their intersection is the **edge** of the angle.

Dihedral angles are expressed in degrees, etc., like ordinary angles.

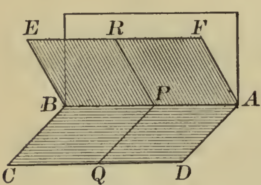


FIG. 152.

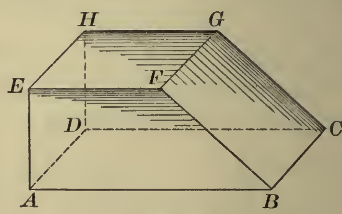


FIG. 153.

10. *Define the plane angle of a dihedral angle.*

The dihedral angle formed by two planes ABC, ABE (Fig. 152), is obviously equal to the angle formed by two lines PQ, PR, drawn one in each face from any point P of the edge and $\perp$ to the edge. The angle of any two lines so drawn is called the **plane angle** of the dihedral angle.

Thus, the angles RPQ, CBE, DAF, are plane angles of the dihedral angle in Fig. 152.

11. *When are two planes perpendicular to each other?*

When their dihedral angle is a right angle.

12. Name in Fig. 153 two parallel planes, two perpendicular planes, two inclined planes. In the last two cases what are plane angles of the dihedral angles?

13. If two parallel planes are cut by a third plane, what is true of their intersections? Illustrate by Fig. 153.

14. If two planes are each $\perp$ to a third plane, what is true of their intersection? Illustrate by Fig. 153.

15. *How are planes represented on paper?*

The part taken for representation is usually a rectangle.

If the plane is to be represented as coincident with, or parallel to, the plane of the paper, we simply draw the rectangle in its true shape. In all other cases we must substitute a parallelogram for the rectangle. The proper way to draw the parallelogram depends on the position of the plane with respect to the eye and the line of sight. By a little practice any one can learn to draw with tolerable accuracy planes in various different positions.

16. Draw two parallel planes joined by a line $\perp$ to them.

17. Draw two perpendicular planes.

18. Draw two planes forming a dihedral angle of about 45°.

19. Draw three planes each $\perp$ to the other two.

20. Draw three parallel planes, and a fourth plane intersecting them.

21. Draw a horizontal plane and three vertical planes, all passing through a line $\perp$ to the horizontal plane.

Lesson 85.

1. *Define a solid (see p. 9, No. 9).*

2. *Define a cube (Fig. 154).*

A **cube** is a solid bounded by six equal squares.

The six squares are called the **faces** of the cube, and their intersections are called its **edges**.

3. Name in Fig. 154 the faces and the edges that are (1) ∥ to the face $ABCD$; (2) ⊥ to the face $ABCD$.

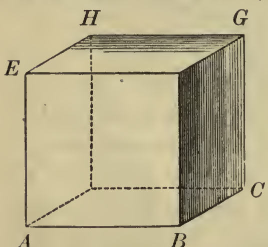


Fig. 154.

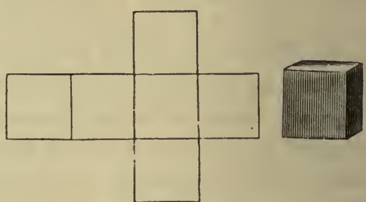


Fig. 155.

4. *How is a model of a cube made?*

Draw six equal squares on stiff cardboard, in one group, as shown in Fig. 155. Cut out the entire group, and then cut half-way through the cardboard, along the interior lines of division. Then fold the squares over these lines into the shape of a cube, and fasten the edges with slips of thin paper and mucilage (or paste).

5. *What is meant by developing a surface?*

To **develop** the surface of a solid is to construct upon a plane surface (as a sheet of paper) the surfaces, plane or curved, which form the boundaries of the solid.

The figure thus obtained is called the **development** of the surface. Fig. 155 shows the development of the cube.

6. Make a model of a cube (edge 3 in.).

7. *Define the volume of a solid.*

The **volume** of a solid is the amount of space it fills.

Volumes, like surfaces and lines, are measured by finding how many times a **unit** of volume is contained in the solid to be measured.

The common units of volume are the **cubic inch**, the **cubic foot**, and the **cubic yard**. Each of these is a cube whose edge is equal to the corresponding unit of length. 1 cub. ft. = 1728 cub. in.; 1 cub. yd. = 27 cub. ft.

8. *How is the volume of a cube found?*

Suppose, for example, that one edge = 10 in. (Fig. 156). The lower face *ABC* may be divided into 10×10, or 100, square inches (p. 000, No. **0**).

Upon each square inch a pile of 10 cubic inches may be formed (like those seen along the edge *DC*).

Hence the entire cube contains $10 \times 10 \times 10$, or 1000, cubic inches.

A similar result is obtained, whatever value the edge may have.

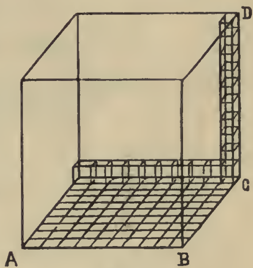


Fig. 156.

Finding the product of a number, when used three times as a factor, is called **cubing** the number; hence the result may be expressed by a formula, thus:

Volume of a cube = cube of one edge.

Find the volume and the entire surface, having given:

9. Edge 8 in.

10. Edge 9 in.

11. Edge 1 ft. 2 in.

12. Edge 2 ft. 3 in.

13. Find one edge if the volume is 8000 cubic feet.

14. How many square feet of lead are needed to line the bottom and sides of a cubical tank $4\frac{1}{2}$ feet deep, and how many cubic feet of water will the tank hold?

Lesson 86.

1. *Define a rectangular solid.*

A **rectangular solid** is a solid bounded by six rectangles. The rectangles are called its **faces**, and their intersections are called its **edges**.

The upper and lower faces are also called the **bases**, and the distance between them the **height**.

2. In the rectangular solid shown in Fig. 157, name the edges equal to AB; to BC; to BF. Also name the faces and edges ∥ to AB; ⊥ to AB.

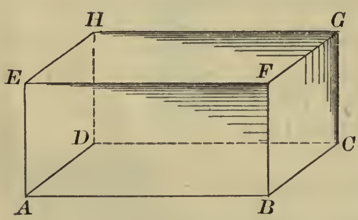


Fig. 157.

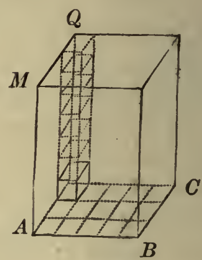


Fig. 158.

3. Name the three dimensions of the solid in Fig. 157.

4. Name in Fig. 157 three pairs of equal and parallel faces.

5. Draw two rectangular solids differing in shape.

6. Make a model of a rectangular solid 6 in. long, 4 in. wide, and 3 in. high (see p. 158, No. **4**).

7. *How is the volume of a rectangular solid found?*

Suppose that the dimensions are: length, 5 in.; breadth, 3 in.; height, 7 in. (see Fig. 158). The base may be divided into 3×5, or 15, square inches. Upon each square inch there may be formed a pile of 7 cubic inches; hence the solid must contain $3 \times 5 \times 7$, or 105, cubic inches. In general:

Volume of a rectangular solid = length × breadth × height.

Find the entire surface and the volume of a rectangular solid, having given:

8. Length 9 in., breadth 7 in., thickness 3 in.

9. Length $6\frac{3}{4}$ ft., breadth $5\frac{1}{2}$ ft., depth $4\frac{1}{4}$ ft.

10. Length $4\frac{1}{2}$ ft., breadth $1\frac{1}{2}$ yds., height 108 in.

11. Length 4 ft. 8 in., breadth 3 ft. 10 in., height 3 ft.

12. How many tons of coal will a bin 20 ft. by 16 ft. by 8 ft. hold, allowing 40 cubic feet to a ton?

13. A cellar which measures 12 ft. by 6 ft. is flooded to a depth of 4 in. Find the weight of the water, supposing that 1 cub. ft. of water weighs 1000 oz. ($62\frac{1}{2}$ lbs.).

14. What weight of water will a rectangular cistern hold, its length being 4 ft., breadth 2 ft. 6 in., depth 3 ft. 3 in., and 1 cub. ft. of water weighing 1000 oz.?

15. How many bricks 9 in. by $4\frac{1}{2}$ in. by 3 in. are needed to build a wall 90 ft. long, 18 in. thick, and 8 ft. high?

16. A book is 8 in. long, 6 in. wide, and $1\frac{1}{4}$ in. thick. Find the depth of a box 3 ft. 4 in. long, and 2 ft. 6 in. wide, that it may hold 400 such books.

17. Marble is 2.716 times as heavy as water, and 1 cub. ft. of water weighs 1000 oz. Find the weight of a block of marble 9 ft. 6 in. long, 2 ft. 3 in. wide, and 2 ft. thick.

18. Iron weighs 7.2 times as much as water, and 1 cub. ft. of water weighs 1000 oz. What will an open cistern made of iron 1 in. thick weigh when empty, if its external dimensions are 5 ft., 4 ft., and 3 ft.?

19. A cistern is 5 ft. 6 in. long, 3 ft. 9 in. wide, and 1 ft. 3 in. deep. How many gallons of water will it hold? What weight of water will it hold? (1 gal. of water = 231 cub. in.; 1 cub. ft. of water weighs 1000 oz.)

Lesson 87.

1. *Define a right prism, and terms related to it.*

A **right prism** is a solid bounded by two equal and parallel polygons called **bases,** and by three or more rectangles called the **lateral faces.** The intersections of the lateral faces are called the **lateral edges.**

The distance between the bases is called the **height** of the prism.

A prism is called **triangular** if the bases are triangles, **hexagonal** if they are hexagons, etc.

A **regular prism** is a right prism whose bases are regular polygons.

The prism in Fig. 159 is a regular hexagonal prism.

2. What positions with respect to the bases do the lateral faces of a right prism have? the lateral edges?

3. Name on a right prism the edges that are parallel.

4. Draw a triangular prism.

5. Draw a hexagonal prism.

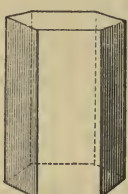


Fig. 159.

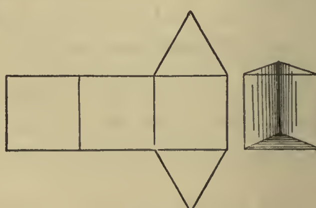


Fig. 160.

6. Make a model of a regular triangular prism (Fig. 160).

7. *How is the volume of a right prism found?*

It can be proved that the volume of a right prism is found by multiplying the area of the base by the height.

Volume of a right prism = base × height.

NOTE.—The word "prism" used alone means here "right prism."

Find the entire surface and the volume of a prism, having given:

8. Area of square base 49 sq. in., height 8 in.

9. Square base, side of base 4 in., height 10 in.

10. Rectangular base 6 ft. by $3\frac{1}{2}$ ft., height 12 ft.

11. Base an equilateral $\triangle$, side 2 ft., height 7 ft.

12. Base a regular hexagon, side 4 ft. 9 in., height $12\frac{1}{2}$ ft.

13. Base a regular hexagon, side 10 in., height 10 ft.

14. Is a cube a right prism? Why? Is a rectangular solid a right prism? Why?

15. What is the volume of a regular four-sided prism if the height is 6 in. and one side of the base is 2 in.? What would be the volume if the height were doubled? if the side of the base were doubled? if both were doubled?

16. The bases of a prism are trapezoids whose parallel sides are 12 ft. and 8 ft., and the altitude is 6 ft. Find the volume of the prism if the height is 32 ft.

17. How many cubic feet of stone are required to build a dam 1000 ft. long, 20 ft. high, 10 ft. wide at the bottom, and 4 ft. wide at the top? (Consider the dam to be a trapezoidal prism like that in No. **16**.)

18. The distance around a reservoir in the shape of a regular hexagon is 360 ft. If the average daily loss from evaporation amounts to a layer of water 2 in. deep, how many cubic feet of water must be supplied daily to maintain the water at a constant level?

19. Pencils are often made in the shape of regular hexagonal prisms. Find the volume of the pencil given you (correct to a hundredth of a cubic inch).

Lesson 88.

1. *Define a right cylinder, and the related terms.*

A **right cylinder** is the solid generated by a rectangle revolving about one of its sides.

A right cylinder is also called a **cylinder of revolution.**

If the rectangle $ABCD$ (Fig. 161) revolve about the side CD, the sides AD, BC, will describe equal and parallel circles, and the side AB will describe a curved surface. A right cylinder, therefore, is a solid bounded by two equal and parallel circles and a curved surface lying between the circles.

The circles are called the **bases** of the cylinder, and the curved surface is called its **lateral surface.**

The **axis** of a right cylinder is the line joining the centres of the bases; it is perpendicular to the bases.

The **height** of a cylinder is the length of its axis.

2. What is the development of the lateral surface? Illustrate by means of a sheet of paper.

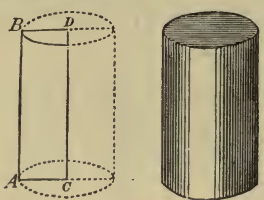


FIG. 161.

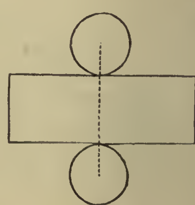


FIG. 162.

3. Draw two right cylinders differing in shape.

4. Make a model of a right cylinder (see Fig. 162)

5. *What are the general formulas for finding the lateral surface and the volume?*

Lateral surface = circumference of base × height.

Volume = base × height.

NOTE.—The word "cylinder" is here used meaning "right cylinder."

Find the lateral surface and the volume of a cylinder, having given:

6. Radius of base 7 in., height 10 in.

7. Radius of base 1 ft. 2 in., height 5 ft.

8. Diameter of base 9 ft. 4 in., height 12 ft.

9. Circumference of base 7 ft. 4 in., height 10 ft.

10. How large a cylinder can be made by rolling up a rectangular sheet of tin 88 in. by 66 in., so that the height of the cylinder is 88 in.? How large, if rolled up so that the height is 66 in.?

11. Find the height of a cylinder if the volume is 114 cub. yds. 2 cub. ft., and the radius of base 7 ft.

12. How many cubic yards of earth must be dug out to make a well 3 ft. in diameter, and 20 ft. deep?

13. The diameter of a well is 4 ft. 8 in., its depth is 30 ft. Find the cost of digging it at $3.75 per cubic yard.

14. How many cubic yards of earth must be dug out in making a tunnel 100 yds. long, whose section is a semicircle with a radius of 10 ft.?

15. What change in the volume of a cylinder is produced by doubling its height? by doubling the diameter of its base? by doubling both?

16. Two cylinders have the same height, but the radius of the base of one cylinder is six times that of the other. Compare their volumes; compare also their lateral surfaces.

17. Find the cost of cementing the side and bottom of a cylindrical tank 20 ft. deep and 18 ft. in diameter, at 32 cents per square foot.

18. How many gallons of water are there in a cylindrical well 7 ft. in diameter, if the water is 10 ft. deep and there are $7\frac{1}{2}$ gals. in each cubic foot?

Lesson 89.

1. *Define a right pyramid, and related terms.*

A **right pyramid** is a solid bounded by a polygon and three or more isosceles triangles which have a common vertex. The polygon is called the **base,** the triangles the **lateral faces,** and their common vertex the **vertex** of the pyramid.

The intersections of the lateral faces are called the **lateral edges.**

The distance from the vertex to the base is called the **height** of the pyramid.

A **regular pyramid** is a right pyramid whose base is a regular polygon. The lateral faces are equal isosceles triangles, and their common altitude is called the **slant height** of the pyramid. Thus, AB is the slant height of the regular pyramid represented in Fig. 163.

2. Draw a triangular pyramid (a pyramid with a △ for the base).

3. Draw a hexagonal pyramid.

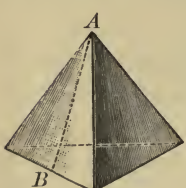


FIG. 163.

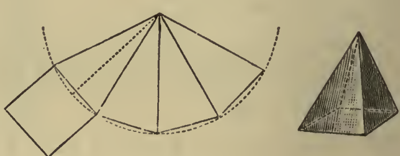


FIG. 164.

4. Make a model of a square pyramid (Fig. 164).

5. *How is the volume of a pyramid found?*

It can be proved that the volume of any pyramid may be found by means of the formula:

$$\textit{Volume of a pyramid} = \tfrac{1}{3} \times \textit{base} \times \textit{height}.$$

NOTE. — The word "pyramid," used alone, means "right pyramid."

6. Find the volume of a pyramid, having given: area of the base $= 64$ sq. in., height of pyramid $= 15$ in.

7. Find the volume of a pyramid if the height is 30 ft., and the base is a regular hexagon whose side is 6 ft.

8. Find the slant height of the pyramid in No. **7**.

9. Find the total surface of the pyramid in No. **7**.

Find the volume, and also the total surface of a square pyramid, having given:

10. Side of base 40 ft., height 48 ft.

11. Side of base 22 ft., height 60 ft.

NOTE. — The **frustum** of a pyramid is the portion of the pyramid contained between the base and a plane parallel to the base.

The base and the section made by the plane are called the **bases** of the frustum.

The distance between the bases is the **height**.

To find the volume of a frustum, *add together the areas of the bases and the square root of their product, and multiply the sum by $\frac{1}{3}$ the height.*

FIG. 165.

12. What kind of figures are the lateral faces of a frustum?

13. Find the total surface of a frustum of a square pyramid (Fig. 165) if the sides of the bases are 12 in. and 4 in., and the slant height (or altitude of each face) is 5 in.

14. Find the volume of the frustum in No. **13** if the height of the frustum is 3 in.

15. Find the volume of the frustum of a square pyramid if the sides of the bases are 21 yds. and 15 yds., and height 84 yds.

16. A church spire has the shape of a frustum of a regular hexagonal pyramid; each side of the base is 5 ft., and of the top 2 ft.; the altitude of each trapezoidal face is 20 ft. How many square feet of tin roofing are required to cover the lateral faces and the top?

Lesson 90.

1. *Define the right cone, and related terms.*

A **right cone** is the solid generated by a right triangle revolving about one of its legs.

A right cone is also termed a **cone of revolution**.

If the right triangle ABC (Fig. 166) revolve about the leg BC, the other leg AB will describe a circle called the **base** of the cone, and the hypotenuse AC will describe a curved surface called the **lateral** or **convex** surface of the cone.

The point C is called the **vertex** of the cone.

The leg BC, about which the triangle revolves, is called the **axis** of the cone.

The length of the axis is called the **height** of the cone.

The length of the hypotenuse AC of the generating triangle is called the **slant height** of the cone.

2. Draw two right cones differing in shape.

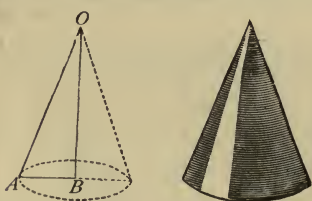


FIG. 166.

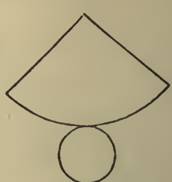


FIG. 167.

3. What kind of a figure is the development of the lateral surface of a right cone (see Fig. 167)?

4. Make a model of a right cone.

5. *What are the formulas for finding the lateral surface and the volume?*

$$\textit{Lateral surface} = \tfrac{1}{2} \times \textit{circumference of base} \times \textit{slant height}.$$

$$\textit{Volume} = \tfrac{1}{3} \times \textit{base} \times \textit{height}.$$

NOTE.—The word "cone," if used alone, means a "right cone."

6. The height of a cone is 6 in., the radius of the base is $2\frac{1}{2}$ in. Find the slant height (p. 126, No. **1**).

7. Find the volume and the lateral surface of a cone if the height is 40 ft., and the radius of the base is 9 ft.

8. Find the volume and the lateral surface if the height is 28 ft., and the radius of the base is 21 ft.

9. A right triangle whose legs are 3 in. and 4 in. generates a cone by revolving about its shorter leg. Find the volume and the lateral surface of this cone.

10. Solve No. **9** if the triangle revolve about the other leg.

11. How much canvas is required to make a conical tent 80 ft. high, and 70 ft. in diameter at the base?

NOTE.—The **frustum** of a cone is the part contained between the base and a plane ∥ to the base.

The bases and height are defined, and the volume found, as in the case of the frustum of a pyramid (p. 167).

The lateral surface = half the product of the circumference of base and the slant height.

FIG. 168.

12. How many square feet of tin will be required to make a funnel with the radii of top and bottom 14 in. and 7 in. respectively, and the height 24 in.?

13. Find the expense of polishing the curved surface of a marble column in the shape of a frustum of a right cone, slant height 12 ft., radii of bases 3 ft. 6 in. and 2 ft. 4 in., at 60 cents per square foot.

14. A round stick of timber is 20 ft. long, 3 ft. in diameter at one end, 2.6 ft. at the other. How many cubic feet does it contain?

15. A bucket is 16 in. deep, 18 in. wide at the top, and 12 in. wide at the bottom. How many gallons of water will it hold, reckoning $7\frac{1}{2}$ gallons to the cubic foot?

Lesson 91.

1. *Define a sphere.*

A **sphere** is the solid generated by a semicircle revolving about its diameter.

If the semicircle $ACBO$ (Fig. 169) revolve about the diameter AB, the semicircle generates a sphere, and the semi-circumference ACB generates a curved surface forming the boundary of the sphere. The centre O of the semicircle is the **centre** of the sphere. All points in a spherical surface are equidistant from the centre.

2. *Define a radius and a diameter of a sphere.*

In Fig. 169 OA is a radius, and AOB is a diameter.

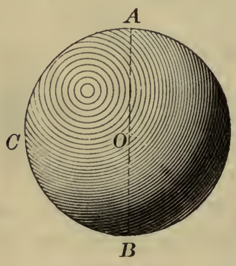


FIG. 169.

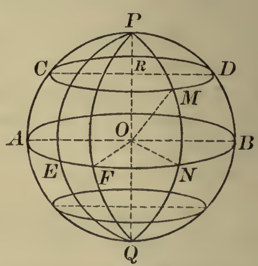


FIG. 170.

3. *Define great circles and small circles.*

Every section of a sphere made by a plane is a circle. If the plane pass through the centre, the circle is called a **great circle**; in all other cases the circle is called a **small circle.**

4. *Define the axis and poles of a circle.*

The **axis** of a circle (great or small) is the diameter which is perpendicular to the plane of the circle.

The **poles** of a circle are the ends of its axis.

5. Illustrate the above definitions by referring to Fig. 170.

6. *State three truths respecting great circles.*

(1) A great circle divides a sphere into two equal parts. These two equal parts are called **hemispheres.**

(2) Two great circles bisect each other.

(3) The poles of a great circle are at the distance of a quadrant from every point in its circumference.

7. *Define parallels.*

If a sphere is cut by a series of parallel planes, the circumferences of the circles thus formed are called **parallels.**

Example: parallels of latitude on the earth's surface.

8. *Define a zone and its altitude.*

A **zone** is the portion of the surface of a sphere contained between two parallel planes. The distance between the two planes is the **altitude** of the zone.

Example: the torrid zone, etc., on the earth's surface.

9. *Define meridians.*

If a sphere is cut by a series of planes all passing through the same diameter, the circumferences of the great circles thus formed are called **meridians.**

Example: meridians of longitude on the earth's surface.

10. *Define a lune.*

A **lune** is the portion of the surface of a sphere contained between two semi-circumferences of great circles.

11. Point out in Fig. 170 parallels, meridians, a zone, its altitude, a lune.

12. Draw Fig. 170 (to a larger scale if you prefer).

13. Draw a sphere cut by three planes passing through the centre, and each perpendicular to the other two.

14. If (Fig. 170) *PEQ* represent the meridian through Greenwich, point out the latitude and longitude of the point *M*.

Lesson 92.

1. *When is a sphere inscribed in a cylinder?*

A sphere is **inscribed** in a cylinder when its surface touches the bases and the lateral surface of the cylinder. Also, the cylinder is **circumscribed** about the sphere (Fig. 171).

2. Compare the radius and diameter of a sphere with the dimensions of the circumscribed cylinder.

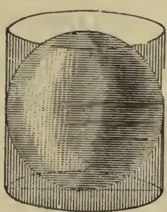


FIG. 171.

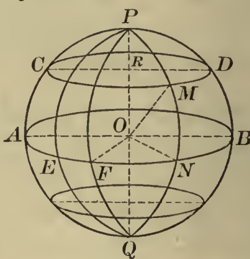


FIG. 172.

3. *How is the surface of a sphere found?*

It can be proved that the surface of a sphere is equal to *the lateral surface of the circumscribed cylinder.*

Let r denote the radius of a sphere. Then r is also equal to the radius of the base of the circumscribed cylinder, and $2r$ is its height; therefore its lateral surface is equal to $2\pi r \times 2r$, or $4\pi r^2$ (p. 164, No. **5**).

Hence if the value of π be taken as $\frac{22}{7}$,

$$\textit{Surface of a sphere} = \tfrac{88}{7} r^2.$$

4. *How is the volume of a sphere found?*

The volume of a sphere is found by multiplying the surface by *one-third* of the radius. Therefore if r denote the radius, the volume $= \frac{1}{3} r \times 4\pi r^2 = \frac{4}{3}\pi r^3$.

Hence if the value of π be taken as $\frac{22}{7}$,

$$\textit{Volume of a sphere} = \tfrac{88}{21} r^3.$$

Find the surface and the volume of a sphere, given:

5. Radius 1 in.

6. Radius 2 in.

7. Radius 3 in.

8. Radius 4 in.

9. Radius 3 ft. 6 in.

10. Diameter 7 in.

11. Diameter 16 in.

12. Diameter 21 in.

13. Diameter 42 ft.

14. Diameter 5 ft. 10 in.

15. A regulation base-ball is $9\frac{1}{4}$ in. in circumference. How many square inches of leather are required in order to make 1000 base-balls?

16. The circumference of a dome in the shape of a hemisphere is 66 ft. How many square feet of tin roofing are required to cover it?

17. If the ball on the top of St. Paul's Cathedral in London is 6 ft. in diameter, what would it cost to gild it at 7 cents per square inch?

18. The area of a zone is equal to *the product of its altitude and the circumference of a great circle.*

Find the area of the upper zone on the sphere in Fig. 172 if the altitude PR is 15 ft. and the radius is 35 ft.

19. The altitude of the torrid zone on the earth is about 3200 miles. What is its area in square miles, assuming the radius of the earth to be equal to 4000 miles?

20. If one cubic inch of iron weighs $4\frac{1}{2}$ oz., what will an iron ball $10\frac{1}{2}$ in. in diameter weigh?

21. Find the weight of a spherical shell 10 in. in diameter and 2 in. thick, composed of a substance 1 cub. ft. of which weighs 216 lbs.

22. How much rubber is there in a tennis ball whose diameter is $3\frac{1}{2}$ in. if the thickness of the rubber is $\frac{1}{4}$ in.?

How much cloth is needed to cover the ball?

How much pasteboard is needed to make a cylindrical box, open at the top, which will just hold the ball?

Lesson 93. Review.

NOTE.—1 cub. ft. of water weighs 1000 oz.; 16 oz. = 1 lb.

1. Review the italicized exercises in Lessons 84–88.

2. Draw two parallel planes cut by a third plane. What is true of the two lines of intersection?

3. Draw two planes each perpendicular to a third plane. What is true of their intersection?

4. Find the volume of the cube given you.

5. Find the volume of the rectangular solid given you.

6. Find the volume of the prism given you.

7. Find the volume of the cylinder given you.

8. What will be the cost of digging a cellar 36 ft. long, 20 ft. wide, and 8 ft. deep, at 5 cents per cubic foot?

9. What will it cost to sink a well 100 ft. deep and $3\frac{1}{2}$ ft. in diameter, at $2 per cubic foot?

10. If a cubic foot of brass is drawn into a wire $\frac{1}{11}$ of an inch in diameter, what will be the length of the wire?

11. How many cylindrical pieces of lead $\frac{3}{4}$ of an inch in diameter and $\frac{1}{8}$ of an inch thick must be melted, in order to form a cube whose edge is 3 in. long?

12. The piston of a pump is 14 in. in diameter, and moves through a space of 3 ft. How many tons of water will be thrown out by 1000 strokes?

13. A body is placed under water in a right cylinder 60 in. in diameter, and the level of water is observed to rise 30 in. Find the volume of the body.

14. How much will a brass cylinder weigh under water, if the height is 10 in. and the diameter of the base is 7 in.? Brass is 7.8 times heavier than water, and a body when immersed in water loses a weight equal to the weight of the water displaced.

Lesson 94. Review.

1. Review the italicized exercises in Lessons 89–92.

2. Find the volume of the pyramid given you.

3. Find the volume of the cone given you.

4. Find the volume of the sphere given you.

5. Make a model of a regular pyramid bounded by four equal equilateral triangles. Make each edge 4 in. long.

Then find the volume and the entire surface.

(The apothem of the base may be found from the table on p. 122.)

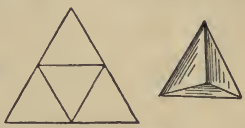


FIG. 173.

6. If the height of a cone and the radius of its base are known, how is the slant height found?

7. Can the surface of a sphere be developed?

8. How much asphalt varnish will be needed to coat 1000 spherical bombs, each $10\frac{1}{2}$ in. in diameter, if 1 pint of varnish will cover 300 square inches of surface?

9. If a cylinder and a cone have equal bases and equal volumes, compare their heights.

10. Find the volume of a cylinder circumscribed about a sphere whose radius is 7 in.

11. A cylindrical pail is partly filled with water. The radius of its base is $3\frac{1}{2}$ in. If a bullet $3\frac{1}{2}$ in. in diameter be dropped in, how much will the level of the water rise?

12. The chimney of a factory has the shape of the frustum of a square pyramid; its height is 180 ft., and the sides of its upper and lower bases are 16 ft. and 10 ft. respectively; the section of the flue is throughout the entire length a square whose side is 7 ft. How many cubic feet of brick does the chimney contain?

Lesson 95. Review.

1. *What are the chief metric units of volume?*

The **cubic meter**, having the abbreviation *cbm.*

The **cubic decimeter**, having the abbreviation *cdm.*

The **cubic centimeter**, having the abbreviation *ccm.*

The cubic meter, when used for measuring wood, is called a **stere**; and the cubic decimeter, when used for measuring fluid substances, is called a **liter.**

2. *How are the metric units of volume related?*

Since 10 centimeters = 1 decimeter, it follows that $10 \times 10 \times 10$, or 1000, cubic centimeters = 1 cubic decimeter.

And since 10 decimeters = 1 meter, it follows that $10 \times 10 \times 10$, or 1000, cubic decimeters = 1 cubic meter.

The reason is evident from a study of Fig. 156, p. 159.

Note.—$1^{cbm} = 35\frac{1}{3}$ cub. ft., nearly; 1 liter $= 1\frac{1}{20}$ qts., nearly.

3. How many cubic centimeters make 1 cubic meter?

4. *Define the gram and the kilogram.*

A **gram** is the weight of 1^{ccm} of pure cold water.

A **kilogram** is 1000 grams, therefore equal to the weight of 1000^{ccm}, or 1 liter, of pure cold water.

Note.—29 grams = 1 oz., nearly; 1 kilogram $= 2\frac{1}{5}$ lbs., nearly.

5. What is the weight of 1^{cbm} of water?

6. A bottle has a capacity of half a liter. How many grams of water will it contain?

7. The volume of a vessel is 2.4^{cbm}. What is the volume in liters? What weight of water will it hold?

8. A gallon jug contains 231 cub. in. Taking 1 inch as equal to $\frac{2}{5}$ of a centimeter, find what weight of water in grams can be put into the jug.

NOTE.—In the following exercises take π equal to 3.1416.

9. Find the total surface of a square pyramid if a side of the base is 3^{m} and the slant height is 15^{m}.

10. Find the total surface of a cone if its height is 12^{m} and the radius of the base 5^{m}.

11. If the radius of a sphere is 8^{m}, find the area of a zone whose height is equal to half the radius.

12. How high must a box 5^{dm} long and 2^{dm} wide be in order that it may hold exactly 30 liters?

13. A cylindrical pail holding just 1 liter is 18^{cm} high. What is the diameter of its base?

14. A vessel has the shape of a frustum of a cone. The height is 1^{m}, and the inside circumferences of the bases are 5^{m} and 4^{m}. How many kilograms of water will the vessel hold?

15. What is the volume of the largest sphere that can be turned from a wooden cube whose edge is 1^{dm}?

16. What is the volume of the largest sphere that can be turned from a wooden cylinder, if the height of the cylinder and diameter of the base are each equal to 12^{cm}?

17. Marble is 2.7 times as heavy as water. What is the weight of a block of marble 20^{dm} by 16^{dm} by 8^{dm}?

18. An immersed body is buoyed up by the weight of its own volume of water. Iron is $7\frac{1}{4}$ times heavier than water. What will an iron ball whose diameter is 10^{cm} weigh under water?

19. Eighty bullets, equal in size, are dropped into a cylindrical vessel 68^{cm} in diameter containing water. The level of the water rises 2^{cm}. Find the diameter of a bullet.

20. Lead is about $11\frac{1}{4}$ times heavier than water. Find the weight of a lead pyramid if the base is a square whose side is 40^{cm}, and the lateral faces are equilateral triangles.

Lesson 96. Review.

NOTE. — When π is used, take π equal to 3.1416.

1. How many square meters of sheet iron are needed to make a cylindrical pipe 12^{m} and 40^{cm} in diameter?

2. How many liters of water are there in a cubical tank if one edge is 24^{m}, and the water is 4^{m} deep?

3. A square field contains 4 hectares. What will it cost to dig all around the field a ditch 2^{m} deep and 1^{m} wide, the terms being 25 cents per cubic meter of earth thrown out?

4. A Dutch windmill, in the shape of the frustum of a cone, is 12^{m} high. The outer diameters of the bases are 16^{m} and 12^{m}; the inner diameters, 12^{m} and 10^{m}. How many cubic meters of stone were needed to build it?

5. The piston of a pump is 36^{cm} in diameter, and moves through a space 50^{cm}. How many kilograms of water are thrown out by 1000 strokes?

6. A body is placed under water in a cylinder 60^{cm} in diameter. The level of the water is observed to rise 30^{cm}. Find the volume of the body.

7. How much will a brass cylinder weigh under water if its height is 64^{cm}, and the diameter of its base 40^{cm}?

Brass is 8.4 times heavier than water. (See p. 177, No. **20.**)

8. When a body floats in water, the weight of the water displaced is just equal to the weight of the body.

Find the weight of a sphere 20^{cm} in diameter, which floats half under water and half above.

9. A cone of loaf-sugar is 1^{m} high, and the diameter of its base is 40^{cm}. How far from the base must the cone be cut by a plane parallel to the base, in order that the two parts may be equal in volume?

DRAWING EXERCISES.

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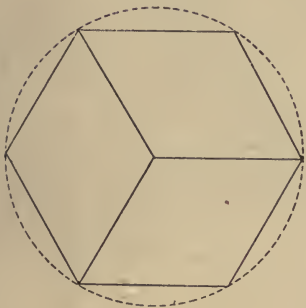

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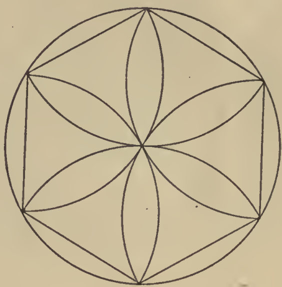

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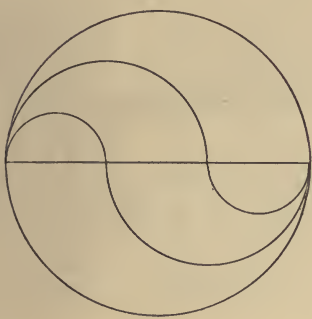

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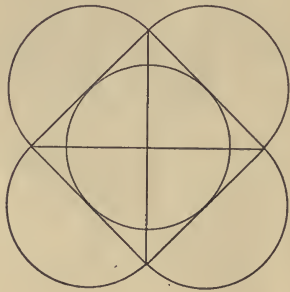

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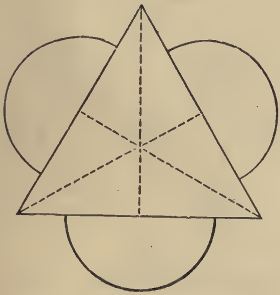

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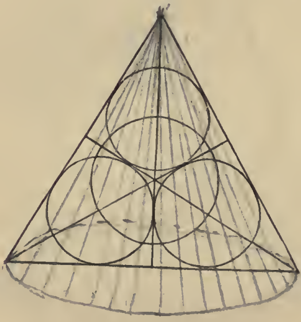

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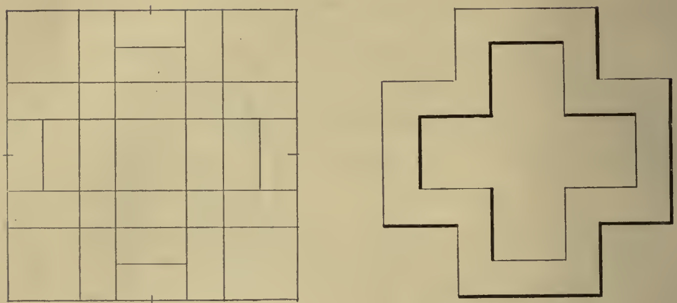

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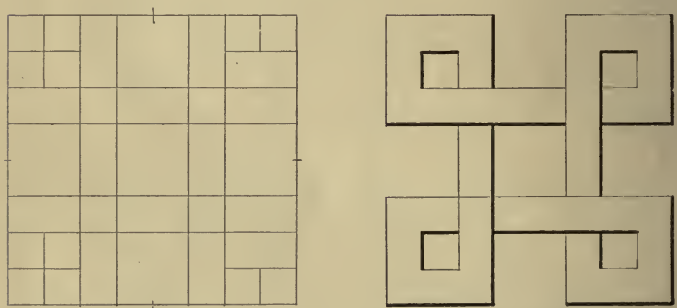

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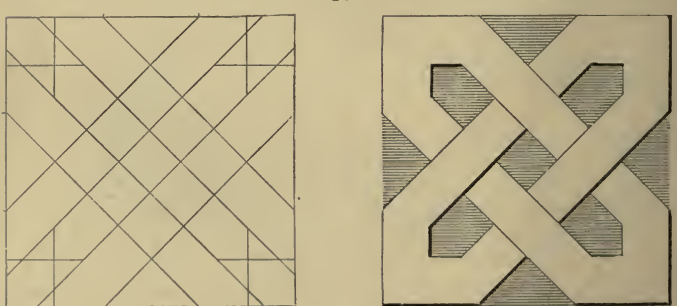

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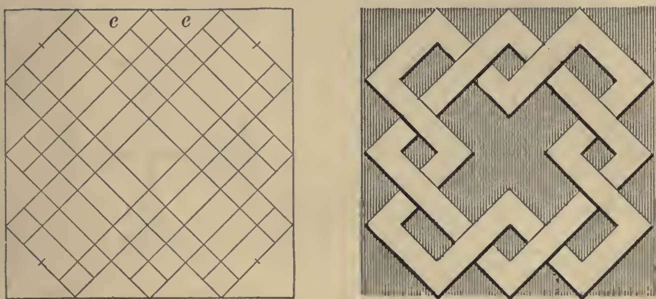

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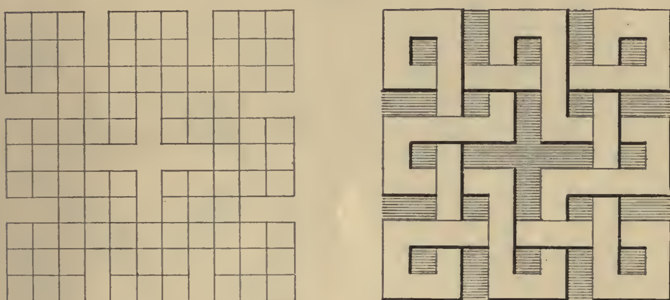

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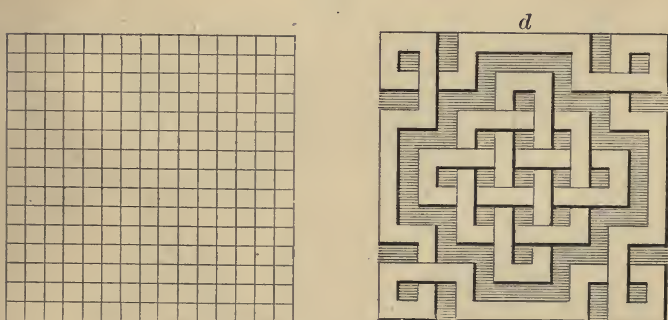
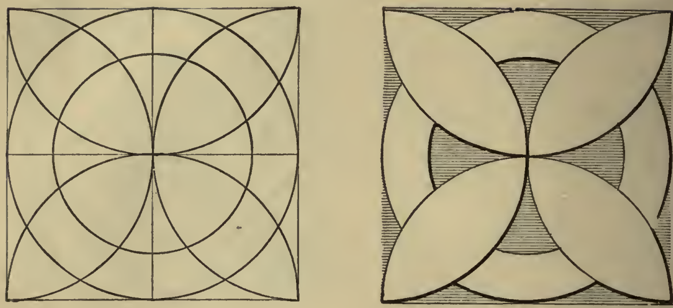


13.



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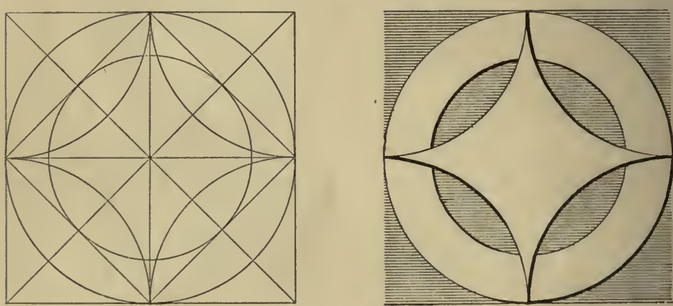

15.